Sudan University of Science and Technology
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Self-Similar Measures with Scaling of Spectra and Beurling Dimension

قياسات التشابه الذاتي مع التدرج الطيفي وبعد بيرلنغ

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

Acknowledgements

I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide. Prof. Dr. Shawgy Hussein AbdAlla.

Abstract

The Singularity Versus Exact Dimensionality, Beurling dimension, Overlaps, Projections of Random Self-Similar measures and sets are studied. The Dimension Conservation and scaling of spectral and a class of random convolution on the real line of self-similar sets and measures with consecutive digits and fractal Percolation are discussed. We obtain the spactra of a Cantor, Moran and Bernoulli measures with Exponential spectra in the Hilbert space. We investigate the self-affine and self-similar measures with dense rotations, singular projection, discrete slices and vector-valued representations. The uniformity, Hausdorff and Packing Measures with Fourier Frames and Slices of Dynamically Defined sets are constructed.

الخلاصة

قمنا بدراسة الشذوذية مقابل الأبعاد المضبوضة وبعد بيرلينج والتداخل وإسقاطات قياسات مماثلة ذاتيا عشوائية والفئات تم مناقشة حفظ البعد والتدريج الطيفي وعائلة الالتفاف العشوائي على الخط الحقيقي لفئات المماثلة الذاتية والقياسات مع الارقام المتتالية والنفاذ الي الكسورية قمنا بالحصول علي طيف كانتور وقياسات موران وبيرنولي مع الطيف الاسي في فضاء هلبرت تم بحث القياسات النسيبية الذاتية والمماثلة الذاتية مع الدورات الكثيفة والإسقاط الشاذ والخط المتقطع والتمثيلات قيّمة المتجه. قمنا بتشخيص القياسات المنتظمة وهاوسدورف والتعبئة مع إطارات فوريير وشرائح الفئات المعرفة ديناميكيا.

Introduction

We study the geometric properties of random multiplicative cascade measures defined on self-similar sets. We show that such measures and their projections and sections are almost surely exact-dimensional, generalizing Feng and Hu's result [11] for self-similar measures. We introduce a technique that uses projection properties of fractal percolation to establish dimension conservation results for sections of deterministic self-similar sets. For example, let K be a self-similar subset of $\mathbb{R}^2$ with Hausdorff dimension $\dim_H K > 1$ such that the rotational components of the underlying similarities generate the full rotation group. Then for all $\epsilon > 0$, writing θ_0 for projection onto the line L_θ in direction θ, the Hausdorff dimensions of the sections satisfy $\dim_H d\left(K \cap \pi_\theta^{-1} x\right) > \dim_H K - 1 - \epsilon$ for a set of $x \in L_\theta$ of positive Lebesgue measure, for all directions θ except for those in a set of Hausdorff dimension 0.

We analyze all orthonormal bases of exponentials on the Cantor set defined by Jorgensen and Pedersen. A complete characterization for all maximal sets of orthogonal exponentials is obtained by establishing a one-to-one correspondence with the spectral labelings of the infinite binary tree. With the help of this characterization we obtain a sufficient condition for a spectral labeling to generate a spectrum (an orthonormal basis). This result not only provides us an easy and efficient way to construct various of new spectra for the Cantor measure but also extends many previous results in the literature. In fact, most known examples of orthonormal bases of exponentials correspond to spectral labelings satisfying this sufficient condition. For $\{\mathcal{D}_k\}_{k=1}^{\infty}$ be a sequence of digit sets in $\mathbb{N}$ and let $\{b_k\}_{k=1}^{\infty}$ be a sequence of integer numbers bigger than 1. We call the family $\left\{f_{k,\mathcal{D}_k}(x) = b_k^{-1}(x+d) \colon d \in \mathcal{D}_k, k \geq 1\right\}$ a Moran iterated function system (IFS), which is a natural generalization of an IFS. For $0 < \rho < 1$ and $N > 1$ an integer, let μ be the self-similar measure defined by $\mu(\cdot) = \sum_{i=0}^{N-1} \frac{1}{N} \mu(\rho^{-1}(\cdot) - i)$. We prove that $L^2(\mu)$ has an exponential orthonormal basis if and only if $\rho = \frac{1}{q}$ for some $q > 0$ and N divides q.

For A be a $d \times d$ integral expanding matrix and let $S_j(x) = A^{-1}\left(x + d_j\right)$ for some $d_j \in \mathbb{Z}^d, j = 1, \dots, m$. The iterated function system (IFS) $\left\{S_j\right\}_{j=1}^{m}$ generates self-affine measures and scale functions. In general this IFS has overlaps, and it is well known that in many special cases the analysis of such measures or functions is facilitated by expressing them in vector-valued forms with respect to another IFS that satisfies the open set condition. We examine Fourier frames and, more generally, frame measures for different probability measures. We prove that if a measure has an associated frame measure, then it must have a certain uniformity in the sense that the weight is distributed quite uniformly on its support. To be more precise, by considering certain absolute continuity properties of the measure and its translation, we recover the characterization on absolutely continuous measures $g\,dx$ with Fourier frames obtained. Moreover, we prove that the frame bounds are pushed away by the essential infimum and supremum of the function g. This also shows that absolutely continuous spectral measures supported on a set Ω, if they exist, must be the standard Lebesgue measure on Ω up to a multiplicative constant. We consider equally-weighted Cantor measures $\mu_{q,b}$ arising from iterated function systems of the form $b^{-1}(x+i), i = 0, 1, \dots, q-1$, where $q < b$. We classify the (q, b) so that they have infinitely many mutually orthogonal exponentials in $L^2\left(\mu_{q,b}\right)$. In particular, if q divides b, the measures have a complete orthogonal exponential system and hence spectral measures. Improving the construction, we characterize all the maximal orthogonal sets Λ when q divides b via a maximal mapping on the q−adic tree in which all elements in Λ are represented uniquely in finite b−adic expansions and we can separate the maximal orthogonal sets into two types: regular and irregular sets. For a regular maximal orthogonal set, we show that its completeness in $L^2(\mu_{q,b})$ is crucially determined by the certain growth rate of non-zero digits in the tail of the b−adic expansions of the elements.

For $1 \leq m < n$ be integers, and let $K \subset \mathbb{R}^n$ be a self-similar set satisfying the strong separation condition, and with $\dim K = s > m$. We study the a.s. values of the $s - m$-dimensional Hausdorff and packing measures of $K \cap V$, where V is a typical $n - m$-dimensional affine subspace. We present some one-parameter families of homogeneous self-similar measures on the line such that, the similarity dimension is greater than 1 for all parameters and the singularity of some of the self-similar measures from this family is not caused by exact overlaps between the cylinders. We construct a planar homogeneous self-similar measure, with strong separation, dense rotations and dimension greater than 1, such that there exist lines for which dimension con-servation does not hold and the projection of the measure is singular.

A spectrum of a probability measure μ is a countable set Λ such that $\{\exp(-2\pi i\lambda \cdot), \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\mu)$. We consider the problem when a countable set become the spectrum of the Cantor measure. Starting from tree labeling of a maximal orthogonal set, we introduce a new quantity to measure minimal level difference between a branch of the labeling tree and its subbranches. Then we use boundedness and linear increment of that level difference measurement to justify whether a given maximal orthogonal set is a spectrum or not. This together with the tree labeling of a maximal orthogonal set provides fine structures of spectra of Cantor measures. Given a Borel probability measure μ on R and a real number p. We call pa spectral eigenvalue of the measure μ if there exists a discrete set Λ such that the sets

$$E(\Lambda) := \left\{e^{2\pi i\lambda x} : \lambda \in \Lambda\right\} \quad \text{and} \quad E(p\Lambda) := \left\{e^{2\pi i p\lambda x} : \lambda \in \Lambda\right\}$$

are both orthonormal basis for Hilbert space $L^2(\mu)$. We consider the equally-weighted Cantor measures $\mu_{p,q}$ generated by the iterated function system (IFS) $\left\{f_i(x) = \frac{x}{p} + \frac{i}{q}\right\}_{i=0}^{q-1}$, where $2 \leq q \in \mathbb{Z}$ and $q < p \in \mathbb{R}$. It is known that if q divides p, then $\mu_{p,q}$ is a spectral measure with a spectrum

$$\Lambda_{p,q} = \{0,1,\dots,q-1\} + p\{0,1,\dots,q-1\} + p^2\{0,1,\dots,q-1\} + \cdots \text{ (finite sum)}$$

(Dai, He and Lai (2013)[362]).

For μ be a Borel probability measure with compact support. We consider exponential type orthonormal bases, Riesz bases and frames in $L^2(\mu)$. We show that if $L^2(\mu)$ admits an exponential frame, then μ must be of pure type. We also classify various μ that admits either kind of exponential bases, in particular, the discrete measures and their connection with integer tiles. We study the Beurling dimension of Bessel sets or sequence and frame spectra of some self-similar measures on $\mathbb{R}^d$.

The Contents

Chapter 1
Exact Dimensionality and Dimension Conservation

We show that a compact group extension argument, enables us to generalize Hochman and Shmerkin's theorems on projections of deterministic self-similar measures [14] to these random measures without requiring any separation conditions on the underlying sets. We give applications to self-similar sets and fractal percolation, including new results on projections, C^1-images and distance sets. For a class of self-similar sets we obtain a similar conclusion for all directions, but with lower box dimension replacing Hausdorff dimensions of sections. We obtain similar inequalities for the dimensions of sections of Mandelbrot percolation sets.

Section (1.1): Projections of Random Self-Similar Measures and Sets:

Relating the Hausdorff dimension of a set $K \subseteq \mathbb{R}^d$ to the dimensions of its projections and sections has a long history. The most basic result, due to Marstrand [22] in the plane and to Mattila [24] more generally, is that if $K \subseteq \mathbb{R}^d$ is Borel or analytic, then, writing $\Pi_{d,k}$ for the family of orthogonal projections from $\mathbb{R}^d$ onto its k-dimensional subspaces,

$$\dim_H \pi K = \min(k, \dim_H K) \tag{1}$$

for almost all $\pi \in \Pi_{d,k}$ with respect to the natural invariant measure on $\Pi_{d,k}$, where $\dim_H$ denotes Hausdorff dimension. We discuss the dimensions of sections or slices of sets and show that for almost all $\pi \in \Pi_{d,k}$, if $\dim_H K > k$, the sections $\pi^{-1}x \cap K$ satisfy

$$\dim_H(\pi^{-1}x \cap K) \le \dim_H K - k \tag{2}$$

for Lebesgue almost all $x \in \pi(K)$, with equality for a set of x of positive Lebesgue measure; see [25] for a good exposition of this material.

The Hausdorff dimension of a probability measure μ is defined as

$$\dim_H \mu = \inf\ \{\dim_H K : \mu(K) > 0\}. \tag{3}$$

The dimension properties of projections and sections of measures directly parallel those for sets; indeed the conclusions for sets generally follow from the measure analogues.

These classical results have been extended beyond recognition, for example to families of generalized projections [29], to obtain estimates on the size of 'exceptional' projections π for which the conclusions (1) or (2) fail [29], and to packing dimensions [10]. Almost all of this work concerns sections and projections of general Borel or analytic sets K for which the possibility of exceptional projections can never be excluded. It has recently been noted that for specific classes of sets and measures the dimensions of projections or sections may be constant for all π, or at least it may be possible to identify the exceptional π. In particular, highly innovative approaches of Hochman and Shmerkin [14] and Furstenberg [12] have addressed this for self-similar sets and measures, we generalise their results to a random setting.

A family of contractions $\mathcal{I} = \{f_i\}_{i=1}^m$ on $\mathbb{R}^d$, referred to as an iterated function system (IFS), defines a unique non-empty compact set K such that

$$K = \bigcup_{i=1}^{m} f_i(K); \tag{4}$$

K is termed the attractor of the IFS, see, for example, [9]. Here we consider an IFS of contracting similarities

$$\mathcal{I} = \{f_i = r_i O_i \cdot + t_i\}_{i=1}^m, \tag{5}$$

where each f_i is a composition of a scaling of ratio $r_i < 1$, an orthonormal rotation O_i and a translation t_i; we call such an attractor K a self-similar set. Our conclusions will depend very much on the nature of the rotation group G of the IFS, that is the closure of the subgroup of $SO(d, \mathbb{R})$ generated by the O_i.

We obtain almost sure properties of projections and sections of random multiplicative cascade measures on self-similar sets. The precise definition is given but for the purposes such a measure will be denoted by $\tilde{\mu}$ and be supported by a self-similar set K. In particular, $\tilde{\mu}$ is statistically self-similar, that is, roughly speaking, the restriction of $\tilde{\mu}$ to each small statistically self-similar, that is, roughly speaking, the restriction of $\tilde{\mu}$ to each small scale component of K has, after scaling, the same random distribution as $\tilde{\mu}$ itself. Our motivation for considering such measures is that they are the natural random generalisations of self-similar measures but also are the natural tools for studying fractal percolation processes. Moreover, random cascade measures provide the classical models for multiplicative chaos theory, an area that has recently attracted attention because of its connection to quantum gravity, see [30].

We give a precise construction of the probability space underlying the random cascade measures, and thus obtain an ergodic random dynamical system on the space of random cascade measures. An application of the compact group extension theorem shows that the skew product of this random dynamical system with the rotation group G is also ergodic.

These ergodicities are used to show that almost surely a random multiplicative cascade measure $\tilde{\mu}$, as well as almost all of its projections and sections (with respect to the Haar measure on G) are exact dimensional, that is the local dimensions exist and are constant almost everywhere. The proofs, which reformulate the measures of small balls as a type of Birkhoff sum, are adapted from the ergodic theoretic approach introduced for the deterministic case in [11]. This sum converges to the conditional entropy with respect to a sub- σ-algebra that captures the overlapping structure of self-similar sets, giving exact-dimensionality without any seperation condition (i.e. without requiring the union in (4) to be disjoint), as well as a formula for the exact dimension in terms of the conditional entropy.

One consequence of this is an almost sure 'dimension conservation' property, relating the dimensions of the projections to those of perpendicular sections. Writing $\pi\tilde{\mu}$ for the measure on $\pi(K)$ obtained by projecting $\tilde{\mu}$ under π, and $\tilde{\mu}_{y,\pi}$ for the section of $\tilde{\mu}$ by the $(d-k)$-dimensional plane $\pi^{-1}y$, we get the following conclusions when the rotation group is finite.

Corollary (1.1.1)[1]: Suppose that G is finite. Then for every projection $\pi \in \Pi_{d,k}$,

$$\dim_H \pi\tilde{\mu} + \dim_H \tilde{\mu}_{y,\pi} = \dim_H \tilde{\mu} \quad \text{for } \pi\tilde{\mu}\text{-almost all } y \in \pi(K) \tag{6}$$

almost surely. In particular, if $\tilde{\mu}$ is deterministic then (6) holds for all π.

Proof. See Corollary (1.1.10) and Corollary (1.1.11).

Note that the deterministic case extends the result of Furstenberg [12] by dispensing with the separation requirement that the union in (4) is disjoint.

We show that if $G = SO(d, \mathbb{R})$ then almost surely all projections of $\tilde{\mu}$ and, indeed, all images of $\tilde{\mu}$ under non-singular C^1-maps, have dimension equal to the 'generic' value. The deterministic results that were proved using CP-processes in [14] follow as a special case. Here we adopt a new approach utilising the skew product dynamical system, leading to results such as the following.

Corollary (1.1.2)[1]: If $G = SO(d, \mathbb{R})$ then almost surely, conditional on non-extinction of the random measure $\tilde{\mu}$,

$$\dim_H \pi\tilde{\mu} = \min(k, \dim_H \tilde{\mu}) \text{ for all } \pi \in \Pi_{d,k}. \quad (7)$$

More generally, for all C^1 maps $h: K \mapsto \mathbb{R}^k$ without singular points,

$$\dim_H h\tilde{\mu} = \min(k, \dim_H \tilde{\mu}). \quad (8)$$

Proof. See Theorem (1.1.21) and Corollary (1.1.23).

We specialise these results to deterministic self-similar sets, and in particular show that conclusions relating to the dimensions of all projections and dimension conservation are valid without any separation condition on the selfsimilar construction, extending work of Hochman and Shmerkin [14] and Furstenberg [12]. Again there are consequences for the dimensions of images of sets under C^1-mappings and also for the dimensions of distances sets.

Recently there has been considerable interest in geometric properties of percolation on self-similar sets, that is random subsets $K_{\mathbb{P}}$ of K obtained by removing components of the iterated construction of K according to a self-similar probability distribution $\mathbb{P}$. Associating the natural measures on $K_{\mathbb{P}}$ with random cascade measures, we obtain new almost sure properties of projections and dimension conservation for these random sets.

Symbolic or code space underlies the structure of selfsimilar sets.

Let $\Lambda = \{1, \cdots, m\}$ be the alphabet on $m \geq 2$ symbols. Denote by $\Lambda^* = \cup_{n\geq 0} \Lambda^n$ the set of finite words, with the convention that $\Lambda^0 = \{\emptyset\}$. Let $\Lambda^{\mathbb{N}}$ be the symbolic space of infinite sequences from the alphabet. For $\underline{i} \in \Lambda^{\mathbb{N}}$ and $n \geq 0$ let $\underline{i}\big|_n \in \Lambda^n$ be the first n digits of $\underline{i}$. For $i \in \Lambda^n$ let $[i] = \left\{\underline{i} \in \Lambda: \underline{i}\big|_n = i\right\}$ be the *cylinder* rooted at i. We may endow $\Lambda^{\mathbb{N}}$ with the standard metric d_ρ with respect to a number $\rho \in (0,1)$, that is for $\underline{i}, \underline{j} \in \Lambda^{\mathbb{N}}, d_\rho(\underline{i}, \underline{j}) = \rho^{\inf\left\{n\geq 0: \underline{i}|_n \neq \underline{j}|_n\right\}}$. Then $\left(\Lambda^{\mathbb{N}}, d_\rho\right)$ is a compact metric space. Let $\overline{\mathcal{B}}$ be its Borel σ-algebra. Define the left-shift map σ by $\sigma(\underline{i}) = (i_{n+1})_{n\geq 1}$ for $\underline{i} = (i_n)_{n\geq 1} \in \Lambda^{\mathbb{N}}$.

Let $\mathcal{I}$ be an IFS as in (5) with non-empty compact attractor $K \subseteq \mathbb{R}^d$ satisfying (4). For $i = i_1 \cdots i_n \in \Lambda^n$ write

$$f_i = f_{i_1} \circ \cdots \circ f_{i_n} = r_i O_i + t_i,$$

where $r_i = r_{i_1} \cdots r_{i_n}, O_i = O_{i_1} \cdots O_{i_n}$ and t_i is the appropriate translation. Throughout the paper, $G = \overline{\langle O_i : i \in \Lambda\rangle}$ will denote the rotation group of the IFS, that is the compact subgroup of $SO(d, \mathbb{R})$ generated by the orthonormal maps $\{O_i, i \in \Lambda\}$.

Let $\Phi: \Lambda^{\mathbb{N}} \mapsto K$ be the canonical projection, that is $\Phi(\underline{i}) = \lim_{n\to\infty} f_{\underline{i}|_n}(x_0)$ for some $x_0 \in K$. Let $R = \max\{|x|: x \in K\}$ and $\rho = \max\{r_i: i \in \Lambda\}$. Then it is easy to see that $\Phi: \left(\Lambda^{\mathbb{N}}, d_\rho\right) \mapsto K$ is R-Lipschitz.

A random multiplicative cascade is essentially a measure on $\Lambda^{\mathbb{N}}$ constructed in a self-similar manner on the successive Λ^n, see [17,4]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let

$$W = (W_i)_{i\in\Lambda} \in [0, \infty)^m$$

be a random vector defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\sum_{i\in\Lambda} \mathbb{E}(W_i) = 1$. Let $\left\{W^{[i]}: i \in \Lambda^*\right\}$ be a sequence of independent and identically distributed random vectors having the same law as W. For $i \in \Lambda^*, n \geq 1$ and $j = j_1 \cdots j_n \in \Lambda^n$ define

$$Q_j^{[i]} = W_{j1}^{[i]} W_{j_2}^{[ij_1]} \cdots W_{j_n}^{[ij_1 \cdots j_{n-1}]},$$

and for $i \in \Lambda^*$ and $n \geq 1$ define $Y_n^{[i]} = \sum_{j \in \Lambda^n} Q_j^{[i]}$. By definition $\{Y_n^{[i]}\}_{n \geq 1}$ is a non-negative martingale. Assume that

$$(a0)\ \ \mathbb{P}(\#\{i \in \Lambda: W_i > 0\} > 1) > 0;$$

$$\text{(a1) There exists } p > 1 \text{ such that } \sum_{i=1}^{m} \mathbb{E}(W_i^p) < 1. \tag{9}$$

Then $Y_n^{[i]}$ converges a.s. to a nontrivial limit which we denote by $Y^{[i]}$, with expectation $\mathbb{E}(Y^{[i]}) = 1$. It is easy to see that $Y^{[i]}, i \in \Lambda^*$ have the same law as $Y = Y^{[\emptyset]}$. Moreover, for $p > 1$ we have $\mathbb{E}(Y^p) < \infty$ if and only if $\sum_{i=1}^m \mathbb{E}(W_i^p) < 1$ (see [6,16]). Since Λ^* is countable, $Y^{[i]}$ is well-defined for all $i \in \Lambda^*$ simultaneously. Moreover, by construction,

$$Y^{[i]} = \sum_{j=1}^{m} W_j^{[i]} Y^{[ij]}. \tag{10}$$

Then for each $i \in \Lambda^*$ we may define a random measure $\mu^{[i]}$ on $\Lambda^{\mathbb{N}}$ by

$$\mu^{[i]}([j]) = Q_j^{[i]} \cdot Y^{[ij]},\ j \in \Lambda^*. \tag{11}$$

The measure $\mu^{[i]}$ is called the random multiplicative cascade measure generated by the sequence $\{W^{[ij]}: j \in \Lambda^*\}$. By definition the sequence $\{\mu^{[i]}: i \in \Lambda^*\}$ has the same law. Moreover, by (10) we have statistical self-similarity in the sense that for $i \in \Lambda^*$ and $j \in \Lambda^n$,

$$\mu^{[i]}\big|_{[j]} = Q_j^{[i]} \cdot \mu^{[ij]} \circ \sigma^{-n}\big|_{[j]}. \tag{12}$$

Sometimes we will write $(\cdot) = (\cdot)^{[\emptyset]}$, in particular $Q_j = Q_i^{[\emptyset]}$ and $\mu = \mu^{[\emptyset]}$. Our main interest will be in random cascade measures on the self-similar set K given by the canonical projection $\Phi\mu$ of μ onto K. For more on random cascade measures, see [4].

We now give a precise definition of the probability space on which the i.i.d. sequence $\{W^{[i]}: i \in \Lambda^*\}$ is defined. First recall that the random vector W is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will work on the countable product space

$$(\Omega^*, \mathcal{F}^*, \mathbb{P}^*) = \bigotimes_{i \in \Lambda^*} (\Omega_i, \mathcal{F}_i, \mathbb{P}_i),$$

where $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) = (\Omega, \mathcal{F}, \mathbb{P})$ for each $i \in \Lambda^*$. For $i \in \Lambda^*$ define the projection

$$\pi_i: \Omega^* \mapsto \Omega_i.$$

Then by letting $W^{[i]} = W \circ \pi_i$ for $i \in \Lambda^*$ we obtain a family of i.i.d. random vectors on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$. For $i \in \Lambda^*$ let $\mu^{[i]} \equiv \mu^{[i]}(\cdot, \omega)$ be the random cascade measure generated by the sequence $\{W^{[ij]}: j \in \Lambda^*\}$, as in (11). For $i \in \Lambda^*$ define

$$\eta_i: \Omega^* \ni (\omega_j)_{j \in \Lambda^*} \mapsto (\omega_{ij})_{j \in \Lambda^*} \in \Omega^*.$$

By definition $W^{[ij]} = W^{[i]} \circ \eta_j$ for all $i, j \in \Lambda^*$, thus

$$\mu^{[ij]}(\cdot, \omega) = \mu^{[i]}(\cdot, \eta_j \omega). \tag{13}$$

Consequently, from (12), for any $B \in \mathcal{B}$,

$$\begin{aligned} \mu^{[i]}(B \cap [j], \omega) \quad &= Q_j^{[i]}(\omega) \cdot \mu^{[ij]}(\sigma^{-n}(B \cap [j]), \omega) \\ &= Q_j^{[i]}(\omega) \cdot \mu^{[i]}(\sigma^{-n}(B \cap [j]), \eta_j \omega). \end{aligned}$$

Let $(\Omega',\mathcal{F}')=(\Lambda^{\mathbb{N}}\times\Omega^*,\mathcal{B}\otimes\mathcal{F}^*)$. Let $\mathbb{Q}$ be the Peyrière measure on $(\Omega',\mathcal{F}')$ with respect to $\mu=\mu^{[\emptyset]}$, that is for all $A\in\mathcal{F}'$,

$$\mathbb{Q}(A)=\int_{\Omega^*}\int_{\Lambda^N}\chi_A(\underline{i},\omega)\mu(\mathrm{d}\underline{i},\omega)\mathbb{P}^*(\mathrm{d}\omega). \tag{14}$$

It is easy to see that $(\Omega',\mathcal{F}',\mathbb{Q})$ is a probability space. Notice that the inside integral is only defined when μ is not trivial. Write $\mathbb{P}_*(A)=\mathbb{P}^*(A\cap\{\|\mu\|>0\})/\mathbb{P}^*(\{\|\mu\|>0\})$ for $A\in\mathcal{F}^*$ for the probability conditional on μ being non-trivial. Thus "for $\mathbb{Q}$-a.e. $(\underline{i},\omega)$ "is equivalent to "for $\mathbb{P}_*$-almost all μ, and μ-a.e. $\underline{i}$". Define the skew product

$$T:\Omega'\ni(\underline{i},\omega)\mapsto\left(\sigma\underline{i},\eta_{\underline{i}|1}(\omega)\right)\in\Omega'.$$

Lemma (1.1.3)[1]: The Peyrière measure $\mathbb{Q}$ is T-invariant.

Proof. For all $B\in\mathcal{F}'$

$$\begin{aligned}\mathbb{Q}(T^{-1}B)&=\int_{\Omega^*}\int_{\Lambda^N}\chi_{T^{-1}B}(\underline{i},\omega)\mu(\mathrm{d}\underline{i},\omega)\mathbb{P}^*(\mathrm{d}\omega)\\&=\int_{\Omega^*}\int_{\Lambda^N}\chi_B\left(\sigma\underline{i},\eta_{i|}\omega\right)\mu(\mathrm{d}\underline{i},\omega)\mathbb{P}^*(\mathrm{d}\omega)\\&=\sum_{j\in\Lambda}\int_{\Omega^*}\int_{[j]}\chi_B\left(\sigma\underline{i},\eta_j\omega\right)\mu(\mathrm{d}\underline{i},\omega)\mathbb{P}^*(\mathrm{d}\omega)\\&=\sum_{j\in\Lambda}\int_{\Omega^*}W_j^{[\emptyset]}(\omega)\int_{[j]}\chi_B\left(\sigma\underline{i},\eta_j\omega\right)\mu\left(\mathrm{d}\sigma\underline{i},\eta_j\omega\right)\mathbb{P}^*(\mathrm{d}\omega)\\&=\sum_{j\in\Lambda}\int_{\Omega^*}W_j^{[\emptyset]}(\omega)\int_{\Lambda^N}\chi_B\left(\underline{i},\eta_j\omega\right)\mu\left(\mathrm{d}\underline{i},\eta_j\omega\right)\mathbb{P}^*(\mathrm{d}\omega)\\&=\sum_{j\in\Lambda}\mathbb{E}\left(W_j\right)\mathbb{Q}(B)\\&=\mathbb{Q}(B).\end{aligned}$$

Proposition (1.1.4)[1]: The dynamical system $(\Omega',\mathcal{F}',\mathbb{Q},T)$ is mixing.

Proof. Let $\mathcal{A}$ be the semi-algebra consisting of sets of the form

$$\underline{i}\big|_k=j,W_a^b\in B_a^b,a\in\Lambda,b\in\cup_{i=1}^k\Lambda^i,$$

for $k\in\mathbb{N},j\in\Lambda^k$ and B_a^b Borel subsets of $[0,\infty)$. It is clear that $\mathcal{A}$ generates $\mathcal{F}'$, so we only need to verify that for $A,B\in\mathcal{A},\lim_{n\to\infty}\mathbb{Q}(T^{-n}A\cap B)=\mathbb{Q}(A)\mathbb{Q}(B)$. This follows from the fact that by the construction of $\mathcal{A}$, given $A,B\in\mathcal{A}$, there exists n_0 such that $T^{-n}A$ and B are independent for all $n\geq n_0$.

For $i\in\Lambda^*$ define

$$\bar{\mu}_{[i]}=\chi_{\{\mu([i])>0\}}\frac{\mu|_{[i]}}{\mu([i])}\quad\text{and}\quad\bar{\mu}^{[i]}=\chi_{\{\|\mu^{[i]}\|>0\}}\frac{\mu^{[i]}}{\|\mu^{[i]}\|},$$

with the convention that $\bar{\mu}=\bar{\mu}^{[\emptyset]}$. Then $\bar{\mu}_{[i]}$ and $\bar{\mu}^{[i]}$ are either probability measures or trivial. If $|i|=n$, then from (12) we have

$$\bar{\mu}_{[i]}\circ\sigma^{-n}=\chi_{\{Q_i>0\}}\bar{\mu}^{[i]}. \tag{15}$$

The measure sequence $\left\{\bar{\mu}^{[\cdot|_n]}\right\}_{n\geq0}$ is a stationary process under the Peyrière measure. This sequence is similar to Furstenberg's CP-processes: Let Δ be the natural partition operator on symbolic space: $\Delta[i]=\{[ij]:j\in\Lambda\}$ for $i\in\Lambda^*$. Starting from $(\bar{\mu},[\emptyset])$ we move to $\left(\bar{\mu}^{[i]},[i]\right)$ with probability $\bar{\mu}([i])$ for $i\in\Lambda$, and from $\left(\bar{\mu}^{[i]},[i]\right)$ we move to $\left(\bar{\mu}^{[ij]},[ij]\right)$

with probability $\bar{\mu}^{[i]}([j])$ for $j \in \Lambda$, and continue in this way. The resulting measure sequence clearly falls into the same sample space as $\{\bar{\mu}^{[\cdot|_n]}\}_{n\geq 0}$, but it seems unlikely they will have the same law unless the random cascade measures degenerate to Bernoulli measures.

Let $G = \overline{\langle O_\iota : \iota \in \Lambda \rangle}$ be the closed subgroup of $SO(d, \mathbb{R})$ generated by the orthogonal maps $\{O_i, i \in \Lambda\}$. For future reference note that G also equals the closed subsemigroup generated by the orthogonal maps $\{O_i, i \in \Lambda\}$; this follows since the inverse of any element in a compact group can be approximated arbitrarily closely by positive powers of the element. Let $\mathcal{B}_G$ be Borel σ-algebra of G and let ξ be its normalized Haar measure. Define the measurable map $\phi : \Omega' \ni (\underline{i}, \omega) \mapsto O_{\underline{i}}\big|_1 \in G$. Let $X = \Omega' \times G$ and define the skew product

$$T_\phi : X \ni (\omega', g) \mapsto \big(T\omega', g\phi(\omega')\big) \in X.$$

It is easy to verify that the product measure $\mathbb{Q} \times \xi$ is T_ϕ-invariant.

Proposition (1.1.5)[1]: The dynamical system $\big(X, \mathcal{F}' \otimes \mathcal{B}_G, \mathbb{Q} \times \xi, T_\phi\big)$ is ergodic.

Proof. From Proposition (1.1.4) we know that $(\Omega', \mathcal{F}', \mathbb{Q}, T)$ is ergodic. Using the compact group extension theorem, see for example [19], T_ϕ is ergodic if and only if the equation

$$F(T\omega') = R\big(\phi(\omega')\big)F(\omega') \text{ for } \mathbb{Q}\text{-a.e. } \omega', \tag{16}$$

where R is an irreducible (unitary) representation (of degree k, say) and $F : \Omega' \mapsto \mathbb{C}^k$ is measurable, has only the trivial solution R, the trivial 1-dimensional representation, with F constant. Let μ_p is the Bernoulli measure on $\Lambda^{\mathbb{N}}$ corresponding to the probability vector $p = \big(\mathbb{E}(W_i)\big)_{i\in\Lambda}$. From the measuable function F in (16) we may construct the following vector measure λ on $\Lambda^{\mathbb{N}}$, defined as

$$\lambda(I) = \int_{\Omega^*} \int_I F\big(\underline{i}, \omega\big)\mu\big(\mathrm{d}\underline{i}, \omega\big)\mathbb{P}^*(\,\mathrm{d}\omega), \forall I \in \mathcal{B}.$$

Then λ is absolutely continuous with respect to μ_p since, for any set $E \in \mathcal{B}$ with $\mu_p(E) = 0$,

$$\begin{aligned} |\lambda(E)| \quad &\leq \limsup_{R\to\infty} \int_{\Omega^*} \int_E \chi_{\{|F|\leq R\}} |F(\underline{i}, \omega)| \mu(\mathrm{d}\underline{i}, \omega)\mathbb{P}^*(\,\mathrm{d}\omega) \\ &\leq \limsup_{R\to\infty} R \cdot \mu_p(E) = 0. \end{aligned}$$

Denote by $f = \mathrm{d}\lambda/\mathrm{d}\mu_p$ the corresponding Radon-Nikodym derivative. In particular

$$f(\underline{i}) = \lim_{n\to\infty} \frac{\lambda\big(\big[\underline{i}|_n\big]\big)}{\mu_p\big(\big[\underline{i}|_n\big]\big)} \text{ for } \mu_p\text{-a.e. } \underline{i} \tag{17}$$

Now fix $I = [i_1 i_2 \cdots i_n]$. From (16)

$$\begin{aligned} R\big(O_{i_1}\big)\lambda([i_1 i_2 \cdots i_n]) \quad &= \int_{\Omega^*} \int_{[i_1 i_2 \cdots i_n]} R\big(O_{i_1}\big)F(\underline{i}, \omega)\mu(\mathrm{d}\underline{i}, \omega)\mathbb{P}^*(\,\mathrm{d}\omega) \\ &= \int_{\Omega^*} \int_{[i_1 i_2 \cdots i_n]} F\big(\sigma\underline{i}, \eta_{i_1}\omega\big)\mu(\mathrm{d}\underline{i}, \omega)\mathbb{P}^*(\,\mathrm{d}\omega) \\ &= \int_{\Omega^*} W_{i_1} \int_{[i_2 \cdots i_n]} F(\underline{i}, \omega)\mu^{[i_1]}(\mathrm{d}\underline{i}, \omega)\mathbb{P}^*(\,\mathrm{d}\omega) \\ &= \mu_p([i_1])\lambda([i_2 \cdots i_n]). \end{aligned}$$

This yields

$$\frac{\lambda([i_2\cdots i_n])}{\mu_p([i_2\cdots i_n])}=R(O_{i_1})\frac{\lambda([i_1 i_2\cdots i_n])}{\mu_p([i_1 i_2\cdots i_n])}.$$

Together with (17) we finally get

$$f(\sigma\underline{i})=R\left(O_{\underline{i}\sqrt{1}}\right)f(\underline{i})\text{ for }\mu_p\text{-a.e. }\underline{i}.$$

From [29, Corollary 4.5] we know that the dynamical system $(\Lambda^{\mathbb{N}}\times G,\mathcal{B}\otimes\mathcal{B}_G,\mu_p\times\xi,\sigma_\phi)$ is ergodic, where $\sigma_\phi(\underline{i},g)=\left(\sigma\underline{i},gO_{\underline{i}|_1}\right)$ is a compact group extension of the Bernoulli full-shift with σ_ϕ having a dense orbit. By using the compact group extension theorem again this implies that R must be the trivial 1-dimensional representation. Applying this to (16) we get that

$$F(T\omega')=F(\omega')\text{ for }\mathbb{Q}\text{-a.e. }\omega',$$

so F is constant using Proposition (1.1.4).

Let $\varphi\colon Y\mapsto Z$ be a continuous mapping between two metric spaces Y and Z. For a Borel measure ν on Y write

$$\varphi\nu=\nu\circ\varphi^{-1}$$

for the pull-back measure of ν on Z through φ.

For a measure ν and $x\in\operatorname{supp}(\nu)$ let

$$D_\nu(x)=\lim_{r\to 0}\frac{\log\nu(B(x,r))}{\log r}$$

whenever the limit exists, where $B(x,r)$ is the closed ball of centre x and radius r. If for some $\alpha\geq 0$ we have $D_\nu(x)=\alpha$ for ν-a.e. x we say that ν is exact-dimensional.

For $0<r<1$ and ν a probability measure supported by a compact subset A of $\mathbb{R}^d$, let

$$H_r(\nu)=-\int_A\log\nu(B(x,r))\nu(\mathrm{d}x)$$

be the r-scaling entropy of ν. Note that, writing $\mathcal{M}$ for the probability measures supported by A, the map $H_r\colon\mathcal{M}\to\mathbb{R}\cup\{\infty\}$ need not be continuous in the weak-star topology. However, H_r is lower semicontinuous as it may be expressed as the limit of an increasing sequence of continuous functions of the form $\nu\mapsto\int\max\{k,\log\left(1/\int f_k(x-y)\nu(dy)\right)\nu(dx)\}$ where f_k is a decreasing sequence of continuous functions approximating $\chi_{B(0,r)}$. The lower entropy dimension of ν is defined as

$$\dim_e\nu=\liminf_{r\to 0}\frac{H_r(\nu)}{-\log r}$$

and the Hausdorff dimension of ν is $\dim_H\nu=\inf\{\dim_H A\colon\nu(A)>0\}$. Then

$$\dim_H\nu\leq\dim_e\nu,$$

with equality when ν is exact-dimensional, see [8,9].

The following result is the conditional measure theorem of Rohlin [31] adapted to symbolic spaces.

Theorem (1.1.6)[1]: Let η be a countable $\mathcal{B}$-measurable partition of $\Lambda^{\mathbb{N}}$ in the sense that the quotient space $\Lambda^{\mathbb{N}}/\eta$ is separated by a countable number of measurable sets in $\mathcal{B}$. Let ν be a Borel probability measure on $\Lambda^{\mathbb{N}}$. Then for every $\underline{i}$ in a set of full ν-measure, there is a probability measure $\nu_{\underline{i}}^{\eta}$ defined on $\eta(\underline{i})$ (the unique element in η that contains $\underline{i}$) such that for any measurable set $B\in\mathcal{B}$, the mapping $\underline{i}\mapsto\nu_{\underline{i}}^{\eta}(B)$ is $\hat{\eta}$-measurable ($\hat{\eta}$ is the sigma-algebra generated by η) and

$$\nu(B)=\int_{\Lambda^{\mathbb{N}}}\nu_{\underline{i}}^{\eta}(B)\nu(\mathrm{d}\underline{i}).$$

These properties imply that for any $f\in L^1(\Lambda^{\mathbb{N}},\mathcal{B},\nu)$ we have $\nu_{\underline{i}}^{\eta}(f)=\mathbb{E}_\nu(f\mid\hat{\eta})$ for ν-a.e. $\underline{i}$, and $\nu(f)=\int\mathbb{E}_\nu(f\mid\hat{\eta})\mathrm{d}\nu$.

For any sub-Borel σ-algebra $\mathcal{A}$ of $\mathcal{B}$, any countable $\mathcal{B}$-measurable partition $\mathcal{P}$ of $\Lambda^{\mathbb{N}}$, and any Borel probability measure ν on $\Lambda^{\mathbb{N}}$ we define the conditional information

$$\mathbf{I}_\nu(\mathcal{P}\mid\mathcal{A})=-\sum_{B\in\mathcal{P}}\chi_B\log\mathbb{E}_\nu(\chi_B\mid\mathcal{A})$$

and the conditional entropy

$$\mathbf{H}_\nu(\mathcal{P}\mid\mathcal{A})=\int_{\Lambda^{\mathbb{N}}}\mathbf{I}_\nu(\mathcal{P}\mid\mathcal{A})(\underline{i})\nu(\mathrm{d}\underline{i})$$

For the trivial σ-algebra $\mathcal{N}=\{\emptyset,\Lambda^{\mathbb{N}}\}$ we use the convention that $\mathbf{I}_\nu(\mathcal{P})=\mathbf{I}_\nu(\mathcal{P}\mid\mathcal{N})$ and $\mathbf{H}_\nu(\mathcal{P})=\mathbf{H}_\nu(\mathcal{P}\mid\mathcal{N})$.

We state, the following result from Feng & Hu [11] which we will need in several places.

Proposition (1.1.7)[1]: Let ν be a Borel probability measure on $\Lambda^{\mathbb{N}}$. Let η and $\mathcal{P}$ be two countable measurable partitions of $\Lambda^{\mathbb{N}}$. Let $\varphi:\Lambda^{\mathbb{N}}\mapsto\mathbb{R}^d$ be a continuous function and denote by $\mathcal{B}_\varphi$ the σ-algebra generated by $\varphi^{-1}\mathcal{B}(\mathbb{R}^d)$. Then for $\nu-a.e.\,\underline{i}\in\Lambda^{\mathbb{N}}$,

$$\lim_{r\to0}\log\frac{\nu_{\underline{i}}^{\eta}\left(\varphi^{-1}\left(B\left(\varphi(\underline{i}),r\right)\right)\cap\mathcal{P}(\underline{i})\right)}{\nu_{\underline{i}}^{\eta}\left(\varphi^{-1}\left(B\left(\varphi(\underline{i}),r\right)\right)\right)}=-\mathbf{I}_m\left(\mathcal{P}\mid\hat{\eta}\vee\mathcal{B}_\varphi\right)(\underline{i}).$$

Moreover, writing

$$h(\underline{i})=-\inf_{r>0}\log\frac{\nu_{\underline{i}}^{\eta}(\varphi^{-1}(B(\varphi(\underline{i}),r))\cap\mathcal{P}(\underline{i}))}{\nu_{\underline{i}}^{\eta}(\varphi^{-1}(B(\varphi(\underline{i}),r)))}$$

and assuming $\mathbf{H}_\nu(\mathcal{P})<\infty$, then $h\geq0$ and $h\in L^1(\Lambda^{\mathbb{N}})$ with $\int_{\Lambda^N}h(\underline{i})\leq\mathbf{H}_\nu(\mathcal{P})+C_d$, where C_d depends only on d.

Proof. This is proved in [11, Proposition 3.5]. The bound for $\int_{\Lambda^N}h(\underline{i})$ is contained within the proof.

We establish the exact-dimensionality of random cascade measures on self-similar sets without any separation condition, as well as of the projections of the measures onto subspaces and of sliced measures for ξ almost all rotations.

Let $\pi\in\Pi_{d,k}$. For $\underline{i}\in\Lambda^{\mathbb{N}}$ define the fibre

$$[\underline{i}]_\pi=(\pi\Phi)^{-1}\left(\pi\Phi(\underline{i})\right),$$

and write $\mathcal{P}_\pi=\left\{[\underline{i}]_\pi:\underline{i}\in\Lambda^{\mathbb{N}}\right\}$. It is a measurable partition since the quotient space $\Lambda^{\mathbb{N}}/\mathcal{P}_\pi$ is separated by $\{(\pi\Phi)^{-1}B_i\}$ where $\{B_i\}$ is the sequence of closed cubes in $\pi(\mathbb{R}^d)$ with rational vertices. Denote by $\hat{\mathcal{P}}_\pi$ the σ-algebra generated by $\mathcal{P}_\pi$. Due to Theorem (1.1.6), given the measurable partition $\mathcal{P}_\pi$, for any probability measure ν on $(\Lambda^{\mathbb{N}},\mathcal{B})$, for every $\underline{i}$ in a set of full ν-measure, there is a probability measure $\nu_{\underline{i}}^{\mathcal{P}\pi}$, which we shortly denote by $\nu_{\underline{i},\pi}$, defined on $\mathcal{P}_\pi(\underline{i})=[\underline{i}]_\pi$ such that for any $B\in\mathcal{B}$, the mapping $\underline{i}\mapsto\nu_{\underline{i},\pi}(B)$ is $\widehat{\mathcal{P}_\pi}$-measurable and

$$\nu(B)=\int_{\Lambda^{\mathbb{N}}}\nu_{\underline{i},\pi}(B)\nu(\mathrm{d}\underline{i}).$$

Furthermore for any $f \in L^1(\Lambda^{\mathbb{N}}, \mathcal{B}, \nu)$ we have $\nu_{\underline{i},\pi}(f) = \mathbb{E}_\nu\big(f \mid \widehat{\mathcal{P}_\pi}\big)$ for ν-a.e. $\underline{i}$, and $\nu(f) = \int \mathbb{E}_\nu\big(f \mid \widehat{\mathcal{P}_\pi}\big)\mathrm{d}\nu$. Moreover, $\nu_{\underline{i},\pi}$ depends only on $[\underline{i}]_\pi$; thus we may write $\nu_{y,\pi} = \nu_{\underline{i},\pi}$ when $y \in \pi(K)$ is such that $\pi\Phi(\underline{i}) = y$. By definition for every Borel set $A \in \overline{\mathcal{B}}$

$$\nu(A) = \int_{\Lambda^{\mathbb{N}}} \nu_{i,\pi}(A)\nu(\mathrm{d}\underline{i}) = \int_{y\in\pi(K)} \nu_{y,\pi}(A)\pi\Phi\nu(\mathrm{d}y). \tag{18}$$

The following lemma, which is a variant of properties stated in [25, Chapter 10], expresses these conditional measures geometrically as limits of measures of narrow slices.

Lemma (1.1.8)[1]: For every set $A \in \mathcal{B}$, for $\pi\Phi\nu$-a.e. $y \in \pi(K)$,

$$\nu_{y,\pi}(A) = \lim_{r\to 0} \frac{\nu(A \cap \Phi^{-1}\pi^{-1}(B(y,r)))}{\nu(\Phi^{-1}\pi^{-1}(B(y,r)))},$$

or equivalently for ν-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\nu_{\underline{i},\pi}(A) = \lim_{r\to 0} \frac{\nu\big(A \cap \Phi^{-1}\pi^{-1}(B(\pi\Phi(\underline{i}),r))\big)}{\nu\big(\Phi^{-1}\pi^{-1}(B(\pi\Phi(\underline{i}),r))\big)}.$$

Proof. Let $f\colon \underline{i} \mapsto \nu_{\underline{i},\pi}(A)$ and $\bar{f}\colon y \mapsto \nu_{y,\pi}(A)$. By (18), for any $B \in \mathcal{B}\left(\pi(\mathbb{R}^d)\right)$,

$$\nu(A \cap \Phi^{-1}\pi^{-1}B) = \int_{\Phi^{-1}\pi^{-1}B} f \,\mathrm{d}\nu = \int_B \bar{f}\,\mathrm{d}\pi\Phi\nu. \tag{19}$$

Define a measure λ on $\pi(K)$ by $\lambda(B) = \nu(A \cap \Phi^{-1}\pi^{-1}B)$ for $B \in \mathcal{B}\left(\pi(\mathbb{R}^d)\right)$. By (19) λ is absolutely continuous with respect to $\pi\Phi\nu$ with

$$\frac{\lambda(B(y,r))}{\pi\Phi\nu(B(y,r))} = \frac{1}{\pi\Phi\nu(B(y,r))}\int_{B(y,r)} \bar{f}\,\mathrm{d}\pi\Phi\nu.$$

Letting $r \to 0$ and applying the differentiation theory of measures, see for example, [25, Theorem 2.12],

$$\lim_{r\to 0} \frac{\nu(A \cap \Phi^{-1}\pi^{-1}(B(y,r)))}{\nu(\Phi^{-1}\pi^{-1}B(y,r))} = \nu_{y,\pi}(A)$$

for $\pi\Phi\nu$-a.e. y, as required.

Let $\pi \in \Pi_{d,k}$ be fixed. Here is the main theorem.

Theorem (1.1.9)[1]: $\mathbb{P}_* - a.s.$,

(i) $\Phi\mu$ is exact-dimensional with dimension

$$\alpha = \frac{\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_\Phi)\right) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i)\log r_i}.$$

(ii) For ξ-a.e. $g \in G, \pi g\Phi\mu$ is exact-dimensional with dimension

$$\beta(\pi) = \frac{\mathbb{E}_{\mathbb{P}^*\times\xi}\left(\mathbf{H}_{\bar{\mu}}\big(\mathcal{P} \mid \mathcal{B}_{\pi g\Phi}\big)\right) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i)\log r_i}.$$

(iii) For ξ-a.e. $g \in G$, for $\pi g\Phi\mu$-a.e. $y \in \pi g(K), \Phi\bar{\mu}_{y,\pi g}$ is exact-dimensional with dimension

$$\gamma(\pi) = \frac{\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_\Phi)\right) - \mathbb{E}_{\mathbb{P}*\times\xi}\left(\mathbf{H}_{\bar{\mu}}\big(\mathcal{P} \mid \mathcal{B}_{\pi g\Phi}\big)\right)}{\sum_{i=1}^m \mathbb{E}(W_i)\log r_i}.$$

'Dimension conservation' for ξ-almost all rotations now follows.

Corollary (1.1.10)[1]: $\mathbb{P}_*$ − a.s. for ξ-a.e. $g \in G$ and $\pi g\Phi\mu - a.e.\, y \in \pi g(K)$,

$$\dim_H \pi g\Phi\bar{\mu} + \dim_H \Phi\bar{\mu}_{y,\pi g} = \dim_H \Phi\bar{\mu}.$$

Proof. If follows from Theorem (1.1.9) that these measures are exact-dimensional and $\alpha = \beta(\pi) + \gamma(\pi)$.

We immediately get the following corollary.

Corollary (1.1.11)[1]: If G is finite then for every projection $\pi \in \Pi_{d,k}$, almost surely

$$\dim_H \pi\Phi\bar{\mu} + \dim_H \Phi\bar{\mu}_{y,\pi} = \dim_H \Phi\bar{\mu} \text{ for } \pi\Phi\bar{\mu} \text{ a.e. } y \in \pi(K), \tag{20}$$

that is π is dimension conserving. In particular, if $\bar{\mu}$ is deterministic (i.e. a selfsimilar measure), then (20) holds for all $\pi \in \Pi_{d,k}$.

Proof of Theorem (1.1.9)(i). The proof is adapted from [11]. Recall that $R = \max\{|x|: x \in K\}$. For $n \geq 0$ and $\underline{i} \in \Lambda^{\mathbb{N}}$ let

$$B_\Phi(\underline{i}, n) = \Phi^{-1}\left(B\left(\Phi(\underline{i}), R \cdot r_{\underline{i}|_n}\right)\right),$$

with the convention that $r_\emptyset = 1$. By definition we have $B_\Phi(\underline{i}, 0) = \Lambda^{\mathbb{N}}$ for all $\underline{i} \in \Lambda^{\mathbb{N}}$. For $n \geq 1$ let

$$f_n: \Lambda^{\mathbb{N}} \times \Omega_* \ni (\underline{i}, \omega) \mapsto -\log \frac{\bar{\mu}\left(B_\Phi(\underline{i}, n) \cap \mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_\Phi(\underline{i}, n)\right)} \in \mathbb{R}.$$

Applying Proposition (1.1.7) in the case of $\eta = \mathcal{N}$ and $\varphi = \Phi$ we have that given any $\omega \in \Omega^*$ such that $\| \mu \| > 0$, for μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} f_n(\underline{i}, \omega) = \mathbf{I}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_\Phi)(\underline{i}) := f(\underline{i}, \omega).$$

Furthermore, as $\mathcal{P}$ is a finite partition of m elements and $\bar{\mu}$ is a probability measure, setting

$$\bar{f}(\underline{i}, \omega) = -\inf_{n\geq 1} \log \frac{\bar{\mu}\left(B_\Phi(\underline{i}, n) \cap \mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_\Phi(\underline{i}, n)\right)},$$

we have

$$\int_{\Lambda^{\mathbb{N}}} \bar{f}(\underline{i}, \omega)\bar{\mu}(\mathrm{d}\underline{i}, \omega) \leq \mathbf{H}_{\bar{\mu}}(\mathcal{P}) + C_d \leq \log m + C_d.$$

This implies that $\bar{f} \in L^1(\mathbb{Q})$.

Next we apply the following ergodic theorem due to Maker [20].

Theorem (1.1.12)[1]: Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and let $\{f_n\}$ be integrable functions on $(X, \mathcal{B}, \mu)$. If $f_n(x) \to f(x)$ a.e. and if $\sup_n |f_n(x)| = \bar{f}(x)$ is integrable, then for a.e. x,

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k} \circ T^k(x) = f_\infty(x),$$

where $f_\infty(x) = \lim \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x)$.

Lemma (1.1.13)[1]: $\mathbb{P}_* -$ a.s. for μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \prod_{k=0}^{n-1} \frac{\bar{\mu}^{[i|k]}\left(B_\Phi(\sigma^k\underline{i}, n-k) \cap \mathcal{P}(\sigma^k\underline{i})\right)}{\bar{\mu}_k^{[i|]}\left(B_\Phi(\sigma^k\underline{i}, n-k)\right)} = \mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_\Phi)\right).$$

Proof. First notice that $-\log \frac{\bar{\mu}^{[i|k]}\left(B_\Phi(\sigma^k\underline{i}, n-k) \cap \mathcal{P}(\sigma^k\underline{i})\right)}{\bar{\mu}^{[i|k]}\left(B_\Phi(\sigma^k\underline{i}, n-k)\right)} = f_{n-k} \circ T^k(\underline{i}, \omega)$. From Theorem (1.1.12), for $\mathbb{Q}$-a.e. $(\underline{i}, \omega)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f_{n-k}\circ T^k(\underline{i},\omega)=f_\infty(\underline{i},\omega),$$

where $f_\infty(\underline{i},\omega)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(\underline{i},\omega)$. But for $\mathbb{Q}$-a.e. $(\underline{i},\omega)\in\Omega', f_\infty(\underline{i},\omega)=\mathbb{E}_{\mathbb{Q}}(f)=\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P}\mid\mathcal{B}_\Phi)\right)$ by Proposition (1.1.4), hence the conclusion.

The next lemma, an analogue of [11, Lemma 5.3] for self-similar sets, relates the shift on symbolic space to its geometric effect on balls in $\mathbb{R}^d$.

Lemma (1.1.14)[1]: For $\underline{i}\in\Lambda^{\mathbb{N}}$ and $r>0$ we have

$$\Phi^{-1}\left(B\left(\Phi(\underline{i}),r_{\underline{i}|1}\cdot r\right)\right)\cap\mathcal{P}\left(\underline{i}\right)=\sigma^{-1}\Phi^{-1}\left(B\left(\Phi\left(\sigma\underline{i}\right),r\right)\right)\cap\mathcal{P}\left(\underline{i}\right).$$

Proof. For $\underline{i}=i_1i_2\cdots$ and $r>0$ we have

$$B\left(\Phi(\underline{i}),r_{i|1}\cdot r\right)=f_{i_1}(B(\Phi(\sigma\underline{i}),r)).$$

Thus

$$\Phi^{-1}\left(B\left(\Phi(\underline{i}),r_{i|1}\cdot r\right)\right)\cap\mathcal{P}\left(\underline{i}\right)=\Phi^{-1}\left(f_{i_1}\left(B\left(\Phi\left(\sigma\underline{i}\right),r\right)\right)\right)\cap\mathcal{P}\left(\underline{i}\right).$$

As

$$\begin{aligned}\underline{j}=j_1j_2\cdots\in\Phi^{-1}\left(f_{i_1}(B(\Phi(\sigma\underline{i}),r))\right)\cap\mathcal{P}(\underline{i})&\\ \Leftrightarrow j_1=i_1,\Phi(\underline{j})\in f_{i_1}(B(\Phi(\sigma\underline{i}),r))&\\ \Leftrightarrow j_1=i_1,f_{j_1}(\Phi(\sigma\underline{j}))\in f_{i_1}(B(\Phi(\sigma\underline{i}),r))&\\ \Leftrightarrow j_1=i_1,\Phi(\sigma\underline{j})\in B(\Phi(\sigma\underline{i}),r)&\\ \Leftrightarrow j_1=i_1,\underline{j}\in\sigma^{-1}\Phi^{-1}(B(\Phi(\sigma\underline{i}),r))&\\ \Leftrightarrow \underline{j}\in\sigma^{-1}\Phi^{-1}(B(\Phi(\sigma\underline{i}),r))\cap\mathcal{P}(\underline{i})&\end{aligned}$$

we get $\Phi^{-1}\left(f_{i_1}(B(\Phi(\sigma\underline{i}),r))\right)\cap\mathcal{P}(\underline{i})=\sigma^{-1}\Phi^{-1}(B(\Phi(\sigma\underline{i}),r))\cap\mathcal{P}(\underline{i})$, hence the conclusion.

For $\underline{i}\in\Lambda^{\mathbb{N}}$ and $n\geq 1$, conditioning on $\mu\left(\left[\underline{i}|_n\right]\right)>0$, we obtain

$$\frac{\mu\left(B_\Phi(\underline{i},n)\right)}{\mu^{[\underline{i}|_n]}\left(B_\Phi\left(\sigma^n\underline{i},0\right)\right)}=\prod_{k=0}^{n-1}\frac{\mu^{\left[\underline{i}|_k\right]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\mu^{[\underline{i}|_{k+1}]}\left(B_\Phi\left(\sigma^{k+1}\underline{i},n-k-1\right)\right)}\tag{21}$$

$$=\prod_{k=0}^{n-1}\frac{\mu^{[\underline{i}|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\mu^{[\underline{i}|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}\frac{\mu^{[i|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}{\mu^{[i]_{k+1}]}\left(B_\Phi\left(\sigma^{k+1}\underline{i},n-k-1\right)\right)}$$

$$=\prod_{k=0}^{n-1}\frac{\mu^{[\underline{i}|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\mu^{[\underline{i}|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}\frac{\mu^{[\underline{i}|_k]}\left(\sigma^{-1}B_\Phi\left(\sigma^{k+1}\underline{i},n-k-1\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}{\mu^{[\underline{i}|_{k+1}]}\left(B_\Phi\left(\sigma^{k+1}\underline{i},n-k-1\right)\right)}$$

$$=\prod_{k=0}^{n-1}\frac{\mu^{[\underline{i}|_k]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\mu^{\left[\underline{i}|_k\right]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}\cdot W_{i_{k+1}}^{[\underline{i}|_k]}$$

$$=\prod_{k=0}^{n-1}\frac{\bar{\mu}^{\left[\underline{i}|_k\right]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}^{\left[\underline{i}|_k\right]}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}\cdot W_{i_{k+1}}^{\left[\underline{i}|_k\right]}.\tag{22}$$

To complete the proof of (i) we need the following lemma.

Lemma (1.1.15)[1]: $\mathbb{P}_* - a.s.$ for μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

(a) $\lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} W_{i_{k+1}}^{[\underline{i}|_k]} = \sum_{i=1}^{m} \mathbb{E}(W_i \log W_i)$;

(b) $\lim_{n\to\infty} \frac{1}{n} \log r_{\underline{i}}\big|_n = \sum_{i=1}^{m} \mathbb{E}(W_i) \log r_i$;

(c) $\lim_{n\to\infty} \frac{1}{n} \log \left\| \mu^{[\underline{i}|_n]} \right\| = 0$.

Proof. (a) and (b) follow from the strong law of large numbers under the Peyrière measure $\mathbb{Q}$. (c) follows from [3, Theorem IV (ii)].

Combining Lemma (1.1.13), (22) and Lemma (1.1.15) we have proved that $\mathbb{P}_*$-a.s. for μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\log \Phi\mu\left(B\left(\Phi(\underline{i}), r_{\underline{i}|_n}\right)\right)}{\log r_{\underline{i}|_n}} = \frac{\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_\Phi)\right) + \sum_{i=1}^{m} \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^{m} \mathbb{E}(W_i) \log r_i},$$

which gives the conclusion.

Proof of Theorem (1.1.9)(ii). The proof is analogous to that of Theorem (1.1.9)(i); we can formally replace Φ by $\pi g \Phi$. Here we only present the differences. For $n \geq 0$, $g \in G$ and $\underline{i} \in \Lambda^{\mathbb{N}}$ let

$$B_{\pi g \Phi}(\underline{i}, n) = (\pi g \Phi)^{-1}\left(B\left(\pi g \Phi(\underline{i}), R \cdot r_{\underline{i}}\big|_n\right)\right).$$

Notice that $B_{\pi g \Phi}(\underline{i}, 0) = \Lambda^{\mathbb{N}}$ for all $\underline{i} \in \Lambda^{\mathbb{N}}$. For $n \geq 1$ let

$$f_n: \Lambda^{\mathbb{N}} \times \Omega_* \times G \ni (\underline{i}, \omega, g) \mapsto -\log \frac{\bar{\mu}\left(B_{\pi g \Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g \Phi}(\underline{i}, n)\right)} \in \mathbb{R}.$$

Using Proposition (1.1.7) again in the case of $\eta = \mathcal{N}$ and $\varphi = \pi g \Phi$ we get that given any $\omega \in \Omega^*$ such that $\| \mu \| > 0$ and given any $g \in G$, for μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} f_n(\underline{i}, \omega, g) = \mathbf{I}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_{\pi g \Phi})(\underline{i}) := f(\underline{i}, \omega, g). \tag{23}$$

Furthermore,

$$\int_{\Lambda^N} \sup_n \left| f_n(\underline{i}, \omega, g) \right| \bar{\mu}(\mathrm{d}\underline{i}, \omega) \leq \mathbf{H}_{\bar{\mu}}(\mathcal{P}) + C_d \leq \log m + C_d.$$

This implies that $\sup_n |f_n| \in L^1(\mathbb{Q} \times \xi)$. By using Theorem (1.1.12) and Proposition (1.1.5) it follows that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$ and μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k} \circ T_\phi^k(\underline{i}, \omega, g) = \mathbb{E}_{\mathbb{Q}\times\xi}(f) = \mathbb{E}_{\mathbb{P}*\times\xi}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_{\pi g \Phi})\right). \tag{24}$$

The following is an analogue of Lemma (1.1.14).

Lemma (1.1.16)[1]: For $\underline{i} \in \Lambda^{\mathbb{N}}$, $g \in G$ and $r > 0$ we have

$$(\pi g \Phi)^{-1}\left(B\left(\pi g \Phi(\underline{i}), r_{i|} \cdot r\right)\right) \cap \mathcal{P}(\underline{i})$$
$$= \sigma^{-1}\left(\pi g O_{\underline{i}|_1} \Phi\right)^{-1}\left(B\left(\pi g O_{\underline{i}|_1} \Phi(\sigma \underline{i}), r\right)\right) \cap \mathcal{P}(\underline{i}).$$

Proof. For $\underline{i} = i_1 i_2 \cdots$ and $r > 0$ we have

$$B\left(\pi g \Phi(\underline{i}), r_{\underline{i}|_1} \cdot r\right) = \pi g f_{i_1}\left(B\left(\Phi(\sigma \underline{i}), r\right)\right).$$

Thus

$$(\pi g \Phi)^{-1}\left(B\left(\pi g \Phi(\underline{i}), r_{\underline{i}|_1} \cdot r\right)\right) \cap \mathcal{P}(\underline{i}) = (\pi g \Phi)^{-1}\left(\pi g f_{i_1}\left(B\left(\Phi(\sigma \underline{i}), r\right)\right)\right) \cap \mathcal{P}(\underline{i}).$$

But

$$
\begin{aligned}
& \underline{j} = j_1 j_2 \cdots \in (\pi g \Phi)^{-1}\big(\pi g f_{i_1}(B(\Phi(\sigma \underline{i}), r))\big) \cap \mathcal{P}(\underline{i}) \\
\Leftrightarrow \quad & j_1 = i_1, \Phi(\underline{j}) \in (\pi g)^{-1}\big(\pi g f_{i_1}(B(\Phi(\sigma \underline{i}), r))\big) \\
\Leftrightarrow \quad & j_1 = i_1, f_{j_1}(\Phi(\sigma \underline{j})) \in (\pi g)^{-1}\big(\pi g f_{i_1}(B(\Phi(\sigma \underline{i}), r))\big) \\
\Leftrightarrow \quad & j_1 = i_1, \pi g f_{j_1}(\Phi(\sigma j)) \in \pi g f_{i_1}(B(\Phi(\sigma \underline{i}), r)) \\
\Leftrightarrow \quad & j_1 = i_1, \pi g O_{j_1}(\Phi(\sigma \underline{j})) \in \pi g O_{i_1}(B(\Phi(\sigma \underline{i}), r)) \\
\Leftrightarrow \quad & j_1 = i_1, \pi g O_{j_1}\Phi(\sigma \underline{j}) \in B\big(\pi g O_{i_1}\Phi(\sigma \underline{i}), r\big) \\
\Leftrightarrow \quad & j_1 = i_1, \underline{j} \in \sigma^{-1}\big(\pi g O_{j_1}\Phi\big)^{-1}\Big(B\big(\pi g O_{i_1}\Phi(\sigma \underline{i}), r\big)\Big) \\
\Leftrightarrow \quad & \underline{j} \in \sigma^{-1}\Big(\pi g O_{\underline{i}|_1}\Phi\Big)^{-1}\Big(B\Big(\pi g O_{\underline{i}|_1}\Phi(\sigma \underline{i}), r\Big)\Big) \cap \mathcal{P}(\underline{i}),
\end{aligned}
$$

which gives the conclusion.

For $\underline{i} \in \Lambda^{\mathbb{N}}$ and $n \geq 1$, conditioning on $\mu\left(\left[\underline{i}|_n\right]\right) > 0$,

$$
\begin{aligned}
\frac{\mu\big(B_{\pi g \Phi}(\underline{i}, n)\big)}{\mu\left[\underline{i}|_n\right]\Big(B_{\pi g O_{i!}\Phi}(\sigma^n \underline{i}, 0)\Big)} &= \prod_{k=0}^{n-1} \frac{\mu^{[i|k]}\Big(B_{\pi g O_{\dot{y}_k}\Phi}\big(\sigma^k \underline{i}, n-k\big)\Big)}{\mu^{[i|_{k+1}]}\Big(B_{\pi g O_{\underline{i}|_{k+1}}\Phi}\big(\sigma^{k+1}\underline{i}, n-k-1\big)\Big)} \\
&= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}\Big(B_{\pi g O_{\perp}}\Phi\big(\sigma^k \underline{i}, n-k\big)\Big)}{\mu^{[i|_k]}\Big(B_{\pi g O_{\underline{i}_k}\Phi}\big(\sigma^k \underline{i}, n-k\big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)} \\
&\qquad \frac{\mu^{[\underline{i}|_k]}\Big(B_{\pi g O_{\dot{\iota}_k}\Phi}\big(\sigma^k \underline{i}, n-k\big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)}{\mu^{[\underline{i}|_{k+1}]}\Big(B_{\pi g O_{\underline{i}_{k+1}}\Phi}\Big(\sigma^{k+1}\underline{i}, n-k-1\Big)\Big)} \\
&= \prod_{k=0}^{n-1} \frac{\mu^{[i|_k]}\Big(B_{\pi g O_{\underline{i}|_k}\Phi}\big(\sigma^k \underline{i}, n-k\big)\Big)}{\mu^{[\underline{i}|_k]}\Big(B_{\pi g O_{\downarrow_k}\Phi}\Big(\sigma^k \underline{i}, n-k\Big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)} \\
&\qquad \frac{\mu^{[\underline{i}|_k]}\Big(\sigma^{-1} B_{\pi g O_{\lambda_{k+1}}\Phi}\big(\sigma^{k+1}\underline{i}, n-k-1\big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)}{\mu^{[i|_{k+1}]}\Big(B_{\pi g O_{\downarrow_{k+1}}\Phi}\Big(\sigma^{k+1}\underline{i}, n-k-1\Big)\Big)} \\
&= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}\Big(B_{\pi g O_{\perp_k}\Phi}\big(\sigma^k \underline{i}, n-k\big)\Big)}{\mu^{[i|_k]}\Big(B_{\pi g O_{\lambda_k}\Phi}\big(\sigma^k \underline{i}, n-k\big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)} \cdot W_{i_{k+1}}^{[\underline{i}|_k]} \\
&= \prod_{k=0}^{n-1} \frac{\bar{\mu}^{[\underline{i}|_k]}\Big(B_{\pi g O_{\perp_k}\Phi}\big(\sigma^k \underline{i}, n-k\big)\Big)}{\bar{\mu}^{[\underline{i}|_k]}\Big(B_{\pi g O_{\underline{i}_k}\Phi}\big(\sigma^k \underline{i}, n-k\big) \cap \mathcal{P}\big(\sigma^k \underline{i}\big)\Big)} \cdot W_{i_{k+1}}^{[\underline{i}|_k]}. \qquad (25)
\end{aligned}
$$

Notice that for $k \geq 0$,

$$f_{n-k}\circ T_{\phi}^{k}(\underline{i},\omega,g)=\log\frac{\bar{\mu}^{[i|_k]}\left(B_{\pi g O_{\underline{\alpha}_k}\Phi}\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}^{[\underline{i}_k]}\left(B_{\pi g O_{\underline{i}|_k}\Phi}\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}.$$

Using (24) we conclude that $\mathbb{P}_*$-a.s. for ξ-a.e. $g\in G$ and μ-a.e. $\underline{i}\in\Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\prod_{k=0}^{n-1}\frac{\bar{\mu}^{[i|_k]}\left(B_{\pi g O_{\underline{i}|_k}\Phi}\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}[\underline{i}_k]\left(B_{\pi g O_{\lambda_k}\Phi}\left(\sigma^k\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}$$
$$=\mathbb{E}_{\mathbb{P}^*\times\xi}\left(\mathbf{H}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)\right).\qquad(26)$$

This completes the proof.

Proof of Theorem (1.1.9)(iii). Given $k\geq 1, g\in G$ and $\bar{\mu}>0$, Lemma (1.1.8) yields that for μ-a.e. $\underline{i}\in\Lambda^N$

$$\bar{\mu}_{\underline{i},\pi g}\left(B_\Phi(\underline{i},k)\cap\mathcal{P}(\underline{i})\right)=\lim_{n\to\infty}\frac{\bar{\mu}\left(B_\Phi(\underline{i},k)\cap B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\right)}.$$

From Lemmas (1.1.14) and (1.1.16) we get

$$\frac{\bar{\mu}\left(B_\Phi(\underline{i},k)\cap B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\right)}$$
$$=\frac{\bar{\mu}\left(B_\Phi(\underline{i},k)\cap B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{\mu})\right)}\frac{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\right)}$$
$$=\frac{\bar{\mu}^{[i\downarrow]_1]}\left(B_\Phi(\sigma\underline{i},k-1)\cap B_{\pi g O_{\mathrm{i}}\Phi}\Phi(\sigma\underline{i},n-1)\right)}{\bar{\mu}^{[i|1]}\left(B_{\pi g O_{\mathrm{i}_1}\Phi}(\sigma\underline{\mu},n-1)\right)}\frac{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\right)}.$$

Since $\bar{\mu}^{[i]^1]}$ is absolutely continuous with respect to $\sigma\bar{\mu}|_{[i]1]}$, we obtain, in a similar way to the proof of Lemma (1.1.8), that for μ-a.e. $\underline{i}\in\Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty}\frac{\bar{\mu}^{[\underline{i}|_1]}\left(B_\Phi(\sigma\underline{i},k-1)\cap B_{\pi g O_{\underline{i}|_1}\Phi}(\sigma\underline{i},n-1)\right)}{\bar{\mu}^{[i]_1]}\left(B_{\pi g O_{\mathrm{i}_1}\Phi}(\sigma\underline{i},n-1)\right)}=\bar{\mu}^{[\underline{i}_1]}_{\sigma\underline{i},\pi g O_{\underline{i}|1}}\left(B_\Phi(\sigma\underline{i},k-1)\right).$$

On the other hand, by (23),

$$\lim_{n\to\infty}\frac{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\cap\mathcal{P}(\underline{i})\right)}{\bar{\mu}\left(B_{\pi g\Phi}(\underline{i},n)\right)}=\exp\left(-\mathbf{I}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)(\underline{i})\right).$$

Hence, for $k\geq 1, \mathbb{P}_*$ a.s. for ξ-a.e. $g\in G$ and μ-a.e. $\underline{i}\in\Lambda^{\mathbb{N}}$,

$$\bar{\mu}_{\underline{i},\pi g}\left(B_\Phi(\underline{i},k)\cap\mathcal{P}(\underline{i})\right)=\bar{\mu}^{[\underline{i}|_1]}_{\sigma\underline{i},\pi g O_{\underline{i}|1}}\left(B_\Phi(\sigma\underline{i},k-1)\right)\cdot\exp\left(-\mathbf{I}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)(\underline{i})\right).$$

This gives, noting that $\bar{\mu}_{\sigma^n}^{[i\underline{i},\pi g}{}_{\underline{i}]_{L_n}}\left(B_\Phi\left(\sigma^n\underline{i},0\right)\right)=1$,

$$\bar{\mu}_{\underline{i},\pi g}\left(B_\Phi(\underline{i},n)\right)$$
$$=\prod_{k=0}^{n-1}\frac{\bar{\mu}^{[\underline{i}|_k]}_{\sigma\underline{i},\pi g O_{\underline{i}|1}}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}^{[\underline{i}|_k]}_{\sigma^k\underline{i},\pi g O_{\underline{i}|_k}}\left(B_\Phi\left(\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}\exp\left(-\mathbf{I}_{\bar{\mu}^{[\underline{i}|_k]}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g O_{\underline{i}|1}\Phi}\right)\left(\sigma^k\underline{i}\right)\right)\qquad(27)$$

We need the following lemma:

Lemma (1.1.17)[1]: $\mathbb{P}_*$ a.s. for ξ-a.e. $g \in G$ and μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\prod_{k=0}^{n-1}\frac{\bar{\mu}^{[\underline{i}|_k]}_{\sigma^k\underline{i},\pi g O_{\underline{i}|_k}}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}^{[\underline{i}|_k]}_{\sigma^k\underline{i},\pi g O_{\underline{i}|_k}}\left(B_\Phi\left(\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}=\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P}\mid\mathcal{B}_\Phi)\right).$$

Proof. For $n \geq 1$ let

$$f_n\left(\underline{i},\omega,g\right)=-\log\frac{\bar{\mu}_{\underline{i},\pi g}\left(B_\Phi\left(\underline{i},n\right)\cap\mathcal{P}\left(\underline{i}\right)\right)}{\bar{\mu}_{\underline{i}\pi g}\left(B_\Phi\left(\underline{i},n\right)\right)}.$$

Applying Proposition (1.1.7) in the case of $\eta = \mathcal{P}_{\pi g}$ and $\varphi = \Phi$ we get that given $g \in G$, for $\mathbb{Q}$-a.e. $(\underline{i},\omega) \in \Omega'$ the sequence f_n converges to

$$f := \mathbf{I}_{\bar{\mu}}\left(\mathcal{P}\mid\widehat{\mathcal{P}_{\pi g}}\vee\mathcal{B}_\Phi\right)=\mathbf{I}_{\bar{\mu}}(\mathcal{P}\mid\mathcal{B}_\Phi),$$

Here we have used that the σ-algebra $\widehat{\mathcal{P}_{\pi g}}$ is a sub- σ-algebra of $\mathcal{B}_\Phi$. Moreover, since $\int \sup_n|f_n|\mathrm{d}\bar{\mu} \leq \mathbf{H}_{\bar{\mu}}(\mathcal{P}) + C_d \leq \log m + C_d$, $\sup_n|f_n|$ is integrable. As

$$\frac{1}{n}\log\prod_{k=0}^{n-1}\frac{\bar{\mu}^{[\underline{i}|_k]}_{\sigma^k\underline{i},\pi g O_{\underline{i}|_k}}\left(B_\Phi\left(\sigma^k\underline{i},n-k\right)\right)}{\bar{\mu}^{[\underline{i}|_k]}_{\sigma^k\underline{i},\pi g}\left(B_\Phi\left(\underline{i},n-k\right)\cap\mathcal{P}\left(\sigma^k\underline{i}\right)\right)}=\frac{1}{n}\sum_{k=0}^{n-1}f_{n-k}\circ T^k_\phi\left(\underline{i},\omega,g\right),$$

the conclusion follows from Theorem (1.1.12) and Proposition (1.1.5). By Proposition (1.1.5) we have $\mathbb{P}_*$ a.s. for ξ-a.e. $g \in G$ and μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{I}_{\bar{\mu}[\underline{i}|_k]}\left(\mathcal{P}\mid\mathcal{B}_{\pi g}O_{\underline{i}|_k}\Phi\right)\left(\sigma^k\underline{i}\right)=\mathbb{E}_{\mathbb{P}*\times\xi}\left(\mathbf{H}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)\right)$$

Combining (27) and Lemma (1.1.17) we get that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$ and μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\bar{\mu}_{\underline{i},\pi g}\left(B_\Phi\left(\underline{i},n\right)\right)=\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P}\mid\mathcal{B}_\Phi)\right)-\mathbb{E}_{\mathbb{P}_*}\times\xi\left(\mathbf{H}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)\right),$$

so that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$ and μ-a.e. $\underline{i} \in \Lambda^{\mathbb{N}}$,

$$\lim_{r\to 0}\frac{\log\Phi\bar{\mu}_{\underline{i},\pi g}(B(\Phi(\underline{i}),r))}{\log r}=\frac{\mathbb{E}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P}\mid\mathcal{B}_\Phi)\right)-\mathbb{E}_{\mathbb{P}*\times\xi}\left(\mathbf{H}_{\bar{\mu}}\left(\mathcal{P}\mid\mathcal{B}_{\pi g\Phi}\right)\right)}{\sum_{i=1}^{m}\mathbb{E}(W_i)\log r_i}. \quad (28)$$

Together with (18) this yields (iii).

We generalize the results of [14] on projections and images under C^1 functions without singularites to random cascade measures.

Let $D = B(0,R)$ where $R = \max\{|x|: x \in K\}$. Denote by $\mathcal{M}$ the family of probability measures on D and let $\mathcal{B}_\star$ be its weak-\star topology. Denote by $C(\mathcal{M})$ the family of all continuous functions on $\mathcal{M}$. We use the separability of $C(\mathcal{M})$ in $\|\cdot\|_\infty$ to get convergence of ergodic averages for all $h \in C(\mathcal{M})$.

Proposition (1.1.18)[1]: $\mathbb{P}_* -$ a.s. for ξ-a.e. g and μ-a.e. $\underline{i}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}h\left(gO_{\underline{i}|_n}\Phi\bar{\mu}^{[\underline{i}|_n]}\right)=\mathbb{E}_{Q\times\xi}(h(g\Phi\bar{\mu}))$$

for all $h \in C(\mathcal{M})$.

Proof. Let $\{h_k\}_{k\geq 1}$ be a countable dense sequence in $C(\mathcal{M})$. If we write

$$M: X \ni (\underline{i}, \omega, g) \mapsto g\Phi\bar{\mu} \in \mathcal{M},$$

then it is easy to verify that for $n \geq 1$

$$M \circ T_\phi^n(\underline{i}, \omega, g) = gO_{\underline{i}|_n} \Phi\bar{\mu}^{[\underline{i}|_n]}.$$

It follows from Proposition (1.1.5) that $\mathbb{P}_*$-a.s. for ξ-a.e. g and μ-a.e. $\underline{i}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h_k\left(gO_{\underline{i}|_n} \Phi\bar{\mu}^{[\mathrm{i}_n]}\right) = \mathbb{E}_{\mathbb{Q}\times\xi}(h_k(g\Phi\bar{\mu})) \text{ for all } k \geq 1.$$

For any $h \in C(\mathcal{M})$, take a subsequence $\{h_k'\}_{k\geq 1}$ of $\{h_k\}_{k\geq 1}$ that converges to h. On the one hand, since $\mathcal{M}$ is compact, h is bounded, so by the uniform convergence in $\|\cdot\|_\infty$,

$$\lim_{k\to\infty} \mathbb{E}_{\mathbb{Q}\times\xi}(h_k'(g\Phi\bar{\mu})) = \mathbb{E}_{\mathbb{Q}\times\xi}\big(h(g\Phi\bar{\mu})\big).$$

On the other hand, for each N,

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} h_k'\big(gO_{\underline{i}|n}\Phi\bar{\mu}^{[i|n]}\big) - \frac{1}{N}\sum_{n=0}^{N-1} h\big(gO_{\underline{i}|n}\Phi\bar{\mu}^{[i|n]}\big)\right| \leq \|h_k' - h\|_\infty.$$

Thus the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h\left(gO_{\underline{i}|_n} \Phi\bar{\mu}^{[i|_n]}\right)$$

exists and equals $\lim_{k\to\infty} \mathbb{E}_{\mathbb{Q}\times\xi}(h_k'(g\Phi\bar{\mu})) = \mathbb{E}_{\mathbb{Q}\times\xi}(h(g\Phi\bar{\mu}))$, $\mathbb{P}_*$-a.s. for ξ-a.e. g and μ-a.e. $\underline{i}$.

We use the ρ-tree method in [14] to obtain close lower bounds for the dimensions of projections of measures. Let $\rho = \max\{r_i : i \in \Lambda\}$ and $c = \min\{r_i : i \in \Lambda\}$. For $i = i_1 \cdots i_n \in \Lambda^*$ write $r_i^- = r_{i_1} \cdots r_{i_{n-1}}$. For each $q \geq 1$ we redefine the alphabet used for symbolic space to obtain one for which the contraction ratios do not vary too much:

$$\Lambda_q = \{i \in \Lambda^* : r_i^- > \rho^q \text{ and } r_i \leq \rho^q\}.$$

By definition $c\rho^q < r_i \leq \rho^q$ for all $i \in \Lambda_q$. The canonical mapping $\Phi_q : (\Lambda_q^{\mathbb{N}}, d_{\rho^q}) \mapsto K$ is R-Lipschitz where $R = \max\{|x| : x \in K\}$. Setting $\left\{W_q^{[j]} = \left(Q_i^{[j]}\right)_{i\in\Lambda_q} : j \in \Lambda_q^*\right\}$ gives a random cascade measure μ_q on $\Lambda_q^{\mathbb{N}}$. Observe that it is the same random cascade measure as μ on embedding $\Lambda_q^{\mathbb{N}}$ into $\Lambda^{\mathbb{N}}$. (The slight ambiguity in notation should not cause any confusion: the subscript q will always refer to the parameter redefining the alphabet, so, for example, $W_{q,i}$ refers to the element of $W_q \equiv W_q^{[\emptyset]}$ with index $\in \Lambda_q$.)

Let $G_q = \overline{\langle O_\iota : \iota \in \Lambda_q\rangle}$ and let ξ_q be its normalised Haar measure. As before, $\Pi_{d,k}$ is the set of orthogonal projections from $\mathbb{R}^d$ onto its k-dimensional subspaces. For $\pi \in \Pi_{d,k}$, $q \in \mathbb{N}$ and ν a measure on $\mathbb{R}^d$, define

$$e_q(\pi, \nu) = \frac{1}{q\log(1/\rho)} H_{\rho^q}(\pi\nu).$$

So $e_q : \Pi_{d,k} \times \mathcal{M} \mapsto [0, k]$ is lower semicontinuous. Let $E_q(\pi) = \mathbb{E}_{\mathbb{P}_*} \times \xi_q\big(e_q(\pi, g\Phi\bar{\mu})\big)$.

Theorem (1.1.19)[1]: $\mathbb{P}_* - a.s.$ for ξ_q-a.e. $g \in G_q$,

$$\dim_H(\pi g\Phi\bar{\mu}) \geq \frac{q\log(1/\rho)}{q\log(1/\rho) - \log c} E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi_{d,k},$$

where the implied constant in $O(1/q)$ only depends on ρ, c, R and k.

Proof. Applying Proposition (1.1.18) to a sequence of continuous functions approximating e_q from below and using the monotone convergence theorem, we have that $\mathbb{P}_*$-a.s. for ξ_q-a.e. g and μ_q-a.e. $\underline{i}$,

$$\liminf \frac{1}{N}\sum_{n=1}^{N} e_q\left(\pi, gO_{\underline{i}|_n}\Phi_q\bar{\mu}_q^{[\underline{i}|_n]}\right) \geq E_q(\pi) \text{for all } \pi \in \Pi_{d,k}. \tag{29}$$

Using the strong law of large numbers we note that $\mathbb{P}_*$-a.s. for μ_q-a.e. $\underline{i} \in \Lambda_q^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\log Q_{\underline{i}}\big|_n}{-n} = -\sum_{i\in\Lambda_q} \mathbb{E}\left(\chi_{\{W_{q,i}>0\}} W_{q,i}\log W_{q,i}\right) \in (0,\infty),$$

so in particular, $\mathbb{P}_*$-a.s. for μ_q-a.e. $\underline{i} \in \Lambda_q^{\mathbb{N}}, Q_{\underline{i}|_n} > 0$ for all $n \geq 1$. Identically,

$$\chi_{\{Q_{i|n}>0\}}\bar{\mu}_q^{[i|n]} = \chi_{\{Q_{\underline{i}|n}>0\}}\chi_{\{\|\mu_q^{[\underline{i}|_n]}\|>0\}} \cdot \frac{\mu_q^{[\underline{i}|_n]}}{\left\|\mu_q^{[\underline{i}|_n]}\right\|} = \sigma^n\bar{\mu}_{q,[\underline{i}|_n]},$$

where

$$\bar{\mu}_{q,[i|n]} = \chi_{\{\mu_q([i|n])>0\}}\frac{\mu_q\big|_{[\underline{i}|_n]}}{\mu_q([\underline{i}\mid_n])},$$

so by (15)

$$\begin{aligned} H_{\rho q}\left(\pi g O_{\underline{i}|_n}\Phi_q\chi_{\{Q_{\underline{i}|_n}>0\}}\bar{\mu}_q^{[\underline{i}|_n]}\right) &= H_{\rho q}\left(\pi g O_{\underline{i}|n}\Phi_q\sigma^n\bar{\mu}_{q,[\underline{i}|_n]}\right) \\ &= H_{\rho^q\cdot r_{\underline{i}|_n}}\left(\pi g\Phi_q\bar{\mu}_{q,[\underline{i}|_n]}\right) \\ &\leq H_{(c\rho^q)^{n+1}}\left(\pi g\Phi_q\bar{\mu}_{q,[\underline{i}|_n]}\right). \end{aligned}$$

Hence, using (29), $\mathbb{P}_*$-a.s. for ξ_q-a.e. g and μ_q-a.e. $\underline{i}$,

$$\frac{1}{q\log(1/\rho)}\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} H_{(c\rho^q)^{n+1}}\left(\pi g\Phi_q\bar{\mu}_{q,[\underline{i}|_n]}\right) \geq E_q(\pi) \text{ for all} \pi \in \Pi_{d,k}.$$

The mapping $f \equiv \pi g\Phi_q: \left((\Lambda^q)^{\mathbb{N}}, d_{\rho q}\right) \mapsto \mathbb{R}^k$ is R-Lipschitz. By [14, Theorem 5.4] there exist a ρ^q-tree $\left(X, d_{\rho^q}\right)$ and maps $(\Lambda^q)^{\mathbb{N}} \overset{h}{\mapsto} X \overset{f'}{\mapsto} \mathbb{R}^k$ such that $f = f'h$, where h is a tree morphism and f' is C-faithful (see [14, Definition 5.1]) for some constant C depending only on R and k. Then, applying [14, Proposition 5.3] to the $c\rho^q$-tree $\left(X, d_{c\rho q}\right)$ (for which f' is $c^{-1}C$-faithful), there is a constant C' depending only on $c^{-1}C$ and k such that for all $n \geq 1$,

$$\left|H_{(c\rho^q)^{n+1}}\left(f\bar{\mu}_{q,[\underline{i}|_n]}\right) - H_{(c\rho^q)^{n+1}}\left(h\bar{\mu}_{q,[\underline{i}|_n]}\right)\right| \leq C'.$$

Consequently, $\mathbb{P}_*$-a.s. for ξ_q-a.e. g and $\bar{\mu}_q$-a.e. $\underline{i}$,

$$\begin{aligned} \frac{1}{q\log(1/\rho)}\liminf_{N\to\infty}\frac{1}{N-1}\sum_{n=1}^{N} & H_{(c\rho^q)^{n+1}}\left(h\bar{\mu}_{q,[\underline{i}|_n]}\right) \\ & \geq E_q(\pi) - O\left(\frac{1}{q}\right) \text{for all } \pi \in \Pi_{d,k}, \end{aligned}$$

where the constant in $O(1/q)$ only depends on ρ and C'. By [14, Theorem 4.4] it follows that $\mathbb{P}_*$-a.s. for ξ_q-a.e. g,

$$\dim_H h\bar{\mu}_q \geq \frac{q\log(1/\rho)}{q\log(1/\rho) - \log c} E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi_{d,k}.$$

Since f' is C-faithful and $f' h\bar{\mu}_q = f\bar{\mu}_q = \pi g \Phi_q \bar{\mu}_q = \pi g \Phi \bar{\mu}$, the conclusion follows from [14, Proposition 5.2].

The projection results in [14] require the strong separation condition on the underlying IFS $\mathcal{I}$. With the approach we avoid the need for any separation condition at all. Moreover, our results apply to random cascade measures as well as deterministic measures on self-similar sets.

We assume that the rotation group $G \equiv \overline{\langle O_\iota : \iota \in \Lambda \rangle}$ is connected and we denote by ξ its normalised Haar measure. We fix $\pi_0 \in \Pi_{d,k}$ and write $\Pi = \pi_0 G$.

We remark that the arguments extend to the more general setting where the orbit πG is of the form $\pi \tilde{G}$, where $\tilde{G}$ is connected. This includes the case of certain restricted families of projections, for example for projections onto the lines lying in certain cones.

Lemma (1.1.20)[1]: If G is connected then $\pi_0 G_q = \Pi$ for each $q \geq 1$.

Proof. For $i = i_1 i_2 \cdots i_l \in \Lambda^*$ (where $i_j \in \Lambda$) let $O_i = O_{i_1} O_{i_2} \cdots O_{i_l}$. It is sufficient to prove that the group $H := \langle O_i : i \in \Lambda_q \rangle$ is dense in G. (Recall that the closed group generated by a set of elements coincides with the closed semigroup generated by them). Write $\Lambda_{<q} = \{i \in \Lambda^* : r_i > \rho^q\}$. Then $\bigcup_{j \in \Lambda_{<q}} O_j H$ is dense in G. By Baire's category theorem, we may choose $j \in \Lambda_{<q}$ such that $\overline{O_j H}$ has nonempty interior in G. Consequently $\bar{H}$ has nonempty interior, so if h is in the interior of $\bar{H}$ then $\bar{H} = h^{-1}\bar{H}$. Thus $\bar{H}$ contains a neighborhood of the identity, so since a compact connected Lie group is generated by any neighbourhood of its identity, $\bar{H} = G$.

Hence for $\pi \in \Pi$ we have

$$E_q(\pi) = \mathbb{E}_{\mathbb{P}^* \times \xi_q}\big(e_q(\pi, g\Phi\bar{\mu})\big) = \mathbb{E}_{\mathbb{P}^* \times \xi}\big(e_q(\pi, g\Phi\bar{\mu})\big).$$

For the same reason we can also deduce from Theorem (1.1.9) (ii) that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in \tilde{G}, \pi_0 g \Phi \mu$ is exact-dimensional with dimension

$$\beta(\pi_0) = \frac{\mathbb{E}_{\mathbb{P}^* \times \xi}\left(\mathbf{H}_{\bar{\mu}}(\mathcal{P} \mid \mathcal{B}_{\pi_0 g \Phi})\right) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i}.$$

Theorem (1.1.21)[1]: Let $\pi_0 \in \Pi_{d,k}$ and let G be connected. Then the limit

$$E(\pi) := \lim_{q \to \infty} E_q(\pi)$$

exists for every $\pi \in \Pi$, and $E : \Pi \mapsto [0, k]$ is lower semi-continuous. Moreover:

(i) $E(\pi_0 g) = \beta(\pi_0)$ for ξ-a.e. g.

(ii) For a fixed $\pi \in \Pi, \mathbb{P}_*$-a.s. for ξ-a.e. g,

$$\dim_e \pi g \Phi \bar{\mu} = \dim_H \pi g \Phi \bar{\mu} = E(\pi).$$

(Recall that $\dim_e$ is the entropy dimension.)

(iii) $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$,

$$\dim_H \pi g \Phi \bar{\mu} \geq E(\pi) \text{ for all } \pi \in \Pi.$$

Proof. The proof is almost the same as that of [14, Theorem 8.2]. By Theorem (1.1.19) and Lemma (1.1.20) we have for each $q \geq 1$ that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$,

$$\dim_H(\pi g \Phi \bar{\mu}) \geq \frac{q\log(1/\rho)}{q\log(1/\rho) - \log c} E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi,$$

where the implied constant in $O(1/q)$ only depends on ρ, c, R and k. This implies that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$,

$$\dim_H(\pi g\Phi\bar{\mu}) \geq \limsup_{q\to\infty} E_q(\pi) \text{ for all } \pi \in \Pi.$$

On the other hand by using Fatou's lemma we have

$$\mathbb{E}_{\mathbb{P}^*\times\xi}(\dim_e(\pi g\Phi\bar{\mu})) \leq \liminf_{q\to\infty} E_q(\pi)$$

This implies that $\lim_{q\to\infty} E_q(\pi)$ exists for all $\pi \in \Pi$. Then (ii) and (iii) follow directly, and (i) follows from Theorem (1.1.9)(ii).

For the lower semicontinuity of E, fix $\pi \in \Pi$ and $\epsilon > 0$. Using that $E_q(\pi) \to E(\pi)$ and E_q is lower semicontinuous, there exist a number q and a neighbourhood $\mathcal{U}(\pi)$ of π in $\Pi_{d,k}$ such that for all $\pi' \in \mathcal{U}(\pi)$,

$$\frac{q\log(1/\rho)}{q\log(1/\rho) - \log c} E_q(\pi') - O(1/q) \geq E(\pi) - \epsilon.$$

This gives that $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$,

$$\dim_H(\pi' g\Phi\bar{\mu}) \geq E(\pi) - \epsilon \text{ for all } \pi' \in \mathcal{U}(\pi).$$

By (ii) this yields that $E(\pi') \geq E(\pi) - \epsilon$ for all $\pi' \in \mathcal{U}(\pi)$, giving the conclusion.

We can now obtain a constant lower bound for the dimension of the projected measure over all $\pi \in \Pi$.

Corollary (1.1.22)[1]: Let G be connected and let $\pi_0 \in \Pi_{d,k}$. Then $\mathbb{P}_* - a.s.$

$$\dim_H \pi\Phi\mu \geq \beta(\pi_0) \text{ for all } \pi \in \Pi = \pi_0 G. \tag{30}$$

Proof. Since E is lower semi-continuous, it follows from Theorem (1.1.21)(i) that for any $\epsilon > 0$ the set

$$\mathcal{U}_\epsilon = \{\pi \in \Pi_{d,k}: E(\pi) > \beta(\pi_0) - \epsilon\}$$

is open and dense in Π. Write $\mathcal{U}_\epsilon g = \{\pi g: \pi \in \mathcal{U}_\epsilon\}$ for $g \in G$. Then from Theorem (1.1.21)(iii) we have $\mathbb{P}_*$-a.s. for ξ-a.e. $g \in G$,

$$\tilde{\mathcal{U}}_\epsilon = \{\pi \in \Pi: \dim_H \pi\Phi\bar{\mu} > \beta(\pi_0) - \epsilon\} \supseteq \mathcal{U}_\epsilon g.$$

Since $\mathcal{U}_\epsilon$ has non-empty interior, we deduce that $\mathbb{P}_*$-a.s. $\tilde{\mathcal{U}}_\epsilon = \Pi$ as required.

Corollary (1.1.23)[1]: If $G = SO(d,\mathbb{R})$, then $\mathbb{P}_*$-a.s.

$$\dim_H \pi\Phi\mu = \min(k, \dim_H \Phi\mu) \text{ for all } \pi \in \Pi_{d,k}. \tag{31}$$

Moreover, with α and $\beta(\pi)$ as in Theorem (1.1.9)(i), (ii), $\beta(\pi) = \min(k,\alpha)$ for all $\pi \in \Pi_{d,k}$.

Proof. If $G = SO(d,\mathbb{R})$, then $\pi_0 G = \Pi_{d,k}$ and for any $\pi \in \Pi_{d,k}$ there exists $g \in G$ such that $\pi_0 g = \pi$. Due to the invariance of Haar meausres this implies that $\beta(\pi) = \beta(\pi_0)$ for all $\pi \in \Pi_{d,k}$, thus a constant. Then by Corollory (1.1.22) we get that $\mathbb{P}_*$-a.s. $\dim_H \pi\Phi\mu \geq \beta(\pi_0)$ for all $\pi \in \Pi_{d,k}$, with equality for almost all π by Theorem (1.1.9)(ii). From the definition of dimension of measures (3), and applying the projection theorems of Marstrand [22] and Mattila [24] to sets E with $\Phi\bar{\mu}(E) > 0$ and $\dim_H E > \dim_H \Phi\bar{\mu} - \epsilon$, for $\epsilon > 0$, it follows that $\mathbb{P}_*$-a.s. $\dim_H \pi\Phi\mu \leq \min(k, \dim_H \Phi\mu) = \min(k,\alpha)$ for all $\pi \in \Pi_{d,k}$ with equality for a.a. $\pi \in \Pi_{d,k}$. The conclusions follow.

As in [14] results on projections may be generalized to C^1-maps without singular points, that is C^1-maps for which the derivative matrix is everywhere non-singular.

Proposition (1.1.24)[1]: Let $\pi \in \Pi = \pi_0$ G. For all C^1-maps $h: B(0,R) \mapsto \mathbb{R}^k$ such that $\sup_{x\in K} \|D_x h - \pi\| < c\rho^q$, we have that $\mathbb{P}_* -$ a.s. for ξ-a.e. $g \in G$,

$$\dim_H hg\Phi\bar{\mu} \geq E_q(\pi) - O(1/q),$$

where the constant in $O(1/q)$ only depends on ρ, c, R and k.

Corollary (1.1.25)[1]: If $G = SO(d,\mathbb{R})$, then $\mathbb{P}_*$-a.s., for all C^1-maps $h: K \mapsto \mathbb{R}^k$ without singular points,

$$\dim_H h\Phi\mu = \mathrm{m}\ \ (k, \dim_H \Phi\mu). \tag{32}$$

Proof. Corollary (1.1.23) together with Theorem (1.1.21)(ii) yields that $E(\pi) = \min(k, \dim_H \Phi\mu) = \min(k, \alpha)$ is a constant for all $\pi \in \Pi_{d,k}$, and it is the maximum possible value since h is a C^1 map. The result follows from Proposition (1.1.24).

Random cascade measures include non-random measures as a special case, so we can apply our results to the fractal geometry of deterministic self-similar sets. We consider an IFS $\mathcal{J}$ of similarities on $\mathbb{R}^d$(5) with rotation group $G = \overline{\langle O_\iota : \iota \in \Lambda\rangle} = SO(d, \mathbb{R})$, with self-similar attractor K the unique non-empty compact subset of $\mathbb{R}^d$ satisfying $K = \cup_{i=1}^m f_i(K)$. Recall that $\mathcal{J}$ satisfies the strong separation condition (SSC) if this union is disjoint and satisfies the open set condition (OSC) if there is a non-empty open set V such that $V \subseteq \cup_{i=1}^m f_i(V)$ with this union disjoint. If either SSC or OSC are satisfied then

$$\dim_H K = s \text{ where } \sum_{i=1}^{m} r_i^s = 1. \tag{33}$$

To transfer our results to sets we need to ensure that the sets support suitable measures. From the definitions, if a probability measure ν is supported by a compact set K then $\dim_H \nu \le \dim_H K$. We say that an IFS $\mathcal{J}$ with self-similar attractor K satisfies the strong variational principle if there is a Bernoulli probability measure μ on $\Lambda^{\mathbb{N}}$ such that $\dim_H \Phi\mu = \dim_H K$. No self-similar set with $G = SO(d, \mathbb{R})$ which does not satisfy the strong variational principle is known, and in particular the principle holds in the cases described in the following lemma.

Lemma (1.1.26)[1]: (a) If the IFS $\mathcal{J}$ satisfies the open set (or strong separation) condition then $\mathcal{J}$ satisfies the strong variational principle.

(b) Given $0 < r_i < \frac{1}{2}$ and O_i, the IFS $\mathcal{J}$ in (5) satisfies the strong variational principle for almost all $(t_1, \dots, t_m)$ in the sense of md-dimensional Lebesgue measure.

Proof. (a) With s given by (33), the Bernoulli probability measure μ on $\Lambda^{\mathbb{N}}$, defined by

$$\mu^{[\emptyset]}([i]) = r_i^s \ (i = 1, \dots, m), \tag{34}$$

has $\dim_H \Phi\mu = \dim_H K$. This fact is the key step in showing that $\dim_H K = s$ when OSC holds, see for example [15].

(b) This follows by applying to self-similar sets the argument used in [7] to find the almost sure dimension of self-affine sets. With μ as in (34), integrating the t-energy of the image measures $\Phi\mu$ over a parameterized family of self-similar sets gives that the energy is bounded for almost all $(t_1, \dots, t_m)$ for all $t < s$, so that $\dim_H K = s$ for almost all $(t_1, \dots, t_m)$.

The following two corollaries, obtained by applying Corollaries (1.1.22) and (1.1.23) to selfsimilar sets, weaken the conditions that guarantee the dimensions of projections and images from those of [14] to just the strong variational principle.

Corollary (1.1.27)[1]: Let K be the self-similar attractor of an IFS I with rotation group $SO(d, \mathbb{R})$ such that the strong variational principle is satisfied. Then

$$\dim_H \pi K = \min(k, \dim_H K) \text{ for all } \pi \in \Pi_{d,k}.$$

Corollary (1.1.28)[1]: Let K be the self-similar attractor of an IFS $\mathcal{J}$ with rotation group $SO(d, \mathbb{R})$ such that the strong variational principle is satisfied. Then for all $C^1 -$ maps $h: K \to \mathbb{R}^k$ without singular points

$$\dim_H h(K) = \min\ \ (k, \dim_H K).$$

The distance set of $A \subseteq \mathbb{R}^d$ is defined as $D(A) = \{|x-y|: x, y \in A\}$ and the pinned distance set of A at a is $D_a(A) = \{|x-a|: x \in A\}$. A general open problem is to relate the Hausdorff dimensions and Lebesgue measures of $D(A)$ and $D_a(A)$ to that of A. For self-similar sets in the plane, Orponen [27] showed that if $\dim_H K > 1$ then $\dim_H D(K) = 1$. We have the following variant.

Corollary (1.1.29)[1]: Let K be the self-similar attractor of an IFS $\mathcal{I}$ with rotation group $SO(d, \mathbb{R})$ such that the strong variational principle is satisfied. Then there exists $a \in K$ such that

$$\min \ (1, \dim_H K) = \dim_H D_a(K) \leq \dim_H D(K) \leq 1.$$

Proof. Take a point $a \in K$, and some $i \in \Lambda$ such that $a \notin f_i(K)$. Then $f_i(K)$ is similar to K, so by scaling, Corollary (1.1.28) applies to C^1-maps $h: f_i(K) \to \mathbb{R}^k$. The mapping $h: f_i(K) \to \mathbb{R}$ given by $h(x) = |x-a|$ is C^1 and has no singular points, so applying Corollary (1.1.28) to $f_i(K)$ gives

$$\dim_H\{|x-a|: x \in f_i(K)\} = \dim_H\{h(f_i(K))\}$$
$$= \min \ (1, \dim_H f_i(K)) = \min \ (1, \dim_H K)$$

since $f_i(K)$ is similar to K. Since $a \in K$ and $f_i(K) \subseteq K, \{|x-a|: x \in f_i(K)\} \subseteq D_a(K)$.
Furstenberg [12] showed that if a self-similar set has finite rotation group finite and satisfies the SSC then all directions are dimension conserving. Here we can dispense with the separation condition.

Corollary (1.1.30)[1]: Let K be the self-similar attractor of an IFS $\mathcal{I}$ with finite rotation group such that the strong variational principle is satisfied. Then every direction is dimension conserving, that is for all $\pi \in \Pi_{d,k}$ there is a number $\Delta > 0$ such that

$$\Delta + \dim_H\{y \in \mathbb{R}^k: \dim_H(K \cap \pi^{-1}y) \geq \Delta\} \geq \dim_H K \tag{35}$$

(we take $\dim \emptyset = -\infty$).

Proof. This follows from Corollary (1.1.11) taking $\Delta = \dim_H \Phi\bar{\mu}_{y,\pi}$ for some measure μ satisfying the strong variational principle.

Examples such as the Sierpiński triangle [18] and the Sierpiński carpet [21] show that the value of Δ in (35) can vary with π.

Whilst fractal percolation or Mandelbrot percolation is most often based on a decomposition of a d-dimensional cube into m^d equal subcubes of sides m^{-1}, random subsets of any self-similar set may be constructed using a similar percolation process. Let $\mathcal{I} = \{f_i = r_i O_i \cdot + t_i\}_{i=1}^m$ be an IFS of similarities with attractor K and let $\mathbb{P}$ be a probability distribution on $\mathcal{P}(\Lambda)$, the collection of all subsets of $\Lambda = \{1, \ldots, m\}$. We define a sequence of random subsets of Λ^n inductively as follows. The random set $S_1 \subseteq \Lambda$ has distribution $\mathbb{P}$. Then, given S_n, let $S_{n+1} = \cup_{i \in S_n} S^i$ where $S^i = \{ij: j \in S_1^i\} \subseteq \Lambda^{n+1}$ and where $S_1^i \subseteq \Lambda$ has the distribution $\mathbb{P}$ independently for each $i \in S_n$. A sequence of random subsets $\{K_n\}_{n=1}^\infty$ of K is given by $K_n = \cup_{i \in S_n} f_i(K)$. We write $K_{\mathbb{P}} = \cap_{n=0}^\infty K_n$ for the resulting random compact subset of K which is known as the percolation set. (Note that standard Mandelbrot percolation on a cubic grid is a particular case of percolation on a self-similar set satisfying OSC.)

We say that $(\mathcal{I}, \mathbb{P})$ satisfies the strong variational principle if there exists a random cascade measure μ on $\Lambda^{\mathbb{N}}$ such that there is a positive probability of $K_{\mathbb{P}} \neq \emptyset$, and such that, conditional on $K_{\mathbb{P}} \neq \emptyset$,

$$\dim_H K_{\mathbb{P}} = \dim_H \Phi\mu = \alpha \tag{36}$$

a.s., where α is given by Theorem (1.1.9)(i). The next lemma gives a condition for $(\mathcal{J}, \mathbb{P})$ to satisfy the strong variational principle, in which case α is given by an expectation equation.

Lemma (1.1.31)[1]: Let $(\mathcal{J}, \mathbb{P})$ be as above with $\mathcal{J}$ satisfying OSC and with $\mathbb{E}\{\text{card } S_1\} > 1$. Then $(\mathcal{J}, \mathbb{P})$ satisfies the strong variational principle with α given by

$$\mathbb{E}\left(\sum_{i \in S_1} r_i^{\alpha}\right) = 1. \tag{37}$$

Proof. By standard branching process theory [2], if $\mathbb{E}\{\text{card } S_1\} > 1$ there is a positive probability that $K_{\mathbb{P}} \neq \emptyset$. Under OSC, conditional on $K_{\mathbb{P}} \neq \emptyset$ the a.s. dimension of $K_{\mathbb{P}}$ is the solution α of (37) The random cascade defined by the random vector

$$W = (W_1, \dots, W_n) = \left(r_1^{\alpha}\chi_{\{1 \in S\}}(\omega), \dots, r_m^{\alpha}\chi_{\{m \in S\}}(\omega)\right). \tag{38}$$

gives rise to a random measure $\Phi\mu$ supported by $K_{\mathbb{P}}$ such that $\mathbb{P}^*(K_{\mathbb{P}} \neq \emptyset) > 0$. Using a potential-theoretic estimate or a direct verification of the formula in Theorem (1.1.9)(i), $\dim_H \Phi\mu = \dim_H K_{\mathbb{P}} = \alpha$ a.s., see [7,26], so the α given by (37) equals that of Theorem (1.1.9)(i).

Investigation of the dimensions of projections of the basic m-adic square-based percolation process goes back some years, see [5] for a survey, and recently Rams and Simon [32] showed using direct geometric arguments that a.s. all orthogonal projections of square-based percolation have Hausdorff dimension $\min\{1, \alpha\}$, where α is the dimension of the percolation set. The following application of Corollary (1.1.22) gives a similar conclusion for percolation on self-similar sets for which the IFS has dense rotations.

Corollary (1.1.32)[1]: If $(\mathcal{J}, \mathbb{P})$ satisfies the strong variational principle and has rotation group $SO(d, \mathbb{R})$, then a.s. conditional on $K_{\mathbb{P}} \neq \emptyset$,

$$\dim_H \pi K_{\mathbb{P}} = \min\ (k, \dim_H K_{\mathbb{P}}) = \min\ (k, \alpha) \text{ for all } \pi \in \Pi_{d,k},$$

where α is given by (36).

Again, Corollary (1.1.25) gives a variant for C^1-maps.

Corollary (1.1.33)[1]: If $(\mathcal{J}, \mathbb{P})$ satisfies the strong variational principle and has rotation group $SO(d, \mathbb{R})$, then a.s. conditional on $K_{\mathbb{P}} \neq \emptyset$,

$$\dim_H h(K_{\mathbb{P}}) = \min\ (k, \dim_H K_{\mathbb{P}}) = \min\ (k, \alpha)$$

for all C^1-maps $h: K \to \mathbb{R}^k$ without singular points, where α is given by (36).

Distance sets of percolation sets have also attracted interest recently, see [32] for the case of square-based percolation. The following result follows from a similar argument to that of Corollary (1.1.29) but in a random setting using Corollary (1.1.33).

Corollary (1.1.34)[1]: Suppose that $(\mathcal{J}, \mathbb{P})$ satisfies the strong variational principle and has rotation group $SO(d, \mathbb{R})$. Then a.s. conditional on $K_{\mathbb{P}} \neq \emptyset$, there exists $a \in K_{\mathbb{P}}$ such that

$$\min\ (1, \alpha) = \dim_H D_a(K_{\mathbb{P}}) \leq \dim_H D(K_{\mathbb{P}}) \leq 1,$$

where α is given by (36).

Section (1.2): Self-Similar Sets and Fractal Percolation:

Relating the Hausdorff dimension $\dim_H K$ of a set $K \subset \mathbb{R}^d$ to the dimensions of its sections and projections has a long history. The best-known result on projections is that, if K is Borel or analytic, then, writing $\pi_V: \mathbb{R}^d \to V$ for orthogonal projection onto the subspace V,

$$\dim_H \pi_V K = \min\ (k, \dim_H K), \tag{39}$$

for almost all k-dimensional subspaces V (with respect to the natural invariant measure on subspaces). For sections of sets, for almost all k-dimensional subspaces V, the dimensions of the sections or slices $\pi_V^{-1}x \cap K$ of K satisfy

$$\dim_H(K \cap \pi_V^{-1}x) \leq \max(0, \dim_H K - k)$$

for Lebesgue almost all $x \in V$ (we take $\dim_H \emptyset = -\infty$). Moreover, for all $\epsilon > 0$ and almost all V, there is a set $W_\epsilon \subset V$ of positive k-dimensional Lebesgue measure such that

$$\dim_H(K \cap \pi_V^{-1}x) \geq \max(0, \dim_H K - k) - \epsilon \qquad (40)$$

for $x \in W_\epsilon$. These inequalities were obtained by Marstrand [48] for subsets of the plane, and extended to general d and k by Mattila [50]. Kaufman [45] introduced the potential theoretic method which is now commonly used in studying dimensions of projections and sections of sets.

These properties are complemented by the fact [49] that, for all k-dimensional subspaces V, for all $0 \leq \Delta \leq d - k$,

$$\Delta + \dim_H\{x \in V: \dim_H(K \cap \pi_V^{-1}x) \geq \Delta\} \leq \dim_H K$$

In particular, if $\dim_H K > k$ then for all V

$$\dim_H(K \cap \pi_V^{-1}x) \leq \dim_H K - k$$

for Lebesgue almost all $x \in V$. A good exposition of this material may be found in [51]. Fursternberg [41] introduced the notion of dimension conservation: given $K \subset \mathbb{R}^d$, a projection π_V is said to be dimension conserving for K if there is a number $\Delta > 0$ such that

$$\Delta + \dim_H\{x \in V: \dim_H(K \cap \pi_V^{-1}x) \geq \Delta\} \geq \dim_H K \qquad (41)$$

We consider a slightly weaker property when $\dim_H K > k$. We say that a projection π_V is weakly dimension conserving if, for all $\epsilon > 0$,

$$\dim_H(K \cap \pi_V^{-1}x) > \dim_H K - k - \epsilon \text{ for all } x \in W, \qquad (42)$$

where W is a 'large' subset of V, either with $\dim_H W = k$ or with $\mathcal{L}^k(W) > 0$, where $\mathcal{L}^k$ denotes k-dimensional Lebesgue measure. It follows from (40) that π_V is weakly dimension conserving for almost every k-dimensional subspace V.

There has been great interest recently in identifying classes of sets, in particular classes of self-similar sets and their variants, for which these various inequalities hold for all, rather than just almost all, subspaces. Several establish (39) for all projections for classes of self-similar sets [41,43,54,58] and for percolation on self-similar sets [40,55,56,57,59]. Here we consider dimensions of sections, and identify sets for which (42), or a similar inequality for box-counting dimension, holds for all subspaces V.

Recall that an iterated function system (IFS) $\mathcal{I} = \{f_i\}_{i=1}^m$ on $\mathbb{R}^d$ is a family of $2 \leq m < \infty$ contractions $f_i: \mathbb{R}^d \to \mathbb{R}^d$. An IFS determines a unique non-empty compact $K \subset \mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^{m} f_i(K), \qquad (43)$$

called the attractor of the IFS, see [39,44]. If the f_i are all similarities then K is self-similar. The IFS satisfies the strong separation condition (SSC) if the union (43) is disjoint, and the open set condition (OSC) if there is a non-empty open set U such that $\cup_{i=1}^m f_i(U) \subset U$ with this union disjoint. If either SSC or OSC hold then $\dim_H K = s$ where s is given by $\sum_{i=1}^m r_i^s = 1$, where r_i is the similarity ratio of f_i.

We may write an IFS of (orientation preserving) similarities as

$$\mathcal{I} = \{f_i = r_i R_i \cdot + a_i\}_{i=1}^m$$

where $R_i \in SO(d, \mathbb{R})$ is a rotation, r_i is the scaling ratio and a_i is a translation. If the group G generated by $\{R_1, \ldots, R_m\}$ is dense in $SO(d, \mathbb{R})$ we say that the IFS has dense rotations.

A number of results on dimension conservation of self-similar sets have been established. Furstenberg [41] showed that (41) holds for projections onto all subspaces V for a class of 'homogeneous' sets. These include self-similar sets where the IFS $\mathcal{J}$ consists of contracting homotheties (i.e. similarities without rotation or reflection so that $R_i = I$ for all) that satisfy SSC or OSC. For example, variants on the Sierpiński carpet are of this type, where the value of Δ in (41) depends on the subspace V. There are detailed analyses of sections of the Sierpiński carpet in [47,46] and of sections of the Sierpiński gasket or triangle in [35]. In the case where the IFS $\mathcal{J}$ satisfies OSC and the group generated by $\{R_1, \ldots, R_m\}$ is finite, then every projection is dimension conserving, that is for all V (41) holds for some number Δ, see [40,42].

We demonstrate that many self-similar sets K are weakly dimension conserving for all, or virtually all, projections π_V. For self-similar sets in $\mathbb{R}^2$ where $\mathcal{J}$ satisfies OSC and has dense rotations and $\dim_H K > 1$, (42) holds with $\mathcal{L}(W) > 0$ for all $\epsilon > 0$ and for projections onto all lines V, except for lines in a set of directions of Hausdorff dimension 0. Provided that we replace Hausdorff dimension by lower box dimension on the left-hand side of the inequality we get (42) for all lines, for a large class of sets that satisfy a projection condition. We also show that, almost surely, (42) is true for all k-dimensional subspaces V for random subsets of $\mathbb{R}^d$ obtained by the Mandelbrot percolation process.

The idea is to demonstrate weak dimension conservation for a deterministic set K by running a percolation-type process on K to 'probe' the dimensions of its sections. We construct random sets $K^\omega \subset K$ such that $k < \dim K^\omega < k + \epsilon/2$ with positive probability. Writing L_x for the $(d-k)$-plane through x and perpendicular to V, if $\dim(K \cap L_x) < \dim K - k - \epsilon$ for some $x \in V$ there is a high probability that $K^\omega \cap L_x = \emptyset$ or equivalently that $x \notin \pi_V K^\omega$. By invoking results on projections of random sets that show that with positive probability $\dim \pi_V K^\omega = k$, we conclude that there must be a significant subset of $x \in V$, indeed a subset of dimension k, for which this does not occur.

We formulate this principle in a general context in Proposition (1.2.1) and Proposition (1.2.2). To apply it in various settings we utilise results on dimensions of projections of percolation sets from [40,55,56,57]. Theorem (1.2.14) and Theorem (1.2.16) depend on the absolute continuity of projections of an alternative type of random measure, and this is established in Theorem (1.2.13) which is a random version of a deterministic result of Shmerkin and Solomyak [58].

We present a general formulation of our method for obtaining lower bounds for the dimensions of sections of a set given a knowledge of the dimensions of projections of related random subsets. The method applies to sets that can be modeled in terms of an infinite rooted tree. These include self-similar sets, where the tree provides a natural description of the hierarchical construction of the set, but extends to a many further fractals.

Let $\Lambda = \{1, \ldots, m\}$ be an alphabet of $m \geq 2$ symbols, with Λ^n denoting the set of words of length $n \geq 0$. Let $\Sigma_* := \cup_{n \geq 0} \Lambda^n$ be the set of finite words and $\Sigma := \Lambda^{\mathbb{N}}$ the corresponding symbolic space of all infinite words. For each $\mathbf{i} \in \Sigma_*$ denote by $[\mathbf{i}] \subset \Sigma$ the set of infinite words that start with $\mathbf{i}$, that is the cylinder rooted at $\mathbf{i}$. We denote the diameter of a set $A \subset \mathbb{R}^d$ by $|A|$.

We consider fractals which are the image of a subset of symbolic space under a continuous mapping $\Phi: \Sigma \mapsto \mathbb{R}^d$ with the usual metrics. For each i $\in \Sigma_*$ we write $B(\Phi[\mathbf{i}])$ for the closed convex hull of $\Phi[\mathbf{i}]$. We also assume throughout that there is a number $d_0 > 0$ such that

$$\frac{\text{inradius } B(\Phi[\mathbf{i}])}{\text{diameter } B(\Phi[\mathbf{i}])} \geq d_0 \text{ for all } \mathbf{i} \in \Sigma_*$$

thus the convex hulls cannot get 'too long and thin'. We assume throughout that Φ satisfies the following two conditions:

(a) There exist $0 < c_0, c_1 < \infty$ such that for all $\rho \in (0, c_0)$, the set

$$\Lambda_\rho = \{\mathbf{i} \in \Sigma_*: \rho \leq |\Phi[\mathbf{i}]| < c_1 \rho\} \tag{44}$$

yields a finite covering of Σ, that is $\#\Lambda_\rho < \infty$ and $\Sigma = \cup_{\mathbf{i} \in \Lambda_\rho} [\mathbf{i}]$;

(b) There exists an integer n_0 such that for all $\rho \in (0, c_0)$ and $x \in \mathbb{R}^n$,

$$\#\{\mathbf{i} \in \Lambda_\rho: x \in B(\Phi[\mathbf{i}])\} \leq n_0. \tag{45}$$

These conditions will certainly be satisfied if Φ codes the attractor of an IFS satisfying OSC.

We may define measures of Hausdorff type on subsets of $\Phi(\Sigma)$ by setting, for all $s > 0, F \subset \Phi(\Sigma)$ and $\delta > 0$,

$$\mathcal{M}_\delta^s(F) = \inf\left\{\sum_{j=1}^{\infty} |\Phi[\mathbf{i}_j]|^s : \Phi^{-1}(F) \subset \bigcup_{j=1}^{\infty} [\mathbf{i}_j], |\Phi[\mathbf{i}_j]| \leq \delta\right\} \tag{46}$$

and

$$\mathcal{M}^s(F) = \lim_{\delta \searrow 0} \mathcal{M}_\delta^s(F).$$

Then $\mathcal{M}^s$ is equivalent to the restriction of s-dimensional Hausdorff measure $\mathcal{M}^s$ to $\Phi(\Sigma)$. Clearly $\mathcal{H}^s(F) \leq \mathcal{M}^s(F)$ for $F \subset \Phi(\Sigma)$. For the opposite inequality (to within a constant multiple), note that the number of sets $\Phi[\mathbf{i}]$ with $\mathbf{i} \in \Lambda_\rho$ that overlap $U \cap \Phi(\Sigma)$ is bounded for all $U \subset \mathbb{R}^n$ with $|U| = \rho < c_0$, from comparing the volumes of maximal inscribed balls of $B(\Phi[\mathbf{i}])$ with that of some ball centered in U of radius $|U|$ and using (45). In particular, $\dim_H F = \inf\{s: \mathcal{M}^s(F) = 0\} = \sup\{s: \mathcal{M}^s(F) = \infty\}$ for $F \subset \Phi(\Sigma)$.

In a similar way, (44) and (45) imply that the box-counting dimension of subsets of $\Phi(\Sigma)$ may be found by counting cylinders. In particular, the lower box-counting dimension of $F \subset \Phi(\Sigma)$ is given by

$$\underline{\dim}_B F = \underline{\lim}_{\rho \to 0} \frac{\log\{\#\mathbf{i} \in \Lambda_\rho: F \cap B(\Phi[\mathbf{i}]) \neq 0\}}{-\log \rho}. \tag{47}$$

Let $\mathcal{B}_\Sigma$ be the σ-field generated by the cylinders of Σ. Let $\mathbb{P}$ be a probability measure on $\mathcal{B}_\Sigma$. Let Σ^ω be a random subset of Σ and let

$$\Sigma_*^\omega := \{\mathbf{i} \in \Sigma_*: [\mathbf{i}] \cap \Sigma^\omega \neq \emptyset\}.$$

We adopt the convention that $A^\omega := A \cap \Sigma^\omega$ if A is a subset of Σ and $A^\omega := A \cap \Sigma_*^\omega$ if A is a subset of Σ_*.

For $\alpha \geq 0$ we say that Σ^ω is an α-random subset of Σ if there exists a constant $c_2 < \infty$ such that for all $\rho \in (0, c_0)$ and all $\mathbf{i} \in \Lambda_\rho$,

$$\mathbb{P}(\mathbf{i} \in \Lambda_\rho^\omega) \leq c_2 \rho^\alpha. \tag{48}$$

For our applications, Σ^ω will typically be the symbolic set underlying fractal percolation on K, so that $\Phi(\Sigma^\omega) = K^\omega$.

Let V be a k-dimensional subspace of $\mathbb{R}^d$ and let $\pi_V: \mathbb{R}^d \to V$ denote orthogonal projection onto V. Write $\mathcal{L}^k$ for k-dimensional Lebesgue measure on V identified with $\mathbb{R}^k$ in the obvious way. (If $k = 1$ then V is a line and we write $\mathcal{L}$ for Lebesgue measure on V.)

The following two propositions are our principal tools. The first, which concerns the Hausdorff measure of sections, has stronger hypotheses on the projection of the random subset but a weaker condition on the projection of the original set, than the second which concerns the lower box dimension of sections.

Proposition (1.2.1)[33]: Let $A \in \mathcal{B}_\Sigma$. Let Σ^ω be an α-random subset of Σ for some $\alpha > 0$, let $\Phi: \Sigma \to \mathbb{R}^d$ satisfy (a) and (b) above, and let V be a k-dimensional subspace of $\mathbb{R}^d$. If $\mathbb{P}\left(\mathcal{L}^k\left(\pi_V(\Phi(A^\omega))\right) > 0\right) > 0$, then

$$\mathcal{L}^k\{x \in V: \dim_H(\Phi(A) \cap \pi_V^{-1}(x)) \geq \alpha\} > 0.$$

Proof. Let

$$S = \{x \in V: \dim_H(\Phi(A) \cap \pi_V^{-1}(x)) < \alpha\}.$$

Let $x \in S$. Using (46), for all $\epsilon > 0$ we may find a set of words $\mathcal{J} \subset \Sigma_*$ such that $\Phi^{-1}(\Phi(A) \cap \pi_V^{-1}(x)) \subset U_{\mathbf{i}\in\mathcal{J}}[\mathbf{i}]$ and $\sum_{\mathbf{i}\in\mathcal{J}}|\Phi[\mathbf{i}]|^\alpha < \epsilon$. Then $\Phi(A^\omega) \cap \pi_V^{-1}(x) \subset \bigcup_{\mathbf{i}\in\mathcal{J}\cap\Sigma_*^\omega} \Phi[\mathbf{i}]$ and

$$\mathbb{E}(\#\{\mathbf{i} \in \mathcal{J} \cap \Sigma_*^\omega\}) = \sum_{\mathbf{i}\in\mathcal{J}} \mathbb{P}(\mathbf{i} \in \Sigma_*^\omega) \leq c_2 \sum_{\mathbf{i}\in\mathcal{J}} |\Phi[\mathbf{i}]|^\alpha < c_2\epsilon,$$

using (48), so $\mathbb{P}(\{\mathbf{i} \in \mathcal{J} \cap \Sigma_*^\omega\} \neq \emptyset) < c_2\epsilon$. Since ϵ is arbitrarily small, we conclude that for all $x \in S$, $\Phi(A^\omega) \cap \pi_V^{-1}(x) = \emptyset$ almost surely.

By Fubini's theorem, almost surely

$$\mathcal{L}^k\left(S \cap \pi_V(\Phi(A^\omega))\right) = \mathcal{L}^k(x \in S: \Phi(A^\omega) \cap \pi_V^{-1}(x) \neq 0) = 0.$$

Hence, with positive probability,

$$0 < \mathcal{L}^k\left(\pi_V(\Phi(A^\omega))\right) = \mathcal{L}^k(\pi_V(\Phi(A^\omega)) \setminus S) \leq \mathcal{L}^k(\pi_V(\Phi(A)) \setminus S).$$

The second general proposition concerns the lower box-counting dimension of sections of sets. Here we require a condition that, for all $\mathbf{i} \in \Sigma_*$, the projection of $\Phi[\mathbf{i}]$ onto the subspace V is the same as that of its convex hull; in particular this will be the case if $\Phi[\mathbf{i}]$ is connected.

Proposition (1.2.2)[33]: Let Σ^ω be an α-random subset of Σ for some $\alpha > 0$, let $\Phi: \Sigma \to \mathbb{R}^d$ satisfy (a) and (b) above, and let V be a line, that is a 1 -dimensional subspace of $\mathbb{R}^d$. Suppose that the projection of $\Phi[\mathbf{i}]$ onto V is the same as that of its convex hull $B(\Phi[\mathbf{i}])$ for all $\mathbf{i} \in \Sigma_*$.. If $\mathbb{P}\left(\dim_H \pi_V\left(\Phi(\Sigma^\omega)\right) = 1\right) > 0$, then for every $\epsilon \in (0, \alpha)$,

$$\dim_H\left\{x \in V: \underline{\dim}_B(\Phi(\Sigma) \cap \pi_V^{-1}(x)) > \alpha - \epsilon\right\} = 1.$$

Proof. To keep the notation simple, we give the proof for $\Phi: \Sigma \to \mathbb{R}^2$ where the sections are intersections with lines perpendicular to the line V. The proof is virtually identical for $\Phi: \Sigma \to \mathbb{R}^d$ where $d > 2$. Write $L_x \equiv \pi_V^{-1}(x)$ for the line through $x \in V$ perpendicular to V. For $x \in V$ and $\rho \in (0, c_0)$ write

$$N(x, \rho) := \#\left\{\mathbf{i} \in \Lambda_\rho: B(\Phi[\mathbf{i}]) \cap L_x \neq \emptyset\right\} \equiv \#\left\{\mathbf{i} \in \Lambda_\rho: \Phi[\mathbf{i}] \cap L_x \neq \emptyset\right\} \quad (49)$$

for the 'box counting numbers', where the equivalence follows as every line that intersects the convex hull $B(\Phi[\mathbf{i}])$ also intersects $\Phi[\mathbf{i}]$.

Here is the first of three subsidiary lemmas within this proof. This enables us to reduce consideration of coverings of subsets of L_x when estimating $N(x,\rho)$ to a small set of x. We identify V with $\mathbb{R}\times\{0\}\subset\mathbb{R}^2$ in the obvious way.

Lemma (1.2.3)[33]: Let $\rho\in(0,c_0)$ and $M>0$. Let $I\subset V$ be an interval with $|I|\le\rho$ such that $N(x,\rho)\le M$ for some $x\in I$. Then there exist $x_1,x_2\in I$ with $x_1\le x_2$ such that

$$N(x_1,\rho),N(x_2,\rho)\le M$$

and such that, if $x\in I$ has $N(x,\rho)\le M$, then, for all i $\in\Lambda_\rho$ such that $B(\Phi[\mathrm{i}])\cap L_x\neq\emptyset$, either $B(\Phi[\mathbf{i}])\cap L_{x_1}\neq\emptyset$ or $B(\Phi[\mathbf{i}])\cap L_{x_2}\neq\emptyset$.

Proof. Let $x_1'=\inf\{x\in I\colon N(x,\rho)\le M\}$. If $N(x_1',\rho)\le M$ then take $x_1=x_1'$. Otherwise take $x_1>x_1'$ sufficiently close to x_1' to ensure that both $N(x_1,\rho)\le M$ and

$$\begin{aligned}\{\mathbf{i}\in\Lambda_\rho\quad&:B(\Phi[\mathbf{i}])\cap L_{x_1}\neq\emptyset\}\\ &=\{\mathbf{i}\in\Lambda_\rho\colon B(\Phi[\mathbf{i}])\cap L_{x_1}\neq\emptyset \text{ and } \pi_V(\operatorname{int}B(\Phi[\mathbf{i}]))\cap[x_1,\infty)\neq\emptyset\}.\end{aligned}$$

In the same way, we may take x_2 to be $\sup\{x\in I\colon N(x,\rho)\le M\}$ or a slightly smaller number if necessary. Clearly we may ensure that $x_1\le x_2$. Since the $B(\Phi[\mathbf{i}])$ with i $\in\Lambda_\rho$ have diameter at least ρ and $x_2-x_1\le\rho$, the conclusion of the lemma follows.

We now write

$$N^\omega(x,\rho)=\#\{\mathbf{i}\in\Lambda_\rho^\omega\colon B(\Phi[\mathbf{i}])\cap L_x\neq\emptyset\}$$

for the random analogue of (49). Fix $\epsilon\in(0,\alpha)$ and for $\rho\in(0,c_0)$ let S_ρ be the deterministic subset of V:

$$S_\rho=\{x\in V\colon N(x,\rho)\le\rho^{-\alpha+\epsilon/2}\}. \tag{50}$$

The second subsidiary lemma shows that if $x\in S_\rho$ then the probability that L_x has non-empty intersection with $\Phi(\Sigma^\omega)$ is small.

Lemma (1.2.4)[33]: Let $\rho\in(0,c_0)$ and let $I\subset V$ be an interval with $|I|\le\rho$ such that $I\cap S_\rho\neq\emptyset$. Then

$$\mathbb{P}(N^\omega(x,\rho)>0 \text{ for some } x\in I\cap S_\rho)\le 2c_2\rho^{\frac{\epsilon}{2}}. \tag{51}$$

Proof. If $I\cap S_\rho=\emptyset$ then (51) is trivial. Otherwise, applying Lemma (1.2.3) to the interval I, taking $M=\rho^{-\alpha+\epsilon/2}$ and noting (50), we may find $x_1,x_2\in I\cap S_\rho$ such that, for all $x\in I\cap S_\rho$, all ω, and all $\mathbf{i}\in\Lambda_\rho^\omega\subset\Lambda_\rho$ with $B(\Phi[\mathbf{i}])\cap L_x\neq\emptyset$, either $B(\Phi[\mathbf{i}])\cap L_{x_1}\neq\emptyset$ or $B(\Phi[\mathbf{i}])\cap L_{x_2}\neq\emptyset$. In particular, for all $x\in I\cap S_\rho$

$$N^\omega(x,\rho)\le N^\omega(x_1,\rho)+N^\omega(x_2,\rho). \tag{52}$$

For $j=1,2$, using (48) and (50),

$$\begin{aligned}\mathbb{E}\left(N^\omega(x_j,\rho)\right)\quad&=\sum\left\{\mathbb{P}\left(\mathbf{i}\in\Lambda_\rho^\omega\right)\colon\mathbf{i}\in\Lambda_\rho,B(\Phi[\mathbf{i}])\cap L_{x_j}\neq\emptyset\right\}\\ &\le\sum\left\{c_2\rho^\alpha\colon\mathbf{i}\in\Lambda_\rho,B(\Phi[\mathbf{i}])\cap L_{x_j}\neq\emptyset\right\}\\ &\le c_2\rho^\alpha N(x_j,\rho)\\ &\le c_2\rho^\alpha\rho^{-\alpha+\epsilon/2},\end{aligned}$$

so

$$\mathbb{P}\left(N^\omega(x_j,\rho)>0\right)\le c_2\rho^{\epsilon/2}.$$

The conclusion (51) follows from (52).

Let

$$S=\{x\in V\colon\underline{\dim}_B(\Phi(\Sigma)\cap L_x)\le\alpha-\epsilon\}.$$

Note that, for all $\rho\in(0,c_0)$, we have

$$\Phi(\Sigma) \cap L_x \subset \bigcup_{\mathbf{i}\in\Lambda_\rho} \Phi[\mathbf{i}] \cap L_x.$$

Thus, from (50), (49) and (47),

$$S \subset \bigcap_{N=N_0}^{\infty} \bigcup_{n=N}^{\infty} S_{2^{-n}},$$

where we choose N_0 so that $0 < 2^{-N_0} < c_0$.

The final subsidiary lemma essentially shows that the Hausdorff dimension of S cannot be too big.

Lemma (1.2.5)[33]: With S as above, $\dim_H\big(\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S\big) \leq 1 - \epsilon/4$ almost surely.

Proof. For $\rho \in (0, c_0)$ write

$$K_\rho^\omega := \bigcup \left\{B(\Phi[\mathbf{i}]) : \mathbf{i} \in \Lambda_\rho^\omega\right\} \supset \Phi(\Sigma^\omega).$$

Let $I \subset V$ be an interval with $|I| = \rho \leq c_0$. If $S_\rho \cap I \neq \emptyset$ then by Lemma (1.2.4)

$$\mathbb{P}\big(\pi_V\big(K_\rho^\omega\big) \cap S_\rho \cap I \neq \emptyset\big) \leq 2c_2\rho^{\epsilon/2}.$$

For $n \geq N_0$, let $\mathcal{C}_n$ be the family of closed binary subintervals of V of lengths 2^{-n}. Thus, for $n \geq N_0$,

$$\mathbb{E}\big(\#j : \pi_V(K_{2^{-n}}^\omega) \cap S_{2^{-n}} \cap I_j \neq \emptyset, I_j \in \mathcal{C}_n\big) \leq 2^{n+1}|\Phi(\Sigma)|2c_2 2^{-n\epsilon/2} = c_3 2^{n(1-\epsilon/2)}.$$

In particular,

$$\sum_{n=N_0}^{\infty} 2^{-n(1-\epsilon/4)}\mathbb{E}\big(\#j : \pi_V(K_{2^{-n}}^\omega) \cap S_{2-n} \cap I_j \neq \emptyset, I_j \in \mathcal{C}_n\big) = c_3 \sum_{n=N_0}^{\infty} 2^{-\epsilon/4} < \infty.$$

Then, for all $N \geq N_0$,

$$\begin{aligned}
\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S \quad &\subset \pi_V\big(\Phi(\Sigma^\omega)\big) \cap \bigcup_{n=N}^{\infty} S_{2^{-n}} \\
&= \bigcup_{n=N}^{\infty} \pi_V\big(\Phi(\Sigma^\omega)\big) \cap S_{2-n} \\
&\subset \bigcup_{n=N}^{\infty} \pi_V(K_{2^{-n}}^\omega) \cap S_{2-n} \\
&\subset \bigcup_{n=N}^{\infty} \bigcup_{I_j\in\mathcal{C}_n} \left\{I_j : \pi_V(K_{2^{-n}}^\omega) \cap S_{2-n} \cap I_j \neq \emptyset\right\}.
\end{aligned}$$

Hence, writing $\mathcal{H}_\delta^s$ for the s-dimensional Hausdorff δ-premeasure, and $\mathcal{H}^s$ for s dimensional Hausdorff measure, it follows on taking these covers of $\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S$ for each N that

$$\begin{aligned}
\mathbb{E}\left(\mathcal{H}^{1-\epsilon/4}\big(\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S\big)\right) &= \mathbb{E}\left(\lim_{N\to\infty} \mathcal{H}_{2^{-N}}^{1-\epsilon/4}\big(\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S\big)\right) \\
&\leq \limsup_{N\to\infty} \mathbb{E}\left(\mathcal{H}_{2^{-N}}^{1-\epsilon/4}\big(\pi_V\big(\Phi(\Sigma^\omega)\big) \cap S\big)\right) \\
&\leq \mathbb{E}\left(\sum_{n=N_0}^{\infty} 2^{-n(1-\epsilon/4)}\big(\#j : \pi_V(K_{2^{-n}}^\omega) \cap S_{2^{-n}} \cap I_j \neq \emptyset, I_j \in \mathcal{C}_n\big)\right) < \infty.
\end{aligned}$$

It follows that almost surely $\mathcal{H}^{1-\epsilon/4}\left(\pi_V\left(\Phi(\Sigma^\omega)\right) \cap S\right) < \infty$ and so $\dim_H\left(\pi_V\left(\Phi(\Sigma^\omega)\right) \cap S\right) \leq 1 - \epsilon/4$.

To complete the proof of Proposition (1.2.2), note that

$$\dim_H \pi_V\left(\Phi(\Sigma^\omega)\right) = \mathrm{m}\ \left\{\dim_H\left(\pi_V\left(\Phi(\Sigma^\omega)\right) \cap S\right), \dim_H\left(\pi_V\left(\Phi(\Sigma^\omega)\right) \setminus S\right)\right\}$$

so that, conditional on $\dim_H \pi_V\left(\Phi(\Sigma^\omega)\right) = 1$, an event of positive probability by the hypothesis of the proposition,

$$1 \leq \max\left\{1 - \epsilon/4, \dim_H\left(\pi_V\left(\Phi(\Sigma^\omega)\right) \setminus S\right)\right\} \leq \max\{1 - \epsilon/4, \dim_H(\pi_V(\Phi(\Sigma)) \setminus S)\}.$$

But this is a deterministic statement, so we conclude that $\dim_H(\pi_V(\Phi(\Sigma)) \setminus S) = 1$.

Next we obtain a weak dimension conservation property for the lower boxcounting dimension of sections for self-similar sets with dense rotations. We also do so for the Hausdorff dimension of sections of Mandelbrot percolation sets.

The best known model of fractal percolation is Mandelbrot percolation, based on a decomposition of the d-dimensional cube into M^d equal subcubes of sides M^{-1}; its topological properties have been studied extensively, see [36,39,57]. Statistically self-similar subsets of any self-similar set may be constructed using a similar percolation process which may be set up in terms of the symbolic space formulation.

Let $\mathcal{I} = \{f_1, \dots, f_m\}$ be an IFS of similarities with attractor K. Intuitively, percolation on K is performed by retaining or deleting components of the natural hierarchical construction of K in a self-similar random manner. Starting with some non-empty compact set D such that $f_i(D) \subset D$ for all i, we select a subfamily of the sets $\{f_1(D), \dots, f_m(D)\}$ according to some probability distribution, and write K^1 for the union of the selected sets. Then, for each selected $f_i(D)$, we choose subsets from $\{f_i f_1(D), \dots, f_i f_m(D)\}$ according to the same probability distribution, independently for each i, with the union of these sets comprising K^2. Continuing in this way, we get a nested hierarchy $K \supset K^1 \supset K^2 \supset \cdots$ of random compact sets, where K^k denotes the union of the components remaining at the k th stage. The random percolation set $K^\omega \subset K$ is then given by $K^\omega = \cap_{k=0}^{\infty} K^k$, see Figure 1.

More formally, percolation on a self-similar set K is defined using the natural representation of K by symbolic space. We take $\Lambda = \{1, \dots, m\}$ with $\Sigma_* = \cup_{n\geq 0} \Lambda^n$ the set of finite words and $\Sigma = \Lambda^N$ the infinite words. The canonical

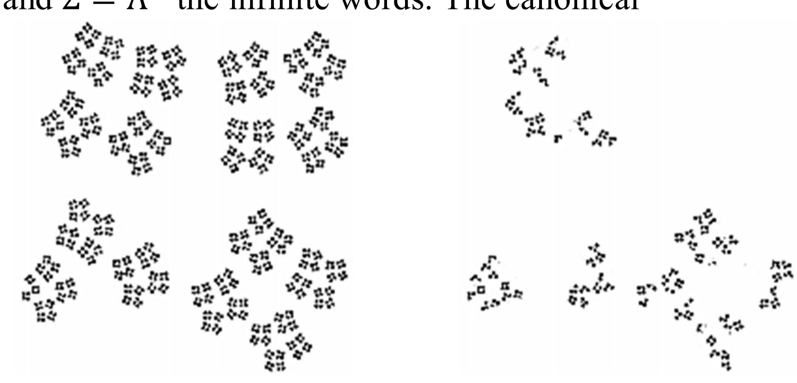

Figure 1[33]: A self-similar attractor of an IFS with rotations and a subset obtained by the percolation process

map $\Phi: \Sigma \to K \subset \mathbb{R}^d$ is given by $\Phi(i_1 i_2 \dots) = \cap_{n=0}^{\infty} f_{i_1} \cdots f_{i_n}(D)$ for any nonempty compact set D such that $f_i(D) \subset D$ for $i = 1, \dots, m$. Then $K = \cup_{\mathbf{i}\in\Sigma} \Phi(\mathbf{i})$, with Φ providing a (not necessarily injective) index to the points of K.

To define percolation on K, let $(\Omega, \mathcal{A}, \mathbb{P})$ bè a probability space. Lĕt $X \equiv (X_1, \dots, X_m)$ be a random vector taking values in $\{0,1\}^m$. Let $\mathcal{X} = \{X^{\mathbf{i}} \equiv (X_1^{\mathbf{i}}, \dots, X_m^{\mathbf{i}})\}_{\mathbf{i} \in \Sigma_*}$ be a family of independent random vectors with values in $\{0,1\}^m$, each having the distribution of X, on the probability space $(\Omega^{\Sigma_*}, \mathcal{A}_{\mathcal{X}}, \mathbb{P} \otimes \Sigma_*)$, where $\mathcal{A}_{\mathcal{X}} \subset \mathcal{A}^{\Sigma_*}$ is the σ-algebra generated by $\mathcal{X}$. This defines a random set $\Sigma^\omega = \left\{ i_1 i_2 \dots \in \Sigma : X_{i_1}^{\emptyset} X_{i_2}^{i_1} X_{i_3}^{i_1 i_2} \cdots = 1 \right\} \subset \Sigma$. The percolation set $K^\omega \subset K$ is the image of Σ^ω under the canonical map, that is the random set $K^\omega = \Phi(\Sigma^\omega)$.

By standard branching process theory [34], if $\mathbb{E}(\#i : X_i = 1) > 1$ there is a positive probability that Σ^ω, and thus K^ω, is non-empty. Provided the IFS defining K satisfies OSC then, conditional on $K^\omega \neq \emptyset$,

$$\dim_B K^\omega = \dim_H K^\omega = s \text{ a.s where s satisfies } \mathbb{E}\left(\sum_{i=1}^{m} X_i r_i^s \right) = 1. \qquad (53)$$

where r_i is the scaling ratio of f_i, see [38,52].

We say that the percolation process is standard with exponent α if the distribution of $X = (X_1, \dots, X_m)$ is defined by $\mathbb{P}(X_i = 1) = r_i^\alpha, \mathbb{P}(X_i = 0) = 1 - r_i^\alpha$ independently for $i = 1, \dots, m$. Then by (53), provided that $\alpha < \dim_H K$, there is a positive probability that $K^\omega \neq \emptyset$, in which case $\dim_H K^\omega = \dim_H K - \alpha$ a.s..

The following theorem on the dimension of projections of percolation subsets of self-similar sets was obtained as a corollary of a more general theorem on projections of random cascade measures on self-similar sets [40].

Theorem (1.2.6)[33]: [40] Let K be the attractor of an IFS of contracting similarities on $\mathbb{R}^d$ with dense rotations and satisfying OSC. Let $\mathbb{P}$ be a probability distribution of a standard percolation process on K with $\mathbb{E}(\#i : X_i = 1) > 1$, so that the percolation set $K^\omega \neq \emptyset$ with positive probability. Then, conditional on $K^\omega \neq \emptyset$, almost surely

$$\dim_H \pi_V(K^\omega) = \min \ (k, \dim_H K^\omega),$$

for every k-dimensional subspace V.

Thus, conditional on non-extinction, the projections of K^ω onto all subspaces have the 'generic' dimension. We now apply Proposition (1.2.2) to sections of selfsimilar sets. The conclusion applies to self-similar sets K such that their projection onto each line is the same as that of the convex hull of K. This includes the case where K is connected as well as many other self-similar sets, see Figure 2.

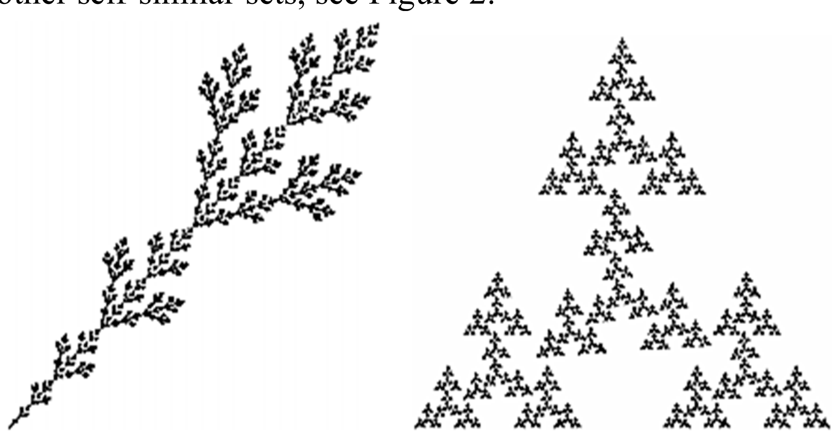

Figure 2[33]: A connected and a totally disconnected self-similar set with dense rotations satisfying the conditions of Theorem (1.2.7)

Theorem (1.2.7)[33]: Let $\mathcal{J}$ be an IFS of contracting similarities on $\mathbb{R}^d$ with dense rotations and satisfying OSC. Let K be the attractor of $\mathcal{J}$ and suppose $s = \dim_H K > 1$ and that the projection of K onto every 1-dimensional subspace equals that of its convex hull. Then for every 1-dimensional subspace V of $\mathbb{R}^d$ and all $\epsilon \in (0, s-1)$,

$$\dim_H\{x \in V : \underline{\dim}_B(K \cap \pi_V^{-1}(x)) > \dim_H K - 1 - \epsilon\} = 1. \qquad (54)$$

Proof. Let K have its symbolic representation $\Phi: \Sigma \to \mathbb{R}^d$. As $\Phi[\mathbf{i}]$ is similar to K for all $\mathbf{i} \in \Sigma_*$, the projection of each $\Phi[\mathbf{i}]$ onto every 1-dimensional subspace is the same as that of its convex hull. We set up standard percolation with exponent $s-1$ on K via its symbolic representation, as above. Then there is a positive probability of non-extinction, conditional on which almost surely, $\dim_H \pi_V(K^\omega) = \min\{1, \dim_H K - (s-1)\} = 1$ for every line V, using Theorem (1.2.6).

A consequence of OSC is that Φ satisfies conditions (a) and (b) (at (44) and (45)) with $c_0 = |K|$ and $c_1 = \max_{1 \le i \le m} r_i^{-1}$. Moreover, if $i_1 \dots i_k \in \Lambda_\rho$ then $\mathbb{P}(i_1 \dots i_k \in \Lambda_\rho^\omega) = r_{i_1}^\alpha \cdots r_{i_k}^\alpha \le |K|^{-\alpha}\rho^\alpha$, so that (48) is satisfied. The conclusion follows by Proposition (1.2.2) since $\Phi(\Sigma) = K$.

It would be desirable to dispense with the requirement in Theorem (1.2.7) that the projections of K are the same as those of its convex hull. Without such a condition it is not hard to show that (54) can be replaced by the conclusion that

$$\dim_H\{x \in V : d(x) > \dim_H K - 1 - \epsilon\} = 1$$

where $d(x) := \underline{\lim}_{\rho \to 0} \log \# N_\rho(L_x^\rho) / -\log \rho$ and where $N_\rho(L_x^\rho)$ denotes the number of $\mathbf{i} \in \Lambda_\rho$ such that $B(\Phi[\mathbf{i}]) \cap L_y \ne \emptyset$ for some $y \in [x-\rho, x+\rho]$. (Here $d(x)$ is a kind of lower box-counting dimension conditioning on fibres that is always no less than the actual lower box-counting dimension of the fibre, with possibility of being strictly larger.)

Next we apply Proposition (1.2.1) to Mandelbrot percolation. Let K be the unit cube in $\mathbb{R}^d$. Fix an integer $M \ge 2$ and a probability $0 < p < 1$. We divide K into M^d subcubes of side $1/M$ in the natural way, and retain each subcube independently with probability p to get a set K^1 formed as a union of the retained subcubes. We repeat this process with the cubes in K^1, dividing each into M^d subcubes of side $1/M^2$ and choosing each with probability p to get a set K^2, and so on. This process, termed Mandelbrot percolation, leads to a percolation set, which we write here as $K_p^\omega = \cap_{k=0}^\infty K^k$ to emphasize the dependence on p.

This may be regarded as percolation on the self-similar set defined by the IFS $\mathcal{J} = \{f^{j_1,\dots,j_d} : 1 \le j_1, \dots, j_d \le M\}$ on $\mathbb{R}^d$ where

$$f^{j_1,\dots,j_d}(x_1, \dots, x_d) = \left(\frac{x_1 + j_1 - 1}{M}, \dots, \frac{x_d + j_d - 1}{M}\right);$$

as before the random construction may be represented in symbolic space, using an alphabet of M^d letters.

If $p > M^{-d}$ then, as above, that there is a positive probability that $K_p^\omega \ne \emptyset$, conditional on which $\dim_H K_p^\omega = d + \log p / \log M$. A useful observation is that for $0 < p, p' < 1$ the intersection of independent realizations of the two random sets K_p^ω and $K_{p'}^\omega$ has the same distribution as that of $K_{pp'}^\omega$.

Rams and Simon [55,56,57] and Simon and Vágó [59] have recently obtained results on the dimensions and Lebesgue measure of projections of Mandelbrot percolation that are almost surely valid for projections onto all subspaces.

Theorem (1.2.8)[33]: [55,59] Let $1 \le k \le d-1$ and let $K_p^\omega \subset \mathbb{R}^d$ be the random set obtained by Mandelbrot percolation on the d-dimensional unit cube, using repeated subdivision into M^d subcubes, and selecting cubes independently with probability $p > 1/M^{d-k}$. Then, conditional on $K_p^\omega \neq \emptyset, \dim_H K_p^\omega = d + \log p/\log M > k$, and for every k-dimensional subspace V we have $\mathcal{L}^k(\pi_V K_p^\omega) > 0$, indeed, $\pi_V K_p^\omega$ contains an open subset of V.

Applying Proposition (1.2.1) to Theorem (1.2.8) we obtain dimension conservation properties for Mandelbrot percolation.

Theorem (1.2.9)[33]: Let $1 \le k \le d-1$. Let $K_p^\omega \subset \mathbb{R}^d$ be the random set obtained by Mandelbrot percolation on the d-dimensional unit cube, using repeated subdivision into M^d subcubes and selecting cubes independently with probability $p > 1/M^{d-k}$. For all $\epsilon > 0$, almost surely conditional on $K_p^\omega \neq \emptyset$, for all k-dimensional subspaces V.

$$\mathcal{L}^k\{x \in V: \dim_H(K_p^\omega \cap \pi_V^{-1}(x)) \ge \dim_H K_p^\omega - k - \epsilon\} > 0.$$

Proof. We may represent the heierarchy of M-ary subcubes of the unit cube in symbolic space Σ with an alphabet Λ of $m = M^d$ letters with $\Phi: \Lambda \to K = [0,1]^d$ the natural cannonical mapping. With notation for percolation as above, let the probability distribution $(X_1, \dots, X_m)$ on Λ be given by $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = 0) = 1 - p$, independently for $i = 1, \dots, m$. This defines a random set $\Sigma_p^\omega \subset \Sigma$ such that $K_p^\omega = \Phi(\Sigma_p^\omega)$ is the Mandelbrot percolation set, with $\dim_H K_p^\omega = d + \log p/\log M$ conditional on non-extinction. Now let $p' = p^{-1} M^{-(d-k-e)}$ and let $\Sigma_{p'}^{\omega'} \subset \Sigma$ be an independent random set defined in the same way but using probability p'; we use $\Sigma_{p'}^{\omega'}$ to 'probe' the dimensions of K_p^ω.

The random set $\sum_p^\omega \cap \Sigma_{p'}^{\omega'}$ has the same distribution as a random set $\Sigma_{pp'}^{\omega''}$, constructed in the same way with probability pp'. Thus, conditional on $\Sigma_p^\omega \cap \Sigma_{p'}^{\omega'} \neq \emptyset$, $\dim_H \Phi\left(\Sigma_p^\omega \cap \Sigma_{p'}^{\omega'}\right) = d + \log pp'/\log M = k + \epsilon$ almost surely, so by Theorem (1.2.8), almost surely,

$$\mathcal{L}^k\left(\pi_V\left(\Phi\left(\Sigma_p^\omega \cap \Sigma_{p'}^{\omega'}\right)\right)\right) > 0 \tag{55}$$

for all k-dimensional subspaces V. Using independence and Fubini's theorem, conditional on $\Sigma_p^\omega \neq \emptyset$, almost surely conditional on $\Sigma_p^\omega \cap \Sigma_{p'}^{\omega'} \neq \emptyset$, inequality (55) holds for all V (Note that, conditional on $\Sigma_p^\omega \neq \emptyset, \mathbb{P}\left(\Sigma_p^\omega \cap \Sigma_{p'}^{\omega'} \neq \emptyset\right) > 0.$)

We may regard $\Sigma_{p'}^{\omega'}$ as an α-random subset of Σ where $= -\log p'/\log M = \log p/\log M + d - k - \epsilon$. Taking $A = \Sigma_p^\omega$ in Proposition (1.2.1) (so in the notation there $A^\omega = \Sigma_p^\omega \cap \Sigma_{p'}^{\omega'}$) we conclude that, conditional on $\Sigma_p^\omega \neq \emptyset$,

$$\mathcal{L}^k\{x \in V: \dim_H(\Phi(\Sigma_p^\omega) \cap \pi_V^{-1}(x)) \ge \alpha\} > 0,$$

and the conclusion follows, noting that $\Phi(\Sigma_p^\omega) = K_p^\omega$.

We now show that we have weak dimension conservation for the Hausdorff dimension of sections of plane self-similar sets in all directions apart from a set of directions of Hausdorff dimension 0. To achieve this we use Proposition (1.2.1) together with a result on the absolute continuity of projections of a class of random measures supported by random subsets of self-similar sets, which is a extension of a result of

Shmerkin and Solomyak [58] for deterministic measures. We do this first for self-similar sets where the defining similarities are translates of each other. Then a device of Peres and Shmerkin [54] enables us to extend the conclusion to general similarities.
Let
$$\mathcal{J} = \{f_i = rR_\theta \cdot +a_i\}_{i=1}^m \tag{56}$$
be an IFS in the plane, where $r \in (0,1)$ and R_θ is the orthogonal rotation with an angle $\theta \in [0,2\pi)$. As before $\Phi: \Sigma \mapsto \mathbb{R}^2$ is the canonical mapping from the symbolic space to the plane.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let
$$X: \Omega \mapsto \left\{(p_1, \dots, p_m) \in [0,1]^m: \sum_{i=1}^{m} p_i = 1\right\} \tag{57}$$
be a random probability vector allowing zero entries. For $n \in \mathbb{N}$ denote by
$$\chi_n: \Omega^{\mathbb{N}} \to \Omega$$
the projection from $\Omega^{\mathbb{N}}$ onto its nth coordinate. Then $\mathcal{X} = \{X^{(n)} = X \circ \chi_n\}_{n\in\mathbb{N}}$ forms a i.i.d. sequence on the probability space $(\Omega^{\mathbb{N}}, \mathcal{A}_{\mathcal{X}}, \mathbb{P} \otimes \mathbb{N})$, where $\mathcal{A}_{\mathcal{X}} \subset \mathcal{A}^{\otimes\mathbb{N}}$ is the σ-algebra generated by $\mathcal{X}$. Let ν be the random probability measure on Σ defined by
$$\nu([i_1 \dots i_k]) = X_{i_1}^{(1)} \cdots X_{i_k}^{(k)} \text{ for all } i_1 \dots i_k \in \Sigma_*. \tag{58}$$

Note that the measure ν is not the same as the random cascade measures studied, for example, in [40]. Here for $k \geq 1$ the ratio $([i_1 \dots i_k i_{k+1}]) : \nu([i_1 \dots i_k])$ is the same for all $i_1 \dots i_k \in \Lambda^k$. The reason why we consider this particular random measure is that its Fourier transform has a convolution structure, which is essential for the proof of absolute continuity in Theorem (1.2.13).

Let $\mathbb{Q}$ be the probability measure on the product space $\Sigma \times \Omega^{\mathbb{N}}$ given by
$$\mathbb{Q}(A) = \int_\Omega \int_\Sigma \mathbf{1}_A(\mathbf{i}, \boldsymbol{\omega}) \nu(\mathrm{di}) \mathbb{P}^{\otimes\mathbb{N}}(d\boldsymbol{\omega}) \text{ for all } A \in \mathcal{B}_\Sigma \otimes \mathcal{A}_{\mathcal{X}}.$$
Denote by $\sigma: \Sigma \times \Omega^{\mathbb{N}} \mapsto \Sigma \times \Omega^{\mathbb{N}}$ the left shift
$$\sigma(i_1 i_2 \dots, \omega_1 \omega_2 \dots) = (i_2 i_3 \dots, \omega_2 \omega_3 \dots).$$
The next proposition and theorem are direct analogues of those obtained in [40] for random cascade measures.

Proposition (1.2.10)[33]: The dynamical system $(\Sigma \times \Omega^{\mathbb{N}}, \mathcal{B}_\Sigma \otimes \mathcal{A}_{\mathcal{X}}, \sigma, \mathbb{Q})$ is mixing.

Proof. The proof is similar to that of [40, Proposition 2.2]. Let $\mathcal{B}$ be the semialgebra of $\mathcal{B}_\Sigma \otimes \mathcal{A}_{\mathcal{X}}$ consisting of sets of the form
$$\left\{(\mathbf{i}, \boldsymbol{\omega}): \mathbf{i}|_k = \mathbf{j}, X_a^{(b)} \in B_a^b\right\}$$
for $k \in \mathbb{N}, \mathbf{j} \in \Lambda^k, b \in \{1, \dots, k\}, a \in \Lambda$ and B_a^b Borel subsets of $[0,1]$. It is clear that $\mathcal{B}$ generates $\mathcal{B}_\Sigma \otimes \mathcal{A}_{\mathcal{X}}$, so we only need to verify that $\lim_{n\to\infty} \mathbb{Q}(\sigma^{-n}(A) \cap B) = \mathbb{Q}(A)\mathbb{Q}(B)$ for $A, B \in \mathcal{B}$. This follows since by the construction of $\mathcal{B}$, given $A, B \in \mathcal{B}$, there exists a positive integer n_0 such that $\sigma^{-n}(A)$ and B are independent for all $n \geq n_0$.

Let $\pi_\beta: \mathbb{R}^2 \mapsto \mathbb{R}^2$ be orthogonal projection onto the line making an angle β with the x-axis. Write $\mu = \Phi\nu$ for the measure defined by $\mu(A) = \nu(\Phi^{-1}A)$. Starting from Proposition (1.2.10) and proceeding just as in [40], we obtain the following projection property.

Theorem (1.2.11)[33]: Suppose that θ/π is irrational. Then almost surely, for all $\beta \in [0, \pi)$
$$\dim_H \pi_\beta \mu = \min\ (1, \dim_H \mu).$$

Proof. When θ/π is irrational, the closed rotation group G generated by R_θ is the whole group $SO(2,\mathbb{R})$. Given this, the proof follows exactly the same lines as in of [40, Sections 2.7&4]. In particular, since $G = SO(2,\mathbb{R})$, the dimension of the projections equals the maximal possible value, just as in [40, Corollary 4.6],

Theorem (1.2.13) below, a random analogue of [58, Theorem B], gives conditions for the projections of the random measure μ to be almost surely absolutely continuous in all directions except for a set E of Hausdorff dimension 0 . First, in the following lemma, we specify the set E and verify that its dimension is 0. We adapt the delicate estimates of [58, Lemmas 3.2&3.4] to our requirements, in particular obtaining estimates for the dimensions of $E_{q,k}(\delta, N)$ that do not depend on q or k.

For $x \in \mathbb{R}$ let $\| x \| = \min\{|x-j|: j \in \mathbb{Z}\}$ and we write $[N] = \{1, \dots, N\}$ for each positive integer N.

Lemma (1.2.12)[33]: Fix $r \in (0,1), \gamma \in \mathbb{R}, b \in (0,\infty)$ and $\theta \in \mathbb{R}$ with θ/π irrational. For $\delta \in \left(0, \frac{1}{2}\right)$ and integers $q, k \geq 1, N \geq 2$, let $E_{q,k}(\delta, N)$ be the set of all $\beta \in [0,\pi)$ such that

$$\max_{\tau\in[1,r^{-qk}]} \frac{1}{N} \# \left\{ n \in [N]: \|b\tau r^{q-qk(N-n)} \cos(\beta+\gamma-nqk\theta)\| \leq \frac{r^{2qk}}{15} \right\} > 1-\delta,$$

$$\text{and let } E = \bigcap_{i\geq 3} \bigcup_{q,k\geq 1} \limsup_{N\to\infty} E_{q,k}(1/i, N).$$

Then $\dim_H E = 0$.

Proof. For the time being we fix the integers $q, k, N \geq 1$ and abbreviate $c := br^q$, $\ell := r^{-qk}$ and $\alpha := qk\theta$. Note that $r^{2qk}/15 = 1/(15\ell^2)$. Let $\tau \in [1,\ell]$. Given $\beta \in [0,\pi)$, for each $n = 1, \dots, N$ write

$$c\tau\ell^{N-n}\cos(\beta+\gamma-n\alpha) = k_n + \epsilon_n, \text{ where } k_n \in \mathbb{Z} \text{ and } \epsilon_n \in \left[-\frac{\frac{1}{2,1}}{2}\right). \quad (59)$$

For $x \in \mathbb{R}$ let $w_x = (\cos x, \sin x)$. Since α/π is irrational, the unique solution of the equation

$$c_1 w_{2\alpha} + c_2 w_\alpha = w_0,$$

is $c_1 = -1$ and $c_2 = 2\cos\alpha$. Clearly $|c_1|, |c_2| \leq 2$.

Applying the formula $\langle w_x, w_{\beta+\gamma-n\alpha}\rangle = \cos(\beta+\gamma-n\alpha-x)$ for $x = 2\alpha, \alpha, 0$ and using (59) we get that

$$c_1\ell^2(k_{n+2}+\epsilon_{n+2}) + c_2\ell(k_{n+1}+\epsilon_{n+1}) = k_n + \epsilon_n. \quad (60)$$

This implies that if

$$\max\{|\epsilon_n|, |\epsilon_{n+1}|, |\epsilon_{n+2}|\} \leq 1/(15\ell^2) \leq 1/\big(3(2\ell^2+2\ell+1)\big),$$

then

$$|c_1\ell^2 k_{n+2} + c_2\ell k_{n+1} - k_n| < \frac{1}{2},$$

which means that k_{n+2} and k_{n+1} uniquely determine k_n. On the other hand,

$$|c_1\ell^2\epsilon_{n+2} + c_2\ell\epsilon_{n+1} - \epsilon_n| \leq \ell^2 + \ell + 1.$$

Hence for fixed k_{n+2} and k_{n+1}, there are at most $\lfloor 2(\ell^2+\ell+1)+1 \rfloor \leq 7\ell^2$ possible values of k_n. Also, from (59), there are at most $(2c\ell+1)(2c\ell^2+1) \leq (2b+1)^2\ell^3$ possible pairs of (k_N, k_{N-1}).

For $\delta \in \left(0, \frac{1}{2}\right)$ denote by $[N]\delta$ the set of all subsets of $[N]$ with cardinality no less than $(1-\delta)N$. For $A \in [N]_\delta$ let $\tilde{A} := \{0 \leq n \leq N-2 : n+2, n+1, n \in A\}$. Then $\#\tilde{A} \geq$

$(1-3\delta)N-3$. This implies that the number of possible sequences $(k_n)_{n=0}^N$ corresponding to $\beta\in[0,\pi)$ for which $|\epsilon_n|\le 1/(15\ell^2)$ in (59) for all $n\in A$, is bounded above by

$$(2b+1)^2\ell^3(7\ell^2)^{3\delta N+3}.$$

Note that once (k_N,k_{N-1}) is given, the possible values of the remaining k_n are determined by (60), hence the value of $\tau\in[1,\ell]$ is irrelevant. Then, by Chernoff's entropy inequality for binomial sums, see [37], or alternatively using Stirling's approximation,

$$\#[N]_\delta\le\sum_{p=0}^{\lfloor\delta N\rfloor}\binom{N}{p}\le 2^{N[-\delta\log\delta-(1-\delta)\log(1-\delta)]}\le \mathrm{e}^{C\sqrt{\delta}N},$$

for all N and $\delta\in\left(0,\frac{1}{2}\right)$, where C is a universal constant.

Combining these estimates, the number of possible sequences $(k_n)_{n=1}^N$ corresponding to $\beta\in[0,\pi)$ satisfying

$$\max_{\tau\in[1,\ell]}\frac{1}{N}\#\{n\in[N]:\|c\tau\ell^{N-n}\cos(\beta+\gamma-n\alpha)\|\le 1/(15\ell^2)\}>1-\delta,$$

is bounded above by

$$\mathrm{e}^{C\sqrt{\delta}N}(2b+1)^2\ell^3(7\ell^2)^{3\delta N+3}.$$

From (59), identically

$$\beta+\gamma-n\alpha=\tan^{-1}\left(\frac{\ell(k_{n+1}+\epsilon_{n+1})}{(k_n+\epsilon_n)\sin\alpha}-\cot\alpha\right).$$

Since α/π is irrational, by estimating the derivatives of the function

$$f(x)=\tan^{-1}((\ell/\sin\alpha)x-\cot\alpha),$$

there is a constant C' depending only on ℓ and α such that

$$\beta\in B\big(j\alpha-\gamma+f(k_{j+1}/k_j),C'\ell^{-N}\big)$$

where j may be 1 or 2 (to ensure that k_1 and k_2 are not both 0 when N is sufficiently large). Hence the set $E_{q,k}(\delta,N)$ can be covered by

$$2\mathrm{e}^{C\sqrt{\delta}N}(2b+1)^2\ell^3(7\ell^2)^{3\delta N+3}=2\mathrm{e}^{C\sqrt{\delta}N}(2b+1)^2r^{-3qk}(7r^{-2qk})^{3\delta N+3}$$

balls of radius $C'\ell^{-N}=C'r^{qkN}$.

Using these coverings, it follows that

$$\begin{aligned}\dim_H\left(\limsup_{N\to\infty}E_{q,k}(\delta,N)\right)&\le\frac{C\sqrt{\delta}+3\delta(\log 7-2qk\log r)}{-qk\log r}\\&\le(6+(C+3\log 7)/-\log r)\sqrt{\delta}.\end{aligned}$$

By countable stability of Hausdorff dimension, for $i\ge 3$,

$$\dim_H\bigcup_{q,k\ge 1}\limsup_{N\to\infty}E_{q,k}(1/i,N)\le(6+(C+3\log 7)/-\log r)/\sqrt{i},$$

giving the conclusion.

Here is the theorem on the absolute continuity of projections of random measures in all but a small set of exceptional directions when the underlying similarities are translates of each other. The proof uses Fourier transforms along the lines of [58, Theorem B].

Theorem (1.2.13)[33]: Suppose that θ/π is irrational and let $\mathcal{I}$ be an IFS of the form (56) satisfying OSC. Then there exists a set $E\subset[0,\pi)$ with $\dim_H E=0$ such that, for every random self-similar measure $\mu = \Phi\nu$ with respect to $\mathcal{J}$ of the form defined by (57)-(58) and satisfying

$$\mathbb{P}\left(\text{there exist } i, j \in \Lambda \text{ such that } X_i, X_j \geq p_*\right) = 1 \tag{61}$$

for some $p_* > 0$ and

$$\mathbb{P}(\dim_H \mu = s) = 1 \tag{62}$$

for some $s > 1$, almost surely for all $\beta \in [0, \pi) \setminus E$, the projected measure $\pi_\beta \mu$ is absolutely continuous with respect to Lebesgue measure.

Proof. We write ang (z) for the angle between the line containing $\{0, z\}$ and the x-axis. For $i, j \in \Lambda$ let $E_{i,j}$ be the set given by Lemma (1.2.12) for the ratio r and angle θ in the IFS (56) with $\gamma = \text{ang}(a_i - a_j) + \theta$ and $b = |a_i - a_j|$. Let $E = \cup_{i,j\in\Lambda} E_{i,j}$. Then $\dim_H E = 0$; we will show that the projected measures $\pi_\beta \mu$ are absolutely continous when $\beta \in [0, \pi) \setminus E$.

With $\mu = \Phi\nu$ as stated, we may, by (61), choose $i, j \in \Lambda$ with $|a_i - a_j| > 0$ such that

$$\mathbb{P}\left(X_i, X_j \geq p_*\right) := p > 0; \tag{63}$$

these i and j will remain fixed throughout the proof.

For each $q \geq 1$, we may regard the attractor K of the IFS (56) as the attractor of the iterated IFS

$$\mathcal{J}_q := \left\{f_{\mathrm{i}} := f_{i_1} \cdots f_{i_q} \equiv r^q R_{q\theta} \cdot + a_{\mathrm{i}} : \mathbf{i} = i_1 \dots i_q \in \Lambda^q\right\},$$

so that $K = \Phi_q(\Sigma_q)$ where $\Sigma_q := \{\mathbf{i}_1 \mathbf{i}_2 \dots : \mathbf{i}_j \in \Lambda^q\}$ and Φ_q is the cannonical map. Let ν_q be the random self-similar measure of the form (57)-(58) with respect to

$$\mathcal{X}_q = \left\{ X^{q,(n)} := \left(X_{\mathrm{i}}^{q,(n)} \equiv \prod_{l=1}^{q} X_{i_l}^{(nq-q+l)} \right)_{\mathrm{i}=i_1\dots i_q \in \Lambda^q} \right\}_{n\geq 1}.$$

Then $\mu = \Phi_q \nu_q$ for all $q \geq 1$. Note that μ satisfies

$$\mu = \sum_{\mathbf{i}\in\Lambda^q} X_{\mathbf{i}}^{q,(1)} f_{\mathbf{i}} \mu^{q,(1)}, \tag{64}$$

where $\mu^{q,(1)}$ is the copy of μ generated by $\{X^{q,(n+1)}\}_{n\in\mathbb{N}}$. In terms of Fourier transforms, writing $T_q = r^q R_{q\theta}$, equation (64) yields that for $\xi \in \mathbb{R}^2$,

$$\hat{\mu}(\xi) = \sum_{\mathbf{i}\in\Lambda^q} X_{\mathbf{i}}^{q,(1)} \mathrm{e}^{\mathrm{i}\pi\langle a_{\mathrm{i}}, \xi\rangle} \widehat{T_q \mu^{q,(1)}}(\xi). \tag{65}$$

Iterating (65) and taking the limit,

$$\hat{\mu}(\xi) = \prod_{n=0}^{\infty} \Psi_n^q(\xi), \tag{66}$$

where, for $n \geq 0$,

$$\Psi_n^q(\xi) = \sum_{\mathbf{i}\in\Lambda^q} X_{\mathbf{i}}^{q,(n+1)} \mathrm{e}^{\mathrm{i}\pi\langle T_q^n a_{\mathrm{i}}, \xi\rangle}.$$

From (66), for $q \geq 1$ and $k \geq 2$, we can write μ as a convolution of two measures $\mu_{q,k} * \eta_{q,k}$, where

$$\widehat{\mu_{q,k}}(\xi) = \prod_{k \setminus n+1} \Psi_n^q(\xi) \text{ and } \widehat{\eta_{q,k}}(\xi) = \prod_{k | n+1} \Psi_n^q(\xi).$$

Notice that $\mu_{q,k}$ is within the class of random self-similar measures of the form (57)-(58); indeed it has the same law as the random self-similar measure with respect to the IFS

$$\left\{T_q^k \cdot + f_{\mathbf{i}_1} \cdots f_{\mathbf{i}_{k-1}}((0,0))\right\}_{\mathbf{i}_1 \ldots \mathbf{i}_{k-1} \in (\Lambda^q)^{k-1}}$$

and the random vector

$$\left\{X_{\mathbf{i}_1}^{q,(1)} \cdots X_{\mathbf{i}_{k-1}}^{q,(k-1)}\right\}_{\mathbf{i}_1 \ldots \mathbf{i}_{k-1} \in (\Lambda^q)^{k-1}}.$$

Thus, almost surely, $\dim_H \mu_{q,k} = \frac{k-1}{k} \dim_H \mu = \frac{k-1}{k} s > 1$ by (62) and for some sufficiently large k which we fix for the remainder of the proof. Applying Theorem (1.2.11) we can find a set Ω_1 with $\mathbb{P}(\Omega_1) = 1$ such that, for all $\omega \in \Omega_1$, for all $\beta \in [0, \pi)$, $q \geq 1$,

$$\dim_H \pi_\beta \mu_{q,k} = 1. \tag{67}$$

The rest of the proof estimates the Fourier transform of $\pi_\beta \eta_{q,k}$ using Lemma (1.2.12). From (63), for $q \geq 1$ and $n \geq 0$ the event

$$A_{q,n} = \left\{X_i^{(qn+h)}, X_j^{(qn+h)} \geq p_* \text{ for some } h = 0, \ldots, q-1\right\}$$

has probability $\mathbb{P}(A_{q,n}) = 1 - (1-p)^q$. Since $\left\{\chi_{A_{q,nk}}\right\}_{n \geq 0}$ are i.i.d. random variables for all $q \geq 1$, by the strong law of large numbers we can find a set Ω_2 with $\mathbb{P}(\Omega_2) = 1$ such that for all $\omega \in \Omega_2$, for all $q \geq 1$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \chi_{A_{q,nk}}(\omega) = 1 - (1-p)^q. \tag{68}$$

By (61) we may also find a set Ω_3 with $\mathbb{P}(\Omega_3) = 1$ such that for all $n \geq 1$,

$$\text{there exists } \ell \in \Lambda \text{ such that } X_\ell^{(n)} \geq p_*. \tag{69}$$

Take $\omega \in \Omega_1 \cap \Omega_2 \cap \Omega_3$. The rest of the proof will be deterministic.

Let $\beta \in [0, \pi) \setminus E$. By Lemma (1.2.12) there exists $i_0 = i_0(\beta)$ such that for all $q \geq 1$ there exists $N_0 = N_0(\beta, q)$ such that $\beta \notin E_{q,k}(1/i_0, N)$ for all $N \geq N_0$. In other words, for all $N \geq N_0$,

$$\max_{\tau \in [1, r^{-qk}]} \frac{1}{N} \# \left\{n \in [N] : \left\| b\tau r^{q-qk(N-n)} \cos(\beta + \gamma - nqk\theta) \right\| > \frac{r^{2qk}}{15}\right\} \geq \frac{1}{i_0}, \tag{70}$$

where $\gamma = \operatorname{ang}(a_i - a_j) + \theta$ and $b = |a_i - a_j|$. Take q large enough so that $(1-p)^q < 1/4i_0$. We show, in a similar manner to [58, Proposition 3.3], that the projected measure $\pi_\beta \eta_{q,k}$ has positive Fourier dimension. (Recall that the Fourier dimension of a measure λ is the supremum of σ such that $\hat{\lambda}(\xi) = O(|\xi|^{-\sigma/2})$.)

Writing $w_\beta = (\cos\beta, \sin\beta)$ as before and applying the formula

$$\widehat{\pi_\beta \lambda}(t) = \hat{\lambda}(t w_\beta) \ (t \in \mathbb{R})$$

for the Fourier transform of the projection of a measure λ on $\mathbb{R}^2$, we obtain

$$\widehat{\pi_\beta \eta_{q,k}}(t) = \prod_{n=1}^{\infty} \Psi_{kn-1}^q(t w_\beta).$$

By (68), we can find an integer N_1 such that for all $N \geq N_1$,

$$\sum_{n=0}^{N} \chi_{A_{q,nk}}(\omega) \geq N(1-2(1-p)^q) \geq N\left(1-\frac{1}{2i_0}\right). \quad (71)$$

We claim that if $\chi_{A_{q,nk}}(\omega)=1$, then there exist distinct $\mathbf{i}_1, \mathbf{i}_2 \in \Lambda^q$ such that

$$X_{\mathbf{i}_1}^{q,(kn)}, X_{\mathbf{i}_2}^{q,(kn)} \geq (p_*)^q;\ \operatorname{ang}(a_{\mathbf{i}_1}-a_{\mathbf{i}_2}) = \operatorname{ang}(a_i - a_j); \quad |a_{\mathbf{i}_1}-a_{\mathbf{i}_2}| = br^{q-1}. \quad (72)$$

To see this, by (69) we can find $i_1, \dots, i_q \in \Lambda$ such that $X_{i_l}^{(qn+l)} \geq p_*$ for all $l=0,\dots,q-1$. Also $X_i^{(qkn+h)}, X_j^{(qkn+h)} \geq p_*$ for some $h \in \{0,\dots,q-1\}$ since $\chi_{A_{q,kn}}(\omega)=1$. Then it is easy to check that $\mathbf{i}_1 = i_1 \dots i_{h-1} i i_{h+1} \dots i_q$ and $\mathbf{i}_2 = i_1 \dots i_{h-1} j i_{h+1} \dots i_q$ satisfy (72). Hence for all $n \geq 0$ such that $\chi_{A_{q,nk}}(\omega)=1$ we can write, for some $d_0, d_{\mathbf{i}} \in \mathbb{R}$,

$$\begin{aligned}\Psi_{nk}^q(tw_\beta) &= \sum_{\mathbf{i}\in\Lambda^q} X_{\mathbf{i}}^{q,(nk)} \mathrm{e}^{\mathrm{i}\pi\langle T_q^{nk-1}a_{\mathbf{i}}, tw_\beta\rangle} \\ &= \mathrm{e}^{\mathrm{i}\pi d_0}\left(X_{\mathbf{i}_1}^{q,(nk)} + X_{\mathbf{i}_2}^{q,(nk)} \mathrm{e}^{\mathrm{i}\pi\langle T_q^{nk-1}(a_{\mathbf{i}_2}-a_{\mathbf{i}_1}), tw_\beta\rangle} + \sum_{\mathbf{i}\neq\mathbf{i}_1,\mathbf{i}_2} X_{\mathbf{i}}^{q,(nk)} \mathrm{e}^{\mathrm{i}\pi d_{\mathbf{i}}}\right).\end{aligned}$$

Let $t=\tau(r^{-qk})^N$, where $\tau \in [1, r^{-qk}]$ and $N \geq N_2 := \max\{N_0, N_1\}$, where $N_0 = N_0(\beta, q)$ is given for (70). Note that ang $(a_{\mathbf{i}_1}-a_{\mathbf{i}_2}) = \gamma - \theta$. Then

$$\begin{aligned}\langle T_q^{nk-1}(a_{\mathbf{i}_2}-a_{\mathbf{i}_1}), tw_\beta\rangle &= br^{q-1}t\langle T_q^{nk-1} w_{-\gamma+\theta}, w_\beta\rangle \\ &= b\tau r^{q-qk(N-n)} \cos(\beta+\gamma-nqk\theta).\end{aligned}$$

Since $\sum_{\mathrm{i}\in\Lambda^q} X_{\mathrm{i}}^{q,(nk)} = 1$ and $X_{\mathrm{i}_1}^{q,(nk)}, X_{\mathrm{i}_2}^{q,(nk)} \geq (p_*)^q$, there is a constant $\rho = \rho(p_*, r, q, k) > 0$ such that

$$|\Psi_{nk-1}^q(tw_\beta)| \leq 1-\rho$$

whenever $\|b\tau r^{q-qk(N-n)} \cos(\beta+\gamma-nqk\theta)\| > r^{2qk}/15$. From (70) and (71) we deduce that

$$\#\{n \in [N] : |\Psi_{nk-1}^q(tw_\beta)| \leq 1-\rho\} \geq \left(\frac{1}{i_0}-\frac{1}{2i_0}\right)N = \frac{1}{2i_0}N.$$

Hence

$$|\widehat{\pi_\beta\eta_{q,k}}(t)| \leq (1-\rho)^{N/2i_0} \leq t^{-\log(1-\rho)/(2i_0 qk \log r)},$$

provided $t \geq r^{-qk(N_2+1)}$, so $\pi_\beta\eta_{q,k}$ has positive Fourier dimension.

It was shown in [58, Lemma 4.3] that the convolution of a measure of full Hausdorff dimension with one of positive Fourier dimension is absolutely continuous with respect to Lebesgue measure. Since $\dim_H \pi_\beta\mu_{q,k} = 1$ by (67), applying [58, Lemma 4.3] to $\pi_\beta\mu_{q,k}$ and $\pi_\beta\eta_{q,k}$ gives that $\pi_\beta\mu$ is absolutely continuous.

We now apply Theorem (1.2.13) to get weak dimension conservation for self-similar sets in $\mathbb{R}^2$ where the IFS consists of similarities with irrational rotations that are translates of each other and satisfy OSC.

Theorem (1.2.14)[33]: Let θ/π be irrational and suppose that the IFS $\mathcal{I} = \{f_i = rR_\theta. + a_i\}_{i=1}^m$ on $\mathbb{R}^2$, with $r > 1/m$ and satisfying OSC, has attractor K, so that $s := \dim_H K = -\log m/\log r > 1$.

Then there is a set $E \subset [0,\pi)$ with $\dim_H E = 0$ such that for all $\beta \in [0,\pi) \setminus E$, for all $E \in (0, s-1)$,

$$\mathcal{L}^1\{x \in \pi_\beta(K) : \dim_H(K \cap \pi_\beta^{-1}(x)) \geq s-1-\epsilon\} > 0.$$

Proof. For each integer $q > \log 2/-\log r$ we may regard K as the attractor of the IFS $\mathcal{I}_q := \left\{f_{i_1} \cdots f_{i_q} : 1 \le i_1, \dots, i_q \le m\right\}$ so that $K = \Phi_q(\Sigma_q)$ where $\Sigma_q := \{\mathbf{i}_1\mathbf{i}_2 \dots : \mathbf{i}_j \in \Lambda^q\}$ and Φ_q is the cannonical map. Let $E_q \subset [0,\pi)$ be the set with $\dim_H E_q = 0$ given by that Theorem (1.2.13) for the IFS $\mathcal{I}_q$. Take $E = \cup_{q > \log 2/-\log r} E_q$ so that $\dim_H E = 0$.
Now fix $\epsilon \in (0, s-1)$. Let $q > \log 2/-\log r$ be an integer to be specified later. Let

$$p_q := \left(r^{q(s-1-\epsilon)} - \frac{2}{m^q}\right)\frac{m^q}{m^q - 2} = \frac{m^{q(1+\epsilon)/s} - 2}{m^q - 2} \in (0,1), \qquad (73)$$

since $r^s = m^{-1}$ and $2 < r^{-q} = m^{q/s}$. Let S_q be a random subset of Λ^q defined as follows. First choose two different symbols from Λ^q with uniform probability, then select each of the remaining $m^q - 2$ symbols with probability p_q, all actions being independent; in this way S_q always contains at least two symbols. Moreover, for each $\mathbf{i} \in \Lambda^q$,

$$\begin{aligned}\mathbb{P}(\mathbf{i} \in S_q) &= \frac{2}{m^q} + \frac{m^q - 2}{m^q} p_q \\ &= r^{q(s-1-\epsilon)}.\end{aligned}$$

Let $\left\{S_q^{(k)} : k \in \mathbb{N}\right\}$ be a sequence of independent copies of S_q. Then the set

$$\Sigma_q^\omega := S_q^{(1)} \times S_q^{(2)} \times \cdots$$

is an α-random set, with $\alpha = \log r^{q(s-1-\epsilon)}/\log r^q = s - 1 - \epsilon$, witn Φ_q satisfying (1) and (b) at (44) and (45).

Define a random vector X_q in a uniform manner, that is,

$$X_q = \left\{\frac{\chi(\mathbf{i} \in S_q)}{\#S_q}\right\}_{\mathbf{i} \in S_q};$$

then $\left(X_q\right)_{\mathbf{i}} \ge 1/m^q := p_*$ for at least two $\mathbf{i} \in S_q$. Let $\left\{X_q^{(k)} : k \in \mathbb{N}\right\}$ be independent copies of X_q which are supported by $S_q^{(k)}$. These random vectors define a random measure ν_q on Σ_q of the form described in (57) and (58) at the start of this section. Then ν_q has support Σ_q^ω, and $\Phi_q \nu_q$ has support $K^\omega = \Phi_q(\Sigma_q^\omega)$. From the strong law of large numbers, and using OSC when mapping the measure under Φ_q, almost surely

$$\dim_H \Phi_q \nu_q = \frac{\mathbb{E}(\log \ \#S_q)}{-\log r^q}.$$

Write $\mathrm{Bin}(n,p)$ to denote the binomial distribution with n points and probability p. Then

$$\mathbb{E}(\log \#S_q) = \mathbb{E}(\log[\mathrm{Bin}(m^q - 2, p_q) + 2]) = \log(m^{q(1+\epsilon)/s}) - o(1)$$

as $q \to \infty$, on using (73) to express p_q in terms of m together with a simple application of Chebyshev's inequality. Thus

$$\dim_H \Phi_q \nu_q = \frac{\log(m^{q(1+\epsilon)/s}) - o(1)}{-\log m^{-q/s}} = 1 + \epsilon - o(q^{-1}) > 1$$

provided we now choose q sufficiently large.

From Theorem (1.2.13), almost surely for all $\beta \in [0,\pi) \setminus E \subset [0,\pi) \setminus E_q$, the projected measure $\pi_\beta \Phi_q \nu_q$ is absolutely continuous with respect to Lebesgue measure, so $\mathcal{L}^1\left(\pi_\beta(K^\omega)\right) > 0$. The conclusion follows from Proposition (1.2.1), taking $A = \Sigma$, $K = \Phi(\Sigma)$ and $\alpha = s - 1 - \epsilon$.

We now extend Theorem (1.2.14) to general sets of similarities using a technique of Peres and Shmerkin [54, Proposition 6]. This allows us to reduce a general plane IFS to one where the similarities are mutual translates with the attractor a subset of that of the original IFS and of arbitraily close dimension to which we may apply Theorem (1.2.14).

Proposition (1.2.15)[33]: Let $\mathcal{I} = \{f_i = r_i R_{\theta_i} \cdot + a_i\}_{i=1}^{m}$ be an IFS on $\mathbb{R}^2$ satisfying OSC with attractor K. For all $\epsilon > 0$ there is an IFS $\mathcal{I}_\epsilon$, satisfying SSC and formed by a collection of compositions of maps from $\mathcal{I}$, such that all the maps in $\mathcal{I}_\epsilon$ have the same rotation R_θ for some angle θ and the same contraction ratio $0 < r < 1$, and with attractor $K_\epsilon \subset K$ such that $\dim_H K_\epsilon > \dim_H K - \epsilon$.

Moreover, if $\mathcal{I}$ has dense rotations then we may take θ/π to be irrational.

Proof. First we may assume that $\mathcal{I}$ satisfies SSC, since there is an IFS formed by compositions of the maps in $\mathcal{I}$ that satisfies SSC with attractor a subset of K and with Hausdorff dimension arbitrarily close to that of K, see, for example, [53]. Next, as in the proof of [54, Proposition 6], we may find integers $n_1, \dots, n_m$ such that the IFS $\mathcal{I}_\epsilon$ formed by all those compositions of the maps of $\mathcal{I}$ taken in any order such that f_i occurs n_i times for each $i = 1, \dots, m$, has an attractor $K_\epsilon \subset K$ with $\dim_H K_\epsilon > \dim_H K - \epsilon$. All the maps in $\mathcal{I}_\epsilon$ have rotation $R_\theta = R_{n_1\theta_1 + \cdots + n_m\theta_m}$ and contraction ratio $r = r_1^{n_1} \cdots r_{m_m}^{n_m}$.

Now suppose that $\mathcal{I}$ has dense rotations. If $(n_1\theta_1 + \cdots + n_m\theta_m)/\pi$ is irrational then there is nothing further to prove. Otherwise, at least one of the θ_i, say θ_1, is an irrational multiple of π. By a slight modification of the proof of [54, Proposition 6] we may conclude that the attractor of the IFS $\mathcal{I}'_\epsilon$ formed by the compositions of the maps of $\mathcal{I}$ such that f_1 occurs $n_1 - 1$ times and f_i occurs n_i times for $= 2, \dots, m$, with attractor $K'_\epsilon \subset K$ has $\dim_H K'_\epsilon > \dim_H K - \epsilon$. (We just note in [54, Proposition 6] that the number of paths ending at a neighboring lattice point to v is comparable to the number of paths ending at v.) Then $\big((n_1 - 1)\theta_1 + \cdots + n_m\theta_m\big)/\pi$ is irrational so the conclusion holds for $\mathcal{I}'_{\epsilon'}$.

Theorem (1.2.16)[33]: Let

$$\mathcal{I} = \{f_i = r_i R_{\theta_i} \cdot + a_i\}_{i=1}^{m}$$

be an IFS on $\mathbb{R}^2$ with dense rotations satisfying OSC, with attractor K and with $s = \dim_H K > 1$, where s is given by $\sum_{i=1}^{m} r_i^s = 1$. Then there is a set $E \subset [0, \pi)$ with $\dim_H E = 0$ such that for all $\beta \in [0, \pi) \setminus E$, for all $\epsilon \in (0, s-1)$,

$$\mathcal{L}^1\{x \in \pi_\beta(K) : \dim_H\big(K \cap \pi_\beta^{-1}(x)\big) \geq s - 1 - \epsilon\} > 0. \qquad (74)$$

Proof. For each $\epsilon > 0$, applying Theorem (1.2.14) to the amended IFS $\mathcal{I}_\epsilon$ with attractor K_ϵ given by Proposition (1.2.15) (replacing ϵ by $\epsilon/2$ in both theorem and proposition), there is a set $E_\epsilon \subset [0, \pi)$ with $\dim_H E_\epsilon = 0$, such that (74) holds for all $\beta \in E_\epsilon$. So that the set of exceptional β does not depend on ϵ, we let $E = \cup_{n=n_0}^{\infty} E_{2^{-n}}$, where $2^{-n_0} < s - 1$, so that $\dim_H E = 0$.

A natural question is whether these results can be strengthened from 'weak dimension conservation' to 'dimension conservation', that is whether the 'ϵ' can be removed in the conclusion of Proposition (1.2.2), and in Theorem (1.2.7), Theorem (1.2.9), Theorem (1.2.14) and Theorem (1.2.16).

Another natural question is whether, in Proposition (1.2.2), the condition on the projection of $B(\Phi[\mathbf{i}])$ can be weakened, with a consequential weakening of the corresponding condition on the projections of K in Theorem (1.2.7). Furthermore, can $\dim_B$ of the sections be replaced by $\dim_H$ in the conclusions of Proposition (1.2.2) and

Theorem (1.2.7)? An alternative approach would be to eliminate the exceptional set of directions in Theorem (1.2.14) and thus Theorem (1.2.16).

This raises the question of whether the box-dimension and Hausdorff dimension of sections of self-similar set are 'typically' equal for all, or perhaps 'nearly all' directions. If $\underline{\dim}_B(K \cap L) = \dim_H(K \cap L)$ for every line L, or at least for a large set of lines, then one might be able to replace lower box dimension by Hausdorff dimension in the conclusion of Theorem (1.2.7). There are plane self-similar sets defined by homotheties with at least some sections having distinct Hausdorff and lower box dimensions, for example for certain horizontal sections of the 1-dimensional Sierpiniski triangle, that is the attractor of the plane IFS with maps $f_1(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right), f_2(x,y) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right), f_3(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3}\right)$ (we are grateful to Thomas Jordan for pointing out this example to us); see also [35]. Is this possible for selfsimilar sets with dense rotations?

Similar conclusions to Proposition (1.2.2) and thus Theorem (1.2.7) might be expected for projections onto k-dimensional subspaces V where $k \geq 2$. However, it seems hard to get an analogue of Lemma (1.2.3) in this case. One would need to show that for any cube $I \subset V$ with $|I| \leq r$ there is a bounded number of points $x_i \in V$ with $N(x_i, r) \leq M$ such that if $N(x,r) \leq M$ for some $x \in I$ then some L_{x_i} intersects every set $B(\Phi[\mathbf{i}])$ such that $\mathbf{i} \in \Lambda_r$ that intersects L_x. (Here $N(x,r)$ is the number of $B(\Phi[\mathbf{i}])$ with $\mathbf{i} \in \Lambda_r$ that intersect L_x, the $(d-k)$-plane through $x \in V$ and perpendicular to V.)

Our results have been presented for self-similar sets defined by orientation preserving similarities. It would be possible to extend them to allow some of the maps to be orientation-reversing, for example by replacing an IFS by one formed by appropriate orientation-preserving compositions of the maps with little reduction in the dimension of the attractor, as in the proof of [54, Proposition 6].

Chapter 2
A Class of Spectral

We obtain two new conditions for a labeling tree to generate a spectrum when other digits (digits not necessarily in $\{0, 1, 2, 3\}$) are used in the base 4 expansion of integers and when bad branches are allowed in the spectral labeling. These new conditions yield new examples of spectra and in particular lead to a surprizing example which shows that a maximal set of orthogonal exponentials is not necessarily an orthonormal basis. We show, under certain conditions in terms of $(b_k, \mathcal{D}_k)$, that the associated Moran measure μ is a spectral measure, i.e., there exists a countable set $\Lambda \subset \mathbb{N}$ such that $\{e^{2\pi i\lambda x} \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. The special case is the Cantor measure with $\rho = \frac{1}{2k}$ and $N = 2$ [125], which was proved recently to be the only spectral measure among the Bernoulli convolutions with $0 < \rho < 1$ [113].

Section (2.1): Spectra of a Cantor Measure:

For certain probability measures μ in $\mathbb{R}^d$ there exist orthonormal bases of countable families of complex exponentials $\{e^{2\pi i\lambda\cdot x} \mid \lambda \in \Lambda\}$ for the Hilbert space $L^2(\mu)$. We called them Fourier series by analogy with the classical example of intervals on the real line. In this case, the measure μ is called a spectral measure and the set Λ is called a spectrum for μ. When $\mu = \frac{1}{|\Omega|}dx$ (where Ω is bounded subset of positive Lebesgue measure $|\Omega| > 0$ and dx is the Lebesgue measure), the existence of a spectrum is closely related to the well-known Fuglede conjecture which asserts that there exists a spectrum for μ if and only if Ω tiles $\mathbb{R}^d$ by translations using discrete set. This conjecture was proved to be false in higher dimensions by Tao [82] and others, but it is still open in dimension 1 and 2. See [76,77,74,71] for some important results and developments related to the spectral pairs with respect to probability measures that are obtained by restricting the Lebesgue measure to bounded sets.

Definition (2.1.1)[60]: Let $e_\lambda(x) := e^{2\pi i\lambda\cdot x}, x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d$. A probability measure μ on $\mathbb{R}^d$ is said to be a spectral measure if there exists a set $\Lambda \subset \mathbb{R}^d$ such that the family $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case Λ is called a spectrum for the measure μ.

There exist other probability measures that are not the restriction of the Lebesgue measure to bounded sets, but they admit spectra. The first example of a singular, non-atomic, spectral measure was constructed by Jorgensen and Pedersen in [68], and Strichartz [78] gave a simplification of part of the proof. These results led to the the spectral theory for fractal measures which has recently become an important topic of research in harmonic analysis. These fractal measures also have very close connections with the theory of multiresolution analysis in wavelet analysis (see e.g., [66, 63]).

The Jorgensen-Pedersen measure is constructed on a slight modification of the Middle Third Cantor set. This can be obtained as follows: consider the interval [0,1]. Divide it into 4 equal intervals, and keep the intervals $\left[0, \frac{1}{4}\right]$, and $\left[\frac{1}{2}, \frac{3}{4}\right]$. Then take each of these intervals and repeat the procedure ad inf. The result is a Cantor set

$$X_4 := \left\{\sum_{k=1}^{\infty} a_k \frac{1}{4^k} \mid a_k \in \{0,2\}\right\}.$$

The probability measure μ_4 on X_4 assigns measure $\frac{1}{2}$ to the sets $X_4 \cap \left[0, \frac{1}{4}\right]$ and $X_4 \cap \left[\frac{2}{4}, \frac{3}{4}\right]$, measure $\frac{1}{4}$ to the four intervals at the next stage, etc. It is the Hausdorff measure of Hausdorff dimension $\frac{\ln 2}{\ln 4} = \frac{1}{2}$.

The set X_4 and the measure μ_4 can be defined also in terms of iterated function systems (see [67]). Consider the iterated function system (IFS)

$$\tau_0(x) = \frac{x}{4},\ \tau_2(x) = \frac{x+2}{4},\ (x \in \mathbb{R}).$$

Then the IFS $\{\tau_0, \tau_2\}$ has a unique attractor X_4, i.e., a unique compact subset of $\mathbb{R}$ with the property that

$$X_4 = \tau_0(X_4) \cup \tau_2(X_4).$$

The measure μ_4 is the unique probability measure on $\mathbb{R}$ which satisfies the invariance equation:

$$\int f(x) d\mu_4(x) = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right) d\mu_4(x) + \int f\left(\frac{x+2}{4}\right) d\mu_4(x)\right),\ \left(f \in C_c(\mathbb{R})\right). (1)$$

Moreover, the measure μ_4 is supported on X_4.

[68], proved that the set

$$\Lambda = \left\{\sum_{k=0}^{n} 4^k d_k \mid d_k \in \{0,1\}, n \geq 0\right\}$$

is a spectrum for μ_4.

The results of Jorgensen and Pedersen were further extended for other measures, and new spectra were found in [79,75,62,64,65,73,72]. Some surprising convergence properties of the associated Fourier series were discovered in [80].

Two approaches to harmonic analysis on Iterated Function Systems have been popular: one based on a discrete version of the more familiar and classical second order Laplace differential operator of potential theory, see [81,70]; and the other is based on Fourier series. The first model in turn is motivated by infinite discrete network of resistors, and the harmonic functions are defined by minimizing a global measure of resistance, but this approach does not rely on Fourier series. the second approach begins with Fourier series, and it has its classical origins in lacunary Fourier series [69].

In general, for a given probability measure μ any of the following possibilities can occur: (i) there exists at most a finite number of orthogonal complex exponentials in $L^2(\mu)$; (ii) there are infinite families of orthogonal complex exponentials and one of them is an orthonormal basis for $L^2(\mu)$, and in this case μ is a spectral measure. The first example satisfying (i) is the Middle Third Cantor set, with its Hausdorff measure of dimension $\frac{\ln 2}{\ln 3}$. In [68] it was proved that for this measure no three exponentials are mutually orhtogonal. Detailed analysis on this was given and many new examples were constructed in [64]. However, for a given measure μ it remains a very difficult problem to "characterize" all the spectra or the maximal families of orthogonal exponentials. Moreover, it is not known whether every such a maximal family must be an orthonormal basis for $L^2(\mu)$. We answer all these questions for the measure μ_4.

We first establish a one-to-one correspondence between the labeling of the infinite binary tree and the base 4 expansions (using the digits $\{0,1,2,3\}$) of the integers. Then we characterize all maximal sets of orthogonal exponentials in $L^2(\mu_4)$ by showing that they correspond to spectral labelings of the binary tree. In Example (2.1.25) we show that there are maximal sets of orthogonal exponentials which are not spectra for μ_4. This is surprising, since in the previous examples in the literature, all maximal sets of orthogonal exponentials were also spectra for the associated fractal measure.

The spectral labeling characterization helps us obtain one sufficient condition for a maximal family of exponentials to an orthonormal basis for $L^2(\mu_4)$. This sufficient condition improves the known results from [68,79,75,62], and, it clarifies why some of the candidates for a spectrum constructed in [75, 79] are incomplete, and how they can be completed to spectra for μ_4.

We consider other digits that can be used for the base 4 expansion of the integers in the candidate set Λ, and give some sufficient conditions when these will generate spectra for μ_4. We construct some examples of spectra and give the example showing that a maximal set of orthogonal exponentials is not necessarily a spectrum. In addition a result of Strichartz in [79] is improved with the help of our Theorem (2.1.21).

In an attempt to obtain a "complete" characterization of all the spectra, we present a few other basic properties of spectra for μ_4 and give another sufficient condition for a spectral labeling to generate a spectrum where limited number of "bad" paths are allowed in the labeling. This new condition allows us to construct an example (Example (2.1.29)) of a spectral labeling that gives us a spectrum even though it does not satisfy the hypothesis of Theorem (2.1.17). Although we were not able to obtain a "complete" characterization for a maximal family to generate a spectrum, we believe that a combination of our results Theorem (2.1.17) and Proposition (2.1.28) might come close.

For the sake of clarity, we focus our discussion on the fractal measure μ_4. We believe that this example has many of the key features that might occur in more general fractal measures, and most of our results can be generalized for other IFS measures.

To define the sets of integers that correspond to families of orthogonal exponentials, we will recall some basic facts about base 4 expansions of integers.

Definition (2.1.2)[60]: Let k be an integer. Define inductively the sequences $(d_n)_{n\geq 0}$ and $(k_n)_{n\geq 0}$, with $d_n \in \{0,1,2,3\}$ and $k_n \in \mathbb{Z}$: $k_0 := k$; using division be 4 with remainder, there exist a unique $d_0 \in \{0,1,2,3\}$ and $k_1 \in \mathbb{Z}$ such that $k_0 = d_0 + 4k_1$. If k_n has been defined, then there exist a unique $d_n \in \{0,1,2,3\}$ and $k_n \in \mathbb{Z}$ such that $k_n = d_n + 4k_{n+1}$.

The infinite string $d_0 d_1 \ldots d_n \ldots$ will be called the base 4 expansion or the encoding of k. We will use the notation

$$k = d_0 d_1 \ldots d_n \ldots .$$

We will denote by $\underline{0}$ the infinite sequence $000 \ldots$, and similarly $\underline{3} = 333 \ldots$ The notation $d_0 d_1 \ldots d_n \underline{0}$ indicates that the infinite string begins with $d_0 \ldots d_n$ and ends in an infinite repetition of the digit 0 . Similarly for the notation $d_0 \ldots d_n \underline{3}$.

Proposition (2.1.3)[60]: Let $k \in \mathbb{Z}$ with base 4 expansion $k = d_0 \ldots d_n \ldots$ If $k \geq 0$ then its base 4 expansion ends in $\underline{0}$, i.e., there exists $N \geq 0$ such that $d_n = 0$ for all $n \geq N$. In this case

$$k = d_0 \ldots d_N \underline{0} = \sum_{n=0}^{N} 4^n d_n. \tag{2}$$

If $k < 0$ then its base 4 expansion ends in $\underline{3}$, i.e., there exists $N \geq 0$ such that $d_n = 3$ for all $n \geq 3$. In this case

$$k = d_0 \dots d_n\underline{3} = \sum_{n=0}^{N} 4^n d_n - 4^{N+1}. \tag{3}$$

Moreover, if k is defined by the formula on the right-hand side of (2) or (3) then its base 4 expansion is $d_0 \dots d_N\underline{0}$, in the first case, or $d_0 \dots d_N\underline{3}$ in the second case.

Proof. For $k \geq 0$, the base 4 expansion is well known. Let us consider the case when $k < 0$ and let $k = d_0 \dots d_n \dots$ be its base 4 expansion. Take $N \geq 0$ such that $k \geq -4^{N+1}$. Let $(k_n)_{n\geq 0}$ be defined as in Definition (2.1.2). Then $0 > k_0 = k \geq -4^{N+1}$. Since $k_1 = \frac{k_0 - d_0}{4}$ it follows that $k_1 \geq \frac{-4^{\bar{N}+1}-3}{4} \geq -4^N$. By induction $0 > k_{N+1} \geq -4^0 = -1$. So $k_{N+1} = -1$. Then $k_{N+2} = \frac{-1-3}{4}$, so $k_n = -1$ and $d_n = 3$ for all $n \geq N+1$. Thus the base 4 expansion of k ends in $\underline{3}$. Moreover, since $k_{N+1} = -1$, we have that $k_N = d_N - 4$, $k_{N-1} = d_{N-1} + 4k_N = d_{N-1} + 4d_N - 4^2$, and, by induction

$$k = k_0 = d_0 + 4d_1 + \cdots + 4^N d_N - 4^{N+1}.$$

Lemma (2.1.4)[60]: Let b be an integer and let $b = b_0 b_1 \dots$ be its base 4 expansion. Let a be another integer that has base 4 expansion ending with the expansion of b, i.e., $a = a_0 \dots a_n b_0 b_1 \dots$ Then

$$a = a_0 + 4a_1 + \cdots + 4^n a_n + 4^{n+1} b. \tag{4}$$

Conversely, if the integers a and b satisfy (4) with $a_0 \dots a_n \in \{0,1,2,3\}$, then the base 4 expansion of a has the form $a = a_0 \dots a_n b_0 b_1 \dots$, where $b = b_0 b_1 \dots$ is the base 4 expansion of b.

The base 4 expansion $d_0 d_1 \dots$ of an integer k is completely determined by the conditions: $d_n \in \{0,1,2,3\}$ for all $n \geq 0$, and

$$\sum_{n=0}^{N} d_n 4^n \equiv k \bmod 4^{N+1},\ (N \geq 0).$$

Proof. The proof follows directly from Proposition (2.1.3) by a simple computation.

We will characterize maximal sets of orthogonal exponentials and give a sufficient condition for such a maximal set to generate an orthonormal basis for $L^2(\mu_4)$.

First we will characterize maximal sets of orthogonal exponentials. These will correspond to sets of integers whose base 4 expansions can be arranged in a binary tree. We will call this arrangement a spectral labeling of the binary tree.

Definition (2.1.5)[60]: Let $\mathcal{T}$ be the complete infinite binary tree, i.e., the oriented graph that has vertices

$$\mathcal{V} := \{\emptyset\} \cup \{\epsilon_0 \dots \epsilon_n \mid \epsilon_k \in \{0,1\}, n \geq 0\},$$

and edges $\mathcal{E}: (\emptyset, 0), (\emptyset, 1), (\epsilon_0 \dots \epsilon_n, \epsilon_0 \dots \epsilon_n \epsilon_{n+1})$ for all $\epsilon_0 \dots \epsilon_n \in \mathcal{V}$, and $\epsilon_{n+1} \in \{0,1\}, n \geq 0$. The vertex $\emptyset$ is the root of this tree.

A spectral labeling $\mathcal{L}$ of the binary tree is a labeling of the edges of $\mathcal{T}$ with labels in $\{0,1,2,3\}$ such that the following properties are satisfied:

(i) For each vertex v in $\mathcal{V}$, the two edges that start from v have labels of different parity.

(ii) For each vertex v in $\mathcal{V}$, there exist an infinite path in the tree that starts from v and ends with edges that are all labeled 0 or all labeled 3.

We will use the notation $\mathcal{T}(\mathcal{L})$ to indicate that we use the labeling $\mathcal{L}$.

Given a spectral labeling, we will identify the vertices $v \in \mathcal{V}$ with the finite word obtained by reading the labels of the edges in the unique path from the root $\emptyset$ to the vertex

v. We will sometimes write $v = d_0 d_1 \dots d_n$, to indicate that the vertex v is the one that is reached from the root by following the labels $d_0 \dots d_n$.

We identify an infinite path in the tree $\mathcal{T}(\mathcal{L})$ from a vertex v with the infinite word obtained by reading the labels of the edges along this path. See Figure 1 [60] for the first few levels in a spectral labeling.

Definition (2.1.6)[60]: Let $\mathcal{L}$ be a spectral labeling of the binary tree. Then the set of integers associated to $\mathcal{L}$ is the set

$$\Lambda(\mathcal{L}) := \{k = d_0 d_1 \dots \dots \mid d_0 d_1 \dots \text{ is an infinite path in the tree starting from } \emptyset \text{ and ending in } \underline{0} \text{ or } \underline{3}\}.$$

Theorem (2.1.7)[60]: Let Λ be a subset of $\mathbb{R}$ with $0 \in \Lambda$. Then $\{e_\lambda \mid \lambda \in \Lambda\}$ is a maximal set of mutually orthogonal exponentials if and only if there exists a spectral labeling $\mathcal{L}$ of the binary tree such that $\Lambda = \Lambda(\mathcal{L})$.

Proof. We will need several lemmas.

Lemma (2.1.8)[60]: The Fourier transform of μ_4 is

$$\hat{\mu}_4(t) = e^{\frac{2\pi i t}{3}} \prod_{j=1}^{\infty} \cos\left(2\pi \frac{t}{4^j}\right), \ (t \in \mathbb{R}). \tag{5}$$

The convergence of the infinite product is uniform on compact subsets of $\mathbb{R}$.

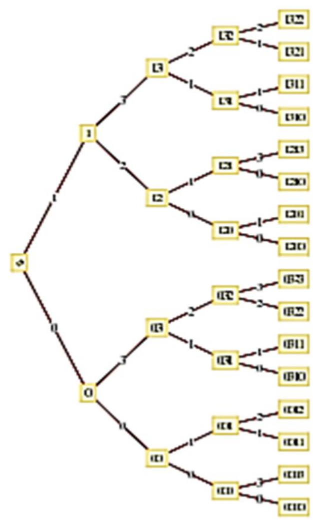

Figure 1[60]: The first levels in a spectral labeling of the binary tree. 0323 is a path in the tree from the root $\emptyset$, 13 is a path in the tree from the vertex 12.

Proof. Applying the invariance equation (1) to the exponential function $e_t, t \in \mathbb{R}$, we get

$$\hat{\mu}_4(t) = \frac{1 + e^{2\pi i 2\frac{t}{4}}}{2} \hat{\mu}_4\left(\frac{t}{4}\right) = e^{2\pi i \frac{t}{4}} \cos\left(2\pi \frac{t}{4}\right) \hat{\mu}_4\left(\frac{t}{4}\right).$$

Since $\hat{\mu}_4(0) = 1$, the cosine function is Lipschitz near 0 , and $\cos 0 = 1$, we can iterate this relation to infinity and obtain

$$\hat{\mu}_4(t) = e^{2\pi i \sum_{j=1}^{\infty} \frac{t}{4^j}} \prod_{j=1}^{\infty} \cos\left(2\pi \frac{t}{4^j}\right).$$

Lemma (2.1.9)[60]: Let $\lambda, \lambda' \in \mathbb{R}$. Then e_λ is orthogonal to $e_{\lambda'}$ in $L^2(\mu_4)$ iff $\lambda - \lambda' \in \mathcal{Z}$, where

$$\mathcal{Z} := \{x \in \mathbb{R} \mid \hat{\mu}_4(x) = 0\} = \{4^j(2k+1) \mid 0 \le j \in \mathbb{Z}, k \in \mathbb{Z}\}. \tag{6}$$

Proof. We have $\langle e_\lambda, e_{\lambda'}\rangle = \int e^{2\pi i(\lambda-\lambda')x} d\mu_4(x) = \hat{\mu}_4(\lambda - \lambda')$. So $e_\lambda \perp e_{\lambda'}$ iff $\lambda - \lambda' \in \mathcal{Z}$. Using the infinite product in (6), we obtain that $\lambda - \lambda' \in \mathcal{Z}$ iff there exists $j \geq 1$ such that $\cos\left(2\pi \frac{\lambda-\lambda'}{4^j}\right) = 0$. So $2\pi(\lambda - \lambda') \in 4^j\pi\left(\mathbb{Z} + \frac{1}{2}\right)$. This implies (6).

Note that, since $0 \in \Lambda$, for any element $a \in \Lambda$, we have $e_a \perp e_0$. Then with Lemma (2.1.9), we must have $a \in \mathcal{Z} \subset \mathbb{Z}$.

For an integer k with base 4 expansion $k = d_0 \ldots d_n \ldots$, we will denote by $d_n(k) := d_n$, the n-th digit of the base 4 expansion of k.
The next lemma follows from an easy computation.
Lemma (2.1.10)[60]: If $n, n' \geq 0, k, k, a, a' \in \mathbb{Z}$ with a, a' not divisible by 4, and $4^n(4k + a) = 4^{n'}(4k' + a')$ then $n = n'$.
Lemma (2.1.11). Let Λ be a subset of $\mathbb{R}$ with $0 \in \Lambda$. Assume $\{e_\lambda \mid \lambda \in \Lambda\}$ is a maximal set of orthogonal exponentials in $L^2(\mu_4)$. Then for $d_0, \ldots, d_{n-1} \in \{0,1,2,3\}$ the set

$$D(d_0 \ldots d_{n-1}) := \{d_n(a) \mid a \in \Lambda, d_0(a) = d_0, \ldots, d_{n-1}(a) = d_{n-1}\}$$

has either zero or two elements of different parity. This means that the n-th digit of the base 4 expansion of elements in Λ with prescribed first $n - 1$ digits, can take only 0 or 2 values, and if it takes 2 values, then these values must have different parity, i.e., $\{0,1\}$, or $\{0,3\}$, or $\{1,2\}$ or $\{2,3\}$.
Proof. Suppose $D(d_0 \ldots d_{n-1})$ has at least one element. Suppose $a, a' \in D(d_0 \ldots d_{n-1})$ with $d_k(a) = d_k(a') = d_k$ for all $0 \leq k \leq n - 1$, and assume $d_n(a) \neq d_n(a')$.
Then (see Lemma (2.1.4)) there exist $b, b' \in \mathbb{Z}$ such that

$$a = 4^{n+1}b + 4^n d_n(a_0) + 4^{n-1}d_{n-1} + \cdots + d_0,\ a' = 4^{n+1}b' + 4^n d_n(a'_0) + 4^{n-1}d_{n-1} + \cdots + d_0$$

Then $a - a' = 4^n\big(4(b - b') + d_n(a) - d_n(a')\big)$. By Lemma (2.1.9), since $a, a' \in \Lambda$, we must have $a - a' \in \mathcal{Z}$, so $a - a' = 4^m(2k + 1) = 4^m(4l + e)$ for some $m \geq 0, k, l \in \mathbb{Z}, e \in \{1,3\}$. Thus, with Lemma (2.1.10), $n = m$ and $d_n(a) - d_n(a')$ is an odd number. In particular, it follows that $D(d_0 \ldots d_{n-1})$ contains at most 2 elements.

Suppose now that $D(d_0 \ldots d_{n-1})$ has just one element. Then for all $a \in \Lambda$, with $d_k(a) = d_k$ for all $0 \leq k \leq n - 1$, one has that $d_n(a)$ is constant d_n.

Let $d'_n := d_n + 1 \bmod 4$ and let $a' := 4^n d'_n + 4^{n-1}d_{n-1} + \cdots + d_0$. We claim that $e_{a'}$ is orthogonal to all $e_a, a \in \Lambda$.

Let $a \in \Lambda$.
Case I: $d_k(a) = d_k$ for all $0 \leq k \leq n - 1$. Then, with Lemma (2.1.4), for some $b \in \mathbb{Z}$,

$$a = 4^{n+1}b + 4^n d_n + 4^{n-1}d_{n-1} + \cdots + d_0$$

so $a - a' = 4^n(4b + d_n - d'_n) \in 4^n(2\mathbb{Z} + 1) \subset \mathcal{Z}$. Therefore, with Lemma (2.1.9), $e_{a'} \perp e_a$.
Case II: There is an integer $0 \leq k \leq n - 1$ such that $d_0(a) = d_0, \ldots, d_{k-1}(a) = d_{k-1}$ and $d_k(a) \neq d_k$. Then for some $b \in \mathbb{Z}$,

$$a = 4^{k+1}b + 4^k d_k(a) + 4^{k-1}d_{k-1} + \cdots + d_0$$

Since $D(d_0 \ldots d_{n-1})$ is not empty, there is a $a'' \in \Lambda$ such that $d_0(a'') = d_0, \ldots, d_k(a'') = d_k$, so

$$a'' = 4^{k+1}b'' + 4^k d_k + 4^{k-1}d_{k-1} + \cdots + d_0,$$

for some $b'' \in \mathbb{Z}$.

Then, as before, since a, a'' are in the tree, and they differ first time at the k-th digit, we have that $d_k - d_k(a)$ is odd.

It follows that $a - a' = 4^k(4b - 4^{n-k}d'_n - 4^{n-k-1}d_{n-1} - \cdots - 4d_{k+1} + d_k(a) - d_k) \in 4^k(2\mathbb{Z} + 1) \subset \mathcal{Z}$. Hence $e_b \perp e_{a'}$.

We construct the spectral labeling $\mathcal{L}$ as follows: we label the root of the tree by $\emptyset$. Using Lemma (2.1.11), the set $D(\emptyset) := \{d(a_0) \mid a_0 \in \Lambda\}$ has two elements d_0 and d'_0. We label the edges from $\emptyset$ by d_0 and d'_0.

By induction, if we constructed the label $d_0 \dots d_n$ for a vertex, this means that there exists an element a of Λ that has base 4 expansion starting with $d_0 \dots d_n$. Therefore, using Lemma (2.1.11), the set $D(d_0 \dots d_n)$ contains exactly two elements of different parity e, e'. We label the edges that start from the vertex $d_0 \dots d_n$ by these elements e, e'. In particular we have that the sets $D(d_0 \dots d_n e)$ and $D(d_0 \dots d_n e')$ are not empty.

Next, we check that, from any vertex in this tree, there exists an infinite path that ends in $\underline{0}$ or $\underline{3}$.

Consider a vertex in this tree, and let $d_0 \dots d_n$ be its label. Then, by construction, the set $D(d_0 \dots d_n)$ is not empty. Therefore there is some a in Λ such that $d_0(a) = d_0, \dots, d_n(a) = d_n$. If we denote $d_k := d_k(a)$ for all $k \geq n$, then by construction the tree contains the vertices labeled $d_0 \dots d_k$ for all $k \geq 0$. Since the string $d_0 d_1 \dots$ is the base 4 expansion of a, it follows that the infinite sequence $d_0 d_1 \dots$ ends in either $\underline{0}$ or $\underline{3}$. Therefore there is an infinite path from the vertex $d_0 \dots d_n$ that ends in either $\underline{0}$ or $\underline{3}$.

Finally, we have to check that $\Lambda = \Lambda(\mathcal{L})$. If $a \in \Lambda$ and it has base 4 expansion $a = d_0 d_1 \dots$, then the vertices $d_0 \dots d_k$ are all in the tree $\mathcal{T}(\mathcal{L})$ so the infinite path $d_0 d_1 \dots$ is a path in this tree starting from the root $\emptyset$. Thus $\Lambda \subset \Lambda(\mathcal{L})$.

For the converse we prove the following:

Lemma (2.1.12)[60]: If $a = d_0 d_1 \dots, a' = d'_0 d'_1 \dots$ are two distinct infinite paths in the binary tree $\Lambda(\mathcal{L})$ starting from the root, that end in either $\underline{0}$ or $\underline{3}$, then $e_a \perp e_{a'}$.

Proof. Let $k \geq 0$ be the first index such that $d_k \neq d'_k$. Then $d_0 = d'_0, \dots, d_{k-1} = d'_{k-1}$ and since $\mathcal{L}$ is a spectral labeling, we have that $d_k - d'_k$ is odd. With Lemma (2.1.4) there exist $b, b' \in \mathbb{Z}$ such that

$$a = 4^{k+1}b + 4^k d_k + 4^{k-1}d_{k-1} + \cdots + d_0,\ a' = 4^{k+1}b' + 4^k d'_k + 4^{k-1}d'_{k-1} + \cdots + d'_0.$$

Then $a - a' = 4^k(4(b - b') + d_k - d'_k) \in 4^k(2\mathbb{Z} + 1) \subset \mathcal{Z}$. So $e_a \perp e_{a'}$.

Lemma (2.1.12) shows that, since $\mathcal{L}$ is a spectral labeling, the set $\{e_\lambda \mid \lambda \in \Lambda(\mathcal{L})\}$ is a set of mutually orthogonal exponentials. Since $\Lambda \subset \Lambda(\mathcal{L})$ and Λ is maximal, it follows that $\Lambda = \Lambda(\mathcal{L})$.

It remains to prove that, if $\mathcal{L}$ is a spectral labeling, then $\Lambda(\mathcal{L})$ corresponds to a maximal set of exponentials. We have seen above that $\Lambda(\mathcal{L})$ corresponds to a family of orthogonal exponentials; we have to prove it is maximal. Suppose there exists $\lambda \in \mathbb{R}$ such that $e_\lambda \perp e_a$ for all $a \in \Lambda(\mathcal{L})$. In particular $e_\lambda \perp e_0$, and with Lemma (2.1.9), we have $\lambda \in \mathbb{Z}$. Let $d_0 d_1 \dots$ be the base 4 expansion of λ. Let $k \geq 0$ be the first index such that $d_0 \dots d_k$ is not in the tree $\mathcal{T}(\mathcal{L})$. One of the labels of the edges from the vertex $d_0 \dots d_{k-1}$ has the same parity as d_k, and is different from d_k. Let d'_k be this label. Then $d_k - d'_k \in \{-2, 2\}$. Using property (ii) in the definition of a spectral labeling, there exists an infinite path a in the tree that starts with $d_0 \dots d_{k-1} d'_k$ and ends with $\underline{0}$ or $\underline{3}$. Then,

$$a = d_0 + \cdots + 4^{k-1}d_{k-1} + 4^k d_k + 4^{k+1}b,\ \lambda = d_0 + \cdots + 4^{k-1}d_{k-1} + 4^k d'_k + 4^{k+1}b',$$

for some $b, b' \in \mathbb{Z}$. Then $a - \lambda = 4^k\big(d_k - d'_k + 4(b - b')\big) \notin \mathcal{Z}$, because $d_k - d'_k$ is even, and not a multiple of 4 (see Lemma (2.1.10)). With Lemma (2.1.9), e_λ is not perpendicular to e_a. This shows that $\Lambda(\mathcal{L})$ corresponds to a maximal set of orthogonal exponentials.

This concludes the proof of Theorem (2.1.7).

Theorem (2.1.7) shows that when a spectral labeling $\mathcal{L}$ of the binary tree is given, it generates a maximal family of mutually orthogonal exponentials, by reading base 4 expansions from the tree. We will give a sufficient condition for a spectral labeling to generate a spectral set, i.e., an orthonormal basis of exponentials.

We will begin by defining certain "good" paths. The restriction on the spectral labeling will require that good paths can be found from any vertex.

Definition (2.1.13)[60]: Let $a \in \mathbb{Z}$ and let $a = d_0 d_1 \ldots$ be its base 4 expansion. We call the length of a the smallest integer n such that either $d_k = 0$ for all $k \geq n$ or $d_k = 3$ for all $k \geq n$. We will use the notation $n = \operatorname{lng}(a)$.

Fix integers $P, Q > 0$. Let $\omega = \omega_0 \omega_1 \ldots$ be an infinite path ending in $\underline{0}$ or $\underline{3}, \omega_n \in \{0,1,2,3\}$ for all $n \geq 0$.

We will say that the path ω is (P, Q)-good (or just good) if the there exists $n \geq 0$ such that the following two conditions are satisfied:

(i) $\omega_0, \ldots, \omega_n \in \{0,2\}$ and the number of occurrences of 2 in $\omega_0 \ldots \omega_n$ is less than P;

(ii) $\operatorname{lng}(\omega_{n+1} \omega_{n+2} \ldots) \leq Q$.

We divide the proof into several lemmas.

Lemma (2.1.14)[60]: ([68]). Let Λ be a set such that $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal family in $L^2(\mu_4)$. Then

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_4(t + \lambda)|^2 \leq 1 \ (t \in \mathbb{R}). \tag{7}$$

The set Λ is a spectrum for μ_4 iff

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_4(t + \lambda)|^2 = 1 \ (t \in \mathbb{R}). \tag{8}$$

Proof. Let $\mathcal{P}$ be the projection onto the span of $\{e_\lambda \mid \lambda \in \Lambda\}$. Then, using Parseval's identity, we have for all $\in \mathbb{R}$:

$$1 \geq \|\mathcal{P} e_{-t}\|^2 = \sum_{\lambda \in \Lambda} |\langle e_\lambda, e_{-t} \rangle|^2 = \sum_{\lambda \in \Lambda} |\hat{\mu}_4(t + \lambda)|^2.$$

This implies (7) and one of the ⇒ part in the last statement. For the converse, if (8) holds, then e_{-t} is in the span of $\{e_\lambda\}_\lambda$, and using the Stone-Weiertrass theorem, this implies that the span is $L^2(\mu_4)$.

Lemma (2.1.15)[60]: Assume that there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for any $y \in [-\epsilon_0, 1 + \epsilon_0]$ and any vertex $v = d_0 \ldots d_{N-1}$ in the binary tree $\mathcal{T}(\mathcal{L})$, there exists an infinite path $\lambda(d_0 \ldots d_{N-1})$ in the tree, starting from v, ending in $\underline{0}$ or $\underline{3}$, such that $\left|\hat{\mu}_4\big(y + \lambda(d_0 \ldots d_{N-1})\big)\right|^2 \geq \delta_0$. Then $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

The main idea of the proof of Lemma (2.1.15) is the same as the one used in a characterization of orthonormal scaling functions in wavelet theory [61], and is similar to the one used in the proof of Theorem 2.8 in [79]. But since 0 is not always present in the branching at a vertex, the details are more complicated.

Proof. With Theorem (2.1.7) we know that $\{e_\lambda \mid \lambda \in \Lambda(\mathcal{L})\}$ is an orthonormal family. We need to check (8). For a finite word $d_0 \ldots d_{N-1}$ with $d_0, \ldots d_{N-1} \in \{0,1,2,3\}$, we write $d_0 \ldots d_{N-1} \in \Lambda(\mathcal{L})$, if $d_0 \ldots d_{N-1}$ is the label of a vertex in the binary tree $\mathcal{T}(\mathcal{L})$.

For $d_0 \ldots d_{N-1}$ in $\Lambda(\mathcal{L})$, let

$$P_x^N(d_0 \dots d_{N-1}) := \prod_{j=1}^{N} \cos^2\left(\frac{2\pi(x + d_0 + \cdots + 4^{N-1}d_{N-1})}{4^j}\right),\ (x \in \mathbb{R}). \quad (9)$$

We claim that for any $N \geq 1$,

$$\sum_{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L})} P_x^N(d_0 \dots d_{N-1}) = 1. \quad (10)$$

For this, note that if $\{e, e'\}$ is any one of the following sets $\{0,1\}, \{0,3\}, \{1,2\}, \{2,3\}$, we have

$$\cos^2\left(\frac{2\pi(x+e)}{4}\right) + \cos^2\left(\frac{2\pi(x+e')}{4}\right) = 1,\ (x \in \mathbb{R}). \quad (11)$$

Then (10) follows from (11) by induction.

Next, fix $x \in \mathbb{R}$. Pick Q_1 such that for $N \geq Q_1, \frac{|x|}{4^N} \leq \epsilon_0$. Then for any $d_0 \dots d_{N-1} \in \Lambda(\mathcal{L})$, the point $y := \frac{x + d_0 + \cdots + 4^{N-1}d_{N-1}}{4^N} \in [-\epsilon_0, 1 + \epsilon_0]$. Therefore there exists a path $\lambda(d_0 \dots d_{N-1})$ starting from the vertex $d_0 \dots d_{N-1}$, ending in $\underline{0}$ or $\underline{3}$ with $\left|\hat{\mu}_4\big(y + \lambda(d_0 \dots d_{N-1})\big)\right|^2 \geq \delta_0$. We have

$$P_x^N(d_0 \dots d_{N-1})$$

$$\leq \frac{1}{\delta_0} P_x^N(d_0 \dots d_{N-1}) \left|\hat{\mu}_4\left(\frac{x + d_0 + \cdots + 4^{N-1}d_{N-1}}{4^N} + \lambda(d_0 \dots d_{N-1})\right)\right|^2$$

$$= \frac{1}{\delta_0} \prod_{j=1}^{N} \cos^2\left(\frac{2\pi\big(x + d_0 + \cdots + 4^{N-1}d_{N-1} + 4^N\lambda(d_0 \dots d_{N-1})\big)}{4^j}\right) \times$$

$$\prod_{j=1}^{\infty} \left|\cos^2\left(\frac{x + d_0 + \cdots + 4^{N-1}d_{N-1} + 4^N\lambda(d_0 \dots d_{N-1})}{4^{N+j}}\right)\right|^2$$

$$= \frac{1}{\delta_0} \left|\hat{\mu}_4\big(x + d_0 + \cdots + 4^{N-1}d_{N-1} + 4^N\lambda(d_0 \dots d_{N-1})\big)\right|^2 = \frac{1}{\delta_0} \left|\hat{\mu}_4\big(x + \eta_x(d_0 \dots d_{N-1})\big)\right|^2,$$

where for all $d_0 \dots d_{N-1} \in \Lambda(\mathcal{L})$, we denote

$$\eta_x(d_0 \dots d_{N-1}) := d_0 + \cdots + 4^{N-1}d_{N-1} + 4^N\lambda(d_0 \dots d_{N-1}) \in \Lambda(\mathcal{L}).$$

Note that the base 4 expansion of $\eta_x(d_0 \dots d_{N-1})$ starts with $d_0 \dots d_{N-1}$.

We claim that for any $\epsilon > 0$ there exists P_ϵ and Q_ϵ such that

$$\sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L}) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon}} P_x^N(d_0 \dots d_{N-1}) < \epsilon,\ (N \geq Q_\epsilon). \quad (12)$$

Fix $\epsilon > 0$. Using (7), there exists $P_\epsilon \geq Q_1 =: Q_\epsilon$ such that

$$\sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) \geq P_\epsilon} |\hat{\mu}_4(x + \lambda)|^2 < \epsilon\delta_0.$$

Then, using the previous calculation, for $N \geq Q_\epsilon$,

$$\sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L}) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon}} P_x^N(d_0 \dots d_{N-1}) \leq \frac{1}{\delta_0} \sum_{\operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon} \left|\hat{\mu}_4\big(x + \eta_x(d_0 \dots d_{N-1})\big)\right|^2$$

$$\leq \frac{1}{\delta_0} \sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) \geq P_\epsilon} |\hat{\mu}_4(x + \lambda)|^2 < \epsilon.$$

This proves (12).

From (12) we get that for all $N \geq Q_\epsilon$,

$$\sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L}) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon}} P_x^N(d_0 \dots d_{N-1})$$

$$= \sum_{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L})} P_x^N(d_0 \dots d_{N-1}) - \sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L}) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon}} P_x^N(d_0 \dots d_{N-1}) \quad (13)$$

$$\overset{\text{by } (10)}{=} 1 - \sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(\mathcal{L}) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) \geq P_\epsilon}} P_x^N(d_0 \dots d_{N-1}) > 1 - \epsilon. \quad (14)$$

We also have for all $\lambda = d_0 d_1 \cdots \in \Lambda(\mathcal{L})$,

$$|\hat{\mu}_4(x + \lambda)|^2 = \lim_{N \to \infty} P_x^N(d_0 \dots d_{N-1}). \quad (15)$$

To prove (15), we consider two cases: if λ ends in $\underline{0}$, then $\lambda = d_0 + \cdots + 4^{p-1} d_{p-1}$ for some $p \geq 0, d_k = 0$ for $k \geq p$, and for $N \geq p$,

$$P_x^N(d_0 \dots d_{N-1}) = \prod_{j=1}^{N} \cos^2\left(\frac{2\pi(x+\lambda)}{4^j}\right) \to |\hat{\mu}_4(x+\lambda)|^2.$$

If λ ends in $\underline{3}$, then $\lambda = d_0 + \cdots 4^{p-1} d_{p-1} - 4^p$, for some $p, d_k = 3$ for $k \geq p$, and for $p \geq N$,

$$P_x^N(d_0 \dots d_{N-1}) = \prod_{j=1}^{N} \cos^2\left(\frac{2\pi\left(x + d_0 + \cdots + 4^{p-1} d_{p-1} + 4^p(3 + \cdots + 3 \cdot 4^{N-1-p})\right)}{4^j}\right) =$$

$$\prod_{j=1}^{N} \cos^2\left(\frac{2\pi\left(x + d_0 + \cdots + 4^{p-1} d_{p-1} - 4^p + 4^N\right)}{4^j}\right) = \prod_{j=1}^{N} \cos^2\left(\frac{2\pi(x+\lambda)}{4^j}\right) \to |\hat{\mu}_4(x+\lambda)|^2.$$

This proves (15).

Now, any $\lambda \in \Lambda(\mathcal{L})$ with $\operatorname{lng}(\lambda) < P_\epsilon$ has base 4 expansion of the form $\lambda = d_0 \dots d_{P_\epsilon - 1}\underline{0}$ or $= d_0 \dots d_{P_\epsilon - 1}\underline{3}$, with $d_0 \dots d_{P_\epsilon} \in \Lambda(\mathcal{L})$. Therefore there are at most $2^{P_\epsilon} \cdot 2 = 2^{P_\epsilon + 1}$ such λ. With (15), for each such λ we can approximate $|\hat{\mu}_4(x+\lambda)|^2$ by $P_x^N(d_0(\lambda) \dots d_{N-1}(\lambda))$, where $d_0(\lambda) d_1(\lambda) \dots$ is the base 4 expansion of λ.

Therefore, using (15), there exists N as large as we want, $N \geq Q_\epsilon$, such that

$$\sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) < P_\epsilon} |\hat{\mu}_4(x+\lambda)|^2 > \sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) < P_\epsilon} P_x^N(d_0(\lambda) \dots d_{N-1}(\lambda)) - \epsilon. \quad (16)$$

But if $d_0 \dots d_{N-1} \in \Lambda(\mathcal{L})$ and $\eta := \eta_x(d_0 \dots d_{N-1})$ has length $\operatorname{lng}(\eta) < P_\epsilon$ then, the first N digits of $\eta_x(d_0 \dots d_{N-1})$ are $d_0(\eta) = d_0, \dots, d_{N-1}(\eta) = d_{N-1}$ and $\eta_x(d_0 \dots d_{N-1})$ is an element of $\Lambda(\mathcal{L})$ such that $\operatorname{lng}(\eta) < P_\epsilon$. Therefore

$$\sum_{\substack{d_0 \dots d_{N-1} \in \Lambda(L) \\ \operatorname{lng}(\eta_x(d_0 \dots d_{N-1})) < P_\epsilon}} P_x^N(d_0 \dots d_{N-1}) \leq \sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) < P_\epsilon} P_x^N\big(d_0 \dots d_{N-1}(\lambda)\big). \quad (17)$$

From (17), and (13), (14) we get

$$\sum_{\lambda \in \Lambda(\mathcal{L}), \operatorname{lng}(\lambda) < P_\epsilon} P_x^N(d_0(\lambda) \dots d_{N-1}(\lambda)) > 1 - \epsilon. \quad (18)$$

Then using (16), we have

$$\sum_{\lambda\in\Lambda(\mathcal{L})}|\hat{\mu}_4(x+\lambda)|^2 \geq \sum_{\lambda\in\Lambda(\mathcal{L}),\mathrm{lng}(\lambda)<P_\epsilon}|\hat{\mu}_4(x+\lambda)|^2 > 1-2\epsilon.$$

Since $\epsilon > 0$ and $x \in \mathbb{R}$ are arbitrary, Lemma (2.1.15) follows from Lemma (2.1.14).

Lemma (2.1.16)[60]: For each $P, Q \geq 0$, there exists $\delta > 0$ depending only on P, Q, such that for all $x \in \left[-\frac{1}{4},\frac{3}{4}\right]$ and all (P,Q)-good paths ω of one of the forms $\omega = \underline{0}$ or $\omega = 0 \dots 02d_0d_1 \dots$, the following inequality holds

$$|\hat{\mu}_4(x+\omega)|^2 \geq \delta.$$

(Note that, unless it is $\underline{0}$, the path ω contains at least one 2 after some zeros. The 2 can be on the first position 2 .... Note also that the path does not have to be in the binary tree.)

Proof. First we prove that for any $n, k \in \mathbb{Z}, n \geq 0$,

$$|\hat{\mu}_4(x+4^nk)|^2 \geq |\hat{\mu}_4(x)|^2\left|\hat{\mu}_4\left(\frac{x}{4^n}+k\right)\right|^2,\ (x\in\mathbb{R}). \tag{19}$$

If $n \geq 1$, we have

$|\hat{\mu}_4(x+4^nk)|^2$

$$= \cos^2\left(\frac{2\pi(x+4^nk)}{4}\right)\dots\cos^2\left(\frac{2\pi(x+4^nk)}{4^n}\right)\prod_{j=n+1}^{\infty}\cos^2\left(\frac{2\pi(x+4^nk)}{4^j}\right)$$

$$= \prod_{j=1}^{n}\cos^2\left(\frac{2\pi x}{4^j}\right)\prod_{j=1}^{\infty}\cos^2\left(\frac{2\pi\left(\frac{x}{4^n}+k\right)}{4^j}\right) \geq |\hat{\mu}_4(x)|^2\left|\hat{\mu}_4\left(\frac{x}{4^n}+k\right)\right|^2.$$

If $n = 0$, then $|\hat{\mu}_4(x+4^0k)|^2 \geq |\hat{\mu}_4(x)|^2\left|\hat{\mu}_4\left(\frac{x}{4^0}+k\right)\right|^2$ simply because $|\hat{\mu}_4(x)| \leq 1$. This proves (19).

The function $|\hat{\mu}_4|^2$ is continuous and its zeros are $\mathcal{Z} = \{4^j(2k+1) \mid j \geq 0, j,k \in \mathbb{Z}\}$ (see Lemma (2.1.9)). This implies in particular that $|\hat{\mu}_4(4k+2)|^2 \neq 0$ for all $k \in \mathbb{Z}$. If an integer a has base 4 expansion $a = a_0a_1 \dots$ of length $\mathrm{lng}(a) \leq Q$ then $|a| \leq 4^Q$. Indeed, if $a = a_0 \dots a_{Q-1}\underline{0}$, then $0 \leq a = a_0 + \dots + 4^{Q-1}a_{Q-1} \leq 3 + \dots + 4^{Q-1}3 = 4^Q - 1$. If $a = a_0 \dots a_{Q-1}\underline{3}$, then $0 \geq a = a_0 + \dots + 4^{Q-1}a_{Q-1} - 4^Q \geq -4^Q$.

Pick $\epsilon_1 > 0$ small (we will need $\epsilon_1 < \frac{7}{48}$). The function $|\hat{\mu}_4|^2$ is continuous and non-zero on the compact set

$$A := [-1+\epsilon_1, 1-\epsilon_1] + \{2+4k \mid |k| \leq 4^Q\}.$$

Therefore, there exists a $\delta_1 > 0$ such that

$$|\hat{\mu}_4(y)|^2 \geq \delta_1,\ (y \in A). \tag{20}$$

Take now $x \in \left[-\frac{1}{4},\frac{3}{4}\right]$ and let ω be a (P,Q)-good path of the forms mentioned in the hypothesis. If $\omega = \underline{0}$ then $x + \omega = x \in A$ and $|\hat{\mu}_4(x+\omega)|^2 \geq \delta_1$. In the other case ω has the form:

$$\omega = 4^{n_1}2 + \dots 4^{n_2}2 + \dots + 4^{n_p}2 + 4^{n_p+1}k,$$

where $0 \leq n_1 < \dots < n_p, 1 \leq p \leq P$ and k is an integer with base 4 expansion of length $\leq Q$, so $|k| \leq 4^Q$. Using (19) we have, by induction:

$$|\hat{\mu}_4(x+\omega)|^2 \geq |\hat{\mu}_4(x)|^2\left|\hat{\mu}_4\left(\frac{x}{4^{n_1}}+2+4^{n_2-n_1}2+\dots+4^{n_p-n_1}2+4^{n_p+1-n_1}k\right)\right|^2$$

$$\geq |\hat{\mu}_4(x)|^2\left|\hat{\mu}_4\left(\frac{x}{4^{n_1}}+2\right)\right|^2\left|\hat{\mu}_4\left(\frac{x}{4^{n_2}}+\frac{2}{4^{n_2-n_1}}+2+4^{n_3-n_2}2+\dots+4^{n_p-n_2}2+4^{n_p+1-n_2}k\right)\right|^2$$

$$|\hat{\mu}_4(x)|^2 \left|\hat{\mu}_4\left(\frac{x}{4^{n_1}}+2\right)\right|^2 \left|\hat{\mu}_4\left(\frac{x}{4^{n_2}}+\frac{2}{4^{n_2-n_1}}+2\right)\right|^2 \cdots \left|\hat{\mu}_4\left(\frac{x}{4^{n_{p-1}}}+\frac{2}{4^{n_{p-1}-n_1}}+\cdots+\frac{2}{4^{n_{p-1}-n_{p-2}}}+2\right)\right|^2 \times$$
$$\left|\hat{\mu}_4\left(\frac{x}{4^{n_p}}+\frac{2}{4^{n_p-n_1}}+\cdots+\frac{2}{4^{n_p-n_{p-1}}}+2+4k\right)\right|^2.$$

We have, when $n_l \geq 1$

$$-1+\epsilon_1 < -\frac{1}{4} \leq \frac{x}{4^{n_l}}+\frac{2}{4^{n_l-n_1}}+\cdots+\frac{2}{4^{n_l-n_{l-1}}} \leq \frac{3}{16}+\frac{2}{4}\frac{1}{1-\frac{1}{4}} = \frac{41}{48} < 1-\epsilon_1.$$

If $n_l = 0$ then $l = 1$ and $-1+\epsilon_1 < \frac{x}{4^0} \leq \frac{3}{4} < 1-\epsilon_1$. Thus we can use (20) on each term in the product above, and we obtain that

$$|\hat{\mu}_4(x+\omega)|^2 \geq \delta_1^p \geq \delta_1^P.$$

This proves Lemma (2.1.16).

Theorem (2.1.17)[60]: Let $\mathcal{L}$ be a spectral labeling of the binary tree. Suppose there exist integers $P, Q \geq 0$ such that for any vertex v in the tree, there exists a (P,Q)-good path starting from the vertex v. Then the set $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

Proof. We will show that the conditions of Lemma (2.1.15) are satisfied. Take $y \in \left[-\frac{1}{4},\frac{5}{4}\right]$ and, take $d_0 \dots d_{N-1}$ to be a vertex in the binary tree $\mathcal{T}(\mathcal{L})$.

We distinguish two cases:

Case I: $y \in \left[-\frac{1}{4},\frac{3}{4}\right]$. We will construct a path λ in the tree starting from the vertex $d_0 \dots d_{N-1}$. For this we follow the even-labeled branches until we reach the first 2 (recall that exactly one of the branches from every vertex is labeled by 0 or 2). If we cannot find a 2, then this means that $\lambda = \underline{0}$ is a path in the tree from the vertex $d_0 \dots d_{N-1}$, and with Lemma (2.1.16), we obtain $|\hat{\mu}_4(y+\lambda)|^2 = |\hat{\mu}_4(y)|^2 \geq \delta$.

Suppose we can find a 2 after finitely many steps from $d_0 \dots d_{N-1}$. Then from the vertex $d_0 \dots d_{N-1}0 \dots 02$, by hypothesis, we can find a (P,Q)-good path γ in the tree. Then $\lambda := 0 \dots 02\gamma$ is a $(P+1,Q)$-good path in the tree from the vertex $d_0 \dots d_{N-1}$. Then with Lemma (2.1.16), $|\hat{\mu}_4(y+\lambda)|^2 \geq \delta$.

Case II: $y \in \left[\frac{3}{4},\frac{5}{4}\right]$. We will construct a path λ from the vertex $d_0 \dots d_{N-1}$. For this we follow the oddlabeled branches until we reach the first 1. If we cannot find a 1 , then this means that $\lambda = \underline{3}$ is a path in the tree from the vertex $d_0 \dots d_{N-1}$; so $\lambda = -1$, and $y + \lambda = y-1 \in \left[-\frac{1}{4},\frac{1}{4}\right]$ so we get $|\hat{\mu}_4(y+\lambda)|^2 \geq \delta$.

If we can find a 1 after finitely may steps from $d_0 \dots d_{N-1}$, then from the vertex $d_0 \dots d_{N-1}3 \dots 31$ there exists a (P,Q)-good path γ in the tree. Then take $\lambda := 3 \dots 31\gamma$, with $p3$ s in the beginning. Then

$$\begin{aligned} y+\lambda &= y+3+4\cdot 3+\cdots+4^{p-1}3+4^p1+4^{p+1}\gamma = y+4^p-1+4^p+4^{p+1}\gamma \\ &= y-1+4^p(2+4\gamma). \end{aligned}$$

But then $y-1 \in \left[-\frac{1}{4},\frac{1}{4}\right]$ and $4^p(2+4\gamma)$ is a $(P+1,Q)$-good path (it is not a path in the tree but that does not matter), that contains at least a 2 (on position p). Therefore, with Lemma (2.1.16), we get $|\hat{\mu}_4(y+\lambda)|^2 \geq \delta$. Thus the hypotheses of Lemma (2.1.15) are satisfied and this implies that $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

As a special consequence of Theorem (2.1.17) we obtain the following corollary, which generalizes the results from [68], where the labels allowed were only $\{0,1\}$.

Corollary (2.1.18)[60]: Suppose $\mathcal{L}$ is a labeling of the binary tree such that for each vertex v in the tree, the two edges that start from v are labeled by either $\{0,1\}$ or $\{0,3\}$. Then $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

Proof. Clearly this is a spectral labeling because for each vertex the path $\underline{0}$ starting at v is in the tree. This is also a (0,0)-good path, so the conditions of Theorem (2.1.17) are satisfied.

We consider the spectral labeling of the binary tree with other digits, not necessarily $\{0,1,2,3\}$. We show that a spectral labeling is a spectrum if the set of digits is uniformly bounded and the zero label is included at each vertex (partially improving a result in [79]). Moreover, we provide the first counterexample for the fractal measure μ_4 of a maximal set of orthogonal exponentials which is not a spectrum for μ_4.

Definition (2.1.19)[60]: Suppose now we want to label the edges in the binary tree with other digits, not necessarily $\{0,1,2,3\}$. At each branching we use different digits, but we obey the rule that at each branching we can use only labels of the type $\{0, a\}$ where $a \in \mathbb{Z}$ is some odd number which varies from one branching to another. Thus, at the root we have a set $A_\emptyset$ of the form $\{0, a\}$ with $a \in \mathbb{Z}$ odd, and inductively, at each vertex $a_0 \dots a_{k-1}$ with $a_0 \in A_\emptyset, \dots, a_{k-1} \in A_{a_0 \dots a_{k-2}}$, we have a set $A_{a_0 \dots a_{k-1}}$ of the form $\{0, a(a_0, \dots, a_{k-1})\}$ with $a(a_0 \dots, a_{k-1}) \in \mathbb{Z}$ odd. We define the set

$$\Lambda := \left\{ \sum_{k=0}^{n} 4^k a_k \mid a_0 \in A_\emptyset, \dots, a_k \in A_{a_0 \dots a_{k-1}}, n \geq 0 \right\}. \tag{21}$$

Definition (2.1.20)[60]: Suppose the sets $A_\emptyset, \dots, A_{a_0 \dots a_{k-1}}$ are given as in Definition (2.1.19). We say that an integer λ has a modified base 4 expansion with digits in A if there exists an infinite sequence $a_0 a_1 \dots$ with the following properties

(i) $a_0 \in A_\emptyset, a_k \in A_{a_0 \dots a_{k-1}}$, for all $k \geq 1$;

(ii) $\sum_{k=0}^{n-1} a_k 4^k \equiv \lambda \bmod 4^n$, for all $n \geq 0$.

We call $a_0 a_1 \dots$ the A-base 4 expansion of λ. We denote by $\Lambda(A)$ the set of all integers that have a modified base 4 expansion with digits in A.

Note that if $\mathcal{L}$ is a spectral labeling and $\lambda \in \mathcal{L}$, then its base 4 expansion coincides with the $\mathcal{L}$-base 4 expansion.

Theorem (2.1.21)[60]: Consider the sets of digits A as in Definition (2.1.19).

(i) For the set Λ in (21), the exponentials $\{e_\lambda \mid \lambda \in \Lambda\}$ form an orthogonal family. There exists a unique spectral labeling $\mathcal{L}$ such that $\Lambda \subset \Lambda(\mathcal{L})$. Moreover $\Lambda(\mathcal{L}) = \Lambda(A)$.

(ii) If the sets $A_{a_0 \dots a_k}$ are uniformly bounded, then $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

Proof. To see that the exponential in $\{e_\lambda\}_{\lambda \in \Lambda}$ are orthogonal, take $\lambda = \sum_{k=0}^{\infty} 4^k a_k, \lambda' = \sum_{k=0}^{\infty} 4^k a'_k$ in Λ, $\lambda \neq \lambda'$, $a_k, a'_k = 0$ for k large. Let n be the first index such that $a_n \neq a'_n$. Then $\lambda - \lambda' = 4^n\big((a_n - a'_n) + 4l\big)$ for some integer l. Since $a_n - a'_n$ is odd, we have $\hat{\mu}_4(\lambda - \lambda') = 0$ (with Lemma (2.1.9)). Therefore $e_\lambda \perp e_{\lambda'}$.

Using Zorn's lemma, there is a maximal set Λ' of orthogonal exponentials such that $\Lambda \subset \Lambda'$. With Theorem (2.1.7), there exists a spectral labeling $\mathcal{L}$ such that $\Lambda(\mathcal{L}) = \Lambda'$. The key fact here is the uniqueness. We can construct the spectral labeling $\mathcal{L}$ as in the proof of Theorem (2.1.7) and Lemma (2.1.11). We consider base 4 expansions of elements in Λ. We want to prove that, if we fix $d_0 \dots d_{n-1} \in \{0,1,2,3\}$ then the set

$$D(d_0 \dots d_{n-1}) := \{d_n(\lambda) \mid \lambda \in \Lambda, d_0(\lambda) = d_0, \dots, d_{n-1}(\lambda) = d_n\}$$

will have 0 or 2 elements, and if it has 2 , then they have different parity. Since $\Lambda \subset \Lambda'$ it is clear that this set can have at most 2 elements, and if there are two then they have different parity. So it remains to prove only that it cannot have exactly one.

Suppose the set contains at least one element. Then there exists $\lambda = \sum_{k=0}^{\infty} 4^k a_k$, with the digits a_k in the sets A, such that the base 4 expansion of λ starts with $d_0 \dots d_{n-1}$. Take now $\lambda' := \sum_{k=0}^{n-1} 4^k a_k + 4^n a_n$ and $\lambda'' = \sum_{k=0}^{n-1} 4^k a_k + 4^n a_n'$ where a_n' is the other digit beside a_n in $A_{a_0 \dots a_{n-1}} = \{a_n, a_n'\}$. Since $\lambda - \lambda'$ and $\lambda - \lambda''$ are multiples of 4^n the base 4 expansions of $\lambda, \lambda', \lambda''$ will have the same first n digits $d_0 \dots d_{n-1}$. The $n+1$-st digits in the base 4 expansion of λ and λ' will be of different parity because $a_n - a_n'$ is odd. Thus $D(d_0 \dots d_{n-1})$ has 0 or 2 elements of different parity and these are completely determined from the set Λ (not just from the maximal one Λ').

Then the construction of the spectral labeling $\mathcal{L}$ proceeds just as in the proof of Theorem (2.1.7).

Next, we prove that an integer λ is in $\Lambda(\mathcal{L})$ iff it has a modified base 4 expansion with digits in A. First, we have that an integer λ with base 4 expansion $d_0 d_1 \dots$ is in the tree iff for all n, there exists $a_0, \dots, a_N$, $a_0 \in A_\emptyset, a_k \in A_{a_0 \dots a_{k-1}}$, such that the base 4 expansion of $\sum_{k=0}^{N} a_k 4^k$ begins with $d_0 \dots d_{n-1}$. But this implies that $\sum_{k=0}^{l} 4^k d_k \equiv \sum_{k=0}^{l} 4^k a_k \bmod 4^{l+1}$ for all $l \leq n-1$. In particular the digits $a_0 \dots a_{n-1}$ are completely determined by the digits $d_0 \dots d_{n-1}$, so they do not change if we increase n.

Thus, if $\lambda = d_0 d_1 \dots$ is in $\Lambda(\mathcal{L})$, there exist $a_0, a_1, \dots$ from A, such that for all $n \geq 0$,

$$\lambda \equiv \sum_{k=0}^{n} 4^k d_k \equiv \sum_{k=0}^{n} 4^k a_k \bmod 4^{n+1}.$$

Therefore λ is in $\Lambda(A)$.

Conversely, let λ be in $\Lambda(A)$, and let $d_0 d_1 \dots$ be its base 4 expansion. Then there exist $a_0, a_1, \dots$ from A such that for all n.

$$\sum_{k=0}^{n} 4^k d_k \equiv \lambda \equiv \sum_{k=0}^{n} 4^k a_k \bmod 4^{n+1}.$$

This implies that the base 4 expansion of $\sum_{k=0}^{n} 4^k a_k$ begins with $d_0 \dots d_n$ so $d_0 \dots d_n$ is a label in the tree $\mathcal{T}(\mathcal{L})$, and letting $n \to \infty$, we get that λ is in $\Lambda(\mathcal{L})$. This completes the proof of (i).

Next we prove (ii), i.e., if the sets $A_{a_0 \dots a_k}$ are uniformly bounded then $\Lambda(\mathcal{L})$ is a spectrum. We will check the conditions of Theorem (2.1.17). Let $Q \geq 0$ such that all the digits a_k used in Λ satisfy $|a_k| \leq 4^Q$.

Take a vertex $d_0 \dots d_{n-1}$ in the tree $d_i \in \{0,1,2,3\}$. This implies that there exists a $\lambda = \sum_{k=0}^{\infty} 4^k a_k$ in Λ, $a_k = 0$ for k large, such that the base 4 expansion of λ starts with $d_0 \dots d_{n-1}$. Take $\lambda' := \sum_{k=0}^{n-1} 4^k a_k \in \Lambda$. Since $\lambda - \lambda' = 4^n l$ for some integer l, the base 4 expansion of λ' starts also with $d_0 \dots d_{n-1}$. But $|\lambda'| \leq \sum_{k=0}^{n} |a_k| 4^k \leq 4^Q \frac{4^n - 1}{4-1} \leq 4^{Q+n}$. Therefore the base 4 expansion of λ' will have $\underline{0}$ or $\underline{3}$ from position $Q+n$ on. Thus, since $\lambda' \in \Lambda$, there exists a $(0, Q)$-good path in the tree that starts at the vertex $d_0 \dots d_{n-1}$. With Theorem (2.1.17), $\Lambda(\mathcal{L})$ is a spectrum for μ_4.

Remark (2.1.22)[60]: In [79], Strichartz analyzed the spectra of a more general class of measures. When restricted to our example, his results (Theorem 2.7 and 2.8 in [79]) cover the case when all vertices at some level n use the same digits $\{0, a_n\}$. In our notation, this means that $A_{a_0,\ldots,a_{n-1}} =: A_n$ depends only on the length n, and not on the digits $a_0 \ldots a_{n-1}$. In [79, Theorem 2.8], an extra condition is needed to guarantee that the set

$$\Lambda = \left\{\sum_{k=0}^{n} b_k 4^k \mid b_k \in \{0, a_k\} n \geq 0\right\}$$

is a spectrum μ_4. The condition requires the set $\frac{1}{4^n}A_0 + \frac{1}{4^{n-1}}A_1 + \cdots + \frac{1}{4}A_{n-1}$ be separated from the zeroes of the function

$$\prod_{k=1}^{n} \cos^2\left(2\pi \frac{x}{4^k}\right)$$

uniformly in k.

Theorem (2.1.21) improves this result by removing this extra condition. Even when the condition is not satisfied we still get a spectrum for μ_4, namely $\Lambda(A)$, but this might be bigger than Λ.

Example (2.1.23)[60]: Let all the sets $A_{a_0 \ldots a_{k-1}}$ in Definition (2.1.19) be equal to $\{0,3\}$. The results in [79] do not apply (since $\sum_{k=0}^{n} \frac{3}{4^k}$ approaches 1). Then the set

$$\Lambda = \left\{\sum_{k=0}^{n} a_k 4^k \mid a_k \in \{0,3\}, n \geq 0\right\},$$

will give an incomplete set of exponentials. To complete it one has to consider the set $\Lambda(A)$ which in this case

$$\Lambda(A) = \Lambda \bigcup \left\{\sum_{k=0}^{n} a_k 4^k - 4^{n+1} \mid a_k \in \{0,3\}, n \geq 0\right\}.$$

The second part comes from the integers with base 4 expansion ending in $\underline{3}$. The set Λ contains only those integers that have a base 4 expansion ending in $\underline{0}$. $\Lambda(A)$ is a spectrum, by Theorem (2.1.21) (ii). The reason for the incompleteness of Λ is that the integers are not read correctly (perhaps thoroughly is the better word) from the labels A.

Example (2.1.24)[60]: Suppose $A_{\emptyset} = \{0,15\}$ and $A_{a_0 \ldots a_{k-1}} = \{0,9\}$ for all $k \geq 1$. Then the set

$$\Lambda := \left\{\sum_{k=0}^{n} a_k 4^k \mid a_0 \in \{0,15\}, a_k \in \{0,9\} \text{ for } k \geq 1, n \geq 0\right\},$$

does not give a maximal set of orthogonal exponentials. e_3 is perpendicular to all $e_\lambda, \lambda \in \Lambda$. Indeed 3 has A-base 4 expansion 15999 ..., so $3 \in \Lambda(A)$, and $\Lambda(A)$ is a spectrum by Theorem (2.1.21).

Example (2.1.25)[60]: In this example we construct a set of digits A which will give a spectral labeling, which is not a spectrum. Thus we will have $\Lambda = \Lambda(A) = \Lambda(\mathcal{L})$ but Λ is not a spectrum. The reason for the incompleteness of $\{e_\lambda \mid \lambda \in \Lambda\}$ is thus more subtle, the set is a maximal set of orthogonal exponentials, but it does not span the entire $L^2(\mu_4)$. Consider the following set

$$\Lambda := \left\{\sum_{k=0}^{N} 4^k \left(4^{10^{k+2}-k} + 1\right) \delta_k \mid \delta_k \in \{0,1\}, N \geq 0\right\}. \tag{22}$$

We will prove the following

Proposition (2.1.26)[60]: There exists a spectral labeling $\mathcal{L}$ such that $\Lambda(\mathcal{L}) = \Lambda$, so, by Theorem (2.1.7) the set $\{e_\lambda \mid \lambda \in \Lambda\}$ forms a maximal family of orthogonal exponentials. Λ is not a spectrum for μ_4.

Proof. The elements in Λ have the form

$$\lambda = \sum_{k=0}^{\infty} \left(4^{10^{k+2}} + 4^k\right) \delta_k, \tag{23}$$

where $\delta_k \in \{0,1\}$ and $\delta_k = 0$ for k larger than some $N \geq 0$.

Let $\lambda = d_0 d_1 \ldots$ be the base 4 expansion of this element. Since $\lambda \geq 0$ the expansion ends in $\underline{0}$. Then, note that

(i) $d_k = 1$ iff one of the following two conditions is satisfied:

(i) k is not of the form 10^{n+2} and $\delta_k = 1$;

(ii) $k = 10^{n+2}$ for some $n \geq 0$, and $\delta_n = 1$ and $\delta_k = 0$.

(ii) $d_k = 2$ iff $k = 10^{n+2}$ for some $n \geq 0$, and $\delta_n = 1$ and $\delta_k = 1$.

(iii) $d_k = 0$ in all other cases.

We construct the spectral labeling $\mathcal{L}$ as follows: First, we consider the spectral labeling $\mathcal{L}_0$ where only the labels $\{0,1\}$ are used at each vertex. We build a new binary tree $\mathcal{T}(\mathcal{L}_0, \mathcal{L})$ with a different kind of labeling. For the vertices we keep the labels from $\mathcal{T}(\mathcal{L}_0)$, but we label the edges differently. We will change the labeling $\{0,1\}$ to $\{1,2\}$ at certain vertices. This will be done in the following way: for all $N \geq 0$ and for all vertices $\delta_0 \ldots \delta_N$ with $\delta_N = 1$, in the subtree with root $\delta_0 \ldots \delta_N$ we will change the labeling at all vertices at level 10^{N+2} from $\{0,1\}$ to $\{1,2\}$. So, at a vertex $\delta_0 \ldots \delta_N \delta_{N+1} \ldots \delta_{10^{N+2}-1}$, the edges are labeled $\{1,2\}$ instead of $\{0,1\}$.

The spectral labeling $\mathcal{L}$ is obtained by relabeling the vertices consistently with the labels of the edges.

We have to check that $\Lambda(\mathcal{L}) = \Lambda$. If $\lambda = d_0 d_1 \cdots \in \Lambda(\mathcal{L})$, ending in $\underline{0}$, then we construct a sequence $\delta_0 \delta_1 \ldots$ by reading the labels of the vertices in $\mathcal{T}(\mathcal{L}_0, \mathcal{L})$ along λ. Then by construction

$$\lambda = \sum_{k=0}^{\infty} 4^k d_k = \sum_{k=0}^{\infty} \left(4^{10^{k+2}} + 4^k\right) \delta_k$$

so $\lambda \in \Lambda$. Conversely, if $\delta_0, \ldots, \delta_N$ are in $\{0,1\}$ it is clear that the base 4 expansion of $\sum_{k=0}^{N} \left(4^{10^{k+2}} + 4^k\right) \delta_k$ is in $\Lambda(\mathcal{L})$.

The labeling $\mathcal{L}$ is a spectral labeling because one can end a path in $\underline{0}$: just follow the zeros in the labeling of the vertices in $\mathcal{T}(\mathcal{L}_0, \mathcal{L})$.

Next we prove that Λ is not a spectrum for μ_4. We will show that

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_4(1+\lambda)|^2 < 1. \tag{24}$$

First, let $\lambda = \lambda(\delta_0 \ldots \delta_N) := \sum_{k=0}^{N} \left(4^{10^{k+2}} + 4^k\right) \delta_k$, with $\delta_N = 1$, and let $\lambda = d_0 d_1 \ldots$ be the base 4 expansion. Then $d_{10^{N+2}} = 1$ and $d_k = 0$ for $k > 10^{N+2}$. Since for $k < N$, we have $10^{k+2} \leq 10^{N+1}$, and $k < 10^{N+1}$, we see that $d_k = 0$ for $10^{N+1} < k < 10^{N+2}$. Thus the base 4 expansion of λ ends with a 1 on position 10^{N+2} and $9 \cdot 10^{N+1}$ zeros before that.

We use the following notation: for $e_0 e_1 \ldots e_n, . e_0 e_1 \ldots e_n := \frac{e_0}{4} + \cdots + \frac{e_n}{4^n}$. Let $m(x) := \cos^2(2\pi x)$. Let the base 4 expansion of $1 + \lambda$ be $b_0 b_1 \ldots$ Then $b_0 = d_0 + 1$ and

$b_n = d_n$ for all $n \geq 1$. Then $\frac{1+\lambda}{4} \equiv .b_0 \bmod \mathbb{Z}$, $\frac{1+\lambda}{4^2} \equiv .b_1 b_0 \bmod \mathbb{Z} \ldots \frac{1+\lambda}{4j} \equiv .b_{j-1} \ldots b_0$. Since $m \leq 1$ we have

$$|\hat{\mu}_4(1+\lambda)|^2 = \prod_{j=1}^{\infty} m\left(\frac{1+\lambda}{4^j}\right) \leq m\left(\frac{1+\lambda}{4^{10^{N+2}+1}}\right).$$

But $\frac{1+\lambda}{4^{10^{N+2}+1}} \equiv y := .b_{10^{N+2}} \ldots b_{10^{N+1}} \ldots b_0 \bmod \mathbb{Z}$. But we saw above that $b_{10^{N+2}} = a_{10^{N+2}} = 1$ and $b_n = a_n = 0$ for $10^{N+1} < n < 10^{N+2}$. So $y - \frac{1}{4} = y - .1$ has at least $9 \cdot 10^{N+1}$ zeros after the decimal point. Therefore $0 \leq y - \frac{1}{4} = y - .1 \leq \frac{1}{4^{9 \cdot 10^{N+1}}}$. Then

$$m(y) = \cos^2\left(2\pi\left(\frac{1}{4} + \left(y - \frac{1}{4}\right)\right)\right) = \sin^2\left(2\pi\left(y - \frac{1}{4}\right)\right)$$
$$\leq 4\pi^2\left(y - \frac{1}{4}\right)^2 \leq \frac{4\pi^2}{4^{18 \cdot 10^{N+1}}}.$$

Therefore

$$|\hat{\mu}_4(1+\lambda)|^2 \leq m\left(\frac{1+\lambda}{4^{10^{N+1}}}\right) = m(y) \leq \frac{4\pi^2}{4^{18 \cdot 10^{N+1}}}.$$

Then

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_4(1+\lambda)|^2 = \sum_{N=0}^{\infty} \sum_{\substack{\delta_0, \ldots, \delta_{N-1} \in \{0,1\} \\ \delta_N = 1}} \left|\hat{\mu}_4\big(1 + \lambda(\delta_0 \ldots \delta_N)\big)\right|^2 \leq \sum_{N=0}^{\infty} 2^N \frac{4\pi^2}{4^{18 \cdot 10^{N+1}}} < 1.$$

With Lemma (2.1.14), this shows that Λ is not a spectrum for μ_4.

We describe some basic properties of spectra for the measure μ_4, and we give an example of a spectral labeling which generates a spectrum but does not satisfy the conditions of Theorem (2.1.17).

Proposition (2.1.27)[60]:

(i) If Λ_1, Λ_2 are spectra for $\mu_4, \Lambda_1, \Lambda_2 \subset \mathbb{Z}$, and e_1, e_2 are two integers of different parity, then the set $\Lambda := (4\Lambda_1 + e_1) \cup (4\Lambda_2 + e_2)$ is a spectrum for μ_4.

(ii) If Λ is a spectrum for $\mu_4, \Lambda \subset \mathbb{Z}$, then there exist $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$ and e_1, e_2 integers of different parity such that

$$\Lambda = (4\Lambda_1 + e_1) \cup (4\Lambda_2 + e_2). \qquad (25)$$

Moreover, for any decomposition of Λ as in (25), the sets Λ_1, Λ_2 are spectra for μ_4.

Proof. (i) We use Lemma (2.1.14). We have for $x \in \mathbb{R}$, using Lemma (2.1.8):

$$\sum_{i=1,2} \sum_{\lambda_i \in \Lambda_i} |\hat{\mu}_4(x + 4\lambda_i + e_i)|^2 = \sum_{i=1,2} \sum_{\lambda_i \in \Lambda_i} \cos^2\left(2\pi \frac{x+e_i}{4} + \lambda_i\right) \left|\hat{\mu}_4\left(\frac{x+e_i}{4} + \lambda_i\right)\right|^2 =$$
$$\sum_{i=1,2} \cos^2\left(2\pi \frac{x+e_i}{4}\right) \sum_{\lambda_i \in \Lambda_i} \left|\hat{\mu}_4\left(\frac{x+e_i}{4} + \lambda_i\right)\right|^2 = \sum_{i=1,2} \cos^2\left(2\pi \frac{x+e_i}{4}\right) = 1.$$

For the next to last equality we used the fact that Λ_i are spectra and Lemma (2.1.14). For the last equality we used the fact that $e_1 - e_2$ is odd.

(ii) We can assume that $0 \in \Lambda$. Otherwise, we work with $\Lambda - \lambda_0$ for some $\lambda_0 \in \Lambda$. Then, since Λ is a spectrum, by Theorem (2.1.7) there is a spectral labeling $\mathcal{L}$ of the binary tree. Take e_1, e_2 to be the labels of the edges that start from the root $\emptyset$, and take Λ_i to be the set of integers that correspond to infinite paths in the subtree with root e_i. Then it is clear that (25) is satisfied.

Assume now that Λ is decomposed as in (25). We want to prove that Λ_1, Λ_2 are spectra. A simple check, that uses Lemma (2.1.9), shows that $\{e_\lambda \mid \lambda \in \Lambda_i\}$ is an orthonormal family, for both $i = 1,2$. With Lemma (2.1.14) and the computation above we have for all $x \in \mathbb{R}$,

$$1 = \sum_{i=1,2} \cos^2\left(2\pi \frac{x+e_i}{4}\right) \sum_{\lambda_i \in \Lambda_i} \left|\hat{\mu}_4\left(\frac{x+e_i}{4} + \lambda_i\right)\right|^2$$
$$=: \sum_{i=1,2} \cos^2\left(2\pi \frac{x+e_i}{4}\right) h_{\Lambda_i}\left(\frac{x+e_i}{4} + \lambda_i\right).$$

Take now $x \notin \mathbb{Z}$. From Lemma (2.1.14), we have $h_{\Lambda_i}\left(\frac{x+e_i}{4} + \lambda_i\right) \leq 1$. Also $\cos^2\left(2\pi \frac{x+e_i}{4}\right) \neq 0$ for $i = 1,2$. If $h_{\Lambda_i}\left(\frac{x+e_i}{4} + \lambda_i\right) < 1$ for one of the 's, then this would contradict the equality above. Thus $h_{\Lambda_i}\left(\frac{x+e_i}{4} + \lambda_i\right) = 1$ for all $x \notin \mathbb{Z}, i = 1,2$. But as in the proof of Lemma (2.1.14), this implies that e_{-x} is in the span of $\{e_\lambda \mid \lambda \in \Lambda_i\}$ for all $x \notin \mathbb{Z}$, and since e_n can be approximated uniformly by e_x with $x \notin \mathbb{Z}$, it follows that e_n is also spaned by exponentials in Λ_i. Then as in the proof of Lemma (2.1.14), it follows that Λ_i is a spectrum.

Applying Proposition (2.1.27) several times, we see that the spectral property is a "tail" property: it does not depend on the labeling of the first few edges. In other words, if all the subtrees, from some level on, correspond to spectra, then the entire tree will correspond to a spectrum.

Proposition (2.1.28)[60]: Let $\mathcal{L}$ be a spectral labeling. For each vertex $d_0 \dots d_{n-1}$, let $\mathcal{L}_{d_0 \dots d_{n-1}}$ be the spectral labeling obtained by reading the labels in the subtree with root $d_0 \dots d_{n-1}$. Suppose there exists a finite set $\mathcal{S}$ of paths in the binary tree $\mathcal{T}(\mathcal{L})$, that start at the root $\emptyset$, and that satisfy the following conditions:

(i) The paths do not end in $\underline{0}$ or $\underline{3}$;

(ii) For any vertex $d_0 \dots d_{n-1}$ that does not lie on any of the paths in $\mathcal{S}$, the spectral labeling $\mathcal{L}_{d_0 \dots d_{n-1}}$ gives a spectrum, i.e. $\Lambda(\mathcal{L}_{d_0 \dots d_{n-1}})$ is a spectrum.

Then $\Lambda(\mathcal{L})$ is a spectrum.

Proof. Let $m(x) := \cos^2(2\pi x), x \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and let $\omega_0\omega_1 \dots$ be a path in $\mathcal{T}(\mathcal{L})$ that does not end in $\underline{0}$ or $\underline{3}$. We prove that

$$\lim_{n\to\infty} \prod_{j=1}^{n} m\left(\frac{x + \omega_0 + \cdots + 4^{n-1}\omega_{n-1}}{4^j}\right) = 0. \tag{26}$$

To prove (26), we will show first that there exists $\epsilon_0 > 0$ and a subsequence $\{n_p\}_{p\geq 0}$ such that

$$\operatorname{dist}\left(\frac{x + \omega_0 + \cdots + 4^{n_p-1}\omega_{n_p-1}}{4^{n_p}}, \left\{0, \frac{1}{2}, 1\right\}\right) \geq \epsilon_0, \ (p \geq 0). \tag{27}$$

If not, then

$$\operatorname{dist}\left(\frac{x + \omega_0 + \cdots + 4^{n-1}\omega_{n-1}}{4^n}, \left\{0, \frac{1}{2}, 1\right\}\right) \to 0, \text{ as } n \to \infty.$$

Take $\epsilon > 0$ small $\epsilon < \frac{1}{4^{10}}$. For n large, $y_n := \frac{x+\omega_0+\cdots+4^{n-1}\omega_{n-1}}{4^n}$ is close to $0, \frac{1}{2}$ or 1.

If $|y_n - 0| < \epsilon$ then $y_{n+1} = \frac{y_n+\omega_n}{4}$ is close to either 0 when $\omega_n = 0$, or $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ when $\omega_n = 1,2$ or 3.

If $\left|y_n - \frac{1}{2}\right| < \epsilon$ then y_{n+1} is close to either $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ or $\frac{7}{8}$, so it cannot be close to $\left\{0, \frac{1}{2}, 1\right\}$.

If $|y_n - 1| < \epsilon$ then y_{n+1} is close to $\left\{0, \frac{1}{2}, 1\right\}$ only when $\omega_n = 3$.

Thus, the only paths that will make y_n stay close to $\left\{0, \frac{1}{2}, 1\right\}$, as $n \to \infty$, are the ones that end in $\underline{0}$ or $\underline{3}$. This proves (27).

If (27) is satisfied then, since $m(y) = 1$ only at $0, \frac{1}{2}$ and 1, for $y \in (-1/4, 5/4)$, there exists some $\delta > 0$, with $\delta < 1$, such that for all $p \geq 0$,

$$m\left(\frac{x + \omega_0 + \cdots + 4^{n_p - 1}\omega_{n_p - 1}}{4^{n_p}}\right) \leq \delta. \tag{28}$$

Then for $n \geq n_p$ we have, since $0 \leq m \leq 1$ and m is $\mathbb{Z}$-periodic,

$$\prod_{k=1}^{n} m\left(\frac{x + \omega_0 + \cdots + 4^{n-1}\omega_{n-1}}{4^j}\right) \leq \prod_{l=1}^{p} m\left(\frac{x + \omega_0 + \cdots + 4^{n_l - 1}\omega_{n_l - 1}}{4^{n_l}}\right) \leq \delta^p.$$

This implies (26).

Let $\mathcal{V}(\mathcal{S})$ be the set of labels of vertices on the paths in $\mathcal{S}$.

To prove Proposition (2.1.28), we use Lemma (2.1.14). Using the computation in the proof of Proposition (2.1.27) we have for all $n \geq 0$:

$$\sum_{\lambda \in \Lambda(\mathcal{L})} |\hat{\mu}_4(x + \lambda)|^2 = \sum_{d_0 \ldots d_{n-1} \in \Lambda(\mathcal{L})} \prod_{j=1}^{n} m\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^j}\right) \times \sum_{\lambda \in \Lambda(\mathcal{L}_{d_0 \ldots d_{n-1}})} \left|\hat{\mu}_4\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^n} + \lambda\right)\right|^2$$

$$\geq \sum_{d_0 \ldots d_{n-1} \in \Lambda(\mathcal{L}) \setminus \mathcal{V}(\mathcal{S})} \prod_{j=1}^{n} m\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^j}\right) \sum_{\lambda \in \Lambda(\mathcal{L}_{d_0 \ldots d_{n-1}})} \left|\hat{\mu}_4\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^n} + \lambda\right)\right|^2 = (*).$$

Since $\Lambda(\mathcal{L}_{d_0 \ldots d_{n-1}})$ is a spectrum for all $d_0 \ldots d_{n-1}$ not in $\mathcal{V}(\mathcal{S})$, with Lemma (2.1.14) we obtain

$$(*) = \sum_{\substack{d_0 \ldots d_n - 1 \\ \in \Lambda(\mathcal{L}) \setminus \mathcal{V}(\mathcal{S})}} \prod_{j=1}^{n} m\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^j}\right)$$

$$= 1 - \sum_{\substack{d_0 \ldots d_{n-1} \\ \in \Lambda(\mathcal{L}) \cap \mathcal{V}(\mathcal{S})}} \prod_{j=1}^{n} m\left(\frac{x + d_0 + \cdots + 4^{n-1}d_{n-1}}{4^j}\right) = (**).$$

We used (10) for the previous equality.

We use the notation $\omega = d_0(\omega)d_1(\omega) \ldots$ We have then with (26),

$$(**) = 1 - \sum_{\omega\in\mathcal{S}} \prod_{j=1}^{n} m\left(\frac{x + d_0(\omega) + \cdots + 4^{n-1}d_{n-1}(\omega)}{4^j}\right) \to 1$$

Example (2.1.29)[60]: We construct an example of a spectral labeling $\mathcal{L}$ such that $\Lambda(\mathcal{L})$ is a spectrum for μ_4 but $\mathcal{L}$ does not satisfy the conditions of Theorem (2.1.17).

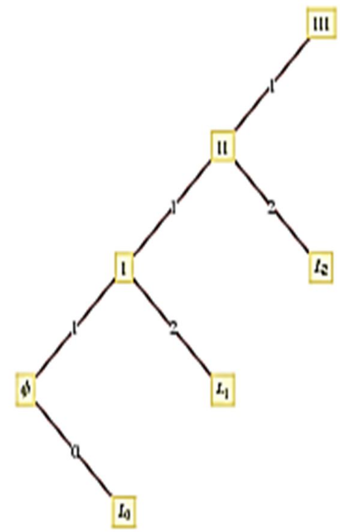

Figure 2[60]: A spectral labeling which gives a spectrum but does not satisfy the conditions of Theorem (2.1.17).

For this pick an infinite path in the binary tree and label it with 111 ....

Let $\mathcal{L}_0$ be the spectral labeling which uses $\{0,1\}$ at each branch. We know $\Lambda(\mathcal{L}_0)$ is a spectrum. Let $\mathcal{L}_n$ be the spectral labeling which uses $\{1,2\}$ for first n levels in the tree and $\{0,1\}$ for the rest. Using Proposition (2.1.27), we have that $\Lambda(\mathcal{L}_n)$ is a spectrum. We label the edges in the binary tree as follows. At the root, we already have one label 1. We use 0 for the other edge, and we label the subtree with root 0 using $\mathcal{L}_0$. At the vertex $\underbrace{1\dots1}_{n\text{ times}}$, we already have one label 1. We use 2 for the other edge, and we label the subtree with root $\underbrace{1\dots1}_{n\text{ times}}$ using $\mathcal{L}_n$.

Doing this for all n, we get a spectral labeling $\mathcal{L}$. Proposition (2.1.28) shows that $\Lambda(\mathcal{L})$ is a spectrum for μ_4. Clearly $\mathcal{L}$ does not satisfy the conditions of Theorem (2.1.17), because for any $P \geq 0$, if we take the vertex $\underbrace{1\dots1}_{n\text{ times}}$, any path from this vertex has to go through a barrage of at least $P + 1$ twos, before it can end times in $\underline{0}$.

Section (2.2): Spectral Moran Measures:

For μ be a Borel probability measure with compact support in $\mathbb{R}^d$. One of basic problems in harmonic analysis associated μ is whether there exists a countable set Λ such that $E(\Lambda) = \{e^{2\pi i\langle x,\lambda\rangle}: \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. If there exists such a set Λ, we say μ is a spectral measure and Λ is a spectrum of μ. This problem has a long history. The first general conjecture on this issue, was due to Fuglede [93].

Conjecture (2.2.1)[83]: Let Ω be a Borel subset of $\mathbb{R}^d$ with finite Lebesgue measure. Then the Lebesgue measure restricted on Ω is a spectral measure if and only if Ω is a translation tile, i.e., there exists a set Γ in $\mathbb{R}^d$ such that $\{\Omega + t: t \in \Gamma\}$ is a partition of $\mathbb{R}^d$ (up to Lebesgue measure zero).

Although the conjecture has been proved to be false in both directions in dimension 3 and in higher dimensions [91,98,108], it is still suggestive in the research of spectral measure theory. In many cases, the existence of spectral measures is equivalent to the existence of tiles. In dimensions 1 and 2, the conjecture is still open in both directions.

The second general result was given by He, Lai and Lau [94]; they showed that if μ is a frame spectral measure, then μ is absolutely continuous or singularly continuous with respect to the Lebesgue measure, or a finite counting measure (spectral measures are frame spectral ones). It is known that the problem with respect to a finite counting measure is either very easy or very difficult [94]. Dutkay and Lai recently studied the frame spectral measures when μ is absolutely continuous [89,102]. We concern the singular measures.

There exists a big difference in the spectral theory between absolutely continuous measures and singular measures (see e.g. [84-86,88,89,93,91,94,99-103,108]). The first example of singularly spectral measure was given by Jorgensen and Pedersen [97]. They showed that the 4th Cantor measure is a spectral measure. Later on, many concentrated their work on investigating self-similar/self-affine measures which are generated by iterated function systems (IFS) [90] and the construction of their spectrums [85-87,94,95,100,104,105,107]. An IFS is a family of contraction functions $\{f_i(x)\}_{i=1}^{n}$, which determines a unique nonempty compact set T, called an attractor, and a Borel probability measure μ supported on T satisfying

$$T = \bigcup_{i=1}^{n} f_i(T),\ \mu(\cdot) = \sum_{i=1}^{n} p_i \mu \circ f_i^{-1}(\cdot),$$

where $\{p_i\}_{i=1}^{n}$ are called probability weights, that is, $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. The 4 th Cantor measure mentioned before is generated by the IFS $f_1(x) = \frac{x}{4}, f_2(x) = \frac{x+2}{4}$ with equal probability weight.

In $\mathbb{R}$, a large class of self-similar measures have been proved to be spectral measures by Laba and Wang [100]. Let $\left\{f_d(x) = \frac{1}{b}(x + d)\right\}_{d\in\mathcal{D}}$ be an IFS, where $b \geqslant 2$ is an integer and $\mathcal{D} \subset \mathbb{Z}$ is a digit set with $0 \in \mathcal{D}$. In this case the corresponding self-similar measure with equal weight probability is denoted by $\mu_{b,\mathcal{D}}$. Eaba and Wang showed that, if $\mathcal{D}$ is an integer tile, i.e., there exists $\Gamma \subset \mathbb{Z}$ such that $\{\Gamma + d\}_{d\in\mathcal{D}}$ is a partition of $\mathbb{Z}$, and the cardinality of $\mathcal{D}$ (denoted by #\mathcal{D}) has no more than two distinct prime factors, then there exists N such that $\#\mathcal{D}$ is a factor of N and $\mu_{N,\mathcal{D}}$ is a spectral measure.

A pair $(b, \mathcal{D})$ is called admissible if there exists a finite set $\mathcal{C} \subset \mathbb{Z}$ with $\#\mathrm{C} = \#\mathcal{D} = q$ such that the matrix $\left[q^{-1/2} e^{2\pi i dc/b}\right]_{d\in\mathcal{D},c\in\mathcal{C}}$ is unitary (usually $(b^{-1}\mathcal{D}, \mathcal{C})$ is called a compatible pair). Eaba and Wang [100] proved that $\mu_{b,\mathcal{D}}$ is a spectral measure if $(b, \mathcal{D})$ is admissible.

We consider the following more general function system: let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integer numbers with all $b_n \geqslant 2$ and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $0 \in \mathcal{D}_n \subset \mathbb{N}$ for each $n \geqslant 1$. We call the function system $\left\{f_{k,d}(x) = b_k^{-1}(x + d): d \in \mathcal{D}_k\right\}_{k=1}^{\infty}$ a Moran IFS, which is a generalization of an IFS.

Let δ_a be the Dirac measure and denote

$$\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$$

for a finite set E. For the completeness, we introduce the following known theorem (see, e.g. [107]).

The measure μ_T in Theorem (2.2.3) is called a Moran measure. In order to construct a spectral measure from a Moran IFS $\{f_{k,d}(x) = b_k^{-1}(x+d): d \in \mathcal{D}_k\}_{k=1}^{\infty}$, we assume that all $(b_k, \mathcal{D}_k)$ are admissible according to Proposition (2.2.2). In this case, it was called an infinite compatible tower by Strichartz [107] (note that the choices of $(b_k, \mathcal{D}_k)$ for $k \geqslant 1$ are finite in Strichartz's case). Under some other conditions, he proved that the measure μ_T in Theorem (2.2.3) is a spectral measure for an infinite compatible tower. Many such examples can be found in [94,107]. However, all these examples are indeed the convolution of a counting measure supported on a finite set and a self-similar spectral measure. Here we show that

If $\sup\{x: x \in b_k^{-1}\mathcal{D}_k, k \geqslant 1\} < \infty$, we believe that the measure μ_T given in Theorem (2.2.3) is a spectral measure if and only if all $(b_k, \mathcal{D}_k)$ are admissible.

To summarize, there are three methods to show a measure to be a spectral one. The first one is due to Jorgensen and Pedersen [97]. They proved it by showing that the Ruelle transfer operator has a unique solution. The second is due to Strichartz [106], who proved it by approximation in terms of the weak convergence of measures. The third is due to Dutkay et al. [86] and Dai et al. [85]. They proved it by constructing a concrete spectral set. The proof of Theorem (2.2.5) is inspired by the ideas from [85,92].

To prove Proposition (2.2.2), we define the mask function of a finite set A in $\mathbb{R}$ by

$$m_A(\xi) = \frac{1}{\#A}\sum_{a\in A} e^{-2\pi i a\xi}.$$

As usually, the Fourier transform of a probability measure μ in $\mathbb{R}$ is defined by

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}\, d\mu(x).$$

Then $m_A(\xi) = \hat{\delta}_A(\xi)$.

Proposition (2.2.2)[83]: Let $\mathcal{D} = \{0,1,\dots,q-1\}$ be a digit set with $q > 1$. Then $(b,\mathcal{D})$ is admissible if and only if q is a factor of b.

Proof. To show the sufficiency, we write $b = qr$ and choose $\mathcal{C} = \{0, r, 2r, \dots, (q-1)r\}$. It is easy to see that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. Hence $(b,\mathcal{D})$ is admissible.

Conversely, let $\gcd(q,b) = d$ and $q = dq_1, b = db_1$. If $(b,\mathcal{D})$ is admissible, then there exists $\mathcal{C} \subset \mathbb{Z}$ with $\#\mathcal{C} = \#\mathcal{D}$ and $0 \in \mathcal{C}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. Moreover $m_{\mathcal{D}}\big(b^{-1}(c_j - c_k)\big) = 0$ for all $0 \leqslant j < k \leqslant q-1$. It is known that, if $\mathcal{C} \equiv \mathcal{C}'(\bmod b)$, then $(b^{-1}\mathcal{D}, \mathcal{C}')$ is a compatible pair too. So we can assume that all c_k satisfy $0 = c_0 < \cdots < c_{q-1} < b$. For $1 \leqslant k < q$, we have

$$|m_{\mathcal{D}}(b^{-1}c_k)| = \left|\frac{\sin b^{-1}qc_k\pi}{q\sin b^{-1}c_k\pi}\right| = 0.$$

This yields $c_k = \alpha_k b_1$ for $1 \leqslant k < q$, where all α_k are integers and $0 < \alpha_1 < \cdots < \alpha_{q-1}$. Hence $\alpha_{q-1} \geqslant q-1$. By the assumption that all $c_k < b$, we have $b > \alpha_{q-1}b_1 \geqslant (q-1)b_1$, and then $d > q-1$. On the other hand, $d = \gcd(q,b)$, thus $d = q$.

For the sake of convenience, we introduce some notations from symbolic dynamical system. Let $\{\mathcal{D}_n\}_{n=0}^{\infty}$ be a sequence of digit sets in $\mathbb{N}$. Denote $D^0 = \{\vartheta\}$ and

$$D^n = \{\sigma_1\sigma_2\cdots\sigma_n : \sigma_j \in \mathcal{D}_j, 1 \leqslant j \leqslant n\}$$

for $n \geqslant 1$. Then the collection of all finite words is $D^* = \bigcup_{n=0}^{\infty} D^n$ and the set of all infinite words is denoted by

$$D^\infty = \{\sigma_1\sigma_2\cdots : \sigma_j \in \mathcal{D}_j, j \geqslant 1\}.$$

Theorem (2.2.3)[83]: Let $\{f_{k,d}(x) = b_k^{-1}(x+d) : d \in \mathcal{D}_k\}_{k=1}^{\infty}$ be a Moran IFS. Suppose that $\sup\{x : x \in b_k^{-1}\mathcal{D}_k, k \geqslant 1\} < \infty$, then the sequence of measures

$$\mu_k = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1b_2\cdots b_k)^{-1}\mathcal{D}_k}$$

converges to a Borel probability measure μ_T with compact support T in a weak sense, where * is the convolution sign and

$$T = \sum_{k=1}^{\infty} (b_1b_2\cdots b_k)^{-1}\mathcal{D}_k = \left\{\sum_{k=1}^{\infty} (b_1b_2\cdots b_k)^{-1}d_k : d_k \in \mathcal{D}_k, k \geqslant 1\right\}.$$

Proof. Let $B(0,r)$ be the open interval $(-r,r)$ in $\mathbb{R}$. Then

$$f_{k,d}(B(0,r)) \subseteq B(0,r), \text{ for } d \in \mathcal{D}_k \text{ and } k \geqslant 1,$$

where r satisfies that $r \geqslant 2\sup\{x : x \in b_k^{-1}\mathcal{D}_k, k \geqslant 1\}$.

For any $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in D^n$, we write $f_\sigma(x) = f_{1,\sigma_1} \circ f_{2,\sigma_2} \circ \cdots \circ f_{n,\sigma_n}(x)$, i.e., $f_\sigma(x)$ is the composition of $\{f_{i,\sigma_i}(x)\}_{i=1}^{n}$. Then it is casy to check that

$$T = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in D^n} f_\sigma(B(0,r))$$

and thus it is a compact set.

Recall that $\mu_n = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}\mathcal{D}_n}$. Then

$$\hat{\mu}_n(\xi) = \hat{\delta}_{\mathcal{D}_1}(b_1^{-1}\xi)\hat{\delta}_{\mathcal{D}_2}((b_1b_2)^{-1}\xi)\cdots\hat{\delta}_{\mathcal{D}_n}((b_1b_2\cdots b_n)^{-1}\xi).$$

By a standard check, the product converges uniformly on each compact set in complex space to an entire function $f(z) = \prod_{n=1}^{\infty}\hat{\delta}_{\mathcal{D}_n}((b_1b_2\cdots b_n)^{-1}z)$. By Levy's continuity theorem [96, p. 167], there exists a probability measure μ such that $\hat{\mu}(x) = f(x)$ and μ_n converges weakly to μ. Moreover, the support of μ is the compact set T.

Let Λ be a countable set. The following well-known lemma, which was proved in [97], gives a criterion for Λ to be a spectrum of a probability measure v.

Theorem (2.2.4)[83]: Let v be a Borel probability measure in $\mathbb{R}$ with compact support. Then a countable set Γ is a spectrum for $L^2(v)$ if and only if

$$Q(\xi) := \sum_{\gamma\in\Gamma} |\hat{v}(\xi+\gamma)|^2 \equiv 1, \text{ for } \xi \in \mathbb{R}.$$

Moreover, if Γ is a bi-zero set of v, i.e., $(\Gamma - \Gamma)\setminus\{0\} \subseteq \mathcal{Z}(\hat{v}) := \{\xi : \hat{v}(\xi) = 0\}$, then $Q(z)$ is an entire function.

We denote $\mathcal{D}_n = \{0,1,\dots,q_n - 1\}$ with $q_n > 1$ and $q_n \mid b_n$.

Theorem (2.2.5)[83]: For any $k \geqslant 1$, let $\mathcal{D}_k = \{0,1,\dots,q_k - 1\}$ with $q_k > 1$ and $q_k \mid b_k$. Let $\{f_{k,d}(x) = b_k^{-1}(x+d) : d \in \mathcal{D}_k\}_{k=1}^{\infty}$ be a Moran IFS. Then the measure μ_T given in Theorem (2.2.3) is a spectral measure.

Proof. Note that, if $q_2 = b_2$, then we have

$$\begin{aligned}\delta_{b_1^{-1}\{0,1,\dots,q_1-1\}} &* \delta_{(b_1b_2)^{-1}\{0,1,\dots,q_2-1\}} \\ &= \delta_{b_1^{-1}\{0,1,\dots,q_1-1\}+(b_1b_2)^{-1}\{0,1,\dots,q_2-1\}} \\ &= \delta_{(b_1b_2)^{-1}\{0,1,\dots,q_1q_2-1\}}.\end{aligned} \tag{29}$$

Hence, we prove Theorem (2.2.5) by considering the following two cases:

Case 1. There are only finitely many integers k such that $b_k > q_k$. In particular, if $b_k = q_k$ for $k \geqslant 1$, then

$$\mu_n = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}\mathcal{D}_n} = \delta_{(b_1b_2\cdots b_n)^{-1}\{0,1,\ldots,b_1b_2\cdots b_n-1\}}$$

tends weakly to the Lebesgue measure restricted on [0,1], which is denoted by $\mathcal{L}|_{[0,1]}$, when n goes to infinity. This shows that the limit measure $\mu = \mathcal{L}|_{[0,1]}$ is a spectral measure. By the assumption that there are only finitely many integers k such that $b_k > q_k$, there exists $\ell \geqslant 1$ such that $b_k = q_k$ for $k > \ell$. In this case, we have $\mu = \mu_\ell * \nu$, where

$$\begin{aligned} \nu(E) &= \delta_{(b_1b_2\cdots b_{\ell+1})^{-1}\mathcal{D}_{\ell+1}} * \delta_{(b_1b_2\cdots b_{\ell+1}b_{\ell+2})^{-1}\mathcal{D}_{\ell+2}} * \cdots (E) \\ &= \mathcal{L}|_{[0,1]}\big((b_1b_2\cdots b_\ell)E\big) \end{aligned}$$

for a Borel set E. Next, we will show that

$$\Lambda := \mathcal{D}_1 + b_1\mathcal{D}_2 + \cdots + b_1b_2\cdots b_{\ell-1}\mathcal{D}_\ell + b_1b_2\cdots b_\ell\mathbb{Z}$$

is a spectrum of μ. Note that $\hat{\mu}_\ell(\xi) = \prod_{k=1}^{\ell}\hat{\delta}_{\mathcal{D}_k}((b_1b_2\cdots b_k)^{-1}\xi)$ and

$$\hat{\nu}(\xi) = \hat{\chi}[0,1]((b_1b_2\cdots b_\ell)^{-1}\xi),$$

where $\hat{\chi}_{[0,1]}(\xi) = \int_0^1 e^{-2\pi i\xi x}dx$ and $\chi_{[0,1]}$ is the characteristic function of $[0,1]$. We have

$$\begin{aligned} \sum_{\lambda\in\Lambda}|\hat{\mu}(\xi+\lambda)|^2 &= \sum_{\lambda\in\mathcal{D}_1+b_1\mathcal{D}_2+\cdots+b_1b_2\cdots b_{\ell-1}\mathcal{D}_\ell}\ \sum_{n\in\mathbb{Z}}|\hat{\mu}_\ell(\xi+\lambda+b_1b_2\cdots b_\ell n)|^2 \\ &\qquad \times|\hat{\chi}[0,1]((b_1b_2\cdots b_\ell)^{-1}(\xi+\lambda)+n)|^2 \\ &= \sum_{\lambda\in\mathcal{D}_1+b_1\mathcal{D}_2+\cdots+b_1b_2\cdots b_{\ell-1}\mathcal{D}_\ell}\ \sum_{n\in\mathbb{Z}}|\hat{\mu}_\ell(\xi+\lambda)|^2 \\ &\qquad \left|\hat{\chi}_{[0,1]}((b_1b_2\cdots b_\ell)^{-1}(\xi+\lambda)+n)\right|^2 \\ &= \sum_{\lambda\in\mathcal{D}_1+b_1\mathcal{D}_2+\cdots+b_1b_2\cdots b_{\ell-1}\mathcal{D}_\ell}|\hat{\mu}_\ell(\xi+\lambda)|^2 \\ &= 1. \end{aligned}$$

Hence, the assertion follows.

Case 2. There exist infinitely many integers $k \geqslant 1$ such that $b_k > q_k$. We divide this case into two subcases:

Subcase I. In this part, we always assume that $b_n > q_n$ for $n \geqslant 1$. Then $b_n/q_n = r_n \geqslant 2$.

Note that the measure we focus on is $\mu = \mu_T = \delta_{b_1}^{-1}\mathcal{D}_1 * \delta_{(b_1b_2)^{-1}}\mathcal{D}_2 * \cdots$, and its Fourier transform is

$$\hat{\mu}(\xi) = \prod_{n=1}^{\infty} m_{\mathcal{D}_n}\left(\frac{\xi}{b_1b_2\cdots b_n}\right). \tag{30}$$

Observe that the zero set of $m_{\mathcal{D}_n}$ is

$$\mathcal{Z}(m_{\mathcal{D}_n}) = \{\xi\in\mathbb{R}: m_{\mathcal{D}_n}(\xi) = 0\} = \left\{\frac{a}{q_n}: a\in\mathbb{Z}\setminus q_n\mathbb{Z}\right\}.$$

Hence, the zero set of $\hat{\mu}$ can be expressed explicitly by

$$\begin{aligned} \mathcal{Z}(\hat{\mu}) &- \left\{\frac{b_1b_2\cdots b_n}{q_n}a: n\geqslant 1, a\in\mathbb{Z}\setminus q_n\mathbb{Z}\right\} \\ &= \{b_1b_2\cdots b_{n-1}r_n a: n\geqslant 1, a\in\mathbb{Z}\setminus q_n\mathbb{Z}\}. \end{aligned} \tag{31}$$

Now we construct a countable set $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ in terms of $(\{q_n\}, \{b_n\})$. For any $n \in \mathbb{N}, n \neq 0$, there exists a unique $\sigma = \sigma_1 \cdots \sigma_k \in D^k, \sigma_k \neq 0$, such that

$$n = \sigma_1 + \sigma_2 q_1 + \cdots + \sigma_k q_1 \cdots q_{k-1} = \sum_{j=1}^{k} \sigma_j q_1 \cdots q_{j-1}. \tag{32}$$

We define $\lambda_0 = 0$ and

$$\lambda_n = \sigma_1 r_1 + \sigma_2 r_2 b_1 + \cdots + \sigma_k r_k b_1 \cdots b_{k-1} = \sum_{j=1}^{k} \sigma_j r_j b_1 b_2 \cdots b_{j-1}, \tag{33}$$

where $\sigma = \sigma_1 \cdots \sigma_k$ is the word given in (32). We call $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ the set determined by $(\{q_n\}, \{b_n\})$.

If the Claim (2.2.9) is true, then Theorem (2.2.5) holds for Subcase I. We will prove the Claim (2.2.9) later.

Subcase II. According to (29) and Subcase I, we can assume without loss of generality that $b_1 = q_1$ and $b_k > q_k$ for $k \geqslant 2$. Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of $v = \delta_{b_2^{-1}\mathcal{D}_2} * \delta_{(b_2 b_3)^{-1}\mathcal{D}_3} * \cdots$ given in Subcase I. Now we prove that $\mathcal{D}_1 + b_1\Lambda$ is a spectrum of μ. Note that $\hat{\mu}(\xi) = m_{\mathcal{D}_1}(b_1^{-1}\xi)\hat{v}(b_1^{-1}\xi)$ and

$$\begin{aligned} Q(\xi) &= \sum_{\lambda \in \mathcal{D}_1 + b_1\Lambda} |\hat{\mu}(\xi + \lambda)|^2 \\ &= \sum_{i=0}^{q_1-1} \sum_{n=0}^{\infty} \left|m_{\mathcal{D}_1}\left(b_1^{-1}(\xi + i + b_1\lambda_n)\right)\right|^2 \left|\hat{v}\left(b_1^{-1}(\xi + i + b_1\lambda_n)\right)\right|^2 \\ &= \sum_{i=0}^{q_1-1} \left|m_{\mathcal{D}_1}(b_1^{-1}(\xi + i))\right|^2 \equiv 1. \end{aligned}$$

By Theorem (2.2.4), Theorem (2.2.5) for Subcase II follows.

We will concentrate on proving the Claim (2.2.9). The following theorem says that $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ is an orthogonal set.

Theorem (2.2.6)[83]: The countable set $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ determined by $(\{q_n\}, \{b_n\})$ is an orthogonal set for $L^2(\mu)$.

Proof. It is equivalent to show that Λ is a bi-zero set. For any $n > n' \geqslant 0$, there exist $\sigma = \sigma_1 \cdots \sigma_k$ and $\sigma' = \sigma_1' \cdots \sigma_{k'}' \in D^*$ such that

$$n = \sum_{j=1}^{k} \sigma_j q_1 \cdots q_{j-1},\ n' = \sum_{j=1}^{k'} \sigma_j' q_1 \cdots q_{j-1}.$$

If $k > k'$, let $\sigma_j' = 0$ for $k' + 1 \leqslant j \leqslant k$. Then, by the definition of Λ, we have

$$\lambda_n = \sum_{j=1}^{k} \sigma_j r_j b_1 b_2 \cdots b_{j-1},\ \lambda_{n'} = \sum_{j=1}^{k} \sigma_j' r_j b_1 b_2 \cdots b_{j-1}.$$

Let s be the first index such that $\sigma_s \neq \sigma_s'$. Then

$$\lambda_n - \lambda_{n'} = b_1 b_2 \cdots b_{s-1}(\sigma_s r_s - \sigma_s' r_s + M b_s) = \frac{b_1 \cdots b_s(\sigma_s - \sigma_s' + M q_s)}{q_s}$$

for some integer M. $\sigma_s, \sigma_s' \in \mathcal{D}_s$ implies that $q_s \nmid (\sigma_s - \sigma_s')$. Hence, $\lambda_n - \lambda_{n'} \in \mathcal{Z}(\hat{\mu})$ by (31).

Next we show the completeness of $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$. Denote $\mu_N = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \cdots * \delta_{(b_1 b_2 \cdots b_N)^{-1}\mathcal{D}_N}, v_N = \delta_{b_{N+1}^{-1}\mathcal{D}_{N+1}} * \delta_{(b_{N+1} b_{N+2})^{-1}\mathcal{D}_{N+2}} * \cdots$, then

$$\hat{\mu}_N(\xi) = \prod_{j=1}^{N} m_{\mathcal{D}_j}\left(\frac{\xi}{b_1 b_2 \cdots b_j}\right),\ \hat{v}_N(\xi) = \prod_{j=1}^{\infty} m_{\mathcal{D}_{N+j}}\left(\frac{\xi}{b_{N+1} b_{N+2} \cdots b_{N+j}}\right).$$

It is known that μ_N converges weakly to μ when N tends to infinity (Theorem (2.2.3)) and

$$\hat{\mu}(\xi) = \hat{\mu}_N(\xi)\hat{v}_N\left(\frac{\xi}{b_1 b_2 \cdots b_N}\right). \tag{34}$$

For the measure μ_N, we have

Theorem (2.2.7)[83]: Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be the orthogonal set determined by $(\{q_n\},\{b_n\})$. Then for all $N \geqslant 1$,

$$\sum_{n=0}^{q_1 \cdots q_N - 1} |\hat{\mu}_N(\xi + \lambda_n)|^2 \equiv 1. \tag{35}$$

Proof. First, we show the mutual orthogonality of $\{\lambda_n\}_{n=0}^{q_1 \cdots q_N - 1}$ for $L^2(\mu_N)$. Note that

$$\hat{\mu}_N(\xi) = \prod_{j=1}^{N} m_{\mathcal{D}_j}\left(\frac{\xi}{b_1 \cdots b_j}\right).$$

For any $n, n' \in \{0,1, \dots, q_1 \cdots q_N - 1\}$ with $n \neq n'$, by (32), we have

$$n = \sum_{j=1}^{N} \sigma_j q_1 \cdots q_{j-1} \text{ and } n' = \sum_{j=1}^{N} \sigma_j' q_1 \cdots q_{j-1},$$

where $\sigma = \sigma_1 \cdots \sigma_N, \sigma' = \sigma_1' \cdots \sigma_N' \in D^N, \sigma_j$ and σ_j' may be zero for $1 \leqslant j \leqslant N$. Let $s \leqslant N$ be the first index such that $\sigma_s \neq \sigma_s'$. Then, we can write

$$\lambda_n - \lambda_{n'} = b_1 b_2 \cdots b_{s-1}(\sigma_s r_s - \sigma_s' r_s + M b_s)$$

for some integer M. It follows from the periodicity of the exponential function $e^{2\pi i x}$ that

$$m_{\mathcal{D}_s}\left(\frac{\lambda_n - \lambda_{n'}}{b_1 b_2 \cdots b_s}\right) = m_{\mathcal{D}_s}\left(\frac{\sigma_s - \sigma_s'}{q_s}\right) = 0.$$

Hence, $\hat{\mu}_N(\lambda_n - \lambda_{n'}) = 0$. Since $\{\lambda_n\}_{n=0}^{q_1 \cdots q_N - 1}$ has exactly $q_1 \cdots q_N$ elements, so it is a spectrum for $L^2(\mu_N)$. The result follows by Theorem (2.2.4).

The next lemma says that $\hat{v}_N(\xi)$ is uniformly bounded in modulus from below for $N \geqslant 1$ and $0 < \xi \leqslant 1$.

Lemma (2.2.8)[83]: For any $0 < \xi \leqslant 1$ and $N \geqslant 1, |\hat{v}_N(\xi)|^2 \geqslant \alpha$ for some positive constant α.

Proof. Recall that

$$|\hat{v}_N(\xi)| = \prod_{j=1}^{\infty} \left| m_{\mathcal{D}_{N+j}}\left(\frac{\xi}{b_{N+1} \cdots b_{N+j}}\right)\right|.$$

By simple calculation, we have $\sin x/x \geqslant 1 - x^2/6$ for $x > 0$. Since $b_k/q_k = r_k \geqslant 2$ for $k \geqslant 1$, then

$$\left|m_{\mathcal{D}_n}(b_n^{-1}\xi)\right| = \frac{\sin b_n^{-1} q_n \pi\xi}{q_n \sin \pi b_n^{-1}\xi} \geqslant \frac{\sin b_n^{-1} q_n \pi\xi}{b_n^{-1} q_n \pi\xi} \geqslant 1 - \frac{1}{6}(b_n^{-1} q_n \pi\xi)^2 \tag{36}$$

and for $j \geqslant 1$

$$\frac{q_{N+j}\pi\xi}{b_{N+1}\cdots b_{N+j}} \leqslant \frac{\pi}{2^j}.$$

Consequently,

$$|\hat{v}_N(\xi)| \geqslant \prod_{j=1}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^j}\right)^2\right) > 0,$$

which follows from the convergence of the sum $\sum_{j=1}^{\infty}\frac{1}{6}\left(\frac{\pi}{2^j}\right)^2$.

From the proof of Lemma (2.2.8), we have the following inequality which will be used in the proof of the Claim (2.2.9):

$$|\hat{v}_N(\xi)| \geqslant \prod_{j=1}^{\infty}\left(1-\frac{1}{6}\left(\frac{q_{N+j}\pi\xi}{b_{N+1}\cdots b_{N+j}}\right)^2\right),\ \forall 0<\xi\leqslant 1. \qquad (37)$$

Now we can prove the Claim (2.2.9). The measure $\mu:=\mu_T$ is given in Theorem (2.2.3). $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$ is the orthogonal set determined by $(\{q_n\},\{b_n\})$, i.e.,

$$\lambda_n=\sum_{j=1}^{k}\sigma_j r_j b_1\cdots b_{j-1},$$

if $n=\sum_{j=1}^{k}\sigma_j q_1\cdots q_{j-1}$, where $\sigma=\sigma_1\cdots\sigma_k\in D^k$

Claim (2.2.9)[83]: The countable set $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$ determined by $(\{q_n\},\{b_n\})$ is an orthonormal basis for $L^2(\mu)$.

Proof. Let

$$Q_N(\xi)=\sum_{n=0}^{q_1\cdots q_N-1}|\hat{\mu}(\xi+\lambda_n)|^2 \text{ and } Q(\xi)=\sum_{n=0}^{\infty}|\hat{\mu}(\xi+\lambda_n)|^2.$$

For any $N\geqslant 1$, using (34), we have the following identity:

$$\begin{aligned} Q_{2N}(\xi) &= Q_N(\xi)+\sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1}|\hat{\mu}(\xi+\lambda_n)|^2 \\ &= Q_N(\xi)+\sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1}|\hat{\mu}_{2N}(\xi+\lambda_n)|^2\left|\hat{v}_{2N}\left(\frac{\xi+\lambda_n}{b_1b_2\cdots b_{2N}}\right)\right|^2. \qquad (38)\end{aligned}$$

Our goal is to prove that $Q(\xi)=\sum_{n=0}^{\infty}|\hat{\mu}(\xi+\lambda_n)|^2\equiv 1$. Since Q is an entire function by Theorem (2.2.6) and Theorem (2.2.4), we only need to determine the value of $Q(\xi)$ for some small values of ξ. Let $0<\xi\leqslant 1$. For $q_1\cdots q_N\leqslant n\leqslant q_1\cdots q_{2N}-1$, we may write λ_n as

$$\lambda_n=\sum_{j=1}^{2N}\sigma_j r_j b_1 b_2\cdots b_{j-1},$$

where $\sigma_j\in\{0,1,\dots,q_j-1\}$. Recall that $b_k=q_k r_k$ and $r_k\geqslant 1$ for $k\geqslant 1$. Then we have

$$\begin{aligned}\frac{\xi+\lambda_n}{b_1b_2\cdots b_{2N}} &\leqslant \frac{r_1+(q_1-1)r_1+(q_2-1)r_2b_1+\cdots+(q_{2N}-1)r_{2N}b_1\cdots b_{2N-1}}{b_1b_2\cdots b_{2N}} \\ &= \frac{b_1+(q_2-1)r_2b_1+\cdots+(q_{2N}-1)r_{2N}b_1\cdots b_{2N-1}}{b_1b_2\cdots b_{2N}} \\ &\leqslant \frac{r_2+(q_2-1)r_2+\cdots+(q_{2N}-1)r_{2N}b_2\cdots b_{2N-1}}{b_2\cdots b_{2N}}\end{aligned}$$

$$\leqslant \frac{r_{2N} + (q_{2N} - 1)r_{2N}}{b_{2N}} = 1. \tag{39}$$

Therefore, using Lemma (2.2.8), Theorem (2.2.7) and the inequality (39), we have

$$\begin{aligned} Q_{2N}(\xi) &\geqslant Q_N(\xi) + \alpha \sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1} |\hat{\mu}_{2N}(\xi + \lambda_n)|^2 \\ &= Q_N(\xi) + \alpha\left(1 - \sum_{n=0}^{q_1\cdots q_N - 1} |\hat{\mu}_{2N}(\xi + \lambda_n)|^2\right). \end{aligned} \tag{40}$$

We now argue by contradiction. Assume that $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ is not a spectrum of μ. Then there exists a $\xi_0, 0 < \xi_0 < 1$, such that $Q(\xi_0) < 1$ and $\hat{\mu}(\xi_0) \neq 0$ since Q is entire and $\hat{\mu}(0) = 1$. Let β be so that $Q(\xi_0) < \beta < 1$. For $0 \leqslant n < q_1 \cdots q_N$, we have, by (39),

$$\frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}} = \frac{1}{b_{N+1}\cdots b_{2N}} \frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_N} \leqslant \frac{1}{b_{N+1}\cdots b_{2N}} \leqslant \frac{1}{4^N}. \tag{41}$$

It follows from (37) that

$$\left|\hat{v}_{2N}\left(\frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}}\right)\right| \geqslant \prod_{j=1}^{\infty}\left(1 - \frac{1}{6}\left(\frac{q_{2N+j}\pi}{b_{2N+1}\cdots b_{2N+j}} \frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}}\right)^2\right). \tag{42}$$

By (41) and the assumption that $b_n \geqslant 4$,

$$\sum_{j=1}^{\infty} \frac{1}{b_{2N+1}\cdots b_{2N+j-1}} \frac{q_{2N+j}\pi}{b_{2N+j}} \frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}} \leqslant \frac{\pi}{2}\frac{1}{4^N}\sum_{j=1}^{\infty} \frac{1}{4j-1} < \frac{\pi}{4^N}.$$

Hence, by the fact that $\ln(1 - x) \geqslant -2x$ for $0 \leqslant x \leqslant 1/2$, (42) and the above inequality, we have

$$\ln\left|\hat{v}_{2N}\left(\frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}}\right)\right| \geqslant -\frac{\pi}{3 \cdot 4^N}$$

uniformly for $0 \leqslant n < q_1 \cdots q_N$. This implies that, if N is large enough, say $N \geqslant M > 0$,

$$\left|\hat{v}_{2N}\left(\frac{\xi_0 + \lambda_n}{b_1 b_2 \cdots b_{2N}}\right)\right|^2 \geqslant \frac{Q(\xi_0)}{\beta}$$

for $0 \leqslant n < q_1 \cdots q_N$. Consequently, for $N \geqslant M$ and $0 \leqslant n < q_1 \cdots q_N$,

$$|\hat{\mu}(\xi_0 + \lambda_n)|^2 = \left|\hat{\mu}_{2N}(\xi_0 + \lambda_n)\hat{v}_{2N}\left(\frac{\xi_0 + \lambda_n}{b_1 \cdots b_{2N}}\right)\right|^2 \geqslant \frac{Q(\xi_0)}{\beta}|\hat{\mu}_{2N}(\xi_0 + \lambda_n)|^2.$$

Combining (40) we have

$$\begin{aligned} Q_{2N}(\xi_0) &\geqslant Q_N(\xi_0) + \alpha\left(1 - \frac{\beta}{Q(\xi_0)} \sum_{n=0}^{q_1\cdots q_N - 1} |\hat{\mu}(\xi_0 + \lambda_n)|^2\right) \\ &\geqslant Q_N(\xi_0) + \alpha(1 - \beta). \end{aligned} \tag{43}$$

Therefore

$$1 \geqslant Q_{2^\ell N}(\xi_0) \geqslant Q_N(\xi_0) + \ell\alpha(1 - \beta) \geqslant \ell\alpha(1 - \beta)$$

for $\ell \geqslant 1$, which is impossible. Hence, Λ must be a spectrum.

Section (2.3): N-Bernoulli Measures:

For μ be a probability measure on $\mathbb{R}^s$ with compact support. For a countable subset $\Lambda \subset \mathbb{R}^s$, we let $e_\Lambda = \{e_\lambda = e^{-2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$. We call μ a spectral measure, and Λ a *spectrum* of μ if e_Λ is an orthogonal basis for $L^2(\mu)$. The existence and nonexistence of a spectrum for μ is a basic problem in harmonic analysis, it was initiated by Fuglede in [122], and has been studied extensively since then [113–119,123,125,128–130,132,133,136,137]. Recently He, Laiand Lau [123] proved that a spectral measure μ must be of pure type, i.e., μ is absolutely continuous or singular continuous with respect to the Lebesgue measure or counting measure supported on a finite set (actually this holds more generally for *frames*).

When μ is the Lebesgue measure restricted on a set K in $\mathbb{R}^s$, it is well-known that the spectral property is closely connected with the tiling property of K, and is known as the Fuglede problem [122,126,128,137]. For continuous singular measures, the first spectral measure was given by Jorgensen and Pedersen [125]: the Cantor measure μ_ρ with contraction ratio $\rho = 1/2k$. There are considerable studies for such measures [114,118,120,124,128, 132,133,136], and a celebrated open problem was to characterize the spectral measures $\mu_\rho, 0 < \rho < 1$ among the Bernoulli convolutions

$$\mu_\rho(\cdot) = 1/2\mu_\rho(\rho^{-1}\,\cdot) + 1/2\mu_\rho(\rho^{-1}\cdot -1)\,.$$

In [124], Hu and Lau showed that μ_ρ admits an infinite orthonormal set if and only if ρ is the n-th root of p/q where p is odd and q is even. The characterization problem was finally completed recently by Dai [113] that the above Cantor measures $\mu_{1/(2k)}$ are the only class of spectral measures among the μ_p.

We study the spectrality of the self-similar measures. Let $0 < \rho < 1, D = \{0, d_1 \dots . d_{N-1}\}$ be a finite set in $\mathbb{R}$, and $\{w_j\}_{j=0}^{N-1}$ a set of probability weights. We call μ a *self-similar measure* generated by (ρ, D) and $\{w_j\}_{j=0}^{N-1}$ if μ is the unique probability measure satisfying

$$\mu(\cdot) = \frac{1}{N}\sum_{j=0}^{N-1} w_j\, \mu\big(\rho^{-1}(\cdot) - d_j\big). \qquad (44)$$

We will use $\mu_{p,N}$ to denote the special case where $\mathcal{D} = \{0 \dots\dots N-1\}$ with uniform weight, i.e.,

$$\mu_{p,N}(\cdot) = \frac{1}{N}\sum_{j=0}^{N-1} \mu_{p,N}\,(\rho^{-1}(\cdot) - j). \qquad (45)$$

The spectral property of such measure was first studied by Dai, He and Lai [114] as a generalization of the Bernoulli convolution in [113] ($\mathcal{D} = \{0,1\}$). Our main result is to extend the characterization of spectral Bernoulli convolution to the class of $\mu_{\rho,N}$ in (45). Our motivation to extend the Bernoulli convolutions to this class of measures is due to a conjecture of Laba and Wang, and also to answer a question on the convolution of spectral measures. We prove

Theorem (2.3.1)[109]: *Let* $0 < \rho < 1$. *Then* $\mu_{\rho,N}$ *is a spectral measure if and only if* $\rho = \frac{1}{q}$ *for some integer* $q > 1$ *and* $N|q$.

The sufficiency of the theorem follows from the same pattern as the Cantor measure in [125] by producing a Hadamard matrix, then construct the canonical spectrum. On the other hand, the proof of the necessity needs more work. We observe that for $\mu_{\rho,N}$ to be a spectral measure, ρ must be an algebraic number. We prove by elimination that each of the following cases can NOT admit a spectrum (for $\frac{p}{q}$, we always assume they have no common factor):

(i) $\rho = \left(\frac{p}{q}\right)^{1/r}$ $for\ some\ r > 1$ (it is an irrational) (Proposition (2.3.5)):

(ii) $\rho \neq \left(\frac{p}{q}\right)^{\frac{1}{r}}$ $for\ any\ r > 1$ and is an irrational (Proposition (2.3.8)):

(iii) ρ $= \frac{p}{q}$ and $1 \leq ged(N.q) < N$ (Proposition (2.3.15)):

(iv) $\rho = \frac{p}{q}$. $p > 1$ and $N|_q$ (Proposition (2.3.19))

Let $\hat{\mu}_{p,N}$ be the Fourier transform of $\mu_{p,N}$, and $\mathcal{Z}(\hat{\mu}_{p,N})$ the zeros of $\hat{\mu}_{\rho,N}$. The proof is based on the criteria in Theorem (2.3.3) and Lemma (2.3.4), and the technique is to make use of some explicit expressions of $\mathcal{Z}(\hat{\mu}_{\rho,N})$, and that $\Lambda - \Lambda \subset \mathcal{Z}(\hat{\mu}_{\rho,N})$ for any exponential orthogonal set Λ.

The most subtle part of the proof is (iv). As is known, there is certain canonical q-adic expansion of λ in a spectrum Λ (see (46)), and there are also others. In [118], Dutkay et al. treated the 4-adic expansions as in a symbolic space Ω_2^*, and considered certain maps on Ω_2^* to $\mathbb{Z}^+$ to preserve the maximal orthogonality property. This idea was refined and investigated by Dai, He and Lai [114] by replacing the q-adic expansion on $\mathbb{Z}$ with digits in $C = \{-1,0,,,,,,,,,,,q-2\}$. Let : $\Omega_N^* \to \mathcal{C}$ be a *selection map* as defined in Definition (2.3.11) (it was called a *maximal map* in [114]), and let $\iota^*(\mathrm{i}) = \sum_{n=1}^{\infty} \iota(\mathrm{i}0^{\infty}|_n)q^{n-1}$. The importance of the selection map is in the following theorem (Theorem (2.3.13)), which also has independent interest.

Theorem (2.3.2)[109]: *Suppose $\rho = p/q$ and $N|q$. Then $\Lambda \subset \mathcal{Z}(\hat{\mu}_{\rho,N})$ defines a maximal exponential orthogonal subset in $L^2(\mu_{\rho,N})$ if and only if there exist $m_0 \geq 1$ and a selection map ι such that* $\Lambda = \rho^{-m_0}N^{-1}(\iota^*(\Omega_N^{\iota}))$.

We set up the notations, the basic criteria of spectrum, and the element properties of the zero set $\mathcal{Z}(\mu_{p,N})$. We settle cases (i), (ii). For the case $\rho = p/q$, we give a detailed study of the maximality of Λ such that $\Lambda - \Lambda \subset \mathcal{Z}(\mu_{\rho,N})$, which is used to consider cases (iii) and (iv). We give some remarks of the spectral measures and the remaining questions.

We assume that μ is a probability measure with compact support. The Fourier transformation of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}\, d\mu(x)\,.$$

Let $\mathcal{Z}(\hat{\mu}) := \{\xi : \hat{\mu}(\xi) = 0\}$ be the set of zeros of $\hat{\mu}$. We denote the complex exponential function $e^{-2\pi i\lambda(\cdot)}$ by e_λ. Note that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set in $L^2(\mu)$ if and only if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$; Λ is called a *spectrum* of μ if $\{e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for $L^2(\mu)$. For $\xi \in \mathbb{R}$, we let

$$Q(\xi) = \sum_{\lambda\in\Lambda} |\, \hat{\mu}(\xi + \lambda)|^2$$

The following theorem is a basic criterion for the spectrality of μ [125].

Theorem (2.3.3)[109]: *Let μ be a probability measure with compact support, and let $\Lambda \subset \mathbb{R}$ be a countable subset. Then*

(i) $\{e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal set of $L^2(\mu)$ if and only if $Q(\xi) \leq 1$ for $\xi \in \mathbb{R}$; and

(ii) it is an orthonormal basis if and only if $Q(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.

We use the notation Λ to denote a subset such that $0 \in \Lambda$ and $\Lambda\backslash\{0\} \subset \mathcal{Z}(\hat{\mu})$. We say that Λ is a *bi-zero set* of μ if $(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu})$, and call it a *maximal* bi-zero set if it is maximal in $\mathcal{Z}(\hat{\mu})$ to have the set difference property. Clearly that Λ is a bi-zero set is equivalent to $\{e_\lambda: \lambda \in \Lambda\}$ is an orthogonal subset of $L^2(\mu)$. An exponential orthonormal basis corresponds to a maximal bi-zero set, but the converse is not true. In fact we will give a characterization of the maximal bi-zero sets of $\mu_{p,N}$ for the case $p = \frac{p}{q}$ and $N|q$, and establish the spectrality through Theorem (2.3.3)(ii).

As a simple consequence of Theorem (2.3.3), we have the following useful lemma.

Lemma (2.3.4)[109]: *Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures $\mu_i, i = 0,1$, and they are not Dirac measures. Suppose that Λ is a bi-zero set of μ_0, then Λ is also a bi-zero of μ, but cannot be a spectrum of μ.*

Proof. Note that μ_i is not a Dirac measure is equivalent to $|\hat{\mu}_i(\xi)| \not\equiv 1$. Since $\hat{\mu}_0(0) = 1$, there exists ξ_0 such that $|\hat{\mu}_0(\xi_0)| \neq 0$ and $|\hat{\mu}_1(\xi_0)| < 1$. Hence by Theorem (2.3.3)(i),

$$Q(\xi_0) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi_0 + \lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_0(\xi_0 + \lambda)|^2 |\hat{\mu}_1(\xi_0 + \lambda)|^2 < \sum_{\lambda\in\Lambda} |\hat{\mu}_0(\xi_0 + \lambda)|^2 \leq 1.$$

The result follows by Theorem (2.3.3)(i) and (ii).

Now we consider the self-similar measure $\mu_{\rho,N}$ in Theorem (2.3.1). It was proved in [114] that if $p = 1/q$ and $N|q$, then $\mu_{\rho,N}$ is a spectral measure. The proof is quite simple. In fact as $N|q$, we write $q = Nr$. If $r = 1$, then μ is just the Lebesgue measure on the unit interval, and the result is trivial. If $r > 1$, observe that for $\mathcal{D} = \{0,\dots, N-1\}$ and $\Gamma = r\{0, \dots, N-1\}$, the matrix

$$H := \left[\frac{1}{\sqrt{N}} e^{2\pi i \frac{\dot{x}k}{q}}\right]_{i\in\mathcal{D},k\in\Gamma} = \left[\frac{1}{\sqrt{N}} e^{2\pi i \frac{ij}{N}}\right]_{0\leq i,j\leq N-1}$$

is a Hadamard matrix (i.e., $HH^* = I$). This shows that $(q^{-1}\mathcal{D}, \Gamma)$ is a *compatible pair*, hence $\mu_{1/q,N}$ is a spectral measure [128], and the canonical spectrum is given by

$$\Lambda = \left\{\sum_{j=0}^{k} a_j\, q^j : a_j \in \Gamma, k \geq 0\right\} \tag{46}$$

(note that the spectrum is not unique). Our main task is to prove the converse. The strategy is to eliminate all the possible cases so that the only admissible case is $\rho = 1/q$ with $N|q$.

Recall that the Fourier transform of $\mu_{\rho,N}$ has the following expression

$$\hat{\mu}_{\rho,N}(\xi) = M_N(\rho\xi)\hat{\mu}_{\rho,N}(\rho\xi) = \prod_{k=1}^{\infty} M_N\,(\rho^k\xi)$$

where $M_N(\xi) = \frac{1}{N}\sum e^{-2\pi i j\xi}$ is the *mask polynomial* of $\mathcal{D}$. It is clear that $|M_N(\xi)| = |\frac{\sin N\pi\xi}{N \sin \pi\xi}|$, and the zeros of $M_N(\xi)$ are $a/N, a \in \mathbb{Z}\backslash\{0\}, N(a$. Let

$$\mathcal{Z}(M_N)=\left\{\frac{a}{N}: a\in\mathbb{Z}\backslash\{0\}, N\middle| a\right\}=\left\{\frac{a}{N}: a\in\mathbb{Z}\backslash N\mathbb{Z}\right\}. \tag{47}$$

It follows from the infinite product expression of $\hat{\mu}_{\rho,N}$ that

$$\mathcal{Z}(\hat{\mu}_{\rho,N})=\left\{\rho^{-k}\frac{a}{N}: k\geq 1, a\in\mathbb{Z}\backslash N\mathbb{Z}\right\}. \tag{48}$$

For distinct $\lambda_1,\lambda_2\in\Lambda\backslash\{0\}$, (48) and the bi-zero property of Λ imply that

$$\rho^{-k_1}\frac{a_1}{N}-\rho^{-k_2}\frac{a_2}{N}=\rho^{-k}\frac{a}{N}.$$

Hence ρ is an algebraic number. Recall that an *algebraic number* is a root of an integer equation of the form $c_0x^n+c_1x^{n-1}+\cdots+c_n\in\mathbb{Z}[x]$, and it is called an *algebraic integer* if $c_0=1$.

For any integer $r\geq 1$, let

$$\mathbb{Q}^{1/r}=\{\rho=u^{1/r}: 0<u<1\ is\ a\ rational\}.$$

We make the convention that the above r is the smallest integer for $\rho=u^{1/r}$ (for example: $\rho=\left(\frac{4}{9}\right)^{1/4}=\left(\frac{2}{3}\right)^{1/2}$, we will take $r=2$). Hence for $\rho\in\mathbb{Q}^{1/r}, r>1$, then ρ is an irrational.

Proposition (2.3.5)[109]: *Let* $\in\mathbb{Q}^{1/r}, r>1$, *then* $\mu_{p,N}$ *is not a spectral measure.*

Proof. Let $\rho=u^{1/r}$ where $0<u<1$ is a rational. We write

$$\hat{\mu}_{p,N}(\xi)=\prod_{k=1}^{\infty}M_N(\rho^k\xi)=\prod_{k=0}^{\infty}\prod_{i=1}^{r}M_N(u^k\rho^i\xi).$$

Define the probability measures $\mu_i(\cdot)=\mu_{u,N}(u\rho^{-i}\cdot), 1\leq i\leq r$. Then

$$\hat{\mu}_i(\xi)=\coprod_{k=0}^{\infty}M_N(u^k\rho^i\xi)$$

for $1\leq i\leq r$. Then μ_β is the convolution of $\mu_i, i=1,2,\ldots,r$. Let Λ be a bi-zero set of $\mu_{\rho,N}$. We claim that Λ is also a bi-zero set of μ_i for some i. Indeed, let $\lambda_j=\rho^{-k_jr-i_j}a_j/N, 1\leq i_1,i_2\leq r, j=1,2$, be any two distinct elements in Λ. The bi-zero property of Λ for μ implies that

$$\rho^{-k_1r-i_1}a_1/N-\rho^{-k_2r-i_2}a_2/N=p^{-kr-i}a/N.$$

Without loss of generality assume $k_1,k_2\geq k$, then we have $u^{(k_1-k)}\rho^{i_1-i}a_1-u^{(k_2-k)}\times\rho^{i_2-i}a_2=a$. This implies $i_1=i_2=i$ because the minimal polynomial of ρ is x^r-u. Hence Λ is a bi-zero set of μ_i, and by Lemma (2.3.4), Λ cannot be a spectrum of μ.

Next we consider $\rho\notin\mathbb{Q}^{1/r}, r>1$. We need two lemmas.

Lemma (2.3.6)[109]: *Suppose* Λ *is an infinite bi-zero set of* $\mu_{p,N}$ *with* $0\in\Lambda$. *Then* $\rho\notin\mathbb{Q}^{1/r}$ *for all* $r\geq 1$ *implies that* ρ *is an algebraic integer.*

Proof. Since $\Lambda\backslash\{0\}\subset\mathcal{Z}(\hat{\mu}_{\rho,N})$, we denote $\Lambda=\{\lambda_k\}_{k=0}^{\infty}$ so that $\lambda_0=0$ and $\lambda_k=p^{-n_k}\frac{a_k}{N}$, where $N|a_k$ for $k\geq 1$. We can assume that $n_k\leq n_{k+1}$ for $k\geq 1$. Fix $\ell\geq 1$. For any integer $G>0$ and $k>\ell$, by the bi-zero property of Λ, we have

$$\lambda_k-\lambda_\ell=\rho^{-n_{k,\ell}}\frac{a_{k,\ell}}{N}, a_{k,\ell}\in\mathbb{Z}\backslash N\mathbb{Z}.$$

We claim $\#\{k: n_{k,\ell}\leq G\}\leq(N-1)G$. Otherwise, by the pigeon hole principle, there exist k_1,k_2 such that $n_{k_1,\ell}=n_{k_2,\ell}\leq G$ and $N|(a_{k_1,\ell}-a_{k_2,\ell})$. Then, by the definition of $Z(\hat{\mu}_{p,N})$ and $\rho\notin\mathbb{Q}^{1/r}$ for all $r\geq 1$, we have

$$\lambda_{k_1}-\lambda_{k_2}=\lambda_{k_1}-\lambda_\ell+\lambda_\ell-\lambda_{k_2}=\rho^{-n_{k_1,\ell}}\frac{a_{k_1},p-a_{k_2,\ell}}{N}\notin Z(\hat{\mu}_{p,N}).$$

Hence the claim follows. Taking any $k>\ell$ such that $n_{k,\ell}>n_\ell$, we conclude from

$$\rho^{-n_k}\frac{a_k}{N}-\rho^{-n_\ell}\frac{a_\ell}{N}=\rho^{-n_{k,\ell}}\frac{a_{k,\ell}}{N}$$

that there exists a polynomial $p(x)=a_\ell x^s+bx^t+c$ with $s>t$ and $p(\rho)=0$. Let $\phi(x)=c_0x^m+c_1x^{m-1}+\cdots+c_m\in\mathbb{Z}[x]$ be the minimal polynomial of ρ. This implies that $\phi(x)|p(x)$, and thus $c_0|a_\ell$. Let ℓ run through all $\lambda_\ell\in\Lambda$. Then

$$\frac{1}{c_0}\Lambda\backslash\{0\}\subseteq\mathcal{Z}(\hat{\mu}_{p,N}). \tag{49}$$

To show that $\frac{1}{c_0}\Lambda$ is a bi-zero set of $\mu_{\rho,N}$ we need to prove that

$$\frac{1}{c_0}(\Lambda-\Lambda)\backslash\{0\}\subseteq\mathcal{Z}(\hat{\mu}_{\rho,N}). \tag{50}$$

For any $\lambda_{k_1}\neq\lambda_{k_2}\in\Lambda$, by the claim there exists k such that $\min\{n_{k,k_1},n_{k,k_2}\}>n_{k_1,k_2}$, thus

$$\begin{aligned}\rho^{-n_{k_1,k_2}}\frac{a_{k_1,k_2}}{N}&=\lambda_{k_1}-\lambda_{k_2}=(\lambda_{k_1}-\lambda_k)-(\lambda_{k_2}-\lambda_k)\\&=\rho^{-n_{k,k_2}}\frac{a_{k,k_2}}{N}-\rho^{-n_{k,k_1}}\frac{a_{k,k_1}}{N}.\end{aligned}$$

Similar to the above, we have $c_0|a_{k_1,k_2}$. Then (50) holds.

By repeating the same argument, we see that $\frac{1}{c_0^k}\Lambda$ is also a bi-zero set of $\mu_{p,N}$ for any $k\geq 1$. This forces $c_0=1$.

For any $x\in\mathbb{R}$, let $\|x\|=|\langle x\rangle|$, where $\langle x\rangle$ is the unique number such that $\langle x\rangle\in(-1/2,1/2]$ and $x-\langle x\rangle\in\mathbb{Z}$. Clearly $\|x\|$ is the distance from x to $\mathbb{Z}$.

Lemma (2.3.7)[109]: *Let ρ be a root of $x^m+c_1x^{m-1}+\cdots+c_{7\gamma\iota}\in\mathbb{Z}[x]$. Then for any $a\in\mathbb{Z}\backslash N\mathbb{Z}$,*

$$\max_{1\leq n\leq m}\left\|\rho^{-n}\frac{a}{N}\right\|\geq\left(N\sum_{n=1}^{m}|c_n|\right)^{-1}:=\alpha>0. \tag{51}$$

Proof. Denote $\rho^{-n}\frac{a}{N}=\langle\rho^{-n}\frac{a}{N}\rangle+k_n, 1\leq n\leq m$. Then

$$\frac{a}{N}+\sum_{n=1}^{m}c_n\langle\rho^{-n}\frac{a}{N}\rangle+\sum_{n=1}^{m}c_n k_n=0. \tag{52}$$

If $|\langle\rho^{-n}\frac{a}{N}\rangle|<\alpha$ for $1\leq n\leq m$, then $|\sum_{n=1}^{m}c_n\langle\rho^{-n}\frac{a}{N}\rangle|<\frac{1}{N}$. This contradicts (52) as $a\in\mathbb{Z}\backslash N\mathbb{Z}$. Hence the result follows.

Some ideas of Feng and Wang [121] are used in the following proof.

Proposition (2.3.8)[109]: *Let ρ be an irrational and $\rho\notin\mathbb{Q}^{1/r}$ for any $r>1$. Then $\mu_{p,N}$ is not a spectral measure.*

Proof. Suppose on the contrary that $\mu_{\rho,N}$ is a spectral measure. Then, by Lemma (2.3.6), p is an algebra integer, and $\varphi(x)=x^m+c_1x^{m-1}+\ldots+c_m\in\mathbb{Z}[x]$ is the minimal polynomial of ρ.

Let Λ be a spectrum of $\mu_{\rho,N}$ with $0 \in \Lambda$. Denote $\Lambda_k = \Lambda \cap \left\{\rho^{-k}\frac{a}{N}: a \in \mathbb{Z}\backslash N\mathbb{Z}\right\}$ for $k \geq 1$. Then $\#\Lambda_k \leq N-1$ for $k \geq 1$ (by the proof of Lemma (2.3.6)). Let $M_N(\xi)$ be the mask polynomial and let $G(\xi) = \sum_{i=1}^{N-1} | M_N\left(\xi + \frac{i}{N}\right)|^2$. Then by applying Theorem (2.3.3) to the point mass measure $\frac{1}{N}\delta_{\{0,\dots,N-1\}}$, we have

$$G(\xi) + |M_N(\xi)|^2 = \sum_{i=0}^{N-1} | M_N\left(\xi + \frac{i}{N}\right)|^2 = 1,$$

and hence $G(0) = 0$. Observing that $G(z)$ is an entire function, then there exist an entire function $H(z)$ and integer $t > 0$ such that $G(z) = z^t H(z)$ and $H(0) \neq 0$. To prove that $Q(\xi) = |\hat{\mu}_{p,N}(\xi)|^2 + \sum_{k=1}^{\infty}\sum_{\lambda\in\Lambda_k} | \hat{\mu}_{\rho,N}(\xi+\lambda)|^2 \not\equiv 1$, we first observe that for any ξ,

$$\begin{aligned}\sum_{\lambda\in\Lambda_k} | \hat{\mu}_{\rho,N}(\xi+\lambda)|^2 &= \sum_{\lambda\in\Lambda_k}\prod_{j=1}^{k} | M_N(\rho^g(\xi+\lambda))|^2 \cdot |\hat{\mu}_{\rho,N}\left(\rho^k(\xi+\lambda)\right)|^2 \\ &\leq \sum_{\lambda\in\Lambda_k} | M_N\left(\rho^k(\xi+\lambda)\right)|^2 \\ &\leq G(\rho^k\xi)\,. \qquad (53)\end{aligned}$$

(The last inequality follows from $\lambda \in \Lambda_k, \rho^k\lambda = \frac{a}{N} \neq 0, a|N.$) Let m and $\alpha(< 1/2)$ be defined as in Lemma (2.3.7), and let $\beta = \min\{1 - |M_N(x)|^2 : \alpha/2 \leq |x| \leq 1 - \alpha/2\}$. Then obviously $\beta > 0$. Note that for each $k > m$ and $\lambda \in \Lambda_k$,

$$\rho^J\lambda = \rho^{-(k-j)}\frac{a}{N}, j = 1,2,\dots,k-1.$$

Hence for $0 \leq \xi \leq \alpha/2, k > m$, by Lemma (2.3.7), there exists $k - m \leq \ell_\lambda \leq k-1$ such that $\|\rho^{\ell_\lambda}(\xi+\lambda)\|^2 \geq \alpha/2$. Hence from (53), we have

$$\begin{aligned}\sum_{\lambda\in\Lambda_k} | \hat{\mu}_{p,N}(\xi+\lambda)|^2 &\leq \sum_{\lambda\in\Lambda_k} | M_N\left(\rho^{\ell_\lambda}(\xi+\lambda)\right)|^2 \cdot |M_N\left(\rho^k(\xi+\lambda)\right)|^2 \\ &\leq (1-\beta)\sum_{\lambda\in\Lambda_k} | M_N\left(\rho^k(\xi+\lambda)\right)|^2 \\ &\leq (1-\beta)G(\rho^k\xi)\,. \qquad (54)\end{aligned}$$

Note that $\Lambda\backslash\{0\} = \cup_{k\in\mathbb{N}}\Lambda_k$, and $\Lambda_{k_1} \cap \Lambda_{k_2} = \emptyset$ when $k_1 \neq k_2$ since $\lambda \notin \mathbb{Q}^{\frac{1}{r}}$ for all $r \in \mathbb{N}$. Hence, by (53) and (54),

$$\begin{aligned}Q(\xi) &= \sum_{\lambda\in\Lambda} | \hat{\mu}_{p,N}(\xi+\lambda)|^2 \\ &= |\hat{\mu}_{\rho,N}(\xi)|^2 + \sum_{k=1}^{\infty}\sum_{\lambda\in\Lambda_k} | \hat{\mu}_{p,N}(\xi+\lambda)|^2 \\ &\leq |\hat{\mu}_{p,N}(\xi)|^2 + \sum_{k=1}^{m} G(\rho^k\xi) + (1-\beta)\sum_{k>m} G(\rho^k\xi)\,. \qquad (55)\end{aligned}$$

On the other hand, recall that $G(z) = z^t H(z)$ and $H(0) = 0$, then $0 < C_1 \leq |H(z)| \leq C_2$ if $|z| \leq \eta \leq \alpha/2$ for some small η. Therefore for $0 \leq \xi \leq \eta$,

$$\frac{C_1\rho^{mt}}{1-\rho^t}\xi^t \le \sum_{k=m}^{\infty} G\left(\rho^k\xi\right) \le \frac{C_2\rho^{mt}}{1-\rho^t}\xi^t, \tag{56}$$

and

$$|\hat{\mu}_{\rho,N}(\xi)|^2 = \prod_{k=1}^{\infty} | M_N(\rho^k\xi)|^2 = \prod_{k=1}^{\infty}\left(1-G(\rho^k\xi)\right)$$
$$\le e^{-\Sigma_{k=1}^{\infty}G(\rho^k\xi)} \le 1-\sum_{k=1}^{\infty} G\left(\rho^k\xi\right)+o\left(\sum_{k=1}^{\infty} G\left(\rho^k\xi\right)\right), \tag{57}$$

where $o(\xi)$ satisfies that $\lim o\,(\xi)/\xi\,=0$. Hence, by (55) and (57), we have

$$Q(\xi) \le 1-\beta\sum_{k=m}^{\infty} G\left(\rho^k\xi\right)+o\left(\sum_{k=1}^{\infty} G\left(\rho^k\xi\right)\right). \tag{58}$$

By (56) this implies $Q(\xi)<1$ for $\xi>0$ small enough. That Λ cannot be a spectrum follows by Theorem (2.3.3).

In view of Proposition (2.3.5) and Proposition (2.3.8), we have to prove that $\mu_{\rho,N}$ cannot be a spectral measure in the remaining cases (iii) and (iv) for $\rho=p/q$. These will be proved latter.

We will consider $\rho=p/q$, we assume p, q are co-primes throughout. Let $\Lambda=\{\lambda_k\}_{k=0}^{\infty} \subseteq \mathcal{Z}\left(\hat{\mu}_{p,N}\right)$ (with $\lambda_0=0$) be a bi-zero set of $\mu_{\rho,N}$. Then by (48),

$$\lambda_k=\left(\frac{q}{p}\right)^{n_k}\frac{a_k}{N} \text{ with } a_k \in \mathbb{Z}\backslash N\mathbb{Z}, k \ge 1. \tag{59}$$

We will give another expression of the λ_k which is more convenient to use here.

Lemma (2.3.9)[109]: *Let Λ be a bi-zero set of $\mu_{p,N}$ with $\rho=\frac{p}{q}$. Then there exists $m_0>0$ such that each $\lambda_k \in \Lambda\backslash\{0\}$ admits an expression*

$$\lambda_k=p^{-m_0}q^{m_k}\frac{c_k}{N} \textit{ with } c_k \in \mathbb{Z}\backslash q\mathbb{Z} \textit{ and } m_k \ge m_0 \tag{60}$$

(*note that N can be a factor of c_k*). *Moreover, if $N|q$, then we can write*

$$\lambda_k=p^{-m_0}q^{m_k}\frac{c_k}{N} \textit{ with } c_k \in \mathbb{Z}\backslash N\mathbb{Z} \textit{ and } m_k \ge m_0.$$

Proof. For the expression of λ_k in (59), we let $a_k=a_k'q^{l_k}$ so that $q|a_k'$. Then we can write λ_k as

$$\lambda_k=\left(\frac{q}{p}\right)^{n_k+l_k}\frac{a_k'p^{l_k}}{N} := \left(\frac{q}{p}\right)^{m_k}\frac{b_k}{N}, \tag{61}$$

where q is not a factor of b_k for $k \ge 1$. Let $m_0 \ge 1$ be the smallest among all such m_k, and denote the corresponding $\lambda_i \in \Lambda$ by $\left(\frac{q}{p}\right)^{m_0}\frac{b_\lambda}{N}$. Then by the bi-zero property, for any $m_k>m_0$,

$$\left(\frac{q}{p}\right)^{m_0}\frac{b_i}{N}-\left(\frac{q}{p}\right)^{m_k}\frac{b_k}{N}=\left(\frac{q}{p}\right)^{m}\frac{b}{N}.$$

It is easy to see that $m=m_0$, and then $p^{m_k-m_0}$ is a factor of b_k. It follows from this that we can rewrite λ_k as

$$\lambda_k=p^{-m_0}q^{m_k}\frac{c_k}{N},$$

where $q|c_k$ for $k \ge 1$.

The second assertion follows by observing that the l_k in (61) is zero (as $q|a_k$ follows by $N|q$ and $N|a_k$). Hence the above $c_k = a_k/p^{m_k-m_0}$ is not divisible by N by (59).

Corollary (2.3.10)[109]: *Let Λ be a bi-zero set of $\mu_{p,N}$ and let $N|q$ and $p = \frac{p}{q}$. Denote $Q = \{qma\colon a \in \mathbb{Z}\backslash N\mathbb{Z}, m \geq 0\}$. Then*

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq \frac{1}{\rho^{m_0}N} Q \subset \mathcal{Z}\left(\hat{\mu}_{p,N}(\xi)\right), \tag{62}$$

where m_0 is as in Lemma (2.3.9).

Proof. It suffices to show that

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq \frac{1}{p^{m_0}N}\{q^m a\colon m \geq m_0, a \in \mathbb{Z}\backslash N\mathbb{Z}\}.$$

If $m_k > m\iota$ for $k \neq l$, we have $\lambda_k - \lambda_l = p^{-m_0}q^{m_l}\frac{q^{m_k-m_l}c_k - c_l}{N} \in \frac{1}{\rho^{m}0N} Q$ because $N|c_l$. If $m_k = m_l$ for $k \neq l$, then by Lemma (2.3.9),

$$\lambda_k - \lambda_l = p^{-m_0}q^{m_k}(c_k - c_\ell)/N = p^{-m_0}q^{m_k+\alpha}\mathrm{c}/N$$

where $q|c$. By the bi-zero property in (59), we have

$$p^{-m_0}q^{m_k+\alpha}c/N = \lambda_k - \lambda_l = \left(\frac{q}{p}\right)^n a/N$$

where $N(a$. Then $q^{m_k+\alpha-n}c = p^{m_0-n}a$, which implies that $m_k + \alpha = n$, and thus $a = cp^{\alpha+m_k-m_0}$. Hence $N|c$ and the claim follows.

It is well-known that every positive integer has a unique q-adic expansion. In order to do this for all integers in $\mathbb{Z}$, we use the q-adic expansion on the set $\mathcal{C} = \{-1,0,\dots,q-2\}$. We will establish a relation of the λ_k in the bi-zero set Λ with such expansion. We characterize the maximal bi-zero set by certain tree-structure. We need the additional condition that $N|q$, and a special selection map to be defined.

Let $\Omega_N = \{0,\dots,N-1\}$ and let $\Omega_N^* = \mathrm{U}_{k=0}^{\infty}\Omega_N^k$ be the set of finite words (by convention $\Omega_N^0 = \{\emptyset\}$). We use $\mathrm{i} = i_1 \cdots i_k$ to denote an element in Ω_N^k, and $|\mathrm{i}| = k$ is the length. For any $\mathrm{i}, \mathrm{j} \in \Omega_N^*$, ij is their natural conjunction. In particular, $\emptyset\mathrm{i} = \mathrm{i}, \mathrm{i}0^\infty = \mathrm{i}00\dots$, and $0^k = 0\cdots 0 \in \Omega_N^k$.

Definition (2.3.11). Suppose N, q are positive integers and $N|q$. We call a map : $\Omega_N^* \to \{-1,0,\dots,q-2\}$ a selection mapping if

(i) $\iota(\mathrm{r}l) = \iota(0^n) = 0$ for all $n \geq 1$;

(ii) for any $\mathrm{i} = i_1 \cdots i_k \in \Omega_N^k, \iota(\mathrm{i}) \in (i_k + N\mathbb{Z})\mathrm{n}C$, where $C = \{-1,0,1,,q-2\}$;

(iii) for any $\mathrm{i} \in \Omega_N^*$, there exists $\mathrm{j} \in \Omega_N^*$ such that ι vanishes eventually on $\mathrm{ij}0^\infty$, i.e., $\iota(\mathrm{ij}0^k) = 0$ for sufficient large k.

Note that $\mathcal{C} \equiv \Omega_N \oplus N\{0,\dots,r-1\}(\mathrm{mod}\ q)$ where $q = rN$, and ι is a selection map on each level k. More explicitly, (ii) means

$$\iota(\mathrm{i}) = \begin{cases} i_k + Nt, & \text{if } 0 \leq i_k \leq N-2, \\ i_k + Nt', & \text{if } i_k = N-1, \end{cases} \tag{63}$$

where $t \in \{0,\dots,r-1\}$ and $t' \in \{-1,0,\dots,r-2\}$. Next we let

$\Omega_N^\iota = \{\mathrm{i} = i_1 \cdots i_k \in \Omega_N^* : i_k \neq 0, \iota(\mathrm{i}0^n) = 0 \text{ for sufficient large } n\} \cup \{\emptyset\}$

and for any $\mathrm{i} \in \Omega_N^\iota$ we define

$$\iota^*(i) = \sum_{n=1}^{\infty} \iota\,(\mathrm{i}0^\infty)|_n q^{n-1}$$

Here we regard $\mathrm{i}0^\infty = \mathrm{i}00\ldots$, and $\mathrm{i}0^\infty|_n$ denotes the word of the first n entries. Clearly $\iota^*(\emptyset) = 0$.

Let $Q = \{q^m a: a \in \mathbb{Z}\backslash N\mathbb{Z}, m \geq 0\}$ be as in Corollary (2.3.10), a subset $L\backslash\{0\} \subset Q$ is called a *D-set* of Q if $0 \in L$ and $L - L \subset Q \cup \{0\}$ (D for difference), and call it a *maximal* D-set if for any $n \in Q\backslash L, L \cup \{n\}$ is not a D-set. The main idea of the proof of the following theorem is in [114] (and the selection map is called a maximal map there) (see also [118]). We provide a simplified proof here for completeness.

Proposition (2.3.12)[109]: *Suppose* $N|q$. *Then* $L \subset Q := \{\mathrm{q}^m\mathrm{a}: m \geq 0, a \in \mathbb{Z}\backslash N\mathbb{Z}\}$ *is a maximal D-set of* Q *if and only if* $L = \iota^*(\Omega_N^\iota)$ *for some selection map* ι.

Proof. We first prove the sufficiency. For a selection map ι, it is direct to check that $L = \iota^*(\Sigma_N^\iota)$ is a D-set of $Q \subseteq \mathbb{Z}$ by the definition of ι. We need only show that L is maximal in Q. Suppose otherwise, there exists $n \notin L$ and $L \cup \{n\}$ is a D-set. We can express n uniquely as

$$n = a_0 + a_1 q + \cdots + a_\ell q^\ell, a_i \in \mathcal{C} = \{-1,0,1,\ldots,q-2\}. \tag{64}$$

We claim that $a_0 = \iota(i_1)$ for some $i_1 \in \Omega_N$. If otherwise, let $j \in \Omega_N = \{0,\ldots,N-1\}$ such that $a_0 \in j + N\mathbb{Z}$. In view of property (ii) of ι (or (63)), $N|\big(a_0 - \iota(j)\big)$. By property (iii) of ι, there exists $\mathrm{i} = i_1 \cdots i_k \in \Omega_N^\iota$ with $i_1 = j$. Then

$$n - \iota^*(\mathrm{i}) \ = a_0 - \iota(j) + qb,$$

where b is an integer. Hence $n - \iota^*(i) \ \notin Q$ (as it has a factor N, and not a factor of q). This contradicts that $L\mathrm{U}\{n\}$ is a D-set of Q, and the claim follows.

Similarly, by considering $n - \iota(i_1) = n - a_0$ in (64), we can show that $a_1 = \iota(i_1 i_2)$ for some $0 \leq i_2 < N-1$, and so on. After finitely many steps, we have $n = \iota^*(\mathrm{i})$ for some $\mathrm{i} \in \Omega_N^\iota$, which contradicts $n \notin L$, and the sufficiency follows.

Conversely, suppose that L is a maximal D-set of Q. Denote $L = \{\lambda_k\}_{k=0}^\infty$ with $\lambda_0 = 0$. Then λ_k can be expressed by

$$\lambda_k = a_{k,0} + a_{k,1} q + \cdots + a_{k,l_k} q^{l_k} = \sum_{n=0}^{\infty} a_{k,n}\, q^n,$$

where $-1 \leq a_{k,n} \leq q-2$ for $0 \leq n \leq l_k$ and $a_{k,n} = 0$ for $n > l_k$. Note that all $a_{0,n}$ are zero. We first consider $\{a_{k,0}: k \geq 0\}$, the first coefficients of the λ_k's. As $a_{k,0}$ can be written uniquely as $i_k + N\alpha_k \in \mathcal{C} = \{-1,0,\ldots,q-2\}$ for some $i_k \in \Omega_N = \{0,\ldots,N-1\}$, we claim that

$$\{a_{k,0}: k \geq 0\} = \{i + N\alpha_i: i \in \Omega_N\} \subseteq \mathcal{C}. \tag{65}$$

(Here α_i depends only on i, but not on k, hence the set has N elements.) Indeed if $\{a_{k,0}: k \geq 0\} \underset{\neq}{\supset} \{i + N\alpha_i: i \in \Omega_N\}$, then there exist k_1 and k_2 such that $N|\big(a_{k_1,0} - a_{k_2,0}\big)$. Hence

$$\lambda_{k_1} - \lambda_{k_2} = a_{k_1,0} - a_{k_2,0} + qb \notin Q$$

(same reasoning as the above), which contradicts that L is a D-set in Q. If $\{a_{k,0}: k \geq 0\} \neq \subset \{i + N\alpha_i: i \in \Omega_N\}$, then there exists $0 \leq i' \leq N-1$ such that $N(\big(a_{k,0} - i'\big)$ for $k \geq 0$. Clearly $L \cup \{i'\}$ is a D-set in Q, which contradicts the maximality of L. This proves the claim. We rewrite (65) as

$$\{a_{k,0}: k \geq 0\} = \{i_0 + N\alpha_{i_0,0}: i_0 \in \Omega_N\} \subseteq \mathcal{C}.$$

From the claim, we can define ι on Ω_N by $\iota(i) = i + N\alpha_{i,0}, i = 0,1,\ldots,N-1$ and in particular $\iota(0) = 0$. Similarly we can show that, for each $0 \leq i_0 \leq N-1$,

$$\left\{a_{k,1}: a_{k,0} = i_0 + N\alpha_{i_0,0}\right\}_{k=0}^{\infty} = \left\{i_1 + N\alpha_{i_1}, 1: i_1 \in \Omega_N\right\} \subseteq \mathcal{C}$$

and define $\iota(i_0 i) = i + N\alpha_{i,1}, i = 0,1, \dots, N-1$. Again, we can show that

$$\{a_{k,2}: a_{k,0} = i_0 + N\alpha_{i_0,0} \textit{ and } a_{k,1} = i_1 + N\alpha_{i_1,1}\}_{k=0}^{\infty} = \left\{i_2 + N\alpha_{i_2,2}: i_2 \in \Omega_N\right\}$$

and define $\iota(i_0 i_1 i) = i + N\alpha_{i,2}, i = 0,1, \dots, N-1$. Inductively, we can define a map ι on Ω_N^* (with $\iota(\emptyset) = 0$). By the construction of ι, it is easy to see that (i) and (ii) in Definition (2.3.11) are satisfied. For any $\mathrm{i} = i_0 i_1 \cdots i_n \in \Omega_N^*$ with $i_n \neq 0$, again by the construction of ι, there exist infinitely many λ_k such that $a_{k,t} = i_t + N\alpha_{i_t,t}$ for $0 \le t \le n$. Fix such a k, if $k \ge l_k$, we have $\lambda_k = \sum_{n=0}^{\infty} a_{k,n} q^n = \iota^*(\mathrm{i})$; If $k < l_k$, there exists $\mathrm{j} = j_{n+1} j_{n+2} \cdots j_{l_k}$ such that $a_{k,t} = \iota(i_0 \cdots i_n j_{n+1} \cdots j_t)$ for $n+1 \le t \le l_k$. Then

$$\lambda_k = \sum_{n=0}^{\infty} a_{k,n} q^n = \iota^*(ij).$$

This implies that (iii) in Definition (2.3.11) holds. Hence, ι is a selection mapping and $L \subseteq \iota^*(\Omega_N^\iota)$. The necessity follows by the maximal property of L and the proof of the sufficiency.

It follows directly from Corollary (2.3.10) and Proposition (2.3.12) that

Theorem (2.3.13)[109]: *Suppose $\rho = p/q$ and $N|q$. Then $\Lambda \subset \mathcal{Z}(\hat{\mu}_{\rho,N})$ is a maximal bi-zero set if and only if there exist $m_0 \ge 1$ and a selection map ι such that =* $p^{-m_0} N^{-1}\left(\iota^*(\Omega_N^\iota)\right)$.

In particular, we see that for $p = 1$, the spectrum Λ in (46) corresponding to the case $m_0 = 1$ and the selection map ι is to take $\iota(\mathrm{i}) = i_k$ in (63). Also by observing that $\iota^*(\Omega_N^\iota)$ is an infinite set, we have

Corollary (2.3.14)[109]: *Suppose $\rho = p/q$ and $N|q$, then $L^2(\mu_{\rho,N})$ admits an infinite exponential orthonormal set.*

We show the necessity of Theorem (2.3.1) when ρ is a rational number.

Proposition (2.3.15)[109]: *Let $\rho = \frac{p}{q}$ and $1 \le \gcd(N, q) < N$, then $\mu_{\rho,N}$ is not a spectral measure.*

Proof. Suppose on the contrary that $\mu_{\rho,N}$ is a spectral measure. Let Λ be a spectrum of $\mu_{p,N}$ with $0 \in \Lambda$. Denote $d = \gcd(N, q)$. If $d = 1$, by Lemma (2.3.9), we have

$$\Lambda \subseteq p^{-m_0}\left\{q^m \frac{a}{N}: m \ge m_0, a \in \mathbb{Z}\backslash q\mathbb{Z}\right\} \cup \{0\}.$$

Denote $\mathcal{D} = \{0,1, \dots, N-1\}$ and let $\mu' = \delta_{\rho^1\mathcal{D}} * \delta_{\rho^2\mathcal{D}} * \dots * \delta_{p^{m_0}\mathcal{D}} * \delta_{\rho^{m_0+2}\mathcal{D}} * \dots$ be the convolution of $\delta_{p^k\mathcal{D}}$ for $k \ge 1$ and $k \neq m_0 + 1$ (here $\delta_A = \frac{1}{\neq A}\sum_{a \in A} \delta_a$ and δ_a is the Dirac measure). Then $\mu_{p,N} = \delta_{\rho^{m_0+1}\mathcal{D}} * \mu'$. We claim that Λ is a bi-zero set of μ'. The claim leads to a contradiction by Lemma (2.3.4). We prove the claim by assuming that $\rho^{-m_0-1}\frac{a}{N} \in \Lambda - \Lambda$ where $a \in \mathbb{Z}\backslash N\mathbb{Z}$. Then there exist k, l such that

$$\rho^{-m_0-1}\frac{a}{N} = p^{-m_0} q^{-m_k}\frac{a_k}{N} - p^{-m_0} q^{-m_l}\frac{a_l}{N},$$

where $a_k, a_l \in (\mathbb{Z}\backslash q\mathbb{Z}) \cup \{0\}$. Then $p|a$. Hence $\rho^{-m_0-1}\frac{a}{N} = \rho^{-m_0}\frac{qa/p}{N} \in \mathcal{Z}\left(M_{p^{m_0}N}\right)$ and the claim follows; If $1 < d < N$, write $N = N'd, q = q'd$. Then $\mathcal{D} = \mathcal{C} + d\mathcal{E}$, where $\mathcal{C} = \{0,1, \dots, d-1\}$ and $\mathcal{E} = \{0,1, \dots, N'-1\}$. Note that $M_N(\xi) = M_d(\xi) M_{N'}(d\xi)$ and

$$\hat{\mu}_{\rho,N}(\xi) = \prod_{k=1}^{\infty} M_N(\rho^k\xi) = \prod_{k=1}^{\infty} M_d(\rho^k\xi) \prod_{k=1}^{\infty} M_{N'}(\rho^k d\xi).$$

Let v be the probability measure such that

$$\hat{v}(\xi) = \coprod_{k=1}^{\infty} M_d\left(\rho^k \xi\right) \prod_{k\geq 1, k\neq m_0+1} M_{N'}\left(\rho^k d\xi\right).$$

Then $\mu = v * \delta_{\rho^{m_0+1} d\mathcal{E}}$. We claim that Λ is a bi-zero set of v. Hence the proposition follows by Lemma (2.3.4) again.

To prove the claim, we let $\eta \in (\Lambda - \Lambda)\backslash\{0\}\left(\subset \mathcal{Z}(\mu_{p,N})\right)$, then either $\eta \in \mathcal{Z}(v)$ or $\eta \in \mathcal{Z}\left(M_{N'}\left(p^{m_0+1}d(\cdot)\right)\right)$. The first case satisfies the claim trivially. Hence we need only consider the second case, i.e., there exists $\eta \in (\Lambda - \Lambda)$ such that $\eta \in \mathcal{Z}\left(M_{N'}\left(p^{m_0+1}d(\cdot)\right)\right)$. By (48), we have $\eta = \frac{1}{\rho^{m_0+1}d}\frac{a}{N'}\left(= \left(\frac{q}{p}\right)^{m_0+1}\frac{a}{N}\right)$ with $N'(a$; also by (60), there exist k, ℓ such that

$$\eta = p^{-m_0} q^{m_k}\frac{c_k}{N} - p^{-m_0} q^{m_\ell}\frac{C\ell}{N},$$

where $qf c_k$ and $q(c_\ell$. Hence we have $\frac{q^{m_0+1}\mathrm{a}}{p} = q^{m_k}c_k - q^{m_\ell}c_\ell$. This implies that $p|a$. By letting $a' = qa/p$, we see that $N(a'$ (as $N'(a)$. Therefore

$$\eta = \left(\frac{q}{p}\right)^{m_0+1}\frac{a}{N} = \left(\frac{q}{p}\right)^{m_0}\frac{a'}{N} \in \mathcal{Z}\left(M_N\left(\rho^{m_0}(\cdot)\right)\right).$$

As $\mathcal{Z}\left(M_N\left(\rho^{m_0}\,(\cdot)\right)\right) \subset \mathcal{Z}(v)$, the claim follows.

Lemma (2.3.16)[109]: *Let ι be a selection mapping. Then*

$$\sum_{\mathrm{i}\in\Omega_N^\iota, |\mathrm{i}|\leq n} |\,\hat{\mu}_{n+m_0-1}\left(\xi + \rho^{-m_0}N^{-1}\iota^*(\mathrm{i})\right)|^2 \leq 1$$

for $n \geq 1$ and $\xi \in \mathbb{R}$.

Proof. First we prove the case for $m_0 = 1$. According to the Bessel inequality, it suffices to show that $\rho^{-1}N^{-1}\iota^*(\{\mathrm{i} \in \Sigma_N^\iota : |\mathrm{i}| \leq n\})$ is a bi-zero set of μ_n. For any $\mathrm{i}, \mathrm{j} \in \Sigma_N^\iota, \mathrm{i} \neq \mathrm{j}$ and $1 \leq |\mathrm{i}|, |\mathrm{j}| \leq n$, we let $\mathrm{i}' = \mathrm{i}0^{n-|\mathrm{i}|} := i_1' \cdots i_n'$ and $j' = j0^{n-|j|} := jí\, j_n'$. Let s be the smallest integer such that $i_s' \neq j_s'$. Then $s \leq n$ and

$$\iota^*(\mathrm{i}) - \iota^*(\mathrm{j}) = \left(\iota(\mathrm{i}'|_s) - \iota(\mathrm{i}'|_s)\right)q^{s-1} + \alpha q^s$$

for some integer α. By (63), $\iota(\mathrm{i}'|_s) - \iota(\mathrm{i}'|_s)$ is not divisible by N. It follows from (47) that,

$$M_N\left(\rho^s \rho^{-1}N^{-1}\left(\iota^*(\mathrm{i}) - \iota^*(\mathrm{j})\right)\right) = M_N\left(\frac{p^{s-1}\left(\iota(\mathrm{i}'|_s) - \iota(\mathrm{i}'|_s)\right)}{N}\right) = 0.$$

This implies that $\hat{\mu}_n\left(\rho^{-1}N^{-1}\left(\iota^*(\mathrm{i}) - \iota^*(\mathrm{i})\right)\right) = 0$. Similarly, we have $\hat{\mu}_n\left(\rho^{-1}N^{-1}\iota^*(\mathrm{i})\right) = 0$ for any $\mathrm{i} \in \Sigma_N^\iota$ and $0 < |\mathrm{i}| \leq n$. By Theorem (2.3.3),

$$\sum_{\mathrm{i}\in\Omega_N^\iota, |\mathrm{i}|\leq n} |\,\hat{\mu}_n\left(\xi + \rho^{-1}N^{-1}\iota^*(\mathrm{i})\right)|^2 \leq 1.$$

This completes the proof for $m_0 = 1$. For $m_0 > 1$, we observe that

$$|\hat{\mu}_{n+rn_0-1}(\xi)| = |\hat{\mu}_{\gamma n_0-1}(\xi)||\hat{\mu}_n(\rho^{m_0-1}\xi)| \leq |\hat{\mu}_n(\rho^{m_0-1}\xi)|$$

and apply the inequality. The result follows.

The following lemma is a simple generalization of Lemma 2.10 in [113].

Lemma (2.3.17)[109]: *Let* $a = \ln p/\ln q$. *Then for any* $\xi > 1$ *there exists* ξ' *such that* $\rho^2\xi^a \le \xi' \le p\xi$ *and*

$$|\hat{\mu}_{\rho,N}(\xi)| \le c|\hat{\mu}_{p,N}(\xi')|,$$

where $c = \max\left\{|M_N(\xi)|: \frac{1}{2q} \le \xi \le \frac{1}{2}\right\} < 1$.

Proof. For any $x \in \mathbb{R}$, denote the unique number $\langle x\rangle$ that satisfies $\langle x\rangle \in (-1/2, 1/2]$ and $x - \langle x\rangle \in \mathbb{Z}$. If $\langle \rho\xi\rangle \notin \left(-\frac{1}{2q}, \frac{1}{2q}\right)$, then

$$|\hat{\mu}_{\rho,N}(\xi)| = |M_N(\rho\xi)||\hat{\mu}_{\rho,N}(\rho\xi)| = |M_N(\langle\rho\xi\rangle)||\hat{\mu}_{\rho,N}(\rho\xi)| \le c|\hat{\mu}_{\rho,N}(\rho\xi)|.$$

Hence we obtain the desired inequality by letting $\xi' = \rho\xi$; If $\langle\rho\xi\rangle \in \left(-\frac{1}{2q}, \frac{1}{2q}\right)$, then

$$\rho\xi - \langle\rho\xi\rangle = r_t q^t + \cdots + r_l q^l, \tag{66}$$

where $0 \le r_j < q$ for $t \le j \le l$ and $r_t > 0$. Then

$$\langle p^{t+2}\xi\rangle = \langle \rho^{t+1}\langle\rho\xi\rangle + \frac{r_t p^{t+1}}{q}\rangle.$$

Note that $|\rho^{t+1}\langle\rho\xi\rangle| < \frac{1}{2q}$ and $\frac{1}{q} \le |\langle\frac{r_t p^{t+1}}{q}\rangle| \le \frac{q-1}{q}$, then $\langle\rho^{t+2}\xi\rangle \notin \left(-\frac{1}{2q}, \frac{1}{2q}\right)$.

By (66), we have $\xi \ge q^t$, which implies $\rho^t \ge \xi^{a-1}$, where $a = \ln p/\ln q$. Let $\xi' = \rho^{t+2}\xi$, then $\xi' \ge \rho^2\xi^a$, and hence

$$\begin{aligned}|\hat{\mu}(\xi)| &= |M_N(\rho\xi)|\,|M_N(\rho^{t+2}\xi)||\hat{\mu}(\rho^{t+2}\xi)| \\ &\le |M_N(\langle\rho^{t+2}\xi\rangle)||\hat{\mu}(\rho^{t+2}\xi)| \\ &\le c|\hat{\mu}(\xi')|.\end{aligned}$$

Lemma (2.3.18)[109]: *Assume* $p > 1$, *then there exist integers* $b \ge 2, n_0 \ge 2$, *and real numbers* $\beta > 1, C > 1$ *such that for any* $\mathrm{i} \in \Omega_N^\iota$ *with* $n^b < |\mathrm{i}| \le (n+1)^b, n \ge n_0$, *we have*

$$|\hat{\mu}_{p,N}\left(\rho^{(n+1)^b+(m_0-1)}\left(\xi + \rho^{-m_0}N^{-1}\iota^*(\mathrm{i})\right)\right)| \le \frac{C}{n^\beta}$$

for $0 \le \xi \le \frac{1}{2_{\beta m}\,{}_0 N}$.

Proof. Note that $p > 1$ implies that $q > 2$. Let b be an integer such that $b > 1 + \frac{\log a}{\log c}$, where $a = \log p/\log q$, and c is as in Lemma (2.3.17). Since $\iota^*(\mathrm{i}) = \sum \iota\left(\mathrm{i}0^\infty|_j\right)q^{j-1}$ for any $\mathrm{i} \in \Sigma^\iota$, let P be the largest index such that $\iota(\mathrm{i}0^\infty|_\ell) \ne 0$. Then $P \ge n^b + 1$, and a direct estimation shows that

$$|\iota^*(\mathrm{i})| \ge q^{\ell-1} - (q-2)\sum_{j=1}^{\ell-1} q^{j-1} \ge q^{\ell-3} + \frac{1}{2}.$$

This together with the assumption on ξ implies that

$$|\rho^{m_0-1}\xi + \frac{\iota^*(\mathrm{i})}{\rho N}| \ge \frac{|\iota^*(\mathrm{i})|}{pN} - \frac{1}{2\rho N} \ge q^{\ell-4} \ge q^{n^b-3}.$$

Let $\eta = \rho^{(n+1)^b}\left(\rho^{m_0-1}\xi + \frac{\iota^*(\mathrm{i})}{pN}\right)$. It is easy to see that if n is large enough, then $(n+1)^b + 3 \le n^b + b^2 n^{b-1}$. Hence if we take a large n_0, then for $n \ge n_0$,

$$|\eta| \ge \frac{p^{(n+1)^b}}{q^{(n+1)^b - n^b + 3}} \ge \frac{p^{(n+1)^b}}{q^{b^2 n^{b-1}}} = \left(\frac{p^{(1+1/n)^b n}}{q^{b^2}}\right)^{n^{b-1}} \ge \left(\frac{p^n}{q^{b^2}}\right)^{n^{b-1}} \ge q^{n^{b-1}}$$

Applying Lemma (2.3.17) to $\eta_i = \rho^i\eta$ recursively, we have

$$|\hat{\mu}(\eta)| \le c|\hat{\mu}(\eta_1)| \le \le c^l|\hat{\mu}(\eta_l)|$$

as long as $|\eta_l| \geq 1$. This is the case if we let $l = \left[\log_a \frac{2n^{1-b}}{1-a}\right]$, because

$$|\eta_l| \geq \rho^2 |\eta_{l-1}|^a \geq \geq \rho^{2+2a+\cdots+2a^{l-1}} |\eta|^{a^l} \geq \rho^{\frac{2}{1-a}} |\eta|^{a^l} > q^{-\frac{2}{1-a}} q^{n^{b-1}a^l} \geq 1.$$

Hence,

$$|\hat{\mu}(\eta)| \leq c^l \leq c^{\log_a \frac{2n^{1-b}}{1-a}} = c^{\log_a \frac{2}{1-a}} n^{-(b-1)\log c/\log a}.$$

The lemma follows by assigning C and β in the obvious way.

Proposition (2.3.19)[109]: *Let* $\rho = \frac{p}{q}$. *If* $q|N$ *and* $p > 1$, *then* $\mu_{p,N}$ *is not a spectral measure*.

The proof of this case is more elaborate. We show that any maximal bi-zero set Λ of $\mu_{p,N}$ does not satisfy the condition on $Q(\xi)$ in Theorem (2.3.3)(ii). To this end, we define

$$\mu_n = \delta_{\rho\Omega_N} * \ldots * \delta_{p^n \Omega_N}$$

for $n \geq 1$. Then

$$\hat{\mu}_n(\xi) = \prod_{i=1}^{n} M_N(\rho^i \xi)\ and \hat{\mu}_{\rho,N}(\xi) = \hat{\mu}_n(\xi) \hat{\mu}_{\rho,N}(\rho^n \xi).$$

Proof. We assume all the parameters in Lemma (2.3.18). To simplify the notations, we write $\mu = \mu_{p,N}, \alpha(\mathrm{i}) = \rho^{-m_0} N^{-1} \iota^*(\mathrm{i}), \mathcal{J}_n = \{\mathrm{i} \in \Omega_N^\iota, |\mathrm{i}| \leq n^b\}, \mathcal{J}_{n,n+1} = \{\mathrm{i} \in \Omega_N^\iota, n^b < |\mathrm{i}| \leq (n+1)^b\}$. Let

$$Q_n(\xi) = \sum_{\mathcal{J}_n} |\hat{\mu}(\xi + \alpha(\mathrm{i}))|^2$$

Then

$$\begin{aligned} Q_{n+1}(\xi) &= Q_n(\xi) + \sum_{\mathcal{J}_{n,n+1}} |\hat{\mu}(\xi + \alpha(\mathrm{i}))|^2 \\ &= Q_n(\xi) + \sum_{\mathcal{J}_{n,n+1}} |\hat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathrm{i}))|^2 |\hat{\mu}\left(p^{(n+1)^b+m_0-1}(\xi + \alpha(\mathrm{i}))\right)|^2 \\ &\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} \sum_{\mathcal{J}_{n,n+1}} |\hat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathrm{i}))|^2 \text{ (by Lemma (2.3.18))} \\ &\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} \left(1 - \sum_{;\Gamma_n} |\hat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathrm{i}))|^2\right) \text{ (by Lemma (2.3.17))} \\ &\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} (1 - Q_n(\xi)). \end{aligned}$$

This implies that $n > n_0$,

$$1 - Q_{n+1}(\xi) \geq (1 - Q_n(\xi)) \left(1 - \frac{C^2}{n^{2\beta}}\right) \geq \cdots \geq \left(1 - Q_{n_0}(\xi)\right) \prod_{k=n_0}^{n} \left(1 - \frac{C^2}{k^{2\beta}}\right).$$

Now let $Q(\xi) = \sum_{i \in \Omega_N^\iota} |\hat{\mu}(\xi + \alpha(\mathrm{i}))|^2$, it is the sum over a maximal bi-zero set (by Theorem (2.3.13)). The above implies

$$1 - Q(\xi) \geq C'\left(1 - Q_{n_0}(\xi)\right),$$

where $C' = \prod_{k=n_0}^{\infty}\left(1 - \frac{c^2}{k^{2\beta}}\right) \neq 0$. This implies that $Q(\xi) \not\equiv 1$, and hence by Theorem (2.3.3) and Theorem (2.3.13), any maximal bi-zero set of $\mu_{\rho,N}$ cannot be a spectrum when $\rho = p/q, p, q$ are co-prime, and $p \neq 1$.

It was proved in [123] that if μ is a spectral self-similar measure with support in $[0,1]$ and u is a probability counting measure support on a finite set in $\mathbb{Z}$, then the convolution $\mu * v$ is a spectral measure if and only if v is a spectral measure. It was pointed out by Gabardo and Lai (private communication) that if both μ and v are two probability measures with $\mu * v = L|_{[0,1]}$, where $L|_{[0,1]}$ is the Lebesgue measure restricted on $[0,1]$, then both μ and v are spectral measures (which is a corollary of the main results in [110] and [131]). It has been asked:

Is the convolution of two spectral self-similar measures with essentially disjoint supports a spectral measure ‘?

The question can be answered by Theorem (2.3.1). Observe that $\{0,1,2,3\} = \{0,1\} \oplus \{0,2\}$, hence

$$\mu_{1/6,4} = \mu_{1/6,2} * \mu_{1/6,\{0,2\}}.$$

It follows that both $\mu_{1/6,2}$ and $\mu_{1/6,\{0,2\}}$ are spectral measures (by [125] or Theorem (2.3.1)), but Theorem (2.3.1) implies that $\mu_{1/6,4}$ is not a spectral measure. As a consequence, convolution of two spectral measures may not be spectral.

One of the challenge questions on the spectral measures is the conjecture of Laba and Wang [128]:

Let μ be a self-similar measure as in (44), *then μ is a spectral measure if and only if* (*i*) $w_j = 1/N$; (ii) $\rho = 1/q$ *for some integer* $q > 1$; *and* (*iii*) *there exist a constant* c *and an integer digit set* $\mathcal{D}'$ *such that* $\mathcal{D} = c\mathcal{D}'$ *and* $\mathcal{D}' \oplus \mathcal{B} \equiv \{0, \dots, q-1\}$ (*mod q*) *for some* $\mathcal{B} \subset \mathbb{Z}$.

In [117], it was shown that (i) is necessary for a spectral measure under the no overlap condition. Our Theorem (2.3.1) settles the case where $\mathcal{D} = \{0, \dots, N-1\}$. The digit set $\mathcal{D}'$ in (iii) is called an *integer tile*. The study of integer tiles has a long history related to the geometry of numbers ([112]), and the spectral property of $\mathcal{D}$ as a discrete set itself is still unsolved [127].

As was proved in [125], the Cantor measure $\mu_{1/k}$ with k an odd integer is not a spectral measure. It is well known that a relaxing of the orthonormal basis is the concept of *frame* introduced by Duffin and Schaeffer in the 50s (see [111]). We call a measure μ an F-*spectral measure* (Γ for frame) if there exists a countable set $\{e_\lambda : \lambda \in \Lambda\}$ and $A, B > 0$ such that for any $f \in L^2(\mu)$,

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2 \leq B\|f\|^2,$$

and call μ an R-*spectral measure* if in addition it is a basis (R for Riesz). The frame structure of $L^2[0,1]$ has been studied in detail in [130,134]; also there are extensive studies of the frames on $L^2(\mu)$ [114,135]. However the basic problem whether $\mu_{1/k}$ with k an odd integer, in particular for $\mu_{1/3}$, is an Γ-(or R-) spectral measure is still unresolved.

Chapter 3
Self-Affine Measures and Uniformity with Spectral Property

We shoe a general theorem on such representation. The proof is constructive; it depends on using a tiling IFS $\{\psi_j\}_{j=1}^{l}$ to obtain a graph directed system, together with the associated probability on the vertices to form some transition matrices. As applications, we study the dimension and Lebesgue measure of a self-affine set, the L^q-spectrum of a self-similar measure, and the existence of a scaling function (i.e., an L^1-solution of the refinement equation). We then investigate affine iterated function systems (IFSs), we show that if an IFS with no overlap admits a frame measure then the probability weights are all equal. We also show that the Laba-Wang conjecture is true if the self-similar measure is absolutely continuous. We will present a new approach to the conjecture of Liu and Wang [206] about the structure of non-uniform Gabor orthonormal bases of the form $\mathcal{G}(g, \Lambda, \mathcal{J})$. We exhibit complete orthogonal exponentials with zero Beurling dimensions. These examples show that the technical condition in Theorem 3.5 of [218] cannot be removed. For an irregular maximal orthogonal set, we show that under some condition, its completeness is equivalent to that of the corresponding regularized mapping.

Section (3.1): Vector-Valued Representations:

We assume that A is a $d \times d$ integral expanding matrix (i.e., all its eigenvalues have moduli > 1) and $\mathcal{D} = \{d_1, \dots, d_m\} \subset \mathbb{Z}^d$. We call $(A, \mathcal{D})$ an integral affine pair. This pair defines an iterated function system (IFS) $\{S_j\}_{j=1}^{m}$ on $\mathbb{R}^d$ by

$$S_j(x) = A^{-1}(x + d_j),\ x \in \mathbb{R}^d.$$

It is known that under a suitable norm on $\mathbb{R}^d$, the expanding property of A implies that the S_j 's are contractive, hence there exists a unique nonempty compact set K satisfying

$$K = \bigcup_{j=1}^{m} S_j(K).$$

Alternatively, K can be written in the form of radix expressions

$$\left\{\sum_{n=1}^{\infty} A^{-n} d_{j_n} : d_{j_n} \in \mathcal{D}\right\}.$$

We call the attractor K a self-affine set, and a self-affine region if $K^\circ \neq \emptyset$. In the case where $|\det A| = m$, a self-affine region K will tile $\mathbb{R}^d$ by certain translations of K (cf., e.g., [157]); we call such a K a self-affine tile. If we associate to the family $\{S_j\}_{j=1}^{m}$ a set of positive probability weights $\{p_j\}_{j=1}^{m}$, then there exists a unique probability measure μ supported on K satisfying

$$\mu(E) = \sum_{j=1}^{m} p_j \mu\left(S_j^{-1}(E)\right) = \sum_{j=1}^{m} p_j \mu\big(A(E) - d_j\big) \tag{1}$$

for any Borel subset E of $\mathbb{R}^d$. This measure is called a self-affine measure. If, in addition, the matrix A is a constant multiple of an orthonormal matrix, (i.e., A is a similarity and $\{S_j\}_{j=1}^{m}$ are similitudes), then in the above terminology we replace self-affine by self-similar.

The above IFS also plays a special role in the refinement equation in wavelet theory:

$$f(x) = \sum_{j=1}^{m} a_j f\left(Ax - d_j\right),\ x \in \mathbb{R}^d, \tag{2}$$

where $a_j \in \mathbb{R}$ and $\sum_{j=1}^{m} a_j = |\det A|$. An L^1-solution of this equation is called a scaling function. It can be seen that the Radon-Nikodym derivative of the μ in (1) satisfies the refinement equation.

One of the most basic assumptions in the study of iterated function systems is the open set condition (OSC): there exists a bounded open set U such that

$$S_j(U) \subset U \ \text{ for each } j \ \text{ and } \ S_i(U) \cap S_j(U) = \emptyset \ \text{ if } i \neq j.$$

Under this condition the attractor K can be identified with a symbolic space and the invariant measure μ can be identified with a product measure on the symbolic space; their geometric and analytic properties are well understood (see, for example, [146],[2],[167],[175]). However, there are many important cases where the OSC is not satisfied (we loosely say that the IFS has overlap), for example when $m(= \#\mathcal{D}) > |\det A|$ in the above $\{S_j\}_{j=1}^{m}$. The overlapping IFS's have very complicated and rich structure; there are many attempts to study them by imposing various conditions such as the transversality condition [173], the weak separation condition ([155],[161],[164],[171]) and the finite type condition [168].

We consider a vector-valued representation of the self-affine measure μ through a new IFS that satisfies the OSC. This approach was first used by Daubechies and Lagarias [141,142] for the refinement equation (2) with $A = [2], \mathcal{D} = \{0, \dots, m-1\}$. The vector form of the equation is

$$F(x) = T_0 F(2x) + T_1 F(2x-1)$$

where $F: [0,1] \to \mathbb{R}^{m-1}$ is defined by $F(x) = [f(x), \dots, f(x+m-2)]^t$ and T_0, T_1 are $(m-1) \times (m-1)$ matrices determined by the coefficients a_j, they are called transfer matrices. This representation initiated the investigation of the joint spectral radius to prove the existence and regularity of scaling functions (e.g., [156],[154],[165]). Another attempt of vector-valued representation was due to Strichartz [174] and Lau and Ngai [161] for the Bernoulli convolution associated with the golden ratio; it was used to give an explicit formula for the L^q-spectrum and verify the multifractal formalism for $q > 0$ in such case. Feng has made a further investigation for $q \leq 0$ and extended this to the Pisot numbers [147,148].

Note that all the established cases are on $\mathbb{R}$. Here we will concentrate on integral self-affine measures on $\mathbb{R}^d$. We will use a tiling IFS (i.e., the attractor is a self-affine tile) to be the new IFS with OSC for the vectorvalued representation. Our main result is

Theorem (3.1.1)[138]: For each self-affine measure μ generated by an integral affine pair $(A, \mathcal{D})$, there exists a self-affine $\mathbb{Z}^d$-tile T such that, for the set $\mathcal{E} = \{e_1, \dots, e_N\} = \{e \in \mathbb{Z}^d : K \cap (T^\circ + e) \neq \emptyset\}$, the vector-valued measure

$$\boldsymbol{\mu}(E) = [\mu((E \cap T) + e_1), \dots, \mu((E \cap T) + e_N)]^t$$

satisfies

$$\boldsymbol{\mu}(\cdot) = \sum_{i=1}^{l} W_i \boldsymbol{\mu}\left(\psi_i^{-1}(\cdot)\right), \tag{3}$$

where $l = |\det(A^{n_0})|$ for some $n_0 \geq 1, \{\psi_i(x) = A^{-n_0}(x + c_i)\}_{i=1}^{l}$ is the associated integral IFS generating T, and $W_i = [W_i(u,v)], 1 \leq i \leq l$, are nonnegative $N \times N$ matrices satisfying: (i) $W := \sum_{i=1}^{l} W_i$ is irreducible; (ii) W is Markov, i.e., the column sums of W are all 1.

The $\mathbb{Z}^d$-tile T in the theorem means T admits $\mathbb{Z}^d$ as a tiling set. The IFS $\{\psi_i\}_{i=1}^{l}$ corresponding to T satisfies the OSC. One of the most important consequences of this representation is that

$$\boldsymbol{\mu}(\psi_\sigma(T)) = W_\sigma \boldsymbol{\mu}(T), \tag{4}$$

where $\sigma = (i_1, \dots, i_n) \in \Sigma_l^n, \Sigma_l = \{1, \dots, l\}, \psi_\sigma = \psi_{i_1} \circ \cdots \circ \psi_{i_n}$ and $W_\sigma = W_{i_1} \cdots W_{i_n}$. The family of $\{\psi_\sigma(T)\}$ generates the Borel sets and the product of the matrices determines the local property of μ.

In the theorem the tile T is generated by A (or A^{n_0} for some n_0) and a suitable choice of the digit set $\mathcal{C} \in \mathbb{Z}^d$ (it has to meet the technical requirement that $\mu(\partial T + e) = 0$ for all $e \in \mathbb{Z}^d$). The set $\mathcal{E} = \{e_1, \dots, e_N\}$ is considered as a set of vertices, and an edge from e_u to e_v exists if there exist $c_i \in \mathcal{C}$ and $d_j \in \mathcal{D}$ such that

$$c_i - d_j + Ae_u = e_v \tag{5}$$

(see (9), Lemma (3.1.10)(iii) and Lemma (3.1.12)). The associated weights of this graph-directed system are the $W_i = [w_i(u,v)]$ with $w_i(u,v) = p_j$ where the j is determined by (5).

The theorem also holds for the refinement equation (2) with some obvious adjustments. With the vector form, all the known theory for the joint spectral radius will go through.

For a given pair $(A, \mathcal{D})$, it is in general difficult to determine whether the self-affine set K is a self-affine region, which is a necessary condition for (2) to have an L^1-solution; in the case $\#\mathcal{D} = |\det A|$, the K is a self-affine tile [157]. This question has been studied in some detail in [152] and an algorithm was given there (see also [168],[175] for self-affine tiles). We make use of the main theorem to give a unified and more satisfactory criterion as follows:

Theorem (3.1.2)[138]: Let $\{W_i\}_{i=1}^{l}$ be the transition matrices in Theorem (3.1.1) corresponding to $p_j = 1/m$. Then the following conditions are equivalent:

(i) K is a self-affine region, i.e., $K^\circ \neq \emptyset$;

(ii) K has positive Lebesgue measure;

(iii) $(W_\sigma \mathbf{1})^\sim \neq 0$ for any $\sigma = (i_1, \dots, i_n), 1 \leq i_j \leq l, n > 0$, where $\tilde{v}$ denotes the vector with 1 in the nonzero entries of v and 0 elsewhere; 1 is the column vector with 1 in all entries.

Let $\mathcal{F} = \{\mathbf{1} = v_1, \dots, v_r\}$ be the set of all distinct $(W_\sigma \mathbf{1})^\sim$. It is easy to see that $r \leq 2^N$. Hence we can determine whether $K^\circ \neq \emptyset$ in at most 2^N steps. It is known that the Lebesgue measure of such a K is a rational number [152], and is an integer if K is a tile [157]. We prove

Theorem (3.1.3)[138]: Let K be a self-affine region generated by an integral affine pair. Then the Lebesgue measure of K is given by

$$\mathcal{L}(K) = \sum_{i=1}^{r} a_i \alpha(v_i),$$

where $\alpha(v_i)$ is the number of nonzero entries of $v_i \in \mathcal{F}$ and $\{a_i\}_{i=1}^{r}$ is defined through the matrix $G = [G(s,t)]_{r \times r}$ with

$$G(s,t) = l^{-1}\#\{i: (W_i v_s)^{\sim} = v_t\},\ 1 \le s,t \le r.$$

The detailed definition of $\{a_i\}_{i=1}^{r}$ is given in Theorem (3.1.23). For the case $K^\circ = \emptyset$, we want to determine its dimension. As a consequence of Theorem (3.1.2), we have $v_r = 0$ in $\mathcal{F}$ (after rearrangement), and the matrix G in Theorem (3.1.3) can be expressed as

$$G = \begin{bmatrix} G_1 & g \\ 0 & 1 \end{bmatrix}.$$

Theorem (3.1.4)[138]: Let K be the self-similar set coming from a pair $(A, \mathcal{D})$, where A is a similarity, and suppose that $K^\circ = \emptyset$. Then $\dim_B K = \dim_H K = d - \log \lambda_1 / \log \varrho < d$, where λ_1 is the maximal eigenvalue of G_1 and ϱ is the contraction ratio of the IFS.

As another application of Theorem (3.1.1), we consider the multifractal structure of the self-similar measure in (1). Let

$$\alpha(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}$$

be the local dimension of μ at x. Let $K_\alpha = \{x \in K: \alpha = \alpha(x)\}$. A classical heuristic principle called the multifractal formalism says that

$$\dim_H K_\alpha = \tau^*(\alpha),$$

where $\tau^*(\alpha)$ is the Legendre transform of $\tau(q)$, the L^q-spectrum of μ. The validity of the formalism has to be considered in individual cases and depends on the differentiability of $\tau(q)$. For example, if the IFS consists of similitudes and satisfies the OSC, then there is an explicit expression of $\tau(q)$ and the formalism holds ([146], [140], [145]). For overlapping IFS, there were extensive investigations of the Bernoulli convolution associated with the golden ratio [161] and the Pisot numbers [148], the convolution of the Cantor measure ([166],[150]) and some other related self-similar measures ([172],[176]). In these cases some extraordinary phenomena were revealed when $q < 0$. There was also a study of the scaling functions where the coefficients are allowed to be negative (e.g., [143]).

By using the vector representation in Theorem (3.1.1), the product of matrices in (4) and the results in [149] and [162], we have

Theorem (3.1.5)[138]: Let μ be the self-similar measure associated with the integral similar pair $(A, \mathcal{D})$. Then

$$\tau(q) = \lim_{n \to \infty} \frac{\log \sum_{|\sigma|=n} \|W_\sigma\|_1^q}{n\log \varrho},\ q > 0,$$

where $|\sigma|$ is the length of σ and $\|W_\sigma\|_1$ is the sum of all entries of W_σ. Moreover, $\tau(q)$ is differentiable and the multifractal formalism holds for $q > 0$.

We prove Theorem (3.1.1); the analog for the scaling function is also described. The vector-valued measure in Theorem (3.1.1) is constructive; we illustrate the construction by some examples. We use a special case of Theorem (3.1.1) to consider self-affine sets; Theorems (3.1.2)-(3.1.4) are proved there. We consider the multifractal structure of integral self-similar measures, and prove Theorem (3.1.5).

Let $(A, \mathcal{D})$ be an integral affine pair as in the last section with $\mathcal{D} = \{d_1, \dots, d_m\}$ and let $\{S_j\}_{j=1}^{m}$ be the associated self-affine IFS. We will use the following symbols throughout: $\Sigma_m = \{1, \dots, m\}$ (or just Σ if there is no confusion) and $\Sigma^* = \bigcup_{n \ge 0} \Sigma^n$. For any $J = j_1 \dots j_n \in \Sigma^n$, let $S_J = S_{j_1} \circ \dots \circ S_{j_n}$ and

$$d_J = d_{j_n} + A d_{j_{n-1}} + \dots + A^{n-1} d_{j_1},\ \mathcal{D}_n = \mathcal{D} + A\mathcal{D} + \dots + A^{n-1}\mathcal{D}.$$

We call a compact set $T \subseteq \mathbb{R}^d$ a tile if there exists a discrete set $\mathcal{T}$ (tiling set) such that $\mathbb{R}^d = \bigcup_{z \in \mathcal{T}} (T + z)$ and $(T^\circ + z) \cap (T^\circ + z') = \emptyset$ for any two distinct $z, z' \in \mathcal{T}$. If the tiling set can be chosen to be $\mathbb{Z}^d$, then we call T a $\mathbb{Z}^d$-tile. It is known that if T is a self-affine tile (i.e., the attractor of an integral affine pair $(A,\mathcal{C})$), then T admits a $\mathbb{Z}^d$-tiling if and only if $\mathcal{L}(T)$ (the Lebesgue measure of T) is 1 ; in this case $\#\mathcal{C} = |\det A| = l$ and $\mathcal{C}$ is a complete residue set, i.e., the set of cosets $\{[d]: d \in \mathcal{C}\}$ equals $\mathbb{Z}^d/A\mathbb{Z}^d$[159].

Using Corollary 5 and Theorem 1 of [177] (or Theorem (3.1.3) of [157]), we have the following lemma which guarantees the existence of a $\mathbb{Z}^d$-tile for a given A.

Lemma (3.1.6)[138]: For any integral expanding matrix A, there exists an integer $k > 0$ and a digit set $\mathcal{C} \subseteq \mathbb{Z}^d$ with $\#\mathcal{C} = |\det(A^k)|$ such that $T := T(A^k, \mathcal{C})$ is a $\mathbb{Z}^d$-tile.

Indeed, according to [177], the k can be chosen such that all eigenvalues of A^k are greater than $3\sqrt{d}$ in modulus. For such k, let $Q = \{A^k x : x = [x_1, \ldots, x_d]^t, x_i \in (-1/2, 1/2]\}$. Then $\mathcal{C} = Q \cap \mathbb{Z}^d$ satisfies the condition of Lemma (3.1.6). In the one- or two-dimensional cases, the bound $3\sqrt{d}$ can be improved to 2. We also remark that the $A^k, k \geq 1$, in the above lemma cannot be taken to be A as there exist expanding integral matrices A (with size $d > 3$) such that $T(A,\mathcal{C})$ is not a $\mathbb{Z}^d$-tile for any integral digit set $\mathcal{C}$ with $\#\mathcal{C} = |\det(A)|$ ([163, corrigendum/addendum] and [170]). So far, for an integral similarity matrix A, no example has been found for which we must choose $k > 1$.

We will introduce an auxiliary IFS $\{\psi_i\}_{i=1}^l$ such that the attractor T is a $\mathbb{Z}^d$-tile; this system satisfies the open set condition automatically and we will reduce the self-affine measure μ to be a vectorvalued self-affine measure $\boldsymbol{\mu}$ of $\{\psi_i\}_{i=1}^l$ in the next section. First we state

Lemma (3.1.7)[138]: Let $\{S_j\}_{j=1}^m$ be the IFS generated by the integral affine pair $(A,\mathcal{D})$, let μ be a self-affine measure, and let K be the attractor of the IFS $\{S_j\}_{j=1}^m$. Let $T = T(A,\mathcal{C})$ be a $\mathbb{Z}^d$-tile and let $V = \bigcup\{T^\circ + z : \mu(T+z) > 0\ z \in \mathbb{Z}^d\}$. Then

(i) V is a nonempty open set and is invariant with respect to $\{S_j\}_{j=1}^m$;

(ii) if $V \cap K \neq \emptyset$, then $\mu(\partial T + z) = 0$ for all $z \in \mathbb{Z}^d$ (∂T is the boundary of T).

Consequently, μ is concentrated on either $\bigcup_{z\in\mathbb{Z}^d}(T^\circ + z)$ or $\bigcup_{z\in\mathbb{Z}^d}(\partial T + z)$.

Proof. Set $S_j(x) = A^{-1}(x + d_j), d_j \in \mathcal{D}$. If $\mu(T+z) > 0$, then $\mu(S_j(T+z)) \geq p_j\mu(T+z) > 0$. Since $T = T(A,\mathcal{C})$ is a $\mathbb{Z}^d$-tile, $\mathcal{C}$ is a complete residue set of A. Hence there exist $c_i \in \mathcal{C}$ and $e \in \mathbb{Z}^d$ such that $z + d_j = c_i + Ae$ and

$$\begin{aligned} S_j(T+z) &= A^{-1}(T + z + d_j) = A^{-1}(T + c_i + Ae) \\ &= A^{-1}(T + c_i) + e \subseteq T + e. \end{aligned}$$

Hence $\mu(T+e) \geq \mu(S_j(T+z)) \geq p_j\mu(T+z) > 0$ and $S_j(T^\circ + z) \subseteq T^\circ + e \subseteq V$. It follows that $S_j(V) \subseteq V$ for all j. This proves (i).

To prove (ii), we assume $V \cap K \neq \emptyset$; then we can find $x_0 \in V \cap K, \varepsilon > 0$ and $J_0 \in \Sigma^k$ such that $S_{J_0}(K) \subseteq B_\varepsilon(x_0) \subseteq V$. We rearrange the distinct S_J 's, $J \in \Sigma^k$, as $\{\phi_j\}_{j=1}^r$ with $\phi_1 = S_{J_0}$ and let $w_j = \sum_{S_J=\phi_j} p_J > 0$. Then we have

(a) $\phi_1(K) = S_{J_0}(K) \subseteq V$ and $\phi_j(V) \subseteq V, j = 1, \ldots, r$;

(b) $K = \bigcup_{j=1}^r \phi_j(K)$

(c) $\mu(\cdot) = \sum_{j=1}^r w_j\mu(\phi_j^{-1}(\cdot))$.

For this new IFS $\{\phi_j\}_{j=1}^r$, let $\tilde{\Sigma}_r = \{2,3,\ldots,r\}$ and let

$$E_n = \bigcup_{J\in\Sigma_r^n\setminus\tilde{\Sigma}_r^n} \phi_J(K).$$

For any $J = j_1 \cdots j_n \in \Sigma_r^n \setminus \tilde{\Sigma}_r^n$, there is an $1 \le s \le n$ such that $j_s = 1$.

Note that $\phi_j(K) \subseteq K$ for all j, so it follows from (a) that $\phi_J(K) \subseteq \phi_{j_1\cdots j_{s-1}}\left(\phi_{j_s}(K)\right) \subseteq \phi_{j_1\cdots j_{s-1}}(V) \subseteq V$ and hence $E_n \subseteq V$. Using (b) and (c), we have

$$\begin{aligned}
1 \quad &\ge \mu(V) \ge \mu(E_n) = \sum_{J\in\Sigma_r^n} w_J\mu\left(\phi_J^{-1}(E_n)\right) \\
&\ge \sum_{J\in\Sigma_r^n\setminus\tilde{\Sigma}_r^n} w_J\mu\left(\phi_J^{-1}(E_n)\right) \ge \sum_{J\in\Sigma_r^n\setminus\tilde{\Sigma}_r^n} w_J\mu\left(\phi_J^{-1}(\phi_J(K))\right) \\
&= \sum_{J\in\Sigma_r^n\setminus\tilde{\Sigma}_r^n} w_J\mu(K) = \sum_{J\in\Sigma_r^n} w_J - \sum_{J\in\tilde{\Sigma}_r^n} w_J \\
&= \left(\sum_{j=1}^{r} w_j\right)^n - \left(\sum_{j=2}^{r} w_j\right)^n = 1-(1-w_1)^n.
\end{aligned}$$

Since $w_1 > 0$, we have $(1-w_1)^n \to 0$ as $n \to \infty$. Therefore $\mu(V) = 1$ and $\mu(\bar{V}) = \mu(K) = 1$. Noting that $\partial V = \bigcup\{\partial T + z: \mu(T+z) > 0, z \in \mathbb{Z}^d\}$, we have $\mu(\partial V) = \mu(\partial T + z) = 0$ for all $z \in \mathbb{Z}^d$.

In view of the above lemma, we need to find a $\mathbb{Z}^d$-tile T such that

$$\mu(\partial T + z) = 0 \ \text{ for all } z \in \mathbb{Z}^d,$$

or equivalently, $(T^\circ + z) \cap K \ne \emptyset$ for some $z \in \mathbb{Z}^d$. This can be achieved by a certain translation of the tile:

Lemma (3.1.8)[138]: Let K be the attractor of the integral affine pair $(A,\mathcal{D})$ and let $T = T(A,\mathcal{C})$ be a $\mathbb{Z}^d$-tile. Then there are $k > 0$ and $e \in \mathcal{C}_k$ such that $T_k := T(A^k, \mathcal{C}_k - e)$ is also a $\mathbb{Z}^d$-tile and $K \cap (T_k^\circ + z) \ne \emptyset$ for some $z \in \mathbb{Z}^d$.

Proof. Let $B_\delta(x_0) \subseteq T^\circ$ and let $a \in \mathbb{N}^d$ be such that $x_0 \in K + a$. Since T is a tile, $a + z \in T$ for some $z \in \mathbb{Z}^d$; hence there exist $c_{i_j} \in \mathcal{C}$ such that $a + z = \sum_{j=1}^{\infty} A^{-j} c_{i_j}$ (recall that $T = \{\sum_{n=1}^{\infty} A^{-n} x_n : x_n \in \mathcal{C}\}$). Let I be the identity matrix and let

$$a_k = -z + (I - A^{-k})^{-1} \sum_{j=1}^{k} A^{-j} c_{i_j}.$$

Note that A^{-k} converges to the zero matrix, hence $\lim_{k\to\infty} a_k = a$. Let k be such that $a_k \in a + B_\delta(0)$. Then $(K + a_k) \cap T^\circ \ne \emptyset$. Let $e = A^k \sum_{j=1}^{k} A^{-j} c_{i_j} = \sum_{j=1}^{k} A^{k-j} c_{i_j}$. We see that $e \in \mathcal{C}_k$ and

$$\begin{aligned}
T_k \quad &= T(A^k, \mathcal{C}_k - e) = T(A^k, \mathcal{C}_k) - \sum_{j=1}^{\infty} A^{-kj} e \\
&= T(A^k, \mathcal{C}_k) - (I - A^{-k})^{-1} A^{-k} e = T - (a_k + z).
\end{aligned}$$

This implies $K \cap (T_k^\circ + z) \ne \emptyset$.

We can now give the main result.

Theorem (3.1.9)[138]: Let $(A,\mathcal{D})$ be an integral affine pair. Then there is $n_0 > 0$ and a digit set $\mathcal{C} \subseteq \mathbb{Z}^d$ with $\#\mathcal{C} = |\det(A^{n_0})|$ such that

(i) $T = T(A^{n_0}, \mathcal{C})$ is a $\mathbb{Z}^d$-tile;

(ii) for any self-affine measure μ associated with $(A, \mathcal{D})$,

$$\mu\left(\bigcup_{z\in\mathbb{Z}^d}(\partial T + z)\right) = 0.$$

Proof. Lemma (3.1.6) implies that there exists an integer $k > 0$ and a digit set $\tilde{\mathcal{C}}$ such that $T(A^k, \tilde{\mathcal{C}})$ is a $\mathbb{Z}^d$-tile. Lemma (3.1.8) shows that there exists an integer $r > 0$ and an integral vector $e \in \tilde{\mathcal{C}}_r$ such that $K \cap \left((T(A^{kr}, \tilde{\mathcal{C}}_r - e))^\circ + z\right) \neq \emptyset$. for some $z \in \mathbb{Z}^d$. Let $n_0 = kr$ and $\mathcal{C} = \tilde{\mathcal{C}}_r - e$. Then $T(A^{n_0}, \mathcal{C})$ is a $\mathbb{Z}^d$-tile and $(T(A^{n_0}, \mathcal{C})^\circ + z) \cap K \neq \emptyset$ for some $z \in \mathbb{Z}^d$. The remaining assertion follows from Lemma (3.1.7).

We will prove Theorem (3.1.1) via several lemmas. For the n_0 and $\mathcal{C}$ defined in Theorem (3.1.9), if $n_0 > 1$, let $\{\phi_j\}_{j=1}^r$ be the distinct S_J 's, $J \in \Sigma^{n_0}$, and $w_j = \sum\{p_J : J \in \Sigma_m^{n_0}, S_J = \phi_j\}$.

Then μ satisfies

$$\mu(\cdot) = \sum_{j=1}^{r} w_j \mu\left(\phi_j^{-1}(\cdot)\right).$$

We can therefore replace $\{S_j\}_{j=1}^m$ and the corresponding probability weights $\{p_j\}_{j=1}^m$ by the IFS $\{\phi_j\}_{j=1}^r$ and $\{w_j\}_{j=1}^r$, respectively. Hence, in order to prove Theorem (3.1.1), we can assume without loss of generality that $n_0 = 1$ in Theorem (3.1.9), i.e., we assume

(H) $T = T(A, \mathcal{C})$ is a $\mathbb{Z}^d$-tile such that $\mu\left(\bigcup_{z\in\mathbb{Z}^d}(\partial T + z)\right) = 0$. This assumption ensures that, for any $z \in \mathbb{Z}^d$, $\mu(T + z) > 0$ if and only if $K \cap (T^\circ + z) \neq \emptyset$.

Let $\psi_i(x) = A^{-1}(x + c_i)$ for some $c_i \in \mathcal{C}$. Since we have two IFS's and so two index sets, to avoid confusion we will use I, J to denote the multi-indices in Σ_m^*, and σ, τ to denote those in $\Sigma_l^* (l = |\det A|)$. Note that $S_I(0) = A^{-n} d_I$ and $S_I(x) = A^{-n}(x + d_I)$. Since S_j and ψ_i are defined by the same matrix A, one can show directly that

$$S_I^{-1}\psi_\sigma(x) = x + c_\sigma - d_I \ \forall I \in \Sigma_m^n, \sigma \in \Sigma_l^n, \tag{6}$$

and

$$\mu(\psi_\sigma(T + e)) = \sum_{I\in\Sigma_m^n} p_I \mu\left(S_I^{-1}(\psi_\sigma(T + e))\right) = \sum_{I\in\Sigma_m^n} p_I \mu(T + e + c_\sigma - d_I) \ \forall e \in \mathbb{Z}^d, \sigma \in \Sigma_l^n.$$

The above reveals the basic relationship of $\{S_j\}_{j=1}^m$ and $\{\psi_i\}_{i=1}^l$ and we make use of this to form a weighted directed graph system. Let

$$\mathcal{E} = \{e_1, \dots, e_N\} = \{e \in \mathbb{Z}^d : K \cap (T^\circ + e) \neq \emptyset\} \tag{7}$$

and

$$\mathcal{B}_n = \{\psi_\sigma(T) + e_u : e_u \in \mathcal{E}, \sigma \in \Sigma_l^n\},\ n \geq 0. \tag{8}$$

Since T is a $\mathbb{Z}^d$-tile by our assumption (H), it is easy to prove

Lemma (3.1.10)[138]: With the above notations, we have

(i) for any $E, F \in \mathcal{B}_n$, $E^\circ \cap F^\circ \neq \emptyset$ if and only if $E = F$;

(ii) $\mathcal{B}_n$ is a partition (with overlaps at the boundary) of the union $\bigcup_{E\in\mathcal{B}_n} E (\supseteq K)$;

(iii) if $z \in \mathbb{Z}^d$ and $E = A^{-n}(T+z)$, then $E \in \mathcal{B}_n$ if and only if there are unique $\sigma \in \Sigma_l^n$ and $e_u \in \mathcal{E}$ such that $c_\sigma + A^n e_u = z$;

(iv) $(T^\circ + e_u) \cap K \neq \emptyset$ for all $e_u \in \mathcal{E}$.

Lemma (3.1.11)[138]: For any $I \in \Sigma_m^n, J \in \Sigma_m^k, \sigma \in \Sigma_l^n, \tau \in \Sigma_l^k$ and $z \in \mathbb{Z}^d$,

$$c_{\sigma\tau} - d_{IJ} + A^{n+k} z \in \mathcal{E} \text{ implies } c_\sigma - d_I + A^n z \in \mathcal{E}.$$

Proof. Observe that $\psi_\sigma(T) + z = A^{-n}(T + c_\sigma + A^n z)$, so (6) implies

$$S_I^{-1}(\psi_\sigma(T) + z) = T + c_\sigma - d_I + A^n z.$$

Since $\mu(S_I^{-1}(E)) = \sum_{J' \in \Sigma_m^k} p_{J'} \mu\big(S_{IJ'}^{-1}(E)\big)$, we have

$$\begin{aligned} \mu(T + c_\sigma - d_I + A^n z) &= \mu\big(S_I^{-1}(\psi_\sigma(T) + z)\big) \geq \mu\big(S_I^{-1}(\psi_{\sigma\tau}(T) + z)\big) \\ &\geq p_J \mu\left(S_{IJ}^{-1}(\psi_{\sigma\tau}(T) + z)\right) \\ &= p_J \mu\big(T + c_{\sigma\tau} - d_{IJ} + A^{n+k} z\big) > 0, \end{aligned}$$

and the lemma follows.

As a crucial step to reformulate the self-affine measure of $\{S_j\}_{j=1}^m$ in terms of the auxiliary IFS$\{\psi_i\}_{i=1}^l$, we have

Lemma (3.1.12)[138]: The family $\{A^{-n}(T+z): z \in \mathbb{Z}^d, n \geq 0\}$ generates the Borel subsets of $\mathbb{R}^d$, and for any $z \in \mathbb{Z}^d$,

$$\mu(A^{-n}(T+z)) = \sum_{v=1}^{N} \left(\sum \left\{p_I : I \in \sum_m^n, z - d_I = e_v\right\}\right) \mu(T + e_v)$$

for some $e_v \in \mathcal{E}$.

Proof. The first part is clear as T is a tile with $\mathbb{Z}^d$ as a tiling set. For the identity we note that

$$\begin{aligned} \mu(A^{-n}(T+z)) &= \sum_{I \in \Sigma_m^n} p_I \mu\big(S_I^{-1}(A^{-n}(T+z))\big) \\ &= \sum_{I \in \Sigma_m^n} p_I \mu(T - d_I + z). \end{aligned}$$

By the definition of $\mathcal{E}, \mu(T - d_I + z) > 0$ if and only if $z - d_I = e_v$ for some $e_v \in \mathcal{E}$. The lemma follows by replacing $z - d_I$ with e_v in the above expression.

It follows from Lemma (3.1.10)(iii) that we only need to consider those sets $A^{-n}(T+z)$ such that $z = c_\sigma + A^n e_u \in \mathcal{E}$. In view of the above lemma, we define $W_i = [w_i(u,v)]_{N \times N}$, $1 \leq i \leq l$, by

$$w_i(u,v) = \begin{cases} p_j, & c_i - d_j + A e_u = e_v \text{ for some } j, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Then we have

Lemma (3.1.13)[138]: For any $\sigma = \sigma_1 \cdots \sigma_n \in \Sigma_l^n$, let $W_\sigma = [w_\sigma(u,v)]$ be the corresponding product matrix. Then

$$w_\sigma(u,v) = \sum \left\{p_I : I \in \sum_m^n, c_\sigma - d_I + A^n e_u = e_v\right\}. \tag{10}$$

Proof. We will use induction. The identity is obviously true for $n = 1$ by the definition of W_i. Assume it is true when $n = k-1$. For $n = k$, let $\sigma = \tau r, \tau \in \Sigma_l^{k-1}, Jj \in \Sigma_m^k$. By

Lemma (3.1.11), $c_\sigma + A^k e_u - d_{Jj} \in \mathcal{E}$ implies $c_\tau + A^{k-1} e_u - d_J \in \mathcal{E}$. Let $W_\tau = [a_{u,v}]_{N\times N}$. By the induction hypothesis we have

$$a_{u,v} = \sum \{p_I : I \in \Sigma_m^{k-1}, c_\tau - d_I + A^{k-1} e_u = e_v\}.$$

This implies that

$$\begin{aligned} w_\sigma(u,v) = & \sum_{t=1}^{N} a_{u,t} w_r(t,v) \\ = & \sum_{t=1}^{N} \left(\sum \{p_I : I \in \Sigma_m^{k-1}, c_\tau - d_I + A^{k-1} e_u = e_t\}\right) \\ & \cdot \left(\sum \{p_i : c_r - d_i + A e_t = e_v\}\right) \\ = & \sum_{t=1}^{N} \sum \{p_{Ii} : I \in \Sigma_m^{k-1}, c_\tau - d_I + A^{k-1} e_u = e_t, c_r - d_i + A e_t = e_v\} \\ = & \sum \{p_J : J \in \Sigma_m^k, c_\sigma - d_J + A^k e_u = e_v\}, \end{aligned}$$

(the last equality follows from Lemma (3.1.11)).

We have assumed in (H) that $n_0 = 1$ in the statement of Theorem (3.1.9). Also we assume all d_j in $\mathcal{D}$ are distinct, otherwise we can combine the corresponding S_j and p_j together. Let $V = \bigcup_{j=1}^{N} (T^\circ + e_j)$. Lemma (3.1.7) implies that V is open and invariant with respect to $\{S_j\}_{j=1}^{m}$, and $\mu(\partial T + z) = 0$ for all $z \in \mathbb{Z}^d$. For

$$\boldsymbol{\mu}(E) = [\mu((E \cap T) + e_1), \dots, \mu((E \cap T) + e_N)]^t$$

we have, by the self-affine identity,

$$\begin{aligned} (\boldsymbol{\mu}(E))_v & = \mu((T \cap E) + e_v) = \sum_{i=1}^{m} p_i \mu(A(T \cap E) + A e_v - d_i) \\ & = \sum_{i=1}^{m} p_i \mu\left(\left(\bigcup_{j=1}^{l} (T + c_j) \cap A(E)\right) + A e_v - d_i\right) \\ & = \sum_{i=1}^{m} p_i \mu\left(\bigcup_{j=1}^{l} \left(T \cap (A(E) - c_j) + c_j\right) + A e_v - d_i\right). \end{aligned}$$

Use $\mu(\partial T + z) = 0$, (9) and the fact that T is a $\mathbb{Z}^d$-tile to obtain

$$\begin{aligned} (\boldsymbol{\mu}(E))_v & = \sum_{i=1}^{m} \sum_{j=1}^{l} p_i \mu\left(T \cap (A(E) - c_j) + c_j + A e_v - d_i\right) \\ & = \sum_{j=1}^{l} \sum_{r=1}^{N} w_j(v,r) \mu\left(T \cap (A(E) - c_j) + e_r\right). \end{aligned}$$

This implies $\boldsymbol{\mu}(E) = \sum_{j=1}^{l} W_j \boldsymbol{\mu}\left(\psi_j^{-1}(E)\right)$.

To prove statement (i), we note that for any $1 \leq u, v \leq N$, our assumption (H) on T implies that $K \cap (T^{\circ} + e_u) \neq \emptyset$, so there exists an integer n and $I \in \Sigma_m^n$ such that $S_I(T + e_v) \subseteq T + e_u$. Since

$$S_I(T + e_v) = A^{-n}(T + e_v + d_I)$$

with $e_v + d_I \in \mathbb{Z}^d$, Lemma (3.1.10)(iii) implies that there exists $\sigma \in \Sigma_l^n$ such that $c_\sigma - d_I + A^n e_u = e_v$; by Lemma (3.1.13), we see that for $W = \sum_{i=1}^m W_i$, the (u, v) entry of W^n is $\geq p_I$ and hence is positive. Thus we have proved that W is irreducible.

For (ii), we first consider the expression $c_i - d_s + Ae_u = e_v$ in (9). We claim that the pair (e_u, c_i) is uniquely determined by e_v and d_s. Indeed, if $c_j - d_s + Ae_{u'} = e_v$, then $A^{-1}(c_i - c_j) = e_{u'} - e_u \in \mathbb{Z}^d$. Since T is a $\mathbb{Z}^d$-tile, $\{c_1, \dots, c_l\}$ is a complete set of residues (mod$A$) [159], and we conclude that $u' = u$ and $i = j$, which yields the claim.

It follows from the claim that distinct pairs (e_u, c_i) and $(e_{u'}, c_j)$ with $w_i(u, v) > 0$ and $w_j(u', v) > 0$ correspond to distinct d_s. Hence

$$\sum_{u=1}^{N} w(u,v) = \sum_{i=1}^{I} \sum_{u=1}^{N} w_i(u,v) \leq \sum_{s=1}^{m} p_s = 1,\ v = 1, \dots, N, \tag{11}$$

i.e., the column sums of W are ≤ 1. On the other hand, by the vector self-affine identity just proved, $[\mu(T + e_1), \dots, \mu(T + e_N)]^t$ is a positive 1eigenvector of W. This implies that all column sums of W must be 1.

The proof is complete.

The above proof yields

Corollary (3.1.14)[138]: With the same assumptions and notations of Theorem (3.1.1), we have

$$\boldsymbol{\mu}(\psi_\sigma(T)) = W_\sigma \boldsymbol{\mu}(T),\ \forall \sigma \in \Sigma^*.$$

We remark that in the above proof, each p_j appears exactly once in each column of W. Also the matrices $\{W_i\}_{i=1}^l$ are not unique, not even the same size. They depend on the choice of $\mathcal{C}$ for the tile T; an example is given for the case $A = [3]$ on $\mathbb{R}$.

For the actual construction of $\boldsymbol{\mu}$ and W_i, we have to find the set $\mathcal{E}$ in the theorem as both the tile T and the attractor K may not be expressed explicitly. We provide an algorithm to construct $\mathcal{E}$ by using the expression in (9).

Proposition (3.1.15)[138]: Let K be the attractor of $(A, \mathcal{D})$, and let $T = T(A, \mathcal{C})$ be a $\mathbb{Z}^d$-tile such that $K \cap (T^{\circ} + z) \neq \emptyset$ for some $z \in \mathbb{Z}^d$ as above. Let $\mathcal{E}_0 = \emptyset$ and let $\emptyset \neq \mathcal{E}_1 \subseteq \mathcal{E}$. Define

$$\mathcal{E}_{n+1} = \mathcal{E}_n \cup \left(\mathbb{Z}^d \cap A^{-1}\big((\mathcal{E}_n \setminus \mathcal{E}_{n-1}) + \mathcal{D} - \mathcal{C}\big)\right),\ n > 0. \tag{12}$$

Then there is an $n > 0$ such that $\mathcal{E}_n = \mathcal{E}_{n+1}$, and for this n we have $\mathcal{E}_n = \mathcal{E}$.

Proof. Since $\mathcal{E}$ is a finite set, we need only prove $\mathcal{E}_n \subseteq \mathcal{E}$ for all $n > 0$, and $\mathcal{E} \subseteq \bigcup_{n>0} \mathcal{E}_n$.

We prove the first inclusion by induction. Assume that $\mathcal{E}_n \subseteq \mathcal{E}$ and let $z \in \mathcal{E}_{n+1} \setminus \mathcal{E}_n$. Then there exist $e \in \mathcal{E}_n$, $d_i \in \mathcal{D}$ and $c_j \in \mathcal{C}$ such that $z = A^{-1}(e + d_i - c_j)$. Hence

$$\mu(T + z) \geq p_i \mu(A(T + z) - d_i) = p_i \mu(AT + e - c_j).$$

Note that $AT \supseteq T + c_j$ and $e \in \mathcal{E}_n \subseteq \mathcal{E}$, so $\mu(T + z) \geq p_i \mu(T + e) > 0$. This implies $\mathcal{E}_{n+1} \subseteq \mathcal{E}$ and induction follows.

For the second inclusion, let $e_v \in \mathcal{E}$. Choose $e_u \in \mathcal{E}_1$. Since $W = \sum_{i=1}^l W_i$ is irreducible, there exist $e_{v_1}, \dots, e_{v_n} \in \mathcal{E}$ with $v_1 = u$ and $v_n = v$ such that $w(v_{j+1}, v_j) >$

0. From the definition of $\{W_i\}_{i=1}^{l}$, we see that $e_{v_{j+1}} \in A^{-1}\left(e_{v_j} + \mathcal{D} - \mathcal{C}\right)$. Therefore $e_v \in \mathcal{E}_n$ from the definition of $\mathcal{E}_j$ and since $e_{v_1} \in \mathcal{E}_1$. Hence $\mathcal{E} \subseteq \bigcup_{n>0} \mathcal{E}_n$.

We illustrate this algorithm by some examples. To conclude, we consider the refinement equation

$$f(x) = \sum_{j=1}^{m} a_j f\left(Ax - d_j\right),\ x \in \mathbb{R}^d, \tag{13}$$

where A and $d_j \in \mathcal{D}$ are as before, and the coefficients $\left\{a_j\right\}_{j=1}^{m}$ are real and satisfy $\sum_{j=1}^{m} a_j = |\det A|$. The L^1-solution of the equation is called a scaling function. In this case, f is supported by K and is unique up to a constant multiple. It is well known that for the scaling function in $\mathbb{R}$ with scaling 2, the analysis depends very much on a vector-valued setup ([141], [142], [154], [165]). For the higher dimensional case, the same technique in the proof of Theorem (3.1.1) can be used for the vector-valued reduction. Below we state such a theorem without proof.

Similarly to the definition of W_i, we define the $N \times N$ matrices $C_i, 1 \leq i \leq l$, by

$$C_i(u, v) = \begin{cases} a_j, & c_i - d_j + Ae_u = e_v \text{ for some } j, \\ 0, & \text{otherwise}. \end{cases}$$

Then $l = |\det A|$ is an eigenvalue of $C := \sum_{j=1}^{l} C_j$. For any function f, we define an N-dimensional vector function $F = (F_1, \ldots, F_N)$ by

$$F_i(x) = \begin{cases} f(x + e_i), & x \in T, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

Theorem (3.1.16)[138]: Let f be a function supported by K, the attractor of $(A, \mathcal{D})$, and let F be defined as above. Then f is an L^1-solution of the refinement equation (13) if and only if F is an L^1-solution of

$$F(x) = \sum_{j=1}^{l} C_j F \circ \psi_j^{-1}(x), \qquad \text{a.e. } x \in \mathbb{R}^d. \tag{15}$$

There is vast literature on scaling functions on $\mathbb{R}$ using the joint spectral radius associated with the above $\{C_i\}_{i=1}^{l}$. Most of the theorems can be generalized directly once the vector-valued form is established. We list one of these as an example ([165],[155]). For any vector $v \in \mathbb{R}^N$, let $H(v)$ be the linear subspace spanned by $\{C_\sigma(I - C_i)v : i = 1, \ldots, l, \sigma \in \Sigma_l^*\}$, where I is the $N \times N$ identity matrix.

Proposition (3.1.17)[138]: With the above notations, let v be a nonzero l-eigenvector of $\sum_{j=1}^{l} C_j$. Then the following three statements are equivalent:

(i) the equation (13) has a nontrivial L^1-solution;

(ii) $\lim_{n\to\infty} l^{-n} \sum_{\sigma\in\Sigma_l^n} \sum_{j=1}^{l} \left\|C_\sigma\left(I - C_j\right)v\right\| = 0$;

(iii) there exists an integer $k > 0$ such that

$$l^{-k} \sum_{\sigma\in\Sigma_l^k} \|C_\sigma w\| < 1\ \forall w \in H(v), \| w \| \leq 1.$$

We will illustrate the construction of the vector form in Theorem (3.1.1). First we consider the well known cases associated with $A = 2$ on $\mathbb{R}$ under our present setting.

Example (3.1.18)[138]: Let $A = 2, \mathcal{D} = \{0, \ldots, m - 1\}$ and let μ be the self-similar measure generated by $(A, \mathcal{D})$ with associated weights $\left\{p_j\right\}_{j=1}^{m}$.

The attractor is $K = [0, m-1]$. According to Theorem (3.1.1), we choose $\mathcal{C} = \{0,1\}$; then $T = [0,1]$. It follows that

$$\mathcal{E} = \{i : \mu(K \cap [i, i+1]) > 0\} = \{0,1, \dots, m-2\}.$$

Let $c_i = i-1, d_j = j-1$ and $e_u = u-1 \in \mathcal{E}$. Then the definition of W_i in (9) implies that $w_i(u,v) = p_j$ if and only if $j = 2u - v + i - 1$. Hence

$$W_1 = [p_{2u-v}] = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ p_3 & p_2 & p_1 & \cdots & 0 \\ p_5 & p_4 & p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_{m-1} \end{bmatrix},$$

$$W_2 = [p_{2u-v+1}] = \begin{bmatrix} p_2 & p_1 & 0 & \cdots & 0 \\ p_4 & p_3 & p_2 & \cdots & 0 \\ p_6 & p_5 & p_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_m \end{bmatrix}.$$

The W_1 and W_2 are uniquely determined regardless of the choice of $\mathcal{C}$, since, for any other digit set $\mathcal{C}'$ such that $T(2, \mathcal{C}')$ is a $\mathbb{Z}$-tile of $\mathbb{R}$, there is an integer k such that $T(2, \mathcal{C}) = T(2, \mathcal{C}') + k$ and the sets $\{T + e_i\}$ are unchanged.

Example (3.1.19)[138]: Let $A = 3, \mathcal{D} = \{0,2,4,6\}$ and let μ be the self-similar measure generated by $(A, \mathcal{D})$ with associated weights $\{p_j\}_{j=1}^{4}$.

The attractor is $K = [0,3]$. If we choose $\mathcal{C} = \{0,1,2\}$, then $T = [0,1]$ and hence $\mathcal{E} = \{0,1,2\}$. Let

$$c_i = i-1,\ d_j = 2j-2,\ e_u = u-1.$$

Then the definition of W_i implies that $w_i(u,v) = p_j$ if and only if $2j = 3u - v + i - 1$. Hence we have

$$W_1 = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ p_4 & 0 & p_3 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & p_1 & 0 \\ p_3 & 0 & p_2 \\ 0 & p_4 & 0 \end{bmatrix}, W_3 = \begin{bmatrix} p_2 & 0 & p_1 \\ 0 & p_3 & 0 \\ 0 & 0 & p_4 \end{bmatrix}.$$

These coincide with the T_0, T_1 and T_2 defined in [166].

If we choose $\mathcal{C} = \{-1,0,1\}$, then $T = [-1/2, 1/2]$ and so $\mathcal{E} = \{0,1,2,3\}$. For this choice,

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ p_2 & 0 & p_1 & 0 \\ 0 & p_3 & 0 & p_2 \\ 0 & 0 & p_4 & 0 \end{bmatrix}, W_2 = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & p_1 \\ p_4 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}, W_3 = \begin{bmatrix} 0 & p_1 & 0 & 0 \\ p_3 & 0 & p_2 & 0 \\ 0 & p_4 & 0 & p_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that, unlike the case in Example (3.1.18), if we choose a different $\mathcal{C}$ (and hence), we may have different $\mathcal{E}$ and W_j.

Also, note that $\mathcal{D} \subset 2\mathbb{Z}$; if we consider $\nu(E) \doteq \mu(2E)$ and choose $\mathcal{C} = \{0,1,2\}$, let $\mu(E) = (\mu(E \cap (2T)), \mu(E \cap (2T) + 2))^t$. Then

$$\boldsymbol{\mu}(E) = \sum_{j=1}^{3} W_j \boldsymbol{\mu}(3E - 2j)$$

with

$$W_1 = \begin{bmatrix} p_1 & 0 \\ p_4 & p_3 \end{bmatrix}, W_2 = \begin{bmatrix} p_2 & p_1 \\ 0 & p_4 \end{bmatrix}, W_3 = \begin{bmatrix} p_3 & p_2 \\ 0 & 0 \end{bmatrix}.$$

This is simpler than the previous two representations.

Example (3.1.20)[138]: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\mathcal{D} = \{[0,0]^t, [1,0]^t, [0,1]^t\}$ and let μ be the self-similar measure generated by $(A, \mathcal{D})$ with associated weights $\{p_j\}_{j=1}^{3}$ and K be the attractor.

Choose $\mathcal{C} = \{[0,0]^t, [1,0]^t\}$. Then $T = T(A, \mathcal{C})$ is a $\mathbb{Z}^2$-tile (the twin dragon). For this example, both K and T are more complicated. Note that $T \subseteq K$ and $\mathcal{D} - \mathcal{C} = \{[0,-1]^t, [1,-1]^t, [0,0]^t, [1,0]^t, [0,1]^t\}$. Let $\mathcal{E}_1 = \{[0,0]^t\}$ By Proposition (3.1.15) we find $\mathcal{E}_i$ inductively and the process stops at the 11th step with

$$\mathcal{E}_{11} = \{[-1,-2]^t, [-2,-1]^t, [-2,0]^t, [0,-2]^t, [-1,-1]^t, [-1,0]^t, [0,-1]^t, [0,0]^t, [-1,1]^t, [1,-1]^t, [1,0]^t, [0,1]^t, [1,1]^t\}.$$

Therefore $\mathcal{E} = \mathcal{E}_{11}$ and there are 13 translates of T° intersecting K. By the definition of W_j in (9), we have

$$W_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_3 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 \\
p_3 & p_2 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_2 & p_3 & p_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 \\
p_2 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & p_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & p_2 & p_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0
\end{bmatrix},$$

$$W_2 = \begin{bmatrix}
0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 \\
p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_2 & 0 & p_3 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & p_2 & 0 & 0 & p_1 & 0 \\
0 & 0 & 0 & p_3 & p_2 & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & p_3 & p_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\
0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

We devoted to the calculation of the Lebesgue measure and Hausdorff dimension of integral selfaffine sets. These problems have been investigated in [175] and [152]. We will make use of the matrix representation to give an alternative approach, which unifies the considerations with the measures and functions and seems to be simpler.

We will use the notations defined with the special set of probabilities $p_1 = \cdots = p_m = 1/m$ (actually any set of positive probabilities $\{p_i\}_{i=1}^{l}$ will do). We also suppose that the assumption (H) holds for the auxiliary affine system.

For any $r \times s$ matrix (or vector if $s = 1$) $B = (b_{ij})$, let $B^{\sim} = (\tilde{b}_{ij})$ be such that $\tilde{b}_{ij}$ equals 1 if $b_{ij} \neq 0$, and equals 0 if $b_{ij} = 0$. For any two nonnegative matrices B and C such that BC is well defined, we have

$$(BC)^{\sim} = (BC^{\sim})^{\sim} \ \forall B, C \geq 0. \tag{16}$$

This follows from the fact that $\sum_s b_{is} c_{sj} \neq 0$ if and only if $\sum_s b_{is} \tilde{c}_{sj} \neq 0$.

We first provide a constructive way to check if K is a self-affine region, i.e., $K^\circ \neq \emptyset$ (see the remark after the theorem). By Theorem (3.1.1), we have

Theorem (3.1.21)[138]: Let K be the attractor generated by the integral affine pair $(A, \mathcal{D})$ and let $T = T(A, \mathcal{C})$ be a $\mathbb{Z}^d$-tile satisfying (H). Then the following statements are equivalent:

(i) K is a self-affine region, i.e., $K^\circ \neq \emptyset$;

(ii) $\mathcal{L}(K) > 0$;

(iii) $W_\sigma \neq 0$ (equivalently $(W_\sigma \mathbf{1})^{\sim} \neq 0$) for any $\sigma \in \Sigma_l^n, n > 0$;

(iv) $T \subseteq \bigcup_{j=1}^N (K - e_j)$, where $\mathcal{E} = \{e_1, \dots, e_N\} = \{e \in \mathbb{Z}^d : \mu(T + e) > 0\}$ as in (8).

Proof. The implications (i)⇒(ii) and (iv)⇒(i) are obviously true. (ii)⇒(iii). If there exist $n > 0$ and $\sigma \in \Sigma_l^n$ such that $W_\sigma = 0$, then the identity in Theorem (3.1.1) implies that

$$\boldsymbol{\mu}(\cdot) = \sum_{\tau \in \Sigma_l^n \setminus \{\sigma\}} W_\tau \boldsymbol{\mu}\left(\psi_\tau^{-1}(\cdot)\right).$$

It follows that $\boldsymbol{\mu}$ is supported by the attractor of the IFS $\{\psi_\tau : \tau \in \Sigma_l^n \setminus \{\sigma\}\}$ which is of Lebesgue measure zero. Since we have $\operatorname{supp} \boldsymbol{\mu} = \bigcup_{j=1}^N \operatorname{supp} \boldsymbol{\mu}_j = \bigcup_{j=1}^N \left(T \cap \left(K - e_j\right)\right)$, it follows that $\mathcal{L}\left(T \cap \left(K - e_j\right)\right) = 0$. Therefore $\mathcal{L}(K) \leq \sum_{j=1}^N \mathcal{L}\left(\left(T + e_j\right) \cap K\right) = 0$, a contradiction.

(iii) ⇒ (iv). Assume that $T \nsubseteq \bigcup_{j=1}^N (K - e_j)$. Then $T^\circ \setminus \bigcup_{j=1}^N (K - e_j)$ is a nonempty open set. Since μ is supported by K, the definition of $\boldsymbol{\mu}$ implies that $\boldsymbol{\mu}$ is supported by $\bigcup_{j=1}^N (K - e_j)$, hence $\boldsymbol{\mu}\left(T^\circ \setminus \bigcup_{j=1}^N \left(K - e_j\right)\right) = 0$. Since T is the attractor of $\{\psi_j\}_{j=1}^l$, there exist $n > 0$ and $\sigma \in \Sigma_l^n$ such that $\psi_\sigma(T) \subseteq T^\circ \setminus \bigcup_{j=1}^N (K - e_j)$. Corollary (3.1.14) implies that $0 = \boldsymbol{\mu}(\psi_\sigma(T)) = W_\sigma \boldsymbol{\mu}(T)$, hence $W_\sigma = 0$, a contradiction.

Note that (16) implies $(W_{\sigma\tau}\mathbf{1})^{\sim} = (W_\sigma (W_\tau \mathbf{1})^{\sim})^{\sim}$, so $(W_\tau \mathbf{1})^{\sim} \in \mathcal{F}_n$ if $|\tau| = n$. Hence there exists an $n > 0$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}$ and, for this $n, \mathcal{F} = \mathcal{F}_n$. Theorem (3.1.21)(iii) can be used to check whether the attractor K has nonvoid interior in at most 2^N steps.

We will use the above setup to consider the Lebesgue measure of a self-affine region. According to the Remark we denote the set of distinct elements of $\mathcal{F} = \{(W_\sigma \mathbf{1})^{\sim} : \sigma \in \Sigma_l^*\}$ by $\{\mathbf{1} = v_1, v_2, \dots, v_r\}$. Let $\alpha(v_t)$ denote the number of nonzero entries of v_t, and let

$$N_{n,t} = \#\{\sigma \in \Sigma_l^n : (W_\sigma \mathbf{1})^{\sim} = v_t\},\ n > 0.$$

Let $\mathcal{B}_n$ be the tile partition defined in (8), and let

$$\mathcal{B}_n^* = \{E \in \mathcal{B}_n : E^\circ \cap K \neq \emptyset\}.$$

It is easy to see that $\bigcap_{n=1}^\infty \bigcup_{E \in \mathcal{B}_n^*} E = K$.

Lemma (3.1.22)[138]: With the above notation, we have

(i) $\mathcal{L}(K) = \lim_{n \to \infty} l^{-n} \# \mathcal{B}_n^*$;

(ii) $\#\mathcal{B}_n^* = \sum_{t=1}^{r} N_{n,t}\alpha(v_t), n = 1,2, \ldots$

Proof. Since T is a $\mathbb{Z}^d$-tile, $\mathcal{L}(T) = 1$. Therefore $\mathcal{L}(E) = l^{-n}\mathcal{L}(T) = l^{-n}$ for $E \in \mathcal{B}_n^*$, and

$$\mathcal{L}(K) = \lim_{n\to\infty} \mathcal{L}\left(\bigcup_{E\in\mathcal{B}_n^*} E\right) = \lim_{n\to\infty} \sum_{E\in\mathcal{B}_n^*} \mathcal{L}(E) = \lim_{n\to\infty} l^{-n}\#\mathcal{B}_n^*.$$

This proves (i).

For each $E \in \mathcal{B}_n$, there exist unique $\sigma \in \Sigma_l^n$ and e_t such that $E = \psi_\sigma(T) + e_t$(Lemma (3.1.10)) and $\boldsymbol{\mu}(\psi_\sigma(\partial T)) = 0$ for all $\sigma \in \Sigma_l^*$(Lemma (3.1.7)) Hence Corollary (3.1.14) implies $\psi_\sigma(T) + e_t \in \mathcal{B}_n^*$ if and only if the t th row of W_σ is nonzero (i.e. the t th coordinate of $(W_\sigma\mathbf{1})^\sim$ is 1). This means that, for any given $\sigma \in \Sigma_l^n$, the number of nonzero rows of W_σ is given by $\#\{\psi_\sigma(T) + e_t \in \mathcal{B}_n^*: 1 \le t \le N\} = \mathbf{1}^t(W_\sigma\mathbf{1})^\sim$. Hence

$$\#\mathcal{B}_n^* = \sum_{\sigma\in\Sigma_l^n} \#\{\psi_\sigma(T) + e_t \in \mathcal{B}_n^*: 1 \le t \le N\} = \sum_{\sigma\in\Sigma_l^n} \mathbf{1}^t(W_\sigma\mathbf{1})^\sim. \tag{17}$$

The identity in (ii) follows directly from this and the definition of $N_{n,t}$. Let G be the $r \times r$ matrix defined by

$$G(s,t) = l^{-1}\#\{i \in \Sigma_l: (W_i v_s)^\sim = v_t\},\ 1 \le s,t \le r, \tag{18}$$

where l is the number of W_i in Theorem (3.1.1) and $\mathcal{F} = \{\mathbf{1} = v_1, v_2, \ldots, v_r\}$. It is clear from the definition that each row sum of G is 1, hence G is a Markov matrix.

Before going on, we will recall some basic facts on the Perron-Frobenius theory on nonnegative matrices [139]. If a Markov matrix B is primitive then it is easy to show that $\lim_{n\to\infty} B^n$ exists. For the Markov matrix G, there is $q > 0$ and a permutation matrix P such that

$$PG^qP^t = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \text{ with } Z = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & R_k \end{bmatrix},$$

where each R_i is primitive with maximal eigenvalue $\varrho(R_i) = 1$, and X has maximal eigenvalue $\varrho(X) < 1$.

It follows that $\lim_{n\to\infty} Z^n = R_0$ exists. Since $\varrho(X) < 1, \lim_{n\to\infty} X^n = 0$ and each R_i is primitive. Therefore $\lim_{n\to\infty} G^{qn} = G_0^{(q)}$ exists and

$$\begin{aligned} PG_0^{(q)}P^t &= \lim_{n\to\infty} PG^{qn}P^t \\ &= \lim_{n\to\infty} \begin{bmatrix} X^n & X^{n-1}Y + X^{n-2}YZ + \cdots + YZ^{n-1} \\ 0 & Z^n \end{bmatrix} \\ &= \begin{bmatrix} 0 & (I-X)^{-1}YR_0 \\ 0 & R_0 \end{bmatrix}. \end{aligned} \tag{19}$$

If in addition G has rational entries, we claim that the limiting matrix $G_0^{(q)}$ also has rational entries. Indeed, in view of (19), we can assume G to be primitive. In that case, 1 is a simple eigenvalue and all the other eigenvalues have moduli < 1. Let $\mathbf{u}$ be the left 1 -eigenvector with $\sum_j u_j = 1$. Then

$$\lim_{n\to\infty} G^n = \begin{bmatrix} \mathbf{u} \\ \vdots \\ \mathbf{u} \end{bmatrix}.$$

Hence the claim will follow if we can show that $\mathbf{u}$ is rational. Note that if G is of order k, then $G - I$ has rank $k - 1$. We can assume

$$G-I=\begin{bmatrix} C & * \\ \mathbf{b} & * \end{bmatrix},$$

where C is a $(k-1)\times(k-1)$ nonsingular matrix. It is checked directly that $[-\mathbf{b}C^{-1},1]$ is a left 1-eigenvector of G by noticing that $G-I$ has rank $k-1$ and has rational coordinates. By uniqueness it equals $\mathbf{u}$ after normalization. This proves the claim.

Theorem (3.1.23)[138]: Let K be the attractor generated by the integral affine pair $(A,\mathcal{D})$. Let G be defined as in (18). Then $\lim_{n\to\infty} G^{nq}$ exists for some q and $\mathcal{L}(K)$ is rational. Furthermore,

$$\mathcal{L}(K)=\sum_{j=1}^{r} a_j\alpha(v_j),$$

where $[a_1,\dots,a_r]$ is the first row of $G_0^{(q)}=\lim_{n\to\infty} G^{qn}$ given by (19).

Proof. First we claim that G satisfies

$$G^n(s,t)=l^{-n}\#\{\sigma\in\Sigma_l^n:(W_\sigma v_s)^\sim=v_t\},\ 1\le s,t\le r. \tag{20}$$

The case $n=1$ follows from the definition. Assume that (20) is true for $n>0$, and consider G^{n+1}. Since $(W_iW_\sigma\mathbf{1})^\sim=(W_i(W_\sigma\mathbf{1})^\sim)^\sim$ by (16), we have

$$\begin{aligned}
G^{n+1}(s,t) &= \sum_{i=1}^{r} G(s,i)G^n(i,t)\\
&= l^{-n-1}\sum_{i=1}^{r}\#\{j\in\Sigma_l:(W_jv_s)^\sim=v_i\}\#\{\sigma\in\Sigma_l^n:(W_\sigma v_i)^\sim=v_t\}\\
&= l^{-n-1}\sum_{i=1}^{r}\#\{\sigma j\in\Sigma_l^{n+1}:(W_jv_s)^\sim=v_i,(W_\sigma v_i)^\sim=v_t\}\\
&= l^{-n-1}\#\{\tau\in\Sigma_l^{n+1}:(W_\tau v_s)^\sim=v_t\},
\end{aligned}$$

proving the claim.

This implies that $N_{n,t}=l^nG^n(1,t), t=1,\dots,r$. By Lemma (3.1.22),

$$\mathcal{L}(K)=\lim_{n\to\infty} l^{-n}\#\mathcal{B}_n^*=\lim_{n\to\infty}\mathbf{e}_1G^n\alpha(v).$$

Now with the choice of q, it follows from the above digression on nonnegative matrices that $G_0^{(q)}=\lim_{n\to\infty}G^{qn}$ exists and $\mathcal{L}(K)$ has the expression as in the theorem follows. That $\mathcal{L}(K)$ is rational also follows from the digression.

We remark that the theorem and (19) allow us to obtain a simple algorithm to calculate $\mathcal{L}(K)$. That $\mathcal{L}(K)$ is rational was proved in [152] using a different method.

Next we consider the case of $K^\circ=\emptyset$. Theorem (3.1.21) implies that $W_\sigma=0$ for some σ. Without loss of generality, let $v_r=0$. Then G has the following expression:

$$G=\begin{bmatrix} G_1 & g \\ 0 & 1 \end{bmatrix},\ g\ne 0.$$

We denote the maximal eigenvalue of G_1 by λ_1.

Lemma (3.1.24)[138]: With the above notations, if $K^\circ=\emptyset$, we have

(i) $0<\lambda_1<1$;

(ii) $\lim_{n\to\infty}\frac{\log\#\mathcal{B}_n^*}{-\log\varrho^n}=d-\frac{\log\lambda_1}{\log\varrho}$, where $\varrho=|\det A|^{-1/d}$.

Proof. (i) For any $1\le s\le r-1$, assume that the t th coordinate of v_s is positive. Since $\sum_j W_j$ is irreducible, there exists W_j such that the t th column of W_j is nonzero, so $W_jv_s\ne 0$. Hence the s th row of G_1 contains at least one nonzero entry, which is $\ge l^{-1}$ by the definition of G_1. This means that each row sum of G_1 is at least l^{-1}, and therefore $\lambda_1 \geq l^{-1} > 0$.

By Theorem (3.1.21), there is a $\sigma \in \Sigma_l^n$ such that $W_\sigma = 0$. (20) implies that all entries in the last column of G^n are positive. This means that all row sums of G_1^n are less than 1. Hence $\lambda_1 < 1$.

(ii) By the definition of $N_{n,i}$, (20) implies $N_{n,s} = l^n G^n(1,s)$ for any $1 \leq s \leq r$. Using Lemma (3.1.22) (ii) and $\alpha(v_r) = 0$, we have

$$\#\mathcal{B}_n^* = l^n[1,0,\dots,0]G_1^n\alpha,$$

where $\alpha = [\alpha(v_1),\dots,\alpha(v_{r-1})]^t$ is positive. Let β be a nonnegative right λ_1-eigenvector of G_1 satisfying $\| \beta \|_1 = 1$. Then $\beta \leq \alpha$ coordinatewise. For any $1 \leq j \leq r-1$, let $\sigma \in \Sigma^*$ be such that $(W_\sigma \mathbf{1})^\sim = v_j$. As (20) implies that there exists $k \geq 0$ such that the $(1,j)$ entry of G_1^k is positive, we have $[1,0,\dots,0]G_1^k\beta = \lambda_1^k[1,0,\dots,0]\beta = \lambda_1^k\beta_1 > 0$. This implies that $\beta_1 = [1,0,\dots,0]\beta > 0$. Hence $\#\mathcal{B}_n^* \geq l^n[1,0,\dots,0]G_1^n\beta = \varrho^{-nd}\lambda_1^n\beta_1 > 0$. It follows that

$$\liminf_{n\to\infty} \frac{\log \#\mathcal{B}_n^*}{-\log \varrho^n} \geq d - \frac{\log \lambda_1}{\log \varrho}.$$

On the other hand, for $\lambda > \lambda_1$, we have $\lambda^{-n}G_1^n \to 0$ as $n \to \infty$. There is a constant $a_\lambda > 0$ such that $\#\mathcal{B}_n^* = l^n[1,0,\dots,0]G_1^n\alpha \leq l^n a_\lambda \lambda^n = \varrho^{-nd}a_\lambda\lambda^n$ $(n > 0)$. Therefore

$$\limsup_{n\to\infty} \frac{\log \#\mathcal{B}_n^*}{-\log \varrho^n} \leq d - \frac{\log \lambda}{\log \varrho}$$

for any $\lambda > \lambda_1$, and (ii) follows by combining the estimations of the limsup and liminf.

Theorem (3.1.25)[138]: Let K be the attractor of an integral affine pair $(A,\mathcal{D})$ with A a similarity. Suppose $K^\circ = \emptyset$. Then

$$\dim_{\mathrm{B}} K = \dim_{\mathrm{H}} K = d - \log\lambda_1/\log\varrho < d.$$

Proof. The theorem follows by showing that

$$d - \log\lambda_1/\log\varrho \geq \overline{\dim}_{\mathrm{B}} K \geq \dim_{\mathrm{H}} K \geq d - \log\lambda_1/\log\varrho. \qquad (21)$$

For $E \subset \mathbb{R}^d$ and $\delta > 0$, let $E_\delta = \{y \in \mathbb{R}^d : \| x - y \| < \delta \text{ for some } x \in E\}$ be the δ-neighborhood of E. Let $\delta_n = \varrho^n$. It is clear that

$$\mathcal{L}(E_{\delta_n}) = \varrho^{nd}\mathcal{L}(T_1) \ \forall E \in \mathcal{B}_n^*, n > 0.$$

(Here T_1 is the 1-neighborhood of T.) It follows from $K_{\delta_n} \subseteq \bigcup_{E\in\mathcal{B}_n^*} E_{\delta_n}$ that

$$\mathcal{L}(K_{\delta_n}) \leq \sum_{E\in\mathcal{B}_n^*} \mathcal{L}(E_{\delta_n}) = (\#\mathcal{B}_n^*)\varrho^{nd}\mathcal{L}(T_1).$$

By Lemma (3.1.24)(ii), we have

$$\liminf_{n\to\infty} \frac{\log \mathcal{L}(K_{\delta_n})}{\log \varrho} \geq \frac{\log \lambda_1}{\log \varrho}.$$

Hence the first inequality in (21) holds in view of [146, Proposition 3.3.2].

The second inequality is well known. For the third, since $(T^\circ + e_i) \cap K \neq \emptyset$ for any $e_i \in \mathcal{E}$, we can find a constant $\varepsilon > 0$ and points $x_i \in K$ such that $B_\varepsilon(x_i) \subseteq T^\circ + e_i, 1 \leq i \leq N$. Choose an invariant open set V of the IFS $\{S_i\}_{i=1}^m$ such that $K \subset V$. Then there exists $k > 0$ such that $|A^{-k}(V)| < \frac{1}{2}\varepsilon$. Hence there exists $I_i \in \Sigma_m^k$ such that

$$S_{I_i}(\bar{V}) \subseteq B_\varepsilon(x_i) \subseteq T^\circ + e_i,\ 1 \leq i \leq N. \qquad (22)$$

For any $E \in \mathcal{B}_n^*$, we can write $E = \psi_\sigma(T) + e_u = S_I(T + e_i)$ for some $\sigma \in \Sigma_l^n, I \in \Sigma_m^n$ and $e_u, e_i \in \mathcal{E}$ (by Lemma (3.1.10)(iii) and the proof of Theorem (3.1.1)). Hence $S_{II_i}(\bar{V}) \subset E^\circ$. Therefore, there exist $I_E \in \Sigma_m^{n+k}$ such that $S_{I_E}(\bar{V}) \subset E$. Let Ψ_n be the set of all those S_{I_E}; they are in one-to-one correspondence with $E \in \mathcal{B}_n^*$, so Ψ_n has cardinality $\#\mathcal{B}_n^*$. We use this class of maps as an IFS; each Ψ_n has contraction ratio ϱ^{n+k}, and from (22), they satisfy the open set condition. Let K_n be the attractor. It follows from the well known identity that

$$\dim_{\mathrm{H}} K_n = -\log(\#\mathcal{B}_n^*)/((n+k)\log \varrho).$$

Since $K_n \subseteq K$, we have $\dim_{\mathrm{II}} K \geq -\log(\#\mathcal{B}_n^*)/((n+k)\log \varrho)$ for all $n > 0$. Hence Lemma (3.1.24) implies that the third inequality in (21) holds.

Example (3.1.26)[138]: Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \mathcal{D} = \{[0,0]^t, [1,0]^t, [0,1]^t, [1,1]^t, [2,-1]^t\}$ and let K be the attractor generated by $(A, \mathcal{D})$. Then $\dim_{\mathrm{H}} K \approx 1.820$.

For this we let

$$\mathcal{C} = \{[0,0]^t, [1,0]^t, [0,1]^t, [-1,0]^t, [0,-1]^t\}.$$

Then, from the remark after Lemma (3.1.6), we see that $T = T(A, \mathcal{C})$ is a $\mathbb{Z}^2$-tile, and $T^\circ \cap K \neq 0$, since $\mathcal{C} = Q \cap \mathbb{Z}^2$. Let $\mathcal{E}_1 = \{[0,0]^t\}$. By using Proposition (3.1.15), we find $\mathcal{E} = \{[0,0]^t, [1,0]^t, [0,1]^t, [1,1]^t\}$.

Let $p_1 = p_2 = p_3 = p_4 = p_5 = 1/5$. By the definition of W_i, we have

$$W_1 = \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ W_2 = \frac{1}{5}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ W_3 = \frac{1}{5}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W_4 = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},\ W_5 = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \mathcal{F}_1 &= \{[1,1,1,1]^t, [1,1,1,0]^t, [0,0,1,1]^t, [1,1,0,0]^t\}, \\ \mathcal{F}_2 &= \mathcal{F}_1 \cup \{[0,0,1,0]^t\},\ \mathcal{F}_3 = \mathcal{F}_2 \cup \{[0,0,0,1]^t\}, \\ \mathcal{F}_4 &= \mathcal{F}_3 \cup \{[0,0,0,0]^t\} = \mathcal{F}_5. \end{aligned}$$

Hence $\mathcal{F} = \mathcal{F}_4$ and $K^0 = \emptyset$. It follows that

$$G_1 = \frac{1}{5}\begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix},$$

so that $\lambda_1 \approx 0.882$ and $\dim_{\mathrm{H}} K = 2 - \log \lambda_1/(-\log 2) \approx 1.820$.

Let $\{B_\delta(x_i)\}_i$ denote a family of disjoint balls with radius δ and centers $x_i \in K$. The L^q – spectrum (or moment scaling exponent) of a self-similar measure μ is defined by

$$\tau(q) = \lim_{\delta \to 0+} \frac{\log\left(\mathrm{s}\ \sum_i \mu\left(B_\delta(x_i)\right)^q\right)}{\log \delta} \tag{23}$$

if the above limit exists, where the supremum is taken over all such families of balls [146,144]. (If the limit does not exist, one can replace the limit by lim inf.)

Proposition (3.1.27)[138]: Let μ be the self-similar measure generated by the integral similar pair $(A, \mathcal{D})$. Then

$$\tau(q)=\lim_{n\to\infty}\frac{\log\sum_{E\in\mathcal{B}_n}\mu(E)^q}{n\log\varrho},\ q>0,$$

where $\mathcal{B}_n=\{\psi_\sigma(T)+e: e\in\mathcal{E},\sigma\in\Sigma_q^n\}$ is a tile-partition of K defined in (8).

Proof. Let $a=1+|T|$ where $|T|$ is the diameter of T. From [169], we know that the limit in the definition of $\tau(q)$ exists for all $q\geq 0$. Hence it suffices to show that

$$\sum_{E\in\mathcal{B}_n}\mu(E)^q\approx\sup\sum_{\mathrm{i}}\mu\left(B_{a\varrho^n}(x_i)\right)^q,\ q,n>0, \tag{24}$$

where the supremum is taken over all families of disjoint balls $\{B_{a\varrho^n}(x_i)\}_i$ with $x_i\in K$. For such a family, let

$$\mathcal{F}_{n,x_i}=\{E\in\mathcal{B}_n: E\cap B_{a\varrho^n}(x_i)\neq\emptyset\},\ \mathcal{G}_{n,E}=\{i: E\cap B_{a\varrho^n}(x_i)\neq\emptyset\}.$$

It is easy to see that there exists a constant $b>0$ such that

$$\max_i\#\mathcal{F}_{n,x_i},\max_{E\in\mathcal{B}_n}\#\mathcal{G}_{n,E}\leq b.$$

Hence

$$\begin{aligned}\sum_i\mu\left(B_{a\varrho^n}(x_i)\right)^q &\leq\sum_i\mu\left(\bigcup\ \{E\in\mathcal{F}_{n,x_i}\}\right)^q\\ &\leq\sum_i b^q\left(\max\ \{\mu(E): E\in\mathcal{F}_{n,x_i}\}\right)^q\\ &\leq b^{q+1}\sum\ \{\mu(E)^q: E\in\mathcal{B}_n\}\ \forall q\geq 0.\end{aligned}$$

It follows that

$$\sup\sum_i\mu\left(B_{a\varrho^n}(x_i)\right)^q\leq b^{q+1}\sum_{E\in\mathcal{B}_n}\mu(E)^q\ \forall q\geq 0. \tag{25}$$

On the other hand, for each $E\in\mathcal{B}_n$ satisfying $\mu(E)>0$, choose a point from $K\cap E$ and denote this set by $\{y_i: i=1,\dots,r\}$. Then we have

$$\sum_{E\in\mathcal{B}_n}\mu(E)^q\leq\sum_{i=1}^r\mu\left(B_{a\varrho^n}(y_i)\right)^q\ \forall q\geq 0,n>0. \tag{26}$$

For the family $\{B_{a\varrho^n}(y_i)\}$, we can choose a disjoint subfamily $\left\{B_{a\varrho^n}\left(y_{i_j}\right)\right\}$ and a number s depending only on T and d such that:

(i) $\#\ \left\{i: B_{a\varrho^n}(y_i)\cap B_{a\varrho^n}\left(y_{i_j}\right)\neq\emptyset\right\}\leq s$ for all i_j (note that $B_{a\varrho^n}(y_i)\cap B_{a\varrho^n}\left(y_{i_j}\right)\neq\emptyset$ implies $E_i\subseteq B_{2a\varrho^n}\left(y_{i_j}\right)$);

(ii) $\mu\left(B_{a\varrho^n}(y_{i_1})\right)=\max_{i\geq 1}\mu\left(B_{a\varrho^n}(y_i)\right)$ and for $j\geq 2,\mu\left(B_{a\varrho^n}\left(y_{i_j}\right)\right)=\max\left\{\mu\left(B_{a\varrho^n}(y_i)\right): B_{a\varrho^n}(y_i)\cap\bigcup_{k=1}^{j-1}B_{a\varrho^n}(y_{i_k})=\emptyset\right\}$;

(iii) any $B_{a\varrho^n}(y_i)$ intersects at least one $B_{a\varrho^n}\left(y_{i_j}\right)$.

Therefore (26) implies

$$\sum_{E\in\mathcal{B}_n}\mu(E)^q\leq s\sum_j\mu\left(B_{a\varrho^n}\left(y_{i_j}\right)\right)^q\leq s\sup\sum_i\mu\left(B_{a\varrho^n}(x_i)\right)^q\ \forall q\geq 0 \tag{27}$$

(the second inequality is by (i)), and (24) follows from (25) and (27).

We can now express $\tau(q)$ in terms of the transition matrices $\{W_i\}_{j=1}^l$ in Theorem (3.1.1).

Theorem (3.1.28)[138]: Let μ be the self-similar measure generated by the integral similar pair $(A, \mathcal{D})$. Then

$$\tau(q) = \lim_{n\to\infty} \frac{\log \sum_{\sigma\in\Sigma_l^n} \|W_\sigma\|_1^q}{n\log \varrho},\ q \geq 0,$$

where W_σ is defined in Theorem (3.1.1), and $\|W_\sigma\|_1$ is the sum of all entries of W_σ.

Proof. Let $\mathbf{e}_i$ be the i th column of the $N \times N$ identity matrix. From Lemma (3.1.10)(iii), Lemma (3.1.12) and Corollary (3.1.14), for all $n > 0$ we have

$$\sum_{E\in\mathcal{B}_n} \mu(E)^q = \sum_{r=1}^{N} \sum_{\sigma\in\Sigma_l^n} \mu(A^{-n}(T + c_\sigma) + e_r)^q = \sum_{r=1}^{N} \sum_{\sigma\in\Sigma_l^n} (\mathbf{e}_r^t W_\sigma \boldsymbol{\mu}(T))^q.$$

Using $\left(\sum_{i=1}^N a_i\right)^q \approx \sum_{i=1}^N a_i^q$ ($N, q > 0$ fixed) for any $a_i \geq 0$, we have

$$\sum_{r=1}^{N} \sum_{\sigma\in\Sigma_l^n} (\mathbf{e}_r^t W_\sigma \boldsymbol{\mu}(T))^q \approx \sum_{\sigma\in\Sigma_l^n} \left(\sum_{r=1}^{N} \mathbf{e}_r^t W_\sigma \boldsymbol{\mu}(T)\right)^q.$$

Therefore

$$\begin{aligned}\tau(q) &= \lim_{n\to\infty} \frac{\log \sum_{\sigma\in\Sigma_l^n} (\sum_{r=1}^N \mathbf{e}_r^t W_\sigma \boldsymbol{\mu}(T))^q}{n\log \varrho} \\ &= \lim_{n\to\infty} \frac{\log \sum_{\sigma\in\Sigma_l^n} \|W_\sigma\|_1^q}{n\log \varrho},\ q \geq 0,\end{aligned}$$

by using the fact that $\sum_{i=1}^N \mathbf{e}_i^t = [1, \dots, 1]$ and $\boldsymbol{\mu}(T)$ is a fixed vector with strictly positive coordinates.

Note that $\tau(q)$ is a concave function. For a concave function g on $\mathbb{R}$, the Legendre transform (or concave conjugate) of g is defined as

$$g^*(\alpha) = \inf\ \{q\alpha - g(q) : q \in \mathbb{R}\}.$$

If g is differentiable at q and $g'(q) = \alpha$, then $g^*(\alpha) = q\alpha - g(q)$.

For a Borel measure μ with support K, we let

$$\alpha(x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$$

be the local dimension of μ at x. Let $K_\alpha = \{x \in K : \alpha(x) = \alpha\}$ be the α-level set of μ. A heuristic principle called multifractal formalism suggests that the dimension spectrum $\dim_{\mathrm{H}} K_\alpha$ should equal the Legendre transform of $\tau(q)$, i.e.,

$$\tau^*(\alpha) = \dim_{\mathrm{H}} K_\alpha.$$

This is the case when the IFS satisfies the OSC ([140],[162]). In the present case, by Theorem (3.1.1), $\sum_{i=1}^l W_i$ is irreducible, hence [149, Theorem (3.1.3)] shows that $\tau(q)$ is differentiable for all $q > 0$. Also the IFS satisfies the weak separation condition under our assumption of integral entries in A and $\mathcal{D}$. Hence [156, Theorem B] implies that the multifractal formalism holds for all $q > 0$:

Theorem (3.1.29)[138]: Let A be an integral similarity matrix. Let μ be the selfsimilar measure generated by the integral pair $(A, \mathcal{D})$. Then the L^q-spectrum $\tau(q)$ of μ is differentiable for all $q > 0$ and

$$\tau^*(\alpha) = \dim_{\mathrm{H}} K_\alpha,\ \forall \alpha = \tau'(q), q > 0.$$

We do not have a complete understanding for $q < 0$. In [163], it is shown that for some special cases the equality of Theorem (3.1.29) also holds for $q < 0$. Note that there is a simple example where $\tau(q), q < 0$, is not differentiable at one point: $A = 3, \mathcal{D} =$

{0,1,2,3} and weights {1/8,3/8,3/8,1/8}[166]; there is a modification of the multifractal formalism for that case [150]. Other interesting cases were considered in [151],[176] and [172].

Section (3.2): Measures with Fourier Frames:

Everyone knows about Fourier series: the exponential functions $\{e^{2\pi inx}: n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2[0,1]$. Perturbations of the set $\mathbb{Z}$ will produce frames for $L^2[0,1]$, or "non-harmonic" Fourier series, see e.g., [187, 208]. This idea was later extended to orthonormal bases or frames of exponentials (Fourier frames) for fractal measures [182, 190, 196, 182, 190, 194, 195, 207, 200, 211, 212, 213, 183, 184, 191].

In [185] the notion of frames of exponentials for an arbitrary measure was extended to that of a frame measure.

Definition (3.2.1)[178]: Let μ be a finite, compactly supported Borel measure on $\mathbb{R}^d$. The Fourier transform of a function $f \in L^1(\mu)$ is defined by

$$\widehat{fd\mu}(t) = \int f(t)e^{-2\pi it\cdot x}d\mu(x),\ (t \in \mathbb{R}^d).$$

Denote by $e_t, t \in \mathbb{R}^d$, the exponential function

$$e_t(x) = e^{2\pi it\cdot x},\ (x \in \mathbb{R}^d).$$

We say that a Borel measure ν is a Bessel measure for μ if there exists a constant $B > 0$ such that for every $f \in L^2(\mu)$, we have

$$\| \widehat{fd\mu} \|^2_{L^2(\nu)} \le B \| f \|^2_{L^2(\mu)}.$$

We call B a (Bessel) bound for ν. We say the measure ν is a frame measure for μ if there exists constants $A, B > 0$ such that for every $f \in L^2(\mu)$, we have

$$A \| f \|^2_{L^2(\mu)} \le \| \widehat{fd\mu} \|^2_{L^2(\nu)} \le B \| f \|^2_{L^2(\mu)}.$$

We call A, B (frame) bounds for ν. We call ν a tight frame measure if $A = B$ and Plancherel measure if $A = B = 1$.

Using the above definitions, we see that a set $E(\Lambda) := \{e_\lambda: \lambda \in \Lambda\}$ is a Fourier frame for $L^2(\mu)$ if and only if the measure $\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a frame measure for μ. $\{e_\lambda: \lambda \in \Lambda\}$ is a tight frame if and only if the measure $\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a tight frame measure for μ. When $E(\Lambda)$ is an orthonormal bases, μ is called a spectral measure and Λ is called a spectrum of μ ([196, 205]).

In [198], Lai proved that for absolutely continuous measures $d\mu = g(x)dx$, if there exists a Fourier frame, then the function g must be bounded above and below on its support. The proof is based on comparing the Beurling densities. We give another approach to prove the theorem. We consider the translates of the original measure μ restricted to some subset F with $\mu(F) > 0$. We denote here by ω the measure $\omega(\cdot) = T_a\mu|_{F+a}(\cdot) = \mu((\cdot + a) \cap (F + a))$ with $a \in \mathbb{R}^d$. We have the following theorem.

Theorem (3.2.2)[178]: Let μ be a finite Borel measure on $\mathbb{R}^d$ and suppose there exists a frame measure for μ, with frame bounds $A, B > 0$. Assume $\omega \ll \mu$. Then

$$\frac{B}{A} \ge \left\|\frac{d\omega}{d\mu}\right\|_\infty.$$

This result shows that the frame bounds control the change of the measure along translations. It will be the key step and it will work also for other general measures which satisfy this translational absolute continuity assumption, not just the Lebesgue measure. First, we will extend the result in [198] by showing that the essential supremum and infimum of the function g will push away the frame bounds of any frame measure for

$d\mu = gdx$. In particular, if g is not bounded below or above on its support, then no such frame measure can exist.

Theorem (3.2.3)[178]: Let $d\mu = gdx$ be an absolutely continuous measure on $\mathbb{R}^d$. If ν is a frame measure for μ with frame bounds $A, B > 0$ then

$$\frac{B}{A} \geq \frac{\mathrm{esssup}_{\mu}(g)}{\mathrm{essinf}_{\mu}(g)}.$$

It has been conjectured that a spectral measure must be uniform on its support. It is known that for discrete measures, spectral measures must have only finitely many atoms and the atoms must have equal weight ([207, 191]). For absolutely continuous measures, spectral measures on finite union of intervals must have uniform density ([207, 181]). Now, an immediate corollary to the inequality in Theorem (3.2.3) is the complete solution to this problem in the case of absolutely continuous spectral measures. More generally, we have

Corollary (3.2.4)[178]: In the hypotheses of Theorem (3.2.3) suppose $\mu = gdx$ admits a tight frame measure. Then g is a characteristic function of its support.

For the case singular measures, the conjecture on spectral self-similar measures of Eaba and Wang in [205] asserts that these spectral measures occur only for equal probability weights and when the digit set $\mathcal{B}$ has a tiling property. We consider the invariant measure associated to an affine iterated function system:

$$\mu_{\mathcal{B}} = \sum_{b \in \mathcal{B}} p_b \mu_{\mathcal{B}} \circ \tau_b^{-1},$$

where $\tau_b(x) = R^{-1}(x + b)$. Assuming also the no overlap condition for $\mu_{\mathcal{B}}$ (i.e. $\mu_{\mathcal{B}}\big(\tau_b(X_B) \cap \tau_{b'}(X_B)\big) = 0$, where $X_{\mathcal{B}}$ is the attractor of the IFS) and checking the translational absolute continuity assumption in Theorem (3.2.10), we prove the following result.

Theorem (3.2.5)[178]: If $\mu_{\mathcal{B}}$ defined above satisfies the no overlap condition and $\mu_{\mathcal{B}}$ admits a frame measure, then all p_b must be equal.

If the affine iterated function system does not satisfy the no overlap condition, it is not known whether we still have the above conclusion. However, with a freedom of choosing the probability weights and the maps, it is of interest to investigate the existence of frame measures in this case. We found that the frame bounds, probability weights and the contraction ratio are closely related. In particular, we can solve the Eaba-Wang conjecture when the self-similar measures is absolutely continuous.

Theorem (3.2.6)[178]: Suppose μ defined in (33) is absolutely continuous with respect to the Lebesgue measure and suppose μ admits a tight frame measure. Then

(i) $p_1 = \cdots = p_N = \lambda$.

(ii) $\lambda = \frac{1}{N}$.

(iii) There exists $\alpha > 0$ such that $\mathcal{D} := \alpha\mathcal{B} \subset \mathbb{Z}$ and $\mathcal{D}$ tiles $\mathbb{Z}$.

To formulate this in another way, Theorem (3.2.6) shows that the only absolutely continuous self-similar measures admitting exponential orthonormal bases/ tight frames/ tight frame measures are the measures supported on a self-similar tile by (ii) and [203]. The statement in (iii) says that tile digit set $\mathcal{D}$ will be a scaled integer tile. This is proved by considering the self-replicating tiling set of the attractor $X_{\mathcal{B}}$.

Our study is based on the translational absolute continuity assumption. We were not able to show that measures with frame measures must have always this property. But from all the examples that we have, this conjecture should be true. We can construct examples of singular measures for which the translational absolute continuity assumption in Theorem (3.2.2) fails.

Our results on frame measures and spectral measures also have applications to Gabor systems (also known as Weyl-Heisenberg systems). Given $g \in L^2(\mathbb{R}^d)$ and a discrete set $\Gamma \in \mathbb{R}^{2d}$, a Gabor system is a set of functions:

$$\mathcal{G}(g,\Gamma) = \{e^{2\pi i a\cdot x} g(x-b): a,b \in \mathbb{R}^d \text{ and } (a,b) \in \Gamma\}.$$

Such a system is called a Gabor frame (Gabor orthonormal basis) if $\mathcal{G}(g,\Gamma)$ is a frame (an orthonormal basis) on $L^2(\mathbb{R}^d)$. If $\Gamma = \Lambda \times \mathcal{J}$, we will write $\mathcal{G}(g,\Lambda,\mathcal{J}) = \mathcal{G}(g,\Lambda \times \mathcal{J})$. Basic theory of Gabor systems can be found in [188].

In [206], the function $g = (\mathcal{L}(\Omega))^{-1/2}\chi\Omega(\mathcal{L}$ denotes the Lebesgue measure) with Λ and $\mathcal{J}$ discrete subsets of $\mathbb{R}^d$ were considered and the following proposition is proved:

Proposition (3.2.7)[178]: [206] Suppose that

(i) $|g| = (\mathcal{L}(\Omega))^{-1/2}\chi_\Omega$ where Ω is a bounded measurable set.

(ii) $\{e_\lambda: \lambda \in \Lambda\}$ is an orthonormal basis of $L^2(\Omega)$ and

(iii) $\mathcal{J}$ is a tiling set of Ω.

Then $\mathcal{G}(g,\Lambda,\mathcal{J})$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^d)$.

$\mathcal{J}$ is a tiling set of Ω means that $\bigcup_{t\in\mathcal{J}}(\Omega+t)$ covers $\mathbb{R}^d$ and the intersection of $\Omega+t$ and $\Omega+t'$ has zero Lebesgue measure for distinct t and t'. In this case, Ω is a translational tile. The proof of this proposition is a standard generalization of the proof that $\mathcal{G}(\chi[0,1],\mathbb{Z},\mathbb{Z})$ is a Gabor orthonormal basis.

In the literature on Gabor systems, there are many examples of functions g that form a Gabor frame with some Γ. For example, if g is a compactly supported function with $|g(x)| \geq c > 0$ on some small cube, then there exists a Γ so that $\mathcal{G}(g,\Gamma)$ is a Gabor frame (see [188, p.125]). However, the requirement for orthonormal bases is more restrictive.

There is no known example of a function g which is not a characteristic function such that its associated Gabor system forms an orthonormal basis with some Γ. Therefore, Liu and Wang conjectured that the converse of the above proposition holds and they proved it in the case when g is supported on an interval. We prove

Theorem (3.2.8)[178]: If the window function g is non-negative, the converse of Proposition (3.2.7) holds.

We will prove Theorem (3.2.2) and Theorem (3.2.3) as Theorem (3.2.10) and Theorem (3.2.12) respectively. Then we give a discussion of the corollaries of Theorem (3.2.3), in particular, Corollary (3.2.4). We consider the affine iterated function system and prove Theorem (3.2.5). We investigate the iterated function system on $\mathbb{R}^1$ and prove Theorem (3.2.6). We present some concluding remarks on frame measures for singular measures. We will focus on Gabor orthonormal bases and prove Theorem (3.2.8).

Definition (3.2.9)[178]: Let μ be a Borel measure on $\mathbb{R}^d$. For a Borel subset E of $\mathbb{R}^d$, we denote by $\mu|_E$ the restriction of μ to the set E, i.e.,

$$\mu|_E(F) := \mu(E \cap F), \text{ for all Borel subsets } F \text{ of } \mathbb{R}^d.$$

For $a \in \mathbb{R}^d$, we denote by $T_a\mu$, the translation by a of the measure μ,i.e.,

$$T_a\mu(F) := \mu(F+a), \text{ for all Borel subsets } F \text{ of } \mathbb{R}^d.$$

This means that

$$\int f dT_a\mu = \int f(x-a)d\mu(x)$$

for all functions $f \in L^1(T_a\mu)$.

We will use the standard notation $\mu \ll \nu$ to indicate that μ is absolutely continuous with respect to ν and we use the notation $\frac{d\mu}{d\nu}$ for its Radon-Nikodym derivative if ν is σ-finite. The following theorem is the key step for the next results.

Theorem (3.2.10)[178]: Let μ be a finite Borel measure on $\mathbb{R}^d$ and suppose there exists a frame measure ν for μ, with frame bounds $A, B > 0$. Assume in addition that there exists a set F of positive measure μ and $a \in \mathbb{R}^d$ such that the measure $T_a(\mu|_{F+a}) \ll \mu$. Then

$$\frac{B}{A} \geq \left\| \frac{dT_a(\mu|_{F+a})}{d\mu} \right\|_\infty . \tag{28}$$

Proof. Let $h := \frac{dT_a(\mu|_{F+a})}{d\mu}$ and let $M := \| h \|_\infty$. Of course, if $M < 1$ there is nothing to prove, so we can assume $M \geq 1$. Restricting to a subset of F we can assume also $M < \infty$. We have for bounded functions :

$$\int f dT_a\mu\Big|_{F+a} = \int f(x-a)d\mu\Big|_{F+a}(x) = \int_{F+a} f(x-a)d\mu(x)$$

and

$$\int f dT_a\mu\Big|_{F+a} = \int f(x)h(x)d\mu(x).$$

Therefore the values of the function f can be ignored outside F and so we can assume f is supported on F and the same is true for h; and we have:

$$\int_{F+a} f(x-a)d\mu(x) = \int_F f(x)h(x)d\mu(x).$$

Since M is the essential supremum of h, given $\epsilon > 0$, we can find a subset E of F, of positive measure μ, such that $M - \epsilon \leq h \leq M$ on E.

Take $f_1 := \frac{1}{\sqrt{\mu(E)}} \chi_E$. We have $\|f_1\|_{L^2(\mu)} = 1$. Also

$$\|f_1(\cdot - a)\|^2_{L^2(\mu)} = \int |f_1(x-a)|^2 d\mu(x) = \int |f_1(x)|^2 h(x) d\mu(x),$$

therefore

$$M - \epsilon \leq \|f_1(\cdot - a)\|^2_{L^2(\mu)} \leq M.$$

We have

$$\left(f_1(\cdot - a)d\mu\right)\hat{}(t) = \int f_1(x-a)e^{-2\pi i t\cdot x}d\mu(x) = e^{-2\pi i t a}\int f_1(x-a)e^{-2\pi i t\cdot(x-a)}d\mu(x)$$

$$= e^{-2\pi i t a}\int f_1(x)e^{-2\pi i t\cdot x}h(x)d\mu(x) = e^{-2\pi i t a}\widehat{f_1 h d\mu}(t)$$

This means that

$$|(f_1(\cdot - a)d\mu)| = \left|\widehat{f_1 h d\mu}\right|.$$

Next we estimate

$$\int |Mf_1 - f_1 h|^2 d\mu = \int |f_1|^2 |h - M|^2 d\mu \leq \epsilon^2 \|f_1\|^2_{L^2(\mu)} = \epsilon^2.$$

Then, using the upper frame bound:

$$\|\widehat{Mf_1 d\mu} - \widehat{fhd\mu}\|^2_{L^2(\nu)} \leq B\|Mf_1 - f_1 h\|^2_{L^2(\mu)} \leq \epsilon^2 B.$$

This implies that

$$\left|\|\widehat{Mf_1d\mu}\|_{L^2(\nu)}^2 - \|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}^2\right| =$$
$$\left|\|\widehat{Mf_1d\mu}\|_{L^2(\nu)} - \|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}\right| \cdot \left(\|\widehat{Mf_1d\mu}\|_{L^2(\nu)} + \|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}\right)$$
$$\leq \|\widehat{Mf_1d\mu} - \widehat{fhd\mu}\|_{L^2(\nu)} \cdot \left(\sqrt{B}M + \sqrt{B}\sqrt{M}\right) \leq \epsilon\sqrt{B}\left(\sqrt{B}M + \sqrt{B}\sqrt{M}\right) =: C\epsilon.$$

Then

$$\frac{\|\widehat{Mf_1d\mu}\|_{L^2(\nu)}^2}{\|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}^2} = 1 + \frac{\|\widehat{Mf_1d\mu}\|_{L^2(\nu)}^2 - \|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}^2}{\|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}^2} \leq 1 + \frac{C\epsilon}{A(M-\epsilon)}.$$

On the other hand

$$\frac{\|\widehat{Mf_1d\mu}\|_{L^2(\nu)}^2}{\|(f_1(\cdot - a)d\mu)\|_{L^2(\nu)}^2} \geq \frac{A\|Mf_1\|_{L^2(\mu)}^2}{B\|f_1(\cdot - a)\|_{L^2(\mu)}^2} \geq \frac{AM^2}{B(M-\epsilon)}.$$

Combining the two inequalities, and letting ϵ go to zero, we obtain

$$\frac{B}{A} \geq M$$

which is the desired inequality.

Definition (3.2.11)[178]: Let μ be a Borel measure on $\mathbb{R}^d$. Let f be a non-negative Borel measurable function. We define the essential supremum of :

$$\operatorname{esssup}_\mu(f) = \| f \|_\infty := \inf \{M \in [0,\infty] : f \leq M, \mu - \text{ a.e. }\}.$$

We define the essential infimum of :

$$\operatorname{essinf}_\mu(f) := \sup \{m \geq 0 : f \geq m, \mu - \text{ a.e. }\}.$$

Theorem (3.2.12)[178]: Let $\mu = gdx$ be an absolutely continuous measure on $\mathbb{R}^d$. If ν is a frame measure for μ with frame bounds $A, B > 0$ then

$$\frac{B}{A} \geq \frac{\operatorname{esssup}_\mu(g)}{\operatorname{essinf}_\mu(g)}.$$

In particular, if $\operatorname{esssup}_\mu(g) = \infty$ or $\operatorname{essinf}_\mu(g) = 0$ then there is no frame measure for μ.

Proof. Let $M := \operatorname{esssup}(g)$ and $m := \operatorname{essinf}(g)$ and assume for the moment that $m > 0$ and $M < \infty$. Take $\epsilon > 0$ arbitrary. Then there exist a set of positive Lebesgue measure C such that $m \leq g(x) \leq m + \epsilon$ for $x \in C$, and a set of positive Lebesgue measure D such that $M - \epsilon \leq g(x) \leq M$ for all $x \in D$.

We need a lemma:

Lemma (3.2.13)[178]: Let C and D be two sets of positive Lebesgue measure in $\mathbb{R}^d$. Then there exists a subset E of C, of positive Lebesgue measure and some $a \in \mathbb{R}^d$ such that $E + a \subset D$.

Proof. Taking subsets we can assume C and D are bounded. Consider the convolution $\chi_C * \chi_{-D}$. We have

$$\chi_C * \chi_{-D}(t) = \int \chi_C(x)\chi_{-D}(t-x)dx = \int \chi_C(x)\chi_{D+t}(x)dx = \mathcal{L}\big(C \cap (D+t)\big).$$

We claim that $\chi_C * \chi_{-D}$ cannot be identically zero. Taking the Fourier transform we have $\widehat{\chi C * \chi_{-D}} = \widehat{\chi_C} \cdot \widehat{\chi_{-D}}$. Both functions are analytic in each variable and not identically zero.

Hence their product cannot be identically zero. Therefore $\chi_C * \chi_{-D}(a) \neq 0$ for some $a \in \mathbb{R}^d$. So $\mathcal{L}(C \cap (D+a)) > 0$. Let $E := C \cap (D+a) \subset C$. Then $E - a \subset D$, and this proves the lemma.

Returning to the proof of the theorem, using Lemma (3.2.13) we find a set E of positive Lebesgue measure and some $a \in \mathbb{R}^d$ such that $m \le g(x) \le m + \epsilon$ and $M - \epsilon \le g(x + a) \le M$ for all $x \in E$.

But the the measure $T_a(\mu|_{E+a}) = g(x + a)dx|_E$ so

$$\left\| \frac{dT_a(\mu|_{E+a})}{d\mu} \right\|_\infty = \left\| \frac{g(x+a)}{g(x)} \Big|_E \right\|_\infty \ge \frac{M - \epsilon}{m + \epsilon}.$$

Letting $\epsilon \to 0$ and using Theorem (3.2.10) we obtain the result.

Assume now $M = \infty$. Then for any N we can find a subset C of positive Lebesgue measure such that $N \le \text{esssup}(g|_C) < \infty$ and $0 < \text{essinf}(g|_C) \le P$ for some fixed P. Take the restriction $\mu|_C$ of the measure μ to C. Then it is clear that ν is also a frame measure for $\mu|_C$ with the same frame bounds. Then we can apply the previous argument to conclude that $B/A \ge N/P$. Letting $N \to \infty$ we obtain a contradiction. A similar argument shows that $\text{essinf}(g) > 0$.

We now give some corollaries of Theorem (3.2.12). The first one is the case when $A = B$.

Corollary (3.2.14)[178]: In the hypotheses of Theorem (3.2.12). suppose $\mu = gdx$ admits a tight frame measure (or Plancherel measure), then g is a constant multiple of a characteristic function.

In other words, if g is not a constant multiple of a characteristic function then the measure $\mu = gdx$ does not admit tight frame measures; in particular it does not admit tight frames of weighted exponential functions $\{w_\lambda e_\lambda : \lambda \in \Lambda\}$, where $w_\lambda \in \mathbb{C}$ for all $\lambda \in \Lambda$.

Proof. From Theorem (3.2.12), we see that if $A = B$ then $\text{esssup}_\mu(g) = \text{essinf}_\mu(g)$ which means that g is a characteristic function. The second statement follows by noting that weighted frames of exponentials correspond to discrete frame measures $\nu = \sum_{\lambda \in \Lambda} |w_\lambda|^2 \delta_\lambda$, where δ_λ is the Dirac measure at λ.

If we replace the Lebesgue measure by general Hausdorff measure, we were not able to prove whether Theorem (3.2.12) will hold since Lemma (3.2.13) cannot be generalized to Hausdorff measures; we have the following example.

Example (3.2.15)[178]: Let C be the set of numbers in $[0,1]$ that can be represented in base 10 using digits $\{0,1\}$ and D be the same as C except the digits are $\{0,2\}$. Then there is no $E \subset C$ with positive Hausdorff dimension (so none of its Hausdorff measures will be positive) such that $E + a \subset D$ for some $a \in \mathbb{R}$.

Proof. It is easy to see that $C - C$ is the set of numbers in $[-1/2,1/2]$ that have a base 10 representation with digits in $\{-1,0,1\}$, while $D - D$ is the set of numbers in $[-1/2,1/2]$ that have a base 10 representation with digits in $\{-2,0,2\}$. Hence, $(C - C) \cap (D - D) = \{0\}$.

Suppose there exists $E \subset C$ with positive Hausdorff dimension such that $E + a \subset D$ for some $a \in \mathbb{R}$. Then $E - E \subset C - C$, and $(E + a) - (E + a) \subset D - D$. But $-E = (E + a) - (E + a)$, this implies that $E - E \subset (C - C) \cap (D - D)$. Hence, $E - E = \{0\}$. This means that E has at most one point, so it has zero Hausdorff dimension. This is a contradiction.

However, as Theorem (3.2.10) holds for general measures, we still have the following corollary.

Corollary (3.2.16)[178]: Let $\mathcal{H}^s$ be the Hausdorff measure of dimension $s > 0$ on $\mathbb{R}^d$. Let $d\mu = g(x)d\mathcal{H}^s(x)$ where g is some non-negative Borel measurable function whose support Ω is a compact set with $0 < \mathcal{H}^s(\Omega) < \infty$. Suppose there exists a Borel set E and some $a \in \mathbb{R}^d$ such that $E, E + a \subset \Omega$ and such that there exist constants $0 < m, M < \infty$ with

$$g(x) \leq m \text{ for all } x \in E \text{ and } g(x) \geq M \text{ for all } x \in E + a.$$

Then for any frame measure ν for μ, its frame bounds A, B satisfy the inequality

$$\frac{B}{A} \geq \frac{M}{m}.$$

Proof. Since $\mathcal{H}^s$ is translation invariant, we have for $\in E$:

$$\frac{dT_a(\mu|_{E+a})}{d\mu}(x) = \frac{g(x+a)}{g(x)} \geq \frac{M}{m}.$$

The conclusion follows from Theorem (3.2.10).

Definition (3.2.17)[178]: Let R be a real $d \times d$ expansive matrix, i.e., all its eigenvalues λ have absolute value $|\lambda| > 1$. Let $\mathcal{B} = \{b_1, \cdots, b_N\}$ be a finite subset of $\mathbb{R}^d$ and let $(p_{b_i})_{i=1}^N$ be a set of positive probability weights, $p_{b_i} > 0$ and $\sum_{i=1}^N p_{b_i} = 1$. We define the affine iterated function system (IFS)

$$\tau_{b_i}(x) := R^{-1}(x + b_i), \ (x \in \mathbb{R}^d, i = 1, \cdots N).$$

According to Hutchinson [193], there exists a unique compact set $X_{\mathcal{B}}$ called the attractor that has the invariance property

$$X_{\mathcal{B}} = \bigcup_{i=1}^N \tau_{b_i}(X_{\mathcal{B}}).$$

Moreover, in this case

$$X_{\mathcal{B}} = \left\{ \sum_{n=1}^{\infty} R^{-n} b_n : b_n \in \mathcal{B} \text{ for all } n \in \mathbb{N} \right\}. \tag{29}$$

Also, there is a unique Borel probability measure $\mu_{\mathcal{B}}$ on $\mathbb{R}^d$ called the invariant measure, such that

$$\mu_{\mathcal{B}}(E) = \sum_{\mathrm{i}=1}^{\mathrm{N}} p_{b_i} \mu_{\mathcal{B}}\left(\tau_{b_i}^{-1}(E)\right), \ \text{ for all Borel sets } E. \tag{30}$$

In addition, the measure $\mu_{\mathcal{B}}$ is supported on the attractor $X_{\mathcal{B}}$. We will write $X = X_{\mathcal{B}}$ and $\mu = \mu_{\mathcal{B}}$ when there is no confusion. We will call the attractor and the invariant measure a self-similar set and a self-similar measure respectively if $R^{-1} = \lambda O$ for some $0 < \lambda < 1$ and orthogonal matrix O.

If for all $i \neq j, i, j \in \{1, \cdots, N\}$, we have $\mu\left(\tau_{b_i}(X) \cap \tau_{b_j}(X)\right) = 0$ then we say that the affine IFS has no overlap.

It is convenient to introduce some multiindex notation for a given affine IFS: let $\Sigma = \{1, \cdots, N\}, \Sigma^n = \underbrace{\Sigma \times \cdots \times \Sigma}_{n}$ and $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma^n$, the set of all finite words. Given $I = i_1 \cdots i_n \in \Sigma^n, \tau_I(x) = \tau_{b_{i_1}} \circ \cdots \circ \tau_{b_{i_n}}(x), p_I = p_{b_{i_1}} \cdots p_{b_{i_n}}$ and $X_I = \tau_I(X)$. By iterating the invariant identity of X, it is easy to see that

$$X = \bigcup_{I \in \Sigma^n} X_I. \tag{31}$$

Finally, we write $I^n = \underbrace{I \cdots I}_{n}$ where IJ denotes concatenation of the words I and J. In this case, $\tau_{I^n}(x) = \tau_I \circ \cdots \tau_I(x) = \tau_I^n(x)$.

We recall that, for the self-similar IFS, the well known open set condition (OSC) states that there exists open set U such that

$$\bigcup_{i=1}^{N} \tau_{b_i}(U) \subset U \text{ and } \tau_{b_i}(U) \cap \tau_{b_j}(U) = \emptyset \text{ for all } i \neq j$$

This condition is fundamental in fractal geometry. Before going to the main theorem in this section, we first clarify the relation between OSC and no overlap condition using theorems in [210] and [202].

Proposition (3.2.18)[178]: If $\mu = \mu_{\mathcal{B}}$ is a self-similar measure, then the open set condition implies the no overlap condition of the measures μ.

Proof. By [210], we can choose an open set U such that $U \cap X \neq \emptyset$. Pick $x \in U \cap X$, then there exists a ball of radius ϵ and centered at x, denoted by $B_\epsilon(x)$, is a subset of U. On the other hand, from (31) we have for all $n > 0$ there exists some $I \in \Sigma^n$ such that $x \in X_I$ (since $\in X$). As the diameter of X_I is tending to 0 as n tends to infinity, it shows that for n large, $X_I \subset B_\epsilon(x) \subset U$. Writing $I = i_1 \cdots i_n$, by iterating the invariance equation (30),

$$\mu(U) \geq \mu(X_I) \geq p_{b_1}\mu\left(X_{i_2 \cdots i_n}\right) \geq \cdots \geq p_{b_{i_1}} \cdots p_{b_{i_n}} > 0.$$

We can then use Theorem 2.3 in [202] to conclude that $\mu(U) = 1$. Writing also $U_b := \tau_b(U)$ with $b \in \mathcal{B}$, by Corollary 2.5 in [202], $\mu(\partial U_b) = 0$, where ∂U_b denotes the boundary of U_b. As $X \subset \bar{U}$, the closure of U, we have $\tau_b(X) \subset \overline{U_b}$ and hence by $U_{b_i} \cap U_{b_j} = \emptyset$ from the OSC,

$$\mu\left(\tau_{b_i}(X) \cap \tau_{b_j}(X)\right) \leq \mu\left(\overline{U_{b_i}} \cap \overline{U_{b_j}}\right) = \mu_B\left(U_{b_i} \cap U_{b_j}\right) = 0.$$

It is not known whether the no overlap condition implies the OSC. We know that the post-critically finite (p.c.f.) fractals (the intersections consist only of finite points) introduced by Kigami [197] satisfy the no overlap condition. However, except for some partial results in [179] and [186], it is still an open question whether all p.c.f. fractals have the OSC.

Much less is known for affine iterated function system. We just know that if the OSC is satisfied, we can also choose U to be an open set with non-empty intersection with the invariant set [189]. However, we do not know whether Proposition (3.2.18) holds in affine IFS.

We can now prove the main theorem using Theorem (3.2.10).

Theorem (3.2.19)[178]: Let $(\tau_b)_{b \in \mathcal{B}}, (p_b)_{b \in \mathcal{B}}$ be an affine iterated function system with no overlap as in Definition (3.2.17). Suppose the invariant measure μ admits a frame measure. Then all the probabilities $p_b, b \in \mathcal{B}$ must be equal.

Lemma (3.2.20)[178]: Pick two elements $b \neq c$ in $\mathcal{B}$ and let $n \in \mathbb{N}$. Define $b^{(n)} := b + Rb + \cdots + R^{n-1}b$ and similarly for $c^{(n)}$. Let $a := R^{-n}\left(c^{(n)} - b^{(n)}\right)$ and $F = \tau_b^n(X)$ (i.e. $\tau_b \circ \cdots \circ \tau_b(X)$ for n compositions).

Consider the measure $T_a(\mu|_{F+a})$ with the notation as in Definition (3.2.9). Then this measure is supported on F, it is absolutely continuous with respect to μ and the Radon-Nikodym derivative is constant on F :

$$\frac{dT_a(\mu|_{F+a})}{d\mu} = \frac{p_c^n}{p_b^n}.$$

Proof. It is easy to see that $\tau_b^n(x) = R^{-n}x + R^{-n}b^{(n)}$ and therefore $\tau_b^n(x) + a = \tau_c^n(x)$ for any $x \in \mathbb{R}^d$. This implies that $F + a = \tau_c^n(X)$, so the measure $T_a(\mu|_{F+a})$ is supported on $\tau_b^n(X)$. Also, we have $\tau_b^{-n}(x) = R^n x - b^{(n)}$.

For any b in $\mathcal{B}$, we consider a arbitrary Borel set E of $\tau_b^n(X)$. We note that $\tau_b^n(X) \subset \tau_b(X)$. By the fact that $\mu \circ \tau_{b'}^{-1}$ is supported on $\tau_{b'}(X)$, the no overlap condition and the invariance identity (30), we get that for all $b' \neq b$,

$$\mu\left(\tau_{b'}^{-1}(E)\right) \leq \mu \circ \tau_{b'}^{-1}(\tau_b(X) \cap \tau_{b'}(X)) \leq p_{b'}^{-1}\mu(\tau_b(X) \cap \tau_{b'}(X)) = 0$$

and hence for any b in B

$$\mu(E) = \sum_{b' \in \mathcal{B}} p_{b'}\mu\left(\tau_{b'}^{-1}(E)\right) = p_b\mu\left(\tau_b^{-1}(E)\right) = \cdots = p_b^n\mu\left(\tau_b^{-n}(E)\right).$$

Now for a Borel subsel E of F, we have that $E + a$ is conlained in $F + a = \tau_c^n(X)$ and thus

$$T_a(\mu|_{F+a})(E) = \mu(E + a) = p_c^n\mu(\tau_c^{-n}(E + a)) = p_c^n\mu\left(R^n(E + a) - c^{(n)}\right) =$$

$$p_c^n\mu\left(R^n E - b^{(n)}\right) = p_c^n\mu(\tau_b^{-n}(E)) = \frac{p_c^n}{p_b^n}\mu(E).$$

This establishes the absolute continuity and also that the density is exactly p_c^n/p_b^n.

Returning to the proof of the theorem, if we have a frame measure with frame bounds A and B, then by Theorem (3.2.10) and Lemma (3.2.20), we have that

$$\frac{B}{A} \geq \frac{p_b^n}{p_c^n} \text{ for all } b, c \in \mathcal{B} \text{ and } n \in \mathbb{N}.$$

This implies that all the probabilities p_b have to be equal.

We focus on affine iterated function systems that do not satisfy the no overlap condition. We will prove some general results on $\mathbb{R}^d$ and then apply them to special cases. From the proof of Theorem (3.2.19), we need to explore the following two questions:

(i) Given any Borel measures μ, is the measure $T_a(\mu|_{F+a})$ absolutely continuous with respect to μ for Borel sets F in the support of μ with positive measure in μ?

(ii) If $\mu = \mu_{\mathcal{B}}$, how to estimate $\mu(\tau_I(X))$?

In answering these questions, we found the results in [192] particularly useful, for the case of self-similar invariant measures. Recall that $\mathcal{H}^\alpha$ denotes the α-Hausdorff measure. We collect their results in the following theorem.

Theorem (3.2.21)[178]: [192] Let $\mu = \mu_{\mathcal{B}}$ be the self-similar measure defined in Definition (3.2.17). Let $R = \lambda^{-1}O$ for some $0 < \lambda < 1$ and orthogonal matrix O. Then

(i) If $\mu \ll \mathcal{H}^\alpha|_X$, then $\mathcal{H}^\alpha|_X \ll \mu$.

(ii) If $\mu \ll \mathcal{L}|_X$ and the Radon-Nikodym derivative has an essential upper bound, then $p_{b_j} \leq \lambda^d$ for all j.

For the first question, when the measure is self-similar, the following is a simple sufficient condition.

Proposition (3.2.22)[178]: Suppose $\mu = \mu_{\mathcal{B}}$ is self-similar and $0 < \mathcal{H}^\alpha(X) < \infty$. If the measure $\mu \ll \mathcal{H}^\alpha|_X$, then for any Borel sets F in the support of μ and for any $a, T_a(\mu|_{F+a}) \ll \mu$.

Proof. By Theorem (3.2.21)(i), $\mathcal{H}^\alpha|_X \ll \mu$ also. Hence, if $E \subset F$ is a Borel set such that $\mu(E) = 0$, then $\mathcal{H}^\alpha|_X(E) = 0$. But $F \subset X$, so $\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha|_X(E) = 0$. As the Hausdorff measure is invariant under translations, $\mathcal{H}^\alpha|_X(E + a) \leq \mathcal{H}^\alpha(E + a) = 0$. Hence, by $\mu \ll \mathcal{H}^\alpha|_X$,

$$T_a(\mu|_{F+a})(E) = \mu(E + a \cap F + a) \leq \mu(E + a) = 0.$$

But $T_a(\mu|_{F+a})$ is supported on F, hence we have established the absolute continuity.

The investigation of the second question is more difficult when there is overlap. For a self-affine measure in (30), we can iterate the invariance identity n times and then identify the maps τ_I, τ_J such that $\tau_I = \tau_J := \tau$. Denote by $\mathcal{A}_n$ the set all equivalence classes under this identification, for the compositions of n maps that coincide, and let p_τ be the sum of the weights in the equivalence class (i.e., for $\in \mathcal{A}_n, p_\tau = \sum\{p_I : \tau_I = \tau\}$). We therefore have

$$\mu = \sum_{\tau \in \mathcal{A}_n} p_\tau \mu \circ \tau^{-1}. \tag{32}$$

Note that if there is no overlap, $\mathcal{A}_n = \{\tau_I : I \in \Sigma^n\}$ and $p_{\tau_I} = p_I$. In this case, $(\tau_I^n(X)) = p_I^n$. To extend our results to IFSs with overlap, we introduce the following definition.

Definition (3.2.23)[178]: Consider the IFS as in Definition (3.2.17). Given $\tau \in \mathcal{A}_n$, we define x_τ to be the fixed point of τ if $x_\tau = \tau(x_\tau)$. We say that the IFS satisfies the fixed point condition if there exists $k > 0$ and $\tau \in \mathcal{A}_k$ such that the fixed point

$$x_\tau \notin \tilde{\tau}(X) \text{ for all } \tilde{\tau} \neq \tau, \tilde{\tau} \in \mathcal{A}_k.$$

The following proposition shows that fixed point condition gives a partial answer to the second question.

Proposition (3.2.24)[178]: Given an IFS and suppose that the fixed point condition is satisfied for some $k \in \mathbb{N}$ and $\tau \in \mathcal{A}_k$. Then there exists n_0 such that for all $n \geq n_0$,

$$\mu(\tau^n(X)) = C p_\tau^n$$

where C is independent of n.

Proof. Writing $\tau = \tau_I$ for some $I = i_1 \cdots i_k \in \Sigma_k, b_I = b_{i_k} + \cdots + R^{k-1} b_{i_1}$ and since $x_\tau = \tau_I(x_\tau) = \tau_I^n(x_\tau)$ for all n, we have

$$x_\tau = \sum_{n=1}^{\infty} R^{-kn} b_I.$$

By (29), $x_\tau \in X$. Moreover, $x_\tau = \tau^n(x_\tau) \in \tau^n(X)$ for all $n \in \mathbb{N}$. Since $\tau^n(X)$ and $\tilde{\tau}(X), \tilde{\tau} \in \mathcal{A}_k$, are compact sets and the diameter of $\tau^n(X)$ tends to 0 , from the fixed point condition, there exists n_0 such that for all $n \geq n_0, \tau^n(X) \cap \tilde{\tau}(X) = \emptyset$ for all $\tilde{\tau} \neq \tau$ and $\tilde{\tau} \in \mathcal{A}_k$.

For all $n \geq n_0$, using the invariance identity (32),

$$\mu(\tau^n(X)) = \sum_{\tau' \in \mathcal{A}_k} p_{\tau'} \mu\big(\tau'^{-1}(\tau^n(X))\big).$$

From the above, we have $\mu\big(\tau'^{-1}(\tau^n(X))\big) = 0$ if $\tau' \neq \tau$. Hence,

$$\mu(\tau^n(X)) = p_\tau \mu\big(\tau^{-1}(\tau^n(X))\big) = p_\tau \mu(\tau^{n-1}(X)).$$

Inductively, $\mu(\tau^n(X)) = p_\tau^{n-n_0} \mu(\tau^{n_0}(X)) = C p_\tau^n$, where $C = p_\tau^{-n_0} \mu(\tau^{n_0}(X))$ is independent of n.

If we assume that the invariant measure is self-similar and is absolutely continuous with respect to the Lebesgue measure, we can use Theorem (3.2.3) and Proposition (3.2.24) to obtain the following:

Theorem (3.2.25)[178]: Let μ be a self-similar measure which is absolutely continuous with respect to the Lebesgue measure supported on X. If μ admits a frame measure then $p_\tau \leq \lambda^{dk}$ for all $\tau \in \mathcal{A}_k$. Suppose furthermore that the fixed point condition is satisfied for some $k \in \mathbb{N}$ and $\tau \in \mathcal{A}_k$, then, for these particular k and $\tau, p_\tau = \lambda^{dk}$.

Proof. Since $\mu \ll \mathcal{L}|_X$, we can write $d\mu = g(x)dx$ with g is supported on X. As the measure is absolutely continuous with respect to the Lebesgue measure, $\mathcal{L}(X) > 0$. By Theorem (3.2.3), g has finite, positive essential upper and lower bounds on X. By Theorem (3.2.21) (ii) applied to (32) (where the expanding matrix now becomes $\lambda^k O^k$), $p_\tau \leq \lambda^{dk}$ for all j. We now establish $p_\tau \geq \lambda^{dk}$.

Suppose now fixed point condition is satisfied but $p_\tau < \lambda^{dk}$. By Proposition (3.2.24), we will then have for all $n \geq n_0$ with n_0 defined in Proposition (3.2.24) that,

$$\frac{\mu(\tau^n(X))}{\mathcal{L}(\tau^n(X))} = \frac{\mu(\tau^n(X))}{\lambda^{dkn}\mathcal{L}(X)} \leq C\left(\frac{p_\tau}{\lambda^{dk}}\right)^n \to 0 \text{ as} n \to \infty.$$

But the density g has a positive essential lower bound $m > 0$, so

$$\frac{\mu(\tau^n(X))}{\mathcal{L}(\tau^n(X))} = \frac{1}{\mathcal{L}(\tau^n(X))}\int_{\tau^n(X)} g(x)dx \geq m > 0.$$

This is a contradiction. Hence, $p_\tau = \lambda^{dk}$. This completes the proof.

If the fixed point condition is satisfied by words which contain all the digits, then all the probability weights are equal.

Corollary (3.2.26)[178]: Let $\mu = \mu_{\mathcal{B}}$ be a self-similar measure which is absolutely continuous with respect to the Lebesgue measure supported on X and assume that μ has a frame measure. Suppose there exists a word I in Σ^n such that I contains all the digits in $\{1, \dots, N\}$ and such that the fixed point x_I of the map τ_I does not belong to any of the sets $\tau_J(X)$ for $J \in \Sigma^n$, $J \neq I$. Then all the probabilities p_{b_i} are equal, $\lambda^d = \frac{1}{N}$, there is no overlap and $\frac{d\mu}{d\mathcal{L}} = \frac{1}{\mathcal{L}(X)}\chi_X$.

Proof. The condition on x_I given in the hypothesis implies that the only word J for which $\tau_J = \tau_I$ is $J = I$. So, if $I = i_1 \dots i_n$ then $p_{\tau_I} = p_I = \prod_{k=1}^n p_{b_{i_k}}$. From Theorem (3.2.25), we have that $p_{b_i} \leq \lambda^d$ for all $i \in \{1, \dots, N\}$. Also, since the fixed point condition is satisfied for n and τ_I, we get that $p_{\tau_I} = \lambda^{dn}$. But then

$$\lambda^{dn} = \prod_{k=1}^{n} p_{b_{i_k}} \leq \lambda^{dn}.$$

This implies that $p_{b_{i_k}} = \lambda^d$ for all $k \in \{1, \dots, n\}$. Since all the digits in $\{1, \dots, N\}$ appear among the elements $\{i_1, \dots, i_N\}$ we obtain that all the probabilities p_{b_i} are equal to λ^d. Since they sum up to 1 , this implies that $\lambda^d = \frac{1}{N}$. Since also $\mathcal{L}(X)$ is positive, X is a self-similar tile [203] on $\mathbb{R}^d$. The rest of the statements will then follow. On the other hand, we can also prove it directly.

Since $\mathcal{L}(X)$ is positive, we apply the Lebesgue measure to the invariance identity of the attractor to get

$$\mathcal{L}(X) \leq \sum_{i=1}^{N} \mathcal{L}\left(\tau_{b_i}(X)\right) = N\lambda^d\mathcal{L}(X) = \mathcal{L}(X).$$

This implies that the sets $\tau_{b_i}(X)$ have overlap of zero Lebesgue measure. Since μ is absolutely continuous, this means that the IFS has no overlap. We can then check that the Lebesgue measure on X, rescaled by $\frac{1}{\mathcal{L}(X)}$ to get a probability measure, is invariant for the IFS. By the uniqueness of the invariant measure we get that $\frac{d\mu}{d\mathcal{L}} = \frac{1}{\mathcal{L}(X)}\chi_X$.

We are not sure whether for any affine IFS there are always fixed points that satisfy the conditions in Theorem (3.2.25) or Corollary (3.2.26). However, the fixed point conditions for finite iterations can many times be checked in concrete situations by an algorithm. We will also see that there are always such fixed points for IFSs on $\mathbb{R}^1$.

We now apply the previous results to some IFSs with overlap to determine whether they have frame measures. Although these results can be applied on $\mathbb{R}^d$, we restrict our attention to $\mathbb{R}^1$ and there is no loss of generality to consider, upon rescaling, IFSs with functions $\tau_{b_i}(x) = \lambda x + b_i$, for $0 < \lambda < 1, i = 1,..,N$ and

$$\mathcal{B} = \{0 = b_1 < \cdots < b_N = 1 - \lambda\}.$$

In this case, the self-similar set X is a subset $[0,1]$. The self-similar measure with weights p_i is the unique Borel probability measure satisfying

$$\mu = \sum_{i=1}^{N} p_i \mu \circ \tau_{b_i}^{-1}. \tag{33}$$

Theorem (3.2.27)[178]: Suppose the measure μ defined in (33) is absolutely continuous with respect to $\mathcal{H}^\alpha|_X$ and $0 < \mathcal{H}^\alpha(X) < \infty$. Then

(i) If μ admits a frame measure, then $p_1 = p_N$.

(ii) If $\alpha = 1$ (i.e. $\mu \ll \mathcal{L}|_X$) and μ admits a frame measure, then $p_j \le \lambda$ for all j and $p_1 = p_N = \lambda$.

Proof. (i) Note that $\tau_1(0) = 0$, so the fixed point of τ_1 is 0 . On the other hand, $\tau_{b_i}(X) \subset [b_i, b_i + \lambda]$. Hence, the fixed point condition holds for $I = 1$. Proposition (3.2.24) implies that there exists n_0 such that

$$\mu(\tau_{1^n}(X)) = C_1 p_1^n, \text{ for all } n \ge n_0. \tag{34}$$

Similarly, as $\tau_N(1) = 1$, we have

$$\mu(\tau_{N^n}(X)) = C_2 p_N^n, \text{ for all } n \ge n_0. \tag{35}$$

Now for any $n \ge n_0$, define $F = \tau_1^n(X)$ and $a = 1 - \lambda^n$. Then $F + a = \tau_{N^n}(X)$. By Proposition (3.2.22), $T_a(\mu|_{F+a}) \ll \mu$. Let $h = dT_a(\mu|_{F+a})/d\mu$. Then by Theorem (3.2.10) and (34),

$$T_a(\mu|_{F+a})(\tau_{1^n}(X)) = \int_{\tau_1^n(X)} h d\mu \le \frac{B}{A}\mu(\tau_{1^n}(X)) = \frac{C_1 B}{A} p_1^n.$$

On the other hand, $F + a = \tau_{N^n}(X)$ and so

$$T_a(\mu|_{F+a})(\tau_{1^n}(X)) = \mu(\tau_{N^n}(X)) = C_2 p_N^n.$$

Combining these, we obtain for all n,

$$\left(\frac{p_N}{p_1}\right)^n \le \frac{C_1 B}{C_2 A}.$$

This is possible only if $p_N \le p_1$. By reversing the role of 1 and N and letting $a = -(1 - \lambda^n)$, we obtain $p_1 \le p_N$.

To prove (ii), from (i) and the given assumption, all the conditions in Theorem (3.2.25) are satisfied. We have $p_1 = p_N = \lambda$.

For an absolutely continuous self-similar measure that admits a frame measure, near the boundary, the measure must behave like a Lebesgue measure and this can only happen when $p_1 = p_N = \lambda$. In the middle part of the attractor, there are overlaps and we cannot conclude whether the weights are equal to λ.

However, if now the measure admits a tight frame measure, then Corollary (3.2.14) applies and we can actually solve the Eaba-Wang conjecture [205] when the measure is absolutely continuous.

Theorem (3.2.28)[178]: Suppose μ defined in (33) is absolutely continuous with respect to the Lebesgue measure on X and suppose μ admits a tight frame measure. Then

(i) $p_1 = \cdots = p_N = \lambda$.

(ii) $\lambda = \frac{1}{N}$.

(iii) There exists $\alpha > 0$ such that $\mathcal{D} := \alpha\mathcal{B} \subset \mathbb{Z}$ and $\mathcal{D}$ tiles $\mathbb{Z}$.

Proof. Since $\mu = gdx$ has a tight frame measure, from Corollary (3.2.14), we have that g is a multiple of a characteristic function, and since μ is supported on the attractor $X_{\mathcal{B}}$, we have that $\mu = c\mathcal{L}|_X$ for some constant $c > 0$.

We will prove by induction that $p_k = \lambda$ and $\tau_{b_k}(X_{\mathcal{B}}) \cap \tau_{b_\ell}(X_B)$ has Lebesgue measure zero for all $\ell > k$. When $k = 1$, we know from Theorem (3.2.27) that $p_1 = \lambda$. From the invariance equation of μ,

$$\mu\big(\tau_{b_1}(X)\big) = \lambda\mu(X) + \sum_{j=2}^{N} p_j\mu\left(\tau_{b_j}^{-1}\left(\tau_{b_1}(X)\right)\right).$$

But $\mu\big(\tau_{b_1}(X)\big) = c\mathcal{L}|_X\big(\tau_{b_1}(X)\big) = \lambda\mu(X)$, so the equation above implies that $\mu\left(\tau_{b_j}^{-1}\big(\tau_{b_1}(X)\big)\right) = 0$. In particular, this shows for all $j \geq 2, \mu\big(\tau_{b_1}(X)\big) \cap \tau_{b_j}(X)\big) = 0$. But since μ is a renormalized the Lebesgue measure on X, this proves the statement for $k = 1$.

Suppose we have proved the statement for all $i \leq k - 1$. We now consider the set $A_k := \tau_{b_k}\tau_{b_1}^n(X)$, where n will be chosen later. This has positive Lebesgue measure and is contained in X so it has positive μ measure. From the rescaling we considered, we have $X \subset [0,1]$. We have that, for $l > k$, (recall that $b_1 = 0$),

$$A_k \cap \tau_{b_\ell}X \subset (\lambda^{n+1}[0,1] + b_k) \cap (b_\ell + \lambda[0,1]).$$

Since $b_k < b_\ell$ for $\ell > k$, we can pick n large enough so that this intersection is empty. In this case, $\mu\left(\tau_{b_\ell}^{-1}(A_k)\right) = 0$. On the other hand, for all $i \leq k - 1$, by the induction hypothesis, $\mu\left(\tau_{b_i}^{-1}(A_k)\right) \leq \mu\left(\tau_{b_i}^{-1}\big(\tau_{b_k}(X)\big)\right) = 0$. In the invariance equation, we have only the k-th term left:

$$\mu(A_k) = p_k\mu\left(\tau_{b_k}^{-1}(A_k)\right).$$

Again, μ is just the Lebesgue measure, so $p_k = \lambda$. Finally, using the induction hypothesis,

$$\mu\big(\tau_{b_k}(X)\big) = \lambda\mu(X) + \sum_{\ell=k+1}^{N} p_\ell\mu\left(\tau_{b_\ell}^{-1}\left(\tau_{b_k}(X)\right)\right).$$

The no overlap follows in the same way as in $k = 1$.

By induction, we have proved (i). (ii) follows immediately from (i). Finally, we now have $\lambda^{-1} = \#\mathcal{B}$ and the attractor X has positive Lebesgue measure. It means that the attractor is a self-similar tile on $\mathbb{R}^1$ [203]. By Theorem 4 in [204], there exists $\alpha > 0$ such that

$$\mathcal{D} = \alpha\mathcal{B} \subset \mathbb{Z}.$$

To prove that $\mathcal{D}$ tiles the integer lattice, we use some known properties of self-similar tiles. We will finish the proof in Proposition (3.2.29) below.

Consider $\mathcal{D} \subset \mathbb{Z}$ and $\#\mathcal{D} = N$. Then if the attractor $X(= X(N, \mathcal{D}))$ of the IFS defined by $\tau_{d_i}(x) = N^{-1}\big(x + d_j\big)$ has positive Lebesgue measure, X is a translational tile. A selfreplicating tiling set of X is a tiling set for X which satisfies

$$\mathcal{J} = N\mathcal{J} \oplus \mathcal{D}. \quad (36)$$

The direct sum here means that every element t in $\mathcal{J}$ can be expressed uniquely as $Nt' + d$ for $t \in \mathcal{J}$ and $d \in \mathcal{D}$.

Proposition (3.2.29)[178]: Let $\tau_{d_i}(x) = \frac{1}{N}(x + d_i)$, with $\mathcal{D} := \{d_i\} \subset \mathbb{Z}$ and $\#\mathcal{D} = N$. If the attractor X of $\{\tau_{d_i}\}$ is a self-similar tile on $\mathbb{R}^1$, then there exists $\mathcal{E} \subset \mathbb{Z}$ such that $\mathcal{D} \oplus \mathcal{E} = \mathbb{Z}$.

Proof. This result actually holds for any dimension [199] by some deeper considerations from the theory of the self-affine tiles. Here, we give another proof in dimension 1 for completeness.

By translation and rescaling, we can assume $\mathcal{D} \subset \mathbb{Z}^+, 0 \in \mathcal{D}$ and g.c.d. $\mathcal{D} = 1$. From Theorem 3.1 in [201], there exists a unique self-replicating tiling set $\mathcal{J}$ that is a subset of $\mathbb{Z}$ (i.e. $\subset \mathbb{Z}$). In the following, we claim that there exists $\mathcal{G}$ such that $\mathcal{J} \oplus \mathcal{G} = \mathbb{Z}$. The proof is the similar to the proof of Theorem 3 in [204].

For $t \in [0,1)$ and a finite subset $\mathcal{G}$ in $\mathbb{Z}$, let

$$\mathcal{G}(t) := \{j \in \mathbb{Z} : t + j \in X\} \text{ and } X_{\mathcal{G}} := \{t \in [0,1) : \mathcal{G}(t) = \mathcal{G}\}. \quad (37)$$

Since X is compact, $\mathcal{G}(t)$ is a finite set and only finitely many $X_{\mathcal{G}}$ are non-empty. Denote these non-empty sets by $\mathcal{G}_1, \cdots, \mathcal{G}_m$, then from the definitions in (37),

$$[0,1) = \bigcup_{j=1}^{m} X_{\mathcal{G}_j} \text{ and } X = \bigcup_{j=1}^{m} \left(X_{\mathcal{G}_j} + \mathcal{G}_j\right).$$

Moreover, $X_{\mathcal{G}_j}$ are mutually disjoint. Thus $\left\{X_{\mathcal{G}_j} + k : j \in \{1, \dots, m\}, k \in \mathbb{Z}\right\}$ is a partition of $\mathbb{R}$; also, since X tiles $\mathbb{R}$ by $\mathcal{J}$, this implies that $\left\{X_{\mathcal{G}_j} + \mathcal{G}_j + k : j \in \{1, \dots, m\}, k \in \mathcal{J}\right\}$ is a partition of $\mathbb{R}$. Then, for any j, the set $X_{\mathcal{G}_j} + \mathcal{G}_j$ tiles $X_{\mathcal{G}_j} + \mathbb{Z}$ using $\mathcal{J}$. Hence,

$$X_{\mathcal{G}_j} + \mathbb{Z} = X_{\mathcal{G}_j} + \mathcal{G}_j \oplus \mathcal{J}.$$

This shows that $\mathbb{Z} = \mathcal{G}_j \oplus \mathcal{J}$.

Add $\mathcal{G}(= \mathcal{G}_j)$ to both sides of (36),

$$\mathbb{Z} = \mathcal{J} \oplus \mathcal{G} = N\mathcal{J} \oplus \mathcal{G} \oplus \mathcal{D}.$$

This means that $\mathcal{D}$ tiles $\mathbb{Z}$ by $\mathcal{E} := N\mathcal{J} \oplus \mathcal{G}$.

We apply our results to IFSs with a small number of maps. The simplest ones are the Bernoulli convolutions.

Example (3.2.30)[178]: Let us consider the biased Bernoulli convolution $\mu = \mu_{p,\lambda}$ with contraction ratio $0 < \lambda < 1$ as follows:

$$\mu = p\mu \circ \tau_1^{-1} + (1 - p)\mu \circ \tau_2^{-1}$$

where $\tau_1(x) = \lambda x$ and $\tau_2(x) = \lambda x + (1 - \lambda)$. Let also

$$A = \{(p, \lambda) : \mu_{p,\lambda} \ll \mathcal{L}\} \text{ and } S = (0,1)^2 \setminus A.$$

Denote $\mathcal{F} = \{(p, \lambda) : \mu_{p,\lambda} \text{ has a frame measure}\}$. It is known that $\{(1/2, 1/2n) : n \in \mathbb{N}\}$ is contained in $\mathcal{F}$. We are interested in the question whether these are all the possible elements in $\mathcal{F}$. Concluding from the above theorems, we have

(i) If $0 < \lambda \le 1/2$, then the IFS satisfies the open set condition and hence has no overlap. This means that if μ has a frame measure, then $p = 1/2$ by Theorem (3.2.19).

(ii) If $1/2 < \lambda < 1$, there is non-trivial overlap. In this case, $p = 1/2 = \lambda$ by Theorem (3.2.27). Hence, we conclude that $A \cap \mathcal{F} = \{(1/2, 1/2)\}$.

Example (3.2.31)[178]: The purpose of this example is to show how Theorem (3.2.25) can be applied to the sets $\tau_I(X)$, so that we can check if more general measures μ have a frame measure. Let

$$\tau_1(x)=\frac{1}{3}x,\tau_2(x)=\frac{1}{3}x+\frac{4}{21},\tau_3(x)=\frac{1}{3}x+\frac{10}{21},\tau_4(x)=\frac{1}{3}x+\frac{2}{3}$$

and consider the self-similar measure μ defined as follows:

$$\mu=\frac{1}{3}\mu\circ\tau_1^{-1}+\frac{1}{6}\mu\circ\tau_2^{-1}+\frac{1}{6}\mu\circ\tau_3^{-1}+\frac{1}{3}\mu\circ\tau_4^{-1}. \qquad (38)$$

Then μ is absolutely continuous with respect to the Lebesgue measure on [0,1], but μ has no frame measure.

Proof. We can rescale the digit of the IFS by a factor $7/2$ so that the IFS becomes

$$\tau_1(x)=\frac{1}{3}x,\tau_2(x)=\frac{1}{3}(x+2),\tau_3(x)=\frac{1}{3}(x+5),\tau_4(x)=\frac{1}{3}(x+7).$$

The absolute continuity is completely determined by its mask polynomial

$$m(\xi)=\frac{1}{3}+\frac{1}{6}e^{2\pi i2\xi}+\frac{1}{6}e^{2\pi i5\xi}+\frac{1}{6}e^{2\pi i7\xi}=\frac{1}{6}\left(2+e^{2\pi i2\xi}+e^{2\pi i5\xi}+2e^{2\pi i7\xi}\right).$$

We note that μ is absolutely continuous if for all $n\in\mathbb{Z}\setminus\{0\}$, there exists k such that $m(3^{-k}n)=0$ (see [180, Theorem 1.1]) The coefficients c_i in this theorem will be $c_i=Np_i$ where p_i are our probabilities and $N=3$ is the scaling factor, $\lambda=3$ and $d_i=b_i$ in the notation of [180]. If g is a solution to the refinement equation in [180] then $\mu=gdx$ is our invariant measure). To check this condition, write $n=\pm 3^r s$ for some $r\geq 0$ and 3 does not divide s. Let $k=r+1$, then $3^{-k}n=\pm s/3$. This implies that

$$\begin{aligned} m(3^{-k}n) &=\frac{1}{6}\left(2+e^{2\pi i2s/3}+e^{2\pi i5s/3}+2e^{2\pi i7s/3}\right)\\ &=\frac{1}{3}\left(1+e^{2\pi is/3}+e^{2\pi i2s/3}\right)=0. (\text{ since 3 does not divide } s)\end{aligned}$$

To see there is no frame measure, we note that we cannot use Theorem (3.2.27) since $p_1=p_4=\frac{1}{3}$ and probability weights are not equal. Now, we iterate (38) one more time so that μ is the invariant measure of the IFS with the following 16 maps (i.e. $\mathcal{A}_2=\{\tau_{ij}: i,j\in\{1,2,3,4\}\}$):

$$\begin{array}{llll} \tau_{11}(x)=\frac{1}{9}x & \tau_{12}(x)=\frac{1}{9}x+\frac{4}{63} & \tau_{13}(x)=\frac{1}{9}x+\frac{10}{63} & \tau_{14}(x)=\frac{1}{9}x+\frac{2}{9}\\ \tau_{21}(x)=\frac{1}{9}x+\frac{4}{21} & \tau_{22}(x)=\frac{1}{9}x+\frac{16}{63} & \tau_{23}(x)=\frac{1}{9}x+\frac{22}{63} & \tau_{24}(x)=\frac{1}{9}x+\frac{26}{63}\\ \tau_{31}(x)=\frac{1}{9}x+\frac{10}{21} & \tau_{32}(x)=\frac{1}{9}x+\frac{34}{63} & \tau_{33}(x)=\frac{1}{9}x+\frac{40}{63} & \tau_{34}(x)=\frac{1}{9}x+\frac{44}{63}\\ T_{41}(x)=\frac{1}{9}x+\frac{2}{3} & \tau_{42}(x)=\frac{1}{9}x+\frac{46}{63} & \tau_{43}(x)=\frac{1}{9}x+\frac{52}{63} & \tau_{44}(x)=\frac{1}{9}x+\frac{56}{63} \end{array}$$

and the weight for τ_{ij} is p_ip_j. Moreover, it is easy to see that the self-similar set X of this IFS is [0,1]. Consider $\tau_{23}(x)$, the fixed point $x_{23}=\frac{11}{28}$. Note that the map that can overlap with $\tau_{23}(X)$ are $\tau_{22}(X)$ and $\tau_{24}(X)$. Since $\tau_{22}(X)=[16/63,23/63]$ and $\tau_{24}(X)=[26/63,31/63]$, a direct calculation shows that x_{23} is not in $\tau_{22}(X)$ nor in $\tau_{24}(X)$. Since also $\tau_{23}(X)\cap\tau_{ij}(X)=\emptyset$ for all other $ij\neq 22$ or $24, x_{23}$ does not belong to all the other $\tau_{ij}(X)$. In particular, if μ has a frame measure, then Theorem (3.2.25) applies which shows that $p_2p_3=\lambda^2=\frac{1}{9}$, but this is not the case since $p_2p_3=\frac{1}{36}$.

The study of frame measures or Fourier frames for singular measures is intriguing and leaves a lot of open problems for us to investigate. We outline the strategies and problems which may be essential towards a full solution for the case of singular measures.

The main strategy exhibited is based on the assumption that measures restricted on a subset are absolutely continuous after translations of that subset. We don't know whether measures with a frame measure must satisfy this assumption. However, there do exist examples for which such translational absolute continuity fails. The following suggests that singular measures supported essentially on positive Lebesgue measurable sets give such examples.

Example (3.2.32)[178]: Let μ be a measure whose support is exactly $[0,1]$. Suppose μ is singular with respect to the Lebesgue measure on $[0,1]$, then there exists $F, F+a \subset [0,1]$ such that $T_a(\mu|_{F+a})$ is singular with respect to μ.

Proof. As the measure is singular with respect to the Lebesgue measure on $[0,1]$, we can find a set $E \subset [0,1]$ such that $\mathcal{L}(E) > 0$ but $\mu(E) = 0$. By decomposing $[0,1]$ into dyadic intervals, we may assume E is in some dyadic interval $F = [i2^{-n}, (i+1)2^{-n}]$ for any n. Let $I = \{x: F + x \subset [0,1]\} = [-i2^{-n}, 1-(i+1)2^{-n}]$. Note that

$$\int_I \mu(E+x)dx = \iint_I \chi_{E+x}(y)dxd\mu(y) = \iint_I \chi_{y-E}(x)dxd\mu(y)$$
$$= \int \mathcal{L}\big((y-E)\cap I\big)d\mu(y).$$

As $-E \subset [-(i+1)2^{-n}, -i2^{-n}]$, we have that $y - E \subset I$ if $y \in [2^{-n}, 1-2^{-n}]$.

$$\int_I \mu(E+x)dx \geq \int_{2^{-n}}^{1-2^{-n}} \mathcal{L}(E)d\mu(y) = \mathcal{L}(E)\mu([2^{-n}, 1-2^{-n}]) > 0.$$

Here, $\mu([2^{-n}, 1-2^{-n}]) > 0$ because μ is supported on $[0,1]$. Hence, there exists a such that $\mu(E+a) > 0$. To complete the proof, we note that $\mu(E) = 0$ but $T_a(\mu|_{F+a})(E) = \mu(E + a \cap F + a) = \mu(E+a) > 0$, this shows the singularity of the measures.

There are many measures that satisfy the condition in Example (3.2.32). In the case of selfsimilar measures, one of the most common examples are the Bernoulli convolutions with overlap and with contraction ratio equal to a Pisot number [209].

It is natural to expect that assumption in Theorem (3.2.10) should be necessary for the existence of frame measures. In particular, we say that a finite Borel measure μ is translationally absolutely continuous if for all Borel sets F in the support of μ and $\mu(F) > 0$ and for all $a \in \mathbb{R}^d, T_a\mu|_{F+a} \ll \mu$.

Another concept that describes, for a given measure μ, the differences in its local distribution is the local dimension at points $x \in \operatorname{supp}(\mu)$. Let $\alpha > 0$

$$K(\alpha) := \left\{x \in \operatorname{supp}\mu : \dim_{\text{loc}} \mu(x) := \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \text{ exists and equals } \alpha\right\}$$

where $B_r(x)$ is the ball of radius r centered at x. If $x \in K(\alpha)$, it means that for all $\epsilon > 0$, we have for all r sufficiently small,

$$r^{\alpha-\epsilon} \leq \mu(B_r(x)) \leq r^{\alpha+\epsilon}.$$

The standard $1/n$-Cantor measure μ_n has only one local dimension $\log 2/\log n$. If μ_n is convolved with a discrete measure of finite number of atoms, it still has only one local dimension. On the other hand, it is known that equal contraction non-overlapping selfsimilar measures with unequal probability weights have more than one local dimension. We do not know examples of measures that have more than one local dimension and that have a frame measure. If there are two local dimensions, the balls around two points scale differently which means the mass around those balls is not evenly distributed. Combining these observations, we propose the following conjecture:

Conjecture (3.2.33)[178]: If μ is a measure with a frame measure, then μ must be translationally absolutely continuous and it has only one local dimension.

In other words, such a measure has only trivial multifractal structure. On the other hand, even if the μ has only one local dimension, we still need to classify the measures for which there exists a Fourier frame. In particular, the following is a famous problem:

(Q1): Does the one-third Cantor measure have a frame measure, Fourier frame or exponential Riesz basis?

It is known that the middle third Cantor measure has no orthogonal spectrum. For some recent approaches, in [183], necessary conditions for the existence of frame spectrum are found in terms of the Beurling dimension. It is also shown that all fractal measures arising from the iterated function systems with equal contraction ratios admit some Bessel exponential sequences of positive Beurling dimension [184]. However, there is still no complete answer to the question. The standard one-third Cantor measure is a measure with only one local dimension $\log 2/\log 3$, so the method we used cannot work. While we contend that it is difficult to answer whether the Cantor measure has frame measures or not, we can ask the following simpler questions:

(Q2): Find a singular measure with a Fourier frame but which is not absolutely continuous with respect to a spectral measure nor a convolution of spectral measures with some discrete measures.

(Q3): Find a self-similar measure admitting a Fourier frame of the type described in Q2.

(Q4): If a measure has a frame measure, does it have a Fourier frame?

We consider the Gabor system of the form

$$\mathcal{G}(g,\Lambda,\mathcal{J}) := \{e^{2\pi i\lambda\cdot x}g(x-p) : \lambda\in\Lambda, p\in\mathcal{J}\}$$

where $g\in L^2(\mathbb{R}^d)$, $\Lambda,\mathcal{J}$ are discrete sets in $\mathbb{R}^d$. We say that $\mathcal{G}(g,\Lambda,\mathcal{J})$ is an orthonormal basis if the functions in the system $\mathcal{G}(g,\Lambda,\mathcal{J})$ is orthonormal and for all $f\in L^2(\mathbb{R}^d)$,

$$\sum_{\lambda\in\Lambda}\sum_{p\in\mathcal{F}}\left|\int f(x)e^{-2\pi i\lambda\cdot x}\overline{g(x-p)}dx\right|^2 = \| f \|_2^2. \quad (39)$$

We also observe that if $\mathcal{G}(g,\Lambda,\mathcal{J})$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^d)$, then for any $(\lambda_0,p_0)\in\mathbb{R}^{2d}$, $\mathcal{G}(g,\Lambda-\lambda_0,\mathcal{J}-p_0)$ is also a Gabor orthonormal basis of $L^2(\mathbb{R}^d)$. Hence, there is no loss of generality to assume $(0,0)\in\Lambda\times\mathcal{J}$.

We recall one proposition due to Jorgensen and Pedersen.

Proposition (3.2.34)[178]: [196] Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Then $\{e_\lambda : \lambda\in\Lambda\}$ is an orthonormal basis on $L^2(\mu)$ if and only if

$$\sum_{\lambda\in\Lambda}|\hat{\mu}(x+\lambda)|^2 \equiv 1.$$

We will now prove the conjecture in [206] when g is non-negative. Our main theorem is as follows,

Theorem (3.2.35)[178]: Let $g\in L^2(\mathbb{R}^d)$ be non-negative function supported on a bounded set Ω with positive Lebesgue measure. Let Λ and $\mathcal{J}$ be discrete sets. Suppose that $\mathcal{G}(g,\Lambda,\mathcal{J})$ is a Gabor orthonormal basis of $L^2(\mathbb{R}^d)$, then

(i) $\mathcal{J}$ is a tiling set of Ω.

(ii) $|g(x)| = \frac{1}{\sqrt{\mathcal{L}(\Omega)}}\chi_\Omega(x)$ a.e. on Ω.

(iii) $\{e_\lambda : \lambda\in\Lambda\}$ is a spectrum of $L^2(\Omega)$.

Proof. We divide the proof into three claims.

Claim (3.2.36)[178]:: If $\mathcal{G}(g,\Lambda,\mathcal{J})$ is complete in $L^2(\mathbb{R}^d)$, then $\mathcal{L}\big(\mathbb{R}^d \setminus \bigcup_{p\in\mathcal{J}}(\Omega+p)\big)=0$.

Suppose that $\mathcal{L}\big(\mathbb{R}^d \setminus \bigcup_{t\in\mathcal{J}}(\Omega+t)\big)>0$, let $K\subset\mathbb{R}^d\setminus\bigcup_{t\in\mathcal{J}}(\Omega+t)$, be such that $0<\mathcal{L}(K)<\infty$. Then $f=\chi_K$, then f is a nonzero L^2 function, but

$$\int f(x)e^{2\pi i\lambda\cdot x}g(x-p)dx=0$$

since $g(\cdot-p)$ is supported on $\Omega+p$ which is disjoint from K. This contradicts the completeness of the system.

Claim (3.2.37)[178]: If $\mathcal{G}(g,\Lambda,\mathcal{J})$ is a Gabor orthonormal basis in $L^2(\mathbb{R}^d)$, then $\mathcal{L}\big((\Omega+p)\cap(\Omega+p')\big)=0$, for all $p\neq p'$ and $p,p'\in\mathcal{J}$.

Suppose for some $p\neq p'$, we have $\mathcal{L}(\Omega_{p,p'})>0$ where $\Omega_{p,p'}=(\Omega+p)\cap(\Omega+p')$. By the orthonormality of the functions represented by $(0,p)$ and $(0,p')$, we have

$$\int_{\Omega_{p,p'}} g(x-p)g(x-p')dx=0.$$

As g is non-negative, $g(\cdot-p)g(\cdot-p')=0$ almost everywhere on $\Omega_{p,p'}$. But $g(\cdot-p)$ and $g(\cdot-p')$ are supported on $\Omega+p$ and $\Omega+p'$ respectively and they are non-zero almost everywhere there. This is a contradiction since $\Omega_{p,p'}$ has positive Lebesgue measure.

Claim (3.2.38)[178]: $\{e_\lambda:\lambda\in\Lambda\}$ is a spectrum of $L^2(|g|^2dx)$.

For any $t\in\mathbb{R}^d$, we let $f_t(x)=g(x)e^{2\pi i\langle t,x\rangle}$. Then $\int|f_t|^2=\int|g|^2<\infty$. We use this in (39) and obtain

$$\sum_{\lambda\in\Lambda}\left|\int |g(x)|^2e^{2\pi i(t-\lambda)\cdot x}dx\right|^2+\sum_{\lambda\in\Lambda}\sum_{p\in\mathcal{J}\setminus\{0\}}\left|\int g(x)g(x-p)e^{2\pi i(t-\lambda)\cdot x}dx\right|^2$$
$$=\int |g(x)|^2dx=1$$

where $\int|g|^2=1$ follows from the orthonormality and $(0,0)\in\Lambda\times\mathcal{J}$. As $g(\cdot)g(\cdot-p)$ is nonzero only on $\Omega\cap\Omega+p$ which is of Lebesgue measure 0 by Claim (3.2.37) , we get that $g(\cdot)g(\cdot-p)=0$ almost everywhere and thus all the integrals in the second sum on the left hand side are zero. Hence,

$$\sum_{\lambda\in\Lambda}|(|g|^2dx)(t-\lambda)|^2\equiv\int |f|^2d\mu. \tag{40}$$

This is equivalent to say Λ is a spectrum of $L^2(|g|^2dx)$ by Proposition (3.2.34).

We can now complete the proof the theorem. Claim (3.2.36) and 2 shows that $\mathcal{J}$ is a tiling set of Ω. This proves (i). By Corollary (3.2.14) and Claim (3.2.38), $|g|=c\chi_\Omega$. As $\int|g|^2dx=1$ and we can see easily that $c=(\mathcal{L}(\Omega))^{-1/2}$. Hence (ii) holds. Finally (iii) follows immediately from Claim (3.2.38).

Section (3.3): Cantor Measures with Consecutive Digits:

For μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. We say that μ is a *spectral measure* if there exists a countable set $\Lambda\subset\mathbb{R}^d$ so that $E(\Lambda):=\{e^{2\pi i\langle\lambda,x\rangle}:\lambda\in\Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ. If $\chi_\Omega dx$ is a spectral measure, then we say that Ω is a *spectral set*. The study of spectral measures was first initiated by B. Fuglede in 1974 [221], when he considered a functional analytic problem of extending some commuting partial differential operators to some dense subspace of L^2 functions. In his first attempt, Fuglede proved that any fundamental domains given by a discrete lattice are spectral sets with its dual lattice as its spectrum. On the other hand, he also proved that triangles and circles on $\mathbb{R}^2$ are not spectral sets, while some examples (e.g. $[0,1] \cup [2,3]$) that are not fundamental domains can still be spectral. From the examples and the relation between Fourier series and translation operators, he proposed a reasonable conjecture on spectral sets: $\Omega \subset \mathbb{R}^s$ *is a spectral set if and only if* Ω *is a translational tile*. This conjecture baffled experts for 30 years until 2004, Tao [237] gave the first counterexample on $\mathbb{R}^d$, $d \geq 5$. The examples were modified later so that the conjecture is false in both directions on $\mathbb{R}^d$, $d \geq 3$ [227,227]. It remains open in dimensions 1 and 2. Despite the counter examples, the exact relationship between spectral measures and tiling is still mysterious.

The problem of spectral measures is as exciting when we consider fractal measures. Jorgensen and Pedersen [223] showed that the standard Cantor measures are spectral measures if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectra self-similar/self-affine measures were also found ([230,219] et al.). The construction of these spectral self-similar measures is based on the existence of the *compatible pairs* (*known also as Hadamard triples*). It is still unknown whether all such spectral measures are obtained from compatible pairs. Having an exponential basis, the series convergence problem was also studied by Strichartz. It is surprising that the ordinary Fourier series of continuous functions converge uniformly for standard Cantor measures [236]. By now there are considerable amount of literatures studying spectral measures and other generalized types of Fourier expansions like the Fourier frames and Riesz bases ([216],[218],[220],[224],[226],[222],[229],[230],[232],[234],[235]).

In [225], Hu and Lau made a start in studying the spectral properties of Bernoulli convolutions, the simplest class of self-similar measures. They classified the contraction ratios with infinitely many orthogonal exponentials. It was recently shown by Dai that the only spectral Bernoulli convolutions are of contraction ratio $\frac{1}{2n}$ [215]. We study another general class of Cantor measures on $\mathbb{R}^1$. Let $b > 2$ be an integer and $q < b$ be another positive integer.

We consider the iterated function system (IFS) with maps

$$f_i(x) = b^{-1}(x+i)\,, i = 0, 1, \ldots, q-1.$$

The IFS arises a natural *self-similar measure* $\mu = \mu_{q,b}$ satisfying

$$\mu(E) = \sum_{i=0}^{q-1} \frac{1}{q} \mu\left(f_j^{-1}(E)\right) \tag{41}$$

for all Borel sets E. Note that we only need to consider equal weight since non-equally weighted self-similar measures here cannot have any spectrum by Theorem 1.5 in [220]. It is also clear that if $q = 2$, μ becomes the standard Cantor measure of b^{-1} contraction. For this class of self-similar measures, we find surprisingly that the spectral properties depend heavily on the number theoretic relationship between q and b. Our first result is to show that $\mu = \mu_{q,b}$ *has infinitely many orthogonal exponentials if and only if* q *and* b *is not relatively prime*. *If moreover*, q *divides* b, *the resulting measure will be a spectral measure*. However, when q does not divide b and they are not relatively prime (e.g. $q =$

4, $b = 6$), variety of cases may occur and we are not sure whether there are spectral measures in these classes.

We then focus on the case when $b = qr$ in which we aim at giving a detailed classification of its spectra. The classification of spectra, for a given spectral measure, was first studied by Lagarias, Reeds and Wang [231]. They considered the spectra of $L^2([0,1)^d)$ (more generally fundamental domains of some lattices) and they showed that the spectra of $L^2([0,1)^d)$ are exactly all the tiling sets of $[0,1)^d$. If $d = 1$, the way of tiling $[0,1)$ is rather rigid, and it is easy to see that the only spectrum (respectively the tiling set) is the translates of the integer lattice $\mathbb{Z}$.

Such kind of rigidity breaks down even on $\mathbb{R}^1$ if we turn to fractal measures. The first attempt of the classification of its spectra was due to [217]; Dutkay, Han and Sun decomposed the maximal orthogonal sets of one-fourth Cantor measure using 4-adic expansion with digits $\{0,1,2,3\}$ and put them into a labeling of the binary tree. The maximal orthogonal sets will then be obtained by reading all the infinite paths with digits ending eventually in 0 (for positive elements) or 3 (for negative elements). They also gave some sufficient conditions on the digits for a maximal orthogonal set to be a spectrum. The condition is not easy to verify.

Turning to our self-similar measures with consecutive digits where the one-fourth Cantor measure is a special case, we will classify all the maximal orthogonal sets using mappings on the standard q-adic tree called *maximal mappings*. This construction improves the tree labeling method in [217] in two ways.

(a) We will choose the digit system to be $\{-1, 0, 1, \dots, b-2\}$ instead of $\{0, 1, \dots, b-1\}$. By doing so, all integers (both positive and negative) can be expanded into b-adic expansions terminating at 0.

(b) We impose restrictions on our labeling position on the tree so that together with (a), all the elements in a maximal orthogonal set can be extracted by reading some specific paths in the tree. These paths are collected in a countable set Γ_q defined in (45).

Having such a new tree structure of a maximal orthogonal set, we discover there are two possibilities for the maximal sets depending on whether all the paths in Γ_q are corresponding to some elements in the maximal orthogonal sets (i.e. the values assigned are eventually 0). If it happens that all the paths in Γ_q behave nicely as said, we call such maximal orthogonal sets *regular*. It turns out that regular sets cover most of the interesting cases and we can give regular sets a natural ordering $\{\lambda_n : n = 0, 1, 2, \dots\}$. If the standard q-adic expansion of n has length k, we define N_n^* to be the number of non-zero digits in the b-adic expansion using $\{-1, 0, \dots, b-2\}$ of λ_n after k. N_n^* is our crucial factor in determining whether the set is a spectrum. We show that if N_n^* *grows slowly enough or even uniformly bounded*, *the set will be a spectrum*, *while if* N_n^* *grows too fast*, *say it is of polynomial rates*, *then the maximal orthogonal sets will not be a spectrum*.

In [218], Dutkay et al. tried to generalize the classical results of Landau [233] about the Beurling density on Fourier frames to fractal settings. They defined the concept of *Beurling dimensions* for a discrete set and showed that all Bessel sequences for an IFS of similitudes with no overlap condition must have Beurling dimension not greater than its Hausdorff dimension of the attractor. Under technical assumption on the frame spectra, they showed that the above two dimensions coincide. They conjectured that the assumption can be removed. However, as we see that N_n^* counts the number of non-zero digits only, we can freely add qb^m for any $m > 0$ on the tree of the canonical spectrum. These additional terms push the λ_n's as far away from each other as wanted and we therefore show that *there exists a spectrum of zero Beurling dimension*.

We discuss the maximal orthogonal sets of $\mu_{q,b}$ and classify all maximal orthogonal sets via the maximal mapping on the q-adic when q divides b. In Section 4, we discuss the regular spectra and prove the growth rate criteria. The examples of the spectra with zero Beurling dimensions will be given. We give a study on the irregular spectra.

Let Λ be a countable set in $\mathbb{R}$ and denote $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ where $e_\lambda(x) = e^{2\pi i\lambda x}$. We say that Λ is a *maximal orthogonal set* (*spectrum*) if $E(\Lambda)$ is a maximal orthogonal set (an orthonormal basis) for $L^2(\mu)$. Here $E(\Lambda)$ is a maximal orthogonal set of exponentials means that it is a mutually orthogonal set in $L^2(\mu)$ such that if $\alpha \notin \Lambda$, e_α is not orthogonal to some e_λ , $\lambda \in \Lambda$. If $L^2(\mu)$ admits a spectrum, then μ is called a *spectral measure*. Given a measure μ, the Fourier transform is defined to be

$$\hat{\mu}(\xi) = \int e^{2\pi i\xi x}\, d\mu(x).$$

It is easy to see that $E(\Lambda)$ is an orthogonal set if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu}) := \{\xi \in \mathbb{R}: \hat{\mu}(\xi) = 0\}.$$

We call such Λ a *bi-zero set* of μ. For $\mu = \mu_{q,b}$, we can calculate its Fourier transform.

$$\hat{\mu}(\xi) = \prod_{j=1}^{\infty}\left[\frac{1}{q}\left(1 + e^{2\pi i b^{-j}\xi} + \cdots + e^{2\pi i b^{-j}(q-1)\xi}\right)\right]. \tag{42}$$

Denote

$$m(\xi) = \frac{1}{q}\left(1 + e^{2\pi i\xi} + \cdots + e^{2\pi i(q-1)\xi}\right) \tag{43}$$

and thus $|m(\xi)| = |\frac{\sin q\pi\xi}{q\sin \pi\xi}|$. The zero set of m is

$$\mathcal{Z}(m) = \left\{\frac{a}{q} : q \dagger a, a \in \mathbb{Z}\right\},$$

where $q \dagger a$ means q does not divide a. We can then write $\hat{\mu}(\xi) = \prod_{j=1}^{\infty} m\left(b^{-j}\xi\right)$, so that the zero set of $\hat{\mu}$ is given by

$$\mathcal{Z}(\hat{\mu}) = \left\{\frac{b^n}{q}a : n \geq 1, q \dagger \mathrm{a}\right\} = r\{b^n a : n \geq 0, q \dagger a\}, \tag{44}$$

where $r = b/q$.

We have the following theorem classifying which $\mu_{q,b}$ possess infinitely many orthogonal exponentials.

We wish to give a classification on the spectra and the maximal orthogonal sets whenever they exist. To do this, it is convenient to introduce some multi-index notations: denote $\Sigma_q = \{0, \ldots, q-1\}, \Sigma_q^0 = \{\vartheta\}$ and $\Sigma_q^n = \underbrace{\Sigma_q \times \ldots \times \Sigma_q}_{n}$.

Let $\Sigma_q^* = \bigcup_{n=0}^{\infty} \Sigma_q^n$ be the set of all finite words and let $\Sigma_q^\infty = \Sigma_q \times \Sigma_q \times \ldots$ be the set of all infinite words. Given $\sigma = \sigma_1\sigma_2 \ldots \in \Sigma^\infty \cup \Sigma^*$, we define $\vartheta\sigma = \sigma$, $\sigma|_k = \sigma_1 \cdots \sigma_k$ for $k \geq 0$ where $\sigma|_0 = \vartheta$ for any σ and adopt the notation $0^\infty = 000 \cdots$, $0^k = \underbrace{0 \ldots 0}_{k}$ and $\sigma\sigma'$ is the concatenation of σ and σ'. We start with a definition.

Definition (3.3.1)[214]: Let Σ_q^* be all the finite words defined as above. We say it is a q-*adic tree* if we set naturally the root is ϑ, all the k-th level nodes are Σ_q^k for $k \geq 1$ and all the offsprings of $\sigma \in \Sigma_q^*$ are σi for $i = 0, 1, \dots, q-1$.

Let τ be a map from Σ_q^* to real numbers. Then the image of τ defines a q-*adic tree labeling*. Define Γ_q

$$\Gamma_q := \{\sigma 0^\infty : \sigma = \sigma 1 \dots \sigma_k \in \Sigma_q^*, \sigma_k \neq 0\}. \tag{45}$$

Γ_q will play a special role in our construction.

Suppose that for some word $\sigma = \sigma' 0^\infty \in \Gamma_q$, $\tau(\sigma|_k) = 0$ for all k sufficiently large, we say that τ is *regular on* σ, otherwise *irregular*. Let b be another integer, if τ is regular on some $\sigma \in \Gamma_q$, we define the projection II_b^τ from Γ_q to $\mathbb{R}$ as

$$\prod_b^\tau (\sigma) = \sum_{k=1}^{\infty} \tau(\sigma|_k) b^{k-1}. \tag{46}$$

The above sum is finite since $\tau(\sigma|_k) = 0$ for sufficiently large k. If τ is regular on any σ in Γ_q, we say that τ is a *regular mapping*.

Example (3.3.2)[214]: Suppose $b = q$, let $C = \{c_0 = 0, c_1, \dots, c_{b-1}\}$ be a residue system mod b where $c_i \equiv i (\mathrm{mod}\ b)$. Define $\tau(\vartheta) = 0$ and $\tau(\sigma) = c_{\sigma_k}$ if $\sigma = \sigma_1 \dots \sigma_k \in \Sigma_q^k \subset \Sigma_q^*$. Then it is easy to see that τ is regular on any $\sigma \in \Gamma_q$ and hence it is regular. Moreover,

$$II_b^\tau(\Gamma_b) \subseteq \mathbb{Z}. \tag{47}$$

When $C = \{0,1, \dots, b-1\}$, then the mapping Π_b^τ is a bijection from Γ_b onto $\mathbb{N} \cup \{0\}$.

In [217], putting their setup in our language, they classified maximal orthogonal sets of standard one-fourth Cantor measure via the mapping τ from Σ_2^* to $\{0,1, 2, 3\}$. However, some maximal orthogonal sets may have negative elements in which those elements cannot be expressed finitely in 4-adic expansions using digits $\{0,1,2,3\}$. In our classification, we will choose the digit system to be $C = \{-1, 0, 1, \dots, b-2\}$ in which we can expand any integers uniquely by finite b-adic expansion. We have the following simple but important lemma.

Lemma (3.3.3)[214]: *Let* $C = \{-1, 0, 1, \dots, b-2\}$ *with integer* $b \geq 3$ *and let* τ *be the map defined in* Example (3.3.2). *Then* II_b^τ *is a bijection between* Γ_b *and* $\mathbb{Z}$.

Proof. For any $n \in \mathbb{Z}$ and $|n| < b$, it is easy to see that there exists unique $\sigma \in \Gamma_b$ such that $n = II_b^\tau(\sigma)$. For example, $n = b-1$, then $n = II_b^\tau(\sigma_1\sigma_2)$ where $\sigma_1 = -1$ and $\sigma_2 = 1$. When $|n| \geq b$, then n can be decomposed uniquely as $n = \ell b + c$ where $c \in C$. *We note that* $|\ell| = |\frac{n-c}{b}| \leq \frac{|n|+b-2}{b} < |n|$. *If* $|\ell| < b$, *we are done.* Otherwise, we further decompose ℓ in a similar way and after finite number of steps, $|\ell| < b$. The expansion is unique since each decomposition is unique.

We now define a q-adic tree labeling which corresponds to a maximal orthogonal set for $\mu_{q,b}$ when $b = qr$. We observe that for $b = qr$, we can decompose $C = \{-1, 0, \dots, b-2\}$ in q disjoint classes according to the remainders after being divided by q: $C = \bigcup_{i=0}^{q-1} C_i$ where

$$C_i = (i + q\mathbb{Z}) \cap C.$$

Definition (3.3.4)[214]: Let Σ_q^* be a q-adic tree and $b = qr$, we say that τ is a *maximal mapping* if it is a map $\tau = \tau_{q,b} : \Sigma_q^* \to \{-1, 0, \dots, b-2\}$ that satisfies

(i) $\tau(\vartheta) = \tau(0^n) = 0$ for all $n \geq 1$,

(ii) for all $k \geq 1$, $\tau(\sigma_1 \dots \sigma_k) \in C_{\sigma_k}$,

(iii) for any word $\sigma \in \Sigma_q^*$, there exists σ' such that τ is regular on $\sigma\sigma'0^\infty \in \Gamma_q$.

We call a tree mapping a *regular mapping* if it satisfies (i) and (ii) in above and is regular on any word in Γ_q. Clearly, regular mappings are maximal.

Given a maximal mapping τ, the following sets will be of our main study (see Fig. 1 [214]).

$$\Lambda(\tau) := \{II_b^\tau(\sigma) : \in \Gamma_q, \tau \text{ is regular on } \sigma\}. \tag{48}$$

From now on, we will assume that $b = qr$, $C = \{-1, 0, 1, \dots, b-2\}$ and $0 \in \Lambda$. The main results are as follows and this is also the reason why τ is called a maximal mapping.

For the proof, (i) in Definition (3.3.4) is to ensure $0 \in \Lambda$. (ii) is to make sure the mutual orthogonality and (iii) is for the maximal orthogonality.

If Λ is a spectrum of $L^2(\mu)$, we call the associated maximal mapping τ a *spectral mapping*. We will restrict our attention to regular mappings (i.e. for all $\sigma \in \Gamma_q$, τ is regular on σ). In this

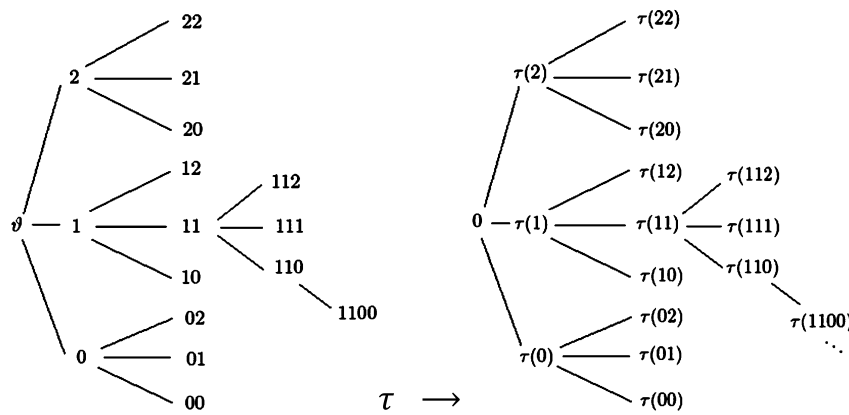

Fig. 1[214]: An illustration of the 3-adic tree and the associated mapping τ.

Case, $\Lambda(\tau) = \{II_b^\tau(\sigma) : \sigma \in \Gamma_q\}$. The advantage of considering regular mappings is that we can give a natural ordering of the maximal orthogonal set (τ). The ordering goes as follows: given any $n \in \mathbb{N}$, we can expand it into the unique finite q-adic expansion,

$$n = \sum_{j=1}^{k} \sigma_j q^{j-1}, \qquad \sigma_j \in \{0, \dots, q-1\}, \quad \sigma_k \neq 0. \tag{49}$$

In this way n is uniquely corresponding to one word $\sigma = \sigma_1 \dots \sigma_k$, which is called the *$q$-adic expansion of* n. For a regular mapping τ, there is a natural ordering of the maximal orthogonal set $\Lambda(\tau)$: $\lambda_0 = 0$ and

$$\lambda_n = II_b^\tau(\sigma 0^\infty) = \sum_{j=1}^{k} \tau(\sigma|_j) b^{j-1} + \sum_{i=k+1}^{N_n} \tau(\sigma 0^{i-k}) b^{i-1} \tag{50}$$

where $\sigma = \sigma_1 \cdots \sigma_k$ is the q-adic expansion of n in (49), $\tau(\sigma 0^{N_n - k}) \neq 0$ and $\tau(\sigma 0^\ell) = 0$ for all $\ell > N_n - k$. Under this ordering, we have $\Lambda(\tau) = \{\lambda_n\}_{n=0}^\infty$. Let

$$N_n^* = \#\{\ell : \sigma = \sigma_1 \cdots \sigma_k, \tau(\sigma 0^\ell) \neq 0\},$$

where we denote $\#A$ the cardinality of the set A. The growth rate of N_n^* is crucial in determining whether $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$. To describe the growth rate, we let $\mathcal{N}_{m,n}^* = \max\{N_k^* : q^m \leq k < q^n\}$, $\mathcal{L}_n^* = \min\{N_k^* : q^n \leq k < q^{n+1}\}$ and $\mathcal{M}_n = \max\{N_k : 1 \leq k < q^n\}$. We have the following two criteria depending on the growth rate of N_n^*.

The following is the most important example of Theorem (3.3.14).

Example (3.3.5)[214]: For a regular mapping τ, if $M := \sup_n\{N_n^*\}$ is finite, then Λ must be a spectrum.

Proof. Note that $\mathcal{N}_{m,n}^* \leq M$ and therefore any strictly increasing sequences α_n will satisfy the second condition in (46). Let $\alpha_1 = 1$ and $\alpha_{n+1} = n + \mathcal{M}_{\alpha_n}$ for $n \geq 1$. Then the first condition holds and hence Λ must be a spectrum by Theorem (3.3.14).

We will also see that when there is some slow growth in $\mathcal{N}_n^*$, Λ can still be a spectrum. The exact growth rate for Λ to be a spectrum is however hard to obtain from the techniques we used.

Now, we can construct some spectra which can have zero Beurling dimension from regular orthogonal sets using Theorem (3.3.14). In fact, they can even be arbitrarily sparse.

We also make a study on the irregular spectra, although most interesting cases are from the regular one. Let τ be a maximal mapping such that it is irregular on $\{I_1 0^\infty, \dots, I_N 0^\infty\}$, where $I_i \in \Sigma^*$ and the last digit in I_i is non-zero, and is regular on the others in Γ_q. We define the corresponding *regularized mapping* τ_R:

$$\tau_R(\sigma) = \begin{cases} 0, & \text{if } \sigma = I_i 0^k \text{ for } k \geq 1; \\ \tau(\sigma), & \text{otherwise.} \end{cases}$$

Our result is as follows.

Theorem (3.3.6)[214]: *Let τ be an irregular maximal mapping of μ. Suppose τ is irregular only on finitely many σ in Γ_q. Then τ is a spectral mapping if and only if a corresponding regularized mapping τ_R is a spectral mapping.*

We will prove this theorem more generally in Theorem (3.3.20) by showing that the spectral property is not affected if we alter only finitely many elements in Γ_q. However, we do not know whether the same holds if the finiteness assumption on irregular elements is removed.

We discuss the existence of orthogonal sets for $\mu_{q,b}$, in particular, Theorem (3.3.7) and Theorem (3.3.9) are proved.

Theorem (3.3.7)[214]: *$\mu = \mu_{q,b}$ has infinitely many orthogonal exponentials if and only if the greatest common divisor between q and b is greater than 1. If q divides b, then $\mu_{q,b}$ is a spectral measure.*

Proof. Let $\gcd(q,b) = d$. Suppose q and b are relatively prime i.e. $d = 1$. Let

$$\mathcal{Z}_n := \{\frac{b^n}{q} a : q \dagger a\}.$$

It is easy to see that $Z(\hat{\mu}_{q,b}) = \bigcup_{n=1}^{\infty} \mathcal{Z}_n$. Note that for any a with $q \dagger a$, we have $q \dagger ba$ since $\gcd(q,b) = 1$. Hence, if $n > 1$,

$$\frac{b^n}{q} a = \frac{b^{n-1}}{q}(ba) \in \mathcal{Z}_{n-1}.$$

This implies that $\mathcal{Z}_1 \supset Z_2 \supset \cdots$ and $Z(\hat{\mu}_{q,b}) = Z_1$. Let

$$\Upsilon_i = \{\frac{b}{q} a : q \dagger a, a \equiv i (\text{mod } q)\},$$

then $Z(\hat{\mu}_{q,b}) = \bigcup_{i=1}^{q-1} \Upsilon_i$. If there exists a mutually orthogonal set Λ for $\mu_{q,b}$ with $\#\Lambda \geq q$, we may assume $0 \in \Lambda$ so that $\Lambda \backslash \{0\} \subset \mathcal{Z}(\hat{\mu}_{q,b})$. Hence there exists $1 \leq i \leq q-1$

such that $Y_i \cap \Lambda$ contains more than 1 element, say λ_1 , λ_2. But then $\lambda_1 - \lambda_2 = \frac{b}{q} r$ where $q|r$. This contradicts the orthogonal property of Λ.

Suppose now $d > 1$, we know $d \leq q$. We first consider $d = q$ and prove that the measure is a spectral measure. This shows also the second statement. Write now $b = qr$ and define $\mathcal{D} = \{0, 1, \dots, q-1\}$ and $S = \{0, r, \dots\ (q-1)r\}$. Then it is easy to see that the matrix

$$H := \left[e^{2\pi i \frac{ijr}{b}}\right]_{0 \leq i,j \leq q-1} = \left[e^{2\pi i \frac{ij}{q}}\right]_{0 \leq i,j \leq q-1}$$

is a Hadamard matrix (i.e. $HH^* = qI$). This shows $\frac{1}{b}\mathcal{D}$ and S form a compatible pair as in [230]. Therefore it is a spectral measure by Theorem 1.2 in [230].

Suppose now $1 < d < q$. We have shown that $\mu_{d,b}$ is a spectral measure and hence $\mathcal{Z}(\hat{\mu}_{d,b})$ contains an infinite bi-zero sets Λ (i.e. $\Lambda - \Lambda \subset \mathcal{Z}(\hat{\mu}_{d,b}) \cup \{0\}$). We claim that $\mathcal{Z}(\hat{\mu}_{d,b}) \subset \mathcal{Z}(\hat{\mu}_{q,b})$ and hence $\mathcal{Z}(\hat{\mu}_{q,b})$ has infinitely many orthogonal exponentials. To justify the claim, we write $q = dt$. Note that for $d \dagger a$,

$$\frac{b^n}{d} a = \frac{b^n}{q}(ta),$$

as q cannot divide ta. Hence, $\frac{b^n}{q}(ta) \in \mathcal{Z}(\hat{\mu}_{q,b})$. This also completes the proof of Theorem (3.3.7).

Remark (3.3.8)[214]: In view of Theorem (3.3.7), we cannot decide whether there are spectral measures when $1 < \gcd(q, b) < q$. In general, $\mu_{q,b}$ is the convolutions of several self-similar measures with some are spectral and some are not spectral. If $q = 4$, $b = 6$, we know that $\{0,1,2,3\} = \{0,1\} \oplus \{0,2\}$ and hence

$$\hat{\mu}_{4,6}(\xi) = \prod_{j=1}^{\infty}\left(\frac{1 + e^{2\pi i 6^{-j}\xi}}{2}\right) \cdot \prod_{j=1}^{\infty}\left(\frac{1 + e^{2\pi i 2 6^{-j}\xi}}{2}\right) = \hat{v}_1(\xi)\hat{v}_2(\xi)$$

where $v_1 = \mu_{2,6}$ and v_2 is the equal weight self-similar measure defined by the IFS with maps $\frac{1}{6}x$ and $\frac{1}{6}(x+2)$. Hence, $\mu_{4,6} = v_1 * v_2$. It is known that both v_1 and v_2 are spectral measures, but we do not know whether $\mu_{4,6}$ is a spectral measure. If $q = 6$ and $b = 10$, then $\{0, 1, \dots, 5\} = \{0,1\} \oplus \{0,2,4\}$ and hence $\mu_{6,10}$ is the convolution of $\mu_{2,10}$ with a non-spectral measure with 3 digits and contraction ratio 1/10. Because of its convolutional structure, it may be a good testing ground for studying the Laba-Wang conjecture [230] and also for finding non-spectral measures with Fourier frame [220,224].

Theorem (3.3.9)[214]: *Λ is a maximal orthogonal set of $L^2(\mu_{q,b})$ if and only if there exists a maximal mapping τ such that $= r\Lambda(\tau)$, where $b = qr$.*

Proof. Suppose $\Lambda = r\Lambda(\tau)$ for some maximal mapping τ. We show that it is a maximal orthogonal set for $L^2(\mu)$. To see this, we first show Λ is a bi-zero set. Pick , $\lambda, \lambda' \in \Lambda$, by the definition of $\Lambda(\tau)$, we can find two distinct σ, σ' in Γ_q such that

$\lambda = \frac{b}{q} II_b^{\tau}(\sigma)$, $\qquad \lambda' = \frac{b}{q} II_b^{\tau}(\sigma')$.

Let k be the first index such that $\sigma|_k \neq \sigma'|_k$. Then for some integer M, we can write

$q\lambda - q\lambda' = b\sum_{i=k}^{\infty}(\tau(\sigma|_i) - \tau(\sigma'|_i))\, b^{i-1} = b^k\left((\tau(\sigma|_k) - \tau(\sigma'|_k)) + bM\right)$.

By (ii) in Definition (3.3.4), $\tau(\sigma|_k)\ and\ \tau(\sigma'|_k)$ are in distinct residue classes of q. This means q does not divide $\tau(\sigma|_k)-\tau(\sigma'|_k)$. On the other hand, q divides b. Hence, q does not divide $\left(\tau(\sigma|_k)-\tau(\sigma'|_k)\right)+bM$. By (44), $\lambda-\lambda'$ lies in $\mathcal{Z}(\hat{\mu})$.

To prove the maximality of the orthogonal set Λ, we show by contradiction. Let $\theta\notin\Lambda$ but θ is orthogonal to all elements in Λ. Since $0\in\Lambda,\ \theta\neq0$ and $\theta=\theta-0\in\mathcal{Z}(\hat{\mu})$. Hence, by (44) we may write

$$\theta=r(b^{k-1}a),$$

where q does not divide a. Expand b^{k-1}a in b-adic expansion using digits $\{-1,0,\dots,b-2\}$

$$b^{k-1}a=\varepsilon_{k-1}b^{k-1}+\varepsilon_k b^k+\cdots+\varepsilon_{k+\ell}b^{k+\ell},$$

q does not divide ε_{k-1}. Note that there exists unique σ_s ,$0\le\sigma_s\le q-1$, such that $\varepsilon_s\equiv\sigma_s\ (mod\ q)$ for $k-1\le s\le k+\ell$. Denote $\sigma_s=\varepsilon_s=0$ for $s>k+\ell$. Since $\theta\notin\Lambda$, we can find the smallest integer α such that $\tau(0^{k-2}\ \ \sigma_{k-1}\sigma_k\cdots\sigma_\alpha)\neq\varepsilon_\alpha$. By (iii) in the definition of τ, we can find $\sigma\in\Gamma_q$ so that $\sigma=0^{k-2}\sigma_{k-1}\sigma_k\cdots\sigma_\alpha\sigma'0^\infty$ and τ is regular on σ, then there exists M' such that

$$\theta-rII_b^\tau(\sigma)=rb^\alpha(\varepsilon_\alpha-\tau(0^{k-1}\sigma_{k-1}\dots\sigma_\alpha)+M'b)\,.$$

By (ii) in the definition of τ, $\tau(0^{k-1}\sigma_{k-1}\cdots\sigma_\alpha)\equiv\sigma_\alpha(\text{mod }q)$, which is also congruent to ε_α by our construction. This implies $\theta-r\Pi_b^\tau(\sigma)$ is not in the zero set of $\hat{\mu}$ since q divides $\varepsilon_\alpha-\tau(0^{k-1}i_{k-1}\cdots i_\alpha)$ and b does not divide it either. It contradicts to θ being orthogonal to all Λ.

Conversely, suppose we are given a maximal orthogonal set Λ of $L^2(\mu)$ with $0\in\Lambda$. Then $\Lambda\subset\mathcal{Z}(\hat{\mu})$. Hence, we can write
$\Lambda=\{ra_\lambda\ :\ \lambda\in\Lambda,a_\lambda=b^{k-1}m$ for some $k\ge1$ and $m\in\mathbb{Z}$ with $q\dagger m\}$,
where $a_0=0$. Now, expand a_λ in b-adic expansion with digits chosen from $C=\{-1\,,0,\dots,b-2\}$.

$$a_\lambda=\sum_{j=1}^{\infty}\varepsilon_\lambda^{(j)}\,b^{j-1}\tag{51}$$

Let $D(\vartheta)=\left\{\varepsilon_\lambda^{(1)}\ :\ \lambda\in\Lambda\right\}$ be all the first coefficients of b-adic expressions (51) of elements in Λ, and let $D(C_1,\dots,c_n)=\left\{\varepsilon_\lambda^{(n+1)}\ :\ \varepsilon_\lambda^{(1)}=C_1,\dots\ \varepsilon_\lambda^{(n)}=c_n,\lambda\in\Lambda\right\}$ be all the $n+1$-st coefficients of elements in Λ whose first n coefficients are fixed, where $c_1\,,c_2,\dots,c_n\in C$. We need the following lemma.

Lemma (3.3.10)[214]: *With the notations above,* $D(\vartheta)$ *contains exactly* q *elements which are in distinct residue class* $(mod\ q)$ *and* $0\in D(\vartheta)$. *Moreover, if* $D(c_1,\dots,c_n)$ *with all* $c_i\in C$ *is non-empty, then it contains exactly* q *elements which are in distinct residue class* $(mod\ q)$ *also. In particular,* $0\in D(c_1,\dots,c_n)$ *if* $c_1=\cdots=c_n=0$ *for* $n\ge1$.

Proof. Clearly, by (51), $0\in D(\vartheta)$ and $0\in D(c_1,\dots,c_n)$ if $c_1=\cdots=c_n=0$ for $n\ge1$.

Suppose the number of elements in $D(\vartheta)$ is strictly less than q. We let $\alpha\in C\backslash D(\vartheta)$ such that α is not congruent to any elements in (ϑ). Then, for any $\lambda\in\Lambda$, by (51) we have

$$r\alpha-\lambda=r\left(\alpha-\sum_{j=1}^{\infty}\varepsilon_\lambda^{(j)}\,b^{j-1}\right)=\frac{b}{q}\left(\alpha-\varepsilon_\lambda^{(1)}+\sum_{j=2}^{\infty}\varepsilon_\lambda^{(j)}\,b^{j-1}\right).$$

Note that $q\ \dagger\left(\alpha-\varepsilon_\lambda^{(1)}\right)$ for all $\lambda\in\Lambda$ by the assumption, this implies $r\alpha$ is mutually orthogonal to Λ but is not in Λ, which contradicts to maximal orthogonality. Hence $D(\vartheta)$

contains at least q elements. If $D(\vartheta)$ contains more than q elements, then there exists $a_{\lambda_1} = \sum_{j=1}^{\infty} \varepsilon_{\lambda_1}^{(j)} b^{j-1}$, $a_{\lambda_2} = \sum_{j=1}^{\infty} \varepsilon_{\lambda_2}^{(j)} b^{j-1}$ such that $\varepsilon_{\lambda_1}^{(1)} \equiv \varepsilon_{\lambda_2}^{(1)} (\text{mod } q)$ and $\varepsilon_{\lambda_1}^{(1)} \neq \varepsilon_{\lambda_2}^{(1)}$. Then $r(a_{\lambda_1} - a_{\lambda_2}) = \frac{b}{q}\left(\varepsilon_{\lambda_1}^{(1)} - \varepsilon_{\lambda_2}^{(1)} + bM\right)$ for some integer M. This means $r\left(a_{\lambda_1} - a_{\lambda_2}\right)$ is not a zero of $\hat{\mu}$. This contradicts to the mutual orthogonality. Hence, $D(\vartheta)$ contains exactly q elements which are in distinct residue class $(\text{mod } q)$.

In general, we proceed by induction. Suppose the statement holds up to $n-1$. If now $D(C_1, \dots, c_n)$ is non-empty, then $D(c_1, \dots, c_k)$ is also non-empty for all $k \leq n$. we now show that $D(c_1, \dots, c_n)$ must contain at least q elements. Otherwise, we consider $\theta = r(c_1 + \cdots + c_n b^{n-1} + \alpha b^n)$ where α is in $C \backslash D(c_1, \dots, c_n)$ and α is not congruent to any elements in $(c_1, \dots, c_n)(\text{mod } q)$. If $\lambda \in \Lambda$ and $\lambda = r(c_1 + \cdots + c_k b^{k-1} + c'_{k+1} b^k + \cdots)$ where $c_{k+1} \neq c'_{k+1}$ and $k \leq n$, then $c_{k+1}, c'_{k+1} \in D(c_1, \dots, c_k)$ and hence θ and λ are mutually orthogonal by the induction hypothesis. If $\lambda \in \Lambda$ is such that the first n digit expansion are equal to θ, the same argument as in the proof of $D(\vartheta)$ shows θ will be orthogonal to this λ. Therefore, θ will be orthogonal to all elements in Λ, a contradiction. Also in a similar way as the above, $D(c_1, \dots, c_n)$ contains exactly q elements can be shown.

Returning to the proof of Theorem (3.3.9), by convention, we define $\tau(\vartheta) = 0$ and on the first level, we define $\tau(\sigma_1)$ to be the unique element in $D(\vartheta)$ such that it is congruent to $\sigma_1(\text{mod } q)$. For $\sigma = \sigma_1 \dots \sigma_n$, we define $\tau(\sigma\sigma_{n+1})$ to be the unique element in $D\left(\tau(\sigma|_1), \dots, \tau(\sigma|_n)\right)$ (it is non-empty from the induction process) that is congruent to $\sigma_{n+1}(\text{mod } q)$. Then $\tau(0^k) = 0$ for $k \geq 1$.

We show that τ is a maximal mapping corresponding to Λ. (i) is satisfied by above. By Lemma (3.3.10), τ is well-defined with (ii) in Definition (3.3.4). Finally, given a node $\sigma \in \Sigma_q^n$, by the construction of the τ we can find λ whose first n digits in the digit expansion (51) exactly equals the value of $\tau(\sigma|_k)$ for all $1 \leq k \leq n$. Since the digit expansion of λ becomes 0 eventually, we continue following the digit expansion of λ so that (iii) in the definition is satisfied.

We now show that $\Lambda = r\Lambda(\tau)$. For each a_λ given in (51), Lemma (3.3.10) with the definition of τ shows that there exists unique path σ such that $\tau(\sigma|_n) = \varepsilon_\lambda^{(n)}$ for all n. As the sum is finite, this means $\Lambda \subset r\Lambda(\tau)$. Conversely, if some $II_b^\tau(\sigma) \in r\Lambda(\tau)$ is not in Λ, then from the previous proof we know $II_b^\tau(\sigma)$ must be orthogonal to all elements in Λ. This contradicts to the maximal orthogonality of Λ. Thus, $\Lambda = r\Lambda(\tau)$.

We study under what conditions a maximal orthogonal set is a spectrum or not a spectrum.

Lemma (3.3.11)[214]: ([223]). *Let μ be a Borel probability measure in $\mathbb{R}$ with compact support. Then a countable set Λ is a spectrum for $L^2(\mu)$ if and only if*

$$Q(\xi) := \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2 \equiv 1, for \xi \in \mathbb{R}.$$

Moreover, if Λ is a bi-zero set, then Q is an entire function.

Note that the first part of Lemma (3.3.11) is well-known. For the entire function property, we just note that the partial sum $\sum_{|\lambda| \leq n} |\dots|^2$ is an entire function and it is locally uniformly bounded by applying Bessel's inequality, hence Q is entire by Montel's theorem in complex analysis. One may refer to [223] for the details of the proof.

Let δ_a be the Dirac measure with center a. We define

$$\delta_{\mathcal{E}} = \frac{1}{\#\mathcal{E}} \sum_{e \in \mathcal{E}} \delta_e$$

for any finite set $\mathcal{E}$. Let μ be the self-similar measure in (41). Write $\mathcal{D} = \{0,1, \dots, q-1\}$ and $D_k = \frac{1}{b}\mathcal{D} + \cdots + \frac{1}{b^k}\mathcal{D}$ for $k \geq 1$. We recall that the *mask function* of $\mathcal{D}$ is

$$m(\xi) = \frac{1}{q}\left(1 + e^{2\pi i \xi} + \cdots + e^{2\pi i (q-1)\xi}\right)$$

and define $\mu_k = \delta_{D_k}$, then

$$\widehat{\mu_k}(\xi) = \prod_{j=1}^{k} m\,(b^{-k}\xi)$$

and it is well-known that μ_k converges weakly to μ when k tends to infinity and we have

$$\hat{\mu}(\xi) = \widehat{\mu_k}(\xi)\hat{\mu}\left(\frac{\xi}{b^k}\right). \tag{52}$$

Lemma (3.3.12)[214]: *Let τ be a regular mapping and let $\Lambda = r\Lambda(\tau) = \{\lambda_k\}_{k=0}^{\infty}$ be the maximal orthogonal set determined by τ. Then for all $n \geq 1$,*

$$\sum_{k=0}^{q^n-1} |\widehat{\mu_n}(\xi + r\lambda_k)|^2 \equiv 1. \tag{53}$$

Proof. We claim that $\{r\lambda_k\}_{k=0}^{q^n-1} = \frac{b}{q}\{\lambda_k\}_{k=0}^{q^n-1}$ is a spectrum of $L^2(\mu_n)$. We can then use Lemma (3.3.11) to conclude our lemma. Since this set has exactly q^n elements, we just need to show the mutual orthogonality. To see this, we note that

$$\hat{\mu}_n(\xi) = m\left(\frac{\xi}{b}\right) \cdots m\left(\frac{\xi}{b^n}\right). \tag{54}$$

Given $l \neq l'$ in $\{0, \dots, q^n - 1\}$, let $\sigma = \sigma_1 \cdots \sigma_n$ and $\sigma' = \sigma_1 \dots \sigma_n'$ be the q-adic expansions of l and l' respectively as in (49), where σ_n and σ_n' may be zero. We let $s \leq n$ be the first index such that $\sigma_s \neq \sigma_s'$. Then we can write

$$\lambda_l - \lambda_{l'} = b^{s-1}(\tau(\sigma|_s) - \tau(\sigma'|_s) + bM)$$

for some integer M. We then have from the integral periodicity of $e^{2\pi i x}$ that

$$m\left(\frac{r(\lambda_l - \lambda_{l'})}{b^s}\right) = m\left(\frac{\tau(\sigma|_s) - \tau(\sigma'|_s)}{q}\right) = 0.$$

It is equal to 0 because (ii) in the definition of maximal mapping implies that q does not divide $(\sigma|_s) - \tau(\sigma'|_s)$. Hence, by (54), $\hat{\mu}_n\big(r(\lambda_l - \lambda_{l'})\big) = 0$.

Now, we let

$$Q_n(\xi) = \sum_{k=0}^{q^n-1} |\hat{\mu}(\xi + r\lambda_k)|^2, and\ Q(\xi) = \sum_{k=0}^{\infty} |\hat{\mu}(\xi + r\lambda_k)|^2.$$

For any n and p, we have the following identity:

$$\begin{aligned} Q_{n+p}(\xi) &= Q_n(\xi) + \sum_{k=q^n}^{q^{n+p}-1} |\hat{\mu}(\xi + r\lambda_k)|^2 \\ &= Q_n(\xi) + \sum_{k=q^n}^{q^{n+p}-1} |\widehat{\mu_{n+p}}(\xi + r\lambda_k)|^2 |\hat{\mu}\left(\frac{\xi + r\lambda_k}{b^{n+p}}\right)|^2. \end{aligned} \tag{55}$$

We see whether $Q(\xi) \equiv 1$. Then by invoking Lemma (3.3.11), we can determine whether we have a spectrum. As Q is an entire function by Lemma (3.3.11), we just need to see the value of $Q(\xi)$ for some small values of ξ. To do this, we need to make a fine estimation of the terms $|\hat{\mu}\left(\frac{\xi+r\lambda_k}{b^{n+p}}\right)|^2$ in the above. Write

$$c_{\min} = \min\left\{\prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 : |\xi| \leq \frac{b-1}{qb}\right\} > 0,$$

where $|m(\xi)| = \frac{|\sin \pi q\xi|}{q|\sin \pi\xi|}$ and $\prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 = |m(\xi)\hat{\mu}(\xi)|^2$. Denote

$$c_{\max} = \max\left\{|m(\xi)|^2 : \frac{1}{b^2} \leq |\xi| \leq \frac{b-1}{qb}\right\} < 1.$$

The following proposition roughly says that the magnitude of the Fourier transform is controlled by the number of non-zero digits in the b-adic expansion in a uniform way. Recall that $b = qr$ with $r \geq 2$.

Proposition (3.3.13)[214]: *Let* $|\xi| \leq \frac{rb-2)}{b-1}$ *and let*

$$t = \xi + \sum_{k=1}^{N} d_i\, b^{n_k},$$

where $d_i \in \{1, 2, \dots, r-1\}$ *and* $1 \leq n1 < \cdots < n_N$. *Then*

$$c_{\min}^{N+1} \leq |\hat{\mu}(t)|^2 \leq c_{\max}^{N}. \tag{56}$$

Proof. First it is easy to check that, for $|\xi| \leq \frac{rb-2)}{b-1}$ and all $d_k \in \{0,1,2,\dots,r-1\}$, we have

$$\left|\frac{\xi + \sum_{k=1}^{n} d_k\, b^k}{b^{n+1}}\right| \leq \frac{1}{b^{n+1}}\left(\frac{r(b-2)}{b-1} + (r-1)(b + b^2 + \cdots + b^n)\right)$$
$$= \frac{r(b-2) + (r-1)(b^{n+1} - b)}{b^{n+1}(b-1)} \leq \frac{b-1}{qb} \tag{57}$$

for $n \geq 1$. The inequality in the last line follows from a direct comparison of the difference and $q \geq 2$. To simplify notations, we let $n_0 = 0$ and $n_{N+1} = \infty$. Then $|\hat{\mu}(t)|^2$ equals

$$\prod_{j=1}^{\infty} |m(b^{-j}t)|^2 = \prod_{i=0}^{N} \prod_{j=n_i+1}^{n_{i+1}} |m(b^{-j}t)|^2. \tag{58}$$

We now estimate the products one by one. By (57), we have

$$\left|\frac{\xi + \sum_{k=1}^{i} d_k\, b^n k}{b^{n_i+1}}\right| \leq \frac{b-1}{qb}$$

Hence, together with the integral periodicity of m and the definition of c_{min}, we have for all $i > 0$,

$$\prod_{j=n_i+1}^{n_{i+1}} |m(b^{-j}t)|^2 = \prod_{j=n_i+1}^{n_{i+1}} \left|m\left(b^{-j}\left(\xi + \sum_{k=1}^{i} d_k\, b^{n_k}\right)\right)\right|^2$$
$$\geq \prod_{j=0}^{\infty} |m(b^{-j}(\frac{\xi + \sum_{k=1}^{i} d_k\, b^n k}{b^{n_i+1}}|||^2 \geq C_{min} \tag{59}$$

For the case $i=0$, it is easy to see that $|\frac{\xi}{b}| \leq \frac{b-2}{q(b-1)} < \frac{b-1}{qb}$. Hence, $\prod_{j=n_0+1}^{n_1} |m(b^{-j}t)|^2 \geq \prod_{j=0}^{\infty} |m\left(b^{-j}\left(\frac{\xi}{b}\right)\right)|^2 \geq c_{min}$. Putting this fact and (59) into (58), we have $|\hat{\mu}(t)|^2 \geq c_{\min}^{N+1}$

We next prove the upper bound. From $|m(\xi)| \leq 1$, (58) and the integral periodicity of m,

$$|\hat{\mu}(t)|^2 \leq \prod_{i=1}^{N} |m(b^{-(n_i+1)}t)|^2$$
$$= \prod_{i=1}^{N} |m\left(b^{-(n_i+1)}\left(\xi + \sum_{k=1}^{i} d_k\, b^n k\right)\right)|^2 \qquad (60)$$

By (57) we have

$$|\xi + \sum_{k=1}^{i} d_k\, b^n k| \geq b^{n_i} - |\xi + \sum_{k=1}^{i-1} d_k\, b^n k| \geq b^{n_i} - \frac{b^{n_{i-1}}(b-1)}{q} \geq b^{n_i-1}$$

By (57), (60), the above and the definition of $c_{\max}$, we obtain that $|\hat{\mu}(t)|^2 \leq c_{\max}^N$.

We now prove Theorem (3.3.14). Write $c_1 = c_{min}$ and $c_2 = c_{\max}$, where c_{min} and $c_{\max}$ are in Proposition (3.3.13). Also recall the quantities defined. For any $n \in \mathbb{N}$, the q-adic expression of n is $\sum_{j=1}^{k} \sigma_j\, q^{j-1}$ with $\sigma_k \neq 0$. Then for the map τ we have

$$\lambda_n = \sum_{j=1}^{k} \tau\left(\sigma_1 \cdots \sigma_j\right) b^{j-1} + \sum_{j=k+1}^{N_n} \tau\left(\sigma_1 \cdots \sigma_k 0^{j-k}\right) b^{j-1}$$

where $\tau(\sigma_1 \cdots \sigma_k\ 0^{N_n-k}) \neq 0$ and $N_n^* = \#\{\tau(\sigma_1 \cdots \sigma_k 0^j) \neq 0 :\ k+1 \leq j \leq N_n\}$. Moreover, $\mathcal{N}_{m,n}^* = \max_{q^m \leq k < q^n}\{N_k^*\}, \mathcal{L}_n^* = \min_{q^n \leq k < q^{n+1}}\{N_k^*\}$ and $\mathcal{M}_n = \max_{1 \leq k < q^n}\{N_k\}$.

Theorem (3.3.14)[214]: *Let* $\Lambda = r\Lambda(\tau)$ *for a regular mapping* τ. *Then we can find* $0 < c_1 < c_2 < 1$ *so that the following holds.*

(i) *If there exists a strictly increasing sequence* α_n *satisfying*

$$\alpha_{n+1} - \mathcal{M}_{\alpha_n} \to \infty, and \sum_{n=1}^{\infty} c_1^{\mathcal{N}_{\alpha_n,\alpha_{n+1}}^*} = \infty, \qquad (61)$$

then Λ *is a spectrum of* $L^2(\mu)$.

(ii) *If* $\sum_{n=1}^{\infty} c_2^{\mathcal{L}_n^*} < \infty$, *then* Λ *is not a spectrum of* $L^2(\mu)$.

Proof. (i) Let α_n be the increasing sequence satisfying (61) and let $|\xi| \leq \frac{b-2}{b-1}$. Recall (55),

$$Q_{\alpha_{n+1}}(q^{-1}\xi) = Q_{\alpha_n}(q^{-1}\xi) + \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}-1}} |\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)|^2 |\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right)|^2$$

For $k = q^{\alpha_n}, \ldots, q^{\alpha_{n+1}} - 1$, we may write λ_k as

$$\lambda_k = \sum_{j=0}^{\alpha_{n+1}-1} c_j\, b^j + \sum_{j=1}^{M_k} d_j\, q b^{n_j},$$

where $c_j \in \{-1, \dots, b-2\}, d_j \in \{1, \dots, r-1\}$ and $\alpha_{n+1} \le n_1 < n_2 < \cdots < n_{M_k}$ with $n_{M_k} = N_k$ and $M_k \le N_k^*$, where N_k, N_k^* were defined in (50) or see the above. Note also that the second term on the right hand of the above is zero whenever $N_k < \alpha_{n+1}$. Now,

$$\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}} = \frac{q^{-1}\xi + q^{-1}\sum_{j=1}^{\alpha_{n+1}} c_j\, b^j}{b^{\alpha_{n+1}}} + \sum_{j=1}^{M_k} d_j\, b^{n_j - \alpha_{n+1}+1}$$

Note that

$$|\frac{\xi}{q} + \frac{1}{q}\sum_{j=1}^{k} c_j\, b^j| \le \frac{b-2}{q(b-1)} + (b-2)\frac{b^{k+1}-b}{q(b-1)} \le \frac{b-2}{q(b-1)} b^{k+1}$$
$$= \frac{r(b-2)}{b-1} b^k$$

for all $k \ge 1$. Hence, Proposition (3.3.13) implies that

$$|\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right)|^2 \ge c_1^{1+M_k} \ge c_1^{1+N_k^*} \ge c_1^{1+N_{\alpha_n,\alpha_{n+1}}^*}$$

for all $q^{\alpha_n} \le k < q^{\alpha_{n+1}}$. Therefore, together with Lemma (3.3.12),

$$Q_{\alpha_{n+1}}(q^{-1}\xi) \ge Q_{\alpha_n}(q^{-1}\xi) + c_1^{1+\mathcal{N}_{\alpha_n,\alpha_{n+1}}^*} \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}}-1} |\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)|^2$$
$$= Q_n(q^{-1}\xi) + c_1^{1+\mathcal{N}_{\alpha_n,\alpha_{n+1}}^*}\left(1 - \sum_{k=0}^{q^{\alpha_n}-1} |\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)|^2\right).$$

From elementary analysis, there exists δ, $0 < \delta < 1$, such that $|\hat{\mu}(\xi)|^2$ is decreasing on $(0,\delta)$ (in fact, since $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\xi)|^2$ is entire in complex plane, there exists $\eta > 0$ such that $|\hat{\mu}(\xi)| < 1$ for all $0 < \xi < \eta$. If $|\hat{\mu}(\xi)|^2$ is not decreasing on $(0,\delta)$ for any $\delta > 0$, we can find a sequence $\xi_n \to 0$ such that $(|\hat{\mu}|^2)'(\xi_n) = 0$ and thus $(|\hat{\mu}|^2)' \equiv 0$ by the entire function property of $|\hat{\mu}|^2$, this is impossible). In the proof, it is also useful to note that $|\hat{\mu}(-\xi)| = |\hat{\mu}(\xi)|$. We now argue by contradiction. Suppose there exists Λ such that Theorem (3.3.14) (i) holds but is not a spectrum, then there exists $t_0 <$ min $\left\{\delta, \frac{b-2}{b-1}\right\}$ such that $Q(q^{-1}t_0) < 1$because Q *is* entire. for $0 \le k \le q^{\alpha_n} - 1$, we have

$$|\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}| \le \frac{1 + rb^{\mathcal{M}_{\alpha_n}}}{b^{\alpha_{n+1}}} := \beta_n.$$

By the assumption that $\alpha_{n+1} - \mathcal{M}_{\alpha_n} \to \infty$, we have for all n large, say $n \ge M$, $\beta_n < \delta$ so that $|\hat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right)|^2 \ge |\hat{\mu}(\beta_n)|^2$ and we can find $\vartheta_0 < 1$ such that

$$|\hat{\mu}(\beta_n)|^{-2} Q(q^{-1}t_0) \le \vartheta_0 < 1, \quad \text{for } n \ge M$$

because β_n tends to zero when n tends to infinity and $\hat{\mu}(0) = 1$. According to $\hat{\mu}(\xi) = \hat{\mu}_{\alpha_{n+1}}(\xi)\hat{\mu}(\xi/b^{\alpha_{n+1}})$, we have

$$|\hat{\mu}(q^{-1}t_0 + r\lambda_k)|^2 = |\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)\hat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right)|^2$$
$$\ge |\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)|^2 |\hat{\mu}(\beta_n)|^2 \ge \frac{Q(q^{-1}t_0)}{\vartheta_0} |\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)|^2.$$

From (55) and for all $n \ge M$,

$$Q_{\alpha_{n+1}}(q^{-1}t_0)$$

$$\geq Q_{\alpha_n}(q^{-1}t_0) + \left(1 - \frac{\vartheta_0}{Q(q-1_{t_0})} \sum_{k=0}^{q^{\alpha_n}-1} |\hat{\mu}(q^{-1}t_0 + r\lambda_k)|^2\right) c_1^{1+\mathcal{N}^*_{\alpha_n,\alpha_{n+1}}}$$
$$\geq Q_{\alpha_n}(q^{-1}t_0) + (1-\vartheta_0) c_1^{1+\mathcal{N}^*_{\alpha_n,\alpha_{n+1}}}$$

Taking summation on n from M to $M+p$ where $p > 0$ and noting that $Q_n(t) \leq 1$ for any n we have

$$1 \geq Q_{\alpha_{M+p+1}}(q^{-1}t_0) \geq Q_{\alpha_M}(q^{-1}t_0) + (1-\vartheta_0) \sum_{n=M}^{M+p} c_1^{1+\mathcal{N}^*_{\alpha_n,\alpha_{n+1}}}$$

As $\sum_{n=M}^{\infty} c_1^{\mathcal{N}^*_{\alpha_n,\alpha_{n+1}}} = \infty$ by the assumption, the right hand side of the above tends to infinity. This is impossible. Hence, Λ must be a spectrum.

(ii) The proof starts again at (55) with $p = 1$, we have

$$Q_{n+1}(q^{-1}\xi) = Q_n(q^{-1}\xi) + \sum_{k=q^n}^{q^{n+1}} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2 |\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{n+1}}\right)|^2$$

Since $N^*_k \geq \mathcal{L}^*_n$ for $q^n \leq k < q^{n+1}$, Proposition (3.3.13) implies that

$$Q_{n+1}(q^{-1}\xi) \leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}^*_n} \sum_{k=q^n}^{q^{n+1}-1} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2$$

Using Lemma (3.3.12) and noting that $|\widehat{\mu_{n+1}}(\xi)|^2 \geq |\hat{\mu}(\xi)|^2$, we have

$$Q_{n+1}(q^{-1}\xi) \leq Q_n(q^{-1}\xi) + c_2^{L^*_n}\left(1 - \sum_{k=0}^{q^n-1} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2\right)$$
$$\leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}^*_n}\left(1 - Q_n(q^{-1}\xi)\right) .$$

Hence,

$$1 - Q_{n+1}(q^{-1}\xi) \geq \left(1 - Q_n(q^{-1}\xi)\right)\left(1 - c_2^{L^*_n}\right)$$
$$\geq \cdots \geq \left(1 - Q_1(q^{-1}\xi)\right) \prod_{k=1}^{n} \left(1^{\mathcal{L}^*_k} - c_2\right). \qquad (62)$$

Since $\sum_n c_2^{L^*_n} < \infty, B := \prod_{k=1}^{\infty}\left(1 - c_2^{L^*_n}\right) > 0$ and hence as n tends to infinity in (62), we have

$$1 - Q(q^{-1}\xi) \geq \left(1 - Q_1(q^{-1}\xi)\right) \cdot B > 0.$$

Therefore, τ is not a spectral mapping.

As known from Example (3.3.5), τ is a spectral mapping if $\sup\{N^*_n\}$ is finite. Now, we give an example of a spectrum with slow growth rate of N^*_n.

Example (3.3.15)[214]: Let τ be a regular mapping so that $N_n \leq \log_q n + \log_{c_1^{-2}} \log_q n$ and $N^*_n \leq \log_{c_1^{-2}} \log_q n$ for ≥ 1 , where c_1 is given in Theorem (3.3.14). Then $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$.

Proof. Take $\alpha_n = n^2$. Recalling $\mathcal{M}_n = \max_{1\leq k<q^n} N_k$, we have

$$\alpha_{n+1} - \mathcal{M}_{\alpha_n} \geq (n+1)^2 - n^2 - \log_{c_1^{-2}} n^2,$$

which tends to infinity when n tends to infinity. Note that

$$\mathcal{N}^*_{\alpha_n,\alpha_{n+1}} = \max_{q^{\alpha_n} \leq k < q^{\alpha_n+1}} \mathcal{N}^*_k \leq \log_{c_1^{-2}} \log_q q^{(n+1)^2} = \log_{c_1^{-2}} (n+1)^2.$$

Then

$$\sum_{n=1}^{\infty} c_1^{\mathcal{N}^*_{\alpha_n,\alpha_{n+1}}} \geq \sum_{n=1}^{\infty} c_1^{\log_{c1^{-2}}(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty .$$

By Theorem (3.3.14) the result follows.

On the other hand, if N_n^* is so that $\mathcal{L}_n^* \geq (1+\varepsilon)\log_{c_2^{-1}} n$, for some $\varepsilon > 0$ and $n \geq 1$, then $r\Lambda(\tau)$ is not a spectrum. This is done by checking the condition of Theorem (3.3.14) (ii) using the similar method as above. Finally, we prove Theorem (3.3.16).

Theorem (3.3.16)[214]: *Let* $\mu = \mu_{q,b}$ *be a measure defined in* (41) *with* $b > q$ and $\gcd(q,b) = q$. *Then given any increasing non-negative function* g *on* $[0,\infty)$, *there exists a spectrum* Λ *of* $L^2(\mu)$ *such that*

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{\#\big(\Lambda \cap (x-R, x+R)\big)}{g(R)} = 0. \tag{63}$$

Proof. For any $n \in \mathbb{N}$, n can be expressed as

$$n = \sum_{j=1}^{k} \sigma_j \, q^{j-1} , \tag{64}$$

where all $\sigma_j \in \{0,1, \dots , q-1\}$ and $\sigma_k \neq 0$. Let $\{m_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $m_1 \geq 2$. We now define a maximal mapping in terms of this sequence by $\tau(\vartheta) = \tau(0^k) = 0$ for $k \geq 1$ and for n as in (64),

$$\tau(\sigma) = \begin{cases} \sigma_k, & \text{if } \sigma = \sigma_1 \cdots \sigma_k, \sigma_k \neq 0; \\ 0, & \text{if } \sigma = \sigma_1 \cdots \sigma_k 0^\ell \text{ , } P \neq m_n; \\ q , & \text{if } \sigma = \sigma_1 \dots \sigma_k 0^\ell \text{and} \ell = m_n. \end{cases}$$

By the definition we have $\lambda_0 = 0$ and

$$\lambda_n = \sum_{j=1}^{k} \tau\left(\sigma_1 \dots \sigma_j\right) b^{j-1} + q b^{m_n},$$

consequently, $N_n^* = 1$ and by Theorem (3.3.14) (i) (see also Example (3.3.5)), $\Lambda := \{\lambda_n\}_{n=0}^{\infty}$ is a spectrum for $L^2(\mu)$.

We now find Λ with density in (48) zero by choosing m_n. To do this, we first note that there exists a strictly increasing continuous function $h(t)$ from $[0,\infty)$ onto itself such that $h(t) \leq g(t)$ for $t \geq 0$ and it is sufficient to replace $g(t)$ by $h(t)$ in the proof. In this way, the inverse of $h(t)$ exists, and we denote it by $h^{-1}(t)$.

Now, note that

$$\lambda_n \leq q \frac{b^k - 1}{b-1} + q b^{m_n} \leq (q+1) b^{m_n}.$$

Hence,

$$\lambda_{n+1} - \lambda_n \geq q b^{m_{n+1}} - (q+1) b^{m_n} \geq b^{m_n+1}. \tag{65}$$

Therefore, we choose m_n so that $b^{m_n} \geq 2h^{-1}(b^{n+1})$ for all $n \geq 1$. For any $h(R) \geq 1$, there exists unique $s \in \mathbb{N}$ such that $b^{s-1} \leq h(R) < b^s$. Then

$$\frac{\sup_{x\in\mathbb{R}} \#\big(\Lambda \cap (x-R, x+R)\big)}{h(R)} \leq \frac{\sup_{x\in\mathbb{R}} \#\Big(\Lambda \cap \big(x - h^{-1}(b^s), x + h^{-1}(bs)\big)\Big)}{b^{s-1}}$$

Note from (65) that the length of the open intervals $\left(x-h^{-1}(b^s), x+h^{-1}(bs)\right)$ is less than $\lambda_{n+1}-\lambda_n$ whenever $n \geq s$. This implies that the set $\Lambda \cap \left(x-h^{-1}(b^s), x+h^{-1}(bs)\right)$ contains at most one λ_n where $n \geq s$. We therefore have

$$\sup_{x\in\mathbb{R}} \#\left(\Lambda \cap \left(x-h^{-1}(b^s), x+h^{-1}(b^s)\right)\right) \leq s+1.$$

Thus the result follows by taking limit.

Let τ be a maximal mapping (not necessarily regular) for $\mu=\mu_{q,b}$ with $b=qr$. Given any $I=\sigma_1\cdots\sigma_k \in \Sigma_q^k=\{0,1,\dots,q-1\}^k$ with $\sigma_k\neq 0$. Define a map τ' by

$$\tau'(\sigma)=\begin{cases}0, & \sigma=I0^\ell \text{ for } \ell\geq 1;\\ \tau(\sigma), & \text{otherwise.}\end{cases}$$

Clearly τ' is a maximal mapping. The main result is as follows.

This result shows that if we arbitrarily change the value of τ along an element in Γ_q as above, the spectral property of τ is unaffected. In particular, Theorem (3.3.6) follows as a corollary because we can alter the irregular elements one by one using Theorem (3.3.20).

Note that we can decompose

$$\Gamma_q=\{\sigma 0^\infty : \sigma\in\Sigma_q^*\}=\bigcup_{I\in\Sigma_q^n} I\Gamma_q \tag{66}$$

for all $n\geq 1$. And recall that

$$\Lambda(\tau)=\{\Pi_b^\tau(J) : \in \Gamma_q, \tau \text{ is regular on } J\},$$

where $\Pi_b^\tau(J)=\sum_{k=1}^\infty \tau(J|_k)b^{k-1}$. Denote naturally $\Pi_b^\tau(I)=\sum_{k=1}^n \tau(I|_k)b^{k-1}$ if $I\in\Sigma_q^n$, and $\Pi_{b,I}^\tau(J)=\sum_{k=1}^\infty \tau(Ij_1\cdots j_k)b^{k-1}$ for $J=j_1j_2\dots\in\Gamma_q$ where IJ is regular for τ. Define also

By (66) we have

$$\Lambda_I(\tau)=\{\Pi_{b,I}^\tau(J) : \in \Gamma_q, \tau \text{ is regular on } J\}.$$

$$\Lambda(\tau)=\bigcup_{I\in\Sigma_q^n}\left(\Pi_b^\tau(I)+b^n\Lambda_I(\tau)\right).$$

The following is a simple lemma which was also observed in [217].

Proposition (3.3.17)[214]: *Let τ be a tree mapping and $n\geq 1$. Then $r\Lambda(\tau)$ is a spectrum for μ if and only if all $r\Lambda_I(\tau)$, $I\in\Sigma_q^n$, are spectra.*

Proof. Recall that μ_k satisfies $\hat{\mu}(\xi)=\hat{\mu}_k(\xi)\hat{\mu}(b^{-k}\xi)$ and $\hat{\mu}_k(\xi)=\prod_{j=1}^k m\left(b^{-j}\xi\right)$. where $m(\xi)=\frac{1}{q}\sum_{j=1}^q e^{2\pi i(j-1)\xi}$. Write $Q_I(\xi)\sum_{\lambda\in\Lambda_I(\tau)}|\hat{\mu}(\xi+r\lambda)|^2$. Note that

$$\begin{aligned}
Q(\xi)&=\sum_{\lambda\in\Lambda(\tau)}|\hat{\mu}(\xi+r\lambda)|^2\\
&=\sum_{I\in\Sigma_q^n,\lambda\in\Lambda_I(\tau)}|\hat{\mu}_n(\xi+r\Pi_b^\tau(I)+rb^n\lambda)|^2\left|\hat{\mu}\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}+r\lambda\right)\right|^2\\
&=\sum_{I\in\Sigma_q^n,\ \lambda\in\Lambda_I(\tau)}|\hat{\mu}_n\left(\xi+r\Pi_b^\tau(I)\right)|^2\left|\hat{\mu}\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}+r\lambda\right)\right|^2\\
&=\sum_{I\in\Sigma_q^n}|\hat{\mu}_n\left(\xi+r\Pi_b^\tau(I)\right)|^2.Q_I\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}\right).
\end{aligned}$$

In a similar proof of Lemma (3.3.12), we have

$$1 \equiv \sum_{I\in\Sigma_q^n} |\,\hat{\mu}_n\big(\xi + r\Pi_b^\tau(I)\big)|^2.$$

Invoking Lemma (3.3.11), the result follows.

Proposition (3.3.17) asserted that spectral property is determined by a finite number of nodes. The following two lemmas show that the spectral property of a particular node σ can be determined by infinitely many of its offsprings and is *independent* of the regularity of $\sigma 0^\infty$. These are the key lemmas to the proof of Theorem (3.3.20).

Lemma (3.3.18)[214]: *Let* $I \in \Sigma_q^* with\ I \neq \vartheta$, *the empty word. If* τ *is regular on* $I0^\infty$, *then*

$$\Lambda_I(\tau) = \{II_{b,I}^\tau(0^\infty)\} \cup \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{q-1}\Big(II_{b,I}^\tau(0^{k-1}j) + b^k \Lambda_{I0^{k-1}j}(\tau)\Big),$$

where $II_{b,I}^\tau(0^{k-1}j) = \tau(I0) + \tau(I0^2)b + \cdots + \tau(I0^{k-1}j)b^{k-1}$. *If* τ *is irregular on* $I0^\infty$, *then*

$$\Lambda_I(\tau) = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{q-1}\Big(II_{b,I}^\tau(0^{k-1}j) + b^k \Lambda_{I0^{k-1}j}(\tau)\Big).$$

Proof. Check it directly.

Lemma (3.3.19)[214]: *Let* τ *be a maximal mapping and let* $I \in \Sigma_q^*$. *Then* $\Lambda_I(\tau)$ *is a spectrum of* μ *if and only if* $\Lambda_{I0^{k-1}j}(\tau)$ *are spectra of* μ *for all* $k \geq 1\ and\ j = 1, \dots, q - 1$.

Proof. The necessity is clear from Proposition (3.3.17). We now prove the sufficiency. Assume that $\Lambda_{I0^{k-1}j}(\tau)$ are spectra for all $k \geq 1\ and\ j = 1, \dots, q-1$. We need to show that $Q_I(\xi) = \sum_{\lambda\in\Lambda(\tau)} |\,\hat{\mu}(\xi + r\lambda)|^2 \equiv 1$.

By the integral periodicity of m and Lemma (3.3.11) which will be used in the second equality below, we have for all $k \geq 2$,

$$\sum_{j=1}^{q-1} |\,\hat{\mu}_k\left(\xi + rII_{b,I}^\tau(0^{k-1}j)\right)|^2$$

$$= \sum_{j=1}^{q-1} |\,\hat{\mu}_{k-1}\Big(\xi + rII_{b,I}^\tau(0^{k-1})\Big)|^2 |\hat{\mu}_1\left(\frac{\xi + rII_{b,l}^\tau(0^{k-1}) + r\tau(0^{k-1}j)b^{k-1}}{b^k}\right)|^2$$

$$= |\hat{\mu}_{k-1}\left(\xi + rII_{b,I}^\tau(0^{k-1})\right)|^2\left(1 - |\hat{\mu}_1\left(\frac{\xi + rII_{b,I}^\tau(0^k)}{b^k}\right)|^2\right)$$

$$= |\hat{\mu}_{k-1}\left(\xi + rII_{b,I}^\tau(0^{k-1})\right)|^2 - |\hat{\mu}_k\left(\xi + rII_{b,I}^\tau(0^k)\right)|^2.$$

If $k = 1$, the above becomes $\sum_{j=1}^{q-1} |\,\hat{\mu}_1\left(\xi + r\Pi_{b,I}^\tau(j)\right)|^2 = 1 - |\hat{\mu}_1\left(\xi + rII_{b,I}^\tau(0)\right)|^2$.

Now we simplify the following terms which is corresponding to the unions of the sets in Lemma (3.3.18),

$$\sum_{k=1}^{\infty}\sum_{j=1}^{q-1}\sum_{\lambda\in\Lambda_{\sigma_{0^{k-1}j}}(\tau)} |\hat{\mu}(\xi + rII_{b,I}^{\tau}(0^{k-1}j) + rb^k\lambda)|^2$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{q-1}\sum_{\lambda\in\Lambda_{\sigma_{0^{k-1}j}}(\tau)} |\hat{\mu}_k\left(\xi + rII_{b,I}^{\tau}(0^{k-1}j)\right)|^2 |\hat{\mu}\left(\frac{\xi + rII_{b,I}^{\tau}(0^{k-1}j)}{b^k} + \lambda\right)|^2$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{q-1} |\hat{\mu}_k\left(\xi + rII_{b,I}^{\tau}(0^{k-1}j)\right)|^2 = 1 - \lim_{N\to\infty} |\hat{\mu}_N\left(\xi + rII_{b,I}^{\tau}(0^{N})\right)|^2$$

$$= 1 - \prod_{j=1}^{\infty} |m\left(\frac{\xi + rII_{b,I}^{\tau}(0^j)}{bj}\right)|^2 \tag{67}$$

We now divide the proof into two cases.

Case (i). If $I0^{\infty}$ is regular, then $|\hat{\mu}\left(\xi + rII_{b,I}^{\tau}(0^{\infty})\right)|^2 = \prod_{j=1}^{\infty} |m\left(\frac{\xi + rII_{b,I}^{\tau}(0^j)}{b^j}\right)|^2$. Hence, by Lemma (3.3.18),

$$Q_I(\xi) = |\hat{\mu}\left(\xi + r\Pi_{b,I}^{\tau}(0^{\infty})\right)|^2 + (67) \equiv 1.$$

This shows $r\Lambda_I(\tau)$ is a spectrum.

Case (ii). If $I0^{\infty}$ is irregular, then Lemma (3.3.18) shows that

$$Q_I(\xi) = (67) = 1 - \prod_{j=1}^{\infty} |m\left(\frac{\xi + rII_{b,I}^{\tau}(0^j)}{b^j}\right)|^2$$

Note that, by (ii) in the definition of the maximal mapping, we may write

$$r\Pi_{b,I}(0^n) = r\sum_{j=1}^{n} \tau\left(I0^j\right)b^{j-1} = r\sum_{j=1}^{n} \left(s_j q\right) b^{j-1} = \sum_{j=1}^{n} s_j\, b^j \tag{68}$$

for $s_j \in \{0, 1, \dots, r-1\}$. Then

$$Q_I(\xi) = 1 - \prod_{j=1}^{\infty} |m\left(\frac{\xi + rII_{b,I}^{\tau}(0^{j-1})}{b^j}\right)|^2. \tag{69}$$

Suppose on the contrary $Q_I(\xi) < 1$ for some $\xi > 0$. Since Q_I is entire, we may assume ξ is small, say $|\xi| < \frac{r-1}{b-1}$. From (69), we must have

$$\prod_{j=1}^{\infty} |m\left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j}\right)|^2 > 0.$$

For those n such that $s_n \neq 0$ in (68).

$$\frac{1}{b} \leq \frac{s_n}{b} \leq \left|\frac{\xi + \sum_{j=1}^{n} s_j\, b^j}{b^{n+1}}\right| \leq \frac{r-1}{b(b-1)} < \frac{1}{q(b-1)} \quad .$$

Hence, letting $c = \max\{|m(\xi)|^2 : \frac{1}{b} \leq |\xi| < \frac{1}{q(b-1)}\} < 1$, we have

$$|m\left(\frac{\xi + rII_{b,I}^{\tau}(0^n)}{b^{n+1}}\right)|^2 = |m\left(\frac{\xi + \sum_{j=1}^{n} s_j\, b^j}{b^{n+1}}\right)|^2 \leq c\,,$$

and

$$\prod_{j=1}^{\infty} | m\left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j}\right)|^2 = \lim_{N\to\infty} \prod_{j=1}^{N} | m\left(\frac{\xi + r\Pi_{b,I}^{\tau}(0^{j-1})}{b^j}\right)|^2$$
$$< \lim_{N\to\infty} c^{\#\{n:s_n\neq 0, n\leq N\}}.$$

As $I0^{\infty}$ is irregular, there exists infinitely many $s_j \neq 0$. The above limit is zero. This is a contradiction and hence $\Lambda_I(\tau)$ must be a spectrum.

Theorem (3.3.20)[214]: *With the notation above*, τ *is a spectral mapping if and only if* τ' *is a spectral mapping*.

Proof. By the definition of τ and τ', $\Lambda_{I'}(\tau) = \Lambda_{I'}(\tau')$ for all $I' \neq I$ and $I' \in \Sigma_q^k$. Moreover, $\Lambda_{I0^{n-1}j}(\tau) = \Lambda_{I0^{k-1}j}(\tau')$ for all $k \geq 1$ and $j = 1, \ldots, q-1$. Therefore, if τ is a spectrum, then $\Lambda_{I'}(\tau')$ are spectra of μ for all $\sigma' \neq \sigma$ and $\sigma' \in \Sigma_q^k$ by Proposition (3.3.17). On the other hand, $\Lambda_{I0^{k-1}j}(\tau')$ are spectra of μ also as τ is a spectrum. By Lemma (3.3.19), $\Lambda_I(\tau')$ is also a spectrum. We therefore conclude that $\Lambda(\tau')$ is a spectrum of μ by Proposition (3.3.17) again. The converse also holds by reversing the role of τ and τ'. This completes the whole proof.

Chapter 4
Hausdorff and Packing Measures with Singularity and Self-Similar Measure with Dense Rotations

We show that for certain numbers $0 < a, b < \frac{1}{2}$, for instance $a = \frac{1}{4}$ and $b = \frac{1}{3}$, if $K = C_a \times C_b$ then typically we have $\mathcal{H}^{s-m}(K \cap V) = 0$. We can obtain such a family as the angle-α projections of the natural measure of the Sierpinski carpet. We present more general one-parameter families of self-similar measures v_α, such that the set of parameters α for which v_α is singular is a dense G_δ set but this "exceptional" set of parameters of singularity has zero Hausdorff dimension. In fact, the set of directions is residual and the typical slices of the measure, perpendicular to these directions, are discrete.

Section (4.1): Slices of Dynamically Defined Sets:

For $1 \le m < n$ be integers, and given $0 \le t \le n$ let H^t and P^t be the t-dimensional Hausdorff and Packing measures respectively. Let $s \in (m, n)$ be a real number, and let $K \subset \mathbb{R}^n$ be compact with $0 < \mathcal{H}^s(K) < \infty$. Denote by μ the restriction of H^s to K, by G the set of all $n-m$-dimensional linear subspaces of $\mathbb{R}^n$, and by ξ_G the natural measure on G. It is well known that $\dim_H(K \cap (x + V)) = s - m$ and $\mathcal{H}^{s-m}(K \cap (x + V)) < \infty$, for $\mu \times \xi G$-a.e. $(x, V) \in K \times G$ (see Theorem 10.11 in [248]). It is also known that if $s = \dim_P K$ then $\dim_P(K \cap (x + V)) \le \max\{0, s - m\}$ for every $V \in G$ and $\mathcal{H}^m - a$.e. $x \in V^\perp$ (see Lemma 5 in [241]), where $dimP$ stands for the packing dimension. K will denote certain self-similar or self-affine sets, in which cases it will be shown that more can be said about the $\mu \times \xi_G$-typical values of $\mathcal{H}^{s-m}(K \cap (x + V))$ and $P^{s-m}(K \cap (x + V))$.

Assume first that K is a self-similar set which satisfies the strong separation condition (SSC). If $m = 1$ and K is rotation-free, then from a result by Kempton (Theorem 6.1 in [247]) it follows that $\mathcal{H}^{s-m}(K \cap (x + V)) > 0$ for $\mu \times \xi_{G^{-a}}$.e. (x, V), if and only if $\frac{dP_{V^\perp \mu}}{d\mathcal{H}^m} \in L^\infty(dP_{V^\perp \mu})$ for $\xi_{G^{-a}}$.e. V where $P_{V^\perp}$is the orthogonal projection onto $V^\perp$. In Theorem (4.1.9) below the case of a general $1 \le m < n$ and a general self-similar set K, satisfying the SSC, will be considered. A necessary and sufficient condition for $\mathcal{H}^{s-m}(K \cap (x + V)) > 0$ to holds for $\mu \times \xi_{G^{-a}}.e. (x, V)$ will be given. In Corollary (4.1.12) this condition is verified when $m = 1, s > 2$ and the rotation group of K is finite. Also given in Theorem (4.1.9), is a necessary and sufficient condition for $\mathcal{H}^{s-m}(K \cap (x + V)) = 0$ to hold for $\mu \times \xi_{G^{-a}}$.e. (x, V).

Continuing to assume that K is a self-similar set with the SSC, it will be shown in Theorem (4.1.10) that $P^{s-m}(K \cap (x + V)) > 0$ for $\mu \times \xi_{G^{-a}}$.e. (x, V). Also given in Theorem (4.1.10), is a sufficient condition for $P^{s-m}(K \cap (x + V)) = \infty$ to hold for $\mu \times \xi_{G^a}$.e. (x, V). By using this condition, it is shown in Corollary (4.1.14) that this is in fact the case when $m = 1$ and $s > 2$. This extends a result of Orponen (Theorem 1.1 in [252]), which deals with the case in which $n = 2, s > m = 1$ and K is rotation-free.

We consider the case in which $n = 2, m = 1$ and K is a certain self-affine set. For $0 < \rho < \frac{1}{2}$ let $C_\rho \subset [0,1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, where $f_{\rho,1}(t) = \rho \cdot t$ and $f_{\rho,2}(t) = \rho \cdot t + 1 - \rho$ for each $t \in \mathbb{R}$. It will be assumed that $K = C_a \times C_b$, where $0 < a, b < \frac{1}{2}$ are such that a^{-1} and b^{-1} are Pisot numbers, $\frac{\log b}{\log a}$ is irrational, and $\dim_H(C_a) + \dim_H(C_b) > 1$. Under these conditions it is shown in [251] that there exists a dense G_δ

set, of 1-dimensional linear subspaces $V \subset \mathbb{R}^2$, such that $P_V\mu$ and H^1 are singular. By using this fact, it will be proven in Theorem (4.1.20) below that $\mathcal{H}^{s-m}(K \cap (x+V)) = 0$ for $\mu \times \xi_{G^{-a}}$.e. (x, V). This result demonstrates some kind of smallness of the slices $K \cap (x+V)$, hence it may be seen as related to a conjecture made by Furstenberg (Conjecture 5 in [243]).. In our setting this conjecture basically says that for $\xi_{G^{-a}}$.e. $V \in G$ we have $\dim_H(K \cap (x+V)) \le \max\{\dim_H K - 1, 0\}$ for each $x \in \mathbb{R}^2$, which demonstrates the smallness of the slices in another manner.

Let $0 < m < n$ be integers, let G be the Grassmann manifold consisting of all $n-m$-dimensional linear subspaces of $\mathbb{R}^n$, let $O(n)$ be the orthogonal group of $\mathbb{R}^n$, and let ξ_O be the Haar measure corresponding to $O(n)$. Fix $U \in G$ and for each Borel set $E \subset G$ define

$$\xi_G(E) = \xi_O\{g \in O(n) | gU \in E\}, \tag{1}$$

then ξ_G is the unique rotation invariant Radon probability measure on G. For a linear subspace V of $\mathbb{R}^n$ let P_V be the orthogonal projection onto V, let $V^\perp$ be the orthogonal complement of V , and set $V_x = x + V$ for each $x \in \mathbb{R}^n$.

Let Λ be a finite and nonempty set. Let $\{\phi_\lambda\}_{\lambda\in\Lambda}$ be a self-similar IFS in $\mathbb{R}^n$, with attractor $K \subset \mathbb{R}^n$ and with $\dim_H K = s > m$. For each $\lambda \in \Lambda$ there exist $0 < r_\lambda < 1$, $h_\lambda \in O(n)$ and $a_\lambda \in \mathbb{R}^n$, such that $\phi_\lambda(x) = r_\lambda \cdot h_\lambda(x) + a_\lambda$ for each $x \in \mathbb{R}^n$. We assume that $\{\phi_\lambda\}_{\lambda\in\Lambda}$ satisfies the strong separation condition. Let H be the smallest closed sub-group of $O(n)$ which contains $\{h_\lambda\}_{\lambda\in\Lambda}$, and let ξ_H be the Haar measure corresponding to H. For each $E \subset \mathbb{R}^n$ set $\mu(E) = \frac{\mathcal{H}^s(K\cap E)}{\mathcal{H}^s(K)}$, then μ is a Radon probability measure which is supported on K. For each $0 \le s < \infty$, ν a Radon probability measure on $\mathbb{R}^n$, and $x \in \mathbb{R}^n$ set

$$\Theta^{*s}(\nu, x) = \limsup_{\epsilon\downarrow 0} \frac{\nu\big(B(x,\epsilon)\big)}{(2\epsilon)^s} \text{ and } \Theta^s_*(\nu, x) = \liminf_{\epsilon\downarrow 0} \frac{\nu\big(B(x,\epsilon)\big)}{(2\epsilon)^s}, \tag{2}$$

where $B(x, \epsilon)$ is the closed ball in $\mathbb{R}^n$ with center x and radios ϱ. It holds that $\Theta^*_s(\nu,\cdot)$ and $\Theta^s_*(\nu,\cdot)$ are Borel functions. For $V \in G$ define $F_V(x, h) = \Theta^m_*(P_{(hV)^\perp}\mu, P_{(hV)^\wedge\perp}(x))$ for $(x, h) \in K \times H$, then FV is a Borel function from $K \times H$ to $[0, \infty]$. In what follows the collection $\{F_V\}_{V\in G}$ will be of great importance for us.

Let V be the set of all $V \in G$ with

$$\xi_H(\mathcal{H}\backslash\{h \in \mathcal{H}: P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0\,.$$

In Lemma (4.1.8) below it will be shown that $\xi_G(G\backslash V) = 0$. First we state our results regarding the Hausdorff measure of typical slices of K.

From Theorem (4.1.9) we can derive the following Corollaries.

Remark (4.1.1)[238]: It is known that under the assumptions of Corollary (4.1.13) we have $dim(P_{V^\perp}\mu) = m$ for each $V \in G$ (see Theorem 1.6 in [245]). It is not known however if $P_{V^\perp}\mu \ll H_m$ for each $V \in G$, which is in fact a major open problem. Hence Corollary (4.1.13) implies that determining whether

$$\mu \times \xi_G\{(x, V) \in K \times G: \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0$$

is probably quite hard.

We state our results regarding the packing measure of typical slices.

From Theorem (4.1.10) the following corollary can be derived.

Assume $n = 2$ and $m = 1$. Given $0 < \rho < \frac{1}{2}$ define $f_{\rho,1}, f_{\rho,1}: \mathbb{R} \to \mathbb{R}$ by

$$f_{\rho,1}(x) = \rho \cdot x \text{ and } f_{\rho,2}(x) = \rho \cdot x + 1 - \rho \text{ for each } x \in \mathbb{R},$$

let $C_\rho \subset [0,1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, set $d_\rho = \dim_H C_\rho$ (so that $d\rho =$

$\frac{\log 2}{\log \rho^{-1}}$), and for each $E \subset \mathbb{R}$ set $\mu_\rho(E) = H^{d_\rho}(C_\rho \cap E)/H^{d_\rho}(C_\rho)$.

The following notations will be used in the proofs of Theorems (4.1.9) and (4.1.10). For each $\lambda \in \Lambda$ set $p_\lambda = r_\lambda^s$. Then μ is the unique self-similar probability measure corresponding to the IFS $\{\phi_\lambda\}_{\lambda\in\Lambda}$ and the probability vector $(p_\lambda)_{\lambda\in\Lambda}$, i.e. μ satisfies the relation $\mu = \sum_{\lambda\in\Lambda} p_\lambda \cdot \mu \circ \phi_\lambda^{-1}$. Given a word $\lambda_1 \cdot \ldots \cdot \lambda_l = w \in \Lambda^*$ we write $p_w = p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_l}, r_w = r_{\lambda_1} \cdot \ldots \cdot r_{\lambda_l}, h_w = h_{\lambda_1} \cdot \ldots \cdot h_{\lambda_l}, \phi_w = \phi_{\lambda_1} \circ \ldots \circ \phi_{\lambda_l}$ and $K_w = \phi_w(K)$. For each $l \geq 1$ and $x \in K$, let $w_l(x) \in \Lambda_l$ be the unique word of length l which satisfies $x \in K_{w_l(x)}$. Set also

$$\rho = \min\left\{d\left(\phi_{\lambda_1}(K), \phi_{\lambda_2}(K)\right): \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 \neq \lambda_2\right\}, \tag{3}$$

then $\rho > 0$ since $\{\phi_\lambda\}_{\lambda\in\Lambda}$ satisfies the strong separation condition. Given $V_1, V_2 \in G$ set $d_G(V_1, V_2) = \|P_{V_1} - P_{V_2}\|$ (where $\|\cdot\|$ stands for operator norm), then d_G is a metric on G.

The following dynamical system will be used in the proofs of Theorems (4.1.9) and (4.1.10). Set $X = K \times H$ and for each $(x, h) \in X$ let $T(x, h) = (\phi_{w_1(x)}^{-1} x, h_{w_1(x)}^{-1} \cdot h)$. It is easy to check that the system $(X, \mu \times \xi_H, T)$ is measure preserving and from corollary 4.5 in [253], and from it follows that it is ergodic. Also, for $k \geq 1$ and $(x, h) \in X$ it is easy to verify that $T^k(x, h) = (\phi_{w_k(x)}^{-1} x, h_{w_1(x)}^{-1} \cdot h)$.

Let $\mathbb{R}$ be the Borel σ-algebra of $\mathbb{R}^n$. For each $V \in G$ set $R_V = P_{V^\perp}^{-1}(R)$, and let $\{\mu_V, x\}_{x\in\mathbb{R}^n}$ be the disintegration of μ with respect to R_V (see section 3 of [242]). For $\mu - a$ .e. $x \in \mathbb{R}^n$ the probability measure $\mu_{V,x}$ is defined and supported on $K \cap V_x$. Also, for each $f \in L^1(\mu)$ the map that takes $x \in \mathbb{R}^n$ to $\int f d\mu_{V,x}$ is R_V -measurable, the formula

$$\int f d\mu = \int\int f(y) d\mu_{V,x}(y) d\mu(x)$$

is satisfied, and for $\mu - a$.e. $x \in V^\perp$ we have

$$\int f d\mu_{V,x} = \lim_{\epsilon\downarrow 0} \frac{1}{P_{V^\perp}\mu(B(x,\epsilon))} \cdot \int_{P_{V^\perp}^{-1}(B(x,\epsilon))} f d\mu$$

For more details on the measures $\{\mu_{V,x}\}_{x\in\mathbb{R}^n}$ see section 3 of [242] and the references therein.

We shall now prove some lemmas that will be needed later on. The following lemma will be used with ξ_H in place of η, when ξ_H is considered as a measure on $O(n)$ (which is supported on H).

Lemma (4.1.2)[238]: Let Q be a compact metric group, and let ν be its normalized Haar measure. Let η be a Borel probability measure on Q, then for each Borel set $E \subset Q$

$$\nu(E) = \int_Q \eta(E \cdot q^{-1})\, d\nu(q).$$

Proof. For each Borel set $E \subset Q$ define $\zeta(E) = \int_Q \eta(E \cdot q^{-1})\, d\nu(q)$. Since ν is invariant it follows that for each $g \in Q$

$$\zeta(E_g) = \int_Q \eta(E \cdot g \cdot q^{-1}) d\nu(q) = \int_Q \eta(E \cdot g \cdot (q \cdot g)^{-1})\, d\nu(q) = \zeta(E).$$

This shows that ζ is a right-invariant Borel Probability measure on Q, hence $\nu = \zeta$ by the uniqueness of the Haar measure, and the lemma follows.

Lemma (4.1.3)[238]: Let $\mathcal{V}$ be the set of all $V \in G$ with

$$\xi_H(H\backslash\{h \in H: P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0,$$

then $\xi_G(G\backslash\mathcal{V}) = 0$.

Proof. Set $L = G\backslash\{V \in G: P_{V^\perp}\mu \ll \mathcal{H}^m\}$. Since $s > m$ it follows that $I_m(\mu) < \infty$ (where $I_m(\mu)$ is the m-energy of μ), hence from Theorem 9.7 and equality (3.10) in [248] we get that $\xi_G(L) = 0$. Let $U \in G$ be as in (1) and set $L' = \{g \in O(n) : gU \in L\}$, then $\xi_O(L') = \xi_G(L) = 0$. Let $B \subset O(n)$ be a Borel set with $L' \subset B$ and $\xi_O(B) = 0$, then from Lemma (4.1.2) it follows that

$$0 = \xi_O(B) = \int \xi_H(B \cdot g^{-1}) d\xi_O(g)$$

From this we get that for ξ_{O}-a.e. $g \in O(n)$

$$0 = \xi_H(B \cdot g^{-1}) \geq \xi_H(L' \cdot g^{-1}) = \xi_H\{h \in H: h_g \in L'\}$$
$$= \xi_H\big(H\backslash\{h \in H : P_{(hgU)^\perp}\mu \ll \mathcal{H}^m\}\big),$$

and so

$$\xi_H(H\backslash\{h \in H: P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0 \text{ for } \xi_{G}\text{-a.e. } V \in G,$$

which proves the lemma.

Lemma (4.1.4)[238]: Let Z be the set of all $(x, V) \in K \times G$ such that $\mu_{V,x}$ is defined and

$$\mu_{V,x}(K_w) = \lim_{\epsilon\downarrow 0} \mu \frac{K_w \cap P_{V^\perp}^{-1}\big(B(P_{V^\perp}x, \epsilon)\big)}{P_{V^\perp}\mu\big(B(P_{V^\perp}x, \epsilon)\big)} \quad \text{for each } w \in \Lambda^*,$$

then for each $V \in G$ we have

$$\mu \times \xi_H\{(x, h) \in X: (x, hV) \notin \mathcal{Z}\} = 0\,.$$

Proof. Fix $V \in G$. It holds that $\mathcal{Z}$ is a Borel set, see section 3 of [249] for a related argument. It follows that the set

$$\mathcal{Z}_V = \{(x, h) \in X: (x, hV) \in \mathcal{Z}\}$$

is also a Borel set. From the properties stated we get that

$$\mu\{x \in K: (x, h) \notin \mathcal{Z}_V\} = 0 \text{ for each } h \in H,$$

and so $\mu \times \xi_H(X\backslash\mathcal{Z}_V) = 0$ by Fubini's theorem. This proves the lemma.

Lemma (4.1.5)[238]: Given a compact set $\widetilde{K} \subset \mathbb{R}^n$ and $0 < t \leq n$, the map that takes $(x, V) \in \widetilde{K} \times G$ *to* $\mathcal{H}^t(\widetilde{K} \cap V_x)$ is Borel measurable.

Proof. For $\delta > 0$ be as defined in section 4.3 of [248]. Let H_δ^t . Let $(x, V) \in \widetilde{K} \times G, \epsilon > 0$ and $\{(x_k, V^k)\}_{k=1}^\infty \subset \widetilde{K} \times G$, be such that $(x_k, V^k) \overset{k}{\to} (x, V)$. Let $W_1, W_2, \ldots \subset \mathbb{R}^n$ be open sets with $\widetilde{K} \cap V_x \subset \cup_{j=1}^\infty W_j$,

$$\sum_{j=1}^\infty \big(\text{diam}(W_j)\big)^t \leq H_\delta^t(\widetilde{K} \cap Vx) + \epsilon$$

and $diam\big(W_j\big) \leq \delta$ for each $j \geq 1$. Since $\widetilde{K}$ is compact and since $(x_k, V^k) \overset{k}{\to} (x, V)$, it follows that $\widetilde{K} \cap V_{x_k}^k \subset \cup_{j=1}^\infty W_j$ for each $k \geq 1$ which is large enough, and so for each such k

$$\mathcal{H}_\delta^t(\widetilde{K} \cap V_{x_k}^k) \leq \sum_{j=1}^\infty \big(\text{diam}(W_j)\big)^t < H_\delta^t(\widetilde{K} \cap V_x) + \epsilon.$$

It follows that the function that maps (x, V) to $\mathcal{H}_\delta^t(\widetilde{K} \cap V_x)$ is upper semi-continuous, and so Borel measurable. Now since $\mathcal{H}^s = \lim_{k\to\infty} \mathcal{H}_{\frac{1}{k}}^s$ the lemma follows.

Lemma (4.1.6)[238]: Given $0 < t \leq n$ and a Radon probability measure ν on $K \times G$, the map that takes $(x, V) \in K \times G$ to $P^t(K \cap V_x)$ is ν-measurable (i.e. this map is universally measurable).

Proof. Let $a \geq 0$ and set $E = \{(x, V) \in K \times G: P_t(K \cap V_x) < a\}$, then in order to prove the lemma it suffice to show that E is ν-measurable. Set $Y = \{C \subset K: C \text{ is compact}\}$, endow Y with the Hausdorff metric, and let G be the σ-algebra of Y which is generated by its analytic subsets. Set

$$\mathcal{E} = \{C \in Y: P^t(C) < a\},$$

then from Theorem 4.2 in [250] it follows that $\mathcal{E} \in G$, and so from Theorem 21.10 in [246] we get that $\mathcal{E}$ is universally measurable.

For each $(x, V) \in K \times G$ set $\psi(x, V) = K \cap V_x$, it will now be shown that $\psi: K \times G \to Y$ is a Borel function. For each $y \in K$ the function that maps $(x, V) \in K \times G$ to $d(K \cap V_x, y)$ is lower semi-continuous, and hence a Borel function. For each $l \geq 1$ let $S_l \subset K$ be finite and l^{-1}-spanning, and set $\psi_l(x, V) = \{y \in S_l: d(K \cap V_x, y) \leq l^{-1}\}$ for each $(x, V) \in K \times G$. It holds that $\psi_l: K \times G \to Y$ is a Borel function and $\psi_l \xrightarrow{l\to\infty} \psi$ pointwise, hence ψ is a Borel function. Note also that $E = \psi^{-1}(\mathcal{E})$. Since $\mathcal{E}$ is universally measurable it is $\nu \circ \psi^{-1}$-measurable, and so there exist A and C, Borel subsets of Y, such that $\mathcal{A} \subset \mathcal{E} \subset \mathcal{C}$ and $\nu \circ \psi^{-1}(C \backslash A) = 0$. It holds that $\psi^{-1}(\mathcal{A})\ and\ \psi^{-1}(\mathcal{C})$ are Borel subsets of $K \times G, \psi^{-1}(\mathcal{A}) \subset \mathcal{E} \subset \psi^{-1}(C)$ and $\nu(\psi^{-1}(C) \backslash \psi^{-1}(\mathcal{A})) = 0$. This shows that E is ν-measurable, and the lemma is proved.

Lemma (4.1.7)[238]: For $(x, h, V) \in K \times H \times G$ set $\psi(x, h, V) = (x, hV)$ and let $B \in K \times G$ be universally measurable. Assume that for $\xi_{G^{-a}}$.e. $V \in G$ it holds for $\xi_{H^{-a}}$.e. $h \in H$ that

$$\mu\{x \in K: \psi(x, h, V) \in B\} = 0$$

then $\mu \times \xi_G(B) = 0$.

Proof. Since B is universally measurable there exist Borel sets $A, C \subset K \times G$ with $A \subset B \subset C$ and $\mu \times \xi_H \times \xi_G(\psi^{-1}(C \backslash A)) = 0$. From the assumption on B and from Fubini's theorem it follows that

$$\begin{aligned}\mu \times \xi_H \times \xi_G\big(\psi^{-1}(C)\big) &= \mu \times \xi_H \times \xi_G\big(\psi^{-1}(A)\big) \\ &= \iint \mu\{x: (x, h, V) \in \psi^{-1}(A)\} d\xi_H(h) d\xi_G(V) \\ &\leq \iint \mu\{x: (x, h, V) \in \psi^{-1}(B)\} d\xi_H(h) d\xi_G(V) = 0\,.\end{aligned}$$

Now from Fubini's theorem, from the definition of ξ_G given in (1), and from Lemma (4.1.2), it follows that

$$\begin{aligned}0 = \mu \times \xi_H \times \xi_G(\psi^{-1}(C)) &= \iint \xi_H\{h: (x, h, V) \in \psi^{-1}(C)\} d\xi_G(V) d\mu(x) \\ &= \iint \xi_H\{h: (x, h, gU) \in \psi^{-1}(C)\} d\xi_O(g) d\mu(x) \\ &= \iint \xi_H\{h: (x, hgU) \in C\} d\xi_O(g) d\mu(x) \\ &= \iint \xi_H(\{h : (x, hU) \in C\} \cdot g^{-1}) d\xi_O(g) d\mu(x) \\ &= \int \xi_O\{g: (x, gU) \in C\} d\mu(x) = \int \xi_G\{V: (x, V) \in C\} d\mu(x) = \mu \times \xi_G(C) \\ &\geq \mu \times \xi_G(B),\end{aligned}$$

which completes the proof of the lemma.

Fix $V \in \mathcal{V}$ for the remainder of this section, set $V = F_V$, and for each $h \in H$ set

$V^h = hV$ and $P_h = P_{(V^h)^\perp}$. Set

$$Q = \{(x,h) \in X: F(x,h) \neq \Theta^{*m}(P_h\mu, P_h(x)) \text{ or } F(x,h) = \infty \; or \; F(x,h) = 0\}$$

where Θ^{*m} is as defined in (2), then Q is a Borel set. it follows that

$$\mu\{x \in K: (x,h) \in Q\} = 0 \quad \text{for each } h \in H \text{ with } P_h\mu \ll \mathcal{H}^m,$$

hence since $V \in \mathcal{V}$ we have

$$\mu \times \xi_H(Q) = \int_H \mu\{x: (x,h) \in Q\} d\xi_H(h) = 0. \tag{4}$$

Let D be the set of all $(x,h) \in X$ such that $P_h\mu \ll \mathcal{H}^m$, μ_{V^h}, x is defined,

$$\mu_{V^h,x}(K_w) = \lim_{\epsilon \downarrow 0} \frac{\mu K_w \cap P_h^{-1}\big(B(P_h x, \epsilon)\big)}{P_h\mu\big(B(P_h x, \epsilon)\big)} \text{ for each } w \in \Lambda^*,$$

and

$$0 < F(x,h) = \lim_{\epsilon \downarrow 0} P_h \mu \frac{B(P_h(x), \epsilon)}{(2\epsilon)^m} < \infty\,.$$

From the choice of , from Lemma (4.1.4) and from (4), it follows that $\mu \times \xi_H(X \backslash D) = 0$. Set $D_0 = \cap_{j=0}^{\infty} T^{-j} D$, then $\mu \times \xi_H(X \backslash D_0) = 0$ since T is measure preserving. The following lemma will be used several times below.

Lemma (4.1.8)[238]: Given $k \geq 1$ and $(x,h) \in D_0$, we have

$$\mu_{V^h,x}(K_{w_k}(x)) = \big(F(x,h)\big)^{-1} \cdot r_{w_k(x)}^{s-m} \cdot F(T^k(x,h))\,.$$

Proof. Set $u = w_k(x)$, then

$$\begin{aligned}
\mu_{V^h,x}(Ku) &= \lim_{\epsilon \downarrow 0} \frac{\mu\Big(K_u \cap P_h^{-1}\big(B(P_h x, \epsilon)\big)\Big)}{P_h\mu\big(B(P_h x, \epsilon)\big)} \\
&= \lim_{\epsilon \downarrow 0} \frac{(2\epsilon)^m}{P_h\mu\big(B(P_h x, \epsilon)\big)} \cdot \frac{\mu\Big(K_u \cap P_h^{-1}\big(B(P_h x, \epsilon)\big)\Big)}{(2\epsilon)^m} \\
&= \big(F(x,h)\big)^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu\Big(K_u \cap P_h^{-1}\big(B(P_h x, \epsilon)\big)\Big)}{(2\epsilon)^m}\,.
\end{aligned}$$

For each $\epsilon > 0$ set $E_\epsilon = P_{h_u^{-1}h}^{-1}(B(P_{h_u^{-1}} h(\phi^{-1} u(x)), \epsilon \cdot r_u^{-1}))$, then since

$$\begin{aligned}
P_h^{-1}\big(B(P_h x, \epsilon)\big) &= x + V^h + B(0, \epsilon) \\
&= \phi_u(\phi_u^{-1}(x) + V^{h_u^{-1}h} + B(0, \epsilon \cdot r_u^{-1})) = \phi_u(E_\epsilon),
\end{aligned}$$

it follows that

$$\begin{aligned}
\mu_{V^h,x}(K_u) &= \big(F(x,h)\big)^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu\big(\phi_u(K \cap E_\epsilon)\big)}{(2\epsilon)^m} \\
&= \big(F(x,h)\big)^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{1}{(2\epsilon)^m} \sum_{w \in \Lambda^*} p_w \cdot \mu(\phi_w^{-1}(\phi_u(K \cap E_\epsilon))).
\end{aligned}$$

Given $w \in \Lambda^k \backslash \{u\}$ we have $\phi_u(K) \cap \phi_w(K) = \emptyset$, so $\phi_w^{-1}(\phi_u(K)) \cap K = \emptyset$, and so

$$\begin{aligned}
\mu_{V^h,x}(K_u) &= \big(F(x,h)\big)^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{p_u \cdot \mu(K \cap E_\epsilon)}{(2\epsilon)^m} \\
&= \big(F(x,h)\big)^{-1} \cdot r_u^{s-m} \cdot \lim_{\epsilon \downarrow 0} \mu \frac{E_\epsilon}{(2\epsilon \cdot r_u^{-1})^m} \\
&= \big(F(x,h)\big)^{-1} \cdot r_u^{s-m} \cdot F(\phi_u^{-1}(x), h_u^{-1}h) \\
&= \big(F(x,h)\big)^{-1} \cdot r_u^{s-m} \cdot F(T^k(x,h)),
\end{aligned}$$

which proves the lemma.

Theorem (4.1.9)[238]:

(i) Given $V \in V$, if $\|FV\|_{L^\infty(\mu\times\xi_H)} < \infty$ then $\mathcal{H}^{s-m}(K \cap (x + hV)) > 0$ for $\mu \times \xi_{H^{-a}}$.e. $(x,h) \in K \times H$.

(ii) Given $V \in V$, if $\|F_V\|_{L^\infty(\mu\times\xi_H)} = \infty$ then $\mathcal{H}^{s-m}(K \cap (x + hV)) = 0$ for $\mu \times \xi_{H^{-a}}$.e. $(x,h) \in K \times H$.

(iii) $\mathcal{H}^{s-m}(K \cap V_x) > 0$ for $\mu \times \xi_{G^{-a}}$.e. $(x,V) \in K \times G$ if and only if $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$ for $\xi_{G^{-a}}$.e. $V \in G$.

(iv) $\mathcal{H}^{s-m}(K \cap V_x) = 0$ for $\mu \times \xi_{G^{-a}}$.e. $(x,V) \in K \times G$ if and only if $\|F_V\|_{L^\infty(\mu\times\xi_H)} = \infty$ for $\xi_{G^{-a}}$.e. $V \in G$.

Proof. Part (i): Assume that V is such that $\|F\|_{L^\infty(\mu\times\xi_H)} < \infty$. Set $M = \|F\|_{L^\infty(\mu\times\xi_H)}$, $E = \{(x,h): F(x,h) \le M\}$ and $E_1 = D_0 \cap (\cap_{j=0}^{\infty} T^{-j}(E))$, then $\mu \times \xi_H(X \backslash E_1) = 0$. For $\xi_{H^{-a}}$.e. $h \in H$ we have

$$\mu\{x \in K: (x,h) \notin E_1\} = 0,$$

fix such $h_0 \in H$. For each $l \ge 1$ set

$$A_l = \{x \in K: (x,h_0) \in E_1 \text{ and } F(x,h_0) \ge l^{-1}\},$$

and fix $l_0 \ge 1$. Set $\kappa = \min\{r_\lambda: \lambda \in \Lambda\}$, it will now be shown that

$$\Theta^{*s-m}\left(\mu_{V^{h_0},x}, x\right) \le (2\rho\kappa)^{m-s} l_0 M \text{ for each } x \in A_{l_0}, \qquad (5)$$

where ρ is as defined in (3). Let $x \in A_{l_0}$ and let $\kappa\rho > \delta > 0$. Let $k \ge 1$ be such that $r_{w_k}(x) \ge \frac{\delta}{\rho} > r_{w_{k+1}(x)}$, and set $u = wk(x)$. From Lemma (4.1.8) and from $T^k(x,h_0) \in E$ we get that

$$\mu_{V^{h_0},x}(K_u) = \left(F(x,h_0)\right)^{-1} \cdot r_u^{s-m} \cdot F(T^k(x,h_0)) \le l_0 \cdot r_u^{s-m} \cdot M,$$

and so

$$\frac{\mu_{V^{h_0},x}\left(B(x,\delta)\right)}{(2\delta)^{s-m}} \le \frac{\mu_{V^{h_0},x}\left(B\left(x, \rho \cdot r_{w_k(x)}\right)\right)}{\left(2\rho \cdot r_{w_{(k+1)}(x)}\right)^{s-m}} \le \frac{\mu_{V^{h_0}}, x(K_u)}{(2\rho^{\kappa} \cdot r_u)^{s-m}}$$

$$\le \frac{l_0 r_u^{s-m} M}{(2\rho\kappa \cdot r_u)^{s-m}} = (2\rho\kappa)^{m-s} l_0 M,$$

which proves (5). It holds that

$$\{x \in K: (x,h_0) \in E_1\} = \cup_{l=1}^{\infty} A_l,$$

hence

$$0 = \mu(K \backslash \cup_{l=1}^{\infty} A_l) = \int \mu_{V^{h_0}}, x(K \backslash \cup_{l=1}^{\infty} A_l)\, d\mu(x),$$

and so for $\mu - a$.e. $x \in K$ there exist $l_x \ge 1$ with $\mu_{V^{h_0},x}\left(A_{l_x} \cap V_x^{h_0}\right) = \mu_{V^{h_0},x}(A_{l_x}) > 0$. Fix such $x_0 \in K$ and let $y \in A_{l_{x_0}} \cap V_{x_0}^{h_0}$, then from (5) we get that

$$\Theta^{*s-m}(\mu_{V^{h_0},x_0}, y) = \Theta^{*s-m}(\mu_{V^{h_0}}, y, y) \le (2\rho\kappa)^{m-s} l_{x_0} M,$$

and so from Theorem 6.9 in [248] it follows that

$$\mathcal{H}^{s-m}\left(K \cap V_{x_0}^{h_0}\right) \ge \mathcal{H}^{s-m}\left(A_{l_{x_0}} \cap V_{x_0}^{h_0}\right)$$

$$\ge 2^{-(s-m)}(2\rho\kappa)^{s-m} l_{x_0}^{-1} M^{-1} \cdot \mu_{V^{h_0},x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) > 0.$$

This proves that if $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$, then for $\xi_{H^{-a}}$.e. $h \in H$ we have

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \text{ for } \mu - a.e.\ x \in K,$$

and so (i) follows from Theorem (4.1.10) and Fubini's theorem.

Part (ii): Assume that V is such that $\|F\|_{L^\infty(\mu\times\xi_H)} = \infty$, then

$$\mu \times \xi_H\{(x,h): F(x,h) > M\} > 0 \text{ for each } 0 < M < \infty\,.$$

For each integer $M \geq 1$ set

$$E_M = \{(x,h) \in X: F(x,h) > M\} \text{ and } E_{0,M} = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E_M),$$

then $\mu \times \xi_H(E_M) > 0$, and so $\mu \times \xi_H(X\backslash E_{0,M}) = 0$ since $\mu \times \xi_H$ is ergodic (see Theorem 1.5 in [254]). Set $\tilde{E} = D_0 \cap (\cap_{M=1}^{\infty} E_{0,M})$, then $\mu \times \xi_H(X\backslash\tilde{E}) = 0$. For ξ_H-a.e. $h \in H$ it holds that $\mu\{x \in K: (x,h) \notin E\} = 0$, fix such $h_0 \in H$ and set

$$A = \{x \in K: (x,h_0) \in \tilde{E}\}\,.$$

Note that since $(x,h_0) \in D_0$ for some $x \in K$, it follows that $P_{h_0}\mu \ll \mathcal{H}^m$. It will now be shown that

$$\Theta^{*s-m}\big(\mu_{V^{h_0},x}, x\big) = \infty \text{ for each } x \in A. \tag{6}$$

Let $x \in A, M \geq 1$ and $N \geq 1$ be given, then there exists $k \geq N$ with $T^k(x,h_0) \in D_0 \cap E_M$, and so $F(T^k(x,h_0)) > M$. Set $u = wk(x)$ and $\beta = \big(F(x,h_0)\big)^{-1}$, then from Lemma (4.1.8)

$$\mu_{V^{h_0},x}(K_u) = \beta \cdot r_u^{s-m} \cdot F(T^k(x,h_0)) \geq \beta \cdot r_u^{s-m} \cdot M$$

Set $d = sup\{|y_1 - y_2|: y_1, y_2 \in K\}$, then

$$\frac{\mu_{V^{h_0},x}\left(B\left(x, d\cdot r_{w_k}(x)\right)\right)}{\left(2d\cdot r_{w_k}(x)\right)^{s-m}} \geq \frac{\mu_{V^{h_0},x}(K_u)}{(2d\cdot r_u)^{s-m}} \geq \frac{\beta\cdot r_u^{s-m}\cdot M}{(2d\cdot r_u)^{s-m}} = \frac{M\beta}{(2d)^{s-m}}.$$

Since $\lim_{k\to\infty} r_{w_k}(x) = 0$ we get that $\Theta^{*s-m}(\mu_{V^{h_0},x}, x) \geq \frac{M\beta}{(2d)^{s-m}}$, and so (6) follows since M can be chosen arbitrarily large. Let $x \in A$ and $y \in A \cap V_x^{h_0}$, then from (6) we get

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, y) = \Theta^{*s-m}(\mu_{V^{h_0},y}, y) = \infty\,.$$

Now from Theorem 6.9 in [248] it follows that for each $M \geq 1$

$$\mathcal{H}^{s-m}\big(A \cap V_x^{h_0}\big) \leq M^{-1} \cdot \mu_{V^{h_0},x}(A \cap V_x^{h_0}) \leq M^{-1},$$

and so $\mathcal{H}^{s-m}(A \cap V_x^{h_0}) = 0$ since M can be chosen arbitrarily large. Also, from $\mu(K\backslash A) = 0$ and Theorem 7.7 in [248] we get that

$$\int_{(V^{h_0})^{\perp}} \mathcal{H}^{s-m}\left((K\backslash A) \cap V_y^{h_0}\right) d\mathcal{H}^m(y) \leq \text{const}\cdot\mathcal{H}^s(K\backslash A) = \text{const}\cdot\mu(K\backslash A) = 0\,.$$

This shows that $\mathcal{H}^{s-m}((K\backslash A) \cap V_y^{h_0}) = 0$ for $\mathcal{H}^m$-a.e. $y \in (V^{h_0})^{\perp}$, and so $H^{s-m}((K\backslash A) \cap V_x^{h_0}) = 0$ for μ-a.e. $x \in K$ since $P_{h_0}\mu \ll \mathcal{H}^m$. It follows that for μ-a.e. $x \in A$ (and so for μ-a.e. $x \in K$) we have

$$\mathcal{H}^{s-m}(K \cap V_x^{h_0}) = \mathcal{H}^{s-m}(A \cap V_x^{h_0}) + \mathcal{H}^{s-m}((K\backslash A) \cap V_x^{h_0}) = 0\,.$$

From this, Lemma (4.1.5) and Fubini's theorem, it follows that $\mathcal{H}^{s-m}(K \cap V_x^h) = 0$ for $\mu \times \xi_H$-a.e. $(x,h) \in K \times H$, which proves (ii).

Part (iii): Assume that $\|F_V\|_\infty < \infty$ for ξ_G-a.e. $V \in G$. From Lemma (4.1.8) and part (i), it follows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that

$$\mathcal{H}^{s-m}(K \cap (x + h_V)) > 0\, for\ \mu - a.e. x \in K\,.$$

Set

$$B = \{(x,V) \in K \times G: H^{s-m}(K \cap V_x) = 0\},$$

then from Lemma (4.1.5) we get that B is a Borel set (hence universally measurable), and so $\mu \times \xi_G(B) = 0$ by Lemma (4.1.7). For the other direction, set $W = \{V \in G: \|F_V\|_\infty = \infty\}$ and assume that $\xi_G(W) > 0$. From part (ii) it follows that for ξ_G-a.e. $V \in W$ we have

$$\mathcal{H}^{s-m}(K \cap (x + h_V)) = 0 \; for \; \mu \times \xi_H - a.e. (x, h) \in X,$$

and so from Lemma (4.1.2)

$$0 < \xi_G(\mathcal{W}) \leq \int \mu \times \xi_H\{(x,h): \mathcal{H}^{s-m}(K \cap (x + hV)) = 0\} d\xi_G(V)$$
$$= \iint \xi_H\{h : \mathcal{H}^{s-m}(K \cap (x + hgU)) = 0\} d\xi_O(g) d\mu(x)$$
$$= \iint \xi_H(\{h : \mathcal{H}^{s-m}(K \cap (x + hU)) = 0\} \cdot g^{-1}) d\xi_O(g) d\mu(x)$$
$$= \int \xi_O\{g: \mathcal{H}^{s-m}(K \cap (x + gU)) = 0\} d\mu(x)$$
$$= \int \xi_G\{V : \mathcal{H}^{s-m}(K \cap Vx) = 0\} d\mu(x) = \mu \times \xi_G\{(x,V): \mathcal{H}^{s-m}(K \cap Vx) = 0\},$$

which completes the proof of (iii). Part (iv) can be proven in a similar manner, and so the proof of Theorem (4.1.9) is complete.

Theorem (4.1.10)[238]::

(i) $P^{s-m}(K \cap V_x) > 0$ for $\mu \times \xi_{G^{-a}}$.e. $(x, V) \in K \times G$.

(ii) Given $V \in V$, if $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu \times \xi_H)} = \infty$ then $P^{s-m}(K \cap (x + hV)) = \infty$ for $\mu \times \xi_{H^{-a}}$.e. $(x, h) \in K \times H$.

(iii) If $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu \times \xi_H)} = \infty$ for $\xi_{G^{-a}}$ .e. $V \in G$, then $P^{s-m}(K \cap \, V_x) = \infty$ for $\mu \times \xi_{G^{-a}}$.e. $(x, V) \in K \times G$.

Proof.

Part (i): Let $M > 0$ be so large such that for

$$E = \{(x,h) \in X: F(x,h) \leq M\}$$

we have $\mu \times \xi_H(E) > 0$. Set $E_0 = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E)$, then $\mu \times \xi_H(X \backslash E_0) = 0$ since $\mu \times \xi_H$ is ergodic. Set $E_1 = E_0 \cap D_0$, then $\mu \times \xi_H(X \backslash E_1) = 0$. For ξ_H-a.e. $h \in H$ it holds that $\mu\{x \in K: (x,h) \notin E_1\} = 0$, fix such $h_0 \in H$. For each $l \geq 1$ set

$$A_l = \{x \in K: (x, h_0) \in E_1 \text{ and } F(x, h_0) \geq l^{-1}\},$$

and fix $l_0 \geq 1$. It will now be shown that

$$\Theta_*^{s-m}(\mu_{V^{h_0},x}, x) \leq (2\rho)^{m-s} l_0 M \text{ for each } x \in A_{l_0}. \tag{7}$$

Let $x \in A_{l_0}$ and let $N \geq 1$ be given, then since $(x, h_0) \in E_1$ it follows that there exist $k \geq N$ with $T^k(x, h_0) \in E \cap D_0$, and so $F(T^k(x, h_0)) \leq M$. Set $u = w_k(x)$, then from Lemma (4.1.8) we have

$$\mu_{V^{h_0},x}(Ku) = (F(x,h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x,h_0)) \leq l_0 r_u^{s-m} M \,,$$

from which it follows that

$$\frac{\mu_{V^{h_0},x}\left(B(x, \rho \cdot r_{w_k(x)})\right)}{\left(2\rho \cdot r_{w_k(x)}\right)^{s-m}} \leq \frac{\mu_{V^{h_0},x}(K_u)}{(2\rho \cdot r_u)^{s-m}} \leq \frac{l_0 r_u^{s-m} M}{(2\rho \cdot r_u)^{s-m}} = (2\rho)^{m-s} l_0 M \,.$$

This proves (7) since $r_{w_k(x)}$ tends to 0 as k tends to ∞.

As in the proof of part (i) of Theorem (4.1.9), from $\mu(K \backslash \cup_{l=1}^{\infty} A_l) = 0$ it follows that for μ-a.e. $x \in K$ there exists $l_x \geq 1$ with $\mu_{V^{h_0},x}(A_{l_x} \cap V_x^{h_0}) > 0$. Fix such an x_0 and let $\in A_{l_{x_0}} \cap V_{x_0}^{h_0}$, then from (7) we get

$$\Theta_*^{s-m}(\mu_{V^{h_0},x_0}, y) = \Theta_*^{s-m}(\mu_{V^{h_0},y}, y) \leq (2\rho)m - s_{l_{x_0}} M,$$

and so from Theorem 6.11 in [248] it follows that

$$\mathcal{P}^{s-m}\big(K \cap V_{x_0}^{h_0}\big) \geq \mathcal{P}^{s-m}\left(A_{l_{x_0}} \cap V_{x_0}^{h_0}\right) \geq (2\rho)^{s-m} l_{x_0}^{-1} M^{-1} \cdot \mu_{V^{h_0},x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) > 0.$$

Since $\xi_G(G \backslash V) = 0$, this shows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that $\mathcal{P}^{s-m}(K \cap (x + hV)) > 0$ for μ-a.e. $x \in K$. Set

$$B = \{(x, V) \in K \times G: \mathcal{P}^{s-m}(K \cap V_x) = 0\},$$

then from Lemma (4.1.6) we get that B is universally measurable, and so the claim stated in (i) follows from Lemma (4.1.7).

Proof of part (ii): Assume V is such that $\left\|\frac{1}{F}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$, then

$$\mu \times \xi_H\{(x, h): F(x, h) < M^{-1}\} > 0 \text{ for each } 0 < M < \infty\,.$$

For each integer $M \geq 1$ set

$$E_M = \{(x, h): F(x, h) < M^{-1}\} \text{ and } E_{0,M} = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E_M),$$

then since $\mu \times \xi_H$ is ergodic and $\mu \times \xi_H(E_M) > 0$ it follows that $\mu \times \xi_H(X \backslash E_{0,M}) = 0$. Set $\tilde{E} = D_0 \cap (\cap_{M=1}^{\infty} E_{0,M})$, then $\mu \times \xi_H(X \backslash \tilde{E}) = 0$. For ξ_H-a.e. $h \in H$ it holds that $\mu\{x \in K: (x, h) \notin \tilde{E}\} = 0$, fix such $h_0 \in H$ and set $A = \{x \in K: (x, h_0) \in \tilde{E})\}$. It will now be shown that

$$\Theta_*^{s-m}\big(\mu_{V^{h_0},x}, x\big) = 0 \text{ for each } x \in A. \tag{8}$$

Let $x \in A, M \geq 1$ and $N \geq 1$ be given, then there exists $k \geq N$ with $T^k(x, h_0) \in D_0 \cap E_M$, and so $F(T^k(x, h_0)) < M^{-1}$. Set $u = w_{k(x)}$, then from Lemma (4.1.8)

$$\mu_{V^{h_0},x}(K_u) = \big(F(x, h_0)\big)^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \leq \big(F(x, h_0)\big)^{-1} \cdot r_u^{s-m} \cdot M^{-1}\,,$$

from which it follows that

$$\frac{\mu_{V^{h_0},x}\left(B\left(x, \rho \cdot r_{w_k}(x)\right)\right)}{\left(2\rho \cdot r_{w_{k(x)}}\right)^{s-m}} \leq \frac{\mu_{V^{h_0},x}(Ku)}{(2\rho \cdot r_u)^{s-m}} \leq \frac{\big(F(x, h_0)\big)^{-1} \cdot r_u^{s-m} \cdot M^{-1}}{(2\rho \cdot r_u)^{s-m}}$$

$$= (2\rho)^{m-s} \cdot \big(F(x, h_0)\big)^{-1} \cdot M^{-1}.$$

This shows that

$$\Theta_*^{s-m}(\mu_{V^{h_0},x}, x) \leq (2\rho)^{m-s} \cdot \big(F(x, h_0)\big)^{-1} \cdot M^{-1},$$

and so (8) holds since M can be chosen arbitrarily large. We have

$$0 = \mu(K \backslash A) = \int \mu_{V^{h_0},x}(K \backslash A)\, d\mu(x)\,,$$

hence $\mu_{V^{h_0},x}(A \cap V_x^{h_0}) > 0$ for μ-a.e. $x \in K$. Fix such $x_0 \in K$ and let $\in A \cap V_{x_0}^{h_0}$, then from (8) we get

$$\Theta_*^{s-m}\big(\mu_{V^{h_0},x_0}, y\big) = \Theta_*^{s-m}\big(\mu_{V^{h_0},y}, y\big) = 0\,.$$

Now from Theorem 6.11 in [248] it follows that for each $\epsilon > 0$

$$\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) \geq \mathcal{P}^{s-m}(A \cap V_{x_0}^{h_0}) \geq \epsilon^{-1} \cdot \mu_{V^{h_0},x_0}(A \cap V_{x_0}^{h_0}),$$

which shows that $\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) = \infty$ since ϵ can be chosen arbitrarily small and $\mu_{V^{h_0},x_0}(A \cap V_{x_0}^{h_0}) > 0$. This proves that if $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$, then for ξ_H-a.e. $h \in H$ we have $\mathcal{P}^{s-m}(K \cap (x + hV\,)) = \infty$ for μ-a.e. $x \in K$, and so (ii) follows from Lemma (4.1.6) and Fubini's theorem. Proof of part (iii): Assume that $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$ for ξ_G-a.e. $V \in G$, then from Lemma (4.1.3) and part (ii) it follows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that $\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty$ for μ-a.e. $x \in K$. Set

$$B = \{(x, V) \in K \times G: \mathcal{P}^{s-m}(K \cap Vx) < \infty\},$$

then from Lemma (4.1.6) we get that B is universally measurable, and so the claim stated in (iii) follows from Lemma (4.1.7). This completes the proof of Theorem (4.1.10).

The following lemma will be used in the proofs of Corollary (4.1.12) and Corollary (4.1.14). For its proof see Lemma 3.2 in [244] and the discussion before it.

Lemma (4.1.11)[238]: Assume $m = 1$ and $s > 2$, then $P_{V^\perp\mu} \ll \mathcal{H}^m$ and $\frac{dP_{V^\perp\mu}}{d\mathcal{H}^m}$ has a continuous version for ξ_G-a.e. $V \in G$.

Corollary (4.1.12)[238]: Assume $m = 1, s > 2$ and $|H| < \infty$, then $\mathcal{H}^{s-m}(K \cap V_x) > 0$ for $\mu \times \xi_{G-a}$.e. $(x, V) \in K \times G$.

Proof. Assuming $m = 1, s > 2$ and $|H| < \infty$, it will be shown that $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$ for ξ_G-a.e. $V \in G$. From this and from part (iii) of Theorem (4.1.9) the corollary will follow. Set

$$E = \{V \in G: P_{V^\perp\mu} \ll \mathcal{H}^m \text{ and } dP_{V^\perp\mu}/d\mathcal{H}^m \text{ is continuous}\},$$

then from Lemma (4.1.11) we get $\xi_G(G\backslash E) = 0$. From this and from Lemma (4.1.2) it now follows that

$$0 = \xi_G(G\backslash E) = \xi_O\{g \in O(n): gU \notin E\} = \int \xi_H\{h: hgU \notin E\}d\xi_O(g)$$
$$= \int \xi_H\{h : hV \notin E\}d\xi_G(V),$$

and so $\xi_H\{h: hV \notin E\} = 0$ for ξ_G-a.e. V . We fix such a $V \in G$. Since $|H| < \infty$, for each $h \in H$ we have $\xi_H\{h\} > 0$, and so $hV \in E$. For each $h \in H$ and $y \in (hV)^\perp$ set $Q_h(y) = \Theta_*^m(P_{(hV)^\perp}\mu, y)$, fix $h_0 \in H$, and set $W = (h_0V)^\perp$. Since $\mathcal{H}^m(B(y, r) \cap W) = (2\epsilon)^m$ for each $y \in W$ and $0 < \epsilon < \infty$, it follows from Theorem 2.12 in [248] that $Q_{h_0}(y) = \frac{dP_W\mu}{d\mathcal{H}^m}(y)$ for $\mathcal{H}^m$a.e. $y \in W$, i.e. the function Q_{h_0} equals a continuous function as members of $L^1(W, \mathcal{H}^m)$. Also, since μ is supported on a compact set it follows that the set $\{y \in W: Q_{h_0}(y) \neq 0\}$ is bounded, so Q_{h_0} equals a continuous function with compact support in $L^1(W, \mathcal{H}^m)$, which shows that $\|Q_{h_0}\|_{L^\infty(W,\mathcal{H}^m)} < \infty$. Since $P_W\mu \ll \mathcal{H}^m$ it follows that $\|Q_{h_0}\|\_(L^\infty(P_W\mu) < \infty$. Now set $M = \max\{\|Q_h\|\_(L^\infty(P_{(hV)^\perp}\mu): h \in H\}$, then $M < \infty$ since $|H| < \infty$. Also, we have

$$0 = \frac{1}{|H|}\sum_{h\in H} P_{(hV)^\perp}\,\mu\{y \in (hV)^\perp: |Q_h(y)| > M\}$$
$$= \frac{1}{|H|}\sum_{h\in H} \mu\{x \in K: |Q_h(P_{(hV)^\perp}(x))| > M\} = \frac{1}{|H|}\, X\, h \in H$$
$$\sum_{h\in H} \mu\{x \in K: |F_V(x, h)| > M\} = \int \mu\{x \in K: |F_V(x, h)| > M\}d\xi_H(h)$$
$$= \mu \times \xi_H\{(x, h) \in K \times H: |F_V(x, h)| > M\},$$

which shows that $\|F_V\|_{L^\infty(\mu\times\xi_H)} \le M < \infty$. This completes the proof of Corollary (4.1.12).

Corollary (4.1.13)[238]:: Assume that $H = O(n)$ and

$$\mu \times \xi_G\{(x, V) \in K \times G: H^{s-m}(K \cap V_x) > 0\} > 0,$$

then there exists $0 < M < \infty$ such that for each $V \in G$ we have $P_{V^\perp}\,\mu \ll H^m$ with

$$\left\|\frac{dP_{V^\perp}\mu}{dH^m}\right\|_{L^\infty(H^m)} \le M.$$

Proof. Assume that $H = O(n)$ and

$$\mu \times \xi_G\{(x,V): \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0 .$$

Let $V \in \mathcal{V}$, then since $\xi_H = \xi_O$ we have

$$\mu \times \xi_H\{(x,h): \mathcal{H}^{s-m}(K \cap (x + hV)) > 0\} > 0 ,$$

and so from part (ii) of Theorem (4.1.9) it follows that $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$. Set $M = \|F_V\|_{L^\infty(\mu\times\xi_H)}$, set

$$E = \{W \in G: P_{W^\perp}\mu \ll \mathcal{H}^m \text{ and } \left\|\frac{dP_{W^\perp}\mu}{d\mathcal{H}^m}\right\| L^\infty(\mathcal{H}^m) \le M\},$$

and for each $h \in \mathcal{H}$ set $P_h = P_{(hV)^\perp}$. We shall first show that $\xi_G(G\backslash E) = 0$. Since $P_{W^\perp\mu} \ll \mathcal{H}^m$ for ξ_G-a.e. $W \in G$ (see the proof of Lemma (4.1.3)), and since $\xi_H = \xi_O$, we have

$$\xi_G(G\backslash E) = \xi_G\big(G\backslash\{W \in G: P_{W^\perp\mu} \ll \mathcal{H}^m\}\big)$$
$$+\xi_G\left\{W \in G: P_{W^\perp\mu} \ll \mathcal{H}^m \text{ and } \left\|\frac{dP_{W^\perp\mu}}{d\mathcal{H}^m}\right\|_{L^\infty(\mathcal{H}^m)} \le M\right\}. \quad (9)$$

Let $h \in H$ be such that $P_h\mu \ll \mathcal{H}^m$ and $\left\|\frac{dP_h\mu}{d\mathcal{H}^m}\right\|_{L^\infty(P_h\mu)} \le M$, then

$$0 = P_h\mu\{y \in (hV)^\perp: \frac{dP_h\mu}{d\mathcal{H}^m}(y) > M\} = (hV)^\perp\, 1_{\left\{\frac{dP_h\mu}{d\mathcal{H}^m} > M\right\}} \cdot \frac{dP_h\mu}{d\mathcal{H}^m} d\mathcal{H}^m$$
$$\ge M \cdot \mathcal{H}^m\left\{y \in (hV)^\perp: \frac{dP_h\mu}{d\mathcal{H}^m}(y) > M\right\},$$

which shows that $\left\|\frac{dP_h\mu}{d\mathcal{H}^m}\right\|_{L^\infty(\mathcal{H}^m)} \le M$. From this and from (9) it follows that

$$\xi_G(G\backslash E) = \xi_H\left\{h: P_h\mu \ll \mathcal{H}^m and \left\|\frac{dP_h\mu}{d\mathcal{H}^m}\right\|_{L^\infty(P_h\mu)} > M\right\}. \quad (10)$$

From Theorem 2.12 in [248] we get that for each $h \in \mathcal{H}$ with $P_h\mu \ll \mathcal{H}^m$

$$F_V(x,h) = \frac{dP_h\mu}{d\mathcal{H}^m}\big(P_h(x)\big) \quad \text{for } \mu - a.e.\, x \in K,$$

and so from (10)

$$\xi_G(G\backslash E) \le \xi_H\{h: \|F_V(\cdot,h)\|_{L^\infty(\mu)} > M\}$$
$$= \xi_H\{h: \mu\{x: F_V(x,h) > \|F_V\|_{L^\infty(\mu\times\xi_H)}\} > 0\} = 0 .$$

Since $\xi_G(W) > 0$ for every non-empty open set $W \subset G$, it follows from $\xi_G(G\backslash E) = 0$ that E is dense in G, and so in order to prove the corollary it suffice to show that E is a closed subset of G. Let $W_0 \in \bar{E}$, let $y \in W_0^\perp$ and let $r \in (0,\infty)$. Given $\epsilon > 0$ there exists $W \in E$ so close to W_0 in G (with respect to the metric d_G defined), such that

$$P_{W_0^\perp}^{-1}(B(y,r)) \cap K \subset P_{W^\perp}^{-1}(B(P_{W^\perp y}, r + \epsilon)).$$

From this and since $W \in E$ it follows that

$$P_{W_0^\perp}\mu\big(B(y,r)\big) = \mu\left(P_{W_0^\perp}^{-1}\big(B(y,r)\big) \cap K\right) \le \mu\left(P_{W^\perp}^{-1}\big(B(P_{W^\perp}y, r+\epsilon)\big)\right)$$
$$= P_{W^\perp}\mu\left(\big(B(P_{W^\perp}y, r+\epsilon)\big)\right) = \int_{B(P_{W^\perp}y, r+\epsilon)\cap W^\perp} \frac{dP_{W^\perp}\mu}{d\mathcal{H}^m} d\mathcal{H}^m$$
$$\le M \cdot \mathcal{H}^m(B(P_{W^\perp}y, r+\epsilon) \cap W^\perp) = M \cdot \big(2 \cdot (r+\epsilon)\big)^m,$$

and since this holds for each $\epsilon > 0$ we have

$$P_{W_0^\perp}\mu(B(y,r) \cap W_0^\perp) \le M \cdot (2r)^m = M \cdot \mathcal{H}^m(B(y,r) \cap W_0^\perp) .$$

This holds for every $y \in W_0^\perp$ and $r \in (0,\infty)$, hence $W_0 \in E$ by Theorem 2.12 in [248], which shows that E is closed in G and completes the proof of the corollary.

Corollary (4.1.14): Assume $m = 1$ and $s > 2$, then $P^{s-m}(K \cap Vx) = \infty$ for $\mu \times \xi G$-a.e. $(x, V) \in K \times G$. (see Lemma 3.2 in [244]).

Proof. Assuming $m = 1$ and $s > 2$, it will be shown that $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$ for ξ_G-a.e. $V \in G$. From this and part (iii) of Theorem (4.1.10) the corollary will follow. Set

$$E = \{V \in G: P_{V^\perp}\mu \ll \mathcal{H}^m \text{ and } \frac{dP_{V^\perp}\mu}{d\mathcal{H}^m} \text{ is continuous}\}$$

then as in the proof of Corollary (4.1.12) it follows from Lemma (4.1.8) and Lemma (4.1.2) that

$$0 = \xi_G(G\backslash E) = \int \xi_H\{h: h_V \notin E\} d\xi_G(V),$$

and so $\xi_H\{h: hV \notin E\} = 0$ for ξ_G-a.e. V . Fix such $V \in G$, let $M > 0$, set

$$A = \{h \in H: hV \in E\},$$

and for each $h \in H$ and $y \in (hV)^\perp$ set $Q_h(y) = \Theta_*^m(P_{(hV)^\perp}\mu, y)$ and

$$L_h = \{y \in (hV)^\perp: 0 < Q_h(y) \le M^{-1}\}.$$

Fix $h_0 \in A$ and set $W = (h_0V)^\perp$. From Theorem 2.12 in [248] it follows that $Q_{h_0}(y) = \frac{dP_W\mu}{d\mathcal{H}^m}(y)$ for $\mathcal{H}^m$-a.e. $y \in W$, hence the function Q_{h_0} equals a continuous function in $L^1(W, \mathcal{H}^m)$. Also, since μ is supported on a compact set, it follows that the set $\{y \in W: Q_{h_0}(y) \ne 0\}$ is bounded. From these two facts it easily follows that $\mathcal{H}^m(L_{h_0}) > 0$, and so $P_W\mu(L_{h_0}) > 0$ since $Q_{h_0} = \frac{dP_W\mu}{d\mathcal{H}^m}$ and $Q_{h_0} > 0$ on L_{h_0}. From this we get that

$$0 < \mu\{x \in K: Q_{h_0}(P_W(x)) \le M^{-1}\} = \mu\{x \in K: F_V(x, h_0) \le M^{-1}\},$$

and so by Fubini's theorem

$$\mu \times \xi_H\{(x,h): \frac{1}{F_V(x,h)} \ge M\} = \int_A \mu\{x \in K: F_V(x,h) \le M^{-1}\} d\xi_H(h) > 0.$$

It follows that $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} \ge M$, and so $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$ since we can choose M as large as we want. This completes the proof of the corollary.

Set $\Lambda = \{1,2\}$. Given $0 < \rho < \frac{1}{2}$, define $f_{\rho,1}, f_{\rho,2}: R \to R$ by $f_{\rho,1}(x) = \rho \cdot x$ and $f_{\rho,2}(x) = \rho \cdot x + 1 - \rho$ for each $x \in R$, let $C_\rho \subset [0,1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, set $d_\rho = \dim_H C_\rho$ (so that $d_\rho = \frac{\log 2}{\log \rho^{-1}}$), and for each $E \subset R$ set $\mu_\rho(E) = \frac{\mathcal{H}^{d_\rho}(C_\rho \cap E)}{H^{d_\rho}(C_\rho)}$. Let $0 < a < b < \frac{1}{2}$ be such that $\frac{1}{a}$ and $\frac{1}{b}$ are Pisot numbers, $\frac{\log b}{\log a}$ is irrational, and $d_a + d_b > 1$. Let $I = [0,1)$ and let L be Lebesgue measure on I. Fix $\tau \in (0,\infty)$, and for each $t \in I$ and $z \in R^2$ define $W^t = \{x \cdot (1, \tau \cdot a^t) : x \in R\}, V^t = (W^t)^\perp$ and $V_z^t = z + V^t$. In order to prove Theorem (4.1.20) we shall first prove the following:

Set $\alpha = \log b \log a$ (so $\alpha \in I\backslash Q$), and for each $t \in I$ set $R(t) = t + \alpha \bmod 1$. Given $0 < \rho < \frac{1}{2}$ and a word $\lambda_1 \cdot \ldots \cdot \lambda_l = w \in \Lambda^*$, write $f_{\rho,w} = f_{\rho,\lambda_1} \circ \ldots \circ f_{\rho,\lambda_l}$ and $C_{\rho,w} = f_{\rho,w}(C_\rho)$. For each $n \ge 1$ and $x \in C_\rho$ let $w_{\rho,n}(x) \in \Lambda^n$ be the unique word of length n which satisfies $x \in C_{\rho,w_\rho,n(x)}$, and let $S_\rho(x) = f^{-1}_{\rho,w_\rho,1(x)}(x)$. We also write $w_{\rho,0}(x) = \emptyset$ and $C_{\rho,\emptyset} = C_\rho$. The following dynamical system will be used in the proof of Theorem (4.1.19). The idea of using this system comes from the partition operator introduced in section 10 of [245]. Set $K = C_a \times C_b, X = K \times I, \mu = \mu_a \times \mu_b, \nu = \mu \times \mathcal{L}$, and for each $(x, y, t) \in X$ define

$$T(x,y,t) = \begin{cases} (\mathrm{x}, \mathrm{S_b}(\mathrm{y}), \mathrm{R}(\mathrm{t})), & \text{if } \mathrm{t} \in [0,1-\alpha) \\ ((\mathrm{S_a}(\mathrm{x}), \mathrm{S_b}(\mathrm{y}), \mathrm{R}(\mathrm{t})), & \text{else} . \end{cases}$$

It is easy to check that the system (X, ν, T) is measure preserving, and from Lemma 2.2 in [240] it follows that it is ergodic. Let $\mathcal{R}$ be the Borel σ-algebra of $\mathbb{R}^2$. For each $t \in I$ let P_t be the orthogonal projection onto W^t, and let $\{\mu_{t,z}\}_{z\in\mathbb{R}^2}$ be the disintegration of μ with respect to $P_t^{-1}(\mathcal{R})$. Also, for each $(z,t) \in X$ define $F(z,t) = \Theta_*^1(P_t\mu, P_t z)$.

Lemma (4.1.15)[238]: It holds that $I_1(\mu) < \infty$, where $I_1(\mu)$ is the 1-energy of μ.

Proof. Set $\delta = 1 - 2b$, then for each $(x,y) \in \mathbb{R}^2$ and $k \geq 1$

$$\begin{aligned} \mu\left(B\left((x,y), \delta \cdot a^k\right)\right) &\leq \mu\left((x - \delta \cdot a^k, x + \delta \cdot a^k) \times (y - \delta \cdot a^k, y + \delta \cdot a^k)\right) \\ &\leq \mu_a(x - \delta \cdot a^k, x + \delta \cdot a^k) \cdot \mu_b(y - \delta \cdot a^k, y + \delta \cdot a^k) \\ &\leq 2^{-k} \cdot 2^{-[k \log_b a]} \leq 2^{-k} \cdot 2^{[-k \log_b a]} = 2 \cdot a^{k(1+\log_b a)\log_a 2^{-1}} \\ &= 2 \cdot a^{k(d_a+d_b)} . \end{aligned}$$

This shows that there exists a constant $M > 0$ with $\mu(B(z,r)) \leq M \cdot r^{d_a+d_b}$ for each $z \in R^2$ and $r > 0$. Since $d_a + d_b > 1$, the lemma follows from the discussion found at the beginning of chapter 8 of [248].

Lemma (4.1.16)[238]: Let $n_1, n_2 \geq 1$, $w_1 \in \Lambda^{n_1}$ and $w_2 \in \Lambda^{n_2}$. For each $(x,y) \in K$ set $g(x,y) = (f_{a,w_1}(x), f_{b,w_2}(y))$, then for each Borel set $B \subset K$

$$\mu\big(g(B)\big) = 2^{-n_1-n_2} \cdot \mu(B) .$$

Proof. We prove this by using the $\pi - \lambda$ theorem (see [239]). Let E be the collection of all Borel sets $B \subset K$ which satisfy $\mu(g(B)) = 2^{-n_1-n_2} \cdot \mu(B)$, then $\mathcal{E}$ is a λ-system. Set $P = \{C_{a,u_1} \times C_{b,u_2} : u_1, u_2 \in \Lambda^*\} \cup \{\emptyset\}$, then P is a π-system, $P \subset \mathcal{E}$ and $\sigma(P)$ equals the collection of all Borel subsets of K. From the $\pi - \lambda$ theorem it follows that $\sigma(P) \subset \mathcal{E}$, hence $\mathcal{E}$ equals the collection of all Borel subsets of K, and the lemma is proven.

Lemma (4.1.17)[238]: It holds that $0 < \mathcal{H}^{d_a+d_b}(K) < \infty$, and $\mu(E) = \frac{\mathcal{H}^{d_a+d_b}(K\cap E)}{\mathcal{H}^{d_a+d_b}(K)}$ for each Borel set $E \subset \mathbb{R}^2$.

Proof. From Theorem 8.10 in [248] it follows that $\mathcal{H}^{d_a+d_b}(K) > 0$, and by an elementary covering argument it can be shown that $\mathcal{H}^{d_a+d_b}(K) < \infty$. The rest of the lemma can be proven by using the $\pi - \lambda$ theorem, as in the proof of Lemma (4.1.16).

Lemma (4.1.18)[238]: Let $0 < M < \infty$ and set $E_M = \{(z,t) \in X : F(z,t) > M\}$, then $\nu(E_M) > 0$.

Proof. Assume by contradiction that $\nu(E_M) = 0$ and set

$$L = \{t \in I : \mu\{z : (z,t) \in E_M\} = 0\},$$

then $\mathcal{L}(I \backslash L) = 0$, and so $\bar{L} = I$. Set

$$A = \{t \in I : P_t\mu \ll \mathcal{H}^1 and \left\|\frac{dP_t\mu}{d\mathcal{H}^1}\right\|_{L^\infty(\mathcal{H}^1)} \leq M\},$$

and let $t \in L$. For $P_t\mu$-a.e. $z \in W_t$ we have $\Theta_*^1(P_t\mu, z) \leq M$, hence from parts (2) and (3) of Theorem 2.12 in [248] it follows that $t \in A$. This shows that $L \subset A$, and so that $\bar{A} = I$. By an argument similar to the one given at the end of the proof of Corollary (4.1.13), it can be shown that A is a closed subset of I, and so $A = I$. In particular it follows that $P_t\mu \ll \mathcal{H}^1$ for each $t \in I$, which is a contradiction to Theorem 4.1 in [251]. This shows that we must have $\nu(E_M) > 0$, and the lemma is proven.

Theorem (4.1.19)[238]: For $\mu_a \times \mu_b \times L$-a.e. $(x,y,t) \in C_a \times C_b \times I$ it holds that

$$\mathcal{H}^{d_a+d_b-1}((C_a \times C_b) \cap V_{x,y}^t) = 0.$$

Proof. Let D be the set of all $(z,t) \in X$ such that $P_t\mu \ll \mathcal{H}^1$, $\mu_{t,z}$ is defined,

$$\mu_{t,z}(C_{a,w_1} \times C_{b,w_2}) = \lim_{\epsilon \downarrow 0} \frac{\mu\left(\left(C_{a,w_1} \times C_{b,w_2}\right) \cap P_t^{-1}\left(B(P_t z, \epsilon)\right)\right)}{P_t\mu\left(B(P_t z, \epsilon)\right)}$$

for each $w_1, w_2 \in \Lambda^*$, and

$$0 < F(z,t) = \lim_{\epsilon \downarrow 0} \frac{P_t\mu\left(B(P_t z, \epsilon)\right)}{2\epsilon} < \infty .$$

From Lemma (4.1.15) and from the same arguments as the ones given at the beginning, it follows that $\nu(X \backslash D) = 0$. Set $D_0 = \cap_{j=0}^{\infty} T^{-j} D$, then $\nu(X \backslash D_0) = 0$ since T is measure preserving. For $0 < M < \infty$ let E_M be as in Lemma (4.1.18), and set $E_{0,M} = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E_M)$. Since $\nu(E_M) > 0$, it follows from the ergodicity of (X, ν, T) $that$ $\nu(X \backslash E_{0,M}) = 0$. Set $D_1 = D_0 \cap (\cap_{M=1}^{\infty} E_0, M)$, then $\nu(X \backslash D_1) = 0$. For $\mathcal{L}$-a.e. $t \in I$ it holds that $\mu\{z \in K : (z,t) \notin D_1\} = 0$, fix such $t_0 \in I$ and set $A = \{z \in K : (z, t_0) \in D_1\}$. Note that from $A \neq \emptyset$ it follows that $P_{t_0}\mu \ll \mathcal{H}^1$. Set $\eta = d_a + d_b - 1$. It will now be shown that

$$\Theta^{*\eta}\left(\mu_{t_0,z}, z\right) = \infty \text{ for each } z \in A \tag{11}$$

Let $(x,y) = z \in A$ and set $\beta = \left(F(z,t_0)\right)^{-1}$, then $0 < \beta < \infty$ since $(z,t_0) \in D_0$. Let $M \geq 1$ and $N \geq 1$ be given, then there exists $k \geq N$ with $T^k(z,t_0) \in D_0 \cap E_M$, and so $F(T^k(z,t_0)) > M$. Set $l = [t_0 + k_\alpha]$, then

$$\begin{aligned}
\mu_{t_0,z}\left(C_{a,w_l(x)} \times C_{b,w_k(y)}\right) &= \lim_{\epsilon \downarrow 0} \frac{\mu\left(\left(C_{a,w_l(x)} \times C_{b,w_k(y)}\right) \cap P_{t_0}^{-1}\left(B\left(P_{t_0} z, \epsilon\right)\right)\right)}{P_{t_0}\mu\left(B\left(P_{t_0} z, \epsilon\right)\right)} \\
&= \lim_{\epsilon \downarrow 0} \frac{2\epsilon}{P_{t_0}\mu\left(B\left(P_{t_0} z, \epsilon\right)\right)} \frac{\mu\left(\left(C_{a,w_l(x)} \times C_{b,w_k(y)}\right) \cap P_{t_0}^{-1}\left(B\left(P_{t_0} z, \epsilon\right)\right)\right)}{2\epsilon} \\
&= \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu\left(\left(C_{a,w_l(x)} \times C_{b,w_k(y)}\right) \cap P_{t_0}^{-1}\left(B\left(P_{t_0} z, \epsilon\right)\right)\right)}{2\epsilon}.
\end{aligned} \tag{12}$$

For each $(x', y') \in \mathbb{R}^2$ set $g(x', y') = (f_{a,w_l(x)}(x'), f_{k,w_k(y)}(y'))$, then

$$C_{a,w_l(x)} \times C_{b,w_k(y)} = f_{a,w_l(x)}(C_a) \times f_{b,w_k(y)}(C_b) = g(C_a \times C_b). \tag{13}$$

Let $\epsilon > 0$, and let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $L(1,0) = (a_l, 0)$ and $L(0,1) = (0, b^k)$. Since L is the linear part of the affine transformation g, we have

$$\begin{aligned}
P_{t_0}^{-1}\left(B\left(P_{t_0} z, \epsilon\right)\right) &= z + V^{t_0} + B(0,\epsilon) \\
&= g \circ g^{-1}(z) + L \circ L^{-1}(V^{t_0}) + L \circ L^{-1}\left(B(0,\epsilon)\right) \\
&= g\left(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}\left(B(0,\epsilon)\right)\right).
\end{aligned} \tag{14}$$

From $a^{-l} \geq a^{-t_0 - k_\alpha + 1} \geq a \cdot b^{-k}$, we obtain

$$L^{-1}\left(B(0,\epsilon)\right) \supset B(0, \epsilon \cdot a \cdot b^{-k}). \tag{15}$$

Also we have

$$\begin{aligned}
L^{-1}(V^{t_0}) &= L^{-1}((W^{t_0})^{\perp}) = L^{-1}(\left((1, \tau \cdot a^{t_0}) \cdot R\right)^{\perp}) = L^{-1}((\tau \cdot a^{t_0}, -1) \cdot \mathbb{R}) \\
&= (\tau \cdot a^{t_0} \cdot a^{-l}, -b^{-k}) \cdot \mathbb{R} = \left(\tau \cdot a^{t_0} \cdot \frac{b^k}{a^l}, -1\right) \cdot \mathbb{R},
\end{aligned}$$

and so since

$$\frac{b^k}{a^l} = a\text{^}(k \cdot \log_a b - l = a^{k\alpha - [t_0 + k\alpha]},$$

it follows that

$$L^{-1}(V^{t_0}) = \left(\tau \cdot a^{t_0 + k\alpha - [t_0 + k\alpha]}, -1\right) \cdot \mathbb{R} = \left(\left(1, \tau \cdot a^{R^k(t_0)}\right) \cdot R\right)^{\perp} = V^{R^k(t_0)}. \quad (16)$$

Set

$$Q_\epsilon = P^{-1}_{R^k(t_0)}(B(P_{R^k(t_0)}(f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)), \epsilon a b^{-k})),$$

then from (14), (15) and (16) it follows that

$$\begin{aligned} P^{-1}_{t_0}(B(P_{t_0} z, \epsilon)) &= g(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}(B(0, \epsilon))) \\ &\supset g((f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)) + V^{R^k(t_0)} + B(0, \epsilon a b^{-k})) = g(Q_\epsilon). \end{aligned}$$

Now from (12), (13) and Lemma (4.1.16) we get that

$$\begin{aligned} \mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)}) &= \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(g(C_a \times C_b) \cap Q_\epsilon)}{2\epsilon} \\ &= \beta \cdot 2^{-l-k} \cdot \frac{a}{b^k} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(g(C_a \times C_b) \cap Q_\epsilon)}{2\epsilon a b^{-k}} \\ &\geq \frac{\beta}{2} \cdot 2^{-k-k\alpha} \cdot \frac{a}{b^k} \cdot F\left(\left(f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)\right), R^k(t_0)\right) \\ &= \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot F\left(T^k(z, t_0)\right) \geq \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M. \end{aligned}$$

Since

$$C_{a,w_l(x)} \times C_{b,w_k(y)} \subset B(z, \frac{2 \cdot b^k}{a}) \text{ and } 2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} = 1,$$

it follows that

$$\begin{aligned} \frac{\mu_{t_0,z}\left(B\left(z, \frac{2 \cdot b^k}{a}\right)\right)}{(4a^{-1} \cdot b^k)^\eta} &\geq \frac{\mu_{t_0,z}\left(C_{a,w_l(x)} \times C_{b,w_k(y)}\right)}{(4a^{-1} \cdot b^k)^\eta} \geq \frac{\frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M}{(4a^{-1} \cdot b^k)^\eta} \\ &\geq \frac{\beta a^2}{8} \cdot M \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} = \frac{\beta a^2}{8} \cdot M\,. \end{aligned}$$

This shows that $\Theta^{*\eta}(\mu_{t_0,z}, z) \geq \frac{\beta a^2}{8} \cdot M$, which proves (1) since $\beta > 0$ and M can be chosen arbitrarily large.

Let $z \in A$ and $u \in A \cap V_z^{t_0}$, then from (1)

$$\Theta^{*\eta}(\mu_{t_0,z}, u) = \Theta^{*\eta}(\mu_{t_0,u}, u) = \infty,$$

and so from Theorem 6.9 in [248] we get that $H^\eta(A \cap V_z^{t_0}) = 0$. Also it holds that $\mu(K \backslash A) = 0$, hence from Theorem 7.7 in [248] and from Lemma (4.1.17) we get that

$$\int_{W^{t_0}} \mathcal{H}^\eta\left((K \backslash A) \cap V_u^{t_0}\right) d\mathcal{H}^1(u) \leq \text{const} \cdot \mathcal{H}^{\eta+1}(K \backslash A) = \text{const} \cdot \mu(K \backslash A) = 0.$$

This shows that $\mathcal{H}^\eta((K \backslash A) \cap V_u^{t_0}) = 0$ for $\mathcal{H}^1$-a.e. $u \in W^{t_0}$, and so $\mathcal{H}^\eta((K \backslash A) \cap V_z^{t_0}) = 0$ for μ-a.e. $z \in K$ since $P_{t_0}\mu \ll \mathcal{H}^1$. It follows that for μ-a.e. $z \in A$, and so for μ-a.e. $z \in K$,

$$\mathcal{H}^\eta(K \cap V_z^{t_0}) = \mathcal{H}^\eta(A \cap V_z^{t_0}) + \mathcal{H}^\eta((K \backslash A) \cap V_z^{t_0}) = 0\,.$$

From this, from Lemma (4.1.10), and from Fubini's theorem it follows that $\mathcal{H}^\eta(K \cap V_z^t) = 0$ for ν-a.e. $(z, t) \in X$, which completes the proof of Theorem (4.1.19).

Theorem (4.1.20)[238]: Let $0 < a < b < \frac{1}{2}$ be such that $\frac{1}{a}$ and $\frac{1}{b}$ are Pisot numbers, $\frac{\log b}{\log a}$ is irrational and $d_a + d_b > 1$, then $H^{d_a+d_b-1}((C_a \times C_b) \cap V_{(x,y)}) = 0$ for $\mu_a \times \mu_b \times \xi_{G^{-a}}$.e. $(x, y, V) \in C_a \times C_b \times G$.

Proof. Let G be the set of all 1-dimensional linear subspaces of $\mathbb{R}^2$, and set

$$E = \{(z, V) \in K \times G: H^{d_a+d_b-1}(K \cap V_z) = 0\}.$$

For each $-\infty \le t_1 < t_2 \le \infty$ set

$$G_{t_1,t_2} = \{V \in G: V = (t, -1) \cdot \mathbb{R}\ with\ t \in (t_1, t_2)\}.$$

Given $k \in \mathbb{Z}$ we can apply Theorem (4.1.19) with $\tau = a^k$, in order to get that $(z, V) \in E$ for $\mu \times \xi_G$-a.e. $(z, V) \in K \times G_{a^{k+1},a^k}$. By doing this for each $k \in \mathbb{Z}$ we get that $(z, V) \in E$ for $\mu \times \xi_G$-a.e. $(z, V) \in K \times G_{0,\infty}$. Now Theorem (4.1.20) follows from the symmetry of K with respect to the map that takes $(x, y) \in K$ to $(1 - x, y)$.

Section (4.2): Versus Exact Overlaps for Self-Similar Measures:

There has been a rapid development in the field of self-similar Iterated Function Systems (IFS) with overlapping construction. Most importantly, Hochman [258] proved for any self-similar measure ν that we can have dimension drop (that is $\dim_H \nu < \min\{1, \dim_S \nu\}$), only if there is a superexponential concentration of cylinders Consequently, for a one-parameter family of self-similar measures $\{\nu_\alpha\}_\alpha$ on $\mathbb{R}$, satisfying a certain non-degeneracy condition the Hausdorff dimension of the measure ν_α is equal to the minimum of its similarity dimension and 1 for all parameters α except for a small exceptional set of parameters E. This exceptional set E is so small that its packing dimension (and consequently its Hausdorff dimension) is zero. The corresponding problem for the singularity versus absolute continuity of self-similar measures was treated by Shmerkin and Solomyak [268]. They considered one-parameter families of self-similar measures constructed by one-parameter families of homogeneous self-similar IFS, also satisfying the non-degeneracy condition of Hochman Theorem. It was proved in [268] that for such families $\{\nu_\alpha\}$ of self-similar measures if the similarity dimension of the measures in the family is greater than 1 then for all but a set of Hausdorff dimension zero of parameters α, the measure να is absolute continuous with respect to the Lebesgue measure. The results presented imply that in this case it can happen that the set of exceptional parameters have packing dimension 1 as opposed to Hochman's Theorem where we remind that the packing dimension of the set of exceptional parameters is equal to 0.

Still, we do not know what causes the drop of dimension or the singularity of a selfsimilar measure on the line of similarity dimension greater than 1. In particular it is a natural question whether the only reason for the drop of the dimension or singularity of self-similar measures having similarity dimension larger than 1 is the "exact overlap". More precisely, let $\{\varphi_i\}_{i=1}^m$ be a self-similar IFS and ν be a corresponding self-similar measure. We say that there is an exact overlap if we can find two distinct $i = (i_1, \dots, i_k)$ and $j = (j_1, \dots, j_\ell)$ finite words such that

$$\phi_{i_1} \circ \dots \circ \phi_{i_k} = \phi_{j_1} \circ \dots \circ \phi_{j_\ell}. \qquad (17)$$

The following two questions have naturally arisen for a long time (e.g. Question 1 below appeared as [263, Question 2.6]):

Question 1: Is it true that a self-similar measure has Hausdorff dimension strictly smaller than the minimum of 1 and its similarity dimension only if we have exact overlap?

Question 2: Is it true for a self-similar measure ν having similarity dimension greater than one, that ν is singular only if there is exact overlap?

Most of the experts believe that the answer to Question 1 is positive and it has been confirmed in some special cases [258]. On the other hand, a result of Nazarov, Peres and Shmerkin indicated that the answer to Question 2 should be negative. Namely, they constructed in [261] a planar self-affine set having dimension greater than one, such that the angle-α projection of its natural measure was singular for a dense G_δ set of parameters α. However, this was not a family of self-similar measures. Question 2 has not yet been answered.

We consider one-parameter families of homogeneous self-similar measures on the line, having similarity dimension greater than 1. We call the set of those parameters for which the measure is singular, set of parameters of singularity.

(a) We point out that the answer to Question 2 above is negative.

(b) We consider one-parameter families of self-similar measures for which the set of parameters of singularity is big in the sense that it is a dense G_δ set but in the same time the parameter set of singularity is small in the sense that it is a set of Hausdorff dimension zero. We call such families antagonistic. We point out that there are many antagonistic families. Actually, we show that such antagonistic families are dense in a natural collection of one parameter families.

(c) As a corollary, we obtain that it happens quite frequently that in Shmerk in Solomyak Theorem (Theorem (4.2.9)) the exceptional set has packing dimension 1. (Corollary (4.2.21).)

(d) We extend the scope of [262, Proposition 8.1] from infinite Bernoulli convolution measures to very general one-parameter families of (not necessarily self-similar, or self-affine) IFS, and state that the parameter set of singularity is a G_δ set (Theorems (4.2.11), (4.2.12)).

We make the observation that the combination of an already existing method of Peres, Schlag and Solomyak [262] and a result due to Manning and [259] yields that the answer to Question 2 is negative.

There are two ingredients of our argument:

(i) The fact that the set of parameters of singularity is a G_δ set in any reasonable one-parameter family of self-similar measures on the line.

(ii) The existence of a one-parameter family of self-similar measures having similarity dimension greater than one (for all parameters) with a dense set of parameters of singularity.

It turned out that both of these ingredients have been a vailable for a while in the literature. Although in an earlier version of their longer proof for (i), we learned from B. Solomyak that (i) has already been proved in [262] in the special case of infinite Bernoulli convolutions. Actually, [262] acknowledged that the short and elegant proof of [262] is due to Elon Lindenstrauss. We extend the scope of [262] to a more general case. Then following the supposition of the anonymous referee we finally got a very general case. So, to prove (i), we will present here a more detailed and very general extension of the proof of [262].

On the other hand (ii) was proved in [259].

First we introduce the Hausdorff and similarity dimensions of a measure and then we present some definitions related to the singularity and absolute continuity of the family of measures considered.

(i) The notion of the Hausdorff and box dimension of a set is well known ([257]).

(ii) Hausdorff dimension of a measure: Let m be a measure on $\mathbb{R}^d$. The Hausdorff dimension of m is defined by

$$\dim_H m := \inf\{\dim_H A : m(A) > 0, \text{and A is a Borel set}\} \quad (18)$$

see [257] for an equivalent definition.

(iii) We will use the following definition of the Packing dimension of a set $H \subset \mathbb{R}^d$ [257]:

$$\dim_P H = \inf\{\sup_i \overline{\dim_B} E_i : H \subset \bigcup_{i=1}^{\infty} E_i\}, \quad (19)$$

where $\overline{\dim_B}$ stands for the upper box dimension. The most important properties of the packing dimension can be found in [257].

(iv) Similarity dimension of a self-similar measure: Consider the self-similar IFS on the line: $\mathcal{F} := \{\varphi(x) := r_i \cdot x + t_i\}_{i=1}^{m}$, where $r_i \in (-1,1)\setminus\{0\}$. Further we are giventhe probability vector $w := (w_1, \ldots, w_m)$. Then there exists a unique measure ν satisfying $\nu(H) = \sum_{i=1}^{m} w_i \cdot \nu\left(\varphi_i^{-1}(H)\right)$. (See [257].) We call $\nu = \nu_{\mathcal{F}}$,w the self-similarmeasure corresponding to $\mathcal{F}$ and w. The similarity dimension of ν is defined by

$$\dim_S(\nu_{\mathcal{F}}, w) := \frac{\sum_{i=1}^{m} w_i \log w_i}{\sum_{i=1}^{m} w_i \log r_i}. \quad (20)$$

Let

$$\mathcal{F}_\alpha := \left\{\varphi_i^\alpha(x) := r_{\alpha,i} \cdot x + t_i^{(\alpha)}\right\}_{i=1}^{m}, \qquad \alpha \in A \quad (21)$$

be a one-parameter family of self-similar IFS on$\mathbb{R}$ and let μ be a measure on the symbolic space $\Sigma := \{1, \ldots, m\}^{\mathbb{N}}$. We write

$$\varphi_{i_1 \ldots i_n}^\alpha := \varphi_{i_1}^\alpha \circ \cdots \circ \phi_{i_n}^\alpha \; and \; r_{\alpha, i_1 \ldots i_n} := r_{\alpha,i_1} \cdots r_{\alpha,i_n}.$$

The natural projection $\Pi_\alpha : \Sigma \to \mathbb{R}$ is defined by

$$\Pi_\alpha(i) := \lim_{n\to\infty} \varphi_{i_1 \ldots i_n}^\alpha(0) = \sum_{k=1}^{\infty} t_{i_k}^{(\alpha)} r_{\alpha, i_1, \ldots, i_{k-1}}, \quad (22)$$

where $r_{\alpha,i_1,\ldots,i_{k-1}} := 1$ when $k = 1$. Let μ be a probability measure on Σ. We study the family of its push forward measures $\{\nu_\alpha\}_{\alpha\in A}$:

$$\nu_\alpha(H) := (\Pi_\alpha)_* \mu(H) := \mu(\Pi_\alpha^{-1}(H)), \quad (23)$$

where H is a Borel subset of Σ. The elements of the symbolic space $\Sigma := \{1, \ldots, m\}^N$ are denoted by $i = (i_1, i_2, \ldots)$. If $w := (w_1, \ldots, w_m)$ is a probability vector and μ is the infinite product of w, that is $\mu = \{w_1, \ldots, w_m\}^N$ then the corresponding one-parameter family of self-similar measures defined in (23) is denoted by $\{\nu_{\alpha,w}\}_{\alpha\in A}$. The set of parameters of singularity and the set of parameters of absolute continuity with L^q-density are denoted by

$$\mathfrak{Sing}(F_{\alpha,\mu}) := \{\alpha \in A : \nu_\alpha \perp \mathcal{L}eb\}. \quad (24)$$

and

$$\mathcal{C}ont_Q(\mathcal{F}_\alpha, \mu) := \{\alpha : \nu_\alpha \ll \mathcal{L}eb \; with \; L^q \; density \; for \; a \; q > 1\}. \quad (25)$$

Definition (4.2.1)[255]: Using the notation introduced in (21)-(24) we say that the family $\{\nu_\alpha\}_{\alpha\in A}$ is antagonistic if both of the two conditions below hold:

$$\dim_H \mathfrak{Sing}(\mathcal{F}_\alpha, \mu) = \dim_H \left(\mathcal{C}ont_Q(\mathcal{F}_\alpha, \mu)\right)^c = 0 \quad (26)$$

and

$$\mathfrak{Sing}(\mathcal{F}_\alpha, \mu) \text{ is a dense } G_\delta \text{ subset of } A. \quad (27)$$

Clearly, $\mathfrak{Sing} \subset \left(\mathcal{C}ont_Q\right)^c$. Our aim is to prove that the angle-α projections of the natural measure of the Sierpiński-carpet is an antagonistic family. This implies that in Shmerkin-Solomyak's Theorem, [268] the exceptional set has packing dimension 1.

Whenever we say that $\{\nu_\alpha\}_{\alpha\in A}$ is a one-parameter family of self-similar IFS we always mean that $\{\nu_\alpha\}_{\alpha\in A}$ is constructed from a pair $(\mathcal{F}_\alpha, \mu)$ as in (23), for a $\mu = w^N$, where $w = (w_1, \dots, w_m)$ is a probability vector.

We always assume that the one-parameter family of self-similar IFS $\{\mathcal{F}_\alpha\}_{\alpha\in A}$ satisfies properties **P1-P4** below:

P1: The parameter domain is a non-empty, proper open interval A.

P2: $0 < r_{min} := \inf_{\alpha\in A, i\le m} |r_{\alpha,i}| \le \sup_{\alpha\in A, i\le m} |r_{\alpha,i}| =: r_{max} < 1$.

P3: $t^*_{max} := \sup_{\alpha\in A, i\le m} |t_i^{(\alpha)}| < \infty$.

P4: Both of the functions $\alpha \mapsto t_i^{(\alpha)}$ and $\alpha \mapsto r_\alpha, \alpha \in A$, can be extended to $\bar{A}$ (the closure of A) such that these extensions are both continuous. Note that P4 implies P3. It follows from properties P2 and P3 that there exists a big $\xi \in \mathbb{R}^+$ such that

$$spt(\nu_\alpha) \subset (-\xi, \xi), \forall \alpha \in A. \quad (28)$$

We always confine ourselves to this interval $(-\xi, \xi)$. In particular, whenever we write H^c for a set $H \subset \mathbb{R}$ we mean $(-\xi, \xi)\backslash H$. It will be our goal to prove that additionally the following properties also hold for some of the families under consideration:

P5A: $\mathfrak{Sing}(\mathcal{F}_\alpha, \mu)$ is dense in A.

P5B: $\mathfrak{Sing}(\mathcal{F}_\alpha, \mu)$ is a G_δ dense subset of A. We will prove below that Properties P5A and P5B are equivalent. Our motivating example, where all of these properties hold is as follows.

Our most important example is the family of angle-α projection of the natural measure of the usual Sierpiński carpet. We will see that the set of angles of singularity is a dense G_δ set which has Hausdorff dimension zero and packing dimension 1. First we define the Sierpiński carpet.

Definition (4.2.2)[255]: Let $t_1, \dots, t_8 \in \mathbb{R}^2$ be the 8 elements of the set $\{\{0,1,2\} \times \{0,1,2\}\backslash\{(1,1)\}\}$ in any particular order. The Sierpiński carpet is the attractor of the IFS

$$S := \left\{\varphi_i(x,y) := \frac{1}{3}(x,y) + \frac{1}{3}t_i\right\}_{i=1}^{8}. \quad (29)$$

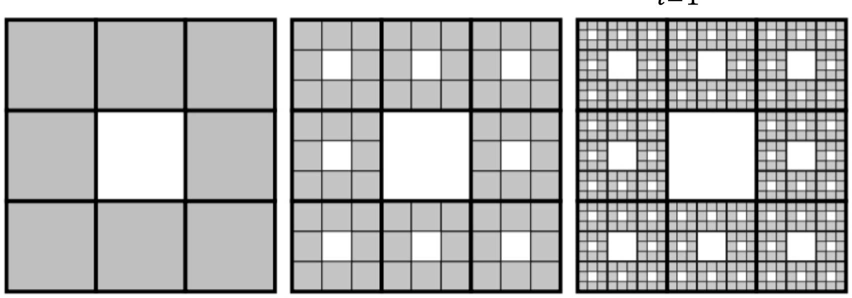

Figure 1[255]: The first three approximations of the Sierpiński carpet

Example (4.2.3)[255]: (Motivating example). Let S be the IFS given in (29). Let $\mu := \left(\frac{1}{8}, \dots, \frac{1}{8}\right)^N$ be the uniform distribution measure on the symbolic space $\Sigma := \{1, \dots, 8\}^N$. Further we write Π for the natural projection from Σ to the attractor Λ. Let $\nu := \Pi_*\mu$. Let $\ell_\alpha \subset \mathbb{R}^2$ be the line having angle α with the positive half of the x-axis (see

Figure 2). Let proj_α be the angle-α projection from $\mathbb{R}^2$ to the line ℓ_α. For each α, identifying ℓ_α with the x-axis, proj_α defines a one parameter family of self-similar IFS on the x-axis:

$$S_\alpha := \left\{\varphi_i^{(\alpha)}\right\}_{i=1}^{8},$$

where $\alpha \in A := (0,\pi)$ and $\phi_i^{(\alpha)}(x) = r_{\alpha,i}x + t_i^{(\alpha)}$ with $r_{\alpha,i} \equiv 1/3$ and $t_i^{(\alpha)} = t_i \cdot (cos(\alpha), sin(\alpha))$. For an $i \in \Sigma$ we define the natural projection $\Pi_\alpha(i)$ as in (22). Clearly,$\Pi_\alpha := proj_\alpha \circ \Pi$. The natural invariant measure for S_α is $\nu_\alpha := (\Pi_\alpha)_*\mu$. Obviously,$\nu_\alpha = (proj_\alpha)_*\nu$.

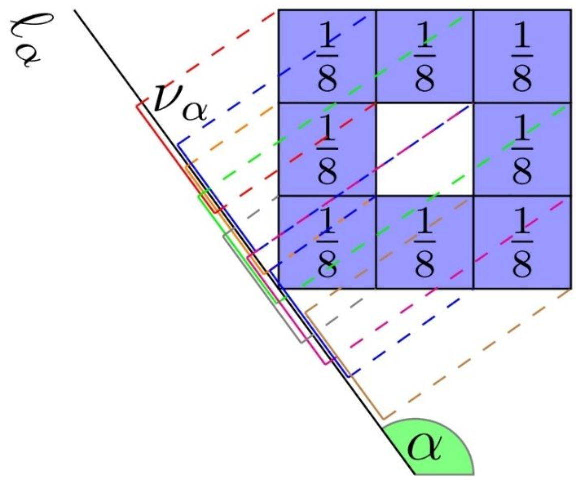

Figure 2[255]: The projected system

The fact that Property P5A holds for the special case in the example was proved in [259]. It follows from the proof of Bárány and Rams [256] that property P5A holds also for the projected family of the natural measure for most of those self-similar carpets, which have dimension greater than one.

Remark (4.2.16)[255]: (The cardinality of parameters of exact overlaps). It is obvious that in the case of the angle-α projections of a general self-similar carpet, exact overlap can happen only for countably many parameters. However, this is not true in general. To see this, we follow the ideas of C_s. Sándor [265] and construct the one parameter family of self-similar IFS $\left\{S_i^{(u)}\right\}_{i=1}^{3}$, $u \in U$, where $S_i^{(u)} := \lambda_i^{(u)}(x+1)$ and $(\lambda_1^{(u)}, \lambda_2^{(u)}, \lambda_3^{(u)}) = (\frac{u}{1+\varepsilon}, u, u+\varepsilon)$,further $U := \frac{1}{3} + \frac{\varepsilon}{3}, \frac{1}{3} + \eta - \varepsilon\}$ for sufficiently small $\eta > 0$ and $0 < \varepsilon < \frac{3}{4}\eta$. Then for all $u \in U$ we have:

(a) there is an exact overlap, namely: $S_{132}^{(u)} \equiv S_{213}^{(u)}$,
(b) the similarity dimension of the attractor is greater than 1,
(c) the Hausdorff dimension of the attractor is smaller than 1.

We collect those theorems. The theorems below are more general as stated here. We confine ourselves to the generality that matters for us.

Theorem (4.2.5)[255]: [258] Given the one-parameter family $\{\mathcal{F}_\alpha\}_{\{\alpha\in A\}}$ in the form as in (21). For $i, j \in \Sigma^n := \{1,\ldots,m\}^n$ we define

$$\Delta_{i,j}(\alpha) := \varphi_i^\alpha(0) - \varphi_i^\alpha(0) \text{ and } \Delta_n(\alpha) := \min_{i,j\in\Sigma^n}\{\Delta_{i,j}(\alpha)\}. \qquad (30)$$

Moreover, we define the exceptional set of parameters $E \subset A$

$$E := \bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n>N} \Delta_n^{-1}(-\varepsilon^n, \varepsilon^n). \tag{31}$$

Then for an $\alpha \in E^c$ and for every probability vector w the Hausdorff dimension of the corresponding self-similar measure $\nu_{\alpha,w}$ is

$$dimH(\nu\alpha, w) = \min\{1, dimS(\nu\alpha, w)\} \tag{32}$$

The following Condition will also be important:

Definition (4.2.6)[255]: We say that for an $\alpha \in A, \mathcal{F}_\alpha$ satisfies Condition H if

$$\exists\rho = \rho(\alpha) > 0, \exists n_k = n_k(\alpha) \uparrow \infty, \qquad \Delta_{n_k}(\alpha) > \rho^{n_k}. \tag{33}$$

Observe that $\alpha \in E^c$ if and only if $\mathcal{F}_\alpha$ satisfies Condition H.

Definition (4.2.7)[255]: We say that the Non-Degeneracy Condition holds if

$$\forall i, j \in \Sigma, i \neq j, \exists\alpha \in A\ s.t. \Pi_\alpha(i) \neq \Pi_\alpha(j). \tag{34}$$

Theorem (4.2.8)[255]: [258] Assume that the Non-Degeneracy Condition holds and the following functions are real analytic:

$$\alpha \longmapsto r_{\alpha,i}, i = 1, \dots, m \text{ and } \alpha \longmapsto t_i^{(\alpha)}. \tag{35}$$

Then

$$\dim_H E = \dim_P E = 0. \tag{36}$$

Theorem (4.2.9)[255]: [268] We assume that the conditions of Theorem (4.2.8) hold. Here we confine ourselves to homogeneous self-similar IFS on the line of the form

$$\mathcal{F}_\alpha := \left\{\varphi_i^\alpha(x) := r_\alpha \cdot x + t_i^{(\alpha)}\right\}_{i=1}^{m}, \alpha \in A. \tag{37}$$

Then there exists an exceptional set $E \subset A$ with $\dim_H E = 0$ such that for any $\alpha \in E^c$ and for any probability vector $w = (w_1, \dots, w_m)$ with $\dim_S(\nu_{\alpha,w}) > 1$ we have

$$\nu_{\alpha,w} \ll \mathcal{L}eb \text{ with } L^q \text{ density, for some } q > 1$$

Lídia Torma realized in her Master's Thesis [269] that the proof of Bárány and Rams [256], related to the projections of general self-similar carpets, works in a much more general setup, without any essential change.

Theorem (4.2.10)[255]: (Extended version of Bárány-Rams Theorem). Given an $a \in \mathbb{R}\backslash\{0\}$. Let $\mathcal{T} = \{n \cdot a\}_{n\in Z}$ be the corresponding lattice on $\mathbb{R}$. Moreover, given the self-similar IFS on the line of the form:

$$S := \left\{S_i(x) := \frac{1}{L} \cdot x + t_i\right\}_{i=1}^{m}, \tag{38}$$

where $L \in \mathbb{N}, L \geq 2$ and $t_i \in \mathcal{T}$ for all $i \in \{1, \dots, m\}$. We are also given a probability vector $w = (w_1, \dots, w_m)$ with rational weights $w_i = p_i/q_i, p_i, q_i \in N\backslash\{0\}$ satisfying

$$L \nmid Q := lcm\{q_1, \dots, q_m\}, s =: \dim_S \nu = \frac{-\sum_{i=1}^m w_i \log w_i}{\log L} > 1, \tag{39}$$

where ν is the self-similar measure corresponding to the weights w. That is $\nu = \sum_{i=1}^m w_i \cdot \nu \circ S_i^{-1}$. Then we have

$$\dim_H \nu < 1. \tag{40}$$

As we have already mentioned the following result appeared as [262] in the special case when the family of self-similar measures is the Bernoulli convolution measures. We extend the original proof of [262] to the following much more general situation.

Theorem (4.2.11)[255]: Let $\mathbb{R} \subset \mathbb{R}^d$ be a non-empty bounded open set. Let U be a metric space (the parameter domain). Let λ be a finite Radon measure with $spt(\lambda) \subset \mathbb{R}$ (the reference measure). For every α we are given a probability Radon measure ν_α such that $spt(\nu_\alpha) \subset \mathbb{R}$. Let

$$C_R := \{f: R \to [0,1]: f \text{ is continuous}\}. \tag{41}$$

For every $F \in C_R$ we define $\Phi_f: U \to R$

$$\Phi_f(\alpha) := \int_R f(x) d\nu_\alpha(x). \tag{42}$$

Finally, we define

$$\mathfrak{Sing}_\lambda(\{\nu_\alpha\}_{\alpha\in U}) := \{\alpha \in U: \nu_\alpha \perp \lambda\}. \tag{43}$$

If $\alpha \longmapsto \Phi_f(\alpha)$ is lower semi-continuous then $\mathfrak{Sing}_\lambda(\{\nu_\alpha\}_{\alpha\in U})$is a G_δ set.

Proof. Recall that ν_α is a probability measure for all α. Note that without loss of generality we may assume that λ is also a probability measure on R. For every $\varepsilon > 0$ we define

$$A_\varepsilon := \left\{f \in C_R: \int f(x) d\lambda(x) < \varepsilon\right\}.$$

We follow the proof of [262] and a suggestion of an unknown referee. First we fix an arbitrary sequence $\varepsilon_n \downarrow 0$ and then define

$$S_\perp := \bigcap_{n=1}^{\infty} \bigcup_{f\in A_{\varepsilon_n}} \{\alpha \in U: \Phi_f(\alpha) > 1 - \varepsilon_n\}.$$

Since we assumed that $\alpha \longmapsto \Phi_f(\alpha)$ is lower semi-continuous, the set$\{\alpha \in U: \Phi_f(\alpha) > 1 - \varepsilon_n\}$ is open. That is $S_\perp$ is a G_δ set. Hence it is enough to prove that

$$\mathfrak{Sing}_\lambda(\{\nu_\alpha\}_{\alpha\in U}) = S_\perp. \tag{44}$$

First we prove that $\mathfrak{Sing}_\lambda(\{\nu_\alpha\}_{\alpha\in U}) \subseteq S_\perp$. Let $\beta \in \mathfrak{Sing}_\lambda(\{\nu_\alpha\}_{\alpha\in U})$. Fix an arbitrary $\varepsilon > 0$. Then by definition we can find a $T \subset R$ such that

$$\nu_\beta(T) = 1, \qquad \lambda(T) = 0. \tag{45}$$

Recall that both λ and ν_β are Radon probability measures. So we can choose a compact $C_\varepsilon \subset T$ such that

$$\nu_\beta(C_\varepsilon) > 1 - \varepsilon, \lambda(C_\varepsilon) = 0. \tag{46}$$

Using that λ is a Radon measure, we can choose an open set $V_\varepsilon \subset \mathbb{R}$ such that $C_\varepsilon \subset V_\varepsilon$ and $\lambda(V_\varepsilon) < \varepsilon$. We can choose an $f_\varepsilon \in C_R$ such that $spt(f_\varepsilon) \subset V_\varepsilon$ and $f_\varepsilon | C_\varepsilon \equiv 1$ (see [264]). Then $\int f_\varepsilon d\lambda(x) \leq \lambda(V_\varepsilon) < \varepsilon$ (that is $f_\varepsilon \in A_\varepsilon$) and $\int f_\varepsilon(x) d\nu_\beta(x) \geq \nu_\beta(C_\varepsilon) > 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary we obtain that $\beta \in S_\perp$. Now we prove that $S_\perp \subseteq \mathfrak{Sing}_\lambda\{\nu_\alpha\}_{\alpha\in U}$. Let $\beta \in S_\perp$. Then for every n there existsan $f_n \in C_R$ such that

$$\int f_n(x) d\nu_\beta(x) > 1 - \varepsilon_n \text{ and } \int f_n d\lambda(x) < \varepsilon_n. \tag{47}$$

Let $C_\beta := spt(\nu_\beta)$. Clearly, C_β is compact and $C_\beta \subset R$. We define $g_n := f_n 1_{C_\beta}, and\ g := 1_{C_\beta}$. Clearly, $0 \leq g_n(x) \leq g(x)$ for all $x \in C_\beta$ and

$$\int g(x) d\nu_\beta(x) = 1, \int g_n(x) d\nu_\beta(x) > 1 - \varepsilon_n \text{ and } \int g_n d\lambda(x) < \varepsilon_n.$$

Hence,

$$g_n \xrightarrow{L_1(\nu_\beta)} g.$$

Thus, we can select a subsequence g_{n_k} such that $g_{n_k}(x) \to g(x)$ for ν_β- almost all $x \in C_\beta$. Let

$$D_\beta := \{x \in C_\beta: g_{n_k}(x) \to g(x)\}.$$

Then on the one hand we have

$$\nu_\beta(D_\beta) = 1. \tag{48}$$

On the other hand using the Lebesgue Dominated Convergence Theorem:

$$\lambda(D_\beta) = \int_{D_\beta} g(x)d\lambda(x) = \int_{D_\beta} \lim_{k\to\infty} g_{n_k}(x)\, d\lambda(x)$$

$$= \lim_{k\to\infty} \int_{D_\beta} g_{n_k}(x)d\lambda(x) \le \lim_{k\to\infty} \varepsilon_{n_k} = 0. \quad (49)$$

Putting together (48) and (49) we obtain that $\beta \in \mathfrak{S}ing_\lambda(\{\nu_\alpha\}_{\alpha\in U})$.

Theorem (4.2.12)[255]: We consider one-parameter families of measures ν_α on $\mathbb{R}^d$ for some $d \ge 1$, which are constructed as follows: The parameter space U is a non-empty compact metric space. We are given a continuous mapping

$$\Pi: U \times \Omega \to R \subset \mathrm{R}^{\mathrm{d}}, \quad (50)$$

where R is an open ball in R^d and Ω is a compact metric space (in our applications U is a compact interval, $\Omega = \Sigma$ and Π_α is the natural projection corresponding to the parameter α). Moreover let μ be a probability Radon measure on Ω. (In our applications μ is Bernoulli measure on Σ.) For every $\alpha \in U$ we define

$$\nu_\alpha := (\Pi_\alpha)_*\mu. \quad (51)$$

Clearly, ν_α is a Radon measure whose support is contained in R. Finally let λ be a Radon (reference) measure whose support is also contained in R. (In our applications λ is the Lebesgue measure Lebd restricted to R.)

Then the set of parameters of singularity

$$\mathfrak{S}ing_\lambda(\Pi_\alpha, \mu) := \{\alpha \in U : \nu_\alpha \perp \lambda\} \quad (52)$$

is a G_δ set.

Proof. This theorem immediately follows from Theorem (4.2.9) if we prove that for every $f \in C_R$ the function $\Phi_f(\cdot)$ is continuous. To see this we set $\psi: U \times \Omega \to R$,

$$\psi(\alpha,\omega) = f(\Pi_\alpha(\omega)), then\ \Phi_f(\alpha) := \int f(x)d\nu_\alpha(x) = \int \psi(\alpha,\omega)d\mu(\omega),$$

where the last equality follows from the change of variables formula. By compactness, ψ is uniformly continuous. Hence for every $\varepsilon > 0$ we can choose $\delta > 0$ such that whenever $dist((\alpha_1,\omega),(\alpha_2,\omega)) < \delta$ then $|\psi(\alpha_1,\omega) - \psi(\alpha_2,\omega)| < \varepsilon$, where $dist\big((\alpha_1,\omega),(\alpha_2,\omega)\big) := \max\{dist_U(\alpha_1,\alpha_2), dist_\Omega(\omega_1,\omega_2)\}$. Using that μ is a probability measure, we obtain that $|\Phi_f(\alpha_1) - \Phi_f(\alpha_2)| < \varepsilon$ whenever $dist_U(\alpha_1,\alpha_2) < \delta$.

Corollary (4.2.13)[255]: Using the notation and assuming our Principal Assumption we obtain that the set of parameters of singularity $\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)$ is a G_δ set. The proof is obvious since our Principal Assumptions imply that the conditions of Theorem (4.2.12) hold. To derive another corollary we need the following fact. It is well known, but we could not look it up in the literature, therefore we include its proof here.

Fact (4.2.14)[255]: Let $H \subset \mathbb{R}^d$ be a G_δ set which is not a nowhere dense set. Then $\dim_P H = d$.

Proof. Since H is not a nowhere dense set, there exist a ball B such that $B \subset \overline{H}$. That is $V := B \cap H$ is a dense G_δ set in B, that is by Banach's Theorem V is not a set of first category. So, if $V \subset \cup_{i=1}^\infty E_i$ then there exists an i such that E_i is not nowhere dense in B. That is there exists a ball $B_0 \subset B$ such that $B_0 \subset E_i$. Then $\dim_B E_i = d$. Hence by (52) we have $\dim_P H \ge \dim_P V = d$. On the other hand, $\dim_P H \le d$ always holds. Applying this for $\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)$ we obtain that

Corollary (4.2.15)[255]: Under the conditions of Theorem (4.2.6), for the set of parameters of singularity $\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)$ the following holds:

(i) Either $\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)$ is nowhere dense or

(ii) $\dim_P(\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)) = d$.

Remark (4.2.16)[255]: Let μ be a compactly supported Borel measure on $\mathbb{R}^2$ with $\dim_H \mu > 1$. Let $\nu_\alpha := (proj_\alpha)_* \mu$. Then Theorem (4.2.12) immediately implies that either the singularity set

$$\mathfrak{S}ing_{\mathcal{L}eb}\{\nu_\alpha\}_{\alpha \in [0,\pi)} = \{\alpha \in [0,\pi) : \nu_\alpha \perp \mathcal{L}eb_1\}$$

or its complement is big in topological sense. More precisely,

(a) Either $\mathfrak{S}ing_{\mathcal{L}eb}\left(\{\nu_\alpha\}_{\alpha \in [0,\pi)}\right)$ is a residual subset of $[0,\pi)$ or

(b) $\left(\mathfrak{S}ing_{\mathcal{L}eb}\left(\{\nu_\alpha\}_{\alpha \in [0,\pi)}\right)\right)^c$ contains an interval.

We remind that a set is called residual if is its complement is a set of first category and residual sets are considered as "big" in topological sense. In contrast we recall that by Kaufman's Theorem (see e.g. [260]) we have

$$\nu_\alpha \ll \mathcal{L}eb_1 \text{ for } \mathcal{L}eb_1 \text{ almost all } \alpha \in [0,\pi). \quad (53)$$

The following theorem shows that there are reasons other than exact overlaps for the singularity of self-similar measures having similarity dimension greater than one.

Theorem (4.2.17)[255]: Using the notation of our Example (4.2.3) (angle-α projections of the Sierpiński carpet), we obtain that

$$\mathfrak{S}ing(S_\alpha, \mu) = \{\alpha \in A : \nu_\alpha \perp \mathcal{L}eb\} is\ a\ dense\ G_\delta\ set \quad (54)$$

and

$$\dim_H(\mathcal{C}ont_Q(S_\alpha, \mu)^c) = 0. \quad (55)$$

That is (S_α, μ) is antagonistic in the sense of Definition (4.2.1).

Proof. The first part follows from Corollary (4.2.13) and from the fact that property P5A holds for the projections of the Sierpiński-carpet. This was proved in [259]. Now we turn to the proof of the second part of the Theorem. This assertion would immediately follow from Shmerkin and Solomyak [268, Theorem A] if we could guarantee that the Non-Degeneracy Condition holds. Unfortunately in this case it does not hold. Still it is possible to gain the same conclusion not from the assertion of [268] but from its proof, combined with [268] as it was explained by P. Shmerkin [266]. For completeness we point out the only two steps of the original proof of [268] where we have to make slight modifications. Let P be the set of probability Borel measures on the line. We write

$$D := \mu \in P : |\hat{\mu}(\xi)| = O_\mu(|\xi|^{-\sigma}) \text{ for some } \sigma > 0. \quad (56)$$

The elements of D are the probability measures on the line with power Fourier-decay. Let $\left\{\phi_i^{(\alpha)}\right\}_{i=1}^{8}$ be the IFS defined in Example (4.2.3). Now we write the projected self-similar natural measure ν_α of the Sierpiński carpet in the infinite convolution form. That is we consider ν_α as the distribution of the following infinite random sum:

$$\nu_\alpha \sim \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} A_n,$$

where A_n are independent Bernoulli random variables with $P(A_n = \phi_i^{(\alpha)}(0)) = 1/8$. For $k \geq 2$ integers we decompose the random sum on the right hand side as

$$\nu_\alpha \sim \sum_{\substack{n=1 \\ k \nmid n}}^{\infty} \left(\frac{1}{3}\right)^{n-1} A_n + \sum_{\substack{n=1 \\ k | n}}^{\infty} \left(\frac{1}{3}\right)^{n-1} A_n .$$

Writing $\eta'_{\alpha,k}$ and $\eta''_{\alpha,k}$ for the distribution of the first and the second random sum, respectively, we get $\nu_\alpha = \eta'_{\alpha,k} * \eta''_{\alpha,k}$. We show that with appropriately chosen k we can apply [268] to $\eta'_{\alpha,k}$ and $\eta''_{\alpha,k}$ which would conclude the proof. To this end it is enough to show that on the one hand

$$\dim_H \eta'_{\alpha,k} = 1 \text{ for every } k \text{ large enough} \tag{57}$$

and on the other hand we have

$$\eta''_{\alpha,k} \in D, \forall k \geq 2. \tag{58}$$

This is the first place where we depart from the proof of [268]. According to [267] if $\dim_S \eta'_{\alpha,k} > 1$ (which holds if k is big enough), then there exists a countable set E'_k such that $\dim_H \eta'_{\alpha,k} = 1$ for all $\alpha \notin E'_k$. Note that the original proof at this point relies on the non-degeneracy condition, what we do not use here.

To get the Fourier decay of $\eta''_{\alpha,k}$ we follow the proof of [268]. In our special case, we may choose the function f in the middle in [268] as

$$f(\alpha) = \frac{proj_\alpha\left(\frac{2}{3},0\right) - proj_\alpha\left(\frac{1}{3},\frac{2}{3}\right)}{proj_\alpha\left(0,\frac{2}{3}\right) - proj_\alpha\left(\frac{1}{3},\frac{2}{3}\right)} = 2\tan(\alpha) - 1.$$

Clearly f is non-constant and f^{-1} preserves the Hausdorff dimension. Hence by [268] there is a set E''_k of Hausdorff dimension 0 such that $\eta''_{\alpha,k}$ has power Fourier-decay for all $\alpha \notin E''_k$. Altogether, setting the 0-dimensional exceptional set of parameters $E = \bigcup_{k=2}^{\infty} E'_k \cup E''_k$, by [268] we have that ν_α is absolutely continuous with an L^q density for some $q > 1$ for all $\alpha \notin E$ exactly as in the proof of [268] with no further modifications.

In Theorem (4.2.17) we have proved that the family of the angle-α projection of the Sierpiński carpet is antagonistic in the sense of Definition (4.2.1). We prove that there are many antagonistic families.

First of all we remark that the Non-Degeneracy Condition does not hold for all families. For example let

$$\mathcal{F}_\alpha := \left\{\frac{1}{2} \cdot x + t_i^{(\alpha)}\right\}_{i=1}^{m}, m \geq 2. \tag{59}$$

Then for every $\alpha, \Pi_\alpha(i) = \Pi_\alpha(j)$ for $i = (1,2,\ldots,2,\ldots)$ and $j = (2,1,\ldots,1,\ldots)$. So, the non-degeneracy condition does not hold. However, if the contraction ratio is the same $\lambda \in (0,\frac{1}{2})$ for all maps of all IFS in the family (the family is equi-homogeneous) and the translations are independent real-analytic functions then the Non-Degeneracy Condition holds:

Proposition (4.2.18)[255]: Given

$$\mathcal{F}_\alpha := \left\{\lambda \cdot x + t_i^{(\alpha)}\right\}_{i=1}^{m}, m \geq 2, \alpha \in A, \tag{60}$$

where

(a) $\lambda \in (0,\frac{1}{2})$ and

(b) For $\ell = 1,\ldots,m$, the functions $\alpha \mapsto t_\ell^{(\alpha)} = \sum_{k=0}^{\infty} a_{\ell,k} \cdot \alpha^k$, are independent real analytic functions:

$$\forall \alpha \in A, \sum_{i=1}^{m} \gamma_i \cdot t_i^{(\alpha)} \equiv 0 \text{ iff } \gamma_1 = \cdots = \gamma_m = 0. \tag{61}$$

Then $\{F_\alpha\}_{\alpha \in A}$ satisfies the Non-Degeneracy Condition.

Proof. Fix two distinct $i, j \in \Sigma$. For every $\ell = 1, \ldots, m$, define $q_\ell := q_\ell(i,j)$ by

$$q_\ell := \sum_{\{k : i_k = \ell\}} \lambda^{k-1} - \sum_{\{k : j_k = \ell\}} \lambda^{k-1}. \tag{62}$$

Then

$$\Pi_\alpha(i) - \Pi_\alpha(j) = \sum_{k=0}^{\infty} \alpha^k \cdot b_k, \tag{63}$$

where

$$b_k := \sum_{\ell=1}^{m} a_{\ell,k} \cdot q_\ell \tag{64}$$

for all $k \in \mathbb{N}^+$,where $\mathbb{N}^+ := \mathbb{N}\backslash\{0\}$. Observe that for $b := (b_0, b_1, \ldots)$ and $\forall \ell = 1, \ldots, m$ for $a_\ell := (a_{\ell,0}, a_{\ell,1}, a_{\ell,2}, \ldots a_{\ell,k}, \ldots)$ we have that (64) can be written as

$$\sum_{\ell=1}^{m} q_\ell \cdot a_\ell = b. \tag{65}$$

Assume that

$$\forall \alpha \in A, \Pi_\alpha(i) - \Pi_\alpha(j) \equiv 0. \tag{66}$$

To complete the proof it is enough to verify that $i = j$. Using (63), we obtain from (66) that $b_k = 0$ for all $k \in \mathbb{N}^+$. Note that (61) states that the vectors$\{a_\ell\}_{\ell=1}^{m}$ are independent. So, from $b = 0$ and from (65) we get that $q_1 = \cdots = q_m = 0$. This and $\lambda \in \left(0, \frac{1}{2}\right)$ implies that $i = j$.

Here we prove the following assertion: The collection of one-parameter families of IFS and self-similar measures are dense in the collection of equi-homogeneous IFS having contraction ratio $1/L$ ($L \in \mathbb{N}^+$) equipped with invariant measures with similarity dimension greater than one.

Definition (4.2.19)[255]: First we consider collections of equi-homogeneous self-similar IFS having at least 4 functions.

(i): Let $\mathfrak{F}_L$ be the collection of all pairs $(\mathcal{F}_\alpha, \mu)$ satisfying the conditions below:

• $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ is of the form:

$$\mathcal{F}_\alpha := \left\{\varphi_i^{(\alpha)}(x) := \frac{1}{L} \cdot x + t_i^{(\alpha)}\right\}_{i=1}^{m}, \alpha \in A, \tag{67}$$

where $m \geq 4$, $A \subset \mathbb{R}$ is a proper interval (A is compact) and

$$L \in \mathbb{N}, 3 \leq L \leq m - 1. \tag{68}$$

Moreover, the functions $\alpha \longmapsto t_\ell^\alpha$ are continuous on A for all $\ell = 1, \ldots, m$.

• Let μ be an infinite product measure $\mu := (w_1, \ldots, w_m)^N$ on $\Sigma := \{1, \ldots, m\}^N$ satisfying:

$$s := -\frac{\sum_{i=1}^{m} w_i \log w_i}{\log L} > 1, \tag{69}$$

(ii): Now we define a rational coefficient sub-collection $\mathfrak{F}_{L,rac} \subset \mathfrak{F}_L$ satisfying a non-resonance like condition (70) below:

• $\alpha \longmapsto t_i^{(\alpha)}$ are polynomials of rational coefficients. We assume that $\left\{t_i^{(\alpha)}\right\}_{i=1}^{m}$ are independent, that is (59) holds. Moreover,

• The weights w_i are rational: $\{w_i\}_{i=1}^{m}, w_i = r_i/q_i$, with $r_i, q_i \in \mathbb{N}\backslash\{0\}$ satisfy-ing:

$$L \nmid lcm\{q_1, \ldots, q_m\}, \tag{70}$$

where lcm is the least common multiple. Let $\nu_\alpha := (\Pi_\alpha)_*\mu$.

Proposition (4.2.20)[255]:

(a) All elements $\{\nu_\alpha\}$ of $\mathfrak{F}_{L,rac}$ are antagonistic.

(b) $\mathfrak{F}_{L,rac}$ is dense in $\mathfrak{F}_L$ in the sup norm.

Proof. (a) It follows from Proposition (4.2.18) that we can apply Shmerkin-Solomyak Theorem (Theorem (4.2.9)). This yield that $Cont_Q$ (defined in (24)) satisfies $\dim_H \left(Cont_Q(\mathcal{F}_\alpha, \mu)\right)^c = 0$. On the other hand, for every rational parameter α, $(\mathcal{F}_\alpha, \mu)$ satisfies the conditions of Theorem (4.2.9). So, for every $\alpha \in Q$ we have $\dim_H \nu_\alpha < 1$. Using this and Corollary (4.2.13) we get that $\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)$ is a dense G_δ set. So, $\{\nu_a\}_{\alpha \in A}$ is antagonistic.

(b) Let $(\tilde{\mathcal{F}}_\alpha, \tilde{\mu}) \in \mathfrak{F}_L$, with $\tilde{\mathcal{F}}_\alpha = \left\{\varphi_i^{(\alpha)}(x) := \frac{1}{L} \cdot x + \tilde{t}_i^{(\alpha)}\right\}_{i=1}^m$ ande $\tilde{\mu} = (\tilde{w}_1, \ldots, \tilde{w}_m)^N$. Fix an $\varepsilon > 0$. We can find independent polynomials $\alpha \longmapsto t_i^{(\alpha)} i = 1, \ldots, m$ of rational coefficients such that $\left\| \tilde{t}_i^{(\alpha)} - t_i^{(\alpha)} \right\| < \varepsilon$ for all $\alpha \in A$ and $i = 1, \ldots, m$. Moreover, wecan find a product measure $\mu = (w_1, \ldots, w_m)^N$ such that for $w = (w_1, \ldots, w_m)$ we have $\|w - \tilde{w}\| < \varepsilon$ and w has rational coefficients $w_i = p_i/q_i$ satisfying (70).

Corollary (4.2.21)[255]: Let $(\mathcal{F}_\alpha, \mu) \in \mathcal{F}_{L,rac}$ Then

$$\dim_P(\mathfrak{S}ing(\mathcal{F}_\alpha, \mu)) = 1. \tag{71}$$

Section (4.3): Singular Projections and Discrete Slices:

For R be a 2×2 rotation matrix, with $R^n \neq Id$ for all $n \geq 1$, and let $r \in (0,1)$. Consider a homogeneous IFS on $\mathbb{R}^2$

$$\{\varphi_i(x) = rRx + a_i\}_{i \in I},$$

with the strong separation condition (SSC), and a self-similar measure

$$\mu = \sum_{i \in I} p_i \cdot \varphi_i \mu.$$

It is among the most basic planar self-similar measures. Hence it is a natural question in fractal geometry to study the dimension and continuity of the projections $\{P_u \mu\}_{u \in S}$ and slices

$$\left\{\{\mu_{u,x}\}_{x \in \mathbb{R}^2} : u \in S\right\}.$$

Here S is the unit circle of $\mathbb{R}^2$, P_u is the orthogonal projection onto the line spanned by u, and $\{\mu_{u,x}\}_{x \in \mathbb{R}^2}$ is the disintegration of μ with respect to $P_u^{-1}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^2$. A more elaborate description of these disintegrations is given. Note that the atoms of $P_u^{-1}(\mathcal{B})$ are lines perpendicular to $\text{span}\{u\}$.

Dimension wise, the behaviour of the projections is as regular as possible. Indeed, Hochman and Shmerkin [279] have proven that $P_u \mu$ is exact dimensional, with

$$\dim P_u \mu = \min \ \{1, \dim \mu\},$$

for each $u \in S$. A version of this, for self-similar sets with dense rotations, was first proven by Peres and Shmerkin [286]. Considering the continuity of the projections, Shmerkin and Solomyak [288] have shown, assuming $\dim \mu > 1$, that the set

$$E = \{u \in S : P_u \mu \text{ is singular }\}$$

has zero Hausdorff dimension.

We discuss the concept of dimension conservation and the dimension of slices. A Borel probability measure ν on $\mathbb{R}^2$ is said to be dimension conserving (DC), with respect to the projection P_u, if

$$\dim_H \nu = \dim_H P_u\nu + \dim_H \nu_{u,x} \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^2,$$

where $\dim_H$ stands for Hausdorff dimension. It always holds that ν is DC with respect to P_u for almost every $u \in S$. This follows from results, valid for general measures, regarding the typical dimension of projections (see [280]) and slices (see [282]). Falconer and Jin [275] have shown that ν is DC, with respect to P_u for all $u \in S$, whenever ν is self-similar with a finite rotation group. An analogous statement, for self-similar sets with the SSC, was first proven by Furstenberg [278]. Another related result for sets is due to Falconer and Jin [276]. They showed that if $K \subset \mathbb{R}^2$ is self-similar, with $\dim K > 1$ and a dense rotation group, then for every $\epsilon > 0$ there exists $N_\epsilon \subset S$, with $\dim_H N_\epsilon = 0$, such that for $u \in S \setminus N_\epsilon$ the set

$$\{x \in \text{span}\{u\}: \dim_H(K \cap P_u^{-1}\{x\}) > \dim K - 1 - \epsilon\}$$

has positive length.

Taking these results into account, it is natural to ask whether the sets E, defined above, and

$$F = \{u \in S: \mu \text{ is not DC with respect to } P_u\},$$

must be empty whenever the dimension of μ exceeds 1. A version, for self-similar sets, of this folklore question regarding E is asked in Section 4 of [271]. We show that E and F are not necessarily empty, and in fact can both be topologically large. The following theorem is our main result. Recall that a measure ν is said to be discrete if it is supported on a countable set.

Theorem (4.3.1)[270]: There exist $r \in (0,1)$, a 2×2 rotation matrix R with $R^n \neq \text{Id}$ for all $n \geq 1$, and a homogeneous planar self-similar IFS

$$\{\varphi_i(x) = rRx + a_i\}_{i\in I}$$

with the SSC, such that the self-similar measure $\mu = \sum_{i\in I} |I|^{-1} \cdot \varphi_i\mu$ satisfies $\dim \mu > 1$, and each of the sets

$$\{u \in S: P_u\mu \text{ is singular }\} \tag{72}$$

And

$$\{u \in S: \mu_{u,x} \text{ is discrete for } \mu\text{-a.e. } x \in \mathbb{R}^2\} \tag{73}$$

contains a dense G_δ subset of S.

If ν is a discrete measure on $\mathbb{R}^2$ then clearly $\dim \nu = 0$, hence we get the following corollary.

Corollary (4.3.2)[270]: Let μ be the self-similar measure from Theorem (4.3.1), then the set

$$\{u \in S: \mu \text{ is not } DC \text{ with respect to } P_u\}$$

contains a dense G_δ subset of S.

Theorem (4.3.1) is related to an example obtained by Nazarov, Peres and Shmerkin [285]. They have presented a planar self-affine measure ν, with the SSC and having dimension greater than 1, such that the set of $u \in S$ for which $P_u\nu$ is singular contains a G_δ subset. We do not pursue this, but our argument can probably be used for showing that, for a residual set of directions $u \in S$, the slices $\{\nu_{u,x}\}_{x\in\mathbb{R}^2}$ are ν-typically discrete. Also related to Theorem (4.3.1), by Simon and Vágó [289], in which certain one-parameter families of self-similar measures ν_α on the line are constructed. For these families it holds that the similarity dimension of ν_α is greater than 1 for every α, but the set of parameters for which ν_α is singular is topologically large.

In our construction of μ the rotational part rR, of the maps in the IFS, comes from a reciprocal of a complex Pisot number. While dealing with parametric families of measures, Pisot numbers have been used before, in several situations, in order to demonstrate the existence of exceptional parameters for which the corresponding measures are singular. This was first done by Erdös [274], who proved that the Bernoulli convolution corresponding to λ, i.e. the distribution of the random sum $\sum_n \pm \lambda^n$, is singular whenever $\lambda^{-1} \in (1,2)$ is Pisot. The example from [285], mentioned above, also utilizes real Pisot numbers. In [290], complex Pisot numbers are used in order to obtain examples of singular complex Bernoulli convolutions.

As a by-product of our construction, we obtain information on the Hausdorff measure of typical slices of self-similar sets at the critical dimension. Let K be a planar self-similar set with the SSC, and denote by $\mathbf{m}$ the Haar measure of S. Write s for the Hausdorff dimension of K, and assume $s > 1$. For $t \geq 0$ denote by $\mathcal{H}^t$ the t-dimensional Hausdorff measure. Given $u \in S$ and $x \in K$ write $K_{u,x}$ for the slice $K \cap (x + \text{span}\{u\})$. Since $0 < \mathcal{H}^s(K) < \infty$, the Hausdorff dimension of $K_{u,x}$ is equal to $s - 1$, with finite $\mathcal{H}^{s-1}$-measure, for $\mathcal{H}^s \times \mathbf{m}$-a.e. $(x, u) \in K \times S$ (see Theorem 10.11 in [284]). However, it was not known whether the set

$$Q = \{(x, u) \in K \times S : \mathcal{H}^{s-1}(K_{u,x}) > 0\}$$

must have positive $\mathcal{H}^s \times$ m-measure. In Corollary 2.3 from [287] the author has shown that if this holds, and K has dense rotations, then $P_u(\mathcal{H}^s|_K)$ is absolutely continuous for all $u \in S$. In our example from Theorem (4.3.1) we shall have $\mu = C \cdot \mathcal{H}^s|_K$, where $C > 0$ is a normalizing constant. Hence we obtain the following corollary.

Corollary (4.3.3)[270]: Let $\{\varphi_i\}_{i\in I}$ be the IFS constructed in Theorem (4.3.1). Denote by K its attractor, and write s for the Hausdorff dimension of K. Then $s > 1$ and,

$$\mathcal{H}^s \times \mathbf{m}\{(x, u) \in K \times S : \mathcal{H}^{s-1}(K_{u,x}) > 0\} = 0.$$

It is interesting to note that, in contrast with Corollary (4.3.3), if $K \subset \mathbb{R}^n$ is self-similar, with the SSC, finite rotation group and dimension s greater than $2m$, then $\mathcal{H}^{s-m}(K \cap V) > 0$ for typical affine $(n - m)$-planes $V \subset \mathbb{R}^n$ (see Corollary 2.2 in [287]).

The measure μ from Theorem (4.3.1) is constructed. We show that the set defined in (72) is residual. We complete the proof by establishing this for the set appearing in (73).

We carry out the construction of the measure μ from Theorem (4.3.1). It will be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We shall use some simple facts from the theory of field extensions, for which we refer to chapters 17 and 18 in [281]. Our example involves complex Pisot numbers, which we now define.

Definition (4.3.4)[270]: An algebraic integer $\theta \in \mathbb{C}$ is called a complex Pisot number if $\theta \notin \mathbb{R}$, $|\theta| > 1$, and all of the Galois conjugates of θ (i.e. the other roots of the minimal polynomial of θ), except $\bar{\theta}$, are less than one in modulus.

Given algebraic numbers $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, we denote by $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ the smallest subfield of $\mathbb{C}$ containing $\alpha_1, \dots, \alpha_n$. If $F \subset E$ are subfields of $\mathbb{C}$, we write $[E: F]$ for the degree of the field extension E/F. The next lemma is probably known, but we could not find an appropriate reference. Hence the proof, which uses a bit of Galois theory, is given at the end.

Now let θ be a complex Pisot number such that

- $\arg \theta \notin \pi\mathbb{Q}$
- $|\theta|$ lies in $(3,4)$;
- the minimal polynomial of θ has constant term 1 or -1.

For example, the polynomial $f(X) = X^3 + X^2 + 10X + 1$ has three roots,

$$\theta_0 \approx -0.45 + 3.11i, \overline{\theta_0} \approx -0.45 - 3.11i \text{ and } \alpha \approx -0.1.$$

Since f doesn't have a root in $\mathbb{Z}$, it follows from Gauss's lemma that f is irreducible over $\mathbb{Q}$. Hence f is the minimal polynomial of θ_0 over $\mathbb{Q}$, and

$$[\mathbb{Q}[\theta_0]:\mathbb{Q}] = \deg f = 3.$$

This shows that θ_0 is a complex Pisot number, and from Lemma (4.3.5) we get that $\arg\theta_0 \notin \pi\mathbb{Q}$. Since $3 < |\theta_0| < 4$ and the constant term of f is 1, the number θ_0 satisfies all of the required properties.

Write $\lambda = \theta^{-1}$ and note that λ may be thought of as a 2×2 matrix rR, where $r = |\lambda|$ and R is a planar rotation by angle $\arg\lambda$. From $\arg\lambda \notin \pi\mathbb{Q}$ it follows $R^n \neq Id$ for all $n \geq 1$. Let $\mathcal{V}$ be the set of all $(a_1, a_2) \in \mathbb{C}^2$ for which the IFS

$$\{z \to \lambda \cdot z + (-1)^k a_j : k, j \in \{1,2\}\}$$

satisfies the strong separation condition (SSC). Since $|\lambda| < \frac{1}{3}$, it is not hard to see that $\left(\frac{2}{3}, \frac{2i}{3}\right) \in \mathcal{V}$ and in particular that $\mathcal{V} \neq \emptyset$. The next lemma is proven at the end of this section.

Clearly $\mathcal{V}$ is open in $\mathbb{C}^2$, hence from $\mathcal{V} \neq \emptyset$ and Lemma (4.3.6) it follows that there exists

$$(a_1, a_2) \in \mathcal{V} \cap (\mathcal{Y} \times \mathcal{Y}).$$

For $k, j \in \{1,2\}$ and $z \in \mathbb{C}$ set

$$\varphi_{k,j}(z) = \lambda \cdot z + (-1)^k \cdot a_j,$$

then the IFS

$$\Phi := \{\varphi_{k,j} : k, j \in \{1,2\}\}$$

satisfies the SSC. Denote by $\mathcal{M}(\mathbb{C})$ the collection of all compactly supported Borel probability measures on $\mathbb{C}$. Let μ be the unique member of $\mathcal{M}(\mathbb{C})$ with

$$\mu = \frac{1}{4}\left(\varphi_{1,1}\mu + \varphi_{1,2}\mu + \varphi_{2,1}\mu + \varphi_{2,2}\mu\right),$$

then

$$\dim_H \mu = \frac{\log 4}{\log|\theta|} > 1.$$

Denote by $\langle \cdot,\cdot \rangle$ the Euclidean inner product on $\mathbb{C}$, i.e. $\langle z, w \rangle = \mathrm{Re}(z \cdot \bar{w})$ for $z, w \in \mathbb{C}$. Write

$$S = \{z \in \mathbb{C} : |z| = 1\},$$

and $P_z w = \langle w, z \rangle$ for $z \in S$ and $w \in \mathbb{C}$. Note that $P_z\mu$ is, up to affine equivalence, the pushforward of μ by the orthogonal projection onto the line $z \cdot \mathbb{R}$. The following proposition is proven.

Denote by $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}$ or $\mathbb{C}$. For $z \in S$ let $\{\mu_{z,w}\}_{w \in \mathbb{C}} \subset \mathcal{M}(\mathbb{C})$ be the disintegration of μ with respect to $P_z^{-1}(\mathcal{B})$, as defined in Theorem 5.14 in [273]. This means that $\mu_{z,w}$ is supported on $P_z^{-1}(P_z w)$ for $w \in \mathbb{C}$, and for each bounded $\mathcal{B}$-measurable $f : \mathbb{C} \to \mathbb{R}$

$$\int f \, d\mu_{z,w} = E_\mu\left(f \mid P_z^{-1}(\mathcal{B})\right)(w) \text{ for } \mu\text{-a.e. } w \in \mathbb{C}.$$

Here the right hand side is the conditional expectation of f given $P_z^{-1}(\mathcal{B})$ with respect to μ. In order to prove Theorem (4.3.1) it remains to establish the following proposition, which is done.

Lemma (4.3.5)[270]: Let θ be a complex Pisot number with $[\mathbb{Q}[\theta]:\mathbb{Q}] = 3$, then $\arg\theta \notin \pi\mathbb{Q}$.

Proof. By the assumptions on θ there exists $\alpha \in \mathbb{C}$, with $|\alpha| < 1$, such that $\bar{\theta}$ and α are the Galois conjugates of θ. Set $E = \mathbb{Q}[\theta, \bar{\theta}, \alpha]$, let $f \in \mathbb{Q}[X]$ be the minimal polynomial of θ over $\mathbb{Q}$, and let G be the Galois group of the field extension $E/\mathbb{Q}$. Note that E is a splitting field for f over $\mathbb{Q}$, and that the roots of f are $\theta, \bar{\theta}$ and α. It follows, by Lemma 18.3 in [281], that the action of G on $\{\theta, \bar{\theta}, \alpha\}$ induces an isomorphism from G into a subgroup of S_3, where S_3 is the symmetric group on 3 letters. It also follows, by Theorem 18.13 in [281], that the extension $E/\mathbb{Q}$ is Galois. Hence, from Corollary 18.19 and Lemma 17.6 in [281], we get

$$|G| = [E:\mathbb{Q}] = [E:\mathbb{Q}(\theta)] \cdot [\mathbb{Q}(\theta):\mathbb{Q}] = [E:\mathbb{Q}(\theta)] \cdot 3,$$

which shows that 3 divides $|G|$. Let $\sigma \in G$ be with $\sigma(\beta) = \bar{\beta}$ for $\beta \in E$, then σ has order 2. This implies that 2 divides $|G|$, and so it must hold that G is isomorphic to S_3. Now assume by contradiction that $\arg\theta \in \pi\mathbb{Q}$, then $\theta^n \in \mathbb{R}$ for some $n \geq 1$. Let $\tau \in G$ be such that $\tau(\theta) = \theta, \tau(\bar{\theta}) = \alpha$ and $\tau(\alpha) = \bar{\theta}$. Since τ and σ are distinct, both have order 2, and G is isomorphic to S_3, it follows that the group generated by τ and σ is G. Clearly $\tau(\theta^n) = \theta^n$ and from $\theta^n \in \mathbb{R}$ we get $\sigma(\theta^n) = \theta^n$, hence $\eta(\theta^n) = \theta^n$ for all $\eta \in G$. Let $\eta \in G$ be with $\eta(\theta) = \alpha$, then

$$\theta^n = \eta(\theta^n) = \eta(\theta)^n = \alpha^n.$$

But we also have $|\theta^n| > 1 > |\alpha^n|$, which yields a contradiction, and so it must holds that $\arg\theta \notin \pi\mathbb{Q}\cdot$

Lemma (4.3.6)[270]: The set $\mathcal{Y} := \{k \cdot \lambda^l : k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Proof. Let $z \in \mathbb{C}$ and $\epsilon > 0$ be given, and let $N \geq 1$ be with $|\lambda^N| < \epsilon$ Since

$$\arg\lambda = -\arg\theta \notin \pi\mathbb{Q},$$

we have that

$$\{l \cdot \arg\lambda \bmod 2\pi\}_{l=N}^{\infty}$$

is dense in $[0,2\pi)$. It follows there exists $l \geq N$ with

$$\left|\exp\left(i \cdot \arg(\lambda^l)\right) - \exp(i \cdot \arg z)\right| < \epsilon. \tag{74}$$

Let $k \geq 0$ be the integer with $k \cdot |\lambda^l| \leq |z| < (k+1) \cdot |\lambda^l|$, then

$$||z| - |k \cdot \lambda^l|| \leq |\lambda^N| < \epsilon.$$

From this, from $\arg(k \cdot \lambda^l) = \arg(\lambda^l)$, and from (74), the lemma follows.

Let $\theta, \lambda, (a_1, a_2), \Phi := \{\varphi_{k,j} : k, j \in \{1,2\}\}$ and μ be as obtained. We shall show that there exists a dense G_δ subset B of S, such that for every $z \in B$ the Fourier transform of $P_z\mu$ does not decay to 0 at infinity.

Lemma (4.3.7)[270]: There exist constants $\rho \in (0,1)$ and $C > 0$, with

$$\operatorname{dist}(2\operatorname{Re}(\theta^n), \mathbb{Z}) \leq C \cdot \rho^{|n|} \text{ for all } n \in \mathbb{Z}. \tag{75}$$

Proof. Let $\theta_3, \ldots, \theta_m$ be the Galois conjugates of θ other than $\bar{\theta}$. Since θ is an algebraic integer,

$$\theta^n + \overline{\theta^n} + \sum_{j=3}^{m} \theta_m^n \in \mathbb{Z} \text{ for all } n \in \mathbb{N}.$$

It follows that (75) holds for $n \in \mathbb{N}$ with

$$\rho = \max\ \{|\theta_j| : 3 \leq j \leq m\} \in (0,1)$$

and $C = m - 2$. Since $|\theta| > 1$ and for each integer $n < 0$

$$\operatorname{dist}(2\operatorname{Re}(\theta^n), \mathbb{Z}) \leq 2|\theta|^n,$$

the lemma follows.

Given $\nu \in \mathcal{M}(\mathbb{C})$ let $\mathcal{F}(\nu)$ be the Fourier transform of ν as a measure on $\mathbb{R}^2$, i.e. for $\xi \in \mathbb{C}$

$$\mathcal{F}(\nu)(\xi) = \int_C e^{i\langle z,\xi\rangle} d\nu(z) = \int_{\mathbb{C}} \exp\left(i \operatorname{Re}\left(z \cdot \bar{\xi}\right)\right) d\nu(z).$$

The proof of the following proposition resembles the argument given by Erdös in [274].

Proposition (4.3.8)[270]: There exists a constant $c > 0$ with $\left|\mathcal{F}(\mu)\left(4\pi\overline{\theta^N}\right)\right| > c$ for all $N \in \mathbb{N}$.

Proof. Let $X_1, X_2, \ldots$ be i.i.d. random variables with

$$\mathbb{P}\left(X_1 = (-1)^k a_j\right) = \frac{1}{4} \text{ for } k, j \in \{1,2\}.$$

Since μ is the unique Borel probability measure on $\mathbb{C}$ with

$$\mu = \frac{1}{4}\left(\varphi_{1,1}\mu + \varphi_{1,2}\mu + \varphi_{2,1}\mu + \varphi_{2,2}\mu\right),$$

it is equal to the distribution of the random sum $\sum_{n=0}^{\infty} \lambda^n \cdot X_n$. Hence for every $\xi \in \mathbb{C}$,

$$\begin{aligned} \mathcal{F}(\mu)(\xi) &= \prod_{n=0}^{\infty} \mathcal{F}\left(\frac{1}{4} \cdot \sum_{j=1}^{2} \left(\delta_{\lambda^n a_j} + \delta_{-\lambda^n a_j}\right)\right)(\xi) \\ &= \prod_{n=0}^{\infty} \frac{1}{4} \cdot \sum_{j=1}^{2} \left(\exp\left(i\operatorname{Re}\left(\lambda^n a_j \cdot \bar{\xi}\right)\right) + \exp\left(i\operatorname{Re}\left(-\lambda^n a_j \cdot \bar{\xi}\right)\right)\right) \\ &= \prod_{n=0}^{\infty} \frac{1}{2} \cdot \left(\cos\left(\operatorname{Re}\left(\lambda^n a_1 \cdot \bar{\xi}\right)\right) + \cos\left(\operatorname{Re}\left(\lambda^n a_2 \cdot \bar{\xi}\right)\right)\right). \end{aligned}$$

Since $a_1, a_2 \in \mathcal{Y}$, where $\mathcal{Y}$ is defined in Lemma (4.3.6), for $j = 1,2$ there exist $k_j, l_j \in \mathbb{N}$ with $a_j = k_j \cdot \theta^{-l_j}$. Hence for $N \in \mathbb{N}$,

$$\begin{aligned} \mathcal{F}(\mu)\left(4\pi\overline{\theta^N}\right) &= \prod_{n=-\infty}^{N} \frac{1}{2} \cdot (\cos(4\pi \operatorname{Re}(\theta^n a_1)) + \cos(4\pi \operatorname{Re}(\theta^n a_2))) \\ &= \prod_{n=-\infty}^{N} \frac{1}{2}(\cos(4\pi k_1 \cdot \operatorname{Re}(\theta^{n-l_1})) + \cos(4\pi k_2 \cdot \operatorname{Re}(\theta^{n-l_2}))). \end{aligned} \quad (76)$$

Let us show that $b_n \neq 0$ for every $n \in \mathbb{Z}$, where

$$b_n := \frac{1}{2}\left(\cos\left(4\pi k_1 \cdot \operatorname{Re}(\theta^{n-l_1})\right) + \cos\left(4\pi k_2 \cdot \operatorname{Re}(\theta^{n-l_2})\right)\right).$$

Recall that the set of algebraic integers is closed under addition, subtraction and multiplication. The product of θ with its Galois conjugates is equal to the constant term of the minimal polynomial of θ, which is ± 1 by assumption. These conjugates are all algebraic integers, hence θ^{-1} is an algebraic integer, and so θ^n is an algebraic integer for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$, then from the identity

$$\cos\beta + \cos\gamma = 2\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right) \text{ for all } \beta, \gamma \in \mathbb{R},$$

we obtain

$$b_n = \cos(2\pi \cdot \operatorname{Re}(k_1\theta^{n-l_1} + k_2\theta^{n-l_2})) \cdot \cos(2\pi \cdot \operatorname{Re}(k_1\theta^{n-l_1} - k_2\theta^{n-l_2})). \quad (77)$$

Since $2\operatorname{Re}(k_1\theta^{n-l_1} + k_2\theta^{n-l_2})$ is equal to

$$k_1\theta^{n-l_1} + k_2\theta^{n-l_2} + k_1\overline{\theta^{n-l_1}} + k_2\overline{\theta^{n-l_2}},$$

it is an algebraic integer, and so it can't be of the form $k+\frac{1}{2}$ with $k\in\mathbb{Z}$. It follows the first term in the product (77) is nonzero. In a similar manner the second term in (77) is nonzero, which shows $b_n\neq 0$.

Fix $n\in\mathbb{Z}$ and $j\in\{1,2\}$, and let $d\in\mathbb{Z}$ be with

$$\left|2\mathrm{Re}\left(\theta^{n-l_j}\right)-d\right|=\mathrm{dist}\left(2\,\mathrm{Re}\left(\theta^{n-l_j}\right),\mathbb{Z}\right).$$

Let C and ρ be the constants from Lemma (4.3.7), and write

$$C_0:=2\pi C\cdot\max\ \{k_1,k_2\}\cdot\rho^{-\max\{l_1,l_2\}}.$$

From Lemma (4.3.7),

$$\begin{aligned}&\left|\cos\left(4\pi k_j\cdot\mathrm{Re}\left(\theta^{n-l_j}\right)\right)-1\right|\\&\qquad=\left|\cos\left(4\pi k_j\cdot\mathrm{Re}\left(\theta^{n-l_j}\right)\right)-\cos\left(2\pi k_j d\right)\right|\\&\qquad\leq 2\pi k_j\cdot\left|2\mathrm{Re}\left(\theta^{n-l_j}\right)-d\right|=2\pi k_j\cdot\mathrm{dist}\left(2\mathrm{Re}\left(\theta^{n-l_j}\right),\mathbb{Z}\right)\\&\qquad\leq 2\pi k_j C\cdot\rho^{|n-l_j|}\leq C_0\cdot\rho^{|n|}.\end{aligned}$$

This shows

$$|b_n|\geq 1-\frac{1}{2}\sum_{j=1}^{2}\left|\cos\left(4\pi k_j\cdot\mathrm{Re}\left(\theta^{n-l_j}\right)\right)-1\right|\geq 1-C_0\cdot\rho^{|n|}.$$

Now let $M\geq 1$ be such that $C_0\cdot\rho^{|n|}<1$ for all $n\in\mathbb{Z}$ with $|n|\geq M$. Then from (76) it follows that for each $N\geq 0$,

$$\begin{aligned}\left|\mathcal{F}(\mu)\left(4\pi\overline{\theta^N}\right)\right|&\geq\prod_{n=-\infty}^{-M}|b_n|\prod_{n=1-M}^{M-1}|b_n|\prod_{n=M}^{\infty}|b_n|\\&\geq\prod_{n=M}^{\infty}\left(1-C_0\cdot\rho^n\right)^2\cdot\prod_{n=1-M}^{M-1}|b_n|>0,\end{aligned}$$

which completes the proof.

Let $\mathcal{M}(\mathbb{R})$ be the collection of all compactly supported Borel probability measures on $\mathbb{R}$. Given $\nu\in\mathcal{M}(\mathbb{R})$ let $\mathcal{F}(\nu)$ be the Fourier transform of ν, i.e.

$$\mathcal{F}(\nu)(r)=\int_{\mathbb{R}}e^{ixr}d\nu(x)\text{ for }r\in\mathbb{R}.$$

Recall that

$$S=\{z\in\mathbb{C}:|z|=1\},$$

and $P_z w=\langle w,z\rangle$ for $z\in S$ and $w\in\mathbb{C}$. For $n\geq 1$ write

$$U_n=\left\{z\in S:\sup_{r\geq n}|\mathcal{F}(P_z\mu)(r)|>c\right\},$$

where c is the constant from Proposition (4.3.8).

Lemma (4.3.9)[270]: Let $n\geq 1$, then U_n is an open and dense subset of S.

Proof. Note that for $z\in S$ and $r\in\mathbb{R}$

$$\mathcal{F}(P_z\mu)(r)=\int_{\mathbb{R}}\exp(ixr)\,dP_z\mu(x)=\int_{\mathbb{R}}\exp(i\langle w,rz\rangle)\,d\mu(w)=\mathcal{F}(\mu)(rz),$$

hence

$$U_n=\left\{z\in S:\sup_{r\geq n}|\mathcal{F}(\mu)(rz)|>c\right\}.$$

Now since $\mathcal{F}(\mu)$ is continuous it follows U_n is open in S. Set $\eta=\exp(-i\arg\theta)$, then from Proposition (4.3.8)

$$\left|\mathcal{F}\left(P_{\eta^k}\mu\right)\left(4\pi|\theta|^k\right)\right| = \left|\mathcal{F}(\mu)\left(4\pi\overline{\theta^k}\right)\right| > c \tag{78}$$

for every integer $k \geq 0$. Let $N \geq 1$ be with $|4\pi\theta^N| \geq n$, then $\{\eta^k\}_{k=N}^{\infty} \subset U_n$ by (78). By assumption $\arg\theta \notin \pi\mathbb{Q}$, hence $\{\eta^k\}_{k=N}^{\infty}$ is dense in S, which proves the lemma.

We can now complete the proof of Proposition (4.3.10).

Proposition (4.3.10)[270]: There exists a dense G_δ subset B of S, such that $P_z\mu$ is singular for all $z \in B$.

Proof. Set $B = \cap_{n=1}^{\infty} U_n$, then B is a dense G_δ subset of S by Lemma (4.3.9) and Baire's theorem. Let $z \in B$, then $\mathcal{F}(P_z\mu)(r)$ does not tend to 0 as $r \to \infty$. Hence, by the Riemann-Lebesgue lemma, $P_z\mu$ is not absolutely continuous. From the law of pure types (see Theorem 3.26 in [272]) it now follows $P_z\mu$ is singular, which completes the proof of the Proposition.

In order to prove Proposition (4.3.16) we shall use the following theorem due to Wiener (see Section VI.2.12 of [283]).

Theorem (4.3.11)[270]: For every $\nu \in \mathcal{M}(\mathbb{R})$,

$$\sum_{x\in\mathbb{R}} (\nu\{x\})^2 = \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} |\mathcal{F}(\nu)(\xi)|^2 d\xi.$$

Let θ, Φ and μ be as above. For $z \in S$ write $z^\perp = e^{-i\pi/2}z$. Given $n, k \in \mathbb{N}$ set

$$J_{n,k} = \left\{z \in S: \left\langle 4\pi\overline{\theta^k}, z^\perp\right\rangle \in (n, n+1)\right\},$$

and let $V_n = \cup_{k\in\mathbb{N}} J_{n,k}$.

Lemma (4.3.12)[270]: Let $n \in \mathbb{N}$, then V_n is a dense open subset of S.

Proof. Since $J_{n,k}$ is open in S for every $k \in \mathbb{N}$ the same holds for V_n. Let $a \in \mathbb{R}$ and $0 < \epsilon < 1$ be given. For $E \subset \mathbb{R}$ write $q(E) = E + 2\pi\mathbb{Z}$. Since $|\theta| > 1$ and $\arg\theta \notin \pi\mathbb{Q}$, there exists $k \in \mathbb{N}$ with

$$\left|4\pi\overline{\theta^k}\right| > \frac{n+1}{\cos\left(\frac{\pi}{2}-\epsilon\right)} \text{ and } \arg\left(4\pi\overline{\theta^k}\right) \in q(a-\epsilon, a+\epsilon).$$

Set $w = 4\pi\overline{\theta^k}$ and for $t \in \mathbb{R}$ write $f(t) = \left\langle w, e^{it}\right\rangle$. It holds that

$$\begin{aligned} f(\arg w^\perp) &= \left\langle w, \frac{w^\perp}{|w^\perp|}\right\rangle = 0, \\ f(\arg w^\perp + \epsilon) &= f\left(\arg w - \frac{\pi}{2} + \epsilon\right) = \left\langle w, \frac{w}{|w|}\cdot e^{i(\epsilon-\pi/2)}\right\rangle \\ &= \mathrm{Re}\left(w \cdot \frac{\bar{w}}{|w|}\cdot e^{i\left(\frac{\pi}{2}-\epsilon\right)}\right) = |w|\cdot\cos\left(\frac{\pi}{2}-\epsilon\right) > n+1, \end{aligned}$$

and

$$[\arg w^\perp, \arg w^\perp + \epsilon] \subset q\left[a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}\right].$$

Hence, since f is continuous and 2π-periodic, there exists $t \in \left[a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}\right]$ with $f(t) \in (n, n+1)$. Set $z = \exp\left(i\left(t + \frac{\pi}{2}\right)\right)$, then

$$\langle w, z^\perp\rangle = \left\langle w, e^{it}\right\rangle = f(t) \in (n, n+1),$$

and so $z \in J_{n,k} \subset V_n$. Now since a and ϵ are arbitrary and

$$\arg z \in q\left\{t + \frac{\pi}{2}\right\} \subset q[a - \epsilon, a + 2\epsilon],$$

it follows that V_n is dense in S, which proves the lemma.

Set $B = \cap_{n\in\mathbb{N}} V_n$, then B is a dense G_δ subset of S by Lemma (4.3.12). Fix $z \in B$ and recall that $\{\mu_{z,w}\}_{w\in\mathbb{C}}$ is the disintegration of μ with respect to $P_z^{-1}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{C}$. In order to prove the proposition, it suffices to show that $\mu_{z,w}$ is discrete for μ-a.e. $w \in \mathbb{C}$. Write

$$\tau_w(\xi) = \xi - w \text{ and } R\xi = \overline{z^\perp} \cdot \xi \text{ for } w, \xi \in \mathbb{C}, \tag{79}$$

for each $w \in \mathbb{C}$ let $\nu_w = R\tau_w\mu_{z,w}$, and note that $\nu_w \in \mathcal{M}(\mathbb{R})$.

Lemma (4.3.13)[270]: Let $c > 0$ be the constant from Proposition (4.3.8), then for each $n \in \mathbb{N}$ there exists $t_n \in (n, n+1)$ with

$$\int |\mathcal{F}(\nu_w)(t_n)|^2 d\mu(w) > c^2. \tag{80}$$

Proof. Let $n \in \mathbb{N}$. Since $z \in \underline{V_n}$ there exists $k_n \in \mathbb{N}$ and $t_n \in (n, n+1)$ with $\langle 4\pi\overline{\theta^{k_n}}, z^\perp\rangle = t_n$. Write $\eta = 4\pi\overline{\theta^{k_n}}$, then by Proposition (4.3.8).

$$c < |\mathcal{F}(\mu)(\eta)| \leq \int \left|\int e^{i\langle\xi,\eta\rangle} d\mu_{z,w}(\xi)\right| d\mu(w). \tag{81}$$

Let $Q_{z^\perp}$ be the orthogonal projection onto $z^\perp\mathbb{R}$, i.e.

$$Q_{z^\perp}\xi = \langle\xi, z^\perp\rangle z^\perp \text{ for } \xi \in \mathbb{C}.$$

From (81) and since $\tau_w\mu_{z,w}$ is supported on $z^\perp\mathbb{R}$ for $w \in \mathbb{C}$,

$$\begin{aligned} c \quad &< \int \left|\int e^{i\langle\xi+w,\eta\rangle} d\tau_w\mu_{z,w}(\xi)\right| d\mu(w) \\ &= \int \left|e^{i\langle w,\eta\rangle}\right| \cdot \left|\int e^{i\langle\xi,\eta\rangle} dQ_z \perp \tau_w\mu_{z,w}(\xi)\right| d\mu(w) \\ &= \int \left|\int \exp(i\langle Q_{z^\perp}\xi, \eta\rangle) d\tau_w\mu_{z,w}(\xi)\right| d\mu(w). \end{aligned}$$

Now since $Q_{z^\perp}$ is self-adjoint, $\langle\eta, z^\perp\rangle$ is equal to t_n, and R from (79) is a rotation,

$$\begin{aligned} c \quad &< \int \left|\int \exp(i\langle\xi, t_n z^\perp\rangle) d\tau_w\mu_{z,w}(\xi)\right| d\mu(w) \\ &= \int \left|\int \exp(i\langle R\xi, t_n \cdot Rz^\perp\rangle) d\tau_w\mu_{z,w}(\xi)\right| d\mu(w) \\ &= \int \left|\int e^{i\xi t_n} d\nu_w(\xi)\right| d\mu(w) = \int |\mathcal{F}(\nu_w)(t_n)| d\mu(w). \end{aligned}$$

From this and Jensen's inequality the lemma follows.

Let us define the set,

$$E_z = \{w \in \mathbb{C}: \mu_{z,w}\{w\} > 0\}. \tag{82}$$

Lemma (4.3.14)[270]: It holds that $\mu(E_z) > 0$.

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be the numbers obtained in Lemma (4.3.13). Since $\text{supp}(\mu)$ is compact and

$$\text{supp}(\mu_{z,w}) \subset \text{supp}(\mu) \text{ for } \mu\text{-a.e. } w \in \mathbb{C},$$

there exists $M > 0$ such that ν_w is supported on $J = [-M, M]$ for μ-a.e. $w \in \mathbb{C}$. Write

$$\mathcal{M}(J) = \{\nu \in \mathcal{M}(\mathbb{R}): \nu \text{ is supported on } J\},$$

then it is easy to see that $\mathcal{F}(\nu)$ is M-Lipschitz for $\nu \in \mathcal{M}(J)$. Hence there exist $\delta > 0$ and intervals $\{A_n\}_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$ it holds

$$t_n \in A_n \subset (n, n+1),$$

A_n has length δ, and for each $\nu \in \mathcal{M}(J)$,

$$||\mathcal{F}(\nu)(t_n)|^2 - |\mathcal{F}(\nu)(x)|^2| < \frac{c^2}{2} \text{ for } x \in A_n.$$

We now get from (80) that for each $N \geq 1$,

$$\begin{aligned} c^2 &\leq \int \frac{1}{N}\sum_{n=0}^{N-1} |\mathcal{F}(\nu_w)(t_n)|^2 d\mu(w) \\ &= \int \frac{1}{\delta N}\sum_{n=0}^{N-1} \int_{A_n} |\mathcal{F}(\nu_w)(t_n)|^2 dx d\mu(w) \\ &\leq \int \frac{1}{\delta N}\sum_{n=0}^{N-1} \int_{A_n} |\mathcal{F}(\nu_w)(x)|^2 + \frac{c^2}{2} dx d\mu(w) \\ &\leq \int \frac{1}{\delta N}\int_{-N}^{N} |\mathcal{F}(\nu_w)(x)|^2 dx d\mu(w) + \frac{c^2}{2}, \end{aligned}$$

which gives

$$\frac{\delta c^2}{4} \leq \int \frac{1}{2N}\int_{-N}^{N} |\mathcal{F}(\nu_w)(x)|^2 dx d\mu(w).$$

Now by Theorem (4.3.11) and the bounded convergence theorem,

$$\begin{aligned} \int \sum_{\xi \in \mathbb{C}} \mu_{z,w}\{\xi\} d\mu(w) &= \int \sum_{x \in \mathbb{R}} \nu_w\{x\} d\mu(w) \\ &= \int \lim_{N\to\infty} \frac{1}{2N}\int_{-N}^{N} |\mathcal{F}(\nu_w)(\xi)|^2 d\xi d\mu(w) \\ &= \lim_{N\to\infty} \int \frac{1}{2N}\int_{-N}^{N} |\mathcal{F}(\nu_w)(\xi)|^2 d\xi d\mu(w) \geq \frac{\delta c^2}{4} > 0. \end{aligned}$$

This gives $\mu(F_z) > 0$, where

$$F_z = \{w \in \mathbb{C}: \mu_{z,w}\{\xi\} > 0 \text{ for some } \xi \in \mathbb{C}\}.$$

Let $w \in F_z$, then there exists

$$\xi \in \operatorname{supp}(\mu_{z,w}) \subset w + z^{\perp}\mathbb{R}$$

with $\mu_{z,w}\{\xi\} > 0$. Since $\mu_{z,\xi} = \mu_{z,w}$ it follows $\xi \in E_z$, where E_z is defined in (82), and so

$$\mu_{z,w}(E_z) \geq \mu_{z,w}\{\xi\} > 0.$$

Now from $\mu(F_z) > 0$ we get

$$\mu(E_z) \geq \int_{F_z} \mu_{z,w}(E_z) d\mu(w) > 0,$$

which proves the lemma.

Write $I = \{1,2\}^2$ and let $\Phi = \{\varphi_i\}_{i\in I}$ be the IFS constructed. Recall that $\lambda = \theta^{-1}$, for each $i \in I$ there exists $a_i \in \mathbb{C}$ with $\varphi(w) = \lambda w + a_i$ for $w \in \mathbb{C}$, and Φ satisfies the SSC. Let $K \subset \mathbb{C}$ be attractor of Φ, write $\mathcal{B}_K$ for the restriction of the Borel σ-algebra $\mathcal{B}$ to K, and let $T: K \to K$ be such that $Tx = \varphi_i^{-1}(x)$ for $i \in I$ and $x \in \varphi_i(K)$.

Lemma (4.3.15)[270]: It holds that $\mu(E_z) = 1$.

Proof. Given $A_1, A_2 \in \mathcal{B}$ with $\mu(A_1 \Delta A_2) = 0$ we write $A_1 = A_2 \operatorname{mod}\mu$. For a σ-algebra $\mathcal{F} \subset \mathcal{B}$ and $A_1 \in \mathcal{B}$ we write $A_1 \in \mathcal{F}\operatorname{mod}\mu$ whenever there exists $A_2 \in \mathcal{F}$ with $A_1 = A_2 \operatorname{mod}\mu$. The system $(K, \mathcal{B}_K, T, \mu)$ is measure preserving and isomorphic to a Bernoulli shift. We shall show that

$$E_z \in \bigcap_{n=0}^{\infty} T^{-n}(\mathcal{B}_K) \qquad \text{mod } \mu,$$

from which the lemma will follow by the zero-one law.

Given a word $i_1 \cdot ... \cdot i_n = \alpha \in I^*$ write $\varphi_\alpha = \varphi_{i_1} \circ ... \circ \varphi_{i_n}$ and $K_\alpha = \varphi_\alpha(K)$. For $n \in \mathbb{N}$ and $w \in K$ let $\alpha_n(w) \in I^n$ be the unique word of length n for which $w \in K_{\alpha_n(w)}$, where $\alpha_0(w)$ is the empty word $\emptyset$ and $K_\emptyset = K$. For $m, n \in \mathbb{N}$ and $w \in K$ set

$$z_m = \frac{\theta^m z}{|\theta^m z|} \in S \text{ and } F_{m,n}(w) = \mu_{z_m,w}\big(K_{\alpha_n(w)}\big).$$

For $m \in \mathbb{N}, w \in \mathbb{C}$ and $\delta > 0$ let

$$V_w^m(\delta) = w + z_m^\perp \cdot \mathbb{R} + B(0,\delta),$$

where $B(0,\delta)$ is the open disk in $\mathbb{C}$ with centre 0 and radius δ. From Lemma (4.3.9) in [277] we get that for each $m \in \mathbb{N}$ and $A \in \mathcal{B}$,

$$\mu_{z_m,w}(A) = \lim_{\delta \downarrow 0} \frac{\mu(V_w^m(\delta) \cap A)}{\mu\big(V_w^m(\delta)\big)} \text{ for } \mu\text{-a.e. } w \in \mathbb{C}.$$

Fix $m, n \in \mathbb{N}$, then for μ-a.e. $w \in K$

$$F_{m,n}(T^m w) = \frac{\mu_{z_m,T_w^m}\big(K_{\alpha_n(T^m w)}\big)}{\mu_{z_m,T^m w}\big(K_{\alpha_0(T^m w)}\big)} = \lim_{\delta \downarrow 0} \frac{\mu\Big(V_{T^m w}^m(\delta) \cap K_{\alpha_n(T^m w)}\Big)}{\mu\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)}. \tag{83}$$

Since μ satisfies the SSC,

$$\varphi_\alpha(K) \cap \varphi_{\alpha_m(w)}(K) = \emptyset \text{ for } \alpha \in I^m \setminus \{\alpha_m(w)\},$$

hence,

$$\mu\big(\varphi_\alpha^{-1}\big(\varphi_{\alpha_m(w)}(K)\big)\big) = 0 \text{ for } \alpha \in I^m \setminus \{\alpha_m(w)\}.$$

From this and (83) it follows that for μ-a.e. $w \in K$,

$$F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\sum_{\alpha \in I^m} \ |I|^{-m} \cdot \mu\Big(\varphi_\alpha^{-1}\Big(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_n(T^m w)}\big)\Big)\Big)}{\sum_{\alpha \in I^m} \ |I|^{-m} \cdot \mu\Big(\varphi_\alpha^{-1}\Big(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)\Big)\Big)}.$$

Now since $\mu = \sum_{\alpha \in I^m} |I|^{-m} \cdot \varphi_\alpha \mu$,

$$F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu\Big(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_n(T^m w)}\big)\Big)}{\mu\Big(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)\Big)} \text{ for } \mu\text{-a.e. } w \in K. \tag{84}$$

For every $\delta > 0$,

$$\begin{aligned}\varphi_{\alpha_m(w)}(V_{T^m w}^{mn}(\delta)) \quad &= \varphi_{\alpha_m(w)}(T^m w) + \lambda^m z_m^\perp \cdot \mathbb{R} + \lambda^m \cdot B(0,\delta) \\ &= w + z_0^\perp \cdot \mathbb{R} + B(0,|\delta\lambda^m|) = V_w^0(|\delta\lambda^m|).\end{aligned}$$

Hence from (84) it follows that for μ-a.e. $w \in K$,

$$F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu\big(V_w^0(|\delta\lambda^m|) \cap K_{\alpha_{m+n}(w)}\big)}{\mu\big(V_w^0(|\delta\lambda^m|) \cap K_{\alpha_m(w)}\big)} = \frac{\mu_{z,w}\big(K_{\alpha_{m+n}(w)}\big)}{\mu_{z,w}\big(K_{\alpha_m(w)}\big)}, \tag{85}$$

where we have used the fact that $\mu_{z,w}\big(K_{\alpha_m(w)}\big) > 0$ for μ-a.e. $w \in K$.

Let $m \in \mathbb{N}$, then from (85) we get that modμ it holds

$$\begin{aligned}E_z \quad &= \left\{w \in K : \lim_n \frac{\mu_{z,w}\big(K_{\alpha_{m+n}(w)}\big)}{\mu_{z,w}\big(K_{\alpha_m(w)}\big)} > 0\right\} \\ &= \left\{w \in K : \lim_n F_{m,n}(T^m w) > 0\right\} \in T^{-m}(\mathcal{B}_K),\end{aligned}$$

which shows

$$E_z \in \cap_{m\in\mathbb{N}} T^{-m}(\mathcal{B}_K) \qquad \text{mod} \quad \mu.$$

Now since $(K, \mathcal{B}_K, T, \mu)$ is isomorphic to a Bernoulli shift, it follows that $\mu(E_z) = 0$ or 1. But by Lemma (4.3.14) we have $\mu(E_z) > 0$, which completes the proof of the lemma.

We can now complete the proof of Proposition (4.3.16).

Proposition (4.3.16)[270]: There exists a dense G_δ subset B of S, such that for each $z \in B$ it holds that $\mu_{z,w}$ is discrete for μ-a.e. $w \in \mathbb{C}$.

Proof. As mentioned above, it suffices to show $\mu_{z,w}$ is discrete for μ-a.e. $w \in \mathbb{C}$. By Lemma (4.3.15) we have $\mu(E_z) = 1$, and so $\mu_{z,w}(E_z) = 1$ for μ-a.e. $w \in \mathbb{C}$. Fix such a $w \in \mathbb{C}$ and let $A = w + z^\perp\mathbb{R}$. Since $\mu_{z,w}(A) = 1$ and $\mu_{z,w} = \mu_{z,\xi}$ for $\xi \in A$,

$$1 = \mu_{z,w}(E_z \cap A) = \mu_{z,w}\{\xi \in A: \mu_{z,w}\{\xi\} > 0\}.$$

This shows that $\mu_{z,w}$ is discrete, which completes the proof.

Corollary (4.3.17)[439]: Let θ_j be a complex Pisot number with $[\mathbb{Q}[\theta_j]:\mathbb{Q}] = 3$, then $\arg\theta_j \notin \pi\mathbb{Q}$.

Proof. By the assumptions on θ_j there exists $\alpha \in \mathbb{C}$, with $|\alpha| < 1$, such that $\bar{\theta}_j$ and α are the Galois conjugates of θ_j. Set $E = \mathbb{Q}[\theta_j, \bar{\theta}_j, \alpha]$, let $f_j \in \mathbb{Q}[X]$ be the minimal polynomial of θ_j over $\mathbb{Q}$, and let G be the Galois group of the field extension $E/\mathbb{Q}$. Note that E is a splitting field for f_j over $\mathbb{Q}$, and that the roots of f_j are $\theta_j, \bar{\theta}_j$ and α. It follows, by Lemma 18.3 in [281], that the action of G on $\{\theta_j, \bar{\theta}_j, \alpha\}$ induces an isomorphism from G into a subgroup of S_3, where S_3 is the symmetric group on 3 letters. It also follows, by Theorem 18.13 in [281], that the extension $E/\mathbb{Q}$ is Galois. Hence, from Corollary 18.19 and Lemma 17.6 in [281], we get

$$|G| = [E:\mathbb{Q}] = [E:\mathbb{Q}(\theta_j)] \cdot [\mathbb{Q}(\theta_j):\mathbb{Q}] = [E:\mathbb{Q}(\theta_j)] \cdot 3,$$

which shows that 3 divides $|G|$. Let $\sigma \in G$ be with $\sigma(\beta) = \bar{\beta}$ for $\beta \in E$, then σ has order 2. This implies that 2 divides $|G|$, and so it must hold that G is isomorphic to S_3. Now assume by contradiction that $\arg\theta_j \in \pi\mathbb{Q}$, then $\theta_j^{1+\epsilon} \in \mathbb{R}$ for some $\epsilon \geq 0$. Let $\tau \in G$ be such that $\tau(\theta_j) = \theta_j, \tau(\bar{\theta}_j) = \alpha$ and $\tau(\alpha) = \bar{\theta}_j$. Since τ and σ are distinct, both have order 2, and G is isomorphic to S_3, it follows that the group generated by τ and σ is G. Clearly $\tau(\theta_j^{1+\epsilon}) = \theta_j^{1+\epsilon}$ and from $\theta_j^{1+\epsilon} \in \mathbb{R}$ we get $\sigma(\theta_j^{1+\epsilon}) = \theta_j^{1+\epsilon}$, hence $\eta(\theta_j^{1+\epsilon}) = \theta_j^{1+\epsilon}$ for all $\eta \in G$.Let $\eta \in G$ be with $\eta(\theta_j) = \alpha$, then

$$\theta_j^{1+\epsilon} = \eta(\theta_j^{1+\epsilon}) = \eta(\theta_j)^{1+\epsilon} = \alpha^{1+\epsilon}.$$

But we also have $|\theta_j^{1+\epsilon}| > 1 > |\alpha^{1+\epsilon}|$, which yields a contradiction, and so it must holds that $\arg\theta_j \notin \pi\mathbb{Q}$·

Corollary (4.3.18)[439]: The set $\mathcal{Y} := \{k \cdot \lambda_j^l : k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Proof. Let $z \in \mathbb{C}$ and $\epsilon > 0$ be given, and let $N \geq 1$ be with $|\lambda_j^N| < \epsilon$. Since

$$\arg\lambda_j = -\arg\theta_j \notin \pi\mathbb{Q},$$

we have that

$$\{l \cdot \arg\lambda_j \bmod 2\pi\}_{l=N}^{\infty}$$

is dense in $[0,2\pi)$. It follows there exists $l \geq N$ with

$$\left|\exp\left(i \cdot \arg(\lambda_j^l)\right) - \exp(i \cdot \arg z)\right| < \epsilon.$$

Let $k \geq 0$ be the integer with $k \cdot |\lambda_j^l| \leq |z| < (k+1) \cdot |\lambda_j^l|$, then

$$\left||z| - |k \cdot \lambda_j^l|\right| \leq |\lambda_j^N| < \epsilon.$$

From this, from $\arg(k \cdot \lambda_j^l) = \arg(\lambda_j^l)$, and from (74), the lemma follows.

Corollary (4.3.19)[439]: There exists a dense G_δ subset B of S, such that $P_z^2\mu_j$ is singular for all $z \in B$.

Proof. Set $B = \cap_{\epsilon=0}^{\infty} U_{1+\epsilon}$, then B is a dense G_δ subset of S by Lemma (4.3.9) and Baire's theorem. Let $z \in B$, then $\mathcal{F}(P_z^2\mu_j)(1+\epsilon)$ does not tend to 0 as $\epsilon \to \infty$. Hence, by the Riemann-Lebesgue lemma, $P_z^2\mu_j$ is not absolutely continuous. From the law of pure types (see Theorem 3.26 in [272]) it now follows $P_z^2\mu_j$ is singular, which completes the proof of the Proposition.

Corollary (4.3.20)[439]: There exists a dense G_δ subset B of S, such that for each $z \in B$ it holds that $(\mu_j)_{z,w}$ is discrete for μ_j-a.e. $w \in \mathbb{C}$.

Proof. As mentioned above, it suffices to show $(\mu_j)_{z,w}$ is discrete for μ_j-a.e. $w \in \mathbb{C}$. By Lemma (4.3.15) we have $\mu_j(E_z) = 1$, and so $(\mu_j)_{z,w}(E_z) = 1$ for μ_j-a.e. $w \in \mathbb{C}$. Fix such a $w \in \mathbb{C}$ and let $A = w + z^\perp\mathbb{R}$. Since $(\mu_j)_{z,w}(A) = 1$ and $(\mu_j)_{z,w} = (\mu_j)_{z,\xi}$ for $\xi \in A$,

$$1 = (\mu_j)_{z,w}(E_z \cap A) = (\mu_j)_{z,w}\{\xi \in A : (\mu_j)_{z,w}\{\xi\} > 0\}.$$

This shows that $(\mu_j)_{z,w}$ is discrete, which completes the proof.

Corollary (4.3.21)[439]: There exist constants $\rho \in (0,1)$ and $\epsilon \geq 0$, with

$$\operatorname{dist}\left(2\operatorname{Re}\left(\theta_j^{1+\epsilon}\right), \mathbb{Z}\right) \leq (1+\epsilon) \cdot \rho^{|1+\epsilon|} \text{ for all } (1+\epsilon) \in \mathbb{Z}.$$

Proof. Let $(\theta_j)_3, \dots, (\theta_j)_m$ be the Galois conjugates of θ_j other than $\bar{\theta}_j$. Since θ_j is an algebraic integer,

$$\theta_j^{1+\epsilon} + \bar{\theta}_j^{1+\epsilon} + \sum_{j_0=3}^{m} \left(\theta_j^{1+\epsilon}\right)_m \in \mathbb{Z} \text{ for all } (1+\epsilon) \in \mathbb{N}.$$

It follows that (75) holds for $(1+\epsilon) \in \mathbb{N}$ with

$$\rho = \max\ \left\{\left|(\theta_j)_{j_0}\right| : 3 \leq j_0 \leq m\right\} \in (0,1)$$

and $\epsilon = m - 3$. Since $|\theta_j| > 1$ and for each integer $\epsilon \geq 0$

$$\operatorname{dist}\left(2\operatorname{Re}\left(\theta_j^{1-\epsilon}\right), \mathbb{Z}\right) \leq 2|\theta_j|^{1-\epsilon},$$

the lemma follows.

Corollary (4.3.22)[439]: There exists a constant $\epsilon \geq 0$ with $\left|\mathcal{F}(\mu_j)\left(4\pi\bar{\theta}_j^N\right)\right| > 1 + \epsilon$ for all $N \in \mathbb{N}$.

Proof. Let $X_1, X_2, \ldots$ be i.i.d. random variables with

$$\mathbb{P}\left(X_1 = (-1)^k a_j\right) = \frac{1}{4} \text{ for } k, j \in \{1,2\}.$$

Since μ_j is the unique Borel probability measure on $\mathbb{C}$ with

$$\mu_j = \frac{1}{4}\left(\varphi_{1,1}\mu_j + \varphi_{1,2}\mu_j + \varphi_{2,1}\mu_j + \varphi_{2,2}\mu_j\right),$$

it is equal to the distribution of the random sum $\sum_{\epsilon=1}^{\infty} \lambda_j^{1-\epsilon} \cdot X_{1-\epsilon}$. Hence for every $\xi \in \mathbb{C}$,

$$\mathcal{F}(\mu_j)(\xi) = \prod_{\epsilon=1}^{\infty} \mathcal{F}\left(\frac{1}{4} \cdot \sum_{j=1}^{2} \left(\delta_{\lambda_j^{1-\epsilon}a_j} + \delta_{-\lambda_j^{1-\epsilon}a_j}\right)\right)(\xi)$$

$$= \prod_{\epsilon=1}^{\infty} \frac{1}{4} \cdot \sum_{j=1}^{2} \left(\exp\left(i\operatorname{Re}\left(\lambda_j^{1-\epsilon}a_j \cdot \bar{\xi}\right)\right) + \exp\left(i\operatorname{Re}\left(-\lambda_j^{1-\epsilon}a_j \cdot \bar{\xi}\right)\right)\right)$$

$$= \prod_{\epsilon=-1}^{\infty} \frac{1}{2} \cdot \left(\cos\left(\mathrm{Re}\left(\lambda_j^{1-\epsilon} a_1 \cdot \bar{\xi}\right)\right) + \cos\left(\mathrm{Re}\left(\lambda_j^{1-\epsilon} a_2 \cdot \bar{\xi}\right)\right)\right).$$

Since $a_1, a_2 \in \mathcal{Y}$, where $\mathcal{Y}$ is defined in Lemma (4.3.6), for $j = 1,2$ there exist $k_j, l_j \in \mathbb{N}$ with $a_j = k_j \cdot \theta_j^{-l_j}$. Hence for $N \in \mathbb{N}$,

$$\mathcal{F}(\mu_j)(4\pi\bar{\theta}_j^N) \quad = \prod_{\epsilon=-\infty}^{N} \frac{1}{2} \cdot \left(\cos\left(4\pi \,\mathrm{Re}\left(\theta_j^{1-\epsilon} a_1\right)\right) + \cos\left(4\pi \,\mathrm{Re}\left(\theta_j^{1-\epsilon} a_2\right)\right)\right)$$

$$= \prod_{\epsilon=-\infty}^{N} \frac{1}{2}\left(\cos\left(4\pi k_1 \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_1}\right)\right) + \cos\left(4\pi k_2 \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_2}\right)\right)\right).$$

Let us show that $b_{1-\epsilon} \neq 0$ for every $(1-\epsilon) \in \mathbb{Z}$, where

$$b_{1-\epsilon} := \frac{1}{2}\left(\cos\left(4\pi k_1 \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_1}\right)\right) + \cos\left(4\pi k_2 \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_2}\right)\right)\right).$$

Recall that the set of algebraic integers is closed under addition, subtraction and multiplication. The product of θ_j with its Galois conjugates is equal to the constant term of the minimal polynomial of θ_j, which is ± 1 by assumption. These conjugates are all algebraic integers, hence θ_j^{-1} is an algebraic integer, and so $\theta_j^{1-\epsilon}$ is an algebraic integer for all $(1-\epsilon) \in \mathbb{Z}$. Let $(1-\epsilon) \in \mathbb{Z}$, then from the identity

$$\cos\beta + \cos\gamma = 2\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right) \text{ for all } \beta, \gamma \in \mathbb{R},$$

we obtain

$$b_{1-\epsilon} = \cos\left(2\pi \cdot \mathrm{Re}\left(k_1\theta_j^{1-\epsilon-l_1} + k_2\theta_j^{1-\epsilon-l_2}\right)\right) \cdot \cos\left(2\pi \cdot \mathrm{Re}\left(k_1\theta_j^{1-\epsilon-l_1} - k_2\theta_j^{1-\epsilon-l_2}\right)\right).$$

Since $2\mathrm{Re}\left(k_1\theta_j^{1-\epsilon-l_1} + k_2\theta_j^{1-\epsilon-l_2}\right)$ is equal to

$$k_1\theta_j^{1-\epsilon-l_1} + k_2\theta_j^{1-\epsilon-l_2} + k_1\bar{\theta}_j^{1-\epsilon-l_1} + k_2\bar{\theta}_j^{\,1-\epsilon-l_2},$$

it is an algebraic integer, and so it can't be of the form $k + \frac{1}{2}$ with $k \in \mathbb{Z}$. It follows the first term in the product (77) is nonzero. In a similar manner the second term in (77) is nonzero, which shows $b_{1-\epsilon} \neq 0$.

Fix $(1-\epsilon) \in \mathbb{Z}$ and $j \in \{1,2\}$, and let $d \in \mathbb{Z}$ be with

$$\left|2\mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right) - d\right| = \mathrm{dist}\left(2\,\mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right), \mathbb{Z}\right).$$

Let $1+\epsilon$ and ρ be the constants from Lemma (4.3.7), and write

$$C_0 := 2\pi(1+\epsilon) \cdot \max\ \{k_1, k_2\} \cdot \rho^{-\max\{l_1, l_2\}}.$$

From Lemma (4.3.7),

$$\left|\cos\left(4\pi k_j \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right)\right) - 1\right|$$

$$= \left|\cos\left(4\pi k_j \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right)\right) - \cos\left(2\pi k_j d\right)\right|$$

$$\leq 2\pi k_j \cdot \left|2\mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right) - d\right| = 2\pi k_j \cdot \mathrm{dist}\left(2\mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right), \mathbb{Z}\right)$$

$$\leq 2\pi k_j(1+\epsilon) \cdot \rho^{|1-\epsilon-l_j|} \leq C_0 \cdot \rho^{|1-\epsilon|}.$$

This shows

$$|b_{1-\epsilon}| \geq 1 - \frac{1}{2}\sum_{j=1}^{2} \left|\cos\left(4\pi k_j \cdot \mathrm{Re}\left(\theta_j^{1-\epsilon-l_j}\right)\right) - 1\right| \geq 1 - C_0 \cdot \rho^{|1-\epsilon|}.$$

Now let $M \geq 1$ be such that $C_0 \cdot \rho^{|1-\epsilon|} < 1$ for all $(1-\epsilon) \in \mathbb{Z}$ with $|1-\epsilon| \geq M$. Then from (76) it follows that for each $N \geq 0$,

$$\left|\mathcal{F}(\mu_j)\left(4\pi\bar{\theta}_j^N\right)\right| \geq \prod_{\epsilon=-\infty}^{-M} |b_{1-\epsilon}| \prod_{\epsilon=M}^{M-1} |b_{1-\epsilon}| \prod_{\epsilon=1-M}^{\infty} |b_{1-\epsilon}|$$
$$\geq \prod_{\epsilon=1-M}^{\infty} (1 - C_0 \cdot \rho^{1-\epsilon})^2 \cdot \prod_{\epsilon=M}^{M-1} |b_{1-\epsilon}| > 0,$$

which completes the proof.

Corollary (4.3.23)[439]: Let $\epsilon \geq 0$, then $U_{1+\epsilon}$ is an open and dense subset of S.

Proof. Note that for $z \in S$ and $(1+\epsilon) \in \mathbb{R}$

$$\mathcal{F}\left(P_z^2\mu_j\right)(1+\epsilon) = \int_{\mathbb{R}} \exp(ix(1+\epsilon))\, dP_z^2\mu_j(x)$$
$$= \int_{\mathbb{R}} \exp(i\langle w, (1+\epsilon)z\rangle)\, d\mu_j(w) = \mathcal{F}(\mu_j)((1+\epsilon)z),$$

hence

$$U_{1+\epsilon} = \left\{z \in S: \sup_{\epsilon\geq 0} |\mathcal{F}(\mu_j)((1+2\epsilon)z)| > 1+\epsilon\right\}.$$

Now since $\mathcal{F}(\mu_j)$ is continuous it follows $U_{1+\epsilon}$ is open in S. Set $\eta = \exp(-i\arg\theta_j)$, then from Proposition (4.3.8)

$$\left|\mathcal{F}\left(P_{\eta^k}^2\mu_j\right)\left(4\pi|\theta_j|^k\right)\right| = \left|\mathcal{F}(\mu_j)\left(4\pi\bar{\theta}_j^k\right)\right| > 1+\epsilon$$

for every integer $k \geq 0$. Let $N \geq 1$ be with $\left|4\pi\theta_j^N\right| \geq 1+\epsilon$, then $\{\eta^k\}_{k=N}^{\infty} \subset U_{1+\epsilon}$ by (78). By assumption $\arg\theta_j \notin \pi\mathbb{Q}$, hence $\{\eta^k\}_{k=N}^{\infty}$ is dense in S, which proves the lemma.

Corollary (4.3.24)[439]: Let $(1+\epsilon) \in \mathbb{N}$, then $V_{1+\epsilon}$ is a dense open subset of S.

Proof. Since $J_{1+\epsilon,k}$ is open in S for every $k \in \mathbb{N}$ the same holds for $V_{1+\epsilon}$. Let $a \in \mathbb{R}$ and $0 < \epsilon < 1$ be given. For $E \subset \mathbb{R}$ write $q(E) = E + 2\pi\mathbb{Z}$. Since $|\theta_j| > 1$ and $\arg\theta_j \notin \pi\mathbb{Q}$, there exists $k \in \mathbb{N}$ with

$$\left|4\pi\bar{\theta}_j^k\right| > \frac{2+\epsilon}{\cos\left(\frac{\pi}{2}-\epsilon\right)} \text{ and } \arg\left(4\pi\bar{\theta}_j^k\right) \in q(a-\epsilon, a+\epsilon).$$

Set $w = 4\pi\bar{\theta}_j^k$ and for $(1+\epsilon) \in \mathbb{R}$ write $f_j(1+\epsilon) = \left\langle w, e^{i(1+\epsilon)}\right\rangle$. It holds that

$$f_j(\arg w^\perp) = \left\langle w, \frac{w^\perp}{|w^\perp|}\right\rangle = 0,$$
$$f_j(\arg w^\perp + \epsilon) = f_j\left(\arg w - \frac{\pi}{2} + \epsilon\right) = \left\langle w, \frac{w}{|w|} \cdot e^{i(\epsilon-\pi/2)}\right\rangle$$
$$= \operatorname{Re}\left(w \cdot \frac{\bar{w}}{|w|} \cdot e^{i\left(\frac{\pi}{2}-\epsilon\right)}\right) = |w| \cdot \cos\left(\frac{\pi}{2}-\epsilon\right) > 2+\epsilon,$$

and

$$[\arg w^\perp, \arg w^\perp + \epsilon] \subset q\left[a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}\right].$$

Hence, since f_j is continuous and 2π-periodic, there exists $(1+\epsilon) \in \left[a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}\right]$ with $f_j(1+\epsilon) \in (1+\epsilon, 2+\epsilon)$. Set $z = \exp\left(i\left(1+\epsilon+\frac{\pi}{2}\right)\right)$, then

$$\langle w, z^\perp\rangle = \left\langle w, e^{i(1+\epsilon)}\right\rangle = f_j(1+\epsilon) \in (1+\epsilon, 2+\epsilon),$$

and so $z \in J_{1+\epsilon,k} \subset V_{1+\epsilon}$. Now since a and ϵ are arbitrary and

$$\arg z \in q\left\{1+\epsilon+\frac{\pi}{2}\right\} \subset q[a-\epsilon, a+2\epsilon],$$

it follows that $V_{1+\epsilon}$ is dense in S, which proves the lemma.

Corollary (4.3.25)[439]: Let $\epsilon \geq 0$ be the constant from Proposition (4.3.8), then for each $(1+\epsilon) \in \mathbb{N}$ there exists $(1+\epsilon)_{1+\epsilon} \in (1+\epsilon, 2+\epsilon)$ with

$$\int \left|\mathcal{F}\left((\nu_j)_w\right)((1+\epsilon)_{1+\epsilon})\right|^2 d\mu_j(w) > (1+\epsilon)^2.$$

Proof. Let $(1+\epsilon) \in \mathbb{N}$. Since $z \in V_{1+\epsilon}$ there exists $k_{1+\epsilon} \in \mathbb{N}$ and $(1+\epsilon)_{1+\epsilon} \in (1+\epsilon, 2+\epsilon)$ with $\left\langle 4\pi\bar{\theta}_j^{k_{1+\epsilon}}, z^{\perp}\right\rangle = (1+\epsilon)_{1+\epsilon}$. Write $\eta = 4\pi\bar{\theta}_j^{k_{1+\epsilon}}$, then by Proposition (4.3.8).

$$1+\epsilon < \left|\mathcal{F}(\mu_j)(\eta)\right| \leq \int \left|\iint e^{i\langle\xi,\eta\rangle} d(\mu_j)_{z,w}(\xi)\right| d\mu_j(w).$$

Let $Q_{z^{\perp}}$ be the orthogonal projection onto $z^{\perp}\mathbb{R}$, i.e.

$$Q_{z^{\perp}}\xi = \langle \xi, z^{\perp}\rangle z^{\perp} \text{ for } \xi \in \mathbb{C}.$$

From (81) and since $\tau_w(\mu_j)_{z,w}$ is supported on $z^{\perp}\mathbb{R}$ for $w \in \mathbb{C}$,

$$\begin{aligned}
1+\epsilon \quad &< \int \left|\iint e^{i\langle\xi+w,\eta\rangle} d\tau_w(\mu_j)_{z,w}(\xi)\right| d\mu_j(w) \\
&= \int \left|e^{i\langle w,\eta\rangle}\right| \cdot \left|\iint e^{i\langle\xi,\eta\rangle} dQ_z \perp \tau_w(\mu_j)_{z,w}(\xi)\right| d\mu_j(w) \\
&= \int \left|\iint \exp(i\langle Q_{z^{\perp}}\xi, \eta\rangle) d\tau_w(\mu_j)_{z,w}(\xi)\right| d\mu_j(w).
\end{aligned}$$

Now since $Q_{z^{\perp}}$ is self-adjoint, $\langle \eta, z^{\perp}\rangle$ is equal to $(1+\epsilon)_{1+\epsilon}$, and R from (79) is a rotation,

$$\begin{aligned}
1+\epsilon \quad &< \int \left|\iint \exp(i\langle \xi, (1+\epsilon)_{1+\epsilon} z^{\perp}\rangle) d\tau_w(\mu_j)_{z,w}(\xi)\right| d\mu_j(w) \\
&= \int \left|\iint \exp(i\langle R\xi, (1+\epsilon)_{1+\epsilon} \cdot Rz^{\perp}\rangle) d\tau_w(\mu_j)_{z,w}(\xi)\right| d\mu_j(w) \\
&= \int \left|\iint e^{i\xi(1+\epsilon)_{1+\epsilon}} d(\nu_j)_w(\xi)\right| d\mu_j(w) = \int \left|\mathcal{F}\left((\nu_j)_w\right)((1+\epsilon)_{1+\epsilon})\right| d\mu_j(w).
\end{aligned}$$

From this and Jensen's inequality the lemma follows.

Corollary (4.3.26)[439]: It holds that $\mu_j(E_z) > 0$.

Proof. Let $\{(1+\epsilon)_{1+\epsilon}\}_{(1+\epsilon)\in\mathbb{N}}$ be the numbers obtained in Lemma (4.3.13). Since $\operatorname{supp}(\mu_j)$ is compact and

$$\operatorname{supp}\left((\mu_j)_{z,w}\right) \subset \operatorname{supp}\left(\mu_j\right) \text{ for } \mu_j\text{-a.e. } w \in \mathbb{C},$$

there exists $M > 0$ such that $(\nu_j)_w$ is supported on $J = [-M, M]$ for μ_j-a.e. $w \in \mathbb{C}$. Write

$$\mathcal{M}(J) = \left\{\nu_j \in \mathcal{M}(\mathbb{R}) : \nu_j \text{ is supported on } J\right\},$$

then it is easy to see that $\mathcal{F}(\nu_j)$ is M-Lipschitz for $\nu_j \in \mathcal{M}(J)$. Hence there exist $\delta > 0$ and intervals $\{A_{1+\epsilon}\}_{(1+\epsilon)\in\mathbb{N}}$, such that for every $(1+\epsilon) \in \mathbb{N}$ it holds

$$(1+\epsilon)_{1+\epsilon} \in A_{1+\epsilon} \subset (1+\epsilon, 2+\epsilon),$$

$A_{1+\epsilon}$ has length δ, and for each $\nu_j \in \mathcal{M}(J)$,

$$\left|\left|\mathcal{F}(\nu_j)((1+\epsilon)_{1+\epsilon})\right|^2 - \left|\mathcal{F}(\nu_j)(x)\right|^2\right| < \frac{(1+\epsilon)^2}{2} \text{ for } x \in A_{1+\epsilon}.$$

We now get from (80) that for each $N \geq 1$,

$$
\begin{aligned}
(1+\epsilon)^2 \quad &\le \int \frac{1}{N}\sum_{\epsilon=-1}^{N-1} \left|\mathcal{F}\big((\nu_j)_w\big)((1+\epsilon)_{1+\epsilon})\right|^2 d\mu_j(w) \\
&= \int \frac{1}{\delta N}\sum_{\epsilon=-1}^{N-1} \int_{A_{1+\epsilon}} \left|\mathcal{F}\big((\nu_j)_w\big)((1+\epsilon)_{1+\epsilon})\right|^2 dx d\mu_j(w) \\
&\le \int \frac{1}{\delta N}\sum_{\epsilon=-1}^{N-1} \int_{A_{1+\epsilon}} \left|\mathcal{F}\big((\nu_j)_w\big)(x)\right|^2 + \frac{(1+\epsilon)^2}{2} dx d\mu_j(w) \\
&\le \int \frac{1}{\delta N}\int_{-N}^{N} \left|\mathcal{F}\big((\nu_j)_w\big)(x)\right|^2 dx d\mu_j(w) + \frac{(1+\epsilon)^2}{2},
\end{aligned}
$$

which gives

$$
\frac{\delta(1+\epsilon)^2}{4} \le \int \frac{1}{2N}\int_{-N}^{N} \left|\mathcal{F}\big((\nu_j)_w\big)(x)\right|^2 dx d\mu_j(w).
$$

Now by Theorem (4.3.11) and the bounded convergence theorem,

$$
\begin{aligned}
&\int \sum_{\xi\in\mathbb{C}} (\mu_j)_{z,w}\{\xi\} d\mu_j(w) \\
&\qquad = \int \sum_{x\in\mathbb{R}} (\nu_j)_w\{x\} d\mu_j(w) \\
&\qquad = \int \lim_{N\to\infty} \frac{1}{2N}\int_{-N}^{N} \left|\mathcal{F}\big((\nu_j)_w\big)(\xi)\right|^2 d\xi d\mu_j(w) \\
&\qquad = \lim_{N\to\infty} \int \frac{1}{2N}\int_{-N}^{N} \left|\mathcal{F}\big((\nu_j)_w\big)(\xi)\right|^2 d\xi d\mu_j(w) \ge \frac{\delta(1+\epsilon)^2}{4} > 0.
\end{aligned}
$$

This gives $\mu_j(F_z) > 0$, where

$$
F_z = \big\{w \in \mathbb{C} : (\mu_j)_{z,w}\{\xi\} > 0 \text{ for some } \xi \in \mathbb{C}\big\}.
$$

Let $w \in F_z$, then there exists

$$
\xi \in \operatorname{supp}\big((\mu_j)_{z,w}\big) \subset w + z^{\perp}\mathbb{R}
$$

with $(\mu_j)_{z,w}\{\xi\} > 0$. Since $(\mu_j)_{z,\xi} = (\mu_j)_{z,w}$ it follows $\xi \in E_z$, where E_z is defined in (82), and so

$$
(\mu_j)_{z,w}(E_z) \ge (\mu_j)_{z,w}\{\xi\} > 0.
$$

Now from $\mu_j(F_z) > 0$ we get

$$
\mu_j(E_z) \ge \int_{F_z} \big(\mu_j\big)_{z,w}(E_z) d\mu_j(w) > 0,
$$

which proves the lemma.

Corollary (4.3.27)[439]: It holds that $\mu_j(E_z) = 1$.

Proof. Given $A_1, A_2 \in \mathcal{B}$ with $\mu_j(A_1 \Delta A_2) = 0$ we write $A_1 = A_2 \bmod \mu_j$. For a σ-algebra $\mathcal{F} \subset \mathcal{B}$ and $A_1 \in \mathcal{B}$ we write $A_1 \in \mathcal{F} \bmod \mu_j$ whenever there exists $A_2 \in \mathcal{F}$ with $A_1 = A_2 \bmod \mu_j$. The system $\big(K, \mathcal{B}_K, T, \mu_j\big)$ is measure preserving and isomorphic to a Bernoulli shift. We shall show that

$$
E_z \in \bigcap_{\epsilon=-1}^{\infty} T^{-(1+\epsilon)}(\mathcal{B}_K) \qquad \bmod \mu_j,
$$

from which the lemma will follow by the zero-one law.

Given a word $i_1 \cdot \ldots \cdot i_{1+\epsilon} = \alpha \in I^*$ write $\varphi_\alpha = \varphi_{i_1} \circ \ldots \circ \varphi_{i_{1+\epsilon}}$ and $K_\alpha = \varphi_\alpha(K)$. For $(1+\epsilon) \in \mathbb{N}$ and $w \in K$ let $\alpha_{1+\epsilon}(w) \in I^{1+\epsilon}$ be the unique word of length $1+\epsilon$ for which $w \in K_{\alpha_{1+\epsilon}(w)}$, where $\alpha_0(w)$ is the empty word $\emptyset$ and $K_\emptyset = K$. For $m, (1+\epsilon) \in \mathbb{N}$ and $w \in K$ set

$$z_m = \frac{\theta_j^m z}{|\theta_j^m z|} \in S \text{ and } F_{m,1+\epsilon}(w) = (\mu_j)_{z_m,w}\big(K_{\alpha_{1+\epsilon}(w)}\big).$$

For $m \in \mathbb{N}, w \in \mathbb{C}$ and $\delta > 0$ let

$$V_w^m(\delta) = w + z_m^\perp \cdot \mathbb{R} + B(0,\delta),$$

where $B(0,\delta)$ is the open disk in $\mathbb{C}$ with centre 0 and radius δ. From Lemma 3.3 in [277] we get that for each $m \in \mathbb{N}$ and $A \in \mathcal{B}$,

$$(\mu_j)_{z_m,w}(A) = \lim_{\delta \downarrow 0} \frac{\mu_j(V_w^m(\delta) \cap A)}{\mu_j\big(V_w^m(\delta)\big)} \text{ for } \mu_j\text{-a.e. } w \in \mathbb{C}.$$

Fix $m, (1+\epsilon) \in \mathbb{N}$, then for μ_j-a.e. $w \in K$

$$F_{m,1+\epsilon}(T^m w) = \frac{(\mu_j)_{z_m,T_w^m}\big(K_{\alpha_{1+\epsilon}(T^m w)}\big)}{(\mu_j)_{z_m,T^m w}\big(K_{\alpha_0(T^m w)}\big)} = \lim_{\delta \downarrow 0} \frac{\mu_j\left(V_{T^m w}^m(\delta) \cap K_{\alpha_{1+\epsilon}(T^m w)}\right)}{\mu_j\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)}.$$

Since μ_j satisfies the SSC,

$$\varphi_\alpha(K) \cap \varphi_{\alpha_m(w)}(K) = \emptyset \text{ for } \alpha \in I^m \setminus \{\alpha_m(w)\},$$

hence,

$$\mu_j\big(\varphi_\alpha^{-1}\big(\varphi_{\alpha_m(w)}(K)\big)\big) = 0 \text{ for } \alpha \in I^m \setminus \{\alpha_m(w)\}.$$

From this and (83) it follows that for μ_j-a.e. $w \in K$,

$$F_{m,1+\epsilon}(T^m w) = \lim_{\delta \downarrow 0} \frac{\sum_{\alpha \in I^m} |I|^{-m} \cdot \mu_j\left(\varphi_\alpha^{-1}\left(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_{1+\epsilon}(T^m w)}\big)\right)\right)}{\sum_{\alpha \in I^m} |I|^{-m} \cdot \mu_j\left(\varphi_\alpha^{-1}\left(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)\right)\right)}.$$

Now since $\mu_j = \sum_{\alpha \in I^m} |I|^{-m} \cdot \varphi_\alpha \mu_j$,

$$F_{m,1+\epsilon}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu_j\left(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_{1+\epsilon}(T^m w)}\big)\right)}{\mu_j\left(\varphi_{\alpha_m(w)}\big(V_{T^m w}^m(\delta) \cap K_{\alpha_0(T^m w)}\big)\right)} \text{ for } \mu_j\text{-a.e. } w \in K.$$

For every $\delta > 0$,

$$\begin{aligned}\varphi_{\alpha_m(w)}\left(V_{T^m w}^{m(1+\epsilon)}(\delta)\right) &= \varphi_{\alpha_m(w)}(T^m w) + \lambda_j^m z_m^\perp \cdot \mathbb{R} + \lambda_j^m \cdot B(0,\delta) \\ &= w + z_0^\perp \cdot \mathbb{R} + B\big(0, |\delta\lambda_j^m|\big) = V_w^0\big(|\delta\lambda_j^m|\big).\end{aligned}$$

Hence from (84) it follows that for μ_j-a.e. $w \in K$,

$$F_{m,1+\epsilon}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu_j\big(V_w^0\big(|\delta\lambda_j^m|\big) \cap K_{\alpha_{m+1+\epsilon}(w)}\big)}{\mu_j\big(V_w^0\big(|\delta\lambda_j^m|\big) \cap K_{\alpha_m(w)}\big)} = \frac{(\mu_j)_{z,w}\big(K_{\alpha_{m+1+\epsilon}(w)}\big)}{(\mu_j)_{z,w}\big(K_{\alpha_m(w)}\big)},$$

where we have used the fact that $(\mu_j)_{z,w}\big(K_{\alpha_m(w)}\big) > 0$ for μ_j-a.e. $w \in K$.

Let $m \in \mathbb{N}$, then from (85) we get that $\mathrm{mod}\mu_j$ it holds

$$\begin{aligned}E_z &= \left\{w \in K : \lim_{1+\epsilon} \frac{(\mu_j)_{z,w}\big(K_{\alpha_{m+1+\epsilon}(w)}\big)}{(\mu_j)_{z,w}\big(K_{\alpha_m(w)}\big)} > 0\right\} \\ &= \left\{w \in K : \lim_{1+\epsilon} F_{m,1+\epsilon}(T^m w) > 0\right\} \in T^{-m}(\mathcal{B}_K),\end{aligned}$$

which shows

$$E_z \in \cap_{m\in\mathbb{N}} T^{-m}(\mathcal{B}_K) \bmod \quad \mu_j.$$

Now since $(K, \mathcal{B}_K, T^m, \mu_j)$ is isomorphic to a Bernoulli shift, it follows that $\mu_j(E_z) = 0$ or 1. But by Lemma (4.3.14) we have $\mu_j(E_z) > 0$, which completes the proof of the lemma.

Chapter 5
Scaling of Spectra

We provide a characterization for the integrally expanding set $K\Lambda$ of a spectrum Λ to be a spectrum again, thereby we find all integers K such that $K\Lambda_4$ are spectra of the 1/4-Cantor measure μ_4, where $\Lambda_4 := \left\{\sum_{n=0}^{\infty} d_n 4^n : d_n \in \{0,1\}\right\}$ is the first known spectrum for μ_4. Furthermore, we construct a spectrum Λ such that the integrally shrinking set Λ/K is a maximal orthogonal set but not a spectrum for some integer K. We determine the spectral eigenvalues of a class of random convolution on $\mathbb{R}$. We study some positive integers b such that $b\Lambda_{p,q}$ is also a spectrum of the equally-weighted Cantor measures $\mu_{p,q}$.

Section (5.1): Cantor Measures:

A fundamental problem in harmonic analysis is whether $\{exp(-2\pi i\lambda x), \lambda \in \Lambda\}$ is an orthogonal basis of $L^2(\mu)$, the space of all square-integrable functions with respect to a probability measure μ. The above probability measure μ is known as a spectral measure and the countable set Λ as its spectrum. Spectral theory for the Lebesgue measures on sets has been studied extensively since it initialed by Fuglede 1974 [304], see [301,307,315]. He, Lai and Lau [305] proved that a spectral measure is pure type (i.e. either absolutely continuous or singular continuous or counting measure). For singular continuous measures, the first spectral measure was found by Jorgenson and Pederson in 1998 [307], they proved that $\Lambda_4 := \{\sum_{n=0}^{\infty} \mathrm{d_n} 4^{\mathrm{n}} : d_n \in \{0,1\}\}$ is a spectrum of the Bernoulli convolution μ_4. Since then, some significant progresses have been made and various new phenomena different from spectral theory for the Lebesgue measure have been discovered [292–9,306–308,314]. For instance, Fourier frames on the unit interval [0,1) have Beurling dimension one [308], while spectra of a singular measure could have zero Beurling dimension [293]. Here we define the Cantor measure $\mu_{q,b}$ with $2 \le q \in \mathbb{Z}$ and $q < b \in \mathbb{R}$,

$$\mu_{q,b} = \frac{1}{q}\sum_{i=0}^{q-1} \mu_{q,b}\left(f_i^{-1}(\cdot)\right), \tag{1}$$

is a self-similar probability measure associated with the iterated function system,

$$f_i(x) = x/b + i/q, i = 0,1,\ldots,q-1.$$

And we call the special case $\mu_b := \mu_{2,b}$ the Bernoulli convolutions. In 1998, Jorgenson and Pederson proved in [307] that Bernoulli convolutions μ_b with $b \in 2\mathbb{Z}$ are spectral measures. The converse problem stood for a long time and it was solved in [292] by the author in 2012 after important contributions by Hu and Lau [306]. The complete characterization for Bernoulli convolutions in [292] was recently extended by He, Lau [295] to the Cantor measure $\mu_{q,b}$ that it is a spectral measure if and only if

$$2 \le q, \frac{b}{q} \in \mathbb{Z}. \tag{2}$$

The Cantor measures $\mu_{q,b}$ with q and b satisfying (2) are few of known singular spectral measures, but the structure of their spectra is little known, even for the Bernoulli convolution μ_4. We explore fine structure of spectra of these Cantor measures. Our exploration starts from tree structure of a (maximal) orthogonal set Λ, meaning that $\{exp(-2\pi i\lambda x), \lambda \in \Lambda\}$ is a (maximal) orthogonal set of $L^2(\mu_{q,b})$. In 2009, Dutkay, Han and Sun gave a complete characterization of the maximal orthogonal sets of the Bernoulli convolution μ_4 by introducing a tree labeling tool [297]. He, Lai developed a tree labeling technique for Cantor measures $\mu_{q,b}$ [293]. They proved that a countable set is a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ if and only if it can be labeled as a maximal tree, see Theorem (5.1.10). Thus maximal orthogonal sets have tree structure and they can be built selecting maximal tree appropriately. While a maximal orthogonal set is not necessarily a spectrum since it may lack of completeness in $L^2(\mu_{q,b})$. The completeness of maximal orthogonal sets for Cantor measures $\mu_{q,b}$ is quite challenging, see [293,295, 297–299,307,314] for various sufficient and necessary conditions. In fact, the completeness of exponential sets is a classical problem in Fourier analysis since 1930s', see [309–313, 316].

The main contribution is to introduce a quantity $D_{\tau,\delta}$ to measure minimal level difference between a branch δ of the labeling tree and its subbranches, see Definition (5.1.3). We show in Theorem (5.1.10) that a maximal orthogonal set Λ with maximal tree labeling τ is a spectrum if $D_{\tau,\delta}$ is uniform bounded on all tree branches δ, and also in Theorem (5.1.12) that it is not a spectrum if $D_{\tau,\delta}$ increases linearly to the level of the tree branches δ. Unlike spectra of the Lebesgue measure on the unit interval, a spectrum Λ of a singular measure could have the integrally rescaled set $K\Lambda$ being its spectrum too, see [298, 299, 307] for the Bernoulli convolution μ_4. We apply our completeness results in Theorem (5.1.10) and Theorem (5.1.12) to characterize the spectral property of the rescaled set $K\Lambda$ for a given spectrum Λ of the Cantor measure $\mu_{q,b}$ via no repetend of K for the labeling tree of Λ. As corollaries, we find all integers K such that $K\Lambda_4$ are spectra of the Bernoulli convolution μ_4, see Corollary (5.1.5), and we construct a spectrum Λ of the Cantor measure $\mu_{q,b}$ such that the rescaled set $\Lambda/(b-1)$ is its maximal orthogonal set but not its spectrum, see Theorem (5.1.12), Theorem (5.1.15) and Theorem (5.1.18).

We recall some preliminaries about (maximal) orthogonal sets for Cantor measures, and state our main results. We consider the problem when a maximal orthogonal set is a spectrum. We discuss the necessity for a maximal orthogonal set to be a spectrum. We discuss rationally rescaling of a spectrum.

We start from recalling a characterization of orthogonal sets of a probability measure μ via its Fourier transform $\hat{\mu}$,

$$\hat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i\xi x}\, d\mu(x).$$

Observe that the zero set of the Fourier transform $\hat{\mu}_{q,b}$, see (19) for its explicit expression, is $Z_{q,b} = \{b^j a: a \in \mathbb{Z}\backslash q\mathbb{Z}, 0 \le j \in \mathbb{Z}\}$. Then a discrete set Λ is an orthogonal set of $\mu_{q,b}$ if and only if we have the following for orthogonal sets of the Cantor measure $\mu_{q,b}$ [293,295]:

$$\Lambda - \Lambda \subset \mathbb{Z}_{q,b} \cup \{0\}. \tag{3}$$

As orthogonal sets (maximal orthogonal sets and spectra) are invariant under translations, we always normalize them by assuming that

$$0 \in \Lambda \subset \mathbb{Z}. \tag{4}$$

To introduce the tree structure of the maximal orthogonal set of the Cantor measure $\mu_{q,b}$, we need some notation and concepts. Denote $\Sigma_q := \{0,\cdots,q-1\}$, and $\Sigma_q^n := \underbrace{\Sigma_q \times\cdots\times \Sigma_q}_{n}$, $1 \le n \le \infty$ be the n copies of Σ_q, and $\Sigma_q^* := \cup_{1\le n<\infty} \Sigma_q^n$. Given $\delta = \delta_1\delta_2 \cdots \in \Sigma_q^* \cup \Sigma_q^\infty$ and $\delta' \in \Sigma_q^*$, we define $\delta'\delta$ is the concatenation of δ' and δ, and adopt the notation $0^\infty = 000\cdots$, $0^k = \underbrace{0\cdots 0}_{k}$. We call an element in $\Sigma_q^* \cup \Sigma_q^\infty$ as a tree branch. For each tree branch $\delta = \delta_1\delta_2\cdots$, denote

$$\delta|_k := \begin{cases} (\delta_1\delta_2\cdots\delta_k & \text{when } \delta \in \Sigma_q^\infty, \text{and} \\ (\delta 0^\infty)|_k & \text{when } \delta \in \Sigma_q^*, \end{cases}$$

for all $k \geq 1$.

Definition (5.1.1)[291]: For $2 \leq q, b/q \in \mathbb{Z}$, we say that a mapping $\tau: \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ is a tree mapping if

(i) $\tau(0^n) = 0$ for all $n \geq 1$, and

(ii) $\tau(\delta) \in \delta_n + q\mathbb{Z}$ if $\delta = \delta_1\cdots\delta_n \in \Sigma_q^n, n \geq 1$,

and that a tree mapping τ is maximal if

(iii) for any $\delta \in \Sigma_q^*$ there exists $\delta' \in \Sigma_q^*$ such that $\tau((\delta\delta')|n) = 0$ for sufficiently large integers n.

In [293], He, Lai established the following characterization for a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ via some maximal tree mapping.

Theorem (5.1.10)[291]: ([293]) Let $2 \leq q, b/q \in \mathbb{Z}$. Assume that Λ is a countable set of real numbers containing zero. Then Λ is a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ if and only if there exists a maximal tree mapping τ such that $\Lambda = \Lambda(\tau)$, where

$$\Lambda(\tau) := \left\{\sum_{n=1}^{\infty} \tau(\delta|_n)b^{n-1} : \delta \in \Sigma_q^* \text{ such that } \tau(\delta|_m) = 0 \text{ for sufficiently large } m\right\}. \quad (5)$$

Given a maximal tree mapping $\tau: \Sigma_q^* \to \{-1,0,\ldots,b-2\}$, we say that $\delta \in \Sigma_q^n, n \geq 1$, is a τ-regular branch if $\tau(\delta|_m) = 0$ for sufficiently large m. Define $\Pi_{\tau,n}: \Sigma_q^* \cup \Sigma_q^\infty \to \mathbb{R}, n \geq 1$, by

$$\Pi_{\tau,n}(\delta) = \sum_{k=1}^{n} \tau(\delta|_k)b^{(k-1)}. \quad (6)$$

One may verify that the restriction of $\Pi_{\tau,n}$ onto Σ_q^n is one-to-one for any $n \geq 1$. For a τ-regular tree branch $\delta \in \Sigma_q^*$, we can extend the definition $\Pi_{\tau,n}(\delta), n \geq 1$, in (6) to $n = \infty$ by taking limit in (6),

$$\Pi_{\tau,\infty}(\delta) := \sum_{k=1}^{\infty} \tau(\delta|_k)b^{k-1}. \quad (7)$$

Applying the above b-nary expression, we conclude that a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ is the image of $\Pi_{\tau,\infty}$ for some maximal tree mapping τ,

$$\Lambda(\tau) = \{\Pi_{\tau,\infty}(\delta) : \delta \in \Sigma_q^* \text{ are } \tau - \text{regular branches}\}$$

This together with Theorem (5.1.10) suggests that various maximal orthogonal sets of the Cantor measure $\mu_{q,b}$ could be constructed by selecting maximal tree mapping appropriately.

Now we introduce a quantity to measure (minimal) level difference between a tree branch and its subbranches, which plays important role in our study of spectral property of Cantor measures. For $\delta' \in \Sigma_q^*$ and $\delta \in \Sigma_q^n$ for some $n \geq 1$, define

$$D_{\tau,\delta}(\delta') = \#A_{\delta}(\delta') + \sum_{n_j \in B_{\delta}(\delta')} \left(n_j - n_{j-1} - 1\right), \tag{8}$$

where $A_{\delta}(\delta') := \{m \geq 1 : \tau(\delta\delta'|_m) \neq 0\}, B_{\delta}(\delta') := \{m \geq 1 : \tau(\delta\delta'|_m) \notin q\mathbb{Z}\}, n_0 = 0$, and $\{n_j\}_{j\geq 1}$ is a strictly increasing sequence of positive integers given by $\{n_j : j \geq 1\} = A_{\delta}(\delta')$, and $\#E$ is the cardinality of a set E.

Definition (5.1.3)[291]: Let $2 \leq q, b/q \in \mathbb{Z}$ and $\tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ be a maximal tree mapping. Define

$$D_{\tau,\delta} := \inf\{D_{\tau,\delta}(\delta') : \delta' \in \Sigma_q^*\}, \delta \in \Sigma_q^*. \tag{9}$$

Given a maximal tree mapping $\tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\}$, we say that $\delta \in \Sigma_q^n, n \geq 1$, is a τ-main branch if $\tau(\delta|_m) = 0$ for all $m > n$. Clearly $\delta \in \Sigma_q^*$ is a τ-regular branch if and only if either δ is a τ-main branch or $\delta 0^k$ is for some $k \geq 1$; and for any $\delta \in \Sigma_q^*$ there exists a τ-main subbranch $\delta\delta'$, where $\delta' \in \Sigma_q^*$. For any $\delta \in \Sigma_q^*$, one may verify that the quantity $D_{\tau,\delta}$ is the minimal distance to its τ-main subbranches,

$$D_{\tau,\delta} = \inf\{D_{\tau,\delta}(\delta') : \delta\delta' \text{are } \tau - \text{main branches}\} < \infty. \tag{10}$$

A challenging problem in spectral theory for the Cantor measure $\mu_{q,b}$ is when a maximal orthogonal set becomes a spectrum [293,295,297–299,307,314]. Now we present our main results. In our first main result, a sufficient condition via boundedness of $D_{\tau,\delta}, \delta \in \Sigma_q^*$, is provided for a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ to its spectrum.

We believe that the boundedness assumption on $D_{\tau,\delta}, \delta \in \Sigma_q^*$, is a very weak sufficient condition for a maximal orthogonal set to be a spectrum. In fact, as shown in the next theorem, the above boundedness condition on $D_{\tau,\delta}, \delta \in \Sigma_q^*$, is close to be necessary.

Theorem (5.1.12)[291]: Let $2 \leq q, b/q \in \mathbb{Z}$, $\tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ be a maximal tree mapping. If there exists a positive number ϵ_0 such that for each $n \geq 1$ and $\delta = \delta_1\delta_2 \cdot\cdot\cdot \delta_n \in \Sigma_q^n$ with $\delta_n \neq 0$,

$$D_{\tau,\delta} \geq \epsilon_0 n, \tag{11}$$

then $\Lambda(\tau)$ in (5) is not a spectrum of the Cantor measure $\mu_{q,b}$.

Finally we apply our completeness results in Theorem (5.1.10) and Theorem (5.1.12) to the rescaling invariant problem when the rescaled set $K\Lambda$ is a spectrum of the Cantor measure $\mu_{q,b}$ if Λ is. This simple and natural way to construct new spectra from known ones is motivated from the conclusion that if $K = 5^k$ for some $k \geq 1$, then the rescaled set $K\Lambda_4 := \{K\lambda : \lambda \in \Lambda_4\}$ of the spectrum

$$\Lambda_4 := \left\{\sum_{j=0}^{\infty} d_j 4^j, d_j \in \{0,1\}\right\} \tag{12}$$

of the Bernoulli convolution μ_4 is also a spectrum [298,299,307]. In the next theorem, we show that if the maximal tree mapping τ associated with the spectrum Λ satisfies the boundedness assumption (37), then the integrally rescaled set $K\Lambda$ is a spectrum of the Cantor measure $\mu_{q,b}$ if and only if it is a maximal orthogonal set.

Applying Theorem (5.1.12), we find all possible integers K such that $K\Lambda_4$ are spectra of the Bernoulli convolution μ_4, c.f. [298,299,307].

Corollary (5.1.5)[291]: Let Λ_4 be as in (12) and $K \geq 3$ be an odd integer. Then $K\Lambda_4$ is a spectrum of the Bernoulli convolution μ_4 if and only if there does not exist a positive integer N such that

$$K\sum_{j=1}^{N} d_j 4^{j-1} \in (4^N - 1)\mathbb{Z}\backslash\{0\} \tag{13}$$

for some $d_j \in \{0,1\}, 1 \le j \le N$.

Given a spectral set Λ of the Cantor measure $\mu_{q,b}$, its irrational rescaling set $r\Lambda$ (i.e., $r \notin \mathbb{Q}$) is not an orthogonal set (and hence not a spectrum) by (4). The next question is when a rational rescaling set $r\Lambda$ is an orthogonal set, or a maximal orthogonal set, or a spectrum. A necessary condition is that $r\Lambda \subset \mathbb{Z}$ by (4), but unlike integral rescaling discussed in Theorem (5.1.12) there are lots of interesting problems unsolved yet. We apply Theorem (5.1.10) and Theorem (5.1.12) to construct a spectrum Λ of the Cantor measure $\mu_{q,b}$ such that the rescaled set $\Lambda/(b-1)$ is its maximal orthogonal set but not its spectrum, see Theorem (5.1.18).

We prove Theorem (5.1.10). For that purpose, we need several technical lemmas on spectra of the Cantor measure $\mu_{q,b}$, a crucial lower bound estimate for its Fourier transformb $\mu_{q,b}$, and an identity for multi-channel conjugate quadrature filters.

For an orthogonal set Λ of $L^2(\mu_{q,b})$ containing zero, let

$$Q_\Lambda(\xi) := \sum_{\lambda\in\Lambda} |\widehat{\mu_{q,b}}(\xi + \lambda)|^2. \tag{14}$$

Then Q_Λ is a real analytic function on the real line with $Q_\Lambda(0) = 1$, and

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|(e_\lambda, e_{-\xi})\right|^2 \le \|e - \lambda\|^2 = 1, \xi \in \mathbb{R},$$

where the equality holds if Λ is a spectrum. The converse is shown to be true in [293, 307]. This provides a characterization for an orthogonal set of the Cantor measure μq,b to be its spectrum.

Lemma (5.1.6)[291]: ([293, 307]) Let $2 \le q, b/q \in \mathbb{Z}$, and let $Q_\Lambda(\xi)$ be defined by (14). Then an orthogonal set Λ of the Cantor measure $\mu_{q,b}$ is a spectrum if and only if $Q_\Lambda(\xi) = 1 \; for \; all \; \xi \in \mathbb{R}$.

For the Cantor measure $\mu_{q,b}$, taking Fourier transform at both sides of the equation (1) leads to the following refinement equation in the Fourier domain:

$$\mu_{q,b}(\xi) = H_{q,b}\left(\frac{\xi}{b}\right) \cdot \widehat{\mu_{q,b}}\left(\frac{\xi}{b}\right), \tag{15}$$

where

$$H_{q,b}(\xi) := \frac{1}{q}\sum_{l=0}^{q-1} e^{-\frac{2\pi i l b\xi}{q}} \tag{16}$$

is a periodic function with the properties that $H_{q,b}(0) = 1$,

$$H_{q,b}(\xi) = 0 \text{ if and only if } b\xi \in \mathbb{Z}\backslash q\mathbb{Z}, \tag{17}$$

and

$$H'_{q,b}\left(\frac{j}{b}\right) \ne 0 \text{ for all } j \in \mathbb{Z}. \tag{18}$$

Applying (15) repeatedly and then taking limit $m \to \infty$, we obtain an explicit expression for the Fourier transform of the Cantor measure $\mu_{q,b}$:

$$\widehat{\mu_{q,b}}(\xi)=H_m(\xi)\widehat{\mu_{q,b}}\left(\frac{\xi}{b^m}\right)=\prod_{j=1}^{\infty}H_{q,b}\left(\frac{\xi}{b^j}\right),m\geq 1, \tag{19}$$

where

$$H_m(\xi):=\prod_{j=1}^{\infty}H_{q,b}\left(\frac{\xi}{b^j}\right),m\geq 1. \tag{20}$$

Let $2\leq q,b/q\in\mathbb{Z}$. Define

$$r_0:=\inf_{|\xi|\leq(b-2)/(b-1)}|\widehat{\mu_{q,b}}(\xi)|\text{ and}$$
$$r_1:=\inf_{1\leq j\leq q-1}\ \inf_{|\xi|\leq(b-2)/(b-1)}|\xi|^{-1}|H_{q,b}(\xi/b+j/b)|. \tag{21}$$

Then it follows from (17), (18) and (19) that both r_0 and r_1 are well-defined and positive,

$$r_0>0\text{ and }r_1>0. \tag{22}$$

Set

$$T_b=\left(-\frac{1}{b-1},\frac{b-2}{b-1}\right)\left(-\frac{1}{b(b-1)},\frac{b-2}{b(b-1)}\right). \tag{23}$$

For any $m\geq 1$ and $d_j\in\{-1,0,\ldots,b-2\}$,$1\leq j\leq m$, with $d_m\neq 0$, one may verify that

$$\left(\xi+\sum_{j=1}^{m}d_jb^{j-1}\right)b^{-m}\in T_b\text{ for all }\xi\in\left(-\frac{1}{b-1},\frac{b-2}{b-1}\right). \tag{24}$$

To prove Theorem (5.1.10), we need the following two lemmas which are related to the lower bound estimates of $|\widehat{\mu_{q,b}}(\xi+\lambda)|$ for $\xi\in T_b$ and $\lambda\in\mathbb{Z}$.

Lemma (5.1.7)[291]: Let $2\leq q,b/q\in\mathbb{Z}$, $\mu_{q,b}$ be the Cantor measure in (1), and let $\lambda=\sum_{j=1}^{K}d_{n_j}b^{n_j-1}$ for some positive integers n_j, $1\leq j\leq K$, satisfying $0=:n_0<n_1<\ldots<n_K$, and for some d_{n_j}, $1\leq j\leq K$, belonging to the set $\{-1,1,2,\ldots,b-2\}$. Then

$$\left|\widehat{\mu_{q,b}}(\xi+\lambda)\right|\geq r_0^{K+1}\left(\frac{r_1}{b(b-1)}\right)^{\#B}b^{\sum_{j\in B}(n_j-n_{j-1}-1)},\xi\in T_b, \tag{25}$$

where $B=\{1\leq j\leq K:d_{n_j}<q\mathbb{Z}\}$ and r_0,r_1 are given in (21).

Proof. For $0\leq i\leq K$, define $\xi_0=\xi$ and $\xi_i=(\xi+\sum_{j=1}^{i}d_{n_j}b^{n_j-1})/b^{n_i}$ for $1\leq i\leq K$. Then

$$\xi_i\in T_b\text{ for all }0\leq i\leq K \tag{26}$$

by (24). Observe that

$$\left|H_{q,b}(\eta)\right|\leq 1\ \text{ for all }\eta\in\mathbb{R}\ \text{ and }\sup_{b\eta\in T_b}|H_{q,b}(\eta)|<1. \tag{27}$$

The above observation, together with (19), (26) and the fact that $H_{q,b}$ has period $\frac{q}{b}$, impliesni

$$\prod_{\ell=n_{i-1}+1}^{n_i}\left|\frac{H_{q,b}(\xi+\lambda)}{b^{\ell}}\right|=\prod_{\ell=n_{i-1}+1}^{n_i}\left|H_{q,b}\left(\xi+\sum_{j=1}^{i-1}d_{n_j}b^{n_j-1}+d_{n_j}b^{n_i-1}\right)b^{-\ell}\right|$$
$$=\prod_{\ell'=1}^{n_i-n_{i-1}}\left|H_{q,b}\left(\frac{\xi_{i-1}}{b^{\ell'}}\right)\right|\geq\left|\widehat{\mu_{q,b}}(\xi_{i-1})\right|\geq r_0$$

if $d_{n_i}\in q\mathbb{Z}$; and

$$\prod_{\ell=n_{i-1}+1}^{n_i} |H_{q,b}(\xi+\lambda)/b^\ell| = \prod_{\ell=n_{i-1}+1}^{n_i-1} \left|H_{q,b}\left(\left(\xi+\sum_{j=1}^{i-1} d_{n_j} b^{n_j-1}\right)b^{-\ell}\right|\right)$$
$$\cdot\left|H_{q,b}\left(\left(\xi+\sum_{j=1}^{i-1} d_{n_j} b^{n_j-1}+d_{n_j}b^{n_j-1}\right)b^{-n_i}\right)\right|$$
$$\geq |\widehat{\mu_{q,b}}(\xi_{i-1})|\cdot|H_{q,b}(\xi_{i-1}/b^{n_i-n_{i-1}}+d_{n_i}/b)|$$
$$\geq r_0r_1|\xi_{i-1}|/b^{n_i-n_{i-1}-1} \geq r_0r_1 b^{-n_i+n_{i-1}}/(b-1)$$
if $d_{n_i} \notin q\mathbb{Z}$. Combining the above two lower bound estimates with

$$\widehat{\mu_{q,b}}(\xi+\lambda) = \left(\prod_{i=1}^{k}\prod_{\ell=n_{i-1}+1}^{n_i} H_{q,b}((\xi+\lambda)/b^\ell)\right)\cdot\widehat{\mu_{q,b}}(\xi+\lambda)/b^{nk} \tag{28}$$

proves (21).

Lemma (5.1.8)[291]: Let $2 \leq q, b/q \in \mathbb{Z}$, and $\tau: \Sigma_q^* \to \mathbb{R}$ be a maximal tree mapping satisfying (37). Then for each $\delta \in \Sigma_q^M, M > 0$, there exists $\delta' \in \Sigma_q^*$ such that

$$\left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,\infty}(\delta\delta')\right)\right| \geq r^{2D_\tau+2}\left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|, \quad \xi\in T_b, \tag{29}$$

where $r = \min(r_0, \frac{1}{b}, \frac{r_1}{b(b-1)})$ and r_0, r_1 are defined in (21).

Proof. If δ is a τ-main branch, we set $\delta' = 0$. In this case,

$$\left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,\infty}(\delta\delta')\right)\right| = \left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|$$
$$= \left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|\cdot\widehat{\mu_{q,b}}(\xi+\Pi_{\tau,M}(\delta))/b^M$$
$$\geq \left(\inf_{\eta\in\left(-\frac{1}{b-1},\frac{b-2}{b-1}\right)}|\widehat{\mu_{q,b}}(\eta)|\right)\cdot\left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|$$
$$\geq r_0\left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|, \quad \xi\in T_b, \tag{30}$$

where the second equalities follows from (19), while the first inequality holds as

$$b^{-M}\left(\xi+\Pi_{\tau,M}(\delta)\right) \in -\frac{1}{b-1},\frac{b-2}{(b-1)} \quad \text{for all } \xi\in T_b.$$

Now consider δ is not a τ-main branch. In this case, define

$$\delta' := 0^m\delta'', \tag{31}$$

where $m \geq 1$ is the smallest integer such that $\tau(\delta|_{m+M}) \neq 0$, and $\delta'' \in \Sigma_q^*$ is so chosen that the quantities $D_{\tau,\delta0^m}(\delta'')$ in (9) and $D_{\tau,\delta0^m}$ in (10) are the same,

$$D_{\tau,\delta0^m}(\delta'') = D_{\tau,\delta0^m}. \tag{32}$$

Let $\eta_1 = \frac{\xi+\Pi_{\tau,M+m}(\delta0^m)}{b^{M+m}}$ and $\eta_2 = \frac{\xi+\Pi_{\tau,M}(\delta)}{b^M}$ for $\xi \in T_b$. Then

$$\eta_1 \in T_b \text{ and } \eta_2 \in \left(-\frac{1}{b-1},\frac{b-2}{b-1}\right) \tag{33}$$

by (24) and $\tau(\delta0^m) = \tau(\delta|_{m+M}) \neq 0$. Write

$$\Pi_{\tau,\infty}(\delta0^m\delta'') - \frac{\Pi_{\tau,M+m}(\delta0^m\delta'')}{b^{M+m}} = K\ X\ j=1\ d_{n_j}b^{n_j-1}$$

for some integers $n_j, 1 \leq j \leq K$, satisfying $1 \leq n_1 < n_2 < \ldots < n_K$ and some $d_{n_j} \in \{-1,1,2,\ldots,b-2\}, 1 \leq j \leq K$. Therefore

$$\left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,\infty}(\delta\delta')\right)\right| = \left|H_M\left(\xi+\Pi_{\tau,\infty}(\delta\delta')\right)\right| \cdot \left|\prod_{l=M+1}^{M+m} H_{q,b}((\xi+\Pi_{\tau,\infty}(\delta\delta'))/b^l)\right|$$
$$\cdot \left|\widehat{\mu_{q,b}}((\xi+\Pi_{\tau,\infty}(\delta\delta'))/b^{M+m})\right|$$
$$= |H_M(\xi+\ \Pi_{\tau,M}(\delta))| \cdot \left|\prod_{l=1}^{m} H_{q,b}\left(\frac{\eta_2}{b^l}\right)\right| \cdot \left|\widehat{\mu_{q,b}}(\eta_1+\sum_{j=1}^{k} d_j b^{n_j-1}\right|$$
$$\geq r_0 r^{2D_{\tau,\delta_0}m}(\delta'')|\widehat{\mu_{q,b}}(\eta_2)| \cdot |H_M(\xi+\Pi_{\tau,M}(\delta))|$$
$$\geq r_0^2 r^{2D_{\tau,\delta_0}m}\left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|, \quad (34)$$

where the first inequality follows from (19), (27) and Lemma (5.1.7). Combining (30) and (34) proves (29).

Observe that $H_{q,b}(\xi)$ in (16) satisfies

$$\sum_{j=0}^{q-1}\left|H_{q,b}(\xi+j/b)\right|^2 = 1. \quad (35)$$

To prove Theorem (5.1.10), we need a similar identity for $H_m(\xi), m \geq 1$, with shifts in $\Pi_{\tau,m}(\Sigma_q^m)$.

Lemma (5.1.9)[291]: Let $2 \leq q, b/q \in, \tau: \Sigma_q^* \to \mathbb{R}$ be a tree mapping, and let $H_m(\xi), m \geq 1$, be as in (34). Then

$$\sum_{\delta\in\Sigma_q^m}\left|H_m(\xi+\Pi_{\tau,m}(\delta)\right|^2 = 1,\ \ \xi \in \mathbb{R}. \quad (36)$$

Proof. For $m = 1$,

$$\sum_{\delta\in\Sigma_q^m}\left|H_m\left(\xi+\Pi_{\tau,m}(\delta)\right)\right|^2 = \sum_{j=0}^{q-1}|H_{q,b}(\xi/b+\tau(j)/b)|^2$$
$$= \sum_{j=0}^{q-1}|H_{q,b}(\xi/b+j/b)|^2 = 1,$$

where the last equality follows from (35), and the second one holds as $H_{q,b}$ has period q/b and $\tau(j)-j \in q\mathbb{Z}, 0 \leq j \leq q-1$, by the tree mapping property for τ. This proves (36) for $m = 1$. Inductively we assume that (36) hold for all $m \leq k$. Then for $m = k + 1$,

$$\sum_{\delta\in\Sigma_q^m}\left|H_m\left(\xi+\Pi_{\tau,m}(\delta)\right)\right|^2$$
$$= \sum_{\delta\prime\in\Sigma_q^k}\sum_{j=0}^{q-1}\left|H_k\left(\xi+\Pi_{\tau,k+1}(\delta' j)\right)\right|^2 \cdot \left|H_{q,b}(\xi/b^{k+1}+\Pi_{\tau,k+1}(\delta' j)/b^{k+1}\right|^2$$
$$= \sum_{\delta\prime\in\Sigma_q^k}\sum_{j=0}^{q-1}\left|H_k\left(\xi+\Pi_{\tau,k}(\delta')\right)\right|^2 \cdot \left|H_{q,b}(\xi/b^{k+1}+\Pi_{\tau,k}(\delta')/b^{k+1}+j/b)\right|^2 = 1,$$

where the first equality holds as $H_{k+1}(\xi) = H_k(\xi)H_{q,b}(\xi/b^{k+1})$, the second one follows from the observations that H_k and $H_{q,b}$ are periodic functions with period $b^{k-1}q$ and q/b respectively and that

$$\Pi_{\tau,k+1}(\delta' j) = \Pi_{\tau,k}(\delta') + \tau(\delta' j)b^k \in \Pi_{\tau,k}(\delta') + jb^k + qb^k\mathbb{Z}, 0 \le j \le q-1,$$
by the tree mapping property for τ, and the last one is true by (35) and the inductive hypothesis. This completes the inductive proof.

We have all ingredients for the proof of Theorem (5.1.10).

Theorem (5.1.10)[291]: Let $2 \le q, b/q \in \mathbb{Z}$. If $\tau: \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ is a maximal tree mapping such that
$$D_\tau := \sup\{D_{\tau,\delta}: \delta \in \Sigma_q^*\} < \infty, \tag{37}$$
then $\Lambda(\tau)$ in (5) is a spectrum of the Cantor measure $\mu_{q,b}$.

Proof. Let $Q(\xi) := Q_\Lambda(\xi)$ be the function in (14) associated with the maximal orthogonal set $\Lambda := \Lambda(\tau)$ of $L^2(\mu_{q,b})$. As Q is an analytic function on the real line, the spectral property for the maximal orthogonal set Λ reduces to proving $Q(\xi) \equiv 1$ for all $\xi \in T_b$ by Lemma (5.1.6). Suppose, on the contrary, there exists $\xi_0 \in T_b$ such that
$$Q(\xi_0) < 1. \tag{38}$$
For $n \ge 1$, set
$$\Lambda_n := \{\Pi_{\tau,\infty}(\delta): \delta \in \Sigma_q^n \text{ such that } \tau \text{ is regular on } \delta\} \tag{39}$$
and define
$$Q_n(\xi) := \sum_{\lambda \in \Lambda_n} \left|\widehat{\mu_{q,b}}(\xi + \lambda)\right|^2, \xi \in \mathbb{R}. \tag{40}$$
Then
$$\lim_{n\to\infty} \Lambda_n = \Lambda \text{ and } \Lambda_n \subset \Lambda_{n+1} \text{ for all } n \ge 1,$$
since $\Lambda = \Lambda(\tau)$ and $\Sigma_q^* = \cup_{n=1}^\infty \Sigma_q^n$. This implies that $Q_n(\xi), n \ge 1$, is an increasing sequence that converges to $Q(\xi)$, i.e.,
$$\lim_{n\to\infty} Q_n(\xi) = Q(\xi), \ \ \xi \in \mathbb{R}. \tag{41}$$
Thus for sufficiently small $\epsilon > 0$ chosen later, there exists an integer N such that
$$Q(\xi_0) - \varepsilon \le Q_N(\xi_0) \le Q_n(\xi_0) \le Q(\xi_0) < 1 \ \text{ for all } n \ge N. \tag{42}$$
For any $\delta \in \Sigma_q^n$ being τ-regular,
$$\lim_{m\to\infty} H_m\left(\xi + \Pi_{\tau,m}(\delta)\right) = \lim_{m\to\infty} H_m\left(\xi + \Pi_{\tau,\infty}(\delta)\right)$$
$$= \widehat{\mu_{q,b}}\left(\xi + \Pi_{\tau,\infty}(\delta)\right), \quad \xi \in \mathbb{R}. \tag{43}$$
For any $\delta \in \Sigma_q^n$ such that δ is not τ-regular, the set $\{m \ge n+1: \tau(\delta|_m) \ne 0\}$ contains infinite many integers. Denote that set by $\{m_j, j \ge 1\}$ for some strictly increasing sequence $\{m_j\}_{j=1}^\infty$. Recall that
$$\tau\left(\delta|_{m_j}\right) \in q\mathbb{Z} \cap \{-1,1,2,\ldots,b-2\} \ \text{ for all } j \ge 1 \tag{44}$$
by the tree mapping property for τ. Therefore for $m_j \le m < m_{j+1}$ with $j \ge 1$,
$$|H_m(\xi + \Pi_{\tau,m}(\delta))| \le |H_{m_j}\left(\xi + \Pi_{\tau,m}(\delta)\right)| = \left|H_{m_j}\left(\xi + \Pi_{\tau,m_j}(\delta)\right)\right|$$
$$\le \prod_{k=1}^{j-1} \left|H_{q,b}(\xi + \Pi_{\tau,m_j}(\delta))/b^{m_k+1})\right|$$
$$= \prod_{k=1}^{j-1} \left|H_{q,b}(\xi + \Pi_{\tau,m_k}(\delta))/b^{m_k+1})\right|$$

$$\leq \left(\sup_{b\eta\in T_b}\left|H_{q,b}(\eta)\right|\right)^{j-1} |j-1,\ \ \xi \in T_b, \tag{45}$$

where the inequalities follow from (24), (27) and (44), and the equalities hold by the tree mapping property τ and the q/b periodicity of the filter $H_{q,b}$. Combining (27) and (45) proves that

$$\lim_{m\to\infty}\left|H_m\left(\xi+\Pi_{\tau,m}(\delta)\right)\right| = 0,\quad \xi \in T_b \tag{46}$$

if $\delta \in \Sigma_q^n$ is not τ-regular. Applying (43) and (46) with n and ξ replaced by N and ξ_0 respectively, we can find a sufficient large integer $M \geq N+1$ such that

$$\sum_{\delta\in\Sigma_q^N}\left|H_M\left(\xi_0+\Pi_{\tau,M}(\delta)\right)\right|^2 \leq \sum_{\lambda\in\Lambda_N}\left|\widehat{\mu_{q,b}}(\xi_0+\lambda)\right|^2+\varepsilon \leq Q(\xi_0)+\varepsilon. \tag{47}$$

This together with Lemma (5.1.9) implies that

$$\sum_{\delta\in\Sigma_q^M\backslash\Sigma_q^N}\left|H_M\left(\xi_0+\ \Pi_{\tau,M}(\delta)\right)\right|^2 > 1-Q(\xi_0)-\varepsilon>0, \tag{48}$$

where

$$\Sigma_q^M\backslash\Sigma_q^N = \delta\in\Sigma_q^M : \delta|_N0^\infty \neq \delta 0^\infty$$

Now, for each $\in \Sigma_q^M\backslash\Sigma_q^N$, let $\lambda(\delta)=\Pi_{\tau,\infty}(\delta\delta')$ with δ' selected as in Lemma (5.1.3). Observe that $\lambda(\delta)-\Pi_{\tau,M}(\delta)\in b^M\mathbb{Z}$ for all $\delta\in\Sigma_q^M\backslash\Sigma_q^N$. This implies that $\lambda(\delta_1)\neq\lambda(\delta_2)$ for two distinct $\delta_1,\delta_2\in\Sigma_q^M\backslash\Sigma_q^N$. Therefore

$$\begin{aligned} Q(\xi_0) = \sum_{\lambda\in\Lambda}\left|\widehat{\mu_{q,b}}(\xi_0+\lambda)\right|^2 &\geq \sum_{\lambda\in\Lambda_x}\left|\widehat{\mu_{q,b}}(\xi_0+\lambda)\right|^2 + \sum_{\delta\in\Sigma_q^M\backslash\Sigma_q^N}\left|\widehat{\mu_{q,b}}\big(\xi_0+\lambda(\delta)\big)\right|^2 \\ &\geq Q(\xi_0)-\varepsilon+r^{4D_\tau+4}\sum_{\delta\in\Sigma_q^M\backslash\Sigma_q^N}\left|H_M\left(\xi+\Pi_{\tau,M}(\delta)\right)\right|^2 \\ &\geq Q(\xi_0)-\varepsilon+r^{4D_\tau+4}(1-Q(\xi_0)-\varepsilon), \end{aligned}$$

where the second inequality follows from (41) and Lemma (5.1.3), and the last holds by (48). This contradicts to (38) by letting ε chosen sufficiently small.

Given a tree mapping τ, define

$$N_\tau(n) := \begin{cases} \inf\limits_{0\neq\delta\in\Sigma_q} D_{\tau,\delta}(0^\infty)\ if\ n=1 \\ \inf\limits_{\delta\in\Sigma_q^n\backslash\Sigma_q^{n-1}} D_{\tau,\delta}(0^\infty)\ if\ n\geq 2' \end{cases} \tag{49}$$

where $\Sigma_q^n\backslash\Sigma_q^{n-1} := \{\delta' j : \delta'\in\Sigma_q^{n-1}, 1\leq j\leq q-1\}$. In this section, we establish the following strong version of Theorem (5.1.12).

Theorem (5.1.11)[291]: Let $2\leq q, b/q\in\mathbb{Z}, \tau:\Sigma_q^*\to\{-1,0,\ldots,b-2\}$ be a maximal tree mapping, and let $N_\tau(n), n\geq 1$, be as in (49). Set

$$r_2 := max\{|H_{q,b}(\xi)| : 1/b \leq b(b-1)|\xi| \leq b-2\}.$$

If $\sum_{n=1}^\infty r_2^{2N_\tau(n)}<\infty$, then $\Lambda(\tau)$ in (5) is not a spectrum of $L^2(\mu_{q,b})$.

Proof. Let $N_0\geq 2$ be so chosen that $N_\tau(n)\geq 1$ for all $n\geq N_0$. The existence follows the series convergence assumption on $N_\tau(n), n\geq 1$. Take $\delta\in\Sigma_q^n\backslash\Sigma_q^{n-1}$ being τ-regular, where $n\geq N_0$. Write $\{m\geq n+1 : \tau(\delta|_m)\neq 0\}=\{n_k : 1\leq k\leq K\}$ for some integers $n<n_1<n_2<\ldots<n_K$, where $K\geq N_\tau(n)$. Therefore for $\xi\in T_b$,

$$\left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,\infty}(\delta)\right)\right| = \left|H_n\left(\xi+\Pi_{\tau,\infty}(\delta)\right)\right|\cdot\left|\widehat{\mu_{q,b}}(\xi+\Pi_{\tau,\infty}(\delta))/b^n)\right|$$

$$\leq \left|H_n\left(\xi+\Pi_{\tau,n}(\delta)\right)\right|\cdot\prod_{k=1}^{k}\left|H_{q,b}(\xi+\Pi_{\tau,n_k}(\delta))/b^{-n_k-1})\right|$$

$$\leq\left(\sup_{\eta\in T_b}|H_{q,b}(\eta/b)|\right)^K\cdot|H_n(\xi+\Pi_{\tau,n}(\delta))|$$

$$\leq r_2^{N_\tau(n)}\left|H_n\left(\xi+\Pi_{\tau,n}(\delta)\right)\right|, \qquad (50)$$

where the first equality holds by (19); the first inequality follows from (19), (27) and $\tau(\delta|_{n_k})\in q\mathbb{Z}, 1\leq k\leq K$, by the tree mapping property for τ; the second inequality is true since $(\xi+\Pi_{\tau,n_k}(\delta))/b^{-n_k}\in T_b$ by (24); and the last inequality follows from the definition of the quality $N_\tau(n)$. Let Λ_n and $Q_n, n\geq 1$, be as in (39) and (40) respectively, and set $\Lambda_0=\{0\}$ and $Q_0(\xi)=\left|\widehat{\mu_{q,b}}(\xi)\right|^2$. Then for $n\geq 1$ and $\xi\in T_b$,

$$1-Q_n(\xi)=1-Q_{n-1}(\xi)-X\ \delta\in\sum_{\Sigma_q^n\backslash\Sigma_q^{n-1} is\ \tau-regular}\left|\widehat{\mu_{q,b}}\left(\xi+\Pi_{\tau,\infty}(\delta)\right)\right|^2$$

$$\geq 1-Q_{n-1}(\xi)-r_2^{2N_\tau(n)}\sum_{\delta\in\Sigma_q^n\backslash\Sigma_q^{n-1}}\left|H_n\left(\xi+\Pi_{\tau,n}(\delta)\right)\right|^2$$

$$\geq 1-Q_{n-1}(\xi)-r_2^{2N_\tau(n)}\sum_{\lambda\in\Lambda_{n-1}}\left|\mu_{q,b}(\xi+\lambda)\right|^2$$

$$=(1-r_2^{2N_\tau(n)}\cdot 1-Q_{n-1}(\xi), \qquad (51)$$

where the first equality holds because

$$\Lambda_n\backslash\Lambda_{n-1}=\left\{\Pi_{\tau,\infty}(\delta):\delta\in\Sigma_q^n\backslash\Sigma_q^{n-1}\ is\ \tau-\text{regular}\right\};$$

the first inequality is true by (48); and the second inequality follows from Lemma (5.1.9) and

$$\sum_{\lambda\in\Lambda_{n-1}}\left|\widehat{\mu_{q,b}}(\xi+\lambda)\right|^2\leq\sum_{\delta\in\Sigma_q^{n-1}}\left|H_n\left(\xi+\Pi_{\tau,n}(\delta)\right)\right|^2, \xi\in\mathbb{R},$$

by (19) and (27). Recall that $\lim_{n\to\infty}Q_n(\xi)=Q(\xi),\xi\in\mathbb{R}$, by (41). Applying (50) repeatedly and using the convergence of $\sum_{n=1}^\infty r_2^{2N_\tau(n)}$ gives

$$1-Q(\xi)\geq\left(\prod_{n=N_0+1}^{\infty}\left(1-r_2^{2N_\tau(n)}\right)\right)\cdot\left(1-Q_{N_0}(\xi)\right),\quad \xi\in T_b. \qquad (52)$$

On the other hand,

$$Q_{N_0}(\xi)=\sum_{\lambda\in\Lambda_{N_0}}\left|\widehat{\mu_{q,b}}(\xi+\lambda)\right|^2<\sum_{\delta\in\Sigma_q^{N_0}}\left|H_{N_0}\left(\xi+\Pi_{\tau,N_0}(\delta)\right)\right|^2=1,\xi\in T_b$$

by (19), (27) and (36). This together with (51) proves that $Q(\xi)<1$ for all $\xi\in T_b$, and hence $\Lambda=\Lambda(\tau)$ is a not a spectrum for $L^2(\mu_{q,b})$ by Lemma (5.1.6).

For a maximal tree mapping τ satisfying (11),

$$\sum_{n=1}^{\infty}r_2^{2N_\tau(n)}\leq\sum_{n=1}^{\infty}r_2^{2\epsilon_0 n}<\infty,$$

where the last inequality holds as $|H_{q,b}(\xi)|<1$ if $b_\xi\notin q\mathbb{Z}$. This together with Theorem (5.1.11) proves Theorem (5.1.12). Now it remains to prove Theorem (5.1.11).

We first prove Theorem (5.1.12). We then consider verification of maximal orthogonality of the rescaled set $K\Lambda$. We show that the rescaled set $K\Lambda$ is not a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ if and only if the labeling tree $\tau(\Sigma_q^*)$ has certain periodic properties (66) and (68).

By Theorem (5.1.12) and Theorem (5.1.15), we see that the rescaled set $K\Lambda$ is a spectrum if and only if the labeling tree of Λ contains no repetend of K. For the spectrum Λ_4 of the Bernoulli convolution μ_4 in (12), the associated maximal tree mapping $\tau_{2,4}$ on Σ_2^* is given by

$$\tau_{2,4}(\delta) = \delta_n \text{ for } \delta = \delta_1 \cdots \delta_n \in \Sigma_2^n, n \geq 1. \qquad (53)$$

Thus $D_{\tau_{2,4},\delta} = 0$ for all $\delta \in \Sigma_2^*$, and the requirement (37) is satisfied for the maximal tree mapping $\tau_{2,4}$. Hence Corollary (5.1.5) follows immediately from Theorem (5.1.12) and Theorem (5.1.15). Finally, we construct a spectrum Λ of the Cantor measure $\mu_{q,b}$ such that $\Lambda/(b-1)$, a seemingly denser set than the spectrum Λ, is its maximal orthogonal set but not its spectrum.

Theorem (5.1.12)[291]: Let $2 \leq q, b/q \in \mathbb{Z}, \tau: \Sigma_q^* \to \{-1,0,\dots,b-2\}$ be a maximal tree mapping satisfying (37), and $\Lambda(\tau)$ be as in (5). Then for any integer K being prime with $b, K\Lambda(\tau)$ is a spectrum of the Cantor measure $\mu_{q,b}$ if and only if it is a maximal orthogonal set.

Proof. The necessity is obvious. Now we prove the sufficiency. Without loss of generality, we assume K is positive since $-\Lambda$ is a spectrum (maximal orthogonal set) if and only if Λ is. Let κ be the maximal tree mapping associated with the maximal orthogonal set $K\Lambda$ of the Cantor measure $\mu_{(q,b)}$. The existence of such a mapping follows from Theorem (5.1.10) and the assumption on $K\Lambda$. Denote the integral part of a real number x by $\lfloor x \rfloor$. By Theorem (5.1.10) and the assumption that $D_\tau < \infty$, it suffices to prove that

$$\inf\{D_{\kappa,\delta}(\delta'), \delta' \in \Sigma_q^*\} \leq (2\lfloor \log_b K \rfloor + 4)(D_\tau + 1), \ \ \delta \in \Sigma_q^*. \qquad (54)$$

Take $\delta \in \Sigma_q^n, n \geq 1$, and let $\delta_1 \in \Sigma_q^*$ be so chosen that $\delta\delta_1$ is κ-regular. As $\Pi_{\kappa,\infty}(\delta\delta_1) \in K\Lambda$, there exists $\zeta \in \Sigma_q^n$ such that

$$K\Pi_{\tau,n}(\zeta) - \Pi_{\kappa,n}(\delta) \in b^n\mathbb{Z}. \qquad (55)$$

Let $\zeta' \in \Sigma_q^*$ be so chosen that $\zeta\zeta'$ is a τ-main subbranch of ζ and

$$D_{\tau,\zeta}(\zeta') = D_{\tau,\zeta}, \qquad (56)$$

where the existence of such a tree branch ζ' follows from (10). Therefore the verification of (54) reduces to showing the existence of $\delta' \in \Sigma_q^*$ such that $\delta\delta'$ is a κ-main branch,

$$K\Pi_{\tau,\infty}(\zeta\zeta') = \Pi_{\kappa,\infty}(\delta\delta'), \qquad (57)$$

and

$$D_{\kappa,\delta}(\delta') \leq (2\lfloor \log_b K \rfloor + 4)\big(D_{\tau,\zeta} + 1\big). \qquad (58)$$

By Theorem (5.1.10), there exists a κ-main branch $\delta_2 \in \Sigma_q^*$ such that

$$\Pi_{\kappa,\infty}(\delta_2) = K\Pi_{\tau,\infty}(\zeta\zeta'). \qquad (59)$$

Then

$$\Pi_{\kappa,n}(\delta_2) - \Pi_{\kappa,n}(\delta) \in K\Pi_{\tau,n}(\zeta) - \Pi_{\kappa,n}(\delta) + b^n\mathbb{Z} = b^n\mathbb{Z} \qquad (60)$$

by (55). This together with one-to-one correspondence of the mapping $\Pi_{\kappa,n}: \Sigma_q^n \to \mathbb{Z}$ proves $\delta_2 = \delta\delta'$ for some $\delta' \in \Sigma_q^*$. The equation (57) follow from (59). Now it remains to prove (58). Without loss of generality, we assume that $\Pi_{\kappa,\infty}(\delta\delta') \neq \Pi_{\kappa,n}(\delta)$, because otherwise $D_{\kappa,\delta}(\delta') = 0$ and hence (58) follows immediately. Thus we may write

$$\Pi_{\kappa,\infty}(\delta\delta') = \Pi_{\kappa,n}(\delta) + \sum_{l=1}^{L} d_l b^{n+m_l-1} \tag{61}$$

for a strictly increasing sequence $\{m_l\}_{l=1}^{L}$ of integers and some $d_l \in \{-1,1,\ldots,b-2\}, 1 \le l \le L$. Also we may assume that $\Pi_{\tau,\infty}(\zeta\zeta') \ne \Pi_{\tau,n}(\zeta)$, because otherwise $K\Pi_{\tau,\infty}(\zeta\zeta') = K\Pi_{\tau,n}(\zeta) \in K(-b^n/(b-1),(b-2)b^n/(b-1))$ and

$$\Pi_{\kappa,\infty}(\delta\delta') < (-b^{n+m_L-1}/(b-1),(b-2)b^{n+mL-1}/(b-1))$$

by (24) and (61). This together with (57) implies that $b^{m_L-1} \le K$ and hence

$$D_{\kappa,\delta}(\delta') \le m_L \le \lfloor \log_b K \rfloor + 1.$$

Therefore we can write

$$\Pi_{\tau,\infty}(\zeta\zeta') = \Pi_{\tau,n}(\zeta) + \sum_{j=1}^{N} c_j b^{n+n_j-1},$$

where $c_j \in \{-1,1,\ldots,b-2\}, 1 \le j \le N$, and $\{n_j\}_{j=1}^{N}$ is a strictly increasing sequence of integers. To prove (58) for the case that $\Pi_{\tau,\infty}(\zeta\zeta') \ne \Pi_{\tau,n}(\zeta)$, we need the following claim:

Claim (5.1.13)[291]: $\{m_l, 1 \le l \le L\} \subset \cup_{j=0}^{N} [n_j, n_j + \lfloor \log_b K \rfloor + 1]$.

Proof. Suppose, on the contrary, that Claim (5.1.13) does not hold. Then there exists $1 \le l \le L$ such that $n_{j_0} + \lfloor \log_b K \rfloor + 1 < m_l < n_{j_0+1}$ for some $0 \le j_0 \le N$, where we set $n_0 = 0$ and $n_{N+1} = +\infty$. Observe that

$$\Pi_{\kappa,n} + m_l(\delta\delta') - K\Pi_{\tau,n+n_{j_0}}(\zeta\zeta') \in b^{n+m_l}\mathbb{Z} \tag{62}$$

by (57) and the assumption $m_l < n_{j_0} + 1$, and

$$\begin{aligned}\Pi_{\kappa,n+m_l}(\delta\delta') - K\Pi_{\tau,n+n_{j_0}}(\zeta\zeta') \\ &\in d_l b^{n+m_l-1} + \frac{b^{n+m_l-1}-1}{b-1}[-1,b-2] - K\frac{b^{n+n_{j_0}}-1}{b-1}[-1,b-2] \\ &\subset d_l b^{n+m_l-1} + (-b^{n+m_l-1}, b^{n+m_l-1})\end{aligned} \tag{63}$$

by the definitions of $\Pi_{\kappa,n+m_l}$ and $\Pi_{\tau,n+n_{j_0}}$ and the assumption $n_{j_0} + \log_b K + 1 < m_l$. Combining (62) and (63) leads to the contradiction that $d_l \in \{-1,1,\ldots,b-2\}$. This completes the proof of Claim (5.1.13).

To prove (58) for the case that $\Pi_{\tau,\infty}(\zeta\zeta') \ne \Pi_{\tau,n}(\zeta)$, we need another claim:

Claim (5.1.14)[291]: If $n_j + \lfloor \log_b K \rfloor + 1 < n_{j+1}$, then there exists l_0 such that $m_{l_0} = n_{j+1}$, $m_{l_0-1} \in [n_j, n_j + \lfloor \log_b K \rfloor + 1]$ and $d_{l_0} \in q\mathbb{Z}$ if and only if $c_{j+1} \in q\mathbb{Z}$.

Proof. Let l_0 be the smallest integer l with $m_l \ge n_{j+1}$. By Claim (5.1.13), $m_{l_0-1} \le n_j + \lfloor \log_b K \rfloor + 1 \le n_{j+1} - 1$. Observe that $\Pi_{\kappa,n+ml_0}(\delta\delta') - K\Pi_{\tau,n+n_{j+1}}(\zeta\zeta') \in b^{n+n_{j+1}}\mathbb{Z}$ by (57); and

$$\begin{aligned}\Pi_{\kappa,n+m_{l_0}}(\delta\delta') - K\Pi_{\tau,n+n_{j+1}}(\zeta\zeta') \\ &\in d_{l_0} b^{n+m_{l_0}-1} - Kc_{j+1}b^{n+n_{j+1}-1} + \frac{b^{n+m_{l_0}-1}}{b-1}(-1,b-2) - \frac{Kb^{n+n_j}}{b-1}(-1,b-2) \\ &\subset d_{l_0} b^{n+m_{l_0}-1} - Kc_{j+1}b^{n+n_{j+1}-1} + b^{n+n_{j+1}-1}(-1,1).\end{aligned} \tag{64}$$

Thus $d_{l_0} b^{m_{l_0}-n_{j+1}} - Kc_{j+1} \in b\mathbb{Z}$. This together, with the assumptions that $c_{j+1} \in \{-1,1,\ldots,b-2\}$ and that K and b are coprime, implies that $m_{l_0} = n_{j+1}$ and $d_{l_0} \in q\mathbb{Z}$ if and only if $c_{j+1} \in q\mathbb{Z}$. From the argument in (64), we see that

$$\Pi_{\kappa,m_{l_0}-1}(\delta\delta') = K\Pi_{\tau,n_j}(\zeta\zeta'). \tag{65}$$

Thus $m_{l_0-1} \geq n_j$, as $\Pi_{\kappa,m_{l_0}-1}(\delta\delta') \in b^{m_{l_0-1}}(-1/(b-1),(b-2)/(b-1)$ and $K\Pi_{\tau,n_j}(\zeta\zeta') \notin Kb^{n_j-1}(-1/(b-1),(b-2)/(b-1)$by (24). This completes the proof of Claim (5.1.14).

Having established the above two claims, let us return to the proof of the inequality (58). Note that if

$$\{k \in \mathbb{Z}: m_{l_0-1} < k < m_{l_0}\} \not\subset \cup_{j=0}^{N} [n_j, n_j + \lfloor \log_b K \rfloor + 1]$$

for some $1 \leq l_0 \leq L$, then by Claim (5.1.13), there exists $1 \leq j_0 \leq N$ such that

$$m_{l_0-1} \leq n_{j_0-1} + \lfloor \log_b K \rfloor + 1 < n_{j_0} \leq m_{l_0}.$$

Then $m_{l_0} = n_{j_0}, m_{l_0-1} \geq n_{j_0-1}$ and $d_{l_0} \in q\mathbb{Z}$ if and only if $c_{j_0} \in q\mathbb{Z}$ by Claim (5.1.14). Thus $\cup_{d_l \notin q\mathbb{Z}} (m_{l-1}, m_l) \subset (\cup_{j=0}^{N} [n_j, n_j + \lfloor \log_b K \rfloor + 1] \cup (\cup_{c_j \notin q\mathbb{Z}} (n_{j-1}, n_j))$, and thus

$$\sum_{d_l \notin q\mathbb{Z}} (m_l - m_{l-1} - 1) \leq (\lfloor \log_b K \rfloor + 2)(N+1) + \sum_{c_j \notin q\mathbb{Z}} (n_j - n_{j-1} - 1).$$

This together with Claim (5.1.13), implies

$$D_{\kappa,\delta}(\delta') \leq 2(\lfloor \log_b K \rfloor + 2)(N+1) + \sum_{c_j \notin q\mathbb{Z}} \left(n_j - n_{j-1} - 1\right)$$
$$\leq (2\lfloor \log_b K \rfloor + 4)\left(D_{\tau,\zeta}(\zeta') + 1\right).$$

We get (58) and hence complete the proof of Theorem (5.1.12).

Theorem (5.1.15)[291]:: Let $2 \leq q, b/q \in \mathbb{Z}, \tau: \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ be a maximal tree mapping, $\Lambda := \Lambda(\tau)$ be as in (5), and let $K > 1$ be an integer coprime with b. Then $K\Lambda$ is not a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ if and only if there exist $\delta \in \Sigma_q^\infty$ and a nonnegative integer M such that $\{\tau(\delta|_n)\}_{n=M+1}^{\infty}$ *is a* periodic sequence with positive period N, i.e.,

$$\tau(\delta|_n) = \tau(\delta|_{n+N}), n \geq M+1, \qquad (66)$$

and that the word $W = \omega_1\omega_2 \cdots \omega_N$ defined by

$$\omega_j = \tau(\delta|_{M+j}), 1 \leq j \leq N, \qquad (67)$$

is a repetend of the recurring b-band decimal expression of i/K for some $i \in \mathbb{Z}\backslash\{0\}$, i.e.,

$$\frac{i}{K} = 0.\omega_N \cdots \omega_2\omega_1\omega_N \cdots \omega_2\omega_1\omega_N \cdots = \sum_{n=1}^{\infty}\sum_{j=1}^{N} \omega_j b^{j-N_{n-1}} = \frac{\sum_{j=1}^{N} \omega_j b^{j-1}}{b^N - 1}. \qquad (68)$$

Proof. ($\Longleftarrow$) Let

$$\lambda_0 = K\Pi_{\tau,M}(\delta) - ib^M, \qquad (69)$$

where $i \in \mathbb{Z}$ is given in (68). Inductively applying (68) proves that

$$\lambda_0 = K\Pi_{\tau,M+N}(\delta) - ib^{M+N} = \cdots = K\Pi_{\tau,M+nN}(\delta) - ib^{M+nN}, n \geq 1. \qquad (70)$$

Take $\lambda \in \Lambda$. Now we show that $exp(-2\pi i\lambda_0 x)$ is orthogonal to $exp(-2\pi iK\lambda x)$. By the maximality of the tree mapping τ, there exists a τ-main branch $\zeta \in \Sigma_q^m$ for some $m \geq 1$ by Theorem (5.1.10) such that

$$\lambda = \Pi_{\tau,\infty}(\zeta). \qquad (71)$$

Also for sufficiently large $n \geq 1$, there exists $\lambda_n \in \Lambda$ by the maximality of the tree mapping τ such that λ_n, λ and

$$\lambda_n - \Pi_{\tau,M+N_n}(\delta) \in b^{M+N_n}\mathbb{Z}. \qquad (72)$$

The reason for $\lambda_n \neq \lambda$ is that $\Pi_{\tau,M+N_n}(\delta) \neq \Pi_{\tau,M+N_n}(\zeta)$ for sufficiently large n by $W = \omega_1 \ldots \omega_N$, 0N by (68). As both $\lambda, \lambda_n \in \Lambda$, there exists a nonnegative integer l and an integer $a \in \mathbb{Z}\backslash q\mathbb{Z}$ by (3) such that

$$\lambda - \lambda_n = ab^l. \tag{73}$$

Now we show that

$$l < M + N_n \tag{74}$$

when n is sufficiently large. Suppose, on the contrary, that $l \geq M + N_n$. Then

$$\lambda - \Pi_{\tau,M+N_n}(\delta) \in b^{M+N_n}\mathbb{Z}. \tag{75}$$

On the other hand,

$$\Pi_{\tau,M+N_n}(\delta) \in b^{M+N_n}[-1/(b-1),(b-2)/(b-1)]$$

by the tree mapping property for τ. Therefore $\lambda = \Pi_{\tau,M+N_n}(\delta)$ for sufficiently large n, which is a contradiction as

$$\Pi_{\tau,M+N_n}(\delta) \notin b^{M+N(n-1)}(-1/(b-1),(b-2)/(b-1))$$

by $W = \omega_1 \dots \omega_N$, 0N and the tree mapping property for τ. Combining (72),(73) and (74) and recalling that K and b are co-prime, we obtain that

$$K\lambda - K\Pi_{\tau,M+M_n}(\delta) = \tilde{a}b^l \tag{76}$$

for some integers $0 \leq l < M + N_n$ and $\tilde{a} \in \mathbb{Z}\backslash q\mathbb{Z}$. Thus the inner product between $exp(-2\pi i\lambda_0 x)$ and $exp(-2\pi iK\lambda x)$ is equal to zero by (3), (70) and (76). This proves that $K\Lambda$ is not a maximal orthogonal set as $\lambda \in \Lambda$ is chosen arbitrarily.

($\Longrightarrow$) By (3) and the assumption on the rescaled set $K\Lambda$, there exists a maximal orthogonal set Θ of the Cantor measure $\mu_{q,b}$ such that

$$K\Lambda \subsetneq \Theta \subset \mathbb{Z}. \tag{77}$$

Take $\vartheta_0 \in \Theta\backslash(K\Lambda)$. Then

$$\vartheta_0 = \Pi_{\kappa,\infty}(\zeta_0) = \Pi_{\kappa,m}(\zeta_0) \tag{78}$$

for some κ-main branch $\zeta_0 \in \Sigma_q^m$, $m \geq 1$, where κ is the maximal tree mapping associated with the maximal orthogonal set Θ. Let τ be the maximal tree mapping in Theorem (4) such that $\Lambda = \Lambda(\tau)$. To establish the necessity, we need the following claim:

Claim (5.1.16)[291]: Let $n \geq 1$. For any $\zeta \in \Sigma_q^n$ there exists a unique $\delta \in \Sigma_q^n$ such that $\Pi_{\kappa,n}(\zeta) - K\Pi_{\tau,n}(\delta) \in b^n\mathbb{Z}$.

Proof. Observe that

$$K\Pi_{\tau,n}(\delta_1) - K\Pi_{\tau,n}(\delta_2) \notin b^n\mathbb{Z} \text{ for all distinct } \delta_1, \delta_2 \in \Sigma_q^n, \tag{79}$$

because $b/q \in \mathbb{Z}, K$ and b are coprime, and $\Pi_{\tau,n}(\delta_1) - \Pi_{\tau,n}(\delta_2) = ab^l$ for some $0 \leq l \leq n-1$ and $a \notin q\mathbb{Z}$. On the other hand,

$$\begin{aligned}\{K\Pi_{\tau,n}(\delta): \delta \in \Sigma_q^n\} + b^n\mathbb{Z} &= K\Lambda + b^n\mathbb{Z} \subset \Theta + b^n\mathbb{Z} \\ &= \{\Pi_{\kappa,n}(\zeta): \zeta \in \Sigma_q^n\} + b^n\mathbb{Z}\end{aligned} \tag{80}$$

by (77). Combining (79) and (80) leads to

$$\{K\Pi_{\tau,n}(\delta): \delta \in \Sigma_q^n\} + b^n\mathbb{Z} = \{\Pi_{\kappa,n}(\zeta): \zeta \in \Sigma_q^n\} + b^n\mathbb{Z}. \tag{81}$$

Then Claim (5.1.16) follows from (81) and (79).

To establish the necessity, we need another claim:

Claim (5.1.17)[291]: $\vartheta_0 \notin K\mathbb{Z}$.

Proof. Suppose, on the contrary, that $\vartheta_0 \in K\mathbb{Z}$. Then for any $\lambda \in \Lambda$, there exist $a \in \mathbb{Z}\backslash q\mathbb{Z}$ and $0 \leq l \in \mathbb{Z}$ by (3) and (77) such that $\vartheta_0 - K\lambda = ab^l$. This together with the co-prime assumption between K and b implies that $a/K \in \mathbb{Z}$ and $0 \neq \vartheta_0/K - \lambda \in \left(\frac{a}{K}\right) b^l$. Thus $\Lambda \cup \{\vartheta_0/K\}$ is an orthogonal set for the measure $\mu_{q,b}$ by (3), which contradicts to the maximality of the set Λ.

Now we continue our proof of the necessity. Let N be the smallest positive integer such that $(b^N-1)\vartheta_0/K \in \mathbb{Z}$, where the existence follows from the co-prime property between K and b. By Claim (5.1.17), there exists $\omega_j \in \{-1,0,\ldots,b-2\}, 1 \le j \le N$, such that the word $W := \omega_1\omega_2 \cdots \omega_N \neq 0$ and

$$\frac{\vartheta_0}{K} = c.\,\omega_N \cdots \omega_2\omega_1\omega_N \cdots \omega_2\omega_1 \cdots = c + \frac{\sum_{j=1}^{N}\omega_j b^{j-1}}{b^N-1} \tag{82}$$

for some integer $c \in Z$. Let $W' = \omega_1'\omega_2' \cdots \omega_N'$ be so chosen that $\omega_j' \in \{-1,0,\ldots,b-2\}, 1 \le j \le N$, and

$$\sum_{j=1}^{N}(\omega_j'+\omega_j)b^{j-1} = \begin{cases} 0 \ \ if \ \sum_{j=1}^{N}\omega_j b^{j-1} \in \frac{b^N-1}{b-1}[-1,1) \\ b^N-1 \ \ if \ \sum_{j=1}^{N}\omega_j b^{j-1} \in \frac{b^N-1}{b-1}[1,b-2] \end{cases}$$

The existence of such a word W' follows from the observation that

$$\left\{\sum_{j=1}^{N}\omega_j b^{j-1}, \omega_j \in \{-1,0,\ldots,b-2\}\right\} = \left(\frac{b^N-1}{b-1}\,[-1,b-2]\right) \cap \mathbb{Z}.$$

Let $n > m/N$ and set $\zeta_{nN} = \zeta_0 0^{nN-m} \in \Sigma_q^{nN}$. By Claim (5.1.16) and the κ-main branch assumption for ζ_0, there exists $\delta_{nN} \in \Sigma_q^{nN}$ such that

$$K\Pi_{\tau,nN}(\delta_{nN}) - \vartheta_0 \in b^{nN}\mathbb{Z}. \tag{83}$$

Combining (82), (5.86) and (83) and recalling that K and b are coprime, we obtain

$$(b^N-1)(\Pi_{\tau,nN}(\delta_{nN}) - \tilde{c}) + \sum_{j=1}^{N}\omega'_j b^{j-1} \in b^{nN}\mathbb{Z},$$

where

$$\tilde{c} = \begin{cases} c \ \ if \ \sum_{j=1}^{N}\omega_j b^{j-1} \in \frac{b^N-1}{b-1}[-1,1) \\ c-1 \ \ if \ \sum_{j=1}^{N}\omega_j b^{j-1} \in \frac{b^N-1}{b-1}[1,b-2]. \end{cases}$$

Therefore

$$\Pi_{\tau,nN}(\delta_{nN}) - \tilde{c} - (\sum_{j=1}^{N}\omega_j b^{j-1})(1 + \ bN \ + \cdots + b(n-1)N \in \ bnNZ. \tag{84}$$

By the construction of $\omega_j', 1 \le j \le N, \sum_{j=1}^{N}\omega'_j b^{j-1} \in \frac{b^N-1}{b-1}(-1,b-2]$. If either $\sum_{j=1}^{N}\omega'_j b^{j-1} \in \frac{b^N-1}{b-1}(-1,b-2)$ or $\sum_{j=1}^{N}\omega'_j b^{j-1} = \frac{b^N-1}{b-1}(b-2)$ and $\tilde{c} \le 0$, then for sufficiently large k,

$$\tilde{c} + (\sum_{j=1}^{N}\omega'_j b^{j-1})(1 + b^N + \cdots + b^{(k-1)N} = \sum_{j=1}^{kN}\theta_j b^{j-1}$$

for some $\theta_j \in \{-1,0,\ldots,b-2\}, 1 \le j \le kN$, as it is contained in $[-(b^{kN}-1)/(b-1), (b^{kN}-1)(b-2)/(b-1)]$. This together with (84) implies that

$$\Pi_{\tau,nN}(\delta_{nN}) = \sum_{j=1}^{kN} \theta_j b^{j-1} + \sum_{j=1}^{N} \omega'_j b^{j-1} \left(b^{kN} + \cdots + b^{(n-1)N}\right)$$

for $n \geq k$. Thus there exists $\delta \in \Sigma_q^\infty$ such that $\delta|_{nN} = \delta_{nN}$ and

$$\tau(\delta|_{nN+j}) = \omega'_j, 1 \leq j \leq N$$

for $n \geq k$, which proves the desired conclusion. Now consider the case that $\sum_{j=1}^{N} \omega'_j b^{j-1} = \frac{b^N-1}{b-1}(b-2)$ and $\tilde{c} > 0$. In this case, $\omega'_j = b-2$ for all $1 \leq j \leq N$ and $N = 1$ by the selection of the integer N. Further we obtain from (84) that

$$\Pi_{\tau,n}(\delta_n) - \tilde{c} + 1 + \sum_{j=1}^{n} b^{j-1} \in b^n\mathbb{Z},$$

which implies that there exists $\delta \in \Sigma_q^\infty$ such that $\delta|_n = \delta_n$ and $\tau(\delta|_n) = -1$ for sufficiently large n, which proves the desired conclusion.

Theorem (5.1.18)[291]: Consider $2 \leq q, b/q \in \mathbb{Z}$ and $b > 4$. Define a tree mapping $\kappa: \Sigma_q^* \to \{-1,0,1,\ldots,b-2\}$ by

$$\kappa(\delta|_{k+1}) = \begin{cases} 0 \; if \; \delta = 0 \text{ and } k \geq 0 \\ \delta \; if \; 1 \leq \delta \leq q-1 \text{ and } k = 0 \\ q \; if \; 1 \leq \delta \leq q-1 \text{ and } k \in \{1,2,\ldots,K_\delta, 2b\} \\ 0 \; if \; 1 \leq \delta \leq q-1 \; and \; K_\delta < k \neq 2b \end{cases} if \; \delta \in \Sigma_q^1 \tag{85}$$

where $0 \leq K_\delta \leq b-2$ is the unique integer such that $q(K_\delta + 1) + \delta \in (b-1)\mathbb{Z}$; and inductively

$$\kappa(\delta|_{k+n}) = \begin{cases} j \; if \; k = 0 \\ q \; if \; k \in \{1,2,\ldots,K_\delta, n+2b-1\} \\ 0 \; if \; k > K_\delta \; and \; k \neq n+2b-1 \end{cases} \tag{86}$$

if $\delta = \delta'$ j for some $\delta' \in \Sigma_q^{n-1}, n \geq 2$ and $j \in \{1,\ldots,q-1\}$, where $K_\delta \in \{0,1,\ldots,b-2\}$ is the unique integer such that

$$\sum_{i=1}^{n-1} \kappa(\delta|_i) + q(K_\delta + 1) + j) \in (b-1)\mathbb{Z}. \tag{87}$$

Then

$$\Lambda_{q,b} := \{\Pi_{\kappa,\infty}(\delta): \delta \in \Sigma_q^*\} \tag{88}$$

is a spectrum of the Cantor measure $\mu_{q,b}$, and the rationally rescaled set $\Lambda_{q,b}/(b-1)$ is its maximal orthogonal set but not its spectrum.

Proof. First we show that $\Lambda_{q,b}$ is a spectrum of the Cantor measure $\mu_{q,b}$. Observe that κ is a maximal tree mapping, every $\delta \in \Sigma_q^*$ is κ-regular, and $\Lambda_{q,b} = \Lambda(\kappa)$. We then obtain from Theorem (5.1.10) that

$$\Lambda_{q,b} \text{ is a maximal orthogonal set of the Cantor measure } \mu_{q,b}. \tag{89}$$

From the definition of the maximal tree mapping κ it follows that

$$D_{\kappa,\delta} \leq D_{\kappa,\delta}(0^\infty) \leq K_\delta + 1 \leq b-1 \text{ for all } \delta \in \Sigma_q^*, \tag{90}$$

where K_δ is given in (17). Therefore the spectral property for $\Lambda_{q,b}$ holds by (89), (90) and Theorem (2). Next we prove that $\Lambda_{q,b}/(b-1)$ is a maximal orthogonal set for the Cantor measure $\mu_{q,b}$. From (3) and the spectral property for the set $\Lambda_{q,b}$ We obtain that

$$\Lambda_{q,b} - \Lambda_{q,b} \subset \{b^j a: 0 \leq j \in \mathbb{Z}, a \in \mathbb{Z}\backslash q\mathbb{Z}\} \cup \{0\}. \tag{91}$$

On the other hand,

$$0 \in \Lambda_{q,b} \subset \mathbb{Z}$$

and for any $\delta \in \Sigma_q^*$,

$$\Pi_{\kappa,\infty}(\delta) = \sum_{j=1}^{\infty} \kappa(\delta|_j) b^{j-1} \in \sum_{j=1}^{\infty} \kappa(\delta|_j) + (b-1)\mathbb{Z} = (b-1)\mathbb{Z} \tag{92}$$

by (25)–(26). Combining (78) and (79) leads to

$$(\Lambda_{q,b} - \Lambda_{q,b})/(b-1) \subset \{b^j a: 0 \le j \in \mathbb{Z}, a \in \mathbb{Z}\backslash q\mathbb{Z}\} \cup \{0\},$$

and hence $\Lambda_{q,b}/(b-1)$ is an orthogonal set for the Cantor measure $\mu_{q,b}$ by (3). Now we establish the maximality of the rescaled set $\Lambda_{q,b}/(b-1)$. Suppose, on the contrary, that there exists $\lambda_0 < \Lambda_{q,b}/(b-1)$ such that $\tilde{\Lambda}_{q,b} := \Lambda_{q,b}/(b-1) \cup \{\lambda_0\}$ is an orthogonal set for the Cantor measure $\mu_{q,b}$. Then

$$(b-1)\tilde{\Lambda}_{q,b} - (b-1)\tilde{\Lambda}_{q,b} \subset (b-1)\{b^j a: 0 \le j \in \mathbb{Z}, a \in \mathbb{Z}\backslash q\mathbb{Z}\} \cup \{0\} \subset \{b^j a : 0 \le j \in \mathbb{Z}, a \in \mathbb{Z}\backslash q\mathbb{Z}\} \cup \{0\}$$

and $(b-1)\tilde{\Lambda}_{q,b}$ is an orthogonal set for the Cantor measure $\mu_{q,b}$ by (3). This contradicts the spectral property for $\Lambda_{q,b}$. Finally we prove that $\Lambda_{q,b}/(b-1)$ is not a spectrum of the Cantor measure $\mu_{q,b}$. Let $\tau_{q,b}: \Sigma_q^* \to \{-1,0,\ldots,b-2\}$ be the maximal tree mapping such that $\Lambda_{q,b}/(b-1) = \Lambda(\tau_{q,b})$. By Theorem (5.1.11), the non-spectral property for the set $\Lambda_{q,b}/(b-1)$ reduces to showing that

$$D_{\tau_{q,b},\delta}(0^\infty) \ge n \tag{93}$$

for all $\delta \in \Sigma_q^n \backslash \Sigma_q^{n-1}, n \ge 2$,being τ_q, b-regular. Recall that $\Lambda_{q,b} = \Lambda(\kappa)$. This together with (25) and (26) implies the existence of $\eta \in \Sigma_q^m, m \ge 1$, such that

$$(b-1)\Pi_{\tau_{q,b},\infty}(\delta) = \Pi_{\kappa,\infty}(\eta) = \sum_{j=1}^{m+b-2} d_j b^{j-1} + q \cdot b^{2m+2b-2}, \tag{94}$$

where $d_j \in \{0,1,\cdots,q\}$ for all $1 \le j \le m+b-2$ and $d_m \in \{1,\ldots,q-1\}$. Write

$$\Pi_{\tau_{q,b},\infty}(\delta) = \sum_{j=1}^{\infty} c_j b^{j-1} = \sum_{j=1}^{M} c_j b^{j-1} \tag{95}$$

where $c_j := \tau_{q,b}(\delta|_j) \in \{-1,0,\ldots,b-2\}$ and $M \ge n$ is so chosen that $c_M \ne 0$. The existence of such an integer follows from $\tau_{q,b}(\delta|_n) \in \mathbb{Z}\backslash q\mathbb{Z}$ and $\tau_{q,b}(\delta|_j) = 0$ for sufficiently large j. Combining (94) and (95) leads to

$$\sum_{j=1}^{M} c_j b^{j-1} = \frac{1}{b-1}\left(\sum_{j=1}^{m+b-2} d_j b^{j-1} + q \cdot b^{m+b-2}\right) + q \sum_{j=m+b-2}^{2m+2b-3} b^j$$
$$\in q \sum_{j=m+b-2}^{2m+2b-3} b^j + \left(0, \frac{b-2}{b-1}\right) b^{m+b-2},$$

where the last inequality follows as $q \le b-3$. This, together with $c_j \in \{-1,0,1,\ldots,b-2\}, 1 \le j \le M$, implies that

$$M = 2m + 2b - 2 \text{ and } c_j = q, m+b-2 < j \le M. \tag{96}$$

On the other hand, for $\delta \in \Sigma_q^n \backslash \Sigma_q^{n-1}$ it follows from the tree mapping property for $\tau_{q,b}$ that $c_n \notin q\mathbb{Z}$. Thus $n \le m+b-2$ according to (96). Therefore

$$D_{\tau_{q,b},\delta}(0^\infty) \ge M - (m+b-2) \ge n.$$

This proves (93) and then the conclusion that $\Lambda_{q,b}$ is not a spectrum of the Cantor set $\mu_{q,b}$ by Theorem (5.1.11).

Section (5.2): A class of Random Convolution on $\mathbb{R}$:

For μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. One fundamental problem in Fourier analysis is to find a sequence $\Lambda \subseteq \mathbb{R}^d$ such that the family of complex exponential functions $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$, the space of all square-integrable functions with respect to the measure μ. In this case, the measure μ is called a spectral measure and Λ is called a spectrum for μ. We also say that (μ, Λ) is a spectral pair. The study on spectral measures has a long history, e.g., see [352], and has been attracted much attention after the work of Fuglede [338].

Jorgensen and Pedersen [347] initiated an investigation of spectral property of the fractal measures. They showed that the infinite Bernoulli convolution μ_ρ is a spectral measure if $\rho = 2k$ for $k \in \mathbb{N}$, and is not a spectral one if $\rho = 2k + 1$ for $k \in \mathbb{N}$. Recently, Dai [321] showed that the scales $2k$ are the only values that generate spectral Bernoulli convolutions. Actually, Jorgensen and Pedersen's example opened up a new area in researching the harmonic analysis on fractals. Many other interesting singular measures which admit orthonormal Fourier series have been constructed, see [318–320,325,326, 330,334,336,339,340,348,353–355], etc.

It is well known that a given singular spectral measure has more than one spectrum which is not obtained by translations of each other. Hence a natural question is: can we construct all spectra for a given spectral measure? It is a very challenging question. Motivated by this question, many found various method to construct spectra for a given spectral measure, see [321–323,325,326,330–332,335,348–350]. Among these results, one basic but most important constructing method is to check that whether the scaling set of a spectrum by a real number is also a spectrum. The first spectrum for the Bernoulli convolution $\mu_{2k}(k \in \mathbb{N})$ given in [347] is

$$\Lambda_0(2k, C) = \left\{\sum_{j=1}^{m} (2k)^{j-1}\, c_j : c_j \in C, m \in \mathbb{N}\right\} \text{ with } C = \left\{0, \frac{k}{2}\right\}.$$

Later on, Jorgensen et al. [343], Li [349,350] provided some conditions on the integer numbers p for the scaling set

$$p\Lambda_0(2k, C) = p\left\{\sum_{j=1}^{m} (2k)^{j-1}\, c_j : c_j \in C, m \in \mathbb{N}\right\} \text{ with } C = \left\{0, \frac{k}{2}\right\}.$$

to be a spectrum for μ_{2k}. In particular, Laba and Wang [348], Dutkay and Jorgensen [332], Dutkay and Haussermann [328] studied for what digits $\{0, p\}$, with p odd, the scaling set

$$p\Lambda_0(4, C) = p\left\{\sum_{j=1}^{m} 4^{j-1}\, c_j : c_j \in C, m \in \mathbb{N}\right\} \text{ with } C = \{0,1\}$$

is a spectrum for μ_4.

Based on these researches, Fu, He and Wen [335] borrowed the notation "eigenvalue" in linear algebra to describe the above phenomena, and discovered a class of new spectra for the measure $\mu_{2k}, k > 1$ mentioned above.

Definition (5.2.1)[317]: Let μ be a Borel probability measure on $\mathbb{R}$. A real number p is called a spectral eigenvalueof μif there exists a discrete set Λ such that both Λ and $p\Lambda$ are spectra for μ. The set Λ is called an eigen-spectrum for μ corresponding to the eigenvalue p.

The study on the spectral eigenvalues of a given singular spectral measure is of interest and helps us find more surprising facts. For example, the eigen-spectra corresponding to a eigenvalue may have the cardinality of the continuum, and the Fourier series associated to the eigen-spectra corresponding to the same eigenvalue may have the same convergence property, see [335]. Other interesting connections between the spectral eigenvalues and Fourier analysis, number theory, dimension theory, operator theory and ergodic theory have been found by Strichartz [354][355], Dutkay and [326][328][333], Jorgensen, Kornelson and Shuman [344][346] and Jorgensen [342], etc. Motivated by the analysis for the Bernoulli convolutions μ_{2k} in [335], we determine the spectral eigenvalues of the following three (singular) spectral measures respectively.

(a) Let $\mu_{\rho,q}$ be a self-similar measure arising from the iterated function system (IFS) $\{\rho^{-1}(x+i): i \in D\}$ on $\mathbb{R}$, where $\rho \in \mathbb{N}$ and $D = \{0, 1,\cdots, q-1\}$. An equivalent expression of the measure $\mu_{\rho,q}$ is the following infinite convolution

$$\mu_{\rho,q} := \delta_{\rho^{-1}D^*}\delta_{\rho^{-2}D^*}\cdots \quad (97)$$

in the weak*-topology (see, e.g. [351]). Here, the symbol $*$denotes the convolution of two measures and δ_E for a finite set E denotes the atomic measure

$$\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$$

where δ_e is the Dirac point mass measure at the point $e, rE = \{re: e \in E\}$ and $\#E$ is the cardinality of E. In 2014, Dai, He and Lau [324] gave the following complete characterization on the spectrality and non-spectrality of $\mu_{\rho,q}$.

Theorem (5.2.2)[317]: The Cantor measure $\mu_{\rho,q}$ is a spectral measure if and only if $q \geq 2$ and $q|\rho$.

(b) Let $\rho > 1$ and let $\{a_k, b_k\}_{k=1}^{\infty}$ be a sequence of integers with bounded from upper and lower, then the measure $\mu_\rho, \{a_k, b_k\}$, called infinite Bernoulli convolution by An, He and Li [320], is defined by

$$\mu_\rho, \{a_k, b_k\} = \delta_{\rho^{-1}\{a_1,b_1\}} * \delta_{\rho^{-2}\{a_2,b_2\}} *\cdots$$

in the weak*-topology.

[320], showed that if $\mu_{\rho,\{a_k,b_k\}}$ is a spectral measure, then ρ is an even integer, which extends the very nice result of Dai [321]. On the contrary, they obtained a sufficient but not necessary condition for a large class of measures $\mu_{\rho,\{a_k,b_k\}}$ to be spectral. More explicitly, let $d_k := b_k - a_k = 2^{l k}m_k$ with all m_k's are odd positive integers, $L = \max_{k\geq 1} l_k < \infty$, their main result [320] showed that

Theorem (5.2.3)[317]: Let $\rho = 2^{l+1}q$ be an integer such that $l > L$if $q = 1$ and $l \geq L$ if the odd number $q > 1$. Then the infinite Bernoulli convolution $\mu_\rho, \{a_k, b_k\}$ is a spectral measure.

(c) Let $\rho > 1$ and let $\{a_k, b_k, c_k\}_k^{\infty} = 1$ be a sequence of integers with bounded from upper and lower, then $\mu_\rho, \{a_k, b_k, c_k\}$ is defined by the infinite convolution

$$\mu_\rho, \{a_k, b_k, c_k\} := \delta_{\rho^{-1}}\{a_1, b_1, c_1\} * \delta_{\rho^{-2}}\{a_2, b_2, c_2\} *\cdots$$

in the weak^*-topology. Under a mild condition, Fu and Wen [337] established the following characterization on the spectral property of the measure $\mu_\rho, \{a_k, b_k, c_k\}$.

Theorem (5.2.4)[317]: With the above notations, if $gcd(b_k - a_k, c_k - a_k) = 1$ for all

$k \in N$, then $\mu_\rho, \{a_k, b_k, c_k\}$ is a spectral measure if and only if $\rho \in 3\mathbb{N}$ and $\{b_k - a_k, c_k - a_k\} \equiv \{1,2\} \pmod 3$ for all $k \in \mathbb{N}$. Corresponding to Theorem (5.2.2), Theorem (5.2.3) and Theorem (5.2.4), our main results stated as follows.

Theorem (5.2.5)[317]: Let p be a real number. Under the assumption of Theorem (5.2.3), the following two statements are equivalent:

(i) p is a spectral eigenvalue of $\mu_\rho, \{a_k, b_k\}$;

(ii) $p = \frac{p_1}{p_2}$ for some $p_1, p_2 \in \mathbb{Z}\backslash\{0\}$, where $gcd(p_1, p_2) = 1$ and p_1, p_2 are odd integers.

Theorem (5.2.6)[317]:: Let p be a real number. Under the assumption of Theorem (5.2.4),

(i) if $\rho = 3$, then $p = \pm 1$ are the only spectral eigenvalues of $\mu_\rho, \{a_k, b_k, c_k\}$;

(ii) if $\rho > 3$, then p is a spectral eigenvalue of $\mu_\rho, \{a_k, b_k, c_k\}$ if and only if $p = \frac{p_1}{p_2}$ where p_1, p_2 and 3 are pairwise coprime.

We have to point out that the above three measures with particular form are a class of random convolution defined by Dutkay and Lai in [334]. Also, see [319,320,335,354] for more examples in which the random convolution is spectral. We introduce some basic definitions and preliminaries on spectral property of spectral measures. We will prove Theorem (5.2.4), (5.2.5) and (5.2.6) respectively.

Given a probability measure μ on $\mathbb{R}$, the Fourier transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x), \quad (\forall \xi \in \mathbb{R}).$$

For a discrete set Λ, we say that Λ is an orthogonal set (or orthogonal system) for μ if

$$\Lambda - \Lambda \subseteq Z(\hat{\mu}) \cup \{0\}$$

where $Z(f)$ denotes the zero set of the function f on $\mathbb{R}$, i.e. $Z(f) = \{\xi \in \mathbb{R}: f(\xi) = 0\}$. Defining

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2, (\forall \xi \in \mathbb{R}).$$

The following theorem provides a universal test which allows us to determine whether a discrete set Λ is an orthogonal set or a spectrum for the measure μ.

Theorem (5.2.7)[317]: (See [347].) Let μ be a compactly supported Borel probability measure on $\mathbb{R}$, and let $\Lambda \subseteq \mathbb{R}$ be a discrete subset. Then

(i) Λ is an orthogonal set for μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}$;

(ii) (μ, Λ) is a spectral pair if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}$;

(iii) Q_Λ is an entire function on the complex plane C if Λ is an orthogonal set for μ; consequently, (μ, Λ) is a spectral pair if and only if $Q_\Lambda(\xi) \equiv 1$ for all $\xi \in B(0, r)$, which is an open ball centered at 0 with radius r.

As a consequence of Theorem (5.2.7), one can easily get that the following lemma.

Lemma (5.2.8)[317]: Let μ be a compactly supported Borel probability measure on $\mathbb{R}$, and let $\Lambda \subseteq \mathbb{R}$ be a countable subset. Then (μ, Λ) is a spectral pair if and only if $(\mu, -\Lambda)$ is a spectral pair.

Proof. In terms of the fact $|\hat{\mu}(\xi)| = |\hat{\mu}(-\xi)|$, we have that

$$\sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2 = \sum_{\lambda \in \Lambda} |\hat{\mu}(-\xi - \lambda)|^2, (\forall \xi \in \mathbb{R}),$$

which yields that the desired result by Theorem (5.2.7)(ii).

Based on the infinite convolution, another useful test for the spectrality and nonspectrality of the measure is given by Dai, He and Lau in [324].

Lemma (5.2.9)[317]: Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures $\mu_i, i = 0,1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set for μ_0 with $0 \in \Lambda$, then Λ is also an orthogonal set for μ, but cannot be a spectrum for μ.

Up to now, many interesting spectral measures in $\mathbb{R}^d$ have been constructed by using the ideas of compatible pairs, please see [329,348,354]. Here we only concern the integral compatible pairs on $\mathbb{R}$.

Definition (5.2.10)[317]: Let ρ be a positive integer and let $D, C \subseteq Z$ be two finite subsets with the same cardinality q. We say that $(\rho^{-1}D, C)$ forms a compatible pair if the $q \times q$ matrix

$$H_{D,C} := q^{-\frac{1}{2}}\left[e^{2\pi i \rho^{-1} dc}\right]_{d\in D, c\in C} \quad (98)$$

is an unitary matrix, i.e., $H_{D,C}H_{D,C}^* = qI$.

The following properties of compatible pairs have been proved in [348,354] and will be used to prove our main theorems.

Lemma (5.2.11)[317]: Let ρ be a positive integer and let $D, C \subseteq \mathbb{Z}$ be two finite subsets such that $(\rho^{-1}D, C)$ forms a compatible pair. Then

(i) $(\delta_{\rho^{-1}D}, C)$ and $(\delta_{\rho^{-1}D}, -C)$ are spectral pairs;

(ii) $(\rho^{-1}(D + a), C + b)$ forms a compatible pair for any $a, b \in \mathbb{Z}$;

(iii) suppose that $C \subseteq \mathbb{Z}$ such that $C \equiv C$ (mod ρ), then $(\rho^{-1}D, C)$ is a compatible pair;

(iv) suppose that $(\rho^{-1}D_k, C)$ are compatible pairs and define $D_k = \rho^{-1}D_1 + \cdots + \rho^{-k}D_k$ and $C_k = w_0C + \rho w_1 C + \cdots + \rho^{k-1}w_{k-1}C$ with $w_i \in \{-1, 1\}$. Then (D_k, C_k) is a compatible pair for each $k \geq 1$.

The concept of compatible pair clearly provides us two useful iterated function system (IFS) $\{\varphi_d(x) = \rho^{-1}(x + d): d \in D\}$ and its dual IFS $\{\psi_c(x) = \rho^{-1}(x + c): c \in C\}$. Since the two pairs of IFSs $\{\varphi_d\}_{d\in D}$ and $\{\psi_c\}_{c\in C}$ are contractive maps, then Hutchinson's theorem can be applied:

Theorem (5.2.12)[317]: (See [341].) For the IFS $\{\varphi_d(x) = \rho^{-1}(x + d): d \in D\}$, there is a unique Borel probability measure $\mu_{\rho,D}$ satisfying the self-similar identity

$$\mu_{\rho,D} = \frac{1}{\#D}\sum_{d\in D} \mu_{\rho,D} \circ \varphi_d^{-1},$$

and the support of the measure $\mu_{\rho,D}$ is the unique compact set $T(\rho, D)$ satisfying that

$$T(\rho, D) = \bigcup_{d\in D} \phi_d(T(\rho, D)).$$

The dual definitions for $\mu_{\rho,C}$ and $T(\rho, C)$ can also be defined by the dual IFS $\{\psi_c(x)\}$. It is well known (e.g., see [353]) that the dual compact set $T(\rho, C)$ plays an important role in the study of spectral property of the measure $\mu_{\rho,D}$. Reminding that each element $x \in T(\rho, C)$ has the following radix expansion in base ρ:

$$x = \sum_{j=1}^{\infty} \rho^{-j}c_j, c_j \in C. \quad (99)$$

Definition (5.2.13)[317]: With the above expansion of $x \in T(\rho, C)$ as in (99), we give definitions:

(i) we say that the expansion (in base ρ) of x is unique if there exists an unique infinite word $c_1c_2 \cdots \in C^{\mathbb{N}}$ such that (99) holds;

(ii) we say that the expansion of x is finite if the infinite word $c_1c_2\cdots$ ends with 0^∞;

(iii) we say that the expansion of x is ultimately periodic (resp., periodic) if the infinite word $c_1c_2\cdots$ is ultimately periodic (resp., periodic), that is, there exist constants $m,\ell\in\mathbb{N}$ such that $c_{j+\ell}=c_j$ for $j\geq m$ (resp., $c_{j+\ell}=c_j$ for all $j\in\mathbb{N}$).

The proof of Theorem (5.2.15) depends on Lemma (5.2.9) and the following theorem.

Theorem (5.2.14)[317]: Let $\rho=qr$ be a positive integer with positive integers $q\geq 2$ and $r\geq 1$. Then

(i) for $r=1$, the set $\Lambda=\mathbb{Z}$ with $0\in\Lambda$ is the only spectrum for $\mu_{\rho,q}$, and hence there is no $p\in\mathbb{R}\backslash\{\pm 1\}$ such that $\Lambda,p\Lambda$ are spectra for $\mu_{\rho,q}$;

(ii) for $r\geq 2$, and for any two finite integers p_1,p_2 with $gcd(p_i,q)=1$ for $i=1,2$, there is a common discrete set Λ (depending on p_1 and p_2) such that $\Lambda,p_1\Lambda,p_2\Lambda$ are spectra for $\mu_{\rho,q}$.

It is clear that Theorem (5.2.14)(i) is trivial, since the measure $\mu_{\rho,q}$ is then the Lebesgue measure restricted on the unit interval $[0,1]$. Before demonstrating Theorem (5.2.14)(ii), we finish the proof of Theorem (5.2.15).

Theorem (5.2.15)[317]: Let p be a real number. Under the assumption of Theorem (5.2.12),

(i) if $\rho=q$, then $p=\pm 1$ are the only spectral eigenvalues of $\mu_{\rho,q}$;

(ii) if $\rho>q$, then p is a spectral eigenvalue of $\mu_{\rho,q}$ if and only if $p=p_1/p_2$ where p_1,p_2 and q are pairwise coprime.

Proof. (i) It is trivial, since Theorem (5.2.14)(i). (ii) Sufficiency. Suppose $p=\frac{p_1}{p_2}$, where $gcd(p_1,p_2)=1$ and $gcd(p_i,q)=1$ for $i=1,2$. By Theorem (5.2.14)(ii), there exists a common discrete set Λ such that Λ, $p_1\Lambda$, $p_2\Lambda$ are spectra for $\mu_{\rho,q}$. Let $\Lambda=p_2\Lambda$. Then $\Lambda,p\Lambda$ are spectra for $\mu_{\rho,q}$, as desired. Necessity.Fix $p\in\mathbb{R}$ and suppose $\Lambda,p\Lambda$ with $0\in\Lambda$ are both spectra for $\mu_{\rho,q}$. Observe that the zero set of the Fourier transform $\hat{\mu}_{\rho,q}$ is

$$Z(\hat{\mu}_{\rho,q})=\bigcup_{k=0}^{\infty}\rho^k r(\mathbb{Z}\backslash q\mathbb{Z}). \tag{100}$$

Let $\Lambda_k=\rho^k r(\mathbb{Z}\backslash q\mathbb{Z}),k\geq 0$.

Claim (5.2.16)[317]: We have $\Lambda_0\cap\Lambda\neq\emptyset$.

Proof. If $\Lambda_0\cap\Lambda\neq\emptyset$, then $\Lambda\subseteq\cup_{k=1}^{\infty}\Lambda_k$, and it is easy to check that $\Lambda-\Lambda\subseteq\cup_{k=1}^{\infty}\Lambda_k\cup\{0\}$. This means that Λ is an orthogonal set for the measure $\mu:=\delta_{\rho^{-2}D}*\delta_{(\rho^{-3}D)}*\cdots$, where $\mu_{\rho,q}=\delta_{\rho^{-1}D}*\mu$. By Lemma (5.2.9), Λ is not a spectrum for the measure $\mu_{\rho,q}$, a contradiction.

Claim (5.2.17)[317]: $r^{-1}(\Lambda_0\cap\Lambda)\cup\{0\}$ contains a complete set of representatives of the set $\mathbb{Z}/q\mathbb{Z}$.

Proof. Suppose on the contrary that there exists a $n_0\in\{1,2,\cdots,q-1\}$ such that

$$r^{-1}(\Lambda_0\cap\Lambda)\cap(n_0+q\mathbb{Z})=\emptyset.$$

This yields, for any $\lambda\in\Lambda_0\cap\Lambda$, that

$$\lambda-rn_0\in r(\mathbb{Z}\backslash q\mathbb{Z})(\subseteq Z(\hat{\mu}_{\rho,q})).$$

Furthermore, since $\Lambda\backslash\{0\}\subseteq Z(\hat{\mu}_{\rho,q})$, then for $\lambda\in\Lambda\backslash\Lambda_0$, there exist $k\in\mathbb{N}$ and $a\in\mathbb{Z}\backslash q\mathbb{Z}$ such that $\lambda=\rho^k ra=(qr)^k ra$, and hence

$$\lambda-rn_0=r(\rho^k a-n_0)\in r(\mathbb{Z}\backslash q\mathbb{Z})(\subseteq Z(\hat{\mu}_{\rho,q})).$$

The above argument shows that rn_0 is orthogonal to every element of the set Λ, that is,

$$\sum_{\lambda\in\Lambda\cup\{rn_0\}} \left|\hat{\mu}_{\rho,q}(\xi+\lambda)\right|^2 \leq 1(\forall\xi\in\mathbb{R}). \quad (101)$$

Since Λ is a spectrum for $\mu_{\rho,q}$, that is,

$$\sum_{\lambda\in\Lambda} \left|\hat{\mu}_{\rho,q}(\xi+\lambda)\right|^2 \equiv 1(\forall\xi\in\mathbb{R}), \quad (102)$$

we conclude, from (101) and (102), that m

$$\left|\hat{\mu}_{\rho,q}(\xi+rn_0)\right|^2 \equiv 0 \ (\forall\xi\in\mathbb{R}).$$

This is impossible since $\hat{\mu}_{\rho,q}(0)=1$. The claim is true.

By the orthogonality of $\Lambda, p\Lambda$, we have $\Lambda, p\Lambda \subseteq Z(\hat{\mu}_{\rho,q}) \subseteq \mathbb{Z}$, hence p is a rational number. Let $p=\frac{p_1}{p_2}$, where $gcd(p_1,p_2)=1$ and

$$gcd(p_i,q)=d_i \text{ for } i=1,2. \quad (103)$$

Moreover, for $i=1,2$, we define

$$p_i=d_ip_i', \text{where } p_i'\in\mathbb{Z}. \quad (104)$$

Next, we use the proof by contradiction to show that $d_1=d_2=1$in (103) and (104).

(a) To prove $d_1=1$. Suppose $d_1>1$. We view p_1 as an element of the cyclic group $\mathbb{Z}_q$, and set p_1 be the cyclic group generated by p_1 in $\mathbb{Z}_q$. The assumption $d_1>1$ means that p_1 is a proper subgroup of $\mathbb{Z}_q$, which yields that $p_1\mathbb{Z}$ does not contain a complete set of the set $\mathbb{Z}/q\mathbb{Z}$. On the other hand, the condition $\Lambda, p\Lambda \subseteq Z(\hat{\mu}_{\rho,q})\cup\{0\} \subseteq r\mathbb{Z}$ implies that $\frac{1}{r}\Lambda\subseteq\mathbb{Z}$ and $\frac{p_1}{rp_2}\Lambda\subseteq\mathbb{Z}$. Since $gcd(p_1,p_2)=1$, we have $\frac{1}{rp_2}\Lambda\subseteq\mathbb{Z}$, and hence

$$\frac{1}{r}(p\Lambda\cap\Lambda_0)\cup\{0\}\subseteq\frac{p_1}{rp_2}\Lambda\cup\{0\}\subseteq p_1\mathbb{Z}.$$

Applying Claim (5.2.17) to the spectrum $p\Lambda$ instead of Λ, we get that the set $p_1\mathbb{Z}$ contains a complete set of the group $\mathbb{Z}/q\mathbb{Z}$. The above discussions imply that $d_1=1$.

(b) To prove $d_2=1$. In fact, if $d_2>1$, by Claim (5.2.17), there exists $\lambda_0\in\Lambda_0\cap\Lambda$ (for example, $\lambda_0\in r(d_2-1+q\mathbb{Z})\cap\Lambda$) such that

$$gcd(d_2,\frac{\lambda_0}{r})=1$$

Since $gcd(d_2,p_1)=1$, we have $gcd(d_2,p_1\lambda_0\, r)=1$. Combining with (104), we have that

$$p\frac{\lambda_0}{r}=\frac{1}{p_2'}\frac{p_1\frac{\lambda_0}{r}}{d_2}\notin\mathbb{Z}. \quad (105)$$

Therefore $p\lambda_0\notin Z(\hat{\mu}_{\rho,q})(\subseteq r\mathbb{Z})$. Thus, $p\Lambda$ is not a spectrum for the measure $\mu_{\rho,q}$, which is a contradiction. This finishes the proof of Theorem (5.2.15).

The following is a direct corollary of Theorem (5.2.15).

Corollary (5.2.18)[317]: Let $\rho=qr$, where q is a prime number and $r\geq 2$. If $\mu_{\rho,q}$ is spectral, then all the values in the set $q\mathbb{Z}+\{1,2,\cdots,q-1\}$ are spectral eigenvalues of $\mu_{\rho,q}$. Consequently, p_1/p_2 are all spectral eigenvalues of $\mu_{\rho,q}$, where $p_1,p_2\in q\mathbb{Z}+\{1,\cdots,q-1\}$. The following is devoted to proving Theorem (5.2.14)(ii). We begin with some lemmas.

Lemma (5.2.19)[317]: Let $\rho=qr$ be a positive integer with $q,r\geq 2$ and $D=\{0,1,\cdots$

$,q-1\}$ and $C_0=rD$. If p is an integer with $gcd(p,q)=1$, then $pC_0\equiv C_0(mod\ \rho)$ and hence for any infinite word $w\in\{-1,1\}^{\mathbb{N}}$, the set

$$p\Lambda_w(\rho,C_0):=p\left\{\sum_{j=1}^{m}w_j\rho^{j-1}c_j:\ c_j\in C_0,w_j\in\{-1,1\},m\in\mathbb{N}\right\}$$

is an orthogonal system for $\mu_{\rho,q}$.

Proof. We identify D with the cyclic group $\mathbb{Z}q$ with neutral element 0. Now the assumption $gcd(p,q)=1$ implies that p is a generator of the group $\mathbb{Z}q$, i.e., $p,2p,\cdots,qp=0$ are distinct elements in $\mathbb{Z}q$, which is equivalent to say that $pD\equiv D(\text{mod }q)$. Thus, $pC_0\equiv C_0(\text{mod }\rho)$, the first statement holds. Let $\nu_n=\delta_{\rho^{-1}D}*\delta_{\rho^{-2}D}*\cdots*\delta^{\rho^{-n}D}$ and $\Lambda_n=pw_1C_0+pw_2\rho C_0+\cdots+pw_n\rho^{n-1}C_0$. Then the second statement follows from the properties of compatible pair $(\rho^{-1}D,pC_0)$ (see Lemma (5.2.11)(i) and (iv)) and the following relationships

$$Z(\hat{\mu}_{\rho,q})=\bigcup_{n=1}^{\infty}Z(\widehat{\nu_n})\ \text{ and }\ p\Lambda_w(\rho,C_0)=\bigcup_{n=1}^{\infty}\Lambda_n.$$

In particular, if $p=1$ in Lemma (5.2.19), Fu, He and Wen [335] obtained a class of spectra for Cantor measure $\mu_{\rho,q}$ via infinite word in $\{-1,1\}^{\mathbb{N}}$, which is listed as follows.

Lemma (5.2.20)[317]: (See [335].) Let $\rho=qr$ be a positive integer with $q,r\geq 2$. Then for any infinite word $w=w_1w_2w_3\cdots\in\{-1,1\}^{\mathbb{N}}$, the set

$$\Lambda_w(\rho,C_0):=\left\{\sum_{j=1}^{m}w_j\rho^{j-1}c_j:\ c_j\in C_0,w_j\in\{-1,1\},m\in\mathbb{N}\right\}\tag{106}$$

is a spectrum for the measure $\hat{\mu}_{\rho,q}$, where $C_0=r_D,D=\{0,1,\cdots,q-1\}$. Fixing $p\neq 1$ be a non-zero positive integer such that $gcd(p,q)=1$. For any positive integers M, N, we define

$$G_{M,N}:=(C_0+\rho C_0+\cdots+\rho^{M-1}C_0)-(\rho^{M}C_0+\cdots+\rho^{M+N-1}C_0),\tag{107}$$

and let $T(\rho^{M+N},pG_{M,N})$ be the attractor of IFS $\{\rho^{-(M+N)}(x+pg):g\in G_{M,N}\}$. More precisely,

$$\begin{aligned}T(\rho^{M+N},pG_{M,N})&=p\left\{\sum_{k=1}^{\infty}\rho^{-(M+N)k}g_k:g_k\in G_{M,N}\right\}\\&=p\left\{\sum_{k=1}^{\infty}(-1)^{(\tau(k))}\rho^{-k}c_k:\ c_k\in C_0\right\},\end{aligned}\tag{108}$$

where τ is a $(M+N)$-periodic function on $\mathbb{Z}$ and $\tau(k)=1$for $1\leq k\leq N$ and $\tau(k)=0$for $N+1\leq k\leq M+N$.

Lemma (5.2.21)[317]: If $gcd(p,q)=1$, then each element x in the compact set $T(\rho,pC_0\cup(-pC_0))$ has a unique expansion in base ρ, and is ultimate periodic. Moreover, the expansion of each element $x\in T(\rho,pC_0\cup(-pC_0))\cap Z(\hat{\mu}_{\rho,q})$ cannot be finite.

Proof. One can easily check it by using the methods and techniques in [320]. The readers can also refer to Lemmas 4.10, 4.11 and 4.12 in [335]. Here we omit the details in the proof.

For $\delta>0$, the closed δ-neighborhood of $T(\rho^{M+N},pG_{M,N})$ is defined by

$$(T(\rho^{M+N}, pG_{M,N}))\delta = \{x \in \mathbb{R}: dist(x, T(\rho^{M+N}, pG_{M,N})) \leq \delta\}.$$

The following lemma essentially gives a characterization on the completeness of the orthogonal system $p\Lambda_w(\rho, C_0)$ in the Hilbert space $L^2(\mu_{\rho,q})$.

Lemma (5.2.22)[317]: With the notations above, for any positive integer p with $gcd(p, q) = 1$ there exist positive integers $M, N \in \mathbb{N}$ (depending on p) such that

$$Z(\hat{\mu}_{\rho,q}) \cap T(\rho^{M+N}, pG_{M,N}) = \emptyset. \quad (109)$$

Consequently, there exist positive constants $\varepsilon, \delta > 0$ such that

$$\left|\hat{\mu}_{\rho,q}(\xi)\right|^2 \geq \varepsilon$$

holds for all $\xi \in (T(\rho^{M+N}, pG_{M,N}))\delta$.

Proof. Since the set $Z(\hat{\mu}_{\rho,q})$ is discrete, then its intersection with $T(\rho, pC_0 \cup (-pC_0))$ is a finite set, say that $A := \{x_1, x_2, \cdots, x_m\}$. Since $C_0 = rD$, then

$$T\big(\rho, pC_0 \cup (-pC_0)\big) = pr\left\{\sum_{j=1}^{\infty} \rho^{-j} c_j : c_j \in D \cup (-D)\right\} = : prT(\rho, D \cup (-D)).$$

Thus, each $x_i \in A$ has the form

$$x_i = pr\sum_{j=1}^{\infty} \rho^{-j} c_{i,j}, c_{i,j} \in \{-(q-1), \cdots, 0, \cdots, q-1\}. \quad (110)$$

By Lemma (5.2.7) and the expansion of x_i in (5.110), the elements in A can be rearranged and divided into the following (at most) three classes:

(a) for $x_i, 1 \leq i \leq s, s \in \mathbb{N}$, there are at least one term $c_{i,j} > 0$ and at least one term $c_{i,j_i'} < 0$. We may assume that $j_i > j_i'$;

(b) for $x_i, s+1 \leq i \leq s+t, t \in \mathbb{N}$, all terms $c_{i,j} \geq 0$ and there are infinitely many $c_{i,j} > 0$. Set $j_i := min\{j: c_{i,j} > 0\}$;

(c) for $x_i, s+t+1 \leq i \leq m$, all terms $c_{i,j} \leq 0$ and there are infinitely many $c_{i,j} < 0$.

For $1 \leq i \leq s+t$, we define $N := max1 \leq i \leq s+t\{j_i\}$. Next, for $s+t+1 \leq i \leq m$ we choose a common positive integer $M \in N$ such that $c_i, j_i < 0$ for some j_i, where $N < j_i < M+N$. Now we define $G_{M,N}$ be as in (107).

We claim that $x_i \notin T(\rho^{M+N}, pG_{M+N})$. Otherwise, by comparing the symbols "+" or " –" of $c_{i,j}$ in the expansion of x_i in the classes (a), (b) and (c) with that in (108), we get that the above x is have two different expansions in base ρ, it is a contradiction to Lemma (5.2.7). Thus the claim is true and this yields the desired result (109). The second statement follows from the continuity of the function $\mu_{\rho,q}$ and the compactness of the set $T(\rho^{M+N}, pG_{M,N})$.

What's more, according to Lemma (5.2.7), one can do the similarly procedure as in the proof of Lemma (5.2.22) to the following intersection $Z\big(\hat{\mu}_{\rho,q}\big) \cap \cup_{i=1}^{2} T(\rho, p_i C_0 \cup (-p_i C_0))$, where $p_1, p_2 > 0$ are integers with $gcd(p_i, q) = 1$ for $i = 1,2$, and get the following lemma.

Lemma (5.2.23)[317]: For any two positive integers p_1, p_2 satisfying that $gcd(p_i, q) = 1$ for $i = 1,2$, there exist positive integers $M, N \in \mathbb{N}$ (depending on p_1, p_2) such that

$$Z(\hat{\mu}_{\rho,q}) \cap T(\rho^{M+N}, p_i G_{M,N}) = \emptyset, i = 1,2. \quad (111)$$

Consequently, there exist positive constants $\varepsilon, \delta > 0$ such that $\left|\hat{\mu}_{\rho,q}(\xi)\right|^2 \geq \varepsilon$ holds for all $\xi \in \left(\cup_{i=1}^{2} T\big(\rho^{M+N}, p_i G_{M,N}\big)\right)_\delta$.

Proof. Fixing integer p_i with $gcd(p_i, q) = 1$ for $i = 1, 2$. Next we need to construct a discrete set Λ such that both $p_1\Lambda$ and $p_2\Lambda$ are spectra for $\mu_{\rho,q}$. From Lemma (5.2.22), one can assume that $p_1 > 0$ and $p_2 > 0$. Let $M, N, \varepsilon, \delta$ be as in Lemma (5.2.23) and define

$$\Lambda_{M,N} := \sum_{k=1}^{\infty} (-1)^{\iota(k)} \rho^{k-1} C_0,$$

where ι is a $(M + N)$-periodic function on $\mathbb{Z}$ such that $\iota(k) = 0$ for $1 \le k \le M$ and $\iota(k) = 1$for $M + 1 \le k \le M + N$.It follows from Lemma (5.2.21) that $\Lambda_{M,N}$ is a spectrum for the measure $\mu_{\rho,q}$. Thus, it is enough to show that $p_1\Lambda_{M,N}$ is a spectrum for the measure $\mu_{\rho,q}$. The case for $p_2\Lambda_{M,N}$ can be proved similarly. By letting $\Lambda^n_{M,N} := \sum_{k=1}^{(M+N)n} (-1)^{\iota(k)} \rho^{k-1} C_0$ and $\nu_n := \delta_{\rho^{-1}D} * \delta_{\rho^{-2}D} * \cdots * \delta_{\rho^{-(M+N)n}D}$, we get, from Lemma (5.2.20) and Lemma (5.2.11), that $(\nu_n, p_1\Lambda^n_{M,N})$ is a spectral pair for all $n \in \mathbb{N}$, which is equivalent to say that

$$\sum_{\lambda \in p_1 \Lambda^n_{M,N}} |\hat{\nu}_n(\xi + \lambda)|^2 = 1 \,, (\forall \xi \in \mathbb{R}). \tag{112}$$

In terms of the facts $\Lambda_{M,N} = \cup_{n=1}^{\infty} \Lambda^n_{M,N}$ and $Z(\hat{\nu}_n) \subseteq Z(\mu_{\rho,q})$, we obtain that $p_1\Lambda_{M,N}$ is an orthogonal set for $\mu_{\rho,q}$, i.e.,

$$\lambda \in p_1\Lambda_{M,N} \left|\mu_{\rho,q}(\xi + \lambda)\right|^2 \le 1, (\forall \xi \in \mathbb{R}). \tag{113}$$

Moreover, for any $\lambda \in p_1\Lambda^n_{M,N}$, we have $\rho^{-(M+N)n}\lambda \in T(\rho^{M+N}, p_1 G_{M,N})$ and by Lemma (5.2.23)

$$\left|\hat{\mu}_{\rho,q}(\xi + \lambda)\right|^2 = |\hat{\nu}_n(\xi + \lambda)|^2 \left|\hat{\mu}_{\rho,q}\left(\frac{\xi + \lambda}{\rho^{(M+N)n}}\right)\right|^2 \ge \varepsilon |\widehat{\nu_n}(\xi + \lambda)|^2 \,,$$

whenever $|\xi| < \delta$ and $\frac{\xi+\lambda}{\rho^{(M+N)n}} \in \left(T(\rho^{M+N}, p_1 G_{M+N})\right)_\delta$.Summing over all $\lambda \in p_1\Lambda_{M,N}$, we obtain, from (113), that

$$1 \ge \sum_{\lambda \in p_1 \Lambda_{M,N}} \varepsilon |\hat{\nu}_n(\xi + \lambda)|^2 \,.$$

Thus, the constant $1/\varepsilon$ is the dominated function, and thus by dominated convergence theorem and (112), we have

$$\sum_{\lambda \in p_1 \Lambda_{M,N}} \left|\hat{\mu}_{\rho,q}(\xi + \lambda)\right|^2 = 1 \,, (|\xi| < \delta \,).$$

Combining with Theorem (5.2.7), we obtain that $p_1\Lambda_{M,N}$ is a spectrum for $\mu_{\rho,q}$. This finishes the proof of Theorem (5.2.14)(ii). In particular, Theorem (5.2.14) implies the following fact which is labeled as Theorem (5.2.14).

Theorem (5.2.24)[317]: Let $\rho = qr$ be a positive integer with $q, r \ge 2$. Then for any integer p with $gcd(p, q) = 1, p$ is a spectral eigenvalue of $\mu_{\rho,q}$, that is, there is a set Λ such that $\Lambda, p\Lambda$ are spectra for $\mu_{\rho,q}$.

More generally, one can similarly get the general version of Theorem (5.2.14). Theorem (5.2.24). Let $\rho = qr$ be a positive integer with $q, r \ge 2$. Then for any finite integers $p_1, \cdots, p_n$ with $gcd(p_i, q) = 1$ for $1 \le i \le n$, there is a common discrete set Λ(depending on pis) such that $\Lambda, p_1\Lambda, \cdots, p_n\Lambda$ are all spectra for $\mu_{\rho,q}$.

We will establish the proof of Theorem (5.2.14) by generalizing the ideas used. Recall that $d_k := b_k - a_k = 2^{l_k} m_k$ with all m_k's are odd positive integers, $L =$

$\max_{k\geq 1} l_k < \infty$ and $\rho = 2^{l+1}q$ is an integer such that $l > L$ if $q = 1$ and $l \geq L$ if $q > 1$. One can easily get that

$$|\hat{\mu}_{\rho,\{a_k,b_k\}}(\xi)| = \prod_{k=1}^{\infty}\left|\hat{\delta}_{\rho^{-k}\{a_k,b_k\}}(\xi)\right| = \prod_{k=1}^{\infty}\left|\hat{\delta}_{\rho^{-k}\{0,d_k\}}(\xi)\right| = |\hat{\mu}_{\rho,\{0,d_k\}}(\xi)|, (\forall\xi \in \mathbb{R}).$$

Thus, for any discrete set Λ,

$$\sum_{\lambda\in\Lambda}\left|\hat{\mu}_{\rho,\{a_k,b_k\}}(\xi+\lambda)\right|^2 = \sum_{\lambda\in\Lambda}\left|\hat{\mu}_{\rho,\{0,d_k\}}(\xi+\lambda)\right|^2 \ (\forall\xi \in \mathbb{R}).$$

This means that the spectrality and non-spectrality of the measures $\mu_{\rho,\{a_k,b_k\}}$ and $\mu_{\rho,\{0,d_k\}}$ are the same. Moreover, $(\mu_{\rho,\{0,d_k\}}, \Lambda)$ is a spectral pair if and only if $(\mu_{\rho,\{0,qd_k\}}, q^{-1}\Lambda)$ is a spectral pair. We will prove the following statement which is equivalent to Theorem (5.2.14).

In order to prove Theorem (5.2.8), we need to make some preparations for it. Since $Z(\hat{\delta}_{\{0,n\}}) = \frac{2\mathbb{Z}+1}{2n}$, $n \in \mathbb{N}$, then

$$Z\left(\hat{\mu}_{\rho,\{0,qd_k\}}\right) = \bigcup_{k=1}^{\infty}\frac{\rho^k}{2qd_k}(2\mathbb{Z}+1). \tag{114}$$

It is easy to see that $(\rho^{-1}\{0, qd_k\}, \{0, 2^{l-l_k}\})$ is a compatible pair for all $k \in \mathbb{N}$. Based on the value of q, we define

$$G = \begin{cases}\{0,1,2,2^2,\cdots,2^l\}, \text{if } q > 1;\\ \{0,2,2^2,\cdots,2^l\}, if\ q = 1 \text{ and } l > L,\end{cases}$$

which contains all different sets $\{0, 2^{l-l_k}\}$ for $k \in \mathbb{N}$. Let D be the set of all different d_k's for $k \in \mathbb{N}$, say that, $D = \{d_{(1)}, d_{(2)},\cdots, d_{(N)}\}$. Here and below in this section we use $\mu_{\rho,\{0,qd_{(i)}\}}$ to denote the self-similar measure generated by the IFS $\{\rho^{-1}x, \rho^{-1}(x + qd_{(i)})\}$. Clearly,

$$Z\left(\hat{\mu}_{\rho,\{0,qd_k\}}\right) \subseteq \bigcup_{i=1}^{N} Z\left(\hat{\mu}_{\rho,\{0,qd_{(i)}\}}\right) := \ Ni = 1 \infty k = 1\bigcup_{i=1}^{N}\bigcup_{k=1}^{\infty}\frac{\rho^k}{2qd_{(i)}}(2\mathbb{Z}+1)\ .$$

Let us use $T(\rho, pG \cup (-pG))$ to denote the attractor of the IFS $\{\rho^{-1}(x + g) : g \in pG\}$, where p is an odd integer. The following Lemma (5.2.11) characterize the finer structure of the compact set $T(\rho, pG \cup (-pG))$, which essentially follows from Lemma 2.3, 3.1 and 3.3 in [320], we state it without a proof.

Lemma (5.2.25)[317]: With the above notations, we get that

(i) each rational number in the set $T(\rho, G \cup (-G))$ has a unique expansion in base ρ and the expansion is ultimate periodic;

(ii) the expansion of the element in $T(\rho, G \cup (-G)) \cap \left(\bigcup_{i=1}^{N} Z\left(\hat{\mu}_{\rho,\{0,qd_{(i)}\}}\right)\right)$ is not finite. For any positive integers $M, N \in \mathbb{N}$, we construct a set as follows

$$\Lambda_{M,N} := \sum_{k=1}^{\infty}(-1)^{\iota(k)}\rho^{k-1}\{0,2^{l-l_k}\} = \left\{\sum_{k=1}^{m}(-1)^{\iota(k)}\rho^{k-1}\{0,2^{l-l_k}\} : m\ \geq 1\right\}, \tag{115}$$

where ι is a $(M + N)$-periodic function on $\mathbb{Z}$ such that $\iota(k) = 0$ for $1 \leq k \leq M$ and $\iota(k) = 1$ for $M + 1 \leq k \leq M + N$.

For any positive integer $n \in \mathbb{N}$ we set

$$\Lambda_{M,N}^{n} := \sum_{k=1}^{(M+N)_n} (-1)^{\iota(k)} \rho^{k-1} \{0, 2^{l-l_k}\},$$

and

$$\nu_{\rho,n} = \delta\_(\rho^{-1}\{0, qd_1\} * \delta\_(\rho^{-2}\{0, qd_2\} * \cdots * \delta_{\rho^{-(M+N)_n}\{0,q^d\ (M+N)_n\}}.$$

Lemma (5.2.26)[317]: For any odd integer p, the set $p\Lambda_{M,N}^{n}$ is a spectrum for the measure $\nu_{\rho,n}$. Consequently, $p\Lambda_{M,N}$ is an orthogonal set for $\mu_{\rho,\{0,qd_k\}}$.

Proof. For an odd integer p, one can easily check that $(\rho^{-1}\{0, qd_k\}, p(-1)^{\iota(k)}\{0,2^{l-l_k}\})$ is a compatible pair for all $k \in \mathbb{N}$. By the property of the compatible pairs (see Lemma (5.2.22)), $(\sum_{k=1}^{(M+N)_n} \rho^{-k}\{0, qd_k\}, p\Lambda_{M,N}^{n})$ is a compatible pair, which yields the first statement. Because $\Lambda_{M,N}^{n}$ is increasing as n and

$$\Lambda_{M,N} = \bigcup_{n=1}^{\infty} \Lambda_{M,N}^{n},\ Z(\hat{\mu}_{\rho,\{0,qd_k\}}) = \bigcup_{n=1}^{\infty} Z(\hat{\nu}_{\rho,n}).$$

The second statement follows. Associated to the above function ι in (115), we define

$$G_{M,N} := \sum_{k=1}^{M+N} (-1)^{\iota(k)} \rho^{k-1} G, \tag{116}$$

and let $T(\rho^{M+N}, G_{M,N})$ be the attractor of the IFS $\{\rho^{-(M+N)}(x+g): g \in G_{M,N}\}$.The following is completeness characterization of the spectral property of the set $\Lambda_{M,N}$ and its scaling set $p\Lambda_{M,N}$ for $p \in \mathbb{Z}$, which is a key ingredient for the proof of Theorem (5.2.9).

Lemma (5.2.27)[317]: With the notations above, for any odd integer p, there are positive integers M, N (depending on p) such that the following two statements hold:

(i) $\left(\bigcup_{i=1}^{N} Z(\mu_{\rho,\{0,qd_{(i)}\}})\right) \cap T(\rho^{M+N}, G_{M,N}) = \emptyset$;

(ii) $\left(\bigcup_{i=1}^{N} Z(\mu_{\rho,\{0,qd_{(i)}\}})\right) \cap T(\rho^{M+N}, pG_{M,N}) = \emptyset$.

Consequently, there are positive constants ε, δ such that

$$\prod_{i=1}^{N} \left|\hat{\mu}_{\rho,\{0,qd_{(i)}\}}(\xi)\right|^2 \geq \varepsilon \tag{117}$$

holds for $\xi \in \left(T(\rho^{M+N}, G_{M,N})\right)_\delta \cup \left(T(\rho^{M+N}, pG_{M,N})\right)_\delta$.

Proof. Suppose p is a positive odd integer. Since $Z(\hat{\mu}_{\rho,\{0,qd_{(i)}\}})$ is uniformly discrete for each $i = 1,2,\cdots,m$ and the sets $T(\rho, G \cup (-G)), T(\rho, pG \cup (-pG))$ are compact, then both

$$\mathcal{A}_1 := \left(\bigcup_{i=1}^{N} Z(\hat{\mu}_{\rho,\{0,qd_{(i)}\}})\right) \cap T(\rho, G \cup (-G))$$

and

$$\mathcal{A}_2 := \left(\bigcup_{i=1}^{N} Z(\hat{\mu}_{\rho,\{0,qd_{(i)}\}})\right) \cap T(\rho, pG \cup (-G))$$

are finite sets. Suppose $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 = \{x_1, x_2, \cdots, x_m\} (m \in \mathbb{N})$. From Lemma (5.2.26), each $x_i \in \mathcal{A}_1$ has a unique expansion in base ρ:

$$xi \ = \sum_{j=1}^{\infty} w_{i,j}\rho^{-j}g_{i,j}\text{, where } w_{i,j} \in \{-1,0,1\}, g_i, j \in G, \quad (118)$$

and each $x_i \in \mathcal{A}_2$ has a unique expansion in base ρ:

$$x_i = \sum_{j=1}^{\infty} w_{i,j}\rho^{-j}g_{i,j}\text{, where } w_{i,j} \in \{-1,0,1\}, g_{i,j} \in G. \quad (119)$$

A useful observation is that each $x_i \in \mathcal{A}_1 \cap \mathcal{A}_2$ will correspond to two different in-finite word $w_{i,1}w_{i,2}w_{i,3} \cdots \in \{-1, 0, 1\}^N$. Without loss of generality, we assume that each $x_i \in \mathcal{A}$ is in this case, and hence the number of infinite words correspond to A is $2m$. Whence, the above infinite words can be rearranged and divided into the following (at most) three classes:

(a) there is s infinite words, written as $w_{i,1}w_{i,2} \cdots, i = 1, 2,\cdots, s$, such that at least one term $w_{i,j_i'} > 0$ and at least one term $w_{i,j_i'} < 0$. We may assume that $j_i > j_i'$;

(b) there is t infinite words, written as $w_{i,1}w_{i,2} \cdots, i = s+1, s+2,\cdots, s+t$ such that all terms $w_{i,j} \geq 0$ and there are infinitely many $w_{i,j} > 0$. Set $j_i = \min\{j : w_{i,j} > 0\}$;

(c) there is $2m-(s+t)$ infinite words, written as $w_{i,1}w_{i,2} \cdots, i = s+t+1, s+t+2,\cdots, 2m$ such that all terms $w_{i,j} \leq 0$ and there are infinitely many $w_{i,j} < 0$. Let $N = max\,\{j_1,\cdots, j_{s+t}\}$ and then choose $M \in \mathbb{N}$ such that $w_{i,j_i} < 0$ for all $s+t+1 \leq i \leq 2m$ and for some j_i, where $N < j_i < M+N$. Now define $G_{M,N}$ as in (119), that is,

$$G_{M,N} = (G + \rho G +\cdots +\rho^{M-1}G) - (\rho^M G + \rho^{M+1}G +\cdots +\rho^{M+N-1}G).$$

The following compact set is generated by the IFS $\{(\rho^{-(M+N)}(x+g): g \in G_{M,N})\}$:

$$T(\rho^{M+N}, G_{M,N}) = \sum_{k=1}^{\infty} \rho^{-(M+N)k}G_{M,N} = \left\{\sum_{k=1}^{m} (-1)^{\tau(k)}\rho^{-k}g_k : g_k \in G\right\}, \quad (120)$$

where τ is a $(M+N)$-periodic function on $\mathbb{Z}$ and $\tau(k) = 1$for $1 \leq k \leq N$ and $\tau(k) = 0$ for $N+1 \leq k \leq M+N$.By comparing the leading $M+N$ words $w_{i,1}w_{i,2} \cdots w_{i,M+N}$ of $x_i \in A_1$ in (118) with that in (120), we get that $A_1 \cap T(\rho^{M+N}, G_{M,N}) = \emptyset$. In the same reason, from (119) and (120) , we get that $\mathcal{A}_2 \cap T(\rho^{M+N}, pG_{M,N}) = \emptyset$. Since $T(\rho^{M+N}, G_{M,N}) \subseteq T(\rho, G \cup (-G))$ and $T(\rho^{M+N}, pG_{M,N}) \subseteq T(\rho, pG \cup (-pG))$, then the statements (i) and (ii) hold. The inequality (117) follows from the continuity of the function

$\mu\{0, qd_{(i)}\}$ for $i = 1,\cdots, N$ and the compactness of the sets $T(\rho^{M+N}, G_{M,N})$ and $T(\rho^{M+N}, pG_{M,N})$.

Theorem (5.2.28)[317]: Under the assumption of Theorem (5.2.14), then for any odd integer p there is a discrete set Λ (depending on p) such that $\Lambda, p\Lambda$ are both spectra for $\mu_{\rho,\{0,qd_k\}}$.

Proof. For any odd positive integer p, let $M, N, \varepsilon, \delta$ be given by Lemma (5.2.27). We will show the sets $\Lambda_{M,N}$ (see (115)), $p\Lambda_{M,N}$ are spectra for the measure $\mu_{\rho,\{0,qd_k\}}$. Fix $\xi \in (-\delta, \delta)$ and let

$$f_n(\lambda) := \begin{cases} |\nu_{\rho,n}(\xi + \lambda)|^2, if\ \lambda \in \Lambda_{M,N}^n; \\ 0, if\ \lambda \in \Lambda_{M,N} \backslash \Lambda_{M,N}^n. \end{cases}$$

By Lemma (5.2.26), we have
$$\sum_{\lambda\in\Lambda_{M,N}^{n}}\left|\hat{\nu}_{\rho,n}(\xi\ +\lambda)\right|^{2}=1 \text{ and } \sum_{\lambda\in\Lambda_{M,N}}\left|\mu_{\rho,\{0,qd_k\}}(\xi+\lambda)\right|^{2}\le 1(\forall\xi\in\mathbb{R}). \qquad (121)$$
Observing that
$$\hat{\mu}_{\rho,\{0,qd_k\}}(\xi)=\hat{\nu}_{\rho,n}(\xi)\prod_{j=1}^{\infty}\hat{\delta}_{\rho^{-j}\{0,qd_{j+(M+N)_n}\}}\left(\frac{\xi}{\rho^{(M+N)_n}}\right).$$
Thus, for $\lambda\in\Lambda_{M,N}$, we have
$$\begin{aligned}\left|\hat{\mu}_{\rho,\{0,qd_k\}}(\xi+\lambda)\right|^{2}&=\left|\hat{\nu}_{\rho,n}(\xi+\lambda)\right|^{2}\prod_{j=1}^{\infty}\left|\hat{\delta}_{\rho^{-j}\{0,qd_{j+(M+N)_n}\}}\left(\frac{\xi}{\rho^{(M+N)_n}}\right)\right|^{2}\\ &\ge\left|\hat{\nu}_{\rho,n}(\xi+\lambda)\right|^{2}\prod_{i=1}^{N}\left|\hat{\mu}_{\rho,\{0,qd_{(i)}\}}\left(\frac{\xi+\lambda}{\rho^{(M+N)_n}}\right)\right|^{2} \qquad (122)\end{aligned}$$
Since
$$\Lambda_{M,N}^{n}\subseteq G_{M,N}+\rho^{M+N}G_{M,N}+\cdots+\rho^{(M+N)(n-1)}G_{M,N},$$
then
$$\rho^{-(M+N)n}\Lambda_{M,N}^{n}\subseteq T(\rho^{M+N},G_{M,N}).$$
Therefore, for any $\lambda\in\Lambda_{M,N}^{n}$, we obtain that $\frac{\lambda}{\rho^{(M+N)n}}\in T(\rho^{M+N},G_{M,N})$. From Lemma (5.2.27) (i) and (122), we get that
$$\left|\hat{\mu}_{\rho,\{0,qd_k\}}(\xi+\lambda)\right|^{2}\ge\varepsilon\left|\hat{\nu}_{\rho,n}(\xi+\lambda)\right|^{2},\forall\xi\in(-\delta,\delta).$$
Summing over all $\lambda\in\Lambda_{M,N}$, we have, from (121), that
$$1\ge\varepsilon\sum_{\lambda\in\Lambda_{M,N}}\left|\hat{\nu}_{\rho,n}(\xi+\lambda)\right|^{2}.$$
Whence, applying dominated convergence theorem to the functions $\{f_n\}_{n=1}^{\infty}$, we obtain that
$$\sum_{\lambda\in\Lambda_{M,N}}\left|\hat{\mu}_{\rho,\{0,qd_k\}}(\xi\ +\lambda)\right|^{2}=\lim_{n\to\infty}\sum_{\lambda\in\Lambda_{M,N}}\left|\hat{\nu}_{\rho,n}(\xi+\lambda)\right|^{2}\equiv 1,\forall\xi\in(-\delta,\delta).$$
By Theorem (5.2.24), the set $\Lambda_{M,N}$ is a spectrum for the measure $\mu_{\rho,\{0,qd_k\}}$. Furthermore, if we do the same procedure for $p\Lambda_{M,N}$ instead of $\Lambda_{M,N}$, we will also get, from Lemma (5.2.27)(ii), that $p\Lambda_{M,N}$ is a spectrum for the measure $\mu_{\rho,\{0,qd_k\}}$. Lemma (5.2.27) and Theorem (5.2.9) can also be generalized as follows and we state them without proofs.

Lemma (5.2.29)[317]: For finite positive odd integers $p_1,p_2,\cdots,p_n$, there are positive integers M,N (depending on $p_1,p_2,\cdots,p_n$) such that the following two statements hold:

(i) $\left(\bigcup_{i=1}^{N}Z(\hat{\mu}_{\rho,\{0,qd_{(i)}\}})\right)\cap T(\rho^{M+N},G_{M,N})=\emptyset$;

(ii) $\left(\bigcup_{i=1}^{N}Z(\hat{\mu}_{\rho,\{0,qd_{(i)}\}})\right)\cap T(\rho^{M+N},pG_{M,N})=\emptyset$. For $j=1,2,\dots,n$.

Theorem (5.2.30)[317]: Under the assumption of Theorem (5.2.14), for any finite positive odd integers $p_1,\cdots,p_n$, there is a common discrete set Λ(depending on $p_i's$) such that Λ, $p_1\Lambda,\cdots,p_n\Lambda$ are spectra for $\mu_\rho,\{0,d_kq\}$.

Theorem (5.2.31)[317]: Let p be a real number. Under the assumption of Theorem (5.2.14), the following two statements are equivalent:

(i) p is a spectral eigenvalue of $\mu_{\rho,\{0,qd_k\}}$;

(ii) $p=\frac{p_1}{p_2}$ for some $p_1,p_2 \in \mathbb{Z}\backslash\{0\}$ where $gcd(p_1,p_2)=1$ and p_1,p_2 are odd integers.

Proof. "(ii) ⇒ (i)". Suppose $p=\frac{p_1}{p_2}$, where p_1,p_2 are odd. By Theorem (5.2.30), there exist a common discrete set Λ such that $\Lambda, p_1\Lambda, p_2\Lambda$ are spectra for $\mu_{\rho,\{0,dk_q\}}$. If $\Lambda = p_2\Lambda$, then $p\Lambda = p_1p_2\Lambda = p_1p_2(p_2\Lambda) = p_1\Lambda$. So $\Lambda, p\Lambda$ are spectra for $\mu_{\rho,q}$. "(i) ⇒ (ii)". Suppose $\Lambda, p\Lambda$ are spectra for the measure $\mu_{\rho,\{0,dk_q\}}$. Then $\Lambda, p\Lambda \subseteq Z\left(\hat{\mu}_{\rho,\{0,dk_q\}}\right) \cup \{0\}$, and hence p is a rational number. Suppose $p = p_1\, p_2$ for some $p_1,p_2 \in Z\backslash\{0\}$ such that $gcd(p_1,p_2)=1$. We prove (ii) by showing that there exist contradiction in the following two cases.

Case I. p_1 is even and p_2 is odd. In this case, we show that $p\Lambda$ is not a spectrum for the measure $\mu_{\rho,\{0,d_kq\}}$. By Lemma (5.2.23), it is enough to show that

$$p\Lambda \subseteq \bigcup_{k=2}^{\infty} \frac{\rho^k}{2qd_k}(2\mathbb{Z}+1). \tag{123}$$

In fact, from (123) and the values of ρ and all d_k's, one can easily check that

$$p\Lambda - p\Lambda \subseteq \bigcup_{k=2}^{\infty} \frac{(\rho^k)}{2qd_k}\ (2\mathbb{Z}+1)\,.$$

which is equivalent to say that $p\Lambda$ is an orthogonal set for the measure $\mu := \delta_{\rho^{-2}\{0,d_2q\}} * \delta_{\rho^{-n}\{0,d_3q\}} * \cdots$, where $\mu_{\rho,\{0,d_kq\}} = \delta_{\rho^{-1}\{0,d_1q\}} * \mu$. Whence, $p\Lambda$ is not a spectrum for the measure $\mu_{\rho,\{0,d_kq\}}$ by Lemma (5.2.23). It remains to prove (123). Let $\Lambda_k = \frac{\rho^k}{2qd_k}(2\mathbb{Z}+1)$. Suppose on the contrary that there is some $\lambda \in \Lambda k$ such that $p\lambda \in \Lambda_1$, then there exist $a, a' \in 2\mathbb{Z}+1$ such that

$$p\lambda = \frac{p_1}{p_2}\frac{\rho^k}{2qd_k}a = \frac{\rho}{2qd_1}a'.$$

This yields that

$$p_1\rho^{k-1}ad_1 = a'p_2d_k \text{ where } k \geq 1. \tag{124}$$

Now we derive contradictions as follows: When $k=1$, the equation (124) becomes $p_1a = p_2a'$, which is a contradiction. When $k>1$, the conditions $d_k = 2^{l_k}m_k$ and $\rho = 2^{l+1}q$, where m_k is an odd integer and $l > l_k$, imply that (124) becomes

$$p_1(\rho^{k-1}2^{-l_k})a2^{l_1}m_1 = a'p_2m_k,$$

which is impossible since $p_1, \rho^{k-1}2^{-l_k}$ are even integers and a', p_2, m_k are odd integers. The desired relation (123) holds.

Case II. p_1 is odd and p_2 is even. In this case, we will show that $p\Lambda$ is not a spectrum for the measure $\mu_\rho, \{0, d_kq\}$ by showing that $p\lambda \notin Z(\hat{\mu}_{\rho,\{0,d_kq\}})$ if $\lambda \in \Lambda_1 \cap \Lambda$. We firstly obtain, by Lemma (5.2.23) again, that $\Lambda_1 \cap \Lambda \neq \emptyset$. Let's just suppose that there exists some $\lambda \in \Lambda_1 \cap \Lambda$ such that $p\lambda \in Z(\hat{\mu}_{\rho,\{0,d_kq\}})$. Then, from (114), there exist $a, a' \in 2\mathbb{Z}+1, k \in \mathbb{N}$ such that

$$p\lambda = p\frac{\rho a}{2qd_1} = \frac{\rho^k}{2qd_k}a' \Leftrightarrow pad_k = \rho^{k-1}a'd_1. \tag{125}$$

Recall that $\rho = 2^{l+1}q$ and $d_k = 2^{l_k}m_k$ for $k \geq 1$, where $l \geq \sup_{k\geq 1} l_k$ and $q, m_k's$ are odd integers. Defining $p_2 = 2^s p_2'$ where $s \geq 1$ and p_2 is odd. Hence, via an explicit calculation, the above equality (125) becomes

$$p_1 a m_k = 2^{(l+1)(k-1)+l_1-l_k+s} q^{k-1} a'_{m_1} p'_2.$$

Since $p_1, a, m_k, q, a', m_1, p'_2$ are odd integers, then

$$(l+1)(k-1)+l_1-l_k+s = 0 . \tag{126}$$

When $k=1$, one gets that $s=0$, which is a contradiction to the assumption of p_2; When $k>1$, one gets, from $l \geq l_k$ and $l \geq l_1$, that

$$(l+1)(k-1)+l_1-l_k+s \geq (l+1)+l_1-l_k+s \geq 1+l_1+s > 0.$$

It is a contradiction to the equation (126). The proof of Theorem (5.2.31) is complete.

We will complete the proof of Theorem (5.2.14). We point out that the techniques and methods used for infinite Bernoulli convolutions. However, due to the special property of $\mu_{\rho,\{a_k,b_k,c_k\}}$, we will show that the proof of Theorem (5.2.14) can be reduced to that of Theorem (5.2.12) for the measure $\mu_{\rho,3} := \mu_{\rho,D_0}$, where $\rho \in 3\mathbb{N}$ and $D_0 = \{0,1,2\}$. For simplicity, let $\rho = 3r$ with $r \geq 1$, and let $D_k = \{a_k, b_k, c_k\}$ for $k \geq 1$ satisfying that $\{b_k - a_k, c_k - a_k\} \equiv \{1,2\}\ (mod\ 3)$ and $gcd(b_k - a_k, c_k - a_k) = 1$ for all $k \in \mathbb{N}$.

Lemma (5.2.32)[317]: Let $\rho = 3r$ be an integer with $r \geq 1$ and let $C_0 = rD_0$ where $D_0 = \{0,1,2\}$. Then

(i) $(\rho^{-1}D_k, C_0)$ is a compatible pair for $k \geq 0$;

(ii) $(\rho^{-1}D_k, pC_0)$ is a compatible pair for $k \geq 0$, where $p \in 3\mathbb{Z} + \{1,2\}$;

(iii) $Z(\hat{\delta}\_(\rho^{-1}D_k) = Z(\hat{\delta}_{\rho^{-1}D_0}) = r(3\mathbb{Z} + \{1,2\})\ for\ k \geq 1$;

(iv) $Z(\hat{\mu}_{\rho,D_k}) = Z(\hat{\mu}_{\rho,D_0}) = \bigcup_{k=1}^{\infty} \rho^{k-1} r(3\mathbb{Z} + \{1,2\})$, where $\mu_{\rho,D_0} := \delta_{\rho^{-1}D_0} * \delta_{\rho^{-2}D_0} * \cdots$ and $\mu_{\rho,D_k} := \delta_{\rho^{-1}D_1} * \delta_{\rho^{-2}D_2} * \delta_{\rho^{-3}D_3} * \cdots$ in the weak*-topology.

Proof. (i) Let $D_k = D_k - a_k = \{0, b_k - a_k, c_k - a_k\}$ for all $k \in \mathbb{N}$. Then it is easy to check that for $k \in \mathbb{N}$ the matrix in (98)

$$H_{\rho^{-1}D_k,C_0} = H_{\rho^{-1}D_0,C_0} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i \frac{1}{3}} & e^{2\pi i \frac{2}{3}} \\ 1 & e^{2\pi i \frac{2}{3}} & e^{2\pi i \frac{4}{3}} \end{pmatrix}, if \begin{cases} b_k - a_k \equiv 1\ (mod\ 3) \\ c_k - a_k \equiv 2\ (mod\ 3) \end{cases}$$

or

$$H_{\rho^{-1}D_k,C_0} = H_{\rho^{-1}D_0,C_0} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i \frac{1}{3}} & e^{2\pi i \frac{1}{3}} \\ 1 & e^{2\pi i \frac{2}{3}} & e^{2\pi i \frac{2}{3}} \end{pmatrix}, if \begin{cases} b_k - a_k \equiv 1\ (mod\ 3) \\ c_k - a_k \equiv 2\ (mod\ 3) \end{cases}$$

is unitary, which means that both $(\rho^{-1}D_k, C_0)$ and $(\rho^{-1}D_0, C_0)$ are compatible pairs. By Lemma (5.2.26)(ii), we get the desired result (i).

(ii) Clearly, $pC_0 \equiv C_0\ (mod\ \rho)$ if $p \in 3\mathbb{Z} + \{1,2\}$. Thus (ii) follows by Lemma (5.2.26) (iii).

(iii) and (iv) can be easily obtained by computation.

The case $\rho = 3$ is interesting and the following lemma gives a proof of Theorem (5.2.3)(i) and also implies that there exists a lot of spectral measures with $\mathbb{Z}$ as a unique spectrum.

Lemma (5.2.33)[317]: If $\mu_{3,D}$ k is a spectral measure, then $\Lambda = \mathbb{Z}$ is the only spectrum with $0 \in \Lambda$ for the measure μ_3, D_k. Consequently, there is no $p \in \mathbb{R}\backslash\{\pm 1\}$ such that the sets Λ and $p\Lambda$ are spectra for μ_{3,D_k}.

Proof. From the proof of [337], we know that

$$\Lambda(3,\{0,-1,1\}) := \{0,-1,1\} + 3\{0,-1,1\} + 3^2\{0,-1,1\} + \cdots = \mathbb{Z}$$

is a spectrum for the measure μ_{3,D_k}, which is equivalent to say that $Q_{\mathbb{Z}}(\xi) := n \in Z$

$\left|\hat{\mu}_{3,D_k}(\xi\ +n)\right|^2 \equiv 1(\ \forall\xi \in \mathbb{R})$. Now we claim that $\mathbb{Z}$ is the unique spectrum for μ_{3,D_k}. Indeed, if a discrete set $\Lambda = \mathbb{Z}$ is also a spectrum for the measure μ_{3,D_k}, then $\Lambda\backslash\{0\} \subseteq Z(\hat{\mu}_{3,D_k}) = \bigcup_{n=0}^{\infty} 3^n((3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)) = \mathbb{Z}$.

Furthermore, for any $n_0 \in \mathbb{Z}\backslash\Lambda$, we can find a $\xi_0 \in (0,1)$ such that $\hat{\mu}_{3,D_k}(\xi_0 + n_0) \neq 0$, and hence

$$Q_\Lambda(\xi_0) := \sum_{\lambda\in\Lambda}\left|\mu_{3,D_k}(\xi_0+\lambda)\right|^2 \leq \sum_{\lambda\in\mathbb{Z}\{n_0\}}\left|\hat{\mu}_{3,D_k}(\xi_0+\lambda)\right|^2$$
$$= Q_Z(\xi_0) - \left|\hat{\mu}_{3,D_k}(\xi_0+n_0)\right|^2 = 1 - \left|\hat{\mu}_{3,D_k}(\xi_0+n_0)\right|^2 < 1.$$

It is a contradiction to Theorem (5.2.24). The proof is complete. The following lemma with Lemma (5.2.32) are two essential ingredients for the reduction of the proof of Theorem (5.2.14)(ii).

Lemma (5.2.34)[317]: Let $\rho = 3r$ be an integer with $r > 1$ and let $C_0 = rD_0$ where $D_0 = \{0,1,2\}$.

(i) For any $w = w_1 w_2 \cdots \in \{-1,1\}^{\mathbb{N}}$, the set

$$p\Lambda_w(\rho, C_0) := p\left\{\sum_{j=1}^{m} w_j\rho^{j-1}c_j : c_j \in C_0, w_j \in \{-1,1\}\right\},$$

is an orthogonal system for μ_{ρ,D_k}, where $p \in 3\mathbb{Z} + \{1,2\}$.

(ii) For any $w = w_1 w_2 \cdots \in \{-1,1\}^{\mathbb{N}}$, the set

$$\Lambda_w(\rho, C_0) := \left\{\sum_{j=1}^{m} w_j\rho^{j-1}c_j\ :\ c_j \in C_0, w_j \in \{-1,1\}\right\}, \tag{127}$$

is a spectrum for the measure μ_{ρ,D_k}.

Proof. (i) If $p \in 3\mathbb{Z} + \{1,2\}$, then $pC_0 \equiv w_j C_0 \pmod{\rho}$, where $w_j \in \{-1,1\}$. Thus, by Lemma (5.2.32)(ii), $(\rho^{-1}D_k, pw_jC_0)$ are compatible pairs. Therefore, it is easy to check, similar to the proof of Lemma (5.2.26), that the desired result follows from Lemma (5.2.34) and the relationship

$$p\Lambda_w(\rho, C_0) = \bigcup_{j=1}^{\infty}(pw_1C_0 + pw_2\rho C_0 + \cdots + pw_j\rho^{j-1}C_0).$$

(ii) Please refer to [335]. Now we can give the proof of Theorem (5.2.14) (ii).

It is enough to construct a discrete set Λ in the form (127) such that $\Lambda, p_1\Lambda$ and $p_2\Lambda$ are all spectra for the measure $\mu_\rho, \{a_k, b_k, c_k\}$, where $p_1, p_2 \in 3\mathbb{Z} + \{1,2\}$and $gcd(p_1,p_2) = 1$. By Lemma (5.2.32), for the above p_1 and p_2, the remaining task is to choose an infinite word $w \in \{-1,1\}^{\mathbb{N}}$ such that $p_1\Lambda_w(\rho, C_0)$ and $p_2\Lambda_w(\rho, C_0)$ are both spectra for the measure μ_{ρ,D_k}. As in the proof of Theorem (5.2.6)(ii), we can see (especially see Lemma (5.2.4) and (5.2.9)) that the choice of w will depend on the intersection of the zero set $Z(\mu_{\rho,D_k})$ and the compact set $T(\rho, p_iC_0 \cup (-p_iC_0))$ for $i = 1,2$. It follows from Lemma (5.2.27)(iv) that the set $Z(\mu_{\rho,D_k})$ can be replaced by $Z(\mu_{\rho,D_0})$. Whence, the proof of the sufficiency has reduced to that of Theorem (5.2.14)(ii) for the measure μ_{ρ,D_0}. Necessity. Fix $p \in \mathbb{R}$ and suppose $\Lambda, p\Lambda$ with $0 \in \Lambda$ are both spectra for μ_{ρ,D_k}. Clearly, the proof for the necessity of Theorem (5.2.12) can be applied successfully to that of the measure μ_{ρ,D_0}. From Lemma (5.2.27) (iii) and (iv), one can repeat this whole proof (in the necessity of Theorem (5.2.2)) for the measure μ_{ρ,D_k} instead of μ_{ρ,D_0}. As a consequence, $p=\frac{p_1}{p_2}$ where p_1, p_2 and 3 are pairwise coprime.

Section (5.3): Self-Similar Measures with Consecutive Digits:

A probability measure μ on $\mathbb{R}$ with compact support is said to be a spectral measure if there exists a countable set Λ of real numbers, called a spectrum of μ, such that $E(\Lambda)=\{e^{-2\pi i\lambda x}: \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. A well-known classic example is the Lebesgue measure on $[0,1]$ for which the set $\mathbb{Z}$ is the only spectrum containing 0. The existence of a spectrum for a probability measure is one of fundamental problems in applied harmonic analysis builded on a measure, and it was initiated by Fuglede [372]. In 1998, Jorgensen and Pedersen [378] discovered the first families of non-atomic singular spectral measures. They showed that the Bernoulli convolutions are spectral measures if the contraction ratios are the reciprocal of an even integer. Recently, Dai [360] showed that the only spectral Bernoulli convolutions are of the above cases. The details on the background of Bernoulli convolutions and recent topics are given in [360,371,383,385,384] and the references therein. Actually, the spectral measure problem attracts more attention due to Jorgensen and Pedersen's examples. Following this discovery, various singular spectral measure on self-similar/self-affine/moran fractal sets have been constructed (see [378,360,361,362,380,382,367,357,358,373]). Usually, the following two types of questions have been considered:

(Q1) When is a Borel probability measure μ spectral?

Until now, there are only a few classes of singular spectral measures that are known. It is still a basic problem to find more spectral measures.

(Q2) For a given spectral measure μ, can we find all the spectra of μ ?

It is quite challenging to characterize all the spectra of a given singular spectral measure μ (no example is known with this property). The first attempt of the classification of spectra was studied by Dutkay, Han and Sun [364]. They gave a complete characterization of the maximal orthogonal sets of the one-fourth standard Cantor measure (denoted by μ_4) by introducing a labeling tool on the infinite binary tree. They gave some sufficient conditions for a maximal orthogonal set to be a spectrum. Later, Dai, He and Lai [362] gave some sufficient conditions and necessary conditions for a maximal orthogonal set of μ_4 to be a spectrum. Generally speaking, for a given singular spectral measure μ, there are two basic problems (call them spectral eigenvalue problems) as follows:

Case I. Let Λ be a spectrum of μ. Find all real numbers b such that $b\Lambda$ is also a spectrum of μ.

Case II. Find all real numbers b for which there exists a set Λ such that both Λ and $b\Lambda$ are spectra of μ.

Let an iterated function system (IFS) on $\mathbb{R}$ of the form $f_i(x)=\frac{x}{p}+\frac{i}{q}, i=0,1,\ldots,q-1$ where $2 \le q \in \mathbb{Z}$ and $p \in \mathbb{R}$. By Hutchinson's theorem [359,369,374], there exists a unique Borel probability measure $\mu_{p,q}$ with compact support $T_{p,q}$ satisfying that

$$\mu_{p,q}(E)=\sum_{i=0}^{q-1}\frac{1}{q}\mu_{p,q}\left(f_i^{-1}(E)\right) \tag{128}$$

for any Borel set $E \subseteq \mathbb{R}$ and $T_{p,q}$ is the unique compact set satisfying that

$$T_{p,q} = \bigcup_{i=0}^{q-1} f_i(T_{p,q}).$$

When $q = 2, \mu_{p,q}$ becomes the standard Cantor measure of contraction ratio $\frac{1}{p}$ denoted by μ_p, i.e., the Bernoulli convolutions. For the $\mu_{p,q}$, there are some known results focused on the above two questions:

- $\mu_{p,q}$ is a spectral measure if and only if $\frac{p}{q} \in \mathbb{Z}$ [363]. If $p = q$, the measure $\mu_{p,q}$ is the Lebesgue measure restricted to the interval [0,1].
- Let b be a real number. If $p > q$ with $\frac{p}{q} \in \mathbb{Z}$, then

  $b \in \{r \in \mathbb{R}$: there exists a set Λ such that both Λ and $r\Lambda$ are spectra of $\mu_{p,q}\}$

  if and only if $b = \frac{b_1}{b_2}$, where gcd $(b_1, b_2) = 1$ and b_1, b_2 are coprime with q respectively [370].

The remaining relatively tractable situation is Case II of the above (Q2) for the measure $\mu_{p,q}$. The special cases are the Bernoulli convolution μ_{2k} with $k \in \mathbb{Z}^+$, i.e., the set of all positive integers. It is known that the simplest spectrum for μ_{2k} [378] is

$$\Lambda_{2k} = \left\{ \sum_{j=0}^{n} a_j (2k)^j : a_j \in \{0,1\}, n \in \mathbb{N} := \{0,1,2,\dots\} \right\}.$$

Later, Laba, Wang, Jorgensen, Dutkay and Li et al. investigated for what $b \in \mathbb{N}$, the scaling set $b\Lambda_4$ or $b\Lambda_{2k}$ is also a spectrum of μ_4, or μ_{2k} respectively [379,366,368,376, 381]. We will investigate those problems for the general case $\mu_{p,q}$.

The study was motivated by the following surprising facts and questions also: (i) There exists a singular spectral measure μ such that the Fourier expansion of any function in $L^2(\mu)$ with respect to a spectrum is uniformly convergent, but it is not convergent with respect to another spectrum for some continuous functions [365,386,387]. (ii) There exists a singular spectral measure with a maximal orthogonal set Λ (not a basis), but $k\Lambda$ is a spectrum of μ for some $k > 1$ [361]. (iii) The ordinary Fourier series of continuous functions converge uniformly for standard Cantor measures with respect to a model spectrum [387]. It is natural to ask whether these phenomena are universal?

In 2013, Dai, He and Lai [362] proved the following:

Theorem (5.3.1)[356]: If q divides p, then $\mu_{p,q}$ is a spectral measure with a spectrum

$$\Lambda_{p,q} = \{0,1,\dots,q-1\} + p\{0,1,\dots,q-1\} + p^2\{0,1,\dots,q-1\} + \cdots \text{ (finite sum).}$$

We often refer to the spectrum $\Lambda_{p,q}$ as the canonical spectrum of $\mu_{p,q}$. The appellation is due to Jorgensen [377].

Assumption (5.3.2)[356]: We will assume $\frac{p}{q} \in \mathbb{Z}$ with $p > q \geq 2$.

We answer when the scaling set $b\Lambda_{p,q}$ is also a spectrum of $\mu_{p,q}$ for $b \in \mathbb{Z}$. Note that if Λ is a spectrum of $\mu_{p,q}$, then we have $-\Lambda$ is also a spectrum of $\mu_{p,q}$. Thus we only need to consider the case that $b \in \mathbb{N}$.

The next theorem gives a general characterization of the integer b by applying the properties of quadratic congruence equations and the order of elements in the finite group.

Here $ord_b(p)$ means the order of p in the multiplicative group $\mathbb{Z}_b^*$ (see Definition (5.3.16)) and $\lfloor x \rfloor$ means the greatest integer number which is no larger than x.

We introduce some basic definitions and lemmas. We prove Theorem (5.3.12). At the end, we provide a direct method to prove $(p-1)\Lambda$ is not a spectrum of $\mu_{p,q}$. The proof of Theorem (5.3.19) is presented. Finally, we give an example to illustrate our theory .

Let μ be a Borel probability measure on $\mathbb{R}$ with compact support. Denote the exponential function $e^{-2\pi i\lambda x}$ by e_λ. We say that a countable set Λ is a maximal orthogonal set (spectrum) of μ if $E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$ is a maximal orthogonal set (an orthonormal basis) for $L^2(\mu)$. Here $E(\Lambda)$ is a maximal orthogonal set of exponentials means that it is a mutually orthogonal set in $L^2(\mu)$ such that if $\alpha \notin \Lambda$, e_α is not orthogonal to some $e_\lambda, \lambda \in \Lambda$. It is easy to show that Λ is an orthogonal set of μ if and only if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}_{\hat{\mu}}. \tag{129}$$

Here $\mathcal{Z}_f := \{\xi : f(\xi) = 0\}$ is the set of the roots of the function $f(\xi)$ on the real line. And the Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}\, \mathrm{d}\mu(x).$$

By the definition of Fourier transform of $\mu_{p,q}$ and (128), one has

$$\hat{\mu}_{p,q}(\xi) = m_{p,q}\left(\frac{\xi}{p}\right)\hat{\mu}_{p,q}\left(\frac{\xi}{p}\right), \tag{130}$$

where $m_{p,q}(\xi) = \frac{1}{q}\left(1 + e^{-2\pi i\frac{p\xi}{q}} + \cdots + e^{-2\pi i\frac{(q-1)p\xi}{q}}\right)$. Iterating (130), we obtain an explicit expression of the Fourier transform of the measure $\mu_{p,q}$:

$$\hat{\mu}_{p,q}(\xi) = \prod_{n=1}^{\infty} m_{p,q}\left(\frac{\xi}{p^n}\right).$$

It is easy to calculate that

$$\mathcal{Z}_{\hat{\mu}_{p,q}} = \bigcup_{n=1}^{\infty} p^n \mathcal{Z}_{m_{p,q}}.$$

Furthermore, we have $\mathcal{Z}_{m_{p,q}} = \left\{\frac{a}{p} : q \nmid a, a \in \mathbb{Z}\right\}$ and $\mathcal{Z}_{\hat{\mu}_{p,q}} = \{p^n a : q \nmid a, a \in \mathbb{Z}, n \geq 0\}$.

Lemma (5.3.3)[356]: Let $b \in \mathbb{N}$. Then $b\Lambda_{p,q}$ is an orthogonal set of $\mu_{p,q}$ if and only if $q \nmid kb$ with $1 \leq k \leq q-1$.

Proof. Since $\mathcal{Z}_{\hat{\mu}_{p,q}} = \{p^j a : a \in \mathbb{Z} \setminus q\mathbb{Z}, j \geq 0\}$, by (129), we obtain that $b\Lambda_{p,q}$ is an orthogonal set of $\mu_{p,q}$ if and only if

$$(b\Lambda_{p,q} - b\Lambda_{p,q}) \setminus \{0\} \subseteq \mathcal{Z}_{\hat{\mu}_{p,q}}. \tag{131}$$

For $\lambda_1 \neq \lambda_2 \in b\Lambda_{p,q}$, we write $\lambda_1 = b(d_0 + pd_1 + \cdots + p^m d_m)$ and $\lambda_2 = b(d_0' + pd_1' + \cdots + p^n d_n')$, where $m \geq n$. Let s be the minimal index such that $d_s \neq d_s'$. Then

$$\lambda_1 - \lambda_2 = p^s(b(d_s - d_s') + pM)$$

for some integer M. By (131) and $d_s, d_s' \in \{0,1,\cdots,q-1\}$, we obtain that $b\Lambda_{p,q}$ is an orthogonal set of $\mu_{p,q}$ if and only if $q \nmid kb$ with $1 \leq k \leq q-1$.

Remark (5.3.4)[356]: Theorem (5.3.12) in [370] tells us that if $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$, then $\gcd(b,q)=1$. Note that the condition $\gcd(b,q)=1$ implies $q\nmid kb$ with $1\le k\le q-1$. In fact, if this is not true, there exists an integer $j\in\{1,2,\dots,q-1\}$ such that $q\mid jb$. Since $\gcd(b,q)=1$ we have $q\mid j$. This is a contradiction.

Definition (5.3.5)[356]: Let $b\in\mathbb{Z}$ with $|b|\ge 2$. Let $D,C\subseteq\mathbb{Z}$ be finite sets of integers with $\#D=\#C$, where $\#D$ means the cardinality of D. We say that the system (b,D,C) forms a Hadamard triple (or $(b^{-1}D,C)$ forms a compatible pair, as it is called in [379]) if the matrix

$$H=\frac{1}{\sqrt{\#D}}\left[e^{2\pi i b^{-1}dc}\right]_{d\in D,c\in C}$$

is unitary, i.e., $H^*H=I$.

Lemma (5.3.6)[356]: Let $D=s\{0,1,\dots,q-1\}$ and $C=b\{0,1,\dots,q-1\}$, where $s=\frac{p}{q}$ and $b\in\mathbb{N}$. Then (p,D,C) forms a Hadamard triple if and only if $q\nmid kb$ with $1\le k\le q-1$.

Proof. Let $H=\frac{1}{\sqrt{q}}\left[e^{2\pi i p^{-1}dc}\right]_{d\in D,c\in C}$ and for simplicity we assume $H^*H=\frac{1}{q}[a_{mn}]$. Then for $m=n$, we have $a_{mm}=\underbrace{e^0+e^0+\cdots+e^0}_{q}=q$. For $m\ne n$, we have

$$a_{mn}=\begin{cases}\dfrac{1-e^{2\pi i b(n-m)}}{1-e^{2\pi i\frac{b}{q}(n-m)}}, & \text{if } \dfrac{sb(n-m)}{p}\notin\mathbb{Z},\\ q, & \text{otherwise.}\end{cases}$$

Therefore, $a_{mn}=0$ is equivalent to $\frac{b}{q}(n-m)\notin\mathbb{Z}$, i.e., $q\nmid kb$ with $1\le k\le q-1$.

Definition (5.3.7)[356]: Let $b\in\mathbb{N}$. We say that a finite set $\{x_0,x_1,\dots,x_{r-1}\}\subseteq\mathbb{R}$ is an extreme cycle for the digit set $b\{0,1,\dots,q-1\}$ (or an extreme cycle, for short) if there exists $\{l_0,l_1,\dots,l_{r-1}\}\subseteq b\{0,1,\dots,q-1\}$ such that

(i) $x_i=\frac{x_{i-1}+l_{i-1}}{p}$ for all $1\le i\le r-1$, and $x_0=\frac{x_{r-1}+l_{r-1}}{p}$,

(ii) $\frac{p}{q}x_i\in\mathbb{Z}$ for all $0\le i\le r-1$.

The points $x_i(0\le i\le r-1)$ are called extreme cycle points.

By Theorem (5.3.19) in [379], Definition (5.3.7) and Lemma (5.3.6), we can obtain the following theorem which is important.

Theorem (5.3.8)[356]: Let $b\in\mathbb{N}$ and $q\nmid kb$ with $1\le k\le q-1$. Then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$ if and only if the only extreme cycle for the digit set $b\{0,1,\dots,q-1\}$ is the degenerate one $\{0\}$.

Remark (5.3.9)[356]: When $b=p-1$, we have $(b-1)\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$ by Theorem (5.3.8). In fact, the set $\{1\}$ is an extreme cycle: $1=\frac{1+(b-1)}{b}$. We will provide another direct method to prove the case. Recall that if $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$, then $\gcd(b,q)=1$. However, the example also implies that the converse is not true.

Depending on the above theorem, our question can be directly turned into a number theory question. That is to say, we are required to find all the possible integers b for which there are non-degenerate extreme cycle for the digit set $b\{0,1,\dots,q-1\}$. This method is originated from the work of Dutkay, Haussermann [366].

We will introduce a couple of propositions, which are introduced in a more general case $\mu_{b,D}$ as follows in comparison with $\mu_{p,q}$. They will play a key role in proving Theorem (5.3.12).

Let b be an integer with $|b| \geq 2$ and let $D \subseteq \mathbb{Z}$ be a finite digit set with $\#D \geq 2$. Then they naturally arise an IFS $\left\{f_d(x) = \frac{1}{b}(x+d): d \in D\right\}$ and the self-similar set $T_{b,D}$, where $T_{b,D}$ can be expressed by

$$T_{b,D} = \left\{\sum_{n=1}^{\infty} d_n b^{-n}: d_n \in D \text{ for all } n \in \mathbb{N}\right\}.$$

They generate the self-similar measure $\mu_{b,D}$, which is the unique probability measure with compact support $T_{b,D}$ satisfying

$$\mu_{b,D}(E) = \sum_{d\in D} \frac{1}{\#D} \mu_{b,D}(f_d^{-1}(E))$$

for any Borel set $E \subseteq \mathbb{R}$. Similarly, one can easily compute the Fourier transform of the measure $\mu_{b,D}$:

$$\hat{\mu}_{b,D} = \prod_{n=1}^{\infty} m_D\left(\frac{\xi}{b^n}\right)$$

where

$$m_D(\xi) = \sum_{d\in D} e^{-2\pi i d\xi}.$$

Thus, we obtain the following relationship

$$\mathcal{Z}_{\hat{\mu}_{b,D}} = \bigcup_{n=1}^{\infty} b^n \mathcal{Z}_{m_D}. \tag{132}$$

The following proposition was proved in [371]. Since the proof is simple, we give it here.

Proposition (5.3.10)[356]: Let b be an integer with $|b| \geq 2$, and let D and A be two finite subsets of $\mathbb{Z}$ such that $0 \in A$ (the cardinality of D and A may not be equal). Then the following two statements are equivalent:

(i) $\mathcal{Z}_{m_{b^{-1}D}} \cap T_{b,A} = \emptyset$,

(ii) $\mathcal{Z}_{\hat{\mu}_{b,D}} \cap T_{b,A} = \emptyset$,

where

$$T_{b,A} = \left\{\sum_{n=1}^{\infty} a_n b^{-n}: a_n \in A \text{ for all } n \in \mathbb{N}\right\}.$$

Proof. (i) ⇒ (ii) Suppose $\mathcal{Z}_{m_{b^{-1}D}} \cap T_{b,A} = \emptyset$. Since $0 \in A$, we have

$$b^{-j} T_{b,A} \subseteq T_{b,A} \text{ for all } j \in \mathbb{N}.$$

This leads to $\mathcal{Z}_{m_{b^{-1}D}} \cap b^{-j} T_{b,A} = \emptyset$ for all $j \in \mathbb{N}$. Notice that $\mathcal{Z}_{m_{b^{-1}D}} = b\mathcal{Z}_{m_D}$. Thus (ii) holds by (132). (ii) ⇒ (i) It follows easily by (132).

The following proposition goes back to the work of Strichartz [386]. Actually, his conclusion is for a more general Borel probability measure on $\mathbb{R}$ which is called Moran measure.

Proposition (5.3.11)[356]: Let b be an integer with $|b| \geq 2$, and let D and C be two finite subsets of $\mathbb{Z}$ such that $0 \in D \cap C$ and (b, D, C) forms a Hadamard triple. Let $\Lambda_{b,C} = \left\{\Sigma_{i=0}^{n} b^i c_k: c_k \in C, n \in \mathbb{N}\right\}$. Suppose that $\mathcal{Z}_{\hat{\mu}_{b,D}} \cap T_{b,C} = \emptyset$, where

$$T_{b,C} = \left\{ \sum_{n=1}^{\infty} c_n b^{-n} : c_n \in C \text{ for all } n \in \mathbb{N} \right\}.$$

Then $\Lambda_{b,C}$ is a spectrum of $\mu_{b,D}$.

We now present the proof of Theorem (5.3.12).

Theorem (5.3.12)[356]: Let $b \in \mathbb{N}$. If $b < \frac{p}{q}$ and $q \nmid kb$ with $1 \le k \le q-1$, i.e., q does not divide kb, then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$.

Proof. Let $D = s\{0,1,\dots,q-1\}$ and $C = b\{0,1,\dots,q-1\}$, where $s = \frac{p}{q}$. It is easy to see that

$$\begin{aligned} T_{p,C} &= \left\{ \sum_{n=1}^{\infty} c_n p^{-n} : c_n \in C \text{ for all } n \in \mathbb{N} \right\} \\ &\subseteq \left[0, \frac{b(q-1)}{p-1} \right], \end{aligned}$$

and

$$\mathcal{Z}_{m_{b^{-1}D}} = p\mathcal{Z}_{m_{p,q}} = \{a : q \nmid a, a \in \mathbb{Z}\}.$$

From the hypothesis $b < s$ it follows that $\frac{b(q-1)}{p-1} < 1$. Thus, $\mathcal{Z}_{m_{b^{-1}}} \cap T_{p,C} = \emptyset$. We then obtain that

$$\mathcal{Z}_{\hat{\mu}_{p,D}} \cap T_{p,C} = \emptyset$$

by Proposition (5.3.10). Since $q \nmid kb$ with $1 \le k \le q-1$, one has (p,D,C) forms a Hadamard triple by Lemma (5.3.6). Consequently, $\Lambda_{p,C} = b\Lambda_{p,q}$ is a spectrum of $\mu_{p,D}$ by Proposition (5.3.11). Note that $\mu_{p,D} = \mu_{p,q}$. This completes the proof of Theorem (5.3.12).

We conclude with a proposition for which we provide another method to prove that $(p-1)\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$.

Proposition (5.3.13)[356]: Let $b \in \mathbb{N}$. If $b = p-1$. Then $b\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$.

Proof. First, we prove that $(p-1)\Lambda_{p,q}$ is an orthogonal set of $\mu_{p,q}$. To see it, for any $\lambda_1 \ne \lambda_2 \in (p-1)\Lambda_{p,q}$, we write $\lambda_1 = (p-1)\sum_{n=0}^{\infty} a_n p^n$ and $\lambda_2 = (p-1)\sum_{n=0}^{\infty} b_n p^n$ (finite sum), where $a_n, b_n \in \{0,1,\dots,q-1\}$ for all $n \ge 0$. Let k be the first index such that $a_k \ne b_k$. Then for some integer M_0, we can write

$$\lambda_1 - \lambda_2 = p^k((p-1)(a_k - b_k) + pM_0).$$

Since $a_k \ne b_k$, we can easy to know $q \nmid (p-1)(a_k - b_k)$. Then $\lambda_1 - \lambda_2$ lies in $\mathcal{Z}_{\hat{\mu}_{p,q}}$. Therefore, $(p-1)\Lambda_{p,q}$ is an orthogonal set of $\mu_{p,q}$ by (129).

Next, we prove $E\big((p-1)\Lambda_{p,q}\big) := \{e^{-2\pi i (p-1)\lambda x}\}_{\lambda \in \Lambda_{p,q}}$ is not complete in $L^2(\mu_{p,q})$, i.e., we need to prove that $\left(\overline{\operatorname{span}} E\big((p-1)\Lambda_{p,q}\big)\right)^{\perp} \ne \emptyset$. Our goal now is to prove that for any $j \in \{1,2,\dots,q-1\}$, $e_{-j} \in \left(\overline{\operatorname{span}} E\big((p-1)\Lambda_{p,q}\big)\right)^{\perp}$. In fact, we take the inner product of $e_{(p-1)\lambda}$ for $\lambda \in \Lambda_{p,q}$ and e_{-j} for $j \in \{1,2,\dots,q-1\}$:

$$\langle e_{(p-1)\lambda}, e_{-j} \rangle = \hat{\mu}_{p,q}\big((p-1)\lambda + j\big).$$

Let $\lambda = \sum_{i=0}^{n} a_i p^i$ with $a_i \in \{0,1,\dots,q-1\}$. Then we see that

$$(p-1)\lambda + j = (p\lambda + j) - \lambda = \left(j + \sum_{i=0}^{n} a_i p^{i+1}\right) - \sum_{i=0}^{n} a_i p^i.$$

Note that $\lambda' := j + \sum_{i=0}^{n} a_i p^{i+1}$ is another element of $\Lambda_{p,q}$. Then, using (129) and since $\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$, we have

$$(p-1)\lambda + j \in \mathcal{Z}_{\tilde{\mu}_{p,q}}.$$

Thus, we obtain that $\langle e_{(p-1)\lambda}, e_{-j}\rangle = 0$ and so $e_{-j} \in \left(\frac{\text{span}}{E}\left((p-1)\Lambda_{p,q}\right)\right)^{\perp}$.

Remark (5.3.14)[356]: To complete $(p-1)\Lambda_{p,q}$, one can consider the set $\bar{\Lambda} = (p-1)\Lambda_{p,q} \cup \left\{(p-1)\sum_{i=0}^{n} a_i p^i - p^{n+1} : a_i \in \{0,1,\dots,q-1\}, n \geq 0\right\}$. Similarly as in [364], by introducing a technique of tree labeling for $\mu_{p,q}$, we can prove that $\bar{\Lambda}$ is a spectrum of $\mu_{p,q}$. This sheds light on the statement that $(p-1)\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$.

We will begin by proving two lemmas and then use them to prove Theorem (5.3.19). We start with some concepts in number theory (e.g., see [375]).

Definition (5.3.15)[356]: Let $b \in \mathbb{N}$. We denote by $\mathbb{Z}_b$ the finite ring of integers modulo $b, \mathbb{Z}/b\mathbb{Z}$. And we denote by $\mathbb{Z}_b^*$ the set of elements in $\mathcal{Z}_b$ that has a multiplicative inverse.

Definition (5.3.16)[356]: Let $b \in \mathbb{N}$ and $gcd(p,b) = 1$. Denote the subgroup G_b of $\mathbb{Z}_b^*$ generated by p,

$$G_b := \{p^n \ (\text{mod} b) : n = 0,1,\dots\}.$$

The order of p in the group Z_6^* is defined to be the smallest positive integer x such that

$$p^x \equiv 1 \ (\text{mod} b).$$

Denote by $\text{ord}_b(p)$ the order of p.

The following two lemmas are motivated by the work of Dutkay, Haussermann [366].

Lemma (5.3.17)[356]: If x^* is an extreme cycle point for some extreme cycle, then $x^* \in \mathbb{Z}$ and x^* has a periodic base p expansion, i.e.,

$$x^* = \frac{a_0}{p} + \frac{a_1}{p^2} + \cdots + \frac{a_{r-1}}{p^r} + \frac{a_0}{p^{r+1}} + \frac{a_1}{p^{r+2}} + \cdots + \frac{a_{r-1}}{p^{2r}} + \cdots,$$

where $a_k \in b\{0,1,\dots,q-1\}$.

Proof. Suppose x^* is an extreme cycle point for $X = \{x_0, x_1, \dots, x_{r-1}\}$. Without loss of generality, we assume $x^* = x_0$. Then by the definition of extreme cycle, we have

$$\begin{aligned} x_0 &= \frac{x_{r-1}}{p} + \frac{l_{r-1}}{p} \\ &= \frac{x_{r-2}}{p^2} + \frac{l_{r-2}}{p^2} + \frac{l_{r-1}}{p} \\ &= \cdots \\ &= \frac{x_0}{p^r} + \frac{l_0}{p^r} + \frac{l_1}{p^{r-1}} + \cdots + \frac{l_{r-1}}{p}, \end{aligned} \tag{133}$$

where $\{l_0, l_1, \dots, l_{r-1}\} \subseteq b\{0,1,\dots,q-1\}$. By iterating the above equality infinitely, one has x_0 is of the periodic base p expansion. For the sake of brevity, we denote $\frac{p}{q} = s$. Since $sx_0 \in \mathbb{Z}$, we have $x_0 \in \frac{\mathbb{Z}}{s}$. If $x_0 = \frac{sm+i}{s}$ with $m \in \mathbb{Z}$ and $i \in \{1,2,\dots,s-1\}$, then

$$x_1 = \frac{x_0 + l_0}{p} = \frac{\frac{sm+i}{s} + l_0}{p},$$

where $l_0 \in b\{0,1,\dots,q-1\}$. We claim $sx_1 = \frac{s(m+l_0)+i}{sq} \notin \mathbb{Z}$. In fact, if $sx_1 = \frac{s(m+l_0)+i}{sq} = k \in \mathbb{Z}$, then $i = s(qk-m-l_0) =: sk_1$, which is contrary to $i \in \{1,2,\dots,s-1\}$. Thus, $sx_1 \notin \mathbb{Z}$, which contradicts the condition (ii) of the definition of extreme cycle. Hence, $x_0 \in \mathbb{Z}$.

Lemma (5.3.18)[356]: Let $C = b\{0,1,\dots,q-1\}$ and $\gcd(b,p) = 1$. Then

$$\{x_0 : x_0 \text{ is an extreme cycle point for some extreme cycle}\} = T_{p,C} \cap \mathbb{Z},$$

where $T_{p,C}$ is the attractor of the iterated function system $\left\{f_i(x) : f_i(x) = \frac{x+i}{p}, i \in b\{0,1,\dots,q-1\}\right\}$.

Proof. If x_0 is an extreme cycle point for some extreme cycle, by Lemma (5.3.17), we have $x_0 \in \mathbb{Z}$ and

$$x_0 = \frac{a_0}{p} + \frac{a_1}{p^2} + \cdots + \frac{a_{r-1}}{p^r} + \frac{a_0}{p^{r+1}} + \frac{a_1}{p^{r+2}} + \cdots + \frac{a_{r-1}}{p^{2r}} + \cdots,$$

where $a_k \in b\{0,1,\dots,q-1\}$. And since $T_{p,C} = \left\{\sum_{n=1}^{\infty} \frac{c_n}{p^n} : c_n \in b\{0,1,\dots,q-1\}\right\}$, one easily sees that

$$x_0 \in T_{p,C} \cap \mathbb{Z}.$$

Conversely, for any $x_0 \in T_{p,C} \cap \mathbb{Z}$, we observe that $T_{p,C} = \bigcup_{i \in b\{0,1,\dots,q-1\}} f_i(T_{p,C})$. So we have $x_0 \in f_i(T_{p,C})$ for some $i \in b\{0,1,\dots,q-1\}$. Then there exists $x_{-1} \in T_{p,C}$ such that $x_0 = \frac{x_{-1}+i}{p}$. Thus $x_{-1} = px_0 - i \in T_{p,C} \cap \mathbb{Z}$ and then we have $x_{-1} \equiv px_0 (\mathrm{mod} b)$. By induction, we obtain $x_{-1}, x_{-2}, \dots \in T_{p,C} \cap \mathbb{Z}$ and $l_0, l_1, \dots \in b\{0,1,\dots,q-1\}$ such that $x_{-j} = \frac{x_{-j-1}+l_j}{p}$ for all $j \geq 1$. Moreover, we have $x_{-j} \equiv p^j x_0$ $(\mathrm{mod} b)$. Since p and b are relatively prime, we have $p^a \equiv 1 (\mathrm{mod} b)$, where $a = \mathrm{ord}_b(p)$. Then

$$x_{-a} \equiv p^a x_{-a} \ (\mathrm{mod}\ b).$$

Since $x_{-a} \equiv p^a x_0 (\mathrm{mod} b)$ and according to the fact that p and b are relatively prime again, it follows that $x_{-a} \equiv x_0 (\mathrm{mod} b)$. Because x_{-a} and x_0 are contained in $T_{p,C} \subseteq \left[0, \frac{b(q-1)}{p-1}\right]$, it follows that $x_{-a} = x_0$. Then x_0 is an extreme cycle point for some extreme cycle. Thus, we complete the proof.

We now present the proof of Theorem (5.3.19).

Theorem (5.3.19)[356]: Let $b \in \mathbb{N}$. Suppose b is a prime with $\gcd(b,p) = 1$. Then the following statements hold:

(i) If ord $_b(p)$ is even, then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$.

(ii) If ord $_b(p)$ is odd with ord $_b(p) > q^{n_0} - 1$ where $n_0 = \left\lfloor \frac{\ln \frac{b(q-1)}{p-1}}{\ln p} \right\rfloor + 1$, then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$.

Claim (5.3.20)[356]: If there exists a number denoted by y in G_b with $y \equiv -1 (\mathrm{mod} b)$, then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$.

Proof. Suppose $b\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$, there exists a non-degenerate extreme cycle $X = \{x_0, x_1, \dots, x_{r-1}\}$ with $L = \{l_0, l_1, \dots, l_{r-1}\} \subseteq b\{0,1,\dots,q-1\}$ by Theorem (5.3.8). Since $x_{i+1} = \frac{x_i + l_i}{p}$ for $i \in \{0,1,\dots,r-2\}$, we have $px_{i+1} \equiv x_i (\mathrm{mod} b)$. Moreover, $x_0 = \frac{x_{r-1}+l_{r-1}}{p}$. Then

$$p^{r-i}x_0 \equiv x_i \ (\text{mod} b) \text{ for } i \in \{0,1,\dots,r-1\}.$$

So for any $m \in \mathbb{N}$, we obtain that $p^m x_0$ is congruent modulo b with an element in X. Combining with the assumption that there exists a number denoted by y in G_b with $y \equiv -1(\text{mod} b)$, we conclude that there exists $j \in \mathbb{N}$ such that

$$yx_0 \equiv x_j \ (\text{mod} b)$$

and

$$yx_0 \equiv -x_0 \ (\text{mod} b).$$

Thus, we obtain that $x_j \equiv -x_0(\text{mod} b)$. By Lemma (5.3.18), we have $0 < x_i \leq \frac{b(q-1)}{p-1}$ for all $i \in \{0,1,\dots,r-1\}$. From these, we obtain that $x_j > \frac{b(q-1)}{p-1}$. In fact, we have

$$b - x_0 - \frac{b(q-1)}{p-1} \geq b - 2\frac{b(q-1)}{p-1} = \frac{b}{p-1}(p - 2q + 1) > 0.$$

Consequently, we have $b - x_0 > \frac{b(q-1)}{p-1}$ and so $x_j > \frac{b(q-1)}{p-1}$. This is a contradiction. Thus, the claim follows.

Returning to the proof of (i), we will use the well-known quadratic congruence equation $x^2 \equiv 1(\text{mod} b)$ [375]. Let a be the smallest positive integer such that $p^a \equiv 1(\text{mod} b)$. By the assumption that a is even, we have $\left(p^{\frac{a}{2}}\right)^2 \equiv 1(\text{mod} b)$ and $p^{\frac{a}{2}} \in G_b$. Since b is a prime, we have

$$p^{\frac{a}{2}} \equiv \pm 1 \ (\text{mod} b).$$

By the minimality of a, we get that $p^{\frac{a}{2}} \not\equiv 1(\text{mod} b)$. Thus $p^{\frac{a}{2}} \equiv -1(\text{mod} b)$. And we then have $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$ by the above claim.

(ii). Suppose $b\Lambda_{p,q}$ is not a spectrum of $\mu_{p,q}$, there exists a non-degenerate extreme cycle $X = \{x_0, x_1, \dots, x_{r-1}\}$ with $L = \{l_0, l_1, \dots, l_{r-1}\} \subseteq b\{0,1,\dots,q-1\}$. By the definition of extreme cycle, it is easy to see that

$$p^r x_i \equiv x_i \ (\text{mod} b) \text{ for all } i \in \{0,1,\dots,r-1\}.$$

Since $0 < x_i \leq \frac{b(q-1)}{p-1} < b$ and b is a prime, one has $p^r \equiv 1(\text{mod} b)$. By the condition of (ii) and the definition of order of p, we have $r \geq \text{ord}_p(b) > q^{n_0} - 1$ (the last inequality is the assumption of Theorem (5.3.19)(ii)).

Next, we consider the set $T_{p,C} := \{\sum_{n=1}^{\infty} c_n p^{-n} : c_n \in C \text{ for all } n \in \mathbb{N}\}$ where $C = b\{0,1,\dots,q-1\}$. We denote the cardinality of $(T_{p,C} \cap \mathbb{Z}) \setminus \{0\}$ by $N_{p,C}$. We will give a better estimate of $N_{p,C}$ in order to reach a contradiction. First, we note that $T_{p,C} \subseteq \left[0, \frac{b(q-1)}{p-1}\right]$ which implies $N_{p,C} \leq \left\lfloor \frac{b(q-1)}{p-1} \right\rfloor$. Furthermore, we have

$$T_{p,C} := bT,$$

where $T = \{\sum_{n=1}^{\infty} c_n p^{-n} : c_n \in \{0,1,\dots,q-1\} \text{ for all } n \in \mathbb{N}\}$. Let $\Theta_q = \{0,1,2,\dots,q-1\}$ and $\Theta_q^n = \{I = i_1 i_2 \cdots i_n : i_k \in \Theta_q, 1 \leq k \leq n\}$. Then the set T can be decomposed into the following form:

$$T = \bigcup_{\sigma \in \Theta_q^n} T_\sigma,$$

where $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ and

$$
\begin{aligned}
T_\sigma &= \left\{\frac{\sigma_1}{p}+\frac{\sigma_2}{p^2}+\cdots+\frac{\sigma_n}{p^n}+\sum_{j=n+1}^{\infty} c_j p^{-j} : c_j \in \{0,1,\dots,q-1\} \text{ for all } j \geq n+1\right\} \\
&\subseteq \left[\frac{\sigma_1}{p}+\frac{\sigma_2}{p^2}+\cdots+\frac{\sigma_n}{p^n}, \frac{\sigma_1}{p}+\frac{\sigma_2}{p^2}+\cdots+\frac{\sigma_n}{p^n}+\frac{(q-1)}{(p-1)p^n}\right] \\
&=: I_\sigma \\
&= \left(\frac{\sigma_1}{p}+\frac{\sigma_2}{p^2}+\cdots+\frac{\sigma_n}{p^n}\right)+\left[0, \frac{(q-1)}{(p-1)p^n}\right]. \qquad (134)
\end{aligned}
$$

With (134), we have $T_{p,C} \subseteq b\bigcup_{\sigma\in\Theta_q^n} I_\sigma$. Then for any $n \in \mathbb{N}$,

$$
\begin{aligned}
N_{p,C} &\leq \left\lfloor\frac{b(q-1)}{(p-1)p^n}\right\rfloor + (q^n-1)\left(\left\lfloor\frac{b(q-1)}{(p-1)p^n}\right\rfloor+1\right) \\
&= q^n\left\lfloor\frac{b(q-1)}{(p-1)p^n}\right\rfloor + q^n - 1. \qquad (135)
\end{aligned}
$$

We will give a better estimation of $N_{p,C}$ below. For any $n \in \mathbb{N}$, one has

$$
\frac{b(q-1)q^n}{(p-1)p^n} + q^n - 1 \geq 2\sqrt{b\frac{(q-1)q^{2n}}{(p-1)p^n}} - 1.
$$

And the equality holds if and only if $\frac{b(q-1)q^n}{(p-1)p^n} = q^n$, i.e., $n = \frac{\ln\frac{b(q-1)}{p-1}}{\ln p}$. Denote $n_0 = \left\lfloor\frac{\ln\frac{b(q-1)}{p-1}}{\ln p}\right\rfloor + 1$. Observe that $\left\lfloor\frac{b(q-1)}{(p-1)p^{n_0}}\right\rfloor = 0$.

Then $N_{p,C} \leq q^{n_0}\left\lfloor\frac{b(q-1)}{(p-1)p^{n_0}}\right\rfloor + q^{n_0} - 1 = q^{n_0} - 1$.

Therefore, $r \geq \text{ord}_{\,p}(b) > q^{n_0} - 1 \geq N_{p,C}$. By Lemma (5.3.18), it is easy to see that $N_{p,C} \geq r$ which is a contradiction. This completes the proof of Theorem (5.3.19).

Through the following example, we illustrate that Theorem (5.3.19) provides a sufficient condition for us to find more spectra. The calculation will be complicated with the increasing of b.

Therefore, we only find the b less than 100 that can be judged by Theorem (5.3.19), so that $b\Lambda_{6,3}$ is still a spectrum of $\mu_{6,3}$.

Example (5.3.21)[356]: Let $p = 6, q = 3$. If $5 < b \leq 100$ with b prime, then $b\Lambda_{6,3}$ is a spectrum of $\mu_{6,3}$.

Proof. The possible choices of b are the following:

$P = \{7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97\}$.

For any $b \in P$,

(i) when $7 \leq b \leq 13$, we have $n_0 = 1$ and $3^{n_0} - 1 = 2$. By a simple calculation, one has $\text{ord}_{\,b}(p) > 2$. Then by Theorem (5.3.19), we obtain that $b\Lambda_{6,3}$ is a spectrum of $\mu_{6,3}$.

(ii) when $17 \leq b \leq 89$, we have $n_0 = 2$ and $3^{n_0} - 1 = 8$.

Case I. $b \in \{19,23,43,47,67,71\} := P_1$. In this case, it is easy to see that $\text{ord}_b(p)$ is odd with $\text{ord}_{\,b}(p) > 8$.

Case II. $b \in P \setminus P_1$. In this case, we have that $\text{ord}_{\,b}(p)$ is even. Then by Theorem (5.3.19), we obtain that $b\Lambda_{6,3}$ is a spectrum of $\mu_{6,3}$.

(iii) when $b = 97$, we have that ord $_b(p)$ is even. Then by Theorem (5.3.19)(i), we obtain that $b\Lambda_{6,3}$ is a spectrum of $\mu_{6,3}$. This finishes the proof.

Dutkay, Haussermann [366] proved that if b is a prime with $b > 3$, then $b\Lambda_4$ is a spectrum of μ_4. Depending on the result and the known examples, we guess the following: let $b > p - 1$, if b is a prime with gcd $(b, p) = 1$, then $b\Lambda_{p,q}$ is a spectrum of $\mu_{p,q}$.

Chapter 6
Exponential and Beurling Dimension

We construct a class of singularly continuous measures that has an exponential Riesz basis but no exponential orthonormal basis. It is the first of such kind of examples. We obtain the exact upper bound of the dimensions, which is the same given by Dutkay. The upper bound is attained in usual cases and some examples are given to explain the theory.

Section (6.1): Exponential Spectra in Hilbert Space:

We assume that μ is a (Borel) probability measure on $\mathbb{R}^d$ with compact support. We call a family $E(\Lambda)=\{e^{2\pi i\lambda x}:\lambda\in\Lambda\}$($\Lambda$ is a countable set) a Fourier frame of the Hilbert space $L^2(\mu)$ if there exist $A,B>0$ such that

$$A\parallel f\parallel^2\leqslant\sum_{\lambda\in A}\left|f,e^{2\pi i\lambda x}\right|^2\leqslant B\parallel f\parallel^2,\ \forall f\in L^2(\mu).\tag{1}$$

Here the inner product is defined as usual,

$$\left\langle f,e^{2\pi i\lambda x}\right\rangle=\int_{\mathbb{R}^d}f(x)e^{-2\pi i\lambda x}d\mu(x).$$

$E(A)$ is called an (exponential) Riesz basis if it is both a basis and a frame of $L^2(\mu)$. Fourier frames and exponential Riesz bases are natural generalizations of exponential orthonormal bases in $L^2(\mu)$. They have fundamental importance in nonharmonic Fourier analysis and close connection with time-frequency analysis [390,396,397]. When (1) is satisfied, $f\in L^2(\mu)$ can be expressed as $f(x)=\sum_{\lambda\in A}c_\lambda e^{2\pi i\lambda x}$, and the expression is unique if it is a Riesz basis.

When $E(\Lambda)$ is an orthonormal basis (Riesz basis, or frame) of $L^2(\mu)$, we say that μ is a spectral measure (R-spectral measure, or F-spectral measure respectively) and A is called a spectrum (R-spectrum, or F-spectrum respectively) of $L^2(\mu)$. We will also use the term orthonormal spectrum instead of spectrum when we need to emphasis the orthonormal property. If $E(\Lambda)$ only satisfies the upper bound condition in (1), then it is called a Bessel set (or Bessel sequence); for convenience, we also call Λ a Bessel set of $L^2(\mu)$.

One of the interesting and basic questions in non-harmonic Fourier analysis is:

What kind of compactly supported probability measures in $\mathbb{R}^d$ belong to the above classes of measures?

When μ is the restriction of the Lebesgue measure on K with positive measure, the question whether it is a spectral measure is related to the well known Fuglede problem of translational tiles (see [395,410,402,414]). While it is easy to show that such μ is an F-measure, it is an open question whether it is an R-spectral measure. If K is a unit interval, its F-spectrum was completely classified in terms of de Brange's theory of entire functions [411]. In another general situation, Lai [401] proved a sharp result that if μ is absolutely continuous with respect to the Lebesgue measure, then it is an F-spectral measure if and only if its density function is essentially bounded above and below on the support.

The problem becomes more intriguing when μ is singular. The first example of such spectral measures was given by Jorgensen and Pedersen [399]. They showed that the Cantor measures with even contraction ratio ($\rho=1/2k$) is spectral, but the one with odd contraction ratio ($\rho=1/(2k+1)$) is not. This raises the very interesting question on the existence of an exponential Riesz basis or a Fourier frame for such measures, and more generally for the self-similar measures [403,404,394,413,398]. In particular Dutkay et al. proposed to use the Beurling dimension as some general criteria for the existence of Fourier frame [392]. They also attempted to find a self-similar measure which admits an exponential Riesz basis or a Fourier frame but not an exponential orthonormal basis [393]. However, no such examples have been found up to now.

We will carry out a detail study of the three classes of spectra mentioned. It is known that a spectral measure must be either purely discrete or purely continuous [404]. Our first theorem is a pure type law for the F-spectral measures.

For the proof, the discrete case is based on the frame inequality, and the two continuous cases make use the concept of lower Beurling density of the F-spectrum.

To complete the previous digression on the continuous measures, we have the following conclusions for finite discrete measures.

To determine such discrete μ to be a spectral measure, we will restrict our consideration on $\mathbb{R}^1$ and let $\mathcal{C} \subset \mathbb{Z}^+$with $0 \in \mathcal{C}$. Then the Fourier transform of μ is

$$\hat{\mu}(x) = p_0 + p_1 e^{2\pi i c_1 x} + \cdots + p_{k-1} e^{2\pi i c_{k-1} x} := m_\mu(x),$$

where $P = \{p_i\}_{i=0}^{k-1}$ is a set of probability weights. We call $m_\mu(x)$ the mask polynomial of μ. Let $\mathcal{Z}_\mu = \{x \in [0,1): m_\mu(x) = 0\}$ be the zero set of $m_\mu(x)$, and Λ is called a bi-zero set if $\Lambda - \Lambda \subset \mathcal{Z}_\mu \cup \{0\}$. Denote the cardinality of E by $\#E$. It is easy to see the following simple proposition.

Proposition (6.1.1)[388]: Let $\mu = \sum_{c \in \mathcal{C}} p_c \delta_c$ with $\mathcal{C} \in \mathbb{Z}^+$and $0 \in \mathcal{C}$. Then μ is a spectral measure if and only if there is a bi-zero set Λ of m_μ and $\#\Lambda = \#\mathcal{C}$. In this case, all the p_c are equal.

The determination of the bi-zero set is, however, non-trivial, as the zeros of a mask polynomial is rather hard to handle. As an implementation of the proposition, we work out explicit expressions of the set $\mathcal{C}$ and the bi-zero set when $\#\, \mathcal{C} = 3,4$. It is difficult to have such expression beyond 4 directly. On the other hand, there are systematic studies of the zeros of the mask polynomials by factorizing the mask polynomial as cyclotomic polynomials (the minimal polynomial of the root of unity). This has been used to study the integer tiles and their spectra (see [391,402,407]). We adopt this approach to a class of self-similar measures (which is continuous) in our consideration:

Let $n > 0$ and let $\mathcal{A} \subset \mathbb{Z}^+$be a finite set with $0 \in \mathcal{A}$, we define a self-similar measure $\mu := \mu_{\mathcal{A},n}$ by

$$\mu(E) = \frac{1}{\#A} \sum_{a \in \mathcal{A}} \mu(nE - a)$$

where E is a Borel subset in $\mathbb{R}$. Note that the Lebesgue measure on $[0,1]$ and the Cantor measures are such kind of measures. The following theorem is a combination of the results in [412,402] and [403] :

Theorem (6.1.2)[388]: Let $\mathcal{A} \subset \mathbb{Z}^+$be a finite set with $0 \in \mathcal{A}$. Suppose there exists $\mathcal{B} \subset \mathbb{Z}^+$such that $\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_n$ where $\mathbb{N}_n = \{0, \dots, n-1\}$. Then $\delta_{\mathcal{A}} = \sum_{a \in \mathcal{A}} \delta_a$ is a spectral measure with a spectrum in $\frac{1}{n}\mathbb{Z}$; the associated self-similar measure $\mu_{\mathcal{A},n}$ is also a spectral measure, and it has a spectrum in $\mathbb{Z}$ if $\gcd \mathcal{A} = 1$.

Note that the $1/4$-Cantor measure $\mu_{\{0,2\},4}$ satisfies the above condition, but not the $1/3$-Cantor measure. In fact, it is an open problem whether the $1/3$-Cantor measure is an F-spectral measure. To a lesser degree we want to know the existence of a singularly continuous measure that admits an R-spectrum but is not a spectral measure. We search for new R-spectral measures and to obtain such an example as corollary.

We let η be a discrete probability measure with support $\mathcal{C} \subset \mathbb{Z}^+$.Let v be another probability measure on $\mathbb{R}$ with support $\Omega \subseteq [0,1]$, and let $\mu = \eta * v$ be the convolution of η and v. Our main result is

Theorem (6.1.3)[388]: Let $\mu = \eta * v$ be as the above, and assume that v is an R-spectral measure with a spectrum in $\mathbb{Z}$. Then μ is an R-spectral measure.

In addition, if $\mathcal{Z}_v \subseteq \mathbb{Z}$. Then μ is a spectral measure if and only if both η and v are spectral measures.

We can modify the theorem slightly with the spectrum Γ and $\mathcal{Z}_v$ to be some subsets of rationals (Theorem (6.1.17), Theorem (6.1.19)), this covers some more interesting cases (e.g., the Cantor measures). Finally by taking η to be a non-uniform discrete measure (Proposition (6.1.1)) and $v = \mu_{\mathcal{A},n}$ in Theorem (6.1.2), we conclude from Theorem (6.1.3) that

Example (6.1.4)[388]: There exists a singularly continuous measure which is an R-spectral measure, but not a spectral measure.

We prove Theorem (6.1.6) and Theorem (6.1.9). We then deal with the discrete spectral measures; Proposition (6.1.1) is proved, and explicit expressions of $\mathcal{C}$ (with # $\mathcal{C} = 3,4$) for $\mu_{\mathcal{C}}$ to be a spectral measure (Example (6.1.12), Example (6.1.13)) are sought. We make a further discussion of the discrete spectral measures in connection with the class of integer tiles. We prove the two statements in Theorem (6.1.3) in two theorems, and Example (6.1.4) follows as a corollary..

Recall that a σ-finite Borel measure μ on $\mathbb{R}^d$ can be decomposed uniquely as discrete, singularly continuous and absolutely continuous measures, i.e., $\mu = \mu_d + \mu_s + \mu_a$. The measure μ is said to be of pure type if μ equals only one of the three components.

In our proof of the pure type property of the F-spectral measures, we need to use the lower Beurling density of an infinite discrete set $\Lambda \subset \mathbb{R}^d$:

$$D^-\Lambda := \lim_{h\to\infty}\inf \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d},$$

where $Q_h(x)$ is the standard cube of side length h centered at x. Intuitively Λ is distributed like a lattice if $D^-\Lambda$ is positive. In [405], Landau gave an elegant and useful necessary condition for Λ to be an F-spectrum on $L^2(\Omega)$: $D^-\Lambda \geqslant \mathcal{L}(\Omega)$ where $\mathcal{L}$ is the Lebesgue measure. The following proposition provides some relationships between the lower Beurling density and the types of the measures.

Proposition (6.1.5)[388]: Let μ be a compactly supported probability measure on $\mathbb{R}^d$, and Λ is an F-spectrum of μ, we have

(i) If $\mu = \sum_{c\in\mathcal{C}} p_c\delta_c$ is discrete, then $\#\Lambda < \infty$ and $\#\mathcal{C} < \infty$;

(ii) If μ is singularly continuous, then $D^-\Lambda = 0$;

(iii) If μ is absolutely continuous, then $D^-\Lambda > 0$.

Proof. (i) By the definition of Fourier frame, we have for all $f \in L^2(\mu)$,

$$\sum_{\lambda\in\Lambda}\left|\sum_{c\in\mathcal{C}} f(c)e^{2\pi i\langle\lambda,c\rangle}p_c\right|^2 \leqslant B\sum_{c\in\mathcal{C}} |f(c)|^2 p_c.$$

Taking $f = \chi_{c_0}$, where $p_{c_0} > 0$, we have $(\#\Lambda)\cdot p_{c_0}^2 \leqslant Bp_{c_0}$. Hence $\#\Lambda \leqslant B/p_{c_0} < \infty$. This implies #C $< \infty$ by the completeness of Fourier frame.

(ii) Suppose on the contrary that $D^{-}\Lambda \geqslant c > 0$. We claim that $\mathbb{Z}^d$ is a Bessel set of $L^2(\mu)$. By the definition of $D^{-}\Lambda$, we can choose a large $h \in \mathbb{N}$ such that

$$\inf_{x\in\mathbb{R}^d}\big(\#(\Lambda \cap Q_h(x))\big) \geqslant ch^d > 1.$$

Taking $x = h\mathbf{n}$, where $\mathbf{n} \in \mathbb{Z}^d$, we see that all cubes of the form $h\mathbf{n} + [-h/2, h/2)^d$ contains at least one points of Λ, say $\lambda_{\mathbf{n}}$. Since Λ is an F-spectrum, $\{\lambda_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a Bessel set. By the stability under perturbation (see e.g.. [392, Proposition 2.3]) and

$$|\lambda_{\mathbf{n}} - h\mathbf{n}| \leqslant \operatorname{diam}([-h/2, h/2)^d) = \sqrt{d}h,$$

we conclude that $h\mathbb{Z}^d$ is also a Bessel set of $L^2(\mu)$. As a Bessel set is invariant under translation, we see that the finite union $\mathbb{Z}^d = \bigcup_{\mathbf{k}\in\{0,\dots,h-1\}^d}(h\mathbb{Z}^d + \mathbf{k})$ is again a Bessel set of $L^2(\mu)$, which proves the claim.

Now consider

$$G(x) := \sum_{\mathbf{n}\in\mathbb{Z}^d} |\hat{\mu}(x + \mathbf{n})|^2.$$

G is a periodic function (mod $\mathbb{Z}^d$). As $\mathbb{Z}^d$ is a Bessel set, applying the definition to $e^{2\pi i(x,\cdot)}$, we see that $G(x) \leqslant B < \infty$. Hence $G \in L^1([0,1)^d)$ and

$$\int_{\mathbb{R}^d} |\hat{\mu}(x)|^2 dx = \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{[0,1)^d} |\hat{\mu}(x + \mathbf{n})|^2 dx = \int_{[0,1)^d} |G(x)| dx < \infty.$$

This means that $\hat{\mu} \in L^2(\mathbb{R}^d)$, which implies μ must be absolutely continuous. This is a contradiction.

(iii) If μ is absolutely continuous, then the density function must be bounded above and below almost everywhere on the support of μ [401, Theorem (6.1.6)]. Hence, Λ is an F-spectrum of $L^2(\Omega)$, where Ω is the support of μ. By Landau's density theorem, $D^{-}\Lambda \geqslant \mathcal{L}(\Omega) > 0$.

Now it is easy to conclude that an F-spectral measure is of pure type.

Theorem (6.1.6)[388]: Let μ be an F-spectral measure on $\mathbb{R}^d$. Then it must be one of the three pure types: discrete (and finite), singularly continuous or absolutely continuous.

Proof. First let us assume that if μ is decomposed into non-trivial discrete and continuous parts, $\mu = \mu_d + \mu_c$. Let Λ be an F-spectrum of μ. As $L^2(\mu_d)$ and $L^2(\mu_s)$ are non-trivial subspaces of $L^2(\mu)$, it is easy to see that Λ is also an F-spectrum of both $L^2(\mu_d)$ and $L^2(\mu_c)$. Then $\#\Lambda < \infty$ by Proposition (6.1.5)(i); but $\#\Lambda = \infty$ since $L^2(\mu_c)$ is an infinite dimensional Hilbert space. This contradiction shows that μ is either discrete or purely continuous.

Suppose μ is continuous and has non-trivial singular part μ_s and absolutely continuous part μ_a. By applying the same argument as the above, Λ is an F-spectrum of $L^2(\mu_s)$ and $L^2(\mu_a)$. This is impossible in view of the Beurling density of Λ in Proposition (6.1.5)(ii) and (iii).

The following corollary is immediate from Theorem (6.1.6).

Corollary (6.1.7)[388]: A spectral measure or an R-spectral measure must be of pure type.

We will show that all discrete measures on $\mathbb{R}^d$ are R-spectral measures. By Proposition (6.1.5)(i), we only need to consider measures with finite number of atoms. Let $\mathcal{C} = \{c_0, \dots, c_{n-1}\} \subset \mathbb{R}^d$ be a finite set and let

$$\mu = \sum_{c\in\mathcal{C}} p_c\delta_c, \text{ with } p_i > 0, \sum_{c\in\mathcal{C}} p_c = 1. \tag{2}$$

For $\lambda \in \mathbb{R}^d$, we denote the vector $\left[e^{2\pi i\langle\lambda,c_0\rangle}, \dots, e^{2\pi i\langle\lambda,c_{n-1}\rangle}\right]^t$ by $\mathbf{v}_\lambda$.

Proposition (6.1.8)[388]: Let $\mathcal{C} = \{c_0, \dots, c_{n-1}\} \subset \mathbb{R}^d$ and let μ be as in (2). Let $\Lambda = \{\lambda_0, \dots, \lambda_{m-1}\} \subset \mathbb{R}^d$ be another finite set. Then

(i) Λ is an F-spectrum of μ if and only if $\operatorname{span}\{\mathbf{v}_{\lambda_0}, \dots, \mathbf{v}_{\lambda_{m-1}}\} = \mathbb{C}^n$.

(ii) Λ is an R-spectrum of μ if and only if $m = n$ in the above identity.

Proof. (i) Suppose first Λ is an F-spectrum of μ. Let $\mathbf{u} = [u_0, \dots, u_{n-1}]^t$ be such that $\langle \mathbf{u}, \mathbf{v}_{\lambda_i}\rangle = 0$ for all i. Consider f as a function defined on $\mathcal{C}$ with $f(c_i) = u_i/p_{c_i}$. By using the lower bound of the Fourier frame, we have

$$A\sum_{c\in\mathcal{C}} |f(c)|^2 p_c \leqslant \sum_{\lambda\in\Lambda} \left|\sum_{c\in\mathcal{C}} f(c)e^{2\pi i\langle\lambda,c\rangle}p_c\right|^2 = \sum_{\lambda\in\Lambda} |\langle \mathbf{u}, \mathbf{v}_\lambda\rangle|^2 = 0.$$

It follows that $f(c) = 0$ for all $c \in \mathcal{C}$, hence $\mathbf{u} = \mathbf{0}$ and the necessity in (i) follows.

Conversely, the assumption implies that the vectors $\mathbf{v}_{\lambda_0}, \dots, \mathbf{v}_{\lambda_{m-1}}$ form a frame on $\mathbb{C}^n$ (see [390, Corollary 1.1.3]), i.e., there exist $A, B > 0$ such that for all $\mathbf{u} = [u_0, \dots, u_{n-1}]^t \in \mathbb{C}^n$

$$A\sum_{i=0}^{n-1} |u_i|^2 \leqslant \sum_{\lambda\in A} |\langle \mathbf{u}, \mathbf{v}_\lambda\rangle|^2 \leqslant B\sum_{i=0}^{n-1} |u_i|^2.$$

For any $f \in L^2(\mu)$, we take $\mathbf{u} = \left[f(c_0)p_{c_0}, \dots, f(c_{n-1})p_{c_{n-1}}\right]^t$, we see that Λ is a frame with lower bound $(\min\ {}_i p_i)A$ and upper bound $(\max_i p_i)B$.

(ii) is clear from (i).

Theorem (6.1.9)[388]: Let $\mu = \sum_{c\in C} p_c\delta_c$ be a discrete probability measure in $\mathbb{R}^d$ with $\mathcal{C}$ a finite set. Then μ is an R-spectral measure.

Proof. Let $\mathcal{C} = \{c_0, \dots, c_{n-1}\}$. We first establish the theorem for $\mathcal{C} \subset \mathbb{R}^1$. Let $W = \operatorname{span}\{\mathbf{v}_\lambda : \lambda \in \mathbb{R}^1\}$, it suffices to show that $W = \mathbb{C}^n$. Then we can select $\{\lambda_0, \dots, \lambda_{n-1}\} \subset \mathbb{R}^1$ so that $\{\mathbf{v}_{\lambda_0}, \dots, \mathbf{v}_{\lambda_{n-1}}\}$ a basis of $\mathbb{C}^n$. The theorem for $\mathbb{R}^1$ will follow from Proposition (6.1.8)(ii).

To see $W = \mathbb{C}^n$, it suffices to show that if $\langle \mathbf{u}, \mathbf{v}_\lambda\rangle = 0$ for all $\lambda \in \mathbb{R}$, then $\mathbf{u} = \mathbf{0}$. To this end, we write $\mathbf{u} = [u_0, \dots, u_{n-1}]^t$, and the given condition is

$$\sum_{i=0}^{n-1} u_i e^{2\pi i\lambda c_i} = 0.$$

We differentiate the expression with respect to λ for k times with $k = 1, \dots, n-1$, then

$$\sum_{i=0}^{n-1} u_i c_i^k e^{2\pi i\lambda c_i} = 0.$$

This means

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_0^{n-1} & c_1^{n-1} & \cdots & c_{n-1}^{n-1} \end{bmatrix} \cdot \begin{bmatrix} u_0 e^{2\pi i\lambda c_0} \\ u_1 e^{2\pi i\lambda c_1} \\ \vdots \\ u_{n-1} e^{2\pi i\lambda c_{n-1}} \end{bmatrix} = 0.$$

As all c_i are distinct, the Vandermonde matrix is invertible. Hence,

$$\left[u_0 e^{2\pi i\lambda c_0}, u_1 e^{2\pi i\lambda c_1}, \dots, u_{n-1} e^{2\pi i\lambda c_{n-1}}\right]^t = \mathbf{0},$$

and thus $\mathbf{u} = \mathbf{0}$. This completes the proof of the theorem for $\mathbb{R}^1$.

On $\mathbb{R}^d$, we note that by Proposition (6.1.8), $\{c_0, \ldots, c_{n-1}\}$ admits an R-spectrum $\{\lambda_0, \ldots, \lambda_{n-1}\}$ if and only if $\{Qc_0, \ldots, Qc_{n-1}\}$ admits an R-spectrum $\{Q\lambda_0, \ldots, Q\lambda_{n-1}\}$, where Q is any orthogonal transformation on $\mathbb{R}^d$. Now given any $\{c_0, \ldots, c_{n-1}\}$ on $\mathbb{R}^d$, we let ℓ_{ij} be the line passes through two points c_i, c_j, and choose a line ℓ such that ℓ is not perpendicular to any ℓ_{ij}. Apply an orthogonal transformation Q so that the first axis coincides with the direction of ℓ. In this way the construction shows that the first coordinates of $Qc_0, \ldots, Qc_{n-1}$ are all distinct.

We then apply the same argument as in $\mathbb{R}^1$ above, using partial differentiation with respect to the first coordinates which are all distinct, the Vandermonde matrix is invertible, hence the theorem follows.

Remark (6.1.10)[388]: If $c_0, \ldots, c_{n-1}$ have rational coordinates, then we can choose elements Λ to have rational coordinates also. To see this, by multiplying an integer, we can assume that $\{c_0, \ldots, c_{n-1}\}$ are in $\mathbb{Z}^d$, we consider the determinant function

$$\varphi(\lambda) = \varphi(\lambda_0, \ldots, \lambda_{n-1}) = \det \begin{bmatrix} e^{2\pi i \langle \lambda_0, c_0 \rangle} & \cdots & e^{2\pi i \langle \lambda_1, c_{n-1} \rangle} \\ & \ddots & \\ e^{2\pi i \langle \lambda_{n-1}, c_0 \rangle} & \cdots & e^{2\pi i \langle \lambda_{n-1}, c_{n-1} \rangle} \end{bmatrix}$$

with $\lambda = (\lambda_0, \ldots, \lambda_{n-1})$ on $\mathbb{R}^{dn}$. Then $\varphi(\lambda)$ is a trigonometric polynomial on $\mathbb{R}^{dn}$, whose zero set is a closed set of Lebesgue measure zero. We can choose λ so that $\varphi(\lambda) \neq 0$ and λ is rational, and Proposition (6.1.8)(ii) shows that $\Lambda = \{\lambda_0, \ldots, \lambda_{n-1}\}$ is an R-spectrum will rational coordinates.

The R-spectrum shown in Theorem (6.1.9) is not explicit. It is also not easy to see whether a given set Λ is an R-spectrum since the invertibility of the matrix is not easy to establish in general. A probabilistic approach of finding such Λ in the case of trigonometric polynomials was given in [389]. The work gave a theoretical background on the theory of reconstruction of multivariate trigonometric polynomials via random sampling sets.

To carry out Theorem (6.1.9) further, we consider the condition that a discrete measure to be an orthogonal spectral measure. We will restrict our consideration on the one-dimensional case, and by translation, we can assume, without loss of generality, that $\mathcal{C} \subset \mathbb{Z}^+$ and $0 \in \mathcal{C}$. The mask polynomial of $\mu = \sum_{c \in \mathcal{C}} p_c \delta_c$ is

$$m_{\mathcal{C},P}(x) = \hat{\mu}(x) = \sum_{c \in \mathcal{C}} p_c e^{2\pi i c x}.$$

In case P is a set of equal probability, then we just use the notation $m_{\mathcal{C}}(x)$. We call a set Λ a bi-zero set of $m_{\mathcal{C},P}$ if $0 \in \Lambda$ and $m_{\mathcal{C},P}(\lambda_i - \lambda_j) = 0$ for distinct $\lambda_i, \lambda_j \in \Lambda$. It is clear that such $E(\Lambda)$ is an orthogonal set in $L^2(\mu)$.

Proposition (6.1.11)[388]: Let $\mathcal{C} \subset \mathbb{Z}$ be a finite set, and let $\mu = \sum_{c \in \mathcal{C}} p_c \delta_c$. Then μ is a spectral measure if and only if there is a bi-zero set Λ of $m_{\mathcal{C},P}$ and $\#\mathcal{C} = \#\Lambda$. In this case, all the p_c 's are equal.

Proof. Note that μ is a spectral measure if and only if there exists a set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $n = \#\mathcal{C}$ such that $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for all $i \neq j$. Since $\hat{\mu}(x) = m_{\mathcal{C},P}(x)$, this is equivalent to Λ is a bi-zero set of $m_{\mathcal{C},P}$ and $\#\mathcal{C} = \#\Lambda$.

To see that all the p_c are equal, we put $f = \chi_c$ into the Parseval's identity.

$$\sum_{\lambda \in \Lambda} \left| \langle f, e^{2\pi i \langle \lambda, x \rangle} \rangle \right|^2 = \| f \|^2.$$

We obtain $\sum_{\lambda \in A} p_c^2 = p_c$. Hence, $p_c = 1/\#\Lambda = 1/\#\mathcal{C}$.

We use Proposition (6.1.11) to obtain explicit expressions of $\mathcal{C}$ with $\#C \leqslant 4$ that are discrete spectral measures. It is trivial to check that when $\#\mathcal{C} = 1,2$, the associated μ is always a spectral measure.

Example (6.1.12)[388]: Let $\mathcal{C} = \{c_0 = 0, c_1, c_2\} \subset \mathbb{Z}^+$with $\gcd(\mathcal{C}) = 1$. Then $\mu = \sum_{c\in\mathcal{C}} \delta_c$ is a spectral measure if and only if $c_2 \equiv 2c_1 (\mathrm{mod}3)$(i.e., $\mathcal{C}$ is a complete residue (mod3)).

Proof. For the sufficiency, we write $c_1 = 3k + i, i = 1,2$ and $k \in \mathbb{N}$. Then $c_2 = 3l + 2i$ and

$$\{0, c_1, c_2\} = \{0,1,2\}\ (\mathrm{mod}3).$$

It is direct to check that $\Lambda = \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ is a bi-zero set of $m_{\mathcal{C}}(x)$ with $\#\Lambda = \#\mathcal{C}$ and hence μ is a spectral measure.

For the necessity, we let $\Lambda = \{0, \lambda_1, \lambda_2\}$ be such that $m_{\mathcal{C}}(b_1) = m_{\mathcal{C}}(b_2) = m_{\mathcal{C}}(b_2 - b_1) = 0$. Note that $m_{\mathcal{C}}(x) = 1 + e^{2\pi i c_1 x} + e^{2\pi i c_2 x}$. Then $m_{\mathcal{C}}(x)$ has roots in (0,1) if and only if $e^{2\pi i c_1 x} = e^{2\pi i/3}, e^{2\pi i c_2} = e^{4\pi i/3}$ (or the other way round). Hence there exists $k, l \in \mathbb{Z}^+$such that

$$2\pi c_1 x = 2k\pi + \frac{2}{3}\pi,\ 2\pi c_2 x = 2l\pi + \frac{4}{3}\pi.$$

It follows that $x = \frac{3k+1}{3c_1} = \frac{3l+2}{3c_2}$. Since $\gcd(c_1, c_2) = 1$, we have $3k + 1 = c_1 m$ and $3l + 2 = c_2 m$. Hence $3 \nmid m$, and $3 \mid (c_2 - 2c_1)$ This implies the sufficiency.

Example (6.1.13)[388]: Let $\mathcal{C} = \{c_0 = 0, c_1, c_2, c_3\} \subset \mathbb{Z}^+$with $\gcd(\mathcal{C}) = 1$. Then μ is a spectral measure if and only if after rearrangement, c_1 is even, c_2, c_3 are odd, and $c_1 = 2^\alpha(2k + 1), c_2 - c_3 = 2^\alpha(2\ell + 1)$ for some $\alpha > 0$.

Proof. We first prove the necessity. The mask polynomial of μ is $m_{\mathcal{C}}(x) = 1 + e^{2\pi i c_1 x} + e^{2\pi i c_2 x} + e^{2\pi i c_3 x}$. That $m_{\mathcal{C}}(x) = 0$ implies

$$\left|1 + e^{2\pi i c_1 x}\right| = \left|1 + e^{2\pi i (c_3 - c_2) x}\right|, \tag{3}$$

which yields (i) $e^{2\pi i c_1 x} = e^{2\pi i (c_3 - c_2) x}$ or (ii) $e^{2\pi i c_1 x} = e^{-2\pi i (c_3 - c_2) x}$. Putting (i) into $m_{\mathcal{C}}(x) = 0$, we have (1+e $2\pi i c_1 x$) $(1 + e^{2\pi i c_2 x}) = 0$. Hence we have two sets of equations:

$$2c_1 x = 2k + 1;\ 2(c_3 - c_2)x = 2l + 1. \tag{4}$$

or

$$2c_2 x = 2k + 1;\ 2(c_3 - c_1)x = 2l + 1. \tag{5}$$

From (4), we have $x = \frac{2k+1}{2c_1} = \frac{2l+1}{2(c_3 - c_2)}$. Let $a = \gcd(c_1, c_2 - c_3)$. It is easy to show that there exists m such that

$$2k + 1 = \frac{c_1 m}{a},\ 2l + 1 = \frac{(c_3 - c_2)m}{a}. \tag{6}$$

Hence $m, c_1/a, (c_3 - c_2)/a$ must be odd. Also note that $\gcd(c_1, c_2, c_3) = 1$, it follows from a direct check of the above that two of the c_1, c_2, c_3 must be odd, and one must be even (all three cases can happen).

The same argument applies to (5) and to (ii). The last statement also follows in the proof.

To prove the sufficiency, we first observe from the above that for c_1 even, c_2, c_3 odd, there are solutions $x_1, x_2 \in (0,1)$ from (i) (see (4)-(6)):

$$x_1 = \frac{2i + 1}{2a},\ 0 \leqslant i < a;\ x_2 = \frac{2j + 1}{2b},\ 0 \leqslant j < b,$$

where $\gcd(c_1, c_2 - c_3)$ as above, and $b = \gcd(c_2, c_3 - c_1)$. Since b is odd, we can take $2j + 1 = b$, so that $x_2 = \frac{1}{2}$ is a solution of $m_{\mathcal{C}}(x) = 0$. Let

$$\lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{2a},\ \lambda_3 = \frac{2^\alpha \gcd(r,s) + 1}{2a},$$

where $c_1 = 2^\alpha r, c_2 = 2^\alpha s$ and r, s are odd integers as in the assumption. We claim that $\Lambda = \{0, \lambda_1, \lambda_2, \lambda_3\}$ is a bi-zero set of $m_{\mathcal{C}}(x)$. Indeed, since $a = 2^\alpha \gcd(r,s), \lambda_1 - \lambda_2 = (a-1)/2a$ is of the form x_1 for $i = 2^{\alpha-1}\gcd(r,s) - 1, \lambda_3 - \lambda_1 = \lambda_2$ and $\lambda_3 - \lambda_2 = \lambda_1$, the claim follows.

For $\#\mathcal{C}$ large, it is difficult to evaluate the zero set of the mask polynomial. However there is a number-theoretical approach to study such zeros related to the spectrum and integer tiling.

We will give a brief discussion of the relationship between discrete spectral measures and integer tiles, and provide the tools we need. Let $\mathcal{A} \subset \mathbb{Z}^+$and assume that $0 \in \mathcal{A}$, we say that $\mathcal{A}$ is an integer tile if there exists $\mathcal{T}$ such that $\mathcal{A} \oplus \mathcal{T} = \mathbb{Z}$, i.e., $\mathcal{A} + [0,1]$ tiles $\mathbb{R}$. Equivalently, $\mathcal{A}$ is a tile if there exists $\mathcal{B}$ and n such that

$$\mathcal{A} \oplus \mathcal{B} \equiv \mathbb{Z}_n \ (\text{mod} n). \qquad (7)$$

Recall that the Fuglede conjecture asserts that for $\Omega \subset \mathbb{R}^d$ with positive measure, Ω is a translational tile if and only if the restriction of the Lebesgue measure $\mathcal{L}|_\Omega$ is a spectral measure. Although the conjecture is proved to be false in either direction [414,400], it remains unanswered for dimension 1 and 2 , and for some special classes of tiles in any dimension.

Let $\mathcal{A}$ be a finite subset in $\mathbb{Z}$, then the Fuglede conjecture reduces to $\mathcal{A}$ is an integer tile if and only if $\mathcal{A} + [0,1]$ is a spectral set, i.e., $\mathcal{L}|_{\mathcal{A}+[0,1]}$ is a spectral measure. It is also known that the latter part is also equivalent $\delta_{\mathcal{A}} = \sum_{a\in\mathcal{A}} \delta_a$ is a discrete spectral measure [410]. This also follows from Theorem (6.1.19). In Example (6.1.12), the spectral condition for $\#\,\mathcal{C} = 3$ is equivalent to $\mathcal{C}$ is a complete residue (mod 3), which trivially satisfies (7). Hence the conjecture is true for $\#\,\mathcal{C} = 3$. In Example (6.1.13), the spectral condition is equivalent to

$$\mathcal{C} = \{0, 2^\alpha(2k+1), 2r+1, 2r+1+2^\alpha(2\ell+1)\}$$

for some non-negative integers k, r, ℓ. If we let $\mathcal{B} = \{0,2\} \oplus \cdots \oplus \{0, 2^{\alpha-1}\}$, then it is direct to check that $\mathcal{C} \oplus \mathcal{B} \equiv \mathbb{Z}_{2^{\alpha+1}}$ (mod $2^{\alpha+1}$). Hence by (7), the conjecture is true for $\#C = 4$. Actually, by using some deeper number-theoretic argument, it can be shown that if $\#\mathcal{C} = p^\alpha q^\beta$ where p, q are distinct primes, then $\mathcal{C}$ is an integer tile implies it is a spectral set [391,402].

The following is a useful sufficient condition of a discrete spectral measure. The condition trivially imply the underlying set is an integer tile.

Theorem (6.1.14)[388]: Let $\mathcal{A} \subset \mathbb{Z}^+$be a finite set with $0 \in \mathcal{A}$. Suppose there exists $\mathcal{B} \subset \mathbb{Z}^+$such that

$$\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_n$$

where $\mathbb{N}_n = \{0, \dots, n-1\}$. Then the discrete measure $\delta_{\mathcal{A}} = \sum_{a\in\mathcal{A}} \delta_a$ (with equal weight) is a spectral measure with a spectrum contained in $\frac{1}{n}\mathbb{Z}$.

The theorem was due to [412] (and also in [394]), and the proof involves an inductive construction of the spectrum. The spectrum is implicit and the proof is long. We will provide an alternative proof using the properties of the root of unity as the zeros of the mask polynomial. The framework is from [391] and the spectrum is explicitly given in [402]. Because of the number-theoretical notations and techniques.

Finally, we state a related theorem of the self-similar measures which follows from the known results.

Theorem (6.1.15)[388]: Let $\mathcal{A} \subset \mathbb{Z}^+$ be a finite set with $0 \in \mathcal{A}$. Suppose there exists $\mathcal{B} \subset \mathbb{Z}^+$ such that $\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_n$. Let μ be the self-similar measure satisfying

$$\mu(\cdot) = \frac{1}{\#A} \sum_{a \in \mathcal{A}} \mu(n \cdot -a).$$

Then μ is a spectral measure. Moreover, if $\gcd(\mathcal{A}) = 1$, then the spectrum Λ of μ can be chosen to be in $\mathbb{Z}$.

Proof. Denote the spectrum of $\delta_{\mathcal{A}}$ in Theorem (6.1.14) by $\mathcal{S}$ with $\mathcal{S} \subset \frac{1}{n}\mathbb{Z}$, then $(\mathcal{A}, \mathcal{S})$ form a compatible pair as in [403], i.e.,

$$\frac{1}{\sqrt{\#\mathcal{A}}} \left[e^{2\pi i a s}\right]_{a \in \mathcal{A}, s \in \mathcal{S}}$$

is a unitary matrix. The theorem follows from [403, Theorem (6.1.9)]. For the last part, since we can change the residue representatives of Γ with $\mathcal{S} = \frac{1}{n}\Gamma$ and $\Gamma \subset \{-(n-2), \dots, n-2\}$. With $\gcd(\mathcal{A}) = 1$, Theorem (6.1.9) in [403] states that

$$\Lambda = \Gamma \oplus n\Gamma \oplus n^2\Gamma \oplus \cdots$$

is a spectrum. This spectrum clearly lies in $\mathbb{Z}$.

Remark (6.1.16)[388]: For the 1/4-Cantor measure, $\mu(\cdot) = \frac{1}{2}\mu(4\,\cdot) + \frac{1}{2}\mu(4 \cdot -2)$. It is easy to compute the Fourier transform is $\hat{\mu}(\xi) = e^{2\pi i \frac{1}{3}\xi} \prod_{j=1}^{\infty} \cos(2\pi\xi/4^j)$ and the zero set of $\hat{\mu}$ is $\mathcal{Z}_\mu = \{4^j a : a \text{ is odd and } j \geqslant 0\}$. Note that $\mathcal{A} = \{0,2\}$ and the condition of the theorem is satisfied, the spectrum of μ can be taken as [399]

$$\Lambda = \{0,1\} \oplus 4\{0,1\} \oplus 4^2\{0,1\} \oplus \cdots \subset \mathcal{Z}_\mu.$$

However the condition $\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_n$ in Theorem (6.1.15) is quite restrictive. For the 1/6-Cantor measure, $(\cdot) = \frac{1}{2}\mu(6\,\cdot) + \frac{1}{2}\mu(6 \cdot -2)$, according to [399], it is again a spectral measure and the spectrum is

$$\Lambda = \frac{3}{2}(\{0,1\} \oplus 6\{0,1\} \oplus 6^2\{0,1\} \oplus \cdots).$$

But for $\mathcal{A} = \{0,2\}$ in this case, we cannot find $\mathcal{B}$ so that $\mathcal{A} \oplus \mathcal{B} = \mathbb{N}_6$. Also, μ does not admit spectrum $\Lambda' \subset \mathbb{Z}$. The proof is as follows: If so, observe that

$$\Lambda' \subset \mathcal{Z}_\mu = \{6^j a/4 : a \text{ is odd and } j \geqslant 1\}.$$

As λ is an integer, we see that for $\lambda \in \Lambda', \lambda = 6^n a/4$, and $n \geqslant 2$ necessarily. Let $x = 3/2$, then $x \in \mathcal{Z}_\mu \setminus \Lambda'$ and $x - \lambda = 6(1 - 6^{n-1}a)/4 \in \mathcal{Z}_\mu$. This means $\sum_{\lambda \in \Lambda} |\hat{\mu}(x - \lambda)|^2 = 0$, which shows that Λ' cannot be a spectrum by Proposition (6.1.18).

Let v be a probability measure with compact support $\Omega \subset [0,1]$ and let η be a discrete probability measure with support on $\mathcal{C} \subset \mathbb{Z}^+$ and probability weight P, i.e. $\eta = \delta_{\mathcal{C},P} = \sum_{c \in \mathcal{C}} p_c \delta_c$. Then $\mu = \eta * v$ has support on $\mathcal{C} + \Omega$. Given a non-negative integer q, we let $\eta_q = \delta_{q\mathcal{C},P}$.

Theorem (6.1.17)[388]: Let v be an R-spectral measure with a spectrum Γ and assume that there exists an integer $q \geqslant 1$ such that $q\Gamma \subseteq \mathbb{Z}$. Then $\mu := \eta_q * v$ is an R-spectral measure.

Proof. We write $\mathcal{A} = q\mathcal{C} = \{0 = a_0, a_1, \dots, a_{k\ 1}\}$. By Theorem (6.1.9), there exists an R-spectrum of $\mathcal{A}$ which we denote it as $\mathcal{S} = \{0 = s_0, s_1, \dots, s_{k-1}\}$. By Proposition (6.1.8)(ii), we see that

$$\det\left[e^{2\pi i a s}\right]_{a\in\mathcal{A},s\in\mathcal{S}} \neq 0.$$
We will show that $\mathcal{S} \oplus \Gamma$ is an R-spectrum of μ.

Since $\Omega = \mathrm{supp}(\nu) \subset [0,1]$, for any $f \in L^2(\mu), f$ is uniquely determined by the vector-valued function $[f(x+a_0),, \dots, f(x+a_{k-1})]^t$ on Ω. Let $M = \left[e^{2\pi i a s}\right]_{a\in\mathcal{A},s\in\mathcal{S}}$, it is invertible. We define
$$[g_0(x), \dots, g_{k-1}(x)]^t = M^{-1}[f(x+a_0), \dots, f(x+a_{k-1})]^t,\ x \in \Omega.$$
Clearly $g_j \in L^2(\nu)$ for $0 \leqslant j \leqslant k-1$. It is easy to see $s_j + \Gamma$ is also an R-spectrum of ν, so that g_j can be uniquely expressed as
$$g_j(x) = \sum_{\gamma\in\Gamma} c_{s_j+\gamma} e^{2\pi i (s_j+\gamma)x}.$$
Hence, we have
$$M[g_0(x), \dots, g_{k-1}(x)]^t = \left[\sum_{j=0}^{k-1} e^{2\pi i a_0 s_j} g_j(x), \dots, \sum_{j=0}^{k-1} e^{2\pi i a_{k-1} s_j} g_j(x)\right]^t.$$
and therefore
$$f(x+a_i) = \sum_{j=0}^{k-1} e^{2\pi i a_i s_j} \sum_{\gamma\in\Gamma} c_{s_j+\gamma} e^{2\pi i (s_j+\gamma)x} = \sum_{j=0}^{k-1}\sum_{\gamma\in\Gamma} c_{s_j+\gamma} e^{2\pi i (s_j+\gamma)(x+a_i)}. \quad (8)$$
Note that the last equality follows from $\gamma a_i = (q\gamma)c_j$ is an integer by the assumption $q\Gamma \subset \mathbb{Z}$. By a change of variable with $y = x + a_i$ for each i, we have
$$f(y) = \sum_{j=0}^{k-1}\sum_{\gamma\in\Gamma} c_{s_j+\gamma} e^{2\pi i (s_j+\gamma)y},\ y \in \mathcal{C} + \Omega = \mathrm{supp}(\mu).$$
It is easy to see that the above representation is unique, this means $E(\mathcal{S}+\Gamma)$ is both a basis and a frame of $L^2(\mu)$. Hence $E(\mathcal{S}+\Gamma)$ is a Riesz basis.

We now recall a general criterion of spectral measures due to Jorgensen and Pedersen [399].

Proposition (6.1.18)[388]: Let μ be a probability measure on $\mathbb{R}^d$ with compact support. Then A is an orthogonal spectrum of μ if and only if
$$Q(x) = \sum_{\lambda\in\Lambda} |\hat{\mu}(x+\lambda)|^2 \equiv 1,\ x \in \mathbb{R}.$$
In particular, if $\mu = \sum_{c\in\mathcal{C}} p_c \delta_c$ is a discrete spectral measure with spectrum Λ, then $p_c = 1\ /\#\mathrm{C}$ by Proposition (6.1.11) and
$$\sum_{\lambda\in\Lambda} \left|m_{\mathcal{C},P}(x+\lambda)\right|^2 \equiv 1.$$
To determine whether μ in Theorem (6.1.17) is a spectral measure, we have the following simple characterization.

Theorem (6.1.19)[388]: Let ν be an R-spectral measure and suppose $q\mathcal{Z}_\nu \subset \mathbb{Z}$. Then $\mu = \eta_q * \nu$ is a spectral measure if and only if both η and ν are spectral measures.

Proof. It is clear that η is a spectral measure if and only if η_q is also a spectral measure. We first prove the sufficiency. Let $\mathcal{A} = q\mathcal{C}$, and let $\mathcal{S} = \{0, s_1, \dots, s_{k-1}\}$ be a bi-zero set of $m_{\mathcal{A},P}$ (note that P is a set of equal weights by Proposition (6.1.11)). Let Γ be a spectrum of ν, then $q\Gamma \subseteq \mathbb{Z}$ by the hypothesis that $q\mathcal{Z}_\nu \subset \mathbb{Z}$. The Fourier transform of μ satisfies
$$\hat{\mu}(\xi) = m_{\mathcal{A},P}(\xi)\hat{\nu}(\xi).$$

By the spectral property of $\mathcal{S}$ and Γ,

$$\begin{aligned}\sum_{0\leqslant j\leqslant k-1}\sum_{\gamma\in\Gamma}\left|\hat{\mu}(x+s_j+\gamma)\right|^2 &= \sum_{0\leqslant j\leqslant k-1}\sum_{\gamma\in\Gamma}\left|m_{\mathcal{A},P}(x+s_j+\gamma)\right|^2\left|\hat{v}(x+s_j+\gamma)\right|^2\\ &= \sum_{0\leqslant j\leqslant k-1}\sum_{\gamma\in\Gamma}\left|m_{\mathcal{A},P}(x+s_j)\right|^2\left|\hat{v}(x+s_j+\gamma)\right|^2\\ &= \sum_{0\leqslant j\leqslant k-1}\left|m_{\mathcal{A},P}(x+s_j)\right|^2=1.\end{aligned}$$

Hence $\mathcal{S}\bigoplus\Gamma$ is an orthogonal spectrum of μ by Proposition (6.1.18).

Conversely, suppose that Λ is a spectrum of μ and without loss of generality assume $0\in\Lambda$. Denote $x=\{x\}+[x]$ where $[x]$ is the maximum integer which is less than or equal to x. We claim that $\mathcal{S}=\{q^{-1}\{q\lambda\}:\lambda\in\Lambda\}$ is a bi-zero set of $m_{\mathcal{A},p}$. Indeed, by writing $\lambda=q^{-1}\{q\lambda\}+q^{-1}[q\lambda]$, we have

$$0=\hat{\mu}(\lambda)=m_{\mathcal{A},P}(q^{-1}\{q\lambda\}+q^{-1}[q\lambda])\hat{v}(\lambda)=m_{\mathcal{A},P}(q^{-1}\{q\lambda\})\hat{v}(\lambda) \qquad (9)$$

for each $\lambda\in\Lambda$. Note that $\hat{v}(\lambda)=0$ implies $q\lambda\in\mathbb{Z}$ (by the assumption $\mathcal{Z}_v\subset\mathbb{Z}$), so that $\{q\lambda\}=0$, (9) implies that either $q^{-1}\{q\lambda\}=0$ or it is a root of $m_{\mathcal{A},p}$. For any given distinct $q^{-1}\{q\lambda_1\},q^{-1}\{q\lambda_2\}\in\mathcal{S}$ and $\{q\lambda_1\}>\{q\lambda_2\}$, we have $q^{-1}\{q(\lambda_1-\lambda_2)\}=q^{-1}(\{q\lambda_1\}-\{q\lambda_2\})$ is a root of $m_{\mathcal{A},P}$.. This proves the claim.

Let us write $\Lambda=\bigcup_{j=0}^{k-1}(s_j+\Lambda_j), s_j\in\mathcal{S}$, where $\Lambda_j=\{q^{-1}[q\lambda]:q^{-1}\{q\lambda\}=s_j\}$. Since Λ is a spectrum of μ, we must have for all $\lambda_1,\lambda_2\in\Lambda_j$,

$$0=\hat{\mu}(\lambda_1-\lambda_2)=m_{\mathcal{A},P}(\lambda_1-\lambda_2)v(\lambda_1-\lambda_2).$$

But $a(q^{-1}[q\lambda])\in\mathbb{Z}$ for all $a\in\mathcal{A}=q\mathcal{C}$, this shows $m_{\mathcal{A},P}(\lambda_1-\lambda_2)\neq 0$ and hence $v(\lambda_1-\lambda_2)=0$. Therefore $E(\Lambda_j), 0\leqslant j\leqslant k-1$, are the orthogonal set of v. By the Bessel inequality, $\sum_{\lambda\in\Lambda_j}\left|\hat{v}(x+s_j+\lambda)\right|^2\leqslant 1$. Note further that $\mathcal{S}$ is a bi-zero set of $m_{\mathcal{A},P}$. By Proposition (6.1.18), we have

$$\begin{aligned}1 &\equiv\sum_{\lambda\in\Lambda}|\hat{\mu}(x+\lambda)|^2=\sum_{j=0}^{k-1}\sum_{\lambda\in\Lambda_i}\left|\hat{\mu}(x+s_j+\lambda)\right|^2\\ &=\sum_{j=0}^{k-1}\sum_{\lambda\in\Lambda_j}\left|m_{\mathcal{A},P}(x+s_j+\lambda)\hat{v}(x+s_j+\lambda)\right|^2\\ &=\sum_{j=0}^{k-1}\sum_{\lambda\in\Lambda_j}\left|m_{\mathcal{A},P}(x+s_j)\hat{v}(x+s_j+\lambda)\right|^2 \text{ (since } a\lambda\in\mathbb{Z})\\ &\leqslant\sum_{j=0}^{k-1}\left|m_{\mathcal{A},P}(x+s_j)\right|^2\leqslant 1.\end{aligned}$$

Hence $\mathcal{S}$ is the orthogonal spectrum of η_q by Proposition (6.1.18) again, so that η is a spectral measure. From the third line of the above, we also have

$$1\equiv\sum_{j=0}^{k-1}\left|m_{\mathcal{A},P}(x+s_j)\right|^2\sum_{\lambda\in\Lambda_j}\left|\hat{v}(x+s_j+\lambda)\right|^2.$$

With $\sum_{j=0}^{k-1}\left|m_{\mathcal{A},P}(x+s_j)\right|^2 \equiv 1$, we must have $\sum_{\lambda\in\Lambda_j}\left|\hat{v}(x+s_j+\lambda)\right|^2 \equiv 1$. Hence, v is a spectral measure and any one of the Λ_i is a spectrum of v.

It has been an open question whether the 1/3-Cantor measure has an F-spectrum (or even an R-spectrum). To a less extend, we do not know a non-trivial singularly continuous R-spectral measure. We can make use of Theorems (6.1.17) and (6.1.19) to construct such measures.

Example (6.1.20)[388]: There exists a singularly continuous R-spectral measure which is not a spectral measure.

Proof. Consider the self-similar measure $v_{\mathcal{A},n}$ in Theorem (6.1.15) with $\mathcal{A}$ satisfying $\mathcal{A}\oplus\mathcal{B}=\mathbb{N}_n$ and $\gcd(\mathcal{A})=1$. It is a spectral measure and has a spectrum $\Gamma\subset\mathbb{Z}$. Moreover we claim that $\mathcal{Z}_v\subset\mathbb{Z}$. Indeed, observe that

$$\hat{v}(\xi)=\prod_{j=1}^{\infty} m_{\mathcal{A}}\left(\frac{\xi}{n^j}\right)$$

where $m_{\mathcal{A}}$ stands for the mask polynomial of $\mathcal{A}$ under equal weight. As $\mathcal{A}\oplus\mathcal{B}=\mathbb{N}_n$, we have

$$m_{\mathcal{A}}(\xi)m_{\mathcal{B}}(\xi)=1+e^{2\pi i\xi}+\cdots+e^{2\pi i(n-1)\xi}.$$

The zero set of $m_{\mathcal{A}}$ on $[0,1)$ is a finite subset $Z\subset\{1/n,\ldots,n-1/n\}$. Let $Z'=Z+\mathbb{Z}$. This shows that $\mathcal{Z}_v=\bigcup_{j=1}^{\infty}n^jZ'$. This proves the claim and the condition in Theorem (6.1.19) holds (taking $q=1$).

Now we let $\eta=\delta_{\mathcal{C},P}$ be a discrete measure with any finite set $\mathcal{C}$ of non-negative integers and non-uniform weight P. $\mu=\eta * v_{\mathcal{A},n}$ is an R-spectral measure but not a spectral measure by Theorem (6.1.17) and Theorem (6.1.19). These measures is clearly singular if $\#\mathcal{A}<n$.

Finally, if E is a Borel set with positive Lebesgue measure, we use $L^2(E)$ to denote the square integrable functions on E. We remark that $L^2(E)$ always have an F-spectrum, and the existence of orthogonal spectrum is related to the translational tile as in Fuglede's conjecture. For R-spectrum, it is not known whether every Borel set E with positive Lebesgue measure, $L^2(E)$ has an R-spectrum. In regard to this we have the following simple result.

Corollary (6.1.21)[388]: If E be a finite union of closed intervals with rational endpoints. Then $L^2(E)$ admits an R-spectrum.

Proof. By the hypothesis and by suitably rescaling and translation, there exist two integers r and s such that

$$rE+s=[0,1]+\mathcal{A}:=F,$$

where $0\in\mathcal{A}$ and $\mathcal{A}\in\mathbb{Z}^+$is a finite set. By Proposition (6.1.1) with v being the Lebesgue measure on $[0,1]$, we see that F has an R-spectrum, which implies $L^2(E)$ also has an R-spectrum.

We remark that similar results were obtained in [409] who considered the problem from the sampling point of view and used techniques in complex analysis. We do not know whether the condition of rational endpoints can be removed. In [406], the case when the end-points lying in certain groups was considered, and the above also follows as a corollary.

Section (6.2): Self-Similar Measures:

For μ be a Borel probability measure with compact support in $\mathbb{R}^d$. We call μ a Fourier-Bessel measure with a Bessel set or Bessel sequence Λ in $\mathbb{R}^d$ if

$$\sum_{\lambda\in\Lambda}|\langle f,e_\lambda\rangle|^2 \le B\|f\|^2, \qquad \forall f\in L^2(\mu),$$

where $e_\lambda = e^{-2\pi i\langle\lambda,x\rangle}$, $\langle x,y\rangle$ is the standard inner product in $\mathbb{R}^d$ and B is a Bessel bound.

Moreover, if in addition there exists $A>0$ such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda}|\langle f,e_\lambda\rangle|^2 \le B\|f\|^2, \quad \forall f\in L^2(\mu).$$

Then μ is called a (Fourier) frame spectral measure with a frame spectrum Λ, and A,B are called the lower and upper frame bounds respectively. In particular, μ is called a Riesz spectral (reps. spectral) measure with Riesz spectrum (resp. spectrum) Λ if $E_\Lambda = \{e_\lambda\}_{\lambda\in\Lambda}$ is both a frame and basis (resp. orthonormal basis) for $L^2(\mu)$. For the frame spectral measure μ, any function in $L^2(\mu)$ can be expanded in terms of the exponentials family E_Λ. The frame theory has become the pillar of applied harmonic analysis, including Gabor analysis, wavelets, compressive sensing, interpolation and sampling theory, signal processing, and has been developed rapidly in recent years in both theory and applications.

Details on the background of general frame theory and recent topics are given in [416,426,427].

Recently He, Lau and Lai [430] proved that a Fourier frame spectral measure μ must be of pure type, that is, μ is one of a discrete measure with finite support, a singular continuous or an absolutely continuous measure with respect to the Lebesgue measure. It has a long history to study a normolized Lebesgue measure restricted to a set to be a frame spectral measure (can be traced back at least to 1967, the work of Landau [434]). The spectral measure problems attract more attention due to the famous Fuglede (spectral set) conjecture. From the beginning of this issue, the Beurling density plays a key role. We will discuss the relationship between the Beurling density (and dimension) and frame spectral measures (and Fourier-Bessel measures) which are singular with respect to the Lebesgue measure. Let Λ be a countable set or sequence in $\mathbb{R}^d$. The r −(upper) Beurling density of Λ is defined by

$$B_r^+(\Lambda) = \lim_{h\to\infty}\sup\sup_{x\in\mathbb{R}^d}\frac{\#\big(\Lambda\cap B(x,h)\big)}{h^r},$$

where $\#E$ is the cardinality of the set E and $B(x,h)$ is the open ball with center x and radius h. And the Beurling dimension of Λ is defined by

$$\dim_B\Lambda = \inf\{r: B_r^+(\Lambda)=0\} = \sup\{r: B_r^+(\Lambda)=\infty\}.$$

According to the results of Landau [434] and Lai [433], the Beurling dimension of a frame spectrum of an absolutely continuous frame spectral measure equals to the space dimension d. In general, there is a conjecture as follows:

Conjecture (6.3.1)[415]: Let μ be a Borel probability measure with compact support $T\subset\mathbb{R}^d$. If μ is a frame spectral measure with a frame spectrum Λ, then

$$\dim_B\Lambda \le \dim_H T,$$

where $\dim_H T$ is the Hausdorff dimension of the set T.

According to Conjecture (6.3.1), we call a Fourier-Bessel measure with compact support T a Bessel spectral one if there exists a Bessel set Λ satisfying $\dim_B\Lambda = \dim_H T$, in this case Λ is called a Maximal Fourier-Bessel set. We prove that Conjecture (6.3.1) holds for a class of Fourier-Bessel measures. It is very surprising and interesting that there exists a Fourier spectrum Λ for a spectral measure μ satisfying that $\dim_B\Lambda = 0 < \dim_H T$[419].

Dutkay et.al. [423] proved that Conjecture (6.3.1) holds for self-similar frame spectral measures with the same contracting ratios, equal probability weight and open set condition (OSC) (see the definition in the following). We will prove that Conjecture (6.3.1) holds for all Bessel sets or sequence of a self-similar measure with the OSC which can have no the same contracting ratio and no equal probability weight.

For $\Phi = \{\varphi_i(x)\}_{i=1}^N$ be an iterated function system (IFS) on $\mathbb{R}^d$ (or a domain D in $\mathbb{R}^d$ in general cases), that is, all $\varphi_i(x)$ are contractive in $\mathbb{R}^d$ with ratio $\rho_i < 1$. Let $P = \{p_i\}_{i=1}^N$ be a positive probability weight, i.e., all $p_i > 0$ and $\sum_{i=1}^N p_i = 1$. By Hutchinson's theorem [429, 431], there exists a unique Borel probability measure $\mu = \mu_{\Phi,P}$ with compact support $T_r = (T_r)_\Phi$ satisfying that

$$\mu(E) = \sum_{i=1}^N p_i \mu\left(\varphi_i^{-1}(E)\right) \tag{10}$$

for any Borel set $E \subset \mathbb{R}^d$ and T is the unique compact set or sequence satisfying that

$$T = \bigcup_{i=1}^N \varphi_i(T).$$

The probability measure μ satisfying (10) is called an invariant measure with respect to the IFS $\Phi = \{\varphi_i(x)\}_{i=1}^N$ and the probability weight P and T_r is called the attractor of the IFS. In particular, if all $\varphi_i(x)$ are self-similar mappings in $\mathbb{R}^d$, that says, $\varphi_i(x) = A_i(x + d_i)$, where $A_i = \rho_i Q_i$, $0 < \rho_i < 1$ and Q_i is an orthonormal matrix for each $1 \le i \le N$, then the IFS is called a self-similar IFS, the measure μ a self-similar measure and T a self-similar set.

We say that an IFS $\{\varphi_i(x)\}_{i=1}^N$ satisfies the open set condition (OSC) if there exists a bounded open set $V \subset \mathbb{R}^d$ such that

$$\bigcup_{i=1}^N \varphi_i(V) \subset V \tag{11}$$

and the union on the left is pairwise disjoint. And we say an IFS satisfies the strong open set condition (SOSC) if there exists a bounded open set V satisfying (11) and $V \cap T \neq \emptyset$, where T is the attractor of the IFS. According to the results of Schief [437], the following three statements are equivalent: (a) the self-similar IFS $\{\varphi_i(x)\}_{i=1}^N$ satisfies the OSC; (b) the self-similar IFS $\{\varphi_i(x)\}_{i=1}^N$ satisfies the SOSC; and (c) $0 < H^s(T) < \infty$, where s is the unique solution of the equation $\sum_{i=1}^N \rho_i^s = 1$, ρ_i is the ratio of the similitude $\varphi_i(x)$ for each $1 \le i \le N$ and H^s denotes the $s-$dimensional Hausdorff measure. In this case the s is called the self-similar dimension of the IFS $\{\varphi_i(x)\}_{i=1}^N$.

Theorem (6.3.2)[415]: Let $\Phi = \{\varphi_i(x) = \rho_i Q_i(x + d_i)\}_{i=1}^N$ be a self-similar IFS satisfying the OSC and let $P = \{p_i\}_{i=1}^N$ be a positive probability weight. Then, for each Bessel set Λ of the self-similar measure $\mu = \mu_{\Phi,\mathrm{P}}$,

$$\dim_B \Lambda \le \dim_H T = s,$$

where T_r is the self-similar set and s is the self-similar dimension. Moreover, if $\dim_B \Lambda = \dim_H T$, then $p_i = \rho_i^s$ for $1 \le i \le N$.

For Φ be of the open set condition. We remark that if $p_i = \rho_i^s$ for $1 \le i \le N$ [437], there exists a constant $c \neq 0$ such that $c\mu_{\Phi,P}$ is the restriction of the $s-$Hausdorff measure to T; In other case, $\mu_{\Phi,P}$ is singular with respect to the $s-$Hausdorff measure [436]. In singular case, we guess that $\mu_{\Phi,P}$ is not a Bessel spectral measure.

To prove Theorem (6.3.2), we study the relationship between the IFS with the OSC and SOSC motivated by the work of Deng et al. [421] and Dutkay et al. [424] and their applications, all results in those two sections are suitable for an IFS consisting of analytic functions on a domain in complex plane.

Let $\Theta_N = \{1, 2, \ldots, N\}$. Denote all the words with length n by $\Theta_N^n = \{I = i_1 i_2 \cdots i_n : \text{all } i_k \in \Theta_N\}$ and all the finite words by $\Theta_N^* = \bigcup_{n=0}^{\infty} \Theta_N^n$, where $\Theta_N^0 = \{\emptyset\}$ is the set containing only empty word. For an IFS $\{\varphi_i(x)\}_{i=1}^N$ and a probability weight $P = \{p_i\}_{i=1}^N$, we denote that, if $I = i_1 i_2 \cdots i_n \in \Theta_N^n$,

$$\varphi_I(x) = \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(x), T_I = \varphi_I(T) \text{ and } p_I = p_{i_1} p_{i_2} \cdots p_{i_n},$$

where the notion $\circ$ means the composition of the functions. The following result is motivated by Dutkay et al. [423].

Theorem (6.3.3)[415]: Let $\Phi = \{\varphi_i(x) = \rho_i Q_i(x + d_i)\}_{i=1}^N$ be a self-similar IFS satisfying the OSC. Let $\mu = \mu_{\Phi,P}$ be the self-similar measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^N$. Let Λ be a frame spectrum of μ. If there exists $I = i_1 i_2 \ldots i_n \in \Theta_N^n$ for some $n \geq 1$ such that

$$\sup_{\lambda \in \Lambda} \inf_{\gamma \in \Lambda} \|\varphi_I(\lambda) - \gamma\| < \infty.$$

Then $\dim_B \Lambda = \dim_H T = s$ if and only if $p_I = \rho_I^s$, where T is the self-similar set, s is the self-similar dimension of the IFS and $\rho_I = \rho_{i_1} \rho_{i_1} \cdots \rho_{i_n}$.

An IFS $\{\varphi_i(x)\}_{i=1}^N$ is called an open mapping IFS if all $\varphi_i(x)$ are open mappings. Here we give several results on an IFS with OSC or SOSC, which will be used in the following.

Theorem (6.3.4)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open mapping IFS on $\mathbb{R}^d$ satisfying the SOSC with a bounded open set or sequence V. Let μ be the invariant measure with respect to the IFS and a probability weight P. Then

(i) $\mu(\partial V) = 0$, where ∂V is the boundary of the open set V;

(ii) The invariant measure μ satisfies the no overlap condition, i.e., $\mu\left(\varphi_i(T) \cap \varphi_j(T)\right) = 0$ for any $i \neq j$;

(iii) $\mu\left(\varphi_i^{-1}\varphi_j(T)\right) = 0$ for any $i \neq j$.

The no overlap condition is also called a measurably separated condition, which was used to study the spectrality of self-similar measures with equal weight probability and equal contracting ratios in [423, 428]. We prove Theorem (6.3.4) by the following three propositions, which are motivated by the work of Deng et al. [421] and Dutkay et al. [424].

Proposition (6.3.5)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an IFS on $\mathbb{R}^d$ satisfying the OSC by a bounded open set V. Let μ be the invariant measure with respect to the IFS and a probability weight P. Then the IFS satisfies the SOSC with the open set V if and only if $\mu(\partial V) = 0$.

Proof. Observe that the attractor T of the IFS satisfies that $T \subset V_r$. Then $\mu(V) = \mu(\overline{V_r}) = 1$ by $\mu(\partial V) = 0$ and the sufficiency follows. Conversely, let $x \in T \cap V$ by the SOSC. Then there exists $\delta > 0$ such that the open ball $B(x, \delta) \subset V$. By the contraction of functions in the IFS, there exist an integer $m \geq 1$ and $I \in \Theta_N^m$ such that $\varphi_I(T) \subset B(x, \delta) \subset V$. Iterating (10) m −times, we obtain that

$$\mu(\cdot) = \sum_{I\in\Theta_N^m} P_I\mu\big(\varphi_I^{-1}(\cdot)\big) \quad (12)$$

and $\{\varphi_I(x)\}_{I\in\Theta_N^m}$ is also an IFS satisfying the SOSC with the same open set V. Therefore, for the simpler notations we can assume that $\varphi_1(T) \subset V$ without loss of generality.

We notice that, for $n \geq 1$ and $I = i_1 i_2 \cdots i_n \in \{1,2,\ldots,N\}^n \backslash \{2,3,\ldots,N\}^n := \Theta_N^n \backslash \Theta_{2,N}^n$, there exists $k, 1 \leq k \leq n$, such that $i_k = 1$. Then

$$\varphi_I(T) \subset \varphi_{i_1 i_2\cdots i_k}(T) \subset \varphi_{i_1 i_2 \cdots i_{k-1}}(V) \subset V.$$

Hence, by (12) we have

$$\begin{aligned} 1 \geq \mu(V) &= \sum_{I\in\Theta_N^n} P_I\mu\big(\varphi_I^{-1}(V)\big) \geq \sum_{I\in\Theta_N^n\backslash\Theta_{2,N}^n} P_I\mu\big(\varphi_I^{-1}(V)\big) \\ &\geq \sum_{I\in\Theta_N^n\backslash\Theta_{2,N}^n} P_I\mu\left(\varphi_I^{-1}\big(\varphi_I(T)\big)\right) = \sum_{I\in\Theta_N^n\backslash\Theta_{2,N}^n} P_I \\ &= \sum_{I\in\Theta_N^n} P_I - \sum_{I\in\Theta_{2,N}^n} P_I \\ &= 1 - \left(\sum_{i=2}^{N} p_i\right)^n. \end{aligned}$$

Consequently, $\mu(V) = 1$ and $\mu(\partial V) = \mu(\bar{V}) - \mu(V) = 0$.

Proposition (6.3.6)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open mapping IFS satisfying the OSC with a bounded open set V. Let μ be the invariant measure with respect to the IFS and a probability weight P. If $\mu(\partial V) = 0$, then the μ satisfies the no-overlap condition.

Proof. By the OSC, one sees that $\varphi_i(V) \cap \varphi_j(V) = \emptyset$ for any $i \neq j$. Since all φ_i are open mappings, then $\varphi_i(V)$ is an open set for each i, and we have

$$\varphi_i(V) \cap \varphi_j(\bar{V}) = \varphi_i(V) \cap \overline{\varphi_j(V)} = \emptyset. \quad (2.2)$$

Then

$$V \cap \varphi_i^{-1}\left(\varphi_j(\partial V)\right) \subset \varphi_i^{-1}\left(\varphi_i(V) \cap \varphi_j(\partial V)\right) \subset \varphi_i^{-1}\left(\varphi_i(V) \cap \varphi_j(V)\right) = \emptyset.$$

Consequently,

$$\bar{V} \cap \varphi_i^{-1}\left(\varphi_j(\partial V)\right) \subset \partial V$$

and by the fact the attractor $T \subset \bar{V}$ one has

$$\mu(\varphi_i^{-1}\left(\varphi_j(\partial V)\right) = 0. \quad (14)$$

Using (13) again, we have

$$\varphi_i(T) \cap \varphi_j(T) \subset \varphi_i(\bar{V}) \cap \varphi_j(\bar{V}) \subset \varphi_i(\partial V) \cap \varphi_j(\partial V) \quad (15)$$

for $i \neq j$. Then, by (14) and (15),

$$\begin{aligned} \mu\left(\varphi_i(T) \cap \varphi_j(T)\right) &\leq \mu\left(\varphi_i(\partial V) \cap \varphi_j(\partial V)\right) \\ &= \sum_{s=1}^{N} p_s\mu\left(\varphi_s^{-1}\left(\varphi_i(\partial V) \cap \varphi_j(\partial V)\right)\right) \leq 0. \end{aligned}$$

Hence the μ satisfies the no-overlap condition.

Proposition (6.3.7)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an IFS on $\mathbb{R}^d$. Let μ be the invariant measure with respect to the IFS and a probability weight P. Then μ satisfies the no-overlap condition if and only if $\mu\left(\varphi_i^{-1}\left(\varphi_j(T)\right)\right) = 0$ for any $i \neq j$.

Proof. Suppose that $\mu\left(\varphi_i^{-1}\left(\varphi_j(T)\right)\right) = 0$ for any $i \neq j$. Then

$$\mu\left(\varphi_i(T) \cap \varphi_j(T)\right) = \sum_{k=1}^{N} p_k \mu\left(\varphi_k^{-1}\left(\varphi_i(T) \cap \varphi_j(T)\right)\right)$$
$$\leq \sum_{1\leq k\leq N, k\neq i,j} p_k \mu\left(\varphi_k^{-1}(\varphi_i(T))\right) + p_j \mu\left(T \cap \varphi_j^{-1}(\varphi_i(T))\right) + p_i \mu\left(T \cap \varphi_i^{-1}\left(\varphi_j(T)\right)\right)$$
$$= 0.$$

That is, μ satisfies the no-overlap condition. Conversely, suppose that $\mu\left(\varphi_i(T) \cap \varphi_j(T)\right) = 0$ for any $i \neq j$. Then

$$0 = \mu\left(\varphi_i(T) \cap \varphi_j(T)\right) = \sum_{k=1}^{N} p_k \mu\left(\varphi_k^{-1}\left(\varphi_i(T) \cap \varphi_j(T)\right)\right).$$

In particular, we have

$$0 = \mu\left(\varphi_i^{-1}\left(\varphi_i(T) \cap \varphi_j(T)\right)\right) = \mu\left(T \cap \varphi_i^{-1}\left(\varphi_j(T)\right)\right) = \mu\left(\varphi_i^{-1}\left(\varphi_j(T)\right)\right).$$

Hence, we complete the proof.

For $\{\varphi_i(x)\}_{i=1}^N$ be an IFS and let T be its attractor. Let $P = \{p_i\}_{i=1}^N$ be a probability weight.

For $i = i_1 i_2 \cdots i_n \in \Theta_N^n$, recall that

$$T_I = \varphi_I(T), \qquad P_I = p_{i_1} p_{i_2} \cdots p_{i_n},$$

where $\varphi_I(x) = \varphi_{i_1} \circ \varphi_{i_2} \cdots \circ \varphi_{i_n}(x)$ is the composition of the mappings $\varphi_{i_k}(x), 1 \leq k \leq n$.

Theorem (6.3.8)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open and bijective mapping IFS in $\mathbb{R}^d$ satisfying the SOSC and let T be its attractor. Let μ be the invariant measure with respect to the IFS and a probability weight $P = \{p_i\}_{i=1}^N$. Then

(i) $\mu(T_I) = P_I$ for $I \in \Theta_N^*$;

(ii) Let $f \in L^1(T_r, \mu)$ and $I \in \Theta_N^*$. Then

$$\int_{T_I} f \circ \varphi_I^{-1}(x) d\mu = P_I \int_{T_r} f d\mu;$$

(iii) Let $f \in L^1(T, \mu)$ and $I \in \Theta_N^*$. Then

$$\int_{T_r} f \circ \varphi_I(x) d\mu = P_I^{-1} \int_{T_I} f d\mu.$$

Proof. (i) For $I = i_1 i_2 \cdots i_n \in \Theta_N^n$ and $n \geq 1$, we have

$$\mu(T)_I = \sum_{i=1}^{N} p_i \mu(\varphi_i^{-1}(T)_I) = \sum p_i \mu\left(\varphi_i^{-1}\left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T)\right)\right).$$

Note that $\sum \varphi_i^{-1}\left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T_r)\right) \subset \sum \varphi_i^{-1} \circ \varphi_{i_1}(T_r)$. By Theorem (6.3.4) and by induction, we have

$$\mu(T)_I = p_{i_1}\mu\left(\varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T)\right) = p_{i_1}p_{i_2}\cdots p_{i_n}\mu(T) = P_I.$$

(ii) By (10) we have

$$\int_{T_I} f \circ \varphi_I^{-1}(x)d\mu = \sum_{i=1}^{N} p_i \int \chi_{T_I}(\varphi_i(x)) f \circ \varphi_{i_n}^{-1} \circ \varphi_{i_{n-1}}^{-1} \circ \cdots \circ \varphi_{i_2}^{-1}\left(\varphi_{i_1}^{-1} \circ \varphi_i(x)\right)d\mu,$$

where $\chi_E(x)$ is the characteristic function of $E \subset \mathbb{R}^d$. Note that $\chi_{T_I}(\varphi_i(x)) = 1$ if and only if $x \in \varphi_i^{-1} \circ \left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \varphi_{i_n}\right)(T)$. Then, by Theorem (6.3.4) again,

$$\begin{aligned}\int_{T_I} f \circ \varphi_I^{-1}(x)d\mu &= p_{i_1}\int \chi_{\varphi_{i_1}^{-1}(T_I)}(x) f \circ \varphi_{i_n}^{-1} \circ \varphi_{i_{n-1}}^{-1} \circ \cdots \circ \varphi_{i_2}^{-1}(x)d\mu \\ &= p_{i_1}p_{i_2}\cdots p_{i_n}\int \chi_{\varphi_{i_n}^{-1}\circ\varphi_{i_{n-1}}^{-1}\circ\cdots\circ\varphi_{i_1}^{-1}(T_r)_I}(x)\, f(x)d\mu \\ &= P_I \int_T f(x)d\mu.\end{aligned}$$

(iii) Firstly,

$$P_I^{-1}\int_{T_I} f(x)d\mu(x) = P_I^{-1}\int_{T_I} f \circ \varphi_I \circ \varphi_I^{-1}(x)d\mu(x).$$

Secondly, by (ii), we get

$$P_I^{-1}\int_{T_I} f \circ \varphi_I \circ \varphi_I^{-1}(x)d\mu(x) = \int_T f \circ \varphi_I(x)d\mu(x).$$

Hence, the assertion (iii) follows.

For μ be a Borel probability measure with compact support. Recall that it is a frame spectral measure if there exists Λ such that

$$A\|f\|_\mu^2 \le \sum_{\lambda\in\Lambda}\left|\langle f, e_\lambda\rangle_\mu\right|^2 \le B\|f\|_\mu^2, \qquad \forall f \in L^2(\mu),$$

where $e_\lambda = e^{-2\pi i\lambda x}$ and $0 < A \le B < \infty$. And it is a Fourier-Bessel measure with a Bessel set Λ if the above inequality holds for the right hand. Moreover, it is a Bessel spectral measure if there exists a Bessel set Λ such that its Beurling dimension is equal to the Haudorff dimension of the support of the measure μ.

Now we give several examples to explain the existence of the frame spectral measures and Fourier-Bessel measures. One of famous self-similar measures is the Bernoulli convolution, which has been studied from 1930s and has several equivalent definitions from analysis, geometry and probability. The simplest definition is the self-similar measure μ_ρ generated by the IFS $\varphi_0(x) = \rho(x-1), \varphi_1(x) = \rho(x+1),\ 0 < \rho < 1$ and the probability $P = \{1/2, 1/2\}$, that is, μ_ρ is the unique probability measure satisfying that

$$\mu_\rho(E) = \frac{1}{2}\mu_\rho\left(\varphi_0^{-1}(E)\right) + \frac{1}{2}\mu_\rho\left(\varphi_1^{-1}(E)\right)$$

for any Borel set or sequence $E \subset \mathbb{R}$. It is known or easy to show the following results on Bernoulli convolution:

- μ_ρ is a spectral measure if and only if $\rho = \frac{1}{2q}$ for some $q \in \mathbb{N}$ [418];

- If μ_ρ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ and $\rho \neq \frac{1}{2}$, then μ_ρ is not a frame spectral measure [433]. This is almost true for ρ in $\left(\frac{1}{2},1\right)$ [433];
- If μ_ρ is absolutely continuous with respect to Lebesgue measure on $[0,1]$ and $\rho \in [2^{-1/2},1)$, then the density of μ_ρ is bounded by [438, Corollary 1]. Similar to the result of Lai [433], it is easy to show that μ_ρ a Maximal Fourier-Bessel measure in this case. And this holds for almost all ρ in $[2-1/2,1)$.

The remaining cases are (a) whether a singular measure μ_ρ for $1/2 < \rho < 1$ is a frame spectral measure (difficult problem); (b) we guess that μ_ρ is a frame spectral measure for $0 < \rho < 1/2$ if and only if $\rho = \frac{1}{2q}$ for some $q \in \mathbb{N}$, and equivalent to that μ_ρ is a Maximal Fourier-Bessel measure for $0 < \rho < 1/2$.

In general, we have that

Theorem (6.3.9)[415]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open and bijective mapping IFS satisfying the SOSC and let T_r be its attractor. Let μ be the invariant measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^N$. Suppose that the E_Λ is a Fourier frame with lower and upper bounds A and $E(A \geq B)$. Then

(i) For $I \in \Theta_N^n, n \geq 1, \mu \circ \varphi_I$ is a Fourier frame spectral measure with lower and upper bounds AP_I and BP_I respectively, i.e.,
$$AP_I\|f\|_{\mu\circ\varphi_I}^2 \leq \sum_{\lambda\in\Lambda}\left|\langle f,e_\lambda\rangle_{\mu\circ\varphi_I}\right|^2 \leq BP_I\|f\|_{\mu\circ\varphi_I}^2,\quad \forall\, f \in L(T_r,\mu\circ\varphi_I);$$

(ii) For $I \in \Theta_N^n, n \geq 1, \mu \circ \varphi_I$ is a Fourier frame spectral measure with lower and upper bounds $P_I^{-1}A$ and $P_I^{-1}B$ respectively, i.e.,
$$AP_I^{-1}\|f\|_{\mu\circ\varphi_I^{-1}}^2 \leq \sum_{\lambda\in\Lambda}\left|\langle f,e_\lambda\rangle_{\mu\circ\varphi_I^{-1}}\right|^2 \leq BP_I^{-1}\|f\|_{\mu\circ\varphi_I^{-1}}^2,\quad \forall\, f \in L(T,\mu\circ\varphi_I^{-1});$$

where $T_I = \varphi_I(T)$.

Proof. Observe that
$$\begin{aligned}\langle f,e_\lambda\rangle_{\mu\circ\varphi_I} &= \int_{T_r} f(x)e^{2\pi\langle\lambda,x\rangle}d\mu\circ\varphi_I(x)\\ &= \int_{T_I} f\left(\varphi_I^{-1}(x)\right)e^{2\pi\langle\lambda,\varphi_I^{-1}(x)\rangle}d\mu(x)\\ &= P_I\int_T f(x)e^{2\pi\langle\lambda,x\rangle}d\mu(x)\ \left(\text{by Theorem (6.3.8)(ii)}\right)\\ &= P_I\langle f,e_\lambda\rangle_u\end{aligned}$$
and by similar calculations,
$$\|f\|_{\mu\circ\varphi_I}^2 = P_I\|f\|_\mu^2.$$
Then (i) follows. To prove (ii), similarly, we have $\langle f,e_\lambda\rangle_{\mu\circ\varphi_I^{-1}} = P_I^{-1}\langle\chi_{T_I}f_r,e_\lambda\rangle_\mu$ and $\|f\|_{\mu\circ\varphi_I^{-1}}^2 = P_I^{-1}\left\|\chi_{T_I}f\right\|_\mu^2$. Then (ii) follows.

Recall that the Fourier transformation of the measure μ is defined by
$$\hat{\mu}(\xi) = \int e^{-2\pi i\langle\xi,x\rangle}d\mu(x).$$

The following corollaries will be used.

Corollary (6.3.10)[415]: With the same hypotheses given in Theorem (6.3.9) and in addition, we assume that $\varphi_i(x) = R_i(x + d_i)$ is a contracting affine mapping for $1 \le i \le N$. Then

$$AP_I^{-1} \le \sum_{\lambda\in\Lambda} \left|\mu\left(R_I^T(\lambda - x)\right)\right|^2 \le BP_I^{-1}, \text{for } I \in \Theta_N^* \text{ and } x \in \mathbb{R}^d.$$

Proof. Recall $\varphi_I(x) = \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(x)$. Thus $\varphi_I(x) = R_I(x + d_I)$ where $R_I = R_{i_1}R_{i_2} \cdots R_{i_n}$, $d_I = d_{i_n} + R_{i_n}^{-1}d_{i_{n-1}} + \cdots + R_{i_n}^{-1}R_{i_{n-1}}^{-1} \cdots R_{i_2}^{-1}d_{i_1}$ and $\varphi_I^{-1}(x) = R_I^{-1}x - d_I$. Choose $f(x) = e_y(x) = e^{-2\pi i\langle y,x\rangle}$ for $y \in \mathbb{R}^d$. Then

$$\begin{aligned}\langle e_y, e_\lambda\rangle_{\mu\circ\varphi_I^{-1}} &= \int_{T_I} e^{2\pi\langle\lambda-y,x\rangle i} d\mu \circ \varphi_I^{-1}(x)\\ &= \int_T e^{2\pi\langle\lambda-y,\varphi_I(x)\rangle i}\, d\mu\\ &= e^{2\pi i\langle\lambda-y,R_I d_I\rangle} \int_T e^{2\pi i\langle R_I^T(\lambda-y),x\rangle} d\mu(x)\\ &= e^{2\pi i\langle\lambda-y,R_I d_I\rangle}\hat{\mu}\left(R_I^T(y-\lambda)\right)\end{aligned}$$

and $\left\|e_y\right\|_{\mu\circ\varphi_I^{-1}}^2 = 1$. Hence, the assertion follows by Theorem (6.3.9)(ii).

Corollary (6.3.11)[415]: With the same hypotheses given in Theorem (6.3.9) and Corollary (6.3.10) respectively, and replacing the frame spectrum Λ with a Bessel set Λ, then the same results hold for the Bessel set.

Proof. The proof is similar to Theorem (6.3.9) and Corollary (6.3.10).

Let Λ be a countable set in $\mathbb{R}^d$. The upper $r-$ density of Λ is defined by

$$D_r^+(\Lambda) = \lim_{h\to\infty} \sup \frac{\#\left(\Lambda \cap B(0,h)\right)}{h^r}$$

and the Beurling upper $r-$ density of Λ is defined by

$$B_r^+(\Lambda) = \lim_{h\to\infty} \sup \sup_{x\in\mathbb{R}^d} \frac{\#\left(\Lambda \cap B(x,h)\right)}{h^r}.$$

Similarly, we can define the (resp. Beurling) lower $r-$ density of Λ denote it by $D_r^-(\Lambda)$ (resp. $B_r^-(\Lambda)$).

It is easy to prove the following result on the right hand. Then we define the Beurling dimension of Λ by

$$\dim_B \Lambda = \inf\{r: B_r^+(\Lambda) = 0\} = \sup\{r: B_r^+(\Lambda) = \infty\}.$$

Clearly $\dim_B a\Lambda = \dim_B \Lambda$ for any nonzero real number a, which will be used of this paper.

Remark (6.3.12)[415]: From the $r-$ density of $\Lambda \subset \mathbb{R}^d$, we can define similarly the Banach dimension of Λ by

$$\dim_D \Lambda = \inf\{r: D_r^+(\Lambda) = 0\} = \sup\{r: D_r^+(\Lambda) = \infty\},$$

which has been used extensively, e.g. see [417, 423, 425]. Clearly, $D_r^+(\Lambda) \le B_r^+(\Lambda)$. We remark that all the results on the Beurling dimension are true if we replace it by the Banach dimension except Theorem (6.3.16). However, for all examples of frame spectrum that we know, the conclusion of Theorem (6.3.16) is also true if we use the Banach dimension.

Theorem (6.3.13)[415]: Let $\Phi = \{\varphi_i(x) = \rho_i Q_i(x + d_i)\}_{i=1}^N$ be a self-similar IFS satisfying the OSC. Let $\mu = \mu_{\Phi,P}$ be the self-similar measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^N$. Suppose that Λ is a Bessel set for μ. Then

(i) $\dim_B \Lambda \le \dim_H T$, where T is the self-similar set;

(ii) if there exists a Bessel set Λ such that $\dim_B \Lambda = \dim_H T = s$, then all $p_i = \rho_i^s$, where s is the self-similar dimension.

Proof. It is known that $\dim_H T = s$ by Schief [437]. We first prove (i). And (ii) follows easily.

Since $\hat{\mu}(0) = 1$, there exist $\delta > 0$ such that

$$|\hat{\mu}(\xi)|^2 \ge \frac{1}{2}, \qquad \text{for } |\xi| \le \delta.$$

Let p be an integer such that $\rho_{\max}^p := \max_{1\le i\le N} \rho_i^p \le \delta$. Suppose that there exists i such that $p_i \ne \rho_i^s$. Then there exists k such that $p_k > \rho_k^s$ by $\sum_{i=1}^N p_i = \sum_{i=1}^N \rho_i^s = 1$. Choose $\varepsilon > 0$ satisfying $p_k \ge r\rho_k^s$. Let $h > 1$ be arbitral, then there exists a natural number n satisfying that $\rho_k^{-n+1} \le h < \rho_k^{-n}$. For any $\lambda \in \Lambda \cap B(x,h)$, we have $\lambda - x \in B(0,h)$ and thus

$$(\rho_k Q_k^T)^{n+p}(\lambda - x) \in B\left(0, \rho_k^{n+p} h\right) \subset B(0,\delta).$$

Consequently,

$$\begin{aligned}\frac{\#\left(\Lambda \cap B(x,h)\right)}{2} &\le \sum_{\lambda\in\Lambda\cap B(x,h)} \left|\hat{\mu}\left((\rho_k Q_k^T)^{n+p}(\lambda - x)\right)\right|^2 \\ &\le \sum_{\lambda\in\Lambda} \left|\hat{\mu}\left((\rho_k Q_k^T)^{n+p}(\lambda - x)\right)\right|^2 \ \left(\text{by Corollary (6.3.10)}\right) \\ &\le B p_k^{-(n+p)} \le B p_k^{-p} r^{-n} \rho_k^{-ns}.\end{aligned}$$

Hence, we obtain that

$$\sum \frac{\#\left(\Lambda \cap B(x,h)\right)}{h^{s+\ln r/\ln\rho_k}} \le 2Bp_k^{-p} \frac{r^{-n}\rho_k^{-ns}}{h^{s+\ln r/\ln\rho_k}} \le Bp_k^{-p}\rho_k^{-ns},$$

where the second inequality follows from $h \ge \rho_k^{-n+1}$ and $s + \ln r / \ln \rho_k \ge 0$ (since $p_k \ge r\rho_k^s$).

Therefore

$$B_{s+\ln r/\ln\rho_k}^{+}(\Lambda) \le 2Bp_k^{-p}\rho_k^{-s} \quad \text{and} \quad \dim_B \Lambda \le s + \ln r / \ln \rho_k < s.$$

Clearly, when all $p_i = \rho_i^s$, then $\dim_B \Lambda \le s$ by the same idea (choose $r = 0$ in the previous proof) and (i) follows. Then (ii) follows easily by (i).

We believe that the converse of Theorem (6.3.13)(ii) is not true in some cases. The Bernoulli convolutions would be counterexamples for some $0 < \rho < 1$. To prove Theorem (6.3.3), we need the following lemmas. The first one was proved in [423, 430]. It can be viewed as the stability of Bessel set under a constant perturbation of a Bessel set and has origin of Duffin and Schaeffer [422]. The following proof was given in [430].

Lemma (6.3.14)[415]: Let $\Lambda = \{\lambda_n\}_{n=0}^\infty$ be a Bessel set or sequence of μ with compact support in $[-P,P]^d$ and Bessel bound B. If there exists L such that $|\lambda_n - \gamma_n| \le L$ for $n \ge 0$, then $\Gamma = \{\gamma_n\}_{n=0}^\infty$ is also a Bessel set or sequence of μ with a Bessel bound $(A+\varepsilon)e^{d(2\pi L)^2 + dP^2}$.

Proof. It is sufficient to show that all $\gamma_n = \left(\gamma_1^{(n)}, \dots, \gamma_d^{(n)}\right)$ differs $\lambda_n = \left(\lambda_1^{(n)}, \dots, \lambda_d^{(n)}\right)$ only on the first component, and the statement follows by induction on the number of components.

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\langle f(x), e^{-2\pi i\gamma_n\cdot x}\rangle\right|^2 &= \sum_{n=0}^{\infty}\left|\langle f(x), e^{2\pi i(\gamma_n-\lambda_n)\cdot x}, e^{-2\pi i\lambda_n\cdot x}\rangle\right|^2 \\ &= \sum_{n=0}^{\infty}\left|\langle f(x), e^{2\pi i\left(\gamma_1^{(n)}-\lambda_1^{(n)}\right)\cdot x}, e^{-2\pi i\lambda_n\cdot x}\rangle\right|^2 \\ &= \sum_{n=0}^{\infty}\left|\sum_{n=0}^{\infty}\frac{\left(2\pi i\left(\gamma_1^{(n)}-\lambda_1^{(n)}\right)\right)^k}{k!}\sum\langle f(x)x_1^k, e^{-2\pi i\lambda_n\cdot x}\rangle\right|^2 \\ &\le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2\pi L)^{2k}}{k!}\sum_{n=0}^{\infty}\frac{\left|\langle f(x)x_1^k, e^{-2\pi i\lambda_n\cdot x}\rangle\right|^2}{k!} \\ &\le e^{(2\pi L)^2}\sum_{n=0}^{\infty}\frac{B\left\|f(x)x_1^k\right\|^2}{k!} \\ &\le Be^{(2\pi L)^2+P^2}\|f\|^2,\end{aligned}$$

where we have used Cauchy-Schwarz inequality at the fourth line above. Hence, the assertion follows.

The following lemma was given in [423]. In order to be complete we give its proof here.

Lemma (6.3.15)[415]: Let μ be a Bessel measure with a Bessel set , Bessel bound B and support in $[-P,P]^d$. Then, for any $\varepsilon \ge 0$,

$$\sum_{\lambda\in\Lambda}\max_{\|x\|\le(1+\varepsilon)}|\hat{\mu}(x+\lambda)|^2 \le Be^{d(2\pi(1+\varepsilon))^2+dP^2} \tag{16}$$

Proof. Since $\hat{\mu}(\xi)$ is a continuous function in $\mathbb{R}^d$, then, for each $\lambda \in \Lambda$, there exists r_λ such that

$$\max_{\|x\|\le 1+\varepsilon}|\hat{\mu}(x+\lambda)| = |\hat{\mu}(r_\lambda)|$$

and $|\lambda - r_\lambda| \le r$. According to Lemma (6.3.14), $\{r_\lambda\}_{\lambda\in\Lambda}$ is also a frame spectrum of μ. Then

$$\sum_{\lambda\in\Lambda}\max_{\|x\|\le(1+\varepsilon)}|\hat{\mu}(x+\lambda)|^2 = \sum_{\lambda\in\Lambda}|\hat{\mu}(r_\lambda)|^2 \le Be^{d(2\pi(1+\varepsilon))^2+dP^2}.$$

Theorem (6.3.16)[415]: Let $\Phi = \{\varphi_i(x) = A_i(x+d_i)\}_{i=1}^N$ be a self-similar IFS satisfying the OSC. Let $\mu = \mu_{\Phi,P}$ be the self-similar measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^N$. Suppose that Λ is a frame spectrum of μ. If there exist a constant L_r and $I \in \Theta_N^n$ such that

$$\sup_{\lambda\in\Lambda}\inf_{\gamma\in\Lambda}\|A_I^T\lambda-\gamma\| \le L.$$

Then $\dim_B \Lambda = \dim_H T = s$ if and only if $p_I = \rho_I^s$, where T is the self-similar set or sequence , s is the self-similar dimension and ρ_i is the ratio of A_i for $1 \le i \le N$.

Proof. The necessity follows by Theorem (6.3.13)(ii). We need to prove the sufficiency. Let the lower and upper frame bounds of the frame E_Λ are $A, A+\varepsilon$ respectively. It is well known that Λ is relative uniformly discrete and thus there exists unique $\beta \in \Lambda$ dependent of λ such that $\|A_I^T\lambda - \beta\| = \inf_{\gamma\in\Lambda}\|A_I^T\lambda-\gamma\|$.

For any $\lambda \in \Lambda$, we define two mappings $\varphi: \Lambda \to \Lambda$ and $\psi: \Lambda \to B(0,L)$ by

$$\|A_I^T\lambda - \varphi(\lambda)\| = \inf_{\gamma\in\Lambda}\|A_I^T\lambda - \gamma\|, \psi(\lambda) = A_I^T\lambda - \varphi(\lambda).$$

Then

$$A_I^T\lambda = \varphi(\lambda) + \psi(\lambda) \tag{17}$$

Iterating (17) n times, we have

$$A_I^{nT}\lambda = \varphi^{(n)}(\lambda) + \psi_n(\lambda), \tag{18}$$

Where $\psi_n(\lambda) = A_I^{n-1}\psi(\lambda) + A_I^{n-2}\psi(\varphi(\lambda)) + \cdots + \psi\left(\varphi^{(n-1)}(\lambda)\right)$. Then

$$\|\psi_n(\lambda)\| \le \|A_I^{n-1}\psi(\lambda)\| + \|A_I^{n-2}\psi(\varphi(\lambda))\| + \cdots + \left\|\psi\left(\varphi^{(n-1)}(\lambda)\right)\right\|$$
$$\le \sum_{i=0}^{n-1}\rho_I^{-i}L \le \frac{L}{1-\rho_I}.$$

If $\varphi^{(n)}(\lambda_1) = \varphi^{(n)}(\lambda_2)$, by (18) one has $\|\lambda_1 - \lambda_2\| \le \sum\frac{2L_r}{1-\rho_I}\rho_I^{-n}$. Then

$$\#(\lambda \in \Lambda: \varphi^{(n)}(\lambda) = \varphi^{(n)}(\lambda_1), \lambda_1 \in \Lambda \le \#\left(\Lambda \cap B\left(\lambda_1, \frac{2L}{1-\rho_I}\rho_I^{-n}\right)\right)$$
$$\le \sup_{x\in\mathbb{R}^d}\#\left(\Lambda \cap B\left(\lambda_1, \frac{2L}{1-\rho_I}\rho_I^{-n}\right)\right).$$

For any $n \in \mathbb{Z}$, by Corollary (6.3.10), (18) and Lemma (6.3.15), we have

$$Ap_I^{-n} \le \sum_{\lambda\in\Lambda}|\hat{\mu}(A_I^{nT}\lambda)|^2$$
$$= \sum_{\lambda'\in\Lambda}\sum_{\varphi^{(n)}(\lambda)=\lambda',\lambda\in\Lambda}\left|\hat{\mu}(\lambda' + \psi_n(\lambda))\right|^2$$
$$\le \sum_{\lambda'\in\Lambda}\sum_{\varphi^{(n)}(\lambda)=\lambda'}\max_{\|x\|\le\frac{L}{1-\rho_I}}|\hat{\mu}(x+\lambda')|^2$$
$$\le \sup_{x\in\mathbb{R}^d}\#\left(\Lambda \cap B\left(x, \frac{2L}{1-\rho_I}\rho_I^{-n}\right)\right)\sum_{\lambda'\in\Lambda}\max_{\|x\|\le\frac{L}{1-\rho_I}}|\hat{\mu}(x+\lambda')|^2$$
$$\le Be^{\left(2\pi\frac{L}{1-\rho_I}\right)^{2+dP^2}}\sup_{x\in\mathbb{R}^d}\#\left(\Lambda \cap B\left(x, \frac{2L}{1-\rho_I}\rho_I^{-n}\right)\right)$$

As $p_I = \rho_I^s$, one has

$$B_s^+(\Lambda) = \lim_{h\to\infty}\sup_{x\in\mathbb{R}^d}\frac{\#(\Lambda\cap B(x,h))}{h^s} \ge \lim_{h\to\infty}\sup\frac{\sup_{x\in\mathbb{R}^d}\frac{\#(\Lambda\cap B(x,h))}{h^s}}{\left(\rho_I^{-n}\frac{2L}{1-\rho_I}\right)^s}$$
$$= \frac{A}{be^{\left(2\pi\frac{L}{1-\rho_I}\right)^{2+dP^2}}\left(\frac{2L}{1-\rho_I}\right)^s}.$$

then $\dim_B \Lambda \ge s$. Hence $\dim_B \Lambda = s$ by Theorem (6.3.13).

For μ be a Borel probability measure with compact support in $\mathbb{R}^d$. We say that $\Lambda \subset \mathbb{R}^d$ is a spectrum of μ if $E_\Lambda = \{e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for $L^2(\mu)$, and in this case (μ, Λ) is called a spectral pair. Define

$$Q_{\mu,\Lambda}(\xi) = \sum_{\lambda\in\Lambda} |\mu(\xi+\lambda)|^2,$$

where

$$\hat{\mu}(\xi) = \int e^{-2\pi i\langle\xi,x\rangle} d\mu(x)$$

is the Fourier transformation of the measure μ. It is well known that (μ,Λ) is a spectral pair if and only if $Q_{\mu,\Lambda}(\xi) \equiv 1$ in $\mathbb{R}^d$ [432].

For b be an integer with $|b| \geq 2$ and let $D \subset \mathbb{Z}$ be a finite digit set with $\#D \geq 2$. Then they naturally generate an IFS $f_d(x) = \frac{1}{b}(x+d), d \in D$, and the self-similar set or sequence $T(b,D)$, where $T(b,D)$ can be expressed by

$$T(b,D) = \left\{\sum_{k=1}^{\infty} d_k b^{-k} : \text{all } d_k \in D\right\} := \sum_{k=1}^{\infty} b^{-k} D.$$

Moreover, for any probability weight $P = \{p_d\}_{d\in D}$, they generate the self-similar measure $\mu = \mu_{b,D,P}$, which is the unique probability measure with compact support $T(b,D)$ satisfying

$$\mu(E) = \sum_{d\in D} p_d \mu\left(f_d^{-1}(E)\right)$$

for any Borel set $E \subset \mathbb{R}$. The advantage of the measure $\mu_{b,D,P}$ is that

$$\hat{\mu}_{b,D,P}(\xi) = \prod_{k=1}^{\infty} M_{D,P}\left(\frac{\xi}{b^k}\right),$$

where $M_{D,P}(\xi) = \sum_{d\in D} p_d e^{-2\pi i d\xi_r}$ is the mask of the digit set D. Then it is easy to show that, for any $0 \neq a \in \mathbb{R}$, (a). $(\mu_{b,D,P},\Lambda)$ is a spectral pair if and only if $(\mu_{b,D-a,P},\Lambda)$ is a spectral pair; And (b). $(\mu_{b,r_\circ D,P},\Lambda)$ is a spectral pair if and only if $\left(\mu_{b,D,P},\frac{1}{r_\circ}\Lambda\right)$ is a spectral pair.

The usual and natural conditions to guarantee that $\mu_{b,D,P}$ is a spectral measure are the following (not necessary): (a) The probability weight P is equal probability weight by Laba and Wang conjecture [435]; And (b) The pair (b,D) is admissible, that is, there exists a finite set $C \subset \mathbb{Z}$ with $\#C = \#D$ such that the matrix $\left[\frac{1}{\sqrt{\#C}} e^{2\pi i\frac{dc}{b}}\right]_{d\in D, c\in C}$ is unitary (usually $(b^{-1}D, C)$ is called a compatible pair). For the sake of brevity we denote $\mu_{b,D}$ be the measure $\mu_{b,D,P}$ with equal weight probability.

To prove Theorem (6.3.19), we need the following two lemmas. The first one was proved in [424, Theorem 2.4] and the second one was proved in [435, Theorem (6.3.2)].

Lemma (6.3.17)[415]: If (b,D) is admissible and $\mu_{b,D,P}$ admits a frame spectral measure, then all p_i must be equal.

Lemma (6.3.18)[415]: Let $(b^{-1}D,S)$ be a compatible pair. Then the self-similar measure $\mu_{b,D}$ is a spectral measure. If moreover $|b| > 2, \gcd(D-D) = 1, 0 \in S$ and $S \subset [2-|b|, |b|-2]$, then $\Lambda(b,S)$ is a spectrum for $\mu_{b,D}$, where $\Lambda(b,S) = \left\{\sum_{i=0}^{n} b^i s_i : n \geq 1, s_i \in S\right\} := \sum_{k=0}^{\infty} b^k S$.

Theorem (6.3.19)[415]: Let (b,D) be admissible and let $P = \{p_d\}_{d\in D}$ be a positive probability weight. Then the self-similar measure $\mu = \mu_{b,D,P}$ is a frame spectral measure if and only if P is an equal probability weight. In the frame spectral case, there exists a frame spectrum Λ such that

$$\dim_B \Lambda = \dim_H T(b,D) = s = \frac{\ln \#D}{\ln|b|},$$

where $T(b,D)$ is the self-similar set or sequence and s is the self-similar dimension.

Proof. By the hypotheses of Theorem (6.3.19) and according to Lemma (6.3.17) and Lemma (6.3.18), the first assertion follows. To show the second assertion, by Theorem (6.3.2), we need to find a frame spectrum Λ such that $\dim_B \Lambda \geq \dim_H T(b,D)$.

According to the basic facts given before Theorem (6.3.19), without loss of generality we assume that $0 \in D, \gcd(D-D) = t$. Define $D' = \frac{1}{t}D$. It is easy to check that $(b^{-1}D', tS)$ is a compatible pair, and $(b^{-1}D', S^*)$ is also a compatible pair if $S^* \equiv tS \pmod b$. So we can choose S^* such that $0 \in S^*$ and $S^* \subset [2-|b|, |b|-2]$ and thus $\Lambda(b,S^*)$ is a spectrum of $\mu_{b,D'}$ if $|b| > 2$, which is equivalent to that $\Lambda\left(b, \frac{1}{t}S^*\right)$ is a spectrum of $\mu_{b,D}$.

Observe that $\dim_H T(b,D) = s = \frac{\ln \#D}{\ln|b|}$ given by Schief [437]. The proof is divided into two cases as follows.

Case I: Suppose $|b| = 2$. Then $\#D = 2$ by the admissible property of (b,D) and $\#D \geq 2$. Notice that $\mu_{2,\{0,1\}}$ (resp. $\mu_{-2,\{0,1\}}$) is the Lebesgue measure restriction on [0, 1](resp. $\left[-\frac{2}{3}, \frac{1}{3}\right]$) and thus $\left(\mu_{2,\{0,1\},\mathbb{Z}}\right)$ (resp. $(\mu_{-2,\{0,1\},\mathbb{Z}})$) is a spectral pair. Then $\left(\mu_{b,\{0,d\}}, \frac{1}{d}\mathbb{Z}\right)$ is a spectral pair and $\dim_B \frac{1}{d}\mathbb{Z} = 1 = \dim_H T(b,D)$.

Case II: Suppose $|b| > 2$. We need to show that $\dim_B \Lambda\left(b, \frac{1}{t}S^*\right) \geq \frac{\ln \#D}{\ln|b|}$. By Theorem (6.3.16), it suffices to show that

$$\sup_{\lambda \in \Lambda\left(b,\frac{1}{t}S^*\right)} \inf_{s \in \Lambda\left(b,\frac{1}{t}S^*\right)} \|b^{-1}\lambda - s\| < \infty \tag{5.1}$$

Since $\Lambda(b,S^*) = S^* + bS^* + \cdots$ and $0 \in S^*$, then for any $\lambda \in \Lambda\left(b, \frac{1}{t}S^*\right)$ we have $\lambda = \frac{1}{t}(s_1 + bs_2 + \cdots + b^k s_{k+1})$ for some $s_i \in S^*, 1 \leq i \leq k+1$. This implies that

$$\inf_{s \in \Lambda\left(b,\frac{1}{t}S^*\right)} \|b^{-1}\lambda - s\| \leq \left\|b^{-1}\lambda - \frac{1}{t}(s_2 + bs_3 \cdots + b^{k-1}s_{k+1})\right\| \leq \|b^{-1}s_1\|$$

$$\leq |b|^{-1} \max_{s \in S^*}|s| \leq \frac{|b|-2}{|b|}.$$

Hence (19) follows and we complete the proof.

Example (6.3.20)[415]: Let $\left\{\varphi_i(x) = \frac{1}{4}(x+i-1)\right\}_{i=1}^{4}$ be an IFS generated by $b = 4$ and $D = \{0,1,2,3\}$ and let $P = \left\{\frac{p}{2}, \frac{p}{2}, \frac{q}{2}, \frac{q}{2}\right\}$ be a probability weight $(p+q = 1, p, q > 0)$. Suppose $\mu = \mu_{4,\{0,1,2,3\},P}$ is the self-similar measure with respect to the IFS and the probability weight. Then μ is a frame spectral measure if and only if $p = q = \frac{1}{2}$. Moreover, if $p \neq q$, then

$$\Lambda = \{0,2\} + 4\{0,2\} + 4^2\{0,2\} + \cdots \text{ (all finite sums)}$$

is a Bessel set or sequence with $\dim_B \Lambda = \frac{1}{2} < \dim_H T(4,\{0,1,2,3\}) = 1$ for μ.

Proof. Let $b = 4, D = \{0, 1, 2, 3\}$. It is easy to see that $(b^{-1}D, D)$ is a compatible pair and thus (b, D) is admissible. According to Theorem (6.3.19), μ is a frame spectral measure if and only if $p = q = \frac{1}{2}$.

Moreover, if $p \neq q$, notice that

$$\delta_{4^{-k}\{0,1,2,3\},P} = \delta_{4^{-k}\{0,2\},\{p,q\}} * \delta_{4^{-k}\{0,1\}},$$

where $\delta_{4^{-k}\{0,1,2,3\},P} = \frac{p}{2}\left(\delta_0 + \delta_{4^{-k}}\right) + \frac{q}{2}\left(\delta_{2\cdot 4^{-k}} + \delta_{3\cdot 4^{-k}}\right)$ and $\delta_{4^{-k}\{0,2\},\{p,q\}} = p\delta_0 + q\delta_{2\cdot 4^{-k}}$. Then one has

$$\begin{aligned}\mu &= \delta_{4^{-1}\{0,1,2,3\},P} * \delta_{4^{-2}\{0,1,2,3\},P} * \cdots \\ &= \mu_{4,\{0,2\},\{p,q\}} * \mu_{4,\{0,1\}}.\end{aligned}$$

Since $\Lambda_r = \{0, 2\} + 4\{0, 2\} + \cdots$ (all finite sum) is a spectrum for $\mu_{4,\{0,1\}}$ [432], for any $f \in C(R)$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda}\left|\langle f, e_\lambda\rangle_\mu\right|^2 &= \sum_{\lambda\in\Lambda}\left|\iint \langle f(x+y), e^{2\pi i\lambda(x+y)}\rangle_{d\mu_{4,\{0,2\},\{p,q\}}}(x)\, d\mu_{4,\{0,1\}}(y)\right|^2 \\ &= \sum_{\lambda\in\Lambda}\left|\int \langle f(x+y), e^{2\pi i\lambda x}\rangle_{\mu_{4,\{0,2\},\{p,q\}}} e^{2\pi i\lambda y}\, d\mu_{4,\{0,1\}}(y)\right|^2 \\ &= \int \left|\langle f(x+y), e^{2\pi i\lambda x}\rangle_{\mu_{4,\{0,2\},\{p,q\}}}\right|^2 d\mu_{4,\{0,1\}}(y) \\ &\leq \iint |f(x+y)|^2 d\mu_{4,\{0,2\},\{p,q\}}(x)\, d\mu_{4,\{0,1\}}(y) \\ &= \|f\|_{L^2(\mu)},\end{aligned}$$

where the third equality holds since Λ is a spectrum for $\mu_{4,\{0,1\}}$. Hence, Λ is a Bessel set or sequence for μ.

By Theorem (6.3.2), we have

$$\dim_B \Lambda \leq \dim_H T(4, \{0, 1\}) = \frac{1}{2}.$$

On the other hand, let $\Lambda_n = \sum_{i=0}^{n} 4^i\{0, 2\}$. Then $\Lambda_n \subseteq \left[0, \frac{2}{3}(4^{n+1} - 1)\right]$ with $\#\Lambda_n = 2^{n+1}$.

Thus, we have

$$\begin{aligned}B^+_{\frac{1}{2}}(\Lambda) &= \overline{\lim_{h\to\infty}} \sup_{x\in\mathbb{R}} \frac{\#\big(\Lambda \cap B(x, h)\big)}{h^{\frac{1}{2}}} \\ &\geq \overline{\lim_{n\to\infty}} \frac{\#\Lambda \cap \left(-\frac{2}{3}4^{n+1}, \frac{2}{3}4^{n+1}\right)}{\left(\frac{2}{3}4^{n+1}\right)^{\frac{1}{2}}} \\ &\geq \lim_{n\to\infty} \frac{2^{n+1}}{\left(\frac{2}{3}4^{n+1}\right)^{\frac{1}{2}}} = \left(\frac{3}{2}\right)^{\frac{1}{2}} > 0.\end{aligned}$$

It follows that $\dim_B(\Lambda) \geq \frac{1}{2}$. Hence, $\dim_B(\Lambda) = \frac{1}{2}$.

Note that E_Λ is an orthogonal family of μ if and only if $\hat{\mu}(\lambda - \lambda') = 0$ for any $\lambda \neq \lambda' \in \Lambda$, which is equivalent to that

$$(\Lambda - \Lambda) \setminus \{0\} \subset \{\xi : \hat{\mu}(\xi) = 0\} := Z(\hat{\mu}). \tag{20}$$

Example (6.3.21)[415]: Let $\mu = \mu_{4,\{0,2\}}$ be the Cantor measure on $T = T(4,\{0,2\})$ with equal weight. Let p, q be two different positive numbers satisfying $p + q = 1$ with $p \neq q$. Suppose that $\nu(E) = p\mu(E) + q\mu(E-2)$ for any Borel set $E \subset \mathbb{R}$, then ν is a non-spectral frame measure.

Proof. It is easy to see that $\nu = \mu * \eta$ where $\eta = p\delta_0 + q\delta_2$. Suppose ν is a spectral measure with a spectrum Λ_r. Since E_Λ is an orthogonal family in $L^2(\nu)$, by (20), one has $(\Lambda - \Lambda)\setminus\{0\} \subset Z(\hat{\nu})$. Note that $Z(\hat{\nu}) = Z(\hat{\mu}) \cup Z(\hat{\eta})$. By a simple calculation, we know $Z(\hat{\eta}) = \emptyset$. Then $(\Lambda - \Lambda)\setminus\{0\} \subset Z(\hat{\mu})$. By (20) again, one sees that E_Λ is an orthogonal family in $L^2(\mu)$. By Lemma 2.2 in [420], Λ must be not a spectrum of ν. This is a contradiction. Then ν is non-spectra. And by Theorem (6.3.19) in [430], one has ν is a frame measure.

Corollary (6.3.22)[439]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an IFS on $\mathbb{R}^d$ satisfying the OSC by a bounded open set or sequence V_r. Let μ be the invariant measure with respect to the IFS and a probability weight P. Then the IFS satisfies the SOSC with the open set or sequence V_r if and only if $\mu(\partial V_r) = 0$.

Proof. Observe that the attractor T_r of the IFS satisfies that $T_r \subset \bar{V}_r$. Then $\sum \mu(V_r) = \sum \mu(\overline{V_r}) = 1$ by $\sum \mu(\partial V_r) = 0$ and the sufficiency follows. Conversely, let $x \in T_r \cap V_r$ by the SOSC. Then there exists $\delta > 0$ such that the open ball $B(x,\delta) \subset V_r$. By the contraction of functions in the IFS, there exist an integer $m \geq 1$ and $I \in \Theta_N^m$ such that $\sum \varphi_I(T_r) \subset B(x,\delta) \subset \sum V_r$. Iterating (10) m −times, we obtain that

$$\mu(\cdot) = \sum_{I \in \Theta_N^m} P_I \mu\left(\varphi_I^{-1}(\cdot)\right)$$

and $\{\varphi_I(x)\}_{I \in \Theta_N^m}$ is also an IFS satisfying the SOSC with the same open set or sequence V_r. Therefore, for the simpler notations we can assume that $\sum \varphi_1(T_r) \subset \sum V_r$ without loss of generality.

We notice that, for $n \geq 1$ and $I = i_1 i_2 \cdots i_n \in \{1,2,\ldots,N\}^n \setminus \{2,3,\ldots,N\}^n := \Theta_N^n \setminus \Theta_{2,N}^n$, there exists $k, 1 \leq k \leq n$, such that $i_k = 1$. Then

$$\sum \varphi_I(T_r) \subset \sum \varphi_{i_1 i_2 \cdots i_k}(T_r) \subset \sum \varphi_{i_1 i_2 \cdots i_{k-1}}(V_r) \subset \sum V_r.$$

Hence, by (12) we have

$$\begin{aligned}
1 \geq \sum \mu(V_r) &= \sum_{I \in \Theta_N^n} \sum P_I \mu\left(\varphi_I^{-1}(V_r)\right) \geq \sum_{I \in \Theta_N^n \setminus \Theta_{2,N}^n} \sum P_I \mu\left(\varphi_I^{-1}(V_r)\right) \\
&\geq \sum_{I \in \Theta_N^n \setminus \Theta_{2,N}^n} \sum P_I \mu\left(\varphi_I^{-1}\left(\varphi_I(T_r)\right)\right) = \sum_{I \in \Theta_N^n \setminus \Theta_{2,N}^n} P_I \\
&= \sum_{I \in \Theta_N^n} P_I - \sum_{I \in \Theta_{2,N}^n} P_I \\
&= 1 - \left(\sum_{i=2}^N p_i\right)^n.
\end{aligned}$$

Consequently, $\sum \mu(V_r) = 1$ and $\sum \mu(\partial V_r) = \sum \mu(\overline{V_r}) - \sum \mu(V_r) = 0$.

Corollary (6.3.23)[439]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open mapping IFS satisfying the OSC with a bounded open set or sequence V_r. Let μ be the invariant measure with respect to the IFS and a probability weight P. If $\sum \mu(\partial V_r) = 0$, then the μ satisfies the no-overlap condition.

Proof. By the OSC, one sees that $\sum \varphi_i(V_r) \cap \sum \varphi_j(V_r) = \emptyset$ for any $i \neq j$. Since all φ_i are open mappings, then $\sum \varphi_i(V_r)$ is an open set for each i, and we have

$$\sum \varphi_i(V_r) \cap \sum \varphi_j(\overline{V_r}) = \sum \varphi_i(V_r) \cap \sum \overline{\varphi_j(V_r)} = \emptyset.$$

Then

$$\sum V_r \cap \varphi_i^{-1}\left(\varphi_j(\partial V_r)\right) \subset \varphi_i^{-1} \sum \left(\varphi_i(V_r) \cap \varphi_j(\partial V_r)\right) \subset \sum \varphi_i^{-1}\left(\varphi_i(V_r) \cap \varphi_j(V_r)\right) = \emptyset.$$

Consequently,

$$\sum \overline{V_r} \cap \varphi_i^{-1}\left(\varphi_j(\partial V_r)\right) \subset \sum \partial V_r$$

and by the fact the attractor $T_r \subset \overline{V}_r$ one has

$$\sum \mu(\varphi_i^{-1}\left(\varphi_j(\partial V_r)\right) = 0.$$

Using (13) again, we have

$$\sum \varphi_i(T_r) \cap \varphi_j(T_r) \subset \sum \varphi_i(\overline{V_r}) \cap \varphi_j(\overline{V}_r) \subset \sum \varphi_i(\partial V_r) \cap \varphi_j(\partial V_r)$$

for $i \neq j$. Then, by (14) and (15),

$$\sum \mu\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right) \leq \sum \mu\left(\varphi_i(\partial V_r) \cap \varphi_j(\partial V_r)\right) = \sum_{s=1}^{N} \sum p_s \mu\,(\varphi_s^{-1}\left(\varphi_i(\partial V_r) \cap \varphi_j(\partial V_r)\right) \leq 0.$$

Hence the μ satisfies the no-overlap condition.

Corollary (6.3.24)[439]: Let $\{\varphi_i(x)\}_{i=1}^{N}$ be an IFS on $\mathbb{R}^d$. Let μ be the invariant measure with respect to the IFS and a probability weight P. Then μ satisfies the no-overlap condition if and only if $\sum \mu\left(\varphi_i^{-1}\left(\varphi_j(T_r)\right)\right) = 0$ for any $i \neq j$.

Proof. Suppose that $\sum \mu\left(\varphi_i^{-1}\left(\varphi_j(T_r)\right)\right) = 0$ for any $i \neq j$. Then

$$\begin{aligned}\sum \mu\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right) &= \sum_{k=1}^{N} \sum p_k \mu\left(\varphi_k^{-1}\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right)\right) \\ &\leq \sum_{1\leq k\leq N, k\neq i,j} \sum p_k\, \mu\left(\varphi_k^{-1}(\varphi_i(T_r))\right) + \sum p_j \mu\left(T_r \cap \varphi_j^{-1}(\varphi_i(T_r))\right) \\ &\quad + \sum p_i \mu\left(T_r \cap \varphi_i^{-1}\left(\varphi_j(T_r)\right)\right) = 0.\end{aligned}$$

That is, μ satisfies the no-overlap condition. Conversely, suppose that $\sum \mu\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right) = 0$ for any $i \neq j$. Then

$$0 = \sum \mu\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right) = \sum_{k=1}^{N} \sum p_k \mu\left(\varphi_k^{-1}\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right)\right).$$

In particular, we have

$$0 = \sum \mu\left(\varphi_i^{-1}\left(\varphi_i(T_r) \cap \varphi_j(T_r)\right)\right) = \sum \mu\left(T_r \cap \varphi_i^{-1}\left(\varphi_j(T_r)\right)\right) = \sum \mu\left(\varphi_i^{-1}\left(\varphi_j(T_r)\right)\right).$$

Hence, we complete the proof.

Corollary (6.3.25)[439]: Let $\{\varphi_i(x)\}_{i=1}^N$ be an open and bijective mapping IFS in $\mathbb{R}^d$ satisfying the SOSC and let T_r be its attractor. Let μ be the invariant measure with respect to the IFS and a probability weight $P = \{p_i\}_{i=1}^N$. Then

(i) $\sum \mu((T_r)_I) = P_I$ for $I \in \Theta_N^*$;

(ii) Let $f_r \in L^1(T_r, \mu)$ and $I \in \Theta_N^*$. Then

$$\int_{(T_r)_I} \sum f_r \circ \varphi_I^{-1}(x) d\mu = P_I \int_{T_r} \sum f_r \; d\mu;$$

(iii) Let $f_r \in L^1(T_r, \mu)$ and $I \in \Theta_N^*$. Then

$$\int_{T_r} \sum f_r \circ \varphi_I(x) d\mu = P_I^{-1} \int_{(T_r)_I} \sum f_r \; d\mu.$$

Proof. (i) For $I = i_1 i_2 \cdots i_n \in \Theta_N^n$ and $n \geq 1$, we have

$$\sum \mu(T_r)_I = \sum_{i=1}^N \sum p_i \mu\left(\varphi_i^{-1}(T_r)_I\right) = \sum p_i \mu\left(\varphi_i^{-1}\left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T_r)\right)\right).$$

Note that $\sum \varphi_i^{-1}\left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T_r)\right) \subset \sum \varphi_i^{-1} \circ \varphi_{i_1}(T_r)$. By Theorem (6.3.4) and by induction, we have

$$\sum \mu(T_r)_I = \sum p_{i_1} \mu\left(\varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(T_r)\right) = \sum p_{i_1} p_{i_2} \cdots p_{i_n} \mu(T_r) = P_I.$$

(ii) By (10) we have

$$\int_{(T_r)_I} \sum f_r \circ \varphi_I^{-1}(x) d\mu$$

$$= \sum_{i=1}^N p_i \int \sum \chi_{(T_r)_I}(\varphi_i(x)) f_r \circ \varphi_{i_n}^{-1} \circ \varphi_{i_{n-1}}^{-1} \circ \cdots \circ \varphi_{i_2}^{-1}\left(\varphi_{i_1}^{-1} \circ \varphi_i(x)\right) d\mu,$$

where $\chi_E(x)$ is the characteristic function of $E \subset \mathbb{R}^d$. Note that $\chi_{(T_r)_I}(\varphi_i(x)) = 1$ if and only if $x \in \varphi_i^{-1} \circ \left(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \varphi_{i_n}\right)(T_r)$. Then, by Theorem (6.3.4) again,

$$\int_{(T_r)_I} \sum f_r \circ \varphi_I^{-1}(x) d\mu = p_{i_1} \int \sum \chi_{\varphi_{i_1}^{-1}(T_r)_I}(x) f_r \circ \varphi_{i_n}^{-1} \circ \varphi_{i_{n-1}}^{-1} \circ \cdots \circ \varphi_{i_2}^{-1}(x) d\mu$$

$$= p_{i_1} p_{i_2} \cdots p_{i_n} \int \sum \chi_{\varphi_{i_n}^{-1} \circ \varphi_{i_{n-1}}^{-1} \circ \cdots \circ \varphi_{i_1}^{-1}(T_r)_I}(x) \, f_r(x) d\mu$$

$$= P_I \int_{T_r} \sum f_r(x) d\mu.$$

(iii) Firstly,

$$P_I^{-1} \int_{(T_r)_I} \sum f_r(x) \, d\mu(x) = P_I^{-1} \int_{(T_r)_I} \sum f_r \circ \varphi_I \circ \varphi_I^{-1}(x) d\mu(x).$$

Secondly, by (ii), we get

$$P_I^{-1} \int_{(T_r)_I} \sum f_r \circ \varphi_I \circ \varphi_I^{-1}(x) d\mu(x) = \int_{(T_r)_I} \sum f_r \circ \varphi_I(x) d\mu(x).$$

Hence, the assertion (iii) follows.

Corollary (6.3.26)[439]: Let $\{\varphi_i(x)\}_{i=1}^{N}$ be an open and bijective mapping IFS satisfying the SOSC and let T_r be its attractor. Let μ be the invariant measure with respect to the IFS and a positive probability weight $P=\{p_i\}_{i=1}^{N}$. Suppose that the $(E_r)_{\Lambda_r}$ is a Fourier frame with lower and upper bounds A and $A+\varepsilon(\varepsilon\geq o)$. Then

(i) For $I\in\Theta_N^n, n\geq 1, \mu\circ\varphi_I$ is a Fourier frame spectral measure with lower and upper bounds AP_I and $(A+\varepsilon)P_I$ respectively, i.e.,

$$AP_I\left\|\sum f_r\right\|_{\mu\circ\varphi_I}^2\leq\sum_{\lambda\in\Lambda_r}\sum\left|\langle f_r,e_\lambda\rangle_{\mu\circ\varphi_I}\right|^2$$

(ii) For $I\in\Theta_N^n, n\geq 1, \mu\circ\varphi_I$ is a Fourier frame spectral measure with lower and upper bounds $P_I^{-1}A$ and $P_I^{-1}(A+\varepsilon)$ respectively, i.e.,

$$AP_I^{-1}\left\|\sum f_r\right\|_{\mu\circ\varphi_I^{-1}}^2\leq\sum_{\lambda\in\Lambda_r}\sum\left|\langle f_r,e_\lambda\rangle_{\mu\circ\varphi_I^{-1}}\right|^2$$
$$\leq(A+\varepsilon)P_I^{-1}\left\|\sum f_r\right\|_{\mu\circ\varphi_I^{-1}}^2, \forall f_r\in L(T_r,\mu\circ\varphi_I^{-1});$$

where $\sum(T_r)_I=\sum\varphi_I(T_r)$.

Proof. Observe that

$$\sum\langle f_r,e_\lambda\rangle_{\mu\circ\varphi_I}=\int_{T_r}\sum f_r(x)\,e^{2\pi\langle\lambda,x\rangle}d\mu\circ\varphi_I(x)$$
$$=\int_{(T_r)_I}\sum f_r\left(\varphi_I^{-1}(x)\right)e^{2\pi\langle\lambda,\varphi_I^{-1}(x)\rangle}d\mu(x)$$
$$=P_I\int_{T_r}\sum f_r(x)\,e^{2\pi\langle\lambda,x\rangle}d\mu(x)\ \text{(by Theorem (6.3.8)(ii))}$$
$$=P_I\left(\sum\langle f_r,e_\lambda\rangle_u\right)$$

and by similar calculations,

$$\left\|\sum f_r\right\|_{\mu\circ\varphi_I}^2=P_I\left\|\sum f_r\right\|_{\mu}^2.$$

Then (i) follows. To prove (ii), similarly, we have $\sum\langle f_r,e_\lambda\rangle_{\mu\circ\varphi_I^{-1}}=\sum P_I^{-1}\langle\chi_{(T_r)_I}f_r,e_\lambda\rangle_\mu$ and $\left\|\sum f_r\right\|_{\mu\circ\varphi_I^{-1}}^2=P_I^{-1}\sum\left\|\chi_{(T_r)_I}f_r\right\|_{\mu}^2$. Then (ii) follows.

Corollary (6.3.27)[439]: With the same hypotheses given in Theorem (6.3.9) and in addition, we assume that $\varphi_i(x)=R_i(x+d_i)$ is a contracting affine mapping for $1\leq i\leq N$. Then

$$AP_I^{-1}\leq\sum_{\lambda\in\Lambda_r}\sum\left|\mu\left(R_I^{T_r}(\lambda-x)\right)\right|^2\leq(A+\varepsilon)P_I^{-1}, \text{for } I\in\Theta_N^* \text{ and } x\in\mathbb{R}^d.$$

Proof. Recall $\varphi_I(x)=\varphi_{i_1}\circ\varphi_{i_2}\circ\cdots\circ\varphi_{i_n}(x)$. Thus $\varphi_I(x)=R_I(x+d_I)$ where $R_I=R_{i_1}R_{i_2}\cdots R_{i_n}, d_I=d_{i_n}+R_{i_n}^{-1}d_{i_{n-1}}+\cdots+R_{i_n}^{-1}R_{i_{n-1}}^{-1}\cdots R_{i_2}^{-1}d_{i_1}$ and $\varphi_I^{-1}(x)=R_I^{-1}x-d_I$. Choose $f_r(x)=e_y(x)=e^{-2\pi i\langle y,x\rangle}$ for $y\in\mathbb{R}^d$. Then

$$\langle e_y, e_\lambda \rangle_{\mu\circ\varphi_I^{-1}} = \int\limits_{(T_r)_I} \sum e^{2\pi\langle \lambda - y, x\rangle i} \ d\mu \circ \varphi_I^{-1}(x)$$
$$= \int\limits_{T_r} \sum e^{2\pi\langle \lambda - y, \varphi_I(x)\rangle i} \ d\mu$$
$$= e^{2\pi i\langle \lambda - y, R_I d_I\rangle} \int\limits_{T_r} \sum e^{2\pi i\langle R_I^{T_r}(\lambda - y), x\rangle} d\mu(x)$$
$$= e^{2\pi i\langle \lambda - y, R_I d_I\rangle} \hat{\mu}\left(\sum \left(R_I^{T_r}(y - \lambda)\right)\right)$$
and $\|e_y\|^2_{\mu\circ\varphi_I^{-1}} = 1$. Hence, the assertion follows by Theorem (6.3.9)(ii).

Corollary (6.3.28)[439]: Let $\Phi = \{\varphi_i(x) = \rho_i Q_i(x + d_i)\}_{i=1}^N$ be a self-similar IFS satisfying the OSC. Let $\mu = \mu_{\Phi,P}$ be the self-similar measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^N$. Suppose that Λ_r is a Bessel set or sequence for μ. Then

(iii) $\dim_B(\sum \Lambda_r) \le \dim_H(\sum T_r)$, where T_r is the self-similar set or sequence;

(iv) if there exists a Bessel set or sequence Λ_r such that $\dim_B(\sum \Lambda_r) = \dim_H(\sum T_r) = s$, then all $p_i = \rho_i^s$, where s is the self-similar dimension.

Proof. It is known that $\dim_H(\sum T_r) = s$ by Schief [437]. We first prove (i). And (ii) follows easily.

Since $\hat{\mu}(0) = 1$, there exist $\delta > 0$ such that
$$\sum |\hat{\mu}(\xi_r)|^2 \ge \frac{1}{2}, \text{for } |\xi_r| \le \delta.$$
Let p be an integer such that $\rho_{\max}^p := \max\limits_{1\le i\le N} \rho_i^p \le \delta$. Suppose that there exists i such that $p_i \neq \rho_i^s$. Then there exists k such that $p_k > \rho_k^s$ by $\sum_{i=1}^N p_i = \sum_{i=1}^N \rho_i^s = 1$. Choose $\varepsilon > 0$ satisfying $p_k \ge (1+\varepsilon)\rho_k^s$. Let $h > 1$ be arbitral, then there exists a natural number n satisfying that $\rho_k^{-n+1} \le h < \rho_k^{-n}$. For any $\lambda \in \Lambda_r \cap B(x,h)$, we have $\lambda - x \in B(0,h)$ and thus
$$\left(\rho_k Q_k^{T_r}\right)^{n+p}(\lambda - x) \in B\left(0, \rho_k^{n+p} h\right) \subset B(0,\delta).$$
Consequently,
$$\sum \frac{\#\left(\Lambda_r \cap B(x,h)\right)}{2} \le \sum_{\lambda\in\Lambda_r\cap B(x,h)} \sum \left|\hat{\mu}\left(\left(\rho_k Q_k^{T_r}\right)^{n+p}(\lambda - x)\right)\right|^2$$
$$\le \sum_{\lambda\in\Lambda_r} \sum \left|\hat{\mu}\left(\left(\rho_k Q_k^{T_r}\right)^{n+p}(\lambda - x)\right)\right|^2 \text{ (by Corollary (6.3.10))}$$
$$\le B p_k^{-(n+p)} \le B p_k^{-p}(1+\varepsilon)^{-n}\rho_k^{-ns}.$$
Hence, we obtain that
$$\sum \frac{\#\left(\Lambda_r \cap B(x,h)\right)}{h^{s+\ln(1+\varepsilon)/\ln\rho_k}} \le 2B p_k^{-p} \frac{(1+\varepsilon)^{-n}\rho_k^{-ns}}{h^{s+\ln(1+\varepsilon)/\ln\rho_k}} \le B p_k^{-p}\rho_k^{-ns},$$
where the second inequality follows from $h \ge \rho_k^{-n+1}$ and $s + \ln(1+\varepsilon)/\ln\rho_k \ge 0$ (since $p_k \ge (1+\varepsilon)\rho_k^s$).

Therefore
$$B^+_{s+\ln(1+\varepsilon)/\ln\rho_k} \sum(\Lambda_r) \le 2B p_k^{-p}\rho_k^{-s} \text{ and } \dim_B \sum \Lambda_r \le s + \ln(1+\varepsilon)/\ln\rho_k < s.$$

Clearly, when all $p_i = \rho_i^s$, then $\dim_B(\sum \Lambda_r) \le s$ by the same idea (choose $\varepsilon = 0$ in the previous proof) and (i) follows. Then (ii) follows easily by (i).

Corollary (6.3.29)[439]: Let $\Lambda_r = \{\lambda_n\}_{n=0}^{\infty}$ be a Bessel set or sequence of μ with compact support in $[-P, P]^d$ and Bessel bound $(A+\varepsilon)$. If there exists L_r such that $|\lambda_n - \gamma_n| \le L_r$ for $n \ge 0$, then $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ is also a Bessel set or sequence of μ with a Bessel bound $(A+\varepsilon)e^{d(2\pi L_r)^2 + dP^2}$.

Proof. It is sufficient to show that all $\gamma_n = \left(\gamma_1^{(n)}, \dots, \gamma_d^{(n)}\right)$ differs $\lambda_n = \left(\lambda_1^{(n)}, \dots, \lambda_d^{(n)}\right)$ only on the first component, and the statement follows by induction on the number of components.

It is easy to see that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum \left|\langle f_r(x), e^{-2\pi i \gamma_n \cdot x}\rangle\right|^2 &= \sum_{n=0}^{\infty}\sum \left|\langle f_r(x), e^{2\pi i(\gamma_n - \lambda_n)\cdot x}, e^{-2\pi i \lambda_n \cdot x}\rangle\right|^2 \\
&= \sum_{n=0}^{\infty}\sum \left|\langle f_r(x), e^{2\pi i\left(\gamma_1^{(n)} - \lambda_1^{(n)}\right)\cdot x}, e^{-2\pi i \lambda_n \cdot x}\rangle\right|^2 \\
&= \sum_{n=0}^{\infty}\left|\sum_{n=0}^{\infty} \frac{\left(2\pi i\left(\gamma_1^{(n)} - \lambda_1^{(n)}\right)\right)^k}{k!} \sum \langle f_r(x) x_1^k, e^{-2\pi i \lambda_n \cdot x}\rangle\right|^2 \\
&\le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum \frac{(2\pi L_r)^{2k}}{k!} \sum_{n=0}^{\infty} \frac{\left|\langle f_r(x) x_1^k, e^{-2\pi i \lambda_n \cdot x}\rangle\right|^2}{k!} \\
&\le \sum e^{(2\pi L_r)^2} \sum_{n=0}^{\infty} \frac{(A+\varepsilon)\left\|f_r(x) x_1^k\right\|^2}{k!} \\
&\le (A+\varepsilon) \sum e^{(2\pi L_r)^2 + P^2} \sum \|f_r\|^2,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality at the fourth line above. Hence, the assertion follows.

Corollary (6.3.30)[439]: Let μ be a Bessel measure with a Bessel set or sequence Λ_r, Bessel bound $A+\varepsilon$ and support in $[-P, P]^d$. Then, for any $\varepsilon \ge 0$,

$$\sum_{\lambda \in \Lambda_r} \max_{\|x\| \le (1+\varepsilon)} |\hat{\mu}(x+\lambda)|^2 \le (A+\varepsilon)e^{d\left(2\pi(1+\varepsilon)\right)^2 + dP^2}$$

Proof. Since $\hat{\mu}(\xi_r)$ is a continuous function in $\mathbb{R}^d$, then, for each $\lambda \in \Lambda_r$, there exists $(1+\varepsilon)_\lambda$ such that

$$\max_{\|x\| \le 1+\varepsilon} |\hat{\mu}(x + \lambda)| = |\hat{\mu}((1+\varepsilon)_\lambda)|$$

and $|\lambda - (1+\varepsilon)_\lambda| \le (1+\varepsilon)$. According to Lemma (6.3.14), $\{(1+\varepsilon)_\lambda\}_{\lambda \in \Lambda_r}$ is also a frame spectrum of μ. Then

$$\sum_{\lambda \in \Lambda_r} \max_{\|x\| \le (1+\varepsilon)} |\hat{\mu}(x+\lambda)|^2 = \sum_{\lambda \in \Lambda_r} |\hat{\mu}((1+\varepsilon)_\lambda)|^2 \le (A+\varepsilon)e^{d\left(2\pi(1+\varepsilon)\right)^2 + dP^2}.$$

Corollary (6.3.31)[439]: Let $\Phi = \{\varphi_i(x) = A_i(x + d_i)\}_{i=1}^{N}$ be a self-similar IFS satisfying the OSC. Let $\mu = \mu_{\Phi,P}$ be the self-similar measure with respect to the IFS and a positive probability weight $P = \{p_i\}_{i=1}^{N}$. Suppose that Λ_r is a frame spectrum of μ. If there exist a constant L_r and $I \in \Theta_N^n$ such that

$$\sup_{\lambda\in\Lambda_r}\inf_{\gamma\in\Lambda_r}\sum\left\|A_I^{T_r}\lambda-\gamma\right\| \le \sum L_r.$$

Then $\dim_B(\sum \Lambda_r) = \dim_H(\sum T_r) = s$ if and only if $p_I = \rho_I^s$, where T_r is the self-similar set or sequence, s is the self-similar dimension and ρ_i is the ratio of A_i for $1 \le i \le N$.

Proof. The necessity follows by Theorem (6.3.13) (ii). We need to prove the sufficiency. Let the lower and upper frame bounds of the frame E_{Λ_r} are $A, A+\varepsilon$ respectively. It is well known that Λ_r is relative uniformly discrete and thus there exists unique $\beta \in \Lambda_r$ dependent of λ such that $\sum\left\|A_I^{T_r}\lambda-\beta\right\| = \inf_{\gamma\in\Lambda_r}\sum\left\|A_I^{T_r}\lambda-\gamma\right\|$.

For any $\lambda \in \Lambda_r$, we define two mappings $\varphi: \Lambda_r \to \Lambda_r$ and $\psi: \Lambda_r \to B(0, L_r)$ by

$$\sum\left\|A_I^{T_r}\lambda-\varphi(\lambda)\right\| = \inf_{\gamma\in\Lambda_r}\sum\left\|A_I^{T_r}\lambda-\gamma\right\|, \psi(\lambda) = \sum A_I^{T_r}\lambda-\varphi(\lambda).$$

Then

$$\sum A_I^{T_r}\lambda = \varphi(\lambda)+\psi(\lambda)$$

Iterating (17) n times, we have

$$\sum A_I^{nT_r}\lambda = \varphi^{(n)}(\lambda)+\psi_n(\lambda),$$

Where $\psi_n(\lambda) = A_I^{n-1}\psi(\lambda) + A_I^{n-2}\psi\big(\varphi(\lambda)\big) + \cdots + \psi\left(\varphi^{(n-1)}(\lambda)\right)$. Then

$$\begin{aligned}\|\psi_n(\lambda)\| &\le \|A_I^{n-1}\psi(\lambda)\| + \left\|A_I^{n-2}\psi\big(\varphi(\lambda)\big)\right\| + \cdots + \left\|\psi\left(\varphi^{(n-1)}(\lambda)\right)\right\| \\ &\le \sum_{i=0}^{n-1}\sum \rho_I^{-i} L_r \le \sum\frac{L_r}{1-\rho_I}.\end{aligned}$$

If $\varphi^{(n)}(\lambda_1) = \varphi^{(n)}(\lambda_2)$, by (18) one has $\|\lambda_1-\lambda_2\| \le \sum\frac{2L_r}{1-\rho_I}\rho_I^{-n}$. Then

$$\begin{aligned}\#(\lambda\in\Lambda_r: \varphi^{(n)}(\lambda) = \varphi^{(n)}(\lambda_1), \lambda_1\in\Lambda_r &\le \#\sum\left(\Lambda_r\cap B\left(\lambda_1, \frac{2L_r}{1-\rho_I}\rho_I^{-n}\right)\right) \\ &\le \sup_{x\in\mathbb{R}^d}\sum\#\left(\Lambda_r\cap B\left(\lambda_1, \frac{2L_r}{1-\rho_I}\rho_I^{-n}\right)\right).\end{aligned}$$

For any $n \in \mathbb{Z}$, by Corollary (6.3.10), (18) and Lemma (6.3.15), we have

$$\begin{aligned}Ap_I^{-n} &\le \sum_{\lambda\in\Lambda_r}\sum\left|\hat{\mu}\left(A_I^{nT_r}\lambda\right)\right|^2 \\ &= \sum_{\lambda'\in\Lambda_r}\sum_{\varphi^{(n)}(\lambda)=\lambda',\lambda\in\Lambda_r}\left|\hat{\mu}\big(\lambda'+\psi_n(\lambda)\big)\right|^2 \\ &\le \sum_{\lambda'\in\Lambda_r}\sum_{\varphi^{(n)}(\lambda)=\lambda'}\max_{\|x\|\le\sum\frac{L_r}{1-\rho_I}}|\hat{\mu}(x+\lambda')|^2 \\ &\le \sup_{x\in\mathbb{R}^d}\sum\#\left(\Lambda_r\cap B\left(x, \frac{2L_r}{1-\rho_I}\rho_I^{-n}\right)\right)\sum_{\lambda'\in\Lambda_r}\max_{\|x\|\le\sum\frac{L_r}{1-\rho_I}}|\hat{\mu}(x+\lambda')|^2 \\ &\le (A+\varepsilon)e^{\left(2\pi\frac{L_r}{1-\rho_I}\right)^{2+dP^2}}\sup_{x\in\mathbb{R}^d}\sum\#\left(\Lambda_r\cap B\left(x, \frac{2L_r}{1-\rho_I}\rho_I^{-n}\right)\right)\end{aligned}$$

As $p_I = \rho_I^s$, one has

$$\sum B_s^+(\Lambda_r) = \lim_{h\to\infty} \sup_{x\in\mathbb{R}^d} \sum \frac{\#(\Lambda_r \cap B(x,h))}{h^s}$$
$$\geq \lim_{h\to\infty} \sup \sum \frac{\sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda_r \cap B(x,h))}{h^s}}{\left(\rho_I^{-n}\frac{2L_r}{1-\rho_I}\right)^s}$$
$$= \sum \frac{A}{(A+\varepsilon)e^{\left(2\pi\frac{L_r}{1-\rho_I}\right)^{2+dP^2}}\left(\frac{2L_r}{1-\rho_I}\right)^s}.$$
then $\dim_B(\sum \Lambda_r) \geq s$. Hence $\dim_B(\sum \Lambda_r) = s$ by Theorem (6.3.13).

Corollary (6.3.32)[439]: Let (b, D) be admissible and let $P = \{p_d\}_{d\in D}$ be a positive probability weight. Then the self-similar measure $\mu = \mu_{b,D,P}$ is a frame spectral measure if and only if P is an equal probability weight. In the frame spectral case, there exists a frame spectrum Λ_r such that
$$\dim_B\left(\sum \Lambda_r\right) = \dim_H\left(\sum T_r\,(b,D)\right) = s = \frac{\ln \#D}{\ln|b|},$$
where $T_r(b, D)$ is the self-similar set or sequence and s is the self-similar dimension.

Proof. By the hypotheses of Theorem (6.3.19) and according to Lemma (6.3.17) and Lemma (6.3.18), the first assertion follows. To show the second assertion, by Theorem (6.3.2), we need to find a frame spectrum Λ_r such that $\dim_B(\sum \Lambda_r) \geq \dim_H(\sum T_r\,(b, D))$.

According to the basic facts given before Theorem (6.3.19), without loss of generality we assume that $0 \in D, \gcd(D-D) = t$. Define $D' = \frac{1}{t}D$. It is easy to check that $(b^{-1}D', tS)$ is a compatible pair, and $(b^{-1}D', S^*)$ is also a compatible pair if $S^* \equiv tS \pmod b$. So we can choose S^* such that $0 \in S^*$ and $S^* \subset [2-|b|, |b|-2]$ and thus $\Lambda_r(b, S^*)$ is a spectrum of $\mu_{b,D'}$ if $|b| > 2$, which is equivalent to that $\Lambda_r\left(b, \frac{1}{t}S^*\right)$ is a spectrum of $\mu_{b,D}$.

Observe that $\dim_H(\sum T_r\,(b, D)) = s = \frac{\ln \#D}{\ln|b|}$ given by Schief [437]. The proof is divided into two cases as follows.

Case I: Suppose $|b| = 2$. Then $\#D = 2$ by the admissible property of (b, D) and $\#D \geq 2$. Notice that $\mu_{2,\{0,1\}}$ (resp. $\mu_{-2,\{0,1\}}$) is the Lebesgue measure restriction on [0, 1] (resp. $\left[-\frac{2}{3}, \frac{1}{3}\right]$) and thus $\left(\mu_{2,\{0,1\},\mathbb{Z}}\right)$ (resp. $(\mu_{-2,\{0,1\},\mathbb{Z}})$) is a spectral pair. Then $\left(\mu_{b,\{0,d\}}, \frac{1}{d}\mathbb{Z}\right)$ is a spectral pair and $\dim_B \frac{1}{d}\mathbb{Z} = 1 = \dim_H\left(\sum T_r\,(b, D)\right)$.

Case II: Suppose $|b| > 2$. We need to show that $\dim_B\left(\sum \Lambda_r\left(b, \frac{1}{t}S^*\right)\right) \geq \frac{\ln \#D}{\ln|b|}$. By Theorem (6.3.16), it suffices to show that
$$\sup_{\lambda\in\Lambda_r\left(b,\frac{1}{t}S^*\right)} \inf_{s\in\Lambda_r\left(b,\frac{1}{t}S^*\right)} \|b^{-1}\lambda - s\| < \infty$$
Since $\sum \Lambda_r(b, S^*) = S^* + bS^* + \cdots$ and $0 \in S^*$, then for any $\lambda \in \sum \Lambda_r\left(b, \frac{1}{t}S^*\right)$ we have $\lambda = \frac{1}{t}(s_1 + bs_2 + \cdots + b^k s_{k+1})$ for some $s_i \in S^*, 1 \leq i \leq k+1$. This implies that

$$\inf_{s\in\Lambda_r\left(b,\frac{1}{t}S^*\right)}\sum\|b^{-1}\lambda-s\|\le\left\|b^{-1}\lambda-\frac{1}{t}(s_2+bs_3\cdots+b^{k-1}s_{k+1})\right\|\le\|b^{-1}s_1\|$$
$$\le|b|^{-1}\max_{s\in S^*}|s|\le\frac{|b|-2}{|b|}.$$
Hence (19) follows and we complete the proof.

Corollary (6.3.33)[439]: Let $\left\{\varphi_i(x)=\frac{1}{4}(x+i-1)\right\}_{i=1}^{4}$ be an IFS generated by $b=4$ and $D=\{0,1,2,3\}$ and let $P=\left\{\frac{p}{2},\frac{p}{2},\frac{q}{2},\frac{q}{2}\right\}$ be a probability weight $(p+q=1,p,q>0)$. Suppose $\mu=\mu_{4,\{0,1,2,3\},P}$ is the self-similar measure with respect to the IFS and the probability weight. Then μ is a frame spectral measure if and only if $p=q=\frac{1}{2}$. Moreover, if $p\ne q$, then
$$\Lambda_r=\{0,2\}+4\{0,2\}+4^2\{0,2\}+\cdots\ \text{(all finite sums)}$$
is a Bessel set or sequence with $\dim_B(\sum\Lambda_r)=\frac{1}{2}<\dim_H\left(\sum T_r(4,\{0,1,2,3\})\right)=1$ for μ.

Proof. Let $b=4,D=\{0,1,2,3\}$. It is easy to see that $(b^{-1}D,D)$ is a compatible pair and thus (b,D) is admissible. According to Theorem (6.3.19), μ is a frame spectral measure if and only if $p=q=\frac{1}{2}$.

Moreover, if $p\ne q$, notice that
$$\delta_{4^{-k}\{0,1,2,3\},P}=\delta_{4^{-k}\{0,2\},\{p,q\}}*\delta_{4^{-k}\{0,1\}},$$
where $\delta_{4^{-k}\{0,1,2,3\},P}=\frac{p}{2}\left(\delta_0+\delta_{4^{-k}}\right)+\frac{q}{2}\left(\delta_{2\cdot4^{-k}}+\delta_{3\cdot4^{-k}}\right)$ and $\delta_{4^{-k}\{0,2\},\{p,q\}}=p\delta_0+q\delta_{2\cdot4^{-k}}$. Then one has
$$\mu=\delta_{4^{-1}\{0,1,2,3\},P}*\delta_{4^{-2}\{0,1,2,3\},P}*\cdots$$
$$=\mu_{4,\{0,2\},\{p,q\}}*\mu_{4,\{0,1\}}.$$
Since $\Lambda_r=\{0,2\}+4\{0,2\}+\cdots$ (all finite sum) is a spectrum for $\mu_{4,\{0,1\}}$ [432], for any $f_r\in C(R)$, we have
$$\sum_{\lambda\in\Lambda_r}\sum\left|\langle f_r,e_\lambda\rangle_\mu\right|^2=\sum_{\lambda\in\Lambda_r}\left|\iint\sum\langle f_r(x+y),e^{2\pi i\lambda(x+y)}\rangle_{d\mu_{4,\{0,2\},\{p,q\}}}(x)d\mu_{4,\{0,1\}}(y)\right|^2$$
$$=\sum_{\lambda\in\Lambda_r}\left|\int\sum\langle f_r(x+y),e^{2\pi i\lambda x}\rangle_{\mu_{4,\{0,2\},\{p,q\}}}e^{2\pi i\lambda y}d\mu_{4,\{0,1\}}(y)\right|^2$$
$$=\int\sum\left|\langle f_r(x+y),e^{2\pi i\lambda x}\rangle_{\mu_{4,\{0,2\},\{p,q\}}}\right|^2d\mu_{4,\{0,1\}}(y)$$
$$\le\iint\sum|f_r(x+y)|^2d\mu_{4,\{0,2\},\{p,q\}}(x)d\mu_{4,\{0,1\}}(y)$$
$$=\sum\|f_r\|_{L^2(\mu)},$$
where the third equality holds since Λ_r is a spectrum for $\mu_{4,\{0,1\}}$. Hence, Λ_r is a Bessel set or sequence for μ.

By Theorem (6.3.2), we have
$$\dim_B\sum\Lambda_r\le\dim_H\left(\sum T_r(4,\{0,1\})\right)=\frac{1}{2}.$$

On the other hand, let $(\Lambda_r)_n = \sum_{i=0}^{n} 4^i\{0,2\}$. Then $(\Lambda_r)_n \subseteq \left[0, \frac{2}{3}(4^{n+1}-1)\right]$ with $\sum \#(\Lambda_r)_n = 2^{n+1}$.

Thus, we have

$$\sum B^+_{\frac{1}{2}}(\Lambda_r) = \overline{\lim_{h\to\infty}} \sup_{x\in\mathbb{R}} \sum \frac{\#\big(\Lambda_r \cap B(x,h)\big)}{h^{\frac{1}{2}}}$$
$$\geq \overline{\lim_{n\to\infty}} \sum \frac{\#\Lambda_r \cap \left(-\frac{2}{3}4^{n+1}, \frac{2}{3}4^{n+1}\right)}{\left(\frac{2}{3}4^{n+1}\right)^{\frac{1}{2}}}$$
$$\geq \lim_{n\to\infty} \frac{2^{n+1}}{\left(\frac{2}{3}4^{n+1}\right)^{\frac{1}{2}}} = \left(\frac{3}{2}\right)^{\frac{1}{2}} > 0.$$

It follows that $\dim_B(\sum(\Lambda_r)) \geq \frac{1}{2}$. Hence, $\dim_B(\sum(\Lambda_r)) = \frac{1}{2}$.

Corollary (6.3.34)[439]: Let $\mu = \mu_{4,\{0,2\}}$ be the Cantor measure on $T_r = T_r(4,\{0,2\})$ with equal weight.

Let p, q be two different positive numbers satisfying $p + q = 1$ with $p \neq q$. Suppose that $\sum \nu(E_r) = \sum p\mu(E_r) + \sum q\mu(E_r - 2)$ for any Borel set $E_r \subset \mathbb{R}$, then ν is a non-spectral frame measure.

Proof. It is easy to see that $\nu = \mu * \eta$ where $\eta = p\delta_0 + q\delta_2$. Suppose ν is a spectral measure with a spectrum Λ_r. Since $(E_r)_{\Lambda_r}$ is an orthogonal family in $L^2(\nu)$, by (20), one has $(\Lambda_r - \Lambda_r)\setminus\{0\} \subset Z(\hat{\nu})$. Note that $Z(\hat{\nu}) = Z(\hat{\mu}) \cup Z(\hat{\eta})$. By a simple calculation, we know $Z(\hat{\eta}) = \emptyset$. Then $(\Lambda_r - \Lambda_r)\setminus\{0\} \subset Z(\hat{\mu})$. By (20) again, one sees that $(E_r)_{\Lambda_r}$ is an orthogonal family in $L^2(\mu)$. By Lemma 2.2 in [420], Λ_r must be not a spectrum of ν. This is a contradiction. Then ν is non-spectra. And by Theorem 5.1 in [430], one has ν is a frame measure.

List of Symbols

Symbol		Page
dim	: dimension	1
min	: minimum	1
IFS	: iterated function system	1
max	: maximum	4
a. e.	: almost everywhere	5
$\otimes$	: Tensor product	6
supp	: support	7
L^1	: Lebesgue integral on the Real line	9
a. s.	: almost sure	11
SSC	: strong separation condition	21
OSC	: open set condition	21
Bin	: Binomial	42
L^2	: Hilbert Space	45
mod	: module	59
gcd	: greatest common divisor	68
$\oplus$	: Orthogonal Sum	83
L^q	: Dual of Lebesgue Space	90
det	: determinant	96
ess	: essential	112
p. c. f	: post-critically finite	119
diam	: diameter	155
spt	: spectrum	171
proj	: projection	172
DC	: dimension conserving	180

Re	: Real	183
arg	: argument	184
dist	: distance	184
inf	: infimum	195
sup	: Supremum	198
ord	: order	234
SOSC	: strong open set condition	253

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