$$\mu(s(Q_0,h)) = \int_{\infty} d\mu(w) \le Ch(w)Ch^{\beta}\int_0 \frac{d\mu(w)}{\left|1-\overline{z_0}w\right|^{\beta}} \le C \quad p(h)h^{\beta-q}$$

Conversely let us take $z_0 = r_0 C^{iQ_0}$ with $r_0 > \frac{3}{4}$, and

Consider $E_0 = S(Q_0,(1-|z_0|))$

$$E_0 = S(Q_0, z^n(1-|z_0|)) - S(Q_0, z^{n-1}(1-|z_0|))$$

An elementary computation shows that for $n \in N$

$$|1-\overline{z_0}w| \ge 2^{n-1}(1-|z_0|) \quad , \quad w \in E_n \quad .$$

Using this estimate and taking $M \in N$ such that $2^M(1-|z_0|) \ge 2$ we have

$$\int_0 \frac{d\mu(w)}{\left|1-\overline{z_0}w\right|^{\beta}} = \sum_{n=0}^{M}\int_{En} \frac{d\mu(w)}{\left|1-\overline{z_0}w\right|^{\beta}}$$

$$\le \frac{\mu(S(Q_0,(1-|z_0|))}{(1-|z_0|)^{\beta}} + \sum_{n=1}^{M}\frac{\mu(S(Q_0,2^{n-1}(1-|z_0|))}{(1-|z_0|)^{\beta}} \le C\sum_{n=1}^{M}\frac{P(2^{n-1}(1-|z_0|))}{(2^{n-1})^{q}(1-|z_0|)q}$$

under the pairing duality given by

$$< f,\phi > = \sum_{n=1}^{m} \alpha_n \alpha_n \quad \text{where}$$

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \quad , \quad \phi(z) = \sum_{n=1}^{m} \alpha_n z^n$$

$$\le C\sum_{n=1}^{M}\int_{2^{n-1}(1-|z_0|)}^{2^n(1-|z_0|)} \frac{\theta(t)}{t^{q+1}}dt \le C\int_{1-|z_0|}^{1}\frac{p(t)}{t^{q+1}}dt = O\left(\frac{p(1-|z_0|)}{1-|z_0|^{q}}\right)$$

Theorem (2-2-3)[1] :-

Let be a finile Borel measure on the disc and a Dini weight such that $P \in b_z$. The following are equivalent

(i) is $p-$ Careson measure

(ii) $|P(\mu)(z)| = O\left(\frac{p(1-|z|)}{1-|z|}\right)$

(iii) $B_1(P) \subset L_1(D,\mu)$ with continuity .

Proof : Lemma (2.2.2) gives (I) of and only if (ii) .

The equivalence between (ii) and (iii) follows from theorem (2.1.11)

Corollary (2.2.4) [1] : Let $\frac{1}{3} < 1$, μ be a finite Borel measure

on , and $\alpha = \frac{1}{p} -$. The following are equivalent :

(i) $H_p(D) \subset L_1(D,\mu)$ with continuity.

(ii) is $\alpha-$ Carleson measure .
proof : use Remark (2.1.11) and observe that $p(t)=t^{\alpha}\in b_2$
Lemma (2.2.5)[1] : Let be a Dini weight such that $p\in b_1$. Let be an analytic function with continuous extenstion at the boundary . The following are equivalent.

(i) $|g'(z)|=O\left(\frac{p(1-|z|}{1-|z|}\right)$ $(|z|\to)$.

(ii) $\int_{|z|=1}\frac{g(\xi)-g(z)}{|1-\bar{\xi}z|^2}d\xi=O\left(\frac{p(1-|z|)}{1-|z|}\right)$ $(|z|\to 1)$.

Proof :-

(ii) (I) obvious from the Cauchy gormula

$$g'(z)=\int_{|\xi|=1}\frac{g(\xi)-f(z)}{(1-\bar{\xi}z)^2}\xi^{-2}d\xi$$.

For the convers , take $\xi=e^{2\pi iQ}$ and $z=|z|e^{2\pi iQ_z}$

Let us first estimate

$$|g(\xi)-g(z)|=|g(\xi)-g(|z|e^{2\pi i t})|+|g(|z|e^{2\pi i t})-g(z)|$$.

On the one hand, using the Dini condition, we have

$$\left|g(\xi)-g\left|z\right|e^{2\pi it}\right|\le\int_{|z|}^{1}\left|g'(Se^{2\pi it})\right|ds\le C\int_{|z|}^{1}\frac{p(1-s)}{1-s}ds\le Cp(1-|z|).$$

On the other hand we Lemma (2.1.7) with $x=c$ and $Q(e^{2\pi it})=g(|z|e^{2\pi it})$ to get

$$|g(|z|e^{2\pi i(Q_z+t)})-g(z)|\le Cp(|t|)$$.

Therefore

$$\int_{|\xi|=1}\frac{|g(\xi)-g(z)|}{|1-\bar{\xi}z|^2}d\xi\le Cp(1-|z|)\int_{|\xi|=1}\frac{d\xi}{|1-\bar{\xi}z|^2}+\int_{1}\frac{|g(|z|e^{2\pi i(Q_z+t)})-g(z)|}{\left|e^{2\pi it}-|z|\right|^2}dt$$

$$\le e\frac{s(1-|z|)}{1-|z|}+C\int_0^1\frac{p(t)}{(1-|z|)^2+2|z|\sin^2(\pi t)}dt$$.

Let us findly use the facts that is nondecreeasing and belongs to to estimate .

$$\int_0^1\frac{p(t)}{(1-|z|)^2+2|z|\sin^2(\pi t)}dt\le C\int_0^1\frac{p(t)}{(1-|z|)^2+Ct^2}dt$$

$$\le C\frac{1}{(1-|z|)^2}\int_0^{1-|z|}\frac{p(t)}{1+\left(\frac{t}{1-|z|}\right)}dt+C\int_{1-|z|}^{1}\frac{p(t)}{t^2}$$

$$\le C\frac{1}{(1-|z|)}\int_0^1\frac{p((1-|z|)s)}{1+S^2}ds+C\frac{p(1-|z|)}{1-|z|}$$

$$\le C\frac{p(1-|z|)}{1-|z|}\left(\int_0^1\frac{1}{1+S^2}ds+1\right)\le C\frac{p(1-|z|}{1-|z|}$$

Theorem (2.2.6) Let p be a Dini weight such that $p \in b_1$ .Let $b \in H^{\infty}(D)$.

The following are equivalent

(i) $H_b = B_1(p) \to H^1$ is bounded .

(ii) $|b'(z)| = O\left(\frac{p(1-|z|}{(1-|z|\log\frac{1}{1-|z|}}\right) \qquad (|z| \to 1)$

Proof :

Denote $F(z) = H_b(K_z)$, and use definition (1-3) to write

$$F'(z)(\xi) = \frac{\xi[\bar{b}(\xi) - \bar{b}(\bar{z})]}{(1-\xi z)^2} - \frac{\bar{b}'(\bar{z})}{1-\xi z} \qquad (12)$$

Let us assume (i) . Applying Corollary (2.1.14) we have

$$\|F'(z)\|_{H^1} = O\left(\frac{P(1-|z|)}{1-|z|}\right) \qquad (13)$$

Now $H_b(f)_{(0)} = \int_{|\xi|=1} \bar{b}(\xi) f(\xi) \frac{d\xi}{\xi}$, so the boundedness of H_b implies

$$\left|\int_{|\xi|=1} \bar{b}(\xi)\frac{d\xi}{\xi}\right| \le \|H_b(f)\|_{H_1} \le C\|f\|_{B_1(p)} .$$

This implies $b \in (B_1(p))^{*}$, which coincides with Λ_δ
According to Corollary (2.1.14).
ffence we can apply Lemma (2.2.5) to obtain

$$\int_{|\xi|=|z|} \frac{|b(\xi) - b(z)|}{|1-\xi\bar{z}|^2} d\xi = O\left(\frac{p(1-|z|)}{1-|z|}\right) , \qquad (|z| \to 1) \qquad (14)$$

from we have

$$|b'(z)| \int_{|\xi|=1} \frac{d\xi}{|1-\xi\bar{z}|} \le \|F'(\bar{z})\|_{Hi} + \int_{|\xi|=1} \frac{|b(\xi) - b(z)|}{|1-\xi\bar{z}|^2} d\xi$$

using $\int_{|\xi|=1} \frac{d\xi}{|1-\xi\bar{z}|} = O\left(Log(\frac{1}{1-|z|})\right)$, (13) and (14) we get (ii).

Let us now assume (ii), From Theorem (2-1-11) we have to show (13)- using (12) again we have :

$$\|F'(z)\|_{H1} \le |b'(z)| \int_{|\xi|=1} \frac{d\xi}{|1-\xi\bar{z}|} + \int_{|\xi|=1} \frac{|b(\xi) - b(z)|}{|1-\xi z|^2} dz$$

Now we estimat (13) follows easily by using (ii) and Lemma (2-2-5) .

Corollary (2-2-7) [1]:

Let $\frac{1}{2}<p<1$ and let $b\in H^{-}$. Then $H_b:\to H^1$ if and only if

$$|b'(z)| = O\left(\frac{1}{(1|Z|)\frac{1}{P} - 2LOG\frac{1}{1-|Z|}}\right)$$

Pproposition (2-2-8) [1]: let be a weight function. Let $\phi D\to D$ be analytic and $O<P<\infty$, then

$$\int_D \frac{p(1-|z|)}{1-|z|}|f(\phi(z)|^p dA(z) \le \frac{1+|\phi(O)|}{1-|\phi(O)|}\int_O \frac{p(1-|z|)}{1-|z|}|f(z)|^p dA(z)$$.

Proof :-

Let $a=\phi(O)$ and consider $\psi(z)=\phi_a(\phi(z))$ where

$$\phi_a(w)=\frac{w-a}{1-\bar{a}w}$$

$\psi D\to D$, $\psi(0)=0$ and $\phi(z)=\phi_a(\psi(z))$ we can we little wood subordination principle to get

$$\int_o |f(\phi(re^{2nie)}|^p d\vartheta \le \int_0^1 |f(\phi_a(re^{2\pi i\vartheta}))|^p de$$

making the change of variable $re^{2\pi it}=\phi_a(re^{2\pi it})$ one gets

$$\int_0^1 |f(\phi(re^{2\pi it}))|^p de \le (1-|a|^2)\int_0^1 \frac{|f(re^{2\pi it})|^p dt.}{|1-\bar{a}re^{2\pi it}|^2}$$

there fore

$$\int_0^1 |f(\phi(re^{2\pi it}))|^p \le \frac{1+|\phi(O)|}{1-|\phi(O)|}\int_0^1 |f(re^{2\pi it})|^p dt$$

hence multiplying :

$\frac{p(1r)}{1-r}$ and integrating one get

$$\|C_C(f)\|_{B_p(f)} \le \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{\frac{1}{p}}\|f\|_{B_p(f)}$$

remark (2-2-9)[1] : We have included the proof , al thought it is very elementary, because the change of variable at the right moment can improve the estimate of the norm

[where $\|C_\phi\|_{\mathcal{L}(B_p,B_p)}$ is estimated by $\left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{2}$] .

we are mainly concerned with analyzing when hankel aperators improve the condition of integrableility. To this purpose we need the following notion

Given $\phi D\to D$ analytic, let us consider the following image measure