GENERALIZED FUNCTIONS IN INFINITE DIMENSIONAL ANALYSIS

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A thesis Submitted in Partial Fulfillment for the Degree of M.Sc. in Mathematics

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2003

CHAPTER 1

BACKGROUND FACTS

In this chapter we present some basic concepts of the L^p-spaces and Tensor products .

Radon-Nikodgm Theorem (1.1) [3] :-

If (X , S , μ) is a σ-finite measure space and v is σ-finite measure on S such that v << μ then there exists a finite valued non-negative measurable function f on X such that for each E ∈ S , v(E) = $\int_E$ f dμ . Also f is unique in the sense that if v(E) = $\int_E$ g dμ for each E ∈ S then f = g a.e(μ) .

Proof :-

We show first that we may assume μ(X) < ∞ and μ (X) < ∞ and v(X) <∞ and so we suppose the result has been proved for that case . We have $X = \bigcup_{n=1}^{\infty} A_n$, μ(A_n) < ∞ and $X = \bigcup_{m=1}^{\infty} B_m$, v(B_m) < ∞ and { A_n} , { B_m} may be supposed to be sequences of disjoint sets . So setting $X = \bigcup_{n,m=1}^{\infty} (A_n \cap B_m)$ we obtain X as the union of disjoint sets on which both μ and v are finite , say $X = \bigcup_{n=1}^{\infty} X_n$.

Let S_n = { E ∩ X_n : E ∈ S } .

A σ-algebra over X_n , and considering μ and v restricted to S_n we obtain a function F_n such that if E ∈ S_n , v(E) = $\int_E f_n \, d\mu$. So if A ∈ S , $A = \bigcup_{n=1}^{\infty} A_n$ where A_n ∈ S_n , defining F = f_n on X_n gives a measurable function on X .and

$$v(A) = \sum_{n=1}^{\infty} \int_{A_n} f_n \, d\mu = \int_A f \, d\mu$$

and the general case follows .

So we need only with finite measures . Let K be the class of non-negative functions measurable with respect to μ and satisfying $\int_E f \, d\mu \leq v(E)$ for all E $\in$ S . Then K is non-empty as $0 \in$ K .

Let α = sup { $\int f \, d\mu$: $f \in K$ } and let $\{f_n\}$ be a sequence in K such that $\lim \int f_n \, d\mu = \alpha$.

If B is any fixed measurable set , n a fixed positive integer and $g_m = \max(f_1 , \ldots f_m)$ then , by induction , B may be written as a union of disjoint measurable sets B_i , i = 1 , .. , n , such that $g_n = f_i$ on B_i , i = 1 , .. , n .

For let n=2 and let $B_1 = \{x : x \in B , f_1(x) \geq f_2(x) \}$, $B_2 = B - B_1$, then $B = B_1 \cup B_2$ has the desired property .

Supposing the result true for n , let $g_{n+1} = \max(f_1 , \ldots f_{n+1}) = \max(g_n , f_{n+1})$, so $B = f_n \cup B_{n+1}$ where $g_{n+1} = f_{n+1}$ on B_{n+1} and $g_{n+1} = g_n$ on f_n and $f_n \cap B_{n+1} = \phi$.

But then by the inductive hypothesis we have $f_n = \bigcup_{i=1}^{n} B_i$ and $g_{n+1}(x) = f_i(x)$ for $x \in B_i$, i = 1 , … , n+1 .

Then

$$\int_B g_n \, d\mu = \sum_{i=1}^{n} \int_{B_i} f_i \, d\mu \leq \sum_{i=1}^{n} v(B_i) = v(B)$$

as each $f_i \in K$.

The Lebegue monotone convergence theorem implies that

$$\int_E f_0 \, d\mu = \lim \int_E g_n \, d\mu \leq v(E)$$

So $f_0 \in K$. Hence

$$\alpha \geq \int f_0 \, d\mu \geq \int g_n \, d\mu \geq \int f_n \, d\mu \quad .$$

So $\alpha = \int f_0 \, d\mu$

Since $\int f_0 \, d\mu \leq v(x) < \infty$, there exists a finite-valued measurable function f , also non-negative , such that $f = f_0$ a.e(μ) .

We will show now that if $v_0(E) = v(E) - \int_E f \, d\mu$, then $v_0(E) = 0$. If v_0 is not identically zero on S , let $C \in S$ and $v_0(C) > 0$. Then for a suitable ε .

$0 < \varepsilon < 1$, $(v_0 - \varepsilon\mu)$ (C) > 0 . We can find A such that $(v_0 - \varepsilon\mu)$ (A) > 0 where A is a positive set with respect to $v_0 - \varepsilon\mu$. Also $\mu(A) > 0$ for otherwise , as $v << \mu$. We would have $v(A) = 0$ and hence $(v_0 - \varepsilon\mu)(A) = 0$ so , for $E \in S$

$$\varepsilon\mu(E \cap A) \le v_0(E \cap A) = v(E \cap A) - \int_{E \cap A} f \, d\mu \quad .$$

Hence if $g = f + \varepsilon \chi_A$, for each $E \in S$ we have

$$\int_E g \, d\mu = \int_E f \, d\mu + \varepsilon\mu(E \cap A) \le \int_{E-A} f \, d\mu + v(E \cap A) \le v(E)$$

and so $g \in K$. But $\int g \, d\mu = \int f \, d\mu + \varepsilon\mu(A) > \alpha$, contradicting the maximality of α . So $v_0 = 0$ on S .

Then by the definition of v_0 , f has the desired properties . Let g also have these properties.

So for $E \in S$, $\int_E (f - g) \, d\mu = 0$.

And taking $E = \{ x : f(x) > g(x) \}$. We get $f \le g$.

And similarly $f \ge g$ a.e . So f is unique in the sense stated .

Definition (1.2) [3] :-

If (X , S , μ) is a measure space and p > 0 , we define $L^p(X, \mu)$, or more briefly $L^p(\mu)$, to be the class of measurable function $\{ f : \int |f|^p \, d\mu < \infty \}$, with the convention that any two functions equal almost every where specify the same element of $L^p(\mu)$. On the real line , if X = (a , b) and μ is Lebesgue measure we will write $L^p(a, b)$ for the corresponding space .

Strictly , the elements of the space $L^p(\mu)$ are not functions but classes of functions such that in each class any two functions are equal almost every where . Since any two functions equal almost every where have the same integrals over each set of S the distinction is not important for many purposes.

We write $f \in L^p(\mu)$ as an abbreviation for :

F measurable and $\int |f|^p \, d\mu < \infty$.

To ask , however , for the value of an element of $L^p(\mu)$ at a particular point is , in general , meaningless .

If p=1 , we obtain the integrable functions which we denoted by $L(X , \mu)$.

We will use the alternative notation $L^1(\mu)$ if we wish to emphasize that the above convention applies .

Definition (1.3) [3] :-

Let $f \in L^p(\mu)$, then the L^p-norm of f , denoted by $\| f \|_p$ is given by

$$\left(\int | f |^p \, d\mu < \infty \right)^{1/p} .$$

Theorem (1.4) [3] :-

Let f , g $\in$ $L^p(\mu)$ and let a , b be constants , then af – bg $\in$ $L^p(\mu)$.

Proof :-

Clearly , if f $\in$ $L^p(\mu)$, and af $\in$ $L^p(\mu)$ for each constant a . Also if f , g $\in$ $L^p(\mu)$ we have f + g $\in$ $L^p(\mu)$ since

$$| f + g |^p \leq 2^p \max (| f |^p , | g |^p) \leq 2^p (| f |^p + | g |^p)$$

giving the result .

If f is the element of $L^p(\mu)$ containing the function F and G that containing g , then if we define aF +bG as the element containing af + bg , this is easily seen to be independent of the particular f in F and g in G . Hence these theorems shows that $L^p(\mu)$ is a vector space . We may use , accordingly , the same notation for elements of $L^p(\mu)$ and for function .

Definition (1.5) [3] :-

If (X , S , μ) is a measure space we define $L^\infty(X , \mu)$ or $L^\infty(\mu)$, to be the class of measurable functions { f : ess sup $|f| < \infty$ } , with the same convention as definition (1.2) . Corresponding to definition (1.3) we have the L^∞-norm :

$$\| f \|_{\infty} = \text{ess sup} \, | f | \quad .$$

Example (1.6) [3] :-

Show that $L^{\infty}(X , \mu)$ is a vector space over the real numbers .

Solution :-

Ess sup $|af + bg| \le |a|$ ess sup $|f|$ + $|b|$ ess sub $|g|$.

Now we show Holder's inequality .

Theorem (1.7) [3] :-

Let $i < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu)$, $g \in L^q(\mu)$ then

$fg \in L^1(\mu)$ and

$$\int f \, g \, d\mu \le \left(\int | f |^p \, d\mu \right)^{1/p} \subset \left(\int | g |^q \, d\mu \right)^{1/q} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Proof :-

If $a > 0$, $b > 0$

$$a^{1/p} \, b^{1/q} \le \frac{a}{p} + \frac{b}{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Now , if $\| f \|_p = 0$ or $\| g \|_q = 0$ then $fg = 0$ a.e and (1) is trivial . If $\| f \|_p > 0$ and $\| g \|_q > 0$ write

$$a = \frac{| f |^p}{(\| f \|_p)^p} \quad , \quad b = \frac{| g |^q}{(\| g \|_q)^q}$$

in (2) to get

$$\frac{| f g |^p}{\| f \|_p \| g \|_q} \le \frac{1}{p} \frac{| f |^p}{(\| f \|_p)^p} + \frac{1}{q} \frac{| g |^q}{(\| g \|_q)^q} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

The right – hand side is integrable so $fg \in L^1(\mu)$. Integrate both sides to get $\| f g \|_1 \le \| f \|_p \| g \|_q$ which is (1) .

The most important special case of Theorem (1.7) occurs when $p = q = 2$ and is called the Schwartz inequality .

Example (1.8) [3] :-

If f and g are non-negative measurable functions show that equality occurs in Holder's inequality if and only if

$$s f^p + t g^q = 0 \quad \text{a.e} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

for some constants s , t not both zero .

Solution :-

Suppose equality occurs (1) . Then if $\|f\|_p > 0$ and $\|g\|_q > 0$ we must have equality in (3) a.e .

But in (2) equality implies a = b so that $f^p = \alpha g^q$ a.e , where $\alpha > 0$, i.e(4) . If say , $\|f\|_p = 0$ then f = 0 a.e and (4) holds . Conversely , if (4) holds we may substitute into (1) to eliminate f or g and we obtain equality .

Theorem (1.9) [3] :-

Every inner product space V is a normd linear space with the norm $\|x\| = (x,x)^{\frac{1}{2}}$.

Proof :-

Since V is a vector space , we need only verify that $\|\cdot\|$ has all the properties of a norm .

All of these properties can be proved , except the triangle inequality .

Suppose x , y $\in$ V Then

$$\|x+y\|^2 = (x,x) + (x,y) + (y,x) + (y,y)$$
$$= (x,x) + 2\mathrm{Re}(x,y) + (y,y)$$
$$\leq (x,x) + 2\,|(x,y)| + (y,y)$$
$$\leq (x,x) + 2(x,x)^{\frac{1}{2}} + 2(y,y)^{\frac{1}{2}} + (y,y)$$

by the Schwartz inequality . Thus

$$\|x+y\|^2 = (\|x\| + \|y\|)^2$$

which proves the triangle inequality .

Definition (1.10) [3] :-

A class of subsets of an arbitrary space X is said to be σ-algebra (sigma algebra) or , by some authors , a σ-field , if X belongs to the class and the class is closed under the formation of countable unions and of complements .

Definition (1.11) [3] :-

If in Definition (1.10) we consider only finite unions we obtain an algebra (or a field) .

We will denote by M the class of Lebesgue – measurable sets .

Theorem (1.12) [3] :-

The class M is σ-algebra .

Proof :-

From definition above c ∈ M , and the symmetry in definition between E and cE implies that if E ∈ M then cE ∈ M . So if $\langle E_i \rangle$ is a sequence of sets of M it remains to be shown that $\bigcup_{i=1}^{\infty} E_i \in M$. Now if A , B ∈ M then A – B = c (cA ∪ B) ∈ M . Also , by induction , gives that the union of any finite collection of sets of M in M . Now we may write

$$\bigcup_{i=1}^{\infty} E_i = E_1 \cup (E_1 \cup E_2) \cup \ldots \cup \left(\bigcup_{i=1}^{n} E_i - \bigcup_{i=1}^{n-1} E_i \right) \cup \ldots$$

a union of disjoint measurable sets and hence without loss of generality we may assume the original sets $\langle E_i \rangle$ disjoint . Then for each set A , since $\bigcup_{i=1}^{n} E_i$ is measurable

$$m^*(A) = m^*\left(A \cap \bigcup_{i=1}^{n} E_i \right) + m^*\left(A \cap \bigcup_{i=1}^{n} E_i \right)$$

But

$$C \bigcup_{i=1}^{n} E_i = \left(\bigcup_{i=n+1}^{\infty} E_i \right) \cup C \bigcup_{i=1}^{\infty} E_i$$

So

$$m^*(A) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i \right) + m^*\left(A \cap C \bigcup_{i=1}^{\infty} E_i \right)$$

$$\geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*\left(A \cap C \bigcup_{i=1}^{\infty} E_i\right)$$

for all n . So

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*\left(A \cap C \bigcup_{i=1}^{\infty} E_i\right)$$

$$\geq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*\left(A \cap C \bigcup_{i=1}^{\infty} E_i\right)$$

Then the result follows .

In the following we describe some aspects of the theory of tensor production of operators on Hilbert spaces . Let A and B be densely defined operators on Hilbert spaces H respectively . We will denote by D(A) ⊗ D(B) the set of finite linear combinations of vectors of the form $\Phi \otimes \Psi$ where $\Phi \in$ D(A) and $\Psi \in$ D(B) .

D(A) ⊗ D(B) is dense in ⊗ . We defined A ⊗ B on D(A) ⊗ D(B) by

$$(A \otimes B)(\Phi \otimes \Psi) = A\Phi \otimes B\Psi \quad \text{………………………… (4)}$$

and extend by linearity .

Proposition (1.13) [4] :-

The operator A ⊗ B is well defined . Further , if A and B are closable , so is A ⊗ B .

Proof :-

Suppose that $\sum c_i \Phi_i \otimes \psi_i$ and $\sum d_j \Phi'_j \otimes \psi'_j$ are two representation of the same vector f ∈ D(A) ⊗ D(B) using Gram-Schmidt orthogonalization we can obtain bases $\{\eta_k\}$ and $\{\theta_t\}$ for the spaces spanned by {Φi} $\{\Phi_i\} \cup \{\Phi'_j\}$ and $\{\psi_i\} \cup \{\psi'_j\}$ respectively so that $\eta_k \in$ D(A) and $\theta_t \in$ D(B) . $\Phi_i \otimes \Psi_i$ and or $\Phi'_j \otimes \psi'_j$ can be expressed

$$\left.\begin{aligned} \Phi_i \otimes \psi_i &= \sum \alpha^i_{kt}\, \eta_k \otimes \theta_t \\ \Phi'_j \otimes \psi'_j &= \sum \beta^j_{kt}\, \eta_k \otimes \theta_t \end{aligned}\right\} \quad \text{…………………………}$$

(5)

Since the two expressions for f give the same vector $\sum_i c_i \alpha_{kt}^i = \sum_j d_j \beta_{kt}^j$ for each pair <k , t> .

Thus ,

$$(A \otimes B) \sum C_i (\Phi_i \otimes \psi_i) = \sum_{kt} \left(\sum_i C_i \alpha_{kt}^i \right) (A\eta_k \otimes B\theta_t)$$

$$= \sum_{kt} \left(\sum_j d_j \beta_{kt}^j \right) (A\eta_k \otimes B\theta_t)$$

$$= (A \otimes B) \sum d_j (\Phi_j \otimes \psi_j)$$

So A ⊗ B is well defined .

If g is any vector in D(A*) ⊗ D(B*) , then

(A ⊗ B f ,g) = (f , A* ⊗ B*g) so

$$D(A^*) \otimes D(B^*) \subset D((A \otimes B)^*)$$

If A and B are closable D(A*) and D(B*) are dense .

Therefore , in that case (A ⊗ B)* is densely defined which proves that A ⊗ B is closable .

Similarly , if A and B are closable then A ⊗ I + I ⊗ B , defined on D(A) ⊗ D(B) , is closable .

Definition (1.14) [4] :-

Let A and B closable operators on Hilbert spaces H_1 and H_2 . The tensor product of A and B is closure of the operator A ⊗ B defined on D(A) ⊗ D(B) .

We will denote the closure by A ⊗ B also .

Usually A + B will denoted the closure of A ⊗ I + I ⊗ B on D(A) ⊗ D(B) .

Proposition (1.15) [4] :-

Let A and B bounded operators on Hilbert spaces H_1 and H_2 . Then $\| A \otimes B \| = \| A \| \, \| B \|$.

Proof :-

Let $\{\Phi_k\}$ and $\{\Psi_k\}$ be orthonormal bases for H_1 and H_2 and suppose $\sum C_{kL}\Phi_k \otimes \psi_L$ is a finite sum .

Then

$$\|(A\otimes I)\sum C_{kL}(\Phi_k\otimes\psi_L)\|^2 = \sum_L \Big\|\sum_k C_{kL} A\Phi_k\Big\|^2$$

$$\leq \sum_L \|A\|^2 \sum_k |C_{kL}|^2$$

$$= \|A\|^2 \, \|\sum C_{kL}(\Phi_k\otimes\psi_L)\|^2$$

Since the set of such finite sums is dense in $H_1 \otimes H_2$ we conclude that $\|A\otimes I\| = \|A\|$. Thus

$$\|A\otimes B\| = \|(A\otimes I)(I\otimes B)\| \le \|A\| \, \|B\| \, .$$

Conversely , given $\varepsilon > 0$, there exist unit vectors $\Phi \in H_1$, $\psi \in H_2$ so that $\|A\Phi\| \ge \|A\| - \varepsilon$ and $\|B\psi\| \ge \|B\| - \varepsilon$. Then

$$\|(A\otimes B)(\Phi\otimes\psi)\| = \|A\Phi\| \, \|B\psi\|$$

$$\ge \|A\| \, \|B\| - \varepsilon\|A\| - \varepsilon\|B\| + \varepsilon^2$$

Since $\varepsilon>0$ is arbitrary $\|A\otimes B\| = \|A\| \, \|B\|$ which concludes the proof .

We remark that propositions (1.13) (1.15) have natural generalizations on arbitrary finite tensor products of operators . This can be proven directly or by using the associativity of the tensor product of Hilbert spaces . We turn now to questions of self-adjointness and spectrum . Let $\{A_k\}_{k=1}^{N}$ be a family of operators , A_k self-adjoint on H_k . We will denote the closure of $I_1 \otimes \dots \otimes A_k \otimes \dots I$ on $D = \otimes D(A_k)$ by Ak also . Let $p(x_1 , \dots , x_N)$ be a polynomial with real coefficients of degree n_k in x_k .

Then , the operator $p(A_1 , \dots , A_N)$ makes sense on $\otimes_k D(A^{nk})$ since $D(A^{nk}) \subset$

$D(A')$ for all $L \le n_n$.

In fact , P is essentially self-adjoint on that domain .

Theorem (1.16) [4] :-

Let A_n be a self – adjoint operator on H_k . Let p (x_1 , … , x_n) be a polynomial with real coefficients of degree n_k in the kth variable and suppose that D_k^e is domain of essential self – adjointness for $A_k^{n_k}$.

Then

(1) p (A_1 , … , A_N) is essentially self – adjoint on

$$D^e = \overset{N}{\underset{k=1}{\otimes}} D_k^e$$

(2) The spectrum of $\overline{p(A_1, \ldots, A_N)}$ is the closure of the range of p on the product of the spectra of the A_k .

That is

$$\overline{\sigma(A_1, \ldots, A_n)} = \overline{p(\sigma(A_1), \ldots, \sigma(A_N))}$$

Corollary (1.17) [4] :-

Let A_1 , … , A_N be self – adjoint operators on H_1 , … , H_N and suppose that , for each k , D_k is domain of essential self-adjointness for A_k . Then

(1) The operators = $A_1 \otimes \ldots \otimes A_N$ and $A_\varepsilon = A_1 + \ldots + A_N$ are essentially self-adjoint on $D = \otimes_{k=1}^{N} D_k$.

(2) $\sigma(A_\pi) = \overline{\prod_{k=1}^{N} \sigma(A'_k)}$ and $\sigma(A'_\varepsilon) = \overline{\sum_{k=1}^{N} \sigma(A'_k)}$.

Example (1.18) [4] :-

Suppose that v(x) is a potential so that H_1 = - $_k$ + v(x) is essentially self-adjoint on (R^3) . Then H_2 = - $_x$ + v(x) + - $_y$ + v(y) is essentially self-adjoint on the set of finite sums of products Φ(x) ψ(y) , with Φ, ψ ∈ (R^3). Further $\sigma(H_2) = \overline{\sigma(H_1) + \sigma(H_1)}$.