Sudan University of Science and Technology
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Generalized and Sharp Trace Gagliardo-Nirenberg-Sobolev Inequalities with Constructive Description of Hardy-Sobolev Spaces

متباينات جاقلياردو – نيرنبيرج – سوبوليف المعممة والأثر القاطع مع وصف البناء لفضاءات هاردي – سوبوليف

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

Acknowledgements

I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

We study the functions of bounded mean ascillation area inequality clavceterigation of Bergman spaces in the unit ball of the complex space, Cauchy-type integrals in several complex variables and best constant in Sobolev trace ine qualities on the half-space. We characteribe amass-transportation approach Gagliardo-Niverberg type inequality with lvitical and sharp Sebder inequalities. We obtain the generalized Gagliardo-Niverberg inequalities using weak lebesgue space, Lorentz spaces ltalder spaces and factienal Soboler spaces. The censtructire description of Hardy-Soboler spaces on ctranglyconver demans in the complex spaces with new shap and sharp trace Gagliardo-Niverberg Sahaler ine pqalities for convex cones and improved Borell-Brascomp-lieb inequality are considered.

الخلاصة

قمنا بدراسة دوال التذبذب المتوسط المحدود وتشخيصات تكامل المساحة لفضاءات بيرجمان في كرة الوحدة للفضاء المركب والتكاملات نوع – كوشي في المتغيرات المركبة المتعددة والثابت الأفضل في متباينات أثر سوبوليف على نصف الفضاء. تم تشخيص مقاربة تنقل الكتلة والمتباينة نوع جاقلياردو – نيرنبيرج مع متباينات سوبوليف الحرجة والقاطعة. تم الحصول على متباينات جاقلياردو – نيرنبيرج المعممة مستخدمين فضاءات لبيغ الضعيفة وفضاءات لورنتز وفضاءات هولدر وفضاءات سوبوليف الكسرية. قمنا باعتبار وصف البناء لفضاءات هاردي – سوبوليف على المجال المحدب القوي في الفضاءات المركبة مع متباينات جاقلياردجو – نيرنبيرج – سوبوليف القاطعة والأثر القاطع لأجل المخاريط المحدبة ومتباينة بوريل – براسكومب – ليب المحسنة.

Introduction

We show an inequality expresses a function can approximated in L^1 mean. We consider the generalized Gagliardo–Nirenberg inequality in $\mathbb{R}^n$ in the homogeneous Sobolev space $\dot{H}^{s,r}(\mathbb{R}^n)$ with the critical differential order $s = n/r,$ which describes the embedding such as $L^p(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{r},r}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ for all q with $p \leqq q < \infty$, where $1 < p < \infty$ and $1 < r < \infty$. We establish the optimal growth rate as $q \to \infty$ of this embedding constant. In particular, we realize the limiting end-point $r = \infty$ as the space of BMO in such a way that $\|u\|_{L^q}(\mathbb{R}^n) \leqq C_n q \|u\|_{L^p}^{\frac{p}{q}}(\mathbb{R}^n)\ \|u\|_{\mathrm{BMO}}^{1-\frac{p}{q}}$ with the constant C^n depending only on n.

We establish real-variable type maximal and area integral characterizations of Bergman spaces in the unit ball of C^n. The characterizations are in terms of maximal functions and area functions on Bergman balls involving the radial derivative, the complex gradient, and the invariant gradient. We present the theory of Cauchy–Fantappié integral operators, with emphasis on the situation when the domain of integration, D, has minimal boundary regularity. Among these operators we focus on those that are more closely related to the classical Cauchy integral for a planar domain, whose kernel is a holomorphic function of the parameter $z \in D$.

We show that mass transportation methods provide an elementary and powerful approach to the study of certain functional inequalities with a geometric content, like sharp Sobolev or Gagliardo–Nirenberg inequalities. Using a mass transportation method, we study optimal Sobolev trace inequalities on the half-space.

Using elementary arguments based on the Fourier transform we show that for $1 \leq q < p < \infty$ and $s \geq 0$ with $s > n(1/2 - 1/p)$, if $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$ and there exists a constant cp,q,s such that $\|f\|_{L^p} \leq c_{p,q,s}\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta}$, where $1/p = \theta/q + (1-\theta)(1/2 - s/n)$. In particular, in $\mathbb{R}^2$ we obtain the generalised Ladyzhenskaya inequality $\|f\|_{L^4} \leq c\|f\|_{L^{2,\infty}}^{\frac{1}{2}}\|f\|_{\dot{H}^1}^{\frac{1}{2}}$. We also show that for $s = n/2$ and $q > 1$ the norm in $\|f\|_{\dot{H}^{\frac{n}{2}}}$ can be replaced by the norm in BMO. We show some generalized Gagliardo–Nirenberg interpolation inequalities involving the Lorentz spaces $L^{p,\alpha}$, BMO and the fractional Sobolev spaces $W^{s,p}$, including also $\dot{C}^{\eta}$ Hölder spaces.

We use the method of pseudoanalytic continuation to obtain the characterization of Hardy-Sobolev spaces on strongly convex domains in terms of polynomial approximations.

We present a simple direct proof of the classical Sobolev inequality in $\mathbb{R}^n$ with best constant from the geometric Brunn–Minkowski–Lusternik inequality. We propose a new Borell–Brascamp–Lieb inequality that leads to novel sharp Euclidean inequalities such as Gagliardo–Nirenberg–Sobolev inequalities in $\mathbb{R}^n$ and in the halfspace $\mathbb{R}^n_+$.

The Contents

Chapter 1
Remarks on Gagliardo–Nirenberg Type Inequality

We make it clear that the well known John–Nirenberg inequality is a consequence of our estimate. Furthermore, it is clarified that the L^∞-bound is established by means of the BMO-norm and the logarithm of the $\dot{H}^{s,r}$-norm with $s > n/r$, which may be regarded as a generalization of the Brezis–Gallouet–Wainger inequality.

Section (1.1): Functions of Bounded Mean Oscillation

We show an inequality, which has been applied by J. Moser. The inequality expresses that a function, which in every subcube C of a cube C_0 can be approximated in the L^1 mean by a constant a_t with an error independent of C, differs then also in the L^{p} mean from a_C in C by an Tor of the same order of magnitude. The measure of the set of points in C, where the function differs from a_c by more than an amount a decreases exponentially as σ increases. We apply Lemma (1.1.2) to denve a result of Weiss and Zygmund [4], and we present an extension of Lemma (1.1.1).

Since for every contiiuooely differentiable function (s) , vanishing at the origin,

$$\int_{C_0} f\left(|u-a_{C_0}|\right)dx = \int_0^\infty \mu(s)df(s)\,,$$

inequality implies that u belongs to L^{p} for every finite $p \geqq 1$, and, in fact for $b' < K^{-1}b$ the function $e^{v'1\mathrm{u}-a}{}_{C_\varphi}^{\int}$ is integrable and

$$\int_{C_0} e^{b'|\mathrm{u}-a_{C_0}|}\,dx \leqq \left(1+\frac{Bb'}{K^{-1}b-b'}\right)m(C_0). \tag{1}$$

A function satisfying for every subcube C of C_0 , for come constant a_G, will be said to have "mean oscillation $\leqq K$ in C_0' Taking for a_C the average of u in C we always have

$$\left(\frac{1}{m(C\rangle}\int_C |u-a_C|dx\right)^2 \leqq \frac{1}{n(C)'}\int_C |u(x)-a\mathrm{d}^2dx$$

$$=\frac{1}{2}\big(m(C)\big)^{-2}\int_C dx\int_C dy|u(x)-u(y)|^2.$$

In particular u has mean oscillation $\leqq K$, if u is bounded and its oscillation $|u(x)-u(y)|$ does not exceed the value $\sqrt{2}K$ in C_0.

Boundedness of u is not necessary for boundedness of its mean oscillation. L^t gindeed $u(x)$ be any integrable function in C_0 with the property that we can associate with every subcube C a value a_C such that the subset S_σ of C, where

$$|u-a_C| \geqq \sigma,$$

has measure

$$\mu(\sigma) \leqq Be^{-b\sigma}m(C)\ for\ \sigma > 0.$$

Then

$$\int_C |u-a_0|dx = \int_0^\infty \mu(\sigma)d\sigma \leqq \frac{B}{b}m(C)$$

so that the mean oscillation of u does not exceed B/b. Take now for u the function $\log|x-y|$, where y is fixed. Let C be any cube of side h, and let ξ and η be points of C for which

$$|\xi-y| = \max_{x\in C}|x-y|, |\eta-y| = \min_{x\in C} |x-y|.$$

Take $a_C = \log|\xi-y|$. Then for x in C

$$|u(x)-a_C| = \log\frac{|\xi-y|}{|x-y|}.$$

S_σ is the subset of C lying in the sphere

$$|x-y| \leqq |\xi-y|e^{-\sigma}.$$

If S_σ is not empty. it must contain η, so that

$$|\xi-y|e^{-\sigma} \geqq |\eta-y| \geqq |\xi-y|-|\xi-\eta| \geqq |\xi-y|-\sqrt{n}h.$$

Thus

$$|\xi-y| \leqq \frac{\sqrt{n}h}{1-e^{-\sigma}}.$$

It follows that S_σ is contained in the sphere

$$|x-y| \leqq \frac{\sqrt{n}h}{e^\sigma-1}$$

and that its measure $\mu(\sigma)$ does not exceed

$$\left(\frac{\sqrt{n}\omega}{e^\sigma-1}n\right)^n m(C),$$

where the volume of the unit sphere in n-space is denoted by $(oj_n)^n$. Since also $\mu(\sigma) \leqq m(C)$ for all $\sigma \geqq 0$, we find that

$$\mu(\sigma) \leqq (1+\sqrt{n}()_n)^n e^{-n\sigma}n(C) = Be^{-b\sigma}m(C)\ for\ \sigma > 0,$$

where B and b do not depend on C. This proves that $\log|x-y|$ is of bounded mean oscillation in every cube C_0. The same holds then for any function $u(x)$ of the form

$u(x) = \int \zeta(y)\log|x-y|dy$ with $\int|\zeta(y)|dy < \infty$. Lemma (1.1.1) will be derived from

Lemma (1.1.1)[1]. Let u(x) be integrable in a cube C_0 and assume that there is a constant κ such that for every parallel subcube C we have

$$\frac{1}{m(C)}\int_C |u-u_C|dx \leqq \kappa, \tag{2}$$

where u_C is the mean vatue of u in C. Then if S_σ is the set of points where $|u-u_{G_\Phi}| > \sigma$, its reassure $m(S_\sigma)$ satisfies

$$m(S_\sigma) \leqq \frac{A}{\kappa}\int_{C_0} |u - \mathrm{u}_{C_\varphi}| dx \cdot e^{-\alpha\sigma\kappa^{-1}}\, for\ \frac{\sigma}{\kappa} \geqq a. \tag{3}$$

Since $n(S_\sigma) \leqq m(C_0)$ *it follows that*

$$m(S_\sigma) \leqq e^{\alpha a} e^{-a\sigma\kappa^{-1}} m(C_0)\, for\ \sigma > 0. \tag{4}$$

Here $A \leqq 1, \alpha, a$ *are positive* $nuvnbers$ *depending only on the dimension* n.

By a standard type of argnment we can derive, as consequences of (4), the following inequalities: for $0 < \beta < \alpha$,

$$\int_{C_0} e^{\beta\kappa^{-1}\mathrm{u}-\mathrm{u}_{C_0}}\, dx \leqq \left(\frac{\alpha}{\infty - p} + e^{\beta u}\right) m(C_0), \tag{5}$$

$$\int_{C_0} \left(e^{\beta\kappa^{-1} 1\mathrm{u} - uc_0 1} - 1\right) dx \leqq \left(\frac{e^{\beta\alpha} - 1}{\beta a} + \frac{A}{\kappa}\frac{\infty}{\alpha - \beta} e^{a(\beta - a)}\right) \int_{C_0} |u - u_{c_0}| dx$$

$$= \tilde{A} \int_{C_0} |u - u_{C_0}| dx \ \leqq 2\tilde{A} \int_{C_0} |u| dx. \tag{6}$$

Lemma (1.1.2)[1]. *Let* $u(x)$ *be an integrable function defined* $;_n$ *a finite cube* $C_0;_{nn}$*-dimensionat space*; $x = (x_1, \cdots . x_n)$. *Assume that there is a constant K such that for every parauel subcube C. artd some constant* a_C. *the inequdity*

$$\frac{1}{m(C)}, \int_C |u - a_C| \mathrm{dx} \leq K \tag{7}$$

holds. Here dx denotes element of volume and n(C) is the Lebesgue ,neasure of C. Then, if μ(σ) is the measure of the set of points where $|\mathrm{u} - \mathrm{a}_{0_\varphi} 1 > \sigma$, we have

$$\mu(\sigma) \leqq B e^{-b\sigma|\mathrm{K}} m(C_0)\, for\ \sigma > 0, \tag{8}$$

where B, b are constants depending only on n.

Proof. We may assume without loss of generality that $u_{C_0} = 0$ and that $\kappa = 1$, by replacing

u by $\left(u - u_{C_l}\right)/\kappa$.

Denote by $F(\sigma)$ the smallest number, depending only on σ and n (and independent of the particular function u or cube C_ϑ) such that

$$m(S_\sigma) \leqq F(\sigma) \int_{G_0} |u| dx;$$

obviously $F(\sigma) \leqq 1/\sigma$. We now prove that, for $\sigma \geqq 2^n$.

$$F(\sigma) \leqq \tfrac{1}{s} F(\sigma - 2^n s) \text{ for } 2^{-n}\sigma \geqq s \geqq 1. \tag{9}$$

The inequality (3) is of interest since, in case $u(x)$ is integrable in an *infinite* cube $\tilde{C}_0$ and satisfies (1) in every finite subcube, we can conclude from (3) that

$$\int_{\overline{C}_0} (e^{p_{\kappa^{-1}} \text{IuI}} - 1)dx \leqq 2_A \text{f} \int_{\zeta_0} |u|dx. \tag{10}$$

If one wishes to prove (7) directly, without proving (7) , then the proof given below can be simplified slightly.

Lemma (1.1.2) follows easily from For (1) implies that $|u_C - a_C| \leqq K$ so that

$$\frac{1}{m(C)} \int_C |u - u_c|dx \leqq 2K.$$

By Lemma (1.1.1) holds, with $K = 2K$, and (2) then folows easily.

The proof of Lemma (1.1.2) is based of integrable functions which, in one dimension, is due to Γ. Riesz, m_d which has been used extensively by Calderon and Zygmund [1] and Hörmander [2]. For completeness we include the proof, in a form suitable for application to Lemma (1.1.2).

For u be an integrable function defined in a cube C_0 and let s be a positive number such that

$$s \geqq \frac{1}{m(C_0)} \int_{C_0} |u|dx. \tag{11}$$

There exists a denumerable number of open disjoint cubes I_k in C_0 such that

i) $|u| \leqq s$ a.e. in $C_0 - \bigcup_k I_k$,

ii) the average value u_k of u in I_k is bounded in absolute value by $2^n s$,

iii) $\sum_k m(I_k) \leqq s^{-1} \int_{G_0} |u1dx$.

Proof. Divide C_0 (by halving each edge) into 2^n equal cubes and let I_{11}, I_{12} , be those open cubes over which the average value of $|u|$ is $\geqq s$. Then

$$sm(I_{1k}) \leqq \int_{I_{1k}} |u1du \leqq 2^n sm(I_{1k})$$

by (4). Next subdivide each remaining cube, over which the average of $|u1$ is $< s$, into 2^n equal cubes, and denote by I_{21} . I_{22}, those cubes thus obtained over which the average of $|u1$ is $\geqq s$. Again subdivide the remaining cubes, etc. In this way we obtain a sequence of cubes I_{ik}, which we rename I_k, such that

$$sm(I_k) \leqq \int_{I_k} |u|dx < 2^n sm(I_k) .$$

Clearly property ii) is satisfied. Furthermore, summing the left inequality over k we obtain iii). We observe finally that a point of C_0 which does not belong to any of the I_k belongs to arbitrarily small cubes over which the average of $|u|$ is $< s$. Hence $|u| \leqq s$a. e. outside all the I_k, venfying i).

To this end we apply the above to the function u in C_0 with

$$2^{-n}\sigma \geqq s \geqq 1 \geqq \frac{1}{m(C_0)} \int_{C_\varphi} |u|dx,$$

the last inequality following from (7). Because of i) we see that if $|u(x)| > \sigma$, then x belongs to one of the I_k (except for a set of measure zero). Hence, since the average u_k of u in I_k is bounded by $2^n s$ in absolute value, we see that

$$m(S_\sigma) = m\{x||u(x)| > \sigma\} \leqq \sum_k \text{r}\, n\{x|u(x) - u_k| > \sigma - 2^n s \; in \; I_k\}.$$

Now in the cube I_k the function $u - u_k$ satisfies the hypotheses of Lemma (1.1.1), in particular it satisfies (1) for every cube in I_*. Hence, using the definition of (σ) , we have

$$\nu\iota\{x|u(x) - u_k| > \sigma - 2^n s \; in \; I_k\} \leqq F(\sigma - 2^n s) \int_{I_k} |u - u_k| dx$$

$$\leqq F(\sigma - 2^n s) m(I_k) .$$

Thus we find

$$m(S_\sigma) \leqq F(\sigma - 2^n s) \sum_\iota \nu\, n(I_k) \leqq \frac{1}{s} F(\sigma - 2^n s) \int_{C_0} |u1 dx$$

by iii) , proving (3). Setting $s = e$ in (3) we see that if $F(\sigma) \leqq Ae^{-a\sigma}$, $\infty = 1/(2^n e)$ for some σ, then

$$F(\sigma + 2^n e) \leqq \frac{1}{e} Ae^{-a\sigma} = Ae^{-\infty(\sigma + 2^n e)}.$$

From this it follows that if on some interval of length $2^n e$ the inequality $F(\sigma) \leqq Ae^{-\alpha\sigma}$ holds, then it holds for all larger σ. But a calculation shows that

$$F(\sigma) \leqq \frac{1}{\sigma} \leqq \frac{12}{10} 2^{-n} e^{-a\sigma} \; for \; \frac{2^n e}{e - 1} \leqq \sigma \leqq \frac{2^n e}{e - 1} + 2^n e.$$

(This interval is the one of length $2^n e$ on which the maximum of $e^{a\sigma}/\sigma$ is as small as possible.) Thus we conclude that

$$m(S_\sigma) \leqq \frac{12}{10} 2^{-n} \ell^{-\alpha\sigma} \int_{C_0} |u| dx \; for \; \propto \sigma \geqq \frac{1}{e - 1} , \alpha = \frac{1}{2^n e},$$

that is, we have proved (8) with

$$A = \frac{12}{10} 2^{-n}, a = \frac{1}{2^n e}, a = \frac{2^n e}{e - 1}. \tag{12}$$

We have made no attempt here to obtain the best constants. The exponent a can be considerably improved, *i.e.* increased, by using the hypothesis (5) again to sharpen the estimate $|u_k| \leqq 2^n s$ that was provided by ii). We mention only that we have proved (8) with a constant α which for large n behaves like $(1/e \log 2)(\log n/kt)$. M. Weiss and A. Zygmund [4] contains the following

Theorem (1.1.3)[1]. If F(x) is φeriodiι and for some β > ξ satisfies

$$F(x + h) + F(x - h) - 2F(x) = 0\left(\frac{h}{|\log h|^\beta}\right) \tag{13}$$

uniformly in x, *then* F *is the indefinite integral of an* f *belonging to every* L_g. They also give an example showing that the result does not hold for $\beta = \frac{1}{2}$.

The proof of the Theorem (1.1.3) in [4] is rather short but it relies on a Theorem (1.1.3) of Littlewood and Paley, and it seems of interest to us to show how it may be derived from our Lemma (1.1.1).

Then u satisfies the conditions of Lemma (1.1.1) with some constant κ depending on K, β and n so that, consequently, u satisfies (7) and (8).

The preceding Theorem (1.1.3) follows easily from this lemma. By convolution of F with a smooth peaked kernel we may suppose that F is infinitely differentiable. It suffices merely to estimate the L_p norm of the denvative f of F. Hypothesis (7) asserts simply that f satisfies (8) for $n = 1$. Applying Lemma (1.1.4) we obtain from (2) or (a) an estimate for the L_p norm of f depending only on K and β, proving the Theorem (1.1.3). From (3) we find, furthermore, that $e^{\alpha'|f1}$ is integtable for some $\alpha' > 0$.

We show

Lemma (1.1.4)[1]. Let u(x) be an integrable function defined in a finite cube C_0 in n-space. Assume that there is a constant K and a constant $\beta > \#$ such that if C_1 and C_a, are any two equal subcubes having a futl $(n - 1)$-dimensional face in common, then

$$|u_{C_1} - u_{C_8}| \leqq \frac{K}{1 + |\log h|}\beta. \tag{14}$$

here u_{C_1} , u_C, are the mean values of u in the cubes C_1 and C_2, and h is the common side length.

Proof. Consider a subcube C, of side length h subdivided into 2^{nN} equal cubes $C_r, r = 1, \cdots, 2^{nN}$, obtained by dividing each edge into 2^N equal parts, and let u_r denote the mean value of u in C_r. Then

$$\frac{1}{m(C)}\int_C |u - u_C|dx = \lim_{N\to\infty} 2^{-nN}\sum |u_r - u_C|.$$

Thus to prove (1) it suffices to show that $2^{-nN}\sum\ |u_r - u_c| \leqq \kappa$, with κ depending only on K, β and n.

By Schwarz inequality,

$$2^{-nN}\sum |u_r - u_C| \leqq \left[2^{-nN}\sum |u_r - u_C|^2\right]^{1/2} = a_{N^{1/2}}.$$

We shall prove that the a_N are uniformly bounded by showing that

$$a_{N+1} \leqq a_N + \left(\frac{nK}{1 + |\log h|^\beta}\right)^2, h = \frac{k}{2^{N+1}}. \tag{15}$$

Since $a_0 = 0$, it follows that

$$a_{N+1} \leqq n^2K^2\sum_{j=1}^{\infty}\left(1+|j\log 2-\ \log k|^{\beta}\right)^{-2} \leqq \kappa^{a}$$

for some constant κ independent of k, convergence being guaranteed by the fact that $\beta > \frac{1}{2}$. Thus to complete the proof we shall establish (9). We observe first that $2^{-nN}\sum u_l. = u_C$ so that using the general identity

$$k\sum_{1}^{k} b_r^2 = \left(\sum b_r\right)^2 + \frac{1}{2}\sum_{r,s}(b_r - b_\varepsilon\rangle^{\mathfrak{B}}$$

for real b_i., we find

$$2^{2nN}a_N = 2^{nN}\sum |u_r - u_C|^2 = \frac{1}{2}\sum_{r.s}|u_r - -u_s 1^2. \tag{16}$$

Now, on the next subdivision of C into $2^{n\{N+1\}}$ cubes each C_r is divided into 2^n equal cubes C_i , $i = 1, \cdots l2^n$, of side length $h = k/2^{N+1}$. If u_{ri} is the mean value of u in C_{ri} we have

$$u_r = 2^{-n}\sum_i u_{rl}. \tag{17}$$

Furthermore, since any two $C_{r:}$, $C_{;}$ can be connected by a chain of at most $n+1$ cubes each having a full face in common with the succeeding one, we find from (6) that

$$|u_{ri} - u_{rJ}| \leqq \frac{nK}{1+|\log h1^p} = M,$$

where M is so defined. This together with (11) implies

$$|u_{ri} - u_r| \leqq M.$$

According to formula (8)

$$2.\,2^{2n(N+1)}a_{N+1} = \sum_{\substack{r,s\leqq 2^{nN}\\ i,j\leqq 2^n}} |u_{li} - u_f 1^2 = \sum\left[\left(u_{ri}^2 + u_{sJ}^2\right) - 2u_{rl}u_{sj}\right]$$

$$= \sum[(u_{ri} - u_r)^2 + \left(u_{ej} - u_\varepsilon\right)^8 + 2u_{ri}u_r + 2u_{s3}u_s - u_r^2 - u_l^2 - 2u_{ri}u_{8j}]$$

$$= \sum_{\substack{r,s\\ i,j}}\left[(u_{ri} - u_r)^2 + \left(u_{ij} - u_\varepsilon\right)^2\right] + \sum_{r.8}[2.\,2^{2n}(u_r^2 + \mathrm{u}_l^2) - 2^{2n}(u_r^2 + u_{\$}^{\mathrm{g}}\}$$

$$- 2.\,2^{\mathfrak{g}n}u, u],$$

by (9),

$$\leqq 2M^2 2^{2nN+2n} + 2^{2n}\sum_{r.\iota}(u_r - u_s)^2 = 2.\,2^{2n(N+1)}M^2 + 2.\,2^{2n(N+1)}a_N,$$

by (8), or

$$a_{N+1} \leqq a_N + M^l.$$

This is the desired inequality (7) and the proof is complete.

We present briefly a generalization of Lemma(1.1.2).

Lemma (1.1.5)[1]. Let u be integrable in a finite cube C_0 and consider a subdivision of C_0 into a denumerable number of cubes C_i, no two having acommon interior point. Assume that for fixed , $1 < p < \infty$, the expression

$$\left\{\sum_i m\,(C_i)^{1-p}\left|\int_{C_\ell} |u - u_{C_l}|dx\right|^p\right\}^{1/p}$$

is tinite. Denote by K_u the lim sup of such expressions for all possible subdivisions of C_0 of this kind: in general $K_u = \infty$. If $K_u < \infty$, the measure $m(S_\sigma)$ of the set S_0., where $|u - u_{\theta_0}| > \sigma$, satisties

$$m(S_\sigma) \leqq A\left|\frac{K_u}{\sigma}\right|^{\Phi} for \sigma > 0,$$

for some constant A depending only on n and p.

The result implies that the function u belongs to $L^{p'}$ for every $p' < p$. For $p = \infty$ the hypothesis of Lemma agrees with that of Lemma (1.1.1).

Proof. We shall not attempt to obtain the best constants. Let $q = pf(p-1\rangle$ be the conjugate exponent to p. We may assume that $u_{C_0} = 0$. Using induction with respect to the integer j we shall prove that if

$$s = \frac{2^{-n}\sigma}{\varphi(q^j - 1) + 1} \geqq \frac{K_u}{n(C_0)^{\frac{1}{p}}}, \tag{18}$$

then

$m(S_\sigma) \leqq 2^{-n} q^{1/q + 2q^2 + \cdots + j/q^j} \left|\frac{2^n\, p(1-g^{1-j})K_u}{\sigma}\right|^{p\langle 1-1/q^{j+1}} \Big) \left(\frac{1}{K_u}\int_{C_0} |u|dx\right)^{1/q^j}$ Since

$$m(S_\sigma) \leqq \frac{1}{\sigma}\int_{C_0} |u|dx, \tag{19}$$

Holds for $i = 0$. Suppose then it is true for $i - 1$, we wish to prove it $i$or$j$. Since

$$\frac{1}{m(C_0)}\int_{C_0} |u|dx \leqq \frac{K_u}{m(C_0)^{1fp}}, \tag{20}$$

we may apply the u, with s equal to its value in (12). Let u_k denote the mean value of u in I_k, and set $v_k = u - u_k$ in I_k. From the definition of K_u we may assert that

$$\sum_k K_{v_k}^p \leqq K_u^p. \tag{21}$$

Setting $a_k = \int_{I_k} |v_k 1 dx$ we note further (as in (14)) that

$$m(I_k)^{1-p} a_k^{\Phi} \leqq K_{v_k}^{\prime p} \tag{22}$$

so that by Hölder's inequality

$$\sum a_k \leqq (\sum w \iota\{I_k)^{1-p} a_k^p)^{1/p} | \sum m\,(I_k)|^1 l^{\mathrm{q}} \leqq \left(\sum K_{v_k}^{g}\right)^{1/p} | \sum \quad m(I_k) 1^{1/q},$$

or

$$\sum a_k \leqq K_{\mathrm{u}} | s^{-1} \int_{C_0} | u | dx |^{\frac{1}{\mathrm{q}}} \tag{23}$$

by (15) and iii).

As in the derivation of (5), we have

$$m(S_\sigma) \leqq \sum_k m\{x \in I_k || v_k 1 > \sigma - 2^n s\}.$$

Applying the induction hypothesis (13), for $j-1$, to the functions v_k in I_k. We find $m(S_\sigma)$

$$\leqq \left[2^{-n} q^{1 \int \mathrm{q} + \cdots + \{l-1)/\alpha^{i-1}} | \frac{2^n p(1-q^{-1})}{\sigma - 2^n s} |^{v(1-1/\mathrm{i})} g \right] . \sum_{\mathrm{k}} K_{v_k}^{v\{1-1/q^j)} \left(\frac{1}{K_{v_k}} \int_{I_k} | v_{\mathrm{k}} 1 dx \right)^{1/\mathrm{q}^{j-1}}$$

$$= [\quad] \sum_k K_{v_{\mathrm{k}}}^{p(1-\mathrm{q}^{1-j})} a_k^{1/\mathrm{q}^{j-1}} \leqq [\quad] \left(\sum a_k\right)^{1/q^{j-1}} \left(\sum K_{v_{\mathrm{k}}}^{p}\right)^{1-a^{1-j}},$$

by Hölder's inequality,

$$\leqq [\quad] K_{\mathrm{u}}^{1/a^{j-1}} | s - 1 \int_{C_0} | u | dx |^{1/\alpha^j} K_{\mathrm{u}}^{p-pq^{1-\prime}}$$

by (17) and (15), so that

$$m(S_\sigma) \leqq \frac{[\quad]}{s^{1/\mathrm{q}^f}} K_{\mathrm{u}}^{v(1-1\prime \mathrm{q}^{j+1})} \left(\frac{1}{K_{\mathrm{u}}} \int_{C_0} | u | dx \right)^{1/q^f}$$

A slightly tedious calculation shows that this inequality is identical with the desired result (*l*3).

Having established (13) we may now express it in a more convenient fonn: if (12) holds, then, in virtue of (14), there is a constant k depending only on n and p such that

$$m(S_\sigma) \leqq k \left(\frac{K_{\mathrm{u}}}{\sigma} \right)^{p(1-1T)} m(C_0)^{1/\mathrm{q}^{j+1}} + 1$$

or

$$m(S_\sigma) \leqq k\left(\frac{K_\mathrm{u}}{\sigma}\right)^\mathrm{p} \cdot |\frac{\sigma m(C_0)^{1/\mathrm{p}}}{K_\mathrm{u}}|^{p/\mathrm{e}^{J+\iota}}$$

If now $2^{\sim n}\sigma \geqq K_\mathrm{u} m(C_0)^{-1/p}$ and we choose the largest integer $j \geqq 0$ so that (12) is satisfied, we have the opposite inequality for $i+1$:

$$\frac{\sigma m(C_0)^{1/p}}{K_\mathrm{u}} \leqq 2^{\cdot}(\varphi(q^{J+1}-1\rangle+1) \leqq 2^n \varphi q^{i+1}.$$

Inserting into the previous inequality we find

$$m(S_\sigma) \leqq k\left(\frac{K_\mathrm{u}}{\sigma}\right)^{p\left(1-\frac{1}{q^{j+1}}\right)} m(C_0)^{1/q^{j+1}}$$

$$\leqq A\left(\frac{K_\mathrm{u}}{\sigma}\right)^p \ for\ \sigma \geqq 2^n K_\mathrm{u} m(C_0\rangle^{-1/\mathrm{p}},$$

for some constant A depending only on n and p. Since $(S_\sigma) \leqq m(C_0)$, the same inequality holds for all $\sigma > \vartheta$. with some other constant A, and the proof of the lemma is complete. Inequality (2) in Lemma (1.1.1) can be replaced by the more general inequality

$$m(S_{2^{n\star}\sigma}) \leqq Ae^{-B\sigma\kappa^{-1}} m(S_\sigma) for\ \sigma > 0 \tag{24}$$

with , B depending only on n. Let $\kappa = 1, u_O. = 0$. For a fixed positive s the cubes I_k shall be defined as in the proof of Lemma (1.1.2). Put

$$\mu_k(\sigma) = m(x||u(x)-u_k| > \sigma - 2^n\ s\ in\ I_k)$$

By definition, $\mu_k(\sigma)$ is non-increasing and does not exceed (I_k) . By (2) applied to I_k,

$$\mu_k(\sigma\rangle \leqq m(x||u(x)-u_k 1 > \sigma - 2^n s\ in\ I_k)$$

$$\leqq e^{\alpha a} e^{-\alpha(\sigma-2^n s)} m(I_k).$$

Then

$$sm(I_k) \leqq \int_{I_k} |u|dx = \int_0^\infty \mu_\mathrm{k}(\sigma)d\sigma$$

$$= \int_0^{s/2} \mu_k(\sigma)d\sigma + \int_{\mathrm{s}/2}^{2^n s} \mu_\mathrm{k}(\sigma)d\sigma + \int_{2^n s}^\infty \mu_k(\sigma)d\sigma$$

$$\leqq \frac{s}{2} m(I_k) + \left(2^n s - \frac{s}{2}\right)\mu_\mathrm{k}\left(\frac{s}{2}\right) + \frac{1}{\alpha} e^{\alpha a} m(I_k).$$

It follows for $s > \alpha^{-1} 2^{\mathrm{u}+2} e^{\alpha a}$ that

$$\mu_k\left(\frac{S}{2}\right) \geqq \frac{1}{2^{n+1}} m(I_k) \geqq \frac{1}{2^{n+1}} e^{-\alpha a} e^{2^n \alpha s} \mu_k(2^{n+1}s).$$

Then also

$$m(S_{s/2}) \geqq \sum_k \mu_k\left(\frac{S}{2}\right) \geqq 2^{-n-1} e^{-\alpha a} e^{2^n as} \sum_k \mu_k(2^{n+1}s)$$

$$= 2^{-n-1}e^{-\alpha a}e^{2^n\alpha s}m(S_{2^{n+1}})\ for\ s > \frac{1}{\alpha}2^{n+2}e^{\alpha a}.$$

Inequality (7) is an immediate consequence.

Section (1.2): Critical Sobolev Space and BMO

We give a systematic treatment to prove the Gagliardo-Nirenberg type inequality at the critical index and its related estimates in the Sobolev space. It is well known that the Sobolev space $H^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$ is continuously imbedded into $L^\infty(\mathbb{R}^n)$ provided $s > n/p$. The case when $s = n/p$ is called the critical exponent, which implies $H^{n/p,p}(\mathbb{R}^n)$ is not imbedded into $L^\infty(\mathbb{R}^n)$, but into $L^q(\mathbb{R}^n)$ for all q with $p \leqq q < \infty$. See, e.g., Adams [7]. Then, Ogawa [19], Ogawa and Ozawa [20] and Ozawa [21] gave a precise investigation for this imbedding and obtained the following optimal growth rate as $q \to \infty$. Indeed, for every p with $1 < p < \infty$, it holds

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq C_{n,p}q^{\frac{1}{p}}\|u\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}}\|(-\triangle)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{(q)}} \tag{25}$$

for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all q with $p \leqq q < \infty$, where $p' = p/(p-1)$ denotes the Hölder conjugate exponent of p and $C_{n,p}$ is a constant depending on n and p, but not on q.

We generalize (25) to the estimate in the homogeneous Sobolev space such as $L^p(\mathbb{R}^n) \cap \dot{H}^{n/r,r}(\mathbb{R}^n)$. In fact, we shall prove that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq C_n r' q^{\frac{1}{r}}\|u\|_{L(\mathbb{R}^n)}^{\frac{p}{q}}p\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r(\mathbb{R}^n)}^{1-\frac{p}{q}} \tag{26}$$

holds for all $u \in L^p(\mathbb{R}^n) \cap \dot{H}^{n/r,r}(\mathbb{R}^n)$ with $1 \leqq p \leqq q < \infty$ and $1 < r < \infty$. Here and in what follows, we denote by $\dot{H}^{s,r}(\mathbb{R}^n)$ the homogeneous Sobolev space defined by

$$\dot{H}^{s,r}(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n);\ \|(-\triangle)^{\frac{s}{2}}u\|_{L^r(\mathbb{R}^n)} < \infty\}.$$

It should be noted that our constant C_n in (26) depends only on n. It seems to be an interesting problem to deal with the limiting end-point of the estimate (26) as $r \to \infty$. Our second purpose is to show the following estimates treating the marginal case in the space of functions of bounded mean oscillation, i.e., *BMO*. It holds

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq C_n q\|u\|_{p}^{\frac{p}{Lq}}(\mathbb{R}^n)\|u\|_{BMO}^{1-\frac{p}{q}} \tag{27}$$

for all $u \in L^p(\mathbb{R}^n) \cap BMO$ with $1 \leqq p < \infty$ and for all q with $p \leqq q < \infty$. Recently, Chen and Zhu [11] obtained (27) by means of a variant of the John-Nirenberg inequality such as

$$\mu(\{x \in \mathbb{R}^n; |u(x)| > t\})$$
$$\leqq C_n\frac{\|u\|_{L^1(\mathbb{R}^n)}}{\|u\|_{BMO}}\exp\left(-\frac{\alpha_n t}{\|u\|_{BMO}}\right) \text{ with } t > \|u\|_{BMO} \tag{28}$$

for all $u \in L^1(\mathbb{R}^n) \cap BMO$ with positive constants C_n and α_n depending on n, where μ denotes the Lebesgue measure on $\mathbb{R}^n$. However, our method is different from theirs. Indeed, instead *of BMO*, we make use of the set W defined by

$$W := \{u \in L^1_{loc}(\mathbb{R}^n)\ ;\ \|u\|_W := \sup\left(u^{**}(t) - u^*(t)\right) < \infty\},$$

where u^* and u^{**} denote the rearrangement of u and the average function of u^*, respectively.

First, we prove (27) with BMO replaced by W. Next, we show the estimate

$$\|u\|_W \leqq C_n \|u\|_{BMO} \text{ for all } u \in BMO.$$

It is well-known that at the critical index, the Gagliardo-Nirenberg inequality and the Trudinger-Moser one are equivalent. Hence, as a consequence of (27) with $p = 1$, we see that there exist two positive constants C_n and α_n such that

$$\int_{\mathbb{R}^n} \left(\exp\left(\alpha_n \frac{|u(x)|}{\|u\|_{BMO}} \right) - 1 \right) dx \leqq C_n \frac{\|u\|_{L^1(\mathbb{R}^n)}}{\|u\|_{BMO}} \tag{29}$$

holds for all $u \in L^1(\mathbb{R}^n) \cap BMO$. Our advantage is to obtain (28) from (29), which implies that the John-Nirenberg type estimate such as (28) is a consequence of (29). As a result, it turns out that the Gagliardo-Nirenberg inequality in $L^1(\mathbb{R}^n) \cap BMO$ is equivalent to the John-Nirenberg type estimate. We also show the Trudinger-Moser inequality in $L^p(\mathbb{R}^n) \cap \dot{H}^{n/r,r}(\mathbb{R}^n)$ which is based on (26).

It is known that the L^∞-norm is dominated by means of the logarithm of the $\dot{H}^{s,q}(\mathbb{R}^n)$-norm with $1 < q < \infty$ and $s > n/q$ provided the norm of the critical Sobolev space $L^p(\mathbb{R}^n) \cap \dot{H}^{n/r,r}(\mathbb{R}^n)$ is added on the right-hand side of the estimate. This is called the Brezis−Gallouet−Wainger inequality. As an application of (27), we show that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqq C_{n,p,q,s} \left(1 + \left(\|u\|_{L(\mathbb{R}^n)^p} + \|u\|_{BMO} \right) \log \left(e + \|(-\triangle)^{\frac{s}{2}} u\|_{L(\mathbb{R}^n)^q} \right) \right) \tag{30}$$

holds for all $u \in L^p(\mathbb{R}^n) \cap \dot{H}^{s,q}(\mathbb{R}^n)$, where $1 \leqq p < \infty$, $1 < q < \infty$ and $s > n/q$. A similar estimate was proved by Kozono et al. [14] by using the Littlewood-Paley theory. On the other hand, based on (27), we derive (30) in terms of the $L^p - L^q$ estimate for the semigroup $\left\{ e^{-t(-\triangle)^{s/2}} \right\}_{t \geqq 0}$, which is different from [14] and Engler [22]. See also Kozono and Taniuchi [21] and Ozawa [16].

The L^p-estimate of difference between the shifted and original Riesz potentials plays an important role for the proof of (26). The relation between the space of BMO and the set W is clarified. The asymptotic behavior of the semigroup $\left\{ e^{-t(-\triangle)^{s/2}} \right\}_{t \geqq 0}$ at $t = 0$ and $t = \infty$ is established. We shall state main results. First, we show the Gagliardo-Nirenberg type inequality which may be regarded as a generalized version of (25). We shall deal with the function defined on $\mathbb{R}^n$. For simplicity, we abbreviate $L^p(\mathbb{R}^n)$ and $H^{s,p}(\mathbb{R}^n)$ to L^p and $H^{s,p}$, respectively.

Theorem (1.2.1)[5]: There exists a constant $\mathrm{C_n}$ depending only on n such that

$$\|u\|_{L^q} \leqq C_n r' q^{\frac{1}{r}} \|u\|_{L^p}^{\frac{p}{q}} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}^{1-\frac{p}{q}} \tag{31}$$

holds for all $u \in L^p \cap H^{n/r,r}$ *with* $1 \leqq p < \infty$, $1 < r < \infty$ *andfor all* q *with* $p \leqq q < \infty$.

Next, we deal with the limiting end-point of Theorem (1.2.1) as $r \to \infty$. Indeed, we may regard the limit of the space $\dot{H}^{n/r,r}$ as $r \to \infty$ as the space *of BMO*.

Proof. We may assume u belongs to the Schwartz class S since $\mathcal{S}$ is dense in the function space $L^p \cap \dot{H}^{n/r,r}$. We set $m := \max\ (n + 1, p, r)$. First, we deal with the case $m \leqq q <$

∞. By taking $\Phi(\xi) = e^{-\pi|\xi|^2}$ and $(\xi) = 1 - \Phi(\xi)$, we have

$$u(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \hat{u}(\xi) d\xi$$

$$= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \hat{u}(\xi)\Phi(\xi/R)d\xi + \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \hat{u}(\xi)\Psi(\xi/R)d\xi$$

$$=: u(x) + u(x)\,,$$

where $R > 0$ is a parameter determined later. Obviously, $\hat{u}\ (\xi) = \mathcal{F}u(\xi)\ :=\ \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, u(x)dx$ represents the Fourier transform of u. Since $\mathcal{F}^{-1}\Phi(x) = e^{-\pi|x|^2}$, it is easy to see that

$$\|\mathcal{F}^{-1}\{\Phi(\cdot/R)\}\|_{L^s} = R^{n\left(1-\frac{1}{s}\right)}\|\mathcal{F}^{-1}\Phi\|_{L^s} = R^{n\left(1-\frac{1}{s}\right)}s^{-\frac{n}{2s}} \leqq R^{n\left(1-\frac{1}{s}\right)}$$

holds for all $R > 0$ and $s \geqq 1$. Taking s so that $1/s = 1/q - 1/p + 1$, we have by the Young inequality that

$$\|u1\|_{Lq} = \|u * \mathcal{F}^{-1}\{\Phi(\cdot/R)\}\|_{Lq} \leqq \|u\|_{Lp}\|\mathcal{F}^{-1}\{\Phi(\cdot/R)\}\|_{L^s}$$

$$\leqq R^{n\left(1-\frac{1}{s}\right)}\|u\|_{L^p} = R^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p}. \tag{32}$$

On the other hand, u_2 can be rewritten as:

$$u_2(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,(2\pi|\xi|)^{\frac{n}{r}}\,\hat{\mathrm{u}}\ (\xi)\frac{\Psi(\xi/R)}{(2\pi|\xi|)^{\frac{n}{r}}}d\xi = (-\triangle)^{\frac{n}{2r}}u * K_R(x)\,, \tag{33}$$

where

$$K_R(x) = \mathcal{F}^{-1}\left\{(2\pi|\cdot|)^{-\frac{n}{r}}\big(1-\Phi(\cdot/R)\big)\right\}(x) = I_{\frac{n}{r}}(x) - I_{\frac{n}{r}} * \mathcal{F}^{-1}\{\Phi(\cdot/R)\}(x)$$

$$= I_{\frac{n}{r}}(x) - I_{\frac{n}{r}} * \big(R^n e^{-\pi|R\cdot|^2}\big)(x) = R^n\int_{\mathbb{R}^n}\left(I_{\frac{n}{r}}(x) - I_{\frac{n}{r}}(x-y)\right)e^{-\pi|Ry|^2}dy.$$

Notice that $R^n \int_{\mathbb{R}^n} e^{-\pi|Ry|^2}\, dy = 1$ for all $R > 0$. Let us take s so that $\frac{1}{s} = \frac{1}{q} - \frac{1}{r} + 1$. Since $q \geqq m$, we have $q \geqq r$ and $q > n$, which yield $s \geqq 1$ and $nr'/(n + r') < s < r'$. Applying the Minkowski inequality and Lemma (1.2.6) with p replaced by r, we see

$$\|K_R\|_{L^s} = R^n\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\left(I_{n/r}(x) - I_{n/r}(x-y)\right)e^{-\pi|Ry|^2}dy\right|^s dx\right)^{\frac{1}{s}}$$

$$\leqq R^n\int_{\mathbb{R}^n} e^{-\pi|Ry|^2}\left(\int_{\mathbb{R}^n}|\,I_{n/r}(x) - I_{n/r}(x-y)|^s dx\right)^{\frac{1}{s}}dy$$

$$= R^n\gamma(n/r)^{-1}\int_{\mathbb{R}^n} e^{-\pi|Ry|^2}\left(\int_{\mathbb{R}^n}|\,\tilde{I}_{n/r}(x) - \tilde{I}_{n/r}(x-y)|^s dx\right)^{\frac{1}{s}}dy$$

$$\leqq R^n\gamma(n/r)^{-1}\int_{\mathbb{R}^n} e^{-\pi|Ry|^2}\left(C_nT|y|^{n\left(1-\frac{S}{r'}\right)}\right)^{\frac{1}{s}}dy$$

$$\leqq C_n\gamma(n/r)^{-1}T^{\frac{1}{s}}R^{-\frac{n}{q}},$$

where

$$T:=\frac{r'}{r'-s}+\frac{r'}{(n+r')s-nr'}.$$

Hence, by (33) and the Young inequality, it holds

$$\|u_2\|_{L^q}\leqq\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}\|K_R\|_{L^s}\leqq C_n\gamma(n/r)^{-1}T^{\frac{1}{s}}R^{-\frac{n}{q}}\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}$$

for all $R>0$. By (32) and the above estimate, we have

$$\|u\|_{Lq}\leqq C_n\left(R^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|u\|_{L^p}+\gamma\left(\frac{n}{r}\right)^{-1}T^{\frac{1}{s}}R^{-\frac{n}{q}}\right\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}\right)\qquad(34)$$

for all $R>0$ with $1/s=1/q-1/r+1$. Taking $R=\left(\frac{\|(-\triangle)^{n/(2r)}u\|_{L^r}}{\|u\|_{L^p}}\right)^{p/n}$ in (34), we have

$$\|u\|_{Lq}\leqq C_n\left(1+y\left(\frac{n}{r}\right)^{-1}T^{\frac{1}{s}}\right)\|u\|_{Lp}^{\frac{p}{q}}\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}1-\frac{p}{q}\qquad(35)$$

for all q with $m=\max\ (n+1,p,r)\leqq q<\infty$.

We next establish a bound $T^{1/s}$ in (35) such as

$$T^{\frac{1}{s}}\leqq 8e^{\frac{1}{e}}q^{\frac{1}{r}}\qquad(36)$$

Indeed, since $1/s=1/q+1/r'$, we have

$$\left(\frac{r'}{r'-s}\right)^{\frac{1}{s}}=\left\{q\left(\frac{1}{q}+\frac{1}{r}\right)\right\}^{\frac{1}{q}+\frac{1}{r}}=q^{\frac{1}{q}}\left(\frac{1}{q}+\frac{1}{r}\right)^{\frac{1}{q}+\frac{1}{r}}q^{\frac{1}{r}}\leqq 4e^{\frac{1}{e}}q^{\frac{1}{r}}$$

Note that $\max_{t>0}t^{1/t}=e^{1/e}$ and $\max_{0\leqq t\leqq 2^{t^t}}=4$. Furthermore, since $\geqq n+1$, we have similarly as above that

$$\left(\frac{r'}{(n+r')s-nr'}\right)^{\frac{1}{s}}=\left(\frac{r'+q}{r'(q-n)}\right)^{\frac{1}{q}+\frac{1}{r}}\leqq\left(\frac{r'+q}{r}\right)^{\frac{1}{q}+\frac{1}{r}}$$

$$=\left\{q\left(\frac{1}{q}+\frac{1}{r}\right)\right\}^{\frac{1}{q}+\frac{1}{r}}\leqq 4e^{\frac{1}{e}}q^{\frac{1}{r}},$$

which yields (36). Since $\Gamma(\alpha)=\mathcal{O}(1/\alpha)$ as $\alpha\to+0$, it holds

$$\gamma(n/r)^{-1}\leqq\pi^{-\frac{\pi}{2}}\frac{\Gamma\left(\frac{n}{(2r')}\right)}{\Gamma\left(\frac{n}{(2r)}\right)}\leqq C_n\frac{1}{r-1}\qquad(37)$$

for all $1<r<\infty$. From (35), (36) and (37), we obtain

$$\|u\|_{L^q} \leqq C_n \frac{1}{r-1} q^{\frac{1}{r}} \|u\|_p^{\frac{p}{Lq}} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r} 1 - \frac{p}{q} \tag{38}$$

for all q with $q \geqq m = \max\ (n+1, p, r)$. We next deal with the case $p \leqq q \leqq m$. By the Hölder inequality, we have

$$\|u\|_{Lq} \leqq \|u\|_p^{\frac{p}{Lq}(1-\theta)} \|u\|_m^{\frac{m}{Lq}\theta},$$

where $0 \leqq \theta \leqq 1$ is given by $q = p(1-\theta) + m\theta$. Moreover, from (38) with $q = m$, we obtain

$$\|u\|_{Lq} \leqq \|u\|_{L^p}^{\frac{p}{q}(1-\theta)} \left(C_n \frac{m^{\frac{1}{r}}}{r-1} \|u\|_p^{\frac{p}{Lm}} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}^{1-\frac{p}{m}} \right)^{\frac{m}{q}\theta}$$

$$\leqq C_n \frac{m^{\frac{1}{r}}}{r-1} \|u\|_{L^p}^{\frac{p}{q}} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r} 1 - \frac{p}{q}$$

$$\leqq \frac{C_n}{r-1} \left(\frac{m}{p}\right)^{\frac{1}{r}} q^{\frac{1}{r}} \|u\|_{L^p} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r} \frac{p}{q} 1 - \frac{p}{q} \tag{39}$$

In the above estimates, notice that $(q/p)^{1/r'} \geqq 1$ and

$$\frac{m^{\frac{1}{r}}}{r-1} \geqq \frac{r^{\frac{1}{r}}}{r-1} = r^{-\frac{1}{r}} \frac{r}{r-1} \geqq e^{-\frac{1}{e}} \frac{r}{r-1} \geqq e^{-\frac{1}{e}},$$

which yields

$$\left(\frac{m^{\frac{1}{r}}}{r-1} \right)^{\frac{m}{q}\theta} \leqq e^{\frac{1}{e}} \frac{m^{\frac{1}{r}}}{r-1}.$$

Now, we investigate the constant $(m/p)^{1/r'}/(r-1)$ in (39). When $p \leqq r$, we see that

$$\frac{1}{r-1} \left(\frac{m}{p}\right)^{\frac{1}{r}} = \frac{1}{r-1} \left(\frac{\max\ (n+1, r)}{p} \right)^{\frac{1}{r}} \leqq \frac{1}{r-1} \left(\frac{r+n+1}{p} \right)^{\frac{1}{r}}$$

$$\leqq \frac{(r+n+1)^{\frac{1}{r}}}{r-1} \leqq C_n r'$$

for all $p \leqq r < \infty$. Next, when $p \geqq r$, we have

$$\frac{1}{r-1} \left(\frac{m}{p}\right)^{\frac{1}{r}} = \frac{1}{r-1} \left(\frac{\max\ (n+1, p)}{p} \right)^{\frac{1}{r}} \leqq \frac{1}{r-1} \left(\frac{p+n+1}{p} \right)^{\frac{1}{r}} \leqq C_n \frac{1}{r-1}$$

for all $1 < r \leqq p < \infty$. Hence, we have by (39) that

$$\|u\|_{L^q} \leqq C_n r^{\gamma^{\frac{p}{q}}{}^{1-\frac{p}{q}}} q^{\frac{1}{r}} \|u\|_{L^p} \|(-\triangle)^{\frac{n}{2r}} u\|_{L^r} \tag{40}$$

for all q with $1 \leqq q \leqq m$. From (38) and (40), we obtain the desired result. This completes the proof of Theorem (1.2.1).

Theorem (1.2.2)[5]. There exists a constant C_n depending only on n such that

$$\|u\|_{L^q} \leqq C_n q \|u\|_{L^p}^{\frac{p}{q}} \|u\|_{BMO}^{1-\frac{p}{q}} \tag{41}$$

holds for all $u \in L^p \cap BMO$ with $1 \leqq p < \infty$ andfor all q with $p \leqq q < \infty$.

An immediate consequence of Lemma (1.2.7) and (1.2.8).

Corollary (1.2.3)[5]. (i) For every $1 < r < \infty$, there exists a positive constant $\alpha_{n,r}$ depending only on n and r such that thefollowing inequality holds. That is, for every $0 < \alpha < \alpha_{n,r}$, there exists a constant $C_{n,r,\alpha}$ which depends only on n, r and α such that

$$\int_{\mathbb{R}^n} \Phi_{p,r}\left(\alpha\left(\frac{|u(x)|}{\|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}}\right)^{r'}\right) dx \leqq C_{n,r,\alpha}\left(\frac{\|u\|_{L^p}}{\|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}}\right)^p$$

holds for all for all $u \in L^p \cap \dot{H}^{n/r,r}$ with $1 \leqq p < \infty$, where $\Phi_{p,r}$ is the function defined by

$$\Phi_{p,r}(t) := \exp t - \sum_{\substack{j < \frac{p}{r'} \\ j \in \mathbb{N} \cup \{0\}}} \frac{t^j}{j} \quad t \in \mathbb{R}.$$

(ii) There exists a positive constant α_n depending only on n such that the following inequality holds. That is, for every $0 < \alpha < \alpha_n$, there exists a constant $C_{n,\alpha}$ depending only on n and α such that

$$\int_{\mathbb{R}^n} \tilde{\Phi}_p\left(\alpha \frac{|u(x)|}{||u||_{BMO}}\right) dx \leqq C_{n,\alpha}\left(\frac{||u||_{Lp}}{||u||_{BMO}}\right)^p$$

holds for all $u \in L^p \cap BMO$ with $1 \leqq p < \infty$, where $\tilde{\Phi}_p$ is defined by

$$\tilde{\Phi}_p(t) := \exp t - \sum_{\substack{j < p, \\ j \in \mathbb{N} \cup \{0\}}} \frac{t^j}{j!}, \xi \in \mathbb{R}.$$

By taking $p = 1$ in Corollary (1.2.3)(ii), we have the following generalized John-Nirenberg type inequality.

Proof. (i) By applying Theorem (1.2.1), we see that

$$\int_{\mathbb{R}^n} \Phi_{p,r}\left(\alpha\left(\frac{|u(x)|}{\|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}}\right)^{r'}\right) dx = \int_{\mathbb{R}^n} \sum_{\substack{j \geqq \frac{p}{r}, \\ j \in \mathbb{N}}} \frac{\alpha^j}{j!}\left(\frac{|u(x)|}{\|(-\triangle)^{\frac{n}{2r}} u\|_{L^r}}\right)^{r' j} dx$$

$$= \sum_{\substack{j\geqq\frac{p}{r}\\ j\in\mathbb{N}}}, \frac{\alpha^j}{j!}\frac{\|u\|_{L^{r'j}}^{r'j}}{\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}^{r'j}}$$

$$\leqq \sum_{\substack{j\geqq\frac{p}{r}\\ j\in\mathbb{N}}}, \frac{\alpha^j}{j!}\frac{\left(C_n r'(r'j)^{\frac{1}{r}}\|u\|^{\frac{p}{L^p rj}1-\frac{p}{rj}}\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}\right)^{r'j}}{\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}^{rj}}$$

$$\leqq \left(\sum_{j=1}^{\infty} a_j\left(\alpha C_n^{r'} r^{rr'+1}\right)^j\right)\left(\frac{\|u\|_{Lp}}{\|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}}\right)^p$$

where $a_j = j^j/(j!)$. Since $\lim_{j\to\infty} a_j/a_{j+1} = e^{-1}$, the power series of the above righthand side converges provided $\alpha C_n^{r'} r^{rr'+1} < e^{-1}$ i.e.,

$$\alpha < \alpha_{n,r} := \left(C_n^{r'} r^{rr'+1} e\right)^{-1}$$

This proves Corollary (1.2.3) (i).

(ii) In the same way as we have derived Corollary (1.2.3)(i) from Theorem (1.2.1), it is easy to see that Theorem (1.2.2) yields Corollary (1.2.3)(ii), so we omit the detail.

Corollary (1.2.4)[5]. There exist two positive constants C_n and α_n depending only on n such that

$$\mu(\{x \in \mathbb{R}^n; |u(x)| > t\}) \leqq C_n \frac{\|u\|_{L^1}}{\|u\|_{BMO}} \frac{1}{\exp\left(\frac{\alpha_n t}{\|u\|BMO}\right) - 1}$$

holds for all $u \in L^1 \cap BMO$ and all $t > 0$. In particular, we have

$\mu(\{x \in \mathbb{R}^n; |u(x)| > t\}) \leqq C_n \frac{\|u\|_{L^1}}{\|u\|_{BMO}} \exp\left(-\frac{\alpha_n t}{\|u\|_{BMO}}\right)$

holds for all $u \in L^1 \cap BMO$ and for all $t > \|u\|_{BMO}$.

By Corollary (1.2.3)(ii) with $= 1$, we have

$$\int_{\mathbb{R}^n}\left(\exp\left(\alpha_n \frac{|u(x)|}{\|u\|_{BMO}}\right) - 1\right)dx \leqq C_n \frac{\|u\|_{L^1}}{\|u\|_{BMO}}$$

for all $u \in L^1 \cap BMO$. Since the distribution function $\lambda_u(t)$ is non-increasing, we obtain from (48) that

$$\int_{\mathbb{R}^n}\left(\exp\left(\alpha_n \frac{|u(x)|}{\|u\|_{BMO}}\right) - 1\right)dx = \sum_{j=1}^{\infty} \frac{\alpha_n^j}{j!} \frac{\|u\|_L^{j_j}}{\|u\|_{BMO}^j}$$

$$= \sum_{j=1}^{\infty} \frac{\alpha_n^j}{j!} \frac{j}{\|u\|_{BMO}^j} \int_0^{\infty} \lambda_u(\tau) \tau^{j-1} d\tau$$

$$\geqq \lambda_u(t) \sum_{j=1}^{\infty} \frac{\alpha_n^j}{j!} \frac{j}{\|u\|_{BMO}^j} \int_0^{t} \tau^{j-1}\, d\tau$$

$$= \lambda_u(t) \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{\alpha_n t}{\|u\|_{BMO}} \right)^j$$

$$= \lambda_u(t) \left(\exp\left(\frac{\alpha_n t}{\|u\|_{BMO}} \right) - 1 \right)$$

for all $t > 0$. Hence, we have

$$\lambda_u(t) \leqq C_n \frac{\|u\|_{L^1}}{\|u\|_{BMO}} \frac{1}{\exp\left(\frac{\alpha_n t}{\|u\|_{BMO}}\right) - 1} \tag{42}$$

for all $\lambda > 0$. In particular, if $t > \|u\|_{BMO}$, then by (42) it holds that

$$\lambda_u(t) \leqq C_n \frac{\|u\|_{L^1}}{\|u\|_{BMO}} \exp\left(-\frac{\alpha_n t}{\|u\|_{BMO}} \right).$$

Finally, we shall show the Brezis−Gallouet−Wainger type inequalities Theorem (1.2.5)(i) and (ii) by applying Theorems (1.2.1) and (1.2.2), respectively. Since those proofs are quite similar, we may only show Theorem (1.2.5)(i).

Theorem (1.2.5)[5]. (i) For every $1 \leqq p < \infty$, $1 < r < \infty$, $1 < q <$ oo and $n/q < s < \infty$, there exists a constant $C = C_{n,p,r,q,s}$ such that

$$\|u\|_{L^\infty} \leqq C \left(1 + \left(\|u\|_{L^p} + \|(-\triangle)^{\frac{n}{2r}} u\|_r \right) \left(\log\left(e + \|(-\triangle)^{\frac{s}{2}} u\|_{L^q} \right) \right)^{\frac{1}{r}} \right)$$

holds for all $u \in L^p \cap \dot{H}^{n/r,r}$ with $(-\triangle)^{s/2} u \in L^q$.

(ii) For every $1 \leqq p < \infty$, $1 < q < \infty$ and $n/q < s < \infty$, there exists a constant $C = C_{n,p,q,s}$ such that

$$\|u\|_{L^\infty} \leqq C \left(1 + (\|u\|_{L^p} + \|u\|_{BMO}) \log\left(e + \|(-\triangle)^{\frac{s}{2}} u\|_{L^q} \right) \right)$$

holds for all $u \in L^p \cap BMO$ with $(-\triangle)^{s/2} u \in L^q$.

Proof. (i) Let $v : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be the solution of the following heat equation with the fractional derivative.

$$\frac{\partial v}{\partial t} = -(-\triangle)^{\frac{s}{2}} v \text{ in } \mathbb{R}^n \times (0, \infty),$$

$$v(\cdot, 0) = \phi \text{ in } \mathbb{R}^n,$$

where ϕ is the given initial data belonging to $\mathcal{S}$. By Lemma (1.2.10), we have the expression of v such as

$$v(\cdot,t) = G_t^s * \phi.$$
Take $u = u(x) \in L^p \cap \dot{H}^{n/r,r}$ with $(-\triangle)^{s/2}u \in L^q$ for $s > n/q$. Since

$$\int_0^t \left(-(-\triangle)^{\frac{s}{2}}u, v(\tau)\right) d\tau = \int_0^t \left(u, -(-\triangle)^{\frac{s}{2}}v(\tau)\right) d\tau = \int_0^t \left(u, \frac{\partial v}{\partial \tau}\right) d\tau$$

$$= \left(u, v(t)\right) - (u, \phi)\,,$$

we have

$$|(u,\phi)| \leqq |\left(u, v(t)\right)| + \int_0^t |\left((-\triangle)^{\frac{s}{2}}u, v(\tau)\right)| d\tau =: J_1(t) + J_2(t)$$

for all $t > 0$. Here, (.,) denotes the usual inner product in L^2. From the Hölder inequality, Lemma (1.2.10) and Theorem (1.2.1), we obtain

$$J_1(t) \leqq \|u\|_{L^{\tilde{q}}} \|v(t)\|_{L^{\tilde{q}'}} = \|u\|_{L^{\tilde{q}}} \| \|G_t^s * \phi\|_{L^{\tilde{q}'}} \leqq C_{n,r,s} \tilde{q}^{\frac{1}{r}} t^{-\frac{n}{s\tilde{q}}}$$

$$\times \left(\|u\|_{Lp} + \|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}\right) \|\phi\|_{L^1} \tag{43}$$

for all $t > 0$ and $p \leqq \tilde{q} < \infty$, where $C_{n,r,s}$ is a constant depending only on n, r and s.
Again by the Hölder inequality and Lemma (1.2.10), we have

$$J_2(t) \leqq \int_0^t \| (-\triangle)^{\frac{s}{2}}u\|_{Lq} \|v(\tau)\|_{L^{q'}} d\tau = \|(-\triangle)^{\frac{s}{2}}u\|_{Lq} \int_0^t \| G_\tau^s * \phi\|_{L^{q'}} d\tau$$

$$\leqq C_{n,q,s} \left\|(-\triangle)^{\frac{s}{2}}u\|_{Lq}\right\| \phi\|_{L^1} \int_0^t \tau^{-\frac{n}{(sq)}} d\tau$$

$$= C_{n,q,s} t^{1-\frac{n}{sq}} \left\|(-\triangle)^{\frac{s}{2}}u\|_{Lq}\right\| \phi\|_{L^1} \tag{44}$$

for all $t > 0$, where $C_{n,q,s}$ is a constant depending only on n, q and s. Hence, from (43), (44) and the duality argument, we obtain

$$\|u\|_{L^\infty}= \|\phi\|_{L^1} \sup |(u,\phi)| \leqq 1$$

$$\leqq C_{n,r,q,s} \left[\tilde{q}^{\frac{1}{r}} t^{-\frac{n}{s\tilde{q}}} \left(\|u\|_{Lp} + \|(-\triangle)^{\frac{n}{2r}}u\|_{L^r}\right) + t^{1-\frac{n}{sq}} \|(-\triangle)^{\frac{s}{2}}u\|_{Lq}\right] \tag{45}$$

for all $t > 0$ and $p \leqq \tilde{q} < \infty$, where $C_{n,r,q,s}$ is a constant depending only on n, r, q and s.

Now we take $\tilde{q} \geqq p$ and $t > 0$ in (45) so that

$$\tilde{q} = \log(1/t)\,, t = \left\{e^p + \|(-\triangle)^{\frac{s}{2}}u\|_{L^q} \left(1 - \frac{n}{sq}\right)^{-1}\right\}^{-1}$$

Since $t^{-n/(s\tilde{q})} = \left(t^{1/\log t}\right)^{n/s} = e^{n/s}$ and since

$$t^{1-\frac{n}{sq}}\|(-\triangle)^{\frac{s}{2}}u\|_{L^q}=\left\{e^p+\|(-\triangle)^{\frac{s}{2}}u\|_{Lq}\left(1-\frac{n}{sq}\right)^{-1}\right\}^{-\left(1-\frac{n}{sq}\right)}\|(-\triangle)^{\frac{s}{2}}u\|_{L^q}\leqq 1,$$

we see that such choice of $\tilde{q}$ and t in (45) yields the desired estimate. We prepare some lemmata for the proof of the main theorems. First, to prove Theorem (1.2.1), we need to estimate the difference of L^s-norm between the translation of the Riesz potential and the usual Riesz potential. We denote by I_α , $0 < \alpha < n$ the Riesz kemel defined by

$$I_\alpha(x):=\frac{1}{\gamma(\alpha)}\tilde{I}_\alpha(x) \text{ with } \tilde{I}_\alpha(x):=|x|^{-(n-\alpha)},$$

where

$$\gamma(\alpha):=\frac{\pi^{\frac{n}{2}}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}.$$

Lemma (1.2.6)[5]. Let $1 < p < \infty$. For every s with $np'/(n+p') < s < p'$, it holds

$$\tilde{I}_{n/p}(\cdot-y)-\tilde{I}_{n/p}(\cdot)\in L^s$$

for all $y \in \mathbb{R}^n$. More precisely, we have

$$\int_{\mathbb{R}^n}|\tilde{I}_{n/p}(x-y)-\tilde{I}_{n/p}(x)|^s dx\leqq C_n\left(\frac{p'}{p'-s}+\frac{p'}{(n+p')s-np'}\right)|y|^{n\left(1-\frac{s}{p'}\right)}$$

for all $y \in \mathbb{R}^n$ with a constant C_n depending only on n.

Proof. We divide the domain of the integration into two parts. For any $\in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n}|\tilde{I}_{n/p}(x-y)-\tilde{I}_{n/p}(x)|^s dx=\int_{\{x\in\mathbb{R}^n;|x|\leqq 2|y|\}}|\tilde{I}_{n/p}(x-y)-\tilde{I}_{n/p}(x)|^s dx$$

$$+\int_{\{x\in\mathbb{R}^n;|x|>2|y|\}}|\tilde{I}_{n/p}(x-y)-\tilde{I}_{n/p}(x)|^s dx$$

$$=:J_1(y)+J_2(y)\,.$$

As for the estimate for $J_1(y)$, we see that

$$J_1(y)\leqq\int_{\{x\in\mathbb{R}^n;|x|\leqq 2|y|\}}\tilde{I}_{n/p}\,(x-y)^s dx$$

$$+\int_{\{x\in\mathbb{R}^n;|x|\leqq 2|y|\}}\tilde{I}_{n/p}\,(x)^s dx\ \leqq 2\int_{\{x\in\mathbb{R}^n;|x|\leqq 3|y|\}}\tilde{I}_{n/p}\,(x)^s dx$$

$$=2\int_{\{x\in\mathbb{R}^n;|x|\leqq 3|y|\}}|x|^{-\frac{ns}{p'}}dx\ \leqq C_n\frac{p'}{p-s}|y|^{n\left(1-\frac{s}{p'}\right)},\quad(46)$$

Here, we note that $\int_{\{x\in\mathbb{R}^n;|x|\leqq 3|y|\}} \tilde{I}_{n/p}(x)^s dx$ is integrable since $s < p'$. On the other hand, for $|x| > 2|y|$, we have

$$|\tilde{I}_{n/p}(x-y) - \tilde{I}_{n/p}(x)| = |\int_0^1 \frac{d}{d\tau}[\tilde{I}_{n/p}(x-\tau y)]d\tau| = |\int_0^1 (\nabla\tilde{I}_{n/p})(x-\tau y)\cdot(-y)d\tau|$$

$$= \frac{n}{p}|y|\int_0^1 |x-\tau y|^{-\frac{n}{p'}-1}d\tau \leqq \frac{n}{p}|y||x|^{-\frac{n}{p'}-1} \times \int_0^1 |1-\tau\frac{|y|}{|x|}|^{-\frac{n}{p'}-1}d\tau$$

$$\leqq \frac{n}{p}|y||x|^{-\frac{n}{p'}-1}\int_0^1 \left(1-\frac{\tau}{2}\right)^{-\frac{n}{p'}-1}d\tau = 2\left(2^{\frac{n}{p'}}-1\right)|y||x|^{-\frac{n}{p'}-1}$$

which yields

$$J_2(y) \leqq 2^s\left(2^{\frac{n}{p'}}-1\right)^s \left|y|^s\int_{\{x\in\mathbb{R}^n;|x|>2|y|\}} |x|^{\left(-\frac{n}{p'}-1\right)s}dx \leqq C_n\frac{p'}{(n+p')s-np'}\right|y|^{n\left(1-\frac{s}{p'}\right)} \quad (47)$$

Notice that since $> np'/(n+p')$, we have $\int_{\{x\in\mathbb{R}^n;|x|>2|y|\}} |x|^{(-n/p'-1)s}dx < \infty$. From (46) and (47), we obtain the desired estimate. For a measurable function u on $\mathbb{R}^n$, let us recall the distribution function $\lambda_u(t)$ and the nonincreasing rearrangement $u^*(t)$ for $t > 0$. For detail, see E.M. Stein-G.Weiss [22]. We denote by u^{**} the average function of u^* defined by

$$u^{**}(t) := \frac{1}{t}\int_0^t u^*(\tau)d\tau \; for \; t > 0.$$

Notice that u^{**} is continuous and nonincreasing on $(0,\infty)$. It is well known that

$$\|u\|_{L^p} = p\int_0^\infty \lambda_u(t)t^{p-1}dt = \int_0^\infty u^*(t)^p dt, 1 \leqq p < \infty \quad (48)$$

and that

$$\|u\|_{L^\infty} = \sup u^*(t) = \lim_{t\downarrow 0} u^*(t) = \sup u^{**}(t) = \lim_{t\downarrow 0} u^{**}(t) .$$

Since

$$\int_0^t u^*(\tau)d\tau \leqq t^{\frac{1}{p}}\|u\|_{Lp}, 1 \leqq p \leqq \infty$$

holds for $t > 0$, we have

$$u^*(t) \leqq u^{**}(t) \leqq t^{-\frac{1}{p}}\|u\|_{L^p}, 1 \leqq p \leqq \infty \quad (49)$$

for all $t > 0$. See, e.g., Stein and Weiss [22]. Moreover, it is easy to see that if $\{u_m\}_{m=1}^\infty$ satisfies $|u_1(x)| \leqq |u_2(x)| \leqq \leqq |u_m(x)| \leqq$ with $\lim_{m\to\infty} |u_m(x)| = |u(x)|$ for a.e. $x \in \mathbb{R}^n$, then it holds $u_1^*(t) \leqq u_2^*(t) \leqq \leqq u_m^*(t) \leqq$ with $\lim u_m^*(t) = u^*(t)$ for all $t > 0$. We define a new function space W by

$$W := \{u \in L^1_{loc}(\mathbb{R}^n)\ ;\ \|u\|_W := \sup\big(u^{**}(t) - u^*(t)\big) < \infty\}.$$

Then, the assertion of Theorem (1.2.2) is an immediate consequence of the following two lemmata.

Lemma (1.2.7)[5]. There exists an absolute constant C such that

$$\|u\|_{Lq} \leqq Cq\|u\|_{L^p}^{\frac{p}{q}}\|u\|_W 1 - \frac{p}{q}$$

holds for all u ∈ L^p ∩ W with 1 ≦ p < ∞ andfor all q with p ≦ q < ∞.

Proof. Let $s > 0$. By (48), it holds that

$$\|u\|_{L^q} = \left(\int_0^\infty u^*(t)^q dt\right)^{\frac{1}{q}} \leqq \left(\int_0^S u^*(t)^q dt\right)^{\frac{1}{q}} + \left(\int_s^\infty u^*(t)^q dt\right)^{\frac{1}{q}} \equiv J_1(s) + J_2(s)\,.$$

Since u^* is non-increasing, we have by (48) and (49) that

$$J_2(s) \leqq \|u^*\|_{L^p(s,\infty)}\|u^*\|_{L^\infty s,\infty)}\frac{p}{q}1 - \frac{p}{q,(}$$

$$\leqq \left\| u^*\|_{L^p(0,\infty)}^{\frac{p}{q}} u^*(s)^{1-\frac{p}{q}} \leqq s^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \right\| u\|_{L^p}. \tag{50}$$

Since

$$\frac{d}{dt}u^{**}(t) = -\frac{d}{dt}\left(\frac{1}{t}\int_0^t u^*(\tau)d\tau\right) = \frac{u^{**}(t) - u^*(t)}{t} \leqq \frac{\|u\|_W}{t} \tag{51}$$

holds for all $t > 0$, we have

$$u^{**}(t) - u^{**}(s) = \int_t^s\left(-\frac{d}{d\tau}u^{**}(\tau)\right)d\tau \leqq \int_t^s \frac{d\tau}{\tau}\|u\|_W = \left(\log\frac{s}{t}\right)\|u\|_W,\ 0 < t \leqq s,$$

which yields with the aid of (49) that

$$u^{**}(t) \leqq u^{**}(s) + \left(\log\frac{S}{t}\right)\|u\|_W = s^{-\frac{1}{p}}\|u\|_{L^p} + \left(\log\frac{S}{t}\right)\|u\|_W\,, 0 < t \leqq s. \tag{52}$$

Hence from (49) and (52) we obtain

$$J_1(s) \leqq \left(\int_0^S u^{**}(t)^q dt\right)^{\frac{1}{q}} \leqq s^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p} + \left(\int_0^S \left(\log\frac{S}{t}\right)^q dt\right)^{\frac{1}{q}}\|u\|_W$$

$$= s^{-\left(\frac{11}{pq}\right)}\|u\|_{L^p} + s^{\frac{1}{q}}\Gamma(q+1)^{\frac{1}{q}}\|u\|_W \leqq Cq\left(s^{-\left(\frac{11}{pq}\right)}\left\|u\|_{L^p} + s^{\frac{1}{q}}\right\|u\|_W\right), \tag{53}$$

where C is an absolute constant independent of q. In the above, we have used the fact that $\Gamma(q+1)^{\frac{1}{q}} = O(q)$ as $q \to \infty$. It follows from (50) and (53) that

$$\|u\|_{Lq} \leqq Cq\left(s^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{Lp} + s^{\frac{1}{q}}\|u\|_W\right)$$

for all $s > 0$ with an absolute constant C independent of q. Now, taking $s = (\|u\|_{L^p}/\|u\|_W)^p$ in this estimate, we have the desired result.

Lemma (1.2.8)[5]. There exists a constant C_n depending only on n such that

$$\|u\|_W \leqq C_n \|u\|_{BMO}$$

holds for all u ∈ BMO.

Proof. For $R > 0$ we denote by Q_R and by $\chi_R := \chi_{Q_R}$ the cube centered at the origin with the side length R and the characteristic function on Q_R, respectively. It is easy to see that

$$(U\chi_R)^{\#}_{Q_R} = u^{\#}_{Q_R}. \qquad (54)$$

For every fixed $t > 0$ we take R so that $\mu(Q_R)/6 > t$. Since $u\chi_R \in L^1(\mathbb{R}^n)$ with supp $u\chi_R \subset \overline{Q_R}$, it follows from (54) and Proposition (1.2.9) that

$$(u\chi_R)^{**}(t) - (u\chi_R)^{*}(t) \leqq C_n\left((u\chi_R)^{\#}_{Q_R}\right)^{*}(t) = C_n\left(u^{\#}_{Q_R}\right)^{*}(t) \leqq C_n\left(u^{\#}_{\mathbb{R}^n}\right)^{*}(t) \,,$$

which yields that

$$(u\chi_R)^{**}(t) \leqq (u\chi_R)^{*}(t) + C_n\left(u^{\#}_{\mathbb{R}^n}\right)^{*}(t) \leqq u^{*}(t) + C_n\left(u^{\#}_{\mathbb{R}^n}\right)^{*}(t) \qquad (55)$$

for all $t > 0$ and all $R > 0$ such that $(6t)^{1/n} < R$. Since $|u\chi_R| \uparrow |u|$ as $R \to \infty$ a.e.in $\mathbb{R}^n$, it holds $(u\chi_R)^* \uparrow u^*$, which yields also $(u\chi_R)^{**} \uparrow u^{**}$ as $R \to \infty$. Since $\|u\#\|_{L^\infty} = \|u\|_{BMO}$, by letting $R \to \infty$ in (55), we have

$$u^{**}(t) - u^{*}(t) \leqq C_n\left(u^{\#}_{\mathbb{R}^n}\right)^{*}(t) \leqq C_n\|u^{\#}_{\mathbb{R}^n}\|_{L^\infty} = C_n\|u\|_{BMO} \qquad (55)$$

for all $0 < t < \infty$. Taking the supremum with respect to $0 < t < \infty$ in (55), we obtain the desired estimate. Finally, for the proof of Theorem (1.2.5) we need the $L^p - L^q$ estimates for the semigroup $\left\{e^{-t(-\triangle)^{s/2}}\right\}_{t\geqq 0}$ defined by $e^{-t(-\triangle)^{s/2}}u = G^s_t * u$ with $G^s_t := \mathcal{F}^{-1}\left(e^{-(2\pi|\cdot|)^s t}\right)$, where $\mathcal{F}^{-1}$ represents the Fourier inverse transformation.

Next, we introduce the sharp function $u^{\#}_{\Omega}$ of u relative to the domain $\Omega \subset \mathbb{R}^n$ defined by

$$u^{\#}_{\Omega}(x) := \begin{cases} \tilde{Q} \subset \Omega \sup \frac{1}{\mu(\tilde{Q})}\int_{\tilde{Q}} |u(y) - u_{\overline{Q}}|dy, & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

with the mean value $u_{\overline{Q}} := \mu(\tilde{Q})^{-1} \int_{\tilde{Q}} u\,(y)dy$ on $\tilde{Q}$, where the supremum is taken over all cubes that contain x and are contained in Ω. For the proof of Lemma (1.2.8), we make use of the following proposition by Bennett-Sharpley [9].

Proposition (1.2.9)[5]. There exists a constant C_n depending only on n such that

$$u^{**}(t) - u^{*}(t) \leqq C_n\left(u^{\#}_{Q}\right)^{*}(t)\,, 0 < t < \mu(Q)/6$$

holds for all cubes Q in $\mathbb{R}^n$ and all u ∈ $L^1(\mathbb{R}^n)$ with supp u⊂ $\overline{Q}$.

Lemma (1.2.10)[5]. For every $0 < s < \infty$, there exists a constant $C_{n,s}$ depending only on n and s such that

$$\|e^{-t(-\triangle)^{s/2}}u\|_{Lq} \leqq c_{n,s^t} - \frac{n}{s}\left(\frac{1}{p} - \frac{1}{q}\right)_{\|u\|_{Lp}}$$

holds for all $u \in L^p$ andfor all $t > 0$, where p and q are any numbers satisfying $1 \leqq p \leqq q \leqq \infty$.

Proof. Let us first recall that $\mathcal{F}^{-1}\left(e^{-(2\pi|\cdot|)^s}\right) \in L^1 \cap L^\infty$, which states

$$\|G_t^s\|_{L^r} = t^{-\frac{n}{s}\left(1-\frac{1}{r}\right)}\|\mathcal{F}^{-1}\left(e^{-(2\pi|\cdot|)^s}\right)\|_{L^r} \leqq C_{n,s}t^{-\frac{n}{s}\left(1-\frac{1}{r}\right)}$$

for all $t > 0$ and all $1 \leqq r \leqq \infty$. See e.g., Bendikov [8] and Jacob [13]. For every $1 \leqq p \leqq q \leqq \infty$, we take r so that $1/r + 1/p = 1/q + 1$. Then it follows from the the Young inequality that

$$\|e^{-t(-\triangle)^{\frac{s}{2}}}u\|_{L^q} \leqq \|G_t^S\|_{L^r}\|u\|_{L^p} \leqq c_{n,s}t^{-\frac{n}{s}\left(1-\frac{1}{r}\right)_{\|u\|_{L^p}=C_{n,s}t^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)_{\|u\|_{L^p}}}}},$$

which yields the desired estimate.

Chapter 2
Maximal and Area Integral and Cauchy-Type Integrals

We show that the proofs utilize some sharp estimates of the Bergman kernel function and Bergman metric. The characterizations extend to cover Besov-Sobolev spaces. A special case of this is a characterization of H^p spaces involving only area functions on Bergman balls in the unit ball of C^n. We show L^p estimates for these operators and, as a consequence, to obtain L^p estimates for the canonical Cauchy–Szegö and Bergman projection operators (which are not of Cauchy–Fantappié type).

Section (2.1): Characterizations of Bergman Spaces in the Unit Ball Of C^n

For $\mathbb{C}$ denote the set of complex numbers. We fix a positive integer n, and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the Euclidean space of complex dimension n. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ componentwise. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where $\overline{w}_k$ is the complex conjugate of w_k. We also write

$$|z| = \sqrt{|z_1|^2 + +|z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathrm{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The boundary of B_n will be denoted by S_n and is called the unit sphere in $\mathbb{C}^n$, i.e.,

$$\mathrm{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}.$$

Also, we denote by $\overline{\mathrm{B}}_n$ the closed unit ball, i.e.,

$$\overline{\mathrm{B}}_n = \{z \in \mathbb{C}^n : |z| \le 1\} = \mathrm{B}_n \cup \mathrm{S}_n.$$

The automorphism group of B^n, denoted by $Aut(B_n)$, consists of all biholomorphic mappings of B^n. Traditionally, bi-holomorphic mappings are also called automorphisms.

Recall that for $\alpha > -1$ and $p > 0$ the (weighted) Bergman space $\mathcal{A}_\alpha^p$ consists of holomorphic functions f in B_n with

$$\|f\|_{p,\alpha} = \left(\int_{\mathrm{B}_n} |f(z)|^p dv_\alpha(z)\right)^{1/p} < \infty,$$

where the weighted Lebesgue measure dv_α on B_n is defined by

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

and $c_\alpha = \Gamma(n + \alpha + 1)/[n!\,\Gamma(\alpha + 1)]$ is a normalizing constant so that dv_α is a probability measure on B_n. Thus,

$$\mathcal{A}_\alpha^p = \mathcal{H}(\mathrm{B}_n) \cap L^p(\mathrm{B}_n, dv_\alpha),$$

where $\mathcal{H}(\mathrm{B}_n)$ is the space of all holomorphic functions in B_n. When $\alpha = 0$ we simply write $\mathcal{A}^p$ for $\mathcal{A}_0^p$. These are the usual Bergman spaces. Note that for $1 \le p < \infty$, $\mathcal{A}_\alpha^p$ is a Banach space under the norm $\|\|_{p,\alpha}$. If $0 < p < 1$, the space $\mathcal{A}_\alpha^p$ is a quasi-Banach space with p-norm $\|f\|_{p,\alpha}^p$.

Recall that $D(z,\gamma)$ denotes the Bergman metric ball at z

$$D(z,\gamma) = \{w \in \mathrm{B}_n : \beta(z,w) < \gamma\}$$

with $\gamma > 0$, where β is the Bergman metric on B_n. It is known that

$$\beta(z,w) = \frac{1}{2} \log \frac{1 + |\phi_z(w)|}{1 - |\phi_z(w)|}, z, w \in \mathrm{B}_n,$$

whereafter ϕ_z is the bijective holomorphic mapping in B_n, which satisfies $\phi_z(0) = z$, $\phi_z(z) = 0$ and $\phi_z \circ \phi_z = id$.

Maximal functions play a crucial role in the real-variable theory of Hardy spaces (cf. [12] we first establish a maximalfunction characterization for the Bergman spaces. To this end, we define for each $\gamma > 0$ and $\in \mathcal{H}(\mathrm{B}_n)$:

$$\left(\mathrm{M}_\gamma f\right)(z) = \sup_{w \in D(z,y)} |f(w)|, \forall z \in \mathrm{B}_n. \quad (1)$$

We begin with the following simple result.

Theorem (2.1.1)[24]: Suppose $\gamma > 0$ and $\alpha > -1$. Let $0 < \mathrm{p} < \infty$. Then for any $\mathrm{f} \in \mathcal{H}(\mathrm{B_n})$, $\mathrm{f} \in \mathcal{A}_\alpha^{\mathrm{p}}$ if and only if $\mathrm{M}_\gamma \mathrm{f} \in \mathrm{L}^{\mathrm{p}}(\mathrm{B_n}, \mathrm{dv}_\alpha)$. Moreover,

$$\|f\|_{p,\alpha} \approx \|\mathrm{M}_\gamma f\|_{p,\alpha}, \quad (2)$$

where ($\zeta \approx$ depends only on γ, α, p, and n.

The norm appearing on the right-hand side of (2) can be viewed an analogue of the so-called nontangential maximal function in Hardy spaces. The proof of Theorem (2.1.1) is fairly elementary (3), using some basic facts and estimates on the Bergman balls.

In order to state the real-variable area integral characterizations of the Bergman spaces, we require some more notation. For any $f \in \mathcal{H}(\mathrm{B}_n)$ and $z = (z_1, \dots, z_n) \in \mathrm{B}_n$ we define

$$\mathcal{R}f(z) = \sum_{k=1}^{n} z_k \frac{\partial f(z)}{\partial z_k}$$

and call it the radial derivative of f at z. The complex and invariant gradients of f at z are respectively defined as

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \dots, \frac{\partial f(z)}{\partial z_n}\right) \text{ and } \overline{\nabla} f(z) = \nabla(f o \phi_z)(0) .$$

Now, for fixed $\gamma > 0$ we define for each $f \in \mathcal{H}(\mathrm{B}_n)$ and $\in \mathrm{B}_n$:

(i) The radial area function

$$A_\gamma(\mathcal{R}f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2)\mathcal{R}f(w)|^2 \frac{dv(w)}{(1 - |w|^2)^{n+1}}\right)^{\frac{1}{2}}$$

(ii) The complex gradient area function

$$A_\gamma(\nabla f)(z)=\left(\int_{D(z,\gamma)}|(1-|w|^2)\nabla f(w)|^2\frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}}$$

(iii) The invariant gradient area function

$$A_\gamma\big(\tilde{\nabla} f\big)(z)=\left(\int_{D(z,\gamma)}|\tilde{\nabla} f(w)|^2\frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}}$$

We state another main result as follows.

Theorem (2.1.2)[24]: Suppose $\gamma > 0$ and $\alpha > -1$. Let $0 < p < \infty$. Then, for any $f \in \mathcal{H}(B_n)$ the following conditions are equivalent:

(i) $f \in \mathcal{A}^p_\alpha$.

(ii) $A_\gamma(\mathcal{R}f)$ is in $L^p(B_n, dv_\alpha)$.

(iii) $A_\gamma(\nabla f)$ is in $L^p(B_n, dv_\alpha)$.

(v) $A_\gamma\big(\tilde{\nabla} f\big)$ is in $L^p(B_n, dv_\alpha)$.

Moreover, the quantities

$$|f(0)|+\|A_\gamma(\mathcal{R}f)\|_{p,\alpha},\ |f(0)|+\|A_\gamma(\nabla f)\|_{p,\alpha},\ |f(0)|+\|A_\gamma\big(\tilde{\nabla} f\big)\|_{p,\alpha},$$

are all comparable to $\|f\|_{p,\alpha}$, where the comparable constants depend only on γ, α, p, and n.

In particular, taking the equivalence of (a) and (b), one obtains

$$\|f\|_{p,\alpha}\approx|f(0)|+\|A_\gamma(\mathcal{R}f)\|_{p,\alpha},$$

which looks tantalizingly simple. There is a mature and powerful real variable Hardy space theory which has distilled some of the essential oscillation and cancellation behavior of holomorphic functions and then found that behavior ubiquitous. A good introduction to that is [33]; a more recent and fuller account is in [25], [35], [36], [37]. However, the real-variable theory of the Bergman space is less well developed, even in the case of the unit disc (cf. [34]). We remark that the first real-variable characterization of the Bergman spaces was presented by Coifman and Weiss in 1970's. Recall that

$$\varrho(z,w)=\begin{cases}||z|-|w||+|1-\dfrac{1}{|z||w|}\langle z,w\rangle|, \text{if } z,w\in B_n\setminus\{0\},\\ |z|+|w|, \text{otherwise}\end{cases}$$

is a pseudo-metric on B_n and $(B_n, \varrho, dv_\alpha)$ is a homogeneous space. By their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [33] can use ϱ to obtain a real-variable atomic decomposition for Bergman spaces. However, since the Bergman metric β underlies the complex geometric structure of the unit ball of $\mathbb{C}^n$, one would prefer to real-variable characterizations of the Bergman spaces in terms of β. Clearly, our results above are such a characterization.

These two real-variable characterizations can be extended to the so-called generalized

Bergman spaces [40]. For $0 < p < \infty$ and $-\infty < \alpha < \infty$ we fix a nonnegative integer k with $pk + \alpha > -1$ and define $\mathcal{A}_\alpha^p$ as the space of all $f \in \mathcal{H}(\mathrm{B}_n)$ such that $(1 - |z|^2)^k \mathcal{R}^k f \in L^p(\mathrm{B}_n, dv_\alpha)$. One then easily observes that $\mathcal{A}_\alpha^p$ is independent of the choice of k and consistent with the traditional definition when $\alpha > -1$. Let N be the smallest nonnegative integer such that $pN + \alpha > -1$ and define

$$\|f\|_{p,\alpha} = |f(0)| + \left(\int_{\mathrm{B}_n} (1 - |z|^2)^{pN} |\mathcal{R}^N f(z)|^p dv_\alpha(z)\right)^{\frac{1}{p}} f \in \mathcal{A}_\alpha^p \quad (3)$$

Equipped with (3), $\mathcal{A}_\alpha^p$ becomes a Banach space when $p \geq 1$ and a quasiBanach space for $0 < p < 1$.

Corollary (2.1.3)[24]. Suppose $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k be a nonnegative integer such that $pk + \alpha > -1$. Then for any $\in \mathcal{H}(\mathrm{B_n})$, $f \in \mathcal{A}_\alpha^p$ if and only if $\mathrm{M}_\gamma(\mathcal{R}^k f) \in L^p(\mathrm{B_n}, dv_\alpha)$, where

$$\mathrm{M}_\gamma(\mathcal{R}^k f)(z) = \sup_{w \in D(z,y)} \left|(1 - |w|^2)^k \mathcal{R}^k f(w)\right|, z \in \mathrm{B}_n. \quad (4)$$

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|\mathrm{M}_\gamma(\mathcal{R}^k f)\|_{p,\alpha}, \quad (5)$$

where ($\zeta \approx$ depends only on γ, α, p, k, and n.

Corollary (2.1.4)[24]. Suppose $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k be a nonnegative integer such that $pk + \alpha > -1$. Then for any $\in \mathcal{H}(\mathrm{B_n})$,

$f \in \mathcal{A}_\alpha^p$ if and only if $A_\gamma(\mathcal{R}^{k+1} f)$ is in $L^p(\mathrm{B_n}, dv_\alpha)$, where

$$A_\gamma(\mathcal{R}^k f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2)^k \mathcal{R}^k f(w)|^2 \frac{dv(w)}{(1 - |w|^2)^{n+1}}\right)^{\frac{1}{2}} \quad (6)$$

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|A_\gamma(\mathcal{R}^{k+1} f)\|_{p,\alpha}, \quad (7)$$

where ($\zeta \approx$ depends only on γ, α, p, k, and n. To prove Corollaries (2.1.3) and (2.1.4), one merely notices that $f \in \mathcal{A}_\alpha^p$ if and only if $\mathcal{R}^k f \in L^p(\mathrm{B}_n, dv_{\alpha+pk})$ and applies Theorems (2.1.1) and (2.1.2) respectively to $\mathcal{R}^k f$ with the help of Lemma (2.1.10). Note that the family of the generalized Bergman spaces $\mathcal{A}_\alpha^p$ covers most of the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, which has been extensively studied before. For example, $B_p^s = \mathcal{A}_\alpha^p$ with $= -(ps + 1)$, where B_p^s is the classical diagonal Besov space consisting of holomorphic functions f in B_n such that $(1 - |z|^2)^{k-s} \mathcal{R}^k f$ belongs to $L^p(\mathrm{B}_n, dv_{-1})$ with

k being any positive integer greater than s. It is clear that $\mathcal{A}_\alpha^p = B_p^s$ with $s = -(\alpha+1)/p$. Thus the generalized Bergman spaces $\mathcal{A}_\alpha^p$ are exactly the diagonal Besov spaces. See Arcozzi-RochbergSawyer [23], [29] and Volberg-Wick [39] for some recent results on such Besov spaces. On the other hand, if k is a positive integer, p is positive, and β is real, then there is the Sobolev space $W_{k,\beta}^p$ consisting of holomorphic functions f in B_n such that the partial derivatives of f of order up to k all belong to $L^p(\mathrm{B}_n, dv_\beta)$ (cf. [26],[27], [30]). It is easy to see that these holomorphic Sobolev spaces are in the scale of the generalized Bergman spaces, i.e., $W_{k,\beta}^p = \mathcal{A}_\alpha^p$ with $= -(pk - \beta + 1)$. There are various characterizations for B_p^s or $W_{k,\beta}^p$ involving complexvariable quantities in terms of fractional differential operators and in terms of higher order (radical) derivatives and/or complex and invariant gradients, see [40]. However, Corollaries (2.1.3) and (2.1.4) can be considered as a unified characterization for such spaces involving real-variable quantities. In particular, $\mathcal{H}_s^p = \mathcal{A}_\alpha^p$ with $\alpha = -2s - 1$, where $\mathcal{H}_s^p$ is the Hardy-Sobolev space defined as the set

$$\{f \in \mathcal{H}(\mathrm{B}_n): \|f\|_{\mathcal{H}_s^p}^p = \sup \int_{\mathrm{S}_n} |(I+\mathcal{R})^s f(r\zeta)|^p d\sigma(\zeta) < \infty\}.$$

Here,

$$(I+\mathcal{R})^s f = \sum_{k=0}^{\infty} (1+k)^s f_k$$

if $f = \sum_{k=0}^{\infty} f_k$ is the homogeneous expansion of f. There are several realvariable characterizations of the Hardy-Sobolev spaces obtained by Ahern. These characterizations are in terms of maximal and area functions on the admissible approach region

$$D_\alpha(\eta) = \{z \in \mathrm{B}_n: |1 - \langle z, \eta\rangle| < \tfrac{\alpha}{2}(1 - |z|^2)\}, \eta \in \mathrm{S}_n, \alpha > 1.$$

Evidently, Corollaries (2.1.3) and (2.1.4) present new real-variable descriptions of the Hardy-Sobolev spaces in terms of the Bergman metric. A special case of this is a characterization of the usual Hardy space $\mathcal{H}^p = \mathcal{A}_{-1}^p$ itself.

We collect a number of auxiliary (and mostly elementary) facts about the Bergman metric and Bergman kernel functions, We show Theorems (2.1.1) and an atomic decomposition of $\mathcal{A}_\alpha^1$ with respect to Carleson tubes, which is of independent interests and crucial for the proof of Theorem (2.1.2) in the case of $p = 1$.

We devoted to the proof of Theorem (2.1.2) in the case of $p \geq 1$, which is somewhat involved and technical. Finally, we show Theorem (2.1.2) in the case of $0 < p < 1$ by using atom decomposition for Bergman spaces due to Coifman and Rochberg [32].

In what follows, C always denotes a constant depending only on $n, \gamma, p,$ and α, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y by $X \approx Y$ we mean that there exists a constant $C > 1$ such that $C^{-1}X \le Y \le CX$. Any notation and terminology not otherwise explained, are as used in [17] for spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$.

Recall that P_α is the orthogonal projection from $L^2(\mathrm{B}_n, dv_\alpha)$ onto $\mathcal{A}^2_\alpha$, which can be expressed as

$$P_\alpha f(z) = \int_{\mathrm{B}_n} K^\alpha (z,w) f(w) dv_\alpha(w) \, , \forall f \in L^1(\mathrm{B}_n, dv_\alpha), \alpha > -1,$$

where

$$K^\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}, z,w \in \mathrm{B}_n.$$

P_α extends to a bounded projection from $L^p(\mathrm{B}_n, dv_\alpha)$ onto $\mathcal{A}^p_\alpha (1 < p < \infty)$.

Also, we let

$$d(z,w) = |1-\langle z,w\rangle|^{\frac{1}{2}}, z,w \in \overline{\mathrm{B}}_n.$$

It is known that d satisfies the triangle inequality and the restriction of d to S_n is a metric. As usual, d is called the nonisotropic metric.

First, we show an inequality for reproducing kernel K^α associated with d, which is essentially borrowed in [38] with d instead of the pseudo-metric ϱ in B_n.

Lemma (2.1.5)[24]. For $\alpha > -1$ there exists a constant $\delta > 0$ such that for all $z, w \in \mathrm{B}_n$, $\zeta \in \mathrm{S}_n$ satisfying $(z,\zeta) > \delta d(w,\zeta)$, we have

$$|K^\alpha(z,w) - K^\alpha(z,\zeta)| \le C_{\alpha,n} \frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}}.$$

Proof. Note that

$$K^\alpha(z,w) - K^\alpha(z,\zeta) = \int_0^1 \frac{d}{dt}\left(\frac{1}{(1-\langle z,\zeta\rangle - t\langle z,w-\zeta\rangle)^{n+1+\alpha}}\right) dt.$$

We have

$$|K^\alpha(z,w) - K^\alpha(z,\zeta)| \le \int_0^1 \frac{(n+1+\alpha)|\langle z,w-\zeta\rangle|}{|1-\langle z,\zeta\rangle - t\langle z,w-\zeta\rangle|^{n+2+\alpha}} dt.$$

Write $z = z_1 + z_2$ and $w = w_1 + w_2$, where z_1 and w_1 are parallel to ζ, while z_2 and w_2 are perpendicular to ζ. Then

$$\langle z,w\rangle - \langle z,\zeta\rangle = \langle z_2,w_2\rangle - \langle z_1, w_1-\zeta\rangle$$

and so

$$|\langle z,w\rangle - \langle z,\zeta\rangle| \le |z_2||w_2| + |w_1-\zeta|.$$

Since $|w_1-\zeta| = |1-\langle w,\zeta\rangle|$,

$$\begin{aligned} |z_2|^2 = |z|^2 - |z_1|^2 < 1 - |z_1|^2 < (1+|z_1|)(1+|z_1|) \\ \le |1-\langle z_1,\zeta\rangle| = 2|1-\langle z,\zeta\rangle|, \end{aligned}$$

and similarly

$$|w_2|^2 \le 2|1-\langle w,\zeta\rangle|,$$

we have

$$|\langle z,w\rangle - \langle z,\zeta\rangle| \le 2|1-\langle z,\zeta\rangle|^{1/2}|1-\langle w,\zeta\rangle|^{1/2} + |1-\langle w,\zeta\rangle|$$

$$= 2d(w,\zeta)[d(z,\zeta) + d(w,\zeta)]$$
$$\leq 2\left(1 + \frac{1}{\delta}\right)\frac{1}{\delta}d^2(z,\zeta)\,.$$

This concludes that there is $\delta > 1$ such that

$$|\langle z, w - \zeta\rangle| < \frac{1}{2}|1 - \langle z,\zeta\rangle|, \forall z, w \in \mathrm{B}_n, \zeta \in \mathrm{S}_n,$$

whenever $(z,\zeta) > \delta d(w,\zeta)$. Then, we have

$$|1 - \langle z,\zeta\rangle - t\langle z, w - \zeta\rangle| > |1 - \langle z,\zeta\rangle| - t|\langle z, \zeta - w\rangle| > \tfrac{1}{2}|1 - \langle z,\zeta\rangle|.$$

Therefore,

$$|K^\alpha(z,w) - K^\alpha(z,\zeta)| \leq \frac{2^{n+3+\alpha}(n+1+\alpha)(1+1/\delta)d(w,\zeta)d(z,\zeta)}{|1 - \langle z,\zeta\rangle|^{n+2+\alpha}}$$
$$\leq C_{\alpha,n}\frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}}$$

and the lemma is proved.

For any $\zeta \in \mathrm{S}_n$ and $r > 0$, the set

$$Q_r(\zeta) = \{z \in \mathrm{B}_n : d(z,\zeta) < r\}$$

is called a Carleson tube with respect to the nonisotropic metric d. We usually write $Q = Q_r(\zeta)$ in short.

As usual, we define the atoms with respect to the Carleson tube as follows: for $1 < q < \infty$, $a \in L^q(\mathrm{B}_n, dv_\alpha)$ is said to be a$(1,q)_\alpha$-atom if there is a Carleson tube Q such that

(i) a is supported in Q;

(ii) $\|a\|_{L^q(\mathrm{B}_n, dv_\alpha)} \leq v_\alpha(Q)^{\frac{1}{q}-1}$;

(iii) $\int_{\mathrm{B}_n} a\,(z)dv_\alpha(z) = 0$.

The constant function 1 is also considered to be a$(1,q)_\alpha$-atom.

By the above lemma, we have the following useful estimates.

Lemma (2.1.6)[24]. For $\alpha > -1$ and $1 < q < \infty$ there exists a constant $C_{q,\alpha} > 0$ such that

$$\|P_\alpha(a)\|_{1,\alpha} \leq C_{q,\alpha}$$

for any $(1, q)_\alpha$-atom a.

Proof. When a is the constant function 1, the result is clear. Thus we may suppose a is a$(1,q)_\alpha$-atom. Let a be supported in a Carleson tuber $Q_r(\zeta)$ and $\delta r \leq \sqrt{2}$, where δ is the constant in Lemma (2.1.5). Since P_α is a bounded operator on $L^q(\mathrm{B}_n, dv_\alpha)$, we have

$$\int_{Q_{\delta r}} |\,P_\alpha(a)|dv_\alpha(z) \leq v_\alpha(Q_{\delta r})^{1-1/q}\|P_\alpha(a)\|_{q,\alpha}$$

$$\leq \|P_\alpha\|_{L^q(\mathbb{B}_n,dv_\alpha)} v_\alpha(Q_{\delta r})^{1-1/q} \|a\|_{q,\alpha}$$

$$\leq \|P_\alpha\|_{L^q(\mathbb{B}_n,dv_\alpha)}.$$

Next, if $d(z,\zeta) > \delta r$ then

$$\left| \int_{\mathbb{B}_n} \frac{a(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} dv_\alpha(w) \right|$$

$$= \left| \int_{Q_r(\zeta)} a(w) \left[\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} - \frac{1}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}} \right] dv_\alpha(w) \right|$$

$$\leq C \int_{Q_r(\zeta)} |a(w)| \frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}} dv_\alpha(w)$$

$$\leq Cr \int_{Q_r(\zeta)} |a(w)| dv_\alpha(w) \frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}} \leq \frac{Cr}{d(z,\zeta)^{2(n+1+\alpha)+1}}.$$

Then

$$\int_{d(z,\zeta)>\delta r} |P_\alpha(a)| dv_\alpha(z) \leq Cr \int_{d(z,\zeta)>\delta r} \frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z)$$

$$= Cr \sum_{k\geq 0} \int_{2^k\delta r < d(z,\zeta) \leq 2^{k+1}\delta r} \frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z) \leq Cr \sum_{k\geq 0} \frac{v_\alpha\left(Q_{2^{k+1}\delta r}\right)}{(2^k\delta r)^{2(n+1+\alpha)+1}}$$

$$\leq Cr \sum_{k=0}^{\infty} \frac{(2^{k+1}\delta r)^{2(n+1+\alpha)}}{(2^k\delta r)^{2(n+1+\alpha)+1}} \leq C,$$

where we have used the fact that $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$ in the third inequality in [41]). Thus, we get

$$\int_{\mathbb{B}_n} |P_\alpha(a)| dv_\alpha(z) = \int_{Q_{\delta r}} |P_\alpha(a)| dv_\alpha(z) + \int_{d(z,\zeta)>\delta r} |P_\alpha(h)| dv_\alpha(z) \leq C,$$

where C depends only on n and α.

Note that for any $(1,q)_\alpha$-atom a,

$$\|a\|_{1,\alpha} = \int_Q |a| dv_\alpha \leq v_\alpha(Q)^{1-1/q} \|a\|_{q,\alpha} \leq 1.$$

Then, we define $\mathcal{A}_\alpha^{1,q}$ as the space of all $f \in \mathcal{A}_\alpha^1$ which admits a decomposition

$$f = \sum_i \lambda_i P_\alpha a_i \quad and \quad \sum_i |\lambda_i| \leq C_q \|f\|_{1,\alpha},$$

where for each i, a_i is an $(1, q)_\alpha$-atom and $\lambda_i \in \mathbb{C}$ so that $\sum_i |\lambda_i| < \infty$. We equip this space with the norm

$$\|f\|_{\mathcal{A}_\alpha^{1,q}} = \inf \left\{ \sum_i |\lambda_i| : f = \sum_i \lambda_i P_\alpha a_i \right\}$$

where the infimum is taken over all decompositions of f described above. It is easy to see that $\mathcal{A}_\alpha^{1,q}$ is a Banach space. By Lemma (2.1.6) we have the contractive inclusion $\mathcal{A}_\alpha^{1,q} \subset \mathcal{A}_\alpha^1$. We will prove that these two spaces coincide. That establishes the "real-variable" atomic decomposition of the Bergman space $\mathcal{A}_\alpha^1$.

In fact, we will show the remaining inclusion $\mathcal{A}_\alpha^1 \subset \mathcal{A}_\alpha^{1,q}$ by duality (Theorem (2.1.14)). Recall that the dual space of $\mathcal{A}_\alpha^1$ is the Bloch space $\mathcal{B}$ defined as follows (see [41]). The Bloch space $\mathcal{B}$ of B_n is defined to be the space of holomorphic functions f in B_n such that

$$\|f\|_B = \sup \{ |\overline{\nabla} f(z)| : z \in \mathrm{B}_n \} < \infty.$$

$\| \|_B$ is a semi-norm on $\mathcal{B}$. $\mathcal{B}$ becomes a Banach space with the following norm

$$\|f\| = |f(0)| + \|f\|_B.$$

It is known that the Banach dual of $\mathcal{A}_\alpha^1$ can be identified with $\mathcal{B}$ (with equivalent norms) under the integral pairing

$$\langle f, g \rangle_\alpha = \lim_{r \to 1-} \int_{\mathrm{B}_n} f(rz) \overline{g(z)} dv_\alpha(z), f \in \mathcal{A}_\alpha^1, g \in \mathcal{B}.$$

(e.g., see Theorem (2.1.14)7 in [17].)
In order to prove the atomic decomposition of the Bergman spaces (cf. Theorem (2.1.14)), we need the following result, which can be found in [41]).
Lemma (2.1.7)[24]. Suppose $\alpha > -1$ and $1 \le p < \infty$. Then, for any $\in \mathcal{H}(\mathrm{B_n})$, f is in $\mathcal{B}$ if and only if there exists a constant $C > 0$ depending only on α and p such that

$$\frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} |f - f_{\alpha, Q_r(\zeta)}|^p dv_\alpha \le C$$

for all $r > 0$ and all $\zeta \in \mathrm{S_n}$, where

$$f_{\alpha, Q_r(\zeta)} = \frac{1}{Q_r(\zeta)} \int_{Q_r(\zeta)} f(z) dv_\alpha(z).$$

Moreover,

$$\|f\|_B \approx \sup \left(\frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} |f - f_{\alpha, Q_r(\zeta)}|^p dv_\alpha \right)^{\frac{1}{p}},$$

where ($\zeta \approx$ depends only on α, p, and n.
The following lemma is elementary.
Lemma (2.1.8)[24]. Suppose $\gamma > 0$ and $\alpha > -1$. If $f \in \mathcal{A}_\alpha^2$, then

$$\int_{\mathrm{B}_n} | A_\gamma(\tilde{\nabla} f)(z)|^2 dv_\alpha \approx \int_{\mathrm{B}_n} | f(z) - f(0)|^2 dv_\alpha,$$

where " ≈ " depends only on γ, α, and n.

Proof. Note that $v_\alpha(D(z,\gamma)) \approx (1-|z|^2)^{n+1+\alpha}$. Then

$$\int_{\mathrm{B}_n} | A_\gamma(\tilde{\nabla} f)(z)|^2 dv_\alpha = \int_{\mathrm{B}_n} \int_{D(z,\gamma)} (1-|w|^2)^{-1-n} |\tilde{\nabla} f(w)|^2 dv(w) dv_\alpha(z)$$

$$= \int_{\mathrm{B}_n} v_\alpha(D(w,\gamma))(1-|w|^2)^{-1-n} |\tilde{\nabla} f(w)|^2 dv(w) \approx \int_{\mathrm{B}_n} |\tilde{\nabla} f(w)|^2 dv_\alpha(w)$$

$$\approx \int_{\mathrm{B}_n} | f(w) - f(0)|^2 dv_\alpha(w).$$

In the last step we have used (ii) in [41].

Let $1 \le p < \infty$ and let $\mathbb{H}$ be a complex Hilbert space. We write $L^p_\alpha(\mathrm{B}_n, \mathbb{H})$ for the Banach space of strongly measurable $\mathbb{H}$-valued functions on B_n such that

$$\left(\int_{\mathrm{B}_n} \| f(z) \|^p dv_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

We write $\mathcal{A}^p_\alpha(\mathrm{B}_n, \mathbb{H})$ for the class of weighted $\mathbb{H}$-valued Bergman space of functions $\in \mathcal{H}(\mathrm{B}_n, \mathbb{H}) \cap L^p_\alpha(\mathrm{B}_n, \mathbb{H})$, where $\mathcal{H}(\mathrm{B}_n, \mathbb{H})$ stands for the space of $\mathbb{H}$-valued holomorphic functions in the unit ball B_n. Also, we define $\mathcal{B}(\mathrm{B}_n, \mathbb{H})$ the space of all $f \in \mathcal{H}(\mathrm{B}_n, \mathbb{H})$ so that

$$\|f\|_B = \sup \|\tilde{\nabla} f(z)\|_{\mathbb{H}} < \infty.$$

$\mathcal{B}(\mathrm{B}_n, \mathbb{H})$ with the norm $\|f\| = \|f(0)\|_{\mathbb{H}} + \|f\|_B$ is the $\mathbb{H}$-valued Bloch space. Then, by merely repeating the proof of the scalar case we have the following interpolation result.

Lemma (2.1.9)[24]. Suppose $\alpha > -1$ and

$$\frac{1}{p} = \frac{1-\theta}{p'}$$

for $0 < \theta < 1$ and $1 \le p' < \infty$. Then

$$\left[\mathcal{A}^{p'}_\alpha(\mathrm{B}_n, \mathbb{H}), \mathcal{B}(\mathrm{B}_n, \mathbb{H})\right]_\theta = \mathcal{A}^p_\alpha(\mathrm{B}_n, \mathbb{H})$$

with equivalent norms.

Finally, we collect some elementary facts on the Bergman metric and holomorphic functions in the unit ball of $\mathbb{C}^n$ as follows.

Lemma (2.1.10)[24]. For each $\gamma > 0$,

$$1 - |a|^2 \approx 1 - |z|^2 \approx |1 - \langle a, z\rangle|$$

for all a and z in B_n with $\beta(a,z) < \gamma$.

Lemma (2.1.11)[24]. Suppose $\gamma > 0$, $p > 0$, and $\alpha > -1$. Then there exists a constant C >

0 such that for any $f \in \mathcal{H}(B_n)$,

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,\gamma)} |f(w)|^p dv_\alpha(w), \forall z \in B_n.$$

Lemma (2.1.12)[24]. For each $\gamma > 0$,

$$|1-\langle z,u\rangle| \approx |1-\langle z,v\rangle|$$

for all z in $\overline{B}_n$ and u, v in B_n with $\beta(u,v) < \gamma$.

We first prove Theorem (2.1.1). The sufficiency is clear. It remains to prove that if $f \in \mathcal{A}_\alpha^p$ then $M_\gamma f \in L_p(B_n, dv_\alpha)$.

Lemma (2.1.13)[24]. For fixed $\gamma > 0$, there exist a positive integer N and a sequence $\{a_k\}$ in B_n such that

(i) $B_n = \bigcup_k D(a_k,\gamma)$, *and*

(ii) *each* $z \in B_n$ *belongs to at most* N *of the sets* $D(a_k, 3\gamma)$.

The following is the proof of Theorem (2.1.1):

Proof. Let $p > 0$. By Lemmas (2.1.13), (2.1.11), and (2.1.10), we have

$$\int_{B_n} |M(f)(z)|^p dv_\alpha(z) \leq \sum_k \int_{D(a_k,\gamma)} |M(f)(z)|^p dv_\alpha(z)$$

$$= \sum_k \int_{D(a_k,\gamma) w\in D(z,\gamma)} \sup |f(w)|^p dv_\alpha(z)$$

$$\leq C \sum_k \int_{D(a_k,\gamma)} \sup \frac{1}{(1-|w|^2)^{n+1+\alpha}} \int_{D(w,\gamma)} |f(u)|^p dv_\alpha(u) dv_\alpha(z)$$

$$\leq C \sum_k \int_{D(a_k,\gamma)} \left(\frac{1}{(1-|a_k|^2)^{n+1+\alpha}} \int_{D(a_k,3\gamma)} |f(u)|^p dv_\alpha(u) \right) dv_\alpha(z)$$

$$\leq C \sum_k \int_{D(a_k,3\gamma)} |f(u)|^p dv_\alpha(u) \leq CN \int_{B_n} |f(u)|^p dv_\alpha(u)$$

where N is the constant in Lemma (2.1.13) depending only on γ and n.

Now we turn to the real-variable atomic decomposition of $\mathcal{A}_\alpha^1 (\alpha > -1)$ with respect to the Carleson tubes. We reproduce this atomic decomposition for the Bergman spaces and then proceed with the proof.

Theorem (2.1.14)[24]. Let $1 < q < \infty$ and $\alpha > -1$. For every $f \in \mathcal{A}_\alpha^1$ there exist a sequence $\{a_i\}$ of $(1, q)_\alpha$-atoms and a sequence $\{\lambda_i\}$ of complex numbers such that

$$f = \sum_i \lambda_i P_\alpha a_i \text{ and } \sum_i |\lambda_i| \leq C_q \|f\|_{1,\alpha}. \quad (8)$$

Moreover,

$$\left\| f \|_{1,\alpha} \approx \inf \sum_i | \lambda_i \right|$$

where the infimum is taken over all decompositions of f described above and ≈ " depends only on α and q. We will prove Theorem (2.1.14) via duality. We first prove the following duality theorem.

Proof. By Lemma (2.1.6) we know that $\mathcal{A}_\alpha^{1,q} \subset \mathcal{A}_\alpha^1$. On the other hand, by Proposition (2.1.15) we have $(\mathcal{A}_\alpha^1)^* = \left(\mathcal{A}_\alpha^{1,q}\right)^*$ Hence, by duality we have $\|f\|_{1,q} \approx \|f\|_{\mathcal{A}_\alpha^{1,q}}$.

The goal prove Theorem (2.1.2) in the case of $p \geq 1$. Note that for any $\in \mathcal{H}(\mathrm{B}_n)$,

$$(1-|z|^2)|\mathcal{R}f(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\tilde{\nabla} f(z)|, \forall z \in \mathrm{B}_n.$$

(e.g., Lemma 2.14 in [17].) We have that (d) implies (c), and (c) implies (b) in Theorem (2.1.2). Then, we need to prove that (b) implies (a), and (a) implies (d).

Proof of (b) ⇒ (a) . Since $\mathcal{R}f(z)$ is holomorphic, by Lemma (2.1.11) we have

$$|\mathcal{R}f(z)|^2 \leq \frac{C}{(1-|z|^2)^{n+1}} \int_{D(z,\gamma)} |\mathcal{R}f(w)|^2 dv(w)$$

$$\leq C_\gamma \int_{D(z,\gamma)} (1-|w|^2)^{-n-1} |\mathcal{R}f(w)|^2 dv(w) .$$

Then,

$$(1-|z|^2)|\mathcal{R}f(z)| \leq C(1-|z|^2)\left(\int_{D(z,\gamma)} (1-|w|^2)^{-n-1} |\mathcal{R}f(w)|^2 dv(w)\right)^{\frac{1}{2}}$$

$$\leq C_\gamma \left(\int_{D(z,\gamma)} (1-|w|^2)^{-n+1} |\mathcal{R}f(w)|^2 dv(w)\right)^{\frac{1}{2}}$$

$$= C_\gamma A_\gamma(\mathcal{R}f)(z) .$$

Hence, for any $p > 0$, if $A_\gamma(\mathcal{R}f) \in L^p(\mathrm{B}_n, dv_\alpha)$ then $(1-|z|^2)|\mathcal{R}f(z)|$ is in $L^p(\mathrm{B}_n, dv_\alpha)$, which implies that $f \in \mathcal{A}_\alpha^p$ (e.g., Theorem 2.16 in [17]).

The proof of (a) ⇒ (d) is divided into two steps. At first we prove the case of $p = 1$ using the atomic decomposition, then we prove the generic case via complex interpolation.

Proof of (a) ⇒ (d) *in the case of* $p = 1$. By Theorem (2.1.14), it suffices to show that for $\gamma > 0$ and $\alpha > -1$ there exists $C > 0$ such that

$$\|A_\gamma(\tilde{\nabla} P_\alpha a)\|_{1,\alpha} \leq C$$

for all $(1,2)_\alpha$-atoms a. Given an $(1,2)_\alpha$-atom a supported in $= Q_r(\zeta)$. By Lemma (2.1.8) we have

$$\int_{2Q} A_\gamma\left(\tilde{\nabla} P_\alpha a\right) dv_\alpha \le v_\alpha(2Q)^{1/2}\left(\int_{2Q}\left[A_\gamma\left(\tilde{\nabla} P_\alpha a\right)\right]^2 dv_\alpha\right)^{1/2}$$

$$\le C v_\alpha(Q)^{1/2}\left(\int_{\mathrm{B}_n} |P_\alpha a(z) - P_\alpha a(0)|^2 dv_\alpha\right)^{1/2}$$

$$\le C v_\alpha(Q)^{1/2}\|a\|_{2,\alpha} \le C,$$

where $2Q = Q_{2r}(\zeta)$. On the other hand,

$$\int_{(2Q)^c} A_\gamma\left(\tilde{\nabla} P_\alpha a\right) dv_\alpha = \int_{(2Q)^c}\left(\int_{D(z,\gamma)} (1-|w|^2)^{-n-1}\, |\tilde{\nabla} P_\alpha a(w)|^2 dv(w)\right)^{1/2} dv_\alpha(z)$$

$$= \int_{(2Q)^c}\left(\int_{D(z,\gamma)} \Big|\int_Q \tilde{\nabla}_w\, [K^\alpha(w,u)$$

$$- K^\alpha(w,\zeta)]a(u) dv_\alpha(u)\Big|^2 \frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}} dv_\alpha(z)$$

$$\le \|a\|_{2,\alpha}\int_{(2Q)^c}\left(\int_{D(z,\gamma)}\int_Q \Big| \tilde{\nabla}_w [K^\alpha(w,u)$$

$$- K^\alpha(w,\zeta)]\Big|^2 dv_\alpha(u) \frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}} dv_\alpha(z)$$

$$\le \int_{(2Q)^c}\left(\int_{D(z,\gamma)} \sup\ |\tilde{\nabla}_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2 \frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}} dv_\alpha(z)\,,$$

where $(2Q)^c = \mathrm{B}_n \backslash 2Q$.
An immediate computation yields that

$$\nabla_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]$$

$$= (n+1+\alpha)\left[\frac{\overline{u}}{(1-\langle w,u\rangle)^{n+2+\alpha}} - \frac{\overline{\zeta}}{(1-\langle w,\zeta\rangle)^{n+2+\alpha}}\right]$$

$$= (n+1+\alpha)\frac{\overline{u}(1-\langle w,\zeta\rangle)^{n+2+\alpha} - \overline{\zeta}(1-\langle w,u\rangle)^{n+2+\alpha}}{(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle w,\zeta\rangle)^{n+2+\alpha}}$$

and

$$\mathcal{R}_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]$$

$$= (n+1+\alpha)\left[\frac{\langle w,u\rangle}{(1-\langle w,u\rangle)^{n+2+\alpha}} - \frac{\langle w,\zeta\rangle}{(1-\langle w,\zeta\rangle)^{n+2+\alpha}}\right]$$

$$= (n+1+\alpha)\frac{\langle w,u\rangle(1-\langle w,\zeta\rangle)^{n+2+\alpha} - \langle w,\zeta\rangle(1-\langle w,u\rangle)^{n+2+\alpha}}{(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle w,\zeta\rangle)^{n+2+\alpha}}.$$

Moreover,

$$|\nabla_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2$$

$$= (n+1+\alpha)^2\{\frac{|u|^2|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)} + |1-\langle w,u\rangle|^{2(n+2+\alpha)}}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$-\frac{(1-\langle w,\zeta\rangle)^{n+2+\alpha}(1-\langle u,w\rangle)^{n+2+\alpha}\langle \zeta,u\rangle}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$-\frac{(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle \zeta,w\rangle)^{n+2+\alpha}\langle u,\zeta\rangle}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}\},$$

and

$$|\mathcal{R}_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2$$

$$= (n+1+\alpha)^2\{\frac{|\langle w,u\rangle|^2|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)} + |\langle w,\zeta\rangle|^2|1-\langle w,u\rangle|^{2(n+2+\alpha)}}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$-\frac{\langle w,u\rangle\langle \zeta,w\rangle(1-\langle w,\zeta\rangle)^{n+2+\alpha}(1-\langle u,w\rangle)^{n+2+\alpha}}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$-\frac{\langle w,\zeta\rangle\langle u,w\rangle(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle \zeta,w\rangle)^{n+2+\alpha}}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}\}.$$

Then, we have

$$|\nabla_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2 - |\mathcal{R}_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2$$

$$= \frac{(n+1+\alpha)^2}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$\times\{(|u|^2 - |\langle w,u\rangle|^2)|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}$$
$$+(1-|\langle w,\zeta\rangle|^2)|1-\langle w,u\rangle|^{2(n+2+\alpha)}$$
$$+(\langle w,u\rangle\langle \zeta,w\rangle - \langle \zeta,u\rangle)(1-\langle w,\zeta\rangle)^{n+2+\alpha}(1-\langle u,w\rangle)^{n+2+\alpha}$$
$$+(\langle w,\zeta\rangle\langle u,w\rangle - \langle u,\zeta\rangle)(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle \zeta,w\rangle)^{n+2+\alpha}\}.$$

Note that for any $\in \mathcal{H}(\mathrm{B}_n)$,

$$|\tilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |\mathcal{R} f(z)|^2)\,, z \in \mathrm{B}_n.$$

It is concluded that

$$|\tilde{\nabla}_w[K^\alpha(w,u) - K^\alpha(w,\zeta)]|^2$$

$$\leq \frac{(n+1+\alpha)^2(1-|w|^2)}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}$$

$$\begin{gathered}\times\{(1-|\langle w,u\rangle|^2)|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)} \\ +(1-|\langle w,\zeta\rangle|^2)|1-\langle w,u\rangle|^{2(n+2+\alpha)} \\ +[(\langle w,u-\zeta\rangle\langle\zeta,w\rangle+(|\langle w,\zeta\rangle|^2-1)+(1-\langle\zeta,u\rangle)] \\ \times(1-\langle w,\zeta\rangle)^{n+2+\alpha}(1-\langle u,w\rangle)^{n+2+\alpha} \\ +[(\langle w,\zeta-u\rangle\langle u,w\rangle+(|\langle w,u\rangle|^2-1)+(1-\langle u,\zeta\rangle)] \\ \times(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle\zeta,w\rangle)^{n+2+\alpha}\}\end{gathered}$$

$$\leq \frac{(n+1+\alpha)^2(1-|w|^2)(M_1+M_2+M_3+M_4)}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}},$$

where

$$\begin{gathered}M_1=|1-\langle w,\zeta\rangle|^{n+2+\alpha}|1-\langle u,w\rangle|^{n+2+\alpha} \\ \times|\langle w,u-\zeta\rangle\langle\zeta,w\rangle+(1-\langle\zeta,u\rangle)|, \\ M_2=|1-\langle w,u\rangle|^{n+2+\alpha}|1-\langle\zeta,w\rangle|^{n+2+\alpha} \\ \times|\langle w,\zeta-u\rangle\langle u,w\rangle+(1-\langle u,\zeta\rangle)|, \\ M_3=(1-|\langle w,u\rangle|^2)|1-\langle\zeta,w\rangle|^{n+2+\alpha} \\ \times|(1-\langle w,\zeta\rangle)^{n+2+\alpha}-(1-\langle w,u\rangle)^{n+2+\alpha}|, \\ M_4=(1-|\langle w,\zeta\rangle|^2)|1-\langle u,w\rangle|^{n+2+\alpha} \\ \times|(1-\langle w,u\rangle)^{n+2+\alpha}-(1-\langle w,\zeta\rangle)^{n+2+\alpha}|,\end{gathered}$$

for $w\in D(z,\gamma)$, $u\in Q_r(\zeta)$ and $z\in \mathrm{B}_n$, $\zeta\in \mathrm{S}_n$.
Hence,

$$\int_{(2Q)^c}A_\gamma\left(\tilde{\nabla}P_\alpha a\right)dv_\alpha$$

$$\leq \int_{(2Q)^c}\left(\int_{D(z,\gamma)}\sup\,|\tilde{\nabla}_w[K^\alpha(w,u)-K^\alpha(w,\zeta)]|^2\frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}}dv_\alpha(z)$$

$$\leq (n+1+\alpha)\int_{(2Q)^c}(I_1+I_2+I_3+I_4)\,dv_\alpha(z)\,,$$

where

$$I_1=\left(\int_{D(z,\gamma)}\sup\frac{(1-|w|^2)^{-n}M_1}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}dv(w)\right)^{\frac{1}{2}}$$

$$I_2=\left(\int_{D(z,\gamma)}\sup\frac{(1-|w|^2)^{-n}M_2}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}}dv(w)\right)^{\frac{1}{2}}$$

$$I_3 = \left(\int_{D(z,\gamma)} \sup \frac{(1-|w|^2)^{-n} M_3}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}} dv(w) \right)^{\frac{1}{2}}$$

$$I_4 = \left(\int_{D(z,\gamma)} \sup \frac{(1-|w|^2)^{-n} M_4}{|1-\langle w,u\rangle|^{2(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{2(n+2+\alpha)}} dv(w) \right)^{\frac{1}{2}}$$

We first estimate I_1. Note that

$$\begin{aligned}
M_1 &\le (|\langle w, u-\zeta\rangle| + |1-\langle \zeta,u\rangle|)|1-\langle w,\zeta\rangle|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
&\le \left(2|1-\langle u,\zeta\rangle|^{\frac{1}{2}}\left(|1-\langle w,\zeta\rangle|^{\frac{1}{2}} + |1-\langle u,\zeta\rangle|^{\frac{1}{2}}\right) + |1-\langle \zeta,u\rangle|\right) \\
&\quad \times |1-\langle w,\zeta\rangle|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
&\le \left(2|1-\langle u,\zeta\rangle|^{\frac{1}{2}}\left(C_\gamma|1-\langle z,\zeta\rangle|^{\frac{1}{2}} + \frac{1}{2}|1-\langle z,\zeta\rangle|^{\frac{1}{2}}\right) + |1-\langle \zeta,u\rangle|\right) \\
&\quad \times |1-\langle w,\zeta\rangle|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
&\le \left(C_\gamma r|1-\langle z,\zeta\rangle|^{\frac{1}{2}} + r^2\right)|1-\langle w,\zeta\rangle|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha},
\end{aligned}$$

where the second inequality is the consequence of the following fact which has appeared in the proof of Lemma (2.1.5)

$$|\langle w, u-\zeta\rangle| \le 2|1-\langle u,\zeta\rangle|^{\frac{1}{2}}\left(|1-\langle w,\zeta\rangle|^{\frac{1}{2}} + |1-\langle u,\zeta\rangle|^{\frac{1}{2}}\right);$$

the third inequality is obtained by Lemma (2.1.12) and the fact

$$|1-\langle u,\zeta\rangle|^{\frac{1}{2}} < r < \frac{1}{2}|1-\langle z,\zeta\rangle|^{\frac{1}{2}}$$

for $u \in Q$ and $z \in (2Q)^c$. Since

$$\begin{aligned}
|1-\langle z,u\rangle|^{\frac{1}{2}} &\ge |1-\langle z,\zeta\rangle|^{\frac{1}{2}} - |1-\langle u,\zeta\rangle|^{\frac{1}{2}} \\
&\ge |1-\langle z,\zeta\rangle|^{\frac{1}{2}} - \frac{1}{2}|1-\langle z,\zeta\rangle|^{\frac{1}{2}} \\
&\ge \frac{1}{2}|1-\langle z,\zeta\rangle|^{\frac{1}{2}},
\end{aligned}$$

by Lemmas (2.1.10) and (2.1.12) we have

$$I_1 \le \left(\int_{D(z,\gamma)} \sup \frac{(1-|w|^2)^{-n}\left[Cr|1-\langle z,\zeta\rangle|^{\frac{1}{2}}+r^2\right]}{|1-\langle w,u\rangle|^{n+2+\alpha}|1-\langle w,\zeta\rangle|^{n+2+\alpha}} dv(w) \right)^{\frac{1}{2}}$$

$$\le C_\gamma \left(\int_{D(z,\gamma)} \sup \frac{(1-|z|^2)^{-n}\left[Cr|1-\langle z,\zeta\rangle|^{\frac{1}{2}}+r^2\right]}{|1-\langle z,u\rangle|^{n+2+\alpha}|1-\langle z,\zeta\rangle|^{n+2+\alpha}} dv(w) \right)^{\frac{1}{2}}$$

$$\le C_\gamma \left(\frac{(1-|z|^2)\left[r|1-\langle z,\zeta\rangle|^{\frac{1}{2}}+r^2\right]}{|1-\langle z,\zeta\rangle|^{2(n+2+\alpha)}} \right)^{\frac{1}{2}} \le C_\gamma \frac{\left[r|1-\langle z,\zeta\rangle|^{\frac{1}{2}}+r^2\right]^{1/2}}{|1-\langle z,\zeta\rangle|^{n+3/2+\alpha}}$$

$$\le C_\gamma \left(\frac{r^{\frac{1}{2}}}{d(z,\zeta)^{2(n+1+\alpha)+1/2}} + \frac{r}{d(z,\zeta)^{2(n+1+\alpha)+1}} \right).$$

Hence,

$$\int_{(2Q)^c} I_1\, dv_\alpha(z) \le C_\gamma \Big(\int_{(2Q)^c} \frac{r^{\frac{1}{2}}}{d(z,\zeta)^{2(n+1+\alpha)+\frac{1}{2}}} dv_\alpha(z)$$

$$+ \int_{(2Q)^c} \frac{r}{d(z,\zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z) \Big) \le C_\gamma,$$

because

$$\int_{d(z,\zeta)>2r} \frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+1/2}} dv_\alpha(z)$$

$$= r^{1/2} \sum_{k\ge 0} \int_{2^k r < d(z,\zeta) \le 2^{k+1} r} \frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1/2}} dv_\alpha(z)$$

$$\le r^{1/2} \sum_{k\ge 0} \frac{v_\alpha\left(Q_{2^{k+1}r}\right)}{(2^k r)^{2(n+1+\alpha)+1/2}}$$

$$\le C r^{1/2} \sum_{k=0}^{\infty} \frac{(2^{k+1} r)^{2(n+1+\alpha)}}{(2^k r)^{2(n+1+\alpha)+1/2}} \le C,$$

where we have used the fact that $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$ in the third inequality in [41] and the second term has been estimated in the proof of Lemma (2.1.6).

By the same argument we can estimate I_2 and omit the details. Next, we estimate I_3.

Note that

$$M_3 \leq (1-|\langle w,u\rangle|^2)|1-\langle w,\zeta\rangle|^{n+2+\alpha}$$
$$\times |(1-\langle w,\zeta\rangle)^{n+2+\alpha}-(1-\langle w,u\rangle)^{n+2+\alpha}|$$
$$\leq 2|1-\langle w,u\rangle||1-\langle w,\zeta\rangle|^{n+2+\alpha}$$
$$\times \left|\int_0^1 \frac{d}{dt}(1-\langle w,t\zeta+(1-t)u\rangle)^{n+2+\alpha}dt\right|$$
$$=2(n+2+\alpha)|1-\langle w,u\rangle||1-\langle w,\zeta\rangle|^{n+2+\alpha}$$
$$\times \left|\langle w,\zeta-u\rangle\int_0^1(1-\langle w,t\zeta+(1-t)u\rangle)^{n+1+\alpha}\,dt\right|$$
$$\leq C_\gamma|1-\langle w,u\rangle||1-\langle w,\zeta\rangle|^{n+2+\alpha}r|1-\langle z,\zeta\rangle|^{n+3/2+\alpha},$$

where the last inequality is achieved by the following estimates

$$|1-\langle w,t\zeta+(1-t)u\rangle| \leq C_\gamma|1-\langle z,t\zeta+(1-t)u\rangle|$$
$$\leq C_\gamma|1-\langle z,u\rangle|+|\langle z,\zeta-u\rangle|$$
$$\leq C_\gamma|1-\langle z,\zeta\rangle|$$

and

$$|\langle w,\zeta-u\rangle| \leq C_\gamma r|1-\langle z,\zeta\rangle|^{\frac{1}{2}},$$

for any $w \in D(z,\gamma)$ and $u \in Q_r(\zeta)$. Thus, by Lemmas (2.1.10) and (2.1.12)

$$I_3 \leq C_\gamma\left(\int_{D(z,\gamma)} \sup \frac{(1-|w|^2)^{-n}r|1-\langle z,\zeta\rangle|^{n+3/2+\alpha}}{|1-\langle w,u\rangle|^{2(n+1+\alpha)+1}|1-\langle w,\zeta\rangle|^{n+2+\alpha}}dv(w)\right)^{\frac{1}{2}}$$
$$\leq C_\gamma\left(\int_{D(z,\gamma)} \sup \frac{(1-|z|^2)^{-n}r|1-\langle z,\zeta\rangle|^{n+3/2+\alpha}}{|1-\langle z,u\rangle|^{2(n+1+\alpha)+1}|1-\langle z,\zeta\rangle|^{n+2+\alpha}}dv(w)\right)^{\frac{1}{2}}$$
$$\leq C_\gamma\left(\frac{(1-|z|^2)r}{|1-\langle z,\zeta\rangle|^{2(n+1+\alpha)+3/2}}\right)^{1/2} \leq C_\gamma\frac{r^{\frac{1}{2}}}{d(z,\zeta)^{2(n+1+\alpha)+1/2}}.$$

Hence,

$$\int_{(2Q)^c} I_3\,dv_\alpha(z) \leq C_\gamma\int_{(2Q)^c}\frac{r^{\frac{1}{2}}}{d(z,\zeta)^{2(n+1+\alpha)+1/2}}dv_\alpha(z) \leq C_\gamma,$$

as shown above.

Similarly, we can estimate I_4 and omit the details. Therefore, combining above estimates we conclude that

$$\int_{(2Q)^c} A_\gamma\left(\tilde{\nabla} P_\alpha a\right) dv_\alpha \leq C,$$

where C depends only on γ, n, and α.

Proof *of* (a) $\Rightarrow$ (d) *for* $p > 1$. Set $\mathbb{H} = L^2(\mathrm{B}_n,\ \chi_{D(0,\gamma)} dv_{-n-1;\mathbb{C}^n})$ with $dv_{-n-1}(w) = \frac{dv(w)}{(1-|w|^2)^{n+1}}$. Consider the operator

$$T(f)(z,w) = \left(\tilde{\nabla} f\right)\left(\phi_z(w)\right), f \in \mathcal{H}(\mathrm{B}_n).$$

Note that $\phi_z\left(D(0,\gamma)\right) = D(z,\gamma)$ and the measure dv_{-n-1} is invariant under any automorphism of B_n we have

$$\|T(f)(z)\|_{\mathbb{H}} = \left(\int_{\mathrm{B}_n} |\left(\tilde{\nabla} f\right)\left(\phi_z(W)\right)|^2 \chi_{D(0,\gamma)}(w) dv_{-n-1}(w)\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathrm{B}_n} |\tilde{\nabla} f(W)|^2 \chi_{D(z,\gamma)}(w) dv_{-n-1}(w)\right)^{\frac{1}{2}} = A_\gamma\left(\tilde{\nabla} f\right)(z).$$

On the other hand,

$$A_\gamma\left(\tilde{\nabla} f\right)(z) \leq \left[C_\gamma (1-|z|^2)^{-n-1} v\left(D(z,\gamma)\right)\right]^{\frac{1}{2}} \|f\|_B \leq C\|f\|_B.$$

Then, we conclude that T is bounded from $\mathcal{B}$ into $\mathcal{B}(\mathrm{B}_n, \mathbb{H})$. Thus, applying Lemma (2.1.9) to this fact with the case of $p = 1$ proved above yields that T is bounded from $\mathcal{A}_\alpha^p$ into $\mathcal{A}_\alpha^p(\mathrm{B}_n, \mathbb{H})$ for any $1 < p < \infty$, that is,

$$\|A_\gamma\left(\tilde{\nabla} f\right)\|_{p,\alpha} \leq C\|f\|_{p,\alpha}, \forall f \in \mathcal{A}_\alpha^p,$$

where C depends only on γ, n, p, and α. The proof is complete.

Proposition(2.1.15)[24]. For any $1 < \mathrm{q} < \infty$ and $\alpha > -1$, we have $\left(\mathcal{A}_\alpha^{1,\mathrm{q}}\right)^* = \mathcal{B}$ isometrically. More precisely,

(i) Every $\mathrm{g} \in \mathcal{B}$ defines a continuous linear functional ϕ_{g} on $\mathcal{A}_\alpha^{1,\mathrm{q}}$ by

$$\phi_g(f) = \lim_{r\to 1-} \int_{\mathrm{B}_n} f(rz)\overline{g(z)} dv_\alpha(z), \forall f \in \mathcal{A}_\alpha^{1,q}. \quad (9)$$

(ii) *Conversely, each* $\phi \in \left(\mathcal{A}_\alpha^{1,q}\right)^*$ *is given as* (9) *by some* $g \in \mathcal{B}$.

Moreover, we have

$$\|\phi_q\| \approx |g(0)| + \|g\|_\beta, \quad \forall g \in \beta. \tag{10}$$

Proof. Let p be the conjugate index of q, i.e., $1/p + 1/q = 1$. We first show $\mathcal{B} \subset \left(\mathcal{A}_\alpha^{1,q}\right)^*$ Let $g \in \mathcal{B}$. For any $(1, q)_\alpha$-atom a, by Lemma (2.1.7) we have

$$|\int_{\mathrm{B}_n} P_\alpha\, a(z)\overline{g(z)}dv_\alpha(z)| = |\langle P_\alpha(a_j), g\rangle_\alpha| = |\int_{\mathrm{B}_n} a\,\overline{g}dv_\alpha| = |\int_{\mathrm{B}_n} a\,\overline{(g - g_Q)}dv_\alpha|$$

$$\le \left(\int_Q |a|^q dv_\alpha\right)^{1/q} \left(\int_Q |g - g_Q|^p dv_\alpha\right)^{1/p} \le \left(\frac{1}{v_\alpha(Q)}\int_Q |g - g_Q|^p dv_\alpha\right)^{1/p}$$

$$\le C\|g\|_B.$$

On the other hand, for the constant function 1 we have $P_\alpha 1 = 1$ and so

$$|\int_{\mathrm{B}_n} P_\alpha\, 1(z)\overline{g(z)}dv_\alpha(z)| = |\int_{\mathrm{B}_n} g\,(z)dv_\alpha(z)| = |g(0)|.$$

Thus, we deduce that

$$|\int_{\mathrm{B}_n} f\,\overline{g}dv_\alpha| \le C\|f\|_{\mathcal{A}_\alpha^{1,q}}(|g(0)| + \|g\|_B)$$

for any finite linear combination f of $(1, q)_\alpha$-atoms. Hence, g defines a continuous linear functional ϕ_g on a dense subspace of $\mathcal{A}_\alpha^{1,q}$ and ϕ_g extends to a continuous linear functional on $\mathcal{A}_\alpha^{1,q}$ such that

$$|\phi_g(f)| \le C(|g(0)| + \|g\|_B)\|f\|_{\mathcal{A}_\alpha^{1,q}}$$

for all $f \in \mathcal{A}_\alpha^{1,q}$.

Next let ϕ be a bounded linear functional on $\mathcal{A}_\alpha^{1,q}$. Note that

$$\mathcal{H}^q(\mathrm{B}_n, dv_\alpha) = \mathcal{H}(\mathrm{B}_n) \cap L^q(\mathrm{B}_n, dv_\alpha) \subset \mathcal{A}_\alpha^{1,q}.$$

Then, ϕ is a bounded linear functional on $\mathcal{H}^q(\mathrm{B}_n, dv_\alpha)$. By duality there exists $g \in \mathcal{H}^p(\mathrm{B}_n, dv_\alpha)$ such that

$$\phi(f) = \int_{\mathrm{B}_n} f\,\overline{g}dv_\alpha, \forall f \in \mathcal{H}^q(\mathrm{B}_n, dv_\alpha).$$

Let $Q = Q_r(\zeta)$ be a Carleson tube. For any $f \in L^q(\mathrm{B}_n, dv_\alpha)$ supported in Q, it is easy to check that

$$a_f = (f - f_Q)\chi_Q/\left[\|f\|_{L^q} v_\alpha(Q)^{1/p}\right]$$

is a$(1,q)$-atom. Then, $|\phi(P_\alpha a_f)| \le \|\phi\|$ and so

$$|\phi(P_\alpha[(f-f_Q)\chi_Q])| \le \|\phi\|\|f\|_{L^q} v_\alpha(Q)^{1/p}.$$

Hence, for any $f \in L^q(\mathrm{B}_n, dv_\alpha)$ we have

$$|\int_Q f\,\overline{(g-g_Q)}dv_\alpha| = |\int_Q (f-f_Q)\,\overline{g}dv_\alpha| = |\int_{\mathrm{B}_n} (f-f_Q)\,\chi_Q\overline{g}dv_\alpha|$$

$$= |\int_{\mathrm{B}_n} P_\alpha\,[(f-f_Q)\chi_Q]\overline{g}dv_\alpha| = |\phi(P_\alpha[(f-f_Q)\chi_Q])|$$

$$\le \|\phi\|\|(f-f_Q)\chi_Q\|_{L^q(\mathrm{B}_n,dv_\alpha)} v_\alpha(Q)^{1/p} \le 2\|\phi\|\|f\|_{L^q(Q,dv_\alpha)} v_\alpha(Q)^{1/p}.$$

This concludes that

$$\left(\frac{1}{v_\alpha(Q)}\int_Q |\,g-g_Q|^p dv_\alpha\right)^{1/p} \le 2\|\phi\|.$$

By Lemma (2.1.7) we have that $g \in \mathcal{B}$ and $\|g\|_B \le C\|\phi\|$. Therefore, ϕ is given as (9) by g with $|g(0)| + \|g\|_B \le C\|\phi\|$.

We prove Theorem (2.1.14).

Remark (2.1.16)[24]. From the proofs of that (b) ⇒ (a) and that (a) ⇒ (d) for $\mathrm{p} > 1$ we find that Theorem (2.1.2) still holds true for the Bloch space of the endpoint case when $\mathrm{p} = \infty$. That is, for any $\mathrm{f} \in \mathcal{H}(\mathrm{B_n})$, $\mathrm{f} \in \mathcal{B}$ if and only if one (or equivalent, all) of $\mathrm{A}_\gamma(\mathcal{R}\mathrm{f})$, $\mathrm{A}_\gamma(\nabla \mathrm{f})$, and $\mathrm{A}_\gamma(\widetilde{\nabla}\mathrm{f})$ is in $\mathrm{L}^\infty(\mathrm{B_n})$. Moreover,

$$\|f\|_B \approx \|A_\gamma(\mathcal{R}f)\|_{L^\infty(\mathrm{B}_n)} \approx \|A_\gamma(\nabla f)\|_{L^\infty(\mathrm{B}_n)} \approx \|A_\gamma(\tilde{\nabla} f)\|_{L^\infty(\mathrm{B}_n)}, \qquad (11)$$

where (ζ ≈ *depends only on* γ *and* n.

We prove Theorem (2.1.2) for the case $0 < p < 1$, through using atom decomposition for Bergman spaces due to Coifman and Rochberg [32],[41],as follows.

Proposition (2.1.17)[24]. Suppose > 0, $\alpha > -1$, and $\mathrm{b} > \mathrm{n}\max\{1, 1/\mathrm{p}\} + (\alpha+1)/\mathrm{p}$. Then there exists a sequence $\{\mathrm{a_k}\}$ in $\mathrm{B_n}$ such that $\mathcal{A}^{\mathrm{p}}_\alpha$ consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{(1-\langle z, a_k\rangle)^b}, z \in \mathrm{B}_n,$$

where $\{\mathrm{c_k}\}$ belongs to the sequence space ℓ^{p} and the series converges in the norm topology of $\mathcal{A}^{\mathrm{p}}_\alpha$. Moreover,

$$\int_{\mathrm{B}_n} |\,f(z)|^p dv_\alpha(z) \approx \inf\left\{\sum_k |\,c_k|^p\right\},$$

where the infimum runs over all the above decompositions.

Also, we need a characterization of Carleson type measures for Bergman spaces as follows, which can be found in [16].

Proposition (2.1.18)[24]. Suppose $n+1+\alpha>0$ and μ is a positive Borel measure on B_n. Then, there exists a constant $C>0$ such that

$$\mu\big(Q_r(\zeta)\big)\leq Cr^{2(n+1+\alpha)},\ \forall\zeta\in S_n\ and\ r>0,$$

if and only if for each $s>0$ there exists a constant $C>0$ such that

$$\int_{B_n}\frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{n+1+\alpha+s}}d\mu(w)\leq C$$

for all $z\in B_n$.

We prove Theorem (2.1.2) in the case of $0<p<1$.

Proof. Since the proof of $(d)\Rightarrow(a)$ valid for $0<p<1$, as noted in the first paragraph of it suffices to prove that $(a)\Rightarrow(d)$, i.e., if $f\in\mathcal{A}_\alpha^p$ then $A_\gamma\big(\tilde{\nabla}f\big)(z)\in\mathcal{A}_\alpha^p$ for $0<p<1$.

To this end, we write

$$f_k(z)=\frac{(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{(1-\langle z,a_k\rangle)^b}.$$

An immediate computation yields that

$$A_\gamma\big(\tilde{\nabla}f_k\big)(z)$$

$$=\left(\int_{D(z,\gamma)}|\tilde{\nabla}f_k(w)|^2\frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{2}}$$

$$=\left(\int_{D(z,\gamma)}[|\nabla f_k(w)|^2-|\mathcal{R}f_k(w)|^2]\frac{dv(w)}{(1-|w|^2)^n}\right)^{\frac{1}{2}}$$

$$=\left(\int_{D(z,\gamma)}b^2\,(1-|a_k|^2)^{2(pb-n-1-\alpha)/p}\,|^{|a_k|^2-|\langle w,a_k\rangle|^2}1-\langle w,a_k\rangle|^{2(b+1)}\frac{dv(w)}{(1-|w|^2)^n}\right)^{\frac{1}{2}}$$

$$\leq b(1-|a_k|^2)^{(pb-n-1-\alpha)/p}\left(\int_{D(z,\gamma)}\frac{1-|\langle w,a_k\rangle|^2}{|1-\langle w,a_k\rangle|^{2(b+1)}}\frac{dv(w)}{(1-|w|^2)^n}\right)^{\frac{1}{2}}$$

$$\leq b(1-|a_k|^2)^{(pb-n-1-\alpha)/p}\left(\int_{D(z,\gamma)}\frac{2|1-\langle w,a_k\rangle|}{|1-\langle w,a_k\rangle|^{2(b+1)}}\frac{dv(w)}{(1-|w|^2)^n}\right)^{\frac{1}{2}}$$

$$\leq b(1-|a_k|^2)^{(pb-n-1-\alpha)/p}\left(\int_{D(z,\gamma)}\frac{4(1-|w|^2)^{-n-1}}{|1-\langle w,a_k\rangle|^{2b}}dv(w)\right)^{\frac{1}{2}}$$

$$\leq \frac{C_\gamma b(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1-\langle z,a_k\rangle|^b}\left((1-|z|^2)^{-n-1}\int_{D(z,\gamma)} d\,v(w)\right)^{\frac{1}{2}}$$

$$\leq \frac{C_\gamma b(1-|a_k|^2)^{(pb-n-1-\alpha)/p}}{|1-\langle z,a_k\rangle|^b},$$

where the last two inequalities are achieved by using Lemmas (2.1.10) and (2.1.12) and the fact $v\big(D(z,\gamma)\big) \approx (1-|z|^2)^{n+1}$. Note that $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$

[41]), by Proposition (2.1.18) we have

$$\int_{\mathrm{B}_n} |\,A_\gamma\big(\tilde{\nabla} f_k\big)(z)|^p dv_\alpha \leq C\mathcal{U}^3 \int_{\mathrm{B}_n} \frac{(1-|a_k|^2)^{(pb-n-1-\alpha)}}{|1-\langle z,a_k\rangle|pb} dv_\alpha(z) \leq C_{p,\alpha}.$$

Hence, for $0<p<1$ we have for $f=\sum_{k=1}^{\infty} c_k\, f_k$ with $\sum_k |\,c_k|^p<\infty$,

$$\int_{\mathrm{B}_n} |\,A_\gamma\big(\tilde{\nabla} f\big)(z)|^p dv_\alpha \leq \int_{\mathrm{B}_n} |\sum_{k=1}^{\infty} c_k\, A_\gamma\big(\tilde{\nabla} f_k\big)(z)|^p dv_\alpha(z)$$

$$\leq \sum_{k=1}^{\infty} |\,c_k|^p \int_{\mathrm{B}_n} |\,A_\gamma\big(\tilde{\nabla} f_k\big)(z)|^p dv_\alpha$$

$$\leq C_{p,\alpha} \sum_{k=1}^{\infty} |\,c_k|^p.$$

This concludes that

$$\int_{\mathrm{B}_n} |\,A_\gamma\big(\tilde{\nabla} f\big)(z)|^p dv_\alpha \leq C_{p,\alpha} \inf\left\{\sum_{k=1}^{\infty} |\,c_k|^p\right\} \leq C_{p,\alpha} \int_{\mathrm{B}_n} |\,f(z)|^p dv_\alpha(z)$$

The proof is complete.

Remark (2.1.19)[24]. We note that Theorem (2.1.2) is valid for so-called q-square area functions. For simplicity, we only consider the case of radical derivatives: For $1<\mathrm{q}<\infty$, define

$$A_\gamma^{(q)}(\mathcal{R}f)(z) = \left(\int_{D(z,\gamma)} |\,(1-|w|^2)\mathcal{R}f(w)|^q \frac{dv(w)}{(1-|w|^2)^{n+1}}\right)^{\frac{1}{q}}$$

It is easy to check that the proofs of both the case $0<\mathrm{p}\leq 1$ above and that (a) $\Rightarrow$ (d) in the case $1<\mathrm{p}<\infty$ can apply to $\mathrm{A}_\gamma^{(\mathrm{q})}$ Thus, we have that

$$\|f\|_{p,\alpha} \approx |f(0)| + \|A_\gamma^{(q)}(\mathcal{R}f)\|_{p,\alpha}, \forall f \in \mathcal{A}_\alpha^p, \tag{12}$$

for $0<\mathrm{p}<\infty$, $\alpha>-1$ and $1<\mathrm{q}<\infty$, where " $\approx$ " depends only on γ, α, p, n, and q.

Section (2.2): Several Complex Variables

We study Cauchy-type integrals in several complex variables and to announce new results concetning these operators. While this is a broad field with a very wide literature, our exposition will be focused more narrowly on achieving the following goal: the construction of such operators and the establishment of their L^p mapping properties under "minimal" conditions of smoothness of the boundary of the domain D in question.

The operators we study are of three interrelated kinds: Cauchy-Fantappié integrals with holomorphic kernels, Cauchy-Szegö projections and Bergman projections. In the case of one complex variable, what happens is by now well-understood. Here the minimal smoothness that can be achieved is "near" C^1 (e.g., the case of a Lipschitz domain). However when the complex dimension is greater than 1 the nature of the Cauchy-Fantappié kernels brings in considerations of pseudo-convexity (in fact strong pseudo-convexity) and these in turn imply that the limit of smoothness should be "near" C^2. We establish L^p-regularity for one or more of these operators in the following contexts:

When D is strongly pseudo-convex and of class C^2;

When D is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$

with p in the range $1 < p < \infty$. for the precise statements.

we briefly review the situation of one complex variable. We devoted to a few generalities about Cauchy-type integrals when n, the complex dimension of the ambient space, is greater than. The Cauchy-Fantappié forms are taken up and the corresponding CauchyFantappié integral operators are set out. Here we adapt the standard treatment in [34], We show that these methods apply when the so-called generating form is merely of class C^1 or even Lipschitz, as is needed in what follows. The Cauchy-Fantappié integrals constructed up to that point may however lack the basic requirement of producing holomorphic functions, whatever the given data is. The kernel of the operator may fail to be holomorphic in the free variable $z \in D$. To achieve the desired holomorphicity requires that the domain D be pseudo-convex, and two specific forms of this property, *strong pseudo-convexity* and *strong* $\mathbb{C}$ -*linear convexity* are discussed.

There are several approaches to obtain the required holomorphicity when the domain is sufficiently smooth and strongly pseudo-convex. The initial methods are due to Henkin [59], [60] and Ramirez [75]; a later approach is in Kerzman-Stein [63], which is the one we adopt here. It requires to start with a "locally" holomorphic kernel, and then to add a correction term obtained by solving a $\bar{\partial}$-problem. These matters are discussed. Note that in the case of strongly $\mathbb{C}$-linearly convex domains, the Cauchy-Leray integral given here requires no correction. So among all the integrals of Cauchy-Fantappié type associated to such domains, the Cauchy-Leray integral is the unique and natural operator that most closely resembles the classical Cauchy integral from one complex variable.

The main L^p estimates for the Cauchy-Leray integral and the Szegö and Bergman projections (for C^2 boundaries). We limit ourselves to highlighting the main points of interest in the proofs. For the last two operators, the L^p results are consequences of estimates that hold for the corrected Cauchy-Fantappié kernels, denoted C_ε and B_ε , that involve also

their respective adjoints. Highlights a further result concerning the Cauchy-Leray integral, also to appear in a separate: the corresponding L^p theorem under the weaker assumption that the boundary is merely of class $C^{1,1}$.
Among matters not covered here are L^p results for the Szegö and Bergman projection and for the Cauchy-Leray integral for other special domains (in particular, with more regularity). For these, see e.g. [44]–[46], [48]–[50], [54], [55], [57], [65], [71], [72], [74], [79]. It is to be noted that several among these works depend in the main on good estimates or explicit formulas for the Szegö or Bergman kernels. we have to proceed via the Cauchy-Fantappié framework.
Euclidean volume measure for $\mathbb{C}^n \equiv \mathbb{R}^{2n} (n \geq 1)$ will be denoted dV. The notation bD will indicate the boundary of a domain $D \subset \mathbb{C}^n (n \geq 1)$ and, for D sufficiently smooth, $d\sigma$ will denote arc-length $(n = 1)$ or Euclidean surface measure $(n \geq 2)$.

In the case of one complex dimension the problem of L^p estimates has a long and illustrious history. Let us review it briefly. (see [52], [58], [66], which contain further citations.)
Suppose D is a bounded domain in $\mathbb{C}$ whose boundary bD is a rectifiable curve. Then the *Cauchy integral* is given by

$$\mathrm{C}(f)(z) = \int_{\mathrm{b}D} f(w) C(w, z) \text{ , } for\ z \in D$$

where $C\ (w, z)$ is the *Cauchy kernel*

$$C(w, z) = \frac{1}{2\pi i} \frac{dw}{w - z}$$

When D is the unit disc, then a classical theorem of M. Riesz says that the mapping $\mapsto \mathrm{C}(f)|_{\mathrm{b}D}$, defined initially for f that are (say) smooth, is extendable to abounded operator on $L^p(\mathrm{b}D)$, for $1 < p < \infty$. Very much the same result holds when the boundary of D is of class $C^{1+\varepsilon}$, with $\varepsilon > 0$, (proved either by approximating to the result when D is the unit disc, or adapting one of the several methods of proof used in the classical case). However in the limiting case when $\varepsilon = 0$, these ideas break down and new methods are needed. The theorems proved by Calderón, Coifman, McIntosh, Meyers and David (between 1977–1984) showed that the corresponding L^p result held in the following list of increasing generality: the boundary is of class C^1; it is Lipschitz (the first derivatives are merely bounded and not necessarily continuous); it is an "Ahlfors-regular" curve.
We pass next to the Cauchy-Szegö projection S, the corresponding orthogonal projection with respect to the Hilbert space structure of $L^2(\mathrm{b}D)$. In fact when D is the unit disc, the two operators C and S are identical. We restrict our attention to the case when D is simply connected and when its boundary is Lipschitz. Here a key tool is the conformal map $: \mathbb{D} \to D$, where $\mathbb{D}$ is the unit disc. We consider the induced correspondence τ given by $\tau(f)(e^{i\theta}) = \left(\Phi'(e^{i\theta})\right)^{\frac{1}{2}} f\left(\Phi(e^{i\theta})\right)$, and the fact that $S = \tau^{-1} S_0 \tau$, where S_0 is the Cauchy-Szegö projection for the disc D. Using ideas of Calderón, Kenig, Pommerenke and

others, one can show that $|\Phi'|^r$ belongs to the Muckenhaupt class A_p, with $r = 1 - p/2$, from which one gets the following. As a consequence, if we suppose that bD has a Lipschitz bound M, then S is bounded on L^p, whenever

· $1 < p < \infty$, if bD is in fact of class C^1.

· $p'_M < p < p_M$. Here p_M depends on M, but $p_M > 4$, and p'_M is the exponent dual to p_M.

There is an alternative approach to the second result that relates the Cauchy-Szegö projection S to the Cauchy integral C. It is based on the following identity, used in [21]

$$\mathrm{S}(I - \mathbb{A}) = \mathrm{C}, where\ \mathbb{A} = \mathrm{C}^* - \mathrm{C}. \quad (13)$$

There are somewhat analogous results for the Bergman projection in the case of one complex dimension. We shall not discuss this further, We shall see that a very different situation occurs when trying to extend to higher dimensions. Here are some new issues that arise when $n > 1$. There is no "universal" holomorphic Cauchy kernel associated to a domain D.

(i) Pseudo-convexity of D, must, in one form or another, play a role.

(ii) Since this condition involves (implicitly) two derivatives, the best" results are to be expected "near" C^2, (as opposed to near C^1 when $n = 1$).

In view of the non-uniqueness of the Cauchy integral(and its problematic existence), it might be worthwhile to set down the minimum conditions that would be required of candidates for the Cauchy integral. We would want such an operator C given in the form

$$\mathrm{C}(f)(z) = \int_{\mathrm{b}D} f\,(w)C(w,z)\,, z \in D,$$

to satisfy the following conditions:

(a) The kernel $C(w,z)$ should be given by a "natural" or explicit formula (at least up to first approximation) that involves D.

(b) The mapping $f \mapsto \mathrm{C}(f)$ should reproduce holomorphic functions. In particular if f is continuous in $\overline{D}$ and holomorphic in D then $\mathrm{C}(f)(z) = f(z)$, for $z \in D$.

(c) $\mathrm{C}(f)(z)$ should be holomorphic in $z \in D$, for any given f that is continuous on bD.

Now there is a formalism (the Cauchy-Fantappié formalism of Fantappié (1943), Leray, and Koppleman (1967)), which provides Cauchy integrals satisfying the requirements (a) and (b) in a general setting. Condition (c) however, is more problematic, even when the domain is smooth. Constructing such Cauchy integrals has been carried out only in particular situations.

The Cauchy-Fantappié formalism that realizes the kernel $C(w,z)$ revolves around the notion of *generating form*: these are a class of differential forms of type $(1,0)$ in the variable of integration whose coefficients may depend on two sets of variables (w and) , and we will accordingly write

$$\eta(w,z) = \sum_{j=1}^{n} \eta_j\,(w,z)dw_j\ with\ (w,z) \in U \times V$$

to designate such a form. The precise definition is given below, where the notation

$$\langle \eta(w,z),\xi\rangle = \sum_{j=1}^{n} \eta_j\,(w,z)\xi_j.$$

is used to indicate the action of η on the vector $\xi \in \mathbb{C}^n$.

Definition (2.2.1)[42]. The form $\eta\ (w,z)$ is *generating at z relative to V* if there is an open set

$$U_z \subseteq \mathbb{C}^n \backslash \{z\}$$

such that

$$\mathrm{b}V \subset U_z \tag{14}$$

and, furthermore

$$\langle \eta(w,z)\,, w-z\rangle = \sum_{j=1}^{n} \eta_j\,(w,z)\big(w_j - Zj\big) \equiv 1\, for\ any\ w \in U_z. \tag{15}$$

We say that η is a generating form for V (alternatively, that V supports a generating form η) if for any z ∈ V we have that η is generating at z relative to V.

Example (2.2.2)[42]. Set

$$\beta(w,z) = |w-z|^2$$

We define the *Bochner-Martinelli generating formto* be

$$\eta(w,z) = \frac{\partial_w \beta}{\beta}(w,z) = \sum_{j=1}^{n} \frac{\overline{w}_{j-\overline{z}j}}{|w-z|^2} dw_j \tag{16}$$

It is clear that η satisfies conditions (14) and (15) for any domain V and for any $z \in V$, with $U_z := \mathbb{C}^n \backslash \{z\}$.

The Bochner-Martinelli generating form has several remarkable features. First, it is “universal” in the sense that it is given by a formula (16) that does not depend on the choice of domain V; secondly, in complex dimension $n = 1$ it agrees (up to a scalar multiple) with the classical Cauchy kernel

$$\frac{1}{2\pi i}\frac{dw}{w-z}, w \in U_z := \mathbb{C}\backslash\{z\}$$

and in particular its coefficient $(w-z)^{-1}$ is holomorphic as a function of $z \in V$ for any fixed $w \in \mathrm{b}V$. On the other hand, it is clear from (16) that for $n \geq 2$ the coefficients of this form are nowhere holomorphic: this failure at holomorphicity is an instance of a crucial, dimension-induced phenomenon which was alluded to in conditions ii. and (c) and will be further discussed in Example (2.2.3).

Suppose now that for each fixed $z\eta(w,z)$ is a form oftype (1, 0) in w with coefficients of class C^1 in each variable. (We are not assuming that η is a generating form). Set

$$\Omega_0(\eta)(w,z) = \frac{1}{(2\pi i)^n}\eta \wedge \left(\overline{\partial}_w \eta\right)^{n-1}(w,z) \tag{17}$$

where $\left(\overline{\partial}_w \eta\right)^{n-1}$ stands for the wedge product: $\overline{\partial}_w \eta \wedge \cdots \wedge \overline{\partial}_w \eta$ performed $(n-1)$-times. We call $\Omega_0(\eta)$ the *Cauchy-Fantappiè form for* η. Note that $\Omega_0(\eta)(w,z)$ is of type $(n, n-1)$

in the variable $w \in U$ while in the variable $z \in V$ it is just a function.

Example (2.2.3)[42]. The Cauchy-Fantappié form for the Bochner-Martinelli generating form or, for short, Bochner-Martinelli CF form is

$$\Omega_0\left(\frac{\partial_w \beta}{\beta}\right)(w,z) = \frac{(n-1)!}{(2\pi i|w-z|^2)^n}\sum_{j=1}^{n}(\overline{w}-\overline{z})\,dw_j \wedge \left(\bigwedge_{v\neq j} d\,\overline{w}_v \wedge dw_v\right).$$

Now the Bochner-Martinelli integral is the operator

$$\mathrm{C_{BM}}f(z) = \int_{w\in \mathrm{b}D} f\,(w)C_{BM}(W,z)\,, z\in D, f\in C(\mathrm{b}D)$$

where the kernel $C_{BM}\ (w,z)$ is the Bochner-Martinelli CΓ form restricted to the boundary; more precisely

$$C_{BM}(W,z) = j^*\Omega_0\left(\frac{\partial_w \beta}{\beta}\right)(w,z)\,, w\in \mathrm{b}D, z\in D$$

where j^* denotes the pullback of forms under the inclusion

$$j\colon \mathrm{b}D \hookrightarrow \mathbb{C}^n$$

It is clear that such operator is "natural" in the sense discussed in condition (a) in and we will see that this operator also satisfies condition (b), see Proposition (2.2.9). On the other hand, the kernel $C_{BM}(w,z)$ is nowhere holomorphic in z: as a result, when $n>1$ the Bochner-Martinelli integral does not satisfy condition (c).

We will now review the properties of Cauchy-Fantappiè forms that are most relevant to us.

Property (2.2.4)[42]. For any function $\mathrm{g}\in \mathrm{C}^1(\mathrm{U})$ we have

$$\Omega_0\big(g(w)\eta(w,z)\big) = g^n(w)\Omega_0(\eta)(w,z)\ for\ any\ w\in U.$$

Proof. The proof is a computation: by the definition (17), we have

$$\Omega_0(g\eta) = g\eta\wedge\left(\overline{\partial}(g\eta)\right)^{n-1}$$

On the other hand, computing $\mathrm{C}\partial^-(g\eta))^{n-1}$ produces two kinds of terms:

(a.) Terms that contain $\overline{\partial}g\wedge\eta$: but these do not contribute to $\Omega_0(g\eta)$ because $g\eta\wedge\overline{\partial}g\wedge\eta=0$ (which follows from $\eta\wedge\eta=0$ since η has degree 1);

(b.) The term $g^{n-1}\overline{\partial}\eta$, which gives the desired conclusion.

Suppose, further, that $\eta\ (w,z)$ is generating at z relative to V. Then the following two properties also hold.

Property (2.2.5)[42]. We have that

$$\left(\overline{\partial}_w\eta\right)^n(w,z) = 0\ for\ any\ w\in U_z. \tag{18}$$

Note that if the coefficients of η (., z) are in $C^2(U_z)$, then as a consequence of the fact that $\overline{\partial}0\overline{\partial}=0$, we have that $\left(\overline{\partial}_w\eta\right)^n(w,z) = d_w\Omega_0(\eta)(w,z)$ and (18) can be formulated equivalently as

$$d_w\Omega_0(\eta)(w,z) = 0, w \in U_z.$$

Proof. We prove (18) in the case: $n = 2$ and leave the proof for general n as an exercise for the reader. Thus, writing

$$\eta = \eta_1 dw_1 + \eta_2 dw_2$$

we obtain

$$\left(\overline{\partial}_w \eta\right)^2 = -2\overline{\partial}_w \eta_1 \wedge \overline{\partial}_w \eta_2 \wedge dw_1 \wedge dw_2 \qquad (19)$$

Now

$$\eta_1(w,z)(w_1 - z_1) + \eta_2(w,z)(\mathrm{w}_2 - z_2) = 1 \; for \; any \; w \in U_z$$

because η is generating at z, and applying $\overline{\partial}_w$ to each side of this identity we obtain

$$(w_1 - z_1)\overline{\partial}_w \eta_1(w,z) + (w_2 - z_2)\overline{\partial}_w \eta_2(w,z) = 0 \; for \; any \; w \in U_z \qquad (20)$$

Recall that $U_z \subset \mathbb{C}^2 \backslash \{z\}$, see Definition (2.2.1), and so

$$U_z \cap U = U_z^1 \cup U_z^2$$

where

$$U_z^1 = \{w = (w,w) \in U_z \cap U, w1 \neq z_1\} \qquad (21)$$

$$U_z^2 = \{w = (w_1, w_2) \in U_z \cap U, w_2 \neq z_2\} \qquad (22)$$

But for any two sets A and B one has $A \cup B = (A\backslash B) \cup (B\backslash A) \cup (A \cap B)$ where $\dot{\cup}$ denotes disjoint union. Now, if $w \in U_z^1 \backslash U_z^2$ then (20) reads

$$\left(w_{1-z_1}\right)\overline{\partial}_w \eta_1(w,z) = 0, w_1 \neq z_1$$

but this implies

$$\overline{\partial}_w \eta_1(w,z) = 0 \; for \; any \; w \in U_z^1 \backslash U_z^2.$$

One similarly obtains

$$\overline{\partial}_w \eta_2(w,z) = 0 \; for \; any \; w \in U_z^2 \backslash U_z^1).$$

We are left to consider the case when $w \in U_z^1 \cap U_z^2$; note that since

$$(w_1 - z_1)(w_2 - z_2) \neq 0 \; for \; any \; w \in U_z^1 \cap U_z^2$$

showing that $\left(\overline{\partial}_w \eta\right)^2 (w,z) = 0$ for any $w \in U_z^1 \cap U_z^2$ is now equivalent to showing that

$$(w_1 - z_1)(w_2 - z_2)\left(\overline{\partial}_w \eta\right)^2 (w,z) = 0 \; for \; any \; w \in U_z^1 \cap U_z^2$$

To this end, combining (19) with (20) we find

$$\left(w_{1-z_1}\right)\left(w_{2-z_2}\right)\left(\overline{\partial}_w \eta\right)^2 (w,z)$$

$$= 2(w_1 - z_1)^2 \overline{\partial}_w \eta_1(w,z) \wedge \overline{\partial}_w \eta_1(w,z) \wedge dw_1 \wedge dw_2$$

and indeed

$$\overline{\partial}_w\eta_1 \wedge \overline{\partial}_w\eta_1 = 0$$

because $\overline{\partial}_w\eta_1$ is a form of degree 1.

Let η (w,z) be a form of type (1, 0) in the variable w (not necessarily generating for V) and with coefficients in $C^1(U \times V)$; set

$$\Omega_1(\eta)(w,z) = \frac{(n-1)}{(2\pi i)^n}\left(\eta \wedge \left(\overline{\partial}_w\eta\right)^{n-2} \wedge \overline{\partial}_z\eta\right)(w,z) \qquad (23)$$

Note that $\Omega_1(\eta)(w,z)$ is of type $(n, n-2)$ in the variable w and of type $(0,1)$ in the variable z. We call $\Omega_1(\eta)$ the *Cauchy-Fantappie 'form of order 1 for* η, and the previous one, $\Omega_0(\eta)$, will now be called *Cauchy-Fantappie 'form of order* 0.

In the previous properties z was fixed; here it is allowed to vary.

Property (2.2.6)[42]. We have (again for η generating at z)

$$(2\pi i)^n\overline{\partial}_z\Omega_0(\eta)(w,z) = -\left(\overline{\partial}_w\eta\right)^{n-1} \wedge \overline{\partial}_z\eta + \eta \wedge \left(\overline{\partial}_w\eta\right)^{n-2} \wedge \overline{\partial}_z\overline{\partial}_w\eta, \quad (24)$$

For any $w \in U_z \cap U$, where U_z is as in (15). Note that if the coefficients are in fact of class C^2 in each variable, then (24) has the equivalentformulation

$$\overline{\partial}_z\Omega_0(\eta)(w,z) = -d_w\Omega_1(\eta)(w,z)\,. \qquad (25)$$

Proof. As before, we specialize to the case: $n = 2$ and leave the proof of the general case as an exercise for the reader. For $n = 2$ identity (24) reads

$$\overline{\partial}_z\left(\eta \wedge \overline{\partial}_w\eta\right) = -\overline{\partial}_w\eta \wedge \overline{\partial}_z\eta + \eta \wedge \overline{\partial}_z\overline{\partial}_w\eta \qquad (26)$$

By the Leibniz rule we have

$$\overline{\partial}_z\left(\eta \wedge \overline{\partial}_w\eta\right) = \overline{\partial}_z\eta \wedge \overline{\partial}_w\eta + \eta \wedge \overline{\partial}_z\overline{\partial}_w\eta$$

and so it is clear that (26) will follow if we can show that

$$\overline{\partial}_w\eta \wedge \overline{\partial}_z\eta = 0, for\ any\ w \in U_z$$

for any generating form η with coefficients of class C^1. Proceeding as in the proof of Basic Property (2.2.5), we decompose

$$U_z \cap U = U_z^1 \cup U_z^2$$

where U_z^1 and U_z^2 are as in (21) and (22), respectively. Again, we have

$$\eta_1(w,z)(w_1 - z_1) + \eta_2(w,z)(\mathrm{w}_2 - z_2) = 1\ for\ any\ w \in U_z$$

because η is generating, and applying $\overline{\partial}_w$ to each side of this identity we find

$$0 = \begin{cases} \left(\mathrm{C}^{\partial\bar{w}\eta_1}\right).(w_1 - z_1) + \left(\overline{\partial}_w\eta_2\right).(w_2 - z_2)\,, \text{if } w \in U_z^1 \cap U_z^2 \\ \left(\overline{\partial}_w\eta_1\right)\cdot(w1 - z_1)\,, \text{if } w \in U_z^1 \backslash U_z^2 \\ \left(\overline{\partial}_w\eta_2\right).\left(w_{2-z_2}\right), \text{if } w \in U_z^2 \backslash U_z^1 \end{cases} \qquad (27)$$

Similarly, applying $\overline{\partial}_z$, we have

$$0=\begin{cases}\left(\overline{\partial}_z\eta_1\right).\left(w_{1-z_1}\right)+\left(\overline{\partial}_z\eta_2\right).\left(w_{2-z_2}\right), \text{if } w\in U_z^1\cap U_z^2\\ \left(\overline{\partial}_z\eta_1\right).\left(w_{1-z_1}\right), \text{if } w\in U_z^1\backslash U_z^2\\ \left(\overline{\partial}_z\eta_2\right).\left(w_2-z_2\right), \text{if } w\in U_z^2\backslash U_z^1\end{cases} \tag{28}$$

Now

$$\overline{\partial}_w\eta\wedge\overline{\partial}_z\eta=\left(\overline{\partial}_w\eta_1\wedge\overline{\partial}_z\eta_2-\overline{\partial}_w\eta_2\wedge\overline{\partial}_z\eta_1\right)\wedge dw_1\wedge dw_2 \tag{29}$$

Note that if $w\in U_z^1\backslash U_z^2$ then $1\neq Z_1$, and so showing that

$$\overline{\partial}_w\eta\wedge\overline{\partial}_z\eta=0\ for\ w\in U_z^1\backslash U_z^2$$

is equivalent to showing that

$$\left(\overline{\partial}_w\eta\wedge\overline{\partial}_z\eta\right).\left(w_{1-z_1}\right)=0$$

that is (using (29))

$$\left(\overline{\partial}_w\eta_1\left(w_1-z_1\right)\wedge\overline{\partial}_z\eta_2-\overline{\partial}_w\eta_2\wedge\overline{\partial}_z\eta_1\left(w_1-z_1\right)\right)\wedge dw_1\wedge dw_2=0$$

but this is indeed true by the generating form hypothesis on η as expressed in (27) and (28). This shows that the desired conclusion is true when $w\in U_z^1\backslash U_z^2$; the case: $w\in U_z^2\backslash U_z^1$ is dealt with in a similar fashion. Finally, if $w\in U_z^1\cap U_z^2$, then $(w_1-z_1)(w_2-z_2)\neq 0\ and$

$$\left(\overline{\partial}_w\eta\wedge\overline{\partial}_z\eta\right).\left(w_{1-z_1}\right)\left(w_2-z_2\right)$$

$$=\left(\left(\overline{\partial}_w\eta_1\right)\left(w_1-z_1\right)\wedge\left(\overline{\partial}_z\eta_2\right)\left(w_2-z_2\right)+-\left(\overline{\partial}_w\eta_2\right)\left(w-z\right)\right.$$

$$\left.\wedge\left(\overline{\partial}_z\eta_1\right)\left(w_1-z_1\right)\right)\wedge dw_1\wedge dw_2$$

but the two terms in the righthand side of this identity cancel out on account of (27) and (28). We highlight the theory of reproducing formulas for holomorphic functions by means of integral operators that arise from the Cauchy-Fantappié formalism. We show that the usual reproducing properties of such operators extend to the situation where the data and the generating form have lower regularity. We begin with a rather specific example: the reproducing formula for the Bochner-Martinelli integral, see Proposition(2.2.9). The proof of this result is a consequence of a recasting of the classical mean value property for harmonic functions in terms of an identity (30) that links the Bochner-artinelli $C\Gamma$ form on a sphere with the sphere's Euclidean surface measure.

Because the Bochner-Martinelli integral of a continuous function is, in general, not holomorphic in z, in fact we need a more general version of Proposition (2.2.9) that applies to integral operators whose kernel is allowed to be any Cauchy-Fantappié form: this is done in Proposition (2.2.10). While the operators defined so far are given by surface integrals over the boundary of the ambient domain, following an idea of Ligocka [68] another family of integral operators can be defined (essentially by ifferentiating the kernels of the operators in the statement of Proposition (2.2.10)) which are realized as solid" integrals over the ambient domain, and we show in Proposition (2.2.11) that such operators, too, have the reproducing property.

Lemma (2.2.7)[42]. Let $z \in \mathbb{C}^n$ be given and consider a ball centered at such z, $B_r(z) = \{w \in \mathbb{C}^n, |w - z| < r\}$.

Then, at the center z andfor any $w \in bB_r(z)$ we have that the Bochner-Martinelli C^F form for the ball $B_r(z)$ has the following representation

$$C_{BM}(w,z) = \frac{d\sigma(w)}{\sigma(b\mathbb{B}_r(z))} \tag{30}$$

where $d\sigma(w)$ is the element of Euclidean surface measure for $bB_r(z)$, and

$$\sigma(b\mathbb{B}_r(z)) = \frac{2\pi^n r^{2n-1}}{(n-1)!}$$

denotes suiface measure of the sphere $b\mathbb{B}_r(z)$.

Proof. We claim that the desired conclusion is a consequence of the following identity

$$\Omega_0(\partial_w \beta)(w,z) = \frac{(n-1)!}{2\pi^n} * \partial_w \beta(w,z) \tag{31}$$

where, as usual, we have set $\beta\ (w,z) = |w - z|^2$, and $*$ denotes the Hodge-star operator mapping forms of type (p,q) to forms of type $(n-q, n-p)$. Let us first prove (30) assuming the truth of (31). To this end, we first note that from (31) and Basic Property (2.2.4) we have

$$\Omega_0\left(\frac{\partial_w \beta}{\beta}\right)(w,z) = \frac{(n-1)!}{2\pi^n \beta^n} * \partial_w \beta(w,z)\ , w \in \mathbb{C}^n$$

But $\partial_w \beta\ (w,z) = \partial\rho(w)$, $w \in \mathbb{C}^n$ with $(w) := \beta(w,z) - r^2$, a defining function for $B_r(z)$. Now recall that $C_{BM}(w,z) = j^*\Omega_0(\partial_w \beta/\beta)$ where j is the inclusion: $bB_r(z) \to \mathbb{C}^n$, see Example (2.2.3), so that $j^*\beta^n = r^{2n}$. Combining these facts we conclude that, for ρ as above

$$C_{BM}(w,z) = \frac{(n-1)!}{2\pi^n r^{2n}} j^*(*\,\partial\rho)(w)\ , w \in b\mathbb{B}_r(z)$$

and since $\|d\rho(w)\| = 2r$ whenever $w \in bB_r(z)$, we obtain

$$C_{BM}(w,z) = \frac{(n-1)!}{2\pi^n r^{2n-1}} \frac{2j^*(*\,\partial\rho)}{\|d\rho\|}(w)\ , w \in b\mathbb{B}_r(z)\ ;$$

but

$$d\sigma(w) = \frac{2j^*(*\,\partial\rho)}{\|d\rho\|}(w)\ , w \in b\mathbb{B}_r(z) \tag{32}$$

see [76], and this gives (30).

We are left to prove (31): to this end, we assume $n = 2$ and leave the case of arbitrary complex dimension as an exercise to for the reader. Since

$$*\, dw_j = \frac{1}{2i^2} dw_j \wedge d\overline{w}_{j'} \wedge dw_{j'}\ , where\ j' = \{1,2\}\backslash\{j\}$$

and

$$\partial_w \beta = (\overline{w}_{1-\bar{z}_1}) dw_1 + (\overline{w}_{2-\bar{z}_2}) dw_2$$

then

$$* \partial_w\beta = \frac{1}{2i^2}(\overline{w}_1 - \overline{z}_1)dw_1 \wedge d\overline{w}_2 \wedge dw_2 + (\overline{w}_2 - \overline{z}_2)dw_2 \wedge d\overline{w}_1 \wedge dw_1$$

On the other hand

$$\overline{\partial}_w\partial_w\beta = d\overline{w}_1 \wedge dw_1 + d\overline{w}_2 \wedge dw_2$$

and so

$$\Omega_0(\partial_w\beta) = \frac{1}{(2\pi i)^2}\partial_w\beta \wedge \overline{\partial}_w\partial_w\beta$$

$$= \frac{1}{(2\pi i)^2}\left(\left(\overline{w}_{1-\overline{z}_1}\right)dw_1 \wedge d\overline{w}_2 \wedge dw_2 + \left(\overline{w}_{2-\overline{z}_2}\right)dw_2 \wedge d\overline{w}_1 \wedge dw_1\right)$$

$$= \frac{1}{2\pi^2} * \partial_w\beta.$$

This shows (31) and concludes the proof of the lemma.
(We remark in passing that identity (30), while valid for the Bochner-Martinelli generating form, is not true for general η.)

Definition (2.2.8)[42]. Given an integer $1 \le k \le \infty$ and a bounded domain $D \subset \mathbb{C}^n$, we say that D *is of class* C^k (alternatively, D is C^k *−smooth*) if there is an open neighborhood U of the boundary of D, and a real-valued function $\rho \in C^k(U)$ such that

$$U \cap D = \{w \in U | \rho(w) < 0\}$$

and

$$\nabla\rho(w) \neq 0 \; for \; any \; w \in U.$$

Any such function is called a *defining function for* D.
From this definition it follows that

$$\mathrm{b}D = \{w \in U | \rho(w) = 0\} \; and \; U \backslash \overline{D} = \{w \in U | \rho(w) > 0\}.$$

Proposition (2.2.9)[42]. For any bounded domain $\mathrm{V} \subset \mathbb{C}^{\mathrm{n}}$ with boundary of class C^1 and for any $\in \vartheta(\mathrm{V}) \cap \mathrm{C}(\overline{\mathrm{V}})$, we have

$$f(z) = \mathrm{C}_{BM}f(z)\,, z \in V.$$

Proof. Given $z \in V$, let $r > 0$ be such that

$$\overline{\mathbb{B}_r(z)} \subset V.$$

Note that the mean value property for harmonic functions:

$$f(z) = \frac{1}{\sigma\left(\mathrm{b}\mathbb{B}_r(z)\right)}\int_{\mathrm{b}\mathbb{B}_r(z)} f(w)d\sigma(w)\,, f \in \mathrm{Harm}\left(\mathbb{B}_r(z)\right) \cap C\left(\overline{\mathbb{B}_r(z)}\right)$$

and identity (30) give

$$f(z) = \int_{w \in \mathrm{b}\mathbb{B}_r(z)} f(w)C_{BM}(w, z) \tag{33}$$

To prove the conclusion, we apply Stokes' theorem on the set

$$V_r(z) := V \backslash \overline{\mathbb{B}_r(z)}$$

and we obtain

$$\int_{w \in V_r(z)} d_w \left(f(w) \Omega_0 \left(\frac{\partial_w \beta}{\beta}(w,z) \right) \right) = \int_{w \in bV_r(z)} f(w) C_{BM}(w,z)$$

But by Basic Property (2.2.5), and since f is holomorphic, we have

$$d_w \left(f(w) \Omega_0 \left(\frac{\partial_w \beta}{\beta}(w,z) \right) \right) = f(w) \bar{\partial}_w \Omega_0 \left(\frac{\partial_w \beta}{\beta}(w,z) \right) = 0$$

and so the previous identity becomes

$$\int_{w \in bV} f(w) C_{BM}(w,z) = \int_{w \in b\mathbb{B}_r(z)} f(w) C_{BM}(w,z)$$

but the lefthand side is $C_{BM} f(z)$, while (33) says that the righthand side equals $f(z)$.

Proposition (2.2.10)[42]. Let $D \subset \mathbb{C}^n$ be a bounded domain of class C^1 and let $z \in D$ be given. Suppose that η (., z) is a generatingform at z relative to D. Suppose, furthermore, that the coefficients of η (., z) are in $C^1(U_z)$, where U_z is as in Definition (2.2.1). Then, we have

$$f(z) = \int_{w \in \mathrm{b}D} f(w) j^* \Omega_0(\eta)(w,z) \; for \; any \; f \in \vartheta(D) \cap C(\overline{D}) . \qquad (34)$$

Proof. Consider a smooth open neighborhood of bD, which we denote $U_z(\mathrm{b}D)$, such that

$$U_z(\mathrm{b}D) \subset U_z \qquad (35)$$

where U_z is as in (14) and (2). Now fix two neighborhoods U' and U'' ofthe boundary of D such that

$$U'' \Subset U' \subset U_z(\mathrm{b}D)$$

and let $\chi_0\ (w,z)$ be a smooth cutoff function such that

$$\chi_0(w,z) = \begin{cases} 1 & \text{if } w \in U^n \\ 0 & \text{if } w \in \mathbb{C}^n \backslash \overline{U'} \end{cases} \qquad (36)$$

Define

$$\eta^\circ(w,z) = \chi_0(w,z)\eta(w,z) + \big(1 - \chi_0(w,z)\big) \frac{\partial_w \beta}{\beta}(w,z)$$

and

$$D^\circ = D \cap U_z(\mathrm{b}D).$$

Then η° is generating at z relative to D° (and the open set U_z of Definition (2.2.1) is the same for η and for η°); furthermore, it follows from (35) that

$$\overline{D^\circ} \subset U_z.$$

Now let $\{\eta_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of (1, 0)-forms with coefficients in $C^2(\overline{D^\circ})$ with the property that

$$\|\eta_\ell^\circ - \eta^\circ(\cdot, z)\|_{C^1(\overline{D^\circ})} \to 0 \; as \; \ell \to \infty.$$

Suppose first that $\in \vartheta\left(U(\overline{D})\right)$. Then by type considerations (and since f is holomorphic in a neighborhood of $\overline{D}$) for any $w \in D^\circ$ and for any P we have

$$d_w\big(f(w)\Omega_0(\eta_\ell^\circ)(w,z)\big) = \overline{\partial}_w\big(f(w)\Omega_0(\eta \quad)(w,z)\big)$$

$$= f(w)\overline{\partial}_w\Omega_0(\eta_\ell^\circ)(w,z) = f(w)\big(\overline{\partial}_w\eta_\ell^\circ\big)^n(w,z)$$

Thus, applying Stokes' theorem on D° we find

$$\int_{w\in D^\circ} f(w)\big(\overline{\partial}_w\eta_\ell^\circ\big)^n(w,z) + \int_{w\in bD} f(w)j^*\Omega_0(\eta_\ell^\circ)(W,z)$$

$$= \int_{w\in D\cap \mathrm{b}(U_z(\mathrm{b}D))} f\,(w)j^*\Omega_0(\eta_\ell^\circ)(w,z)$$

Letting $P \to \infty$ in the identity above we obtain

$$\int_{w\in D^\circ} f\,(w)\big(\overline{\partial}_w\eta^\circ\big)^n(w,z) + \int_{w\in \mathrm{b}D} f\,(w)j^*\Omega_0(\eta^\circ)(w,z)$$

$$= \int_{w\in D\cap \mathrm{b}(U_z(\mathrm{b}D))} f\,(w)j^*\Omega_0(\eta^\circ)(w,z)$$

Since η° is generating at z, by Basic Property (2.2.5) this expression is reduced to

$$\int_{w\in \mathrm{b}D} f\,(w)j^*\Omega_0(\eta^\circ)(w,z) = \int_{w\in D\cap \mathrm{b}(U_z(\mathrm{b}D))} f\,(w)j^*\Omega_0(\eta^\circ)(w,z) \qquad (37)$$

But

$$\eta^\circ(w,z) = \begin{cases} \eta\,(w,z)\,, & \text{for } w \text{ in an open neighborhood of } \mathrm{b}D \\ \dfrac{\partial_w\beta}{\beta}\,(w,z)\,, & \text{for } w \text{ in an open neighborhood of } \mathrm{b}\big(U_z(\mathrm{b}D)\big) \end{cases}$$

as a result, (37) reads

$$\int_{w\in \mathrm{b}D} f\,(w)j^*\Omega_0(\eta)(w,z) \; = \int_{w\in D\cap \mathrm{b}(U_z(\mathrm{b}D))} f\,(w)C_{BM}(w,z)$$

On the other hand, $D \cap \mathrm{b}\big(U_z(\mathrm{b}D)\big) = \mathrm{b}V$ for a(smooth) open set $V \subset D$, and using Proposition (2.2.9) we conclude that (34) holds in the case when $\in \vartheta\left(U(\overline{D})\right)$. To prove the conclusion in the general case: $\in \vartheta(D) \cap C(\overline{D})$, we write $D = \{\rho(w) < 0\}$, so that $\rho(z) < 0$ (since $z \in D$) and furthermore

$$z \in D_k := \{w\, \rho(w) < -\frac{1}{k}\}\ for\ any\ k \geq k(z)\,. \tag{38}$$

But $D_k \subset D$ and so $\in \vartheta\left(U(\overline{D}_k)\right)$; moreover

$$\mathrm{b}D_k \subset U_z\ for\ k = k(z)\ sufficiently\ large.$$

Thus, (34) grants

$$\int_{w \in \mathrm{b}D_k} f\,(w) j_k^* \Omega_0(\eta)(w,z) = f(z)\ for\ any\ k \geq k(z)$$

where j_k^* denotes the pullback under the inclusion $j_k : \mathrm{b}D_k \hookrightarrow \mathbb{C}^n$.
The conclusion now follows by letting $k \to \infty$.
We remark that in the case when η is the Bochner-Martinelli generating form $:= \partial_w \beta / \beta$, Proposition (2.2.10) is simply a restatement of Proposition (2.2.9). However, since the coefficients of the Bochner-Martinelli CΓ form are nowhere holomorphic in the variable z, Proposition (2.2.9) is of limited use in the investigation of the Cauchy-Szegö and Bergman projections, and Proposition (2.2.10) will afford the use of more specialized choices of η.
The following reproducing formula is inspired by an idea of Ligocka [68].
Proposition (2.2.11)[42]. With same hypotheses as in Proposition (2.2.10), we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{w \in D} f\,(w) \left(\overline{\partial}_w \overline{\eta}\right)^n (w,z)\,, f \in \vartheta(D) \cap L^1(D)$$

for any (1, 0)-form $\overline{\eta}$(., z) with coefficients in $C^1(\overline{D})$ such that

$$j^* \Omega_0(\tilde{\eta})(\cdot,z) = j^* \Omega_0(\eta)(\cdot,z) \tag{39}$$

where j^* denotes the pullback under the inclusion j : bD $\hookrightarrow \mathbb{C}^n$.

Note that if one further assumes that the coefficients of $\overline{\eta}$ (., z) are in $C^2(D) \cap C^1(\overline{D})$ then,

as a consequence of the fact that $\overline{\partial} 0 \overline{\partial} = 0$, we have

$$\frac{1}{(2\pi i)^n} \left(\overline{\partial}_w \overline{\eta}\right)^n = \overline{\partial}_w \Omega_0(\eta\gamma.$$

Proof. Fix $z \in D$ arbitrarily and let $\{\overline{\eta}_\ell\}_{\ell \in \mathbb{N}} \subset C^2_{1,0}(\overline{D})$ be such that

$$\|\tilde{\eta}_\ell - \tilde{\eta}(\cdot,z)\|_{C^1(\overline{D})} \to 0\ as\ \ell \to \infty. \tag{40}$$

Suppose first that $\in \vartheta\left(U(\overline{D})\right)$. Applying Stokes' theorem to the $(n, n-1)$-form with coefficients in $C^1(\overline{D})$

$$f \cdot \Omega_0(\tilde{\eta}_\ell)$$

we find

$$\int_{w\in D} f(w)\bar{\partial}\Omega_0(\tilde{\eta}_\ell)(w) = \int_{w\in bD} f(w)j^*\Omega_0(\tilde{\eta}_\ell)(w)\ for\ any\ \ell.$$

On the other hand, since the coefficients of $\overline{\eta}\ell$ are in $C^2(D)$, we have

$$\bar{\partial}\Omega_0(\overline{\eta}_\ell) = \frac{1}{(2\pi i)^n}\left(\bar{\partial}\overline{\eta}p\right)^n\ for\ any\ \ell$$

and so the previous identity can be written as

$$\frac{1}{(2\pi i)^n}\int_{w\in D} f(w)\left(\bar{\partial}\overline{\eta}\ell\right)^n(w) = \int_{w\in bD} f(w)j^*\Omega_0(\overline{\eta}p)(w)\ for\ any\ \ell.$$

Letting $\ell \to \infty$ in the identity above and using (40) we obtain

$$\frac{1}{(2\pi i)^n}\int_{w\in D} f(w)\left(\bar{\partial}\overline{\eta}\right)^n(w,z) = \int_{w\in bD} f(w)j^*\Omega_0(\overline{\eta})(w,z)\,.$$

Combining the latter with the hypothesis (39) we obtain

$$\frac{1}{(2\pi i)^n}\int_{w\in D} f(w)\left(\bar{\partial}_w\overline{\eta}\right)^n(w,z) = \int_{w\in bD} f(w)j^*\Omega_0(\eta)(w,z) = f(z)$$

where the last identity is due to Proposition (2.2.10).

If $f\in\vartheta(D)\cap L^1(D)$ then $\in\vartheta\left(U\left(\overline{D}_k\right)\right)$, where D_k is as in (38); moreover, $\mathrm{b}D_k\subset U_z$ for any $k\geq k(z)$, so by the previous case we have

$$f(z) = \int_{w\in D_k} f(w)\left(\bar{\partial}_w\overline{\eta}\right)^n(w,z)\ for\ any\ k\geq k(z)\,.$$

The conclusion now follows by observing that

$$\int_{w\in D_k} f(w)\left(\bar{\partial}_w\overline{\eta}\right)^n(w,z) \to \int_{w\in D} f(w)\left(\bar{\partial}_w\overline{\eta}\right)^n(w,z)$$

as $k\to\infty$, by the Lebesgue dominated convergence theorem.

Note that the extension $\overline{\eta}\,(w,z) := \chi_0(w,z)\eta(w,z)$, with χ_0 as in (36), satisfies a stronger condition than (39), namely the identity

$$\overline{\eta}(\cdot,z) = \eta(\cdot,z)\ for\ any\ w\in U_z'(\mathrm{b}D)\,. \tag{41}$$

On the other hand, it will become clear in the sequel that this simple-minded extension is not an adequate tool for the investigation of the Bergman projection, and more ad-hoc constructions are presented. In order to obtain operators that satisfy the crucial condition (c) one would need generating forms whose coefficients are holomorphic. However, in contrast with the situation for the planar case (where the Cauchy ketnel plays the role of a universal generating form with holomorphic coefficient) in higher dimension there is a large class of domains $V\subset\mathbb{C}^n$ that cannot support generating forms with holomorphic coefficients. This dichotomy is related to the notion of *domain of holomorphy*, that is, the property that for any boundary point $w\in \mathrm{b}V$ there is a holomorphic function $f_w\in\vartheta(D)$ that cannot be continued holomorphically in a neighborhood of w. It is clear that every planar domain $V \subset \mathbb{C}$ is a domain of holomorphy, because in this case one may take $f_w(z) := (w-z)^{-1}$ where $w \in bV$ has been fixed. On the other hand the following

$$V = \{z \in \mathbb{C}^2 | 1/2 < |z| < 1\}$$

is a simple example of a smooth domain in $\mathbb{C}^2$ that is not a domain of holomorphy; other classical examples are discussed e.g., in [76]. A necessary condition for the existence of a generating form η whose coefficients are holomorphic in the sense described above is then that V be a domain of holomorphy. To prove the necessity of such condition, it suffices to observe that as a consequence of (15) and (14) one has

$$\sum_{j=1}^{n} \eta_j (w,z)(w_j - z_j) = 1 \; for\ any\ w \in bV, z \in V. \tag{42}$$

It is now clear that for each fixed $w \in bV$, at least one of the $\eta_j\ (w,z)$'s blows up as $z \to w$

(and it is well known that this is strong enough to ensure that V be a domain of holomorphy). In its current stage of development, the Cauchy-Fantappié framework is most effective in the analysis of two particular categories of pseudo-convex domains: these are the *strongly pseudo-convex* domains and the related category of *strongly* $\mathbb{C}$*-linearly convex* domains.

Definition (2.2.12)[42]. We say that a domain $D \subset \mathbb{C}^n$ is *strongly pseudo-convex* if D is of class C^2 and if *any* defining function ρ for D satisfies the following inequality

$$L_w(\rho)(\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 \rho(w)}{\partial \zeta_j \partial \overline{\zeta}_k} \xi_j \overline{\xi}_k > 0 \; for\ any\ w \in bD, \xi \in T_w^{\mathbb{C}}(bD) \tag{43}$$

where $T_w^{\mathbb{C}}$ denotes the *complex tangent space to* bD *at* w, namely

$$T_w^{\mathbb{C}}(bD) = \{\xi \in \mathbb{C}^n | \langle \partial\rho(w), \xi \rangle = 0\},$$

see [76].

If D is of class C^k with $k \geq 1$, and if ρ_1 and ρ_2 are two distinct defining functions for D, it can be shown that there is a positive function h of class C^{k-1} in a neighborhood U of the boundary of D, such that

$$\rho_1(w) = h(w)\rho_2(w), w \in U,$$

and

$$\nabla \rho_1(w) = h(w) \nabla \rho_2(w) \; for\ any\ w \in U \cap bD, \tag{44}$$

see [76]. As a consequence of (44), if condition (43) is satisfied by *one* defining function then it will be satisfied by *every* defining function. The hermitian form $L_w(\rho)$ defined by (43) is called *the Levi form*, *or complex Hessian*, *of* ρ at w. We remark that in fact there is a "special" defining function ρ for D that is strictly plurisubharmonic on a neighborhood U

of $\overline{D}$, that is

$$L_w(\rho)(\xi) > 0 \; for\ any\ w \in U\ and\ any\ \xi \in \mathbb{C}^n \setminus \{0\}, \tag{45}$$

see [76], and we will assume that ρ satisfies this stronger condition.

We should point out that there is another notion of strong pseudo-convexity that includes the domains of Definition (2.2.12) as a subclass (this notion does not require the gradient of

ρ to be non-vanishing on bD); the domains of Definition (2.2.12) are sometimes referred to as "strongly Levi-pseudo-convex see [76].

Definition (2.2.13)[42]. We say that $D \subset \mathbb{C}^n$ is *strongly* $\mathbb{C}$*-linearly convex* if D is of class C^1 and if any defining function for D satisfies this inequality:

$$|\langle \partial\rho(w), w - z\rangle| \geq C|w - z|^2 \; for\; any\; w \in \mathrm{b}D, z \in \overline{D}. \tag{46}$$

We call those domains that satisfy the following, weaker condition

$$|(\partial\rho(w)\,, w - z\rangle| > 0 \; for\; any\; w \in \mathrm{b}D\; and\; any\; z \in \overline{D}\backslash\{w\} \tag{47}$$

strictly $\mathbb{C}$*-linearly convex*. This condition is related to certain separation properties of the domain from its complement by (real or complex) hyperplanes, see [43], [62]: that this must be so is a consequence of the assertion that, for w and z as in (46), the quantity $|\langle \partial\rho(w)\,, w - z\rangle|$ is comparable to the Euclidean distance of z to the complex tangent space $T_w^{\mathbb{C}}(\mathrm{b}D)$; It is not difficult to check that

$$D := \{z \in \mathbb{C}^n |\; \mathrm{Im}\, z_n > \; (|z_1|^2 + \; +|z_{n-1}|^2)^2\}$$

is strictly, but not strongly, $\mathbb{C}$-linearly convex.

Lemma (2.2.14)[42]: If D is strictly $\mathbb{C}$-linearly convex then for any $z \in D$ there is an open set $U_z \subset \mathbb{C}^n\backslash\{z\}$ such that $\mathrm{b}D \subset U_z$ and inequality (47) holds for any w in U_z. Furthermore, if D is strongly $\mathbb{C}$-linearly convex then the improved inequality (46) will hold for any $w \in U_z$.

Proof. Suppose that D is strictly $\mathbb{C}$-linearly convex and fix $z \in D$. By the continuity of the function $h(\zeta) := |\langle \partial\rho(\zeta)\;,\; \zeta - z)\;|$, if (47) holds at $w \in \mathrm{b}D$ then there is an open neighborhood $U_z(w)$ such that $h(\zeta) > 0$ for any $\zeta \in U_z(w)$ and so we have that $h(\zeta) > 0$ whenever

$$\zeta \in U_z := \bigcup_{w \in \mathrm{b}D} U_z\,(w)\,.$$

It is clear that $\mathrm{b}D \subset U_z$; furthermore, since $h(z) = 0$ then $U_z(w) \subset \mathbb{C}^n\backslash\{z\}$ for any $w \in \mathrm{b}D$ and so $U_z \subset \mathbb{C}^n\backslash\{z\}$.

If D is strongly $\mathbb{C}$-linearly convex then the conclusion will follow by considering the function $h(\zeta) := |\langle \partial\rho(\zeta)\,, \zeta - z\rangle| - C|\zeta - z|^2$.

Lemma (2.2.15)[42]: Any strongly $\mathbb{C}$-linearly convex domain of class C^2 is strongly pseudoconvex.

The key point in the proof of this lemma is the observation that, as a consequence of (46), the real tangential Hessian of any defining function for a domain as in Lemma (2.2.15) is positive definite when restricted to the complex tangent space $T_w^{\mathbb{C}}(\mathrm{b}D)$ (viewed as a vector space over the *real* numbers). The converse of Lemma (2.2.15) is not true: we have the following (smooth) domain

$$D := \{z = (z_1, z_2) \in \mathbb{C}^2 |\mathrm{Im}\, z_2 \; > 2(\,\mathrm{Re}\, z_1)^2 - (\,\mathrm{Im}\, z_1)^2\}$$

is strongly pseudo-convex but not strongly $\mathbb{C}$-linearly convex.

We remark that while the designations "strongly" and "strictly" indicate distinct families of $\mathbb{C}$-linearly convex domains (and of convex domains), for pseudo-convex domains there is no such distinction, and in fact in the literature the terms "*strictly* pseudo-convex" and "*strongly* pseudo-convex" are often interchanged: this is because the positivity condition (45) implies the seemingly stronger inequality

$$L_w(\rho,\xi) \geq c_0|\xi|^2 \; for\ any\ w \in U' \; and\ for\ any\ \xi \in \mathbb{C}^n \tag{48}$$

Indeed, if (45) holds then the function $\gamma(w) := \min\{L_w(\rho,\xi)||\xi| = 1\}$ is positive, and by bilinearity it follows that $L_w\,(\rho,\xi) \geq \gamma(w)|\xi|^2$ for any $\xi \in \mathbb{C}^n$; since ρ is of class C^2 (and D is bounded) we may further take the minimum of $\gamma(w)$ over, say, $w \in U' \subset U$ and thus obtain (48), see [76]

A first step in the study of the Bergman and Cauchy-Szegö projections is the construction of integral operators with kernels given by Cauchy-Fantappié forms that are (at least) *locally* holomorphic in z, that is for z in a neighborhood of each (fixed) w: it is at this juncture that the notion of strong pseudo-convexity takes center stage. We show how to construct such operators in the case when D is a bounded, strongly pseudo-convex domain, and we then proceed to prove the reproducing property.

we fix a strictly plurisubharmonic defining function for D; that is, we fix

$$\rho: \mathbb{C}^n \to \mathbb{R}, \rho \in C^2(\mathbb{C}^n)$$

such that $D = \{\rho < 0\}$; $\nabla\rho(w) \neq 0$ for any $w \in \mathrm{b}D$ and

$$L_w(\rho, w-z) \geq 2c_0|w-z|^2, w, z \in \mathbb{C}^n$$

where L_w denotes the Levi form for ρ, see (43) and (48). Consider the *Levi polynomial of* ρ *at* w:

$$\triangle\,(w,z) := \langle\partial\rho(w), w-z\rangle - \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial^2\rho(w)}{\partial\zeta_j\partial\zeta_k}(w_j - z_j)(w_k - z_k)$$

Lemma (2.2.16)[42]. Suppose $\mathrm{D} = \{\rho(\mathrm{w}) < 0\}$ is bounded and strongly pseudo-convex. Then, there is $\overline{\varepsilon}_0 = \overline{\varepsilon}_0(\mathrm{c}_0) > 0$ such that

$$2\ \mathrm{Re}\ \triangle\,(w,z) \geq \rho(w) - \rho(z) + c|w-z|^2$$

whenever $\mathrm{w} \in \mathrm{D}_c0 = \{\mathrm{w}|\rho(\mathrm{w}) < \mathrm{c}_0\}$, and $\mathrm{z} \in \overline{\mathrm{B}_{\overline{\varepsilon}}0(\mathrm{w})}$.

Here c_0 is as in (49). We leave the proof of this lemma, along with the corollary below, as an exercise for the reader. Now let $\chi_1\,(w,z)$ be a smooth cutoff function such that

$$\chi_1(w,z) = \begin{cases} 1, \text{if } |w-z| < \overline{\varepsilon}_0/2 \\ 0, \text{if } |w-z| > \overline{\varepsilon}_0 \end{cases} \tag{49}$$

where $\overline{\varepsilon}_0$ is as in Lemma (2.2.16) and set

$$g(w,z) = \chi_1(w,z)\triangle\,(w,z) + \big(1-\chi_1(w,z)\big)|w-z|^2, w, z \in \mathbb{C}^n \tag{50}$$

Lemma (2.2.17)[42]. Sup-pose $\mathrm{D} = \{\rho(\mathrm{w}) < 0\}$ is strongly pseudo-convex and of class C^2.

Then, there is $\delta_0 = \overline{\delta}_0(\overline{\varepsilon}_0, \mathrm{c}_0) > 0$ such that

$$2\ \mathrm{Re}\ g(w,z) \geq \begin{cases} \rho(w) - \rho(z) + c|w-z|^2, \text{if } |w-z| \leq \overline{\varepsilon}_0/2 \\ \rho(w) + 2\overline{\delta}_0, \text{if } \overline{\varepsilon}_0/2 \leq |w-z| < \overline{\varepsilon}_0 \\ \overline{\varepsilon}_0^2, \text{if } |w-z| > \overline{\varepsilon}_0 \end{cases}$$

whenever

$$w \in D_{c_0} = \{w|\rho(w) < c_0\} \tag{51}$$

and

$$z \in D_{2\overline{\delta}_0} = \{w|\rho(w) < 2\overline{\delta}_0\}.$$

Proof. It suffices to choose $0 < \overline{\delta}_0 < c_0\overline{\varepsilon}_0^2/16$: the desired inequalities then follow from Lemma (2.2.16).

Corollary (2.2.18)[42]. Let $D = \{\rho(w) < 0\}$ be a bounded, strongly pseudo-convex domain. Let

$$\triangle_j\,(w,z) := \frac{\partial\rho}{\partial\zeta_j}(w) - \frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2\rho(w)}{\partial\zeta_j\partial\zeta_k}\left(w_{k-z_k}\right), j = 1,\ldots n,$$

Define

$$\eta_j(w,z) := \frac{1}{g(w,z)}\Big(\chi_1(w,z)\triangle_j\,(w,z) + \big(1-\chi_1(w,z)\big)(\overline{w}-\overline{z})\Big)$$

where χ_1 and g are as in (49) and (50), and set

$$\eta(w,z) := \sum_{j=1}^{n}\eta_j\,(w,z)dw_j\ for\ (w,z) \in D_{c_0}\times D$$

with D_c0 as in (51). Then we have that $\eta(w,z)$ is a generating form for D, and one may take for U_z in Definition (2.2.1) the set

$$U_z := \{w|\max\{\rho(z),-\overline{\delta}_0\} < \rho(w) < \min\{|\rho(z)|,c_0\}\}. \tag{52}$$

Note, however, that the coefficients of η in this construction are only continuous in the variable w and so the Cauchy-Fantappié form $\Omega_0(\eta)$ cannot be defined for such η because doing so would require differentiating the coefficients of η with respect to w, see (17). For this reason, proceeding as in [34], we refine the previous construction as follows. For $\overline{\varepsilon}_0$ as in Lemma (2.2.16) and for any $0 < \varepsilon < \overline{\varepsilon}_0$, we let $\tau_{j,k}^{\varepsilon} \in C^\infty(\mathbb{C}^n)$ be such that

$$\max_{w\in\overline{D}}\quad \left|\frac{\partial^2\rho(w)}{\partial\zeta_j\partial\zeta_k} - \tau_{j,k}^{\varepsilon}(w)\right| < \varepsilon, j,k = 1,\ldots,n$$

We now define the following quantities:

$$\triangle_j^{\varepsilon}\,(w,z) := \frac{\partial\rho}{\partial\zeta_j}(w) - \frac{1}{2}\sum_{k=1}^{n}\tau_{j,k}^{\varepsilon}\,(w)(w_k-z_k)\,, j = 1,\ldots,n; \tag{53}$$

$$\triangle^{\varepsilon}\,(w,z) := \sum_{j=1}^{n}\triangle_j^{\varepsilon}\,(w,z)\big(w_j-z_j\big)\,;$$

and, for χ_1 as in (49):

$$g^{\varepsilon}(w,z) := \chi_1(w,z) \triangle^{\varepsilon}(w,z) + \left(1 - \chi_1(w,z)\right)|w-z|^2; \qquad (54)$$

$$\eta_j^{\varepsilon}(w,z) := \frac{1}{g^{\varepsilon}(w,z)} \left(\chi_1(w,z) \triangle_j^{\varepsilon}(w,z) + \left(1-\chi_1(w,z)\right)\left(\overline{w}_j - \overline{z}_j\right)\right)$$

and finally

$$\eta^{\varepsilon}(w,z) := \sum_{j=1}^{n} \eta_j^{\varepsilon}(w,z) dw_j.$$

Lemma (2.2.19)[42]. Let $D = \{\rho(w) < 0\}$ be a bounded strongly pseudo-convex domain. Then, there is $\varepsilon_0 = \varepsilon_0(c_0) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and for any $z \in D$, we have that $\eta^{\varepsilon}(w,z)$ defined as above is generating at z relative to D with an open set U_z (see Definition (2.2.1)) that does not depend on ε. Furthermore, we have that for each (fixed) $z \in D$ the coefficients of $\eta^{\varepsilon}(.,z)$ are in $C^1(U_z)$.

Proof. We first observe that $\triangle^{\varepsilon}$ can be expressed in terms of the Levi polynomial $\triangle$, as follows

$$\triangle^{\varepsilon}(w,z) := \triangle(w,z) + \frac{1}{2}\sum_{j,k=1}^{n}\left(\frac{\partial^2\rho(w)}{\partial\zeta_j\partial\zeta_k} - \tau_{j,k}^{\varepsilon}(w)\right)(w-z)(w_k - z_k)$$

and so by Lemma (2.2.16) we have

$$2 \operatorname{Re} \triangle^{\varepsilon}(w,z) \geq \rho(w) - \rho(z) + c_0|w-z|^2$$

for any

$$0 < \varepsilon < \varepsilon_0 := \min\left\{\overline{\varepsilon}_0, 2_{C_0}/n^2\right\}$$

whenever $w \in D_c 0 = \{\rho(w) < c_0\}$ and $z \in \overline{B_{\varepsilon} 0(w)}$. Proceeding as in the proof of Lemma (2.2.17) we then find that

$$2 \operatorname{Re} g^{\varepsilon}(w,z) \geq \begin{cases} \rho(w) - \rho(z) + c_0|w-z|^2, & \text{if } |w-z| \leq \varepsilon_0/2 \\ \rho(w) + \mu_0, & \text{if } \varepsilon_0/2 \leq |w-z| < \overline{\varepsilon}_0 \\ \overline{\varepsilon}_0^2, & \text{if } |w-z| \geq \overline{\varepsilon}_0 \end{cases}$$

for any $0 < \varepsilon < \varepsilon_0$ whenever

$$w \in D_{c_0} = \{w|\rho(w) < c_0\}$$

and

$$z \in D_{\mu_0} = \{w|\rho(w) < \mu_0\}$$

as soon as we choose

$$0 < \mu_0 < c0\varepsilon_0^2/8. \qquad (55)$$

We then define the open set $U_z \subset \mathbb{C}^n \backslash \{z\}$ as in (52) but now with δ_0 in place of $\overline{\delta}_0$ (note that U_z does not depend on ε). Then, proceeding as in the proof of Corollary (2.2.18) we find that

$$\inf_{w\in U_z} \operatorname{Re} g^{\varepsilon}(w,z) > 0 \; for\ any\ 0 < \varepsilon < \varepsilon_0.$$

From this it follows that η^{ε} is a generating form for D; it is clear from (53) that the coefficients of η^{ε} are in $C^1(U_z)$.

Lemma (2.2.19) shows that η^{ε} satisfies the hypotheses of Proposition (2.2.10); as a consequence we obtain the following results:

Proposition (2.2.20)[42]. Let D be a bounded strongly pseudo-convex domain. Then, for any $0 < \varepsilon < \varepsilon_0$ we have

$$f(z) = \int_{w\in bD} f(w) j^*\Omega_0(\eta^{\varepsilon})(w,z) \; for\ any\ f \in \vartheta(D) \cap C(\overline{D}),\, z \in D$$

where ε_0 and η^{ε} are as in Lemma (2.2.19).

Proposition (2.2.21)[42]. Let $D = \{\rho(w) < 0\}$ be a bounded strongly pseudo-convex domain. Let

$$\tilde{\eta}^{\varepsilon}(w,z) := \frac{g^{\varepsilon}(w,z)}{g^{\varepsilon}(w,z) - \rho(w)} \eta^{\varepsilon}(w,z)\,, w \in \overline{D}, z \in D.$$

where η^{ε} is as in Lemma (2.2.19). Then, for any $0 < \varepsilon < \varepsilon_0$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{w\in D} f(w) \left(\overline{\partial}_w \overline{\eta}^{\varepsilon}\right)^n (w,z) \; for\ any\ f \in \vartheta(D) \cap L^1(D)\,, z \in D.$$

Proof. We claim that $\overline{\eta}^{\varepsilon}$ satisfies the hypotheses of Proposition (2.2.11) for any $0 < \varepsilon < \varepsilon_0$. Indeed, proceeding as in the proof of Lemma (2.2.19) we find that

$$\operatorname{Re}\left(g^{\varepsilon}(w,z) - \rho(w)\right) > 0 \; for\ any\ w \in \overline{D}, for\ any\ z \in D$$

and for any $0 < \varepsilon < \varepsilon_0$; from this it follows that

$$\tilde{\eta}^{\varepsilon}(.,z) \in C^1_{1,0}(\overline{D}) \; for\ any\ 0 < \varepsilon < \varepsilon_0.$$

Moreover, as a consequence of Basic Property (2.2.4) we have

$$\Omega_0(\tilde{\eta}^{\varepsilon})(\cdot,z) = \left(\frac{g^{\varepsilon}(\cdot,z)}{g^{\varepsilon}(\cdot,z) - \rho(\cdot)}\right)^n \Omega_0(\eta^{\varepsilon})(\cdot,z) \; for\ any\ 0 < \varepsilon < \varepsilon_0,$$

but this grants

$$j^*\Omega_0(\tilde{\eta}^{\varepsilon})(.,z) = j^*\Omega_0(\eta^{\varepsilon})\,(.,z) \; for\ any\ 0 < \varepsilon < \varepsilon_0.$$

The conclusion now follows from Proposition (2.2.11).

Fundamental limitation: it is that these propositions employ kernels, namely $j^*\Omega_0(\eta^{\varepsilon})(w,z)$ and $\left(\overline{\partial}_w \tilde{\eta}^{\varepsilon}\right)^n (w,z)$, that are only *locally* holomorphic as functions of z, that is, they are holomorphic only for $z \in B_{\varepsilon_0/2}(w)$. We address this issue by constructing for each of these kernels a "correction" term obtained by solving an ad-hoc $\overline{\partial}$-problem in the z-variable.

We shift our focus from the w-variable to z, that is: we fix $w \in \overline{D}$, we regard z as a variable and we define the "parabolic" region

$$\mathcal{P}_w := \{z | \rho(z) + \rho(w) < c_0 |w - z|^2\}.$$

The region $\mathcal{P}_w$ has the following properties:

$$w \in \overline{D} \Rightarrow D \subset \mathcal{P}_w;$$

$$w \in \mathrm{b}D \Rightarrow z := w \in \mathrm{b}\mathcal{P}_w.$$

As a consequence of these properties we have that

$$\mathcal{P}_w \cap \mathrm{B}_{\varepsilon_0/2}(w) \neq \emptyset$$

Lemma (2.2.22)[42]. Let $\mathrm{D} = \{\mathrm{z} | \rho(\mathrm{z}) < 0\}$ be a bounded strongly pseudo-convex domain. Then, there is $\mu_0 = \mu_0(c0) > 0$ such that

$$D_{\mu_0} = \{z | \rho(z) < \mu_0\} \subset \mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w) \tag{55}$$

for any (fixed) $\mathrm{w} \in \overline{\mathrm{D}}$. Furthermore, there is a bounded strongly pseudo-convex Ω of class C^∞ such that

$$D_{\mu_0/2} = \{z | \rho(z) < \mu_0/2\} \subset \Omega \subset D_{\mu_0} = \{z | \rho(z) < \mu_0\}$$

where $\mu_0 > 0$ *is as in* (55).

Proof. For the first conclusion, we claim that it suffices to choose $\mu_0 = \mu_0(c_0)$ as in (55). Indeed, given $\in D_{\mu_0}$, if $|w - z| \geq \varepsilon_0/2$ then $\rho(z) \leq c_0 |w - z|^2/2$ and since $\rho(w) \leq 0$ (as $w \in \overline{D}$) it follows that $z \in \mathcal{P}_w$. On the other hand, if $|w - z| < \varepsilon_0/2$ then of course $\in \mathrm{B}_{\varepsilon_0/2}(w)$.

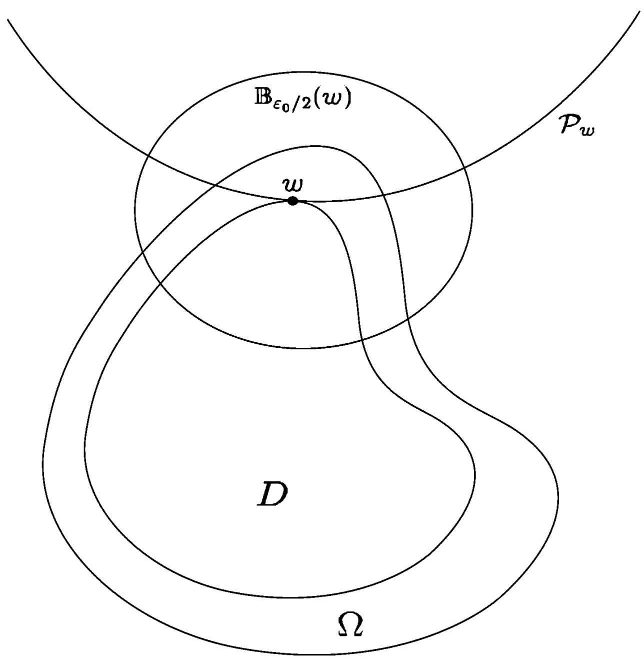

Fig. 1 [42] The region $\wp_w$ in the case when $w \in \mathrm{b}D$

To prove the second conclusion note that, since ρ (the defining function of D) is of class C^2 and is strictly plurisubharmonic in a neighborhood of $\overline{D}$, there is $\overline{\rho} \in C^\infty\left(U\left(\overline{D}\right)\right)$ such that

$$\|\overline{\rho} - \rho\|_{C^2\left(U\left(\overline{D}\right)\right)} \le \mu_0/8$$

and

$$L_z(\overline{\rho}, \xi) > 0 \ for\ any\ z \in U'\left(\overline{D}\right)\ and\ for\ any\ \xi \in \mathbb{C}^n,$$

see (43) and (45). Define

$$\Omega := \{z\ \overline{\rho}(z) - \frac{3\mu_0}{4} < 0\}$$

Then Ω is smooth and strongly pseudo-convex; we leave it that Ω satisfies the desired inclusions: $D_{\mu_0/2} \subset \Omega \subset D_{\mu_0}$.

Lemma (2.2.22) shows that (the smooth and strongly pseudo-convex domain) Ω has the following properties, see Fig. 1:

$$\overline{D} \subset \Omega\ , and\ \overline{\Omega} \subset \mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)\ , for\ every\ w \in \overline{D}.$$

We now set up two $\overline{\partial}$-problems on Ω. For the first $\overline{\partial}$-problem, we begin by observing that if w is in $\mathrm{b}D$ and z is in $\mathcal{P}_w$ then Re $g^\varepsilon\ (w,z) > 0$ (that this must be so can be seen from the inequalities for Re $g^\varepsilon\ (w,z)$ that were obtained in the proof of Lemma (2.2.19)), and so the coefficients of $\eta^\varepsilon(w,)$ are in $C^\infty(\mathcal{P}_w)$ whenever $w \in \mathrm{b}D$. Now fix $w \in \mathrm{b}D$ arbitrarily and denote by $H(w,z) = H_\varepsilon(w,z)$ the following double form, which is of type $(0,1)$ in z, and of type $(n, n-1)$ in w

$$H(w,z) = \begin{cases} -\overline{\partial}_z \Omega_0(\eta^\varepsilon)(w,z)\,, & \text{if } z \in \mathcal{P}_w \\ 0, & \text{if } z \in \mathrm{B}_{\varepsilon_0/2}(w) \end{cases} \tag{56}$$

Now for each fixed $w \in \mathrm{b}\,D$, the coefficients of $\Omega_0(w,z)$ are holomorphic in z for $z \in \mathrm{B}_{\varepsilon_0/2}(w)$ and so $H(w,z)$ is defined consistently in $\mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)$. It is also clear that H (w,z) is C^∞ for $\in \mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)$, and as such it depends continuously on $w \in \mathrm{b}D$. Moreover we have that $\overline{\partial}_z H(w,z) = 0$, for $z \in \mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)$, $w \in \mathrm{b}D$. For the second $\overline{\partial}$-problem, we begin by observing that if w is in $\overline{D}$ and z is in $\mathcal{P}_w$ then Re $\left(g^\varepsilon(w,z) - \rho(w)\right) > 0$ (that this must be so can again be seen from the inequalities for Re $g^\varepsilon\ (w,z)$ in the proof of Lemma (2.2.19)), and so the coefficients of $\overline{\eta}^\varepsilon(w,\cdot)$ are in

$C^\infty(\mathcal{P}_w)$ whenever $w \in \overline{D}$. Fixing $w \in \overline{D}$ arbitrarily, we denote by $F\ (w,z) = F_\varepsilon(w,z)$ the following double form, which is of type $(0,1)$ in z, and of type (n,n) in w

$$F(w,z) = \begin{cases} -\overline{\partial}_z\left(\overline{\partial}_w\tilde{\eta}^\varepsilon\right)^n(w,z), & \text{if } z \in \mathcal{P}_w \\ 0, & \text{if } z \in \mathrm{B}_{\varepsilon_0/2}(w) \end{cases}$$

Now for each fixed $w \in \overline{D}$, the coefficients of $\overline{\eta}^\varepsilon\ (w,z)$ are holomorphic in z for $z \in \mathrm{B}_{\varepsilon_0/2}(w)$ and so $F(w,z)$ is defined consistently in $\mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)$. It is also clear that $F(w,z)$ is C^∞ for $\in \mathcal{P}_w \cup \mathrm{B}_{\varepsilon_0/2}(w)$, and as such it depends continuously on $w \in \overline{D}$. Moreover we have that $\overline{\partial}_z F\ (w,z) = 0$, for $z \in \mathcal{P}_w \cup \mathrm{B}_{\varepsilon/2}0(w)$, $w \in \overline{D}$.

Now let $S = S_z$ be the solution operator, giving the normal solution of the problem $\overline{\partial} u = \alpha$ in Ω, via the $\overline{\partial}$-Neumann problem, so that $u = \mathcal{S}(\alpha)$ satisfies the above whenever α is a(0,1)-form with $\overline{\partial}\alpha = 0$. We set

$$C_\varepsilon^2(w,z) = S_z\big(H\ (w,)\big)\,, w \in \mathrm{b}D \tag{57}$$

and

$$B_\varepsilon^2(w,z) = S_z\big(F(w,\cdot)\big)\,, w \in \overline{D}.$$

Then by the regularity properties of S, for which see e.g., [51], or [56], we have that C_ε^2 (w,z) is in $C^\infty\big(\overline{\Omega}\big)$, as a function of z, and is continuous for $w \in \mathrm{b}D$. Moreover $\overline{\partial}_z$ $\big(C_\varepsilon^2\ (w,z)\big) = -\overline{\partial}_z\Omega_0(\eta^\varepsilon)(w,z) = 0$, for $z \in D$ (recall that $D \subset \mathcal{P}_w$) so $\overline{\partial}_z(\Omega_0(\eta^\varepsilon) + C_\varepsilon^2))(w,z) = 0$ for $z \in D$ and $w \in \mathrm{b}D$. We similarly have that $B_\varepsilon^2(w,z)$ is in $C^\infty\big(\overline{\Omega}\big)$, as a function of z, and is continuous for $w \in \overline{D}$ and, furthermore

$$\overline{\partial}_z\left(\left(\overline{\partial}_w\tilde{\eta}^\varepsilon\right)^n + B_\varepsilon^2\right))(w,z) = 0\ for\ z \in D\ and\ w \in \overline{D}.$$

We complete the construction of a number of integral operators that satisfy all three of the fundamental conditions $(\mathrm{a}) - (\mathrm{c})$ that were presented. The crucial step in all these constructions is to produce integral kernels that are globally holomorphic in D as functions of z. For strongly pseudo-convex domains, this goal is achieved by adding to each of the (locally holomorphic) Cauchy-Fantappié forms that were produced the ad-hoc "correction" term that was constructed in the resulting two families of operators are denoted $\{\mathrm{C}_\varepsilon\}_\varepsilon$ (acting on $C(\mathrm{b}D)$) and $\{\mathrm{B}_\varepsilon\}_\varepsilon$ (acting on $L^1(D)$). In the case of strongly $\mathbb{C}$-linearly convex domains of class C^2, there is no need for "correction": a natural, globally holomorphic CauchyFantappié form is readily available that gives rise to an operator acting on $C(\mathrm{b}D)$ (even on $L^1(\mathrm{b}D)$) , called the Cauchy-Leray Integral C_L and, in the more restrictive setting of strongly convex domains, also to an operator B_L that acts on $L^1(D)$. In the special case when the domain is the unit ball, the Cauchy-Leray integral C_L agrees with the Cauchy-Szegö projection S, while the operator B_L agrees with the Bergman projection B.) All the operators that are produced satisfy, by their very construction, conditions (a) and (c) and we show that they also satisfy condition (b) (the reproducing property for holomorphic functions).

For η_ε is as in Proposition (2.2.20) we now write

$$C_\varepsilon^1(w,z) = \Omega_0(\eta^\varepsilon)(w,z)$$

and let

$$C_\varepsilon(w,z) = j^*\left(C_\varepsilon^1(w,z) + C_\varepsilon^2(w,z)\right)$$

and we define the operator

$$\mathrm{C}_\varepsilon f(z) = \int_{w\in \mathrm{b}D} f(w) C_\varepsilon(w,z)\,, z\in D, f\in C(\mathrm{b}D)\,. \tag{58}$$

Proposition (2.2.23)[42]. Let D be a bounded strongly pseudo-convex domain. Then, for any $0 < \varepsilon < \varepsilon_0$ we have

$$f(z) = \mathrm{C}_\varepsilon f(z)\,, for\ any\ f \in \vartheta(D)\cap C(\overline{D})\,, z\in D.$$

Proof. By Proposition (2.2.20), for any $f \in \vartheta(D)\cap C(\overline{D})$ we have

$$\int_{w\in \mathrm{b}D} f(w) C_\varepsilon(w,z) = f(z) + \int_{w\in \mathrm{b}D} f(w) j^* C_\varepsilon^2(w,z)\ for\ any\ z\in D,$$

and so it suffices to show that

$$\int_{w\in \mathrm{b}D} f(w) j^* C_\varepsilon^2(w,z) = 0\ for\ any\ z\in D.$$

By Fubini's theorem and the definition of C_ε^2, see (57), we have

$$\int_{w\in \mathrm{b}D} f(w) j^* C_\varepsilon^2(w,z) = S_z\left(\int_{w\in \mathrm{b}D} f(w) j^* H(w,\cdot)\right)$$

where $H(w,\cdot)$ is as in (56). Since the solution operator S_z is realized as a combinations of integrals over Ω and $\mathrm{b}\Omega$, the desired conclusion will be a consequence of the following claim:

$$\int_{w\in bD} f(w) j^* H(w,\zeta) = 0 \; for \; any \; \zeta \in \overline{\Omega},$$

and since $\overline{\Omega} \subset \mathcal{P}_w$ for any $w \in bD$, proving the latter amounts to showing that

$$\int_{w\in M_\zeta} f(w) j^* \overline{\partial}_\zeta \Omega_0(\eta^\varepsilon)(w,\zeta) = 0 \; for \; any \; \zeta \in \overline{\Omega}, \tag{59}$$

where we have set

$$M_\zeta = \{w \in bD \,||w-\zeta| \geq \varepsilon_0/2\}, \tag{60}$$

see (56) and Fig. 2 below. To this end, we fix $\zeta \in \overline{\Omega}$ arbitrarily; we claim that there is a sequence of forms $\left(\eta_\ell^\varepsilon(.,\zeta)\right)_\ell$ with the following properties:

a. $\eta_\ell^\varepsilon(.,\zeta)$ is generating at ζ relative to D;

b. $\eta_\ell^\varepsilon(.,\zeta)$ has coefficients in $C^2(U_\zeta)$ with U_ζ as in Definition (2.2.1); c. as $P \to \infty$, we have that

$$j^*\Omega_0(\eta_\ell^\varepsilon)(\cdot,\zeta) \to j^*\Omega_0(\eta^\varepsilon)(\cdot,\zeta) \; uniformly \; on \; bD;$$

d. the coefficients of $\eta_\ell^\varepsilon(w,\zeta)$ are holomorphic in $\zeta \in B_{\varepsilon_0/2}(w)$ for any $w \in bD$.

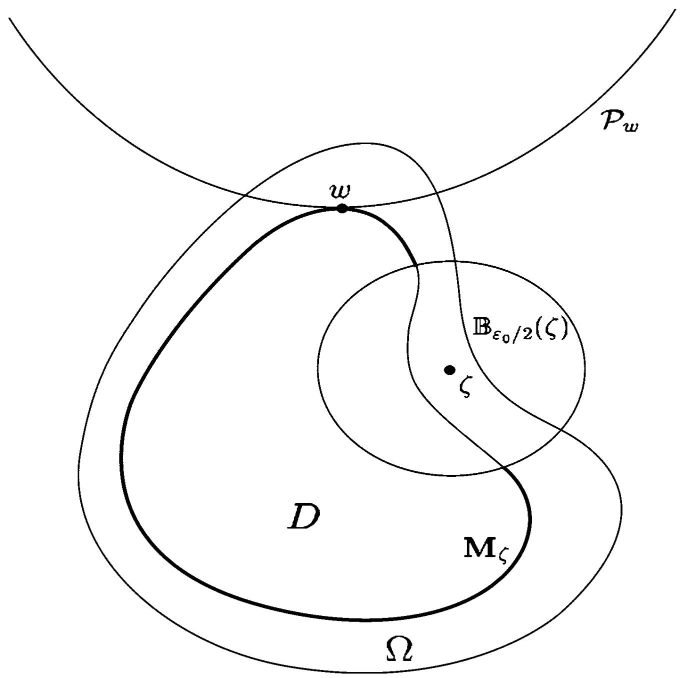

Fig. 2 **[42]**The manifold M_ζ in the proof of Proposition (2.2.23)

Note that (59) will follow from item c. above if we can prove that

$$\int_{w\in M_\zeta} f(w) j^* \overline{\partial}_\zeta \Omega_0(\eta_\ell^\varepsilon)(w,\zeta) = 0 \; for \; any \; \ell. \tag{61}$$

We postpone the construction of $\eta_\ell^\varepsilon(.,\zeta)$ to later below, and instead proceed to proving (61) assuming the existence of the $\{\eta_p^\varepsilon(.,\zeta)\}_\ell$. On account of items $a.$ and $b.$ above along with Basic Property (2.2.6) as stated in (25), proving (61) is equivalent to showing that

$$\int_{w\in M_\zeta} f(w) j^*\overline{\partial}_w \Omega_1(\eta_\ell^\varepsilon)(w,\zeta) = 0 \; for\; any\; \ell.$$

To this end, we first consider the case when $\in \vartheta(D)\cap C^1(\overline{D})$, as in this case we have that

$$f(w) j^*\overline{\partial}_w \Omega_1(\eta_\ell^\varepsilon)(w,\zeta) = j^*\overline{\partial}_w\big(f\Omega_1(\eta_\ell^\varepsilon)\big)(w,\zeta) = j^* d_w\big(f\Omega_1(\eta_\ell^\varepsilon)\big)(w,\zeta)$$

(where in the last identity we have used the fact that $\overline{\partial}_w\Omega_1 = d_w\Omega_1$ because $\Omega_1(\eta_\ell^\varepsilon)$ is of type $(n, n-2)$ in w). But the latter equals

$$\mathrm{d}_w j^*\big(f\Omega_1(\eta_\ell^\varepsilon)\big)(w,\zeta)$$

where d_w denotes the exterior derivative operator for M_ζ viewed as a real manifold of dimension $2n-1$. Applying Stokes' theorem on M_ζ to the form $\alpha(w) := j^*\big(f\Omega_1(\eta_\ell^\varepsilon)\big)(w,\zeta) \in C^1_{n,n-2}(M_\zeta)$ we obtain

$$\int_{w\in M_\zeta} f(w) j^*\overline{\partial}_w \Omega_1(\eta_\ell^\varepsilon)(w,\zeta) = \int_{w\in \mathrm{b}M_\zeta} f(w) j^*\Omega_1(\eta_\ell^\varepsilon)(w,\zeta)$$

but

$$j^*\Omega_1(\eta_\ell^\varepsilon)(w,\zeta) = 0 \; for\; any\; w \in \mathrm{b}M_\zeta = \mathrm{b}D \cap \{|w-\zeta| = \varepsilon_0/2\}$$

because the coefficients of $\eta_\ell^\varepsilon(w,\zeta)$ are holomorphic in $\zeta \in \mathrm{B}_{\varepsilon/2}(w)$ for any $\mathrm{b}D$, see (23) and item $d.$ above. This concludes the proof of Proposition (2.2.23) in the case when $\in \vartheta(D)\cap C^1(\overline{D})$. To prove the proposition in the case when $\in \vartheta(D)\cap C^0(\overline{D})$, we fix $z\in D$ and choose $\delta = \delta(z) > 0$ such that

$$z \in D_{-\delta} = \{\rho < -\delta\} \; for\; any\; \delta \le \delta(z).$$

Then we have that

$$f \in \vartheta(D_{-\delta}) \cap C^1(\overline{D}_{-\delta}) \; for\; any\; \delta \le \delta(z)$$

and so by the previous argument we have

$$\int_{w\in \mathrm{b}D-\delta} f(w) j^*_{-\delta} C_\varepsilon^2(w,z) = 0 \; for\; any\; \delta \le \delta(z), \tag{62}$$

where $j^*_{-\delta}$ denotes the pullback under the inclusion: $\mathrm{b}D_{-\delta} \hookrightarrow \mathbb{C}^n$. For δ sufficiently small there is a natural one-to-one and onto projection along the inner normal direction:

$$\Lambda_\delta \colon \mathrm{b}D \to \mathrm{b}D_{-\delta},$$

and because D is of class C^2 one can show that this projection tends in the C^1-norm to the identity $1_{\mathrm{b}D}$, that is we have that

$$\|1_{\mathrm{b}D} - \Lambda_\delta\|_{C^1(\mathrm{b}D)} \to 0 \ as \ \delta \to 0.$$

Using this projection one may then express the integral on $\mathrm{b}D_{-\delta}$ in identity (62) as an integral on $\mathrm{b}D$ for an integrand that now also depends on Λ_δ and its Jacobian, and it follows from the above considerations that

$$\int_{w\in \mathrm{b}D_{-\delta}} f(w) j^*_{-\delta} C^2_\varepsilon(w,z) \to \int_{w\in \mathrm{b}D} f(w) j^* C^2_\varepsilon(w,z) \ as \ \delta \to 0.$$

We are left to construct, for each fixed $\zeta \in \overline{\Omega}$, the sequence $\{\eta^\varepsilon_\ell(.,\zeta)\}_\ell$ that was invoked earlier on. To this end, set

$$U := D \cup \bigcup_{z\in D} U_z$$

where U_z is the open neighborhood of $\mathrm{b}D$ that was determined in Lemma (2.2.19). Consider a sequence of real-valued functions $\{\rho_\ell\}p \subset C^3(\mathbb{C}^n)$ such that

$$\|\rho\ell - \rho\|_{C^1(U)} \to 0 \ as \ \ell \to \infty,$$

and, for $\zeta \in \overline{\Omega}$ fixed arbitrarily, set

$$\triangle^\varepsilon_{j,\ell}(w,\zeta) := \frac{\partial \rho_\ell}{\partial \zeta_j}(w) - \frac{1}{2}\sum_{k=1}^{n} \tau^\varepsilon_{j,k}(w)(w_k - \zeta_k)\,, j = 1,\dots,n;$$

$$\triangle^\varepsilon_\ell(w,\zeta) := \sum_{j=1}^{n} \triangle^\varepsilon_{jp},(w,\zeta)\big(w_j - \zeta_j\big)\,;$$

and, for χ_1 as in (49):

$$g^\varepsilon_\ell(w,\zeta) := \chi_1(w,\zeta)\,\triangle^\varepsilon_\ell(w,\zeta) + \big(1 - \chi_1(w,\zeta)\big)|w - \zeta|^2;$$

$$\eta^\varepsilon_{jp},(w,\zeta) := \frac{1}{g^\varepsilon_\ell(w,\zeta)}\Big(\chi_1(w,\zeta)\,\triangle^\varepsilon_{j,\ell}(w,\zeta) + \big(1 - \chi_1(w,\zeta)\big)\big(\overline{w}_j - \overline{\zeta}_j\big)\Big)$$

and, finally

$$\eta^\varepsilon_\ell(w,\zeta) := \sum_{j=1}^{n} \eta^\varepsilon_{j,\ell}(w,\zeta) dw_j,$$

We leave it that $\{\eta^\varepsilon_\ell(\cdot,\zeta)\}_\ell$ has the desired properties.
Next, for $\tilde{\eta}^\varepsilon$ is as in Proposition (2.2.21), we write

$$B_\varepsilon^1(w,z) = \frac{1}{(2\pi i)^n}\left(\overline{\partial}_w \tilde{\eta}^\varepsilon\right)^n$$

and

$$B_\varepsilon(w,z) := (B_\varepsilon^1 + B_\varepsilon^2)(w,z)\,, w \in \overline{D}, z \in \overline{\Omega},$$

and we define the operator

$$\mathrm{B}_\varepsilon f(z) = \int_{w\in D} f\,(w) B_\varepsilon(w,z)\,, z \in D, f \in L^1(D)\,. \tag{63}$$

Proposition (2.2.24)[42]. Let D be a bounded strongly pseudo-convex domain. Then, for any $0 < \varepsilon < \varepsilon_0$ we have

$$f(z) = \mathrm{B}_\varepsilon f(z)\,, for\ any\ f \in \vartheta(D) \cap L^1(D)\,, z \in D.$$

Proof. By Proposition (2.2.21), for any $f \in \vartheta(D) \cap L^1(D)$ we have

$$\int_{w\in D} f\,(w) B_\varepsilon(w,z) = f(z) + \int_{w\in D} f\,(w) B_\varepsilon^2(w,z)\ for\ any\ z \in D,$$

and so it suffices to show that

$$\int_{w\in D} f\,(w) B_\varepsilon^2(w,z) = 0\ for\ any\ z \in D.$$

For the proof of this assertion we refer to [67].

Let D be a bounded, strictly $\mathbb{C}$-linearly convex domain. We claim that if ρ is (any) defining function for such a domain, and if U is an open neighborhood of bD such that $\nabla\rho(w) \neq 0$ for any $w \in U$, then

$$\eta(w,z) := \frac{\partial\rho(w)}{\langle\partial\rho(w), w - z\rangle} \tag{64}$$

is a generating form for D; indeed, by Lemma (2.2.14) for any $z \in D$ there is an open set $U_z \subset \mathbb{C}^n\backslash\{z\}$ such that $\langle\partial\rho(w)\ ,\ w - z\rangle \neq 0$ for any $w \in U_z$ and b$D \subset U_z$; thus the coefficients of η (., z) are in $C(U_z)$ and (14) holds. It is clear from (64) that $\langle\eta\ (w,z)\ , w - z\rangle = 1$ for any $w \in U_z$, so (15) holds for any $z \in D$, as well. It follows that Proposition (2.2.10) applies to any strictly $\mathbb{C}$-linearly convex domain D with η chosen as above under the further assumption that D be of class C^2 (which is required to ensure that the coefficients of η (., z) are in $C^1(U_z)$). The form

$$C_L(w,z) = j^*\Omega_0\left(\frac{\partial\rho(w)}{\langle\partial\rho(w), w - z\rangle}\right) = j^*\left(\frac{\partial\rho(w) \wedge \left(\overline{\partial}\partial\rho\right)^{n-1}(w)}{(2\pi i\langle\partial\rho(w), w - z\rangle)^n}\right) \tag{65}$$

is called the *Cauchy-Leray kernel for* D. It is clear that the coefficients of the CauchyLeray kemel are *globally* holomorphic with respect to $z \in D$: indeed the denominator $j^*\langle\partial\rho(w)\ ,$ $w - z\rangle^n$ is polynomial in the variable z, and by the strict $\mathbb{C}$-linear convexity of D we have that $j^*\langle\partial\rho(w)\ ,\ w - z\rangle^n \neq 0$ for any $z \in D$ and for any $w \in$ bD, see (47). The resulting integral operator:

$$\mathrm{C}_L f(z) = \int_{w\in \mathrm{b}D} f(w) C_L(w,z) z \in D, \tag{66}$$

is called the *Cauchy-Leray Integral*. Under the further assumption that *D* be strictly convex (as opposed to strictly ℂ-*linearly* convex), for each fixed $z \in D$ one may extend η (., *z*) holomorphically to the interior of *D* as follows

$$\tilde{\eta}(\cdot,z) := \left(\frac{\langle \partial\rho(\cdot), \cdot - z\rangle}{\langle \partial\rho(\cdot), \cdot - z\rangle - \rho(\cdot)}\right)\eta(\cdot,z) = \frac{\partial\rho(\cdot)}{\langle \partial\rho(\cdot), \cdot - z\rangle - \rho(\cdot)} \tag{67}$$

The following lemma shows that if *D* is sufficiently smooth (again of class C^2) then $\overline{\eta}$ satisfies the hypotheses of Proposition (2.2.11), and so in particular the operator

$$\mathrm{B}_L f(z) = \int_{w\in D} f(w) B_L(w,z)$$

with

$$B_L(w,z) = \frac{1}{(2\pi i)^n}\left(\overline{\partial}_w \tilde{\eta}\right)^n (w,z) \tag{68}$$

and $\overline{\eta}$ given by (67), reproduces holomorphic functions.

Lemma (2.2.25)[42]. If $\mathrm{D} = \{\rho < 0\} \subset \mathbb{C}^n$ is strictly convex and of class C^2, then for each fixed z ∈ D we have that $\overline{\eta}$ (., z) given by (67) has coefficients in $\mathrm{C}^1(\overline{\mathrm{D}})$ and satisfies the hypotheses of Proposition (2.2.11).

Proof. In order to prove the first assertion it suffices to show that

$$\mathrm{Re}\,(\langle\partial\rho(w), w - z\rangle) - \rho(w) > 0 \, for\ any\ w \in \overline{D}, z \in D. \tag{69}$$

Indeed, one first observes that if *D* is strictly convex and sufficiently smooth then

$$\mathrm{Re}\,\langle\partial\rho(w)\,, w - z\rangle > 0\ for\ any\ w \in \overline{D}\setminus\{z\}$$

(see [62] for the proof of this fact) so that Re $\langle\partial\rho(w)\,,\, w - z\rangle$ is non-negative in $\overline{D}$ and it vanishes only at $w = z$. On the other other hand the term $-\rho(w)$ is non-negative for any $w \in \overline{D}$, and if $w = z \in D$ then $-\rho(w) = -\rho(z) > 0$. This proves (69) and it follows that the coefficients of $\overline{\eta}$ (., *z*) are in $C^1(\overline{D})$. By Basic Property (2.2.4) we have

$$\Omega_0(\tilde{\eta})(\cdot,z) = \left(\frac{\langle \partial\rho(\cdot), \cdot - z\rangle}{\langle \partial\rho(\cdot), \cdot - z) - \rho(\cdot)}\right)^n \Omega_0(\eta)(\cdot,z)\,;$$

it is now immediate to verify that $j^*\Omega_0(\tilde{\eta})\ (.,z) = j^*\Omega_0(\eta)$ (., *z*), so that $\overline{\eta}$ satisfies (39), as desired. We summarize these results in the following two propositions:

Proposition (2.2.26)[42]. Suppose that D is a bounded, strictly ℂ-linearly convex domain of class C^2. Then, with same notations as above we have

$$f(z) = \mathrm{C}_L f(z)\,, z \in D, f \in \vartheta(D) \cap C(\overline{D})\,.$$

Proposition (2.2.27)[42]. Suppose that D is a bounded, strictly convex domain of class C^2. Then, with same notations as above we have that

$$f(z) = \mathrm{B}_L f(z)\,, z \in D, f \in \vartheta(D) \cap L^1(D)\,.$$

we discuss L^p-regularity of the Cauchy-Leray integral and of the Cauchy-Szegö and Bergman projections for the domains under consideration. Detailed proofs of the results concerning the Bergman projection, Theorem (2.2.32) and Corollary (2.2.34) below, can be found in [67]. The statements concerning the Cauchy-Leray integral and the Cauchy-Szegö projection Theorem (2.2.37) is the subject of a senes of forthcoming here we will limit ourselves to presenting an outline of the main points of interest in their proofs.
We begin by recalling the defining properties of the Bergman and Cauchy-Szegö projections and of their corresponding function spaces.

Let $D \subset \mathbb{C}^n$ be a bounded connected open set.

Definition (2.2.28)[42]. For any $1 \le q < \infty$ the *Bergman space* $\vartheta L^q(D)$ is

$$\vartheta L^q(D) = \vartheta(D) \cap L^q(D, dV)\,.$$

The following inequality

$$\sup_{z \in \mathcal{K}} |F(z)| \le C(\mathcal{K}) \|F\|_{L^p(D,dV)}$$

which is valid for any compact subset $\mathcal{K} \subset D$ and for any holomorphic function $F \in \vartheta(D)$, shows that the Bergman space is a closed subspace of $L^q(D, dV)$. This inequality also shows that the point evaluation:

$$ev_z(f) := f(z)\,, z \in D$$

is a bounded linear functional on the Bergman space (take $\mathcal{K} := \{z\}$). In the special case $q = 2$, classical arguments from the theory of Hilbert spaces grant the existence of an orthogonal projection, called the *Bergman projection for D*

$$\mathrm{B}: L^2(D) \to \vartheta L^2(D)$$

that enjoys the following properties

$$\mathrm{B} f(z) = f(z)\,, f \in \vartheta L^2(D), z \in D$$

$$\mathrm{B}^* = \mathrm{B}$$

$$\|\mathrm{B} f\|_{L^2(D,dV)} \le \|f\|_{L^2(D,dv)}, f \in L^2(D, dV)$$

$$\mathrm{B} f(z) = \int_{w \in D} f(w)\mathcal{B}(w,z) dV(w)\,, z \in D, f \in L^2(D, dV)$$

where dV denotes Lebesgue measure for $\mathbb{C}^n$. The function $\mathcal{B}\,(w,z)$ is holomorphic with respect to $z \in D$; it is called the *Bergman kernel function*. The Bergman kernel function depends on the domain and is known explicitly only for very special domains, such as the unit ball, see e.g. [77]:

$$\mathcal{B}(w,z) = \frac{n!}{\pi^n (1 - [z,w])^{n+1}}\,, (w,z) \in \mathbb{B}_1(0) \times \mathbb{B}_1(0) \qquad (70)$$

here $[z, w] := \sum_{j=1}^{n} z_j \cdot \overline{w}_j$ is the hermitian product for $\mathbb{C}^n$.

Let $D \subset \mathbb{C}^n$ be a bounded connected open set with sufficiently smooth boundary. For such a domain, various notions of Hardy spaces of holomorphic functions can be obtained by considering (suitably interpreted) boundary values of functions that are holomorphic in D and whose restriction to the boundary of D has some integrability, see [78]. While a number of such definitions can be given, here we adopt the following

Definition (2.2.29)[42]. For any $1 \le q < \infty$ the *Hardy Space* $H^q(\mathrm{b}D, d\sigma)$ is the closure in $L^q(\mathrm{b}D, d\sigma)$ of the restriction to the boundary of the functions holomorphic in a neighborhood of $\overline{D}$. In the special case when $q = 2$ the orthogonal projection

$$\mathrm{S} : L^2(\mathrm{b}D, d\sigma) \to H^2(\mathrm{b}D, d\sigma)$$

is called the *The Cauchy-Szegö Projection for* D.

The Cauchy-Szegö projection has the following basic properties:

$$\mathrm{S}^* = \mathrm{S}$$

$$\|\mathrm{S}f\|_{L^2(\mathrm{b}D, d\sigma)} \le \|f\|_{L^2(\mathrm{b}D, d\sigma)}, f \in L^2(\mathrm{b}D, d\sigma)$$

$$\mathrm{S}f(z) = \int_{w \in \mathrm{b}D} S\,(w, z) f(w) d\sigma(w)\,, z \in \mathrm{b}D.$$

The function (w, z) , initially defined for $z \in \mathrm{b}D$, extends holomorphically to $z \in D$; it is called the *Cauchy-Szegö kernel function*. Like the Bergman ketnel function, the Cauchy-Szegö kernel function depends on the domain D; for the unit ball we have [77]

$$S\,(w, z) = \frac{(n-1)!}{2\pi^n (1 - [z, w])^n}\,, (w, z) \in \mathrm{b}\mathbb{B}_1(0) \times \mathrm{b}\mathbb{B}_1(0)\,. \quad (71)$$

We may now state the main results.

Theorem (2.2.30)[42]. Suppose D is a bounded domain of class C^2 which is strongly $\mathbb{C}$-linearly convex. Then the Cauchy-Leray integral (2.67), initially defined for $\in \mathrm{C}^1(\mathrm{bD})$, extends to a bounded operator on $\mathrm{L^p(bD, d\sigma)}$, $1 < \mathrm{p} < \infty$.

It is only the weaker notion of *strict* $\mathbb{C}$-linear convexity that is needed to define the Cauchy-Leray integral, but to prove the L^p results one needs to assume *strong* $\mathbb{C}$-linear convexity.

Theorem (2.2.31)[42]. Under the assumption that the bounded domain D has a C^2 boundary and is strongly pseudo-convex, one can assert that S extends to a bounded mapping on $\mathrm{L^p(bD, d\sigma)}$, when $1 < \mathrm{p} < \infty$.

Theorem (2.2.32)[42]. Under the same assumptions on D itfollows that the operator B extends to a bounded operator on $\mathrm{L^p(D, dV)}$ for $1 < \mathrm{p} < \infty$.

The following additional results also hold.

Corollary (2.2.33)[42]. The result of Theorem (2.2.31) extends to the case when the projection S is replaced by the corresponding orthogonal projection S_ω, with respect to the Hilbert space $\mathrm{L}^2(\mathrm{bD}, \omega \mathrm{d}\sigma)$ where ω is any continuous strictly positive function on bD.

A similar variant of Theorem (2.2.32) holds for B_ω, the orthogonal projection on the subspace of $L^2(D, \omega dV)$. Here ω is any strictly positive continuous function on $\overline{D}$.

Corollary (2.2.34)[42]. One also has the L^p boundedness of the operator $|B|$, whose kernel is $|B(z,w)|dV(w)$, where $B(z,w)$ is the Bergman kernel function.
Cauchy-type integrals

We begin by making the following remarks to clarify the background of these results.
(i) The proofs make use of the whole *family* of operators $\{C_\varepsilon\}_\varepsilon, 0 < \varepsilon < \varepsilon_0$: in order to obtain L^p estimates for p in the full range (1, ∞) one needs the flexbility to choose $\varepsilon = \varepsilon(p)$ sufficiently small. (A single choice, as in [76], of C_ε for a fixed ε, will not do.)
(ii) There is no simple and direct relation between S and S_ω, nor between B and B_ω. Thus the results for general ω are not immediate consequences of the results for $\omega \equiv 1$.
(iii) When bD and ω are smooth (i.e. C^k for sufficiently high k), the above results have been known for a long time (see e.g., the remarks that were made concerning the case when D is the unit ball). Moreover when bD and ω are smooth (and bD is strongly pseudo-convex), there are analogous asymptotic formulas for the ketnels in question due to [55],[74].
(4) Another approach to Theorem (2.2.32) in the case of smooth strongly pseudo-convex domains is via the $\overline{\partial}$-Neumann problem [51] and [56], but we shall not say anything more about this here.

A further point of interest is to work with the "Levi-Leray" measure $d\mu_\rho$ for the boundary of D, which we define as follows. We take the linear functional

$$\ell(f) = \frac{1}{(2\pi i)^n} \int_{\mathrm{b}D} f\,(w) j^* \left(\partial\rho \wedge \left(\overline{\partial}\partial\rho\right)^{n-1}\right) \tag{72}$$

and write $\ell f = \int_{\mathrm{b}D} f\, d\mu_\rho$. We then have that $d\mu_\rho(w) = \mathcal{D}(w) d\sigma(w)$ where $\mathcal{D}(w) = c|\nabla\rho(w)| \det L_w(\rho)$ via the calculation in [76] in the case ρ is of class C^2, and we observe that $\mathcal{D}(w) \approx 1$, via (48).
With this we have that the Cauchy-Leray integral becomes

$$C_L(f)(z) = \int_{\mathrm{b}D} \frac{f(w) d\mu_\rho(w)}{\{\partial\rho(w), w - z\rangle^n} \tag{73}$$

Thus the reason for isolating the measure $d\mu_\rho$ is that the coefficients of the kernel of each of C_L and its adjoint (computed with respect to $L^2(\mathrm{b}D, d\mu_\rho)$), are C^1 functions in both variables. This would not be the case if we replaced $d\mu_\rho$ by the induced Lebesgue measure $d\sigma$ (and had taken the adjoint of C_L with respect to $L^2(\mathrm{b}D, d\sigma)$). In studying (73) we apply the T(1)-theorem"technique [53], where the underlying geometry is determined by the quasi-metric

$$|\langle \partial\rho(w), w-z\rangle|^{\frac{1}{2}}$$

(It is at this juncture that the notion of *strong* ℂ-linear convexity, as opposed to *strict* ℂ-linear convexity, is required.) In this metric, the ball centered at w and reaching to z has $d\mu_\rho$ −measure ≈ $|\{\partial\rho(w)\ , w-z\rangle|^n$.

The study of (73) also requires that we verify cancellation properties in terms of its action on "bump functions These matters again differ from the case $n = 1$, and in fact there is an unexpected favorable twist: the kernel in (73) is an appropriate derivative, as can be surmised by the observation that on the Heisenberg group one has $(|z|^2 + it)^{-n} = c'\frac{d}{dt}(|z|^2 + it)^{-n+1}$, if $n > 1$. (However for $= 1$, the corresponding identity involves the logarithm!). Indeed by an integration-by-parts argument that is presented in (77) below, we see that when $n > 1$ and f is of class C^1,

$$\mathrm{C}_L(f)(z) = c\int_{\mathrm{b}D} \frac{df(w)\wedge j^*\left(\bar{\partial}\partial\rho\right)^{n-1}}{\langle \partial\rho(w), w-z\rangle^{n-1}} + \mathrm{E(f)(z)}\,,$$

where

$$\mathrm{E(f)(z)} = \int_{\mathrm{b}D} \mathcal{E}\,(z,w)f(w)d\sigma(w)$$

with

$$\mathcal{E}(z,w) = O(|z-w||\langle \partial\rho(w), w-z\rangle|^{-n})$$

so that the operator E is a negligible term.

A final point is that the hypotheses of Theorem (2.2.30) are in the nature of best possible. In fact, [46] gives examples of Reinhardt domains where the L^2 result for the CauchyLeray integral fails when a condition near C^2 is replaced by $C^{2-\varepsilon}$, or "strong" pseudoconvexity is replaced by its "weak" analogue.

One more observation concerning the Cauchy-Leray integral is in order. In the special case when D is the unitball $\mathrm{B}(0)$, we claim that the operators C_L and B_L agree, respectively, with the Cauchy-Szegö and Bergman projections for $\mathrm{B}_1(0)$. Indeed, for such domain the calculations apply with $U_z = \mathbb{C}^n\backslash\{z\}$ and

$$\rho(w) := |w|^2 - 1 \tag{74}$$

and by the Cauchy-Schwarz inequality we have Re $(\langle \partial\rho(w), w-z\rangle) \geq |w|(|w|-|z|)$ for any w , $z \in \mathbb{C}^n$. Using (74) and (32) we find that

$$C_L(w,z) = \frac{(n-1)!}{2\pi^n}\frac{d\sigma(w)}{(1-[z,w])^n} = S(w,z)d\sigma$$

which is the Cauchy-Szegö kemel for the ball, see (71) Next, we observe that, again for $D = \mathrm{B}_1(0)$ and with ρ as in (74), we have that

$$\langle \partial\rho(w), w-z\rangle - \rho(w) = 1-[z,w]\; for\ any\ w,z \in \mathbb{C}^n$$

and from this it follows that (68) now reads

$$B_L(w,z)=\frac{n!\,dV(w)}{\pi^n(1-[z,w])^{n+1}}=\mathcal{B}(w,z)dV(w)$$

which is the Bergman kernel of the ball, see (70).

There are three main steps in the proof of Theorem (2.2.31).

(i) Construction of a family of bounded Cauchy Fantappié-type integrals C_ε (ii) Estimates for $\mathrm{C}_\varepsilon-\mathrm{C}_\varepsilon^*$

(iii) Application of a variant of identity (13)

Step (i): The construction of C_ε was given see (58). One notes that the kernel $C_\varepsilon^2(w,z)$ of the correction term that was produced is "harmless" since it is bounded as (w,z) ranges over $\mathrm{b}D\times\overline{D}$. Using a methodology similar to the proof of Theorem (2.2.30) one then shows

$$\|\mathrm{C}_\varepsilon(f)\|_{L^p}\le c_{\varepsilon,p}\|f\|_{L^p},1<p<\infty.$$

However it is important to point out, that in general the bound $c_{\varepsilon,p}$ grows to infinity as $\varepsilon\to 0$, so that the C_ε can not be genuine approximations of S. Nevertheless we shall see below that in a sense the C_ε gives us critical information about S.

Step (ii): Here the goal is the following splitting:

Proposition (2.2.35)[42]. Given $0<\varepsilon<\varepsilon_0$, we can write

$$\mathrm{C}_\varepsilon-\mathrm{C}_\varepsilon^*=\mathrm{A}_\varepsilon+\mathrm{R}_\varepsilon$$

where

$$\|\mathrm{A}_\varepsilon\|_{L^p\to L^p}\le\varepsilon c_p\,,1<p<\infty\qquad(75)$$

and the operator R_ε has a bounded kernel, hence R_ε maps $\mathrm{L}^1(\mathrm{bD})$ to $\mathrm{L}^\infty(\mathrm{bD})$.

We note that in fact the bound of the kernel of R_ε may grow to infinity as $\varepsilon\to 0$. To prove Proposition (2.2.35) we first verify an important "symmetry" condition: for each ε, there is a δ_ε , so that

$$|g^\varepsilon(w,z)-\overline{g^\varepsilon}(z,w)|\le\varepsilon c|w-z|^2, if\ |w-z|<\delta_\varepsilon.\qquad(76)$$

Here $g_\varepsilon(w,z)$ is as in (54). With this one proceeds as follows. Suppose $H_\varepsilon(z,w)$ is the kernel of the operator $\mathrm{C}_\varepsilon-\mathrm{C}_\varepsilon^*$. Then we take A_ε and R_ε to be the operators with kernels respectively $\chi_\delta(w-z)H_\varepsilon(w,z)$ and $\big(1-\chi_\delta(w-z)\big)H_\varepsilon(w,z)$, where $\chi_\delta(w-z)$ is as in (54) and $\delta=\delta_\varepsilon$, chosen acccording to (76).

Step (iii): We conclude the proof of Theorem (2.2.31) by using an identity similar to (13):

$$\mathrm{S}\big(\mathrm{I}-(\mathrm{C}_\varepsilon^*-\mathrm{C}_\varepsilon)\big)=\mathrm{C}_\varepsilon$$

Hence

$$\mathrm{S}(\mathrm{I}-\mathrm{A}_\varepsilon)=\mathrm{C}_\varepsilon+\mathrm{SR}_\varepsilon$$

Now for each p, take $\varepsilon>0$ so that for the bound c_p as in (2.76)

$$\varepsilon c_p \leq \frac{1}{2}.$$

Then $\mathrm{I} - \mathrm{A}_\varepsilon$ is invertible and we have

$$\mathrm{S} = (\mathrm{C}_\varepsilon + \mathrm{SR}_\varepsilon)(\mathrm{I} - \mathrm{A}_\varepsilon)^{-1}$$

Since $(\mathrm{I} - \mathrm{A}_\varepsilon)^{-1}$ is bounded on L^p, and also C_ε , it sufficies to see that SR_ε is also bounded on L^p. Assume for the moment that $p \leq 2$. Then since R_ε maps L^1 to L^∞, it also maps L^p to L^2 (this follows from the inclusions of Lebesgue spaces, which hold in this setting because D is bounded), while S maps L^2 to itself, yielding the fact that SR_ε is bounded on L^p. The case $2 \leq p$ is obtained by dualizing this argument.

The proof of Theorem (2.2.32) can be found in [67]: it has an outline similar to the proof of Theorem (2.2.31) with the operators B_ε, see (63), now in place of the C_ε , but the details are simpler since we are dealing with operators that converge absolutely (as suggested by Corollary (2.2.34)). Thus one can avoid the delicate $T(1)$-theorem machinery and make instead absolutely convergent integral estimates.

For domains with boundary regularity below the C^2 category there is no canonical notion of strong pseudo-convexity- much less a working analog of the Cauchy-type operators C_ε and B_ε that were introduced. By contrast, the Cauchy-Leray integral can be defined for less regular domains, but the definitions and the proofs are substantially more delicate than the C^2 framework of Theorem (2.2.30).

Definition (2.2.36)[42]. Given a bounded domain $D \subset \mathbb{C}^n$, we say that D *is of class* $C^{1,1}$ if D has a defining function (in the sense of Definition (2.2.8)) that is of class $C^{1,1}$ in a neighborhood U of bD; that is, ρ is of class C^1 and its (real) partial derivatives $\partial\rho/\partial x_j$ are Lipschitz functions with respect to the Euclidean distance in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$:

$$\left|\frac{\partial\rho}{\partial x_j}(w) - \frac{\partial\rho}{\partial x_j}(\zeta)\right| \leq C|w-\zeta| w, \zeta \in U, j = 1, \ldots, 2n.$$

Theorem (2.2.37)[42]. Suppose D is a bounded domain ofclass $\mathrm{C}^{1,1}$ which is strongly $\mathbb{C}$-linearly convex. Then there is a natural definition of the Cauchy-Leray integral (66), so that the mapping $\mathrm{f} \mapsto \mathrm{C_L(f)}$ initially defined for $\in \mathrm{C}^1(\mathrm{bD})$, extends to a bounded operator on $\mathrm{L^p(bD, d\sigma)}$ for $1 < \mathrm{p} < \infty$.

Here our hypotheses about the nature of convexity are stronger, but the regularity of the boundary is weaker.

First, we explain the main difficulty in defining the Cauchy-Leray integral in the case of $C^{1,1}$ domains. It arises from the fact that the definitions (65) and (72) involve *second* derivatives of the defining function ρ. However ρ is only assumed to be of class $C^{1,1}$, so that these derivatives are L^∞ functions on $\mathbb{C}^n$, and as such not defined on bD which has $2n$-dimensional Lebesgue measure zero. What gets us out of this quandary is that here in effect not all second derivatives are involved but only those that are “tangential” to bD. Matters are made precise by the following “restriction” principle and its variants.

Suppose $F \in C^{1,1}(\mathbb{C}^n)$ and we want to define $\bar\partial\partial F\big|_{bD}$. We note that if F were of class C^2

we would have

$$\int_{\mathrm{b}D} j^*\left(\bar{\partial}\partial F\right) \wedge \Psi = -\int_{\mathrm{b}D} j^*\left(\partial F\right) \wedge d\Psi, \tag{77}$$

where Ψ is any $2n-3$ form of class C^1 , and here j^* is the induced mapping to forms on $\mathrm{b}D$.

Proposition (2.2.38)[42]. For $\in \mathrm{C}^{1,1}(\mathbb{C}^{\mathrm{n}})$, there exists a unique 2-form $\mathrm{j}^*\left(\bar{\partial}\,\partial \mathrm{F}\right)$ in bD with $\mathrm{L}^{\infty}(\mathrm{d}\sigma)$ coefficients so that (77) holds.

This is a consequence of an approximation lemma: There is a sequence $\{F_n\}$ of C^∞ functions on $\mathbb{C}^n$, that are uniformly bounded in the $C^{1,1}(\mathbb{C}^n)$ norm, so that $F_k \to F$ and $\nabla F_k \to \nabla F$ uniformly on $\mathrm{b}D$, and moreover $\nabla_T^2 F_n$ converges $(d\sigma)$ a.e. on $\mathrm{b}D$. Here $\nabla_T^2 F$ is the "tangential" restriction of the Hessian $\nabla^2 F$ of F. Moreover the indicated limit, which we may designate as $\nabla_T^2 F$, is independent of the approximating sequence $\{F_n\}$.

Chapter 3
A Mass-Transportation Approach and Best Constant in Sobolev Trace Inequalities

The Euclidean structure of $\mathbb{R}^n$ plays no role in our approach: we establish the inequalities, together with cases of equality, for an arbitrary norm. We show a conjecture made by J.F. Escobar in 1988 about the minimizers.

Section (3.1): Sharp Sobolev and Gagliardo–Nirenberg Inequalities

We discuss a new approach for the study of certain geometric functional inequalities, namely Sobolev and Gagliardo- Nirenberg inequalities with *sharp* constants. We wish to

(a) give a unified and elementary treatment of sharp Sobolev and GagliardoNirenberg inequalities (within a certain range of exponents);

(b) illustrate the efficiency of mass transportation techniques for the study of such inequalities, and by this method reveal in a more explicit manner their geometrical nature;

(c) show that the treatment of these sharp Sobolev-type inequalities does not even require the Euclidean structure of $\mathbb{R}^n$, but can be performed for arbitrary norms on $\mathbb{R}^n$;

(d) exhibit a new duality for these problems;

(e) as a by-product of our method, determine all cases of equality in the sharp Sobolev inequalities.

Whenever $n \geq 1$ is an integer and $p \geq 1$ is a real number, define the Sobolev space

$$W^{1,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n); \nabla f \in L^p(\mathbb{R}^n)\}.$$

Here $L^p(\mathbb{R}^n)$ is the usual Lebesgue space of order p, and ∇ stands for the gradient operator, acting on the distribution space $\mathcal{D}'(\mathbb{R}^n)$. When $p \in [1, n)$, define

$$p^\star = \frac{np}{n-p}. \tag{1}$$

Then the (critical) Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$ asserts the existence of a positive constant $S_n(p)$ such that for every $f \in W^{1,p}(\mathbb{R}^n)$

$$||f||_{ii^\star} \leq S_n(p) \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p} \tag{2}$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. For the great majority of applications, it is not necessary to know more about the Sobolev embedding, apart maybe from explicit bounds on $S_n(p)$. However, in some circumstances one is interested in the exact value of the smallest admissible constant $S_n(p)$ in (2). There are usually two possible motivations for this: either because it provides some geometrical insights (as we recall below, a sharp version of (2) when $p = 1$ is equivalent to the Euclidean isoperimetric inequality), or for the computation of the ground-state energy in a physical model. Most often, the determination of $S_n(p)$ is in fact not as important as the identification of *extremal functions* in (2).

Similar problems have been studied at length for very many variants of (2): one example discussed by Del Pino and Dolbeault, which we also consider here, is the Gagliardo-Nirenberg inequality:

$$||f||_{L^r} \leq G_n(p,r,s)||\nabla f||_{L^p}^{\theta}||f||_{L^s}^{1-\theta}, \tag{3}$$

where $n \geq 2, p \in (1,n), s < r \leq p^{\star}$, and $\theta = \theta(n,p,r,s) \in (0,1)$ is determined by scaling invariance. Note that inequality (3) can be deduced from (2) with the help of Hölder's inequality.

The identification of the best constant $S_n(p)$ in (2) for $p > 1$ goes back to Aubin [82] and Talenti [110]. The proofs by Aubin and Talenti rely on rather standard techniques (symmetrization, solution of a particular one-dimensional problem). For $p = 1$, it has been known for a very long time that (2) is equivalent to the classical Euclidean isoperimetric inequality which asserts that, among Borel sets in $\mathbb{R}^n$ with given volume, Euclidean balls have minimal surface area (see [108], [109]). Also the case $p = 2$ is particular, due to its conformal invariance, as exploited in Beckner [85]. In Lieb [101], this case was derived by (rather technical) rearrangement arguments. Carlen and Loss have pointed out the crucial role of "competing symmetries" in this problem and used it to give a simpler proof [91], reproduced in [102]. Recently, Lutwak et al. [103] and Zhang [112] combined the co-area formula and a generalized version of the Petty projection inequality (related to the new concept of affine L^p surface area) to obtain an affine version of the Sobolev inequalities, which implies the Euclidean version (2).

Considerable effort has been spent recently on the problem of optimal Sobolev inequalities on Riemannian manifolds, see the survey [97]. We shall concentrate on the situation where the problem is set on $\mathbb{R}^n$. We do not know whether our methods would still be as efficient in a Riemannian setting. Note however that nonsharp Sobolev Riemannian inequalities can easily be derived by mass transportation techniques, as shown in [92].

For inequality (3), the computation of sharp constants $G_n(p,r,s)$ is still an open problem in general. Del Pino and Dolbeault [95], [96] made the following breakthrough: they obtained sharp forms of (3) in the case of the oneparameter family of exponents:

$$\begin{cases} p(s-1) = r(p-1) \text{ when } r,s > p, \\ p(r-1) = s(p-1) \text{ when } r,s < p. \end{cases} \tag{4}$$

Inequality (2) is actually a limit case of (3) when $r = p^{\star}$ (in which case $\theta = 1$). Note that an L^p version of the usual logarithmic Sobolev inequality also arises as a limit case of (3) when $r = s = p$ (see [96]; the usual inequality would be $p = 2$).

The proofs by Del Pino and Dolbeault for (3) rely on quite sophisticated results from calculus of variations, including uniqueness results for nonnegative radially symmetric solutions of certain nonlinear elliptic or p-Laplace equations. This work by Del Pino and Dolbeault has been the starting point of our investigation. We shall show how their results can be recovered (also in sharp form) by completely different methods.

Unlike the above-mentioned approaches, our arguments do not rely on conformal invariance or symmetrization, nor on Euler-Lagrange partial differential equations for related variational problems. Instead, we shall use the tools of mass transportation, which combine analysis and geometry in a very elegant way. We recall some relevant facts from the theory of mass transportation. If μ and v are two nonnegative Borel measures on $\mathbb{R}^n$ with same total mass (say 1), then a Borel map $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to *push-forward*(or *transport*) μ

onto v if, whenever B is a Borel subset of $\mathbb{R}^n$, one has

$$v[B] = \mu[T^{-1}(B)], \tag{5}$$

or equivalently, for every nonnegative Borel function $b: \mathbb{R}^n \to \mathbb{R}+$,

$$\int b(y)dv(y) = \int b\big(T(x)\big)d\mu(x). \tag{6}$$

The central ingredient in our proofs is the following result of Brenier [6], refined by McCann [105]:

Theorem (3.1.1)[80]. If μ and v are two probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to Lebesgue measure, then there exists a convex function ϕ such that ∇ϕ transports μ onto v. Furthermore, ∇ϕ is uniquely determined dμ almost everywhere.

Observe that ϕ is differentiable almost everywhere on its domain since it is convex; in particular, it is differentiable $d\mu$ almost everywhere. The (monotone) map $T = \nabla\phi$ will be referred to as the *Brenier map*. By construction, it is known to solve the Monge-Kantorovich minimization problem with quadratic cost between μ and v, but here we shall not need this optimality property explicitly. See [111] for a review, and discussion of existing proofs.

From now on, we assume that μ and v are absolutely continuous, with respective densities F and G. Then (6) takes the form

$$\int b(y)G(y)dy = \int b\big(\nabla\phi(x)\big)F(x)dx, \tag{7}$$

for every nonnegative Borel function $b: \mathbb{R}^n \to \mathbb{R}+\cdot$ If ϕ is of class C^2, the change of variables $y = \nabla\phi(x)$ in (7) shows that ϕ solves the *Monge-Ampère equation*

$$F(x) = G\big(\nabla\phi(x)\big)\det D^2\phi(x). \tag{8}$$

Here $D^2\phi(x)$ stands for the Hessian matrix of ϕ at point x. Caffarelli's deep regularity theory [88]−[90] asserts the validity of (8) in classical sense when F and G are Hölder-continuous and strictly positive on their respective supports and G has convex supportwe shall use a much simpler measure theoretical observation, due to McCann [106] which asserts the validity of (8) in the $F(x)dx$ almost everywhere sense, without further assumptions on F and G beyond integrability. In Eq. (8), $D^2\phi$ should then be interpreted in Aleksandrov sense, i.e. as the absolutely continuous part of the distributional Hessian of the convex function ϕ. Of course, $D^2\phi$ is only defined almost everywhere. An alternative, equivalent way of defining $D^2\phi$ is to note (see [98]) that a convex function ϕ admits almost everywhere a second-order Taylor expansion

$$\phi(x+h) = \phi(x) + \nabla\phi(x)\cdot h + \frac{1}{2}D^2\phi(x)(h)\cdot h + o(|h|^2).$$

Where defined, the matrix $D^2\phi$ is symmetric and nonnegative, since ϕ is convex.

Mass transportation (or *parameterization*) techniques have been used in geometric analysis for quite a time. They somehow appear in all known proofs of the Brunn-Minkowski inequality,

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{9}$$

where $A,\ B \subset \mathbb{R}^n$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$ (see [99], [109]). The isoperimetric inequality easily follows from (9). An important source of inspiration for us has been the direct mass transportation proof by Gromov [107] of the (functional) isoperimetric inequality, namely inequality (2) in the case $p = 1$; we shall recall his argument below. Closely related to our work is also the mass transportation proof by McCann [106] of functional versions of (9) known as Prékopa-Leindler and Borell−Brascamp−Lieb inequalities (see [99]). Barthe has exploited all the power of Brenier's theorem to prove deep Gaussian inequalities (see [4] or the reviews [99], [111]). Our proof has many common points with Barthe's work, which is surprising since the inequalities under study here and there look quite different. As far as tools and methods are concerned, be seen as the continuation at [93], [94]. Until recently, it was believed that those techniques could not be adapted to general Sobolev-type inequalities besides the $p = 1$ case. Here we shall demonstrate that this guess was wrong.
Among the main advantages of our proof, we note that it is extremely simple (apart from nonessential technical subtleties linked to the lack of smoothness of the Brenier map). In addition to the existence of the Brenier map, our proof makes use of just two ingredients: the *arithmetic−geometric inequality* on one hand (domination of the geometric mean by the arithmetic mean), and on the other hand the standard *Young inequality* for convex conjugate functions, in the very particular case of Eq. (10) below, or equivalently *Hlöder's inequality* (11).
The proof avoids any compactness argument, and has the great merit to allow room for quantitative versions, which are often important in problems coming from physics: for instance, if a function is far enough from the optimizers in (2), how to give a lower bound on how far the ratio $||\nabla f||_{II}/||f||_{Iy\star}$ departs from the optimal value $S_n(p)$? Here we will not investigate such questions (to do so, it would be desirable to have a more precise formulation of the problem), but it will be clear from our arguments that their constructive nature makes them a plausible starting point for such an investigation, at least when f is strictly positive on $\mathbb{R}^n$.
Finally, our proof will cover non-Euclidean norms. It clearly shows that the treatment of optimal Sobolev inequalities, and the resulting extremal functions, do not depend on the Euclidean structure of $\mathbb{R}^n$. As far as Sobolev inequalities are concerned, such versions for arbitrary norms are not new. The $p = 1$ case was contained in Gromov's treatment. For $p > 1$, the inequalities can be obtained by using a symmetrization procedure and Aubin and Talenti's argument; this was done recently by Alvino et al. [81]. As mentioned, our approach is completely different since we will not solve any variational problem and since our proof will be carried on $\mathbb{R}^n$ till the end.
As we just discussed, the only two ingredients which lie behind our proof of Sobolev inequalities are the arithmetic-geometric inequality, and Hölder's inequality. By tracing carefully cases of equality in these two inequalities, we shall manage to identify all cases of equality in the Sobolev inequalities. Though this problem has been solved in the case of the Euclidean norm, the result seems to be new in the case of arbitrary norms; in [81] this

problem was left open. And even in the Euclidean case, we believe that our approach is simpler than the classical one based on sharp rearrangement inequalities.
We give a proof of optimal Sobolev inequalities. We shall give the adaptations which enable to turn this proof into a proof of optimal Gagliardo-Nirenberg inequalities. Even though we could have treated directly the general case of Gagliardo-Nirenberg inequalities with general norms, we have chosen to present Sobolev inequalities separately because they are popular and of independent interest.
We consider general norms from the beginmng. Let $(E, ||\cdot||)$ be an n-dimensional normed space, with dual space $(E^*, ||\cdot||_*)$. Let λ be an invariant Haar measure on E (unique up to a multiplicative constant). We shall prove a sharp version of the Sobolev inequality

$$\left(\int_E |f|^{p^\star} d\lambda\right)^{1/p^\star} \leq S_{E,\lambda}(p)\left(\int_E ||df||_*^p d\lambda\right)^{1/p}$$

Here $: E \to E^*$ denotes the differential map of $f : E \to \mathbb{R}$.
We assume that $E = (\mathbb{R}^n, ||\cdot||)$ where $||\cdot||$ is an arbitrary norm on $\mathbb{R}^n$. Then the dual space is $E^* = (\mathbb{R}^n, ||\cdot||_*)$ where, for $X \in E^*$,

$$||X||_* := \sup X\cdot \mathrm{Y}$$

and $X\cdot \mathrm{Y} := \sum X_i \mathrm{Y}_i$. The duality can also be expressed through *Young's inequality*

$$X\cdot \mathrm{Y} \leq \frac{\lambda^{-p}}{p}||X||_*^p + \frac{\lambda^q}{q}||\mathrm{Y}||^q \tag{10}$$

for $\lambda > 0$. $q = p/(p-1)$ denotes the dual exponent of $p > 1$ (we hope this notation will avoid confusions with $p^\star$ defined in (3.1)). For $X: \mathbb{R}^n \to E^*$ in L^p and $\mathrm{Y}: \mathbb{R}^n \to E$ in L^q, integration of (10) and optimization in λ gives *Hölder's inequality* in the form

$$\int X\cdot \mathrm{Y} \leq \left(\int ||X||_*^p\right)^{\frac{1}{p}}\left(\int ||\mathrm{Y}||^q\right)^{\frac{1}{q}} \tag{11}$$

This inequality expresses the well-known fact that the dual space of $L^p(\mathbb{R}^n, E)$ coincides with $L^q(\mathbb{R}^n, E^*)$.
The norm $||\cdot||$ is Lipschitz and therefore differentiable almost everywhere. Whenever $x \in \mathbb{R}^n\backslash\{0\}$ is a point of differentiability, the gradient of the norm at x is the unique vector $x^* = \nabla(||\cdot||)(x)$ such that

$$||x^*||_* = 1, x\cdot x^* = ||x|| = \sup x\cdot y. \tag{12}$$

In the usual case of the Euclidean norm $|\cdot|$, $x^* = x/|x|$.

For $1 \leq p < n$, we define the function h_p as follows:

$$(h_1(x) := h_p(x) := \frac{1}{\left(\sigma_p + ||x||^q\right)^{\frac{n-p}{p}}, |B|^{\frac{n-1}{n}} 1_B(x)} \ (p > 1)\,, \tag{13}$$

where $\sigma_p > 0$ is determined by the condition

$$||h_p||_{L^{p}} \star = 1, \tag{14}$$

and B stands for the unit ball of $(\mathbb{R}^n, ||\cdot||)$,

$$B := \{x \in \mathbb{R}^n; ||x|| \leq 1\}.$$

These functions will turn out to be extremal in the Sobolev inequalities. This property is well-known in the Euclidean case $(||\cdot|| = |\cdot|)$: for $p > 1$ it is due to Aubin and Talenti and for $p = 1$ it is the classical isoperimetric inequality. As mentioned, the case of arbitrary norms was considered in [81].

The natural space to look for extremal functions in the Sobolev inequality is the *homogeneous* Sobolev space

$$\dot{W}^{1p}(\mathbb{R}^n) := \{f \in L^{p^\star}(\mathbb{R}^n); \nabla f \in L^p(\mathbb{R}^n)\}.$$

This space coincides with the space of functions f whose distributional gradient lies in L^p and verifying that $|\{|f| \geq a\}|$ is finite for every $a > 0$. It is homogeneous in the same sense inequality (2) is homogeneous under the rescaling $\mapsto f_\lambda \equiv f(\cdot/\lambda)$. This space is better adapted to the study of inequality (2) than W^{1p}; indeed, for $p > 1$, extremal functions will always exist in $\dot{W}^{1p}(\mathbb{R}^n)$ but will not belong to $W^{1p}(\mathbb{R}^n)$ when $p \geq \sqrt{n}$.

If $\in \dot{W}^{1p}(\mathbb{R}^n)$, it is natural to consider the dual norm of the ∇f. Thus, we define

$$||\nabla f||_{i_I} := \left(\int ||\nabla f||_*^p\right)^{1/p} \tag{15}$$

For notational reasons, we will separate the case $p = 1$ from the rest. Let us start with $p > 1$.

Theorem (3.1.2)[80]. *Let $p \in (1, n)$ and $= p/(p-1)$. Whenever $f \in \dot{W}^{1p}(\mathbb{R}^n)$ and $g \in L^{p^*}(\mathbb{R}^n)$ are two functions with $||f||_{Iy^\star} = ||g||_{II^\star}$, then*

$$\frac{\int |g|^{p^\star(1-1/n)}}{\left(\int ||y||^q |g(y)|^{p^\star} dy\right)^{1/q}} \leq \frac{p(n-1)}{n(n-p)} ||\nabla f||_{L^p} \tag{16}$$

with equality if $f = g = h_p$.

As immediate consequences we have

(i) *The duality principle*

$$\sup \frac{\int |g|^{p^\star(1-1/n)}}{\left(\int ||y||^q |g(y)|^{p^\star} dy\right)^{1/q}} = \frac{p(n-1)}{n(n-p)} \inf ||\nabla f||_{L^p} \tag{17}$$

with h_p extremal in both variational problems;

(ii) *The sharp Sobolev inequality*: *if $f \neq 0$ lies in $\dot{W}^{1,p}(\mathbb{R}^n)$, then*

(18)

$$\frac{||\nabla f||_{L^1}}{||f||_{L f^*}} \geq \left\|\nabla h_g\right\|_{L^p}$$

The variant for $p = 1$ of (18), for general norms, can be found in Gromov [107]. Below we shall shortly reproduce his argument, with minor modifications which will make it look just like the proof of Theorem (3.1.2) above. Extremal functions for $p = 1$ do not exist in $W^{1,1}(\mathbb{R}^n)$, and should rather be searched for in the space of functions with bounded variation.

Proof. First of all, it is well-known that whenever $\in \dot{W}^{1,J}(\mathbb{R}^n)$, then $\nabla|f| = \pm\nabla f$ almost everywhere, so f and $|f|$ have equal Sobolev norms. Thus, without loss of generality, we may assume that f and g are nonnegative and, by homogeneity, satisfy $||f||_{II^\star} = ||g||_{Iy^\star} = 1$. Moreover, we shall prove (16) only in the special case when f and g are smooth functions with compact support; the general case will follow by density.
Introduce the two probability densities

$$F(x) = f^{p^\star}(x)\,, G(y) = g^{p^\star}(y)$$

on $\mathbb{R}^n$; let $\nabla\phi$ the Brenier map which transports $F(x)dx$ onto $G(y)dy$. In a first step, we shall establish that

$$\int G^{1-\frac{1}{n}} \leq \frac{1}{n}\int F^{1-\frac{1}{n}}\Delta\phi, \tag{18}$$

where $\Delta\phi(x) := \mathrm{tr}D^2\phi(x)$ appears as the absolutely continuous part of the distributional Laplacian.
As explained in the introduction (8), we have, for $F(x)dx$ almost every $x \in \mathbb{R}^n$,

$$F(x) = G\big(\nabla\phi(x)\big)\det D^2\phi(x)\,. \tag{19}$$

Therefore, for $F(x)dx$ almost every x,

$$\begin{aligned} G^{-1/n}\big(\nabla\phi(x)\big) &= F^{-1/n}(x)\big(\det D^2\phi(x)\big)^{1/n} \\ &\leq F^{-\frac{1}{n}}(x)\frac{\Delta\phi(x)}{n}, \end{aligned} \tag{20}$$

where we used the arithmetic-geometric inequality. By integrating inequality (23) with respect to $F(x)dx$, we find

$$\int G^{-1/n}\big(\nabla\phi(x)\big)F(x)dx \leq \frac{1}{n}\int F(x)^{1-\frac{1}{n}}(x)\Delta\phi(x)dx.$$

The proof of (21) is completed by using the definition of mass transport (7).
Here we shall go a little bit into nonessential technical subtleties. In the inequality (21), $\Delta\phi = \mathrm{tr}D^2\phi$ is to be understood in the almost everywhere sense. It is wellknown that $\Delta\phi$ can be bounded above by $\Delta_{D'}\phi$, which denotes the distributional Laplacian of ϕ, viewed as a nonnegative measure on the set where ϕ is finite (see [98] or [93]). On the other hand, since f and g are compactly supported, we know that $\nabla\phi$ is bounded on $\mathrm{supp}(f)$, the

support of f, since $\nabla\phi(\mathrm{supp}(f)) \subset \mathrm{supp}(g)$ (see [111]). Extending ϕ if necessary outside of the support off, we can assume that the support off lies within an open set where ϕ is finite, and then we can apply the integration by parts formula

$$\frac{1}{n}\int F^{1-\frac{1}{n}}\Delta\phi \leq \frac{1}{n}\int F^{1-\frac{1}{n}}\Delta_{\mathcal{D}},\phi = -\frac{1}{n}\int \nabla\left(F^{1-\frac{1}{n}}\right)\cdot\nabla\phi. \quad (21)$$

Back to our original notations $F = f^{p^\star}$ and $G = g^{p^\star}$, we have just shown, combining (21) and (24), that

$$\int g^{p^\star\left(1-\frac{1}{n}\right)} \leq -\frac{p(n-1)}{n(n-p)}\int f^{\frac{n(p-1)}{n-p}}\nabla f\cdot\nabla\phi$$

$$= \frac{p(n-1)}{n(n-p)}\int f^{p^\star/q}\nabla f\cdot\nabla\phi. \quad (22)$$

We now apply our second crucial tool: Hölder's inequality (11) with the choice $X = -\nabla f$ and $Y = f^{p^\star/q}\nabla\phi$. This gives

$$-\int f^{p^\star lq}\nabla f\cdot\nabla\phi \leq ||\nabla f||_{i_I}\left(\int f^{p^\star}||\nabla\phi||^q\right)^{1/q} \quad (23)$$

But, by definition of mass transport (7), $\int f^{p^\star}||\nabla\phi||^q = \int||y||^q g^{p^\star}(y)dy$. Therefore, the combination of (25) and (26) concludes the proof of inequality (16).

We now choose $f = g = h_p$, and check that equality holds at all the steps of the proof, and therefore in (16). This function is not compactly supported, but in this particular case the Brenier map reduces to the identity map $\nabla\phi(x) = x$, and all the steps can be checked explicitly. Indeed $\nabla\phi(x) = x$ leads to an equality in (21) and in (24) (via integration by parts). Then Eq. (20) ensures the equality in (26). This ends the proof of Theorem (3.1.2).

Theorem (3.1.3)[80]. *If $f \neq 0$ is a smooth compactly supported function, then*

$$\frac{||\nabla f||_{L^1}}{||f||_{L^{n/(n-1)}}} \geq n|B|^{\frac{1}{n}}.$$

This inequality extends to functions with bounded variation, with equality if $f = h_1$.

Proof. Gromov's original proof [107] relied on the Knothe map [100], but the proof also works with the Brenier map as it was pointed out to us some time ago by Michael Schmuckenschläger.

We prove the theorem only when f is a nonnegative function, such that $||f||_{L^{n/(n-1)}} = 1$. We introduce the Brenier map $\nabla\phi$ which pushes forward $F(x)dx = f^{n/(n-1)}(x)dx$ onto $G(y)dy = h_1^{n/(n-1)}(y)dy$. Reasoning as in the proof of Theorem (3.1.2), we write, after (21),

$$|B|^{1/n} \leq \frac{1}{n}\int f\,\Delta\phi \leq -\frac{1}{n}\int \nabla f \cdot \nabla\phi.$$

The justification of the integration by parts goes as in (24). By definition of h_1, for almost every x in the support of f, $\nabla\phi(x) \in B$. In particular $-\nabla f \cdot \nabla\phi \leq ||\nabla f||_*$, and thus

$$n|B|^{\frac{1}{n}} \leq \int ||\nabla f||_* = ||\nabla f||_{L^1}. \qquad (24)$$

By a standard approximation argument, one can express this inequality in terms of an isoperimetric inequality: whenever A is some closed (say) subset of $\mathbb{R}^n$, we have

$$m^+(\partial A) \geq n|B|^{\frac{1}{n}}|A|^{\frac{n-1}{n}}, \qquad (25)$$

where m^+ stands for the surface measure with respect to the metric $||\cdot||$ (not necessarily Euclidean),

$$m^+(\partial A) := \lim_{\varepsilon\to 0}\,\inf \frac{|A+\varepsilon B| - |A|}{\varepsilon}.$$

Note that $A + \varepsilon B$ is the ε-neighborhood of A with respect to the metric $||\cdot||$. Now, there is equality in (28) when A is an affine image of B. So this inequality has to be sharp, and so has to be (27).

Remark (3.1.4)[80]. (i) Inequality (16) is interesting only when $\int ||y||^q|g(y)|^{p^\star}dy < +\infty$, in which case (16) forces g to belong to $L^{p^\star(1-1/n)}(\mathbb{R}^n)$.

(ii) The crucial property of h_p here is that, for almost every x, there is *equality* in Young's inequality (10) when $X = -\nabla h_p(x)$, $Y = \mu_p^{J/q}(x)x\star$ and

$$\lambda = \lambda_p := \left(\frac{n-p}{p-1}\right)^{1/q} \qquad (26)$$

Indeed, after a few computations and using (12), we are led to the straightforward equality

$$\left(\frac{n-p}{p-1}\right)\frac{||x||^q}{\left(\sigma_p + ||x||^q\right)^n} = \frac{1}{p\lambda_p^p}\left(\frac{n-p}{p-1}\right)^p\frac{||x||^q}{\left(\sigma_p + ||x||^q\right)^n} + \frac{\lambda_p^q}{q}\frac{||x||^q}{\left(\sigma_p + ||x||^q\right)^n}.$$

As a consequence (or by a direct computation), the same choice of X and Y gives an equality in Hölder's inequality (11):

$$-\int \nabla h_p(x)\cdot\left[M_p^{J/q}(x)x\right]dx \star = ||\nabla h_p||_{If}\left(\int ||x||^q h_p^{p^\star}(x)dx\right)^{1/q} \qquad (27)$$

Let us now give the proof of Theorem (3.1.2).

Remark (3.1.4). Following the terminology of McCann [106], inequality (21) can be rephrased by saying that the functional

$$\rho \mapsto -\int \rho\,(x)^{1-\frac{1}{n}}dx$$

is *displacement convex*. This fact is well-known to specialists, and rests on the concavity of the map $M \mapsto (\det M)^{1/n}$, defined on the set of nonnegative symmetric matrices; see in particular [111].

We conclude with a few remarks about the way we have proven and stated our results.

Remark (3.1.5)[80]. (i) A classical way to attack the problem of optimal constants for Sobolev inequalities is to look at the Euler-Lagrange equation and to identify its solutions. Here, on the contrary, we have established that h_p is an optimizer without establishing any Euler-Lagrange equation. Neither did we use the co-area formula or a rearrangement procedure.

(ii) The best constant $\tilde{S}_n(p) := ||\nabla h_p||_{Lp}^{-1}$ in the sharp Sobolev inequality (18) can easily be expressed as a function of $|B|$ since h_p is radially symmetric with respect to the norm $||\cdot||$. In particular, we have

$$\tilde{S}_n(p) = \left(\frac{|B_2^n|}{|B|}\right)^{1/n} S_n(p)\,,$$

where B_2^n is the standard Euclidean ball and $S_n(p)$ is the best constant in the Euclidean Sobolev inequality (2). We stress however that the extremal function h_p depends on $||\cdot||$ and not just on $|B|$.

(iii) If we exploit the left-hand side maximization in (17), we immediately obtain, after setting $h = g^{p^\star}$, the following sharp inequality: there exists a constant $C_n(p) > 0$ such that for every $h \in L^1$,

$$\int |h|^{1-1/n} \le C_n(p)\left(\int ||y||^q |h(y)| dy\right)^{1/q}\left(\int |h|\right)^{1/p^\star}$$

with equality if $h(y) = h_p^{p^\star}(y) = \left(\sigma_p + ||y||^q\right)^{-n}$. It would be interesting to understand why this inequality appears as a dual of the Sobolev inequality.

The right-hand side of (16) is invariant under dilations and translations (for fixed $L^{p^\star}$ norm), whereas the left-hand side is only invariant under dilations. If we define $\mathrm{Var}_q(G) := \inf_{y_0} \int ||y - y_0||^q G(y)dy$, then inequality (16) can obviously be replaced by the following dilation-translation invariant version: for $||f||_{II^\star} = ||g||_{ii^\star} = 1$,

$$\frac{\int |g|^{p^\star(1-1/n)}}{\mathrm{Var}_q(|g|^{p^\star})^{1/q}} \le \frac{p(n-1)}{n(n-p)} ||\nabla f||_{ii} \qquad (28)$$

with equality if $f = g = h_p$.

What happens if in the proof of Theorem (3.1.2), in Eq. (26), we use, instead of Hölder's inequality (11), the simpler Young inequality (10)? In view of the Remark (3.1.5) before (19), we obtain the following (equivalent) form of the theorem: *whenever* $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$ *are two functions with* $||f||_{II^\star} = ||g||_{II^\star} = 1$, *then, for all* $\lambda > 0$,

$$\frac{n(n-p)}{p(n-1)} \int |g|^{p^\star(1-1/n)} - \frac{\lambda^q}{q} \int ||y||^q |g(y)|^{p^\star} dy \leq \frac{1}{p\lambda^p} \int ||\nabla f||_*^p \quad (29)$$

with equality if $f = g = h_p$ *and* $\lambda = \lambda_p(19)$. *As a consequence we have the duality principle*

$$\sup \left[\frac{n(n-p)}{p(n-1)} \int |g|^{p^\star(1-1/n)} - \frac{\lambda_p^q}{q} \int ||y||^q |g(y)|^{p^\star} dy \right] = \frac{1}{p\lambda_p^p} \inf \int ||\nabla f||_*^p,$$

with h_p *extremal in both variational problems.*

This formulation was our original one. Clearly, the duality seems to be expressed in a much more satisfactory way in (16) than in (30). Furthermore, the extremal function h_p appears more naturally via (16), and one need not choose $\lambda = \lambda_p$ in a seemingly arbitrary manner.

On the other hand, (30) has the advantage to separate the integrals in an additive way, and this form will appear more convenient to deal with more sophisticated integral expressions. We give a treatment of some Gagliardo-Nirenberg inequalities in sharp form. As we explained, the results in the Euclidean case were recently obtained, with a different method, by Del Pino and Dolbeault (the case $p = 2$ is treated in [95] and the general case in [96]). Here again, we shall consider an arbitrary norm $|| \cdot ||$ on $\mathbb{R}^n$.

We introduce a new family of functions, which will turn out to be optimal in a more general family of inequalities: for $\alpha \geq 0$, we define

$$h_{\alpha,p}(x) := \left(\sigma_{\alpha,p} + (\alpha - 1)||x||^q\right)_+^{\frac{1}{1-\alpha}}.$$

As before we write $= p/(p-1)$, and $\sigma_{\alpha,p} > 0$ is chosen in such a way as to turn $h_{\alpha,p}^{\alpha p}$ into a probability density. Note that for $\alpha < 1, h_{\alpha,p}$ has compact support, while for $\alpha > 1$ it is positive everywhere, decaying polynomially at infinity. The L^p norm of the gradient is again considered in the sense of (15). We stress that the statement will include $L^r(\mathbb{R}^n)$-spaces with $\in (0,1)$, for which $|| \cdot ||_{L^r}$ is no longer a norm. We shall prove

Theorem (3.1.6)[80]. *Let* $n \geq 2, p \in (1, n)$ *and* $\alpha \in (0, \frac{n}{n-p}]$, $\alpha \neq 1$. *Let* f *and* g *be such that* $||f||_{L^{\alpha p}} = ||g||_{L^{\alpha p}} = 1$. *Then, for all* $\mu > 0$,

$$\frac{\alpha p}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int\ |g|^{\alpha(p-1)+1}-\frac{\mu^q}{q}\int\ ||y||^q|g(y)|^{\alpha p}dy$$

$$\leq\frac{1}{p\mu^p}\int\ ||\nabla f||_*^p+\frac{\alpha p-n(\alpha-1)}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int\ |f|^{\alpha(p-1)+1}\quad(30)$$

Moreover, when

$$\mu=\mu_p:=q^{1/q},\quad(31)$$

then

(i) *equality holds in* (31) *when* $f=g=h_{\alpha p}$; *in particular, one has the duality principle*

$$\sup_{||g||_{L^{\alpha p}=1}}\left[\frac{\alpha p}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int\ |g|^{\alpha(p-1)+1}-\frac{\mu_p^q}{q}\int\ ||y||^q|g(y)|^{\alpha p}dy\right]$$

$$=\inf_{||f||_{L^{\alpha p}=1}}\left[\frac{1}{p\mu_p^p}\int||\nabla f||_*^p+\frac{\alpha p-n(\alpha-1)}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int|f|^{\alpha(p-1)+1}\right]\quad(32)$$

and $h_{\alpha p}$ *is extremal in both variational problems*;

(ii) *as a corollary*, *whenever* $f\neq 0$ *lies in* $\dot{W}^{1,p}(\mathbb{R}^n)$, *then*
for $\alpha>1$,

$$\frac{||\nabla f||_{L^p}^{\theta}||f||_{L^{\alpha(p,-1)+1}}^{1-\theta}}{||f||_{L^{\alpha p}}}\geq||\nabla h_{\alpha,p}||_{L^p}^{\theta}||h_{\alpha,p}||_{L^{\alpha(p-1)+1}}^{1-\theta},\quad(33)$$

where

$$\theta=\frac{n(\alpha-1)}{\alpha\big(np-(\alpha p+1-\alpha)(n-p)\big)}=\frac{p^\star(\alpha-1)}{\alpha p(p^\star-\alpha p+\alpha-1)};$$

for $\alpha<1$,

$$\frac{||\nabla f||_{L^p}^{\theta}||f||_{L^{\alpha p}}^{1-\theta}}{||f||_{L^{\alpha(p-1)+1}}}\geq\frac{||\nabla h_{\alpha,p}||_{L^p}^{\theta}}{||h_{\alpha p}||_{L^{\alpha(p-1)+1}}},\quad(34)$$

where

$$\theta=\frac{n(1-\alpha)}{(\alpha p+1-\alpha)\big(n-\alpha(n-p)\big)}=\frac{p^\star(1-\alpha)}{(p^\star-\alpha p)(\alpha p+1-\alpha)}.$$

Proof. The proof will follow the same scheme as in the the basic inequality replacing (21) will be the following: whenever $\gamma\geqslant 1-1/n,\gamma\neq 1$,

$$\frac{1}{1-\gamma}\int g^\eta\leqslant\frac{1-N(1-\gamma)}{1-\gamma}\int f^\eta+\int f^\eta\Delta\varphi.\quad(35)$$

Inequality (21) corresponds to the case $\gamma = 1 - 1/n$. Here and as in the proof of Theorem (3.1.2), F and G denote two probability densities, and $\nabla\varphi$ is the Brenier map pushing $F(x)dx$ forward to $G(y)dy$. In [111], inequality (36) is shown to be an immediate consequence of the displacement convexity (in the terminology of McCann [106]) of the functional

$$\rho \mapsto \frac{1}{1-\gamma}\int \rho^{\gamma}(x)dx. \tag{36}$$

Again, for the sake of completeness we shall give a short proof which does not rely explicitly on this concept. It proceeds exactly in the same way that we followed to prove (21). From the Monge–Ampe`re equation (8) we deduce that for $F(x)dx$ almost every $x \in \mathbb{R}^n$ we have

$$G^{\eta-1}\big(\nabla\varphi(x)\big) = F^{\eta-1}(x)\big(\det D^2\,\varphi(x)\big)^{1-\gamma}. \tag{37}$$

The function $M \mapsto (\det M)^k$ is concave (resp. convex) on the set of nonnegative symmetric $n \times n$ matrices when $k \in [0, 1/n]$ (resp. $k < 0$). In other words, the function

$$M \mapsto \frac{1}{1-\gamma}(\det M)^{1-\gamma}$$

is concave on the set of nonnegative symmetric $n \times n$ matrices whenever $\gamma \geqslant 1 - 1/n$. (The case $\gamma = 1$, not needed here, is defined in the limit as the log-concavity of the determinant and can be used for provinglog arithmic Sobolev inequalities [93]).
Thus, for a nonnegative symmetric matrix M, we have

$$\begin{aligned}(1-\gamma)^{-1}(\det M)^{1-\gamma} &= (1-\gamma)^{-1}(\det(I + (M - I)))^{1-\gamma}\\ &\leqslant (1-\gamma)^{-1} + tr(M - I)\\ &= (1-\gamma)^{-1}\big(10n(1-\gamma)\big) + tr(M).\end{aligned}$$

We then deduce from (38) that

$$\frac{1}{1-\gamma}G^{\gamma-1}\big(\nabla\phi(x)\big) \leq \frac{1-n(1-\gamma)}{1-\gamma}F^{\gamma-1}(x) + F^{\gamma-1}(x)\Delta\phi(x)\,.$$

Integrating this inequality with respect to $F(x)dx$ and using the definition of mass transport (7), we conclude to (36).
We go on with the proof of (31). Define

$$\gamma := \frac{\alpha(p-1)+1}{\alpha p} = 1 - \frac{\alpha - 1}{\alpha p}$$

and note that $\gamma \geq 1 - 1/n$ precisely when $\alpha \in (0, n/(n-p)]$. Reasoning exactly as in Theorem (3.1.2), we deduce from (36) that whenever F and G are two smooth, compactly supported probability densities and $\nabla\phi$ is the corresponding Brenier map, then

$$\frac{\alpha p}{\alpha - 1}\int G^{\gamma} \leq \frac{\alpha p - n(\alpha - 1)}{\alpha - 1}\int F^{\gamma} - \int \nabla F^{\gamma}\cdot\nabla\phi. \tag{38}$$

Choosing $F = f^{\alpha p}$ and $G = g^{\alpha p}$ in this inequality, we obtain

$$\frac{\alpha p}{\alpha-1}\int g^{\alpha(p-1)+1}$$

$$\leq \frac{\alpha p-n(\alpha-1)}{\alpha-1}\int f^{\alpha(p-1)+1}-(\alpha(p-1)+1)\int f^{\alpha(p-1)}\nabla f\cdot\nabla\phi.$$

Next we apply Young's inequality (10) with $X=-\nabla f(x)$ and $=f^{\alpha(p-1)}(x)\nabla\phi(x)$, to find

$$\frac{\alpha p}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int g^{\alpha(p-1)+1}\leq\frac{\alpha p-n(\alpha-1)}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int f^{\alpha(p-1)+1}$$

$$+\frac{1}{p\mu^p}\int ||\nabla f||_*^p+\frac{\mu^q}{q}\int f^{\alpha p}\,||\nabla\phi||^q.$$

To conclude the proof of (31), it suffices to apply the identity (7) to the last integral.
We now turn to the proof of part (i) of Theorem (3.1.6). Just as in Theorem (3.1.2), it is a direct consequence of the observation that if we set $f=g=h_{\alpha,p}$, and thus $\nabla\phi(x)=x,$ in the previous proof, then all the steps can be computed explicitly and lead to equalities. The crucial point here, which ensures a pointwise equality in Young's inequality (10) is that, for almost all $x\in\mathbb{R}^n$,

$$-\nabla h_{\alpha_1 p}(x)\cdot\left[h_\alpha^{\alpha(d^{J-1})}J^J(x)x\right]=\frac{1}{pf_p}||\nabla h_{\alpha,p}(x)||_*^p+\frac{\mu_p^q}{q}||h_\alpha^{\alpha(p-1)}J^J(x)x||^q.$$

Indeed, after a little bit of computation, this identity reduces to the straightforward equality

$$q\frac{||x||^q}{\big(\sigma_{\alpha,p}+(\alpha-1)||x||^q\big)^{\frac{\alpha p}{\alpha-1}}}=\frac{q^p}{pf_p}\frac{||x||^q}{(\sigma_\alpha J^J+(\alpha-1)||x||^q)^{\frac{\alpha p}{\alpha-1}}}$$

$$+\frac{\mu_p^q}{q}\frac{||x||^q}{(\sigma_\alpha J^J+(\alpha-1)||x||^q)^{\frac{\alpha p}{\alpha-1}}}.$$

Finally, let us prove part (ii) of Theorem (3.1.6). To show that part (i) of the theorem implies part (ii), we use a scaling argument, more or less standard in problems of this kind. Assume for instance $\alpha>1$, and let us see how to establish (34). From part (i) we have the inequality, when $||f||_{L^{\alpha p}}=1$,

$$\frac{1}{p\mu_p^p}\int ||\nabla f||_*^p+\frac{\alpha p-n(\alpha-1)}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int |f|^{\alpha(p-1)+1}$$

$$\geq C:=\left[\frac{\alpha p}{(\alpha-1)\big(\alpha p-(\alpha-1)\big)}\int |h_\alpha J^J|^{\alpha(p-1)+1}-\frac{\mu_p^q}{q}\int ||y||^q|h_{\alpha p}(y)|^{\alpha p}dy\right], \quad (39)$$

with equality when $f=h_\alpha J^j$. Thus, for every $\in\dot{W}^{1_1 p}(\mathbb{R}^n)$,

$$\frac{||f||_{L^{\alpha(p-1)+1}}^{\alpha(p-1)+1}}{||f||_{L^{\alpha p}}^{\alpha(p-1)+1}} + C_1 \frac{||\nabla f||_{II}}{||f||_{L^{\alpha p}}^{p}} \geq C_2, \tag{40}$$

where C_1 and C_2 are positive constants. Here we do not write down the precise values of C_1 and C_2; anyway this is not necessary, to carry on the argument till the end it will be sufficient to know that $h_{\alpha:p}$ is optimal in this inequality.

Next, we apply (41) with f replaced by $f_\lambda = f(\cdot/\lambda)(\lambda > 0)$. We find

$$\lambda^{\frac{n(\alpha-1)}{\alpha p}} \frac{||f||_{L^{\alpha(p-1)+1}}^{\alpha(p-1)+1}}{||f||_{L^{\alpha p}}^{\alpha(p-1)+1}} + C_1 \lambda^{\frac{\alpha(n-p)-n}{\alpha}} \frac{||\nabla f||_{Iy}}{||f||_{L^{\alpha p}}^{p}} \geq C_2, \tag{41}$$

and we can now optimize with respect to $\lambda > 0$, to recover

$$||f||_{L^{\alpha p}} \leq C||\nabla f||_{L^p}^{\theta} ||f||_{L^{\alpha(p-1)+1}}^{1-\theta},$$

with equality when $f = (h_\alpha J^J)_\lambda$, with the optimal choice of λ. As expected, θ is determined by scaling invariance. The same scaling invariance guarantees that there is also equality when $f = h_\alpha J^2$, which is the content of (34).

The case $\alpha < 1$ is obtained exactly in the same way. This concludes the proof of Theorem (3.1.6).

The mass transportation method appears to be extremely efficient in the treatment of sharp Gagliardo-Nirenberg inequalities, as illustrated by the short length and simplicity of the proofs above. Among the other advantages of our method, we note that it provides a common framework to *all* the family of Sobolev inequalities, making the link with isoperimetric estimates clearer. It also emphasizes a strong connection between the Brunn-Minkowski inequality (9) (and more generally convex geometry) and sharp Sobolev inequalities. Finally, we should mention that the use of the Brenier map is not compulsory: we could as well have worked with the Knothe map [100].

Certainly, one of the most irritating open problems remaining in the field, is the fact that we do not understand how to get sharp inequalities and extremal functions in the rest of the range of the Gagliardo-Nirenberg family (3).

Another natural problem is that of the identification of *all* cases of equality in Sobolev or Gagliard +Nirenberg inequalities. In the case of a Euclidean norm, it is known that the functions h_p are the only minimizers, up to translation, dilation and multiplication by a constant. But even in this case, the known proofs of this result are far from being straightforward; they first use the Brothers-Ziemer theorem [87] to reduce to the one-dimensional case, after which a somewhat tedious analysis is performed. From our proof, it is possible to determine all cases of equality, even when dealing with arbitrary norms. We restrict the discussion to the sharp Sobolev inequalities. A similar proof would solve the problem for the Gagliardo-Nirenberg inequalities, at least in the case $\alpha > 1$.

Theorem (3.1.7)[80]. *A function $f \in \dot{W}^{1:p}(\mathbb{R}^n)$ is optimal in the Sobolev inequality* (18) *if and only if there exist $C \in \mathbb{R}$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that*

$$f(x) = Ch_p\big(\lambda(x - x_0)\big)\,. \tag{42}$$

It is enough to prove Theorem (3.1.7) for nonnegative functions f. Indeed, for an arbitrary optimal function f, $|f|$ will also be optimal and then the conclusion of the theorem will force f to have constant sign on $\mathbb{R}^n$.

Let f and g be two nonnegative measurable functions; we say that f is a *dilationtranslation image* of g if there exists $C > 0$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that $(x) = Cg\big(\lambda(x - x_0)\big)$. If $\int f^k = \int g^k$ for some $k > 0$, then necessarily $f(x) = |\lambda|^{n/k} g(\lambda(x - x_0))$. This is equivalent to saying that the Brenier map $\nabla\phi$ pushing $f^k(x)dx$ forward to $g^k(y)dy$ is a *dilation-translation map*, in the sense that $= \lambda(\mathrm{Id} - x_0)$. Note that f is a dilation-translation image of g if and only if g is a dilation-translation image of f.

The Sobolev inequality is invariant under dilation ⊣ranslation maps. Thus, it suffices to prove Theorem (3.1.7) when $f^{p^\star}$ is a probability density. In view of Theorem(3.1.2), we just have to set $g = h_p$, and prove that all f's which achieve equality in (16) are dilation-translation images of h_p. Then, Theorem (3.1.7) is an immediate consequence of

Proposition (3.1.8)[80]. Let $\Omega \in (1, \mathrm{n})$, and letf and g be two nonnegative functions satisfying the assumptions of Theorem (3.1.2). If equality holds in (16), then f is a dilation-translation image of g.

proof. will not rely on any sharp rearrangement inequality, but on rather standard tools from distribution theory, combined with careful approximation procedures. Let us start with an informal discussion. Our derivation of the optimal Sobolev inequality only relied on (i) Theorem (3.1.1), together with the Monge-Ampère equation (22) and the definition of mass transport;

(ii) the arithmetic-geometric inequality (23), $(\det D^2\phi)^{1/n} \leq \Delta\phi/n$, integrated with respect to $f^{p^\star(1-1/n)}(x)dx$;

(iii) the integration by parts formula (24);

(iv) Hölder's inequality (11), in the form of Eq. (26).

If ϕ was smooth and f positive everywhere, equality in the arithmetic- geometric inequality (ii) would imply that $D^2\phi$ is a pointwise multiple of the identity, from which it could be shown that it is in fact a constant multiple of the identity, so that $\nabla\phi$ is a dilation-translation map. However, we do not know a priori that ϕ is smooth, neither that f is positive almost everywhere. Moreover, it is definitely not clear that the integration by parts formula (24) applies to the minimizer: we proved it only in the case when f and g are compactly supported! This restriction on f and g had no consequence on the generality of the final inequality, since a density argument could be applied; but it prevents us to go anywhere as far as equality cases are concerned. Therefore, our proof will be performed in two steps: (i)

generalize the proof of (16) in order to directly obtain the inequality for all admissible f's and g's, *not necessarily smooth and compactly supported*; (ii) trace back cases of equality in the proof of this inequality, without assuming extra smoothness on f, g or ϕ.
To carry out step 1, it is sufficient to generalize the proof of the integration by parts (24) to more general functions f and g.
With the notations of Theorem (3.1.2), let us fix nonnegative functions f and g for which there is equality in (3.16). We will trace back the equality cases in the proof of (16). Recall that $\nabla\phi$ denotes the Brenier map pushing $f^{p^\star}(x)dx$ forward to $g^{p^\star}(y)dy$. Our goal is to prove that $\nabla\phi$ is a dilation -translation map. As before, we denote by Ω the interior of the convex set where $\phi < +\infty$; we recall that $\overline{\Omega}$ contains $\text{supp}(f)$, and that $\partial\Omega$ is of zero measure.
The proof will be done in three steps:
Step i: The function f is positive on Ω; *Step* ii: D_{D}^{2}, ϕ has no singular part on Ω; Step iii**:** $\nabla\phi$ is a dilation-translation map.
Let us first show that f is positive everywhere, or more rigorously that *for every compact subset K of* Ω, *there exists a positive constant* α_K *such that*

$$\forall x \in K, f(x) \geq \alpha_K > 0. \tag{43}$$

Here, of course $\forall x$" should be understood as "for almost all x" A proof was suggested to us by Almut Burchard; we reproduce her argument almost verbatim below.
For equality to hold in Hölder's inequality (11) it is necessary that, for some positive constant $k > 0$,

$$||X||_*^p = k||\text{Y}||^q \ almost\ everywhere \tag{44}$$

Therefore, equality in (26) implies

$$||\nabla f(x)||_*^p = kf^{p^\star}(x)||\nabla\phi(x)||^q \tag{45}$$

for almost every $x \in \Omega$.
Let us introduce $f_m(x) = \max(f(x), 1/m)$. We know that $\nabla f_m \in L^p$ and that in fact $\nabla f_m = \nabla f 1_{f>1/m}$. It follows that

$$||\nabla f_m(x)||_*^p \leq ||\nabla f(x)||_*^p = kf^{p^\star}(x)||\nabla\phi\langle x)||^q \leq kf_m^{p^\star}(x)||\nabla\phi(x)||^q.$$

As a consequence,

$$||\nabla\left(f_m^{-p/(n-p)}\right)||_* \leq k^{1/p}\left(\frac{p}{n-p}\right)||\nabla\phi||^{1/(p-1)}. \tag{46}$$

Since $||\nabla\phi||$ is locally bounded on Ω, it follows from (51) that the functions $f_m^{-p/(n-p)}$ are uniformly (in m) locally Lipschitz on Ω. Taking m to infinity shows that $f^{-p/(n-p)}$ is locally Lipschitz, and therefore locally bounded, on Ω. From this we deduce that f is positive, locally bounded away from 0 on Ω, in the sense of (48). This implies in particular that the support of f is $\overline{\Omega}$.

We now prove that $D^2_{\mathcal{D}'}\phi$ has no singular part. Since this is a nonnegative matrixvalued measure, it is enough to prove that its trace $\Delta_{\mathcal{D}'}\phi$ is itself absolutely continuous in Ω. Let $\Delta_s\phi$ be the singular part of $\Delta_{\mathcal{D}}\phi$; recall that $\Delta_s\phi$ is a nonnegative measure and that $\Delta_{\mathcal{D}'}\phi = \Delta\phi + \Delta_s\phi$. Since there should be equality in (44), we deduce from the proof of Lemma (3.1.9) that

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf \langle (f_\varepsilon^\delta)^{\frac{p(n-1)}{n-p}}, \Delta_s\phi\rangle_{\mathcal{D}'} = 0. \tag{47}$$

Without loss of generality, we assume that $0 \in \Omega$. Let K be an arbitrary convex compact subset of Ω containing 0 in its interior. For $d_K := d(K, \Omega^c)$, let $K' = \{x \in \Omega; d(x, K) \le d_K/2\}$. From its definition K' is a convex compact subset of Ω whose interior is a neighborhood of K. By (48) we know that there exists $\alpha = \alpha_{K'} > 0$ such that $f \ge \alpha 1_{K'}$, where $1_{K'}$ stands for the indicator function of K'. If ε is small enough, we can make sure that $K/(1-\varepsilon)^2 \subset K'$; then, with the notation of Lemma (3.1.9) we have $f_\varepsilon(x) \ge \alpha 1_{K/(1-\varepsilon)}(x)$. If δ is small enough, this implies

$$f_\varepsilon^\delta \ge \alpha 1_K.$$

As a consequence, when both ε and δ are small enough we see that

$$\langle (f_\varepsilon^\delta)^{p^\star\left(1-\frac{1}{n}\right)}, \Delta_s\phi\rangle_{\mathcal{D}'} \ge \alpha^{p^\star\left(1-\frac{1}{n}\right)}\Delta_s\phi[K]. \tag{48}$$

Combining this with (52) and the positivity of α, we find that $\Delta_s\phi[K] = 0$. Since K is arbitrary, we conclude that $\Delta_s\phi$ vanishes. As announced above, this means that $D^2_{\mathcal{D}'}\phi$ is absolutely continuous. Since we have equality in the arithmetic-geometric inequality (23) for $f^{p^\star(1-1/n)}(x)dx$ almost every $x \in \Omega$, and therefore for almost every $x \in \Omega$, we conclude that $D^2\phi$, which can be identified with $D^2_{\mathcal{D}'}\phi$, is proportional to the identity matrix at almost every $x \in \Omega$. Let κ be a smooth regularizing kernel with support included in a small ball of radius ε. Since $D^2(\phi * \kappa) = D^2\phi * \kappa$, we deduce that the smooth function $\phi * \kappa$ is such that its Hessian is also pointwise proportional to the identity matrix on $\Omega_\varepsilon := \{x \in \Omega; d(x, \partial\Omega) > \varepsilon\}$. From this one easily shows that $D^2(\phi * \kappa)$ is a constant multiple of the identity. By making κ tend to a Dirac mass, we see that $D^2_{\mathcal{D}'}\phi$ is also a constant multiple of the identity on the whole of Ω, and therefore $\nabla\phi$ is a dilationtranslation map on Ω. This concludes the proof of Proposition (3.1.8).

This is the content of the following:

Lemma (3.1.9)[80]. *Let $f \in \dot{W}^{1p}(\mathbb{R}^n)$ and $g \in L^{p^\star}(\mathbb{R}^n)$ be two nonnegative functions such that $||f||_{II^\star} = ||g||_{II^\star} = 1$ and $\int g^{p^\star}(y)||y||^q dy < +\infty$. Let $\nabla\phi$ denote the Brenier map pushing $f^{p^\star}(x)dx$ forward to $g^{p^\star}(y)dy$. Then, $f^{p^\star/q}\nabla\phi \in L^q(\mathbb{R}^n)$ and*

$$\int f^{p^\star\left(1-\frac{1}{n}\right)}\Delta\phi \leq -\int \nabla\left[f^{p^\star\left(1-\frac{1}{n}\right)}\right].\nabla\phi = \frac{p(n-1)}{(n-p)}\int f^{p^\star/q}\,\nabla f.\nabla\phi, \quad (49)$$

where $\Delta\phi = \mathrm{tr}D^2\phi \geq 0$ denotes the absolutely continuous part of the distributional Laplacian.

To achieve step 2, and eventually prove Proposition (3.1.8), we shall have to overcome a few more technical difficulties. We establish that f is positive; the proof of this fact was given to us by Almut Burchard, As we shall see, the argument eventually relies on the fact that there should be equality in Hölder's inequality (iv) above. From this strict positivity we shall deduce that the distributional Hessian $D^2_{\mathcal{D}},\phi$ is absolutely continuous, and therefore coincides with $D^2\phi$, defined almost everywhere. Once we have introduced the distributional Hessian in our problem, we will use a standard regularization argument to conclude the proof.

A subtle point in the argument is the following: for our proof to work out, it is not sufficient to prove that f is positive almost everywhere. Indeed, if f would vanish at some place, then we could not exclude the possibility that $D^2\phi$ has some singular part, living precisely on the set where f vanishes. On the other hand, f is not a priori continuous, so discussing the positivity off *everywhere* does not seem to make much sense. To avoid this contradiction, we shall show that f is positive everywhere in the sense that it is, locally, bounded from below almost everywhere by a positive constant.

After these explanations, we can go on with the proofs of Lemma (3.1.9) and of Proposition (3.1.8).

Proof. By definition of mass transport (7), we know that $\int f^{p^\star}\,||\nabla\phi||^q = \int g^{p^\star}\,||y||^q dy$ and so $f^{p^\star/q}\nabla\phi \in L^q(\mathbb{R}^n)$. The proof of (44) will be done by approximation and regularization; there is no fundamental difficulty, but one has to be careful enough.

Let Ω be the interior of the convex set where $\phi < +\infty$. Note that $\overline{\Omega}$ contains the support of f, and that $\partial\Omega$ is of zero measure. Without loss of generality we assume that $0 \in \Omega$. Whenever $\varepsilon > 0$ is a (small) positive number, we define

$$f_\varepsilon(x) = \min\left[f\left(\frac{x}{1-\varepsilon}\right), f(x)\chi(\varepsilon x)\right], \quad (50)$$

where χ is a C^∞ cut-off function with $0 \leq \chi \leq 1$, $\chi(x) \equiv 1$ for $|x| \leq 1/2$, $\chi(x) \equiv 0$ for $|x| \geq 1$. Note that the support of f_ε is compact and contained within Ω (here we use the fact that Ω is starshaped with respect to 0).

Both functions in the right-hand side of (45) are bounded in $\dot{W}^{1,p}(\mathbb{R}^n)$, uniformly in ε. This is clear for the first one; for the second one this is a consequence of

$$\int_{\mathbb{R}^n} f^p(x)|\nabla[\chi(\varepsilon x)]|^p dx = \varepsilon^p\int_{\mathbb{R}^n} f^p\,|\nabla\chi(\varepsilon x)|^p dx$$

$$\leq \left(\int_{\mathbb{R}^n} f^{p^\star}\right)^{1-\frac{p}{n}} \left(\int_{\mathbb{R}^n} \varepsilon^n \, |\nabla\chi(\varepsilon x)|^n dx\right)^{\frac{p}{n}}$$
$$= \left(\int_{\mathbb{R}^n} f^{p^\star}\right)^{1-\frac{p}{n}} \left(\int_{\mathbb{R}^n} |\nabla\chi(x)|^n dx\right)^{\frac{p}{n}},$$

where we used Hölder's inequality and the change of variables $x \to \varepsilon x$. Thus (by the formula $\min(f, g) = (f + g)/2 - |f - g|/2$), f_ε lies in $\dot{W}^{1,p}$, and in fact ∇f_ε is bounded in L^p as $\varepsilon \to 0$.

We now fix $\varepsilon > 0$, and let Ω_ε be a bounded open set whose closure is contained within Ω, and which contains the support of f_ε. It is standard that f_ε can be approximated in $\dot{W}^{1,p}(\mathbb{R}^n)$ by a sequence $f_\varepsilon^\delta \to f_\varepsilon$ of smooth nonnegative functions compactly supported inside Ω_ε; for this one just has to regularize f_ε by convolution with a kernel whose support is contained within a ball of radius δ, δ small enough and going to 0. Then we can use the fact that $\Delta\phi$ (in the sense of Aleksandrov) is bounded above by the distributional Laplacian of ϕ in Ω (see [98] or [92]), and write

$$\int (f_\varepsilon^\delta)^{p^\star\left(1-\frac{1}{n}\right)} \Delta\phi \leq -\int \nabla\left[(f_\varepsilon^\delta)^{p^\star\left(1-\frac{1}{n}\right)}\right] . \nabla\phi = -c_{np} \int (f_\varepsilon^\delta)^{p^\star/q} \, \nabla f_\varepsilon^\delta \cdot \nabla\phi \quad (51)$$

where $c_{np} := p(n-1)/(n-p) > 0$. We know that f_ε^δ converges to f_ε in $L^{p^\star}$ (by convergence in $\dot{W}^{1,p}(\mathbb{R}^n)$) and since $\nabla\phi$ remains essentially bounded within Ω_ε, we conclude that $(f_\varepsilon^\delta)^{p^\star/q} \nabla\phi$ converges to $(f_\varepsilon)^{p^\star/q} \nabla\phi$ in L^q. On the other hand we know that $\nabla f_\varepsilon^\delta$ converges to ∇f_ε in L^p. We then deduce from (46) by Fatou's lemma that

$$\int (f_\varepsilon)^{p^\star(1-1/n)} \Delta\phi \leq -c_{n:p} \int (f_\varepsilon)^{p^\star/q} \, \nabla f_\varepsilon \cdot \nabla\phi. \quad (52)$$

It now remains to pass to the limit in (47) as $\varepsilon \to 0$. For this we argue as follows. First of all we note that, up to possible extraction of a subsequence $\varepsilon = (\varepsilon_k)_{k\in\mathbb{N}}$, f_ε converges almost everywhere to f as $\varepsilon \to 0$. To prove this, it is sufficient to show that $g_\varepsilon(x) := f\big(x/(1-\varepsilon)\big)$ converges almost everywhere to $f(x)$ as $\varepsilon \to 0$. Clearly, g_ε is bounded in $\dot{W}^{1,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$, and it also converges to f in the sense of distributions, since for all compactly supported test-functions ϕ one can write

$$\int g_\varepsilon \phi = (1-\varepsilon)^n \int f(x)\phi\big((1-\varepsilon)x\big)dx \to \int f\,\phi.$$

So g_ε converges weakly to f in $\dot{W}^{1,j}$, and therefore locally strongly in L^r for any $r \in (1, p^\star)$. It follows that (up to extraction of a subsequence) $g_\varepsilon \to f$ almost everywhere. As a consequence, f_ε converges to f almost everywhere. Since $f_\varepsilon \leq f \in L^{p^\star}$, by dominated convergence theorem $f_\varepsilon \to f$ in $L^{p^\star}$ Similarly (or as a consequence of the L^p convergence of f_ε to f) ∇f_ε converges to ∇f in distributional sense on $\mathbb{R}^n$, and is also bounded in L^p, so ∇f_ε converges weakly in L^p to ∇f. On the other hand, again because $f_\varepsilon \leq f$, we know that $(f_\varepsilon)^{p^\star/q}||\nabla\phi|| \in L^q$. So, by dominated convergence, $(f_\varepsilon)^{p^\star/q}\nabla\phi$ converges (strongly) in L^q to $f^{p^\star/q}\nabla\phi$. Thus we can pass to the limit as $\varepsilon \to 0$ in the right-hand side of (47), and by Fatou's lemma we obtain

$$\int f^{p^\star(1-1/n)}\,\Delta\phi \leq -c_{n,p}\lim_{\varepsilon\to 0}\int (f_\varepsilon)^{p^\star/q}\,\nabla f_\varepsilon\,.\nabla\phi = -c_{n,p}\int f^{p^\star/q}\,\nabla f\cdot\nabla\phi.$$

This concludes the proof of (44).

Remark (3.1.10)[80]. No strict convexity of the norm is required for (49), as shown by the following short argument. Let $\lambda > 0$ satisfy

$$\left(\int ||X||_*^p\right)^{1/p}\left(\int ||\mathrm{Y}||^q\right)^{1/q} = \frac{\lambda^{-p}}{p}\left(\int ||X||_*^p\right) + \frac{\lambda^q}{q}\left(\int ||\mathrm{Y}||^q\right).$$

Then, equality in Hölder's inequality (11) implies a pointwise (almost everywhere) equality in Young's inequality (10). When there is equality in Young's inequality, the function $\alpha(t) := (X\cdot Y)t - \left(\lambda^{-p}||X||_*^p/p\right)t^p$ achieves its maximum at $t = 1$, and therefore $\lambda^{-p}||X||_*^p = \lambda^q||\mathrm{Y}||^q$. This implies (49) with $k = \lambda^{p+q}$.

In the case $g = h_p > 0$, once the strict positivity of f has been proven, it is possible to appeal to Caffarelli's interior regularity results [90] for solutions of the Monge-Ampère equation, in order to conclude directly that $\phi \in W^{2,\alpha}_{1\mathrm{oc}}(\alpha > 1)$. This argument also implies that $D^2_{\mathcal{D}},\phi$ has no singular part; it has however the drawback to rely on very sophisticated results.

If we look for extremal g's in (17), we can set $f = h_p$ in (16) and check for equality cases there. From Proposition (3.1.8) we know that g has to be a dilationtranslation image of h_p, and that $\nabla\phi$ is a dilation ⊣ranslation map. But, as in the proof of Proposition (3.1.8), equality in Hölder's inequality with $f = h_p$ implies $||\nabla\phi(x)|| = \lambda||x||$ almost everywhere, for some $\lambda > 0$ (see (50)). Therefore $\nabla\phi(x) = \pm\lambda x$, and the only cases of equality are dilations of h. In (29) with $f = h_p$, the only equality cases will again be the dilation ⊣ranslation images of h_p, as in Theorem (3.1.7).

Replacing Hölder's inequality by Young's inequality—in fact, we eventually used the cases of equality in Young's inequality! – in the proof of Proposition (3.1.8), we can conclude that for equality to hold in (30), it is necessary that f be a translationdilation image of g.
It was pointed out to us by Maggi [104] that the technicalities encountered above can be greatly simplified if one restricts to radially symmetric functions. Indeed, in this case we have to deal with a one-dimensional transportation problem, which is completely elementary. The interest of this Remark (3.1.10) lies in the fact that it is often possible, for many variational problems, to show a priori that optimal functions have to be radially symmetric around some point, by sharp rearrangement inequalities (in this case, the Brothers-Ziemer theorem). Once this reduction has been performed, the classical procedure for the identification of extremals is still somewhat subtle, and even in this context the mass transportation argument leads to substantial simplifications. On the other hand, these sharp rearrangement inequalities are in general nontrivial. A proof of the Brothers-Ziemer theorem for general norms has been recently announced by Ferone and Volpicelli (after a similar result for strictly convex norms, by Esposito and Trombetti); by combining this with Maggi's Remark (3.1.10), one can devise an alternative proof of Theorem (3.1.7).

Section (3.2): On the Half-Space

For $n \geq 3$ be an integer and p a real in (1,n) . The classical Sobolev inequality in $\mathbb{R}^n$ asserts the existence of an universal positive constant $K_n(p)$, such that, for all function f in the Sobolev space $\mathrm{W}^{1,p}(\mathbb{R}^n)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq K_n(p)\|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{53}$$

with $p^* = \frac{np}{n-p}$.

In fact, the space $\mathrm{W}^{1,p}$ is not well adapted to study Sobolev inequalities on $\mathbb{R}^n$, and should be replaced by the homogeneous Sobolev space

$$\dot{\mathrm{W}}^{1,p}(\mathbb{R}^n) = \{f \in L^{p^*}(\mathbb{R}^n), \mathrm{s.t.}\, \nabla f \in L^p(\mathbb{R}^n)\},$$

or equivalently (see [23]), the space of functions vanishing at infinity with their gradient in $L^p(\mathbb{R}^n)$. We will say that a measurable function: $\mathbb{R}^n \to \mathbb{R}$ vanishes at infinity if

$$\forall a > 0, mes\ (\{x \in \mathbb{R}^n; |f(x)| \geq a\}) < +\infty. \tag{54}$$

The best constant in (53), together with extremal functions, was found in 1974 independently by Aubin [82] and Talenti [126] by classical variational methods. By a symmetrization argument, the problem is reduced to the dimension one, and then solved by ODE techniques. The extremal functions, up to dilations and translations, take then the following form,

$$f(y) = (\varepsilon^q + |y|^q)^{1-\frac{n}{p}}, \varepsilon > 0, \frac{1}{p} + \frac{1}{q} = 1. \tag{55}$$

Consider now a smooth bounded domain $\Omega \subset \mathbb{R}^n$. The inequality (53) does not hold anymore, because of the constant functions. Then, a natural strategy is to replace the condition (54) on the functions by a vanishing condition on the boundary, that is to work on the space $W_0^{1,p}(\Omega)$. In that setting, Guedda and Veron proved in [122] that, for a bounded and starpshaped domain Ω, the corresponding inequality does not admit extremal function. In [118], F. Demengel and E. Hebey studied a modified version:

$$I_p(\Omega) = \inf_{u\in W_0^{1,p}(\mathbb{R}^n), \int_\Omega f(x)|u(x)|dx=1} \left(\int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)\, dx \right). \quad (55)$$

They proved that the existence of extremal functions to (55) is related to the following condition

$$I_p(\Omega) K_n(p)^p \left(\max_{x\in\Omega} f(x) \right)^{1-\frac{n}{p}} < 1. \quad (56)$$

The value of $I_p(\Omega)$ being unknown, they use the extremals in (53) in order to get explicit conditions on the domain and the data , f, by localization around a point of maximum of f. A generalization of (55) by Demengel leads to the subject of the present Replacing the constraint in (55) by

$$u \in W^{1,p}(\Omega)\,; \int_{\partial\Omega} f(x)|u(x)|^{\tilde{p}} dx = 1,$$

where $\tilde{p} = \frac{(n-1)p}{n-p}$ is the critical exponent for Sobolev imbeddings into trace spaces, an existence condition similar to (56) is found, the reference constant $K_n(p)$ being replaced by the best constant in the following Sobolev inequality on $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$,

$$\|f\|_{L^{\tilde{p}}(\partial\mathbb{R}_+^n)} \leq Q_n(p) \|\nabla f\|_{L^p(\mathbb{R}_+^n)}. \quad (57)$$

If we know the extremals in (57), the work done in [118] can be adapted to get explicit existence conditions of minimizers. Notice that the existence of extremals for (57) has been established by Lions in [124] using the concentration-compactness principle. In the case where $p = 2$, these extremals (and then the best constant) have been computed by Escobar in [120], and further by Beckner in [85]. Unfortunately, their proofs both deeply used the conformal invariance of the associated variational problem, then cannot be generalized to other values of p and the problem was still open. Escobar conjectured that the functions

$$\forall (t,x) \in \mathbb{R}_+^n, f_\lambda(t,x) = \frac{1}{((t+\lambda)^2 + |x|^2)^{\frac{n-p}{2(p-1)}}}, \lambda > 0 \quad (58)$$

are optimal in (57). This is natural since they solve the PDE

$$\begin{cases} -\mathrm{div}(|\nabla f|^{p-2}\nabla f) = 0 & in\ \mathbb{R}_+^n \\ |\nabla f|^{p-2}\nabla f \cdot \mathrm{n} = -Q_n'(p) f^{\frac{n(p-1)}{n-p}} & on\ \partial\mathbb{R}_+^n. \end{cases}$$

The difficulty is to prove that $Q_n'(p) = Q_n(p)^{-p}$. It is impossible to reduce the problem to the dimension 1, since symmetrization arguments only apply with respect to the x variable, and don't really simplify it (see [123] for such a construction). Using the particular form of the expected extremals, the proof of this result uses a mass transportation method, and is deeply inspired by [80] and more recently [125]. In [125], Maggi and Villani proved an optimal inequality valid on all locally Lipschitz domains Ω:

$$\|f\|_{L^{p^*}(\Omega)} \leq K_n(p)\|\nabla f\|_{L^p(\Omega)} + T_n(p)^{-1}\|f\|_{L^{\bar{p}}(\partial\Omega)}, \tag{59}$$

where $\tilde{p} = \frac{(n-1)p}{n-p}$ (this exponent is the critical ones for the Sobolev embedding into L^q space on the boundary), and $T_n(p)^{-\overline{p}}$ is the isoperimetric constant. In addition, they showed that (59) is sharp on balls. This generalizes in particular a result of Brezis and Lieb in [116], correponding to $p = 2$.
The results of [125] stand in the continuation of the considerable efforts which have been spent in the last years to get a better understanding of Sobolev type inequalities, with the introduction of mass transportation concepts in the field. These methods are not new and, for example, are somehow used in all known proofs of the Brünn-Minkowski inequality:

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}, \tag{60}$$

for all measurable sets A and B in $\mathbb{R}^n$. [125], Gromov gave a proof of the isoperimetric inequality based on the Knothe transport map between two probability densities. After the work of Brenier on the polar factorization [115] and more especially of McCann on the Monge-Kantorovitch optimal transportation problem [125], Otto and Villani found deep links between $\log$-Sobolev inequality and the asymptotic profiles in the FokkerPlanck equation. In [95], Dolbeault and Del Pino used the entropy-entropy production method to prove that asymptotics in nonlinear diffusion equations were exactly the extremals of a one-parameter family of Gagliardo-Nirenberg inequalities, with (53) as a limit case, and obtained these extremals by similar methods as Aubin and Talenti. Inspired on the one hand by this work, and on the other hand by the direct proof using a mass transportation method of Gaussian inequalities obtained by Barthe in [84], Cordero-Erausquin, Villani exhibited in [80] a completely new proof of (53) in its optimal form (together with the whole family of Gagliardo-Nirenberg inequalities of [95]). This result is derived from a new duality principle generalized later by Agueh, Ghoussoub and Kang in [114]. The proof of this duality principle is direct, by writing a sequence of elementary inequalities, each of them being optimal. The extremals are then recovered by tracing back the cases of equality at each step. In addition, except from the Brenier map, it only involves arithmetic-geometric and Hölder inequalities. For that reason, the result is obtained for an arbitrary norm on $\mathbb{R}^n$, and it is possible to prove that, up to translations and dilations, the functions given by (55) are the only minimizers in (53), avoiding the use of the moving plane method [121], or competting symmetries of Carlen and Loss [117]. Contrary to the inequalities covered by [80], which have already been proved by variational approach, the inequalities studied in [20] and this work have no other proof. This shows that mass transportation techniques are particularly

adapted to inequalities involving trace terms. Actually, since the proof applies for any norm on $\mathbb{R}_+^n$, it generalizes in the case $p = 2$ the result of [120] and [85], showing that neither conformal invariance nor the Euclidean structure of $\mathbb{R}^n$ play any role in the problem of sharp Sobolev inequalities with trace terms.
We devoted to the proof of Theorem (3.2.1), after a summary of the mass transportation ingredients and some basic considerations about norms on Euclidean spaces we shall give some comments about uniqueness and results which can be derived from the proof, especially when interesting to arbitrary domains.
Theorem (3.2.1)[113]. *Let $n \geq 3$ and p a real in $(1, n)$. Let $\| \,.\, \|$ be a norm on $\mathbb{R}^n$, its dual norm being denoted by $\| \cdot \|_*$. Then, for all f in $W^{1,p}(\mathbb{R}_+^n)$,*

$$\|f\|_{L^{\frac{(n-1)p}{n-p}}(\partial\mathbb{R}_+^n)} \leq Q_n(p)\left(\int_{\mathbb{R}_+^n} \| \nabla f(y)\|_*^p dy\right)^{\frac{1}{p}} \tag{61}$$

with equality in (61) *as soon as, for some $\lambda > 0$ and $x_0 \in \mathbb{R}^{n-1}$,*

$$\forall (t,x) \in \mathbb{R}_+^n, f(t,x) = \left(\frac{\lambda^{\frac{1}{p}}}{\|(t+\lambda, x-x_0)\|}\right)^{\frac{n-p}{(p-1)}} \tag{62}$$

It contains the conjecture of [120], in the particular case of the Euclidean norm.
Proof. First, we give a brief presentation of the key ingredients of the analysis, and recall the main properties of the Brenier map. Let F and G some probability densities on $\mathbb{R}_+^n$. By a result of Brenier [5], further refined by McCann [21], there exists a function $= \nabla\phi : \mathbb{R}_+^n \to \mathbb{R}_+^n$, with ϕ convex, such that T maps the measure $F(y)dy$ onto $G(y)dy$, which means by definition that, for all measurable set B,

$$\int_B G\,(y)dy = \int_{T^{-1}(B)} F\,(y)dy, \tag{63}$$

or, equivalently, for all measurable function $\psi : \mathbb{R}_+^n \to \mathbb{R}$,

$$\int_{\mathbb{R}_+^n} \psi\,(y)G(y)dy = \int_{\mathbb{R}_+^n} \psi\,\big(T(y)\big)F(y)dy. \tag{64}$$

With this properties, T is uniquely determined F-a.e. Note, even if it is not crucial in this work, that T realizes the optimal transportation in the Monge-Kantorovich problem with quadratic cost (see [126]) from $F(y)dy$ to $G(y)dy$. Assuming that the function T is C^1 and realizes a diffeomorphism, which is in general completely unrealistic, a change of variables in (64) leads to the following Monge-Ampère equation:

$$F(y) = G\big(\nabla\phi(y)\big)\mathrm{Det}D^2\phi(y)\,. \tag{65}$$

In the general setting of probability densities, McCann proved in [124] that (65) remains valid at least F almost everywhere, $D^2\phi$ being undestood as the absolutely continuous part of the of the distributional Hessian matrix of ϕ (or in the Alexandrov sense, that is, the quadratic part obtained in a second order Taylor expansion). The only more general regularity result known is due to Caffarelli [88], [89], [90] for $C^{0,\alpha}$ densities with compact convex support, in which case the potential ϕ becomes of classe $C^{2,\alpha}$ and then equation (65) holds in the classical sense. Note, as done in [125], that if the arrival density G is compactly supported in $\mathbb{R}^n_+$, then the potential ϕ can be assumed to have its domain the whole $\mathbb{R}^n$, since it is possible to extend it as a convex function outside $\mathbb{R}^n_+$. The main consequence of this fact is that we can assume that $\nabla\phi$ has bounded variations up to the boundary.

Now, we introduce the underlying structure. Consider the vector space $\mathbb{R}^n$ endowed with some arbitrary norm denoted by $\|\cdot\|$. Then, the dual norm on $(\mathbb{R}^n)^* = \mathbb{R}^n$ is defined for all $X \in \mathbb{R}^n$ by

$$\|X\|_* = \sup_{\|Y\|=1} \; X \cdot Y.$$

It means that $\|\cdot\|_*$ is the conjugate function of the convex function $\|\cdot\|$. With this setting, the Hölder inequality holds: $\forall f : (\mathbb{R}^n)^* \to \mathbb{R}$ in L^p, and $\forall g\colon \mathbb{R}^n \to \mathbb{R}$ in L^q, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{\mathbb{R}^n} f \cdot g \leq \left(\int_{\mathbb{R}^n} \|f\|_*^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \|g\|^q\right)^{\frac{1}{q}} \tag{66}$$

which expresses the fact that $L^p((\mathbb{R}^n)^*, \|\cdot\|_*)$ is the dual space of $L^q(\mathbb{R}^n, \|\cdot\|)$. In addition, the norm function is differentiable at all $X \in \mathbb{R}^n$, $x \neq 0$, and for all such x, $\nabla(\|\cdot\|)(x)$ is the unique vector x^* such that

$$x^* \cdot x = \|x\|.$$

More generally, for a given differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, its gradient with respect to the norm $\|\cdot\|$ lives in $(\mathbb{R}^n)^*$ and its $\|\cdot\|_*$ norm corresponds to the norm of the derivative of f as a linear map.

We set, for $p \in (1,n)$, and for $y = (t,x) \in \mathbb{R}^n_+ = \mathbb{R} +\times \mathbb{R}^{n-1}$,

$$\overline{f}(y) = C_n(p)\|y - e\|^{\frac{p-n}{p-1}}, \tag{67}$$

where $e = (-1,0)$ and

$$C_n(p) = \left(\frac{n}{(p-1)I_n(p)}\right)^{\frac{1}{p}-\frac{1}{n}} \; I_n(p) = \int_{\mathbb{R}^{n-1}} \|(1,x)\|^{-nq} dx. \tag{68}$$

This normalization makes the function $\overline{f}^{\frac{np}{n-p}}$ to be a probability density. Let us remark that it is sufficient to prove that $\overline{f}$ is extremal in (59), in order to get that all the functions given by (62) are also extremals by the scaling invariance. The key point will be that $\overline{f}$ realizes the equality in (66) with

$$f(y) = \overline{f}(y)^{\frac{n(p-1)}{n-p}}(y-e) \; and \; g(y) = -\nabla\overline{f}(y)\,. \tag{69}$$

Note that it is sufficient to prove (59) for non negative functions by taking $|f|$ in the inequality. We begin by establishing with the techniques of [80], [125] an inequality valid

on two arbitrary smooth and compactly supported probability densities F and G. Then, in a second part, we apply the inequality to $F = f^{\frac{np}{n-p}}$ and $G = g^{\frac{np}{n-p}}$, where f and g are still supposed to be non negative, smooth and compactly supported in $\mathbb{R}^n_+$. At this stage, we shall remove the assumptions by a density argument and the final step will then lead to the conclusion.
Consider F and G two smooth and compactly supported probability densities on $\mathbb{R}^n_+$, and let $\nabla\phi$ the Brenier transportation map between $F(y)dy$ and $G(y)dy$. What follows is deeply inspired from [80], [125], and only differs by the treatment of the trace term. Then, we write

$$\int_{\mathbb{R}^n_+} G\,(y)^{1-\frac{1}{n}}dy = \int_{\mathbb{R}^n_+} G\left(\nabla\phi(y)\right)^{-\frac{1}{n}}F(y)dy,$$

by the definition of the Brenier map. Then, using the Monge-Ampère equation (65), we have

$$\int_{\mathbb{R}^n_+} G\,(y)^{1-\frac{1}{n}}dy \;=\; \int_{\mathbb{R}^n_+} \left(\mathrm{Det}D^2\phi(y)\right)^{\frac{1}{n}} F(y)^{1-\frac{1}{n}}dy,$$

$$\leq \frac{1}{n}\int_{\mathbb{R}^n_+} \triangle\,\phi(y)F(y)^{1-\frac{1}{n}}dy,$$

thanks to the arithmetic-geometric inequality. In the previous inequality, the Laplacian is understood as the absolutely continuous part of the distributional Laplacian. But, since ϕ is convex, its distributional Hessian is a measure with values in the set of the semi-definite positive matrix. Then, it follows that

$$\int_{\mathbb{R}^n_+} G\,(y)^{1-\frac{1}{n}}dy \leq \frac{1}{n}\int_{\mathbb{R}^n_+} \triangle_{D'}\,\phi(y)F(y)^{1-\frac{1}{n}}dy.$$

We introduce $\psi(y) = \phi(y) - e\cdot y$. Then, ϕ and ψ share the same Laplacian. We have that $\in \mathrm{B}V_{1oc}(\mathbb{R}^n_+)$, and since F is smooth, we can use an integration by parts formula for BV functions. This leads to

$$n\int_{\mathbb{R}^n_+} G\,(y)^{1-\frac{1}{n}}dy \leq -\int_{\mathbb{R}^n_+} \nabla\,\psi\cdot\nabla\left(F^{1-\frac{1}{n}}\right)(y)dy + \int_{\partial\mathbb{R}^n_+} F\,(y)^{1-\frac{1}{n}}\nabla\psi\cdot \mathrm{n}, \quad (70)$$

where n denotes the exterior normal vector, and in this case we have $\mathrm{n} = e$ at each point of the boundary. In addition, by the definition of the mass transportation, $\nabla\phi(y) \in \mathbb{R}^n_+$, for all y, which exactly means that $\nabla\phi$ satisfies $\nabla\phi(y)\cdot \mathrm{n} \leq 0$. Since, $e\cdot e = 1$, we get

$$\int_{\partial\mathbb{R}^n_+} F\,(y)^{1-\frac{1}{n}}dy + n\int_{\mathbb{R}^n_+} G\,(y)^{1-\frac{1}{n}}dy \leq -\int_{\mathbb{R}^n_+} (\nabla\phi(y) - e)\cdot\nabla\left(F^{1-\frac{1}{n}}\right)(y)dy. \quad (71)$$

The next step consists in considering two non negative, smooth and still compactly supported functions f and g such that

$$\int_{\mathbb{R}_+^n} f(y)^{\frac{np}{n-p}} dy = \int_{\mathbb{R}_+^n} g(y)^{\frac{np}{n-p}} dy = 1,$$

and setting $F = f^{\frac{np}{n-p}}$ and $G = g^{\frac{np}{n-p}}$. Then, (3.72) becomes

$$\int_{\partial\mathbb{R}_+^n} f(y)^{\frac{(n-1)p}{n-p}} dy$$

$$\leq -\frac{(n-1)p}{n-p}\int_{\mathbb{R}_+^n} f(y)^{\frac{n(p-1)}{n-p}} (\nabla\phi(y) - e)\cdot\nabla f(y)dy - n\int_{\mathbb{R}_+^n} g(y)^{\frac{(n-1)p}{n-p}} dy.$$

Using the definition of the dual norm and then applying Hölder inequality, it follows that

$$\int_{\partial\mathbb{R}_+^n} f(y)^{\frac{(n-1)p}{n-p}} dy \leq \frac{(n-1)p}{n-p}\left(\int_{\mathbb{R}_+^n} \| \nabla\phi(y) - e\|^q f(y)^{\frac{np}{n-p}} dy\right)^{1/q}\left(\int_{\mathbb{R}_+^n} \| \nabla f(y)\|_*^p dy\right)^{\frac{1}{p}}$$

$$-n\int_{\mathbb{R}_+^n} g(y)^{\frac{(n-1)p}{n-p}} dy,$$

and actually, by the definition of the transport map,

$$\int_{\partial\mathbb{R}_+^n} f(y)^{\frac{(n-1)p}{n-p}} dy \leq \frac{(n-1)p}{n-p}\left(\int_{\mathbb{R}_+^n} \| y - e\|^q g(y)^{\frac{np}{n-p}} dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}_+^n} \| \nabla f(y)\|_*^p dy\right)^{\frac{1}{p}}$$

$$-n\int_{\mathbb{R}_+^n} g(y)^{\frac{(n-1)p}{n-p}} dy. \qquad (72)$$

At this stage, it is necessary to remove the compactness and smoothness hypothesis. Let us make two remarks. First, (72) has been established only for compactly supported functions, but since the transport map $\nabla\phi$ does not appear anymore, by density arguments, it remains valid as soon as each term is defined, typically for $f \in \dot{W}^{1,p}(\mathbb{R}_+^n)$ and $g \in L^{p^*}(\mathbb{R}_+^n)$ (Indeed, the compactness hypothesis on g was necessary to get sufficient regularity on $\nabla\phi$ for the integration by parts). On the other hand, this beeing done, if we set $f = g = \overline{f}$, then, the transport map becomes the identity, so each step can be performed and provides an equality. As a conclusion for this step, (72) holds for any functions $f \in \dot{W}^{1,p}(\mathbb{R}_+^n)$ and $\in L^{p^*}(\mathbb{R}_+^n)$, and is an equality for $f = g = \overline{f}$.

Now, we can finish the proof. Let us set $g = \overline{f}$ in (3.73). This leads to, for all non negative f in $\dot{W}^{1,p}(\mathbb{R}_+^n)$ such that $\|f\|_{L^{p^*}(\mathbb{R}_+^n)} = 1$,

$$\int_{\mathbb{R}_+^n} f(y)^{\frac{(n-1)p}{n-p}} dy \le A_n(p)\|\nabla f\|_{L^p(\mathbb{R}_+^n)} - B_n(p), \tag{73}$$

with

$$A_n(p) = \frac{(n-1)p}{n-p}\left(\int_{\mathbb{R}_+^n} \| y - e\|^q \bar{f}(y)^{\frac{np}{n-p}} \mathrm{dy}\right)^{\frac{1}{q}} = \frac{(n-1)p}{n-p}\left(\frac{p-1}{n-p}\right)^{\frac{1}{q}} C_n(p)^{\frac{n(p-1)}{n-p}} J_n(p)^{\frac{1}{q}}$$

and

$$B_n(p) = n\int_{\mathbb{R}_+^n} \bar{f}(y)^{\frac{(n-1)p}{n-p}} dy = \frac{n(p-1)}{n-p} C_n(p)^{\frac{(n-1)p}{n-p}} J_n(P),$$

where

$$J_n(p) = \int_{\mathbb{R}^{n-1}} \| (1, x)\|^{-q(n-1)} dx.$$

Here, we remove the normalization. For all $f \in \dot{\mathrm{W}}^{1,p}(\mathbb{R}_+^n)$, the inequality (73) reads

$$K(f)^{\frac{(n-1)p}{n-p}} Q(f) \le A_n(p)K(f) - B_n(p)\,,$$

where we set

$$K(f) = \frac{\|\nabla f\|_{L^p(\mathbb{R}_+^n)}}{\|f\|_{L^{p^*}(\mathbb{R}_+^n)}}, and\ Q(f) = \frac{\|f\|^{\bar{p}}_{L^{\bar{p}}(\partial\mathbb{R}_+^n)}}{\|\nabla f\|^{\tilde{p}}_{L^p(\mathbb{R}_+^n)}},$$

hence

$$Q(f) \le K(f)^{-\frac{(n-1)p}{n-p}}\big(A_n(p)K(f) - B_n(p)\big)\,.$$

The function $H: k \mapsto k^{-\frac{(n-1)p}{n-p}}\big(A_n(p)k - B_n(p)\big)$ achieves its maximum on $\mathbb{R}^+$ at the point

$$k_0 = \frac{p(n-1)}{n(p-1)}\,\frac{B_n(p)}{A_n(p)}$$

$$= \left(\frac{n-p}{p-1}\right)^{\frac{1}{q}} C_n(P)J_n(P)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}_+^n} \| \nabla\bar{f}\|_*^p\right)^{\frac{1}{p}} = K(\bar{f})\,,$$

by a simple computation. As a conclusion, for all $f \in \dot{\mathrm{W}}^{1,p}(\mathbb{R}_+^n)$,

$$Q(f) \le H(k_0)\,,$$

with equality for $f = \bar{f}$, which ends the proof. A natural question arising here concerns the identification of all the minimizers in (60). Following [80], the guess would be that, up to dilations and multiplication by constants, the function given by (62) are the unique

minimizers in (59). Unfortunately, it supposes that we are able to perform the integration by parts (70), only assuming that $f \in W^{1,p}(\mathbb{R}^n_+)$, which is not really a problem, but also $g \in L^{p^*}(\mathbb{R}^n_+)$ not necessarily *compactly supported*. In that case, the normal derivative of ϕ has no reason even to exist on the boundary.
If we don't introduce the vector e in (70), then the trace term does not appear anymore, and, replacing $\overline{f}$ by the minimizers in the corresponding sharp Sobolev inequality in $\mathbb{R}^n$, we can end the proof in the same way as in [80] and get the Sobolev inequality (53) for $\mathbb{R}^n_+$, which means that $\mathbb{R}^n_+$ is a *gradient domain* in the sense of [125] (This was already proved in [124]). It is not the only one, and a natural criterion can be directly derived from the proof. Indeed, it is easy to see that, if $\Omega \subset \mathbb{R}^n$ satisfies that there exists some $y_0 \in \Omega$ such that, $\forall y \in \Omega$ and $x \in \partial\Omega$,

$$(y - y_0) \cdot \mathrm{n}_x \leq 0, and\ (x - y_0) \cdot \mathrm{n}_x = 0,$$

where n stands for the exterior normal of the boundary at the point x, then Ω is a gradient domain, and the minimizers takes the form

$$f(y) = (\|y - y_0\|^q + \sigma^q)^{1-\frac{p}{n}}, \sigma > 0.$$

In particular, this is the case for the following conical subsets of $\mathbb{R}^n_+$,

$$\Omega = \left\{(t, x) \in \mathbb{R}^n_+; F\left(\frac{x}{t}\right) \leq 0\right\},$$

for any convex function $F: \mathbb{R}^{n-1} \to \mathbb{R}$.

Chapter 4
Generalised Gagliardo–Nirenberg Inequalities

We provide a brief primer of some basic concepts in harmonic analysis, including weak spaces, the Fourier transform, the Lebesgue Differentiation Theorem, and Calderon–Zygmund decompositions. We obtained by using interpolation spaces, the precise form of the inequalities stated here appears to be novel and, moreover, the proofs given in the present are self-contained (save for the use of the John–Nirenberg inequality for the BMO result) in contrast to the other mentioned approach. The use of $\dot{C}^{\eta}$ Hölder spaces in such Gagliardo–Nirenberg inequalities seems to be new.

Section (4.1): Weak Lebesgue Spaces and BMO

The Gagliardo-Nirenberg interpolation inequality (Nirenberg [23])

$$\|f\|_{L^p} \leq c\|f\|_{L^q}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta}, 1 \leq q < p < \infty, \frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right) \quad (1)$$

is an extremely useful tool in the analysis of many partial differential equations. In particular, in the mathematical theory of the two-dimensional Navier-Stokes equations it is frequently encountered in the form of Ladyzhenskaya's inequality (Ladyzhenskaya [141])

$$\|f\|_{L^4} \leq c\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} \quad (2)$$

provides an introduction to some of the basic ideas of harmonic analysis, as a means of generalising the Gagliardo-Nirenberg inequality in two directions.

First, using only simple properties of the weak L^p spaces and the Fourier transform we show that one can replace the L^q norm on the right-hand side of (1) by the norm in the weak L^q space:

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta} \quad (3)$$

Along the way we also provide a proof of various forms of Young's inequality for convolutions and the endpoint Sobolev embedding $\dot{H}^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $s = n(1/2 - 1/p)$, $2 < p < \infty$.

We note that, in particular, (3) provides the following generalisation of the 2D Ladyzhenskaya inequality:

$$\|f\|_{L^4} \leq c\|f\|_{L^2}^{1/2\infty}\|\nabla f\|_{L^2}^{1/2} \quad (4)$$

How this inequality is relevant for an analysis of the coupled system

$$-\triangle u + \nabla p = (B \cdot \nabla)B, \nabla \cdot u = 0,$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} + \eta \triangle B + (u \cdot \nabla)B = (B \cdot \nabla)u, \nabla \cdot B = 0,$$

on a two-dimensional domain (for full details see McCormick et al. [143]). This system arises from the theory of magnetic relaxation for the generation of stationary Euler flows (see Moffatt [144]), and was our original motivation for pursuing generalisations of (2) and then of (1).

Related to the case $s = n/2$ in (1), Chen & Zhu [11] (see also Azzam & Bedrossian [11]; Dong & Xiao [132]; Kozono & Wadade [5]) obtain the inequality

$$\|f\|_{L^p} \leq c\|f\|_{L^q}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}, \tag{5}$$

where BMO is the space of functions with bounded mean oscillation. This inequality Grafakos [136]) is stronger than (1) since $\|f\|_{\mathrm{BMO}} \leq c\|f\|_{\dot{H}^{n/2}}$. In fact for $q > 1$ one can obtain a stronger inequality still, weakening the L^q norm on the right-hand side as we did in our transition from (1) to (3):

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}. \tag{6}$$

We adapt the proof used in [11] for (5) to prove (6); their argument makes use of the John-Nirenberg inequality for functions in BMO, which is proved via a Calderon-Zygmund type decomposition This decomposition in turn makes use of the Lebesgue Differentiation Theorem.

One can prove (6), and a slightly stronger inequality involving Lorentz spaces,

$$\|f\|_{L^{p,1}} \leq c\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}, 1 < q < p < \infty,$$

using the theory of interpolation spaces (as in McCormick et al. [143]); see Corollary (4.1.17) (and also Kozono et al. [140]). We briefly recall the theory of interpolation spaces and give a proof of this inequality.

Since it provides one of the main applications of weak L^p spaces, we include a final that contains a statement of the Marcinkiewicz interpolation theorem and some of its consequences, including a strengthened form of Young's inequality. A very readable account of all the harmonic analysis included here can be found in the two books by Grafakos [135], [136].

We attempt to find the optimal constants for our inequalities, and throughout we treat functions defined on the whole of $\mathbb{R}^n$. Similar results for functions on bounded domains are more involved, since one requires carefully tailored extension theorems (see Azzam & Bedrossian [128]).

We begin with the definition of the weak L^p spaces and quick proofs of some of their properties. For more details see Chapter 1 of Grafakos [135].

For a measurable function $f: \mathbb{R}^n \to \mathbb{R}$ define the *distribution function of* f by

$$d_f(\alpha) = \mu(\{x: |f(x)| > \alpha\}),$$

where $\mu(A)$ (or later $|A|$) denotes the Lebesgue measure of a set A. It follows using Fubini's Theorem that

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x = p\int_{\mathbb{R}^n}\int_0^{|f(x)|} \alpha^{p-1}\,\mathrm{d}\alpha\mathrm{d}x = p\int_0^\infty \alpha^{p-1}\,d_f(\alpha)\mathrm{d}\alpha. \tag{7}$$

For $1 \leq p < \infty$ set

$$\|f\|_{L^{p,\infty}} = \inf \left\{C: d_f(\alpha) \le \frac{C^p}{\alpha^p}\right\}$$
$$= \sup \{\gamma d_f(\gamma)^{1/p}: \gamma > 0\}.$$

The space $L^{p,\infty}(\mathbb{R}^n)$ consists of all those f such that $\|f\|_{L^{p,\infty}} < \infty$. It follows immediately from the definition that

$$f \in L^{p,\infty}(\mathbb{R}^n) \Rightarrow d_f(\alpha) \le \|f\|_{L^{p,\infty}}^p \alpha^{-p} \tag{8}$$

and that for any f and g

$$d_{f+g}(\alpha) \le d_f(\alpha/2) + d_g(\alpha/2), \tag{9}$$

which implies that

$$\|f+g\|_{L^{p,\infty}} \le 2(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}). \tag{10}$$

The following simple lemma (the proof is essentially that of Chebyshev's inequality) is fundamental and shows that any function in L^p is also in $L^{p,\infty}$

Lemma (4.1.1)[127]. *If* $\in L^p(\mathbb{R}^n)$, *then* $f \in L^{p,\infty}(\mathbb{R}^n)$ *and* $\|f\|_{L^p}, \infty \le \|f\|_{L^p}$.

Proof. This follows since

$$d_f(\alpha) = \int_{\{x:} |f(x)| > \alpha\}^{1 dx \le} \int_{\{x:} |f(x)| > \alpha\} \frac{|f(x)|^p}{\alpha^p} dx \le \|f\|_{L^p}^p \alpha^{-p}.$$

While $L^p \subset L^{p,\infty}$, clearly $L^{p,\infty}$ is a larger space than L^p: for example,

$$|x|^{-n/p} \in L^{p,\infty}(\mathbb{R}^n) \tag{11}$$

but this function is not an element of $L^p(\mathbb{R}^n)$.

An immediate indication of why these spaces are useful is given in the following simple result, which shows that in the L^p interpolation inequality

$$\|f\|_{L^r} \le \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}, \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

one can replace the Lebesgue spaces on the right-hand side by their weak counterparts.

Lemma (4.1.2)[127]. *Take* $1 \le p < r < q \le \infty$. *If* $f \in L^{p,\infty} \cap L^{q,\infty}$, *then* $f \in L^r$ *and*

$$\|f\|_{L^r} \le c_{p,r,q} \|f\|_{L^{p,\infty}}^\theta \|f\|_{L^{q,\infty}}^{1-\theta},$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

If $q = \infty$, *we interpret* $L^{\infty,\infty}$ *as* L^∞

Proof. We give the proof when $q < \infty$; the proof when $q = \infty$ is slightly simpler. If $f \in L^{p,\infty}$, then $d_f(\alpha) \le \|f\|_{L^{p,\infty}}^p \alpha^{-p}$, so for any x we have

$$\|f\|_{L^r}^r = r \int_0^\infty \alpha^{r-1} d_f(\alpha) d\alpha$$

$$\le r\int_0^x \alpha^{r-1}\,\|f\|_{L^{p,\infty}}^p\alpha^{-p}\mathrm{d}\alpha + r\int_x^\infty \alpha^{r-1}\,\|f\|_{L^{q,\infty}}^q\alpha^{-q}\mathrm{d}\alpha$$

$$\le \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p x^{r-p} + \frac{r}{r-q}\|f\|_{L^{q,\infty}}^q x^{q-r}$$

Now choose

$$x^{p-q} = \frac{||f||_{L^{p,\infty}}^p}{||f||_{L^{q,\infty}}^q}$$

to equalise the dependence of the two terms on the right-hand side on the weak norms.

The Schwartz space $\mathcal{S}$ of rapidly decreasing test functions consists of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup|x^\beta\partial^\alpha\varphi| \le M_{\alpha,\beta} \text{ for all } \alpha, \beta \ge 0,$$

where α, β are multi-indices.
For any $f \in \mathcal{S}$ one can define the Fourier transform1

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x \tag{12}$$

It is straightforward to check that

$$\mathcal{F}[\partial^\alpha f](\xi) = (2\pi\mathrm{i})^{|\alpha|}\xi^\alpha\hat{f}(\xi) \quad \text{and} \quad \mathcal{F}\left[x^\beta f\right](\xi) = (-2\pi\mathrm{i})^{|\beta|}\left[\partial^\beta\hat{f}\right](\xi)\,,$$

from which it follows that $\mathcal{F}$ maps $\mathcal{S}$ into itself.

Given the Fourier transform of f, one can reconstruct f by essentially applying the Fourier transform operator once more:

$$f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}\xi\cdot x}\hat{f}(\xi)\mathrm{d}\xi. \tag{13}$$

If we define $\sigma(f)$ by $(f)(x) = f(-x)$, then we can write the inversion formula more compactly as $f = \sigma\circ\mathcal{F}(\hat{f})$. We define $\mathcal{F}^{-1} = \sigma\circ\mathcal{F}$, the point being that when we can meaningfully extend the definition of $\mathcal{F}$ and σ we will retain this inversion formula.

An obvious extension of the Fourier transform is to any function $\in L^1(\mathbb{R}^n)$, using the integral definition in (12) directly. Since

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\mathrm{d}x = \|f\|_{L^1}$$

it follows that $\mathcal{F}$ maps L^1 into L^∞. Furthermore, there is a natural definition of the Fourier transform for $\in L^2(\mathbb{R}^n)$. Given $f \in \mathcal{S}$,

$$\|\hat{f}\|_{L^2} = \int_{\mathbb{R}^n}\overline{\hat{f}(x)}\left(\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x}f(\xi)\mathrm{d}\xi\right)\mathrm{d}x$$

$$= \int_{\mathbb{R}^n} f(\xi)\left(\int_{\mathbb{R}^n}\overline{\hat{f}(x)\mathrm{e}^{2\pi\mathrm{i}\xi\cdot x}}\,\mathrm{d}x\right)\mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \overline{f(\xi)}\, f(\xi) \mathrm{d}\xi = \|f\|_{L^2}^2.$$

Now given any $f \in L^2$, one can write $f = \lim_{n\to\infty} f_n$, where $f_n \in \mathcal{S}$ and the limit is taken in L^2.

It follows that $\hat{f}_n$ is Cauchy in L^2, and we identify its limit as $\hat{f}$. So we can define $\mathcal{F}: L^2 \to L^2$, with $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

The Fourier transform can therefore be defined (by linearity) for any $f \in L^1 + L^2$; f can be recovered from $\hat{f}$ using $\mathcal{F}^{-1}$ if $\hat{f} \in L^1 + L^2$, and if $f \in L^1$ (in particular if $\hat{f} \in \mathcal{S}$) , then we can use the integral form of the Fourier inversion formula (13) to give f pointwise as an integral involving $\hat{f}$.

Given this, we can in fact define the Fourier transform if $f \in L^{r,\infty}$ for some $1 < r < 2$ (and in particular if $f \in L^r$), by splitting f into two parts, one in L^1 and one in L^2. The following lemma gives a more general version of this, which will be useful later. We use χ_P to denote the characteristic function of the set $\{x : P \text{ holds}\}$.

Lemma (4.1.3)[127]. *Take* $1 \le t < r < s \le \infty$, *and suppose that* $g \in L^{r,\infty}$ *For any* $M > 0$ *set*

and

$$g_{M-} = g\chi_{|g|\le M} \; g_{M+} = g\chi_{|g|>M}.$$

Then $g = g_{M-} + 9M +$, *where* $g_{M-} \in L^s$ *with*

$$\|g_{M-}\|_{L^s}^s \le \frac{s}{s-r} M^{s-r} \|g\|_{L^{r,\infty}}^r - M^s d_g(M) \tag{14}$$

if $s < \infty$ *and* $\|g_{M-}\|_{L^\infty} \le M$, *and* $g_{M+} \in L^t$ *with*

$$\|g_{M+}\|_{L^t}^t \le \frac{r}{r-t} M^{t-r} \|g\|_{L^{r,\infty}}^r. \tag{15}$$

Proof. Simply note that

$$d_{g_{M-}}(\alpha) = \begin{cases} 0 & \alpha \ge M \\ d_g(\alpha) - d_g(M) & \alpha < M \end{cases} \tag{16}$$

and

$$d_{9M+}(\alpha) = \begin{cases} d_g(\alpha) & \alpha > M \\ d_g(M) & \alpha \le M. \end{cases} \tag{17}$$

Then using (7), (16), and (8) it is simple to show (14), and (15) follows similarly, using (17) in place of (16).

It is natural to ask what one can say about $\hat{f}$ when $f \in L^p$. We will that $\hat{f} \in L^q$ with (p, q) conjugate, provided that $1 \le p \le 2$. Note, however, that for any $p > 2$ one can find a function in L^p whose Fourier transform is not even a locally integrable function in Grafakos [128]).

One can extend the definition of the Fourier transform further to the space of tempered distributions $\mathcal{S}'$. We say that a sequence $\{\varphi_n\} \in \mathcal{S}$ converges to $\varphi \in \mathcal{S}$ if

$$\sup \left|x^\alpha \partial^\beta (\varphi_n - \varphi)\right| \to 0 \quad \text{for all} \quad \alpha, \beta \ge 0,$$

and a linear functional F on $\mathcal{S}$ is an element of $\mathcal{S}'$ if $\langle F, \varphi_n \rangle \to \langle F, \varphi \rangle$ whenever $\varphi_n \to \varphi$ in $\mathcal{S}$. It is easy to show that for any $\varphi, \psi \in \mathcal{S}$

$$\langle \varphi, \hat{\psi} \rangle = \langle \hat{\varphi}, \psi \rangle,$$

and this allows us to define the Fourier transform for $F \in \mathcal{S}'$ by setting

$$\langle \hat{F}, \psi \rangle = \langle F, \hat{\psi} \rangle \text{ for every } \psi \in \mathcal{S}.$$

Since one can also extend the definition of σ to $\mathcal{S}'$ via the definition $\langle \sigma(F), \psi \rangle = \langle F, \sigma(\psi) \rangle$, the identity $F = \mathcal{F}^{-1}\hat{F}$ still holds in this generality.

Expressions given by convolutions, i.e.,

$$[f \star g](x) = \int_{\mathbb{R}^n} f(y) g(x-y) \mathrm{d}y,$$

occur frequently. It is a fundamental result that $[f \star g]^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$; for $f, g \in \mathcal{S}$ this is the result of simple calculation, which can be extended to $f \in \mathcal{S}, g \in \mathcal{S}'$ via the definition $\langle f \star g, \varphi \rangle = \langle g, \sigma(f) \star \varphi \rangle$.

One of the primary results for convolutions is Young's inequality. Following Grafakos (Theorem 1.2.12 in [11]) we give an elementary proof that uses only Hölder's inequality.

Lemma (4.1.4)[127]. Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}.$$

Then for all $\in L^q$, $g \in L^r$, we have $f \star g \in L^p$ with

$$\|f \star g\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}. \tag{18}$$

Proof. We use p' to denote the conjugate of p. Then we have

$$\tfrac{1}{r} + \tfrac{1}{p} + \tfrac{1}{q} = 1, \ \tfrac{q}{p} + \tfrac{q}{r} = 1, \text{ and } \tfrac{r}{p} + \tfrac{r}{q} = 1.$$

First use Hölder's inequality with exponents r', p, and q':

$$|(f \star g)(x)| \leq \int |f(y)||g(x-y)| \mathrm{d}y$$

$$= \int |f(y)|^{q/r'} \big(|f(y)|^{q/p} |g(x-y)|^{r/p}\big) |g(x-y)|^{r/q'} \mathrm{d}y$$

$$\leq \|f\|_{L^q}^{q/r'} \left(\int |f(y)|^q |g(x-y)|^r \mathrm{d}y\right)^{1/p} \left(\int |g(x-y)|^r \mathrm{d}y\right)^{1/q'}$$

$$= \|f\|_{L^q}^{q/r'} \left(\int |f(y)|^q |g(x-y)|^r \mathrm{d}y\right)^{1/p} \|g\|_{L^r}^{r/q'}$$

Now take the L^p norm (with respect to x):

$$\|f \star g\|_{L^p} \leq \left\| f\|_{L^q}^{\frac{q}{r'}} \right\| g\|_{L^r}^{\frac{r}{q'}} \left(\int \int |f(y)|^q |g(x-y)|^r \mathrm{d}y \mathrm{d}x\right)^{1/p}$$

$$= \|f\|_{L^q}^{q/r'} \|g\|_{L^r}^{r/q'} \|f\|_{L^q}^{q/p} \|g\|_{L^r}^{r/p}$$

$$= \|f\|_{L^q} \|g\|_{L^r}.$$

We will need a version of this inequality that allows L^q on the right-hand side to be replaced by $L^{q,\infty}$. The price we have to pay for this (at least initially) is that we also weaken the left-hand side; and note that we have also lost the possibility of some endpoint values ($r = \infty$ and $p, q = 1, \infty$) that are allowed in (18). In fact one can keep the full L^p norm on the left, provided that $r > 1$; but this requires Proposition (4.1.5) as an intermediate step and the Marcinkiewicz Interpolation Theorem.

Proposition (4.1.5)[127]: Suppose that $1 \leq r < \infty$ and $1 < p, q < \infty$. If $f \in L^{q,\infty}$ and $g \in L^r$ with

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r},$$

then $f \star g \in L^{p,\infty}$ *with*

$$\|f \star g\|_{L^{p,\infty}} \leq c_{p,q,r} \|f\|_{L^{q},\infty} \|g\|_{L^r}. \tag{19}$$

Proof. We follow the proof in Grafakos [135], skipping some of the algebra. We have already introduced the main step, the splitting of f in Lemma (4.1.3). For a fixed $M > 0$ we set $f = f_{M-} + f_{M+}$. Using (14) and Hölder's inequality we obtain

$$|(f_{M-} \star g)(x)| \leq \|f_{M-}\|_{L^{r'}} \|g\|_{L^r} \leq \left(\frac{r'}{r' - q} M^{r'-q} \|f\|^q_{L^{q,\infty}}\right)^{1/r'} \|g\|_{L^r},$$

where (r, r') are conjugate; the right-hand side reduces to $M\|g\|_{L^1}$ if $r = 1$. Note in particular that if

$$M = \left(\alpha^{r'} 2^{-r'} q p^{-1} \|f\|^{-q}_{L^q \infty}, \|g\|^{-r'}_{L^r}\right)^{1/(r'-q)}$$

(or $\alpha/2\|g\|_{L^1}$ if $r = 1$), then $d_{f_{M-}\star g}(\alpha/2) = 0$.

For f_{M+} we can use (15) and apply Young's inequality to yield

$$\|f_{M+} \star g\|_{L^r} \leq \|f_{M+}\|_{L^1} \|g\|_{L^r} \leq \frac{q}{q-1} M^{1-q} \|f\|^q_{L^{q,\infty}} \|g\|_{L^r}.$$

Choosing M as above and using (9) it follows that

$$\begin{aligned} d_{f\star g}(\alpha) &\leq d_{f_{M+}\star g}(\alpha/2) \\ &\leq (2\|f_{M+} \star g\|_{L^r} \alpha^{-1})^r \\ &\leq \left(2qM^{1-q} \|f\|^q_{L^{q,\infty}} \|g\|_{L^r} (q-1)^{-1} \alpha^{-1}\right)^r \\ &= C\|f\|^p_{L^{q,\infty}} \|g\|^p_{L^r} \alpha^{-p}, \end{aligned}$$

which yields (19).

This result has implications, among other things, for the regularity of solutions of elliptic equations. It was mentioned that our study of generalised Gagliardo-Nirenberg inequalities was motivated by the study of a particular coupled system in two dimensions, namely

$$-\triangle u + \nabla p = (B \cdot \nabla)B, \nabla \cdot u = 0,$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} + \eta \triangle B + (u \cdot \nabla)B = (B \cdot \nabla)u, \nabla \cdot B = 0.$$

Formal energy estimates (which can be made rigorous via a suitable regularisation) yield

$$\frac{1}{2}\|B(t)\|_{L^2}^2+\eta\int_0^t\|\nabla B\|_{L^2}^2+\int_0^t\|\nabla u\|_{L^2}^2\leq\frac{1}{2}\|B(0)\|_{L^2}^2,$$

showing in particular that $B\in L^\infty(0,T;L^2)$ when $B(0)\in L^2$. To obtain a similar uniform estimate on u we need to understand the regularity of solutions of the Stokes problem

$$-\triangle u+\nabla p=(B\cdot\nabla)B\ \nabla\cdot u=0$$

when $B\in L^2$. A slightly simpler problem with the same features is
$-\triangle\varphi=\partial_i f$, (3) with $f\in L^1$. It is well known that the solution of $-\triangle\varphi=g$ in $\mathbb{R}^2$ is given by $E\star g$, where

$$E(x)=-\frac{1}{2\pi}\log|x|.$$

Noting (after an integration by parts) that the solution of (3) is given by $\partial_i E\star f$, and that $\partial_i E\in L^{2,\infty}$, it follows from Proposition (4.1.5) that $f\in L^1$ implies that $\varphi\in L^{2,\infty}$. The stronger version of Young's inequality given in Theorem (4.1.20) does not apply when $f\in L^1$, so would not improve the regularity here. Thus to obtain further estimates (in particular on the time derivative of B) we required a version of the Ladyzhenskaya inequality that replaced the L^2 norm of u with the norm of u in $L^{2,\infty}$. Further details can be found in McCormick et al. [143].
In our proof of the inequality

$$\|f\|_{L^p}\leq c\|f\|_{L^{q,\infty}}^{\alpha}\|f\|_{\dot H^s}^{1-\alpha}$$

we will use the endpoint Sobolev embedding $\dot H^s(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)$ for $s=n(1/2-1/p)$ when $2<p<\infty$. We prove this here, in Chemin et al. [131]. Since the Fourier transform maps L^2 isometrically into itself, and
$\mathcal{F}[\partial^\alpha f](\xi)=(2\pi\mathrm{i})^{|\alpha|}\xi^\alpha\hat f(\xi)$,
it is relatively straightforward to show that when s is a non-negative integer

$$\sum_{|\alpha|=s}\|\partial^\alpha f\|_{L^2}^2\simeq\int_{\mathbb{R}^n}|\xi|^{2s}|\hat f(\xi)|^2\mathrm{d}\xi, \tag{20}$$

where we write $a\simeq b$ if there are constants $0<c\leq C$ such that $ca\leq b\leq Ca$.

For any $s\geq 0$, even if s is not an integer, we can $define$[3] the homogeneous Sobolev space $\dot H^s(\mathbb{R}^n)$ using (20):

$$\dot H^s(\mathbb{R}^n)=\{f\in\mathcal{S}':\hat f\in L^1_{1\mathrm{oc}}(\mathbb{R}^n)\text{ and }\textstyle\int_{\mathbb{R}^n}|\xi|^{2s}|\hat f(\xi)|^2\mathrm{d}\xi<\infty\}.$$

For $s<n/2$ this is a Hilbert space with the natural norm

$$\|f\|_{\dot H^s}=\left(\int_{\mathbb{R}^n}|\xi|^{2s}|\hat f(\xi)|^2\mathrm{d}\xi\right)^{1/2}$$

and one can therefore also define $\dot H^s(\mathbb{R}^n)$ in this case as the completion of $\mathcal{S}$ with respect to the $\dot H^s$ norm (that $\dot H^s(\mathbb{R}^n)$ is complete iff $s<n/2$ is shown in Bahouri et al. [2]; the simple example showing that $\dot H^s(\mathbb{R}^n)$ is not complete when $s\geq n/2$ can also be found in

Chemin et al. [5]).

Theorem (4.1.6)[127]. *For* $2 < p < \infty$ *there exists a constant* $c = c_{n,p}$ *such that if* $f \in \dot{H}^s(\mathbb{R}^n)$ *with* $= n(1/2 - l/p)$, *then* $f \in L^p(\mathbb{R}^n)$ *and*

$$\|f\|_{L^p} \le c\|f\|_{\dot{H}^s}. \tag{21}$$

Proof. First we prove the result when $\|f\|_{\dot{H}^s} = 1$. For such an f, write $f = f_{<R} + f_{>R}$, where

$$f_{<R} = \mathcal{F}^{-1}\big(\hat{f}\chi_{\{|\xi|\le R\}}\big) \text{ and } f_{>R} = \mathcal{F}^{-1}\big(\hat{f}\chi_{\{|\xi|>R\}}\big). \tag{22}$$

In both expressions the Fourier inversion formula makes sense: for $f_{>R}$ we know that $\hat{f}\chi > R \in L^2(\mathbb{R}^n)$, and $\mathcal{F}$ (and likewise $\mathcal{F}^{-1}$) is defined on L^2; while for $f_{<R}$ we know that $\hat{f} \in L^1_{1\text{oc}}(\mathbb{R}^n)$, and so $\hat{f}\chi \le R \in L^1(\mathbb{R}^n)$ which means that we can write $f_{<R}$ using the integral form of the inversion formula (13):

$$f_{<R}(x) = \int_{|\xi|\le R} e^{2\pi i\xi\cdot x}\,\hat{f}(\xi)\mathrm{d}\xi.$$

Thus

$$\|f_{<R}\|_{L^\infty} \le \int_{|\xi|\le R} |\xi|^{-s}|\xi|^{s}|\hat{f}(\xi)|\mathrm{d}\xi$$

$$\le \left(\int_{|\xi|\le R} |\xi|^{-2s}\mathrm{d}\xi\right)^{1/2} \|f\|_{\dot{H}^s} = C_s R^{n/2-s} = C_s R^{n/p},$$

since we took $\|f\|_{\dot{H}^s} = 1$ and $s = n\left(\frac{1}{2} - \frac{1}{p}\right)$. Now, since for any choice of R

$$d_f(\alpha) \le d_{f<R}(\alpha/2) + d_{f>R}(\alpha/2)$$

(using (9)), we can choose R to depend on α, $R = R_\alpha := (\alpha/2C_s)^{p/n}$, and then we have

$$d_{f<R_\alpha}(\alpha/2) = 0,$$

it follows that $d_f(\alpha) \le d_{f>R_\alpha}(\alpha/2)$. Thus, using the fact that the Fourier transform is an isometry from L^2 into itself,

$$\|f\|_{L^p}^p \le p\int_0^\infty \alpha^{p-1}\, d_{f>R_\alpha}(\alpha/2)\mathrm{d}\alpha$$

$$\le p\int_0^\infty \alpha^{p-1}\frac{4}{\alpha^2}\|f_{>R_\alpha}\|_{L^2}^2\mathrm{d}\alpha$$

$$= C\int_0^\infty \alpha^{p-3}\,\|\mathcal{F}(f_{>R_\alpha})\|_{L^2}^2\mathrm{d}\alpha$$

$$= C\int_0^\infty \alpha^{p-3}\int_{|\xi|\geq R_\alpha} |\hat{f}(\xi)|^2 \mathrm{d}\xi \mathrm{d}\alpha$$

$$= C\int_{\mathbb{R}^n}\left(\int_0^{2C_s|\xi|^{n/p}} \alpha^{p-3}\,\mathrm{d}\alpha\right)|\hat{f}(\xi)|^2 \mathrm{d}\xi$$

$$\leq C\int_{\mathbb{R}^n} |\xi|^{n(p-2)/p}|\hat{f}(\xi)|^2 \mathrm{d}\xi$$

$$= C,$$

since $n(p-2)/p = 2s$ and we took $\|f\|_{\dot{H}^s} = 1$.

Thus for $f \in \dot{H}^s$ with $\|f\|_{\dot{H}^s} = 1$ we have $\|f\|_{L^p} \leq C$, and (21) follows for general $f \in \dot{H}^s$ on applying this result to

$$gf\|f\|_{\dot{H}^s}$$

We will require a result, known as Bernstein's inequality, that provides integrability of f assuming localisation of its Fourier transform: if $\hat{f}$ is supported in $B(0,R)$ (the ball of radius R), then for any $1 \leq p \leq q \leq \infty$ if $f \in L^q(\mathbb{R}^n)$, then

$$\|f\|_{L^p} \leq c_{p,q}R^{n(1/q-1/p)}\|f\|_{L^q}. \tag{23}$$

For our purposes we will require a version of this inequality that replaces L^q by $L^{q,\infty}$ on the right-hand side.

As in the standard proof of (23), we make use of the following simple result. We use the notation $\mathfrak{D}_h f(x) = h^{-n}f(x/h)$; note that $\widehat{\mathfrak{D}_h}(x) = \hat{f}(hx)$. The support of $g \in \mathcal{S}'$ is the of all closed sets K such that $\langle g, \varphi\rangle = 0$ whenever the support of $\varphi \in \mathcal{S}$ is disjoint from K.

Lemma (4.1.7)[127]. *There is a fixed $\varphi \in \mathcal{S}$ such that if $\hat{f}$ is supported in $(0,R)$, then $f = (\mathfrak{D}_{1/R}\varphi) \star f$.*

Proof. Take $\varphi \in \mathcal{S}$ so that $\hat{\varphi} = 1$ on $(0,1)$. Then

$$\overline{\mathfrak{D}_{1/R}\varphi}(\xi) = \hat{\varphi}(\xi/R)$$

which is equal to 1 on $B(0,R)$. Thus $(\mathfrak{D}_{1/R}\varphi) \star f - f$ has Fourier transform zero, and the lemma follows.

For use in the proof of our next lemma, note that

$$\|\mathfrak{D}_{1/R}\varphi\|_{L^r} = R^{n(1-1/r)}\|\varphi\|_{L^r}. \tag{24}$$

Lemma (4.1.8) [127]. Let $1 \leq q < \infty$ and suppose that $f \in L^{q,\infty}(\mathbb{R}^n)$ and that $\hat{f}$ is supported in $(0,R)$. Then for each p with $q < p < \infty$ there exists a constant $c_{p,q}$ such that

$$\|f\|_{L^p} \leq cR^{n(1/q-1/p)}\|f\|_{L^q},\infty. \tag{25}$$

Proof. We follow the standard proof, replacing Young's inequality by its weak form, and making use of the interpolation result of Lemma (4.1.2). First we prove the weak weak version

$$\|f\|_{L^{p,\infty}} \leq cR^{n(1/q-1/p)}\|f\|_{L^{q,\infty}}$$

valid for all $1 \leq q \leq p < \infty$. To do this we simply apply the weak form of Young's inequality (Proposition (4.1.5)) to $f = (\mathfrak{D}_{1/R}\varphi) \star f$:

$$\|f\|_{L^{p,\infty}} = \|(\mathfrak{D}_{1/R}\varphi) \star f\|_{L^{p,\infty}}$$

$$\leq c\|\mathfrak{D}_{1/R}\varphi\|_{L^r}\|f\|_{L^{q,\infty}},$$

where

$$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$$

with $1 \leq q < \infty$ and $1 < p, r < \infty$. It follows using (24) that

$$\|f\|_{L^{1,\infty}} \leq cR^{n(1/q-1)}\|f\|_{L^q,\infty} \text{ and } \|f\|_{L^{2p,\infty}} \leq cR^{n(1/q-1/2p)}\|f\|_{L^q,\infty},$$

and we then obtain (25) by interpolation of L^p between $L^{1,\infty}$ and $L^{2p,\infty}$ (Lemma (4.1.2)),

$$\|f\|_{L^p} \leq c\|f\|_{L^{1,\infty}}^{1/(2p-1)}\|f\|_{L^{2p,\infty}}^{(2p-2)/(2p-1)}$$

$$\leq cR^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^{q,\infty}} .$$

We now prove our first generalisation of the Gagliardo-Nirenberg inequality, replacing the L^q norm on the right-hand side of (1) by the norm in $L^{q,\infty}$. The new part of the following result is when $s \geq n/2$, with the case $s = n/2$ particularly interesting: in the range $n(1/2 - l/p) < s < n/2$ the inequality follows using weak-Lp interpolation from Lemma (4.1.2) coupled with the Sobolev embedding $\dot{H}^{n(1/2-1/p)} \subset L^p$ from Theorem (4.1.6).

Theorem (4.1.9)[127]. *Take $1 \leq q < p$ and $s \geq 0$ with $> n(1/2 - l/p)$. There exists a constant $c_{p,q,s}$ such that if $\in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_{L^p} \leq c_{p,q,s}\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta} \text{ for every } f \in L^{q,\infty} \cap \dot{H}^s, \tag{26}$$

where

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right) \tag{27}$$

Proof. First we prove the theorem in the case $p \geq 2$. As in the proof of Theorem (4.1.6) we write

$$f = f_{<R} + f_{>R},$$

where $f_{<R}$ and $f_{>R}$ are defined in (22).

Using the endpoint Sobolev embedding $\dot{H}^{n(1/2-1/p)}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ from Theorem(4.1.6) (taking $\dot{H}^0 = L^2$ when $p = 2$) we can estimate

$$\|f_{>R}\|_{L^p} \leq c\|f_{>R}\|_{\dot{H}^{n(1/2-1/p)}}$$

$$= c\left(\int_{|\xi|\geq R} |\xi|^{2n(1/2-1/p)}|\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$\leq \frac{c}{R^{s-n(1/2-1/p)}}\left(\int_{|\xi|\geq R}|\xi|^{2s}|\hat{f}(\xi)|^2\mathrm{d}\xi\right)^{1/2}$$

$$\leq \frac{c}{R^{s-n(1/2-1/p)}}\|f\|_{\dot{H}^s},$$

while

$$\|f_{<R}\|_{L^p} \leq cR^{n(1/q-1/p)}\|f_{<R}\|_{L^{q,\infty}} \leq cR^{n(1/q-1/p)}\|f\|_{L^{q,\infty}}$$

using the weak-strong Bernstein inequality from Lemma (4.1.8) and (10).

Thus

$$\|f\|_{L^p} \leq c\left(R^{n(1/q-1/p)}\|f\|_{L^q,\infty} + R^{-s+n(1/2-1/p)}\|f\|_{\dot{H}^s}\right).$$

Choosing

$$R^{s+n(1/q-1/2)} = \frac{||f||_{\dot{H}^s}}{\|f\|_{L^{q,\infty}}}$$

we obtain

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta}, \tag{28}$$

where

$$\theta = 1 - n\frac{1/q-1/p}{s+n(1/q-1/2)},$$

which on rearrangement yields the condition (27).

If $1 \leq q < p < 2$, then we first interpolate L^p between $L^{q,\infty}$ and L^2, and then use the above result with $p = 2$. Setting $\frac{1}{2} = \frac{\theta'}{q} + (1-\theta')\left(\frac{1}{2}-\frac{s}{n}\right)$ we have

$$\|f\|_{L^p} \leq c\|f\|_{L^{q,\infty}}^{q(2-p)/p(2-q)}\|f\|_{L^2}^{2(p-q)/p(2-q)}$$

$$\leq c\|f\|_{L^{q,\infty}}^{q(2-p)/p(2-q)}\left(c\|f\|_{L^{q,\infty}}^{\theta'}\|f\|_{\dot{H}^s}^{1-\theta'}\right)^{2(p-q)/p(2-q)}$$

$$= c\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{\dot{H}^s}^{1-\theta},$$

with θ given by (27), as required.

For any set $A \subset \mathbb{R}^n$ we write

$$f_A = \frac{1}{|A|}\int_A f\,\mathrm{d}x$$

for the average of f over the set A. The space of functions with bounded mean oscillation, $\mathrm{BMO}(\mathbb{R}^n)$, consists of those functions f for which

$$\|f\|_{\mathrm{BMO}} := Q \subset \mathbb{R}^n \sup \frac{1}{|Q|}\int_Q |f - f_Q|\mathrm{d}x$$

is finite, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Note that this is a not a norm (any constant function has $\|c\|_{\mathrm{BMO}} = 0$), but BMO is a linear space, i.e., if f, $g \in \mathrm{BMO}$, then $f + g \in \mathrm{BMO}$ and

$$\|f+g\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}} + \|g\|_{\mathrm{BMO}}.$$

This space was introduced by John & Nirenberg [1]; more details can be found in BMO is a space with the same scaling as L^∞, but is a larger space. Indeed, if $\in L^\infty(\mathbb{R}^n)$, then clearly for any cube Q

$$\int_Q |f - f_Q| \mathrm{d}x \leq 2 \int_Q |f| \leq 2|Q| \|f\|_{L^\infty}, \tag{29}$$

and so

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}. \tag{30}$$

However, the function $\log|x| \in \mathrm{BMO}(\mathbb{R}^n)$ but is not bounded on $\mathbb{R}^n$ (Example (4.1.9), in Grafakos [136]).

The endpoint Sobolev embedding from Theorem (4.1.6) fails when $s = n/2$, but at this endpoint we still have $\dot{H}^{n/2}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. This is simple to show, if we note that for any $x \in Q$

$$|f(x) - f_Q| = |\frac{1}{|Q|}\int_Q \big(f(x) - f(y)\big)\,\mathrm{d}y| \leq \sqrt{n}|Q|^{1/n}\|\nabla f\|_{L^\infty(Q)}.$$

Lemma (4.1.10)[127]. *If* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n)$, *then* $f \in \mathrm{BMO}(\mathbb{R}^n)$ *and there exists a constant* $C = C(n)$ *such that*

$$\|f\|_{\mathrm{BMO}} \leq C\|f\|_{\dot{H}^{n/2}} \text{ for all } f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n).$$

Proof. We write $f = f_{<R} + f_{>R}$ as in the proof of Theorem (4.1.6) and then, recalling (29),

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq \sqrt{n}|Q|^{1/n}\|\nabla f_{<R}\|_{L^\infty(Q)} + \frac{1}{|Q|}\int_Q |f_{>R} - (f_{>R})_Q|$$

$$\leq c_n|Q|^{1/n}\int_{|\xi|\leq R}|\xi||\hat{f}(\xi)|\mathrm{d}\xi + \frac{2}{|Q|^{1/2}}\left(\int_Q |f_{>R}|^2\right)^{1/2}$$

$$\leq c_n|Q|^{1/n}R\left(\int_{\mathbb{R}^n}|\xi|^n|\hat{f}(\xi)|^2\mathrm{d}\xi\right)^{1/2} + \frac{2}{|Q|^{1/2}}\left(\int_{|\xi|\geq R}|\hat{f}(\xi)|^2\mathrm{d}\xi\right)^{1/2}$$

$$\leq c_n\big[|Q|^{1/n}R + |Q|^{-1/2}R^{-n/2}\big]\|f\|_{\dot{H}^{n/2}}.$$

Choosing $R = |Q|^{-1/n}$ yields

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq C\|f\|_{\dot{H}^{n/2}};$$

taking the supremum over all cubes $Q \subset \mathbb{R}^n$ yields $\|f\|_{\mathrm{BMO}} \leq C\|f\|_{\dot{H}^{n/2}}$. We now want to prove a result, due to John & Nirenberg [16], that gives an important property of functions in BMO which will be crucial in the proof of the inequality

$$\|f\|_{L^p} \leq C\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}, q < p < \infty,$$

To prove the John-Nirenberg inequality we will need a Calderon-Zygmund type decomposition of $\mathbb{R}^n$ into a family of cubes with certain useful properties. The proof that such a decomposition is possible uses the Lebesgue Differentiation Theorem, which we now state (without proof). We define the uncentred cubic maximal function by

$$\mathfrak{M}f(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x. The proof of the Lebesgue Differentiation Theorem uses the fact that $\mathfrak{M}$ maps L^1 into $L^{1,\infty}$; see in Grafakos [11].

Theorem (4.1.11)[127]. If $\in L^1_{1oc}(\mathbb{R}^n)$, then

$$\lim_{|Q|\to 0} \frac{1}{|Q|}\int_Q f(y)\mathrm{d}y = f(x) \tag{31}$$

for almost every $x \in \mathbb{R}^n$, where Q is a cube containing x. As a consequence, $|f(x)| \le \mathfrak{M}f(x)$ almost everywhere.

Proposition (4.1.12)[127]. Let Q be any cube in $\mathbb{R}^n$. Given $f \in L^1(Q)$ and $M > 0$ there exists a countable collection $\{Q_j\}$ of disjoint open cubes such that $|f(x)| \le M$ for almost every $x \in Q \setminus \bigcup_j Q_j$ and

$$M < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\mathrm{d}x \le 2^n M \tag{32}$$

for every Q_j.

Note that it follows from (32) that

$$\sum_j |Q_j| \le \frac{1}{M}\int_Q |f|. \tag{33}$$

Proof. Decompose Q, by halving each side, into a collection $\mathcal{Q}_0$ of 2^n equal cubes. Select one of these cubes $\hat{Q}$ if

$$\frac{1}{|\hat{Q}|}\int_{\hat{Q}} |f(x)|\mathrm{d}x > M. \tag{34}$$

Call the selected cubes $\mathcal{C}_1$ and let $\mathcal{Q}_1 = \mathcal{Q}_0 \setminus \mathcal{C}_1$. Repeat this process inductively, to produce a set $\mathcal{C} = \bigcup_j \mathcal{C}_j$ of selected cubes, on which (34) holds. Note that if $\hat{Q}$ was selected at step k, then it is contained in a cube $Q' \in \mathcal{Q}_{k-1}$, and so

$$M < \frac{1}{|\hat{Q}|}\int_{\hat{Q}} |f(x)|\mathrm{d}x \le 2^n \frac{1}{|Q|}\int_{Q'} |f(x)|\mathrm{d}x \le 2^n M.$$

Enumerate the countable set $\mathcal{C}$ of cubes as $\{Q_j\}_{j=1}^{\infty}$.

Finally, if $x \in Q \setminus \bigcup_j Q_j$, then there exists a sequence of cubes Q_k containing x with sides shrinking to zero and such that

$$\frac{1}{|Q_k|}\int_{Q_k} |f(x)|\mathrm{d}x \le M.$$

It follows from the Lebesgue Differentiation Theorem that $|f(x)| \le M$ for almost every $x \in Q \setminus \bigcup_j Q_j$.

Lemma (4.1.13)[127]. There exist constants c and C (depending only on n) such that if $\in \mathrm{BMO}(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$

$$|\{x \in Q: |f - f_Q| > \alpha\}| \le \frac{C}{\|f\|_{\mathrm{BMO}}} \mathrm{e}^{-c\alpha/\|f\|_{\mathrm{BMO}}} \int_Q |f - f_Q| \tag{35}$$

for all $\alpha \ge \|f\|_{\mathrm{BMO}}$.

Proof. We prove the result assuming that $\|f\|_{\mathrm{BMO}} = 1$; we then obtain (35) by applying the resulting inequality to $f/\|f\|_{\mathrm{BMO}}$. Let $F(\alpha)$ be the infimum of all numbers such that the inequality

$$|\{x \in Q : |f(x)| > \alpha\}| \le F(\alpha)\int_Q |f| \tag{36}$$

holds for all $f \in L^1(Q)$ and all cubes Q; note (cf. Lemma (4.1.1)) that $F(\alpha) \le 1/\alpha$. Following the original proof of John & Nirenberg [16] we show that for all $\alpha \ge 2^n$,

$$F(\alpha) \le \tfrac{1}{M} F(\alpha - 2^n M) \text{ for all } 1 \le M \le 2^{-n}\alpha. \tag{37}$$

Given M in this range we decompose f using Proposition (4.1.12). Now, if $|f(x)| > \alpha \ge 2^n$, then $x \in Q_k$ for some k, and we know that $|f_{Q_k}| \le 2^n M$ from (32). So then

$$|\{x \in Q: |f(x)| > \alpha\}| \le \sum_k |\{x \in Q_k: |f(x) - f_{Q_k}| > \alpha - 2^n M\}|.$$

We can now use (36) on the cube Q_k for the function $f - f_{Q_k}$, so that

$$|\{x \in Q_k: |f(x) - f_{Q_k}| > \alpha - 2^n M\}| \le F(\alpha - 2^n M)\int_{Q_k} |f - f_{Q_k}|\mathrm{d}x$$

$$\le F(\alpha - 2^n M)|Q_k|$$

(recall that we took $\|f\|_{\mathrm{BMO}} = 1$). It follows using (33) that

$|\{x \in Q : |f(x)| > \alpha\}| \le (\sum_k |Q_k|)F((y - 2^n M) \le \frac{1}{M} F(\alpha - 2^n M)\int_Q |f|\mathrm{d}x,$

which is (37).

To finish the proof we iterate (37) in a suitable way. We remarked above that $F(\alpha) \le 1/\alpha$; now observe that

$$\frac{1}{\alpha} \le C\mathrm{e}^{-\alpha/2^n\mathrm{e}} \quad \text{for all} \quad 1 \le \alpha \le 1 + 2^n\mathrm{e},$$

$F(\alpha + 2^n\mathrm{e}) \le \frac{\le 1}{\mathrm{e}} F(\alpha)$, weobtainfor $C = \max_{1\alpha \le 1+2^n\mathrm{e}} \alpha^{-1}\mathrm{e}^{\alpha/2^n\mathrm{e}}$. Iterating (37) with $M =$ e, which implies that $F(\alpha) \le C\mathrm{e}^{-c\alpha}$ for all $\alpha \ge 1$,
where $c = 1/2^n\mathrm{e}$, which gives (35).

The more usually quoted form of this inequality,

$$|\{x \in Q: |f - f_Q| > \alpha\}| \le C|Q|\mathrm{e}^{-c\alpha/\|f\|_{\mathrm{BMO}}},$$

follows immediately from the definition of $\|f\|_{\mathrm{BMO}}$.

We now adapt the very elegant argument of Chen & Zhu [11] to prove the following stronger version of the inequality in (26) in the case $s = n/2$; they proved the inequality for $f \in L^q \cap \mathrm{BMO}$, but the changes required to take $f \in L^{q,\infty} \cap$ BMO are in fact straightforward. Another proof for $f \in L^q \cap \mathrm{BMO}$, which still relies on the John-Nirenberg inequality (but less explicitly), is given by Azzam & Bedrossian [1], and a sketch of an alternative proof of the result for $f \in L^{q,\infty} \cap \mathrm{BMO}$ can be found by Kozono et al. [140].

Theorem (4.1.14)[127]. *For any* $1 < q < p < \infty$, *if* $f \in L^{q,\infty}(\mathbb{R}^n)\mathrm{nBMO}(\mathbb{R}^n)$, *then* $f \in L^p(\mathbb{R}^n)$ *and there exists a constant* $C = C(q,p,n)$ *such that*

$$\|f\|_{L^p} \le C\|f\|_{L^{q,\infty}}^{q/p}\|f\|_{\mathrm{BMO}}^{1-q/p}. \tag{38}$$

Proof. First we note that it is a consequence of the John-Nirenberg inequality from Lemma (4.1.13) that if $f \in \mathrm{BMO} \cap L^1$, then

$$d_f(\alpha) \le C\mathrm{e}^{-C\alpha/\|f\|_{\mathrm{BMO}}}\|f\|_{L^1} \tag{39}$$

for all $\alpha > \|f\|_{\mathrm{BM}}0$; this follows by taking $|Q| \to \infty$ in (35), since when $f \in L^1$,

$$|f_Q| \le \frac{1}{|Q|}\int |f| \to 0 \text{ as } |Q| \to \infty,$$

and $\int_Q |f - f_Q|\mathrm{d}x \le 2\int_Q |f|\mathrm{d}x$.

Now take $f \in \mathrm{BMO}$ with $\|f\|_{\mathrm{BMO}} = 1$. Split $f = f_{1-} + f_{1+}$ as in Lemma (4.1.3). Since $f_{1-} \in L^\infty$, $\|f_{1-}\|_{\mathrm{BMO}} \le 2\|f_{1-}\|_{L^\infty} \le 2$ (using (30)); thus $f_{1+} = f - f_{1-} \in \mathrm{BMO}$ and

$$\|fi+\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}} + \|fi-\|_{\mathrm{BMO}} \le 3.$$

Using Lemma (4.1.3) we know that

$$\|f_{1-}\|_{L^p}^p \le C\|f_{1-}\|_{L^{q,\infty}}^q. \tag{40}$$

Also, for (q, q') conjugate,

$$\|f_{1+}\|_{L^1} = \int |f_{1+}| \le \int |f_{1+}|^{1+1/q'} = \|f_{1+}\|_{L^{1+1/q'}}^{1+1/q'} \le c\|f_{1+}\|_{L^1}^{1/q'}\|f_{1+}\|_{L^{q,\infty}}$$

(since $1 < 1 + 1/q' < q$ we can use weak-olation), which yields

$$\|f_{1+}\|_{L^1} \le c\|f_{1+}\|_{L^{q,\infty}}^q.$$

Now we calculate

$$\|f_{1+}\|_{L^p}^p = p\int_0^\infty \alpha^{p-1}\, d_{f1+}(\alpha)\mathrm{d}\alpha$$

$$= p\int_0^1 \alpha^{p-1}\, d_f(1)\mathrm{d}\alpha + p\int_1^\infty \alpha^{p-1}\, d_{f1+}(\alpha)\mathrm{d}\alpha$$

$$\le d_f(1) + p\left(\int_1^\infty \alpha^{p-1}\, C\mathrm{e}^{-C\alpha/3}\mathrm{d}\alpha\right)\|f_{1+}\|_{L^1},$$

where we have used (17), (39), and the fact that $\|f_{1+}\|_{\mathrm{BMO}} \le 3$. Thus

$$\|f_{1+}\|_{L^p}^p \le \|f\|_{L^{q,\infty}}^q + C\|f_{1+}\|_{L^{q,\infty}}^q \le C\|f\|_{L^{q,\infty}}^q. \tag{41}$$

Adding (40) and (41) we obtain

$$\|f\|_{L^p} \le C\|f\|_{L^{q,\infty}}^{q/p};$$

(38) follows.

So far we have avoided defining the two-parameter Lorentz spaces $L^{p,r}$, which involve decreasing rearrangements. We will obtain an inequality involving such spaces

$$\|u\|_{L^{p,1}} \le C_{n,p,q}\|u\|_{L^{q,\infty}}^{q/p}\|u\|_{\mathrm{BMO}}^{1-q/p}, \tag{42}$$

from which (at least for $q > 1$) our two previous inequalities follow (we require $1 < q < p < \infty$ in (42), see Theorem (4.1.17)). We will do this via the theory of interpolation spaces. Here we will not provide detailed proofs of any of the results, for the most part merely providing statements of the relevant general theory.

Given a measurable function $f: \mathbb{R}^n \to \mathbb{R}$, we have already defined and made much use of its distribution function d_f. We now define its decreasing rearrangement $f^*: [0,\infty) \to [0,\infty]$ as

$$f^*(t) = \inf\{\alpha : d_f(\alpha) \le t\},$$

with the convention that $\inf\emptyset = \infty$. The point of this definition is that f and f^* have the same distribution function,

$$d_{f^*}(\alpha) = d_f(\alpha),$$

but f^* is a positive non-increasing scalar function. Since their distribution functions agree, we can use the identity in (7) to show that the L^p norm of f is equal to the L^p norm of f^*:

$$\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x = p\int_0^\infty \alpha^{p-1}\, d_f(\alpha)\mathrm{d}\alpha$$

$$= p\int_0^\infty \alpha^{p-1}\, d_{f^*}(\alpha)\mathrm{d}\alpha = \int_0^\infty f^*(t)^p \mathrm{d}t.$$

Given $1 \le p, q \le \infty$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ consists of all measurable functions f for which the quantity

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

(for $q < \infty$) or

$$\|f\|_{L^p,\infty} := \sup t^{1/p} f^*(t)$$

(for $q = \infty$) is finite. It is simple to show in Grafakos [135]) that this definition agrees with our previous definition of $L^{p,\infty}$, that $L^{\infty,\infty} = L^\infty$, and that $L^{p,p} = L^p$ (the last of these, at least, is immediate).

If $r < s$, then $L^{p,r} \subset L^{p,s}$; so the largest space in this family for fixed p is the weak space $L^{p,\infty}$, and the smallest is $L^{p,1}$ To see that $L^{p,r} \subset L^{p,\infty}$ for every r, simply observe that

$$t^{1/p} f^*(t) = \left\{\frac{r}{p}\int_0^t \left[s^{1/p} f^*(t)\right]^r \frac{\mathrm{d}s}{s}\right\}^{1/r}$$

$$\le \left\{\frac{r}{p}\int_0^\infty \left[s^{1/p} f^*(s)\right]^r \frac{\mathrm{d}s}{s}\right\}^{1/r}$$

$$\le (r/p)^{1/r}\|f\|_{L^{p,r}},$$

which yields $\|f\|_{L^p,\infty} \le (r/p)^{1/r}\|f\|_{L^{p,r}}$ on taking the supremum over $t > 0$. Given this, if $r < q < \infty$, then, using Hölder's inequality,

$$\|f\|_{L^{p,q}} = \left\{\int_0^t \left[t^{1/p} f^*(t)\right]^{q-r+r} \frac{\mathrm{d}t}{t}\right\}^{1/r} \le \|f\|_{L^{p,\infty}}^{(q-r)/q}\|f\|_{L^{p,r}}^{r/q} \le C_{p,q,r}\|f\|_{L^{p,r}}.$$

We now very briefly outline the theory of interpolation spaces; the general theory is modelled on the definition of the Lorentz spaces given above. For sustained expositions of the theory see Bennett & Sharpley [9], Bergh & Löfström [130], or Lundari [142].

Given two Banach spaces X_0 and X_1 that embed continuously into some parent Hausdorff topological vector space, which we term "a compatible pair we define the K-functional for each $x \in X_0 + X_1$ and $t > 0$ by

$$K(x,t) = \inf\left\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x_0 + x_1 = x, x_0 \in X_0, x_1 \in X_1\right\}.$$

Then for $0 < \theta < 1$ and $1 \le q < \infty$ we define the interpolation space $(X_0, X_1)_{\theta,q}$ as the space of all $x \in X_0 + X_1$ for which

$$\|x\|_{\theta,q} := \left(\int_0^\infty \left[t^{-\theta} K(f,t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

is finite. Similarly, for $0 \le \theta \le 1$ and $q = \infty$, the space $(X_0, X_1)_{\theta,\infty}$ is the space of all $x \in X_0 + X_1$ such that

$$\|x\|_{\theta,\infty} = \sup t^{-\theta} K(f,t)$$

is finite. For all these spaces $(1 \le q \le \infty)$ we have the interpolation inequality

$$\|f\|_{\theta,q} \le C_{\theta,q} \|f\|_{X_0}^{1-\theta} \|f\|_{X_1}^{\theta} \tag{43}$$

Löfström [130].

Given the definitions of Lorentz spaces and of the interpolation spaces, it is not surprising that

$$(L^1, L^\infty)_{1-1/p,r} = L^{p,r}$$

for $1 < p < \infty$, $1 \le r \le \infty$. That one can replace L^∞ here by BMO is much less obvious, but key to the 'quick' proof of (42).

Theorem (4.1.15)[127]. For $1 < p < \infty$ and $1 \le r \le \infty$,

$$L^{p,r} = (L^1, \mathrm{BMO})_{1-1/p,r}.$$

Proof. One can also find a proof of this result by Hanks [137], and of a similar but slightly weaker result (with L^p on the left-hand side) using complex interpolation spaces by Janson& Jones [139].

We note here that the key step in the proof of this result given in Bennett & Sharpley [9] (and in Hanks [137]) is a relationship between the sharp function of f,

$$f_Q^{\#}(x) := Q' \subset Q, Q' \ni x \sup \frac{1}{|Q|} \int_{Q'} |f - f_{Q'}|,$$

its decreasing rearrangement f^*, and the function $f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \mathrm{d}s$:

$$f^{**}(t) - f^*(t) \le C\left(f_Q^{\#}\right)^*(t) \quad 0 < t < |Q|$$

This also forms the main ingredient in the proof of (38) in Kozono & Wadade [5] (and the proof of (44) in Kozono et al. [140]).

The inequality (38) in fact follows simply from Theorem (4.1.15) using the following 'Reiteration Theorem', which allows one to identify interpolants between two interpolation spaces in terms of the original 'endpoints'.

Theorem (4.1.16)[127]. *Let* (X_0, X_1) *be a compatible pair of Banach spaces, and let* $0 \le \theta_0 < \theta_1 \le 1$ *and* $1 \le q_0, q_1 \le \infty$. *Set*

$$Y_0 = (X_0, X_1)_{\theta_0,q_0} \quad \textit{and} \quad Y_1 = (X_0, X_1)_{\theta_1,q_1}.$$

If $0 < \theta < 1$ *and* $1 \le q \le \infty$, *then*

$$(Y_0, Y_1)_{\theta,q} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1,q}.$$

Corollary (4.1.17)[127]. If $u \in L^{q,\infty} \cap \mathrm{BMO}$ for some $q > 1$ and $q < p < \infty$, then $u \in L^{p,1}$

and there exists a constant $C_{n,p,q}$ such that

$$\|u\|_{L^{p,1}} \leq C_{n,p,q} \|u\|_{L^{q,\infty}}^{q/p} \|u\|_{\mathrm{BMO}}^{1-q/p}. \tag{44}$$

Note that given the ordering of Lorentz spaces, $L^{p,1} \subset L^{p,p} = L^p$ and so this result implies Theorem (4.1.14) in the case $q > 1$.

Proof. Using Theorem (4.1.15), since $q > 1$ we have

$$L^{q,s} = (L^1, \mathrm{BMO})_{1-1/q,s};$$

set $\mathrm{B} = (L^1, \mathrm{BMO})_{1,\infty}$. Note that from (43) $\|f\|_{\mathrm{B}} \leq C\|f\|_{\mathrm{BMO}}$. Now simply use the Reiteration Theorem to obtain

$$L^{p,r} = (L^{q,s}, \mathrm{B})_{1-q/p,r},$$

from which the inequality (44) follows immediately using (43).

(One can use interpolation spaces to provide a proof of Theorem (4.1.14) that does not involve Lorentz spaces by using interpolation only with $q = \infty$ and then interpolation between weak L^p spaces, see McCormick et al. [143].)

Although we have not needed it here, one of the main uses of weak spaces arises due to the powerful Marcinkiewicz interpolation theorem, in which bounds in weak spaces at the endpoints lead to bounds in strong spaces in between. We include here a statement of the $theorem^4$ and some straightforward consequences.

We say T is sublinear if

$$|T(f+g)| \leq |Tf| + |Tg| \text{ and } |T(\lambda f)| \leq |\lambda||Tf|$$

almost everywhere.

Theorem (4.1.18)[127]. *Suppose that* $q_0 < q_1$ *and that* T *is a sublinear map defined on* $L^{q_0} + L^{q}$' *such that for some* p_0, p_1

$$\|Tf\|_{L^{p_0},\infty} \leq A_0\|f\|_{L^{q_0}} \text{ and } \|Tf\|_{L^{p_1},\infty} \leq A_1\|f\|_{L^{q_1}}.$$

If $0 < t < 1$,

$$\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1} \text{ and } \frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \tag{45}$$

and $p \geq q$, *then* $T\colon L^q \to L^p$ *and there exists a constant* A_t *such that*

$$\|Tf\|_{L^p} \leq A_t\|f\|_{L^q}. \tag{46}$$

With the restriction that $p_0 \geq q_0$ and $p_1 \geq q_1$ one can find an elementary proof of this theorem in Folland[134]

We now give some interesting consequences of this theorem that $\mathcal{F}$ maps L^1 into L^∞ and L^2 into L^2, so the following result is immediate.

Corollary (4.1.19)[127]. For $1 \leq \mathrm{p} \leq 2$ the Fourier transform is a bounded linear map from $\mathrm{L^p}$ into $\mathrm{L^q}$, where (p, q) are conjugate.

Another application is the improved version of Young's inequality that was promised.

Theorem (4.1.20)[127]. *Suppose that* $1 < p, q, r < \infty$. *If* $f \in L^{q,\infty}$ *and* $g \in L^r$ *with*

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r},$$

then $f \star g \in L^p$ *with*

$$\|f \star g\|_{L^p} \le c_{p,q,r}\|f\|_{L^{q,\infty}}\|g\|_{L^r}. \quad (47)$$

Proof. Note that it follows from the conditions on p, q, r that $p > r$. Fix $f \in L^{q,\infty}$ with $\|f\|_{L^q},\infty = 1$, and consider the linear operator $T(g) = f \star g$. Since $1 < p, r < \infty$ we can find $p_0 < p < p_1$, $r_0 < r < r_1$, and $0 < t < 1$ such $p_0 \ge r_0$, $p_1 \ge r_1$, and (45) holds. Now using the weak form of Young's inequality from Proposition (4.1.5),

$$\|f \star g\|_{L^{p_0}},\infty \le C\|g\|_{L^{r_0}} \quad \text{and} \quad \|f \star g\|_{L^{p_1}},\infty \le C\|g\|_{L^{r_1}}.$$

We can now use the Marcinkiewicz interpolation theorem to guarantee that

$$\|f \star g\|_{L^{p}} \le C\|g\|_{L^r}.$$

Since $f \star g$ is also linear in f, we obtain (47).

Using Theorem (4.1.20) and the fact that if $P_\alpha(x) = |x|^{-\alpha}$, then $[\hat{P}_\alpha](\xi) = c_{n},{}_\alpha P_{n-\alpha}(\xi)$ (this can be checked by simple calculation) we can give a very quick alternative proof of the endpoint Sobolev embedding, after in Bahouri et al. [129].

Theorem (4.1.21)[127]. For $2 < p < \infty$ there exists a constant $c = c_{n,p}$ such that if $f \in \dot{H}^s(\mathbb{R}^n)$ with $= n(1/2 - 1/p)$, then $f \in L^p(\mathbb{R}^n)$ and $\|f\|_{L^p} \le c\|f\|_{\dot{H}^s}$.

Proof. We make the pointwise definition $(\xi) = |\xi|^s \hat{f}(\xi)$; since $f \in \dot{H}^s(\mathbb{R}^n), \gamma \in L^2(\mathbb{R}^n)$. If we set $g = \mathcal{F}^{-1}\gamma$, then $g \in L^2(\mathbb{R}^n)$ and $\|g\|_{L^2} = \|\gamma\|_{L^2} = \|f\|_{\dot{H}^s}$. Now,

$$\hat{f}(\xi) = \frac{|\xi|^s \hat{f}(\xi)}{|\xi|^s} = \hat{g}(\xi)|\xi|^{-s},$$

and so $f = g \star c_n^{-1}{}_{n-s}P_{n-s}$. Since $P_{n-s} \in L^{n/(n-s),\infty}$ and $g \in L^2$ it follows from Theorem (4.1.20) that $\in L^p(\mathbb{R}^n)$.

Corollary (4.1.22)[222]: *If* $f_j \in L^{1+\epsilon}(\mathbb{R}^n)$, *then* $f_j \in L^{1+\epsilon,\infty}(\mathbb{R}^n)$ *and* $\|\sum_j f_j\|_{L^{1+\epsilon,\infty}} \le \sum_j \|f_j\|_{L^{1+\epsilon}}$.

Proof. This follows since

$$\sum_j d_{f_j}(\alpha) = \int_{\{x:\sum_j |f_j(x)|>\alpha\}} 1\,\mathrm{d}x \le \int_{\{x:|f_j(x)|>\alpha\}} \sum_j \frac{|f_j(x)|^{1+\epsilon}}{\alpha^{1+\epsilon}}\mathrm{d}x$$

$$\le \sum_j \|f_j\|_{L^{1+\epsilon}}^{1+\epsilon}\alpha^{-(1+\epsilon)}.$$

While $L^{1+\epsilon} \subset L^{1+\epsilon,\infty}$, clearly $L^{1+\epsilon,\infty}$ is a larger space than $L^{1+\epsilon}$: for example,

$$|x|^{-\frac{1}{4}} \in L^{1+\epsilon,\infty}(\mathbb{R}^n) \quad (48)$$

but this function is not an element of $L^{1+\epsilon}(\mathbb{R}^n)$.

An immediate indication of why these spaces are useful is given in the following simple result, which shows that in the $L^{1+\epsilon}$ interpolation inequality

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \le \sum_j \|f_j\|^{\theta}_{L^{1+\epsilon}} \|f_j\|^{1-\theta}_{L^{1+\epsilon}}, \qquad \epsilon = 0,$$

one can replace the Lebesgue spaces on the right-hand side by their weak counterparts.

Corollary (4.1.23)[222]: [127] *Take* $0 \le \epsilon \le \infty$. *If* $f_j \in L^{1+\epsilon,\infty} \cap L^{1+3\epsilon,\infty}$, *then* $f_j \in L^{1+2\epsilon}$ *and*

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \le c_{1+\epsilon,1+2\epsilon,1+3\epsilon} \sum_j \|f_j\|^{\theta}_{L^{1+\epsilon,\infty}} \|f_j\|^{1-\theta}_{L^{1+3\epsilon,\infty}},$$

where

$$\frac{1}{1+2\epsilon} = \frac{\theta}{1+\epsilon} + \frac{1-\theta}{1+3\epsilon}.$$

If $\epsilon = \infty$, *we interpret* $L^{\infty,\infty}$ *as* L^{∞}

Proof. We give the proof when $\epsilon < \infty$; the proof when $\epsilon = \infty$ is slightly simpler. If $f_j \in L^{1+\epsilon,\infty}$, then $\sum_j d_{f_j}(\alpha) \le \sum_j \|f_j\|^{1+\epsilon}_{L^{1+\epsilon,\infty}} \alpha^{-(1+\epsilon)}$, so for any x we have

$$\|\sum_j f_j\|^{1+2\epsilon}_{L^{1+2\epsilon}} = (1+2\epsilon)\int_0^\infty \sum_j \alpha^{2\epsilon} d_{f_j}(\alpha) d\alpha$$

$$\le (1+2\epsilon)\int_0^x \sum_j \alpha^{2\epsilon} \|f_j\|^{1+\epsilon}_{L^{1+\epsilon,\infty}} \alpha^{-(1+\epsilon)} d\alpha + (1+2\epsilon)\int_x^\infty \sum_j \alpha^{2\epsilon} \|f_j\|^{1+3\epsilon}_{L^{1+3\epsilon,\infty}} \alpha^{-(1+3\epsilon)} d\alpha$$

$$\le \frac{1+2\epsilon}{\epsilon}\sum_j \|f_j\|^{1+\epsilon}_{L^{1+\epsilon,\infty}} x^{\epsilon} - \frac{1+2\epsilon}{\epsilon}\sum_j \|f_j\|^{1+3\epsilon}_{L^{1+3\epsilon,\infty}} x^{\epsilon}$$

Now choose

$$x^{2\epsilon} = \sum_j \frac{\|f_j\|^{1+\epsilon}_{L^{1+\epsilon,\infty}}}{\|f_j\|^{1+3\epsilon}_{L^{1+3\epsilon,\infty}}}$$

to equalise the dependence of the two terms on the right-hand side on the weak norms.

Corollary (4.1.24)[222]: *Take* $0 \le \epsilon \le \infty$, *and suppose that* $g_j \in L^{1+2\epsilon,\infty}$. *For any* $M > 0$ *set*

$$\sum_j (g_j)_{M-} = \sum_j g_j \chi_{|g_j| \le M} \quad \text{and} \quad \sum_j (g_j)_{M+} = \sum_j g_j \chi_{|g_j| > M}.$$

Then $g_j = (g_j)_{M-} + (g_j)_{M^+}$, *where* $(g_j)_{M-} \in L^{1+3\epsilon}$ *with*

$$\|\sum_j (g_j)_{M-}\|_{L^{1+3\epsilon}}^{1+3\epsilon}$$

$$\leq \frac{1+3\epsilon}{\epsilon} M^{\epsilon} \sum_j \|g_j\|_{L^{1+2\epsilon,\infty}}^{1+2\epsilon} - M^{1+3\epsilon} \sum_j d_{g_j}(M) \tag{49}$$

if $\epsilon < \infty$ *and* $\|(g_j)_{M-}\|_{L^\infty} \leq M$, *and* $(g_j)_{M+} \in L^{1+\epsilon}$ *with*

$$\|\sum_j (g_j)_{M+}\|_{L^{1+\epsilon}}^{1+\epsilon} \leq \frac{1+2\epsilon}{\epsilon} M^{-\epsilon} \sum_j \|g_j\|_{L^{1+2\epsilon,\infty}}^{1+2\epsilon}. \tag{50}$$

Proof. Simply note that

$$d_{(g_j)_{M-}}(\alpha) = \begin{cases} 0 & \alpha \geq M \\ d_{g_j}(\alpha) - d_{g_j}(M) & \alpha < M \end{cases} \tag{51}$$

and

$$d_{(g_j)_{M^+}}(\alpha) = \begin{cases} d_{g_j}(\alpha) & \alpha > M \\ d_{g_j}(M) & \alpha \leq M. \end{cases} \tag{52}$$

Then using (7), (51), and (8) it is simple to show (49), and (50) follows similarly, using (52) in place of (51).

It is natural to ask what one can say about $\hat{f}_j$ when $f_j \in L^{1+\epsilon}$. We will show that $\hat{f}_j \in L^{1+3\epsilon}$ with $(1+\epsilon, 1+3\epsilon)$ conjugate, provided that $0 \leq \epsilon \leq 1$ (Corollary (4.1.19)). Note, however, that for any $\epsilon > 0$ one can find a function in $L^{2+\epsilon}$ whose Fourier transform is not even a locally integrable function (see Exercise 2.3.13 in Grafakos [128]).

One can extend the definition of the Fourier transform further to the space of tempered distributions $\mathcal{S}'$. We say that a sequence $\left\{\varphi_{\frac{1+\epsilon}{2}}\right\} \in \mathcal{S}$ converges to $\varphi \in \mathcal{S}$ if

$$\sup \left|x^\alpha \partial^\beta \left(\varphi_{\frac{1+\epsilon}{2}} - \varphi\right)\right| \to 0 \quad \text{for all} \quad \alpha, \beta \geq 0,$$

and a linear functional F on $\mathcal{S}$ is an element of $\mathcal{S}'$ if $\langle F, \varphi_{\frac{1+\epsilon}{2}}\rangle \to \langle F, \varphi\rangle$ whenever $\varphi_{\frac{1+\epsilon}{2}} \to \varphi$ in $\mathcal{S}$. It is easy to show that for any $\varphi, \psi \in \mathcal{S}$

$$\langle \varphi, \hat{\psi}\rangle = \langle \hat{\varphi}, \psi\rangle,$$

and this allows us to define the Fourier transform for $F \in \mathcal{S}'$ by setting

$$\langle \hat{F}, \psi\rangle = \langle F, \hat{\psi}\rangle \text{ for every } \psi \in \mathcal{S}.$$

Since one can also extend the definition of σ to $\mathcal{S}'$ via the definition $\langle \sigma(F), \psi\rangle = \langle F, \sigma(\psi)\rangle$, the identity $F = \mathcal{F}^{-1}\hat{F}$ still holds in this generality.

Corollary (4.1.25)[222]: (Young's inequality). *Let* $0 \leq \epsilon \leq \infty$ *satisfy* $\epsilon = 0$.

Then for all $f_j \in L^{1+2\epsilon}$, $g_j \in L^{1+3\epsilon}$, we have $f_j \star g_j \in L^{1+\epsilon}$ with

$$\|\sum_j (f_j \star g_j)\|_{L^{1+\epsilon}} \leq \sum_j \|f_j\|_{L^{1+2\epsilon}} \|g_j\|_{L^{1+3\epsilon}}. \tag{53}$$

Proof. We use $\frac{1+\epsilon}{\epsilon}$ to denote the conjugate of $1+\epsilon$. Then we have

$$\frac{1}{1+3\epsilon} + \frac{1}{1+\epsilon} + \frac{1}{1+2\epsilon} = 1, \quad \frac{1+2\epsilon}{1+\epsilon} + \frac{1+2\epsilon}{1+3\epsilon} = 1, \text{ and } \frac{1+3\epsilon}{1+\epsilon} + \frac{1+3\epsilon}{1+2\epsilon} = 1.$$

First use Hölder's inequality with exponents $\frac{1+3\epsilon}{3\epsilon}$, $1+\epsilon$, and $\frac{1+2\epsilon}{2\epsilon}$:

$$\left|\sum_j (f_j \star g_j)(x)\right| \leq \int \sum_j |f_j(y)||g_j(x-y)| \mathrm{d}y$$

$$= \int \sum_j |f_j(y)|^{\frac{3\epsilon(1+2\epsilon)}{1+3\epsilon}} \left(|f_j(y)|^{\frac{1+2\epsilon}{1+\epsilon}} |g_j(x-y)|^{\frac{1+3\epsilon}{1+\epsilon}}\right) |g_j(x-y)|^{\frac{2\epsilon(1+3\epsilon)}{1+2\epsilon}} \mathrm{d}y$$

$$\leq \sum_j \|f_j\|_{L^{1+2\epsilon}}^{\frac{3\epsilon(1+2\epsilon)}{1+3\epsilon}} \left(\int |f_j(y)|^{1+2\epsilon} |g_j(x-y)|^{1+3\epsilon} \mathrm{d}y\right)^{\frac{1}{1+\epsilon}} \left(\int |g_j(x-y)|^{1+3\epsilon} \mathrm{d}y\right)^{\frac{2\epsilon}{1+2\epsilon}}$$

$$= \sum_j \|f_j\|_{L^{1+2\epsilon}}^{\frac{3\epsilon(1+2\epsilon)}{1+3\epsilon}} \left(\int |f_j(y)|^{1+2\epsilon} |g_j(x-y)|^{1+3\epsilon} \mathrm{d}y\right)^{\frac{1}{1+\epsilon}} \|g_j\|_{L^{1+3\epsilon}}^{\frac{2\epsilon(1+3\epsilon)}{1+2\epsilon}}$$

Now take the $L^{1+\epsilon}$ norm (with respect to x):

$$\|\sum_j (f_j \star g_j)\|_{L^{1+\epsilon}}$$

$$\leq \sum_j \|f_j\|_{L^{1+2\epsilon}}^{\frac{3\epsilon(1+2\epsilon)}{1+3\epsilon}} \|g_j\|_{L^{1+3\epsilon}}^{\frac{2\epsilon(1+3\epsilon)}{1+2\epsilon}} \left(\int \int |f_j(y)|^{1+2\epsilon} |g_j(x - y)|^{1+3\epsilon} \mathrm{d}y \mathrm{d}x\right)^{\frac{1}{1+\epsilon}} = \sum_j \|f_j\|_{L^{1+2\epsilon}}^{\frac{3\epsilon(1+2\epsilon)}{1+3\epsilon}} \|g_j\|_{L^{1+3\epsilon}}^{\frac{2\epsilon(1+3\epsilon)}{1+2\epsilon}} \|f_j\|_{L^{1+2\epsilon}}^{\frac{1+2\epsilon}{1+\epsilon}} \|g_j\|_{L^{1+3\epsilon}}^{\frac{1+3\epsilon}{1+\epsilon}}$$

$$= \sum_j \|f_j\|_{L^{1+2\epsilon}} \|g_j\|_{L^{1+3\epsilon}}.$$

We will need a version of this inequality that allows $L^{1+2\epsilon}$ on the right-hand side to be replaced by $L^{1+2\epsilon,\infty}$. The price we have to pay for this (at least initially) is that we also weaken the left-hand side; and note that we have also lost the possibility of some endpoint values ($\epsilon = \infty$ and $1+\epsilon$, $\epsilon = 0, \infty$) that are allowed in (53). In fact one can keep the full $L^{1+\epsilon}$ norm on the left, provided that $\epsilon > 0$; but this requires Proposition (4.1.26) as an intermediate step and the Marcinkiewicz Interpolation Theorem.

Corollary(4.1.26)[222]: *Suppose that* $0 \le \epsilon < \infty$. *If* $f_j \in L^{1+2\epsilon,\infty}$ *and* $g_j \in L^{1+\epsilon}$ *with*

$$\frac{2+\epsilon}{1+\epsilon} = \frac{1}{1+2\epsilon} + \frac{1}{1+\epsilon},$$

then $f_j \star g_j \in L^{1+\epsilon,\infty}$ *with*

$$\|\sum_j (f_j \star g_j)\|_{L^{1+\epsilon,\infty}} \le c_{1+\epsilon,1+2\epsilon,1+\epsilon} \sum_j \|f_j\|_{L^{1+2\epsilon,\infty}} \|g_j\|_{L^{1+\epsilon}}. \quad (54)$$

Proof. We follow the proof in Grafakos [135], skipping some of the algebra. We have already introduced the main step, the splitting of f_j in Lemma (4.1.24). For a fixed $M > 0$ we set $f_j = (f_j)_{M-} + (f_j)_{M+}$. Using (49) and Hölder's inequality we obtain

$$|\sum_j \big((f_j)_{M-} \star g_j\big)(x)| \le \sum_j \|(f_j)_{M-}\|_{L^{\frac{1+3\epsilon}{3\epsilon}}} \|g_j\|_{L^{1+\epsilon}}$$

$$\le \sum_j \left(\frac{1+3\epsilon}{1+6\epsilon^2} M^{\frac{1+6\epsilon^2}{3\epsilon}} \|f_j\|^{1+2\epsilon}_{L^{1+2\epsilon,\infty}}\right)^{\frac{3\epsilon}{1+3\epsilon}} \|g_j\|_{L^{1+\epsilon}},$$

where $\left(1+\epsilon, \frac{1+3\epsilon}{3\epsilon}\right)$ are conjugate; the right-hand side reduces to $M\|g_j\|_{L^1}$ if $\epsilon = 0$. Note in particular that if

$$M = \sum_j \left(\alpha^{\frac{1+3\epsilon}{3\epsilon}} 2^{-\frac{1+3\epsilon}{3\epsilon}} (1+2\epsilon)(1+\epsilon)^{-1} \|f_j\|^{-(1+2\epsilon)}_{L^{1+2\epsilon,\infty}} \|g_j\|^{-\frac{1+3\epsilon}{3\epsilon}}_{L^{1+\epsilon}}\right)^{\frac{3\epsilon}{1+6\epsilon^2}}$$

(or $\alpha/2\|g_j\|_{L^1}$ if $\epsilon = 0$), then $d_{(f_j)_{M-}\star g_j}(\alpha/2) = 0$.

For $(f_j)_{M+}$ we can use (50) and apply Young's inequality to yield

$$\|\sum_j \left(\left(f_j\right)_{M+} \star g_j\right)\|_{L^{1+\epsilon}} \le \sum_j \|(f_j)_{M+}\|_{L^1} \|g_j\|_{L^{1+\epsilon}}$$

$$\le \frac{1+2\epsilon}{2\epsilon} M^{-2\epsilon} \sum_j \|f_j\|^{1+2\epsilon}_{L^{1+2\epsilon,\infty}} \|g_j\|_{L^{1+\epsilon}}.$$

Choosing M as above and using (9) it follows that

$$d_{\sum_j (f_j\star g_j)}(\alpha) \le \sum_j d_{(f_j)_{M+}\star g_j}(\alpha/2) \le \left(2\sum_j \|(f_j)_{M+}\star g_j\|_{L^{1+\epsilon}}\alpha^{-1}\right)^{1+\epsilon}$$

$$\le \left(2(1+2\epsilon)M^{-\epsilon}\sum_j \|f_j\|_{L^{1+2\epsilon,\infty}}^{1+2\epsilon}\|g_j\|_{L^{1+\epsilon}}(2\epsilon)^{-1}\alpha^{-1}\right)^{1+\epsilon}$$

$$= C\sum_j \|f_j\|_{L^{1+2\epsilon,\infty}}^{1+\epsilon}\|g_j\|_{L^{1+\epsilon}}^{1+\epsilon}\alpha^{-(1+\epsilon)},$$

which yields (54).

Corollary (4.1.27)[222]: [127] *For* $0<\epsilon<\infty$ *there exists a constant* $c=c_{n,2+\epsilon}$ *such that if* $f_j\in\dot{H}^{1+\epsilon}(\mathbb{R}^n)$ *with* $s=n\left(\frac{\epsilon}{2+\epsilon}\right)$, *then* $f_j\in L^{2+\epsilon}(\mathbb{R}^n)$ *and*

$$\left\|\sum_j f_j\right\|_{L^{2+\epsilon}} \le c\sum_j \|f_j\|_{\dot{H}^{1+\epsilon}}. \tag{55}$$

Proof. First we prove the result when $\|f_j\|_{\dot{H}^{1+\epsilon}}=1$. For such an f_j, write $f_j=(f_j)_{<R}+(f_j)_{>R}$, where

$$(f_j)_{<R}=\mathcal{F}^{-1}\big(\hat{f}_j\chi_{\{|\xi|\le R\}}\big) \text{ and } (f_j)_{>R}=\mathcal{F}^{-1}\big(\hat{f}_j\chi_{\{|\xi|>R\}}\big). \tag{56}$$

In both expressions the Fourier inversion formula makes sense: for $(f_j)_{>R}$ we know that $\hat{f}_j\chi>R\in L^2(\mathbb{R}^n)$, and $\mathcal{F}$ (and likewise $\mathcal{F}^{-1}$) is defined on L^2; while for $(f_j)_{<R}$ we know that $\hat{f}_j\in L^1_{1oc}(\mathbb{R}^n)$, and so $\hat{f}_j\chi\le R\in L^1(\mathbb{R}^n)$ which means that we can write $(f_j)_{<R}$ using the integral form of the inversion formula (13):

$$\sum_j (f_j)_{<R}(x)=\int_{|\xi|\le R}\sum_j e^{2\pi i\xi\cdot x}\hat{f}_j(\xi)d\xi.$$

Thus

$$\|\sum_j (f_j)_{<R}\|_{L^\infty}\le\int_{|\xi|\le R}\sum_j |\xi|^{-(1+\epsilon)}|\xi|^{1+\epsilon}|\hat{f}_j(\xi)|d\xi$$

$$\le\left(\int_{|\xi|\le R}|\xi|^{-2(1+\epsilon)}d\xi\right)^{\frac{1}{2}}\sum_j \|f_j\|_{\dot{H}^{1+\epsilon}}=C_{1+\epsilon}R^{\frac{n}{2}-(1+\epsilon)}=C_{1+\epsilon}R^{\frac{n}{2+\epsilon}},$$

since we took $\|f_j\|_{\dot{H}^{1+\epsilon}} = 1$ and $1+\epsilon = n\left(\frac{\epsilon}{2+\epsilon}\right)$. Now, since for any choice of R

$$d_{f_j}(\alpha) \leq d_{f_{j<R}}(\alpha/2) + d_{f_{j>R}}(\alpha/2)$$

(using (9)), we can choose R to depend on α, $R = R_\alpha := (\alpha/2C_{1+\epsilon})^{\frac{2+\epsilon}{n}}$, and then we have

$$d_{f_{j<R_\alpha}}(\alpha/2) = 0,$$

it follows that $d_{f_j}(\alpha) \leq d_{f_{j>R_\alpha}}(\alpha/2)$. Thus, using the fact that the Fourier transform is an isometry from L^2 into itself,

$$\begin{aligned}
\|f_j\|_{L^{2+\epsilon}}^{2+\epsilon} &\leq (2+\epsilon)\int_0^\infty \sum_j \alpha^{1+\epsilon}\, d_{f_{j>R_\alpha}}\left(\frac{\alpha}{2}\right) \mathrm{d}\alpha \\
&\leq (2+\epsilon)\int_0^\infty \sum_j \alpha^{1+\epsilon}\frac{4}{\alpha^2}\|(f_j)_{>R_\alpha}\|_{L^2}^2 \mathrm{d}\alpha \\
&= C\int_0^\infty \sum_j \alpha^{\epsilon-1}\,\|\mathcal{F}\left((f_j)_{>R_\alpha}\right)\|_{L^2}^2 \mathrm{d}\alpha \\
&= C\int_0^\infty \alpha^{\epsilon-1}\int_{|\xi|\geq R_\alpha}\sum_j |\hat{f}_j(\xi)|^2 \mathrm{d}\xi \mathrm{d}\alpha \\
&= C\int_{\mathbb{R}^n}\sum_j \left(\int_0^{2C_{1+\epsilon}|\xi|^{\frac{n}{2+\epsilon}}} \alpha^{\epsilon-1}\,\mathrm{d}\alpha\right)|\hat{f}_j(\xi)|^2 \mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}^n}\sum_j |\xi|^{\frac{n(\epsilon)}{2+\epsilon}}|\hat{f}_j(\xi)|^2 \mathrm{d}\xi = C,
\end{aligned}$$

since $\frac{n(\epsilon)}{2+\epsilon} = 2(1+\epsilon)$ and we took $\|f_j\|_{\dot{H}^{1+\epsilon}} = 1$.

Thus for $f_j \in \dot{H}^{1+\epsilon}$ with $\|f_j\|_{\dot{H}^{1+\epsilon}} = 1$ we have $\|f_j\|_{L^{2+\epsilon}} \leq C$, and (55) follows for general $f_j \in \dot{H}^{1+\epsilon}$ on applying this result to $g_j = f_j/\|f_j\|_{\dot{H}^{1+\epsilon}}$

Corollary (4.1.28)[222]: [127] *There is a fixed* $\varphi \in \mathcal{S}$ *such that if* $\hat{f}_j$ *is supported in* $(0,R)$, *then* $f_j = (\mathfrak{D}_{1/R}\varphi) \star f_j$.

Proof. Take $\varphi \in \mathcal{S}$ so that $\hat{\varphi} = 1$ on $(0,1)$. Then

$$\widehat{\mathfrak{D}_{1/R}\varphi}(\xi) = \hat{\varphi}(\xi/R)$$

which is equal to 1 on $B(0,R)$. Thus $\left(\mathfrak{D}_{1/R}\varphi\right) \star f_j - f_j$ has Fourier transform zero, and the Corollary follows. (57)

Corollary (4.1.29)[222]: (Weak-strong Bernstein inequality). *Let* $0 \leq \epsilon < \infty$ *and suppose that* $f_j \in L^{1+\epsilon,\infty}(\mathbb{R}^n)$ *and that* $\hat{f}_j$ *is supported in* $(0,R)$. *Then for each* $1+2\epsilon$ *with* $\epsilon < \infty$ *there exists a constant* $c_{1+2\epsilon,1+\epsilon}$ *such that*

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \leq cR^{n\left(\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}\right)} \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}. \tag{58}$$

Proof. We follow the standard proof, replacing Young's inequality by its weak form, and making use of the interpolation result of Lemma (4.1.23). First we prove the weak version

$$\|\sum_j f_j\|_{L^{1+2\epsilon,\infty}} \leq cR^{n\left(\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}\right)} \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}$$

valid for all $0 \leq \epsilon < \infty$. To do this we simply apply the weak form of Young's inequality (Proposition (4.1.26)) to $f_j = \left(\mathfrak{D}_{1/R}\varphi\right) \star f_j$:

$$\|\sum_j f_j\|_{L^{1+2\epsilon,\infty}} = \sum_j \|\left(\mathfrak{D}_{1/R}\varphi\right) \star f_j\|_{L^{1+2\epsilon,\infty}} \leq c\sum_j \|\mathfrak{D}_{1/R}\varphi\|_{L^{1+\epsilon}} \|f_j\|_{L^{1+\epsilon,\infty}},$$

where

$$\frac{2+\epsilon}{1+\epsilon} = \frac{2}{1+\epsilon} + \frac{1}{1+2\epsilon}$$

with $0 < \epsilon < \infty$. It follows using (57) that

$$\|\sum_j f_j\|_{L^{1,\infty}} \leq cR^{n\left(-\frac{\epsilon}{1+\epsilon}\right)} \sum_j \|f_j\|_{L^{1+\epsilon,\infty}} \text{ and } \|\sum_j f_j\|_{L^{2(1+\epsilon),\infty}}$$

$$\leq cR^{n\left(\frac{1}{2}\right)} \sum_j \|f_j\|_{L^{1+\epsilon,\infty}},$$

and we then obtain (58) by interpolation of $L^{1+\epsilon}$ between $L^{1,\infty}$ and $L^{2(1+\epsilon),\infty}$ (Lemma (4.1.23)),

$$\|\sum_j f_j\|_{L^{1+\epsilon}} \leq c\sum_j \|f_j\|_{L^{1,\infty}}^{\frac{1}{1+2\epsilon}} \|f_j\|_{L^{2(1+\epsilon),\infty}}^{\frac{1+\epsilon}{1+2\epsilon}} \leq c\sum_j \|f_j\|_{L^{1+\epsilon,\infty}} .$$

Corollary (4.1.30)[222]: *Take* $\epsilon > 0$ *and* $\epsilon \geq -1$ *with* $n < \frac{(1+\epsilon)(2\epsilon-1)}{1+2\epsilon}$. *There exists a constant* $c_{1+2\epsilon,1+\epsilon,1+\epsilon}$ *such that if* $f_j \in L^{1+\epsilon,\infty}(\mathbb{R}^n) \cap \dot{H}^{1+\epsilon}(\mathbb{R}^n)$, *then* $f_j \in L^{1+2\epsilon}(\mathbb{R}^n)$ *and*

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \leq c_{1+2\epsilon,1+\epsilon,1+\epsilon}\sum_j \|f_j\|^{\theta}_{L^{1+\epsilon,\infty}}\|f_j\|^{1-\theta}_{\dot{H}^{1+\epsilon}} \text{ for every } f_j \in L^{1+\epsilon,\infty}\cap \dot{H}^{1+\epsilon}, \quad (59)$$

where

$$\frac{1}{1+2\epsilon} = \frac{\theta}{1+\epsilon} + (1-\theta)\left(\frac{1}{2} - \frac{1+\epsilon}{n}\right) \quad (60)$$

Proof. First we prove the theorem in the case $\epsilon \geq 0$. As in the proof of Theorem (4.1.27) we write

$$f_j = (f_j)_{<R} + (f_j)_{>R},$$

where $(f_j)_{<R}$ and $(f_j)_{>R}$ are defined in (56).

Using the endpoint Sobolev embedding $\dot{H}^{n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}(\mathbb{R}^n) \subset L^{2+\epsilon}(\mathbb{R}^n)$ from Theorem (4.1.27) (taking $\dot{H}^0 = L^2$ when $\epsilon = 0$) we can estimate

$$\|\sum_j (f_j)_{>R}\|_{L^{2+\epsilon}} \leq c\sum_j \|(f_j)_{>R}\|_{\dot{H}^{n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}} = c\left(\int_{|\xi|\geq R}\sum_j |\xi|^{2n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}|\hat{f}_j(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq \frac{c}{R^{1+\epsilon-n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}}\left(\int_{|\xi|\geq R}\sum_j |\xi|^{2(1+\epsilon)}|\hat{f}_j(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq \frac{c}{R^{1+\epsilon-n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}}\sum_j \|f_j\|_{\dot{H}^{1+\epsilon}},$$

while

$$\|\sum_j (f_j)_{<R}\|_{L^{2+\epsilon}} \leq cR^{n\left(\frac{1}{(2+\epsilon)(1+\epsilon)}\right)}\sum_j \|(f_j)_{<R}\|_{L^{1+\epsilon,\infty}} \leq cR^{n\left(\frac{1}{(2+\epsilon)(1+\epsilon)}\right)}\sum_j \|f_j\|_{L^{1+\epsilon,\infty}}$$

using the weak-strong Bernstein inequality from Lemma (4.1.29) and (10).

Thus

$$\|\sum_j f_j\|_{L^{2+\epsilon}} \leq c\sum_j \left(R^{n\left(\frac{1}{(2+\epsilon)(1+\epsilon)}\right)}\|f_j\|_{L^{1+\epsilon,\infty}} + R^{-(1+\epsilon)+n\left(\frac{\epsilon}{2(2+\epsilon)}\right)}\|f_j\|_{\dot{H}^{1+\epsilon}}\right).$$

Choosing

$$R^{1+\epsilon+n\left(\frac{1-\epsilon}{1+\epsilon}\right)} = \sum_j \frac{\|f_j\|_{\dot{H}^{1+\epsilon}}}{\|f_j\|_{L^{1+\epsilon,\infty}}}$$

we obtain

$$\|\sum_j f_j\|_{L^{2+\epsilon}} \le c \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\theta} \|f_j\|_{\dot{H}^{1+\epsilon}}^{1-\theta}, \tag{61}$$

where

$$\theta = 1 - n \frac{\frac{1}{(2+\epsilon)(1+\epsilon)}}{1+\epsilon+n\left(\frac{1-\epsilon}{1+\epsilon}\right)},$$

which on rearrangement yields the condition (60).

If $0 \le \epsilon < 1$, then we first interpolate $L^{1+2\epsilon}$ between $L^{1+\epsilon,\infty}$ and L^2, and then use the above result with $\epsilon = \frac{1}{2}$. Setting $\frac{1}{2} = \frac{\theta'}{1+\epsilon} + (1-\theta')\left(\frac{1}{2} - \frac{1+\epsilon}{n}\right)$ we have

$$\begin{aligned}\|\sum_j f_j\|_{L^{1+2\epsilon}} &\le c \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\frac{(1+\epsilon)(1-2\epsilon)}{(1+2\epsilon)(1-\epsilon)}} \|f_j\|_{L^2}^{\frac{2(\epsilon)}{(1+2\epsilon)(1-\epsilon)}} \\ &\le c \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\frac{(1+\epsilon)(1-2\epsilon)}{(1+2\epsilon)(1-\epsilon)}} \left(c\|f_j\|_{L^{1+\epsilon,\infty}}^{\theta'} \|f_j\|_{\dot{H}^{1+\epsilon}}^{1-\theta'}\right)^{\frac{2(\epsilon)}{(1+2\epsilon)(1-\epsilon)}} \\ &= c \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\theta} \|f_j\|_{\dot{H}^{1+\epsilon}}^{1-\theta},\end{aligned}$$

with θ given by (60), as required.

Corollary (4.1.31)[222]: [127] *If* $f_j \in L^1_{1\text{oc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n)$, *then* $f_j \in \mathrm{BMO}(\mathbb{R}^n)$ *and there exists a constant* $C = C(n)$ *such that*

$$\|\sum_j f_j\|_{\mathrm{BMO}} \le C \sum_j \|f_j\|_{\dot{H}^{n}/2} \; for\ all\ f_j \in L^1_{1\text{oc}}(\mathbb{R}^n) \cap \dot{H}^{n/2}(\mathbb{R}^n) .$$

Proof. We write $f_j = (f_j)_{<R} + (f_j)_{>R}$ as in the proof of Theorem (4.1.27) and then, recalling (29),

$$\begin{aligned}&\frac{1}{|Q|}\int_Q \sum_j |f_j - (f_j)_Q| \\ &\le \sqrt{n}|Q|^{1/n} \sum_j \|\nabla (f_j)_{<R}\|_{L^\infty(Q)} + \frac{1}{|Q|}\int_Q \sum_j |(f_j)_{>R} - \left((f_j)_{>R}\right)_Q| \\ &\le c_n |Q|^{\frac{1}{n}} \int_{|\xi|\le R} \sum_j |\xi||\hat{f}_j(\xi)| \mathrm{d}\xi + \frac{2}{|Q|^{1/2}}\left(\int_Q \sum_j |(f_j)_{>R}|^2\right)^{1/2}\end{aligned}$$

$$\leq c_n|Q|^{1/n}R\left(\int_{\mathbb{R}^n}\sum_j |\xi|^n|\hat{f}_j(\xi)|^2 \mathrm{d}\xi\right)^{1/2} + \frac{2}{|Q|^{1/2}}\left(\int_{|\xi|\geq R}\sum_j |\hat{f}_j(\xi)|^2 \mathrm{d}\xi\right)^{1/2}$$

$$\leq c_n\left[|Q|^{\frac{1}{n}}R + |Q|^{-\frac{1}{2}}R^{-\frac{n}{2}}\right]\sum_j \|f_j\|_{\dot{H}^{n/2}}.$$

Choosing $R = |Q|^{-1/n}$ yields

$$\frac{1}{|Q|}\int_Q\sum_j |f_j - (f_j)_Q| \leq C\sum_j \|f_j\|_{\dot{H}^{n/2};}$$

taking the supremum over all cubes $Q \subset \mathbb{R}^n$ yields $\|\sum_j f_j\|_{\mathrm{BMO}} \leq C\sum_j \|f_j\|_{\dot{H}^{n/2}}$. □

Corollary(4.1.32)[222]: [127] *Let Q be any cube in $\mathbb{R}^n$. Given $f_j \in L^1(Q)$ and $M > 0$ there exists a countable collection $\{Q_{j_0}\}$ of disjoint open cubes such that $|\sum_j f_j(x)| \leq M$ for almost every $x \in Q \setminus \bigcup_{j_0} Q_{j_0}$ and*

$$M < \frac{1}{|Q_{j_0}|}\int_{Q_{j_0}}\sum_j |f_j(x)|\mathrm{d}x \leq 2^n M \tag{62}$$

for every Q_{j_0}.

Note that it follows from (62) that

$$\sum_{j_0} |Q_{j_0}| \leq \frac{1}{M}\int_Q\sum_j |f_j|. \tag{63}$$

Proof. Decompose Q, by halving each side, into a collection $\mathcal{Q}_0$ of 2^n equal cubes. Select one of these cubes $\hat{Q}$ if

$$\frac{1}{|\hat{Q}|}\int_{\hat{Q}}\sum_j |f_j(x)|\mathrm{d}x > M. \tag{64}$$

Call the selected cubes $\mathcal{C}_1$ and let $\mathcal{Q}_1 = \mathcal{Q}_0 \setminus \mathcal{C}_1$.

Repeat this process inductively, to produce a set $\mathcal{C} = \bigcup_j \mathcal{C}_j$ of selected cubes, on which (64) holds. Note that if $\hat{Q}$ was selected at step k, then it is contained in a cube $Q' \in \mathcal{Q}_{k-1}$, and so

$$M < \frac{1}{|\hat{Q}|}\int_{\hat{Q}}\sum_j |f_j(x)|\mathrm{d}x \leq 2^n\frac{1}{|Q|}\int_{Q'}\sum_j |f_j(x)|\mathrm{d}x \leq 2^n M.$$

Enumerate the countable set $\mathcal{C}$ of cubes as $\{Q_j\}_{j=1}^{\infty}$.

Finally, if $x \in Q \setminus \bigcup_j \; Q_j$, then there exists a sequence of cubes Q_k containing x with sides shrinking to zero and such that

$$\frac{1}{|Q_k|}\int_{Q_k}\sum_j \; |f_j(x)|\mathrm{d}x \leq M.$$

It follows from the Lebesgue Differentiation Theorem that $|f_j(x)| \leq M$ for almost every $x \in Q \setminus \bigcup_j Q_j$.

Corollary (4.1.33)[222]: (See [127]) (John-Nirenberg inequality). *There exist constants c and C (depending only on n) such that if $f_j \in \mathrm{BMO}(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$*

$$|\{x \in Q: \sum_j \; |f_j - (f_j)_Q| > \alpha\}| \leq \sum_j \frac{C}{\|f_j\|_{\mathrm{BMO}}} \mathrm{e}^{-c\alpha/\|f_j\|_{\mathrm{BMO}}} \int_Q \sum_j \; |f_j - (f_j)_Q| \quad (65)$$

for all $\alpha \geq \|f_j\|_{\mathrm{BMO}}$.

Proof. We prove the result assuming that $\|f_j\|_{\mathrm{BMO}} = 1$; we then obtain (65) by applying the resulting inequality to $f_j/\|f_j\|_{\mathrm{BMO}}$. Let $F(\alpha)$ be the infimum of all numbers such that the inequality

$$|\{x \in Q : \sum_j \; |f_j(x)| > \alpha\}| \leq F(\alpha) \int_Q \sum_j \; |f_j| \quad (66)$$

holds for all $f_j \in L^1(Q)$ and all cubes Q; note (cf. Lemma (4.1.22)) that $F(\alpha) \leq 1/\alpha$. Following the original proof of John & Nirenberg [1] we show that for all $\alpha \geq 2^n$,

$$F(\alpha) \leq \frac{1}{M} F(\alpha - 2^n M) \text{ for all } 1 \leq M \leq 2^{-n}\alpha. \quad (67)$$

Given M in this range we decompose f_j using Proposition (4.1.32). Now, if $|f_j(x)| > \alpha \geq 2^n$, then $x \in Q_k$ for some k, and we know that $|(f_j)_{Q_k}| \leq 2^n M$ from (62). So then

$$|\{x \in Q: \sum_j \; |f_j(x)| > \alpha\}| \leq \sum_k |\{x \in Q_k: \sum_j \; |f_j(x) - (f_j)_{Q_k}| > \alpha - 2^n M\}|.$$

We can now use (66) on the cube Q_k for the function $f_j - (f_j)_{Q_k}$, so that

$$|\{x \in Q_k: \sum_j |f_j(x) - (f_j)_{Q_k}| > \alpha - 2^n M\}| \leq F(\alpha - 2^n M) \int_{Q_k} \sum_j |f_j - (f_j)_{Q_k}| dx$$
$$\leq F(\alpha - 2^n M)|Q_k|$$

(recall that we took $\|f_j\|_{\mathrm{BMO}} = 1$). It follows using (63) that

$$|\{x \in Q : \sum_j |f_j(x)| > \alpha\}| \leq \left(\sum_k |Q_k|\right) F((y - 2^n M)$$
$$\leq \frac{1}{M} F(\alpha - 2^n M) \int_Q \sum_j |f_j| dx,$$

which is (67).

To finish the proof we iterate (67) in a suitable way. We remarked above that $F(\alpha) \leq 1/\alpha$; now observe that

$$\frac{1}{\alpha} \leq C e^{-\alpha/2^n e} \quad \text{for all} \quad 1 \leq \alpha \leq 1 + 2^n e,$$

$F(\alpha + 2^n e) \leq \frac{\leq 1}{e} F(\alpha)$, we obtain for $C= \max_{1\alpha \leq 1+2^n e} \alpha^{-1} e^{\alpha/2^n e}$. Iterating (67) with $M =$ e, which implies that $F(\alpha) \leq C e^{-c\alpha}$ for all $\alpha \geq 1$,
where $c = 1/2^n e$, which gives (65).

The more usually quoted form of this inequality,

$$|\{x \in Q: \sum_j |f_j - (f_j)_Q| > \alpha\}| \leq C \sum_j |Q| e^{-c\alpha/\|f_j\|_{\mathrm{BMO}}},$$

follows immediately from the definition of $\|f_j\|_{\mathrm{BMO}}$.

Corollary (4.1.34)[222]: (See [127])*For any* $0 < \epsilon < \infty$, *if* $f_j \in L^{1+\epsilon,\infty}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, *then* $f_j \in L^{1+2\epsilon}(\mathbb{R}^n)$ *and there exists a constant* $C = C(1+\epsilon, 1+2\epsilon, n)$ *such that*

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \leq C \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\frac{1+\epsilon}{1+2\epsilon}} \|f_j\|_{\mathrm{BMO}}^{\frac{\epsilon}{1+2\epsilon}}. \tag{68}$$

Proof. First we note that it is a consequence of the John-Nirenberg inequality from Lemma (4.1.33) that if $f_j \in \mathrm{BMO} \cap L^1$, then

$$d_{f_j}(\alpha) \leq C e^{-C\alpha/\|f_j\|_{\mathrm{BMO}}} \|f_j\|_{L^1} \tag{69}$$

for all $\alpha > \|f_j\|_{\mathrm{BMO}}$; this follows by taking $|Q| \to \infty$ in (65), since when $f_j \in L^1$,

$$|\sum_j (f_j)_Q| \leq \frac{1}{|Q|}\int \sum_j |f_j| \to 0 \text{ as } |Q| \to \infty,$$

and $\int_Q |f_j - (f_j)_Q| \mathrm{d}x \leq 2\int_Q |f_j| \mathrm{d}x$.

Now take $f_j \in \mathrm{BMO}$ with $\|f_j\|_{\mathrm{BMO}} = 1$. Split $f_j = (f_j)_{1-} + (f_j)_{1+}$ as in Lemma (4.1.24). Since $(f_j)_{1-} \in L^\infty$, $\|\sum_j (f_j)_{1-}\|_{\mathrm{BMO}} \leq 2\sum_j \|(f_j)_{1-}\|_{L^\infty} \leq 2$ (using (8.2)); thus $(f_j)_{1+} = f_j - (f_j)_{1-} \in \mathrm{BMO}$ and

$$\|\sum_j (f_j)_{i^+}\|_{\mathrm{BMO}} \leq \sum_j \left(\|f_j\|_{\mathrm{BMO}} + \left\|f_{j_{i^-}}\right\|_{\mathrm{BMO}}\right) \leq 3.$$

Using Lemma (4.1.24) we know that

$$\sum_j \|(f_j)_{1-}\|_{L^{1+2\epsilon}}^{1+2\epsilon} \leq C\sum_j \|(f_j)_{1-}\|_{L^{1+\epsilon,\infty}}^{1+\epsilon}. \tag{70}$$

Also, for $\left(1+\epsilon, \frac{1+2\epsilon}{2\epsilon}\right)$ conjugate,

$$\|\sum_j (f_j)_{1+}\|_{L^1} = \int \sum_j |(f_j)_{1+}| \leq \int \sum_j |(f_j)_{1+}|^{\frac{1+4\epsilon}{1+2\epsilon}} = \sum_j \|(f_j)_{1+}\|_{L^{\frac{1+4\epsilon}{1+2\epsilon}}}^{\frac{1+4\epsilon}{1+2\epsilon}}$$

$$\leq c\sum_j \|(f_j)_{1+}\|_{L^1}^{\frac{2\epsilon}{1+2\epsilon}} \|(f_j)_{1+}\|_{L^{1+\epsilon,\infty}}$$

(since $0 < \frac{2\epsilon}{1+2\epsilon} < 1$ we can use weak-$L^{1+2\epsilon}$ interpolation), which yields

$$\|\sum_j (f_j)_{1+}\|_{L^1} \leq c\sum_j \|(f_j)_{1+}\|_{L^{1+\epsilon,\infty}}^{1+\epsilon}.$$

Now we calculate

$$\|\sum_j (f_j)_{1+}\|_{L^{1+2\epsilon}}^{1+2\epsilon} = (1+2\epsilon)\int_0^\infty \sum_j \alpha^{2\epsilon} d_{(f_j)_{1^+}}(\alpha)\mathrm{d}\alpha$$

$$= (1+2\epsilon)\int_0^1 \sum_j \alpha^{2\epsilon} d_{f_j}(1)\mathrm{d}\alpha + (1+2\epsilon)\int_1^\infty \sum_j \alpha^{2\epsilon} d_{(f_j)_{1^+}}(\alpha)\mathrm{d}\alpha$$

$$\leq \sum_j d_{f_j}(1) + (1+2\epsilon)\left(\int_1^\infty \alpha^{2\epsilon} C\mathrm{e}^{-C\alpha/3}\mathrm{d}\alpha\right)\sum_j \|(f_j)_{1+}\|_{L^1},$$

where we have used (52), (69), and the fact that $\|(f_j)_{1+}\|_{\mathrm{BMO}} \le 3$. Thus

$$\|\sum_j (f_j)_{1+}\|_{L^{1+2\epsilon}}^{1+2\epsilon} \le \sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{1+\epsilon} + C\sum_j \|(f_j)_{1+}\|_{L^{1+\epsilon,\infty}}^{1+\epsilon} \le C\sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{1+\epsilon}. \quad (71)$$

Adding (70) and (69) we obtain

$$\|\sum_j f_j\|_{L^{1+2\epsilon}} \le C\sum_j \|f_j\|_{L^{1+\epsilon,\infty}}^{\frac{1+\epsilon}{1+2\epsilon}};$$

(68) follows.

Corollary (4.1.35)[222]: (Bennett & Sharpley). *For* $0 \le \epsilon \le \infty$,

$$L^{1+\epsilon,1+\epsilon} = (L^1, \mathrm{BMO})_{\frac{\epsilon}{1+\epsilon},1+\epsilon}.$$

Proof. See Chapter 5, Theorem (4.1.31)1, in Bennett & Sharpley [9]. One can also find a proof of this result in the paper by Hanks [137], and of a similar but slightly weaker result (with $L^{1+\epsilon}$ on the left-hand side) using complex interpolation spaces in the paper by Janson& Jones [139].

We note here that the key step in the proof of this result given in Bennett & Sharpley [9] (and in Hanks [137]) is a relationship between the sharp function of f_j,

$$(f_j)_Q^{\#}(x) := Q' \subset Q, Q' \ni x sup\, \frac{1}{|Q|}\int_{Q'} \sum_j |f_j - (f_j)_{Q'}|,$$

its decreasing rearrangement f_j^*, and the function $f_j^{**}(1+\epsilon) := \frac{1}{1+\epsilon}\int_0^{1+\epsilon} \sum_j f_j^*(1+3\epsilon)\mathrm{d}(1+3\epsilon)$:

$$\sum_j f_j^{**}(1+\epsilon) - \sum_j f_j^*(1+\epsilon) \le C\sum_j \left((f_j)_Q^{\#}\right)^*(1+\epsilon)\ 0 < 1+\epsilon < |Q|$$

Corollary (4.1.36)[222]: (Reiteration Corollary). *Let* (X_0, X_1) *be a compatible pair of Banach spaces, and let* $0 \le \theta_0 < \theta_1 \le 1$ *and* $0 \le \epsilon \le \infty$. *Set*

$$Y_0 = (X_0, X_1)_{\theta_0,1+\epsilon} \quad \text{and} \quad Y_1 = (X_0, X_1)_{\theta_1,1+2\epsilon}.$$

If $0 < \theta < 1$ *and* $0 \le \epsilon \le \infty$, *then*

$$(Y_0, Y_1)_{\theta,1+\epsilon} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1,1+\epsilon}.$$

Proof. See Theorem 2.4 of Chapter 5 in Bennett & Sharpley [9], or Theorem 3.5.3 in Bergh & Löfström [130].

Corollary (4.1.37)[222]: (Generalised Gagliardo-Nirenberg with Lorentz spaces). *If* $u \in L^{1+\epsilon,\infty} \cap \mathrm{BMO}$ *for some* $0 < \epsilon < \infty$, *then* $u \in L^{1+2\epsilon,1}$ *and there exists a constant* $C_{n,1+2\epsilon,1+\epsilon}$ *such that*

$$\|u\|_{L^{1+2\epsilon,1}} \le C_{n,1+2\epsilon,1+\epsilon}\|u\|_{L^{1+\epsilon,\infty}}^{\frac{1+\epsilon}{1+2\epsilon}}\|u\|_{\mathrm{BMO}}^{\frac{\epsilon}{1+2\epsilon}}. \tag{72}$$

Note that given the ordering of Lorentz spaces, $L^{1+2\epsilon,1} \subset L^{1+2\epsilon,1+2\epsilon} = L^{1+2\epsilon}$ and so this result implies Theorem (4.1.34) in the case $\epsilon > 0$.

Proof. Using Theorem (4.1.35), since $\epsilon > 0$ we have

$$L^{1+\epsilon,1+3\epsilon} = (L^1, \mathrm{BMO})_{\frac{\epsilon}{1+\epsilon},1+3\epsilon};$$

set $\mathrm{B} = (L^1, \mathrm{BMO})_{1,\infty}$. Note that from (43) $\|\sum_j f_j\|_{\mathrm{B}} \le C\sum_j \|f_j\|_{\mathrm{BMO}}$. Now simply use the Reiteration Theorem to obtain

$$L^{1+2\epsilon,1+\epsilon} = (L^{1+\epsilon,1+3\epsilon}, \mathrm{B})_{\frac{\epsilon}{1+2\epsilon},1+\epsilon},$$

from which the inequality (72) follows immediately using (43).

(One can use interpolation spaces to provide a proof of Theorem (4.1.34) that does not involve Lorentz spaces by using interpolation only with $\epsilon = \infty$ and then interpolation between weak $L^{1+2\epsilon}$ spaces, see McCormick et al. [143].)

Corollary (4.1.38)[222]: [127] *Suppose that* $0 < \epsilon < \infty$. *If* $f_j \in L^{1+2\epsilon,\infty}$ *and* $g_j \in L^{1+3\epsilon}$ *with*

$$\frac{2+\epsilon}{1+\epsilon} = \frac{1}{1+2\epsilon} + \frac{1}{1+3\epsilon},$$

then $f_j \star g_j \in L^{1+\epsilon}$ *with*

$$\|\sum_j f_j \star g_j\|_{L^{1+\epsilon}} \le c_{1+\epsilon,1+2\epsilon,1+3\epsilon}\sum_j \|f_j\|_{L^{1+2\epsilon,\infty}}\|g_j\|_{L^{1+3\epsilon}}. \tag{73}$$

Proof. Note that it follows from the conditions on $1+\epsilon, 1+2\epsilon, 1+3\epsilon$ that $\epsilon > 0$. Fix $f_j \in L^{1+2\epsilon,\infty}$ with $\|f_j\|_{L^{1+2\epsilon,\infty}} = 1$, and consider the linear operator $T(g_j) = f_j \star g_j$. Since $0 < \epsilon < \infty$ we can find $\epsilon \ge 0$, and (11.1) holds. Now using the weak form of Young's inequality from Proposition (4.1.26),

$$\|\sum_j (f_j \star g_j)\|_{L^{1+2\epsilon,\infty}} \le C\sum_j \|g_j\|_{L^{1+\epsilon}} \quad and \quad \|\sum_j (f_j \star g_j)\|_{L^{1+3\epsilon,\infty}} \le C\sum_j \|g_j\|_{L^{1+2\epsilon}}.$$

We can now use the Marcinkiewicz interpolation theorem to guarantee that

$$\|\sum_j (f_j \star g_j)\|_{L^{1+\epsilon}} \le C\sum_j \|g_j\|_{L^{1+\epsilon}}.$$

Since $f_j \star g_j$ is also linear in f_j, we obtain (73).

Corollary (4.1.39)[222]: *For* $0 < \epsilon < \infty$ *there exists a constant* $c = c_{n,2+\epsilon}$ *such that if* $f_j \in \dot{H}^{1+3\epsilon}(\mathbb{R}^n)$ *with* $s = n\left(\frac{\epsilon}{2(2+\epsilon)}\right)$, *then* $f_j \in L^{2+\epsilon}(\mathbb{R}^n)$ *and* $\| \sum_j f_j\|_{L^{2+\epsilon}} \le c \sum_j \|f_j\|_{\dot{H}^{1+3\epsilon}}$.

Proof. We make the pointwise definition $\sum_j f_j(\xi) = |\xi|^{1+3\epsilon} \sum_j \hat{f}_j(\xi)$; since $f_j \in \dot{H}^{1+3\epsilon}(\mathbb{R}^n)$, $\gamma \in L^2(\mathbb{R}^n)$. If we set $g_j = \mathcal{F}^{-1}\gamma$, then $g_j \in L^2(\mathbb{R}^n)$ and $\|g_j\|_{L^2} = \|\gamma\|_{L^2} = \|f_j\|_{\dot{H}^{1+3\epsilon}}$. Now,

$$\sum_j \hat{f}_j(\xi) = \sum_j \frac{|\xi|^{1+3\epsilon}\hat{f}_j(\xi)}{|\xi|^{1+3\epsilon}} = \sum_j \hat{g}_j(\xi)|\xi|^{-(1+3\epsilon)},$$

and so $f_j = g_j \star c_n^{-1}{}_{n-(1+3\epsilon)} P_{n-(1+3\epsilon)}$. Since $P_{n-(1+3\epsilon)} \in L^{\frac{n}{n-(1+3\epsilon)},\infty}$ and $g_j \in L^2$ it follows from Theorem (4.1.38) that $f_j \in L^{2+\epsilon}(\mathbb{R}^n)$.

Section (4.2): Lorentz Spaces and BMO with Hölder Spaces and Fractional Sobolev Spaces

We prove some generalized Gagliardo-Nirenberg interpolation inequalities involving the Lorentz spaces $L^{p,\alpha}$, BMO, and the fractional Sobolev spaces $W^{s,p}$, including also $\dot{C}^\eta$ Hölder spaces.

It is well known that the Gagliardo-Nirenberg inequality plays an important role in the analysis of PDEs, and the references therein. Thus, any possible improvement of this one could be relevant for many purposes. We recall some previous results involving the Gagliardo-Nirenberg inequalities that we shall improve later:

For any $1 \le q < p < \infty$, the following interpolation inequality holds (see Nirenberg [134])

$$\|f\|_{L(\mathbb{R}^n)}p \le c\|f\|_{L(\mathbb{R}^n)}^{\theta_q}\|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}, \frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right). \quad (74)$$

In [133], McCormick et al. proved a stronger version of (74) involving the weak L^q space (denoted as $L^{q,\infty}$) as follows:

$$\|f\|_{L(\mathbb{R}^n)}p \le c\|f\|_{L(\mathbb{R}^n)}^{\theta_{q,\infty}}\|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}. \quad (75)$$

Concerning the critical case $s = n/2$, McCormick et al. [133] obtained

$$\|f\|_{L(\mathbb{R}^n)}p \le C\|f\|_{L,(\mathbb{R}^n)}^{q/_qp}\infty\|f\|_{BMO(\mathbb{R}^n)}^{1-q/p}. \quad (76)$$

Note that (76) is better than (75) since $\|f\|_{BMO(\mathbb{R}^n)} \le c\|f\|_{\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)}$. Furthermore, they also showed a stronger version of inequality (76) for the norm $\|f\|_{L^{p,1}}(\mathbb{R}^n)$ instead of $\|f\|_{L(\mathbb{R}^n)}p$ when $q > 1$.

Another version of (74), in the critical case, was proved by Kozono and Wadade [98] (see also [128]):

$$\|f\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L(\mathbb{R}^n)}^{\frac{q}{p}} \|f\|_{\dot{H}^{\frac{n}{r},r}(\mathbb{R}^n)}^{1-\frac{q}{p}}, \tag{77}$$

for any $1 \leq q < p < \infty$, and for $1 < r < \infty$.
we enhance the results of McCormick et al., [9]. We shall prove a stronger version of (75)

$$\|f\|_{L^{p,\alpha}}(\mathbb{R}^n) \leq c\|f\|_{L(\mathbb{R}^n)}^{\theta_{q,\infty}} \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{1-\theta}. \tag{78}$$

In fact, we shall prove an interpolation inequality which implies (78) After that, we shall prove that (76) holds for $\|f\|_{L^{p,\alpha}}$ instead of $\|f\|_{Lp}$, for any $\alpha > 0$. Finally, we shall study the Gagliardo-Nirenberg type inequality for the case $sp > n$ of the fractional Sobolev space $W^{s,p}$, and also the Lipschitz and Hölder continuous space. We point out that although some of the results can be alternatively obtained by using interpolation spaces (specifically, the reiteration theorem), the precise forms of the inequalities stated here appear to be novel and, moreover, the proofs given in the present are self-contained (save for the use of the John-Nirenberg inequality for the BMO result, which will be recalled later) in contrast to the other mentioned approach.
For the reader convenience, We recall here the definition of the functional spaces that we use throughout We define

$$\|g\|_{L^{q,\alpha}}(\mathbb{R}^n) := \begin{cases} (q\int_0^\infty (\lambda^q |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|)^{\frac{\alpha}{q}} \frac{d\lambda}{\lambda})^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup\limits_{\lambda>0} \lambda(|\{x \in \mathbb{R}^n : |g(x)| > \lambda\}|)^{1/q} & \text{if } \alpha = \infty. \end{cases}$$

The Lorentz spaces $L^{q,\alpha}(\mathbb{R}^n)$ includes all measurable functions $g: \mathbb{R}^n \to \mathbb{R}$ such that $\|g\|_{L^{q,\alpha}}(\mathbb{R}^n) < \infty$. For a definition of Lorentz spaces using rearrangement techniques see [150].
On the other hand, we recall that the space $\dot{H}^{s,r}(\mathbb{R}^n)$, the homogeneous Sobolev space, is defined by

$$\dot{H}^{s,r}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \|(-\Delta)^{\frac{s}{2}} f\|_{L^r} < \infty\}.$$

In particular, we shall denote $\dot{H}^s(\mathbb{R}^n) = \dot{H}^{s,2}(\mathbb{R}^n)$ (see, [136]).
We denote the space of Lipschitz (or Hölder) continuous functions of order $\eta \in (0,1]$ on $\mathbb{R}^n$ by $\dot{C}^\eta(\mathbb{R}^n)$: i.e. functions f such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} < \infty.$$

It is useful to introduce the notation

$$\|f\|_{\dot{C}(\mathbb{R}^n)}\eta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}$$

On the other hand, if $s \in (0,1)$, then we recall $W^{s,p}(\mathbb{R}^n)$ the fractional Sobolev space, endowed with the norm:

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \left(\|f\|^p_{L(\mathbb{R}^n)}p + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \right)^{\frac{1}{p}}$$

When $s > 1$, and s is not an integer, we write $s = m + \sigma$, with m is an integer and $\sigma \in (0,1)$. Then, $W^{s,p}(\mathbb{R}^n)$ is endowed with the norm (see, e.g. [147]):

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \left(\|f\|^p_{W^{m,p}(\mathbb{R}^n)} + \sum_{|\alpha|=m} \| D^\alpha f\|^p_{W^{\sigma,p}(\mathbb{R}^n)} \right)^p$$

Note that if $s \geq 0$ is an integer, then $W^{s,p}(\mathbb{R}^n)$ is the usual Sobolev space.
Finally, concerning the space of functions with bounded mean oscillation (denoted as BMO) see [4].
First of all, we show an interpolation inequality which is regarded as a generalized version of (75).

Theorem (4.2.1)[146]. *Let* $0 < q < p < r \leq \infty$ *and* $\alpha > 0$. *If* $\in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$, *then* $f \in L^{p,\alpha}(\mathbb{R}^n)$ *and*

$$\|f\|_{Lp,\alpha}(\mathbb{R}^n) \leq C\|f\|^{\theta}_{L(\mathbb{R}^n)}q,\infty\|f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^n)}, \tag{79}$$

where $C = C(q,r,p,\alpha) > 0$, *and*

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}.$$

Proof. Let us write

$$\|f\|^{\alpha}_{L^{p,\alpha}}(\mathbb{R}^n) = p\int_0^{\lambda_0} \lambda^\alpha\, |\{|f| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda} + p\int_{\lambda_0}^{\infty} \lambda^\alpha\, |\{|f| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda}. \tag{80}$$

Since $\in L^{q,\infty}(\mathbb{R}^n) \cap L^{r,\infty}(\mathbb{R}^n)$, we have

$$\int_0^{\lambda_0} \lambda^\alpha\, |\{|f| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda} \leq \int_0^{\lambda_0} \lambda^\alpha \left(\frac{\|f\|^q_{L(\mathbb{R}^n)}q,\infty}{\lambda^q} \right)^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{\|f\|^{\alpha q/p}_{L(\mathbb{R}^n)}q,\infty}{\alpha(1-q/p)}\lambda_0^{\alpha(1-q/p)},$$

and

$$\int_{\lambda_0}^{\infty} \lambda^\alpha\, |\{|f| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda} \leq \int_{\lambda_0}^{\infty} \lambda^\alpha \left(\frac{\|f\|^r_{L^{r,\infty}(\mathbb{R}^n)}}{\lambda^r} \right)^{\alpha/p} \frac{d\lambda}{\lambda} = \frac{\|f\|^{\alpha r/p}_{L^{r,\infty}(\mathbb{R}^n)}}{\alpha(r/p-1)}\lambda_0^{\alpha(1-r/p)}.$$

$$\|f\|^{\alpha}_{L^{p,\alpha}}(\mathbb{R}^n) \leq p\left(\frac{\|f\|^{\alpha q/p}_{L(\mathbb{R}^n)}q,\infty}{\alpha(1-q/p)}\lambda_0^{\alpha(1-q/p)} + \frac{\|f\|^{\alpha r/p}_{L^{r,\infty}(\mathbb{R}^n)}}{\alpha(r/p-1)}\lambda_0^{\alpha(1-r/p)} \right)$$

Now, we equalize the right hand side of the above inequality by choosing

$$\lambda_0^{r-q} = \frac{\|f\|^r_{L^{r,\infty}(\mathbb{R}^n)}}{\|f\|^{q_{q,\infty}}_{L(\mathbb{R}^n)}}.$$

As a consequence of Theorem (4.2.1), we have

Corollary (4.2.2)[146]. Let $1 \le q < p$, and $s \ge 0$ with $s > n(1/2 - 1/p)$. For any $\alpha > 0$, there is a constant $C = C(n, p, q, s, \alpha)$ such that if $f \in L^{q,\infty}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ then $f \in L^{p,\alpha}(\mathbb{R}^n)$ and

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \le C\|f\|^{\theta_{q,\infty}}_{L(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{H}^s(\mathbb{R}^n)}, \tag{81}$$

with

$$\frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{n}\right).$$

Concerning the critical case $s = n/2$, we have the following result.

Proof. For any $r > 2$, it follows from the Sobolev embedding theorem

$$\|f\|_{L^r(\mathbb{R}^n)} \le C(n,r)\|f\|_{\dot{H}^s(\mathbb{R}^n)}, with\ s = n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Take $r > p$, we have $s > n\left(\frac{1}{2} - \frac{1}{p}\right)$. By noting that $\|f\|_{L^{r,\infty}(\mathbb{R}^n)} \le \|f\|_{L^r(\mathbb{R}^n)}$, and inserting the last inequality into the right hand side of (79) yield the result.

Theorem (4.2.3)[146]. *Let* $1 < q < p < \infty$ *and* $\alpha > 0$. *If* $\in L^{q,\infty}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$, *then* $f \in L^{p,\alpha}(\mathbb{R}^n)$, *and there is a constant* $C = C(q,p,n,\alpha) > 0$ *such that*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} \le C\|f\|^{\frac{q}{p}}_{L(\mathbb{R}^n)}q,\infty\|f\|^{1-\frac{q}{p}}_{BMO(\mathbb{R}^n)}. \tag{82}$$

Finally, we obtain the Gagliardo-Nirenberg inequality for the η-Hölder space, and the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ with $s \in (0,1)$ such that $sp > n$.

Proof. It suffices to prove that holds for $\|f\|_{BMO(\mathbb{R}^n)} = 1$, then we obtain (82) by applying the resulting inequality to $f/\|f\|_{BMO(\mathbb{R}^n)}$. Let us split $f = f_{1-} + f_{1+}$, with $f_{1-} = f\chi_{\{|f|\le 1\}}$, and $f_{1+} = f\chi_{\{|f|>1\}}$, with $\chi_A(x) = \{^1_0$, if $x \notin A$if$x \in A$.

Then,

$$\|f_{1-}\|^{\alpha}_{L^{p,\alpha}(\mathbb{R}^n)} = p\int_0^\infty \lambda^\alpha\, |\{|f_{1-}| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda} = p\int_0^1 \lambda^\alpha\, |\{|f_{1-}| > \lambda\}|^{\alpha/p}\frac{d\lambda}{\lambda}. \tag{83}$$

On the other hand, we have from $f \in L^{q,\infty}(\mathbb{R}^n)$

$$|\{|f_{1-}| > \lambda\}| \le |\{|f| > \lambda\}| \le \frac{\|f\|^q_{L(\mathbb{R}^n)}q,\infty}{\lambda^q}. \tag{84}$$

Combining (83) and (84) yields

$$\|f_{1-}\|^{\alpha}_{L^{p,\alpha}(\mathbb{R}^n)} \le p\int_0^1 \lambda^\alpha\left(\frac{\|f\|^q_{L(\mathbb{R}^n)}q,\infty}{\lambda^q}\right)^{\alpha/p}\frac{d\lambda}{\lambda} = \frac{p}{\alpha(1-q/p)}\|f\|^{\alpha q/p(\mathbb{R}^n)}_{L^{q,\infty}}.$$

Or

$$\|f_{1-}\|_{L^{p,\alpha}(\mathbb{R}^n)} \le C_1(p,q,\alpha)\|f\|^{q/_q p}_{L,(\mathbb{R}^n)}\infty. \tag{85}$$

Next, we estimate

$$\left\|f_{1+}\right\|_{L^{p,\alpha}(\mathbb{R}^n)}^{\alpha} = p\int_0^\infty \lambda^\alpha \left|\{|f_{1+}| > \lambda\}\right|^{\frac{\alpha}{p}} \frac{d\lambda}{\lambda}$$

$$= p\int_1^\infty \lambda^\alpha\, |\{|f_{1+}| > \lambda\}|^{\alpha/p} \frac{d\lambda}{\lambda}. \quad (86)$$

To estimate the level set $\{|f_{1+}| > \lambda\}$ involving $\|f\|_{BMO}$, we recall the following result, which is a consequence of John-Nirenberg inequality, [6] (see also Corollary 2.2, [8]):

Theorem (4.2.4)[146]. *Let* $0 < q < p < \infty,\ \alpha > 0,$ *and* $\eta \in (0,1)$. *If* $\in L^{q,\infty}(\mathbb{R}^n) \cap \dot{C}^\eta(\mathbb{R}^n)$, *then* $f \in L^{p,\alpha}(\mathbb{R}^n)$ *and*

$$\|f\|_{Lp,\alpha}(\mathbb{R}^n) \le C\|f\|^{\frac{\eta+\frac{n}{p}}{Lq\eta+\frac{n}{\infty q}}}\|f\|^{\frac{\frac{n}{q}-\frac{n}{p}}{\dot{C}(\mathbb{R}^n\frac{n}{q\eta}+\eta}}. \quad (87)$$

We point out that the use of Hölder spaces in such Gagliardo-Nirenberg's inequalities seems to be new in the literature. As a consequence of Theorem (4.2.4) and the embedding $W^{s,r}(\mathbb{R}^n) \subset \dot{C}^\eta(\mathbb{R}^n)$, with $\eta = \frac{sr-n}{r}$ (see [129]), we get the following result.

Proof. For any $\varepsilon > 0$, let us put

$$\rho_{\xi j}(x) := \frac{\chi_{\{|x|\le\varepsilon\}}(x)}{|B_\in(0)|}.$$

Step 1: If $q > 1$, by [133], we have

$$\|f * \rho_\varepsilon\|_{L^{p,\infty}(\mathbb{R}^n)} \le C\|f\|_{L^{q,\infty}(\mathbb{R}^n)}\|\rho_\varepsilon\|_{L^{p_0}(\mathbb{R}^n)},$$

where

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{p_0}.$$

Note that $\|\rho_\varepsilon\|_{L^{p_0}(\mathbb{R}^n)} = C(n,p_0)\varepsilon^{-n\left(\frac{1}{q}-\frac{1}{p}\right)}$. Thus,

$$\|f * \rho_\varepsilon\|_{L^{p,\infty}(\mathbb{R}^n)} \le C\varepsilon^{-n\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^{q,\infty}(\mathbb{R}^n)}. \quad (88)$$

Similarly, we also have

$$\|f * \rho_{\xi j}\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\|f\|_{L^{q,\infty}(\mathbb{R}^n)}. \quad (89)$$

Applying (79) to the function $(f - f * \rho_\varepsilon)$ in Theorem 2.1 yields

$$\|f - f * \rho_{\in \mathrm{i}}\|_{L(\mathbb{R}^n)p} \le C\|f - f * \rho_\in\|_{Lq,\infty}^{\frac{q}{p}}\|f - f * \rho_\varepsilon\|_{L^\infty}^{1-\frac{q}{p}}$$

$$\le C_1\left(\|f\|_{L^{q,\infty}(\mathbb{R}^n)} + \|f * \rho_\varepsilon\|_{L^{q,\infty}(\mathbb{R}^n)}\right)^{\frac{q}{p}}\|f - f * \rho_{\varepsilon;}\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{p}}.$$

It follows from the last inequality and (89) that there exists a constant, still denoted by C such that

$$\|f-f*\rho_\varepsilon\|_{L^p(\mathbb{R}^n)}p \le C\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}}\|f-f*\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{q}{p}}. \quad (90)$$

Since for all $x \in \mathbb{R}^n$,

$$|f(x)-f*\rho_\varepsilon(x)| \le \oint_{B_\varepsilon(0)} |f(x-y)-f(x)|dy$$

$$\le \oint_{B_\varepsilon(0)} |y|^\eta \|f\|_{\dot{C}(\mathbb{R}^n)}\eta dy$$

$$\le C\varepsilon^\eta\|f\|_{\dot{C}(\mathbb{R}^n)}\eta.$$

Or

$$\|f-f*\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon^\eta\|f\|_{\dot{C}(\mathbb{R}^n)}\eta.$$

By inserting the last inequality into (90), we obtain

$$\|f-f*\rho_{\xi j}\|_{L^p(\mathbb{R}^n)}p \le C\varepsilon^{\eta\left(1-\frac{q}{p}\right)}\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}}\|f\|_{\dot{C}(\mathbb{R}^n)}^{1-\frac{q}{p}}\eta. \quad (91)$$

Combining (91) with (88) yields

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} \le C\left(\in^{-n\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^{q,\infty}(\mathbb{R}^n)} + \varepsilon^{\eta\left(1-\frac{q}{p}\right)}\|f\|_{L^{q,\infty}(\mathbb{R}^n)}^{\frac{q}{p}}\|f\|_{\dot{C}(\mathbb{R}^n)}^{1-\frac{q}{p}}\eta\right). \quad (92)$$

By choosing $\varepsilon = \left(\frac{\|f\|_{L^{q,\infty}(\mathbb{R}^n)}}{\|f\|_{C\eta(\mathbb{R}^n)}}\right)^{\frac{1}{\frac{n}{q}+\eta}}$ in (92), we obtain with $\alpha = \infty$ and $q > 1$.

Step 2: If $0 < q \le 1$, we set for any $\delta \in (0,q)$:

$$g = |f|^{q-1-\delta}f, p_1 = \frac{p}{q-\delta}, q_1 = \frac{q}{q-\delta}, \eta_1 = \eta(q-\delta)\,.$$

Thanks to Step 1, we obtain

$$\|g\|_{L^{p_1,\infty}(\mathbb{R}^n)} \le C\|g\|_{L^{q_1}\eta_1+\frac{n}{\infty q_1}(\mathbb{R}^n)}^{\frac{\eta_1+\frac{n}{p_1}}{}}\|g\|^{\frac{\frac{n}{q_1}-\frac{n}{p_1}}{\dot{C}(\mathbb{R}^n\frac{n}{q_1\eta_1}+\eta_1}})\,. \quad (93)$$

Next, we have the following inequality

$$||a|^{s-1}a-|b|^{s-1}b| \le C_s|a-b|^s, \forall a,b \in \mathbb{R}^n, s \in (0,1)\,.$$

By applying this inequality with $a = f(x)$, $b = f(y)$, and $s = q-\delta$, we obtain

$$||f(x)|^{q-1-\delta}f(x)-|f(y)|^{q-1-\delta}f(y)| \le |f(x)-f(y)|^{q-\delta},$$

which implies

$$\|g\|_{C^{\eta_1}(\mathbb{R}^n)} = \||f|^{q-1-\delta}f\|_{C^{\eta_1}(\mathbb{R}^n)} \le \|f\|_{\dot{C}(\mathbb{R}^n)}^{q-\delta}\eta. \quad (94)$$

Note that

$$\|f\|_{L^{m,\infty}(\mathbb{R}^n)} = \||f|^{s-1}f\|_{L^{\frac{ms}{s},\infty}(\mathbb{R}^n)}^{1/}, \forall m,s > 0. \quad (95)$$

Then, a combination of (93), (94), and (95) yields

$$\|f\|^{q-\delta}_{L^{p,\infty}(\mathbb{R}^n)} \leq C\|f\|q,\infty\|f\|_{\dot{C}(\mathbb{R}^n}(q-\delta)\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{)q},\mathbb{R}^n}(q-\delta)\frac{\frac{n}{q}-\frac{n}{p}}{)\frac{n}{q}+\eta}$$

Or holds for $\alpha=\infty$, and $0<q\leq 1$.

Step 3: Finally, holds for any $\alpha>0$ and $q>0$.

In fact, let $q<p_2<p<p_3<\infty$ be such that

$$\frac{1}{p}=\frac{\theta}{p_2}+\frac{1-\theta}{p_3}.$$

We have

$$\|f\|_{Lp,\alpha}(\mathbb{R}^n) \leq C\|f\|^{\theta}_{L^{p_2,\infty}(\mathbb{R}^n)}\|f\|^{1-\theta\infty}_{L^{p}3,(\mathbb{R}^n)}. \tag{96}$$

By Step 1 and Step 2, we proved that (87) is true with $=\infty$, so

$$\|f\|_{L^{p_i,\alpha}(\mathbb{R}^n)} \leq C\|f\|^{\frac{\eta+\frac{n}{p_i}}{Lq\eta+\frac{n}{\infty q}}}\|f\|^{\frac{\frac{n}{q}-\frac{n}{p_i}}{\dot{C}(\mathbb{R}^n\frac{n}{\eta q}+\eta}}, i=2,3.$$

From the last inequality and (96), we deduce

$\|f\|_{L^{p,\alpha}}(\mathbb{R}^n)$

$$\leq C\left(\|f\|_{L^{q,\alpha}}\frac{\eta+\frac{n}{p_2}}{\eta+\frac{n}{q}}\|f\|\frac{\frac{n}{q}-\frac{n}{p_2}}{\dot{C}(\mathbb{R}^n)\frac{n}{q}+\eta}\right)^{\theta}\left(\|f\|_{L_{q,\alpha}(\mathbb{R}^n)}\frac{\eta+\frac{n}{p_3}}{\eta+\frac{n}{q}}\|f\|\dot{C}(\mathbb{R}^n)\frac{\frac{n}{q}-\frac{n}{p_3}}{\frac{n}{q}+\eta}\right)^{1-\theta}$$

$$=C\|f\|_{L_{q,\alpha}}^{\frac{\eta+\frac{n}{p}}{\eta+\frac{n}{q}}}\|f\|\dot{C}(\mathbb{R}^n)^{\frac{\frac{n}{q}-\frac{n}{p}}{\frac{n}{q}+\eta}},$$

which implies (87).

Finally, follows immediately from Theorem(4.2.4) using the embedding $W^{s,r}\subset\dot{C}^{\eta}$, with $\eta=\frac{sr-n}{r}$. Then we leave it to the reader.

Corollary (4.2.5)[146]. *Let* $0<q<p<\infty$, *and* $\alpha>0$. *Let* $s,r>0$ *be such that* $sr>n$, *and* $\eta=\frac{sr-n}{r}$. *If* $\in L^{q,\infty}(\mathbb{R}^n)\cap W^{s,r}(\mathbb{R}^n)$, *then* $f\in L^{p,\alpha}(\mathbb{R}^n)$ *and*

$$\|f\|_{Lp,\alpha}(\mathbb{R}^n) \leq C\|f\|^{\frac{\eta+\frac{n}{p}}{Lq\eta+\frac{n}{\infty q}}}_{(\mathbb{R}^n)}\|f\|^{\frac{\frac{n}{q}-\frac{n}{p}}{W^{s,r}\frac{n}{q}+\eta}}_{(\mathbb{R}^n)}. \tag{97}$$

We also note that there is an alternative approach using interpolation spaces (specifically, the reiteration theorem). We emphasize that in contrast to that approach, the proofs given in are self-contained.

Lemma (4.2.6)[146]. *If* $\in BMO(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$, *then there exists a constant* $C=C(n)>0$ *such that*

$$|\{|g|>\lambda\}|\leq Ce^{-C\lambda/\|g\|_{BMO(\mathbb{R}^n)}}\|g\|_{L^1(\mathbb{R}^n)},$$

for all $\lambda > \|g\|_{BMO(\mathbb{R}^n)}$.

We refer the reader in[8] (see also in [9]).

Now, we apply to f_{1+} to get

$$|\{|f_{1+}| > \lambda\}| \leq Ce^{-C\lambda/\|\|_{BMO}} f_1 + \|f_{1+}\|_{L^1(\mathbb{R}^n)}.$$

Note that $\|f_{1-}\|_{BMO} \leq 2\|f_{1-}\|_{L^\infty(\mathbb{R}^n)} \leq 2$, so

$$\|f_{1+}\|_{BMO(\mathbb{R}^n)} \leq \|f_1\|_{BMO(\mathbb{R}^n)} + \|f_{1-\|_{BMO(\mathbb{R}^n)}} \leq 3.$$

This leads to

$$|\{|f_{1+}| > \lambda\}| \leq Ce^{-C\lambda/3}\|f_{1+}\|_{L^1(\mathbb{R}^n)}. \tag{98}$$

Moreover, we have from Chapter 1 in Grafakos, [136]

$$\|f_{1+}\|_{L^1(\mathbb{R}^n)} \leq \frac{q}{q-1} 1^{1-q}\|f\|^q_{L^{q,\infty}(\mathbb{R}^n)}. \tag{99}$$

By (86), (98) and (99), there is a constant $C_2 = C_2(p, q, r, n) > 0$ such that

$$\|f_{1+}\|^\alpha_{L(\mathbb{R}^n)}p,\alpha \leq C_2 \int_1^\infty \lambda^\alpha\, e^{-C\lambda\alpha/3p}\|f\|^{\alpha q/p}_{L(\mathbb{R}^n)}q,\infty \frac{d\lambda}{\lambda} \leq C_3\|f\|^{cxq/p}_{Lq,\infty}(\mathbb{R}^n)\ .$$

Thus,

$$\|f_{1+}\|_{Lp,\alpha}(\mathbb{R}^n) \leq C_4\|f\|^{q/_q p_\infty}_{L,(\mathbb{R}^n)}. \tag{100}$$

It follows from (85) and (100) that

$$\|f\|_{Lp,\alpha}(\mathbb{R}^n) \leq C(p, q, \alpha, n)\|f\|^{q/_q p_\infty}_{L,(\mathbb{R}^n)},$$

which completes the proof of Theorem (4.2.3).

Chapter 5
Constructive Description of Hardy-Sobolev Spaces

We study the polynomial approximations in Hardy-Sobolev spaces on for convex domains. We use the method of pseudoanalytical continuation to obtain the characterization of these spaces in terms of polynomial approximations.

Section (5.1): Hardy-Sobolev Spaces in C^n

We give an alternative characterizations of Hardy-Sobolev (see. [25]) spaces

$$H_p^l(\Omega) = \{f \in H(\Omega): \|f\|_{H^p(\Omega)} + \sum_{|\alpha| \leq l} \| \partial^\alpha f\|_{H^p(\Omega)} < \infty\} \qquad (1)$$

on strongly convex domain $\Omega \subset \mathbb{C}^n$. We continue the research started in [163] and devoted to description of basic spaces of holomorphic functions of several variables in terms of polynomial approximations and pseudoanalytical continuation. In particular, we show that for $1 < p < \infty$ and $l \geq 1$ a holomorphic on a strongly convex domain Ω function f is in the Hardy-Sobolev space $H_p^l(\Omega)$ if and only if there exist a sequence of 2^k-degree polynomials P_{2^k} such that

$$\int_{\partial\Omega} d\,\sigma(z) \left(\sum_{k=1}^{\infty} |\, f(z) - P_{2^k}(z)|^2 2^{2lk} \right)^{p/2} < \infty. \qquad (2)$$

In the one variable case this condition follows from the characterization obtained by E.M. Dynkin [155] for Radon domains.

We devoted to the Cauchy-Leray-Fantappiè integral formula, the polynomial approximations and estimates of its kernel. We also define internal and external Korányi regions, the multidimensional analog of Lusin regions. we introduce the method of pseudoanalytical continuation and three constructions of the continuation with different estimates. We use these constructions to obtain the characterization of Hardy-Sobolev spaces in terms of estimates of the pseudoanalytical continuation. To prove this result we use the special analog of the Krantz-Li area-integral inequality [158] contains the proof of characteristics (2).

For $\mathbb{C}^n$ be the space of n complex variables, $n \geq 2$, $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$;

$$\partial_j f = \frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \bar{\partial}_j f = \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right),$$

$$\partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k, \bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k, df = \partial f + \bar{\partial} f.$$

The notation

$$\langle \partial f(z), w\rangle = \sum_{k=1}^{n} \frac{\partial f(z)}{\partial z_k} w_k.$$

is used to indicate the action of ∂f on the vector $w \in \mathbb{C}^n$, and

$$|\bar{\partial} f| = |\frac{\partial f}{\partial z_1}| + \ +|\frac{\partial f}{\partial z_n}|.$$

The euclidean distance form the point $z \in \mathbb{C}^n$ to the set $D \subset \mathbb{C}^n$ we denote as dist $(z, D) = \inf\{|z-w| : w \in D\}$. Lebesgue measure in $\mathbb{C}^n$ we denote as $d\mu$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{IN}^n$ we set $|\alpha| = \alpha_1 + +\alpha_n$ and $! = \alpha_1! \ldots \alpha_2 !$, also

$$z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n} \text{ and } \partial^\alpha f = \frac{\partial^{|.\alpha.|} f}{\partial \bar{z}_1^{\alpha_1} . \partial \bar{z}_n^{\alpha_n}}.$$

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a strongly convex domain with a C^3-smooth defining function. We need to consider a family of domains

$$\Omega_t = \{z \in \mathbb{C}^n : \rho(z) < t\}$$

that are also strongly convex for each $|t| < \varepsilon_j$, where $\varepsilon_j > 0$ is small enough, that is $d^2\rho(z)$

is positive definite when $|\rho(z)| \leq \varepsilon_j$. For $z \in \Omega_\varepsilon \backslash \Omega_{-\varepsilon}$ we denote the nearest point on $\partial\Omega$ as

$\mathrm{pr}_{\partial\Omega}(z)$. Then the mapping

$$\mathrm{pr}_{\partial\Omega} : \Omega_\varepsilon \backslash \Omega_{-\varepsilon} \to \partial\Omega$$

is well defined, C^2-smooth on $\Omega_\varepsilon \backslash \Omega$ and $|z - \mathrm{pr}_{\partial\Omega}(z)| = \mathrm{dist}(z, \partial\Omega)$.
For $\xi \in \partial\Omega_t$ we define the complex tangent space

$$T_\xi = \{z \in \mathbb{C}^n : \langle \partial\rho(\xi), \xi - z\rangle = 0\}.$$

The space of holomorphic functions we denote as $H(\Omega)$ and consider the Hardy space (see [166], [156])

$$H^p(\Omega) := \{f \in H(\Omega) : \|f\|_{H^p(\Omega)}^p = \sup_{-\varepsilon < t < 0} \int_{\Omega_t} |f(z)|^p d\sigma_t(z) < \infty\},$$

where $d\sigma_t$ is induced Lebesgue measure on the boundary of Ω_t. We also denote $d\sigma = d\sigma_0$.

Hardy-Sobolev spaces $H_p^l(\Omega)$ are defined by (1).

We use notations $\lesssim, \asymp$. We let $f \lesssim g$ if $f \leq cg$ for some constant $c > 0$, that doesn't depend on main arguments of functions f and g and usually depend only on dimension n and domain Ω. Also $f \asymp g$ if $c^{-1}g \leq f \leq cg$ for some $c > 1$.

In the context of theory of several complex variables there is no unique reproducing formula formula, however we could use the Leray theorem, that allows us to construct holomorphic reproducing kernels ([152], [161], [76]). For convex domain $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ this theorem brings us Cauchy-LerayFantappiè formula, and for $f \in H^1(\Omega)$ and $z \in \Omega$ we have

$$f(z)=K_{\Omega}f(z)=\frac{1}{(2\pi i)^n}\int_{\partial\Omega}\frac{f(\xi)\partial\rho(\xi)\wedge\left(\bar{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle\partial\rho(\xi),\xi-z\rangle^n}=\int_{\partial\Omega}f(\xi)K(\xi,z)\omega(\xi)\,,(3)$$

where $\omega(\xi)=\frac{1}{(2\pi i)^n}\partial\rho(\xi)\wedge\left(\bar{\partial}\partial\rho(\xi)\right)^{n-1}$, and $K(\xi,z)=\langle\partial\rho(\xi),\xi-z\rangle^{-n}$

The $(2n-1)$-form ω defines on $\partial\Omega_t$ Leray-Levy measure dS, that is equivalent to Lebesgue surface measure $d\sigma_t$ (see [152], [42], [160]). This allows us to identify Lebesgue, Hardy and Hardy-Sobolev spaces defined with respect to measures $d\sigma_t$ and dS. Also note, that measure dV defined by the $2n$-form $d\omega=\left(\partial\bar{\partial}\rho\right)^n$ is equivalent to Lebesgue measure $d\mu$ in $\mathbb{C}^n$.

By [162] the integral operator K_{Ω} defines a bounded mapping on $L^p(\partial\Omega)$ to $H^p(\Omega)$ for $1<p<\infty$.

The function $d(w,z)=|\langle\partial\rho(w),w-z\rangle|$ defines on $\partial\Omega$ quasimetric, and if $B(z,\delta)=\{w\in\partial\Omega:\ d(w,z)<\delta\}$ is a quasiball with respect to d then $\sigma\left(B(z,\delta)\right)-\wedge\ \delta^n$, see for example [162]. Therefore $\{\partial\Omega,d,\sigma\}$ is a space of homogeneous type.

Note also the crucial role in the forthcoming considerations of the following estimate that is proved in [163].

Lemma (5.1.1)[151]. *Let Ω be strongly convex, then*

$$d(w,z)-\wedge\ \rho(w)+d(\mathrm{pr}_{\partial\Omega}(w),z),w\in\mathbb{C}^{□}\backslash\Omega,z\in\partial\Omega.$$

In lemma (5.1.3) here we construct a polynomial approximations of Cauchy-Leray-Fantappié kernel based on theorem by V.K. Dzyadyk about estimates of Cauchy kernel on domains on complex plane. The approximation is choosed similarly to [16]. This construction allows us in Theorem (5.1.9) to get polynomials that approximate holomorphic function with desired speed.

Lemma (5.1.2)[151]: *Let Ω be a strongly convex domain with $0\in\Omega$, then for every $\xi\in\Omega_{\varepsilon}\backslash\Omega$ the value of $\lambda=\frac{\langle\partial\rho(\xi),z\rangle}{\langle\partial\rho(\xi),\xi\rangle}$ for $z\in\Omega$ lies in domain $L(t)$, bounded by the bigger arc of the circle $|\lambda|=R=R(\Omega)$ and the chord $\{\lambda\in\mathbb{C}:\lambda=1+e^{it}s,s\in\mathbb{R},|\lambda|\le R\}$, where $t=\frac{\pi}{2}-\arg\left(\langle\partial\rho(\xi),\xi\rangle\right)$.*

Proof. For $\xi\in\partial\Omega$ define

$$\Lambda(\xi)=\left\{\lambda\in\mathbb{C}:\lambda=\frac{\langle\partial\rho(\xi),z\rangle}{\langle\partial\rho(\xi),\xi\rangle},z\in\Omega\right\}.$$

The convexity of Ω with $0\in\Omega$ implies that

$$|\langle\partial\rho(\xi),\xi\rangle|\gtrsim|\partial\rho(\xi)||\xi|\gtrsim 1,\qquad(4)$$

$$\mathrm{Re}\,\langle\partial\rho(\xi),z-\xi\rangle\le 0,z\in\bar{\Omega},\xi\in\Omega_{\varepsilon}\backslash\Omega.\qquad(5)$$

The domain $\Lambda(\xi)\subset\mathbb{C}$ is also convex and contains 0, thus the equality

$$\frac{\langle\partial\rho(\xi),z\rangle}{\langle\partial\rho(\xi),\xi\rangle}=1+\frac{\langle\partial\rho(\xi),z-\xi\rangle}{\langle\partial\rho(\xi),\xi\rangle}$$

with estimates (4), (5) completes the proof of the lemma.

Lemma (5.1.3)[151]. *Let Ω be a strongly convex domain and $r>0$. Then for every $k\in$ IN there exist function $K_k^{glob}(\xi,z)$ defined for $\xi\in\Omega_\varepsilon\backslash\Omega$ and polynomial in $z\in\Omega$ with* $\deg K_k(\xi,\cdot)\le k$ *and following properties*:

$$|K(\xi,z)-K_k^{glob}(\xi,z)|\lesssim\frac{1}{k^r}\frac{1}{d(\xi,z)^{n+r}},d(\xi,z)\ge\frac{1}{k}; \tag{6}$$

$$|K_k^g\iota_{ob}(\xi,z)|\lesssim k^n,d(\xi,z)\le\frac{1}{k}. \tag{7}$$

Proof. Due to [153] and [165] for any $j\in$ IN there exists a function $T_j(t,\lambda)$ polynomial in λ with $\deg T_j(t,\cdot)\le j$ such that

$$|\frac{1}{1-\lambda}-T_j(t,\lambda)|\lesssim\frac{1}{j^r}\frac{1}{|1-\lambda|^{1+r}} \tag{8}$$

for $\lambda\in L(t)\backslash\{\lambda:|1-\lambda|<\frac{1}{j}\}$ and coefficients of polynomials $T_j(t,\lambda)$ continuously depend on t. Note also that by maximum principle

$$T_j(t,\lambda)\lesssim j,\lambda\in L(t)\cap\{\lambda:|1-\lambda|<\frac{1}{j}\}. \tag{9}$$

Let $t(\xi)=\frac{\pi}{2}-$ arg $(\langle\partial\rho(\xi),\xi\rangle$ and for $j\in$ IN and $(j-1)<k\le jn$ define

$$K_k^{glob}(\xi,z)=K_{jn}^{glob}(\xi,z)=\frac{1}{\langle\partial\rho(\xi),\xi\rangle^n}T_j^n\left(t(\xi),\frac{\langle\partial\rho(\xi),z\rangle}{\langle\partial\rho(\xi),\xi\rangle}\right).$$

Due to definition of T_j polynomials $K_k^g\iota_{ob}(\xi,\cdot)$ satisfy relations (6), (7).

For $\xi\in\partial\Omega$ and $\varepsilon j>0$ we define the *inner Korányi region* as

$$D^i(\xi,\eta,\varepsilon j)=\{\tau\in\Omega:\mathrm{pr}_{\partial\Omega}(\tau)\in B\big(\xi,-\eta\rho(\tau)\big),\rho(\tau)>-\varepsilon j\}.$$

The strong convexity of Ω implies that area-integral inequality by S. Krantz and S.Y. Li [158] for $\in H^p(\Omega)$, $0<p<\infty$, could be expressed as

$$\int_{\partial\Omega}d\,\sigma(z)\left(\int_{D^i(z,\eta\varepsilon)}|\partial f(\tau)|^2\frac{d\mu(\tau)}{\big(-\rho(\tau)\big)^{n-1}}\right)^{p/2}\le c(\Omega,p)\int_{\partial\Omega}|f|^p d\sigma. \tag{10}$$

Consider the decomposition of vector $\tau\in\mathbb{C}^n$ as $=w+tn(\xi)$, where $w\in T_\xi$, $t\in\mathbb{C}$, and $n(\xi)=\frac{\bar\partial\rho(\xi)}{|\bar\partial\rho(\xi)|}$ is a complex normal vector at ξ. We define the *external Korányi region* as

$$D^e(\xi,\eta,\varepsilon j)=\{\tau\in\mathbb{C}^n\backslash\Omega:\tau=w+tn(\xi),$$

$$w \in T_\xi, t \in C, |w| < \sqrt{\eta\rho(\tau)}, |\operatorname{Im}(t)| < \eta\rho(\tau), \rho(\tau) < \varepsilon\}. \quad (11)$$

We will proof the area-integral inequality similar to (10) for external regions $D^e(\xi, \eta, \varepsilon j)$. We point out two rules for integration over regions $D^e(\xi, \eta, \varepsilon j)$. First, for every function F we have

$$\int_{\Omega_\Xi \setminus \Omega} |F(z)| d\mu(z) \asymp \int_{\partial\Omega} d\,\sigma(\xi) \int_{D^e(\xi,\eta,\varepsilon)} |F(\tau)| \frac{d\mu(\tau)}{\rho(\tau)^n}.$$

Second, if $F(w) = \tilde{F}(\rho(w))$ then

$$\int_{D^e(\xi,\eta,\varepsilon)} |F(\tau)| d\mu(\tau) \asymp \int_0^\varepsilon |\tilde{F}(t)| t^n dt.$$

Similar rules are valid for regions $D^i(\xi, \eta, \varepsilon j)$.
We could clarify the estimate of $d(\tau, w)$ in lemma (5.1.1) for $\in D^e(z, \eta, \varepsilon j)$.
Lemma (5.1.4)[151]. *Let Ω be a strongly convex domain and* $j, \eta > 0$, *then*

$$d(\tau, w) \asymp \rho(\tau) + d(z, w)\,, z, w \in \partial\Omega, \tau \in D^e(z, \eta, \varepsilon j)\,. \quad (12)$$

Proof. For $\tau \in D^e(z, \eta, \varepsilon j)$ we denote $\hat{\tau} = \mathrm{pr}_{\partial\Omega}(\tau)$, then $d(\hat{\tau}, z) \sim< \eta\rho(\tau)$.

$$d(\tau, w) \lesssim \rho(\tau) + d(\hat{\tau} \lesssim \rho(\tau) + d(\hat{\tau}, z) + d(z, w) \lesssim \rho(\tau) + d(z, w)\,.$$

On the other hand,

$$\rho(\tau) + d(z, w) \lesssim \rho(\tau) + \big(d(z, \hat{\tau}) + d(\hat{\tau}, w)\big) \lesssim (1+\eta)\rho(\tau) + d(\hat{\tau}, w)$$

$$\lesssim \rho(\tau) + d(\hat{\tau}, w) \lesssim d(\tau, w)\,.$$

We the method of continuation of function $f \in H(\Omega)$ outside the domain Ω. Let $f \in H^1(\Omega)$ and let the boundary values of f almost everywhere coincide with the boundary values of some function $\mathrm{f} \in C^1_{loc}(\mathbb{C}^n \setminus \overline{\Omega})$ such that $|\overline{\partial}\mathrm{f}| \in L^1(\mathbb{C}^n \setminus \Omega)$. Then by Stokes formula for $z \in \Omega$ we have

$$f(z) = \lim_{r \to 0+} \frac{1}{(2\pi i)^n} \int_{\partial\Omega_r} \frac{\mathrm{f}(\xi)\partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n} =$$

$$\lim_{r \to 0+} \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \setminus \Box_r} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n}$$

$$= \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \setminus \Omega} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n},$$

since (for details see [13])

$$d_\xi \left(\frac{\partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n} \right) = 0, z \in \Omega, \xi \in \mathbb{C}^n \setminus \Omega.$$

This formula allows us to study properties of function $f \in H(\Omega)$ relying on estimates of its continuation.

Definition (5.1.5)[151]. *We call the function* $\mathrm{f} \in C^1_{loc}\left(\mathbb{C}^n \backslash \overline{\Omega}\right)$ *the pseudoanalytic continuation of the function* $f \in H(\Omega)$ *if*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \backslash \Omega} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z \rangle^n}, z \in \Omega. \quad (13)$$

Note that it is not necessary for the function f to be a continuation in terms of coincidence of boundary values.

For $z \in \Omega_\varepsilon \backslash \Omega$ we define the symmetric along $\partial\Omega$ point $z^* \in \Omega$ by

$$z^* - z = 2(\mathrm{pr}_{\partial\Omega}(z) - z).$$

Theorem(5.1.6)[151]. *Let* $f \in H^1_p(\Omega)$ *and* $1 < p < \infty$, $m \in \mathbb{N}$. *There exist a pseudoanalytical continuation* $\mathrm{f} \in C^1_{loc}\left(\mathbb{C}^n \backslash \overline{\Omega}\right)$ *of function* f *such that* supp f$\subset \Omega_\varepsilon$, $|\overline{\partial}\mathrm{f}(z)| \in L^p\left(\Omega_\varepsilon \backslash \overline{\Omega}\right)$ *and*

$$|\overline{\partial}\mathrm{f}(z)| \lesssim \max_{|\alpha|=m} |\partial^\alpha f(z^*)| \rho(z)^{m-1}, z \in \Omega_\varepsilon \backslash \Omega. \quad (14)$$

Proof. Define

$$\mathrm{f}_0(z) = \sum_{|\alpha| \le m-1} \partial^\alpha f(z^*) \frac{(z - z^*)^\alpha}{\alpha!}, z \in \Omega_\varepsilon \backslash \Omega. \quad (15)$$

Let $\alpha \pm e_k = (\alpha_1, \dots, \alpha_k \pm 1, \alpha_n)$ and define $(z - z^*)^{\alpha - e_k} = 0$ if $\alpha_k = 0$. In these notations we have

$$\overline{\partial}_j \mathrm{f}_0 = \sum_{k=1}^{\infty} \sum_{|\alpha| \le m-1} \left(\partial^{\alpha + e_k} f(z^*) \frac{(z - z^*)^\alpha}{\alpha!} - \partial^\alpha f(z^*) \frac{(z - z^*)^{\alpha - e_k}}{(\alpha - e_k)!} \right) \overline{\partial}_j z_k^*$$

$$= \sum_{k=1}^{\infty} \sum_{|\alpha| = m-1} \partial^{\alpha + e_k} f(z^*) \frac{(z - z^*)^\alpha}{\alpha!} \overline{\partial}_j z_k^*, \quad (16)$$

hence,

$$|\overline{\partial}\mathrm{f}_0(z)| \lesssim \max |\partial^\alpha f(z^*)| \rho(z)^{m-1}, z \in \mathbb{C}^n \backslash \Omega.$$

Consider function $\chi \in C^\infty(0, \infty)$ such that $\chi(t) = 1$ for $t \le \varepsilon/2$ and $\chi(t) = 0$ for $t \ge \varepsilon j$.

The function $\mathrm{f}(z) = \mathrm{f}_0(z)\chi(\rho(z))$ satisfies the condition (15) and supp f$\subset \Omega_\varepsilon$.

Let $d = \mathrm{dist}(z^*, \partial\Omega)/10$, then for every mutiindex α such that $|\alpha| = m$ by Cauchy maximal inequality we have

$$|\partial^\alpha f(z^*)| \lesssim d^{-m+1} \sup_{|\tau - z^*| < d} |\partial f(\tau)| \lesssim \rho(z)^{-m+1} \sup_{\tau \in D^i(pr_{\partial\Omega}(z), c_0 d, \varepsilon)} |\partial f(\tau)|,$$

for some $c_0 > 0$. Finally, from [8] we get

$$\int_{\Omega_\varepsilon \backslash \Omega} \left(|\bar{\partial} f(z)|\right)^p d\mu(z) \lesssim \int^p_{\int_{\Omega \backslash \Omega_{-\varepsilon}}} d\mu(z) \left(\sup_{\tau \in D^i(pr_{\partial\Omega}(z), c_0 d, \varepsilon)} |\partial f(\tau)| \right)^p$$

$$\lesssim \|\partial f\|^p_{H^p(\Omega)} < \infty.$$

Let $f \in H^1(\Omega)$ and consider a polynomial sequence P_1, P_2, converging to f in $L^1(\partial\Omega)$. Define

$$\lambda(z) = \rho(z)^{-1} |P_{2^{k+1}}(z) - P_{2^k}(z)|, 2^{-k} < \rho(z) \leq 2^{-k+1}$$

Theorem (5.1.7)[151]. Assume that $\lambda \in L^p(\mathbb{C}^n \backslash \Omega)$ for some $p \geq 1$. Then there exist a pseudoanalytical continuation f of function f such that

$$|\bar{\partial} f(z)| \lesssim \lambda(z), z \in \mathbb{C}^n \backslash \Omega. \tag{17}$$

Proof. Consider function $\chi \in C^\infty(0, \infty)$ such that $\chi(t) = 1$ for $t \leq \frac{5}{4}$ and $\chi(t) = 0$ for $t \geq \frac{7}{4}$. We let

$$f_0(z) = P_{2^k}(z) + \chi\left(2^k \rho(z)\right)\left(P_{2^{k+1}}(z) - P_{2^k}(z)\right), 2^{-k} < \rho(z) < 2^{-k+1}, k \in \mathbb{N},$$

and define the continuation of a function f by formula $f = \chi(2\rho(z)) f_0(z)$.

Now f is C^1-function on $\mathbb{C}^n \backslash \overline{\Omega}$ and $|\bar{\partial} f(z)| \sim< \lambda(z)$. We define a function $F_k(z)$ as $F_k(z) = f(z)$ for $\rho(z) > 2^{-k}$ and as $F_k(z) = P_{2^{k+1}}(z)$ for $\rho(z) < 2^{-k}$. The the function F_k is smooth and holomorphic in Ω_{2-k}, and $|\bar{\partial} F_k(z)| \sim< \lambda(z)$ for $z \in \mathbb{C}^n \backslash \Omega_{2-k}$. Thus similarly we get

$$P_{2^{k+1}}(z) = F_k(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \backslash \Omega} \frac{\bar{\partial} F_k(\xi) \wedge \partial\rho(\xi) \wedge \left(\bar{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z \rangle^n}, z \in \Omega,$$

We can pass to the limit in this formula by the dominated convergence theorem; hence, function f satisfies the formula (13) and is a pseudoanalytical continuation of function f.

Theorem (5.1.8)[151]. Let Ω be a strongly convex domain, $1 < p < \infty$, $l \in$ IN and $\in H^p(\Omega)$. Then $f \in H^l_p(\Omega)$ if and only if there exists such pseudoanalytical continuation f that for some $\eta > 0$

$$\int_{\partial\Omega} d\,\sigma(z) \left(\int_{D^e(z,\eta,\varepsilon)} |\bar{\partial} f(\tau) \rho(\tau)^{-l}|^2 d\iota/(\tau) \right)^{p/2} < \infty, \tag{18}$$

where $dv(\tau) = \frac{d\mu(\tau)}{\rho(\tau)^{n-1}}$.

Proof. Let $\in H_p^l(\Omega)$. By Theorem (5.1.5) we could construct pseudoanalytical continuation f such that

$$|\bar{\partial}\mathrm{f}(z)| \lesssim \max_{|\alpha|=l+1} |\partial^\alpha f(z^*)|\rho(z)^l, z \in \mathbb{C}^n\backslash\Omega.$$

Note that the symmetry $(z \mapsto z^*)$ with respect to $\partial\Omega$ maps the external sector $D^e(z,\eta,\varepsilon j)$ into some internal Korányi sector. Indeed, for every $\eta > 0$ there exists η_1, $\varepsilon_1 > 0$ such that

$$\{\tau^*: \tau \in D^e(z,\eta,\varepsilon j)\} \subseteq D^i(z,\eta_1,\varepsilon_1) .$$

Applying area-integral inequality (10) we obtain

$$\int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^e(z,\eta,\varepsilon)} |\,\bar{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2 d\nu(\tau)\right)^{p/2}$$

$$\lesssim \max_{|\alpha|=l+1} \int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^e(z,\eta,\varepsilon)} |\,\partial^\alpha f(\tau^*)|^2 d\iota/(\tau)\right)^{p/2}$$

$$\lesssim \max_{|\alpha|=l+1} \int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^i(z,\eta_1\varepsilon_1)}, |\partial^\alpha f(\tau)|^2 \frac{d\mu(\tau)}{(-\rho(\tau))^{n-1}}\right)^{p/2} < \infty$$

To prove the sufficiency, assume that function $f \in H^1(\Omega)$ admits the pseudoanalytical continuation f with the estimate (18) We will prove that for every function $\in L^{p'}(\partial\Omega)$, $\frac{1}{p} + \frac{1}{p} = 1$, and every multiindex α, $|\alpha| \le l$,

$$|\int_{\Omega} g\,(z)\partial^\alpha f(z) dS(z)| \le c(f)\|g\|_{L^{p'}(\partial\Omega)}.$$

Assume, without loss of generality, that $= (l, 0, \dots, 0)$. By representation (13) we have

$$f(z) = \int_{\mathbb{C}^n\backslash\Omega} \frac{\bar{\partial}\mathrm{f}(\xi) \wedge \omega(\xi)}{\langle \partial\rho(\xi), \xi - z\rangle^n}$$

and with $C_{nl} = \frac{(n+l-1)!}{(n-1)!}$

$$\int_{\partial\Omega} g\,(z)\partial^\alpha f(z) dS(z)$$

$$= C_{nl} \int_{\partial\Omega} g\,(z)\left(\int_{\mathbb{C}^n\backslash\Omega} \left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \frac{\bar{\partial}\mathrm{f}(\xi) \wedge \omega(\xi)}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}}\right) dS(z)$$

$$= C_{nl} \int_{\mathbb{C}^n\backslash\Omega} \left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \bar{\partial}\mathrm{f}(\xi) \wedge \omega(\xi) \int_{\partial\Omega} \frac{g(z) dS(z)}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}}.$$

Define $\Phi_l(\xi)=\int_{\partial\Omega}\frac{g(z)dS(z)}{\langle\partial\rho(\xi),\xi-z\rangle^{n+l}}$. Applying Hölder inequality twice we have

$$|\int_{\Omega} g\,(z)\partial^{\alpha}f(z)dS(z)|\lesssim\int_{\mathbb{C}^n\backslash\Omega}|\,\bar{\partial}\mathrm{f}(\xi)||\Phi_l(\xi)|d\mu(\xi)$$

$$\lesssim\int_{\partial\Omega}d\,S(\xi)\int_{D^e(\xi,\eta,\varepsilon)}|\,\bar{\partial}\mathrm{f}(\tau)||\Phi_l(\tau)|\frac{d\mu(\tau)}{\rho(\tau)^n}$$

$$\lesssim\int_{\partial\Omega}d\,\sigma(\xi)\left(\int_{D^e(\xi,\eta,\varepsilon)}|\,\bar{\partial}\mathrm{f}(\tau)|^2\rho(\tau)^{-2l}\frac{d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{1/2}\times$$

$$\times\left(\int_{D^e(\xi,\eta\varepsilon)},|\Phi_l(\tau)|^2\rho(\tau)^{2l-2}\frac{d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{1/2}$$

$$\lesssim\left(\int_{\partial\Omega}d\,S(\xi)\left(\int_{D^e(\xi,\eta,\varepsilon)}|\,\bar{\partial}\mathrm{f}(\tau)|^2\rho(\tau)^{-2\iota}d\iota/(\tau)\right)^{p/2}\right)^{1/p}\times$$

$$\times\left(\int_{\partial\Omega}d\,S(\xi)\left(\int_{D^e(\xi,\eta,\varepsilon)}|\,\Phi_l(\tau)|^2\rho(\tau)^{2l-2}d\nu(\tau)\right)^{p'/2}\right)^{1/p'}$$

The first product term is bounded by (18), and the second one by the areaintegral inequality (28), that we will prove in Theorem (5.1.12).

Theorem (5.1.9)[151]. *Let $f\in H^1(\Omega)$ and $1<p<\infty$, $l\in$ IN. Then $f\in H_p^l(\Omega)$ iff there exists sequence of 2^k-degree polynomials P_{2^k} such that*

$$\int_{\partial\Omega}d\,\sigma(z)\left(\sum_{k=1}^{\infty}|\,f(z)-P_{2^k}(z)|^2 2^{2\iota k}\right)^{p/2}<\infty. \qquad (19)$$

Proof. Assume that condition (19) holds, then polynomials P_{2^k} converge to function f in $L^p(\partial\Omega)$ and by the theorem (5.1.7) we could construct pseudoanalytical continuation f such that

$$|\bar{\partial}\mathrm{f}(z)|\lesssim|P_{2^{k+1}}(z)-P_{2^k}(z)|\rho(z)^{-1},z\in\mathbb{C}^n\backslash\Omega,2^{-k}\le\rho(z)<2^{-k+1}$$

Consider the decomposition of region $D^e(z,\eta,\varepsilon j)$ to sets $D_k(z)=\{\tau\in D^e(z,\eta,\varepsilon j):2^{-k}\le\rho(\tau)<2^{-k+1}\}$, and define functions

$$a_k(z)=|P_{2^{k+1}}(z)-P_{2^k}(z)|2^{-kl},$$

$$b_k(z)=\left(\int_{D_k(z)}|\,\bar{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2d\iota/(\tau)\right)^{1/2},z\in\partial\Omega.$$

Prove the necessity. Now $f \in H_p^l(\Omega)$ with $1 < p <$ oo and $l \in \mathbb{N}$. By Theorem (5.1.8) we could construct continuation f of function f with estimate (18). Applying the approximation of Cauchy-Leray-Fantappiè kernel from Lemma (5.1.3) to function f we define polynomials

$$P_{2^k}(z) = \int_{\mathbb{C}^n \setminus \Omega} \overline{\partial}\, \mathrm{f}(\xi) \wedge \omega(\xi) K^g_{2^k \iota_{ob}}(\xi, z) .$$

We will prove that these polynomials satisfy the condition (19). From Lemma (5.1.3) we obtain

$$|f(z) - P_{2^k}(z)| \lesssim \int_{\mathbb{C}^n \setminus \Omega} |\overline{\partial}\mathrm{f}(\xi)|\left|\frac{1}{\langle \partial\rho(\xi), \xi - z\rangle^d} - K^g_{2^k \iota_{ob}}(\xi, z)\right| d\mu(\xi)$$
$$\lesssim U(z) + V(z) + W_1(z) + W_2(z) ,$$

where

$$U(z) = \int_{d(\tau,z)<2-k} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} d\mu(\tau) ,$$

$$V(z) = 2^{kn} \int_{d(\tau,z)<2-k} |\overline{\partial}\mathrm{f}(\tau)| d\mu(\tau) ,$$

$$W_1(z) = 2^{-kr} \int_{\substack{d(\tau,z)>2^{-k} \\ \rho(\tau)<2^{-k}}} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^{n+r}} d\mu(\tau) ,$$

$$W_2(z) = 2^{-kr} \int_{\rho(\tau)>2^{-k}} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^{n+r}} d\mu(\tau) .$$

The parameter $r > 0$ will be chosen later.
Note that $V(z) \lesssim cU(z)$ and estimate the contribution of $U(z)$ to the sum. For some $c_1, c_2 > 0$ we have

$$U(z) \leq \int_{w\in\partial\Omega} d\,(w,z) < c_1 2^{-k} d\sigma(w) \sum_{j>c_2 k_D} \int_{j(w)} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} \frac{d\iota/(\tau)}{\rho(\tau)}$$

$$\leq \int_{w\in\partial\Omega} d\,(w,z) < c_1 2^{-k} d\sigma(w) \sum_{j>c_2 k} \left(\int_{D_j(w)} |\overline{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2 d\iota/(\tau) \right)^{1/2} \times$$

$$\times \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} \right)^{1/2} = \sum_{j>c_2 k_{d(w,z)}} \int_{<c_1 2-j} b_j\,(w) m_j(w) d\sigma(w)$$

Consider the integral $m_j(w)$. Since $\tau \in D_j(w)$ then by estimates from lemma (5.1.4) $d(\tau, z) -\wedge\, \rho(\tau) + d(w, z) > 2^{-j}$ and

$$m_j(w) = \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\iota/(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^n}\right)^{1/2} \lesssim \frac{2^{-j(l-1)}}{2^{-jn}} 2^{-j} = 2^{jn-jl} \quad (20)$$

Thus

$$2^{kl} U(z) \lesssim \sum_{j>c_1 k} 2^{-(j-k)l}\, 2^{jn} \int_{d(w,z)<c_2 2^j} b_j\,(w) \square\sigma(w)$$

$$\lesssim \sum_{j>c_1 k} 2^{-(j-k)l}\, Mb_j(z)\,. \quad (21)$$

Now estimate the value $W_1(z)$. Similarly to the previous we have

$$W_1(z) \leq 2^{-kr} \sum_{j>k} \int_{d(w,z)\geq c_1 2-k} b_j\,(w) m_j^r(w) d\sigma(w)$$

$$\leq 2^{-kr} \sum_{j>k} \sum_{t=c_2 2^{-t} \leq d(w}^{k} \int_{zc_1,,)\leq c_1 2-t+1} b_j\,(w) m_j^r(w) d\sigma(w)\,,$$

where

$$m_j^r(w) = \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\iota/(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^{2(n+r)}}\right)^{1/2}$$

Applying the estimate $d(\tau, z) \wedge -\rho(\tau) + d(w, z) \sim> 2^{-t}$, we obtain

$$m_j^r(w) \lesssim 2^{-jl+t(n+r)}.$$

Finally

$$t = c_2 \sum_{d(w,z)}^{k} \int_{\leq c_1 2-t+1} b_j\,(w) m_j^r(w) d\sigma(w) \lesssim \sum_{t=c_2}^{k} 2^{-jl+tr}\, Mb_j(z) \lesssim 2^{-jl+kr} Mb_j(z)$$

and

$$2^{kl} W_1(z) \lesssim \sum_{j>k} 2^{-l(j-k)}\, Mb_j(z)\,. \quad (22)$$

Similarly, estimating the contribution of $W_2(z)$, we obtain

$$2^{k\iota} W_2(z) \lesssim 2^{-k(r-l)} \sum_{j=0_\partial}^{k} \int_{\Omega} b_j\,(w) m_j^r(w) d\sigma(w)\,. \quad (23)$$

Since $d(\tau, z) \sim> 2^{-j} + d(w, z)$ for $\tau \in \partial\Omega$, $\square \in D_j(z)$ then

$$m_j^r(w) \lesssim \frac{2^{-jl}}{\left(2^{-j} + d(w, z)\right)^{n+r}} \leq \min\,(2^{j(n+r-l)}, 2 - j\iota_{d(w,z)^{-n-r}}).$$

Thus

$$\int_{\partial\Omega} b_j(w) m_j^r(w) d\sigma(w) \lesssim \int_{d(w,z)\leq 2-j} \frac{2^{-j\iota}}{2^{-j(n+r)}} b_j(w) d\sigma(w)$$

$$+\sum_{\leq d}^{j-1} \int_{(w,z)\leq 2-t} t = 1_{2-t-1} \frac{2^{-j\iota}}{2^{-t(n+r)}} b_j(w) d\sigma(w)$$

$$\lesssim \sum_{t=1}^{j} 2^{-jl} 2^{tr} M b_j(z) \lesssim 2^{-jl} 2^{jr} M b_j(z) .$$

Choosing $r = 2l$, we have

$$W_2(z) 2^{kl} \lesssim \sum_{j=1}^{k} 2^{-(k-j)(r-l)} M b_j(z) \leq \sum_{j=1}^{k} 2^{-(k-j)l} M b_j(z) . \qquad (24)$$

Combining the estimates (21, 22, 24) we finally obtain

$$|f(z) - P_{2^k}(z)| 2^{kl} \lesssim \sum_{j=1}^{k} 2^{-(k-j)l} M b_j(z) + \sum_{j>k} 2^{-(j-k)l} M b_j(z) ,$$

which similarly to [5] implies

$$\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^2 2^{2kl} \lesssim \sum_{k=1}^{\infty} \left(M b_k(z)\right)^2$$

Then, by Fefferman-Stein theorem

$$\int_{\partial\Omega} d\,\sigma(z) \left(\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^2 2^{2\iota k}\right)^{p/2} \leq \int_{\partial\Omega} \left(\sum_{k=1}^{\infty} b_k^2(z)\right)^{p/2} d\sigma(z)$$

$$\leq \int_{\partial\Omega} d\,\sigma(z) \left(\int_{D^e(z,\eta,\varepsilon)} |\bar{\partial} \mathrm{f}(\xi) \rho(\xi)^{-l}|^2 d\nu(\xi)\right)^{p/2} < \infty.$$

This completes the proof of the theorem and it remains to prove Lemma (5.1.10).

Lemma (5.1.10)[151]. $\mathrm{b_k(z)} \sim< \mathrm{Ma_k(z)}$, where $\mathrm{Ma_k}$ is the maximal function with respect to centred quasiballs on $\partial\Omega$

$$M a_k(z) = \sup_{r>0} \frac{1}{\sigma\left(B(z,r)\right)} \int_{B(z,r)} |a_k(\xi)| d\sigma(\xi) .$$

Assume, that this lemma holds, then by Fefferman-Stein maximal theorem (see [157], [156]) we have

$$\int_{\partial\Omega} \left(\sum_{k=1}^{\infty} b_k(z)^2\right)^{p/2} d\sigma(z) \lesssim \int_{\partial\Omega} \left(\sum_{k=1}^{\infty} a_k(z)^2\right)^{p/2} d\sigma(z) .$$

The right-hand side of this inequality is finite by the condition (19), also we have

$$\sum_{k=1}^{\infty} b_k(z)^2 = \int_{D^e(z,\eta,\varepsilon)} |\bar{\partial} f(\xi)\rho(\xi)^{-l}|^2 d\nu(\xi),$$

which completes the proof of the sufficiency in the theorem.

Proof. Define $g_k(z) := 2^{-kl}\left(P_{2^{k+1}}(z) - P_{2^k}(z)\right)$.

Let $z \in \partial\Omega$ and $\in S_k(z)$. Consider complex normal vector $n(z) = \frac{\bar{\partial}\rho(z)}{|\bar{\partial}\rho(z)|}$ at z, complex tangent hyperplane $T_z = \{w \in \mathbb{C}^n : \langle \partial\rho(z), w - z\rangle = 0\}$ and complex plane $T_{z,\tau}^{\perp}$, orthogonal to T_z and containing the point τ

$$T_{z,\tau}^{\perp} := \{\tau + sn(z) : s \in \mathbb{C}\}.$$

Projection of vector $\tau \in \mathbb{C}$ to $\partial\Omega \cap T_{z,\tau}^{\perp}$ we will denote as $\pi_z(\tau)$.

Define $\Omega_{z,\tau} = \Omega \cap T_{z,\tau}^{\perp}$ and $\gamma_{z,\tau} = \partial\Omega_{z,\tau}$. There exist a conformal map $\phi_{z,\tau} : T_{z,\tau}^{\perp}\backslash\Omega_{z,\tau} \to \mathbb{C}\backslash\{w \in \mathbb{C} : |w| = 1\}$ such that $\phi_{z,\tau}(\text{oo}) = \infty$, $\phi_{z,\tau}'(\text{oo}) > 0$, and we could consider analytical in $T_{z,\tau}^{\perp}\backslash\Omega_{z,\tau}$ function $G_k(s) := \frac{g_k(s)}{\phi_{z,\tau}^{2^{k+1}}(s)}$.

Applying to function G_k Dyn'kin maximal estimate from [4] for domain $T_{z,\tau}^{\perp}\backslash\Omega(z,\tau)$ we obtain the estimate

$$|G_k(\tau)| \lesssim \frac{1}{\rho(\tau)}\int_{s\in I_{z,\tau}} |G_k(s)||ds| + \int_{\partial\Omega_{z,\tau}\backslash I_{z,\tau}} |G_k(s)|\frac{\rho(\tau)^m}{|s-\pi_z(\tau)|^{m+1}}|ds|,$$

where $I_{z,\tau} = \{s \in \gamma_{z,\tau} : |s - \pi_z(\tau)| < \text{dist}(\tau, \partial\Omega_{z,\tau})/2\}$, and $m > 0$ could be chosen arbitrary large.

Note that $|\phi_{z,\tau}(s)| - 1 \wedge -\text{dist}(s, \partial\Omega_{z,\tau}) \wedge -2^{-k}$, thus $|g_k(s)| \wedge -|G_k(s)|$ for $s \in D_k(z) \cap T_{z,\tau}^{\perp}$. Hence,

$$|g_k(\tau)| \lesssim \sum_{j=1}^{\infty} 2^{-jm}\frac{1}{2^j\rho(\tau)}\int_{\substack{s\in\partial\Omega_{z,\tau}\\ |s-\pi_z(\tau)|<2^j\rho(\tau)}} |g_k(s)||ds|. \tag{25}$$

Since the boundary of the domain Ω is C^3-smooth, we can assume that the constant in this inequality (25) does not depend on $z \in \partial\Omega$ and $\tau \in \Omega_\varepsilon\backslash\Omega$.

Note that function $g_k(\tau + z - w)$ is holomorphic in $w \in T_z$, then estimating the mean we obtain

$$|g_k(\tau)| \le \frac{1}{\rho(\tau)^{n-1}}\int_{|w-z|<\sqrt{\rho(\tau)}} |g_k(\tau + z - w)|d\mu_{2n-2}(w)$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-jm}\frac{1}{\rho(\tau)^{n-1}}\int_{|w-z|<\sqrt{\rho(\tau)}}\frac{d\mu_{2n-2}(w)}{2^j\rho(\tau)}\int_{\substack{s\in\partial\Omega_{z,\tau}\\ |s-\pi_z(\tau+z-w)|<2^j\rho(\tau)}} |g_k(s)||ds|$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-j(m-n+1)} \int_{B\left(z,2^{j}\rho(\tau)\right)} |\, g_k(w)| d\sigma(w)\,, \tag{26}$$

where $d\mu_{2n-2}$ is Lebesgue measure in T_z

Assume that $m > n-1$, then $|g_k(\tau)| \sim< Mg_k(z), z \in \partial\Omega, \tau \in D_k(z)$. Finally,

$$b_k(z) = \int_{D_k(z)} |\,\bar{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2 d\iota/(\tau) \lesssim \int_{D_k(z)} |\, g_k(\tau)\rho(\tau)^{-l-1}|^2 d\iota/(\tau)$$

$$\lesssim \left(Ma_k(z)\right)^2 \int_{D_k(z)} \frac{d_{lJ}(\tau)}{\rho(\tau)^2} \lesssim \left(Ma_k(z)\right)^2$$

and the lemma is proved.

Let $\Omega \subset \mathbb{C}^n$ be a strongly convex domain and , $\varepsilon j > 0$. For function $g \in L^1(\partial\Omega)$ and $l \in$ IN we define a function

$$I_l(g,z) = \left(\int_{D^e(z,\eta\varepsilon)} \left| \int_{\partial\Omega} \frac{g(w)dS(w)}{\langle \partial\rho(\tau), \tau - w\rangle^{n+l}} \right|^2 d\nu_l(\tau) \right)^{1/2} \tag{27}$$

where $dS(w) = \frac{1}{(2\pi i)^n} \partial\rho(w) \wedge \left(\bar{\partial}\partial\rho(w)\right)^{n-1}$ (see (3)) and $d\iota/l(\tau) = \frac{d\mu_{2n}(\tau)}{\rho(\tau)^{n-2l-1}}$.

Theorem (5.1.11)[151]: *Let Ω be strongly convex domain and $g \in L^p(\partial\Omega)$, $1 < p < \infty$, Then*

$$\int_{\partial\Omega} I_l(g,z)^p d\sigma(z) \lesssim \int_{\partial\Omega} |\, g(z)|^p d\sigma(z)\,. \tag{28}$$

Note that in the one-variable case the integral (27) gives the holomorphic function and the result of the theorem follows from [155].

Definition (5.1.12)[151]: Assume, that defining function ρ for strongly convex domain Ω has the following form near $0 \in \partial\Omega$

$$\rho(z) = 2\,\mathrm{Re}\,(z_n) + \sum_{j,k=1}^{n} A_{jk}\, z_j \bar{z}_k + O(|z|^3) \tag{29}$$

with positive definite form $A_{jk}z_j\bar{z}_k$. We define a set

$$D_0(\eta,\varepsilon) = \{\tau \in \mathbb{C}^n \backslash \Omega : |\tau_1|^2 + \cdots + |\tau_{n-1}|^2 < \eta\,\mathrm{Re}\,(\tau_n)\,, \\ |\,\mathrm{Im}\,(\tau_n)| < \eta\,\mathrm{Re}\,(\tau_n), |\,\mathrm{Re}\,(\tau_n)| < \varepsilon\}. \tag{30}$$

Proof. Since operators T_j with kernels K_j verify the conditions of T_1-theorem, we have $T_j \in$

$\mathcal{L}(L^p(\partial\Omega), L^p(\partial\Omega, L^2(D_{0l))}d\iota/$ and

$$\sum_{j=1\partial}^{N}\int_{\Omega} \| T_j g(z)\|^p dS(z)$$

$$= \sum_{j=1\partial}^{N}\int_{\Omega} d\,S(z)\left(\int_{D_0} |\int_{\partial\Omega} \frac{g(w)\chi_j^{1/2}(z)J_j(z,\tau)dS(w)}{\langle \partial\rho\left(\psi_j(z,\tau)\right), \psi_j(z,\tau)-w\rangle^{n+1}}|^2 \frac{d\mu(\tau)}{\text{Re }(\tau_n)^{n-1}}\right)^p$$

$$\lesssim \|g\|_{L^p(\partial\Omega)}^p.$$

Thus by decomposition (35) $\int_{\partial\Omega} I_l\,(g,z)^p d\sigma(z) \lesssim \int_{\partial\Omega} |\,g(z)|^p d\sigma(z)$, which proves the theorem.

Lemma (5.1.13)[151]: Suppose, that ρ has the form (29). There exist constants c, $\varepsilon_0 > 0$ such that

$$D^e(0,\eta,\varepsilon) \subset D_0(c\eta, c\varepsilon j), D_0(\eta,\varepsilon) \subset D^e(0,c\eta,c\varepsilon j)\ for\ 0<\eta,\varepsilon<\varepsilon_0.$$

Proof. For the function ρ of the form (29) the Korányi sector (11) could be expressed as follows

$$D^e(0,\eta,\varepsilon) = \{\tau \in \mathbb{C}^n\backslash\Omega: |\tau_1|^2+\cdots+|\tau_{n-1}|^2 \le \eta\rho(\tau)\,,\\ |\text{ Im }(\tau_n)| \le \eta\rho(\tau), \rho(\tau)<\varepsilon\}$$

and

$$\rho(\tau) \le 2\text{ Re }(\tau_n) + c_0(|\tau_1|^2+\ +|\tau_{n-1}|^2+\text{ Im }(\tau_n)^2+\text{ Re }(\tau_n)^2)$$
$$\le \left(2+c_0\text{ Re }(\tau_n)\right)\text{Re }(\tau_n) + c_0\left(1+\eta\rho(\tau)\right)\eta\rho(\tau), \tau \in D^e(0,\eta,\varepsilon)\,.$$

Thus for $\eta < \eta_0 = \frac{1}{8c_0}$ we have $(\tau) \le c\text{ Re }(\tau_n)$.

It is easy to see, that $|\tau| \to 0$ when $\rho(\tau)\to 0, \tau \in D^e(0,\eta,\varepsilon)$. Then by convexity of Ω

$$2\text{ Re }(\tau_n) = \rho(\tau) - \sum_{j,k=1}^{n} A_{jk}\,\tau_j\overline{\tau}_k + O(|\tau|^3) \le \rho(\tau), \tau \in D^e(0,\eta,\varepsilon_0)$$

for some $\varepsilon_0 \in (0,\eta_0)$.

Finally $D^e(0,\eta,\varepsilon) \subset D_0(c\eta,\varepsilon j)$ and analogously $D_0(\eta,\varepsilon) \subset D^e(0,\eta,\varepsilon)$ for $0<\eta,\ \varepsilon<\varepsilon_0$.

Theorem (5.1.14)[151]: There exists such covering of the set $\overline{\Omega}_\varepsilon\backslash\Omega_{-\varepsilon}$ by open sets Γ_j such that for every $\xi \in \Gamma_j$ we can find a holomorphic change of coordinates $\phi_j(\xi,\cdot) : \mathbb{C}^n \to \mathbb{C}^n$ such that

i. The mapping $\phi_j(\xi,\cdot)$ transforms function ρ to the type (29) and could be expressed as follows

$$\phi_j(\xi,z) = \Phi_j(\xi)(z-\xi) + (z-\xi)^{\perp}B_j(\xi)(z-\xi)e_n, \tag{31}$$

where matrices $\Phi_j(\xi)$, $B_j(\xi)$ are C^1-smooth on Γ_j, and $e_n = (0, \dots, 0,1)$.

ii. Let $\psi_j(\xi,\cdot)$ be an inverse map of $\phi_j(\xi,\cdot)$, and let $J_j(\xi,\cdot)$ be a complex Jacobian of ψ_j. Then

$$\sup_{\tau\in\Omega_\varepsilon\setminus\bar{\Omega}_\varepsilon} |J_j(\xi,\cdot) - J_j(\xi',\cdot)| \lesssim |\xi - \xi'|, \tag{32}$$

$$\sup_{\tau\in\Omega_\varepsilon\setminus\bar{\Omega}_\varepsilon} |\psi_j(\xi,\cdot) - \psi_j(\xi',\cdot)| \lesssim |\xi - \xi'|. \tag{33}$$

Note that real Jacobian is then equal to $|J_j(\xi,\cdot)|^2 = J_j(\xi,\cdot)\overline{J_j(\xi,\cdot)}$.

iii. *There exist constants* c, $\varepsilon_0 > 0$ *such that for* $0 < \eta$, $\varepsilon < \varepsilon_0$

$$\phi_j\big(\xi, D^e(\xi,\eta,\varepsilon)\big) \subseteq D_0(c\eta, c\varepsilon)\,, \psi_j(\xi, D_0(\eta,\varepsilon) \subseteq D^e(\xi, c\eta, c\varepsilon)\,. \tag{34}$$

Proof. Let $\xi \in \partial\Omega$, by linear change of coordinates $z' = (z-\xi)\Phi(\xi)$ we could obtain the following form for function ρ

$$\rho(z) = \rho(\xi + \Phi^{-1}(\xi)z')$$

$$= 2\ \mathrm{Re}\ (z_n') + \sum_{j,k=1}^{n} A_{jk}^1\,(\xi) z_j \bar{z}_k' + \ \mathrm{Re}\ \sum_{j,k=1}^{n} A_{jk}^2\,(\xi) z_j z_k' + O(|z'|^3)\,.$$

Setting $z_n'' = z_n' + A_{jk}^2 z_j' z_k'$ and $z_j'' = z_j'$, $1 \le j \le n-1$, we have (see [13])

$$\rho(z'') = 2\ \mathrm{Re}\ (z_n') + \sum_{j,k=1}^{n} A_{jk}^1\,(\xi) z_j'' \bar{z}_k'' + O(|z''|^3)\,.$$

Denote $(\xi) = \Phi(\xi)^\perp A^2(\xi)\Phi(\xi)$, then

$$\phi(\xi, z) = \Phi(\xi)(z-\xi) + (z-\xi)^\perp B(\xi)(z-\xi)e_n.$$

We choose Γ_j such that the matrix $\Phi(\xi)$ could be defined on Γ_j smoothly, this choice we denote as Φ_j, and the change corresponding to this matrix as ϕ_j

$$\phi_j(\xi, z) = \Phi_{\square}(\xi)(z-\xi) + (z-\xi)^\perp B_j(\xi)(z-\xi)e_n.$$

Thus mappings ϕ_j satisfy the first condition. Easily, the second condition also holds.

The last condition (34) follows immediately from Lemma (5.1.15). This ends the proof of the theorem.

Further we will assume, that the covering $\overline{\Omega}_\varepsilon\setminus\Omega_{-\varepsilon} \subset \cup_{j=1}^{N} \Gamma_j$ and maps ϕ_j, ψ_j. For covering

$\{\Gamma_j\}$ we consider a smooth decomposition of identity on :

$$\chi_j \in C^\infty(\Gamma_j), 0 \le \chi_j \le 1, supp\ \chi_j \subset \mathrm{T}_j, \sum_{j=1}^{N} \chi_j\,(z) = 1, z \in \partial\Omega.$$

Fix parameters $0 < \varepsilon, \eta < \varepsilon_0$, denote $D_0 = D_0(\eta, \varepsilon)$. Then by (34)

$$D^e(z) = \phi_j\big(z, D^e(z, \eta/c, \varepsilon j/c)\big) \subset D_0$$

and

$$I_l(g,z)^2$$

$$= \sum_{j=1}^{N} \chi_j\,(z) \int_{D^e(z)} |\int_{\partial\Omega} \frac{g(w)J_j(z,\tau)dS(w)}{\langle \partial\rho\left(\psi_j(z,\tau)\right), \psi_j(z,\tau) - w\rangle^{n+l}}|^2 \frac{d\mu(\tau)}{\mathrm{Re}\ (\tau_n)^{n-2l+1}}$$

$$\lesssim \sum_{j=1_D}^{N} \int_0 |\int_{\partial\Omega} \frac{g(w)\chi_j^{1/2}(z)J_j(z,\tau)dS(w)}{\langle \partial\rho\left(\psi_j(z,\tau)\right), \psi_j(z,\tau) - w\rangle^{n+l}}|^2 \frac{d\mu(\tau)}{\mathrm{Re}\ (\tau_n)^{n-2l+1}}. \quad (35)$$

We will consider the function

$$K_j(z,w)(\tau) = \frac{\chi_j^{1/2}(z)J_j(z,\tau)}{\langle \partial\rho\left(\psi_j(z,\tau)\right), \psi_j(z,\tau) - w\rangle^{n+1}} \quad (36)$$

as a map $\partial\Omega \times \partial\Omega \to \mathcal{L}(\mathbb{C}, L^2(D_{0l))}d\iota/$, such that its values are operator of multiplication from $\mathbb{C}$ to $L^2(D, d\iota/)$, where $d\iota/l(\tau) = \frac{d\mu(\tau)}{\mathrm{Im}\ (\tau_n)^{n-2l+1}}$ is a measure on the region D_0. Throughout the proof of the Theorem (5.1.11) j, l will be fixed integers and the norm of function F in the space $L^2(D_{0l)}d\iota/$ will be denoted as $\|F\|$.

We will show that integral operator defined by kernel K_j is bounded on L^p. To prove this we apply $T1$-theorem for transformations with operator-valued kernels formulated by Hytönen and Weis in [159], taking in account that in our case concerned spaces are Hilbert. Some details of the proof are similar to the proof of the boundedness of operator Cauchy-Leray-Fantappiè K_Ω for lineally convex domains introduced in [14]. Below we formulate the $T1$-theorem.

Definition (5.1.15)[151]. We say that the function $\mathrm{f} \in \mathrm{C}_0^\infty(\partial\Omega)$ is a normalized bumpfunction, associated with the quasiball $\mathrm{B}(\mathrm{w}_0, \mathrm{r})$ if supp $\mathrm{f} \subset \mathrm{B}(\mathrm{z}, \mathrm{r})$, $|\mathrm{f}| \le 1$, and

$$|f(\xi) - f(z)| \le \frac{d(\xi, z)^\gamma}{r^\gamma}.$$

The set of bump-functions associated with $\mathrm{B}(\mathrm{w}_0, \mathrm{r})$ is denoted as $(\gamma, \mathrm{w}_0, \mathrm{r})$.

Theorem (5.1.16)[151]. Let $\mathrm{K}: \partial\Omega \times \partial\Omega \to \mathcal{L}\big(\mathbb{C}, \mathrm{L}^2(\mathrm{D}d\iota/)\big)$ verify the estimates

$$\|K(z,w)\| \lesssim \frac{1}{d(z,w)^n}; \quad (37)$$

$$\|K(z,w)-K(\xi,w)\| \lesssim \frac{d(z,\xi)^\gamma}{d(z,w)^{n+\gamma}}, d(z,w) > Cd(z,\xi); \quad (38)$$

$$\|K(z,w)-K(z,w')\| \lesssim \frac{d(w,w')^\gamma}{d(z,w)^{n+\gamma}}, d(z,w) > Cd(w,w'). \quad (39)$$

Assume that operator $\mathrm{T}: \mathcal{S}(\partial\Omega) \to \mathcal{S}'\left(\partial\Omega, \mathcal{L}\left(\mathbb{C}, \mathrm{L}^2(\mathrm{D}, \mathrm{d}\iota/)\right)\right)$ with kernel K verify the following conditions. $\cdot$ $T1, T'1 \in \mathrm{BMO}(\partial\Omega, L^2(D_{0l))}d\iota/$,

where T' is formally adjoint operator. · Operator T satisfies the weak boundedness property, that is for every pair of normalized bump-functions $\mathrm{f}, \mathrm{g} \in \mathrm{A}(\gamma, \mathrm{w}_0, \mathrm{r})$ we have

$$\|\langle g, Tf\rangle\| \le Cr^{-n}$$

Then $T \in \mathcal{L}(L^p(\partial\Omega), L^p\left(\partial\Omega, L^2(D_0, d\nu_l)\right)$ *for every* $\in (1,\infty)$.

In the following three lemmas we will prove that kernels K_j and corresponding operators T_j satisfy the conditions of the $T1$-theorem.

Lemma (5.1.17)[151]. The kernel $\mathrm{K_j}$ verify estimates

Proof. By lemma (5.1.4) we have $|\langle\partial\rho(\tau), \tau - w\rangle| \wedge -\rho(\tau) + |\langle\partial\rho(z), z - w\rangle|$, $z, w \in \partial\Omega$, $\tau \in D^e(z, c\eta, c\varepsilon j)$. Thus

$$\|K_j(z,w)\|^2 = \int_{D_0} |K_j(z,w)(\tau)|^2 d\nu_l(\tau) \lesssim \int_{D^{\square}(z,c\eta,c\varepsilon)} \frac{\mathrm{dv_l}(\tau)}{|\langle\partial\rho(\tau), \tau - w\rangle|^{2n+2l}}$$

$$\lesssim \int_{D^e(z,c\eta,c\varepsilon)} \frac{1}{(\rho(\tau) + |\langle\partial\rho(z), z-w\rangle|)^{2n+2l}} \frac{d\mu(\tau)}{\rho(\tau)^{n-2l+1}}$$

$$\lesssim \int_0^\infty \frac{t^{2l-1}dt}{(t+|\langle\partial\rho(z), z-w\rangle|)^{2n+2l}} \sim< \frac{1}{|\langle\partial\rho(z), z-w\rangle|^{2n}} \sim< \frac{1}{d(z,w)^{2n}}.$$

Similarly,

$$\|K_j(z,w) - K_j(z,w')\|^2$$

$$\lesssim \int_{D^e(z,c\eta,c\varepsilon)} \left|\frac{1}{\langle\partial\rho(\tau), \tau - w\rangle^{n+l}} - \frac{1}{\langle\partial\rho(\tau), \tau - w'\rangle^{n+l}}\right|^2 d\nu_l(\tau)$$

Denote $\hat{\tau} = \mathrm{pr}_{\partial\Omega}(\tau)$, then

$$|\langle\partial\rho(\tau), \tau - w\rangle| \lesssim \rho(\tau) + |\langle\partial\rho(\hat{\tau}), \hat{\tau} - w\rangle|$$
$$\lesssim \rho(\tau) + |\langle\partial\rho(z), z-w\rangle| + |\langle\partial\rho(\hat{\tau}), \hat{\tau} - z\rangle| \lesssim \rho(\tau) + |\langle\partial\rho(z), z-w\rangle|,$$

which combined with Lemma (5.1.4) and condition

$$d(w,w') = |\langle\partial\rho(w), w-w'\rangle| < C|\langle\partial\rho(z), z-w\rangle| = Cd(z,w)$$

implies

$$|\langle \partial\rho(\tau), \tau - w\rangle| \wedge -\rho(\tau) + |\langle \partial\rho(z), z - w\rangle| \wedge -\rho(\tau) + |\langle \partial\rho(z), z - w'\rangle| \wedge -|\langle \partial\rho(\tau), \tau - w'\rangle|.$$

Next, we have

$$|\langle \partial\rho(\tau), \tau - w'\rangle - \langle \partial\rho(\tau), \tau - w\rangle| = |\langle \partial\rho(\tau), \hat{\tau} - w\rangle - \langle \partial\rho(\tau), \hat{\tau} - w'\rangle|$$
$$\leq |\langle \partial\rho(\tau) - \partial\rho(\hat{\tau}), w - w'\rangle| + |\langle \partial\rho(\hat{\tau}), \hat{\tau} - w\rangle - \langle \partial\rho(\hat{\tau}), \hat{\tau} - w'\rangle|$$
$$\lesssim \rho(\tau)|\langle \partial\rho(w), w - w'\rangle|^{1/2} + |\langle \partial\rho(\hat{\tau}), \hat{\tau} - w\rangle|^{1/2}|\langle \partial\rho(w), w - w'\rangle|^{1/2}$$
$$\lesssim |\langle \partial\rho(\tau), \tau - w\rangle|^{1/2}|\langle \partial\rho(w), w - w'\rangle|^{1/2}$$

Hence,

$$\|K_j(z,w) - K_j(z,w')\|^2 \lesssim \int_{D^e(z,c\eta,c\varepsilon)} \frac{|\langle \partial\rho(w), w - w'\rangle|}{|\langle \partial\rho(\tau), \tau - w\rangle|^{2n+2l+1}} d_{L/}l(\tau)$$

$$\lesssim \int_0^\infty \frac{|\langle \partial\rho(w), w - w'\rangle| t^{2l-1} dt}{(t + |\langle \partial\rho(z), z - w\rangle|)^{2n+2l+1}} \lesssim \frac{|\langle \partial\rho(w), w - w'\rangle|}{|\langle \partial\rho(z), z - w\rangle|^{2n+1}} = \frac{d(w,w')}{d(z,w)^{n+1}}.$$

The last inequality (39) is a bit harder to prove.
Let $z, \xi, w \in \partial\Omega$, $C(z,\xi) < d(z,w)$, and estimate the value

$$A = |\langle \partial\rho\left(\psi_j(z,\tau)\right), \psi_j(z,\tau) - w\rangle - \langle \partial\rho\left(\psi_j(\xi,\tau)\right), \psi_j(\xi,\tau) - w\rangle|.$$

Denote $\tau_z = \psi_j(z,\tau)$, $\tau_\xi = \psi_j(\xi,\tau)$, then by (31)

$$\tau = \Phi(z)(\tau_z - z) + i(\tau_z - z)^T B(z)(\tau_z - z)e_n$$
$$= \Phi(\xi)(\tau_\xi - \xi) + i(\tau_\xi - \xi)^T B(\xi)(\tau_\xi - \xi)e_n,$$

whence denoting $\Psi(z) = \Phi(z)^{-1}$ and introducing $L(z,\xi,\tau)$ we obtain

$$\tau_z = z + \Psi(z)\tau - (\tau_z - z)^T B(z)(\tau_z - z)\Psi(z)e_n,$$
$$\tau_\xi = \xi + \Psi(\xi)\tau - (\tau_\xi - \xi)^T B(\xi)(\tau_\xi - \xi)\Psi(\xi)e_n,$$
$$\tau_z - \tau_\xi = z - \xi + \left(\Psi(z) - \Psi(\xi)\right)\tau + L(z,\xi,\tau)e_n.$$

Note, that norms of matrices $\|\Psi(\xi)\|$ are bounded, thus

$$|L(z,\xi,\tau)| \leq |(\tau_z - z)^T B(z)(\tau_z - z)\left(\Psi(z) - \Psi(\xi)\right)|$$
$$+|(\tau_z - z)^T B(z)(\tau_z - z) - (\tau_\xi - \xi)^T B(\xi)(\tau_\xi - \xi)|\|\Psi(\xi)\|$$
$$\lesssim |z - \xi||\tau_z - z|^2 + |(\tau_z - z - \tau_\xi + \xi)^T B(z)(\tau_z - z)|$$
$$+|(\tau_\xi - \xi)^T B(z)(\tau_z - z) - (\tau_\xi - \xi)^T B(\xi)(\tau_\xi - \xi)|$$
$$\lesssim |z - \xi||\tau_z - z|^2 + |z - \xi||\tau| + |\left(((z) - \Psi(\xi))\tau + L(z,\xi,\tau)e_n\right)^T B(z)(\tau_z - z)|$$

$$+|(\tau_\xi-\xi)^T(B(z)-B(\xi))(\tau_z-z)|+|(\tau_\xi-\xi)^T B(\xi)(\tau_z-z-\tau_\xi-\xi)|$$
$$\lesssim |z-\xi||\tau_z-z|^2+|z-\xi||\tau|+|\tau||L(z,\xi,\tau)|+|z-\xi||\tau|^2+|\tau|L(z,\xi,\tau)\,.$$
Choosing $\varepsilon j>0$ small enough we get $|\tau|\le\eta|\operatorname{Im}(\tau_n)|+(1+\eta)|\operatorname{Im}(\tau_n)|\le 3\varepsilon$ and $|L(z,\xi,\tau)|\lesssim d(z,\xi)^{1/2}|\tau|$, for $\tau\in D_0=D_0(\eta,\varepsilon)$. Hence,

$$A\le|\langle\partial\rho(\tau_z)-\partial\rho(\tau_\xi),\tau_z-w\rangle|+|\langle\partial\rho(\tau_\xi),\tau_z-w\rangle|$$

$$\lesssim|\tau_z-\tau_\xi|\left(\rho(\tau_z)+d(z,w)^{1/2}\right)+|\langle\partial\rho(\tau_z)-\partial\rho(\tau_\xi),z-\xi\rangle|+|\langle\partial\rho(z),z-\xi\rangle|$$

$$+|\langle\partial\rho(\tau_\xi),$$

$$(\Psi(z)-\Psi(\xi))\tau\rangle|+|\langle\partial\rho(\tau_\xi),$$

$$L(z,\xi,\tau)\rangle|\lesssim d(z,\xi)^{1/2}d(\tau_z,w)+|\tau_z-\xi||z-\xi|+d(z,\xi)+|z-\xi||\tau|+|L(z,\xi,\tau)|$$
$$\lesssim d(z,\xi)+d(z,\xi)^{1/2}d(z,w)^{1/2}$$
$$\lesssim d(z,\xi)^{1/2}d(z,w)^{1/2}$$

Combining this estimate with inequality $|\langle\partial\rho(\tau_z),\ \tau_z-w\rangle|\wedge-|\langle\partial\rho(\tau_\xi),\ \tau_\xi-w\rangle|$ we obtain

$$\|K_j(z,w)-K_j(\xi,w)\|^2\lesssim\int_{D^e(z,c\eta,c\varepsilon)}\frac{|\chi_j(z)^{1/2}-\chi_j(\xi)^{1/2}|^2}{|\langle\partial\rho(\tau),\tau-w\rangle|^{2n+2l}}\frac{d\mu(\tau)}{\rho(\tau)^{n-2l+1}}$$

$$+\chi_j(\xi)\int_{D_0}\frac{|\langle\partial\rho(z),z-\xi\rangle||\langle\partial\rho(z),z-w\rangle|}{|\langle\partial\rho(\tau_z),\tau_z-w\rangle|^{2n+4}}\frac{d\mu(\tau)}{\operatorname{Re}(\tau_n)^{n-2l+1}}$$
$$\lesssim\frac{|\langle\partial\rho(z),z-\xi\rangle|}{|\langle\partial\rho(z),z-w\rangle|^{2n}}+\frac{|\langle\partial\rho(z),z-\xi\rangle|}{|\langle\partial\rho(z),z-w\rangle|^{2n+1}}\lesssim\frac{|\langle\partial\rho(z),z-\xi\rangle|}{|\langle\partial\rho(z),z-w\rangle|^{2n+1}}$$
$$\lesssim\frac{d(z,\xi)}{d(z,w)^{2n+1}}.$$

Lemma (5.1.18)[151]: $T_j(1)=0$ *and* $\|T_j(1)\|\sim<1$.

Proof. Introduce the notation $\tau_z=\psi_j(z,\tau)$. The function $\langle\partial\rho(\tau_z),\tau_z-w\rangle$ is holomorphic in Ω with respect to w, then the form $\langle\partial\rho(\tau_z),\tau_z-w\rangle^{-n-\iota}dS(w)$ is closed in Ω and

$$T_j(1)(\tau)=\chi_j(z)^{1/2}J_j(z,\tau)\int_{\partial\Omega}\frac{dS(w)}{\langle\partial\rho(\tau_z),\tau_z-w\rangle^{n+l}}=0.$$

It remains to estimate the value of formally-adjoint operator T' on $f\equiv 1$.

$$T_j'(1)(w)(\tau)=\int_{\partial\Omega}\frac{\chi_j(z)^{1/2}J_j(z,\tau)dS(z)}{\langle\partial\rho(\tau_z),\tau_z-w\rangle^{n+l}}$$

$$= \int_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau)\big(dS(z) - dS(\tau_z)\big)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}} + \int_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dS(\tau_z)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+\iota}} = L_1 + L_2.$$

Note that $|z - \tau_z| \lesssim \operatorname{Re}(\tau_n)$, therefore $|dS(z) - dS\big(\psi(z,\tau)\big)| \lesssim \operatorname{Re}(\tau_n) d\sigma(z)$ and

$$|L_1| \lesssim \int_{\partial\Omega} \frac{\operatorname{Re}(\tau_n) d\sigma(z)}{|\langle \partial\rho(\tau_z), \tau_z - w\rangle|^{n+l}} \lesssim \frac{\operatorname{Re}(\tau_n) d\sigma(z)}{(\operatorname{Re}(\tau_n) + |\langle \partial\rho(z), z - w\rangle|)^{n+l}}$$

$$\lesssim \int_0^\infty \frac{\operatorname{Re}(\tau_n) v^{n-1} dv}{(\operatorname{Re}(\tau_n) + v)^{n+l}} \lesssim \frac{1}{\operatorname{Re}(\tau_n)^{l-1}}.$$

Thus we get

$$\int_{D_0} |L_1|^2 d\iota/l(\tau) \lesssim \int_{D_0} \frac{1}{\operatorname{Re}(\tau_n)^{2l-2}} \frac{d\mu(\tau)}{\operatorname{Re}(\tau_n)^{n-2l+1}} \lesssim \int_0^\varepsilon \frac{t^n dt}{t^{n-1}} \sim< 1 \quad (40)$$

To estimate L_2 we recall that $d_\xi \frac{dS(\xi)}{\langle \partial\rho(\xi), \xi - z\rangle^n} = 0, z \in \partial\Omega, \xi \in \mathbb{C}^n \setminus \Omega$, and consequently

$$d \frac{dS(\xi)}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}} = \frac{\left(\bar\partial\partial\rho(\xi)\right)^n}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}}$$

$$-(n+l) \frac{\left(\bar\partial_\xi(\langle \partial\rho(\xi), \xi - z\rangle) \wedge \bar\partial\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}} = -\frac{l}{n} \frac{dV(\xi)}{\langle \partial\rho(\xi), \xi - z\rangle^{n+l}}.$$

By Stokes' theorem we obtain

$$L_2 = \int_{\partial\Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dS(\tau_z)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}}$$

$$= \int_{\Omega_{\varepsilon_1} \setminus \Omega} \frac{\bar\partial_z\left(\chi_j(z)^{1/2} J_j(z,\tau)\right) \wedge dS(\tau_z)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}} - \frac{l}{n} \int_{\Omega_{\varepsilon_1} \setminus \Omega} \frac{\chi_j(z)^{1/2} J_j(z,\tau) dV(\tau_z)}{\langle \partial\rho(\square_z), \tau_z - z\rangle^{n+l}}$$

Analogously to Lemma (5.1.4) we have $|\langle \partial\rho(\tau_z), \ \tau_z - w\rangle| -\wedge \operatorname{Im}(\tau_n) + \rho(z) + |\langle \partial\rho(\hat z), \hat z - w\rangle|$, where $\hat z = \mathrm{pr}_{\partial\Omega}(z)$. Hence,

$$|L_2| \lesssim \int_{\Omega_{\varepsilon_1} \setminus \Omega} \frac{d\mu(z)}{|\langle \partial\rho(\tau_z), \tau_z - w\rangle|^{n+l}}$$

$$\lesssim \int_0^\varepsilon \square t \int_{\partial\Omega_t} \frac{d\sigma_t}{(t + \operatorname{Im}(\tau_n) + |\langle \partial\rho(\hat z), \hat z - w\rangle|)^{n+l}}$$

$$\lesssim \int_0^\varepsilon d\,t \int_0^\infty \frac{v^{n-1} dv}{(t + \operatorname{Re}(\tau_n) + v)^{n+l}} \sim< \int_0^\varepsilon \frac{dt}{\big(t + \operatorname{Re}(\tau_n)\big)^l}$$

$$\lesssim \left(\operatorname{Re}(\tau_n)\right)^{1-\iota} \ln\left(1+\frac{1}{\operatorname{Re}(\tau_n)}\right),$$

and

$$\int_{D_0} |L_2|^2 d\iota/l(\tau) \lesssim \int_{D_0} \left(\operatorname{Re}(\tau_n)\right)^{2-2\iota} \ln^2\left(1+\frac{1}{\operatorname{Re}(\tau_n)}\right) d\nu_l(\tau)$$

$$\lesssim \int_0^{\varepsilon} \ln^2\left(1+\frac{1}{s}\right) s ds_{\sim} < 1,$$

which with the estimate (40) completes the proof of the lemma.

Lemma (5.1.19)[151]: *Operator T_j is weakly bounded.*

Proof. Let $f, g \in A\left(\frac{1}{2}, w_0, r\right)$, denote again $\tau_z = \psi_j(z,\tau)$, then

$$\|\langle g, Tf\rangle\|^2 \lesssim \int_{D_0} \mathrm{dv_l}(\tau)\left(\int_{B(w_0,r)} |g(z)| dS(z)\right| \int_{B(w_0,r)} \frac{f(w)dS(w)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+1}} \left|\right)^2$$

Denote $t := \inf |\langle \partial\rho(\tau_z), \tau_z - w\rangle|$ and introduce the set

$$W(z,\tau,r) := \{w \in \partial\Omega : |\langle \partial\rho(\tau_z), \tau_z - w\rangle| < t + r\}.$$

Note that $B(w_0,r) \subset W(z,\tau,cr) \subset B(z,c^2r)$ for some $c > 0$, therefore,

$$\left|\int_{B(w_0,r)} \frac{f(w)dS(w)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}}\right|$$

$$= \left|\int_{W(z,\tau,cr)} \frac{f(w)dS(w)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}}\right| \lesssim \int_{W(z,\tau,cr)} \frac{|f(z) - f(w)| dS(w)}{|\langle \partial\rho(\tau_z), \tau_z - w\rangle|^{n+l}}$$

$$+ |f(z)| \left|\int_{\partial\Omega \setminus W(z\tau,cr)} \frac{dS(w)}{\langle \partial\rho(\tau_z), \tau_z - w\rangle^{n+l}}\right| = L_1(z,\tau) + |f(z)| L_2(z,\tau).$$

It follows from the estimate $|f(z) - f(w)| \le \sqrt{v(w,z)}/r$ that

$$L_1(z,\tau) \lesssim \frac{1}{\sqrt{r}} \int_{B(z,c^2r)} \frac{v(w,z)^{1/2}}{\left(\operatorname{Re}(\tau_n) + v(w,z)\right)^{n+l}} \lesssim \frac{1}{\sqrt{r}} \int_0^{c^2r} \frac{t^{n-1/2} dt}{(\operatorname{Re}(\tau_n) + t)^{n+l}}$$

$$\lesssim \frac{1}{\sqrt{r}} \int_0^{c^2r} \frac{dt}{(\operatorname{Re}(\tau_n) + t)^{l+1/2}} \lesssim \frac{1}{\sqrt{r}}\left(\frac{1}{\operatorname{Re}(\tau_n)^{l-1/2}} - \frac{1}{(\operatorname{Re}(\tau_n) + r)^{l-1/2}}\right)$$

$$= \frac{1}{\sqrt{r}} \frac{(\operatorname{Re}(\tau_n) + r)^{\iota-1/2} - r^{l-1/2}}{\operatorname{Re}(\tau_n)^{\iota-1/2}(\operatorname{Re}(\tau_n) + r)^{l-1/2}} \lesssim \frac{1}{\sqrt{r}} \frac{(\operatorname{Re}(\tau_n) + r)^{2l-1} - r^{2l-1}}{\operatorname{Im}(\tau_n)^{\iota-1/2}(\operatorname{Re}(\tau_n) + r)^{2l-1}}$$

$$\lesssim \frac{1}{\sqrt{r}}\frac{r\mathrm{Re}\,(\tau_n)^{2l-2}+r^{2l-1}}{\mathrm{Re}\,(\tau_n)^{l-1/2}(\,\mathrm{Re}\,(\tau_n)+r)^{2l-1}}$$

Estimating the $L^2(D, dv_l)$-norm of the function $L_1(z,\tau)$, we obtain

$$\int_{D_0(\tau)} L_1\,(z,\tau)^2 \mathrm{dv}_l(\tau)$$

$$\lesssim \int_{D_0(\tau)} \left(\frac{r\ \mathrm{Re}\,(\tau_n)^{2l-3}}{(\,\mathrm{Re}\,(\tau_n)+r)^{4l-2}}+\frac{r^{4l-3}}{\mathrm{Re}\,(\tau_n)^{2l-1}(\,\mathrm{Re}\,(\tau_n)+r)^{4l-2}}\right)\frac{d\mu(\tau)}{\mathrm{Re}\,(\tau_n)^{n-2l+1}}$$

$$\lesssim r\int_0^{\infty}\frac{s^{4l-4}}{(s+r)^{4l-2}}ds+r^{4l-3}\int_0^{\infty}\frac{ds}{(s+\square)^{4l-2}}\lesssim 1 \qquad (41)$$

To estimate the second summand L_2 we apply the Stokes theorem to the domain

$$\{w\in\Omega\colon |\langle\partial\rho(\tau_z),\tau_z-w\rangle|>t+cr\}$$

and to the closed in this domain form $\frac{dS(w)}{\langle\partial\rho(\tau_z),\tau_z-w\rangle^{n+l}}$

$$\int_{\partial\Omega\setminus W(z,\tau,cr)}\frac{dS(w)}{\langle\partial\rho(\tau_z),\tau_z-w\rangle^{n+l}}=-\int_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\frac{dS(w)}{\langle\partial\rho(\tau_z),\tau_z-w\rangle^{n+l}}$$

$$=-\frac{1}{(t+cr)^{2n+l}}\int^{-}_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\,dS(w).$$

Applying Stokes' theorem again, now to the domain

$$\{w\in\Omega\colon |\langle\partial\rho(\tau_z),\tau_z-w\rangle|<t+cr\},$$

we obtain

$$L_3:=\int_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\,dS(w)$$

$$=-\int_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\,dS(w)$$

$$+\int_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\bar\partial_w\left(\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\right)\wedge dS(w)$$

$$+\int_{\substack{w\in\Omega\\ |v(\tau_z,w)|=t+cr}}\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\,dV(w)\,.$$

Since $|\bar\partial_w\left(\overline{\langle\partial\rho(\tau_z),\tau_z-w\rangle}^{n+l}\right)\wedge dS(w)|\lesssim|\langle\partial\rho(\tau_z),\tau_z-w\rangle|^{n+l-1}$ we get

$$|L_3| \lesssim \int_t^{t+cr} (s^{n+l}s^{n-1} + s^{n+l} + s^{n+l-1}s^n)ds \lesssim \int_t^{t+cr} s^{2n+l-1}ds \lesssim r(t+r)^{2n+l-1.}$$

Note that $t -\wedge \rho(\tau_z) -\wedge \text{ Im } (\tau_n)$ and consequently

$$\int_{D_0} L_2(z,\tau)^2 d\nu_l(\tau) \lesssim \int_{D_0} \left(\frac{r(\text{ Re } (\tau_n) + r)^{2n+l-1}}{(\text{ Re } (\tau_n) + r)^{2n+2l}}\right)^2 d\nu_l(\tau)$$

$$\lesssim \int_0^\infty \frac{r^2}{(t+r)^{2l+2}} \frac{t^n dt}{t^{n-2l+1}} = r^2 \int_0^\infty \frac{t^{2l-1}}{(t+r)^{2l+2} t^{n-2l+1}} \lesssim r^2 \int_0^\infty \frac{dt}{(r+t)^3} \lesssim 1.$$

(16) Summarizing estimates (15) and condition $|f(z)| \leq 1, z \in \partial\Omega$, we obtain

$$\|\langle g, Tf\rangle\|^2 \leq \int_{D_0} \mathrm{d}\nu_l(\tau) \left(\int_{B(w_0,r)} |g(z)|(L_1(z,\tau) + L_2(z,\tau)|f(z)|)dS(z)\right)^2$$

$$\lesssim \|g\|^2_{L^1(\partial\Omega)} \sup_{z\in\partial\Omega} \int_{D_0} (L_1(z,\tau)^2 + L_2(z,\tau)^2)\, d\nu_l(\tau)$$

$$\lesssim \|g\|^2_{L^1(\partial\Omega)} \lesssim |B(w_0,r)|^2$$

The last estimate implies weak boundedness of operator T and completes the proof of the lemma.

Section (5.2): Strongly Convex Domains in C^n

We continue the research started in [164] and devoted to the description of some fundamental spaces of holomorphic functions of several complex variables in terms of polynomial approximations and the pseudoanalytic continuation. We give alternative characterizations of Hardy-Sobolev spaces (see [25], [158])

$$H_p^l(\Omega) = \{f \in H(\Omega): \|f\|_{H^p(\Omega)} + \sum_{|\alpha|\leq l} \|\partial^\alpha f\|_{H^p(\Omega)} < \infty\} \tag{42}$$

on the strongly convex domain $\Omega \subset \mathbb{C}^n$.

We show that for $1 < p <$ oo and $l \geq 1$ a holomorphic on a strongly convex domain Ω function f is in the Hardy-Sobolev space $H_p^l(\Omega)$ if

and only if there exists a sequence of 2^k-degree polynomials P_{2^k} such that

$$\int_{\partial\Omega} d\,\sigma(z) \left(\sum_{k=1}^\infty |f(z) - P_{2^k}(z)|^2 2^{2\iota k}\right)^{p/2} < \infty. \tag{43}$$

In the one variable case this characterization was obtained by E.M. Dynkin [155] for Radon domains.

We devoted to the Cauchy-Leray-Fantappiè integral formula, polynomial approximations and estimates of its kernel. We also define internal and external Korányi regions, the multidimensional analog of Lusin regions. we introduce the method of pseudoanalytic continuation and two constructions of the continuation with different estimates. We use these constructions to obtain the characterization of Hardy-Sobolev spaces in terms of estimates of the pseudoanalytic continuation. To prove this result we use the special analog of the Krantz-Li area-integral inequality [158] for external Korányi regions established in [169]. Finally, contains the proof of characterization (43).

For $\mathbb{C}^n$ be the space of n complex variables, $n \geq 2$, $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$;

$\partial_j f = \frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \bar{\partial}_j f = \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right),$

$$\partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k, \bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k, df = \partial f + \bar{\partial} f.$$

We use the notation

$$\langle \partial f(z), w \rangle = \sum_{k=1}^{n} \frac{\partial f(z)}{\partial z_k} w_k.$$

to indicate the action of ∂f on the vector $w \in \mathbb{C}^n$ and define

$$|\bar{\partial} f| = |\frac{\partial f}{\partial \bar{z}_1}| + +|\frac{\partial f}{\partial \bar{z}_n}|.$$

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{IN}^n$ we set $|\alpha| = \alpha_1 + +\alpha_n$ and $! = \alpha_1! \ldots \alpha_2$!, also $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and $\partial^\alpha f = \frac{\partial^{|.\alpha.|} f}{\partial z_1^{\alpha_1} . \partial z_n^{\alpha_n}}$.

We denote the euclidean distance from the point $z \in \mathbb{C}^n$ to the set $D \subset \mathbb{C}^n$ as $\operatorname{dist}(z, D) = \inf\{|z - w| : w \in D\}$ and the Lebesgue measure in $\mathbb{C}^n$ as $d\mu$. Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a strongly convex domain with a C^3-smooth defining function. We will also consider the family of domains

$$\Omega_t = \{z \in \mathbb{C}^n : \rho(z) < t\}$$

and assume that they are strongly convex for each $|t| < \xi_j$, where $\varepsilon_i > 0$ is small enough.

This is equivalent to the assumption that $d^2\rho(z)$ is positive definite when $|\rho(z)| \leq \varepsilon$. For $z \in \Omega_\varepsilon \backslash \Omega_{-\varepsilon:}$ we denote the nearest point on $\partial\Omega$ as $\operatorname{pr}_{\partial\Omega}(z)$. Then the mapping

$$\operatorname{pr}_{\partial\Omega} : \Omega_\varepsilon \backslash \Omega_{-\epsilon} \to \partial\Omega$$

is well defined, C^2-smooth on $\Omega_\varepsilon \backslash \Omega$ and $|z - \operatorname{pr}_{\partial\Omega}(z)| = \operatorname{dist}(z, \partial\Omega)$.

For $\xi \in \partial\Omega_t$ we define the complex tangent space

$$T_\xi = \{z \in \mathbb{C}^n : \langle \partial\rho(\xi), \xi - z \rangle = 0\}$$

and complex normal vector

$$n(\xi) = \frac{1}{|\bar{\partial}\rho(\xi)|}\left(\bar{\partial}_1\rho(\xi), \ldots, \bar{\partial}_n\rho(\xi)\right). \tag{44}$$

We denote the space of holomorphic functions as $H(\Omega)$ and consider the Hardy space (see

[167], [156])

$$H^p(\Omega) := \{f \in H(\Omega) : \|f\|^p_{H^p(\Omega)} = \sup \int_{\Omega_t} |f(z)|^p d\sigma_t(z) < \infty\},$$

where $d\sigma_t$ is induced Lebesgue measure on the boundary of Ω_t. We also denote $d\sigma = d\sigma_0$. We are interested in Hardy-Sobolev spaces $H^l_p(\Omega)$ defined by (42).

we use notations $\lesssim, \asymp$.We write $f <\sim g$ if $f \leq cg$ for some constant $c > 0$, that doesn't depend on main arguments of functions f and g and usually depend only on the dimension n and the domain Ω. Also $f^{\vee}_{\wedge} g$ if $c^{-1}g \leq f \leq cg$ for some $c > 1$.

In the theory of several complex variables there is no canonical reproducing formula, however we could use the Leray theorem that allows us to construct holomorphic reproducing kernels ([152], [76], [162]). For convex domain $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ this theorem gives us the Cauchy-Leray-Fantappiè formula, and for $f \in H^1(\Omega)$ and $z \in \Omega$ we have

$$f(z) = K_\Omega f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} \frac{f(\xi)\partial\rho(\xi) \wedge \left(\bar{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n}$$

$$= \int_{\partial\Omega} f(\xi) K(\xi, z)\omega(\xi), \tag{45}$$

where $\omega(\xi) = \frac{1}{(2\pi i)^n} \partial\rho(\xi) \wedge \left(\bar{\partial}\partial\rho(\xi)\right)^{n-1}$, *and* $K(\xi, z) = \langle \partial\rho(\xi), \xi - z\rangle^{-n}$

The $(2n-1)$-form ω defines the Leray-Levy measure on $\partial\Omega_t$ which is equivalent to the induced Lebesgue surface measure $d\sigma_t$ (see [152], [160], [161]). This allows us to identify Lebesgue, Hardy and Hardy-Sobolev spaces defined with respect to these measures. Also note, that the measure defined by the $2n$form $d\omega = \left(\partial\bar{\partial}\rho\right)^n$ is equivalent to the Lebesgue measure $d\mu$ in $\mathbb{C}^n$ in $\Omega_{\varepsilon:} \backslash \Omega_{-\varepsilon:}$. By [163] the integral operator K_Ω defines a bounded mapping from $L^p(\partial\Omega)$ to $H^p(\Omega)$ for $1 < p < \infty$.

The function $d(w, z) = |\langle \partial\rho(w), w - z\rangle|$ defines on $\partial\Omega$ a quasimetric, and if $B(z, \delta) = \{w \in \partial\Omega : d(w, z) < \delta\}$ is a quasiball with respect to d then $\sigma\left(B(z, \delta)\right) \vee\wedge \delta^n$, see for example [163].

In Lemma (5.2.2) here we construct polynomial approximations of CauchyLeray-Fantappié kernel based on the theorem by V.K. Dzyadyk [153] about estimates of the Cauchy kernel on domains in the complex plane. We choose approximations similarly to [165]. This construction allows us in Theorem (5.2.9) to get polynomials that approximate the holomorphic function with the desired speed.

Lemma (5.2.1)[167]: *Let Ω be a strongly convex domain with* $0 \in \Omega$, *then for every* $\xi \in \Omega_{\xi j} \backslash \Omega$ *the value of* $\lambda = \frac{\langle \partial\rho(\xi), z\rangle}{\langle \partial\rho(\xi), \xi\rangle}$ *for* $z \in \Omega$ *lies in the domain* $L(t)$ *bounded* *by the bigger arc of some circle* $|w| = R = R(\Omega)$ *and the chord* $\{w \in \mathbb{C} : w = 1 + e^{it}s, s \in \mathbb{R}, |w| \leq R\}$, *where* $t = \frac{\pi}{2} - \arg(\langle \partial\rho(\xi), \xi \rangle)$.

Proof. For $\xi \in \partial\Omega$ define

$$\Lambda(\xi) = \left\{ \lambda \in \mathbb{C} : \lambda = \frac{\langle \partial\rho(\xi), z \rangle}{\langle \partial\rho(\xi), \xi \rangle}, z \in \Omega \right\}.$$

The convexity of Ω with $0 \in \Omega$ implies that $|\langle \partial\rho(\xi), \xi \rangle|_{\sim} > |\partial\rho(\xi)||\xi|_{\sim} > 1$, thus for some $R = R(\Omega) > 0$

$$\frac{|\langle \partial\rho(\xi), z \rangle|}{|\langle \partial\rho(\xi), \xi \rangle|} < R, \tag{46}$$

$$\operatorname{Re} \langle \partial\rho(\xi), z - \xi \rangle \leq 0, z \in \overline{\Omega}, \xi \in \Omega_{\varepsilon} \backslash \Omega. \tag{47}$$

The domain $\Lambda(\xi) \subset \mathbb{C}$ is also convex and contains 0, thus the equality

$$\frac{\langle \partial\rho(\xi), z \rangle}{\langle \partial\rho(\xi), \xi \rangle} = 1 + \frac{\langle \partial\rho(\xi), z - \xi \rangle}{\langle \partial\rho(\xi), \xi \rangle}$$

with estimates (46), (47) completes the proof of the lemma.

Lemma (5.2.2)[167]: *Let* Ω *be a strongly convex domain and* $r > 0$. *Then for every* $k \in$ IN *there exists a function* $K_k^{g}\iota_{ob}(\xi, z)$ *which is defined for* $\xi \in \Omega_{\xi j} \backslash \Omega$, *is polynomial in* $z \in \Omega$ *with* $\deg K_k^{glob}(\xi, \cdot) \leq k$ *and satisfies the following properties*:

$$|K(\xi, z) - K_k^{glob}(\xi, z)| \lesssim \frac{1}{k^r} \frac{1}{d(\xi, z)^{n+r}}, d(\xi, z) \geq \frac{1}{k}; \tag{48}$$

$$|K_k^{glob}(\xi, z)| \lesssim k^n, d(\xi, z) \leq \frac{1}{k}. \tag{49}$$

Proof. Due to [153] and [166] for any $j \in$ IN there exists a function $T_j(t, \lambda)$ polynomial in λ with $\deg T_j(t, \cdot) \leq j$ such that

$$\left| \frac{1}{1 - \lambda} - T_j(t, \lambda) \right| \lesssim \frac{1}{j^r} \frac{1}{|1 - \lambda|^{1+r}} \tag{50}$$

for $\lambda \in L(t) \backslash \{\lambda : |1 - \lambda| < \frac{1}{j}\}$ and coefficients of polynomials $T_j(t, \lambda)$ depend continuously on t. Note also that by the maximum principle

$$T_j(t, \lambda) \lesssim j, \lambda \in L(t) \cap \{\lambda : |1 - \lambda| < \frac{1}{j}\}. \tag{51}$$

Let $t(\xi) = \frac{\pi}{2} - \arg(\langle \partial\rho(\xi), \xi \rangle$ and for $j \in$ IN and $(j - 1)n < k \leq jn$ define

$$K_k^{glob}(\xi, z) = K_{jn}^{glob}(\xi, z) = \frac{1}{\langle \partial\rho(\xi), \xi \rangle^n} T_j^n \left(t(\xi), \frac{\langle \partial\rho(\xi), z \rangle}{\langle \partial\rho(\xi), \xi \rangle} \right).$$

Due to the definition of T_j polynomials $K_k^g \iota_{ob}(\xi,\cdot)$ satisfy relations (48), (49).

Following [158] for $\xi \in \partial\Omega$ and $\varepsilon > 0$ we define the *inner Korányi region* as

$$D^i(\xi,\eta,\varepsilon) = \{\tau \in \Omega : \mathrm{pr}_{\partial\Omega}(\tau) \in B(\xi, -\eta\rho(\tau)), \rho(\tau) > -\varepsilon\}.$$

The strong convexity of Ω and the area-integral inequality by S. Krantz and S.Y. Li [158] imply that for $f \in H^p(\Omega)$, $0 < p < \infty$,

$$\int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^i(z,\eta\varepsilon)}, |\partial f(\tau)|^2 \frac{d\mu(\tau)}{(-\rho(\tau))^{n-1}}\right)^{p/2} \leq c(\Omega,p)\int_{\partial\Omega} |f|^p d\sigma. \quad (52)$$

Consider the decomposition of a vector $\tau \in \mathbb{C}^n$ as $= w + tn(\xi)$, where $w \in T_\xi$, $t \in \mathbb{C}$, and $n(\xi)$ is a complex normal vector (44). We define the *external Korányi region* as
$D^e(\xi,\eta,\varepsilon) = \{\tau \in \mathbb{C}^n \backslash \Omega : \tau = w + tn(\xi)$,

$$w \in T_\xi, t \in \mathbb{C}, |w| < \sqrt{\eta\rho(\tau)}, |\operatorname{Im}(t)| < \eta\rho(\tau), \rho(\tau) < \varepsilon\}. \quad (53)$$

For function $g \in L^1(\partial\Omega)$ and $l \in \mathrm{IN}$ we define *the area integral function* (compare to (44))

$$I_l(g,z) = \left(\int_{D^e(z,\eta,\varepsilon:)} \left|\int_{\partial\Omega} \frac{g(w)\omega(w)}{\langle\partial\rho(\tau), \tau - w\rangle^{n+l}}\right|^2 d\nu_l(\tau)\right)^{1/2}, z \in \partial\Omega, \quad (54)$$

where $d\nu_l(\tau) = \frac{d\mu(\tau)}{\rho(\tau)^{n-2l-1}}$. It is proven in [169 that

Theorem (5.2.3)[167]: *Let Ω be a strongly convex domain and $g \in L^p(\partial\Omega)$, $1 < p < \infty$. Then*

$$\int_{\partial\Omega} I_l\,(g,z)^p d\sigma(z) \lesssim \int_{\partial\Omega} |g(z)|^p d\sigma(z)\,. \quad (55)$$

We point out two estimates for integration over regions $D^e(\xi,\eta,\varepsilon)$. First, for every function F we have

$$\int_{\Omega_\varepsilon\backslash\Omega} |F(z)|d\mu(z) - \int_{\partial\Omega} d\,\sigma(\xi)\int_{D^e(\xi,\eta,\varepsilon)} |F(\tau)|\frac{d\mu(\tau)}{\rho(\tau)^n}.$$

Second, if $F(w) = \tilde{F}(\rho(w))$ then

$$\int_{D^e(\xi,\eta,\varepsilon)} |F(\tau)|d\mu(\tau)_\wedge \vee \int_0^\varepsilon |\tilde{F}(t)|t^n dt.$$

Analogous estimates are valid for regions $D^i(\xi,\eta,\varepsilon)$.
Recall the following lemma (see [165]).
Lemma (5.2.4)[167]: *Let Ω be a strongly convex domain and* , $\eta > 0$, *then*

$$d(\tau,w) = \rho(\tau) + d(z,w)\,, z,w \in \partial\Omega, \tau \in D^e(z,\eta,\varepsilon)\,. \quad (55)$$

We give the definition and exhibit two constructions of pseudoanalytic continuation. These constructions allow us to relate different properties of a holomorphic function. We use the continuation by symmetry (Theorem (5.2.6)) to obtain the description of Hardy-Sobolev spaces by pseudoanalytic continuation (Theorem (5.2.8)) and the continuation by global polynomial approximations (Theorem (5.2.8)) to prove characterization (43) in Theorem (5.2.9).

Let $f \in H^1(\Omega)$ and let boundary values of f coincide almost everywhere with boundary values of some function $\mathrm{f} \in C^1(\mathbb{C}^n \backslash \overline{\Omega})$ such that $|\overline{\partial}\mathrm{f}| \in L^1(\mathbb{C}^n \backslash \Omega)$ and supp $\mathrm{f} \subset \Omega_\varepsilon \backslash \Omega$ for some $\varepsilon > 0$. Then by Stokes formula for $z \in \Omega$ we have $f(z) = \lim_{r \to 0+} \frac{1}{(2\pi i)^n} \int_{\partial \Omega_r} \frac{\mathrm{f}(\xi)\partial\rho(\xi) \wedge (\overline{\partial}\partial\rho(\xi))^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n} =$

$$\lim_{r \to 0+} \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \backslash \Omega_r} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n}$$

$$= \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \backslash \Omega} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n}, \tag{56}$$

since (for details see [162])

$$d_\xi \left(\frac{\partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n} \right) = 0, z \in \Omega, \xi \in \mathbb{C}^n \backslash \Omega.$$

Definition (5.2.5)[167]: *We call the function* $\mathrm{f} \in C^1(\mathbb{C}^n \backslash \overline{\Omega})$ *the pseudoanalytic continuation of the function* $f \in H(\Omega)$ *if*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n \backslash \Omega} \frac{\overline{\partial}\mathrm{f}(\xi) \wedge \partial\rho(\xi) \wedge \left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle \partial\rho(\xi), \xi - z\rangle^n}, z \in \Omega. \tag{57}$$

Note that in this definition we do not assume that boundary values of functions f and f coincide. This definition allows us to study properties of a holomorphic function using estimates of its continuation.

For $z \in \Omega_{\varepsilon:} \backslash \Omega$ we define the point z^* symmetric to z with respect to $\partial\Omega$ by

$$z^* - z = 2(\mathrm{pr}_{\partial\Omega}(z) - z)\,. \tag{58}$$

Theorem (5.2.6)[167]. *Let* $f \in H_p^1(\Omega)$ *and* $1 < p < \infty$, $m \in \mathrm{IN}$. *Then there exist a pseudoanalytic continuation* $\mathrm{f} \in C^1(\mathbb{C}^n \backslash \overline{\Omega})$ *of the function* f *such that* supp $\mathrm{f} \subset \Omega_\varepsilon$, $|\overline{\partial}\mathrm{f}(z)| \in L^p(\Omega_\varepsilon \backslash \Omega)$ *and*

$$|\overline{\partial}\mathrm{f}(z)|_\sim < \max |\partial^\alpha f(z^*)| \rho(z)^{m-1}, z \in \Omega_\varepsilon \backslash \Omega. \tag{59}$$

Proof. Define

$$f_0(z) = \sum_{|\alpha|\leq 7n-1} \partial^{\alpha} f(z^*)\frac{(z-z^*)^{\alpha}}{\alpha!}, z \in \Omega_{\xi j}\backslash\Omega. \quad (60)$$

Let $\alpha \pm e_k = (\alpha_1, \dots, \alpha_k \pm 1, \alpha_n)$ if $\alpha_k \neq 0$ and $(z-z^*)^{\alpha-e_k} = 0$ if $\alpha_k = 0$. With these notations we have

$$\bar{\partial}_j f_0 = \sum_{k=1}^{\infty}\sum_{|\alpha|\leq rn-1}\left(\partial^{\alpha+e_k}f(z^*)\frac{(z-z^*)^{\alpha}}{\alpha!} - \partial^{\alpha}f(z^*)\frac{(z-z^*)^{\alpha-e_k}}{(\alpha-e_k)!}\right)\bar{\partial}_j z_k^*$$

$$= \sum_{k=1}^{\infty}\sum_{|\alpha|=m-1} \partial^{\alpha+e_k} f(z^*)\frac{(z-z^*)^{\alpha}}{\alpha!}\bar{\partial}_j z_k^*, \quad (61)$$

hence,

$$|\bar{\partial} f_0(z)| \sim< \max |\partial^{\alpha} f(z^*)|\rho(z)^{7n-1}, z \in \mathbb{C}^n\backslash\Omega.$$

Consider a function $\chi \in C^{\infty}(0,\infty)$ such that $\chi(t) = 1$ for $t \leq \varepsilon/2$ and $\chi(t) = 0$ for $t \geq \varepsilon$.

The function $f(z) = f_0(z)\chi(\rho(z))$ satisfies condition (59) and supp f$\subset \Omega_{\varepsilon}$.

Let $d = \operatorname{dist}(z^*, \partial\Omega)/10$. Then by the Cauchy maximal inequality for every multiindex α such that $|\alpha| = m$ we have

$$|\partial^{\alpha} f(z^*)| \lesssim d^{-m+1} \sup |\partial f(\tau)| \lesssim \rho(z)^{-m+1} \sup |\partial f(\tau)|,$$

for some $c_0 > 0$. Finally, by [158] we get

$$\int_{\Omega_{\in}\backslash\Omega} |\bar{\partial} f(z)|^p d\mu(z) \lesssim \int_{\Omega\backslash\Omega_{-\varepsilon}} d\,\mu(z)(\sup |\partial f(\tau)|)^p$$

$$\lesssim \|\partial f\|_{H^p(\Omega)}^p < \infty.$$

Thus $|\bar{\partial} f(z)| \in L^p(\Omega_{\xi j}\backslash\Omega)$ and this finishes the proof of the theorem.

By [20] strict convexity of domain Ω implies that functions holomorphic in neighbourhood of Ω are dense in $H^1(\Omega)$. Also every holomorphic in neighbourhood of Ω function can be approximated on $\bar{\Omega}$ by polynomials since Ω is Runge ([159]). Thus there exists a polynomial sequence P_1, P_2, converging to f in $L^1(\partial\Omega)$. Define

$$\lambda(z) = \rho(z)^{-1}|P_{2^{k+1}}(z) - P_{2^k}(z)|, 2^{-k} < \rho(z) \leq 2^{-k+1}$$

Theorem (5.2.7)[167]: Assume that $\lambda \in L^p(\mathbb{C}^n\backslash\Omega)$ for some $p \geq 1$. Then there exist a pseudoanalytic continuation f of the function f such that

$$|\bar{\partial} f(z)| \lesssim \lambda(z), z \in \mathbb{C}^n\backslash\Omega. \quad (62)$$

Proof. Consider a function $\chi \in C^{\infty}(0,\infty)$ such that $\chi(t) = 1$ for $t \leq \frac{5}{4}$ and $\chi(t) = 0$ for

$t \geq \frac{7}{4}$. We let

$\mathrm{f}_0(z) = P_{2^k}(z) + \chi\left(2^k\rho(z)\right)\left(P_{2^{k+1}}(z) - P_{2^k}(z)\right), 2^{-k} < \rho(z) < 2^{-k+1}, k \in \mathbb{N},$ and define the continuation of the function f by formula $\mathrm{f} = \chi(2\rho(z))\mathrm{f}_0(z)$.

Now f is C^1-function on $\mathbb{C}^n\backslash\overline{\Omega}$ and $|\overline{\partial}\mathrm{f}(z)| \lesssim \lambda(z)$. We define a function $F_k(z)$ as $F_k(z) = \mathrm{f}(z)$ for $\rho(z) > 2^{-k}$ and as $F_k(z) = P_{2^{k+1}}(z)$ for $\rho(z) < 2^{-k}$. The function F_k is smooth and holomorphic in Ω_{2-k}, and $|\overline{\partial}F_k(z)| \leq \lambda(z)$ for $z \in \mathbb{C}^n\backslash\Omega_{2-k}$. Thus similarly to (56) we get

$$P_{2^{k+1}}(z) = F_k(z) = \frac{1}{(2\pi i)^n}\int_{\mathbb{C}^n\backslash\Omega}\frac{\overline{\partial}F_k(\xi)\wedge\partial\rho(\xi)\wedge\left(\overline{\partial}\partial\rho(\xi)\right)^{n-1}}{\langle\partial\rho(\xi),\xi - z\rangle^n}, z \in \Omega,$$

We can pass to the limit in this formula by the dominated convergence theorem; hence, the function f satisfies identity (57) and is a pseudoanalytic continuation of the function f.

Theorem (5.2.8)[167]. *Let Ω be a strongly convex domain*, $1 < p < \infty$, $l \in$ IN *and* $\in H(\Omega)$. *Then $f \in H_p^l(\Omega)$ if and only if there exists such pseudoanalytic continuation* f *that for some* $\varepsilon j, \eta > 0$

$$\int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^e(z,\eta,\varepsilon)}|\,\overline{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2 d\nu(\tau)\right)^{p/2} < \infty, \qquad (63)$$

where $D^e(z) = D^e(z,\eta,\varepsilon j)$ and $d\nu(\tau) = \frac{d\mu(\tau)}{\rho(\tau)^{n-1}}$.

Proof. Let $f \in H_p^l(\Omega)$. By Theorem (5.2.6) we can construct a pseudoanalytic continuation f such that

$$|\overline{\partial}\mathrm{f}(z)|_{\sim} < \max|\partial^\alpha f(z^*)|\rho(z)^l, z \in \mathbb{C}^n\backslash\Omega.$$

Note that the symmetry with respect to $\partial\Omega$ ($z \mapsto z^*$ defined by (58)) maps the external sector $D^e(z) = D^e(z,\eta,\varepsilon)$ into some internal Korányi sector. Indeed, for every sufficiently small $\varepsilon, \eta > 0$ there exist $\eta_1, \varepsilon_1 > 0$ such that
$\{\tau^*: \tau \in D^e(z,\eta,\varepsilon j)\} subseteq D^i(z,\eta_{11}\xi j) = D^i(z)$.
Applying the area-integral inequality (52) we obtain

$$\int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^{\mathrm{e}}(z)} |\,\bar{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^2 d\nu(\tau)\right)^{p/2}$$

$$\lesssim \max \int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^{e}(z)} |\,\partial^{\alpha} f(\tau^{*})|^2 di/(\tau)\right)^{p/2}$$

$$\lesssim \max \int_{\partial\Omega} d\,\sigma(z)\left(\int_{D^{i}(z)} |\,\partial^{\alpha} f(\tau)|^2 \frac{d\mu(\tau)}{\left(-\rho(\tau)\right)^{n-1}}\right)^{p/2} < \infty$$

To prove the sufficiency, assume that the function $f \in H^1(\Omega)$ admits the pseudoanalytic continuation f with estimate (63). We will prove that for every function $\in L^{p'}(\partial\Omega), \frac{1}{p}+\frac{1}{p'}=1$, and every multiindex $\alpha, |\alpha| \leq l,$

$$|\int_{\partial\Omega} g\,(z)\partial^{\alpha} f(z)\omega(z)| \leq c(f)\|g\|_{L^{p'}(\partial\Omega)}.$$

Assume, without loss of generality, that $a=(l,0,\dots,0)$. By representation (57) we have $f(z)=\int_{\mathbb{C}^n\setminus\Omega} \frac{\bar{\partial}\mathrm{f}(\xi)\wedge\omega(\xi)}{\langle\partial\rho(\xi),\xi-z\rangle^n}$ and with $C_{nl}=\frac{(n+l-1)!}{(n-1)!}$

$$\int_{\partial\Omega} g\,(z)\partial^{\alpha} f(z)\omega(z) = C_{nl}\int_{\partial\Omega} g\,(z)\left(\int_{\mathbb{C}^n\setminus\Omega}\left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \frac{\bar{\partial}\mathrm{f}(\xi)\wedge\omega(\xi)}{\langle\partial\rho(\xi),\xi-z\rangle^{n+l}}\right)\omega(z)$$

$$= C_{nl}\int_{\mathbb{C}^n\setminus\Omega}\left(\frac{\partial\rho(\xi)}{\partial\xi_1}\right)^l \bar{\partial}\mathrm{f}(\xi)\wedge\omega(\xi)\int_{\partial\Omega}\frac{g(z)\omega(z)}{\langle\partial\rho(\xi),\xi-z\rangle^{n+l}}.$$

Define $\Phi_l(\xi)=\int_{\partial\Omega}\frac{g(z)\omega(z)}{\langle\partial\rho(\xi),\xi-z\rangle^{n+l}}, \xi\in\mathbb{C}^n\setminus\Omega$. Applying Hölder inequality twice we have

$$|\int_{\partial\Omega} g\,(z)\partial^{\alpha} f(z)\omega(z)| \lesssim \int_{\mathbb{C}^n\setminus\Omega} |\,\bar{\partial}\mathrm{f}(\xi)||\Phi_l(\xi)|d\mu(\xi)$$

$$\lesssim \int_{\partial\Omega}\omega\,(\xi)\int_{D^{\mathrm{e}}(\xi)} |\,\bar{\partial}\mathrm{f}(\tau)||\Phi_l(\tau)|\frac{d\mu(\tau)}{\rho(\tau)^n}$$

$$\lesssim \int_{\partial\Omega} d\,\sigma(\xi)\left(\int_{D^{e}(\xi)} |\,\bar{\partial}\mathrm{f}(\tau)|^2\frac{\rho(\tau)^{-2\iota}d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{\frac{1}{2}}\left(\int_{D^{\mathrm{e}}(\xi)} |\,\Phi_l(\tau)|^2\frac{\rho(\tau)^{2l-2}d\mu(\tau)}{\rho(\tau)^{n-1}}\right)^{\frac{1}{2}}$$

$$\leq\left(\int_{\partial\Omega}\omega(\xi)\left(\int_{D^{e}(\xi)}|\bar{\partial}\mathrm{f}(\tau)|^{2}\rho(\tau)^{-2l}d_{l/}(\tau)\right)^{p/2}\right)^{1/p}\times$$

$$\times\left(\int_{\partial\Omega}\omega(\xi)\left(\int_{D^{e}(\xi)}|\Phi_{l}(\tau)|^{2}\rho(\tau)^{2l-2}d_{l/}(\tau)\right)^{p'/2}\right)^{1/p'}$$

The first product term is finite by (63) and the second one is estimated by $\|g\|_{L^{p'}(\partial\Omega)}$ in the view of area-integral inequality (55) in Theorem (5.2.3).

Theorem (5.2.9)[167]: *Lest* $f\in H(\Omega)$ *and* $1<p<\infty, l\in \mathrm{IN}$. *Then* $f\in H_{p}^{l}(\Omega)$ *if and only if there exists a sequence of* 2^{k}*-degree polynomials* $P_{2^{k}}$ *such that*

$$\int_{\partial\Omega}d\sigma(z)\left(\sum_{k=1}^{\infty}|f(z)-P_{2^{k}}(z)|^{2}2^{2\iota k}\right)^{p/2}<\infty. \tag{64}$$

Proof. Assume that condition (64) holds, then polynomials $P_{2^{k}}$ converge to the function f in $L^{p}(\partial\Omega)$ and by Theorem (5.2.7) we can construct pseudoanalytic continuation f such that

$$|\bar{\partial}\mathrm{f}(z)|\lesssim|P_{2^{k+1}}(z)-P_{2^{k}}(z)|\rho(z)^{-1}, z\in\mathbb{C}^{n}\backslash\Omega, 2\text{^}(-k)\leq\rho(z)<2^{-k+1}$$

Consider the decomposition of the region $D^{e}(z,\eta,\varepsilon)$ into sets

$$D_{k}(z)=\{\tau\in D^{e}(z,\eta,\varepsilon): 2^{-k}\leq\rho(\tau)<2^{-k+1}\},$$

and define functions

$$a_{k}(z)=|P_{2^{k+1}}(z)-P_{2^{k}}(z)|2^{-kl},$$

$$b_{k}(z)=\left(\int_{D_{k}(z)}|\bar{\partial}\mathrm{f}(\tau)\rho(\tau)^{-l}|^{2}d\nu(\tau)\right)^{1/2}, z\in\partial\Omega.$$

Lemma (5.2.10)[167]: $b_{k}(z)_{\sim}<Ma_{k}(z)$, *where* Ma_{k} *is the maximal function with respect to centred quasiballs on* $\partial\Omega$

$$Ma_{k}(z)=\sup\frac{1}{\sigma(B(z,r))}\int_{B(z,r)}|a_{k}(\xi)|d\sigma(\xi).$$

Assume, that this lemma holds, then by the Fefferman-Stein maximal theorem (see [157], [156]) we have

$$\int_{\partial\Omega}\left(\sum_{k=1}^{\infty}b_{k}(z)^{2}\right)^{p/2}d\sigma(z)\lesssim\int_{\partial\Omega}\left(\sum_{k=1}^{\infty}a_{k}(z)^{2}\right)^{p/2}d\sigma(z).$$

The right-hand side of this inequality is finite by condition (64), also we have

$$\sum_{k=1}^{\infty}b_{k}(z)^{2}=\int_{D^{e}(z,\eta,\varepsilon)}|\bar{\partial}\mathrm{f}(\xi)\rho(\xi)^{-l}|^{2}d\nu(\xi),$$

which completes the proof of the sufficiency in the theorem.

Let us prove the necessity. Now $f \in H_p^l(\Omega)$ with $1 < p < \infty$ and $l \in$ IN. By Theorem (5.2.8) we could construct a continuation f of the function f with estimate (63). Applying the approximation of the Cauchy-Leray-Fantappiè kernel from Lemma (5.2.2) to the function f we define polynomials

$P_{2^k}(z) = \int_{\mathbb{C}^n \setminus \Omega} \overline{\partial}\, \mathrm{f}(\xi) \wedge \omega(\xi) K^g_{2^k lob}(\xi, z)$.

We will prove that these polynomials satisfy condition (64).
From Lemma (5.2.2) we obtain

$$|f(z) - P_{2^k}(z)| \lesssim \int_{\mathbb{C}^n \setminus \Omega} |\overline{\partial} \mathrm{f}(\xi)| \left| \frac{1}{\langle \partial\rho(\xi), \xi - z \rangle^d} - K^g_{2^k lob}(\xi, z) \right| d\mu(\xi)$$

$\lesssim U(z) + V(z) + W_1(z) + W_2(z)$,
where

$U(z) = \int_{d(\tau,z)<2-k} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} d\mu(\tau)$,

$V(z) = 2^{kn} \int_{d(\tau,z)<2-k} |\overline{\partial}\mathrm{f}(\tau)| d\mu(\tau)$,

$W_1(z) = 2^{-kr} \int_{\rho(\tau)<2^{-k}} d\,(\tau, z) > 2^{-k} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^{n+r}} d\mu(\tau)$,

$W_2(z) = 2^{-kr} \int_{\rho(\tau)>2-k} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^{n+r}} d\mu(\tau)$.

The parameter $r > 0$ will be chosen later.
Note that $V(z) \leq cU(z)$ and estimate the contribution of $U(z)$ to the sum. For some $c_1, c_2 > 0$ we have

$$U(z) \leq \int_{w \in \partial\Omega} d\,(w, z) < c_1 2^{-k} d\sigma(w) \sum_{j > c_2 k_D} \int_{j(w)} \frac{|\overline{\partial}\mathrm{f}(\tau)|}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} \frac{d\nu(\tau)}{\rho(\tau)}$$

$$\leq \int_{w \in \partial\Omega} d\,(w, z) < c_1 2^{-k} d\sigma(w) \sum_{j > c_2 k} \left(\int_{D_j(w)} |\overline{\partial}\mathrm{f}(\tau) \rho(\tau)^{-l}|^2 di/(\tau) \right)^{1/2} \times$$

$$\times \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d_{IJ}(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^n} \right)^{1/2} = \sum_{j > c_2 k_{d(w,z)}} \int_{<c_1 2^{-j}} b_j\,(w) m_j(w) d\sigma(w)$$

By Lemma (5.2.4) we have $d(\tau, z)^{\vee}_{\wedge} \rho(\tau) + d(w, z) > 2^{-j}$ and $\nu\left(D_j(w)\right)^{\vee}_{\wedge} 2^{-2j}$ for $\in D_j(w)$. Hence,

$$m_j(w) = \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^n}\right)^{1/2} \lesssim \frac{2^{-j(l-1)}}{2^{-jn}} 2^{-j} = 2^{jn-jl}$$

and

$$2^{k\iota} U(z) \leq \sum_{j>c_1 k} 2^{-(j-k)l}\, 2^{jn} \int_{d(w,z)<c_2 2-j} b_j\,(w) d\sigma(w)$$

$$\leq \sum_{j>c_1 k} 2^{-(j-k)\iota}\, M b_j(z)\,. \tag{65}$$

Now we estimate the value $W_1(z)$. Similarly to the previous we have

$$W_1(z) \leq 2^{-kr} \sum$$

$$j > k \int_{d(w,z)\geq c_1 2-k} b_j\,(w) m_j^r(w) d\sigma(w)$$

$\leq 2^{-kr} \sum_{j>k} \sum_{t=c_2 2^{-t} \leq d(w}^{k} \int_{zc_{1,,}) \leq c_1 2-t+1} b_j\,(w) m_j^r(w) d\sigma(w)$,

where

$$m_j^r(w) = \left(\int_{D_j(w)} \frac{\rho(\tau)^{2(l-1)} d\nu(\tau)}{|\langle \partial\rho(\tau), \tau - z\rangle|^{2(n+r)}}\right)^{1/2}$$

Applying the estimate $d(\tau, z) = \rho(\tau) + d(w, z) \sim> 2^{-t}$, we obtain

$$m_j^r(w) \lesssim 2^{-jl+t(n+r)}.$$

Finally

$$\sum_{t=c2}^{k} \int_{\leq c_1 2-t+1 \atop d(w,z)} b_{j(w) m_j^r(w) d\sigma(w)} \lesssim \sum_{t=c_2}^{k} 2^{-\mathrm{jl+tr}} \mathrm{M\ b_j\ (z)} \lesssim\ 2^{-\mathrm{jl+kr}} \mathrm{M\ b_j\ (z)}$$

and

$$2^k W_1(z) \lesssim \sum_{j>k} 2^{-l(j-k)}\, M b_j(z)\,. \tag{66}$$

Similarly, estimating the contribution of $W_2(z)$, we obtain

$$2^{kl} W_2(z) \lesssim 2^{-k(r-l)} \sum_{j=0}^{k} \int_{\partial\Omega} b_j\,(w) m_j^r(w) d\sigma(w)\,. \tag{67}$$

Since $d(\tau, z) \gtrsim 2^{-j} + d(w, z)$ for $\tau \in \partial\Omega, \tau \in D_j(z)$ then

$m_j^r(w) \lesssim \frac{2^{-jl}}{\left(2^{-j}+d(w,z)\right)^{n+r}} \leq\ \min\left(2^{j(n+r-l)}, 2^{-jl} d(w,z)^{-n-r}\right)$.

Thus

$$\int_{\partial\Omega} b_j(w) m_j^r(w) d\sigma(w) \lesssim \int_{d(w,z)\leq 2-j} \frac{2^{-jl}}{2^{-j(n+r)}} b_j(w) d\sigma(w)$$

$$+ \sum_{t=1_{2-t-1} \leq d}^{j-1} \int_{(wz)\leq 2-t}, \frac{2^{-jl}}{2^{-t(n+r)}} b_j(w) d\sigma(w)$$

$$\lesssim \sum_{t=1}^{j} (Z)_{\sim}.$$

Choosing $r = 2l$ and applying estimate (67)

$$W_2(z) 2^{kl} \lesssim \sum_{j=1}^{k} 2^{-(k-j)(r-l)} M b_j(z) \leq \sum_{j=1}^{k} 2^{-(k-j)l} M b_j(z). \tag{68}$$

Combining estimates (25, 26, 28) we finally obtain

$|f(z) - P_{2^k}(z)| 2^{kl} \lesssim \sum_{j=1}^{k} 2^{-(k-j)\iota} M b_j(z) + \sum_{j>k} 2^{-(j-k)\iota} M b_j(z)$,

which similarly to [155] implies

$$\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^{22kl}\, 2 \lesssim \sum_{k=1}^{\infty} \left(M b_k(z)\right)^2$$

Then, by the Fefferman-Stein theorem ([156], [157])

$$\int_{\partial\Omega} d\,\sigma(z) \left(\sum_{k=1}^{\infty} |f(z) - P_{2^k}(z)|^2 2^{2lk} \right)^{p/2} \leq \int_{\partial\Omega} \left(\sum_{k=1}^{\infty} b_k^2(z) \right)^{p/2} d\sigma(z)$$

$$\leq \int_{\partial\Omega} d\,\sigma(z) \left(\int_{D^e(z,\eta,\varepsilon)} |\bar{\partial} \mathrm{f}(\xi) \rho(\xi)^{-l}|^2 d\nu(\xi) \right)^{p/2} < \infty.$$

This completes the proof of the theorem and it remains to prove Lemma (5.2.10).

Define $g_k(z) := P_{2^{k+1}}(z) - P_{2^k}(z)$. Let $z \in \partial\Omega$ and $\tau \in D_k(z)$. Consider the complex normal vector $n(z)$ defined by (44), the complex tangent hyperplane T_z and the complex plane $T_{z,\tau}^{\perp}$, orthogonal to T_z and containing the point τ $T_{z,\tau}^{\perp} := \{\tau + sn(z): s \in \mathbb{C}\}$. Projection of the vector $\tau \in \mathbb{C}$ to $\partial\Omega \cap T_{z,\tau}^{\perp}$ we will denote as $\pi_z(\tau)$.

Define $\Omega_{z,\tau} = \Omega \cap T_{z,\tau}^{\perp}$ and $\gamma_{z,\tau} = \partial\Omega_{z,\tau}$. There exist a conformal map $\phi_{z,\tau}: T_{z,\tau}^{\perp} \backslash \Omega_{z,\tau} \to \mathbb{C} \backslash \{w \in \mathbb{C}: |w| < 1\}$ such that $\phi_{z,\tau}(\infty) = \infty$, $\phi_{z,\tau}'(\infty) > 0$. We consider an auxiliary function $G_k(s) := \frac{g_k(s)}{\phi_{z,\tau}^{2^k+1}(s)}$ that is holomorphic in $T_{z,\tau}^{\perp} \backslash \Omega_{z,\tau}$. Applying the maximal estimate from [4] to this function we have

$|G_k(\tau)| <\sim \frac{1}{\rho(\tau)} \int_{s \in I_{z,\tau}} |G_k(s)||ds| + \int_{\partial\Omega_{z,\tau} \backslash I_{z,\tau}} |G_k(s)| \frac{\rho(\tau)^m}{|s - \pi_z(\tau)|^{7rt+1}} |ds|$, where $I_{z,\tau} =$

$\{s \in \gamma_{z,\tau} : |s - \pi_z(\tau)| < \operatorname{dist}(\tau, \partial\Omega_{z,\tau})/2\}$, and $m > 0$ could be chosen arbitrary large.

Note that $|\phi_{z,\tau}(s)| - 1 \vee\wedge \operatorname{dist}(s, \partial\Omega_{z,\tau}) \vee\wedge 2^{-k}$, thus $|g_k(s)| \vee\wedge |G_k(s)|$ for $s \in D_k(z) \cap T^{\perp}_{z,\tau}$. Hence,

$$|g_k(\tau)| \lesssim \sum_{j=1}^{\infty} 2^{-jm} \frac{1}{2^j \rho(\tau)} \int_{s \in \partial\Omega_{z,\tau}, |s-\pi_z(\tau)|<2^j\rho(\tau)} |\, g_k(s)||ds|. \tag{69}$$

Since the boundary of the domain Ω is C^3-smooth, we can assume that the constant in this inequality (69) does not depend on $z \in \partial\Omega$ and $\tau \in \Omega_\varepsilon \backslash \Omega$.
Note that the function $g_k(\tau + z - w)$ is holomorphic in $w \in T_z$, then estimating the mean we obtain

$$|g_k(\tau)| \leq \frac{1}{\rho(\tau)^{n-1}} \int_{|w-z|<\sqrt{\rho(\tau)}} |\, g_k(\tau + z - w)| d\mu_{2n-2}(w)$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-j_7 n} \frac{1}{\rho(\tau)^{n-1}} \int_{|w-z|<\sqrt{\rho(\tau)}} \frac{d\mu_{2n-2}(w)}{2^j \rho(\tau)} \int_{s \in \partial\Omega_{z,\tau}} |g_k(s)||ds||s - \pi_z(\tau + z - w)|$$

$$< 2^j \rho(\tau) \lesssim \sum_{j=1}^{\infty} 2^{-j(m-n+1)} \int_{B\left(z, 2^j \rho(\tau)\right)} |\, g_k(w)| d\sigma(w)\,, \tag{70}$$

where $d\mu_{2n-2}$ is the Lebesgue measure in T_z
Assume that $m > n - 1$, then $|g_k(\tau)| \lesssim M g_k(z)$, $z \in \partial\Omega$, $\tau \in D_k(z)$. Finally,

$$b_k(z)^2 = \int_{D_k(z)} |\, \bar{\partial} \mathrm{f}(\tau) \rho(\tau)^{-l}|^2 d\nu(\tau)_\sim < \int_{D_k(z)} |\, g_k(\tau) \rho(\tau)^{-l-1}|^2 d\iota/(\tau)$$

$$\lesssim \left(2^{-kl} M g_k(z)\right)^2 \int_{D_k(z)} \frac{d\nu(\tau)}{\rho(\tau)^2} \lesssim \left(M a_k(z)\right)^2$$

and the lemma is proved.

Chapter 6
Brunn–Minkowski and Sharp Gagliardo–Nirenberg–Sobolev Inequalities

We give a new bridge between the geometric point of view of the Brunn– Minkowski inequality and the functional point of view of the Sobolev-type inequalities. We find a new sharp trace Gagliardo-Nirenberg-Sobolev inequality on convex cones, as well as a sharp weighted trace Sobolev inequality on epigraphs of convex functions. By using a generalized Borell-Brascamp-Lieb inequality, coming from the Brunn-Minkowski theory.

Section (6.1): Sharp Sobolev Inequalities

The classical Sobolev inequality in $\mathbb{R}^n, n \geq 3$, indicates that there is a constant $C_n > 0$ such that for all smooth enough (locally Lipschitz) functions $f : \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity,

$$\|f\|_q \leq C_n \|\nabla f\|_2 \tag{1}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$. Here $\|f\|_q$ denotes the usual L^q-norm of f with respect to Lebesgue measure on $\mathbb{R}^n$, and, for $p \geq 1$,

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p \mathrm{d}x\right)^{1/p}$$

where $|\nabla f|$ is the Euclidean norm of the gradient ∇f of f.
Inequality (1) goes back to Sobolev [199], as a consequence of a Riesz type rearrangement inequality and the Hardy-Littlewood-Sobolev fractional-integral convolution inequality. Other approaches, including the elementary Gagliardo-Nirenberg argument [105], [145], are discussed in (cf. e.g. [82], [194], [102] ...). The best possible constant in the Sobolev inequality (1) was established independently by Aubin [82] and Talenti [110] in 1976 using symmetrization methods of isoperimetric flavor, together with the study of the one-dimensional extremal problem. Rearrangements arguments have been developed extensively in (cf. [87], [102] ...). The optimal constant C_n is achieved on the extremal functions $(x) = (\sigma + |x|^2)^{(2-n)/2}, x \in \mathbb{R}^n, \sigma > 0$. Building on early ideas by Rosen [197], Lieb [196] determined the best constant and the extremal functions in dimension 3. According to [198], the result seems to have been known before, at least back to the early sixties, in unpublished notes by Rodemich.
The geometric Brunn-Minkowski inequality, and its isoperimetric consequence, is a well-known argument to reach Sobolev type inequalities. It states that for every non-empty Borel measurable bounded sets A, B in $\mathbb{R}^n$,

$$\mathrm{vo1}_n(A+B)^{1/n} \geq \mathrm{vo1}_n(A)^{1/n} + \mathrm{vo1}_n(B)^{1/n} \tag{2}$$

where $\mathrm{vo1}_n(\cdot)$ denotes Euclidean volume. The Brunn-Minkowski inequality classically implies the isoperimetric inequality in $\mathbb{R}^n$. Choose namely for B a ball with radius $\varepsilon > 0$ and let then $\varepsilon \to 0$ to get that for any bounded measurable set A in $\mathbb{R}^n$,

$$\mathrm{vo1}_{n-1}(\partial A) \geq n\omega_n^{1/n} \mathrm{vo1}_n(A)^{(n-1)/n}$$

where $\mathrm{vo1}_{n-1}(\partial A)$ is understood as the outer-Minkowski content of the boundary of A and ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. By means of the co-area formula [182], [194], the isoperimetric inequality may then be stated equivalently on functions as the L^1-Sobolev inequality

$$\|f\|_q \leq \frac{1}{n\omega_n^{1/n}}\|\nabla f\|_1 \tag{3}$$

where $\frac{1}{q} = 1 - \frac{1}{n}$. Changing $f \geq 0$ into f^r for some suitable r and applying Hölder's inequality yields the L^{2-} Sobolev inequality (1), however not with its best constant. In the same way, the argument describes the full scale of Sobolev inequalities

$$\|f\|_q \leq C_n(p)\|\nabla f\|_p, \tag{4}$$

$1 \leq p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, : $\mathbb{R}^n \to \mathbb{R}$ smooth and vanishing at infinity. According to Gromov [107], the L^1-case of the Sobolev inequality appears in Brunn's work from 1887.

We show that the Brunn-Minkowski inequality may actually be used to also reach the optimal constants in the Sobolev inequalities (1) and (4). This new approach thus completely bridges the geometric Brunn-Minkowski inequalities and the functional Sobolev inequalities.

Inequality (2) was first proved by Brunn in 1887 for convex sets in dimension 3, then extended by Minkowski (cf. [109]). Lusternik [191] generalized the result in 1935 to arbitrary measurable sets. Lusternik's proof was further analyzed and extended in the works of Hadwiger and Ohmann [186] and Henstock and Macbeath [187] in the fifties.

Note in particular that the one-dimensional case is immediate: assume that A and B𝔅 Springer are non-empty compact sets in $\mathbb{R}$, and after a suitable shift, that $\sup A = 0 = \inf B$. Then $A \cap B = \{0\}$ and $A + B \supset A \cup B$.

Starting with the contribution [187], integral inequalities have been developed throughout the last century in the investigation of the geometric Brunn-MinkowskiLusternik theorem. The idea of the following elementary, but fundamental, Lemma (6.1.1) goes back to Bonnesen's proof of the Brunn-Minkowski inequality (cf. [176]) and may be found already by Henstock and Macbeath [187]. The result appears in this form independently in the works of Dancs and Uhrin [178] and Das Gupta [179]. We enclose a proof for completeness. As a result, the proof below only relies on the one-dimensional Brunn-Minkowski-Lusternik inequality, which is the only basic ingredient in the argument. All the further developments and applications to Sobolev inequalities are consequences of this elementary Lemma (6.1.1).

Lemma (6.1.1)[18]. *Let $\theta \in [0,1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,*

$$w(\theta x + (1-\theta)y) \geq \min\left(u(x), v(y)\right).$$

Then, if $\sup_{x\in\mathbb{R}} u(x) = \sup_{x\in\mathbb{R}} v(x) = 1$,

$$\int w\,\mathrm{d}x \geq \theta \int u\,\mathrm{d}x + (1-\theta)\int v\,\mathrm{d}x.$$

Proof. Define, for $t > 0$, $E_u(t) = \{x \in \mathbb{R}; u(x) > t\}$ and similarly $E_v(t)$, $E_w(t)$. Since $\sup_{x\in\mathbb{R}} u(x) = \sup_{x\in\mathbb{R}} v(x) = 1$, for $0 < t < 1$, both $E_u(t)$ and $E_v(t)$ are non-empty, and $E_w(t) \supset \theta E_u(t) + (1-\theta)E_v(t)$. By the one-dimensional Brunn-Minkowski-Lusternik inequality (2), for every $0 < t < 1$,

$$\lambda\big(E_w(t)\big) \geq \theta\lambda\big(E_u(t)\big) + (1-\theta)\lambda\big(E_v(t)\big)$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Hence,

$$\int w\,\mathrm{d}x \geq \int_0^1 \lambda\left(E_w(t)\right)\mathrm{d}t$$

$$\geq \theta \int_0^1 \lambda\left(E_u(t)\right)\mathrm{d}t + (1-\theta)\int_0^1 \lambda\left(E_v(t)\right)\mathrm{d}t$$

$$= \theta \int u\,\mathrm{d}x + (1-\theta)\int v\,\mathrm{d}x$$

which is the conclusion.

As discussed in [178], the preceding Lemma (6.1.1) may be extended to more general means by elementary changes of variables. For $\alpha \in [-\infty, +\infty]$, denote by $M_\alpha^{(\theta)}(a,b)$ the α-mean of the non-negative numbers a, b with weights θ, $1-\theta \in [0,1]$ defined as

$$M_\alpha^{(\theta)}(a,b) = (\theta a^\alpha + (1-\theta)b^\alpha)^{1/\alpha}$$

(with the convention that $M_\alpha^{(\theta)}(a,b) = \max(a,b)$ if $\alpha = +\infty$, $M_\alpha^{(\theta)}(a,b) = \min(a,b)$ if $\alpha = -\infty$ and $M_\alpha^{(\theta)}(a,b) = a^\theta b^{1-\theta}$ if $\alpha = 0$) if $ab > 0$, and $M_\alpha^{(\theta)}(a,b) = 0$ if $ab = 0$.

Note the extension of the usual arithmetic-geometric mean inequality as

$$M_{\alpha_1\alpha}^{(\theta)}(a_1,b_1)M_2^{(\theta)}(a_2,b_2) \geq M_\alpha^{(\theta)}(a_1a_2,b_1b_2) \tag{5}$$

if $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha 2}$, $\alpha_1 + \alpha 2 > 0$.

Corollary (6.1.2)[18]. *Let* $-\infty \leq \alpha \leq +\infty$, $\theta \in [0,1]$ *and* u, v, w *be non-negative measurable functions on* $\mathbb{R}$ *such that for all* $x, y \in \mathbb{R}$,

$$w(\theta x + (1-\theta)y) \geq M_\alpha^{(\theta)}\big(u(x), v(y)\big).$$

Then, if $a = \sup_{x\in\mathbb{R}} u(x) < \infty$, $b = \sup_{x\in\mathbb{R}} v(x) < \infty$,

$$\int w\,\mathrm{d}x \geq M_\alpha^{(\theta)}(a,b)M_1^{(\theta)}\left(\frac{1}{a}\int u\,\mathrm{d}x, \frac{1}{b}\int v\,\mathrm{d}x\right).$$

The statement still holds if a or $b = +\infty$ with the convention that $0 \times \infty = 0$.

Proof. Assume first that $-\infty < \alpha < +\infty$. For $\rho = M_\alpha^{(\theta)}(a,b) > 0$, set

$$U(x) = \frac{1}{a} u\left(\frac{a^{\alpha x}}{\rho^\alpha}\right) \; and \; (y) = \frac{1}{b} v\left(\frac{b^\alpha y}{\rho^\alpha}\right).$$

Then, if $\eta = \theta a^\alpha / \rho^\alpha (\in [0,1])$,

$$w(\eta x + (1-\eta)y) \geq M_\alpha^{(\theta)}(a,b) \min\left(U(x), V(y)\right)$$

for all $x, y \in \mathbb{R}$. Since $\sup_{x\in\mathbb{R}} U(x) = \sup_{x\in\mathbb{R}} V(x) = 1$, by the Lemma (6.1.1),

$$\int w \,\mathrm{d}x \geq M_\alpha^{(\theta)}(a,b)\left(\eta \int U \,\mathrm{d}x + (1-\eta)\int V \,\mathrm{d}x\right)$$

$$= M_\alpha^{(\theta)}(a,b)\left(\frac{\theta}{a}\int u \,\mathrm{d}x + \frac{1-\theta}{b}\int v \,\mathrm{d}x\right)$$

by definition of η. The cases $\alpha = -\infty$ and $\alpha = +\infty$ may be proved by standard limit considerations. The corollary is thus established.

By the Hölder inequality (5), the preceding corollary implies the more classical Prékopa-Leindler theorem [189], [186], [196], as well as its generalized form put forward by Borell [174] and Brascamp and Lieb [175], in which the supremum norms of u and v do not appear. Namely, under the assumption of Corollary (6.1.2) and provided that $-1 \leq \alpha \leq +\infty$,

$$\int w \,\mathrm{d}x \geq M_\alpha^{(\theta)}(a,b) M_1^{(\theta)}\left(\frac{1}{a}\int u \,\mathrm{d}x, \frac{1}{b}\int v \,\mathrm{d}x\right)$$

$$\geq M_\beta^{(\theta)}\left(\int u \,\mathrm{d}x, \int v \,\mathrm{d}x\right)$$

where $\beta = \alpha/(1+\alpha)$.

The preceding generalized Prékopa-Leindler theorem is easily tensorisable in $\mathbb{R}^n$ by induction on the dimension to yield that whenever $-\frac{1}{n} \leq \alpha \leq +\infty$, $\theta \in [0,1]$ and $u, v, w : \mathbb{R}^n \to \mathbb{R}+$ are measurable such that

$$w(\theta x + (1-\theta)y) \geq M_\alpha^{(\theta)}\left(u(x), v(y)\right)$$

for all $x, y \in \mathbb{R}^n$, then

$$\int w \,\mathrm{d}x \geq M_\beta^{(\theta)}\left(\int u \,\mathrm{d}x, \int v \,\mathrm{d}x\right)$$

where $\beta = \alpha/(1+\alpha n)$. Namely, assuming the result in dimension $n-1$, for x_1 , y_1 , $z_1 = \theta x_1 + (1-\theta)y_1 \in \mathbb{R}$ fixed,

$$\int_{\mathbb{R}^{n-1}} w\left(z_1, t)dr \geq M_{\alpha/(1+\alpha(n-1))}^{(\emptyset)}(\int_{\mathbb{R}^{n-1}} u(x_1,t)dt, \int_{\mathbb{R}^{n-1}} v\,(y_1,t)dt\right).$$

Since $\alpha \geq -\frac{1}{n}$ implies that $\overline{\alpha} = \alpha/\big(1+\alpha(n-1)\big) \geq -1$, the one-dimensional result applied to $\int_{\mathbb{R}^{n-1}} u\,(x_1,t)\mathrm{d}t$, $\int_{\mathbb{R}^{n-1}} v\,(y_1,t)\mathrm{d}t$, $\int_{\mathbb{R}^{n-1}} w\,(z_1,t)\mathrm{d}t$ yields the conclusion since $\overline{\alpha}/(1+\overline{\alpha}) = \beta$. The case $\alpha = 0$ corresponds to the Prékopa-Leindler theorem. When applied to the characteristic functions $u = \chi_A$, $v = \chi_B$ of the bounded non-empty sets A, B in $\mathbb{R}^n$ with $\alpha = +\infty$, we immediately recover the Brunn-Minkowski-Lusternik inequality (2).

Most of the proofs of the preceding integral inequalities rely in one way or another on integral parametrizations. They may be proved either first in dimension one together with induction on the dimension as above, or by suitable versions of the parametrizations by multidimensional measure transportation. see [172], [179], [99], [193] for complete accounts on these various approaches.

As presented in [178], Corollary (6.1.2) may also be turned in dimension n, as a consequence of the generalized Prékopa-Leindler theorem. Re resulting statement will be the essential step in the proof of the sharp Sobolev inequalities. In particular, the possibility to use α up to- $\frac{1}{n-1}$ will turn out to be crucial.

For a non-negative function $f : \mathbb{R}^n \to \mathbb{R}$, and $i = 1, , n$, set

$$m_i(f) = \sup_{x_i \in \mathbb{R}} \int_{n-1} f\,(x)\mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n.$$

Corollary (6.1.3)[18]. *Let* - $\frac{1}{n-1} \leq \alpha \leq +\infty$, $\theta \in [0,1]$ *and* u, v, w *be non-negative measurable functions on* $\mathbb{R}^n$ *such that for all* $x, y \in \mathbb{R}^n$,

$$w(\theta x + (1-\theta)y) \geq M_\alpha^{(\theta)}\big(u(x), v(y)\big).$$

If, for some $i = 1, , n$, $m_i(u) = m_i(v) < \infty$, *then*

$$\int w\,\mathrm{d}x \geq \theta \int u\,\mathrm{d}x + (1-\theta) \int v\,\mathrm{d}x.$$

Proof. Apply the generalized Prékopa-Leindler theorem in $\mathbb{R}^{n-1}$ (thus with- $\frac{1}{n-1} \leq \alpha \leq +\infty$) to the functions $u(x), v(y), w(z)$ with $x_i, y_{i,Z_i} = \theta x_i + (1-\theta)y_i$ fixed, and conclude with the Lemma (6.1.1) applied to $\tilde{u}(x_i) = \int_{\mathbb{R}^{n-1}} u\,(x)\mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n$, $\tilde{v}(y_i)$ and $\widetilde{w}(Z_i)$ being defined similarly.

Under the assumption $m_i(u) = m_i(v)$, the conclusion of Corollary (6.1.3) does not depend on α and is thus sharpest for $\alpha = -\frac{1}{n-1}$ (the statement for - $\frac{1}{n-1} < \alpha \leq +\infty$ being actually a consequence of this case). Following the proof of Corollary (6.1.2), the complete form of Corollary (6.1.3) actually states that (cf. [178]), for every $i = 1, . n$,

$$\int w \, dx \geq M_\beta^{(\theta)}(m_i(u), m_i(v)) M_1^{(\theta)}\left(\frac{1}{m_i(u)}\int u \, dx, \frac{1}{m_i(v)}\int v \, dx\right)$$

with $\beta = \alpha/(1+\alpha(n-1))$.

Recently, mass transportation arguments have been developed to simultaneously reach the Brunn-Minkowski-Lusternik inequality and the sharp Sobolev inequalities (cfi [99] [172], [193], [201], [202] ...). In particular, Cordero-Erausquin et al. [80] provide a complete treatment of the classical Sobolev inequalities with their best constants by this tool (see also [114]). Their approach covers in the same way the family of GagliardoNirenberg inequalities put forward by Del Pino and Dolbeault [180] of non-linear diffusion equations (see also [201]). By means of Hölder's inequality, the Sobolev inequality (1) implies the family of so-called GagliardoNirenberg inequalities [183], [145],

$$\|f\|_r \leq C\|\nabla f\|_2^{\lambda}\|f\|_s^{1-\lambda} \tag{6}$$

For r,some constant > 0 and $\frac{1}{r} = \frac{C\lambda}{q} + \frac{1-\lambda 0}{s}, \lambda >$and all smooth enough functions $f\in$ $[0,1]$. Re optimal constants $f: \mathbb{R}^n \to \mathbb{R}$whereare notpreserved through Hölder's inequality. However, it was shown by Del Pino and Dolbeault [180] that optimal constants and extremal functions may be described for a sub-family of Gagliardo-Nirenberg inequalities, namely the one for which $r = 2(s-1)$ when $r, s > 2$ and $s = 2(r-1)$ when $r, s < 2$. The extremal functions turn out to be of the form $f(x) = (\sigma + |x|^2)^{2/(2-r)}$ in the first case, whereas in the second case they are given by $f(x) = ([\sigma - |x|^2]_+)^{1/(2-r)}$ (being thus compactly supported). The limiting case $r, s \to 2$ gives rise to the logarithmic Sobolev inequality (in its Euclidean formulation) with the Gaussian kernels as extremals.

While mass transport arguments may be offered to directly reach the n-dimensional Prékopa-Leindler theorem (cfi [172], [201] ...), we do not know if Corollary (6.1.3) admits an n-dimensional optimal transportation proof.

On the other hand, the Prékopa-Leindler theorem was shown in [173], following the early ideas by Maurey [192] (cL [188]) , to imply the logarithmic Sobolev inequality for Gaussian measures [185] which, in its Euclidean version [177], corresponds to the limiting case $r, s \to 2$ in the scale of Gagliardo-Nirenberg inequalities. We demonstrate that the extended Prékopa-Leindler theorem in the form of Corollary (6.1.3) above may be used to prove in a simple direct way the classical Sobolev inequality (1) with sharp constant. The argument only relies on a suitable choice of functions u, v, w. The varying parameter α in Corollary (6.1.3) allows us to cover in the same way precisely the preceding sub-family of Gagliardo-Nirenberg inequalities with optimal constants, justifying thus this particular subset of functional inequalities. As in [80], we may deal as simply with the L^p-versions of the Sobolev and Gagliardo-Nirenberg inequalities (cf. (4)), and even replace the Euclidean norm on $\mathbb{R}^n$ by some arbitrary norm. The extension of the Sobolev inequalities to arbitrary norms on $\mathbb{R}^n$ was known previously [81] by symmetrization methods. With respect to earlier developments (notably the recent [80], which provides a new and complete treatment in this respect), the approach presented here does not provide any type of characterization of

extremal functions and their uniqueness, which have to be hinted in the choice of the functions u, v, w.
We presents an outline of the direct proof of the sharp Sobolev inequality (1) from Corollary (6.1.3). We then discuss variations on the basic principle which lead to the sharp Sobolev and Gagliardo-Nirenberg inequalities (4) and (6).
The last describes, with standard technical arguments, the rigorous and detailed proof of the Sobolev inequality.
We follow the strategy put forward in [173] (see also [184]) on the basis of Corollary (6.1.3) rather than the more classical Prékopa-Leindler theorem. For $g: \mathbb{R}^n \to \mathbb{R}$ and $t > 0$, recall the infimum-convolution of g with the quadratic cost defined by

$$Q_t g(x) = \inf \left\{ g(y) + \frac{1}{2t}|x-y|^2 \right\}, x \in \mathbb{R}^n$$

(with $Q_0 g = g$). It is a standard fact (cL e.g. [5], [181] ...) that, for suitable C^1 functions g,

$$\partial_t Q_t g|_{t=0} = -\frac{1}{2}|\nabla g|^2. \tag{7}$$

Actually, if g is Lipschitz continuous, the family $\rho = \rho(x,t) = Q_t g(x)$, $t > 0$, $x \in \mathbb{R}^n$, represents the solution of the Hamilton-Jacobi initial value problem $\partial_t \rho + \frac{1}{2}|\nabla \rho|^2 = 0$ in $\mathbb{R}^n \times (0, \infty)$, $\rho = g$ on $\mathbb{R}^n \times \{t = 0\}$.
For $\sigma > 0$, set

$$v_\sigma(x) = \sigma + \frac{|x|^2}{2}, x \in \mathbb{R}^n.$$

Let $\sigma > 0$ to be determined and let $g: \mathbb{R}^n \to \mathbb{R}+$ be smooth and such that $m_1(g^{1-n}) < \infty$. In order not to obscure the main idea, we refer for a precise description of the class of functions g that should be considered in order to justify the technical differential arguments freely used below.
By definition of the infimum-convolution operator, we may apply Corollary (6.1.3) with $\alpha = -\frac{1}{n-1}$ to the set of (positive) functions

$$u(x) = g(\theta x)^{1-n},$$
$$v(y) = v_\sigma\left(\sqrt{\theta} y\right)^{1-n}$$
$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta} g(z)]^{1-n}$$

Note that $m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and $m_1(v) = (\sigma\theta)^{(1-n)/2} m_1(v_1^{1-n}) < \infty$. Choose thus $\sigma = \kappa\theta > 0$ such that $m_1(u) = m_1(v)$ where $\kappa = \kappa(n,g) = \left(m_1(v_1^{1-n}) / m_1(g^{1-n})\right)^{2/(n-1)}$. Set $= 1-\theta \in (0,1)$. Hence, by Corollary (6.1.3), for every $\in (0,1)$,

$$\int (\kappa s + Q_s g)^{1-n} \, dx \geq \int g^{1-n} \, dx + s\kappa^{(2-n)/2} \int v_1^{1-n} \, dx.$$

Taking the derivative at $s = 0$ yields, by (7),

$$(1-n)\int g^{-n}\left(\kappa-\frac{1}{2}|\nabla g|^2\right)\mathrm{d}x \geq \kappa^{(2-n)/2}\int v_1^{1-n}\,\mathrm{d}x. \tag{8}$$

Set $g = f^{2/(2-n)}$ so that

$$\frac{2}{(n-2)^2}\int |\nabla f|^2\mathrm{d}x \geq \kappa\int f^q\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-2)/2}}\int v_1^{1-n}\,\mathrm{d}x$$

where we recall that $q = 2n/(n-2)$. In particular,

$$\int |\nabla f|^2\mathrm{d}x \geq \inf_{\kappa>0}\frac{(n-2)^2}{2}\left(\kappa\int f^q\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-2)/2}}\int v_1^{1-n}\,\mathrm{d}x\right). \tag{9}$$

This infimum is precisely $C_n^{-2}\|f\|_q^2$ where C_n is the optimal constant in the Sobolev inequality (1). Actually, if $(x) = v_1(x) = 1+\frac{|x|^2}{2}$, the preceding argument develops with equalities at each step with $\kappa = \kappa(n,g) = 1$. Moreover, the infimum on the right-hand side of (9) is attained at $\kappa = 1$ if and only if

$$\int f^q\,\mathrm{d}x = \int v_1^{-n}\,\mathrm{d}x = \frac{n-2}{2(n-1)}\int v_1^{1-n}\,\mathrm{d}x$$

which is easily checked by elementary calculus. Thus (9) is an equality in this case and the conclusion follows.

As emphasized, the same proof, with the varying parameter α in Corollary (6.1.3), yields the sub-family of Gagliardo-Nirenberg inequalities recently put forward in [180]. We briefly emphasize the modifications in the argument. (It is somewhat surprising that these optimal Gagliardo-Nirenberg inequalities follow from Corollary (6.1.3) with$-\frac{1}{n-1} < \alpha \leq +\infty$ which is a consequence of the $\alpha = -\frac{1}{n-1}$ case, whereas they are not direct consequences of the sharp Sobolev inequality.)

For- $\frac{1}{n-1} \leq \alpha < 0$, apply Corollary (6.1.3) to

$$u(x) = g(\theta x)^{1/\alpha},$$

$$v(y) = v_\sigma\left(\sqrt{\theta}y\right)^{1/\alpha}$$

$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta}g(z)]^{1/\alpha}$$

to get that for all $s \in (0,1)$,

$$\int [\kappa s(1-s)^a + (1-s)Q_s g]^{1/\alpha}\,\mathrm{d}x$$

$$\geq (1-s)^{1-n}\int g^{1/\alpha}\,\mathrm{d}x + \kappa^c s(1-s)^b\int v_1^{1/\alpha}\,dx.$$

Here $a > 0$, $b,c < 0$, $\kappa > 0$ depending on n and α (and g), are such that $m_1(u) = m_1(v)$ for some suitable choice of σ. Taking the derivative at $s = 0$,

$$\frac{1}{\alpha}\int g^{(1/\alpha)-1}\left(\kappa - g - \frac{1}{2}|\nabla g|^2\right)\mathrm{d}x \geq (n-1)\int g^{1/\alpha}\,\mathrm{d}x + \kappa^c\int v_1^{1/\alpha}\,dx.$$

Set $f = g^p, 2p - 2 = \frac{1}{\alpha} - 1$, so that

$$-\frac{1}{2\alpha p^2}\int |\nabla f|^2 \mathrm{d}x - \left[(n-1) + \frac{1}{\alpha}\right]\int f^r \,\mathrm{d}x \geq -\frac{\kappa}{\alpha}\int f^s \,\mathrm{d}x + \kappa^c \int v_1^{1/\alpha}\,\mathrm{d}x$$

where $r = 2(1-\alpha)/(1+\alpha)$ and $s = 2/(1+\alpha)$. Note that $r, s > 2, r = 2(s-1)$. Take the infimum over $\kappa > 0$ on the right-hand side, and rewrite then the inequality by homogeneity to get the Gagliardo-Nirenberg inequality

$$\|f\|_r \leq C\|\nabla f\|_2^{\lambda}\|f\|_s^{1-\lambda},$$

$\frac{1}{\mathfrak{B}r} = \frac{\lambda}{\mathrm{ri}q} + \frac{1-\lambda}{s}$Spnger , with optimal constant C.

To reach the sub-family $r, s < 2, s = 2(r-1)$, work now with $0 < \alpha < +\infty$ and replace v_σ by the compactly supported function $\left[\sigma - \frac{|x|^2}{2}\right]_+$, $|x| < \sqrt{2\sigma}$. Actually, only the values $0 < \alpha < 1$ are concerned in the argument. We do not know what type of functional information is contained in the interval $\alpha \geq 1$. The case $\alpha = 0$ leading to the logarithmic Sobolev inequality has been studied in [173], [184] and follows here as a limiting case.

We can work more generally with the L^p-Sobolev inequalities (4), $1 < p < n$, and similarly with the corresponding sub-family of Gagliardo-Nirenberg inequalities. It is also possible to equip $\mathbb{R}^n$ with an arbitrary norm $\|\cdot\|$ instead of the Euclidean one $|\cdot|$, and to consider

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p \mathrm{d}x$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$. To these tasks, consider as in [184],

$$Q_t g(x) = \inf\left\{g(y) + tV^*\left(\frac{x-y}{t}\right)\right\}, t > 0, x \in \mathbb{R}^n,$$

where $V^*(x) = \frac{1}{p^*}\|x\|^{p^*}$ with p^* is the Hölder conjugate of p, i.e. $(1/p) + (1/p^*) = 1$. Then $\rho = \rho(x,t) = Q_t g(x)$ is the solution of the Hamilton-Jacobi equation $\partial_t \rho + V(\nabla \rho) = 0$ with initial condition g, where $V(x) = \frac{1}{p}\|x\|_*^p$ is the Legendre transform of V^* (cf. [181]). The proof then follows along the same lines as before. The general statement obtained in this way is the following (cf. [80], [96]). For $1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, s < r \leq q, \lambda \in [0,1]$,

$$\|f\|_r \leq C_n(p,r)\|\nabla f\|_p^{\lambda}\|f\|_s^{1-\lambda}$$

with $\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, $p(s-1) = r(p-1)$ if $r, s > p$, $p(r-1) = s(p-1)$ if $r, s < p$, and the optimal constant $C_n(r,p)$ is achieved on the extremal functions $\left(\sigma + \|x\|^{p^*}\right)^{p/(p-r)}, x \in \mathbb{R}^n$, $\sigma > 0$, in the first case and $\left(\left[\sigma - \|x\|^{p^*}\right]_+\right)^{(p-1)/(p-r)}$, $x \in \mathbb{R}^n$, $\sigma > 0$, in the second case. The optimal Sobolev inequality (4) corresponds to the limiting case $\lambda \to 1, r \to q, s \to r$. We collect the technical details necessary to fully justify the proof of the Sobolev

inequality outlined . Although the case $p = 2$ is a bit more simple, we can actually easily handle in the same way the more general case of $1 < p < n$ and of an arbitrary norm $\|\ .\ \|$ on $\mathbb{R}^n$. The arguments are easily modified so to deal similarly with the Gagliardo-Nirenberg inequalities discussed.
Consider thus on $\mathbb{R}^n$ the Sobolev inequality

$$\|f\|_q \le C_n(p)\|\nabla f\|_p \tag{10}$$

in the class of all locally Lipschitz functions f vanishing at infinity, with parameters p, q satisfying $1 < p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. The right-hand side in (10) is understood with respect to the given norm $\|\cdot\|$ on $\mathbb{R}^n$. More precisely,

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p \mathrm{d}x$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. We show that the best constant $c_n\mathrm{C}p)$ in (10) corresponds to the family of extremal functions

$$f(x) = \left(\sigma + \|x\|^{p^*}\right)^{\mathrm{C}p-n)/p}, x \in \mathbb{R}^n, \sigma > 0,$$

where p^* is the conjugate of p. We may assume that the norm $x \mapsto \|x\|$ is continuously differentiable in the region $x \neq 0$. In this case, $\|\nabla\|x\|\|_* = 1$ for all $x \neq 0$, and all the extremal functions belong to the class $C^1(\mathbb{R}^n)$.
The associated infimum-convolution operator is constructed for the cost function $V^*(x) = \frac{1}{p^*}\|x\|^{p^*}$, that is,

$$Q_t g(x) = \inf_{y\in\mathbb{R}^n} \left\{g(y) + tV^*\left(\frac{x-y}{t}\right)\right\}, t > 0, x \in \mathbb{R}^n.$$

The dual (Legendre transform) of V^* is $V(x) = \sup_{y\in\mathbb{R}^n}[\{x, y\rangle - V^*(y)] = \frac{1}{p}\|x\|_*^p$ (and conversely).
See (such as [171], [181] . . .) for general facts about infimum-convolution operators and solutions to Hamilton-Jacobi equations, and only concentrate below on the aspects relevant to the proof of the Sobolev inequality. What follows is certainly classical, but we could not find appropriate references.
Lemma (6.1.4)[18]. *If a function g on $\mathbb{R}^n$ is bounded from below and is differentiable at the point $x \in \mathbb{R}^n$, then*

$$\lim_{t\to 0}\frac{1}{t}[Q_t g(x) - g(x)] = -V\big(\nabla g(x)\big) = -\frac{1}{p}\|\nabla g(x)\|_*^p.$$

Proof. Fix $x \in \mathbb{R}^n$. By Taylor's expansion, $g(x-h) = g(x) - \langle\nabla g(x), h) + |h|\varepsilon(h)$ with $\varepsilon(h) = \varepsilon_x(h) \to 0$ as $|h| \to 0$. Hence, for vectors $h_t = th$ with fixed $h \in \mathbb{R}^n$,

$$\lim_{t\to 0}\frac{1}{t}[g(x-h_t) - g(x)] = -\{\nabla g(x), h) .$$

Since we always have $Q_t g(x) \le g(x-h_t) + tV^*(h)$,

$$\lim_{t\to 0}\ \sup\ \frac{1}{t}[Q_t g(x)-g(x)] \le \lim_{t\to 0}\frac{1}{t}[g(x-h_t)-g(x)]+V^*(h)$$
$$= -\{\nabla g(x),h\rangle + V^*(h)\,.$$
The left-hand side of the preceding does not depend on h. Hence, taking the infimum on the right-hand side over all $h \in \mathbb{R}^n$, we get
$$\lim_{t\to 0}\ \sup\ \frac{1}{t}[Q_t g(x)-g(x)] \le -V\big(\nabla g(x)\big)\,.$$
Now, we need an opposite inequality for the lim inf. Assume without loss of generality that $g \ge 0$. Since $Q_t g(x) \le g(x)$, it is easy to see that for any $t > 0$,
$$Q_t g(x) = \inf_{tV^*(h)\le g(x)} \{g(x-h_t)+tV^*(h)\}.$$
Hence, recalling Taylor's expansion,
$$\frac{1}{t}[Q_t g(x)-g(x)] = \inf_{tV^*(h)\le g(x)} \{-\langle \nabla g(x),h\rangle + |h|\varepsilon(th)+V^*(h)\}. \tag{11}$$
Note first that the argument in $\varepsilon(\cdot)=\varepsilon_x(\cdot)$ is small uniformly over all admissible h since, as is immediate,
$$\sup\ \{t|h|; tV^*(h)\le g(x)\} \to 0\ as\ t\to 0.$$
Thus removing the condition $tV^*(h)\le g(x)$ in (11), we get that, given $\eta > 0$, for all t small enough,
$$\frac{1}{t}[Q_t g(x)-g(x)] \ge \inf_{\mathrm{h}}\ \{-\{\nabla g(x),h\rangle - |h|\eta + V^*(h)\}. \tag{12}$$
Now, to get rid of η on the right-hand side for t approaching zero, note that the infimum in (12) may be restricted to the ball $|h| \le r$ for some large r. Indeed, the left-hand side in (12) is non-positive. But if $|h|$ is large enough and $0<\eta<1$, the quantity for which we take the infimum will be positive for $V^*(h)\ge C|h| > \{\nabla g(x),h\rangle + |h|\eta$ with C taken in advance to be as large as we want. Finally, restricting the infimum to $|h|\le r$, we get that
$$\frac{1}{t}[Q_t g(x)-g(x)] \ge \inf_{|h|\le r}\ \{-(\nabla g(x),h\rangle + V^*(h)\} - r\eta = -V\big(\nabla g(x)\big) - r\eta.$$
It remains to take the lim inf on the left for $t\to 0$, and then to send η to 0. The proof of Lemma (6.1.4) is complete.

Our next step is to complement the above convergence with a bound on $|Q_t g(x)-g(x)|/t$ in terms of $\|\nabla g(y)\|_*$ with vectors y that are not far from x. So, given a C^1 function g on $\mathbb{R}^n$, for every point $x\in\mathbb{R}^n$ and $r>0$, define $Dg(x,r)=\sup_{\|x-y\|\le r}\|\nabla g(y)\|_*$. Note that $Dg(x,r)\to\|\nabla g(x)\|_*$ as $r\to 0$. Assume $g\ge 0$ and write once more
$$Q_t g(x) = \inf_{h\in\mathbb{R}^n} \left\{g(x-h)+\frac{||h||^{p^*}}{p^* t^{p^*-1}}\right\}, t>0.$$

Again, since $Q_t g(x) \le g(x)$, the infimum may be restricted to the ball $\left(\|h\|^{p^*}/p^* t^{p^*-1}\right) \le g(x)$. Hence, replacing h with th and applying the Taylor formula in integral form, we get that with $r = \left(p^* g(x)\right)^{1/p^*}$, for any $t > 0$,

$$\frac{1}{t}[g(x) - Q_t g(x)] \le \sup_{t\|h\|\le r} \left\{\frac{1}{t}[g(x) - g(x - th)] - \left(\|h\|^{p^*}/p^*\right)\right\}$$
$$\le \sup_{t\|h\|\le r} \left\{Dg(x, t\|h\|)\|h\| - \left(\|h\|^{p^*}/p^*\right)\right\}$$
$$\le \sup\nolimits_{\mathrm{h}} \left\{Dg(x, r)\|h\| - \left(\|h\|^{p^*}/p^*\right)\right\}$$
$$= \frac{1}{p} Dg(x, r)^p. \quad (13)$$

In applications, we need to work with functions $g(x) = O\left(|x|^{p^*}\right)$ as $|x| \to \infty$. So, let us define the class $\mathcal{F}_p *, p^* > 1$, of all C^1 functions g on $\mathbb{R}^n$ such that

$$\lim_{|x|\to\infty} \sup \frac{|\nabla g(x)|}{|x|^{p^*-1}} < \infty.$$

If $f \in \mathcal{F}_p *$, then, for some C, $|\nabla g(x)| \le C|x|^{p^*-1}$ as long as $|x|$ is large enough, and hence $|g(x)|^{1/p^*} \le C'|x|$ for $|x|$ large. It easily follows that $Dg\left(x, \left(p^* g(x)\right)^{1/p^*}\right) \le C''\left(1 + |x|^{p^*-1}\right)$ for all x. As a consequence of (13), we may conclude that for any $g \ge 0$ in $\mathcal{F}_p *$, $p^* > 1$, there is a constant $C > 0$ such that

$$\sup_{t>0} \frac{1}{t}[g(x) - Q_t g(x)] \le C\left(1 + |x|^{p^*}\right), x \in \mathbb{R}^n. \quad (14)$$

We may now start the proof of the Sobolev inequality according to the scheme outlined Given a parameter $\sigma > 0$, define

$$v_\sigma(x) = \sigma + \frac{\|x\|^{p^*}}{p^*}, x \in \mathbb{R}^n.$$

For a positive C^1 function g on $\mathbb{R}^n$, and $\theta \in (0,1)$, define the three (positive, continuous) functions

$$u(x) = g(\theta x)^{1-n}$$
$$v(y) = v_\sigma(\theta^{1/p^* j)^{1-n}},$$
$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta} g(z)]^{1-n}$$

The function w is chosen as the optimal one satisfying

$$w(\theta x + (1-\theta)y)^\alpha \le \theta u(x)^\alpha + (1-\theta)v(y)^\alpha$$

for $\alpha = -\frac{1}{n-1}$ and all $x, y \in \mathbb{R}^n$. Assume that

$$m_1(g^{i-n}) = \sup_{x_1 \in \mathbb{R}} \int_{n-1} g\,(xx, \cdots, x_n)^{1-n} \mathrm{d}_{X2} \ldots \mathrm{d}x_n < \infty.$$

By homogeneity, $m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and
$m_1(v) = \theta^{(1-n)/p^*} \sigma^{(1-n)/p} m_1(v_1^{1-n})$.
Note that $m_1(v_1^{1-n}) < \infty$. Hence, we may choose σ such that $m_1(u) = m_1(v)$, that is,

$$\sigma = \kappa\theta, where\ \kappa = \kappa(n, g) = \left(\frac{m_1(v_1^{1-n})}{m_1(g^{1-n})}\right)^{p/(n-1)}$$

By Corollary (6.1.3) (with $\alpha = -\frac{1}{n-1}$), we have

$$\int w\,\mathrm{d}x \geq \theta \int u\,\mathrm{d}x + (1-\theta) \int v\,\mathrm{d}x,$$

that is,

$$\int [(1-\theta)\sigma + \theta Q_{1-\theta} g(x)]^{1-n}\,\mathrm{d}x \geq \theta \int g\,(\theta x)^{1-n} \mathrm{d}x + (1-\theta) \int v_\sigma \left(\theta^{1/p^*} x\right)^{1-n} \mathrm{d}x.$$

After a change of variable in the last two integrals, and since $\sigma = \kappa\theta$, we get, setting $s = 1 - \theta$,

$$\int (\kappa s + Q_s g)^{1-n}\,\mathrm{d}x \geq \int g^{1-n}\,\mathrm{d}x + s\kappa^{(p-n)/p} \int v_1^{1-n}\,\mathrm{d}x. \qquad (15)$$

Inequality (15) holds true for all $0 < s < 1$, and formally there is equality at $s = 0$. Re next step is to compare the derivatives of both sides at this point. To do this, assume $g \in \mathcal{F}_p *$ and

$$g(x) \geq c\left(1 + \|x\|^{p^*}\right) \qquad (16)$$

for some constant $c > 0$. (Recall that the functions in $\mathcal{F}_p *$ satisfy an opposite bound $g(x) \leq C\left(1 + \|x\|^{p^*}\right)$ which will not be used.) Due to (16), $Q_s g(x) \geq c'\left(1 + \|x\|^{p^*}\right)$ (where $c' > 0$ is independent of s). In particular, $m_1(g^{1-n}) < \infty$, and the first and second integrals in (15) are finite and uniformly bounded over all $s \in (0,1)$. Rewrite (15) as

$$\kappa^{(p-n)/p} \int v_1^{1-n}\,\mathrm{d}x \leq \int \frac{1}{s}[(\kappa s + Q_s g)^{1-n} - g^{1-n}]dx. \qquad (17)$$

Now we can use a general inequality

$$|a^{1-n} - b^{1-n}| \leq (n-1)|a-b|(a^{-n} + b^{-n})\,, a, b > 0,$$

to see that, uniformly in s,

$$\frac{1}{s}[(\kappa s + Q_s g)^{1-n} - g^{1-n}] \leq 2(n-1)\left(\kappa + \frac{1}{s}[g - Q_s g]\right)(Q_s g)^{-n}$$

$$\leq C'\left(1 + \|x\|^{p^*}\right)^{1-n}$$

for some constant $C' > 0$. On the last step, we used that $Q_s g(x) \geq c'(1 + \|x\|^{p^*})$ together with the bound (14) for functions from the class $\mathcal{F}_p *$. Since the function $\left(1 + \|x\|^{p^*}\right)^{1-n}$ is integrable (for $p < n$), we can apply the Lebesgue dominated convergence theorem in order to insert the limit lim inside the integral in (17), and to thus get together with Lemma (6.1.4),

$$\kappa^{(p-n)/p} \int v_1^{1-n}\,\mathrm{d}x \leq (1-n) \int g^{-n} \left(\kappa - \frac{\|\nabla g\|_*^p}{p}\right) \mathrm{d}x,$$

or equivalently,

$$\frac{1}{p} \int g^{-n}\,\|\nabla g\|_*^p \mathrm{d}x \geq \kappa \int g^{-n}\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n}\,\mathrm{d}x. \quad (18)$$

Now, let us take a non-negative, compactly supported C^1 function f on $\mathbb{R}^n$, and for $\varepsilon > 0$, define C^1 functions

$$g_\varepsilon(x) = \left(f(x) + \varepsilon\phi(x)\right)^{p/(p-n)} + \varepsilon\left(1 + \|x\|^{p^*}\right)$$

where $\phi(x) = \left(1 + \|x\|^{p^*}\right)^{\mathrm{C}p-n)/p}$. Clearly, all g_ε satisfy (16). The first partial derivatives of f are continuous and vanishing for large values of $|x|$. $g_\varepsilon(x) = c_\varepsilon\left(1 + \|x\|^{p^*}\right)$ for $|x|$ large enough, so all g_ε belong to the class $\mathcal{F}_p *$. Thus, we can apply (18) to get

$$\frac{1}{p} \int g_\varepsilon^{-n}\,\|\nabla g_\varepsilon\|_*^p \mathrm{d}x \geq \kappa \int g_\varepsilon^{-n}\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n}\,\mathrm{d}x. \quad (19)$$

Note that $g_\varepsilon^{-n} \leq (f + \varepsilon\phi)^q$ and $\int \phi^q\,\mathrm{d}x < \infty$ (where we recall that $q = pn/(n-p)$). Hence, by the Lebesgue dominated convergence theorem again, $\int g_\varepsilon^{-n}\,\mathrm{d}x$ is convergent, as $\varepsilon \to 0$, to $\int f^q\,\mathrm{d}x$. By a similar argument, recalling that $\|\nabla\|x\|^{p^*}\|_* = p^*\|x\|^{p^*-1}, x \in \mathbb{R}^n$, we see that there is a finite limit for the left integral in (19). As a result, we arrive at

$$\frac{p^{p-1}}{(n-p)^p} \int \|\nabla f\|_*^p \mathrm{d}x \geq \kappa \int f^q\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{(n-p)/p}} \int v_1^{1-n}\,\mathrm{d}x, \quad (20)$$

which implies

$$\frac{p^{p-1}}{(n-p)^p} \int \|\nabla f\|_*^p \mathrm{d}x \geq \inf \left(\kappa \int f^q\,\mathrm{d}x + \frac{1}{(n-1)\kappa^{\frac{(n-p)}{p}}} \int v_1^{1-n}\,\mathrm{d}x\right) \quad (21)$$

As we will see with the case of equality below, this is precisely the desired Sobolev

inequality (10) with optimal constant. It is now easy to remove the assumption on the compact support of f and thus to extend (21) to all C^1 and furthermore locally Lipschitz functions $f(\geq 0)$ on $\mathbb{R}^{\boldsymbol{n}}$ vanishing at infinity.
To conclude the argument, we investigate the case of equality. To this task, let us return to the beginning of the argument and check the steps where equality holds true. Take $g = v_1$ so that $\kappa = \kappa(n, g) = 1$ and $\sigma = \theta$. In addition, the right-hand side of (15) automatically turns into $(1 + s) \int v_1^{1-n}\, \mathrm{d}x$. By direct computation,

$$Q_s v_1(x) = 1 + \frac{\|x\|^{p^*}}{p^*(1+s)^{p^*-1}},$$

so the left-hand side of (15) is

$$\int (\kappa s + Q_s g)^{1-n}\, \mathrm{d}x = \int \left((1+s) + \frac{\|x\|^{p^*}}{p^*(1+s)^{p^*-1}} \right)^{1-n} \mathrm{d}x$$

$$= (1+s) \int \left(1 + \frac{\|y\|^{p^*}}{p}* \right)^{1-n} \mathrm{d}y = (1+s) \int v_1^{1-n}\, \mathrm{d}y$$

where we used the change of the variable $x = (1+s)y$. Thus, for $g = v_1$ there is equality in (15), and hence in (18) and (20) as well.
As for (21), first note that, given parameters $A, B > 0$, the function $A\kappa + B\kappa^{(p-n)/p}, \kappa > 0$, attains its minimum on the positive half-axis at $\kappa = 1$ if and only if $A = B(n-p)/p$. In the situation of the particular functions $= v_1$, $f^q = g^{-n} = v_1^{-n}$, we have

$$A = \int v_1^{-n}\, \mathrm{d}x, B = \frac{1}{n-1} \int v_1^{1-n}\, \mathrm{d}x.$$

Hence, the infimum in (20) is attained at $\kappa = 1$ if and only if

$$\int v_1^{-n}\, \mathrm{d}x = \frac{n-p}{p(n-1)} \int v_1^{1-n}\, \mathrm{d}x.$$

But this equality is easily checked by elementary calculus.
We may thus summarize our conclusions. In the class of all locally Lipschitz functions f on $\mathbb{R}^n$, vanishing at infinity and such that $0 < \|f\|_q < \infty$, the quantity

$$\frac{\|\nabla f\|_p}{\|f\|_q},$$

$1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, is minimized for the functions

$$f(x) = \left(\sigma + \|x\|^{p^*}\right)^{(p-n)/p}, x \in \mathbb{R}^n, \sigma > 0.$$

Here $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\|\cdot\|$ is a given norm on $\mathbb{R}^n$, and

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \| \nabla f(x)\|_*^p \mathrm{d}x$$

where $\| \cdot \|_*$ is the dual norm to $\| \cdot \|$.
From Brunn-Minkowski to sharp Sobolev inequalities.

Section (6.2): An Improved Borell–Brascamp–Lieb Inequality

Sharp inequalities are interesting not only because they correspond to exact solutions of variational problems but also because they encode, in general, deep geometric information on the underneath space. We are interested in new functional inequalities of Sobolev type, and their links with the Brunn-Minkowski inequality

$$\mathrm{vo1}_n(A+B)^{1/n} \geq \mathrm{vo1}_n(A)^{\frac{1}{n}} + \mathrm{vo1}_n(B)^{\frac{1}{n}} \tag{22}$$

for nonempty Borel sets A, B in $\mathbb{R}^n$; here $\mathrm{vo1}_n(\cdot)$ denotes the n-dimensional Lebesgue measure. It is known since [171] that sharp Sobolev and Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$ may be derived using Brunn –Minkovvski type inequalities, we will see that a new functional version of (22) provides a more direct and simple answer, that allows to tackle both the cases of $\mathbb{R}^n$ and the half-space $\mathbb{R}_+^n$.
Before presenting this new functional inequality, We discuss new sharp Sobolev type inequalities in $\mathbb{R}^n$.

Let $\|f\|_p = \|f\|_{L(\mathbb{R}^n)}p$ denote the L^p-norm with respect to Lebesgue measure. The sharp classical Sobolev inequalities state that for $n \geq 2$, $p \in [1, n)$, $p^* = \frac{np}{n-p}$, and any smooth enough function f on $\mathbb{R}^n$ (i.e., for f belonging to the correct Sobolev space ensuring that both integrals are finite),

$$\|f\|_{p^*} \leq \frac{\|h_p\|_{p^*}}{\left(\int_{\mathbb{R}^n} |\nabla h_p|^p\right)^{1/p}} \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p} ; \tag{23}$$

here

$$h_p(x) := \left(1 + |x|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}$$

The optimal constants in the Sobolev inequalities have been first exhibited in [82], [215]. Quite naturally, these inequalities admit a generalization when the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ is replaced by any norm or quasi-norm $\| \cdot \|$ on $\mathbb{R}^n$. Indeed, if we use a norm $\| \cdot \|$ to compute the size of the differential in (2), then the result remains true, namely

$$\|f\|_{p^*} \leq \frac{\|h_p\|_{p^*}}{\left(\int_{\mathbb{R}^n} \|\nabla h_p\|_*^p\right)^{1/p}} \left(\int_{\mathbb{R}^n} \|\nabla f\|_*^p\right)^{1/p} \tag{24}$$

where $\|y\|_*:=\sup_{\|x\|\le 1} x\cdot y$. In this case, $h_p(x):=\left(1+||x||^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}$

In turn, a natural extension of this problem may then be the minimization, under integrability constraints on a function g, of more general quantities like

$$\int_{\mathbb{R}^n} F(\nabla g)g^{\alpha},$$

where $:\mathbb{R}^n\to\mathbb{R}$ is a convex function ($F=W^*$ below). We have to allow a g^{α} term, $\alpha\in\mathbb{R}$ since it can no longer be absorbed in the gradient term when F is not homogeneous.
We have the following optimal Sobolev type inequality.

Theorem (6.2.1)[203]. Let $n\ge 2$ and $:\mathbb{R}^n\to(0,+\infty)$ such that $\liminf_{+\infty}\frac{W(x)}{|x|^{\gamma}}>0$ for some $\gamma>\frac{n}{n-1}$. For any $g:\mathbb{R}^n\to(0,+\infty)$ with $g^{-n}|\nabla g|^{V/(y-1)}\in L^1$ and

$$\int g^{-n}=\int W^{-n}=1,$$

one has

$$\int W^*(\nabla g)g^{-n}\ge\frac{1}{n-1}\int W^{1-n}. \tag{25}$$

Moreover, equality holds in (25) when g is equal to W and is convex.
Here W^* is the Legendre transform of the function W, explained below. This result admits auconcave" analog, as we shall see.
We shall see that the sharp Sobolev inequalities (24), for $p\in(1,n)$, easily follow from this theorem when applied to $W(x)=C(1+\|x\|^q/q)$, $q=p/(p-1)>n/(n-1)(\gamma=q$ in the assumptions), and to $g=f^{p/(p-n)}$. Let us mention that the coefficients n and $n-1$ in this theorem are not arbitrary at all; in some aspects, they are the ugood" ones to reach the Sobolev inequality, as we shall see. This may be compared to of [171] that was derived via a more involved formulation of the Prékopa-Leindler inequality, leading to a less direct proof of the Sobolev inequalities.
As mentioned above, our work is inspired by the Brunn−Minkowski-Borell theory. We will propose a new functional viewpoint on this theory. As already said, it has been observed by S. Bobkov and M. Ledoux in [174], [171] that Sobolev inequalities can be reached through a functional version of the Brunn−Minkowski inequality, the so-called Borell − Brascamp −Lieb [BBL) inequality, due to C. Borell and H. J. Brascamp-E. H. Lieb [207], [176].
The standard BBL inequality states that, for $n\ge 1$, given $s\in[0,1]$, $t=1-s$, and three nonnegative functions $u,v,w:\mathbb{R}^n\to[0,+\infty]$ such that $\int u=\int v=1$ and

$$\forall x, y \in \mathbb{R}^n, w(sx+ty) \geq \left(su^{-1/n}(x) + tv^{-1/n}(y)\right)^{-n},$$

then

$$\int w \geq 1.$$

This is the ustrongest" version of BBL inequality, see for example, [99]. By a simple change of functions, the result can be restated as follows: let three nonnegative functions g, $W, H: \mathbb{R}^n \to [0, +\infty]$ be such that

$$\forall x, y \in \mathbb{R}^n, H(sx+ty) \leq sg(x) + tW(y)$$

and $\int W^{-n} = \int g^{-n} = 1$. Then

$$\int H^{-n} \geq 1. \tag{26}$$

One observes that (26) is not well adapted to the Sobolev inequality, but that a version with $n-1$ instead of n would do the job. To solve this issue, in [171] S. Bobkov and M. Ledoux cleverly used a classical geometric strengthening of the Brunn−Minkowski inequality, for sets having an hyperplane of same volume.

A natural question raised by S. Bobkov and M. Ledoux is whether the Sobolev inequality can be proved directly from a new BBL inequality, which moreover would be well adapted to a monotone mass transport argument. We propose an answer in the following form.

Theorem (6.2.2)[203]: Let $n \geq 2$. Let $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0,1]$, $t = 1 - s$ be such that

$$\forall x, y \in \mathbb{R}^n, H(sx+ty) \leq sg(x) + tW(y)$$

and $\int W^{-n} = \int g^{-n} = 1$. Then

$$\int H^{1-n} \geq s\int g^{1-n} + t\int W^{1-n}. \tag{27}$$

We shall that, for small t, the optimal H satisfies $H = g - tW^*(\nabla g) + o(t)$, so that (6) gives the above (4) in Theorem (6.2.1) and therefore the Sobolev inequalities (24) at the 1st order for $t \to 0$; as mentioned the Sobolev inequalities correspond to the case $W(x) = C(1 + \|x\|^q/q)$, $q = p/(p-1)$, $g = f^{p/(p-n)}$. We shall see that sharp (classical and trace) Sobolev inequalities and new (trace) Gagliardo-Nirenberg inequalities follow from it. Moreover, it can be easily proved using a mass transport argument, and we believe that this is a way of closing the circle of ideas relating BrunnMinkowski and Sobolev inequalities.

The Sobolev inequalities in $\mathbb{R}^n$ belong to the larger family of Gagliardo-Nirenberg inequalities

$$\|f\|_\alpha \leq C\|\nabla f\|_p^\theta \|f\|_\beta^{1-\theta}$$

Here the coefficients α, β, p belong to an adequate range and $\theta \in [0,1]$ is fixed by scaling

invariance. These inequalities have attracted much attention these past years. Sharp inequalities are known for a certain family of parameters since the pioneering work of M. del Pino and J. Dolbeault [181]; namely, for $p > 1, \alpha = ap/(a-p)$, and $\beta = p(a-1)/(a-p)$, where $a > p$ is a free parameter.
This family can be recovered from Theorem (6.2.1), or rather an extension of it (see Theorem (6.2.3) and its uconcave" counterpart Theorem (6.2.4)). In fact this extension turns out not only to be a natural way of recovering this family, but also allows to extend the family to parameters $a < p$ leading to new sharp Gagliardo-Nirenberg inequalities with negative powers

$$\|f\|_{p\frac{a-1}{a-p}} \leq C\|\nabla f\|_p^{\theta}\|f\|_{\frac{1ap-\theta}{a-p}}$$

Here $p > a$ if $\geq n+1$, or $p \in \left(a, \frac{n}{n+1-a}\right)$ if $a \in [n, n+1)$, and θ is fixed by a scaling condition. We note that partial results for a narrower range of such $a < p$ have been proved by V.-H. Nguyen [212], by another approach.
It can be applied to reach a new family of sharp trace Gagliardo-Nirenberg inequalities that extend the trace Sobolev inequality proved by B. Nazaret [113]. Indeed, letting $\mathbb{R}_+^n = \{(u,x)\ , u \geq 0, x \in \mathbb{R}^{n-1}\}$ we obtain the sharp family of inequalities

$$\|f\|_{L^{\alpha}(\partial\mathbb{R}_+^n)} \leq C\|\nabla f\|_{L(\mathbb{R}_+^n)}^{\theta_p}\|f\|_{L^{\beta}(\mathbb{R}_+^n)}^{1-\theta}.$$

Here $p > 1, \alpha = p(a-1)/(a-p)$, and $\beta = p(a-1)/(a-p)$, where $a > p$ is a free parameter and again $\theta \in [0,1]$ is fixed by a scaling argument. This is thus the analog of the del Pino-Dolbeaut family in the trace case.
We state and prove the main results, namely generalizations we show how these results lead to Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$, including and extending the del Pino-Dolbeault family, whereas in we follow the same procedure to reach trace Gagliardo-Nirenberg inequalities. We devoted to limit forms of the BBL and Gagliardo-Nirenberg inequalities, namely the classical Prékopa-Leindler inequality and classical or new trace logarithmic Sobolev inequalities. Finally, deals with a general result on the infimum convolution, which is a crucial tool for our proofs.
When the measure is not mentioned, an integral is understood with respect to Lebesgue measure. For $x, \mathrm{y} \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and $x \cdot \mathrm{y}$ the Euclidean scalar product. As already used, $\|f\|_p$ stands for the $IP(\mathbb{R}^n)$ norm of a function f.

Our results have two formulations, as a convex (or concave) Sobolev-type inequality illustrated by Theorem (6.2.1), and as a Borell−Brscamp−Lieb type inequality like Theorem (6.2.2).

Our setting splits in two separate cases, the origin of which will be explained below. We shall measure the gradient using a function W on $\mathbb{R}^n$ in one of the following two categories:
i. Either $W: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a *convex* fonction, with Legendre transform W^* defined by

$$W^*(\mathrm{y}) = \sup\{x \cdot \mathrm{y} - W(x)\}.$$

The function W is differentiable at almost every x in its domain, with

$$W^*(\nabla W(x)) + W(x) = x \cdot \nabla W(x)\,. \tag{28}$$

ii. Either W is a nonnegative function that is *concave* on its support $\Omega_W = \{W > 0\}$. More precisely, W is a nonnegative function such that the function $\widetilde{W}$ defined on $\mathbb{R}^n$ by $\widetilde{W}(x) = W(x)$ if $x \in \Omega_W$ and $-\infty$ otherwise, is concave. In particular Ω_W is a convex set. The corresponding Legendre transform is defined by

$$W_*(\mathrm{y}) = \inf\{x\,.\mathrm{y} - W(x))\} = \inf\{x\,.\mathrm{y} - \widetilde{W}(x)\}. \tag{29}$$

Likewise, W is differentiable at almost every $x \in \Omega_W$, with

$$W_*(\nabla W(x)) + W(x) = x \cdot \nabla W(x). \tag{30}$$

We will later assume that W is continuous on $\mathbb{R}^n$ to avoidjumps on $\partial\Omega_W$.
See [214] for instance for these classical definitions and properties.
One rather naturally comes to such a setting if one has in mind the Brunn−Minkowski theory of convex measures on $\mathbb{R}^n$ as put forward by C. Borell. We briefly recall it to put our results in perspective, although we will not explicitly use it. A nonnegative function G on $\mathbb{R}^n$ is said to be κ-*concave* with $\kappa \in \mathbb{R}$ if κG^κ is concave on its support. In other words,
i. If $\kappa < 0$, then $G = W^{1/\kappa}$ with W convex on $\mathbb{R}^n$. The Brunn−Minkowski−Borell theory shows that one should consider the range $\kappa \in [-\frac{1}{n}, 0)$. Below we shall let $\kappa = -1/a$ for $a \geq n$ with the typical examples $W(x) = 1 + |x|^q$, $q \geq 1$ and then $G(x) = (1 + |x|^q)^{-a}$. The results above correspond to the extremal case $a = n$.
ii. If $\kappa > 0$, $G = W^{1/\kappa}$ with W concave on its support. Below we shall let $\kappa = 1/a$ for $a > 0$ with the typical examples $W(x) = (1 - |x|^q)_+$, $q \geq 1$ and $G(x) = (1 - |x|^q)_+^a$.
The limit case $\kappa = 0$ is defined as the $\log$-concavity of G.
A central tool in our work will be monotone transportation, which by now has become a cornerstone of many proofs of functional inequalities. So let us briefly describe the mathematical setting and notation on this topic we shall use below, see [202], [216] for instance.
Given μ and v two (Borel) probability measure on $\mathbb{R}^n$ with μ absolutely continuous with respect to Lebesgue measure, a result of Brenier [86], in a form improved by McCann [105], states that there exists a convex function ϕ {the so-called Brenier map) on $\mathbb{R}^n$ such that v is the image measure $\nabla\phi\#\mu$ of μ by $\nabla\phi$, that is, for any positive or bounded Borel function H on $\mathbb{R}^n$,

$$\int H\,\mathrm{d}v = \int H\,(\nabla\phi)\mathrm{d}\mu.$$

Assuming that $\mathrm{d}\mu = f\mathrm{d}x$ and $\mathrm{d}v = g\mathrm{d}x$ then [211] ensures that the Monge-Ampère equation

$$f(x) = g(\nabla\phi(x)) \det(\nabla^2\phi(x)) \tag{31}$$

holds $f dx$-almost surely. Here $\nabla^2\phi$ is the Alexandrov Hessian of ϕ that is the absolutely continuous part of the distributional Hessian of the convex function ϕ (but below ϕ will belong to $W^{2,1}_{loc}$ so there will be no singular part).

and classical and elementary tool will be the convexity of the determinant of nonnegative symmetric matrices, such as $\nabla^2\phi(x)$. This splits in two cases, in accordance to the cases discussed above.

For every $k \in (0,1/n]$, the map $H \to \det^k H$ is concave over the set of positive symmetric matrices. Concavity inequality around the identity implies

$$\det^k H \leq 1 - nk + k \operatorname{tr} H \tag{32}$$

for all positive symmetric matrix H.

For every $k < 0$, the map $H \to \det^k H$ is convex over the set of positive symmetric matrices. Convexity inequality around the identity implies

$$\det^k H \geq 1 - nk + k \operatorname{tr} H \tag{33}$$

for all positive symmetric matrix H.

We start with a generalization of Theorem (6.2.1) and we will next establish its uconcave" counterp art.

The result involves a umeasurement" function: $\mathbb{R}^n \to \mathbb{R}^+$ that will be convex in applications, and actually of the form

$$W(x) = 1 + \|x\|^q/q \tag{34}$$

for a norm $\|\cdot\|$ on $\mathbb{R}^n$ and $q > 1$; its Legendre transform is $W^*(\mathrm{y}) = \|\mathrm{y}\|_*^p/p - 1$ with $p = q/(q-1)$ and $\|\cdot\|_*$ the dual norm. We assume that negative povvers of W are integrable, so when W is convex this implies already that W is greater than $|x|$ at infinity. We actually require a slightly stronger super-linearity, which is trivially fulfilled in the applications of type (34).

Theorem (6.2.3) [203]: Let $n \geq 1$. Let $a \geq n$ {and $a > 1$ if $n = 1$) and let $W: \mathbb{R}^n \to (0, +\infty)$ such that

$$\int W^{-a} = 1$$

and

$$\exists\gamma > \max\left\{\frac{n}{a-1}{}'1\right\}, \qquad \lim_{X\to+} \inf \frac{W(x)}{|x|^\gamma} > 0. \tag{35}$$

For any positive function $g \in W^{1,1}_{loc}$ such that $g^{-a}|\nabla g|^{y/(\gamma-1)}$ is integrable and

$$\int g^{-a} = 1,$$

one has

$$(a-1)\int W^*(\nabla g)g^{-a}+(a-n)\int g^{1-a}\geq\int W^{1-a}. \quad (36)$$

Moreover, there is equality in (36) if $g=W$ and is convex.
Theorem (6.2.1) and the classical Sobolev inequalities correspond to the extremal case $a=n$. Much could be said regarding the assumptions on W and g in the theorem.

First, the condition {14) and $\int W^{-a}<+\infty$ ensure that $\int W^{1-a}<+\infty$. Actually, $W>0$ continuous (for instance convex) and (35) ensure that $\int W^{1-a}$ and $\int W^{-a}$ are finite.

Next, the integrability assumption $g^{-a}|\nabla g|^{V/(\gamma-1)}\in L^1(\mathbb{R}^n)$ is here for technical reasons, in order to justify an integration by parts; we believe that the correct assumption should simply be that $\int W^*(\nabla g)g^{-a}<+\infty$. Note that a convex W itself has no reason to match this integrability assumption (although it is $W_{loc}^{1,1}$). When we write that there is equality in (36) for $g=W$, it is by direct computation and integration by parts, as we shall see; then the assumption (35) appears as the natural requirement to justify the computation.
Note that the condition $\gamma>1$ in (35), already needed for the condition on g to make sense, ensures that W^* is well defined (i.e., finite) on $\mathbb{R}^n$.
Analogously, we assume that W is finite (i.e., the convex function W has a domain equal to $\mathbb{R}^n$) ; this prevents us from reaching the 1-homogeneous case $W^*(x)=C+\|x\|_*$, which corresponds to the L^1 Sobolev inequality. In this case, extremal functions are given by indicators of sets (given by the domain of W), and it requires to work with functions of bounded variation and related notions of capacity. Therefore, it is to be expected that this degenerate case should be treated separately when it comes to identifying the extremal functions.
Proof. Let ϕ be Brenier's map such that $\nabla\phi\# g^{-a}=W^{-a}$. Then, from (10), almost everywhere

$$W(\nabla\phi)=g(\det\nabla^2\phi)^{1/a}$$

Moreover, since $a\geq n$, from {11) with $k=1/a$ we have almost everywhere

$$(\det\nabla^2\phi)^{1/a}\leq 1-\frac{n}{a}+\frac{1}{a}\triangle\phi,$$

where here and below $\triangle\phi=\operatorname{tr}(\nabla^2\phi)$. Integrating with respect to the measure $g^{-a}dx$ we get

$$\int W(\nabla\phi)g^{-a}\leq\left(1-\frac{n}{a}\right)\int g^{1-a}+\frac{1}{a}\int\triangle\phi g^{1-a}.$$

Let us assume we can integrate by parts the and term; this only requires to put some suitable condition on g^{1-a} (in our situation ϕ is at least $W_{loc}^{2,1}$, see e.g., [208]). Actually, we can for instance establish, when $a>\gamma/(\gamma-1)$, the following sufficient inequality:

$$\int \triangle \phi g^{\mathrm{i}-a} \leq (a-1) \int \nabla \phi \cdot \nabla g g^{-a}. \tag{37}$$

Assuming (16) we have

$$a \int W(\nabla\phi) g^{-a} \leq (a-n) \int g^{1-a} + (a-1) \int \nabla g \cdot \nabla\phi g^{-a}.$$

But by definition of Legendre's transform

$$\nabla g \cdot \nabla \phi \leq W(\nabla\phi) + W^*(\nabla g)$$

so collecting terms we have

$$\int W(\nabla\phi) g^{-a} \leq (a-1) \int W^*(\nabla g) g^{-a} + (a-n) \int g^{1-a}.$$

Finally, $\int W(\nabla\phi) g^{-a} = \int W^{1-a}$ since $\nabla\phi \# g^{-a} = W^{-a}$, so we have

$$(a-1) \int W^*(\nabla g) g^{-a} + (a-n) \int g^{1-a} \geq \int W^{1-a} \tag{38}$$

as claimed.

This ends the proof of the inequality in the Theorem when $\gamma' := \gamma/(\gamma-1) < a$, provided we justify the integration by parts (16). For this, we extend the argument in [80] that is given for $W(x) = 1 + \|x\|^V$ and $a = n$. We introduce the function $g_\varepsilon^{1-\frac{a}{\gamma}}(x) := \min \left\{ g^{1-\frac{a}{\gamma}}(x/(1-\varepsilon)), g^{1-\frac{a}{V}}(x)\chi(\varepsilon x) \right\}$ for a cutoff function χ , for instance such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1/2$ and $\chi(x) = 0$ is $|x| \geq 1$. The argument is then identical to the one in [80]; we firstjustify (37) for the function g_ε instead of g, then we let ε tend to 0. For this, a key fact is that the sequence $\nabla g_\varepsilon^{1-\frac{a}{\gamma}}$ is bounded in $L^{\gamma'}$. To see this fact we observe that the sequence $\nabla\left(g^{1-\frac{a}{\gamma}}(x/(1-\varepsilon))\right)$ is bounded in $L^{y'}$ by change of variable $\mathrm{y} = x/(1-\varepsilon)$. So is the sequence $\nabla\left(g^{1-\frac{a}{\gamma}}(x)\chi(\varepsilon x)\right)$ since

$$2^{1-y'} \int |\nabla\left(g^{\mathrm{i}-\frac{a}{\gamma}}(x)\chi(\varepsilon x)\right)|^{\gamma'} \mathrm{d}x$$

$$\leq \int |\nabla g^{1-\frac{a}{\gamma}}(x)|^{\gamma'} |\chi(\varepsilon x)|^{y'} \mathrm{d}x + \varepsilon^{\gamma'} \int g^{y'-a}(x) |\nabla\chi|^{\gamma'}(\varepsilon x) \mathrm{d}x.$$

There, for the 1st term, $|\chi| \leq 1$ and $\nabla g^{1-\frac{a}{\gamma}} \in L^{\gamma'}$ since $g^{-a}|\nabla g|^{y'} \in L^1$ by assumption.

Moreover, by Hölder's inequality for the power $a/(a-\gamma')$ with $a>\gamma'$ and then change of variable $y=\varepsilon x$ the 2nd term is bounded by

$$\varepsilon^{\gamma'\left(1-\frac{n}{a}\right)}\left(\int g^{-a}\right)^{1-\frac{\gamma'}{a}}\left(\int |\nabla\chi|^{a}\right)^{\frac{\gamma'}{a}}$$

and hence uniformly bounded in ε for $a\geq n$.
Next, we extend the result to the case $\gamma'\geq a$ by reducing to the previous case as follows. Fix any $s>a/(a-1)$, that is, $1<s' := s/(s-1)<a$. Define $W_\varepsilon(x) := Z_\varepsilon(W(x)+\varepsilon|x|^{S})$ with Z_ε such that $\int W_\varepsilon^{-a}=1$. Since $s'\leq\gamma'$, Hölder's inequality and the integrability of g^{-a} ensure that $g^{-a}|\nabla g|^{s'}$ is integrable. Therefore, g and W_ε match the hypotheses of the previous case, so (38) gives

$$(a-1)\int (W_\varepsilon)^*(\nabla g)g^{-a}+(a-n)\int g^{1-a}\geq\int W_\varepsilon^{1-a}.$$

Note that $Z_\varepsilon\to 1$ and $W_\varepsilon\to W$. The right-hand side converges to $\int W^{1-a}$ by dominated or monotone convergence. For the left-hand side, since $W_\varepsilon\geq Z_\varepsilon W$, we have $\int (W_\varepsilon)^*(\nabla g)g^{-a}\leq Z_\varepsilon\int W^*\left(\frac{\nabla g}{Z_\varepsilon}\right)g^{-a}$ that converges to $\int W^*(\nabla g)g^{-a}$ by dominated convergence. This gives the desired inequality (38) for W and g.
Finally, it is easily proved that equality holds in (38) when $g=W$ with W convex. In this case $\nabla\phi(x)=x$ in the argument above. The growth condition {14) allows to perform the integration by parts $(a-1)\int (x\cdot\nabla W)W^{-a}=n\int W^{1-a}$, which means equality in (37); together with the crucial relation 6.7), this ensures equality in the argument above and in (15). The companion uconcave" case is as follows. The notation are those given.For any nonnegative W we let $W_*(y)=\inf_{W(x)>0}\{x\cdot y-W(x)\}$. Note that W_* is a negative function in our case of interest when W is a nonnegative continuous function concave on its support.
Theorem (6.2.4)[203]: Let $n\geq 1$, $a>0$, and $W:\mathbb{R}^n\to[0,+\infty)$. Then for any compactly supported function $g:\mathbb{R}^n\to[0,+\infty)$ with $g^{a+1}\in W^{1,1}$ such that

$$\int g^{a}=\int W^{a}=1$$

we have

$$(a+1)\int (-W_*)(\nabla g)g^{a}-(a+n)\int g^{1+a}\geq\int W^{1+a}. \tag{39}$$

Moreover, there is equality if $g=W$, with W continuous on $\mathbb{R}^n$ and concave on its support.
Proof. The proof follows the previous one. Let ϕ be Brenier's map such that $\nabla\phi\#g^{a}=W^{a}$. Then, from {12), g^{a}-almost everywhere

$$W(\nabla\phi) = g(\det \nabla^2\phi)^{-1/a} \geq \left(1+\frac{n}{a}\right)g - \frac{1}{a}g \triangle \phi.$$

Integrating with respect to the measure $g^a \mathrm{d}x$ and then by parts, we find

$$\int W(\nabla\phi)g^a \geq \left(1+\frac{n}{a}\right)\int g^{a+1} + \frac{a+1}{a}\int g^a \nabla g \cdot \nabla\phi.$$

We obtain inequality {18) using the g^a- $a.e.$ inequality

$$\nabla g \cdot \nabla\phi \geq W(\nabla\phi) + W_*(\nabla g),$$

which is valid since $W(\nabla\phi(x))^a > 0$ for g^a-almost all x, and the fact that $\nabla\phi\#g^a = W^a$.

When $g = W$ and is continuous and concave on its support, the proof above with $\nabla\phi(x) = x$ gives equality at all steps. Note that integration by parts is valid because W is continuous and therefore equal to zero on $\partial\{W > 0\}$ and that we can invoke (30) in the last step.

We convex or concave generalizations of Theorem (6.2.1) (which is Theorem (6.2.3) for $a = n$), we now present two generalizations of Theorem (6.2.2).

The 1st one concerns the convex case.

Theorem (6.2.5)[203]: Let $a \geq n \geq 1$ (and $a > 1$ if $n = 1$) and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function.

Let also $g, W, H: \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0,1], t = 1 - s$, be such that

$$\forall x, y \in \mathbb{R}^n, H(sx + ty) \leq sg(x) + tW(y) \tag{40}$$

and $\int W^{-a} = \int g^{-a} = 1$. Then

$$\int \Phi(H)H^{-a} \geq s\int \Phi(g)g^{-a} + t\int \Phi(W)W^{-a}. \tag{41}$$

Observe that Theorem (6.2.2) is Theorem (6.2.5) in the case when $\Phi(x) = x$ and $a = n$, while the classical BBL inequality (26) is recovered for $\Phi(x) \equiv 1$ and $a = n$. There is a hierarchy between all the inequalities (41), and inequality (27) (when $a = n$) appears as the strongest one.

Proof. The theorem can be proved in two ways, following the ideas from R. J. McCann's or Γ. Barthe's PhDs [81], [210].

Let ϕ be Brenier's map such that $\nabla\phi\#g^{-a} = W^{-a}$. Then from the Monge-Ampère equation {10), we have that almost everywhere

$$W(\nabla\phi) = g \det(\nabla^2\phi)^{1/a}.$$

Moreover, it follows from the assumptions that Φ is non-decreasing and $x \mapsto \frac{\Phi(x)-\Phi(0)}{X}$ is non-increasing, so that $x \mapsto \Phi(x)x^{-a}$ is non-increasing.

This proof is a little bit formal since we use a change of variables formula without proof. However, it is useful to fix the ideas and helpful to follow the rigorous proof below. So, by the change of variable $z = sx + t\nabla\phi(x)$, and using both assumptions on Φ we have

$$\int \Phi(H)H^{-a} = \int \Phi\left(H\big(sx + t\nabla\phi(x)\big)\right) H^{-a}\big(sx + t\nabla\phi(x)\big) \det\big(s\mathrm{Id} + t\nabla^2\phi(x)\big)\mathrm{d}x$$

$$\geq \int \Phi\left(sg + tW(\nabla\phi)\right)\big(sg + tW(\nabla\phi)\big)^{-a} \det\left(s\mathrm{Id} + t\nabla^2\phi\right).$$

$$\geq \int \left[s\Phi(g) + t\Phi\big(W(\nabla\phi)\big)\right]\left(s + t\det\left(\nabla^2\phi\right)^{1/a}\right)^{-a} \det\left(s\mathrm{Id} + t\nabla^2\phi\right)g^{-a}.$$

Since $a \geq n$, the concavity of $\det^k$ with $k = 1/a$, recalled before {11), yields

$$\det\left(s\mathrm{Id} + t\nabla^2\phi\right) \geq \left(s + t\det\left(\nabla^2\phi\right)^{1/a}\right)^a \tag{42}$$

Finally, $\int \Phi\big(W(\nabla\phi)\big)g^{-a} = \int \Phi(W)W^{-a}$ by image measure property since $\nabla\phi\#g^{-a} = W^{-a}$. This concludes the argument, as

$$\int \Phi(H)H^{-a} \geq \int \left[s\Phi(g) + t\Phi\big(W(\nabla\phi)\big)\right] g^{-a} = s\int \Phi(g)g^{-a} + t\int \Phi(W)W^{-a}.$$

We use the idea of R. J. McCann. From [210], let $(\rho_t)_{t\in[0,1]}$ be the density of the path between g^{-a} and W^{-a} defined as follows: for each t, ρ_t is the density of the image measure of ρ_0 under $s\mathrm{Id} + tT = \nabla\phi_t$, where $\phi_t(x) = s\frac{|x|^2}{2} + t\phi(x)$, $x \in \mathbb{R}^n$. Then, using twice the associated Monge-Ampère equations (31) for $\rho_1 = \nabla\phi\#\rho_0$ and $\rho_t = \nabla\phi_t\#\rho_0$, together with the determinant inequality {21), we find that ρ_0-almost everywhere

$$\rho_t(\nabla\phi_t) \leq \big(sg + tW(\nabla\phi)\big)^{-a}.$$

Multiplying the inequality by $\big(sg + tW(\nabla\phi)\big)$, then $\rho_0 - \mathrm{a.e.}$

$$\Phi\big(sg + tW(\nabla\phi)\big)\rho_t(\nabla\phi_t) \leq \Phi\big(sg + tW(\nabla\phi)\big)\big(sg + tW(\nabla\phi)\big)^{-a}.$$

Hence, using that Φ is concave and $x \mapsto \Phi(x)x^{-a}$ nonincreasing, we get that $\rho_0 - \mathrm{a.e.}$

$$\left[s\Phi(g) + t\Phi\big(W(\nabla\phi)\big)\right]\rho_t(\nabla\phi_t) \leq \Phi\big(H(sx + \nabla\phi)\big)H(sx + \nabla\phi)^{-a}$$

$$= \Phi(H\big(\nabla\phi_t(x)\big)H\big(\nabla\phi_t(x)\big)^{-a}.$$

Since $\rho_t\big(\nabla\phi_t(x)\big) > 0$ for ρ_0-almost every x, we can rewrite the previous inequality as

$$s\Phi\big(g(x)\big) + t\Phi\left(W\big(\nabla\phi(x)\big)\right) \leq \frac{\Phi(H\big(\nabla\phi_t(x)\big)H\big(\nabla\phi_t(x)\big)^{-a}}{\rho_t\big(\nabla\phi_t(x)\big)}1_{\rho_t(\nabla\phi_t(x))>0}, \rho_0(x) - a.e.$$

Integrating with respect to $\rho_0 = g^{-a}$ we find, using $\nabla\phi\#g^{-a} = W^{-a}$, for the left-hand side

$$\int \left[s\Phi(g(x)) + t\Phi\left(W(\nabla\phi(x))\right)\right]\rho_0(x)\mathrm{d}x = s\int \Phi(g)g^{-a} + t\int \Phi(W)W^{-a}$$

and, using $\nabla\phi_t\#\rho_0 = \rho_t$, for the right-hand side

$$\int \left[\frac{\Phi(H(\nabla\phi_t(\mathrm{x}))H(\nabla\phi_t(x))^{-a}}{\rho_t(\nabla\phi_t(x))}1_{\rho_t(\nabla\phi_t(x))>0}\right]\rho_0(x)\mathrm{d}x = \int_{\{\rho_t>0\}}\Phi(H)H^{-a}\mathrm{dy}$$

$$\leq \int \Phi(H)H^{-a}.$$

This concludes the argument. $blacksquare$

The concave inequality in Theorem (6.2.4) also has a BBL formulation. We only state it for power functions Φ since the general case seems less appealing.

Theorem (6.2.6)[203]. Let $n \geq 1$ and $a > 0$. Let also $g, W, H : \mathbb{R}^n \to [0,+\infty]$ be Borel functions and $t \in [0,1]$ and $s = 1 - t$ be such that

$$\forall x, \mathrm{y} \in \mathbb{R}^n, H(sx + t\mathrm{y}) \geq sg(x) + tW(\mathrm{y}) \tag{43}$$

and $\int W^a = \int g^a = 1$. Then

$$\int H^{1+a} \geq s^{n+a+1}\int g^{1+a} + s^{n+a}t\int W^{1+a} + (n+a)s^{n+a}t\int g^{1+a}. \tag{44}$$

Inequality (41) is optimal in the sense that if $g = W$ and is convex, then one can exhibit a map H that depends on s such that inequality (41) is an equality. This is not the case for inequality (44) that is less powerful than (20). The linearization of (23), for t going to 0, becomes optimal and gives optimal Gagliardo-Nirenberg inequalities in the concave case.

Proof. We start as in the proof of Theorem (6.2.5), sticking to the 1st formal argument for size limitation. As above, the argument can be made rigorous following McCann's argument.

Let ϕ be Brenier's map such that $\nabla\phi\#g^a = W^a$. Then almost surely,

$$g^a = W(\nabla\phi)^a \det(\nabla^2\phi).$$

By assumption on ϕ and the concavity inequality (42) we have

$$\int H^{1+a} = \int H^{1+a}\left(sx + t\nabla\phi(x)\right)\det\left(s\mathrm{Id} + t\nabla^2\phi(x)\right)\mathrm{d}x$$

$$\geq \int \left(sg + tW(\nabla\phi)\right)^{1+a}\det\left(s\mathrm{Id} + t\nabla^2\phi\right)$$

$$\geq \int \left(sg + tW(\nabla\phi)\right)^{1+a}\left(s + t(\det\nabla^2\phi)^{1/n}\right)^n$$

Now we keep only the order zero and one terms in the Taylor expansion in t of both terms above;

$$(sg+tW(\nabla\phi)^{1+a}=(sg)l+a\left(1+\frac{t}{s}\frac{W(\nabla\phi)}{g}\right)^{1+a}$$

$$\geq s^{1+a}g^{i+a}+(a+1)s^{a}tg^{a}W(\nabla\phi)\,;$$

$$\left(s+t(\det\nabla^{2}\phi)^{1/n}\right)^{n}=s^{n}\left(1+\frac{t}{s}\left(\frac{g}{W(\nabla\phi)}\right)^{a/n}\right)^{n}\geq s^{n}+ns^{n-1}t\left(\frac{g}{W(\nabla\phi)}\right)^{a/n}$$

Hence,

$$\int H^{1+a}\geq s^{n+a+1}\int g^{1+a}+(1+a)s^{n+a}t\int g^{a}W(\nabla\phi)$$

$$+ns^{n+a}t\int g^{a}W(\nabla\phi)\left(\frac{g}{W(\nabla\phi)}\right)^{\frac{n+a}{n}}$$

Then in the last term we apply the inequality

$$nX^{\frac{n+a}{n}}\geq(n+a)X-a,X\geq0$$

with $X=g/W(\nabla\phi)$. We obtain the desired inequality.

BBL inequalities admit an equivalent dynamical formulation given by the largest possible function H given g and W. For that, consider the following inf-convolution, defined for functions $W,g:\mathbb{R}^{n}\to(0,+\infty]$, $h\geq0$, and $x\in\mathbb{R}^{n}$ by

$$O_{h}^{W}(g)(x)=\begin{cases}\inf\left\{g(\mathrm{y})+hW\left(\frac{x-\mathrm{y}}{h}\right)\right\} & \text{if } h>0,\\ g(x) & \text{if } h=0\end{cases}\tag{45}$$

or equivalently

$$O_{h}^{W}(g)(x)=\inf\{g(x-hz)+hW(z)\}.$$

Then the constraint (40) implies that the inf-convolution

$$H(x)=sO_{\mathrm{r}/s}^{W}(g)(x/s)\,,x\in\mathbb{R}^{n}$$

if the largest function H satisfying (40). From this observation, the Φ-BBL inequality (41) can be rewritten as follows.

Theorem (6.2.7) [203]: Let $a\geq n\geq1$ (and $a>1$ if $n=1$) and let $\Phi:\mathbb{R}^{+}\to\mathbb{R}^{+}$ be a concave function. Let also g, $W:\mathbb{R}^{n}\to[0,+\infty]$ be Borel functions such that $\int W^{-a}=\int g^{-a}=1$.

Then for any $h\geq0$ the Φ-BBL inequality (41) is equivalent to

$$(1+h)^{a-n}\int\Phi\left(\frac{1}{1+h}O_{h}^{W}(g)\right)o_{h}^{W}(g)^{-a}$$

$$\geq \frac{1}{1+h}\int \Phi(g)g^{-a} + \frac{h}{1+h}\int \Phi(W)W^{-a}. \quad (46)$$

In particular, when $a = n$ and $\Phi(x) = x$, the extended BBL inequality (27) is equivalent to

$$\forall h \geq 0, \int o_h^W(g)^{1-n} \geq \int g^{1-n} + h\int W^{1-n}. \quad (47)$$

Moreover, equality holds in inequalities (46) and (47) when $g = W$ and is convex.
For the equality case, note from (45) that

$$O_h^W(g)(x) = (1+h)W\left(\frac{X}{h+1}\right), \mathrm{x} \in \mathbb{R}^n$$

when $g = W$ and is convex. Hence, equality holds in (46) and (47) in this case, as claimed. Inequalities (25) and (47) are equalities when $h = 0$. Moreover, for $h \to 0$ we have in general that

$$O_h^W g = g - hW^*(\nabla g) + o(h)$$

so that Theorem (6.2.7) admits a linearization as a convex inequality. With the same conditions on the function Φ as in Theorem (6.2.7), from inequality (25) we obtain

$$\int W^*(\nabla g)\left(a\frac{\Phi(g)}{g} - \Phi'(g)\right)g^{-a} + \int \big((a-n+1)\Phi(g) - g\Phi'(g)\big)\, g^{-a}$$

$$\geq \int \Phi(W)W^{-a} \quad (48)$$

for a class of functions g and W (which we do not try to carefully describe for a general Φ) . Of course again inequality (48) is optimal; equality holds when $g = W$ and is convex. In the case $\Phi(x) = x$, it is shown how to deduce the inequality (15) of Theorem (6.2.3) (and therefore Theorem (6.2.1)) from (25) for a restricted class $\mathcal{F}^a$ of functions (g, W) , inspired by [171] and given and Definition (6.2.22). In the case of interest of the Sobolev inequality (44) for $W(x) = C(1 + \|x\|^q/q)$, $q = p/(p-1)$, it is shown in [171] how to recover the Sobolev inequality from this restricted class. For, it is classical to be sufficient to prove (44) for C^1 , nonnegative and compactly functions f, and this case can be recovered by using

$$g_\varepsilon(x) = \big(f(x) + \varepsilon(1 + \|x\|^q)^{(p-n)/p}\big)^{p/(p-n)} + \varepsilon(1 + \|x\|^q)$$

that is in the restricted class.

Remark (6.2.8)[203]. Likewise, the classical BBL inequality (6.5) admits the following dynamical formulation: if $W, g: \mathbb{R}^n \to (0, +\infty)$ are such that $\int W^{-n} = \int g^{-n} = 1$, then

$$\int o_h^W(g)^{-n} \geq 1, h \geq 0.$$

For h tending to 0 we recover the convexity inequality (36) with $a = n + 1$, namely

$$\int_{\mathbb{R}^n} \frac{W^*(\nabla g)}{g^{n+1}} \geq 0, \tag{49}$$

which had been derived in [206]. As can be seen from, this inequality implies the Gagliardo-Nirenberg inequalities only for the parameters $a \geq n+1$. In particular, it does not imply the Sobolev inequality, as pointed out in [171].
It has recently been proved in [217] that the two formulations(5) and (49) are in fact equivalent. The *concave* BBL inequality (44) also admits a dynamical formulation with the supconvolution instead of the inf-convolution. For $W, g : \mathbb{R}^n \to [0, +\infty)$ and $h \geq 0$ we let

$$R_h^W(g)(x) = \begin{cases} \sup_{x\in\mathbb{R}^n} \left\{ g(\mathrm{y}) + hW\left(\frac{x-\mathrm{y}}{h}\right)\right\} & \text{if } h > 0, \\ g(x) & \text{if } h = 0, \end{cases}$$

Then the constraint (22) implies that the best function H if given by the sup -convolution,

$$H(x) = sR_{t/s}^W(g)(x/s)\,, x \in \mathbb{R}^n.$$

From this observation, the uconcave" BBL inequality (23) admits the equivalent following dynamical formulation: if $\int W^a \,\mathrm{d}x = \int g^a \,\mathrm{d}x = 1$ then for all $h \geq 0$,

$$\int R_h^W(g)^{1+a} \geq \int g^{1+a} + h\int W^{1+a} + (n+a)h\int g^{1+a}. \tag{50}$$

Similarly to the convex case, inequality (39) can be recovered from (50) by taking the derivative in h, at $h = 0$.

A family of sharp Gagliardo-Nirenberg inequalities in $\mathbb{R}^n$ was first obtained by M. del Pino and J. Dolbeault in [181]. The family was generalized to an arbitrary norm in [80] by using the mass transport method proposed in [93].
The del Pino-Dolbeault Gagliardo-Nirenberg family of inequalities, which includes the Sobolev inequality, is a consequence. We prove in a rather direct and easy way that our extended BBL inequality (48) implies the del PinoDolbeault Gagliardo-Nirenberg family of inequalities, but also a new family. As recalled, S. Bobkov and M. Ledoux [171] have also derived the Sobolev inequality from the Brunn−Minkowski inequality, but we believe that our method is more intuitive than theirs.

Below, $||\cdot||$ denotes an arbitrary norm in $\mathbb{R}^n$ and for $\mathrm{y} \in \mathbb{R}^n$ we let $||\mathrm{y}||_* = \sup_{||\mathrm{x}||\leq 1} x \cdot \mathrm{y}$ its dual norm. Recall that the Legendre transform of $x \mapsto ||x||^q/q$ (with $q > 1$) is the function $\mathrm{y} \mapsto ||\mathrm{y}||_*^p/p$ for $1/p + 1/q = 1$.

Let $n \geq 1$, $a \geq n$ ($a > 1$ if $n = 1$), and $q > 1$. Let W be defined by

$$W(x) = \frac{||x||^q}{q} + C, x \in \mathbb{R}^n,$$

where the constant $C > 0$ is such that $\int W^{-a} = 1$. Then

$$W^*(\mathrm{y}) = \frac{||\mathrm{y}||_*^p}{p} - C, \mathrm{y} \in \mathbb{R}^n,$$

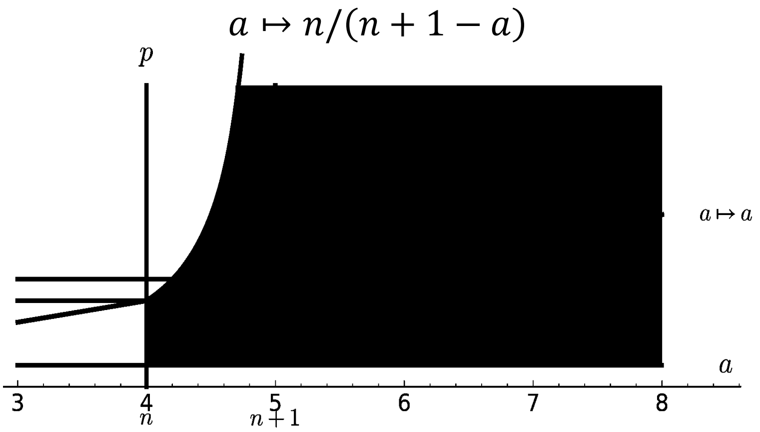

Fig. 1.[203] Ranges of admissible parameters (a, p) with $n = 4$.

where $1/p + 1/q = 1$.

We apply Theorem (6.2.3) with this fixed function W. First, let us notice that C is well defined and $\int W^{1-a}$ is finite whenever

$$\begin{cases} \text{If } a \geq n+1 \text{ then } p > 1 \\ \text{If } a \in [n, n+1) \text{ then } 1 < p < \dfrac{n}{n+1-a} = \overline{p}(\overline{p} = n \text{ when } a = n). \end{cases} \tag{51}$$

These constraints are illustrated in Figure 1 in the case $n = 4$; Equation (51) is satisfied whenever the couple (a, p) is in the gray or black area.

Let us note that, under (30), the condition (14) on W in Theorem (6.2.3) is satisfied with $\gamma = q$. Assuming that the parameters a and p are in this admissible set, then for any function $g: \mathbb{R}^n \to]0, +\infty[$ such that $\int g^{-a} = 1$ and $\nabla g^{1-\frac{a}{p}} \in IP$, inequality (36) in Theorem (6.2.3) becomes

$$D \leq \frac{a-1}{p} \int \frac{||\nabla g||_*^p}{g^a} + (a-n) \int g^{1-a}. \tag{52}$$

Here $D = (a-1)C + \int W^{1-a}$ is well defined, W and $a > 1$ being fixed. This inequality is the cornerstone.

Sobolev inequalities: As a warm up, let us consider the case $a = n, n \geq 2$, and $p \in (1, n)$. Then inequality (31) becomes

$$\frac{Dp}{n-1} \leq \int \frac{||\nabla g||_*^p}{g^n}$$

for any positive function g such that $\int g^{-n} = 1$ and $\nabla g^{1-\frac{n}{p}} \in IP$. Letting $f = g^{\frac{p-n}{p}}$, this gives

$$\frac{Dp}{n-1}|\frac{n-p}{p}|^p \leq \int ||\nabla f||_*^p$$

for any positive function f such that $\int f^{\frac{np}{n-p}} = 1$ and $\nabla f \in L^p$. Removing the normalization we get

$$\frac{Dp}{n-1}|\frac{n-p}{p}|^p \left(\int f^{\frac{np}{n-p}}\right)^{\frac{n-p}{n}} \leq \int ||\nabla f||_*^p.$$

The inequality is of course optimal since equality holds when $g = W$ or equivalently when $f(x) = \left(C + \frac{||x||^q}{q}\right)^{\frac{p-n}{p}}$. Classically removing the sign condition we recover

Theorem (6.2.9)[203]. Let $n \geq 2, p \in (1,)$, and $p^* = np/(n-p)$. The inequality

$$\left(\int |f|^{p^*}\right)^{\frac{1}{p^*}} \leq C_{n,p}\left(\int ||\nabla f||_*^p\right)^{\frac{1}{p}}$$

holds for any function $f \in L^{p^*}$ with $\nabla f \in IP$; here $C_{n,p}$ is the optimal constant reached by the function $x \mapsto (1 + ||x||^q)^{\frac{p-n}{p}}$

Gagliardo-Nirenberg inequalities: Consider now the case $a > n$ and $p \neq a$ satisfying conditions (51). Letting $h = g^{1-\frac{a}{p}} = g^{\frac{p-a}{p}}$, inequality (52) becomes

$$1 \leq D_2 \int ||\nabla h||_*^p + (a-n)\int h^{p\frac{a-1}{a-p}}$$

for any positive function h such that $\int h^{\frac{ap}{a-p}} = 1$ and $\nabla h \in IP$, where D_2 is an explicit positive constant. Removing the normalization, the inequality becomes

$$\left(\int h\frac{ap}{a-p}\right)^{\frac{a-p}{a}} \leq D_2\int ||\nabla h||_*^p + (a-n)\int h^{p\frac{a-1}{a-p}}\left(\int h\frac{ap}{a-p}\right)^{\frac{1-p}{a}}$$

for any positive h for which the integrals are finite.

To obtain a compact form of this inequality, we replace $h(x) = f(\lambda x)$ and optimize over $\lambda > 0$. For another explicit constant D_3 we get

$$\left(\int f^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}\left(1-\frac{1-p}{a-p}\omega\right)} \leq D_3\left(\int ||\nabla f||_*^p\right)^{\frac{1-\omega}{p}}\left(\int f^{p\frac{a-1}{a-p}}\right)^{\frac{a-p}{p(a-1)}\frac{a-1}{a-p}\omega}$$

where $=\left(1-\frac{1-p\omega}{a-p}\omega\right) = \frac{+n^{\in(0,1}(a-n-1)p+n)}{p(a-n)+n}$ and $\frac{a-1re}{a-p}\omega$.If $p < a$ the n both coefficients arepositive $\frac{p(a-n}{p(a\frac{-n)a}{a-p}}$). Thearenowtwocases,depending onthesignof as one can check by

considering the cases $a < n+1$ and $a \geq n+1$; this leads to the 1st case in Theorem (6.2.10) below. If $p > a$, then under the constraints (30) both coefficients are negative; this leads to the 2nd case below.
Removing the sign condition we have obtained the following:
Theorem (6.2.10)[203]. Let $n \geq 1$ and $a > n$.
For any $1 < p < a$, the inequality

$$\left(\int |f|^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}} \leq D^{+}_{n,p,a}\left(\int ||\nabla f||_*^p\right)^{\frac{\theta}{p}}\left(\int |f|^{p\frac{a-1}{a-p}}\right)^{\frac{a-p}{p(a-1)}(1-\theta)} \tag{53}$$

holds for any function f for which the integrals are finite. Here $\theta \in]0,1[$ is given by

$$\frac{a-p}{a} = \theta\frac{n-p}{n} + (1-\theta)\frac{a-p}{a-1} \tag{54}$$

and $D^{+}_{n,p,a}$ is the optimal constant given by the extremal function $x \mapsto (1+||x||^q)^{\frac{p-a}{p}}$

· If $p > a$ when $a \geq n+1$, or if $p \in \left(a, \frac{n}{n+1-a}\right)$ when $a \in [n, n+1)$, then the inequality

$$(\int [f|^{p\frac{a-1}{a-p}})^{\frac{a-p}{p(a-1)}} \leq D_{\overline{n},p,a}\left(\int ||\nabla f||_*^p\right)^{\frac{\theta'}{p}}\left(\int |f|^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}(1-\theta')} \tag{55}$$

holds for any function f for which the integrals are finite. Here $\theta' \in]0,1[$is given by

$$\frac{p-a}{a-1} = \theta'\frac{p-n}{n} + (1-\theta')\frac{p-}{a}a$$

and $D_{\overline{n},p,a}$ is the optimal constant given by the extremal function $x \mapsto (1+||x||^q)^{\frac{p-a}{p}}$

Let $n \geq 1$. Let $a > 0$ and > 1 , and define

$$W(x) = \frac{C}{q}(1-||x||^q)_+, x \in \mathbb{R}^n,$$

where C is such that $\int W^a = 1$. From definition {8}, we have

$$W_*(\mathrm{y}) = \begin{cases} -\frac{C^{1-p}}{p}||\mathrm{y}||_*^p - \frac{C}{q} & \text{if } ||\mathrm{y}||_* \leq C \\ -||\mathrm{y}||_* & \text{if } ||\mathrm{y}||_* \geq C, \end{cases} \quad \mathrm{y} \in \mathbb{R}^n,$$

where $1/p + 1/q = 1$. In particular from the Young inequality

$$W_*(\mathrm{y}) \geq -\frac{C^{1-p}}{p}||\mathrm{y}||_*,^{p}{}^{-\frac{C}{q}} \mathrm{y} \in \mathbb{R}^n. \tag{55}$$

Then the inequality (39) with this function W gives

$$(a+n)\int g^{1+a} \leq (a+1)\frac{C^{1-p}}{p}\int ||\nabla g||_*^p g^a + \frac{C}{q}(a+1) - \int W^{a+1}$$

for any nonnegative and compactly supported function g such that $\int g^a = 1$ and $g^{1+a} \in$

$W^{1,1}$. Let us notice that $\frac{C}{q}(a+1)-\int \quad W^{a+1}\,\mathrm{d}x>0$. Letting now $f=g^{\frac{a+p}{p}}$ we obtain

$$\int \quad f^{p\frac{1+a}{a+p}}\leq D_1\int \quad ||\nabla f||_*^p+D_2,$$

for any nonnegative and compactly supported function f such that $\int \quad f^{\frac{ap}{a+p}}=1$ and $f^{p\frac{1+a}{a+p}}\in W^{1,1}$; here D_1 and D_2 are explicit constants. Removing the normalization, this gives

$$\int \quad f^{p\frac{1+a}{a+p}}\leq D_1\int \quad ||\nabla f||_*^p\left(\int \quad f^{\frac{ap}{a+p}}\right)^{\frac{1-p}{a}}+D_2\left(\int \quad h^{\frac{ap}{a+p}}\right)^{\frac{1+a}{a}}$$

We can now remove the sign condition and optimize by scaling to recover the following result of [181] (and [80] for an arbitrary norm).

Theorem (6.2.11)[203]. Let $n\geq 1$. For any $p>1$ and $a>0$ the inequality

$$(\int \quad [f|^{p\frac{a+1}{a+p}})^{\frac{a+p}{p(a+1)}}\leq D_{n,p,a}\left(\int \quad ||\nabla f||_*^p\right)^{\frac{\theta}{p}}\left(\int \quad |f|^{\frac{ap}{a+p}}\right)^{\frac{a+p}{ap}(1-\theta)}$$

holds for any compactly supported function f with $\nabla f\in L^p$. Here $\theta\in]0,1[$is given by

$$\frac{a+p}{a+1}=\theta\frac{n-p}{n}+(1-\theta)\frac{a+p}{a}$$

and $D_{n,p,a}$ is the optimal constant given by the extremal function $x\mapsto(1-||x||^q)^{\frac{a+p}{+p}}$

The obtained inequality is optimal since (39) is an equality when $g=W$. When $g=W$, then almost surely $||\nabla g||_*\leq C$, so that (55) is an equality. We explain how our framework allows to recover known and obtain new trace Sobolev and Gagliardo-Nirenberg inequalities on $\mathbb{R}_+^n$, in sharp form. In the above denomination we shall restrict to the convex case. As before, we have two possible, equivalent, routes. One is to establish an abstract convex Sobolev type inequality using mass transport, and the other one is to establish a new functional Brunn-Minkowski type inequality on $\mathbb{R}_+^n$, and derive Sobolev inequalities from it, by linearization. Since the and one is formally more general although it requires technical, non-essential, assumptions on the functions), we will favor it.

Let us fix some notation. For any $n\geq 2$, we let

$$\mathbb{R}_+^n=\{z=(u,x),u\geq 0,x\in\mathbb{R}^{n-1}\}.$$

Then $\partial\mathbb{R}_+^n=\{(0,x),x\in\mathbb{R}^{n-1}\}=\mathbb{R}^{n-1}$. For $e=(1,0)\in\mathbb{R}\text{x}\mathbb{R}^{n-1}$ and $h\in\mathbb{R}$ we let

$$\mathbb{R}_{+he}^n=\mathbb{R}_+^n+he=\{(u,x),u\geq h,x\in\mathbb{R}^{n-1}\}.$$

The BBL inequality (41) with $\Phi(x)=x$ takes the following form in $\mathbb{R}_+^n$.

Theorem (6.2.12)[203]. Let $a\geq n$, $g:\mathbb{R}_+^n\to(0,+\infty)$ and $:\mathbb{R}_{+e}^n\to(0,+\infty)$ such that $\int_{\mathbb{R}_+^n}g^{-a}=\int_{\mathbb{R}_{+e}^n}W^{-a}=1$. Then, for all $h>0$,

$$(1+h)^{a-n}\int_{\mathbb{R}_{+he}^n}o_h^W(g)^{1-a}\geq\int_{\mathbb{R}_+^n}g^{1-a}+h\int_{\mathbb{R}_{+e}^n}W^{1-a},\qquad(56)$$

where, for $(u,x) \in \mathbb{R}^n_{+he}$,

$$O_h^W(g)(u,x) = \inf_{(v,y)\in\mathbb{R}^n_+, 0\le v\le u-h} \left\{g(v,\mathrm{y}) + hW\left(\frac{u-v}{h}, \frac{x-y}{h}\right)\right\}.$$

Moreover, (36) is an equality when $g(z) = W(z+e)$ for any $z \in \mathbb{R}^n_+$ and is convex.

Proof. Let $\tilde{g} : \mathbb{R}^n \to (0,+\infty]$ and $\widetilde{W} : \mathbb{R}^n \to (0,+\infty]$ be defined by

$$\tilde{g}(\mathrm{x}) = \begin{cases} g(x) \text{ if } x \in \mathbb{R}^n_+ \\ +\infty \text{ if } x \notin \mathbb{R}^n_+ \end{cases} \textit{and } \widetilde{W}(x) = \begin{cases} W(x) \text{ if } x \in \mathbb{R}^n_{+e} \\ +\infty \text{ if } x \notin \mathbb{R}^n_{+e}. \end{cases} \tag{57}$$

Then $\int_{\mathbb{R}^n} \tilde{g}^{-a} = \int_{\mathbb{R}^n} \widetilde{W}^{-a} = 1$. Hence, we can apply the dynamical formulation (25) of Theorem (6.2.5) with $\Phi(x) = x$ and the functions $\tilde{g}$, $\widetilde{W}$. For any $h \ge 0$ we obtain

$$(1+h)^{a-n} \int_{\mathbb{R}^n} o_h^{\widetilde{W}}(\tilde{g})^{1-a} \ge \int_{\mathbb{R}^n_+} g^{1-a} + h\int_{\mathbb{R}^n_{+e}} W^{1-a},$$

where

$$O_h^{\widetilde{W}}(\tilde{g})(u,x) = \inf\left\{\tilde{g}(V,y) + h\widetilde{W}\left(\frac{u-v}{h}, \frac{x-\mathrm{y}}{h}\right)\right\}, (u,x) \in \mathbb{R}^n.$$

From the definition of $\tilde{g}$ and $\widetilde{W}$, the infimum can be restricted to $0 \le v \le u-h$, so that $O_h^{\widetilde{W}}(\tilde{g})(u,x)$ is equal to $+\infty$ when $u < h$, and to $o_h^W(g)(x)$ otherwise. It implies

$$\int_{\mathbb{R}^n} o_h^{\overline{W}}(\overline{g})^{1-a} = \int_{\mathbb{R}^n_{+he}} o_h^{\overline{W}}(\tilde{g})^{1-a} = \int_{\mathbb{R}^n_{+he}} o_h^W(g)^{1-a},$$

which gives inequality (56).

When $g(z) = W(z+e)$ and W is convex, then by convexity

$$O_h^W(g)(u,x) = (h+1)W\left(\frac{u+1}{h+1}, \frac{X}{h+1}\right)$$

for any $(u,x) \in \mathbb{R}^n_{+he}$. Then inequality (36) is an equality. $blacksquare$ BBL type inequality on $\mathbb{R}^n$ implies a convex inequality. It is also the case on $\mathbb{R}^n_+$, by computing the derivative of (56) at $h = 0$ and using the identity

$$\int_{\mathbb{R}^n_{+he}} o_h^W(g)^{1-a} = \int_h^\infty \int_{\mathbb{R}^{n-1}} o_h^W(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x.$$

Assume now that (g,W) is in $\mathcal{F}^a_+$ as in Definition (6.2.18). Then by Theorem (6.2.21),

$$\frac{d}{dh}|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} o_h^W(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x = -\int_{\partial\mathbb{R}^n_+} g^{\mathrm{i}-a}\,\mathrm{d}x + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}\mathrm{d}z,$$

where we recall the definition of the Legendre transform

$$W^*(\mathrm{y}) = \inf_{x\in\mathbb{R}^n_{+e}} \{x\cdot \mathrm{y} - W(x)\}, \mathrm{y} \in \mathbb{R}^n. \tag{58}$$

So we have obtained the following:

Proposition (6.2.13)[203]. Let $a \ge n$. Let : $\mathbb{R}^n_+ \to (0,+\infty)$ and $W : \mathbb{R}^n_{+e} \to (0,+\infty)$ belong to $\mathcal{F}_+^a$ (see Definition (6.2.18)) with W convex and $\int_{\mathbb{R}_+^n} g^{-a} = \int_{\mathbb{R}_{+e}^n} W^{-a} = 1$. Then

$$(a-1)\int_{\mathbb{R}_+^n} \frac{W^*(\nabla g)}{g^a} + (a-n)\int_{\mathbb{R}_+^n} g^{\mathrm{i}-a} \geq \int_{\mathbb{R}_{+e}^n} W^{1-a}\,\mathrm{dz} + \int_{\partial\mathbb{R}_+^n} g^{1-a}. \tag{59}$$

Moreover, (39) is an equality when $g(z) = W(z+e)$ for $z \in \mathbb{R}_+^n$ and is convex.

Remark (6.2.14)[203]: Inequality (39) can also be directly proved by mass transport and integration by parts.

We follow to get trace Gagliardo-Nirenberg inequalities from Proposition (6.2.13). Let $a \geq n, p \in (1, \mathrm{n})$, and $q = p/(p-1)$. Let also $W(z) = C\frac{||z||^q}{q}$ for $z \in \mathbb{R}_{+e}^n$, where the constant $C > 0$ is such that $\int_{\mathbb{R}_{+e}^n} W^{-a} = 1$. We first observe that Conditions (Cl) and ($C2$) of Definition (6.2.18) hold with $\gamma = q$ since $q > n/(a-1)$ for $a \geq n$. Moreover, for $\mathrm{y} \in \mathbb{R}^n$,

$$W^*(\mathrm{y}) = \sup_{x\in\mathbb{R}_{+e}^n} \left\{x\cdot\mathrm{y} - C\frac{||x||^q}{q}\right\} \leq \sup_{x\in\mathbb{R}_{+e}^n} \left\{x\cdot\mathrm{y} - C\frac{||x||^q}{q}\right\} = C^{1-p}\frac{||\mathrm{y}||_*^p}{p}. \tag{60}$$

Hence, Proposition (6.2.13) implies

$$C^{1-p}\frac{a-1}{p}\int_{\mathbb{R}_+^n} \frac{||\nabla g||_*^p}{g^a} + (a-n)\int_{\mathbb{R}_+^n} g^{\mathrm{i}-a} \geq \int_{\mathbb{R}_{+e}^n} W^{1-a} + \int_{\partial\mathbb{R}_+^n} g^{\mathrm{i}-a} \tag{61}$$

for any function g satisfying $\int_{\mathbb{R}_+^n} g^{-a} = 1$ and ($C3$) and ($C4$) with $\gamma = q$, so that (g, W) belongs to $\mathcal{F}_+^a$.

It has to be mentioned that inequality (61) is still optimal, despite inequality {40). For, when $g(x) = W(x+e)$ for $\in \mathbb{R}_+^n$, then the minimum in (6.59) at the point $\nabla g(x)$ is reached in $\mathbb{R}_{+e}^n$ and then (40) is an equality.

Inequality (61) is again the cornerstone.

Trace Sobolev inequalities: Again, as a warm up, let us assume that $a = n$. Then (61) gives

$$\int_{\partial\mathbb{R}_+^n} g^{1-n} \leq C^{1-p}\frac{n-1}{p}\int_{\mathbb{R}_+^n} \frac{||\nabla g||_*^p}{g^n} - \int_{\mathbb{R}_{+e}^n} W^{1-n}$$

for any function g satisfying $\int_{\mathbb{R}_+^n} g^{-n} = 1$ and ($C3$) and (C4) for $\gamma = q$. For $f = g^{\frac{p-n}{p}}$, so that $\int_{\mathbb{R}_+^n} f^{\frac{pn}{n-p}} = 1$, this inequality becomes

$$\int_{\partial\mathbb{R}_+^n} f\frac{p(n-1)}{n-p} \leq C^{1-p}\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p\int_{\mathbb{R}_+^n} ||\nabla f||_*^p - \int_{\mathbb{R}_{+e}^n} W^{1-n}. \tag{62}$$

We now need to extend this inequality to all C^1 and compactly supported functions f in $\mathbb{R}_+^n$ (it does not mean that f vanishes in $\partial\mathbb{R}_+^n$) . For this, consider a C^1 and compactly supported function f in $\mathbb{R}_+^n$ and let

$$f_\varepsilon(x) = \varepsilon|x+e|^{-\frac{n-p}{p-1}} + c_\varepsilon f(x)\,,$$

where c_ε is such that $\int_{\mathbb{R}^n_+} f_\varepsilon^{\frac{pn}{n-p}} = 1$. Then $g_\varepsilon = f_\varepsilon^{-\frac{p}{n-p}}$ satisfies (*C3*) and (*C4*). Moreover, $c_\varepsilon \to 1$ when ε goes to 0 and then inequality (62) holds for the function f. Removing the normalization in (62) we have, for any f,

$$\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}}\,\mathrm{d}x \le A\int_{\mathbb{R}^n_+} ||\nabla f||_*^p\,\mathrm{d}z\beta^{\overline{p}-p} - B\beta^{\overline{p}},$$

where

$$\tilde{p} = \frac{p(n-1)}{n-p}\quad,\quad \beta = \left(\int_{\mathbb{R}^n_+} f^{\frac{pn}{n-p}}\mathrm{d}z\right)^{\frac{n-p}{np}}\quad A = C^{1-p}\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p \quad \text{and} \quad B = \int_{\mathbb{R}^n_{+e}} W^{1-n}\,\mathrm{d}z.$$

Equivalently, with $u = \frac{\tilde{p}}{p} = \frac{n-1}{n-p}$ and $v = \frac{\tilde{p}}{\tilde{p}-p}$ (which satisfy $u, v > 1$, and $1/u + 1/v = 1$),

$$\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}} \le B_V\left[\frac{A}{BV}\int_{\mathbb{R}^n_+} ||\nabla f||_*^p\beta^{\tilde{p}-p} - \frac{1}{V}\beta^{\tilde{p}}\right].$$

Now the Young inequality $xy \le x^u/u + y^V/v$ with

$$x = \frac{A}{B_V}\int_{\mathbb{R}^n_+} ||\nabla f||_*^p \;and\; y = \beta^{\overline{p}-p}$$

yields

$$\left(\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}}\right)^{1/\tilde{p}} \le \frac{A^{1/p}}{(BV)^{\frac{p-1}{p(n-1)}}}\left(\frac{n-p}{n-1}\right)^{\frac{n-p}{p(n-1)}}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p\right)^{1/p}$$

The proof of optimality it is a little bit technical and will be given below in the more general case of Theorem (6.2.16). It is also given in [113]. Equality holds when $g(z) = W(z+e)$ or equivalently when $f(z) = \left(C\frac{||z+e||^q}{q}\right)^{-\frac{n-p}{p}} = C'||z+e||^{-\frac{n-p}{p-1}}$ for $z \in \mathbb{R}^n_+$. Removing the sign condition we have thus obtained the following result by B. Nazaret [113], who promoted the idea of adding a vector e to the map W. We proved the main inequality for C^1 and compactly supported functions, but by approximation it is possible to extend it to the appreciate space.

Theorem (6.2.15)[203]: For any $1 < p < n$ and for $\tilde{p} = p(n-1)/(n-p)$ the Sobolev inequality

$$\left(\int_{\partial\mathbb{R}^n_+} |f|^{\tilde{p}}\mathrm{d}x\right)^{1/\tilde{p}} \le D_{n,p}\left(\int_{\mathbb{R}^n_+} ||\nabla f||_*^p\,\mathrm{d}z\right)^{1/p}$$

holds for any C^1 and compactly supported function f on $\mathbb{R}^n_+$. Here $D_{n,p}$ is the optimal constant given by the extremal function

$$h_p(z) = ||z+e||^{-\frac{n-p}{p-1}}, z \in \mathbb{R}^n_+.$$

Trace Gagliardo-Nirenberg inequalities: Assume now that $a \geq n > p > 1$ and let $h = g^{\frac{p-a}{p}}$. Then the inequality (41) can be written as

$$\int_{\partial\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}}\,\mathrm{d}x$$

$$\leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \mathrm{d}z + (a-n)\int_{\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}}\,\mathrm{d}z$$

$$-\int_{\mathbb{R}^n_{+e}} W^{1-a}\,\mathrm{d}z$$

for any C^1 and compactly supported function h in $\mathbb{R}^n_+$ such that $\int_{\mathbb{R}^n_+} h^{\frac{ap}{a-p}} = 1$. In this case we use the same argument as for the Sobolev inequality above to replace the Conditions (*C3*) and (*C4*) of Definition (6.2.18).
Removing the normalization, we get

$$\int_{\partial\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} \leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \beta^{p\frac{p-1}{a-p}}$$

$$-\int_{\mathbb{R}^n_{+e}} W^{1-a}\,\beta^{p\frac{a-1}{a-p}} + (a-n)\int_{\mathbb{R}^n_+} h^{p\frac{a-1}{a-p}} \tag{63}$$

with now

$$\beta = \left(\int_{\mathbb{R}^n_+} h^{\frac{pa}{a-p}}\,\mathrm{d}z\right)^{\frac{a-p}{ap}}$$

Let $u = \frac{a-1}{a-p}$ and $v = \frac{a-1}{p-1}$ that satisfy $u,\ v > 1$ and $1/u + 1/v = 1$. As for the Sobolev inequality we rewrite the right-hand side of (63) as

$$C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\mathbb{R}^n_+} ||\nabla h||_*^p \beta^{p\frac{p-1}{a-p}} - \int_{\mathbb{R}^n_{+e}} W^{1-a}\,\beta^{p\frac{a-1}{a-p}}$$

$$= BV\left[\frac{A}{BV}\int_{\mathbb{R}^n_+} ||\nabla h||_*^p \beta^{p\frac{p-1}{a-p}} - \frac{1}{V}\beta^{p\frac{a-1}{a-p}}\right],$$

with

$$A = C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \textit{ and } B = \int_{\mathbb{R}^n_{+e}} W^{1-a}.$$

From the Young inequality applied to the parameters u, v and

$$x=\frac{A}{BV}\int_{\mathbb{R}_+^n}||\nabla h||_*^p\ and\ \mathrm{y}=\beta^{p\frac{p-1}{a-p}} \tag{64}$$

we get

$$C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p\int_{\mathbb{R}_+^n}||\nabla h||_*^p\beta^{p\frac{p-1}{a-p}}-\int_{\mathbb{R}_{+e}^n}W^{1-a}\,\beta^{p\frac{a-1}{a-p}}$$

$$\leq\frac{A^{\frac{a-1}{a-p}}}{(B_V)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}_+^n}||\nabla h||_*^p\right)^{\frac{a-1}{a-p}} \tag{65}$$

and then

$$\int_{\partial\mathbb{R}_+^n}h^{p\frac{a-1}{a-p}}\,\mathrm{d}x\leq\frac{A^{\frac{a-1}{a-p}}}{(B_V)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}_+^n}||\nabla h||_*^p\mathrm{d}z\right)^{\frac{a-1}{a-p}}+(a-n)\int_{\mathbb{R}_+^n}h^{p\frac{a-1}{a-p}}\,\mathrm{d}z \tag{66}$$

from (43). For any $\lambda>0$, we replace $h(z)=f(\lambda z)$ for $z\in\mathbb{R}_+^n$. We obtain

$$\int_{\partial\mathbb{R}_+^n}f^{p\frac{a-1}{a-p}}\,\mathrm{d}x$$

$$\leq\lambda^{\frac{(a-n)(p-1)}{a-p}}\frac{A^{\frac{a-1}{a-p}}}{(B_V)^{\frac{p-1}{a-p}}}\frac{a-p}{a-1}\left(\int_{\mathbb{R}_+^n}||\nabla f||_*^p\mathrm{d}\mathcal{Z}\right)^{\frac{a-1}{a-p}}$$

$$+\lambda^{-1}(a-n)\int_{\mathbb{R}_+^n}f^{p\frac{a-1}{a-p}}\,\mathrm{d}z.$$

Taking the infimum over $\lambda>0$ gives

$$\left(\int_{\partial\mathbb{R}_+^n}f^{p\frac{a-1}{a-p}}\,\mathrm{d}x\right)^{\frac{a-p}{p(a-1)}}\leq D\left(\int_{\mathbb{R}_+^n}||\nabla f||_*^p\mathrm{d}\mathcal{Z}\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}_+^n}f^{p\frac{a-1}{a-p}}\,\mathrm{d}z\right)^{(1-\theta)\frac{a-p}{p(a-1)}}$$

for an explicit constant D and $\theta\in]\,0,1]$ being the unique parameter satisfying

$$\frac{n-1}{n}\frac{a-p}{a-1}=\theta\frac{n-p}{n}+(1-\theta)\frac{a-p}{a-1}. \tag{67}$$

Removing the sign condition, we have obtained the following:

Theorem (6.2.16)[203]: For any $a\geq n>p>1$, the Gagliardo-Nirenberg inequality

$$\left(\int_{\partial\mathbb{R}_+^n}|f|^{p\frac{a-1}{a-p}}\mathrm{d}x\right)^{\frac{a-p}{p(a-1)}}$$

$$\leq D_{n,p,a}\left(\int_{\mathbb{R}^n_+}||\nabla f||_*^p \mathrm{dz}\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}\mathrm{dz}\right)^{(1-\theta)\frac{a-p}{p(a-1)}} \tag{68}$$

holds for any C^1 and compactly supported function f on $\mathbb{R}^n_+$. Here θ is defined in (67) and $D_{n,p,a}$ is the optimal constant, reached when

$$f(z) = h_p(z) = ||z+e||^{-\frac{a-p}{p-1}}, z \in \mathbb{R}^n_+.$$

When $a = n$, then $\theta = 1$ and we recover the trace Sobolev inequality of Theorem (6.2.15).

Proof. From the above computation we only have to prove that the inequality (48) is optimal.

First, it follows from Proposition (6.2.13) that inequality {43) is an equality when

$$\forall z \in \mathbb{R}^n_+, h(z) = h_p(z) = ||z+e||^{-\frac{a-p}{p-1}},$$

the function h_p not needing to be normalized. Moreover, if inequality (65) is an equality, then it is also the case for (66) and then (68). So, we only have to prove that (65) is an equality when $h = h_p$ that sums up to the fact that the Young inequality is an equality. This is the case when $x = \mathrm{y}^{V-1}$ in (44), that is,

$$\frac{A}{BV}\int_{\mathbb{R}^n_+}||\nabla h||_*^p \mathrm{dz} = \left(\beta^{p\frac{p-1}{a-p}}\right)^{V-1}$$

or equivalently

$$\frac{A}{BV}\int_{\mathbb{R}^n_+}||\nabla h||_*^p \mathrm{dz} = \left(\int_{\mathbb{R}^n_+}||z+e||^{-\frac{ap}{p-1}}\right)^{\frac{a-p}{a}}$$

Let now $\mathcal{J}_\alpha = \int_{\mathbb{R}^n_+}||z+e||^{-\alpha}\mathrm{d}z$ for $\alpha > 0$. Then

$$C = \frac{p}{p-1}\mathcal{J}_{\frac{1/aap}{p-1}}, B = \mathcal{J}^{\frac{1-a}{p-1apa}}_{-}\mathcal{J}_{p\frac{a-1}{p-1}}, and \int_{\mathbb{R}^n_+}||\nabla h||_*^p \mathrm{dz} = \left(\frac{a-p}{p-1}\right)^p \mathcal{J}_{p\frac{a-1}{p-1}}$$

from their respective definition. Then, from the definition of A, equality in the Young inequality indeed holds. This finally gives equality for the map h. It has to be mentioned that the case $a = n$ gives the optimality of the trace Sobolev inequality of Theorem (6.2.15).

In their work [96] on Gagliardo-Nirenberg inequalities where only the Euclidean norm is considered), M. del Pino and J. Dolbeault observed that when the parameter a goes to infinity, the sharp Gagliardo-Nirenberg inequality (32) in $\mathbb{R}^n$ yields the IP-Euclidean logarithmic Sobolev inequality

$$\mathrm{Ent}_{dx}(f^p) \leq \frac{n}{p} \int_{\mathbb{R}^n} f^p \,\mathrm{d}x \log \left(\mathcal{L}_p \frac{\int \; ||\nabla f||_*^p \mathrm{d}x}{\int \; f^p \,\mathrm{d}x} \right) \tag{69}$$

for any positive function f. Here $1/p + 1/q = 1$, $\mathcal{L}_p$ is the optimal constant attained for $f(x) = e^{-\|x\|^q}$ and

$$\mathrm{Ent}_{dx}(f^p) := \int_{\mathbb{R}^n} f^p \; \log \frac{f^p}{\int \; f^p} \mathrm{d}x.$$

This bound is an instance, when $V(x) = ||x||^q + C$, of the following general inequality of [185], [209]; for any $V, f : \mathbb{R}^n \to (0, +\infty)$ such that $\int \; e^{-f} = \int \; e^{-V} = 1$, there holds

$$\int_{\mathbb{R}^n} \big(f + V^*(\nabla f)\big)\, e^{-f} \geq n, \tag{70}$$

with equality when $f = V$ and is convex. Inequality (70) has been derived in [185], [209] from the Prékopa-Leindler inequality, which is a consequence of the classical BBL inequality (5). It says that for $F, V, f : \mathbb{R}^n \to \mathbb{R}$ and $u \in [0,1]$ such that $\int e^{-f} = \int e^{-V} = 1$ and

$$\forall x, \mathrm{y} \in \mathbb{R}^n, F\big((1-u)x + uy\big) \leq (1-u)f(x) + uV(\mathrm{y}), \tag{71}$$

then

$$\int_{\mathbb{R}^n} e^{-F} \geq 1. \tag{72}$$

As above for the BBL inequalities, (72) can be rewritten in a dynamical form;

$$\int_{\mathbb{R}^n} e^{-\frac{1}{1+h} \mathrm{O}_h^V(\cdot)} \geq (1+h)^n, h \geq 0. \tag{73}$$

Then, as for above inequalities, this formulation can be linearized as $h \to 0$, recovering (70).
Our new BBL inequality (20) also yields the Prékopa-Leindler inequality (71) and (72) for $\Phi = 1$ and a going to infinity. For that, it suffices to apply (61) with $g = Z_g^{-1/a}(1 + f/a)$, $W = Z_V^{-1/a}(1 + V/a)$ for $Z_g = \int \; (1 + g/a)^{-a}$ and $Z_V = \int \; (1 + V/a)^{-a}$, $s = uZ_g^{1/a}/(uZ_g^{1/a} + (1-u)Z_V^{1/a})$ and $= (1 + F/a)/\big(uZ_g^{1/a} + (1-u)Z_V^{1/a}\big)$, and then to let$a$ go to infinity.
In the derivation of (49) from the sharp Gagliardo-Nirenberg inequality, the argument in

[96] is based on the key fact that the exponent θ in Equation (54) goes to 0 when $a \to +\infty$. In the case of the half-space $\mathbb{R}^n_+$, the exponent θ in Equation(47) goes to $1/p$ when $a \to +\infty$; hence, this method does not seem to adapt easily to the $\mathbb{R}^n_+$ case. Hence, to get a trace logarithmic Sobolev inequality in $\mathbb{R}^n_+$ we rather resort to the argument, as follows.

Let then $W: \mathbb{R}^n_{+e} \to \mathbb{R}$ and $g : \mathbb{R}^n_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^n_{+e}} e^{-W} = \int_{\mathbb{R}^n_+} e^{-g} = 1$, and define $\widetilde{W}$ and $\tilde{g}$ as in (57). Then by (73)

$$\int_{\mathbb{R}^n} e^{-\frac{1}{1+h} \mathrm{O}_h^{\widetilde{W}}(\tilde{g})}(z)\mathrm{d}z = \int_h^\infty \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{1+h} o_h^W(g)(u,x)} \mathrm{d}u\mathrm{d}x \geq (1+h)^n.$$

In the limit $h \to 0$ we get the bound corresponding to (70) in the trace case, namely

$$\int_{\mathbb{R}^n_+} \big(g + W^*(\nabla g)\big) e^{-g} \geq n + \int_{\partial\mathbb{R}^n_+} e^{-g} \tag{74}$$

whenever the function g is in an appropriate set of functions. We will not give here more details. As in again, let now > 1 , $||\cdot||$ be a norm in $\mathbb{R}^n$, and let $W(z) = C||z||^q/q$ for $z \in \mathbb{R}^n_{+e}$, where $C = \left(\int_{\mathbb{R}^n_+} e^{-\frac{||x||^q}{q}} \mathrm{d}x\right)^{q/n}$ is such that $\int_{\mathbb{R}^n_{+e}} e^{-W} = 1$. Then $W^*(\mathrm{y}) \leq C^{1-p}||\mathrm{y}||_*^p/p$ for $\mathrm{y} \in \mathbb{R}^n_+$, with $1/p + 1/q = 1$. Let then f be a positive function on $\mathbb{R}^n_+$ such that $\int_{\mathbb{R}^n_+} f^p = 1$, and apply inequality {54) to $g = -p\log f$. After removing the normalization we obtain

$$\mathrm{Ent}_{dx}(f^p) \leq \left(\frac{C}{p}\right)^{1-p} \int_{\mathbb{R}^n_+} ||\nabla f||_*^p \mathrm{d}x - n\int_{\mathbb{R}^n_+} f^p \,\mathrm{d}x - \int_{\partial\mathbb{R}^n_+} f^p \,\mathrm{d}x. \tag{75}$$

Inequality (75) is a trace logarithmic Sobolev inequality. It does not have a compact expression as does inequality (69) in the case of $\mathbb{R}^n$, where the scaling optimization can be performed. In $\mathbb{R}^n_+$, it improves upon the usual (69) if we consider functions on $\mathbb{R}^n_+$.

The time derivative of the Hopf-Lax formula (65) has been treated in different contexts, namely for Lipschitz (as in [182]) or bounded as in [216]) initial data. In our case the function g grows as $|x|^p$ with $p > 1$ at infinity and thus these classical results cannot be applied. We will thus follow the method proposed by S. Bobkov and M. Ledoux [171], extending it to more general functions W and also to the half-space $\mathbb{R}^n_+$.
We give all the details for the half-space $\mathbb{R}^n_+$ that are more intricate.

Let $a \geq n$ and let $g: \mathbb{R}^n_+ \to (0, +\infty)$, $W: \mathbb{R}^n_{+e} \to (0, +\infty)$ such that $\int_{\mathbb{R}^n_+} g^{-a}$ and $\int_{\mathbb{R}^n_{+e}} W^{-a}$ are finite. The functions g and W are assumed to be C^1 in the interior of their respective domain of definition. Moreover, we assume that W goes to infinity faster that linearly

$$\lim_{x\in\mathbb{R}^n_{+e},|x|\to\infty}\frac{W(x)}{|x|}=+\infty. \tag{76}$$

Our objective is to give sufficient conditions such that the derivative at $h=0$ of the function

$$\mathbb{R}^+\ni h\mapsto\int_h^\infty\int_{\mathbb{R}^{n-1}}o_h^W\,(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x$$

is equal to

$$-\int_{\partial\mathbb{R}^n_+}g^{\mathrm{i}-a}\,\mathrm{dz}+(a-1)\int_{\mathbb{R}^n_+}\frac{W^*(\nabla g)}{g^a}\mathrm{dz},$$

where

$$W^*(\mathrm{y})=\sup_{x\in\mathbb{R}^n_{+e}}\{x\cdot\mathrm{y}-W(x)\},\mathrm{y}\in\mathbb{R}^n. \tag{77}$$

For this, let us first recall the definition of $O_h^W g$; for $x\in\mathbb{R}^n_{+he}$,

$$O_h^W g(x)=\begin{cases}\inf_{y\in\mathbb{R}^n_+,x-y\in\mathbb{R}^n_{+he}}\left[g(\mathrm{y})+hW\left(\frac{x-\mathrm{y}}{h}\right)\right] & \text{if } h>0,\\ g(x) & \text{if } h=0\end{cases}$$

or equivalently, for $h>0$ and $x\in\mathbb{R}^n_{+he}$,

$$O_h^W g(x)=\inf_{y\in\mathbb{R}^n_{+e},x-hz\in\mathbb{R}^n_+}\{g(x-hz)+hW(z)\}$$

$$=\inf_{y\in\mathbb{R}^n_{+he},x-z\in\mathbb{R}^n_+}\left\{g(x-z)+hW\left(\frac{Z}{h}\right)\right\}$$

First, we have the following:

Lemma (6.2.17)[203]: In the above notation and assumptions, for all $x\in\mathbb{R}^n_+$ ∘

$$\frac{\partial}{\partial h}|_{h=0}o_h^W g(x)=-W^*\big(\nabla g(x)\big). \tag{78}$$

Proof. We follow and adapt the proof proposed in [171]. Let $x\in\mathbb{R}^n_+$ be∘ fixed.

By definition of $O_h^W g$, for any $z\in\mathbb{R}^n_{+e}$ and $h>0$ small enough so that $x-hz\in\mathbb{R}^n_+$, one has

$$\frac{O_h^W g(x)-g(x)}{h}\le\frac{g(x-hz)-g(x)}{h}+W(z).$$

Since g is C^1 , then for all $z\in\mathbb{R}^n_{+e}$

$$\lim_{h\to 0}\sup\frac{O_h^W g(x)-g(x)}{h}\le-\nabla g(x).z+W(z).$$

Then, from the definition (77) of W^*,

$$\lim_{h\to 0}\sup\frac{O_h^W g(x)-g(x)}{h}\le-W^*\big(\nabla g(x)\big).$$

We now prove the converse inequality. Let

$$A_{x,h}=\{z\in\mathbb{R}^n_{+e},hW(z)\le g(x-he)+hW(e)\}.$$

For a small enough $h>0$ such that $x-he\in\mathbb{R}^n_+$ we have $O_h^W g(x)\le g(x-he)+hW(e)$, so

$$O_h^W g(x) = \inf_{z\in A_x, x-hz\in\mathbb{R}_+^n} \{g(x-hz)+hW(z)\}.$$

Hence,

$$\frac{O_h^W g(x)-g(x)}{h} = \inf_{z\in A_x, x-hz\in\mathbb{R}_+^n} \left\{\frac{g(x-hz)-g(x)}{h}+W(z)\right\}$$

$$= \inf_{z\in A_x, x-hz\in\mathbb{R}_+^n}\{-\nabla g(x)\,.z+z\varepsilon_X(hz)\ +W(z)\},$$

where $\varepsilon_X(hz)\to 0$ when $hz\to 0$. It implies

$$\frac{O_h^W g(x)-g(x)}{h}\geq \inf_{z\in A_{x,h}} \{-\nabla g(x)\cdot z+z\varepsilon_X(hz)+W(z)\}.$$

By the coercivity condition {A1) on W and since g is locally bounded, the set $A_{\mathrm{x},h}$ is bounded by a constant C, uniformly in $h\in(0,1)$. In particular for every $\eta>0$, there

New Borell-Brascamp-Lieb inequalities and applications exists $h_\eta>0$ such that for all $h\leq$

h_η and $\in A_{x,h}$, $|\varepsilon_X(hz)|\leq\eta$. Moreover, for all $h\leq h_\eta$,

$$\frac{O_h^W g(x)-g(x)}{h}\geq \inf_{z\in A_{x,h}} \{-\nabla g(x)\,.z+W(z)\}-C\eta$$

$$\geq \inf_{z\in\mathbb{R}_{+e}^n} \{-\nabla g(x)\,.z+W(z)\}-C\eta$$

$$=-W^*\big(\nabla g(x)\big)-C\eta.$$

Let us take the limit when h goes to 0,

$$\lim_{h\to 0}\ \inf \frac{O_h^W g(x)-g(x)}{h}\geq -W^*\big(\nabla g(x)\big)-C\eta.$$

As η is arbitrary, we finally get equality (78). $blacksquare$

Our assumptions on the couple (g,W) are summarized in the following definition.

Definition (6.2.18)[203]: Let $n\geq 2$, $g:\mathbb{R}_+^n\to(0,+\infty)$, and $W:\mathbb{R}_{+e}^n\to(0,+\infty)$. We say that the couple (g,W) belongs to $\mathcal{F}_+^a$ with $a\geq n$ if the following four conditions are satisfied for some y:

(C1) $>$ max $\{\frac{n}{a-1}, 1\}$.

(C2) There exists a constant $A>0$ such that $W(x)\geq A|x|^\gamma$ for all $x\in\mathbb{R}_{+e}^n$.

(C3) There exists a constant $B>0$ such that $|\nabla g(x)|\leq B(|x|^{\gamma-1}+1)$ for all $x\in\mathbb{R}_+^n$. (C4) There exists a constant C such that $C(|x|^V+1)\leq g(x)$ for all $x\in\mathbb{R}_+^n$.

In the following, we let C_j denote several constants that are independent of $h>0$ and $x\in$

$\mathbb{R}_{+he}^n$, but may depend on γ , A, B.

Lemma (6.2.19)[203]: Assume $(Cl)\sim(C4)$. Then, we find a constant $h_1>0$ such that, for all $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$

$$-C_1 h(1 + |x|^\gamma) \le O_h^W g(x) - g(x) \le C_2 h(|x|^{y-1} + 1)\,.$$

Proof. i. Let us first consider the easier upper bound. For any $h > 0$ and $x \in \mathbb{R}^n_{+he}$ then $-he \in \mathbb{R}^n_+$, so that

$$O_h^W g(x) - g(x) \le g(x - he) - g(x) + hW(e)\,.$$

On the other hand, for any $x \in \mathbb{R}^n_+$ and $\mathrm{y} \in \mathbb{R}^n$ such that $x + \mathrm{y} \in \mathbb{R}^n_+$ we have from (*C3*),

$$|g(x + \mathrm{y}) - g(x)|$$

$$= |\int_0^1 \nabla g(x + \theta \mathrm{y}).\mathrm{y}\mathrm{d}\theta| \le |\mathrm{y}| \int_0^1 |\nabla g(x + \theta \mathrm{y})|\mathrm{d}\theta$$

$$\le C_3|\mathrm{y}|(|x|^{V-1} + |\mathrm{y}|^{\gamma-1} + 1) \quad (79)$$

From this remark applied to $\mathrm{y} = -he$ with $h \in (0,1)$, one gets for any $x \in \mathbb{R}^n_{+he}$

$$O_h^W g(x) - g(x) \le C_4 h(|x|^{\gamma-1} + 1) + hW(e) \le C_5 h(|x|^{V-1} + 1)\,. (80)$$

ii. For the lower bound, we first need some preparation. Thus, fix $h \in (0,1)$ and $x \in \mathbb{R}^n_{+he}$ arbitrarily. Let $\hat{\mathrm{y}} \in \mathbb{R}^n_{+he}$ be a minimizer of the infimum convolution

$$O_h^W g(x) = \inf \left[g(x - \mathrm{y}) + hW\left(\frac{\mathrm{y}}{h}\right)\right] = g(x - \hat{\mathrm{y}}) + hW\left(\frac{\hat{\mathrm{y}}}{h}\right).$$

Such a $\hat{y}$ surely exists by (*C2*) and (*C4*). From (80) and (*C2*) we have (recall that $h < 1$),

$$\frac{A}{hV - 1}|\hat{\mathrm{y}}|^V \le hW\left(\frac{\hat{\mathrm{y}}}{h}\right) \le g(x) - g(x - \hat{\mathrm{y}}) + C_5(|x|^{\gamma-1} + 1)\,. \quad (81)$$

From inequality {A4),

$$|g(x) - g(x - \hat{\mathrm{y}})| \le C_6|\hat{\mathrm{y}}|[|x|^{\gamma-1} + |\hat{\mathrm{y}}|^{V-1} + 1]. \quad (82)$$

From (82) and (81),

$$\frac{A}{hy - 1}|\hat{\mathrm{y}}|^\gamma \le C_6|\hat{\mathrm{y}}|(|x|^{y-1} + |\hat{\mathrm{y}}|^{\gamma-1} + 1) + C_5(|x|^{V-1} + 1)\,.$$

Choose a small constant $0 < h_1 \le 1$ so that

$$1 < \frac{A}{h_1^{\gamma-1}} - C_6.$$

When $0 < h < h_1$, we have

$$\frac{|\hat{\mathrm{y}}|^y}{|\hat{\mathrm{y}}| + 1} \le C_7[1 + |x|^{\gamma-1}]$$

so that

$$|\hat{\mathrm{y}}| \le C_8(1 + |x|)\,.$$

iii**.** Then, fix $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$ arbitrarily, where h_1 is the constant defined in Step By the arguments in Step 2, we see that

$$O_h^W g(x) - g(x) = \inf_{\mathrm{y}\in\mathbb{R}^n_{+he}, x-\mathrm{y}\in\mathbb{R}^n_+ |\mathrm{y}|\le C_8(1+|x|)} \left[g(x - \mathrm{y}) - g(x) + hW\left(\frac{\mathrm{y}}{h}\right)\right]. (83)$$

we have

$$g(x) - g(x - \mathrm{y}) \le |\mathrm{y}| \int_0^1 |\nabla g(x - \theta \mathrm{y})|\mathrm{d}\theta. \quad (84)$$

When $|y| \leq C_8(1+|x|)$ and $0 < \theta < 1$, we have $|x-\theta y| \leq (1+C_8)(1+|x|)$, so that $|\nabla g(x-\theta y)| \leq C_9(1+|x|^{V-1})$ by (*C3*), uniformly in $0 < \theta < 1$. Thus, when $|y| \leq C_8(1+|x|)$,
we have, by (A9),

$$g(x) - g(x-y) \leq C_9(1+|x|^{V-1})|y|.$$

Hence, by (83) and (*Cl*), we obtain

$$O_h^W g(x) - g(x) \geq \inf_{y \in \mathbb{R}^n_{+he}, |y| \leq C_8(1+|x|)} \left[-C_9(1+|x|^{\gamma-1})|y| + hW\left(\frac{y}{h}\right)\right]$$

$$\geq \inf_{y \in \mathbb{R}^n_{+he}, |y| \leq C_8(1+|x|)} \left[-C_9(1+|x|^{y-1})|y| + \frac{A}{h\gamma - 1}|y|^V\right]$$

$$\geq \inf_{y \in \mathbb{R}^n} \left[-C_9(1+|x|^{V-1})|y| + \frac{A}{hV - 1}|y|^\gamma\right]$$

$$= -C_{10}h(1+|x|^{V-1})^{\frac{\gamma}{\gamma-1}}.$$

The last equality is a direct computation. Therefore, we conclude that

$$O_h^W g(x) - g(x) \geq -C_{11}h(1+|x|^V) .$$

The proof is complete.

Lemma (6.2.20)[203]: Assume $(\mathrm{C1}) \sim (\mathrm{C4})$. Then, we find constants $C_0, h_2 > 0$ such that for all $h \in (0, h_2)$ and $x \in \mathbb{R}^n_{+he}$

$$|\frac{O_h^W g(x)^{1-a} - g(x)^{1-a}}{h}| \leq \frac{C_0}{1+|x|^{\gamma(a-1)}}.$$

Proof. First, for any $\alpha, \beta > 0$ and > 1 , then

$$|\alpha^{1-a} - \beta^{1-a}| \leq (a-1)|\alpha-\beta|(\alpha^{-a} + \beta^{-a}) . \quad (85)$$

Indeed, if for instance $\beta > \alpha > 0$, then for some $\theta \in (\alpha, \beta)$ we have

$$\alpha^{1-a} - \beta^{1-a} = (a-1)(\beta-\alpha)\theta^{-a} \leq (a-1)(\beta-\alpha)\alpha^{-a}.$$

By inequality (85) and Lemma (6.2.19), we have

$$|\frac{O_h^W g(x)^{1-a} - g(x)^{1-a}}{h}| \leq (a-1)|\frac{O_h^W g(x) - g(x)}{h}|[O_h^W g(x)^{-a} + g(x)^{-a}]$$

$$\leq K_1(1+|x|^\gamma)[O_h^W g(x)^{-a} + g(x)^{-a}]$$

for all $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$.

On the other hand, by (*C4*) and Lemma (6.2.19), we have for all $h \in (0, h_1)$ and $x \in \mathbb{R}^n_{+he}$

$$O_h^W g(x) \geq g(x) - C_1 h(1+|x|^\gamma) \geq (C - C_1 h)(|x|^V + 1) .$$

Choose a small constant h_3 so that

$$\frac{C}{2} \leq C - C_1 h_3$$

and let h_2 min $\{h_1, h_3\}$. Then, for all

$$O_h^W g(x) \geq \frac{C}{2}(|x|^V + 1)$$

whence, again using (*C4*),

$$|\frac{O_h^W g(x)^{1-a}-\left(g(x)\right)^{1-a}}{h}|\leq C_2(1+|x|^{\gamma})^{1-a}$$

for all $h\in(0,h_2)$ and $x\in\mathbb{R}^n_{+he}$.

We can now state and prove the main result.

Theorem (6.2.21)[203]: In the above notation, assume that the couple (g,W) is in $\mathcal{F}^a_+$. Then

$$\frac{d}{dh}|_{h=0}\int_h^{\infty}\int_{\mathbb{R}^{n-1}} o_h^W\,(g)^{1-a}(u,\mathrm{x})\mathrm{d}u\mathrm{d}\mathrm{x}=-\int_{\partial\mathbb{R}^n_+} g^{\mathrm{i}-a}\,\mathrm{d}x+(a-1)\int_{\mathbb{R}^n_+}\frac{W^*(\nabla g)}{g^a}\mathrm{d}z.$$

Proof. One can write the h-derivative as follows:

$$\frac{1}{h}\left(\int_h^{\infty}\int_{\mathbb{R}^{n-1}} o_h^W\,(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x-\int_{\mathbb{R}^n_+} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x\right)$$

$$=\frac{1}{h}\left(\int_h^{\infty}\int_{\mathbb{R}^{n-1}} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x-\int_{\mathbb{R}^n_+} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x\right)$$

$$+\frac{1}{h}\left(\int_h^{\infty}\int_{\mathbb{R}^{n-1}} o_h^W\,(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x-\int_h^{\infty}\int_{\mathbb{R}^{n-1}} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x\right).$$

First

$$\frac{1}{h}\left(\int_h^{\infty}\int_{\mathbb{R}^{n-1}} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x-\int_{\mathbb{R}^n_+} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x\right)=-\frac{1}{h}\int_0^{h}\int_{\mathbb{R}^{n-1}} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x,$$

which goes to $-\int_{\mathbb{R}^{n-1}} g^{1-a}\,(0,x)\mathrm{d}x=-\int_{\partial\mathbb{R}^n_+} g^{1-a}$ when h goes to 0. Secondly,

$$\frac{1}{h}\left(\int_h^{\infty}\int_{\mathbb{R}^{n-1}} o_h^W\,(g)^{1-a}(u,x)\mathrm{d}u\mathrm{d}x-\int_h^{\infty}\int_{\mathbb{R}^{n-1}} g^{1-a}\,(u,x)\mathrm{d}u\mathrm{d}x\right)$$

$$=\int_{\mathbb{R}^n_+}\left[\frac{O_h^W(g)^{1-a}(u,x)-g^{1-a}(u,x)}{h}\right]1_{u\geq h}\mathrm{d}u\mathrm{d}x. \quad (86)$$

By Lemma (6.2.17) the function in the right-hand side of (All) converges pointwisely to $W^*(\nabla g)g^{-a}$ as $h\to 0$. Moreover, since $\gamma(a-1)>n$, by Lemma (6.2.20) it is bounded uniformly in h by an integrable function. Hence, by the Lebesgue-dominated convergence Theorem the left-hand side of (All) converges (when $h\to 0$) to

$$(a-1)\int_{\mathbb{R}^n_+} W^*\,(\nabla g)g^{-a}.$$

The proof is complete.

We only give the result and conditions for the $\mathbb{R}^n$ case.

We let $g,W:\mathbb{R}^n\to(0,+\infty)$ such that g is C^1 and $\int\ g^{-n}=\int\ W^{-n}=1$.

Definition (6.2.22)[203]. Let $n:\mathbb{R}^n\to(0,+\infty)$ and $W:\mathbb{R}^n\to(0,+\infty)$. We say that the couple (g, W) belongs to $\mathcal{F}^n$ with $a \geq n(a > 1$ if $n = 1)$ if the following four conditions are satisfied for some y:

(C1) $\gamma > \max\{\frac{n}{a-1}, 1\}$.

(C2) There exists a constant $A > 0$ such that $W(x) \geq A|x|^\gamma$ for all $x \in \mathbb{R}^n$.
(C3) There exists a constant $B > 0$ such that $|\nabla g(x)| \leq B(|x|^{\gamma-1} + 1)$ for all $x \in \mathbb{R}^n$. (C4) There exists a constant C such that $C(|x|^\gamma + 1) \leq g(x)$ for all $x \in \mathbb{R}^n$.
Theorem (6.2.23)[203]. Assume that the couple (g, W) is in $\mathcal{F}^a$. Then, the derivative at $h = 0$ of the map

$$(0, +\infty) \ni h \mapsto \int \ o_h^W\ (g)^{1-a}$$

is equal to

$$(1-a)\int \ \frac{W^*(\nabla g)}{g^a}.$$

Section (6.3): Convex Cones and Convex Domains

The classical Sobolev inequality states that, for any function f sufficiently smooth and decaying fast enough at infinity, defined on the Euclidean space $\mathbb{R}^n$ with $n \geq 2$ (for instance, $f \in C_c^\infty(\mathbb{R}^n)$), and for any $p \in [1, n)$,

$$\|f\|_{L^{p^*}(\mathrm{R}^n)} \leq C\|\nabla f\|_{L^p(\mathrm{R}^n)}, p^* = \frac{pn}{n-p}, \tag{88}$$

Furthermore, equality is reached in inequality (87) if f can be written

$$f(x) = \left(1 + \|x\|^{p/(p-1)}\right)^{\frac{p-n}{p}}$$

up to a translation, a rescaling, and multiplication by a constant, where $\|.\|$ is the Euclidean norm. This was proved by Talenti [178] and Aubin [170] independently for $p = 2$. The Sobolev inequality can be seen as a corollary of a more general inequality, the Gagliardo-Nirenberg inequality, which states that

$$\|f\|_{L^q(\mathrm{R}^n)} \leq C\|\nabla f\|_{L^p(\mathrm{R}^n)}^\theta \|f\|_{L^r(\mathrm{R}^n)}^{1-\theta}, \tag{88}$$

for any $p \in [1, n)$, $q, r \in [1, +\infty]$, $\theta \in [0,1]$ such that

$$\frac{1}{q} = \left(\frac{1}{p} \ \ -\frac{1}{n}\right)\theta + \frac{1-\theta}{r};$$

whence the case $\theta = 1$ is exactly the Sobolev inequality. This family of inequalities has been notably investigated by del Pino and Dolbeault [8], who, studying the 1-parameter sub-family given by $p = 2$ and $r = q/2 + 1$, have not only found an explicit sharp constant, but also proved that there is equality if, and only if, f has the form

$$f(x) = (1 + \|x\|^2)^{\frac{2}{2-q}}$$

up to, once again, a translation, a rescaling, and multiplication by a constant.

As Bobkov and Ledoux [114] showed, these sharp inequalities can be reached within the framework of the Brunn-Minkovski theory [177]. With this approach, the sharp inequality follows in the more general case where the Euclidean norm is replaced by a generic norm on $\mathbb{R}^n$, which is a result already proved by Cordero-Erausquin, Nazaret, and Villani using optimal transport [173]. This makes sense, since the Brunn-Minkovski inequality directly implies the isoperimetric inequality, which is famously equivalent to the sharp Sobolev inequality with $p = 1$ (for a nice overview on this subject, see Osserman's article on the isoperimetric inequality [87]).

The key tool Bobkov and Ledoux use is an extended Borell-Brascamp-Lieb inequality, a quick proof of which using optimal transport is given by Bolley, Cordero-Erausquin, Fujita, Gentil and Guillin [81]. Let us state the Brunn-Minkoski inequality: for any compact nonempty subsets A and B in $\mathbb{R}^n$, and any $t \in [0,1]$

$$|tA + (1-t)B|^{1/n} \geq t|A|^{1/n} + (1-t)|B|^{1/n},$$

where $|.|$ denotes the Lebesgue measure on $\mathbb{R}^n$. This is to say that the volume, to the power $1/n,$ is concave with respect to the Minkowski sum, defined by $A + B = \{a + b, (a,b) \in A \times B\}$. The classical Borell-Brascamp-Lieb inequality [4], [5], just like the isoperimetric inequality, follows from the Brunn-Minkowski inequality. It is, in some sense, its functional counterpart: let $t \in [0,1]$ and $u, v, w\colon \mathbb{R}^n \to (0; +\infty]$ such that for all $x, y \in \mathbb{R}^n$,

$$w\big((1-t)x + ty\big) \leq \left((1-t)\big(u(x)\big)^{-1/n} + t\big(v(y)\big)^{-1/n}\right)^{-n}$$

then

$$\int w \geq \min\left(\int u, \int v\right).$$

Playing with the exponents and normalizing this inequality gives the following reformulation of the Borell-Brascamp-Lieb inequality: let , W, and $H : \mathbb{R}^n \to (0, +\infty]$, and $t \in [0,1]$, such that $\int g^{-n} = \int W^{-n} = 1$ and

$$\forall x, y \in \mathbb{R}^n, H\big((1-t)x + ty\big) \leq (1-t)g(x) + tW(y)$$

then

$$\int H^{-n} \geq 1. \tag{89}$$

Applying this inequality to the greatest function H meeting these criteria allows us to prove that

$$\int W^*(\nabla g) g^{-n-1} \geq 0, \tag{90}$$

where W^* is the Legendre transform of W. This inequality, turns out to be equivalent to the Borell-Brascamp-Lieb inequality we use here. This might look like it is to be expected, because of the semigroup structure that underlies the theorem, but is actually a little bit surprising, because said semigroup is not quite linear.

Inequality (90) can, in turn, be used to prove sharp Sobolev-type inequalities, but in the end proves to be limited as it does not allow to reach the full range of Gagliardo-Nirenberg inequalities showcased by del Pino and Dolbeault [174]. Thus, a better inequality to work with is the following extension of the Borell-Brascamp-Lieb inequality, which was proved by Bolley et al. [81].

Theorem (6.3.1)[218]: *Let* $n \geq 2$, *and* $t \in [0,1]$. *Let* , W, *and* H : $\mathbb{R}^n \to (0, +\infty]$ *be measurable functions such that* $\int g^{-n} = \int W^{-n} = 1$ *and*

$$\forall x, y \in \mathbb{R}^n, H\big((1-t)x + ty\big) \leq (1-t)g(x) + tW(y) \tag{91}$$

then

$$\int H^{1-n} \geq (1-t)\int g^{1-n} + t\int W^{1-n}.$$

With this theorem, we are able to prove sharp trace-Sobolev inequalities on convex domains. We prove sharp trace Sobolev in some convex domains, and sharp trace Gagliardo-Nirenberg inequalities in convex cones. In what follows, $\|.\|$ is a norm on $\mathbb{R}^n$, and $\|.\|_*$ is the dual norm, defined by $\|x\|_* = \sup_{\|y\|=1} x \cdot y$. In L^q norms of vector functions, the dual norm $\|.\|_*$ will be used. Let φ: $\mathbb{R}^{n-1} \to \mathbb{R}$ be a convex function such that $\phi(0) = 0$. We consider functions defined on ϕ's epigraph, that is $\Omega = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_2 \geq \phi(x_1)\}$. We say that Ω is a convex cone whenever ϕ is positive homogeneous of degree 1: for all $t > 0$ and $x_1 \in \mathbb{R}^{n-1}, \phi(tx_1) = t\phi(x_1)$.

Theorem (6.3.2))[218]: *Let* $a \geq n > p > 1$, *and* $\Omega = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_2 \geq \phi(x_1)\}$ *be a convex cone. There exists a positive constant* $D_{n,p,a}(\Omega)$ *such that for any non-negative function* $f \in C_c^\infty(\Omega)$,

$$\left(\int_{\mathrm{R}^{n-1}} f^q\left(x, \phi(x)\right)\mathrm{d}x\right)^{1/q} \leq D_{n,p,a}(\Omega)\|\nabla f\|_{L^p(\Omega)}^{\theta}\|f\|_{L^q(\Omega)}^{1-\theta}, \tag{92}$$

where

$$\theta = \frac{a-p}{p(a-n-1)+n}, q = p\frac{a-1}{a-p}.$$

Furthermore, *when* $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{a-p}{p-1}}$, *then* (92) *is an equality.*

The fact that there exists a function for which the equality is reached means that the constant $D_{n,p,a}(\Omega)$ may be computed explicitly. Choosing $a = n$, Theorem (6.3.2) immediately yields the sharp trace Sobolev inequality as a corollary:

Corollary (6.3.3))[218]: *Let* $n > p > 1$, *and* $\Omega = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_2 \geq \phi(x_1)\}$ *be a convex cone. There exists a positive constant* $D_{n,p}(\Omega) = D_{n,p,n}(\Omega)$ *such that for any non-negative function* $f \in C_c^\infty(\Omega)$,

$$\left(\int_{\mathrm{R}^{n-1}} f^{p\frac{n-1}{n-p}}\left(x,\phi(x)\right)\mathrm{d}x\right)^{\frac{n-\mathrm{p}}{\mathrm{p}(n-1)}} \leq D_{n,p}(\Omega)\|\nabla f\|_{L^p(\Omega)}, \qquad (93)$$

Furthermore, when $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{n-\mathrm{p}}{\mathrm{p}-1}}$, *then* (93) *is an equality.*

The case $\Omega = \mathbb{R}_+^n$ has already been studied by Nazaret [10].

If we only assume Ω to be convex, we prove, under some growth criteria on Ω, the following sharp weighted trace Sobolev inequality:

Theorem (6.3.4)[218]: *Let* $n > p > 1$, *and* $\Omega = \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R},\ x_2 \geq \phi(x_1)\}$ *be a convex set. There exists a positive constant* $D'_{n,p}(\Omega)$ *such that for any nonnegative function* $f \in C_c^\infty(\Omega)$,

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{n-1}{n-p}}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1 \leq D'_{n,p}(\Omega)\left(\int_{\Omega} \| \nabla f\|_*^p\right)^{\frac{n-1}{n-p}} \qquad (94)$$

where $(x_1) = 1 + \phi(x_1) - x_1 \cdot \nabla\phi(x_1)$. *Furthermore, when* $f(x) = \|(x_1, x_2 + 1)\|^{-\frac{n-\mathrm{p}}{p-1}}$, *then* (94) *is an equality.*

Once again, $D'_{n,p}(\Omega)$ can be computed explicitly. This inequality may be surprising, since the weight P can (and usually is, whenever Ω is not a cone) negative outside a compact neighbourhood of 0, but it is still sharp. For instance, with the set defined by $\phi(x) = \|x\|^2$, the weight becomes $P(x) = 1 - \|x\|^2$, which happens to be negative outside the unit ball. One may define $\partial\Omega_+ \subset \partial\Omega$ such that $\partial\Omega_+ = \{\left(x_1, \phi(x_1)\right), P(x_1) > 0\}$. In that case, inequality (94) restricted to functions $f \in C_c^\infty(\cup\, \partial\Omega_+)$ becomes a regular weighted inequality, with a positive weight. We first study the infimal convolution, which is the key tool in the proof of Theorems (6.3.2) and (6.3.4). Once some crucial properties are established, we prove the Claim (6.3.19)ed equivalence between the classical Borell-Brascamp-Lieb inequality (89) and its differentiated formulation (90), within some limitations. we move on to prove the main Theorems (6.3.2) and (6.3.4), starting from an improved version of the Borell-Brascamp-Lieb inequality. The technical details, which will be glided, can be found in the comprehensive

Let $t \in [0,1)$. To use Theorem (6.3.1), instead of considering any H such that

$$\forall x, y \in \mathbb{R}^n, H\left((1-t)x + ty\right) \leq (1-t)g(x) + tW(y) ,$$

we may well choose the greatest such function. That is,

$$H(z) = \inf_{(1-i)x+ty=zxy\in\mathrm{R}^n} \{(1-t)g(x) + tW(y)\},$$

or, writing $h = t/(1-t)$,

$$\frac{H(z)}{1-t} = \inf\left\{g\left(\frac{z}{1-t} - hy\right) + hW(y)\right\}.$$

This formula, being explicit, allows for some properties to be brought to light. It motivates the definition, and the study, of the so-called infimal convolution:

Definition (6.3.5)[218]: Let , $g:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. Their infimal convolute $f\square g:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$(fg)(x) = \inf\{f(y)+g(z), y+z=x\} = \inf\{f(y)+g(x-y)\}.$$

The infimal convolution of f with g is said to be *exact at* x if the infimum is achieved, and *exact* if it is exact everywhere.

With this definition, and whenever $h = t/(1-t) > 0$, the greatest function H in Theorem (6.3.1) is given by

$$H(z) = (1-t)\inf\left\{g\left(\frac{z}{1-t}-y\right)+hW(y/h)\right\} = (1-t)\big(g\square hW(./h)\big)\big(z/(1-t)\big),$$

we thus define

$$Q_h^W(g) = g\square hW(./h) = x \mapsto \inf\{g(x-y)+hW(y/h)\}.$$

Using Q_h^W in Theorem (6.3.1), inequality (91) becomes

$$\int Q_h^W(g)^{1-n} \geq \int g^{1-n} + h\int W^{1-n} \qquad (95)$$

but there exists a slightly more general version of this inequality, namely Theorem (6.3.16). To begin with, let us first showcase some properties of the infimal convolution. Here to build some intuition about infimal convolution, before proving specific results useful for the study of Q_h^W.

Definition (6.3.6)[218]: With any function $f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$, we associate its

· essential domain (usually shortened to domain), dom $f = \{x\in\mathbb{R}^n, f(x) < +\infty\}$;

· epigraph, epi $f = \{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}, f(x)\leq\alpha\}$;

· strict epigraph, $\mathrm{epi}_s f = \{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}, f(x)<\alpha\}$.

Furthermore, the function f is said to be proper if it is not equal to the constant $+\infty$.

With these definitions, we highlight in the next proposition the link between infimal convolution of functions and Minkowski sum of sets, classically defined for two sets A, B by $A+B=\{a+b,(a,b)\in A\times B\}$.

Proposition (6.3.7)[218]: *Let* , $g:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. *Then*

· dom $f\square g$ = dom f + dom g;

· $\mathrm{epi}_s f\square g = \mathrm{epi}_s f + \mathrm{epi}_s g$;

· epi $f\square g \supset$ epi f + epi g, *and equality holds if, and only if, the infimal convolution is exact at each* $x\in$ dom $f\square g$.

Proof of this proposition and more in-depth details on infimal convolutions can be found in Thomas Strömberg's thesis [13]. The more delicate question of regularity of the infimal convolution is only addressed in the particular study of $Q_h^W(g)$. That is because there is not *one* natural set of assumptions ensuring regularity, so it really depends on the goal, which, here, is that $Q_h^W(g)$ should be smooth enough to prove Sobolev inequalities. We only prove the following lemma in the most general case, since it is very useful.

Lemma (6.3.8)[218]: *Let* $f, g:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *be lower semicontinuous functions. If* f *is nonnegative and* g *is coercive, that is,*

$$\lim_{\|x\|\to+\infty} g(x) = +\infty,$$

then $f\square g$ is exact.

Proof. Fix $x \in \mathbb{R}^n$. Consider $\psi: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, y \mapsto f(x-y) + g(y)$ and assume that there exists y_0 such that $\psi(y_0) < +\infty$: ψ is lower semicontinuous, and greater than g, thus tends to $+\infty$ as $\|y\|$ goes to $+\infty$. As such, $\{y \in \mathbb{R}^n, \psi(y) \le \psi(y_0)\}$ is closed and bounded, thus compact. Now, let $(y_n) \subset \{\psi \le \psi(y_0)\}$ be a minimizing sequence, $\lim_{n\to+\infty} \psi(y_n) = \inf_{y\in\mathbb{R}^n}\{\psi(y)\}$. By compactness, we can assume that the sequence (y_n) converges towards $z \in \mathbb{R}^n$, and by lower semicontinuity, $-\infty < \psi(z) \le \lim_{n\to+\infty} \psi(y_n) = \inf_{y\in\mathbb{R}^n}\{\psi(y)\}$, thus the infimum is finite and is actually a minimum. If such a y_0 does not exist, then $f\square g(x) = +\infty$, and the infimum is also reached.

We begin here the specific study of $Q_h^W(g) = g\square hW(./h)$. The study of the regularity of $Q_h^W(g)$ with respect to $h > 0$ is crucial, because we would like to differentiate inequality (95) with respect to h. Let us first state some classical results about the Legendre transform. The proofs can be found in Evans' book, [9], and Brézis' book, [6].

Definition (6.3.9)[218]: The Legendre transform of W is defined by

$$W^*(y) = \sup\{x\cdot y - W(x)\} \in \overline{\mathbb{R}}.$$

By definition, W^* is a lower semicontinuous convex function, but it is not always proper. For W^* to be well behaved, we have to assume a little bit more about W. In fact, it is enough to assume W to be lower semicontinuous: indeed, if $W: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function, then W^* is also a lower semicontinuous proper convex function, and $(W^*)^* = W$. The infimal convolution is not only closely related to Minkovski sums, but also to Legendre transforms, as the next lemma shows.

Lemma (6.3.10)[218]: *Let $g, W: \mathbb{R}^n \to (-\infty, +\infty]$ be two measurable functions. If g is nonnegative and almost everywhere differentiable on its domain* dom $g = \Omega_0$ (*with nonempty interior*), *and W grows superlinearly,*

$$\lim_{|x|\to+\infty} \frac{W(x)}{|x|} = +\infty,$$

then for almost every $x \in \Omega_0^{\circ}$, $h \mapsto Q_h^W(g)(x)$ is differentiable at $h = 0$, and

$$\frac{\partial}{\partial h}|_{h=0} Q_h^W(g)(x) = -W^*(\nabla g(x)),$$

where W^* is the Legendre transform of W.

Proof. Let $\Omega_1 = \text{dom } W$, and fix $x \in \Omega_0^{\circ}$ such that the differential of g at x exists. Let $y \in \Omega_1$. For $h > 0$ sufficiently small, $x - hy \in \Omega_0$, and we get, by definition of $Q_h^W(g)$,

$$\frac{Q_h^W(g)(x) - g(x)}{h} \le \frac{g(x-hy) - g(x)}{h} + W(y) .$$

Taking the superior limit when $h \to 0$ yields

$$\lim_{h\to}\sup \frac{Q_h^W(g)(x)-g(x)}{h} \leq -\nabla g(x)\cdot y + W(y)\,.$$

This being true for any $y \in \Omega_1$, we may take the infimum to find that

$$\lim_{h\to}\sup \frac{Q_h^W(g)(x)-g(x)}{h} \leq -W^*\big(\nabla g(x)\big)\,.$$

Conversely, fix $e \in \Omega_1$, and $h_0 > 0$ such that $\overline{B(x,h_0\|e\|)} \in$. For $h \in (0,h_0)$, define

$$\Omega_{x,h} = \{y \in \Omega_1, hW(y) \leq g(x-he) + hW(e)\};$$

note that $e \in \Omega_{x,h}$. We Claim (6.3.19) that $\lim_{h\to\infty} \sup_{h\to 0}\left\{h\|y\|, y \in \Omega_{x,h}\right\} = 0$. Indeed, if $y \in \Omega_{x,h}$, then

$$h\|y\|\frac{W(y)}{\|y\|} \leq g(x-he) + hW(e) \leq \sup_{z\in\overline{B(x,h_0\|e\|)}} g(z) + h_0W(e)\,.$$

Now, when h goes to 0, either $\lim\sup\|y\| < +\infty$, or $\lim\sup\|y\| = +\infty$; in both cases, since $\lim_{|y|\to+\infty}\frac{W(y)}{\|y\|} = +\infty$, the Claim (6.3.19) is proved. Notice now that for all $h \in (0,h_0)$, $Q_h^W(g)(x) \leq g(x-he) + hW(e)$, hence $Q_h^W(g)(x) = \inf_{y\in\Omega_{x,h}}\{...\}$. Thus,

$$\frac{Q_h^W(g)(x)-g(x)}{h} = \inf_{y\in\Omega_{x,k}}\left\{\frac{g(x-hy)-g(x)}{h} + W(y)\right\}$$

$$= \inf_{y\in\Omega_{x,k}}\{-\nabla g(x)\cdot y + y\cdot\varepsilon_x(hy) + W(y)\}$$

where $\varepsilon_x(z) \to 0$ when $\|z\| \to 0$. Let $1 \geq \eta > 0$; the Claim (6.3.19) proves that there exists $h_\eta \in (0,h_0)$ such that for all $0 < h < h_\eta$, $\forall y \in \Omega_{x,h}$, $\|\varepsilon_x(hy)\| \leq \eta$. Thus,

$$\frac{Q_h^W(g)(x)-g(x)}{h} \geq \inf_{y\in\Omega_{x,k}}\{-\nabla g(x)\cdot y - \eta\|y\| + W(y)\}$$

$$= \inf_{\substack{y\in\Omega_{x,k}\\ y\in B(0,R)}}\{\dots\}$$

$$\geq \inf_{y\in\Omega_{x,k}}\{-\nabla g(x)\cdot y + W(y)\} - R\eta$$

$$\geq -W^*\big(\nabla g(x)\big) - R\eta,$$

where R was chosen such that $\|y\| \geq R \Rightarrow W(y) \geq (\|\nabla g(x)\|+1)\|y\| + W(e) - \nabla g(x)\cdot e$. Finally, taking the inferior limit of this inequality, and noticing that the result stays true for any $0 < \eta \leq 1$, we may conclude (since R is independent from η) that

$$\lim_{h\to 0}\frac{Q_h^W(g)(x)-g(x)}{h} = -W^*\big(\nabla g(x)\big)\,.$$

This differentiation result is enough to prove the main theorems contained but we can go a little bit further with more assumptions on g and W. Assuming W to be convex bestows upon Q_h^W a semigroup structure:

Lemma (6.3.11)[218]: *Assume that* $g: \mathbb{R}^n \to [0, +\infty]$ *is lower semicontinuous, and that* W *is a lower semicontinuous proper convex function such that* $\lim W(x) = +\infty$. *Then, for all* $x \in \mathbb{R}^n$ *and* $0 < s < h$,

$$Q_h^W(g)(x) = \min_{y \in \mathbb{R}^n} \{g(x - hy) + hW(y)\}$$

$$= Q_{h-s}^W\big(Q_s^W(g)\big)(x)\,.$$

Proof. Exactness was already proved in Lemma (6.3.8). Notice that

$$Q_{h-s}^W\big(Q_s^W(g)\big)(x) = \inf_{y \in \mathbb{R}} \inf_{z \in \mathbb{R}} \{g(x - (h-s)y - sz) + (h-s)W(y) + sW(z)\}$$

$$\le \inf_{y \in \mathbb{R}} \{g(x - hy) + hW(y)\} = Q_h^W(g)(x)\,.$$

Conversely, let $y \in \mathbb{R}^n$, and choose $z \in \mathbb{R}^n$ such that

$$Q_s^W(g)(x - (t-s)y) = g(x - sz) + sW(z)\,.$$

Then, by convexity,

$$Q_t^W(g)(x) \le g(x - (t-s)y - sz) + tW\left(\frac{t-s}{t} y + \frac{s}{t} z\right)$$

$$\le g(x - (t-s)y - sz) + (t-s)W(y) + sW(z)$$

$$= (t-s)W(y) + Q_s^W(g)(x - (t-s)y)\,.$$

Taking the infimum over $y \in \mathbb{R}^n$ proves that $Q_t^W(g)(x) \le Q_{h-s}^W\big(Q_s^W(g)\big)(x)$, and thus there is equality.

We want to investigate if some kind of regularity is preserved under the operation of infimal convolution. The answer is yes, under certain specific conditions. We will also provide an example showcasing regularity loss, emphasizing the delicate nature of this question. Work on this subject already exists, notably in Evans' book [175], where there is a global Lipschitz assumption, or in Villani's book [179], where functions are bounded. However, such assumptions are at odds with the goals we aim for here, as ultimately, we want $g^{-\alpha}$ to be integrable for some exponant $\alpha > 0$.

We study the case where g and W are finite *everywhere*.

Lemma (6.3.12)[218]: *Let* $g, W : \mathbb{R}^n \to \mathbb{R}$. *If* g *is nonnegative, locally Lipschitz continuous, and* W *is convex and coercive, then* $(h, x) \mapsto Q_h^W(g)$ *is locally Lipschitz continuous.*

Proof. In order to prove the full local Lipschitz continuity, we must first localize the arginf of the infimal convolution. Fix $p > 0$, $\eta > 0$, and let $x,\ x' \in B(0, \rho)$ and $0 < h < \eta$. Consider the set

$$\Omega_{x,h} := \{y \in \mathbb{R}^n, g(x - y) + hW(y/h) \le g(x) + hW(0)\}.$$

We Claim (6.3.19) that, by positivity of g, and convexity of W, the set is bounded. Indeed, since W is convex and coercive, there exists $R > 0$ and $m > 0$ such that

$$\|y\| > R \Rightarrow W(y) \geq m\|y\|.$$
If $y \in \Omega_{x,h}$, then either $\|y\| \leq hR \leq \eta R$, or $\|y\| > hR$ and then $g(x) + hW(0) \geq hW(y/h) \geq m\|y\|$. Invoking continuity of g, we may prove the Claim (6.3.19), and conclude that there exists $R_{\rho,\eta}$, independent from x and h, such that $\Omega_{x,h} \subset B\left(0, R_{\rho,\eta}\right)$.

We prove the local Lipschitz continuity with respect to x. The functions g and W are assumed continuous, and so the infimal convolution is exact, and there exists $y \in \mathbb{R}^n$ such that $Q_h^W(g)(x) = g(x-y) + hW(y/h)$. Necessarily, $\|y\| \leq R_{\rho,\eta}$, so

$$\begin{aligned} Q_h^W(g)(x') - Q_h^W(g)(x) &= \inf\{g(x'-y') + hW(y'/h)\} - g(x-y) - hW(y/h) \\ &\leq g(x'-y) - g(x-y) \\ &\leq \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g\right) \|x - x'\|, \end{aligned}$$

where $\mathrm{Lip}_A f := \sup_{x \neq x \in A}\{|f(x) - f(x')|/\|x - x'\|\}$. By symmetry, we conclude that

$$|Q_h^W(g)(x') - Q_h^W(g)(x)| \leq \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g\right) \|x - x'\|,$$

hence the local Lipschitz continuity with respect to x.
Now,

$$\begin{aligned} Q_h^W(g)(x) - g(x) &= \inf_{y \in B(0,R_{\rho,n})} \{g(x-y) - g(x) + hW(y/h)\} \\ &\geq \inf_{y \in B(0,R_{\rho,n})} \left\{-\left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g\right) \|y\| + hW(y/h)\right\} \\ &= h \inf_{z \in B(0,R_{\rho,n}/h)} \{-\lambda\|z\| + W(z)\} \\ &\geq -h \sup_{t \in (B(0,\lambda)} \{\lambda\|z\| - W(z)\} \end{aligned}$$

$$\geq -h \sup W^*(t),$$

where $\lambda = \mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g$. Conversely, by definition,

$$Q_h^W(g)(x) - g(x) \leq hW(0),$$

and thus $|Q_h^W(g)(x) - g(x)| \leq Ch$, where $C = \max\left\{W(0), \sup_{t \in B(0,\lambda)} W^*(t)\right\}$. Note that C is finite because W^* is, by definition, convex and finite on $\mathbb{R}^n$, thus continuous. Finally, using the semigroup property $Q_{h+s}^W(g) = Q_h^W\left(Q_s^W(g)\right)$ and the fact that the Lipschitz constant with respect to x is uniformly bounded by $\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}$ for $0 < h < \eta$, we may conclude for the full local Lipschitz continuity.

The above lemma is a slight generalization of the following proposition:

Proposition (6.3.13)[218]: *Let f, g : $\mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous functions. If f is nonnegative, locally Lipschitz continuous, and g is coercive, then $f \square g$ is locally Lipschitz continuous.* Here, we do not need any convexity assumption, which was only used to prove

Lipschitz continuity with respect to the $(n+1)$th variable, h. Also, note here that it is important for f and g to be finite *everywhere*, which will not. In order for $f\square g$ to be locally Lipschitz continuous, further assumptions are needed on f and g, in particular on their domain. For example, if dom $f=\{x_0\}$, then $f\square g=f(x_0)+g(.-x_0)$, so it already seems necessary that both f and g be at least locally Lipschitz continuous. However, this is not sufficient. Consider for example the following functions f and g, defined on $\mathbb{R}^2$ by

$$f(x_1,x_2)=\begin{cases}1 & \text{if } x_1\in[0,1], x_2=0,\\ 1-x_2 & \text{if } x_1=0, x_2\in[0,1],\\ +oo & \text{otherwise,}\end{cases}$$

And

$$g(x_1,x_2)=\begin{cases}0 & \text{if } x_1\in[0,1], x_2=0,\\ +\infty & \text{otherwise,}\end{cases}$$

then

$$(f\square g)(x_1,x_2)=\begin{cases}1 & \text{if } x_1\in(0,1], x_2\in[0,1],\\ 1-x_2 & if\ x_1= \ x_2\ \in[0,1]\\ 0 & if\ x_1= \ x_2\ \in[1,2]\\ +\infty & otherwise\end{cases}$$

is not a continuous function. This example can easily be adapted to obtain a discontinuous infimal convolution for smooth functions f and g. We conjecture that if the domain is assumed convex, and if both functions are Lipschitz continuous, and their domain is of non-empty interior, then their infimal convolution is Lipschitz continuous.

Lemma (6.3.12), together with Lemma (6.3.10) and Rademacher's theorem, prove the following proposition:

Proposition (6.3.14))[218]: *Let* $g,W:\mathbb{R}^n\to\mathbb{R}$. *If* g *is nonnegative, locally Lipschitz continuous, and* W *is convex and grows superlinearly,*

$$\lim_{|x|\to+\infty}\frac{W(x)}{|x|}=+\infty,$$

then, for almost every $h\geq 0$ *and* $x\in\mathbb{R}^n$,

$$\frac{\partial}{\partial h}Q_h^W(g)(x)=-W^*\left(\nabla Q_h^W g(x)\right).$$

We prove an interesting equivalence between the classical Borell-BrascampLieb inequality and its differentiated expression, as announced in the introduction. It is also a good presentation of what is to come.

Proposition (6.3.15)[218]: *Let* $g,\ W\ :\mathbb{R}^n\to\mathbb{R}$. *If* g *is nonnegative, locally Lipschitz continuous, and* W *is convex and grows superlinearly,*

$$\lim_{|x|\to+\infty}\frac{W(x)}{|x|}=+\infty,$$

and are such that $\int g^{-n}=\int W^{-n}=1$, *and if* (g,W) *is admissible in the sense of Definition (6.3.21), then the following statements are equivalent:*

a. *The Borell-Brascamp-Lieb inequality holds: for every* $t\in[0,1]$ *and* $H:\mathbb{R}^n\to\mathbb{R}$ *such that*

$$\forall x, y \in \mathbb{R}^n, H\big((1-t)x+ty\big) \le (1-t)g(x)+tW(y)\,,$$

there holds

$$\int H^{-n} \ge 1.$$

b. The following inequality stands:

$$\int \frac{W^*(\nabla g)}{g^{n+1}} \ge 0.$$

Proof. By definition of the infimal convolution $Q_h^W(g)$, it is actually sufficient to only consider the function $H = (1-t)Q_h^W(g)(./(1-t))$, where $h = t/(1-t)$, in statement a. In fact, this leads to the statement a

$$\int Q_h^W\,(g)^{-n} \ge 1,$$

which we prove is equivalent to b. Let us consider the function $\varphi\colon h \mapsto \int Q_h^W\,(g)^{-n}$, which is continuous and almost everywhere differentiable in light of Lemma (6.3.12) and Theorem (6.3.22). Its derivative is given by

$$\varphi'(h) = n\int \frac{W^*(\nabla g)}{g^{n+1}}.$$

The implication $a'. \Rightarrow b.$ follows from the fact that $\varphi(0)=1$, and $\varphi(h) \ge 1$ for $h \ge 0$. Then, necessarily, $\varphi'(0) \ge 0$. Conversely, assume that b. holds. Then, whenever $h > 0$ is such that $\varphi(h) = \int Q_h^W\,(g)^{-n} = 1$, statement b. applied to the function $\tilde{g} = Q_h^W(g)$ and the corresponding function $\tilde{\varphi}$ implies that $\tilde{\varphi}'(0) = \varphi'(h) \ge 0$ thanks to the semigroup property proved in Lemma (6.3.11). This, together with the fact that $\varphi(0)=1$, proves that φ stays above 1, which is exactly statement a. Once again, we insist on the fact that the semigroup Q_h^W is not linear, and not Markov, which means, in particular, that there is no mass conservation. As such, this result stands as a bit unusual among similar results. Bolley et al. [3], the dynamical formulation of Borell-Brascamp-Lieb inequality.

Theorem (6.3.16)[218]: *Let $a > 1$ and $n \in \mathbb{N}^*$ such that $a \ge n$, and $g, W : \mathbb{R}^n \to (0,+\infty]$ be measurable functions such that $\int g^{-a} = \int W^{-a} = 1$. Then, for any $h \ge 0$,*

$$(1+h)^{a-n}\int_{\mathrm{R}^n} Q_h^W\,(g)^{1-a} \ge \int_{\mathrm{R}^n} g^{1-a} + h\int_{\mathrm{R}^n} W^{1-a}, \qquad (96)$$

where

$$Q_h^W(g)(x) = \inf\{g(x-hy)+hW(y)\} \in (0,+\infty].$$

Furthermore, when g is equal to W and is convex, there is equality.

To see that there is equality whenever $g = W$ is convex, fix $x \in \mathbb{R}^n$. For any $y \in \mathbb{R}^n$, since $\frac{x}{1+h} = \frac{1}{1+h}(x-hy) + \frac{h}{1+h}y$,

$$(1+h)\left(\frac{W(x-hy)}{1+h}+\frac{h}{1+h}W(y)\right)\geq(1+h)W\left(\frac{x}{1+h}\right).$$

Conversely, $Q_h^W(g)(x)$ is achieved at $= x/(1+h)$. In particular, for all $x\in\mathbb{R}^n$, $h\geq 0$,

$$Q_h^W(W)(x)=(1+h)W\left(\frac{x}{1+h}\right),$$

and equality in (96) is a straightforward computation. In [81], Bolley, Cordero-Erausquin, Fujita, Gentil, and Guillin use Theorem (6.3.16) to prove optimal Sobolev and Gagliardo-Nirenberg-Sobolev type inequalities in the half-space $\mathbb{R}_n^+=\mathbb{R}^{n-1}\times\mathbb{R}_+$. We want to extend these results to more general domains Ω in $\mathbb{R}^n$, where $n\geq 2$. Let us assume that Ω is the epigraph of a continuous function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ such that $\phi(0)=0$. In other words,

$$\Omega=\{(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R}, x_2\geq\phi(x_1)\}.$$

Let $e=(0,1)\in\mathbb{R}^{n-1}\times\mathbb{R}$, and for $h\geq 0$, define

$$\Omega_h=\Omega+\{he\}=\{\,(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R}, x_2\geq\phi(x_1)+h\}.$$

Let $a\geq n$, and consider $g:\Omega\to(0,+\infty)$ and $W:\Omega_1\to(0,+\infty)$, two measurable functions such that $\int_\Omega g^{-a}=\int_{\Omega_1}W^{-a}=1$. After extending these functions by $+\infty$ outside of their respective domain, inequality (97) yields

$$(1+h)^{a-n}\int_{B_h}Q_h^W(g)^{1-a}\geq\int_\Omega g^{1-a}+h\int_{\Omega_1}W^{1-a}\qquad(97)$$

where

$$B_h=\mathrm{dom}\left(Q_h^W(g)\right).$$

When $g(x)=W(x+e)$ and W is convex, then

$$Q_h^W(g)(x)=(1+h)W\left(\frac{x+e}{1+h}\right)$$

and equality is reached in the inequality above. To get a sense of what is to follow, notice that there is equality in inequality (97) when $h=0$. Now, when $\Omega=\mathbb{R}_+^n$, the interesting fact that $\Omega_h=B_h$ allows us, under certain admissibility criteria for W and g, to compute the derivative of inequality (97) with respect to h, at $h=0$. By doing so, the term $\int_{\partial\mathrm{R}_+^n}Q_0^W(g)^{1-a}=\int_{\partial\mathrm{R}_+^n}g^{1-a}$ appears in the left hand side, thus leading to trace inequalities.

Before going any further, let us investigate under which condition the two sets Ω_h and B_h coincide. We have the following lemma:

Lemma (6.3.17)[218]: *There exists $h_0>0$ such that for all $h\in(0,h_0)$, $B_h=\Omega_h$ if, and only if, Ω is a convex cone. In that case, B_h and Ω_h coincide for all $h\geq 0$.*

Proof. First, note that $Q_h^W(g)(x)<+\infty$ if, and only if, there exists $y\in\Omega_1$ such that $x-hy\in\Omega$. By definition of Ω, this is equivalent to

$$\exists(y_1,y_2)\in\mathbb{R}^{n-1}\times\mathbb{R}\,\mathrm{s.t.}\begin{cases}y_2\geq\phi(y_1)+1\\x_2-hy_2\geq\phi(x_1-hy_1)\end{cases}$$

$$\Leftrightarrow(\exists y_1\in\mathbb{R}^{n-1}\mathrm{s.t.}\,x_2\geq\phi(x_1-hy_1)+h\phi(y_1)+h)).$$

If $x \in \Omega_h$, then choosing $y_1 = 0$ proves that $x \in B_h$, so $\Omega_h \subset B_h$. If $h > 0$, $\Omega_h = B_h$ if, and only if, for all $x_1, y_1 \in \mathbb{R}^{n-1}$,

$$\phi\left(\frac{x_1 - y_1}{h}\right) \geq \frac{\phi(x_1) - \phi(y_1)}{h}. \tag{98}$$

Indeed, if $\Omega_h \supset B_h$, then, for any $x_1, y_1 \in \mathbb{R}^{n-1}$,

$$x_2 := \phi(x_1 - hy_1) + h\phi(y_1) + h \geq \phi(x_1) + h$$

and thus, replacing y_1 by $(x_1 - y_1)/h$, we get the stated inequality. The reciprocal is immediate. Now, let $z \in \mathbb{R}^{n-1}$, $|z| = 1$. Inequality (98), for $y_1 = 0$, becomes

$$\phi(z) \geq \frac{1}{h}\phi(hz)$$

for any h smaller than h_0. Let $\alpha = \limsup_{h\to 0} \phi(hz)/h$. Using inequality (98) once again, we get, for any $s \geq 0$,

$$\phi(sz) \geq \frac{s}{sh}\phi(shz)\,,$$

for any sufficiently small $h > 0$. Taking the inferior limit when $h \to 0$ proves that for any $s \geq 0$

$$\phi(sz) \geq s\alpha. \tag{99}$$

The set $\{s \geq 0, \phi(sz) = s\alpha\}$ is non-empty because it contains 0, and it is closed by continuity. Let $s \geq 0$ be such that $\phi(sz) = s\alpha$. Then, invoking inequality (98), and then inequality (99), we get

$$\begin{aligned}\phi\left(\frac{(1+h)sz - sz}{h}\right) = \phi(sz) = s\alpha &\geq \frac{\phi\big((1+h)sz\big) - \phi(sz)}{h} \\ &= \frac{\phi\big((1+h)sz\big) - s\alpha}{h} \\ &\geq \frac{(1+h)s\alpha - s\alpha}{h} = s\alpha\end{aligned}$$

so there is actually equality, and $\phi\big((1+h)sz\big) = (1+h)s\alpha$ for any sufficiently small $h > 0$. This shows that the connected component of $\{s \geq 0, \phi(sz) = s\alpha\}$ containing 0 is open in $\mathbb{R}_+$. Since it is also closed, it is the half real line $\mathbb{R}_+$. Thus, ϕ is linear over half-lines with initial point 0. Inequality (98) then becomes

$$\phi(x_1 - y_1) \geq \phi(x_1) - \phi(y_1)$$

for any $x_1, y_1 \in \mathbb{R}^{n-1}$. Let $t \in [0,1]$; replacing x_1 by $(1-t)x_1 + ty_1$ and y_1 by ty_1, and using linearity, the inequality becomes exactly the convexity inequality, that is

$$\phi\big((1-t)x_1 + ty_1\big) \leq (1-t)\phi(x_1) + t\phi(y_1)\,.$$

The reciprocal is trivial. It is also clear that in this case, $B_h = \Omega_h$ for *any* $h \geq 0$.

This lemma will be used to prove the trace Sobolev and the trace Gagliardo Nirenberg-Sobolev inequalities in convex cones. We can go a bit further, and impose only ϕ to be convex.

Lemma (6.3.18)[218]: *If ϕ is convex, then*

$$B_h=\left\{(x_1,x_2)\in\mathbb{R}^n, x_2\geq h+(1+h)\phi\left(\frac{x_1}{1+h}\right)\right\}.$$

Proof. One may notice that setting $\omega(x)=0$ if $x\in\Omega$ and $+\infty$ if $x\in\Omega^c$, and $W(x)=\omega(x-e)$, then ω is convex, thus

$$B_h=\operatorname{dom}\left(Q_h^W(\omega)\right)=\operatorname{dom}\left(x\mapsto(1+h)W\left(\frac{x+e}{1+h}\right)\right),$$

and

$$W\left(\frac{x+e}{1+h}\right)<+\infty\Leftrightarrow\frac{x+e}{1+h}-e\in\Omega$$
$$\Leftrightarrow x_2\geq h+(1+h)\phi\left(\frac{x_1}{1+h}\right).$$

We assume that Ω is a convex cone. In that case, invoking Lemma (6.3.17), inequality (96) becomes

$$(1+h)^{a-n}\int_{\Omega_h}Q_h^W(g)^{1-a}\geq\int_\Omega g^{1-a}+h\int_{\Omega_1}W^{1-a},\tag{100}$$

for any $h>0$, and there is equality when $h=0$. Taking the derivative of this inequality with respect to h, under the admissibility conditions for g and W exposed in full details, and evaluating at $h=0$, we prove that

$$(a-n)\int_\Omega g^{1-a}+(a-1)\int_\Omega\frac{W^*(\nabla g)}{g^a}$$
$$-\int_{\mathrm{R}^{n-1}}g^{1-a}\left(x_1,\phi(x_1)\right)\mathrm{d}x_1\geq\int_{\Omega_1}W^{1-a}.\tag{102}$$

There, we used Lemma (6.3.10), and the fact that

$$\frac{1}{h}\left(\int_{\Omega_h}Q_h^W(g)^{1-a}-\int_\Omega g^{\mathrm{i}-a}\right)=\int_{\Omega_h}\frac{Q_h^W(g)^{1-a}-g^{1-a}}{h}+\frac{1}{h}\left(\int_{\Omega_h}g^{\mathrm{i}-a}-\int_\Omega g^{\mathrm{i}-a}\right)$$
$$=\int_{\Omega_h}\frac{Q_h^W(g)^{1-a}-g^{1-a}}{h}-\frac{1}{h}\left(\int_{\mathrm{R}^{n-1}}\int_{\phi(x_1)}^{h+\phi(x_1)}g^{1-a}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1\right)$$
$$\overrightarrow{h\to0}(1-a)\int_\Omega\frac{W^*(\nabla g)}{g^a}-\int_{\mathrm{R}^{n-1}}g^{1-a}\left(x_1,\phi(x_1)\right)\mathrm{d}x_1,$$

see Theorem (6.3.22). Let $p\in(1,n)$, and q its conjugate exponent, $1/p+1/q=1$. Applying inequality (102) to the function W defined by $W(x)=C\|x\|^q/q$, where $C>0$ is such that $\int W^{-a}=1$, which happens to be admissible for this choice of q, in the sense of Definition (6.3.21) We find

$$(a-n)\int_{\Omega} g^{1-a} + C^{1-p}\frac{a-1}{p}\int_{\Omega}\frac{\|\nabla g\|_*^p}{g^a} - \int_{\mathrm{R}^{n-1}} g^{\mathrm{i}-a}\left(x_1,\phi(x_1)\right)\mathrm{d}x_1 \geq \int_{\Omega_1} W^{1-a}$$

for any admissible g, where $\|x\|_* = \sup_{\|y\|=1} x \cdot y$ is the dual norm of x. Next, we extend the above inequality to all functions g such that $f = g^{(p-a)/p} \in C_c^\infty(\Omega)$. This can be done by approximation by admissible functions, Rewriting the quantities in terms of $f = g^{-(a-p)/p}$ yields

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}}\left(x,\phi(x)\right)\mathrm{d}x$$

$$\leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \int_{\Omega} \| \nabla f \|_*^p - \int_{\Omega_1} W^{1-a} + (a-n)\int_{\Omega} f^{p\frac{a-1}{a-p}}.$$

We may then remove the normalization to find that inequality (102) becomes

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}}\left(x,\phi(x)\right)\mathrm{d}x$$

$$\leq C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p \left(\int_{\Omega} \| \nabla f \|_*^p\right)\beta^{p\frac{p-1}{a-p}} - \left(\int_{\Omega_1} W^{1-a}\right)\beta^{p\frac{a-1}{a-p}}$$

$$+(a-n)\int_{\Omega} f^{p\frac{a-1}{a-p}} \tag{103}$$

where

$$\beta = \left(\int_{\Omega} f\,\frac{pa}{a-p}\right)^{\frac{a-p}{ap}}$$

Now, define $u = \frac{a-1}{a-p}$ and $v = u' = \frac{a-1}{p-1}$, so that $u, v > 1$ and $1/u + 1/v = 1$. By Young's inequality, we find

$$A\int_{\Omega} \| \nabla f \|_*^p \beta^{p\frac{p-1}{a-p}} - \left(\int_{\Omega_1} W^{1-a}\right)\beta^{p\frac{a-1}{a-\mathrm{p}}}$$

$$= Bv\left(\frac{A}{Bv}\int_{\Omega} \| \nabla f \|_*^p \beta^{p\frac{p-1}{a-p}} - \frac{1}{v}\beta^p\,\frac{a-1}{a-p}\right) \tag{104}$$

$$\leq D\left(\int_{\Omega} \| \nabla f \|_*^p\right)^u$$

where

$$A = C^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p B = \int_{\Omega_1} W^{1-a} \text{ and } D = \frac{A^u}{(Bv)^{u-1}}\frac{1}{u}.$$

In order to find a more compact inequality, we consider, for $\lambda > 0$, $f_\lambda : \mapsto f(\lambda x)$. By linearity of ϕ, applying (104) to f_λ leads to

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}}(x,\phi(x))\mathrm{d}x \le \lambda^{(a-n)\frac{p-1}{a-p}} \frac{A^u}{(Bv)^{u-1}}\frac{1}{u}\left(\int_\Omega \| \nabla f\|_*^p\right)^u + \frac{a-n}{\lambda}\int_\Omega f^{p\frac{a-1}{a-\mathrm{p}}}.$$

optimizing this inequality with respect to $\lambda > 0$ finally yields inequality (92) of Theorem (6.3.2). It remains to show that inequality (92) is optimal. The function for which equality is reached does not have compact support, but this technicality does not bear much relevance. To prove optimality, note that there is equality in (102) when $g(x) = W(x+e)$, which implies equality in (103) when $f(x) = \|x+e\|^{-\frac{a-p}{p-1}}$. If Young's inequality (104) is an equality, then the optimization with respect to parameter λ necessarily preserves the equality. Thus, it is enough to show that for $f(x) = \|x+e\|^{-\frac{a-p}{p-1}}$, there is equality in (104). This is the case if, and only if,

$$\frac{A}{Bv}\int_\Omega \| \nabla f\|_*^p = \left(\beta^{p\frac{p-1}{a-p}}\right)^{v-1}$$

Let us now write, for $\alpha > 0$

$$I_\alpha := \int_\Omega \| x+e\|^{-\alpha}$$

Then,

$$C = q\left(\int_\Omega \| x+e\|^{-qa}\right)^{\frac{1}{a}} = \frac{p}{p-1} I_{ap/(p-1)}^{1/a}$$

hence

$$A = \frac{(a-1)(p-1)^{p-1}}{(a-p)^p} I_{ap/(p-1)}^{(1-p)/a}, B = I_{ap/(p-1)}^{(1-a)/a} I_{p(a-1)/(p-1)},$$

$$\text{and } \left(\beta^{p\frac{p-1}{a-p}}\right)^{v-1} = I_{ap/(p-1)}^{(a-p)/a}.$$

Claim (6.3.19)[218]: For $\gamma \in \mathbb{R}$, let $\delta : \mathbb{R}^n \setminus \{0\} \to]\ 0, +\infty[$, $x \mapsto \|x\|^\gamma$. Then, almost everywhere, δ is differentiable, and $\|\nabla\delta(x)\|_*^p = |\gamma|\|x\|^{\gamma-1}$

Using this, we conclude that there is indeed equality in (104), since then

$$\int_\Omega \| \nabla f\|_*^p = \left(\frac{a-p}{p-1}\right)^p I_{p(a-1)/(p-1)}.$$

Proof. Consider φ: $x \mapsto \|x\|$ and $\psi : \rho \mapsto \rho^\gamma$. φ is convex, hence almost everywhere differentiable by Rademacher's theorem, and ψ smooth on $]\ 0, +\infty[$, hence the Claim (6.3.19) regularity of $\delta = \psi \circ \varphi$. For almost every , $\nabla\delta(x) = \gamma\nabla\varphi(x)\|x\|^{\gamma-1}$, so

$$\|\nabla\delta(x)\|_* = |\gamma|\|x\|^{\gamma-1}\|\nabla\varphi(x)\|_*$$

If $x \neq 0$ is a point of differentiability of φ, and $t > 0$, then

$$1=\frac{\|x+tx/\|x\|\|-\|x\|}{t}\xrightarrow{t\to 0}\nabla\varphi(x)\cdot\frac{x}{\|x\|},$$

so $\|\nabla\varphi(x)\|_*\geq 1$. Conversely, if $\|v\|=1$, then

$$\nabla\varphi(x)\cdot v=\lim_{t\to 0+}\frac{\|x+tv\|-\|x\|}{t}\leq\lim_{t\to 0+}\|v\|=1,$$

so $\|\nabla\varphi(x)\|_*=1$ and the Claim (6.3.19) is proved.

We assume that Ω is the epigraph of a convex function ϕ, with $\phi(0)=0$. Then, according to Lemma (6.3.18), for $h\geq 0$,

$$B_h=\operatorname{dom}\left(Q_h^W(g)\right)=\left\{(x_1,x_2)\in\mathbb{R}^n,x_2\geq h+(1+h)\phi\left(\frac{x_1}{1+h}\right)\right\}.$$

Inequality (96) becomes

$$(1+h)^{a-n}\int_{B_h}Q_h^W(g)^{1-a}\geq\int_\Omega g^{1-a}+h\int_{\Omega_1}W^{1-a},\tag{105}$$

and there still is equality for all $h>0$ whenever $g(x)=W(x+e)$ and is convex. However, it is slightly trickier to compute the derivative at $h=0$, since $B_h\neq\Omega_h$, and their symmetric difference depends heavily on ϕ. Effectively, a third term appears when trying to differentiate $\int_{B_h}Q_W^h(g)^{1-a}$:

$$\frac{1}{h}\left(\int_{B_h}Q_h^W(g)^{1-a}-\int_\Omega g^{\mathrm{i}-a}\right)$$

$$=\int_{\Omega_h}\frac{Q_h^W(g)^{1-a}-g^{1-a}}{h}-\frac{1}{h}\int_{\Omega\backslash\Omega_h}g^{\mathrm{i}-a}+\frac{1}{h}\int_{B_h\backslash\Omega_h}Q_h^W(g)^{1-a}.$$

Taking the derivative at $h=0$, when possible, yields

$$(a-n)\int_\Omega g^{1-a}+(a-1)\int_\Omega\frac{W^*(\nabla g)}{g^a}$$

$$-\int_{\mathrm{R}^{n-1}}g^{1-a}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1\geq\int_{\Omega_1}W^{1-a},\tag{106}$$

where

$$P(x_1)=1+\phi(x_1)-x_1\cdot\nabla\phi(x_1)\,.$$

Using inequality (106) with $W=C\|.\|^q/q$, and extending it for all $f=g^{-(a-p)/p}\in C_c^\infty(\Omega)$ just like we did for convex cones, and finally invoking Young's inequality, we get the theorem

Theorem (6.3.20)[218]: *Let* $a\geq n>p>1$, *and* $\Omega=\{(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R},x_2\geq\phi(x_1)\}$ *be a convex set. There exists a positive constant* $D'_{n,p,a}(\Omega)$ *such that for any positive function* $f\in C_c^\infty(\Omega)$,

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1$$

$$\leq D'_{n,p,a}(\Omega)\left(\int_\Omega \| \nabla f\|_*^p\right)^{\frac{a-1}{a-p}} + (a-n)\int_\Omega f^{p\frac{a-1}{a-p}}, \qquad (107)$$

where $P(x_1) = 1 + \phi(x_1) - x_1 \,.\, \nabla\phi(x_1)$. *Furthermore, when* $f(x) = \|x+e\|^{-\frac{a-p}{p-1}}$, *then* (107) *is an equality*. Applying this theorem for $a = n$, we find a new version of the trace Sobolev inequality, Theorem (6.3.4), with $D'_{n,p}(\Omega) = D'_{n,p,n}(\Omega)$. It is important to note that in Theorem (6.3.4), as well as in Theorem (6.3.20), the left-hand side can be negative. The weight P itself generally is negative outside of a compact neighbourhood of the origin, but the inequality is still optimal. We prove that the results are true for a class of admissible functions, and we extend these results to the appropriate, more general setting, by approximation by admissible functions. The difficulty here lies in that g must not be bounded or even Lipschitz, since g^{-a} has to be integrable. The case of the half-plane has already been investigated (in [3]), and easily extends to convex cones. Here, we will only tackle convex sets, which, although more technical, follows the same general idea. Throughout, $\phi : \mathbb{R}^{n-1} \to [0,+\infty)$ is a convex function such that $\phi(0) = 0, g\colon \Omega \to (0,+\infty)$ is assumed to be locally Lipschitz continuous, and $W\colon \Omega_1 \to (0,+\infty)$ is convex. Inequality (105),

$$(1+h)^{a-n}\int_{B_h} Q_h^W(g)^{1-a} \geq \int_\Omega g^{1-a} + h\int_{\Omega_1} W^{1-a},$$

is trivially an equality for $h = 0$, we thus ask compute its derivative. Let us first give a nonrigorous proof for clarity. The most difficult part is computing the derivative of $\int_{B_h} Q_h^W(g)^{1-a}$, so let us start with that. Notice that $\Omega_h \subset B_h \cap \Omega$, thus

$$\frac{1}{h}\left(\int_{B_h} Q_h^W(g)^{1-a} - \int_\Omega g^{\mathrm{i}-a}\right)$$

$$= \underbrace{\int_{\Omega_h} \frac{Q_h^W(g)^{1-a} - g^{1-a}}{h}}_{(i)} \underbrace{- \frac{1}{h}\int_{\Omega\backslash\Omega_h} g^{\mathrm{i}-a}}_{(ii)} + \underbrace{\frac{1}{h}\int_{B_h\backslash\Omega_h} Q_h^W(g)^{1-a}}_{(iii)}$$

Recalling Lemma (6.3.10), almost everyhere,

$\lim\limits_{h\to 0}\frac{Q_h^W(g)(x)-g(x)}{h} = -W^*\left(\nabla g(x)\right)$,

thus (i) should converge towards

$$(a-1)\int_{\Omega}\frac{W^*(\nabla g)}{g^a}.$$

Next, (*ii*) can be rewritten in a way such that the convergence is quite clear:

$$(ii)=\int_{\mathrm{R}^{n-1}}\left(\frac{1}{h}\int_{\phi(x_1)}^{\phi(x_1)+h} g^{1-a}\,(x_1,x_2)\mathrm{d}x_2\right)\mathrm{d}x_1\overrightarrow{h\to 0}\int_{\mathrm{R}^{n-1}} g^{1-a}\left(x_1,\phi(x_1)\right)\mathrm{d}x_1$$

as $h\to 0$. Finally, giving (*iii*) the same treatment,

$$(iii)=\int_{\mathrm{R}^{n-1}}\left(\frac{1}{h}\int_{h+(1+h)\phi(x_1/(1+h))}^{h+\phi(x_1)} Q_h^W\,(g)^{1-a}(x_1,x_2)\mathrm{d}x_2\right)\mathrm{d}x_1$$

$$\overrightarrow{h\to 0}\int_{\mathrm{R}^{n-1}} g^{1-a}\left(x_1,\phi(x_1)\right)\left(x_1\cdot\nabla\phi(x_1)-\phi(x_1)\right)\mathrm{d}x_1,$$

since $Q_0^W(g)=g$ and

$$\lim_{h\to 0}\frac{1}{h}\left(\phi(x_1)-(1+h)\phi\left(\frac{x_1}{1+h}\right)\right)=x_1\cdot\nabla\phi(x_1)-\phi(x_1)\,.$$

Summing these results up, we find the Claim (6.3.19) derivative at $h=0$. Whenever Ω is a convex cone, $B_h\setminus\Omega_h=\emptyset$, and thus (*iii*) $=0$. In that case, the argument is much more succinct, but since it is also a corollary of the more general case, we will not address it. The conditions for the convergence to play out nicely are summed up in the following definition. They are mostly growth conditions on g and W, and will come into play later on.

Definition (6.3.21)[218]. The couple of functions (g,W) is said to be admissible if the following conditions are satisfied for some constant γ:

$$(\mathrm{CO})>\ \max\left(\tfrac{a}{n-1},1\right)\ ;$$

(C1) there exists $A_1>0$ such that $W(x)\geq A_1\|x\|^\gamma$ for all $x\in\Omega_1$;
(C2) there exists $A_2>0$ such that $W(x)\leq A_2(1+\|x\|^\gamma)$ for all $x\in\Omega_1$;
(C3) there exists $A_3>0$ such that $g(x)\geq A_3(1+\|x\|^\gamma)$ for all $x\in\Omega$;
(C4) there exists $A_4>0$ such that $\|\nabla g(x)\|\leq A_4(1+\|x\|^{\gamma-1})$ for all $x\in\Omega$.

The challenge is to prove that under these conditions, $Q_h^W(g)$ converges towards g in a controlled manner as $h\to 0$. The main result the following:

Theorem (6.3.22)[218]: *Assume that the couple* (g,W) *is admissible, and that there exist some constants* $C>0$ *and* $R>0$ *such that*

$$\forall\|x_1\|>R,|x_1\cdot\nabla\phi(x_1)|\leq C\|\left(x_1,\phi(x_1)\right)\|. \tag{108}$$

Then

$$\lim_{h\to 0}\frac{1}{h}\left(\int_{B_h} Q_h^W\,(g)^{1-a}(g)-\int_{\Omega} g^{1-a}\right)$$

$$=(a-1)\int_{\Omega}\frac{W^*(\nabla g)}{g^a}-\int_{\mathrm{R}^{n-1}} g^{1-a}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1, \tag{109}$$

where $P(x_1) = 1 + \phi(x_1) - x_1 \cdot \nabla\phi(x_1)$. In what follows, we will use a good number of different positive constants, which will all be written C for convenience. They will not depend on $x \in \mathbb{R}^n$, or $h > 0$, but might depend on $A_i, i \in \{1,2,3,4\}, \gamma$.

Lemma (6.3.23)[218]: *If* (g, W) *is admissible, there exist constants* $C > 0$ *and* $h_0 > 0$, *such that for all* $0 < h < h_0$, *and* $x \in \Omega_h$,

$$|Q_h^W(g)(x) - g(x)| \le Ch(1 + \|x\|^\gamma) .$$

Proof. First, let $x', x \in \Omega$. Then, we may estimate $|g(x') - g(x)|$ using hypothesis (C4):

$$|g(x') - g(x)| \le \int_0^1 \| \frac{\partial}{\partial\theta} g\big(x + \theta(x' - x)\big)\| \mathrm{d}\theta$$

$$\le \|x' - x\| \int_0^1 A_4 \,(1 + \|x + \theta(x' - x)\|^{\gamma-1})\mathrm{d}\theta$$

$$\le C\|x' - x\|(1 + \|x\|^{\gamma-1} + \|x' - x\|^{\gamma-1}). \tag{110}$$

Now, let $0 < h \le 1$ and $x \in \Omega_h$. Then, $x - he \in \Omega$, so

$$\begin{aligned} Q_h^W(g)(x) - g(x) &\le g(x - he) + hW(e) - g(x) \\ &\le Ch(1 + \|x\|^{\gamma-1} + h^{\gamma-1}) + hW(e) \\ &\le Ch(1 + \|x\|^{\gamma-1}) . \end{aligned}$$

For the converse inequality, we will of course use hypotheses (C1) and (C3), but we first have to localize the point where the infimum $Q_h^W(g)(x)$ is reached. Let $y \in \Omega_1$ be such that $Q_h^W(g)(x) = g(x - hy) + hW(y)$. Then, invoking hypothesis (C1) and inequality (110),

$$\begin{aligned} hA_1\|y\|^\gamma &\le hW(y) = Q_h^W(g)(x) - g(x - hy) \\ &= Q_h^W(g)(x) - g(x) + g(x) - g(x - hy) \\ &\le Ch(1 + \|x\|^{\gamma-1}) + Ch\|y\|(1 + \|x\|^{\gamma-1} + (h\|y\|)^{\gamma-1}) . \end{aligned}$$

We thus choose $h_0 \in (0,1)$ such that for any $h \in (0, h_0)$, $A_1 - Ch^{\gamma-1} > h^{\gamma-1}$. Then, for any $h \in (0, h_0)$,

$$\begin{aligned} h^{\gamma-1}\|y\|^\gamma &< (A_1 - Ch^{\gamma-1})\|y\|^\gamma \\ &\le C(1 + \|y\|)(1 + \|x\|^{\gamma-1}) , \end{aligned}$$

which implies that

$$h^{\gamma-1}\|y\|^{\gamma-1} \le C(1 + \|x\|^{\gamma-1}) ,$$

since $\|y\|^{\gamma-1} \le \max\left(1, 2\frac{\|y\|^\gamma}{1+\|y\|}\right)$. Now, using inequality (110) once again,

$$\begin{aligned} |g(x - hy) - g(x)| &\le Ch\|y\|(1 + \|x\|^{\gamma-1} + h^{\gamma-1}\|y\|^{\gamma-1}) \\ &\le Ch\|y\|(1 + \|x\|^{\gamma-1}) . \end{aligned}$$

Plugging this in the definition of $Q_h^W(g)(x)$, we find

$$Q_h^W(g)(x) - g(x) \ge \inf_{y\in\Omega_1} \{-Ch\|y\|(1 + \|x\|^{\gamma-1}) + hA_1\|y\|^\gamma\}$$

$$\ge \inf_{y\in\mathbb{R}^n} \{\ldots\} = -Ch(1 + \|x\|^\gamma) .$$

To conclude, it is enough to notice that $1 + \|x\|^{\gamma-1} \le 2 + \|x\|^\gamma$ since $\gamma > 1$.

Now that we have this estimation, we may estimate the speed of convergence of $Q_h^W(g)^{1-a}$ towards g^{1-a}.

Proposition (6.3.24)[218]: *If* (g, W) *is admissible, there exist constants* $C > 0$ *and* $h_0 > 0$, *such that for all* $0 < h < h_0$, *and* $x \in \Omega_h$,

$$\frac{|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)|}{h} \leq \frac{C}{1 + \|x\|^{\gamma(a-1)}}.$$

Proof. First, let , $\beta > 0$. Then,

$$|\int_\alpha^\beta t^{-a}\,\mathrm{d}t| = |\frac{1}{1-a}(\beta^{1-a} - \alpha^{1-a})| \leq \max(\alpha^{-a}, \beta^{-a})|\alpha - \beta|,$$

implying that

$$|\alpha^{1-a} - \beta^{1-a}| \leq (a-1)|\alpha - \beta|(\alpha^{-a} + \beta^{-a}). \qquad (111)$$

Then, according to Lemma (6.3.23), there exists $h_0 > 0$ such that for any $h \in (0, h_0)$, and any $x \in \Omega_h$,

$$\frac{|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)|}{h} \leq C\frac{|Q_h^W(g)(x) - g(x)|}{h}\left(Q_h^W(g)^{-a}(x) + g^{-a}(x)\right)$$
$$\leq C(1 + \|x\|^\gamma)\left(Q_h^W(g)^{-a}(x) + g^{-a}(x)\right). \qquad (112)$$

Now, hypotheses (C1) and (C3) and a straightforward computation yield

$$Q_h^W(g)(x) \geq \inf\{A_3(1 + \|x - hy\|^\gamma) + hA_1\|y\|^\gamma\}$$
$$\geq \inf\{A_3(1 + |\|x\| - h\|y\||^\gamma) + hA_1\|y\|^\gamma\}$$
$$\geq C(1 + \|x\|^\gamma).$$

Using (C3) once again, we know that

$$g^{-a}(x) \leq \left(A_3(1 + \|x\|^\gamma)\right)^{-a};$$

putting these two inequalities together with inequality (112), we finally obtain

$$\frac{|Q_h^W(g)^{1-a}(x) - g^{1-a}(x)|}{h} \leq C\frac{1 + ||x||^\gamma}{(1 + \|x\|^\gamma)^a}$$
$$\leq \frac{C}{1 + \|x\|^{\gamma(a-1)}}.$$

Proposition (6.3.24), together with Lemma (6.3.10), proves the dominated convergence, and

$$\lim_{h\to 0}(i) = (a-1)\int_\Omega \frac{W^*(\nabla g)}{g^a},$$

as Claim (6.3.19)ed. The convergence of (*ii*) is straightforward, as it is a direct implication of the local Lipschitz continuity of g and hypothesis (C3).

This term is a bit trickier, because comparing $Q_h^W(g)$ to g is not possible on the entirety of B_h, g being defined only on Ω. For many functions ϕ, $B_h \not\subset \Omega$ as is showcased on figure 1 below. Thus, we prove the following result:

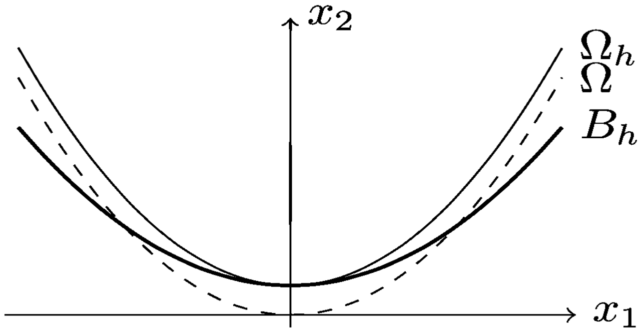

Figure 1[218]: Graph of , Ω_h, and B_h for $\phi(x_1) = \|x_1\|^2$ and $h = 0.5$

Lemma (6.3.25)[218]: *If* (g, W) *is admissible, there exist constants* $C > 0$ *and* $h_1 > 0$, *such that for atl* $0 < h < h_1$, *and* $(x_1, x_2) \in B_h \setminus \Omega_h$,

$$\left|Q_h^W(g)(x_1, x_2) - g\big(x_1, \phi(x_1)\big)\right|$$

$$\le hC\big(1 + \|\big(x_1, \phi(x_1)\big)\|^\gamma + |x_1 \cdot \nabla\phi(x_1)|^\gamma\big)\,. \tag{113}$$

The proof follows the same logic as the proof of Lemma (6.3.23).

Proof. Recall that, according to Lemma (6.3.18)

$$\Omega_h = \{\, (x_1, x_2) \in \mathbb{R}^n, x_2 \ge h + \phi(x_1)\}\,, B_h = \left\{(x_1, x_2) \in \mathbb{R}^n, x_2 \ge h + (1 + h)\phi\left(\frac{x_1}{1 + h}\right)\right\},$$

and that

$|g(x') - g(x)| \le C\|x' - x\|(1 + \|x\|^{\gamma-1} + \|x' - x\|^{\gamma-1})$ (23 revisited) for any x', $x \in \Omega$.

Fix $h \in (0,1)$, $x = (x_1, x_2) \in B_h \setminus \Omega_h$, and define $p(x_1, x_2) = \big(x_1, \phi(x_1)\big)$, its projection onto $\partial\Omega$. Letting $= \left(\frac{x_1}{1+h}, 1 + \phi\left(\frac{x_1}{1+h}\right)\right)$, we find that $y \in \Omega_1$, and also that $x - hy \in \Omega$, thus, with hypothesis (C2) and inequality (110),

$$Q_h^W(x) - g\big(p(x)\big) \le g(x - hy) - g\big(p(x)\big) + hW(y)$$

$$\le C\|x - hy - p(x)\|(1 + \|p(x)\|^{\gamma-1} + \|x - hy - p(x)\|^{\gamma-1}) + hA_2(1 + \|y\|^\gamma)\,.$$

For brevity, let us write $u = \|p(x)\| = \|(x_1, \phi(x_1)\|$, and $v = x_1\ .\ \phi(x_1) = |x_1 \cdot \phi(x_1)|$. Now, notice that

$$\|x - hy - p(x)\| = h\|\left(\frac{x_1}{1 + h}, \frac{x_2 - h - \phi(x_1)}{h} - \phi\left(\frac{x_1}{1 + h}\right)\right)\|.$$

From the definition of Ω_h and B_h, we find out that

$$0 \le \frac{h + \phi(x_1) - x_2}{h} \le \frac{\phi(x_1) - (1 + h)\phi\big(x_1/(1 + h)\big)}{h} \le x_1 \cdot \nabla\phi(x_1) = v,$$

since ϕ is convex and nonnegative. Thus,

$$\|x - hy - p(x)\| \leq h\left\|\left(\frac{x_1}{1+h}, \phi\left(\frac{x_1}{1+h}\right)\right)\right\| + h|x_1 \cdot \nabla\phi(x_1)|$$

$\leq h(\|p(x)\| + |x_1 \cdot \nabla\phi(x_1)|) = h(u+v)$,
so, since $h < 1$,

$$1 + \|p(x)\|^{\gamma-1} + \|x - hy - p(x)\|^{\gamma-1} \leq 1 + u^{\gamma-1} + \big(h(u+v)\big)^{\gamma-1}$$
$$\leq C(1 + u^{\gamma-1} + v^{\gamma-1}) .$$

Finally,

$$A_2(1 + \|y\|^\gamma) = A_2\left(1 + \left\|\frac{x_1}{1+h}, 1 + \phi\left(\frac{x_1}{1+h}\right)\right\|^\gamma\right) \leq C(1 + u^\gamma) .$$

Putting all these inequalities together, we find

$$Q_h^W(x) - g\big(p(x)\big) \leq hC(u+v)(1 + u^{\gamma-1} + v^{\gamma-1}) + hC(1 + u^\gamma)$$
$$\leq hC(1 + u^\gamma + v^\gamma) . \tag{114}$$

Conversely, let $y \in \Omega_1$ be such that $Q_h^W(g)(x) = g(x - hy) + hW(y)$. As before, we localize y. Using hypothesis A_1 and inequalities (110) and (114),

$$hA_1\|y\|^\gamma \leq hW(y) = Q_h^W(g)(x) - g\big(p(x)\big) + g\big(p(x)\big) - g(x - hy)$$
$$\leq hC(1 + u^\gamma + v^\gamma) + C\|x - hy - p(x)\|(1 + u^{\gamma-1} + \|x - hy - p(x)\|^{\gamma-1})$$
$$\leq hC(1 + u^\gamma + v^\gamma) + hC(\|y\| + v)(1 + u^{\gamma-1} + h^{\gamma-1}(\|y\| + v)^{\gamma-1})$$
$$\leq hC(1 + u^\gamma + v^\gamma) + hC(\|y\| + v)(1 + u^{\gamma-1} + h^{\gamma-1}\|y\|^{\gamma-1} + v^{\gamma-1}) .$$

Rearranging the terms and dividing by h yields

$$A_1\|y\|^\gamma - Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\| + v) \leq C(1 + u^\gamma + v^\gamma) + C(\|y\| + v)(1 + u^{\gamma-1} + v^{\gamma-1})$$
$$\leq C(1 + u + v + \|y\|)(1 + u^{\gamma-1} + v^{\gamma-1}) .$$

We must now split the reasoning in two cases: either $\|y\| \leq v$, in which case the conclusion follows, or $\|y\| \geq v$, and then $A_1\|y\|^\gamma - Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\| + v) \geq A_1\|y\|^\gamma - 2Ch^{\gamma-1}\|y\|^\gamma$. We thus choose $0 < h_1 < 1$ such that for all $h \in (0, h_1)$, $A_2 - 2Ch^{\gamma-1} \geq h^{\gamma-1}$. Then, we have, for any $h \in (0, h_1)$,

$$\frac{h^{\gamma-1}\|y\|^\gamma}{1 + u + v + \|y\|} \leq C(1 + u^{\gamma-1} + v^{\gamma-1}) .$$

Once again, either $\|y\| \leq 1 + u + v$, or

$$h^{\gamma-1}\|y\|^{\gamma-1} \leq \frac{2h^{\gamma-1}\|y\|^\gamma}{1 + u + v + \|y\|}.$$

Taking the greatest of the constants in those two cases, we may conclude that

$$h^{\gamma-1}\|y\|^{\gamma-1} \leq C(1 + u^{\gamma-1} + v^{\gamma-1}) . \tag{115}$$

We may now proceed with the converse inequality. Invoking once again inequality (110), and then inequality (115),

$$|g(x - hy) - g\big(p(x)\big)| \leq hC(\|y\| + v)(1 + u^{\gamma-1} + h^{\gamma-1}\|y\|^{\gamma-1} + v^{\gamma-1})$$
$$\leq hC(\|y\| + v)(1 + u^{\gamma-1} + v^{\gamma-1}) .$$

Finally,

$$Q_h^W(g)(x) - g\big(p(x)\big) = g(x-hy) - g\big(p(x)\big) + hW(y)$$

$$\geq -hC(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1}) + hA_2\|y\|^\gamma$$
$$\geq h\inf\{-C(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1}) + A_2\|y\|^\gamma\}$$
$$\geq -hC(1+u^{\gamma-1}+v^{\gamma-1})^{\gamma/(\gamma-1)}$$

and we may conclude. We may now prove Theorem (6.3.22): using the same notations as in the proof above, that is $u = \|p(x)\| = \|(x_1,\phi(x_1)\|$, and $v = x_1\cdot\phi(x_1) = |x_1\cdot\phi(x_1)|$, hypothesis (C2) immediately yields, for all $h>0$ and all $x \in B_h\backslash\Omega_h$,

$$g^{-a}\big(p(x)\big) \leq \frac{C}{(1+u^\gamma)^a}.$$

Furthermore, inequality (113) and hypothesis (C2) yield

$$Q_h^W(g)(x) \geq -hC(1+u^\gamma+v^\gamma) + C(1+u^\gamma)$$

for all $x\in B_h\backslash\Omega_h$ and $0<h<h_1$. Now, assumption (108) reads: for all $x_1\in\mathbb{R}^{n-1}$ such that $\|x_1\|>R$,

$$v \leq Cu.$$

Since both u and v are bounded functions of x on the set $\{(x_1,x_2)\in B_h\backslash\Omega_h, \|x_1\|\leq R\}$, there exists $h_2>0$ such that, for all $0<h<h_2$,

$$Q_h^W(g)(x) \geq \begin{cases} C>0 & \text{whenever } \|x_1\|\leq R \\ C(1+u^\gamma) & \text{whenever } \|x_1\|>R \end{cases}$$

Thus, for all $0<h<h_2$ and all $x\in B_h\setminus\Omega_h$,

$$Q_h^W(g)^{-a}(x) \leq \frac{C}{(1+u^\gamma)^a}.$$

Finally, invoking inequality (111) together with assumption (108) yields, for any $0<h<h_2$ and $x=(x_1,x_2)\in B_h\backslash\Omega_h$,

$$\frac{|Q_h^W(g)^{1-a}(x) - g^{\mathrm{i}-a}\big(p(x)\big)|}{h} \leq C\frac{|Q_h^W(g)(x)-g\big(p(x)\big)|}{h}\Big(Q_h^W(g)^{-a}(x) + g^{-a}\big(p(x)\big)\Big)$$
$$\leq C(1+u^\gamma+v^\gamma)\frac{1}{(1+u^\gamma)^a}$$
$$\leq C\frac{1}{1+u^{(a-1)\gamma}}.$$

Note that $u\geq\|x_1\|$, and we chose a such that $(a-1)\gamma>n$, hence $q(a-1)\gamma-q > q(n-1)$, thus the dominated convergence theorem applies, and we may conclude that

$$\lim_{h\to 0}\int_{B_h\backslash\Omega_h}\frac{1}{h}Q_h^W(g)^{1-a} = \lim_{h\to 0}\int_{\mathrm{R}^{n-1}}\left(\frac{1}{h}\int_{h+(1+h)\phi(x_1/(1+h))}^{h+\phi(x_1)} g^{\mathrm{i}-a}\left(x_1,\phi(x_1)\right)\mathrm{d}x_2\right)\mathrm{d}x_1$$
$$= \int_{\mathrm{R}^{n-1}}\big(x_1\cdot\nabla\phi(x_1)-\phi(x_1)\big)\, g^{1-a}\big(x_1,\phi(x_1)\big)\mathrm{d}x_1,$$

this last equality also being a dominated convergence result, using the hypotheses on g.

We just proved that whenever (g,W) is admissible, with $\int_\Omega g^{-a} = \int_{\Omega_1} W^{-a} = 1$, and ϕ satisfies the asymptotic growth condition (108), then

$$(a-n)\int_{\Omega} g^{1-a}+(a-1)\int_{\Omega}\frac{W^*(\nabla g)}{g^a}$$

$$-\int_{\mathrm{R}^{n-1}} g^{1-a}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1 \geq \int_{\Omega_1} W^{1-a}. \qquad (116)$$

Let $q>1$. We want to use this inequality with $W(x)=C\|x\|^q/q$, where $C>0$ is such that $\int_{\Omega_1} W^{-a}=1$. The goal being to prove Sobolev-type inequalities, we may consider only the real q such that their conjugate exponent $p=q/(q-1)$, which will appear in W^*, is strictly less than n. Thus, we assume that $q>n/(n-1)$, and conditions (CO), (C1) and (C2) are automatically satisfied with $\gamma=q$.

We now compute W^*:

$$W^*(y)=\sup_{x\in\Omega_1}\ \{x\cdot y-C\|x\|^q/q\}\leq \sup_{x\in\mathbb{R}^n}\ \{x\cdot y-C\|x\|^q/q\} \qquad (117)$$

$$=\sup_{R\geq 0}\ \sup_{\|x\|=R}\ \{x\cdot y-C\|x\|^q/q\}$$

$$=\sup_{R\geq 0}\ \{R\|y\|_*-CR^q/q\}$$

$$=C^{1-p}\|y\|_*^p/p.$$

It is important to note that (117) becomes an equality for $y=\nabla g(z)$ whenever $g(.)=W(.+e)$, since in that case,

$$W^*\big(\nabla g(z)\big)=\sup_{x\in\Omega_1}\{x\cdot\nabla g(z)-W(x)\}=\sup_{x\in\Omega}\{(x+e)\cdot\nabla g(z)-g(x)\}$$

$$=e\cdot\nabla g(z)+g^*\big(\nabla g(z)\big)$$

and the supremum is indeed reached inside the right set. Optimality is not lost, and inequality (116) then becomes

$$(a-n)\int_{\Omega} g^{1-a}+C^{1-p}\left(\frac{a-1}{p}\right)\int_{\Omega}\frac{\|\nabla g\|^p}{g^a}-\int_{\mathrm{R}^{n-1}} g^{1-a}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1\geq\int_{\Omega_1} W^{1-a}. \ (118)$$

The next step is to lift the restrictions on the function g, extending the results to more general functions. Our tool here will be approximation by admissible functions. Let $f\in C_c^\infty(\Omega)$ be a nonnegative function such that $\int_\Omega f^{ap/(a-p)}=1$. Let us fix some $\gamma>\max\{1,a/(n-1)\}$ and consider, for $\varepsilon>0$,

$$f_\varepsilon(x)=\left(\varepsilon\|x+e\|^{-\gamma(a-p)/p}+C_\varepsilon f\right),$$

where C_ε is such that $\int_\Omega f_\varepsilon^{ap/(a-p)}=1$, whenever ε is small enough for $C_{\xi j}$ to exist. It is not difficult to see that the corresponding functions $g_\in=f_\varepsilon^{p/(p-a)}$ satisfy conditions (C3) and

(C4), and that $\int_\Omega g_\varepsilon^{-a} = 1$. Furthermore, C_ε increases strictly as ε decreases towards 0, and an argument of continuity shows that $\lim_{\varepsilon\to 0} C_\varepsilon = 1$, meaning that, pointwise, $\lim_{\varepsilon\to 0} g_\varepsilon = f^{(p-a)/p} =: g$. Finally, the dominated convergence theorem, applied to $g_{\xi j}^{1-a} = f_\varepsilon^{(a-1)p/(a-p)}$, proves that inequality (118) is indeed valid for g. Rewriting it with f yields

$$(a-n)\int_\Omega f^{p\frac{a-1}{a-p}} + C^{1-p}\left(\frac{a-1}{p}\right)\left(\frac{p}{a-p}\right)^p \int_\Omega \| \nabla f\|^p$$

$$-\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1$$

$$\geq \int_{\Omega_1} W^{1-a}. \tag{119}$$

Finally, we may lift the growth condition on $\phi(21)$ and prove Theorem (6.3.4) in its full generality. Let ϕ be a convex function with $\phi(0)=0$, and let Ω be its epigraph. The subdifferential of ϕ at point $x \in \mathbb{R}^{n-1}$ is the convex set

$$\partial\phi(x) = \{v \in \mathbb{R}^{n-1} | \forall x' \in \mathbb{R}^{n-1}, \phi(x) - \phi(x') \geq v\cdot(x-x')\}.$$

Whenever ϕ is differentiable, the subdifferential coincides with the gradient. Next, given $x \in \mathbb{R}^{n-1}$ and $\in \partial\phi(x)$, we consider the tangent half-space

$$H_{x,v} = \{(y_1,y_2) \in \mathbb{R}^{n-1}\times\mathbb{R}, y_2 - \phi(x) \geq v\cdot(y_1 - x)\}.$$

For $R > 0$, define

$$\Omega_R = \cap_{\substack{x\in B(0,R)\\ v\in\partial\phi(x)}} H_{x,v}.$$

Ω_R is the epigraph of a convex function ϕ_R that coincides with the function ϕ on the ball $B(0,R) \in \mathbb{R}^{n-1}$, and its gradient is uniformly bounded by $\sup_{x\in B(0,R)}|\nabla\phi(x)| < +\infty$, so that it verifies the condition (108). Now, fix a function $\in C_c^\infty(\Omega)$. The support of f is inside a ball of radius R_0, so for any $R \geq R_0$, we may apply inequality (119) to the function f:

$$(a-n)\int_{\Omega_R} f^p\frac{a-1}{a-p} + A_R\int_{\Omega_R}\| \nabla f\|^p - \int_{\mathrm{R}^{n-1}} f^p\frac{a-1}{a-p}\left(x_1,\phi_R(x_1)\right)P_R(x_1)\mathrm{d}x_1$$

$$= (a-n)\int_\Omega f^p\frac{a-1}{a-p} + A_R\int_\Omega \| \nabla f\|^p - \int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-\mathrm{p}}}\left(x_1,\phi(x_1)\right)P(x_1)\mathrm{d}x_1 \geq \int_{\Omega_{R,1}} W_R^{1-a}.$$

where P_R, $\Omega_{R,1}$, and W_R are the usual definition of P, Ω, and W respectively, with function ϕ_R instead of ϕ. The constants are given by $A_R = C_R^{1-p}\frac{a-1}{p}\left(\frac{p}{a-p}\right)^p$ and $C_R > 0$ is such that

$\int_{\Omega_{R,1}} W_R^{-a} = 1$, i.e.

$$C_R = q^a \int_{\Omega_{R,1}} \| x\|^{-qa} \mathrm{d}x.$$

It is now easy to verify that $\lim_{R\to+\infty} C_R = q^a \int_{\Omega_1} \| x\|^{-qa} \mathrm{d}x = C; A_R$, and $\int_{\Omega_{1,R}} W_R^{1-a}$ also converge towards the right constants, so that equation (119) is still valid for the function ϕ, without any growth condition. Optimality remains to be shown. Let f be the optimal function (which does not have compact support) given in Theorem (6.3.20). First, note that

$$\lim_{R\to+\infty} \int_{\Omega_R} f^{p\frac{a-1}{a-\mathrm{p}}} = \int_{\Omega} f^{p\frac{a-1}{a-p}}, \lim_{R\to+\infty} \int_{\Omega_R} \| \nabla f\|^p = \int_{\Omega} \| \nabla f\|^p,$$

and also that for all R, f is an optimal function for inequality (119) on domain Ω_R. Then, by approximation by smooth functions with compact support, inequality (119) is true for f, so that, writing $A = \lim_{R\to+\infty} A_R$, and putting these facts together,

$$\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}} \left(x_1, \phi(x_1)\right) P(x_1) \mathrm{d}x_1 \leq (a-n) \int_{\Omega} f^p \frac{a-1}{a-p} + A \int_{\Omega} \| \nabla f\|^p - \int_{\Omega_R} W^{1-a}$$

$$= \lim_{R\to+\infty} \left((a-n) \int_{\Omega_R} f^{p\frac{a-1}{a-p}} + A_R \int_{\Omega_R} \| \nabla f\|^p - \int_{\Omega_{R,1}} W_R^{1-a} \right)$$

$$= \lim_{R\to+\infty} \left(\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-p}} \left(x_1, \phi_R(x_1)\right) P_R(x_1) \mathrm{d}x_1 \right).$$

We can decompose that last integral as a sum of the two following integrals

$$\int_{\mathrm{R}^{n-1}} f^p \frac{a-1}{a-p} \left(x_1, \phi_R(x_1)\right) \mathrm{d}x_1$$
$$- \int_{\mathrm{R}^{n-1}} f^p \frac{a-1}{a-p} \left(x_1, \phi_R(x_1)\right) \left(x_1 \cdot \nabla\phi_R(x_1) - \phi_R(x_1)\right) \mathrm{d}x_1.$$

By monotone convergence, the first term converges to the integral of the pointwise limit of its integrand. Furthermore, by convexity, for all $x_1 \in \mathbb{R}^{n-1}$, $x_1 \cdot \nabla\phi_R(x_1) - \phi_R(x_1) \geq 0$, so Fatou's lemma applied to the second term yields

$$\lim_{R\to+\infty} \left(\int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-\mathrm{p}}} \left(x_1, \phi_R(x_1)\right) P_R(x_1) \mathrm{d}x_1 \right) \leq \int_{\mathrm{R}^{n-1}} f^{p\frac{a-1}{a-\mathrm{p}}} \left(x_1, \phi(x_1)\right) P(x_1) \mathrm{d}x_1,$$

which finishes to prove equality in the previous inequalities, whence optimality.

Corollary (6.3.26)[222]; *Let* f_j, $g_j\colon \mathbb{R}^{1+2\epsilon} \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous* *functions. If f_j is nonnegative and g_j is coercive, that is,*

$$\lim_{\|x\|\to+\infty} g_j(x) = +\infty,$$

then $f_j \square g_j$ is exact.

Proof. Fix $x \in \mathbb{R}^{1+2\epsilon}$. Consider ψ: $\mathbb{R}^{1+2\epsilon} \to \mathbb{R} \cup \{+\infty\}$, $y \mapsto f_j(x-y) + g_j(y)$ and assume that there exists y_0 such that $\psi(y_0) < +\infty$: ψ is lower semicontinuous, and greater than g_j, thus tends to $+\infty$ as $\|y\|$ goes to $+\infty$. As such, $\{y \in \mathbb{R}^{1+2\epsilon}, \psi(y) \le \psi(y_0)\}$ is closed and bounded, thus compact. Now, let $(y_{1+2\epsilon}) \subset \{\psi \le \psi(y_0)\}$ be a minimizing sequence, $\lim_{1+2\epsilon\to+\infty} \psi\,(y_{1+2\epsilon}) = \inf_{y\in\mathrm{R}^{1+2\epsilon}}\{\psi(y)\}$. By compactness, we can assume that the sequence $(y_{1+2\epsilon})$ converges towards $z \in \mathbb{R}^{1+2\epsilon}$, and by lower semicontinuity, $-\infty < \psi(z) \le \lim_{1+2\epsilon\to+\infty} \psi\,(y_{1+2\epsilon}) = \inf_{y\in\mathrm{R}^{1+2\epsilon}}\{\psi(y)\}$, thus the infimum is finite and is actually a minimum. If such a y_0 does not exist, then $f_j \square g_j(x) = +\infty$, and the infimum is also reached.

Corollary (6.3.27)[222]; *Let g_j, $W: \mathbb{R}^{1+2\epsilon} \to (-\infty, +\infty]$ be two measurable functions. If g_j is nonnegative and almost everywhere differentiable on its domain* dom $g_j = \Omega_0$ (*with nonempty interior*), *and W grows superlinearly,*

$$\lim_{|x|\to+\infty} \frac{W(x)}{|x|} = +\infty,$$

then for almost every $x \in \Omega_0^{\mathrm{o}}$, $h \mapsto Q_h^W(g_j)(x)$ is differentiable at $h = 0$, and

$$\frac{\partial}{\partial h}|_{h=0} Q_h^W(g_j)(x) = -W^*\left(\nabla g_j(x)\right),$$

where W^* is the Legendre transform of W.

Proof. Let $\Omega_1 = \mathrm{dom}\, W$, and fix $x \in \Omega_0^{\mathrm{o}}$ such that the differential of g_j at x exists. Let $y \in \Omega_1$. For $h > 0$ sufficiently small, $x - hy \in \Omega_0$, and we get, by definition of $Q_h^W(g_j)$,

$$\frac{Q_h^W(g_j)(x) - g_j(x)}{h} \le \frac{g_j(x-hy) - g_j(x)}{h} + W(y)\,.$$

Taking the superior limit when $h \to 0$ yields

$$\lim_{h\to} \sup \frac{Q_h^W(g_j)(x) - g_j(x)}{h} \le -\nabla g_j(x)\cdot y + W(y)\,.$$

This being true for any $y \in \Omega_1$, we may take the infimum to find that

$$\lim_{h\to} \sup \frac{Q_h^W(g_j)(x) - g_j(x)}{h} \le -W^*\left(\nabla g_j(x)\right).$$

Conversely, fix $e \in \Omega_1$, and $h_0 > 0$ such that $\overline{B(x, h_0\|e\|)} \in$. For $h \in (0, h_0)$, define

$$\Omega_{x,h} = \{y \in \Omega_1, hW(y) \le g_j(x - he) + hW(e)\};$$

note that $e \in \Omega_{x,h}$. We claim that $\lim_{h\to\infty} \sup_{h\to 0} \{h\|y\|, y \in \Omega_{x,h}\} = 0$. Indeed, if $y \in \Omega_{x,h}$, then

$$h\|y\| \frac{W(y)}{\|y\|} \le g_j(x - he) + hW(e) \le \sup_{z\in\overline{B(x,h_0\|e\|)}} g_j(z) + h_0W(e) .$$

Now, when h goes to 0, either $\lim\ \sup\ \|y\| < +\infty$, or $\lim\ \sup\ \|y\| = +\infty$; in both cases, since $\lim_{|y|\to+\infty} \frac{W(y)}{\|y\|} = +\infty$, the claim is proved. Notice now that for all $h \in (0, h_0)$, $Q_h^W(g_j)(x) \le g_j(x - he) + hW(e)$, hence $Q_h^W(g_j)(x) = \inf_{y\in\Omega_{x,h}}\{. . .\}$. Thus,

$$\frac{Q_h^W(g_j)(x) - g_j(x)}{h} = \inf_{y\in\Omega_{x,k}} \left\{\frac{g_j(x - hy) - g_j(x)}{h} + W(y)\right\}$$

$$= \inf_{y\in\Omega_{x,k}} \{-\nabla g_j(x) \cdot y + y \cdot \varepsilon_x(hy) + W(y)\}$$

where $\varepsilon_x(z) \to 0$ when $\|z\| \to 0$. Let $1 \ge \eta > 0$; the claim proves that there exists $h_\eta \in (0, h_0)$ such that for all $0 < h < h_\eta$, $\forall y \in \Omega_{x,h}$, $\|\varepsilon_x(hy)\| \le \eta$. Thus,

$$\frac{Q_h^W(g_j)(x) - g_j(x)}{h} \ge \inf_{y\in\Omega_{x,k}} \{-\nabla g_j(x) \cdot y - \eta\|y\| + W(y)\}$$

$$= \inf_{\substack{y\in\Omega_{x,k} \\ y\in B(0,R)}} \{\dots\}$$

$$\ge \inf_{y\in\Omega_{x,k}} \{-\nabla g_j(x) \cdot y + W(y)\} - R\eta$$

$$\ge -W^*\left(\nabla g_j(x)\right) - R\eta,$$

where R was chosen such that $\|y\| \ge R \Rightarrow\ W(y) \ge \left(\left\|\nabla g_j(x)\right\| + 1\right)\|y\| + W(e) - \nabla g_j(x) \cdot e$. Finally, taking the inferior limit of this inequality, and noticing that the result stays true for any $0 < \eta \le 1$, we may conclude (since R is independent from η) that

$$\lim_{h\to 0}\frac{Q_h^W(g_j)(x)-g_j(x)}{h} = -W^*\left(\nabla g_j(x)\right).$$

This differentiation result is enough to prove the main theorems contained in section 3, but we can go a little bit further with more assumptions on g_j and W. Assuming W to be convex bestows upon Q_h^W a semigroup structure:

Corollary (6.3.28)[222]; *Assume that* g_j: $\mathbb{R}^{1+2\epsilon} \to [0,+\infty]$ *is lower semicontinuous, and that* W *is a lower semicontinuous proper convex function such that* $\lim W(x) = +\infty$. *Then, for all* $x \in \mathbb{R}^{1+2\epsilon}$ *and* $0 < s < h$,

$$Q_h^W(g_j)(x) = \min_{y\in\mathbb{R}^{1+2\epsilon}} \{g_j(x-hy)+hW(y)\}$$

$$= Q_{h-s}^W\left(Q_s^W(g_j)\right)(x).$$

Proof. Exactness was already proved in Lemma (6.3.26). Notice that

$$Q_{h-s}^W\left(Q_s^W(g_j)\right)(x) = \inf_{y\in\mathbb{R}}\inf_{z\in\mathbb{R}} \{g_j(x-(h-s)y-sz)+(h-s)W(y)+sW(z)\}$$

$$\le \inf_{y\in\mathbb{R}} \{g_j(x-hy)+hW(y)\} = Q_h^W(g_j)(x).$$

Conversely, let $y \in \mathbb{R}^{1+2\epsilon}$, and choose $z \in \mathbb{R}^{1+2\epsilon}$ such that

$$Q_s^W(g_j)(x-(t-s)y) = g_j(x-sz)+sW(z).$$

Then, by convexity,

$$Q_t^W(g_j)(x) \le g_j(x-(t-s)y-sz)+tW\left(\frac{t-s}{t}y+\frac{s}{t}z\right)$$

$$\le g_j(x-(t-s)y-sz)+(t-s)W(y)+sW(z)$$

$$= (t-s)W(y)+Q_s^W(g_j)(x-(t-s)y).$$

Taking the infimum over $y \in \mathbb{R}^{1+2\epsilon}$ proves that $Q_t^W(g_j)(x) \le Q_{h-s}^W\left(Q_s^W(g_j)\right)(x)$, and thus there is equality.

Corollary (6.3.29)[222]; *Let* g_j, W : $\mathbb{R}^{1+2\epsilon} \to \mathbb{R}$. *If* g_j *is nonnegative, locally Lipschitz continuous, and* W *is convex and coercive, then* $(h,x) \mapsto Q_h^W(g_j)$ *is locally Lipschitz continuous.*

Proof. In order to prove the full local Lipschitz continuity, we must first localize the arginf of the infimal convolution. Fix $\rho > 0$, $\eta > 0$, and let x, $x' \in B(0,\rho)$ and $0 < h < \eta$. Consider the set

$$\Omega_{x,h} := \{y \in \mathbb{R}^{1+2\epsilon}, g_j(x-y) + hW(y/h) \leq g_j(x) + hW(0)\}.$$

We claim that, by positivity of g_j, and convexity of W, the set is bounded. Indeed, since W is convex and coercive, there exists $R > 0$ and $m > 0$ such that

$$\|y\| > R \Rightarrow W(y) \geq m\|y\|.$$

If $y \in \Omega_{x,h}$, then either $\|y\| \leq hR \leq \eta R$, or $\|y\| > hR$ and then $g_j(x) + hW(0) \geq hW(y/h) \geq m\|y\|$. Invoking continuity of g_j, we may prove the claim, and conclude that there exists $R_{\rho,\eta}$, independent from x and h, such that $\Omega_{x,h} \subset B(0, R_{\rho,\eta})$.

Let us now prove the local Lipschitz continuity with respect to x. The functions g_j and W are assumed continuous, and so the infimal convolution is exact, and there exists $y \in \mathbb{R}^{1+2\epsilon}$ such that $Q_h^W(g_j)(x) = g_j(x-y) + hW(y/h)$. Necessarily, $\|y\| \leq R_{\rho,\eta}$, so

$$\sum_j (Q_h^W(g_j)(x') - Q_h^W(g_j)(x))$$

$$= \inf \sum_j \{g_j(x'-y') + hW(y'/h)\} - g_j(x-y) - hW(y/h)$$

$$\leq \sum_j g_j(x'-y) - g_j(x-y)$$

$$\leq \sum_j \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g_j\right) \|x - x'\|,$$

where $\mathrm{Lip}_A f_j := \sup_{x \neq x \in A}\{|f_j(x) - f_j(x')|/\|x - x'\|\}$. By symmetry, we conclude that

$$\sum_j |Q_h^W(g_j)(x') - Q_h^W(g_j)(x)| \leq \sum_j \left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g_j\right) \|x - x'\|,$$

hence the local Lipschitz continuity with respect to x.
Now,

$$\sum_j (Q_h^W(g_j)(x) - g_j(x)) = \inf_{y \in B(0,R_{\rho,1+2\epsilon})} \sum_j \{g_j(x-y) - g_j(x) + hW(y/h)\}$$

$$\geq \inf_{y \in B(0,R_{\rho,1+2\epsilon})} \sum_j \left\{-\left(\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g_j\right) \|y\| + hW(y/h)\right\}$$

$$= h \inf_{z\in B(0,R_{\rho,1+2\epsilon}/h)} \{-\lambda\|z\| + W(z)\} \geq -h \sup_{t\in(B(0,\lambda)} \{\lambda\|z\| - W(z)\} \geq -h \sup W^*(t)\,,$$

where $\lambda = \mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})} g_j$. Conversely, by definition,

$$Q_h^W(g_j)(x) - g_j(x) \leq hW(0)\,,$$

and thus $|Q_h^W(g_j)(x) - g_j(x)| \leq Ch$, where $C = \max\{W(0), \sup_{t\in B(0,\lambda)} W^*(t)\}$. Note that C is finite because W^* is, by definition, convex and finite on $\mathbb{R}^{1+2\epsilon}$, thus continuous. Finally, using the semigroup property $Q_{h+s}^W(g_j) = Q_h^W\left(Q_s^W(g_j)\right)$ and the fact that the Lipschitz constant with respect to x is uniformly bounded by $\mathrm{Lip}_{B(0,\rho+R_{\rho,\eta})}$ for $0 < h < \eta$, we may conclude for the full local Lipschitz continuity.

Corollary (6.3.30)[222]; *Let g_j, W : $\mathbb{R}^{1+2\epsilon} \to \mathbb{R}$. If g_j is nonnegative, locally Lipschitz continuous, and W is convex and grows superlinearly,*

$$\lim_{|x|\to+\infty} \frac{W(x)}{|x|} = +\infty,$$

and are such that $\int g_j^{-(1+2\epsilon)} = \int W^{-(1+2\epsilon)} = 1$, and if (g_j, W) is admissible in the sense of Definition (6.3.21), then the following statements are equivalent:

a. The Borell-Brascamp-Lieb inequality holds: for every $t \in [0,1]$ and H: $\mathbb{R}^{1+2\epsilon} \to \mathbb{R}$ such that

$$\forall x, y \in \mathbb{R}^{1+2\epsilon}, H\big((1-t)x + ty\big) \leq (1-t)g_j(x) + tW(y)\,,$$

there holds

$$\int H^{-(1+2\epsilon)} \geq 1.$$

b. *The following inequality stands:*

$$\int \frac{W^*(\nabla g_j)}{g_j^{2+2\epsilon}} \geq 0.$$

Proof. By definition of the infimal convolution $Q_h^W(g_j)$, it is actually sufficient to only consider the function $H = (1-t)Q_h^W(g_j)(./(1-t))$, where $h = t/(1-t)$, in statement a. In fact, this leads to the statement a

$$\int Q_h^W(g_j)^{-(1+2\epsilon)} \geq 1,$$

which we prove is equivalent to b.

Let us consider the function $\varphi\colon h \mapsto \int Q_h^W\left(g_j\right)^{-(1+2\epsilon)}$, which is continuous and almost everywhere differentiable in light of Lemma (6.3.29) and Theorem (6.3.22). Its derivative is given by

$$\varphi'(h) = (1+2\epsilon)\int \frac{W^*\left(\nabla g_j\right)}{g_j^{2(1+\epsilon)}}.$$

The implication $a'.$ ⇒b. follows from the fact that $\varphi(0) = 1$, and $\varphi(h) \geq 1$ for $h \geq 0$. Then, necessarily, $\varphi'(0) \geq 0$.

Conversely, assume that b. holds. Then, whenever $h > 0$ is such that $\varphi(h) = \int Q_h^W\left(g_j\right)^{-(1+2\epsilon)} = 1$, statement b. applied to the function $\tilde{g}_j = Q_h^W\left(g_j\right)$ and the corresponding function $\tilde{\varphi}$ implies that $\tilde{\varphi}'(0) = \varphi'(h) \geq 0$ thanks to the semigroup property proved in Lemma (6.3.28). This, together with the fact that $\varphi(0) = 1$, proves that φ stays above 1, which is exactly statement a.

Once again, we insist on the fact that the semigroup Q_h^W is not linear, and not Markov, which means, in particular, that there is no mass conservation. As such, this result stands as a bit unusual among similar results.

Corollary (6.3.31)[222]; *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$, $B_h = \Omega_h$ if, and only if, Ω is a convex cone. In that case, B_h and Ω_h coincide for all $h \geq 0$.*

Proof. First, note that $Q_h^W\left(g_j\right)(x) < +\infty$ if, and only if, there exists $y \in \Omega_1$ such that $x - hy \in \Omega$. By definition of Ω, this is equivalent to

$$\exists (y_1, y_2) \in \mathbb{R}^\epsilon \times \mathbb{R} \text{s.t.} \begin{cases} y_2 \geq \phi(y_1) + 1 \\ x_2 - hy_2 \geq \phi(x_1 - hy_1) \end{cases}$$
$$\Leftrightarrow \left(\exists y_1 \in \mathbb{R}^\epsilon \text{s.t.}\, x_2 \geq \phi(x_1 - hy_1) + h\phi(y_1) + h)\right).$$

If $x \in \Omega_h$, then choosing $y_1 = 0$ proves that $x \in B_h$, so $\Omega_h \subset B_h$. If $h > 0$, $\Omega_h = B_h$ if, and only if, for all $x_1, y_1 \in \mathbb{R}^\epsilon$,

$$\phi\left(\frac{x_1 - y_1}{h}\right) \geq \frac{\phi(x_1) - \phi(y_1)}{h}. \tag{120}$$

Indeed, if $\Omega_h \supset B_h$, then, for any $x_1, y_1 \in \mathbb{R}^\epsilon$,

$$x_2 := \phi(x_1 - hy_1) + h\phi(y_1) + h \geq \phi(x_1) + h$$

and thus, replacing y_1 by $(x_1 - y_1)/h$, we get the stated inequality. The reciprocal is immediate. Now, let $\in \mathbb{R}^\epsilon$, $|z| = 1$. Inequality (120), for $y_1 = 0$, becomes

$$\phi(z) \geq \frac{1}{h}\phi(hz)$$

for any h smaller than h_0. Let $\alpha = \limsup_{h\to 0} \phi(hz)/h$. Using inequality (120) once again, we get, for any $s \geq 0$,

$$\phi(sz) \geq \frac{s}{sh}\phi(shz)\,,$$

for any sufficiently small $h > 0$. Taking the inferior limit when $h \to 0$ proves that for any

$s \geq 0$

$$\phi(sz) \geq s\alpha. \tag{121}$$

The set $\{s \geq 0, \phi(sz) = s\alpha\}$ is non-empty because it contains 0, and it is closed by continuity. Let $s \geq 0$ be such that $\phi(sz) = s\alpha$. Then, invoking inequality (120), and then inequality (121), we get

$$\phi\left(\frac{(1+h)sz - sz}{h}\right) = \phi(sz) = s\alpha \geq \frac{\phi\big((1+h)sz\big) - \phi(sz)}{h} = \frac{\phi\big((1+h)sz\big) - s\alpha}{h}$$

$$\geq \frac{(1+h)s\alpha - s\alpha}{h} = s\alpha$$

so there is actually equality, and $\phi\big((1+h)sz\big) = (1+h)s\alpha$ for any sufficiently small $h > 0$. This shows that the connected component of $\{s \geq 0, \phi(sz) = s\alpha\}$ containing 0 is open in $\mathbb{R}_+$. Since it is also closed, it is the half real line $\mathbb{R}_+$. Thus, ϕ is linear over half-lines with initial point 0. Inequality (120) then becomes

$$\phi(x_1 - y_1) \geq \phi(x_1) - \phi(y_1)$$

for any $x_1, y_1 \in \mathbb{R}^\epsilon$. Let $t \in [0,1]$; replacing x_1 by $(1-t)x_1 + ty_1$ and y_1 by ty_1, and using linearity, the inequality becomes exactly the convexity inequality, that is

$$\phi\big((1-t)x_1 + ty_1\big) \leq (1-t)\phi(x_1) + t\phi(y_1)\,.$$

The reciprocal is trivial. It is also clear that in this case, $B_h = \Omega_h$ for *any* $h \geq 0$.

This lemma will be used to prove the trace Sobolev and the trace Gagliardo-Nirenberg-Sobolev inequalities in convex cones. We can go a bit further, and impose only ϕ to be convex.

Corollary (6.3.32)[222]; *If ϕ is convex, then*

$$B_h = \left\{ (x_1, x_2) \in \mathbb{R}^{1+\epsilon}, x_2 \geq h + (1+h)\phi\left(\frac{x_1}{1+h}\right)\right\}.$$

Proof. One may notice that setting $\omega(x) = 0$ if $x \in \Omega$ and $+\infty$ if $x \in \Omega^c$, and $W(x) = \omega(x - e)$, then ω is convex, thus

$$B_h = \operatorname{dom}\left(Q_h^W(\omega)\right) = \operatorname{dom}\left(x \mapsto (1+h)W\left(\tfrac{x+e}{1+h}\right)\right),$$

and

$$W\left(\frac{x+e}{1+h}\right) < +\infty \Leftrightarrow \frac{x+e}{1+h} - e \in \Omega$$

$$\Leftrightarrow x_2 \geq h + (1+h)\phi\left(\frac{x_1}{1+h}\right).$$

Corollary (6.3.33)[222]; *If $\big(g_j, W\big)$ is admissible, there exist constants $C > 0$ and $h_0 > 0$, such that for all $0 < h < h_0$, and $x \in \Omega_h$,*

$$|Q_h^W\big(g_j\big)(x) - g_j(x)| \leq Ch(1 + \|x\|^\gamma)\,.$$

Proof. First, let $x', x \in \Omega$. Then, we may estimate $|g_j(x') - g_j(x)|$ using hypothesis (C4):

$$\sum_j |g_j(x') - g_j(x)| \le \int_0^1 \sum_j \|\frac{\partial}{\partial\theta} g_j\big(x + \theta(x' - x)\big)\| \mathrm{d}\theta$$

$$\le \|x' - x\| \int_0^1 A_4 (1 + \|x + \theta(x' - x)\|^{\gamma-1}) \mathrm{d}\theta$$

$$\le C\|x' - x\|(1 + \|x\|^{\gamma-1} + \|x' - x\|^{\gamma-1}) . \tag{122}$$

Now, let $0 < h \le 1$ and $x \in \Omega_h$. Then, $x - he \in \Omega$, so

$$Q_h^W\big(g_j\big)(x) - g_j(x) \le g_j(x - he) + hW(e) - g_j(x)$$

$$\le Ch(1 + \|x\|^{\gamma-1} + h^{\gamma-1}) + hW(e)$$

$$\le Ch(1 + \|x\|^{\gamma-1}) .$$

For the converse inequality, we will of course use hypotheses (C1) and (C3), but we first have to localize the point where the infimum $Q_h^W\big(g_j\big)(x)$ is reached. Let $y \in \Omega_1$ be such that $Q_h^W\big(g_j\big)(x) = g_j(x - hy) + hW(y)$. Then, invoking hypothesis (C1) and inequality (122),

$$hA_1\|y\|^\gamma \le hW(y) = Q_h^W\big(g_j\big)(x) - g_j(x - hy)$$

$$= Q_h^W\big(g_j\big)(x) - g_j(x) + g_j(x) - g_j(x - hy)$$

$$\le Ch(1 + \|x\|^{\gamma-1}) + Ch\|y\|(1 + \|x\|^{\gamma-1} + (h\|y\|)^{\gamma-1}) .$$

We thus choose $h_0 \in (0,1)$ such that for any $h \in (0, h_0)$, $A_1 - Ch^{\gamma-1} > h^{\gamma-1}$. Then, for any $h \in (0, h_0)$,

$$h^{\gamma-1}\|y\|^\gamma < (A_1 - Ch^{\gamma-1})\|y\|^\gamma$$

$$\le C(1 + \|y\|)(1 + \|x\|^{\gamma-1}) ,$$

which implies that

$$h^{\gamma-1}\|y\|^{\gamma-1} \le C(1 + \|x\|^{\gamma-1}) ,$$

since $\|y\|^{\gamma-1} \le \max\left(1,2\frac{\|y\|^\gamma}{1+\|y\|}\right)$. Now, using inequality (122) once again,

$$|g_j(x - hy) - g_j(x)| \le Ch\|y\|(1 + \|x\|^{\gamma-1} + h^{\gamma-1}\|y\|^{\gamma-1})$$

$$\le Ch\|y\|(1 + \|x\|^{\gamma-1}) .$$

Plugging this in the definition of $Q_h^W\big(g_j\big)(x)$, we find

$$Q_h^W\big(g_j\big)(x) - g_j(x) \ge \inf_{y\in\Omega_1} \{-Ch\|y\|(1 + \|x\|^{\gamma-1}) + hA_1\|y\|^\gamma\}$$

$$\ge \inf_{y\in\mathbb{R}^{1+2\epsilon}} \{\ldots\} = -Ch(1 + \|x\|^\gamma) .$$

To conclude, it is enough to notice that $1+\|x\|^{\gamma-1} \leq 2+\|x\|^{\gamma}$ since $\gamma > 1$.

Now that we have this estimation, we may estimate the speed of convergence of $Q_h^W(g_j)^{-3\epsilon}$ towards $g_j^{-3\epsilon}$.

Corollary (6.3.34)[222]; *If (g_j, W) is admissible, there exist constants $C > 0$ and $h_0 > 0$, such that for all $0 < h < h_0$, and $x \in \Omega_h$,*

$$\sum \frac{|Q_h^W(g_j)^{-3\epsilon}(x) - g_j^{-3\epsilon}(x)|}{h} \leq \frac{C}{1+\|x\|^{\gamma(3\epsilon)}}.$$

Proof. First, let , $\beta > 0$. Then,

$$\left|\int_\alpha^\beta t^{-(1+3\epsilon)}\,\mathrm{d}t\right| = \left|\frac{1}{-3\epsilon}(\beta^{-3\epsilon} - \alpha^{-3\epsilon})\right| \leq \max\left(\alpha^{-(1+3\epsilon)}, \beta^{-(1+3\epsilon)}\right)|\alpha - \beta|,$$

implying that

$$|\alpha^{-3\epsilon} - \beta^{-3\epsilon}| \leq (3\epsilon)|\alpha - \beta|\left(\alpha^{-(1+3\epsilon)} + \beta^{-(1+3\epsilon)}\right). \tag{123}$$

Then, according to Corollary (6.3.33), there exists $h_0 > 0$ such that for any $h \in (0, h_0)$, and any $x \in \Omega_h$,

$$\sum_j \frac{|Q_h^W(g_j)^{-3\epsilon}(x) - g_j^{-3\epsilon}(x)|}{h}$$

$$\leq C \sum_j \frac{|Q_h^W(g_j)(x) - g_j(x)|}{h}\left(Q_h^W(g_j)^{-(1+3\epsilon)}(x) + g_j^{-(1+3\epsilon)}(x)\right)$$

$$\leq C(1+\|x\|^{\gamma}) \sum_j \left(Q_h^W(g_j)^{-(1+3\epsilon)}(x) + g_j^{-(1+3\epsilon)}(x)\right). \tag{124}$$

Now, hypotheses (C1) and (C3) and a straightforward computation yield

$$Q_h^W(g)(x) \geq \inf y \in \Omega_1 \ \{A_3(1+\|x-hy\|^{\gamma}) + hA_1\|y\|^{\gamma}\}$$
$$\geq \ \inf\{A_3(1+|\|x\| - h\|y\||^{\gamma}) + hA_1\|y\|^{\gamma}\}$$
$$\geq C(1+\|x\|^{\gamma}).$$

Using (C3) once again, we know that

$$g_j^{-(1+3\epsilon)}(x) \leq \left(A_3(1+\|x\|^{\gamma})\right)^{-(1+3\epsilon)};$$

putting these two inequalities together with inequality (124), we finally obtain

$$\sum \frac{|Q_h^W(g_j)^{-3\epsilon}(x) - g_j^{-3\epsilon}(x)|}{h} \leq C\frac{1+||x||^{\gamma}}{(1+\|x\|^{\gamma})^{1+3\epsilon}}$$
$$\leq \frac{C}{1+\|x\|^{\gamma(3\epsilon)}}.$$

Corollary (6.3.34), together with Corollary (6.3.27), proves the dominated convergence, and

$$\lim_{h\to 0}(i) = (3\epsilon)\int_{\Omega} \sum \frac{W^*(\nabla g_j)}{g_j^{1+3\epsilon}},$$

as claimed. The convergence of (*ii*) is straightforward, as it is a direct implication of the local Lipschitz continuity of g_j and hypothesis (C3).

Corollary (6.3.35)[222]; *If* (g_j, W) *is admissible, there exist constants* $C > 0$ *and* $h_1 > 0$, *such that for at* $0 < h < h_1$, *and* $(x_1, x_2) \in B_h \setminus \Omega_h$,

$$\left|Q_h^W(g_j)(x_1,x_2) - g_j(x_1,\phi(x_1))\right| \leq hC\left(1 + \|(x_1,\phi(x_1))\|^\gamma + |x_1 \cdot \nabla\phi(x_1)|^\gamma\right). \quad (125)$$

The proof follows the same logic as the proof of Corollary (6.3.33).

Proof. Recall that, according to Corollary (6.3.32)

$$\Omega_h = \{ (x_1,x_2) \in \mathbb{R}^n, x_2 \geq h + \phi(x_1)\}, B_h = \left\{(x_1,x_2) \in \mathbb{R}^n, x_2 \geq h + (1+h)\phi\left(\frac{x_1}{1+h}\right)\right\},$$

and that

$$|g_j(x') - g_j(x)| \leq C\|x' - x\|(1 + \|x\|^{\gamma-1} + \|x' - x\|^{\gamma-1}) \qquad \text{(23 revisited)}$$

for any $x', x \in \Omega$.

Fix $h \in (0,1)$, $x = (x_1,x_2) \in B_h/\Omega_h$, and define $p(x_1,x_2) = (x_1,\phi(x_1))$, its projection onto $\partial\Omega$. Letting $y = \left(\frac{x_1}{1+h}, 1 + \phi\left(\frac{x_1}{1+h}\right)\right)$, we find that $y \in \Omega_1$, and also that $x - hy \in \Omega$, thus, with hypothesis (C2) and inequality (122),

$$\begin{aligned} Q_h^W(x) - g_j(p(x)) &\leq g_j(x - hy) - g_j(p(x)) + hW(y) \\ &\leq C\|x - hy - p(x)\|(1 + \|p(x)\|^{\gamma-1} + \|x - hy - p(x)\|^{\gamma-1}) \\ &\quad + hA_2(1 + \|y\|^\gamma). \end{aligned}$$

For brevity, let us write $u = \|p(x)\| = \|(x_1,\phi(x_1)\|$, and $v = x_1 \,.\, \phi(x_1) = |x_1 \cdot \phi(x_1)|$. Now, notice that

$$\|x - hy - p(x)\| = h\left\|\left(\frac{x_1}{1+h}, \frac{x_2 - h - \phi(x_1)}{h} - \phi\left(\frac{x_1}{1+h}\right)\right)\right\|.$$

From the definition of Ω_h and B_h, we find out that

$$\begin{aligned} 0 \leq \frac{h + \phi(x_1) - x_2}{h} &\leq \frac{\phi(x_1) - (1+h)\phi(x_1/(1+h))}{h} \\ &\leq x_1 \cdot \nabla\phi(x_1) = v, \end{aligned}$$

since ϕ is convex and nonnegative. Thus,

$$\|x-hy-p(x)\| \le h\left\|\left(\frac{x_1}{1+h},\phi\left(\frac{x_1}{1+h}\right)\right)\right\| + h|x_1\cdot\nabla\phi(x_1)|$$

$$\le h(\|p(x)\| + |x_1\cdot\nabla\phi(x_1)|) = h(u+v)\,,$$

so, since $h<1$,

$$1+\|p(x)\|^{\gamma-1}+\|x-hy-p(x)\|^{\gamma-1} \le 1+u^{\gamma-1}+\big(h(u+v)\big)^{\gamma-1}$$
$$\le C(1+u^{\gamma-1}+v^{\gamma-1})\,.$$

Finally,

$$A_2(1+\|y\|^\gamma) = A_2\left(1+\left\|\tfrac{x_1}{1+h},1+\phi\left(\tfrac{x_1}{1+h}\right)\right\|^\gamma\right) \le C(1+u^\gamma)\,.$$

Putting all these inequalities together, we find

$$Q_h^W(x)-g_j\big(p(x)\big) \le hC(u+v)(1+u^{\gamma-1}+v^{\gamma-1})+hC(1+u^\gamma)$$
$$\le hC(1+u^\gamma+v^\gamma)\,. \tag{126}$$

Conversely, let $y\in\Omega_1$ be such that $Q_h^W\big(g_j\big)(x)=g_j(x-hy)+hW(y)$. As before, we localize y. Using hypothesis A_1 and inequalities (122) and (126),

$$hA_1\|y\|^\gamma \le hW(y) = Q_h^W\big(g_j\big)(x)-g_j\big(p(x)\big)+g_j\big(p(x)\big)-g_j(x-hy)$$
$$\le hC(1+u^\gamma+v^\gamma)+C\|x-hy-p(x)\|(1+u^{\gamma-1}+\|x-hy-p(x)\|^{\gamma-1})$$
$$\le hC(1+u^\gamma+v^\gamma)+hC(\|y\|+v)(1+u^{\gamma-1}+h^{\gamma-1}(\|y\|+v)^{\gamma-1})$$
$$\le hC(1+u^\gamma+v^\gamma)+hC(\|y\|+v)(1+u^{\gamma-1}+h^{\gamma-1}\|y\|^{\gamma-1}+v^{\gamma-1})\,.$$

Rearranging the terms and dividing by h yields

$$A_1\|y\|^\gamma - Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\|+v) \le C(1+u^\gamma+v^\gamma)+C(\|y\|+v)(1+u^{\gamma-1}+v^{\gamma-1})$$
$$\le C(1+u+v+\|y\|)(1+u^{\gamma-1}+v^{\gamma-1})\,.$$

We must now split the reasoning in two cases: either $\|y\|\le v$, in which case the conclusion follows, or $\|y\|\ge v$, and then $A_1\|y\|^\gamma - Ch^{\gamma-1}\|y\|^{\gamma-1}(\|y\|+v) \ge A_1\|y\|^\gamma - 2Ch^{\gamma-1}\|y\|^\gamma$. We thus choose $0<h_1<1$ such that for all $h\in(0,h_1)$, $A_2-2Ch^{\gamma-1}\ge h^{\gamma-1}$. Then, we have, for any $h\in(0,h_1)$,

$$\frac{h^{\gamma-1}\|y\|^\gamma}{1+u+v+\|y\|} \le C(1+u^{\gamma-1}+v^{\gamma-1})\,.$$

Once again, either $\|y\|\le 1+u+v$, or

$$h^{\gamma-1}\|y\|^{\gamma-1} \le \frac{2h^{\gamma-1}\|y\|^\gamma}{1+u+v+\|y\|}.$$

Taking the greatest of the constants in those two cases, we may conclude that

$$h^{\gamma-1}\|y\|^{\gamma-1} \le C(1+u^{\gamma-1}+v^{\gamma-1})\,. \tag{127}$$

We may now proceed with the converse inequality. Invoking once again inequality (122), and then inequality (127),

$$|g_j(x-hy)-g_j\big(p(x)\big)| \le hC(\|y\|+v)(1+u^{\gamma-1}+h^{\gamma-1}\|y\|^{\gamma-1}+v^{\gamma-1})$$

$$\leq hC(\|y\| + v)(1 + u^{\gamma-1} + v^{\gamma-1})\,.$$

Finally,

$$Q_h^W(g_j)(x) - g_j(p(x)) = g_j(x - hy) - g_j(p(x)) + hW(y)$$
$$\geq -hC(\|y\| + v)(1 + u^{\gamma-1} + v^{\gamma-1}) + hA_2\|y\|^\gamma$$
$$\geq h\inf\{-C(\|y\| + v)(1 + u^{\gamma-1} + v^{\gamma-1}) + A_2\|y\|^\gamma\}$$
$$\geq -hC(1 + u^{\gamma-1} + v^{\gamma-1})^{\gamma/(\gamma-1)}$$

and we may conclude.

We may now prove Theorem (6.3.22): using the same notations as in the proof above, that is $u = \|p(x)\| = \|(x_1, \phi(x_1))\|$, and $v = x_1 \cdot \phi(x_1) = |x_1 \cdot \phi(x_1)|$, hypothesis (C2) immediately yields, for all $h > 0$ and all $x \in B_h \backslash \Omega_h$,

$$g_j^{-(1+3\epsilon)}(p(x)) \leq \frac{C}{(1 + u^\gamma)^{1+3\epsilon}}.$$

Furthermore, inequality (125) and hypothesis (C2) yield

$$Q_h^W(g_j)(x) \geq -hC(1 + u^\gamma + v^\gamma) + C(1 + u^\gamma)$$

for all $x \in B_h \backslash \Omega_h$ and $0 < h < h_1$. Now, assumption (21) reads: for all $x_1 \in \mathbb{R}^{2\epsilon}$ such that $\|x_1\| > R$,

$$v \leq Cu.$$

Since both u and v are bounded functions of x on the set $\{(x_1, x_2) \in B_h \backslash \Omega_h, \|x_1\| \leq R\}$, there exists $h_2 > 0$ such that, for all $0 < h < h_2$,

$$Q_h^W(g_j)(x) \geq \begin{cases} C > 0 & \text{whenever } \|x_1\| \leq R \\ C(1 + u^\gamma) & \text{whenever } \|x_1\| > R \end{cases}$$

Thus, for all $0 < h < h_2$ and all $x \in B_h \setminus \Omega_h$,

$$Q_h^W(g_j)^{-(1+3\epsilon)}(x) \leq \frac{C}{(1 + u^\gamma)^{1+3\epsilon}}.$$

Finally, invoking inequality (123) together with assumption (21) yields, for any $0 < h < h_2$ and $x = (x_1, x_2) \in B_h \backslash \Omega_h$,

$$\sum_j \frac{|Q_h^W(g_j)^{-3\epsilon}(x) - g_j^{i-(1+3\epsilon)}(p(x))|}{h}$$
$$\leq C\sum_j \frac{|Q_h^W(g_j)(x) - g_j(p(x))|}{h}\left(Q_h^W(g_j)^{-(1+3\epsilon)}(x) + g_j^{-(1+3\epsilon)}(p(x))\right)$$
$$\leq C(1 + u^\gamma + v^\gamma)\frac{1}{(1 + u^\gamma)^{1+3\epsilon}}$$
$$\leq C\frac{1}{1 + u^{(3\epsilon)\gamma}}.$$

Note that $u \geq \|x_1\|$, and we chose $1 + 3\epsilon$ such that $(3\epsilon)\gamma > 1 + 2\epsilon$, hence $(1 + \epsilon)(3\epsilon)\gamma - (1 + \epsilon) > 2\epsilon(1 + \epsilon)$, thus the dominated convergence theorem applies, and we may conclude that

$$\lim_{h\to 0}\int_{B_h\setminus\Omega_h}\sum \frac{1}{h} Q_h^W(g_j)^{-3\epsilon}$$

$$= \lim_{h\to 0}\int_{\mathrm{R}^{2\epsilon}}\left(\frac{1}{h}\int_{h+(1+h)\phi(x_1/(1+h))}^{h+\phi(x_1)}\sum g_j^{\mathrm{i}-(1+3\epsilon)}\left(x_1,\phi(x_1)\right)\mathrm{d}x_2\right)\mathrm{d}x_1$$

$$= \int_{\mathrm{R}^{2\epsilon}}\sum \left(x_1\cdot\nabla\phi(x_1)-\phi(x_1)\right) g_j^{-3\epsilon}\left(x_1,\phi(x_1)\right)\mathrm{d}x_1,$$

this last equality also being a dominated convergence result, using the hypotheses on g_j.

List of Symbols

Symbol		Page
BMO:	Bounded mean oscillation	1
$\dot{H}^{s,r}$:	Sobolev space	1
L^1:	Lebesgue integral in the line	1
L^p:	Lebesgue space	2
max:	maximum	2
min:	minimum	4
$a.e$:	Almost every where	8
sup:	supremum	11
L^∞:	Essential Lebesgue space	11
L^q:	Dual of Lebesgue space	11
loc:	local	19
L^2:	Hilbert space	25
Aut:	Automorphism	25
A_α^p:	Weighted Bergman space	25
B_p^s:	Besov space	29
$W_{k,\beta}^p$:	Sobolev space	29
inf:	infimum	33
H^q:	Dual of Hardy space	40
ℓ^p:	Dual of Banach space of sequence	45
Harm:	Hormonic	57
Im:	imaginary	64
Re:	real	64
det:	determinant	80
$W^{1,p}$:	Sobolev space	84
tr:	trace	90
$supp$:	support	91
var:	variance	94
mes:	measurable	105
$L^{p,q}$:	Lorentz space	150
arg:	Argument	160
$glob$:	Global	161
deg:	degree	161
$Voln$:	Euchidean volume	195
BBL:	Borell – Brqscap – Lieb	211
dom:	domain	246
epi:	epigraph	246
Lip:	Lipschitz	250
co:	condition	260

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