Sudan University of Science and Technology
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Hadamard Determinant Inequality with Wigner Theorem and Convergent Star Products on Hilbert Spaces

متباينة محددة هادامارد مع مبرهنة ويغنر و حاصل الضرب النجمي المتقارب على فضاءات هلبرت

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

The Jenson operator inequality and for spectral order with submajorization, transformations on the set of all n-dimensional subspaces of a Hilbert space, orthogonality and in metric-projective geometry are discussed. The complete positivity of Rieffel deformation quantization by actions of the Euclidean space and nuclear Weyl algebra are presented. The Fuglede-Kadison and Hadamard determinants and inequalities with determinants of perturbed positive matrices and extensions in operators on Hilbert space are characterized. We show the isometries and the geometric version of Wigner theorem on Grassmann spaces and for Hilbert Grassmannians. We investigate C^*-completion, the DFR-algebra and the convergent star products for the projective limits of Hilbert spaces.

الخلاصة

قمنا بمناقشة متباينة مؤثر جنسون والرتبة الطيفية مع التخصص الجزئي والتحويلات على فئة الفضاءات الجزئية ذات البعد – n لفضاء هلبرت والتعامدية وفي هندسة الأسقاط – المترية. تم تقديم الموجيية التامة لتكميم تشوه ريفل بواسطة الأفعال للفضاء الأقليدي وجبر ويل النووي. تم تشخيص متباينات ومحددات فيقليد – كاديسون وهادامارد مع محددات المصفوفات الموجبة الأرتجاج والتمديدات في المؤثرات على فضاء هلبرت. قمنا بتوضيح الايزوميتريس والاصدارة الهندسية لمبرهنة ويغنر على فضاءات غراسمان وغراسمان هلبرت. تم تقصي تمام – C^* وجبر – DFR وحاصل ضرب النجم المتقارب لنهايات الاسقاط لفضاءات هلبرت.

Introduction

We establish what we consider to be the definitive versions of Jensen operator inequality and Jensen trace inequality for real functions defined on an interval. This is accomplished by the introduction of genuine non-commutative convex combinations of operators, as opposed to the contractions considered in earlier versions of the theory, [10] & [4]. For A be a C^*-algebra and $\varphi : A \to L(H)$ be a positive unital map. Then, for a convex function $f : I \to R$ defined on some open interval and a self-adjoint element $a \in A$ whose spectrum lies in I , we obtain a Jensen-type inequality $f\left(\varphi(a)\right) \leq \varphi(f\ (a))$ where $\leq$ denotes an operator preorder (usual order, spectral preorder, majorization) and depends on the class of convex functions considered, i.e., monotone convex or arbitrary convex functions.

Wigner classical theorem on symmetry transformations plays a fundamental role in quantum mechanics. It can be formulated, for example, in the following way: Every bijective transformation on the set L of all 1-dimensional subspaces of a Hilbert space H which preserves the angle between the elements of L is induced by either a unitary or an antiunitary operator on H. In an n-dimensional projective space with a polarity two k-subspaces are ortho-adjacent if they are adjacent and one intersects the polar of the other.

We consider C^*-algebraic deformations by actions of $\mathbb{R}^d$ à la Rieffel and show that every state of the undeformed algebra can be deformed into a state of the deformed algebra in the sense of a continuous field of states. A bilinear form on a possibly graded vector space V defines a graded Poisson structure on its graded symmetric algebra together with a star product quantizing it. This gives a model for the Weyl algebra in an algebraic framework, only requiring a field of characteristic zero. When passing to $\mathbb{R}$ or $\mathbb{C}$ one wants to add more: the convergence of the star product should be controlled for a large completion of the symmetric algebra. Assuming that the underlying vector space carries a locally convex topology and the bilinear form is continuous, we establish a locally convex topology on the Weyl algebra such that the star product becomes continuous.

We review the definition of determinants for finite von Neumann algebras, due to Fuglede and Kadison [Fuglede B, Kadison R (1952)], and a generalization for appropriate groups of invertible elements in Banach algebras, from a paper by Skandalis and the author (1984). We show two inequalities regarding the ratio $\det(A + D)/ \det A$ of the determinant of a positive-definite matrix A and the determinant of its perturbation $A + D$. A generalization of classical determinant inequalities like Hadamard inequality and Fischer's inequality is studied. For a version of the inequalities originally proved by Arveson for positive operators in von Neumann algebras with a tracial state, we give a different proof. We also improve and generalize to the setting of finite von Neumann algebras, some 'Fischer-type' inequalities by Matic for determinants of perturbed positive-definite matrices.

Botelho, Jamison, and Moln´ar have recently described the general form of surjective isometries of Grassmann spaces on complex Hilbert spaces under certain dimensionality assumptions. We provide a new approach to this problem which enables us first, to give a shorter proof and second, to remove dimensionality constraints completely. In one of the low dimensional cases, which was not covered by Botelho, Jamison, and Molnar, an exceptional possibility occurs. As a byproduct, we are able to handle the real case as well. Furthermore, in finite dimensions we remove the surectivity assumption. Wigner celebrated theorem, which is particularly important in the mathematical foundations of quantum mechanics, states that every bijective transformation on the set of all rank-one projections of a complex Hilbert space which preserves the transition probability is induced by a unitary or an antiunitary operator. This vital theorem has been generalised in various ways by several scientists. In 2001, Molnár provided a natural generalisation, namely, he provided a characterisation of (not necessarily bijective) maps which act on the Grassmann space of all rank-n projections and leave the system of Jordan principal angles invariant (see [34] and [259]). For H be a complex Hilbert space of dimension not less than 3 and let $\mathcal{G}_k(H)$ be the Grassmannian formed by k-dimensional subspaces of H. Suppose that dim $H \geq 2k > 2$. We show that the transformations of $\mathcal{G}_k(H)$ induced by linear or conjugate-linear isometries can be characterized as transformations preserving some of principal angles (corresponding to the orthogonality, adjacency and orthoadjacency relations).

We present the construction of a general family of C^*- algebras which includes, as a special case, the "quantum spacetime algebra" introduced by Doplicher, Fredenhagen, and Roberts. It is based on an extension of the notion of C^*-completion from algebras to bundles of algebras, compatible with the usual C^*-completion of the appropriate algebras of sections, combined with a novel definition for the algebra of the canonical commutation relations using Rieffel's theory of strict deformation quantization. Given a locally convex vector space with a topology induced by Hilbert seminorms and a continuous bilinear form on it we construct a topology on its symmetric algebra such that the usual star product of exponential type becomes continuous.

The Contents

Chapter 1
Jenson Operator Inequality

We show how this relates to the pinching inequality of Davis [5], and how Jensen trace inequality generalizes to C^*-algebras. Some extensions of Jensen's-type inequalities to the multi-variable case are considered.

Section (1.1): Jensen Operator Inequality

If f is a continuous, real function on some interval I in ℝ, we can use spectral theory to define an operator function

$$f: \mathbb{B}(\mathfrak{H})_{sa}^{I} \to \mathbb{B}(\mathfrak{H})_{sa} \quad \text{where} \quad f(x) = \int f(\lambda) dE_x(\lambda). \tag{1}$$

Here $\mathbb{B}(\mathfrak{H})_{sa}^{I}$ denotes the convex set of self-adjoint operators on the Hilbert space $\mathfrak{H}$ with spectra in I, and E_x denotes the spectral measure of x. It is somewhat dangerous to use the same symbol for the two rather different functions, but the usage is sanctified by time. Whenever necessary we shall try to distinguish between the two by referring either to the function f or to the operator function f. As pointed out by C. Davis in [4] a general operator function $F: \mathbb{B}(\mathfrak{H})_{sa}^{I} \to \mathbb{B}(\mathfrak{H})$ arises from a spectral function, i.e. $F(x) = f(x)$, if and only if for every unitary u

$$F(u^* x u) = u^* F(x) u \quad \text{and} \quad F\begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} F(y) & 0 \\ 0 & F(z) \end{pmatrix} \tag{2}$$

for every operator $x = y + z$ that decomposes in block form by multiplication by a projection p in its commutant. (We do not demand that p and $1 - p$ are equivalent.) There is a slight ambiguity in this statement – easily compensated for by its versatility – since by F(y) we really mean F evaluated at y, but now regarded as an operator function on $\mathbb{B}(p\mathfrak{H})_{sa}^{I}$. We demand that $pF(y + z) = F(y + z)p$ and that it is independent of z. Thus, $pF(y + z) = pF(y + s(1 - p))$ for some, hence any scalar s in I. (Davis tacitly assumes that $0 \in I$ and takes $s = 0$.)

A continuous function $f: I \to \mathbb{R}$ is said to be operator convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{3}$$

for each λ in [0, 1] and every pair of self-adjoint operators x, y on an infinite dimensional Hilbert space $\mathfrak{H}$ with spectra in I. The function is said to be matrix convex of order n if the same conditions are satisfied for operators on a Hilbert space of finite dimension n. It is well known, cf. [4] that a function is operator convex if and only if it is matrix convex of arbitrary orders.

Just because the function f is convex there is no guarantee that the operator function f is convex. In fact, as shown by Bendat and Sherman in [4], f is operator convex on the interval]-1, 1[if and only if it has a (unique) representation

$$f(t) = \beta_0 + \beta_1 t + \frac{1}{2}\beta_2 \int_{-1}^{1} t^2 (1 - \alpha t)^{-1} d\mu(\alpha), \tag{4}$$

for $\beta_2 \geq 0$ and some probability measure μ on [−1, 1]. f must be analytic with $f(0) = \beta_0, f'(0) = \beta_1$ and $f''(0) = \beta_2$. Lowner's theory of operator monotonicity can be found in [11].

An unexpected phenomenon turns up in relation with convexity in $\mathbb{B}(\mathfrak{H})_{sa}$. If $(a_1, \ldots, a_n)$ is an n−tuple of operators with $\sum_{k=1}^{n} a_k^* a_k = 1$, we may think of the element $\sum_{k=1}^{n} a_k^* x_k a_k$ as a non-commutative convex combination of the n−tuple $(x_1, \ldots, x_n)$ in $\mathbb{B}(\mathfrak{H})_{sa}$. The remarkable fact is that when f is an operator convex function, then the operator function f respects this new structure in the sense that we have the Jensen operator inequality:

$$f\left(\sum_{k=1}^{n} a_k^* x_k a_k\right) \leq \sum_{k=1}^{n} a_k^* f(x_k) a_k. \tag{5}$$

This result was found in [8], and used to give a review of Lowner's and Bendat-Sherman's theory of operator monotone and operator convex functions in [11]. We must admit that we unfortunately proved and used the contractive form $f(a^*xa) \leq a^*f(x)a$ for $a^*a \leq 1$, this being the seemingly most attractive version at the time. This necessitated the further conditions that $0 \in I$ and $f(0) \leq 0$, conditions that have haunted the theory since then. The Jensen inequality for a normal trace on a von Neumann algebra, now for an arbitrary convex function f, was found by Brown and Kosaki in [4], still in the contractive version.
We rectify our omissions and prove the full Jensen inequality, both with and without a trace.
Corollary (1.1.1)[1]: Let f be a convex, continuous function defined on an interval I, and suppose that $0 \in I$ and $f(0) \leq 0$. Then for all natural numbers m and n we have the inequality (7) for every n−tuple $(x_1, \ldots, x_n)$ of self-adjoint $m \times m$ matrices with spectra contained in I and every n−tuple $(a_1, \ldots, a_n)$ of $m \times m$ matrices with $\sum_{k=1}^{n} a_k^* a_k \leq 1$.
Remark (1.1.2)[1]: Let n=1 in (7). If f is convex, 0∈I and f(0)≤0 we have

$$Tr\left(f(a^*xa)\right) \leq Tr(a^*f(x)a) \tag{6}$$

for every self-adjoint $m \times m$ matrix x with spectrum in I and every $m \times m$ contractive matrix a. This is Jensen's trace inequality (for matrices) of Brown and Kosaki [4].
This inequality alone is not sufficient to ensure convexity of f, even if $m > 1$ (unless $f(0) = 0$ is specified in advance). For $n > 1$ the inequality gives convexity of f as we see from Theorem (1.1.7). In each case we must assume that $0 \in I$, otherwise the inequality does not make sense. This fact, together with the irrelevant information about $f(0)$, makes the contractive versions of Jensen's inequality less desirable. When we eventually pass to the theory of several variables, cf. [11], the contractive versions mean that 0 belongs to the cube where f is defined, so that part of the coordinate axes must belong to the domain of definition for f, and on these we must assume that $f \leq 0$.

If the trace τ is unbounded, but lower semi-continuous and densely defined, the inequality (9) is still valid if $f \geq 0$, although now some of the numbers may be infinite.

An n−tuple $a = (a_1, \ldots, a_n)$ of operators in $\mathbb{B}(\mathfrak{H})$ is called a contractive column (respectively a unital column) if $\sum_{k=1}^{n} a_k^* a_k \leq 1$ (respectively $\sum_{k=1}^{n} a_k^* a_k = 1$). Contractive rows and unital rows are defined analogously by the conditions $\sum_{k=1}^{n} a_k a_k^* \leq 1$ and $\sum_{k=1}^{n} a_k a_k^* = 1$. We say that $a = (a_1, \ldots, a_n)$ is a unitary column if there is a unitary $n \times n$ operator matrix $U = (u_{ij})$, one of whose columns is $(a_1, \ldots, a_n)$. Thus, $u_{ij} = a_i$ for some j and all i. Unitary rows are defined analogously, cf. [1] Note that an n−tuple $(a_1, \ldots, a_n)$ is a contractive/unital/unitary row if and only if the adjoint tuple $(a_1^*, \ldots, a_n^*)$ is a contractive/unital/unitary column. Even for a finite-dimensional Hilbert space $\mathfrak{H}$ it may happen that an n−tuple $\underline{a}$ is a unitary (unital or contractive) column in $\mathbb{B}(\mathfrak{H})$, while $\underline{a}$ is not a unitary (unital or contractive) row in $\mathbb{B}(\mathfrak{H})$. Evidently every unitary column is also a unital column (and similarly every unitary row is a unital row). On the other hand, if $(s_1, \ldots, s_n)$ is an n−tuple of co-isometries such that $\sum_{k=1}^{n} s_k^* s_k = 1$ (these are the canonical generators for the Cuntz algebra $\mathcal{O}_n$), then we have a simple example of a unital column that is not unitary. If we insist that a unital column of elements in a unital C^*−algebra $\mathcal{A}$ should be called a unitary column only if we can choose the unitary in $\mathbb{M}_n(\mathcal{A})$, then already for $\mathcal{A}$

commutative, viz. $\mathcal{A} = C(\mathbb{S}^5)$, we have a unital 3−column that is not a unitary column in $\mathbb{M}_3(\mathcal{A})$, cf. [18].
Given a unital column $(a_1, \dots, a_n)$ we may regard it as an isometry $\underline{a}: \mathfrak{H} \to \mathfrak{H}^n$, where $\mathfrak{H}^n = \bigoplus_{i=1}^n \mathfrak{H}$. Better still we may regard it as a partial isometry $V: \mathfrak{H}^n \to \mathfrak{H}^n$, where $V|\mathfrak{H}^{n-1} = 0$. Evidently the column is unitary precisely when V extends to a unitary operator on $\mathfrak{H}^n$, and this happens if and only if the index of V is 0, in the generalized sense that dim ker $V^* = (n-1)\dim \mathfrak{H}$. Here $V^*(\xi_1, \dots, \xi_n) = a_1^*\xi_1 + \dots + a_n^*\xi_n$ in $\mathfrak{H}$. It follows from [1] that this holds if just one of the operators a_i has (generalized) index zero, since in this case $a_i = u|a_i|$ for some unitary u on $\mathfrak{H}$. We are then reduced to the situation where one of the operators, say a_n, is positive, so that with $\underline{b} = (a_1, \dots, a_{n-1})$ we can extend V to the unitary operator

$$U = \begin{pmatrix} \left(1 - \underline{b}(\underline{b})^*\right)^{\frac{1}{2}} & \underline{b} \\ -(\underline{b})^* & a_n \end{pmatrix}. \quad (7)$$

It follows that every contractive n−column can be enlarged to a unitary (n+1)– column simply by setting $a_{n+1} = (1 - \sum_{k=1}^n \ a_k^* a_k)^{\frac{1}{2}}$. In particular, every unital n−column can be enlarged to a unitary (n+1)−column with $a_{n+1} = 0$. As usual we shall refer to this as a unitary dilation of the unital (or contractive) column.

It may sometimes be desirable to know exactly the terms in a unitary dilation of some unital column $\underline{a} = (a_1, \dots, a_n)$. If n=1, so that $\underline{a} = a$ for some isometry a, the canonical dilation is given by a 2×2−matrix U having $(a, 0)$ as the second column. For a general unital n−column we may regard it as an isometry $\underline{a}: \mathfrak{H} \to \mathfrak{H}^n$, and the unitary dilation U_n on $\mathfrak{H} \oplus \mathfrak{H}^n$ then has the same form as U; in fact

$$U = \begin{pmatrix} 1 - aa^* & a \\ -a^* & 0 \end{pmatrix} \quad \text{and} \quad U_n = \begin{pmatrix} p & \underline{a} \\ -(\underline{a})^* & 0 \end{pmatrix}, \quad (8)$$

where $p = 1 - \underline{a}(\underline{a})^*$ is the $n \times n$ projection in $\mathfrak{H}^n$ with $p_{ii} = 1 - a_i a_i^*$ and $p_{ij} = -a_i a_j^*$ for $i \neq j$. Thus, the canonical dilation of $(a_1, \dots, a_n)$ has the form:

$$U_n = \begin{pmatrix} 1 - a_1 a_1^* & -a_1 a_2^* & \dots & -a_1 a_m^* & a_1 \\ -a_2 a_1^* & 1 - a_2 a_2^* & \dots & -a_2 a_n^* & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ -a_n a_1^* & -a_n a_2^* & \dots & 1 - a_n a_n^* & a_n \\ -a_1^* & -a_2^* & \dots & -a_n^* & 0 \end{pmatrix} \quad (9)$$

As seen from (7), the formula for the canonical dilation of a contractive column is only marginally more complicated, cf. [1]

Lemma (1.1.3)[1]: Define the unitary matrix $E = diag(\theta, \theta^2, \dots, \theta^{n-1}, 1)$ in $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathfrak{H}^n)$, where $\theta = \exp(2\pi\, i/n)$. Then for each element $A = (a_{ij})$ in $\mathbb{B}(\mathfrak{H}^n)$ we have

$$\frac{1}{n}\sum_{k=1}^{n} \ E^{-k} A E^k = diag\ (a_{11}, a_{22}, \dots, a_{nn}). \quad (10)$$

Proof. By computation

$$\left(\frac{1}{n}\sum_{k=1}^{n} \ E^{-k} A E^k\right)_{ij} = \frac{1}{n}\sum_{k=1}^{n} \ \left(\theta^{j-i}\right)^k a_{ij}, \quad (11)$$

and this sum is zero if $i \neq j$, otherwise it is a_{ii}.

Corollary (1.1.4)[1]: Let P denote the projection in $\mathbb{M}_n(\mathbb{C})$ given by $P_{ij} = n^{-1}$ for all i and j, so that P is the projection of rank one on the subspace spanned by the vector $\xi_1 + \cdots + \xi_n$ in $\mathbb{C}^n$, where $\xi_1, \ldots, \xi_n$ are the standard basis vectors. Then with E as in Lemma (1.1.3) we obtain the pairwise orthogonal projections $P_k = E^{-k} P\, E^k$, for $1 \le k \le n$, with $\sum_{k=1}^{n} P_k = 1$.

Proof. (Cf. [8]) Evidently each P_k is a projection of rank one. Moreover, by Lemma (1.1.3),

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n} E^{-k} P\, E^k = n \;\; diag(n^{-1}, \ldots, n^{-1}) = 1, \qquad (12)$$

from which it also follows that the projections are pairwise orthogonal,

Theorem (1.1.5)[1]:

For a continuous function f defined on an interval I the following conditions are equivalent:

(i) f is operator convex.

(ii) For each natural number n we have the inequality

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) \le \sum_{i=1}^{n} a_i^* f(x_i) a_i \qquad (13)$$

for every n−tuple $(x_1, \ldots, x_n)$ of bounded, self-adjoint operators on an arbitrary Hilbert space $\mathfrak{H}$ with spectra contained in I and every n−tuple $(a_1, \ldots, a_n)$ of operators on $\mathfrak{H}$ with $\sum_{k=1}^{n} a_k^* a_k = 1$.

(iii) $f(v^{*x} v) \le v^* f(x) v$ for each isometry v on an infinite-dimensional Hilbert space $\mathfrak{H}$ and every self-adjoint operator x with spectrum in I.

(vi) $pf(pxp + s(1-p))p \le pf(x)p$ for each projection p on an infinite-dimensional Hilbert space $\mathfrak{H}$, every self-adjoint operator x with spectrum in I and every s in I.

Proof. (i) ⇒ (ii) Assume that we are given a unitary n−column $(a_1, \ldots, a_n)$, and choose a unitary $U_n = \left(u_{ij}\right)$ in $\mathbb{B}(\mathfrak{H}^n)$ such that $u_{kn} = a_k$. Let $E = diag\,(\theta, \theta^2, \ldots, 1)$ as in Lemma (1.1.3) and put $X = diag(x_1, \ldots, x_n)$, both regarded as elements in $\mathbb{B}(\mathfrak{H}^n)$. Using Lemma (1.1.3) and the operator convexity of f we then get the desired inequality:

$$f\left(\sum_{k=1}^{n} a_k^* x_k a_k\right) = f((U_n^* X U_n)_{nn}) = f\left(\left(\sum_{k=1}^{n} \frac{1}{n} E^{-k} U_n^* X U_n E^k\right)_{nn}\right)$$

$$= \left(f\left(\sum_{k=1}^{n} \frac{1}{n} E^{-k} U_n^* X U_n E^k\right)\right)_{nn} \le \left(\frac{1}{n}\sum_{k=1}^{n} f(E^{-k} U_n^* X U_n E^k)\right)_{nn}$$

$$= \left(\frac{1}{n}\sum_{k=1}^{n} E^{-k} U_n^* f(X) U_n E^k\right)_{nn} = (U_n^* f(X) U_n)_{nn} = \sum_{k=1}^{n} a_k^* f(x_k) a_k.$$

Note that for the second equality we use that $f(y_n) = \left(f\left(\, diag\,(y_1, \ldots, y_n)\right)\right)_{nn}$ because $f\left(diag\,(y_1, \ldots, y_n)\right) = diag\left(f(y_1), \ldots, f(y_n)\right)$.

In the general case where the column is just unital, we enlarge it to the unitary (n+1)−column $(a_1, \ldots, a_n, 0)$ and choose x_{n+1} arbitrarily, but with spectrum in I. By the first part of the proof we therefore have

$$f\left(\sum_{k=1}^{n} a_k^* x_k a_k\right) = f\left(\sum_{k=1}^{n+1} a_k^* x_k a_k\right) \le \sum_{k=1}^{n+1} a_k^* f(x_k) a_k$$

$$= \sum_{k=1}^{n} a_k^* f(x_k) a_k. \qquad (14)$$

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (iv) Take any self-adjoint operator x with spectrum in I and let p be an infinite-dimensional projection. Then we can find an isometry v (i.e. $v * v = 1$) such that $p = vv^*$. By assumption $f(v^*xv) \le v^*f(x)v$, whence also

$$vf(v^*xv)v^* \le vv^*f(x)vv^* = pf(x)p. \qquad (15)$$

For any monomial $g(t) = t^m$ and any s in I we have

$$pvg(v^*xv)v^*p = pv(v^*xv)^m v^*p = p(vv^*xvv^*)^m p = pg(pxp)p$$
$$= pg\big(pxp + s(1-p)\big)p. \qquad (16)$$

Since f is continuous, it can be approximated by polynomials on compact subsets of I, and therefore also $pvf(v^*xv)v^*p = pf\big(pxp + s(1-p)\big)p$. Combined with (14) this gives the pinching inequality

$$pf(pxp + s(1-p))p \le pf(x)p. \qquad (17)$$

If p is a projection of finite rank we can define the infinite dimensional projection $q = p \otimes 1_\infty$ on $\mathfrak{H}^\infty$. Similarly we let $y = x \otimes 1_\infty$ for any given self-adjoint operator x with spectrum in I. Since $f(a) \otimes 1_\infty = f(a \otimes 1_\infty)$ for any operator a we get by (17) that

$$pf\big(pxp + s(1-p)\big)p \otimes 1_\infty = qf\big(qyq + s(1-q)\big)q \le qf(y)q$$
$$= pf(x)p \otimes 1_\infty, \qquad (18)$$

which shows that (17) is valid also for projections of finite rank.

(iv) ⇒ (1) Given self-adjoint operators x and y with spectra in I and λ in [0, 1], define the three elements

$$X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \qquad U = \begin{pmatrix} \lambda^{\frac{1}{2}} & (1-\lambda)^{\frac{1}{2}} \\ -(1-\lambda)^{\frac{1}{2}} & \lambda^{\frac{1}{2}} \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (19)$$

in $\mathbb{B}(\mathfrak{H}^2)$. Then for some s in I we have by the pinching inequality in (iv) that

$$Pf\big(P\,U^*XUP + s(1-P)\big)P \le P\,f(U^*XU)P = P\,U^*f(X)UP. \qquad (20)$$

Since

$$U^*XU = \begin{pmatrix} \lambda x + (1-\lambda)y & (\lambda - \lambda^2)^{\frac{1}{2}}(y-x) \\ (\lambda - \lambda^2)^{\frac{1}{2}}(y-x) & \lambda y + (1-\lambda)x \end{pmatrix}, \qquad (21)$$

it follows that

$$\begin{pmatrix} f(\lambda x + (1-\lambda)y) & 0 \\ 0 & 0 \end{pmatrix} = P\,f(P\,U^*XUP + s(1-P))P \le P\,U^* \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} UP$$
$$= \begin{pmatrix} \lambda f(x) + (1-\lambda)f(y) & 0 \\ 0 & 0 \end{pmatrix}. (25)$$

Corollary (1.1.6)[1]:

Let f be a continuous function defined on an interval I and suppose that 0∈I. Then f is operator convex and f(0)≤0 if and only if for some, hence every natural number n, the inequality (13) is valid for every n−tuple $(x_1, \ldots, x_n)$ of bounded, self-adjoint operators on a Hilbert space $\mathfrak{H}$ with spectra contained in I, and every n−tuple $(a_1, \ldots, a_n)$ of operators on $\mathfrak{H}$ with $\sum_{k=1}^{n} a_k^* a_k \le 1$.

Setting n=1 we see that f is operator convex on an interval I containing 0 with f(0)≤0 if and only if

$$f(a^*xa) \le a^*f(x)a \qquad (22)$$

for every self-adjoint x with spectrum in I and every contraction a. This is the original Jensen operator inequality from [11].

Proof . If $\sum_{k=1}^{n} a_k^* a_k = b \leq 1$, put $a_{n+1} = (1-b)^{\frac{1}{2}}$. Then we have a unital (n+1)−tuple, so with $x_{n+1} = 0$ we get

$$f\left(\sum_{k=1}^{n} a_k^* x_k a_k\right) = f\left(\sum_{k=1}^{n+1} a_k^* x_k a_k\right) \leq \sum_{k=1}^{n+1} a_k^* f(x_k) a_k$$

$$= \sum_{k=1}^{n} a_k^* f(x_k) a_k + a_{n+1}^* f(0) a_{n+1} \leq \sum_{k=1}^{n} a_k^* f(x_k) a_k. \quad (26)$$

Conversely, if (13) is satisfied for all contractive n−tuples, then – a fortiori – it holds for unital n−tuples, so f is operator convex; and with $a = x = 0$ we see that $f(0) \leq 0 \cdot f(0) \cdot 0 = 0$.

Theorem (1.1.7)[1]:

Let f be a continuous function defined on an interval I and let m and n be natural numbers. If f is convex we then have the inequality

$$Tr\left(f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right)\right) \leq Tr\left(\sum_{i=1}^{n} a_i^* f(x_i) a_i\right) \quad (23)$$

for every n−tuple $(x_1, \ldots, x_n)$ of self-adjoint $m \times m$ matrices with spectra contained in I and every n−tuple $(a_1, \ldots, a_n)$ of $m \times m$ matrices with $\sum_{k=1}^{n} a_k^* a_k = 1$.

Conversely, if the inequality (23) is satisfied for some n and m, where n>1, then f is convex.

Proof. Let $x_k = \sum_{sp(x_k)} \lambda E_k(\lambda)$ denote the spectral resolution of x_k for $1 \leq k \leq n$. Thus, $E_k(\lambda)$ is the spectral projection of x_k on the eigenspace corresponding to λ if λ is an eigenvalue for x_k; otherwise $E_k(\lambda) = 0$. For each unit vector ξ in $\mathbb{C}^m$ define the (atomic) probability measure

$$\mu_\xi(S) = \left(\sum_{k=1}^{n} a_k^* E_k(S) a_k \xi | \xi\right) = \sum_{k=1}^{n} (E_k(S) a_k \xi | a_k \xi) \quad (24)$$

for any (Borel) set S in ℝ. Note now that if $y = \sum_{k=1}^{n} a_k^* x_k a_k$ then

$$(y\xi|\xi) = \left(\sum_{k=1}^{n} a_k^* x_k a_k \xi | \xi\right) = \left(\sum_{k=1}^{n} \sum_{sp(x_k)} \lambda E_k(\lambda) a_k \xi | a_k \xi\right)$$

$$= \int \lambda \, d\mu_\xi(\lambda). \quad (25)$$

If a unit vector ξ is an eigenvector for y, then the corresponding eigenvalue is $(y\xi|\xi)$, and ξ is also an eigenvector for f(y) with correponding eigenvalue $(f(y)\xi|\xi) = f((y\xi|\xi))$. In this case we therefore have

$$\left(f\left(\sum_{k=1}^{n} a_k^* x_k a_k\right)\xi|\xi\right) = (f(y)\xi|\xi) = f\big((y\xi|\xi)\big) = f\left(\int \lambda \, d\mu\xi(\lambda)\right)$$

$$\leq \int f(\lambda) d\mu_\xi(\lambda) = \sum_{k=1}^{n} \left(\sum_{sp(x_k)} f(\lambda) E_k(\lambda) a_k \xi | a_k \xi\right)$$

$$= \sum_{k=1}^{n} (a_k^* f(x_k) a_k \xi | \xi), \qquad (26)$$

where we used (25) and the convexity of f – in form of the usual Jensen inequality – to get the inequality in (26). The result in (23) now follows by summing over an orthonormal basis of eigenvectors for y.

Conversely, if (23) holds for some pair of natural numbers n, m, where $n > 1$, then taking $a_i = 0$ for $i \geq 2$ we see that the inequality holds for n=2. Given s, t in I and λ in [0, 1] we define $x = s1_m$ and $y = t1_m$ in $\mathbb{M}_m(\mathbb{C})$. Then with $a = \lambda^{\frac{1}{2}} 1_m$ and $b = (1-\lambda)^{\frac{1}{2}} 1_m$ we get by (23) that

$$mf(\lambda x + (1-\lambda)t) = Tr(f(\lambda x + (1-\lambda)t)1_m) = Tr\big(f(a^*xa + b^*yb)\big)$$
$$\leq Tr(a^*f(x)a + b^*f(y)b) = Tr\Big(\big(\lambda f(s) + (1-\lambda)f(t)\big)1_m\Big)$$
$$= m\big(\lambda f(s) + (1-\lambda)f(t)\big), \qquad (27)$$

which shows that f is convex.

Let $\mathcal{A}$ be a C^*-algebra of operators on some Hilbert space $\mathfrak{H}$ and T a locally compact Hausdorff space. We say that a family $(a_t)_{t\in T}$ of operators in the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$, i.e. the C^*-algebra $\{a \in \mathbb{B}(\mathfrak{H}) \,|\forall x \in \mathcal{A}: xa + ax \in \mathcal{A}\}$, is a continuous field, if the function $t \to a_t$ is norm continuous. If μ is a Radon measure on T and the function $t \to \|a_t\|$ is integrable, we can then form the Bochner integral $\int_T a_t \, d\mu(t)$, which is the unique element in $M(\mathcal{A})$ such that

$$\varphi\left(\int_T a_t \, d\mu(t)\right) = \int_T \varphi(a_t) d\mu(t) \qquad \varphi \in \mathcal{A}^*. \qquad (28)$$

If all the a_t's belong to $\mathcal{A}$ then also $\int_T a_t d\mu(t)$ belongs to $\mathcal{A}$. If $(a_t^* \, a_t)_{t\in T}$ is integrable with integral 1 we say that $(a_t)_{t\in T}$ is a unital column field.

The transition from sums to continuous fields is prompted by the nature of the proof of Theorem (1.1.8), we can easily verify that also Theorem (1.1.5) is valid for continuous fields. We finally note that the restriction to continuous fields is handy, but not necessary. In [11] we shall generalize the setting to arbitrary weak* measurable fields.

The centralizer of a positive functional φ on a C^*-algebra $\mathcal{A}$ is the closed $*-$subspace $\mathcal{A}^\varphi = \{y \in \mathcal{A} \,|\forall x \in \mathcal{A} : \varphi(xy) = \varphi(yx)\}$. In general this is not an algebra, but if $y_1, \ldots, y_n$ are pairwise commuting, self-adjoint elements in $\mathcal{A}^\varphi$ then the C^*-algebra they generate is contained in $\mathcal{A}^\varphi$. Evidently the size of $\mathcal{A}^\varphi$ measures the extent to which φ is a trace. The fact we shall utilize is that even if an element x is outside $\mathcal{A}^\varphi$ the functional will behave ”trace-like” on the subspace spanned by $\mathcal{A}^\varphi x \mathcal{A}^\varphi$.

If φ is unbounded, but lower semi-continuous on $\mathcal{A}_+$ and finite on the minimal dense ideal $K(\mathcal{A})$ of $\mathcal{A}$, we define $\mathcal{A}^\varphi = \{y \in \mathcal{A} \mid \forall x \in K(\mathcal{A}): \varphi(xy) = \varphi(yx)\}$.

Theorem (1.1.8)[1]: Let $(x_t)_{t\in T}$ be a bounded, continuous field on a locally compact Hausdorff space T consisting of self-adjoint elements in a C^*-algebra $\mathcal{A}$ with $sp(x_t) \subset I$. Furthermore, let $(a_t)_{t\in T}$ be a unital column field in $M(\mathcal{A})$ with respect to some Radon measure μ on T. Then for each continuous, convex function f defined on I and every positive functional φ that contains the element $y = \int_T a_t^* x_t a_t d\mu(t)$ in its centralizer $\mathcal{A}^\varphi$, i.e. $\varphi(xy) = \varphi(yx)$ for all x in $\mathcal{A}$, we have the inequality:

$$\varphi\left(f\left(\int_T a_t^* x_t a_t \, d\mu(t)\right)\right) \leq \varphi\left(\int_T a_t^* f(x_t) a_t d\mu(t)\right). \tag{29}$$

If φ is unbounded, but lower semi-continuous on $\mathcal{A}_+$ and finite on the minimal dense ideal $K(\mathcal{A})$ of $\mathcal{A}$, the result still holds if $f \geq 0$, even though the function may now attain infinite values.

Proof. Let $C = C_o(S)$ denote the commutative C^*-subalgebra of $\mathcal{A}$ generated by y, and let μ_φ be the finite Radon measure on the locally compact Hausdorff space S defined, via the Riesz representation theorem, by

$$\int_S z(s) d\mu_\varphi(s) = \varphi(z) \qquad z \in C = C_o(S). \tag{30}$$

Since for all (x, z) in $M(\mathcal{A})_+ \times C_+$ we have $\varphi(xz) = \varphi\left(z^{\frac{1}{2}} x z^{\frac{1}{2}}\right)$ it follows that

$$0 \leq \varphi(xz) \leq \|x\| \varphi(z). \tag{31}$$

Consequently the functional $z \to \varphi(xz)$ on C defines a Radon measure on S dominated by a multible of μ_φ, hence determined by a unique element Φ(x) in $L^\infty_{\mu_\varphi}(S)$. By linearization this defines a conditional expectation $\varphi: M(\mathcal{A}) \to L^\infty_{\mu_\varphi}(S)$ (i.e. a positive, unital module map) such that

$$\int_S z(s) \Phi(x)(s) d\mu_\varphi(s) = \varphi(zx), \qquad z \in C \qquad x \in M(\mathcal{A}). \tag{32}$$

Inherent in this formulation is the fact that if $z \in C = C_o(S)$, then Φ(z) is the natural image of z in $L^\infty_{\mu_\varphi}(S)$. In particular, $z(s) = \Phi(z)(s)$ for almost all s in S.

Observe now that since the C^*-algebra $C_o(I)$ is separable, we can for almost every s in S define a Radon measure μ_s on I by

$$\int_I g(\lambda) d\mu_s(\lambda) = \Phi\left(\int_T a_t^* g(x_t) a_t d\mu(t)\right)(s) \qquad g \in C(I). \tag{33}$$

As $\int_T a_t^* \, a_t d\mu(t) = 1$, this is actually a probability measure. If we take $g(\lambda) = \lambda$, then

$$\int_I \lambda \, d\mu_s(\lambda) = \Phi\left(\int_T a_t^* x_t a_t d\mu(t)\right)(s) = \Phi(y)(s) = y(s). \tag{34}$$

Since $y \in C$ we get by (34) and (33) – using also the convexity of f in form of the standard Jensen inequality – that

$$\begin{aligned} f(y)(s) = f(y(s)) = f\left(\int_I \lambda \, d\mu_s(\lambda)\right) &\leq \int f(\lambda) d\mu_s(\lambda) \\ &= \Phi\left(\int a_t^* f(x_t) a_t d\mu(t)\right)(s). \end{aligned} \tag{35}$$

Integrating over s, using (32), now gives the desired result:

$$\varphi(f(y)) = \int_S f(y)(s) d\mu_\varphi(s) \leq \int_S \Phi\left(\int_T a_t^* f(x_t) a_t d\mu(t)\right)(s) d\mu_\varphi(s)$$

$$= \int_T \int_S \Phi(a_t^* f(x_t) a_t)(s) d\mu_\varphi(s) d\mu(t) = \int_T \varphi(a_t^* f(x_t) a_t) d\mu(t)$$

$$= \varphi\left(\left(\int_T a_t^* f(x_t) a_t d\mu(t)\right)\right). \qquad (36)$$

Having proved the finite case, let us now assume that φ is unbounded, but lower semi-continuous on $\mathcal{A}_+$ and finite on the minimal dense ideal $K(\mathcal{A})$ of $\mathcal{A}$. This – by definition – means that $\varphi(x) < \infty$ if $x \in \mathcal{A}_+$ and $x = xe$ for some e in $\mathcal{A}_+$, because $K(\mathcal{A})$ is the hereditary $*-$subalgebra of $\mathcal{A}$ generated by such elements, cf. [18]. Restricting φ to C we therefore obtain a unique Radon measure μ_φ on S such that

$$\int_S z(t) d\mu_\varphi(t) = \varphi(z) \qquad y \in C. \qquad (37)$$

Inspection of the proof above now shows that the Jensen trace inequality still holds if only $f \geq 0$, even though ∞ may now occur in the inequality.

Theorem (1.1.9)[1]:

Let f be a convex, continuous function defined on an interval I and let $\mathcal{A}$ be a C^*-algebra with a finite trace τ. Then the inequality

$$\tau\left(f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right)\right) \leq \tau\left(\sum_{i=1}^{n} a_i^* f(x_i) a_i\right)$$

is valid for every n−tuple $(x_1, \ldots, x_n)$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I and every n−tuple $(a_1, \ldots, a_n)$ in $\mathcal{A}$ with $\sum_{k=1}^{n} a_k^* a_k = 1$.

Corollary (1.1.10)[288]: Define the unitary matrix $E = \mathrm{diag}(\theta, \theta^2, \ldots, \theta^{n-1}, 1)$ in $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathfrak{H}^n)$, where $\theta = \exp(2\pi\, i/n)$. Then for each element $A_r = \left(a_{ij}^r\right)$ in $\mathbb{B}(\mathfrak{H}^n)$ we have

$$\frac{1}{n}\sum_{k=1}^{n}\sum_{r} E^{-k} A_r E^k = \sum_{r} \mathrm{diag}\,(a_{11}^r, a_{22}^r, \ldots, a_{nn}^r). \qquad (38)$$

Proof. By computation

$$\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{r} E^{-k} A_r E^k\right)_{ij} = \frac{1}{n}\sum_{k=1}^{n}\sum_{r} \left(\theta^{j-i}\right)^k a_{ij}^r, \qquad (39)$$

and this sum is zero if $i \neq j$, otherwise it is a_{ii}^r.

Corollary (1.1.11)[288]: Let P^2 denote the projection in $\mathbb{M}_n(\mathbb{C})$ given by $P_{ij}^2 = n^{-1}$ for all i and j, so that P^2 is the projection of rank one on the subspace spanned by the vector $\xi_1 + \cdots + \xi_n$ in $\mathbb{C}^n$, where $\xi_1, \ldots, \xi_n$ are the standard basis vectors. Then with E as in Corollary (1.1.10) we obtain the pairwise orthogonal projections $P_k^2 = E^{-k} P^2 E^k$, for $1 \leq k \leq n$, with $\sum_{k=1}^{n} P_k^2 = 1$.

Proof. (Cf. [8]) Evidently each P_k^2 is a projection of rank one. Moreover, by Corollary (1.1.10),

$$\sum_{k=1}^{n} P_k^2 = \sum_{k=1}^{n} E^{-k} P^2 E^k = n \ \ \mathrm{diag}(n^{-1}, \ldots, n^{-1}) = 1, \qquad (40)$$

from which it also follows that the projections are pairwise orthogonal.

Corollary (1.1.12)[288]: (Jensen's Operator Inequality).

For a continuous functions f_r defined on an interval I the following conditions are equivalent:

(i) f_r is operator convex.

(ii)For each natural number n we have the inequality

$$\sum_r f_r\left(\sum_{i=1}^n a_i^{*r}x_ia_i^r\right) \le \sum_{i=1}^n\sum_r a_i^{*r}f_r(x_i)a_i^r \tag{41}$$

for every $n-$tuple $(x_1,\dots,x_n)$ of bounded, self-adjoint operators on an arbitrary Hilbert space $\mathfrak{H}$ with spectra contained in I and every $n-$tuple $(a_1^r,\dots,a_n^r)$ of operators on $\mathfrak{H}$ with $\sum_{k=1}^n\sum_r a_k^{*r}a_k^r = 1$.

(iii) $\sum_r f_r(v^*xv) \le \sum_r v^*f_r(x)v$ for each isometry v on an infinite-dimensional Hilbert space $\mathfrak{H}$ and every self-adjoint operator x with spectrum in I.

(vi) $\sum_r p^2f_r(p^2xp^2+s(1-p^2))p^2 \le \sum_r p^2f_r(x)p^2$ for each projection p^2 on an infinite-dimensional Hilbert space $\mathfrak{H}$, every self-adjoint operator x with spectrum in I and every s in I.

Proof. (i) ⇒ (ii) Assume that we are given a unitary n−column $(a_1^r,\dots,a_n^r)$, and choose a unitary $U_n^r=(u_{ij}^r)$ in $\mathbb{B}(\mathfrak{H}^n)$ such that $u_{kn}^r=a_k^r$. Let $E=\text{diag}\,(\theta,\theta^2,\dots,1)$ as in Corollary (1.1.10) and put $X=\text{diag}(x_1,\dots,x_n)$, both regarded as elements in $\mathbb{B}(\mathfrak{H}^n)$. Using Corollary (1.1.10) and the operator convexity of f_r we then get the desired inequality:

$$\sum_r f_r\left(\sum_{k=1}^n a_k^{*r}x_ka_k^r\right) = \sum_r f_r((U_n^{*r}XU_n^r)_{nn}) = \sum_r f_r\left(\left(\sum_{k=1}^n \frac{1}{n}E^{-k}U_n^{*r}XU_n^rE^k\right)_{nn}\right)$$

$$= \left(\sum_r f_r\left(\sum_{k=1}^n \frac{1}{n}E^{-k}U_n^{*r}XU_n^rE^k\right)\right)_{nn}$$

$$\le \sum_r \left(\frac{1}{n}\sum_{k=1}^n f_r(E^{-k}U_n^{*r}XU_n^rE^k)\right)_{nn} = \left(\frac{1}{n}\sum_{k=1}^n\sum_r E^{-k}U_n^{*r}\, f_r(X)U_n^rE^k\right)_{nn}$$

$$= \sum_r (U_n^{*r}f_r(X)U_n^r)_{nn} = \sum_{k=1}^n\sum_r a_k^{*r}f_r(x_k)a_k^r. \tag{42}$$

Note that for the second equality we use that $f_r(y_n) = \left(f_r\left(\text{ diag }(y_1,\dots,y_n)\right)\right)_{nn}$ because $f_r\left(\text{diag }(y_1,\dots,y_n)\right) = \text{diag}\left(f_r(y_1),\dots,f_r(y_n)\right)$.

In the general case where the column is just unital, we enlarge it to the unitary $(n+1)-$column $(a_1^r,\dots,a_n^r,0)$ and choose x_{n+1} arbitrarily, but with spectrum in I. By the first part of the proof we therefore have

$$\sum_r f_r\left(\sum_{k=1}^n a_k^{*r}x_ka_k^r\right) = \sum_r f_r\left(\sum_{k=1}^{n+1} a_k^{*r}x_ka_k^r\right) \le \sum_{k=1}^{n+1}\sum_r a_k^{*r}f_r(x_k)a_k^r$$

$$= \sum_{k=1}^n\sum_r a_k^{*r}f_r(x_k)a_k^r. \tag{43}$$

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (iv) Take any self-adjoint operator x with spectrum in I and let p^2 be an infinite-dimensional projection. Then we can find an isometry v (i.e. $v^*v=1$) such that $p^2=vv^*$. By assumption $\sum_r f_r(v^*xv) \le \sum_r v^*f_r(x)v$, whence also

$$\sum_r vf_r(v^*xv)v^* \le \sum_r vv^*f_r(x)vv^* = \sum_r p^2f_r(x)p^2. \qquad (44)$$

For any monomial $g(t) = t^m$ and any s in I we have

$$\sum_r p^2vg(v^*xv)v^*p^2 = \sum_r p^2v(v^*xv)^mv^*p^2 = \sum_r p^2(vv^*xvv^*)^mp^2$$
$$= \sum_r p^2g(p^2xp^2)p^2$$
$$= \sum_r p^2g\big(p^2xp^2 + s(1-p^2)\big)p^2. \qquad (45)$$

Since f_r is continuous, it can be approximated by polynomials on compact subsets of I, and therefore also $\sum_r p^2vf_r(v^*xv)v^*p^2 = \sum_r p^2f_r\big(p^2xp^2 + s(1-p^2)\big)p^2$. Combined with (43) this gives the pinching inequality

$$\sum_r p^2f_r(p^2xp^2 + s(1-p^2))p^2 \le \sum_r p^2f_r(x)p^2. \qquad (46)$$

If p^2 is a projection of finite rank we can define the infinite dimensional projection $q^2 = p^2 \otimes 1_\infty$ on $\mathfrak{H}^\infty$. Similarly we let $y = x \otimes 1_\infty$ for any given self-adjoint operator x with spectrum in I. Since $f_r(a^r) \otimes 1_\infty = f_r(a^r \otimes 1_\infty)$ for any operator a^r we get by (46) that

$$\sum_r p^2f_r(p^2xp^2 + s(1-p^2))p^2 \otimes 1_\infty = \sum_r q^2f_r(q^2yq^2 + s(1-q^2))q^2$$
$$\le \sum_r q^2f_r(y)q^2 = \sum_r p^2f_r(x)p^2 \otimes 1_\infty, \qquad (47)$$

which shows that (46) is valid also for projections of finite rank.

(iv) ⇒ (1) Given self-adjoint operators x and y with spectra in I and λ_r in [0, 1], define the three elements

$$X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \qquad U^r = \begin{pmatrix} \lambda_r^{\frac{1}{2}} & (1-\lambda_r)^{\frac{1}{2}} \\ -(1-\lambda_r)^{\frac{1}{2}} & \lambda_r^{\frac{1}{2}} \end{pmatrix}, \qquad P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (48)$$

in $\mathbb{B}(\mathfrak{H}^2)$. Then for some s in I we have by the pinching inequality in (iv) that

$$\sum_r P^2f_r\big(P^2\, U^{*r}XU^rP^2 + s(1-P^2)\big)P^2 \le \sum_r P^2\, f_r(U^{*r}XU^r)P^2$$
$$= \sum_r P^2\, U^{*r}f_r(X)U^rP^2. \qquad (49)$$

Since

$$U^{*r}XU^r = \begin{pmatrix} \lambda_rx + (1-\lambda_r)y & (\lambda_r - \lambda_r^2)^{\frac{1}{2}}(y-x) \\ (\lambda_r - \lambda_r^2)^{\frac{1}{2}}(y-x) & \lambda_ry + (1-\lambda_r)x \end{pmatrix}, \qquad (50)$$

it follows that

$$\sum_r \begin{pmatrix} f_r(\lambda_r x + (1-\lambda_r)y) & 0 \\ 0 & 0 \end{pmatrix} = \sum_r P^2 f_r\left(P^2 U^{*r} X U^r P^2 + s(1-P^2)\right)P^2$$
$$\leq \sum_r P^2 U^{*r} \begin{pmatrix} f_r(x) & 0 \\ 0 & f_r(y) \end{pmatrix} U^r P^2$$
$$= \sum_r \begin{pmatrix} \lambda_r f_r(x) + (1-\lambda_r) f_r(y) & 0 \\ 0 & 0 \end{pmatrix}. \quad (51)$$

Corollary (1.1.13)[288]: (Contractive Version). Let f_r be a continuous functions defined on an interval I and suppose that 0∈I. Then f_r is operator convex and $f_r(0) \leq 0$ if and only if for some, hence every natural number n, the inequality (41) is valid for every $n-$tuple $(x_1, \dots, x_n)$ of bounded, self-adjoint operators on a Hilbert space $\mathfrak{H}$ with spectra contained in I, and every $n-$tuple $(a_1^r, \dots, a_n^r)$ of operators on $\mathfrak{H}$ with $\sum_{k=1}^n \sum_r a_k^{*r} a_k^r \leq 1$.

Setting $n = 1$ we see that f_r is operator convex on an interval I containing 0 with $f_r(0) \leq 0$ if and only if
$$\sum_r f_r(a^{*r} x a^r) \leq \sum_r a^{*r} f_r(x) a^r \quad (52)$$
for every self-adjoint x with spectrum in I and every contraction a^r. This is the original Jensen operator inequality from [10] (see [22]).

Proof. If $\sum_{k=1}^n a_k^{*r} a_k^r = b^r \leq 1$, put $a_{n+1}^r = (1-b^r)^{\frac{1}{2}}$. Then we have a unital (n+1)−tuple, so with $x_{n+1} = 0$ we get
$$\sum_r f_r\left(\sum_{k=1}^n a_k^{*r} x_k a_k^r\right) = \sum_r f_r\left(\sum_{k=1}^{n+1} a_k^{*r} x_k a_k^r\right) \leq \sum_{k=1}^{n+1} \sum_r a_k^{*r} f_r(x_k) a_k^r$$
$$= \sum_{k=1}^n \sum_r a_k^{*r} f_r(x_k) a_k^r + \sum_r a_{n+1}^{*r} f_r(0) a_{n+1}^r \leq \sum_{k=1}^n \sum_r a_k^{*r} f_r(x_k) a_k^r. \quad (53)$$
Conversely, if (41) is satisfied for all contractive n−tuples, then – a fortiori – it holds for unital n−tuples, so f_r is operator convex; and with $a^r = x = 0$ we see that $f_r(0) \leq 0 \cdot f_r(0) \cdot 0 = 0$.

Corollary (1.1.14)[288]: (Jensen's Trace Inequality).
Let f_r be a continuous functions defined on an interval I and let m and n be natural numbers. If f_r is convex we then have the inequality
$$Tr\left(\sum_r f_r\left(\sum_{i=1}^n a_i^{*r} x_i a_i^r\right)\right) \leq Tr\left(\sum_{i=1}^n \sum_r a_i^{*r} f_r(x_i) a_i^r\right) \quad (54)$$
for every n−tuple $(x_1, \dots, x_n)$ of self-adjoint $m \times m$ matrices with spectra contained in I and every n−tuple $(a_1^r, \dots, a_n^r)$ of $m \times m$ matrices with $\sum_{k=1}^n \sum_r a_k^{*r} a_k^r = 1$.

Proof. Let $x_k = \sum_{\mathrm{sp}(x_k)} \sum_r \lambda_r E_k(\lambda_r)$ denote the spectral resolution of x_k for $1 \leq k \leq n$. Thus, $E_k(\lambda_r)$ is the spectral projection of x_k on the eigenspace corresponding to λ_r if λ_r is an eigenvalue for x_k; otherwise $E_k(\lambda_r) = 0$. For each unit vector ξ in $\mathbb{C}^m$ define the (atomic) probability measure
$$\mu_\xi(S) = \left(\sum_{k=1}^n \sum_r a_k^{*r} E_k(S) a_k^r \xi | \xi\right) = \sum_{k=1}^n \sum_r (E_k(S) a_k^r \xi | a_k^r \xi) \quad (55)$$
for any (Borel) set S in ℝ. Note now that if $y = \sum_{k=1}^n \sum_r a_k^{*r} x_k a_k^r$ then

$$(y\xi|\xi) = \left(\sum_{k=1}^{n}\sum_{r} a_k^{*r} x_k a_k^r \xi|\xi\right) = \left(\sum_{k=1}^{n} \sum_{\mathrm{sp}(x_k)} \sum_{r} \lambda_r E_k(\lambda_r) a_k^r \xi | a_k^r \xi\right)$$
$$= \int \sum_{r} \lambda_r \, d\mu_\xi(\lambda_r). \quad (56)$$

If a unit vector ξ is an eigenvector for y, then the corresponding eigenvalue is $(y\xi|\xi)$, and ξ is also an eigenvector for $f_r(y)$ with correponding eigenvalue $\sum_r (f_r(y)\xi|\xi) = \sum_r f_r((y\xi|\xi))$. In this case we therefore have

$$\sum_{r}\left(f_r\left(\sum_{k=1}^{n} a_k^{*r} x_k a_k^r\right)\xi|\xi\right) = \sum_{r} (f_r(y)\xi|\xi) = \sum_{r} f_r\big((y\xi|\xi)\big)$$
$$= \sum_{r} f_r\left(\int \lambda_r \, d\mu_\xi(\lambda_r)\right) \le \int \sum_{r} f_r(\lambda_r) d\mu_\xi(\lambda_r)$$
$$= \sum_{k=1}^{n}\left(\sum_{\mathrm{sp}(x_k)}\sum_{r} f_r(\lambda_r) E_k(\lambda_r) a_k^r \xi | a_k^r \xi\right)$$
$$= \sum_{k=1}^{n}\sum_{r} (a_k^{*r} f_r(x_k) a_k^r \xi|\xi), \quad (57)$$

where we used (56) and the convexity of f_r – in form of the usual Jensen inequality – to get the inequality in (57). The result in (54) now follows by summing over an orthonormal basis of eigenvectors for y.

Conversely, if (54) holds for some pair of natural numbers n, m, where $n > 1$, then taking $a_i^r = 0$ for $i \geq 2$ we see that the inequality holds for n=2. Given s, t in I and λ_r in [0, 1] we define $x = s1_m$ and $y = t1_m$ in $\mathbb{M}_m(\mathbb{C})$. Then with $a^r = \lambda_r^{\frac{1}{2}} 1_m$ and $b^r = (1-\lambda_r)^{\frac{1}{2}} 1_m$ we get by (54) that

$$\sum_{r} m f_r(\lambda_r x + (1-\lambda_r)t) = \sum_{r} Tr(f_r(\lambda_r x + (1-\lambda_r)t)1_m)$$
$$= \sum_{r} Tr\big(f_r(a^{*r} x a^r + b^{*r} y b^r)\big) \le \sum_{r} Tr(a^{*r} f_r(x) a^r + b^{*r} f_r(y) b^r)$$
$$= \sum_{r} Tr\Big(\big(\lambda_r f_r(s) + (1-\lambda_r) f_r(t)\big)1_m\Big)$$
$$= \sum_{r} m\big(\lambda_r f_r(s) + (1-\lambda_r) f_r(t)\big), \quad (58)$$

which shows that f_r is convex.

Corollary (1.1.15)[288]: Let $(x_t)_{t\in T}$ be a bounded, continuous field on a locally compact Hausdorff space T consisting of self-adjoint elements in a C^*−algebra $\mathcal{A}$ with $\mathrm{sp}(x_t) \subset I$. Furthermore, let $(a_t^r)_{t\in T}$ be a unital column field in $M(\mathcal{A})$ with respect to some Radon measure μ on T. Then for each continuous, convex functions f_r defined on I and every positive functional φ that contains the element $y = \int_T \sum_r a_t^{*r} x_t a_t^r d\mu(t)$ in its centralizer $\mathcal{A}^\varphi$, i.e. $\varphi(xy) = \varphi(yx)$ for all x in $\mathcal{A}$, we have the inequality:

$$\varphi\left(\sum_r f_r\left(\left(\int_T a_t^{*r} x_t a_t^r \, d\mu(t)\right)\right)\right) \le \varphi\left(\int_T \sum_r a_t^{*r} f_r(x_t) a_t^r d\mu(t)\right). \qquad (59)$$

If φ is unbounded, but lower semi-continuous on $\mathcal{A}_+$ and finite on the minimal dense ideal $K(\mathcal{A})$ of $\mathcal{A}$, the result still holds if $f_r \ge 0$, even though the function may now attain infinite values.

Proof. Let $C = C_o(S)$ denote the commutative C^*-subalgebra of $\mathcal{A}$ generated by y, and let μ_φ be the finite Radon measure on the locally compact Hausdorff space S defined, via the Riesz representation theorem, by

$$\int_S z(s) d\mu_\varphi(s) = \varphi(z) \qquad z \in C = C_o(S). \qquad (60)$$

Since for all (x, z) in $M(\mathcal{A})_+ \times C_+$ we have $\varphi(xz) = \varphi\left(z^{\frac{1}{2}} x z^{\frac{1}{2}}\right)$ it follows that

$$0 \le \varphi(xz) \le \|x\| \varphi(z). \qquad (61)$$

Consequently the functional $z \to \varphi(xz)$ on C defines a Radon measure on S dominated by a multible of μ_φ, hence determined by a unique element Φ(x) in $L^\infty_{\mu_\varphi}(S)$. By linearization this defines a conditional expectation $\varphi: M(\mathcal{A}) \to L^\infty_{\mu_\varphi}(S)$ (i.e. a positive, unital module map) such that

$$\int_S z(s) \Phi(x)(s) d\mu_\varphi(s) = \varphi(zx), \qquad z \in C \qquad x \in M(\mathcal{A}). \qquad (62)$$

Inherent in this formulation is the fact that if $z \in C = C_o(S)$, then Φ(z) is the natural image of z in $L^\infty_{\mu_\varphi}(S)$. In particular, $z(s) = \Phi(z)(s)$ for almost all s in S.

Observe now that since the C^*-algebra $C_o(I)$ is separable, we can for almost every s in S define a Radon measure μ_s on I by

$$\int_I \sum_r g(\lambda_r) d\mu_s(\lambda_r) = \Phi\left(\int_T \sum_r a_t^{*r} g(x_t) a_t^r d\mu(t)\right)(s) \qquad g \in C(I). \qquad (63)$$

As $\int_T \sum_r a_t^{*r} a_t^r d\mu(t) = 1$, this is actually a probability measure. If we take $g(\lambda_r) = \lambda_r$, then

$$\int_I \sum_r \lambda_r \, d\mu_s(\lambda_r) = \Phi\left(\int_T \sum_r a_t^{*r} x_t a_t^r d\mu(t)\right)(s) = \Phi(y)(s) = y(s). \qquad (64)$$

Since $y \in C$ we get by (64) and (63) – using also the convexity of f_r in form of the standard Jensen inequality – that

$$\sum_r f_r(y)(s) = \sum_r f_r(y(s)) = \sum_r f_r\left(\int_I \lambda_r \, d\mu_s(\lambda_r)\right) \le \int \sum_r f_r(\lambda_r) d\mu_s(\lambda_r)$$

$$= \Phi\left(\int \sum_r a_t^{*r} f_r(x_t) a_t^r d\mu(t)\right)(s). \qquad (65)$$

Integrating over s, using (62), now gives the desired result:

$$\sum_r \varphi(f_r(y)) = \int_S \sum_r f_r(y)(s) d\mu_\varphi(s) \le \int_S \Phi\left(\int_T \sum_r a_t^{*r} f_r(x_t) a_t^r d\mu(t)\right)(s) d\mu_\varphi(s)$$
$$= \int_T \int_S \sum_r \Phi(a_t^{*r} f_r(x_t) a_t^r)(s) d\mu_\varphi(s) d\mu(t) = \int_T \sum_r \varphi(a_t^{*r} f_r(x_t) a_t^r) d\mu(t)$$
$$= \varphi\left(\int_T \sum_r a_t^{*r} f_r(x_t) a_t^r d\mu(t)\right). \tag{66}$$

Having proved the finite case, let us now assume that φ is unbounded, but lower semi-continuous on $\mathcal{A}_+$ and finite on the minimal dense ideal $K(\mathcal{A})$ of $\mathcal{A}$. This – by definition – means that $\varphi(x) < \infty$ if $x \in \mathcal{A}_+$ and $x = xe$ for some e in $\mathcal{A}_+$, because $K(\mathcal{A})$ is the hereditary $*-$subalgebra of $\mathcal{A}$ generated by such elements, cf. [18]. Restricting φ to C we therefore obtain a unique Radon measure μ_φ on S such that

$$\int_S z(t) d\mu_\varphi(t) = \varphi(z) \qquad y \in C. \tag{67}$$

Inspection of the proof above now shows that the Jensen trace inequality still holds if only $f_r \ge 0$, even though ∞ may now occur in the inequality.

Corollary (1.1.16)[288]: (Jensen's Trace Inequality for C^*-Algebras).

Let f_r be a convex, continuous functions defined on an interval I and let $\mathcal{A}$ be a C^*-algebra with a finite trace τ. Then the inequality

$$\tau\left(\sum_r f_r\left(\sum_{i=1}^n a_i^{*r} x_i a_i^r\right)\right) \le \tau\left(\sum_{i=1}^n \sum_r a_i^{*r} f_r(x_i) a_i^r\right)$$

is valid for every n−tuple $(x_1, \dots, x_n)$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I and every n−tuple $(a_1^r, \dots, a_n^r)$ in $\mathcal{A}$ with $\sum_{k=1}^n \sum_r a_k^{*r} a_k^r = 1$.

Proof. Evidently this (like Corollary (1.1.14)) is a special case of Corollary (1.1.15), where the continuous field is replaced by a finite sum and the functional φ is a trace, so that $\mathcal{A}^\varphi = \mathcal{A}$.

Section (1.2): Spectral Order and Submajorization

Jensen's inequality is the continuous version of the usual definition of convex function and it can be stated in the following way: let I be an open interval and $f: I \to \mathbb{R}$ a convex map. Then, for every probability space (X, P) and every integrable map $g: X \to I$,

$$f\left(\int_X g\, dP\right) \le \int_X f\, o g dP.$$

the context of C^*-algebras the simplest generalization of Jensen's inequality can be made by taking a state (ℓ and a selfadjoint element a of a C^*-algebra A such that $\sigma(a) \subseteq I$. In this case,

$$f(\phi(a)) \le \phi(f(a)), \tag{68}$$

because the state ϕ restricted to the C^*-algebra generated by a can be represented as an integral with respect to a probability measure. If one replaces ϕ by a unital positive map between two C^*-algebras, inequality (68) is only true for operator convex functions f, as one can prove from the well-known characterizations of operator convexity (see [26], [27], [29], [1]), using the Stinespring's theorems. Previous works on the matter, such as Brown-

Kosaki [4] and Hansen-Pedersen [1] suggest the idea of studying Jensen's type inequalities with respect to other preorders, such as the spectral order and submajorization in order to consider convex (but not operator convex) functions.
We study different Jensen's type inequalities for a positive (unital) map between two C^*-algebras, with respect to the above mentioned preorders. We mention the main results, which are stated. Let $A, \mathcal{B}$ be unital C^*-algebras, $\varphi: A \to \mathcal{B}$ a positive unital map, f a convex function defined on an open interval I and $a \in A$, such that $a = a^*$ and $\sigma(a) \subseteq I$.
(i) If f is monotone and $\mathcal{B}$ is a von Neumann algebra, then
$f(\varphi(a)) \leq \varphi(f(a))$ (spectral preorder).
(ii) If $\mathcal{B}$ is abelian or, more generally, if $\varphi(f(a))$ and $\varphi(a)$ commute, then
(iii) If $\mathcal{B}$ a finite factor, then $f(\varphi(a)) <_w \varphi(f(a))$ (submajorization).

$$f(\varphi(a)) \leq \varphi(f(a)) .$$

We remark that all these inequalities still hold for *contractive* positive maps, under the assumption that $0 \in I$ and $f(0) \leq 0$ we briefly describe the multi-variable functional calculus and obtain similar results to those by using essentially the same techniques. we apply the results obtained to the finite dimensional case, where there exist fairly simple expressions for the spectral preorder and for the submajorization. Several results were included in a Los Alamos preprint version in 2004 (see [23]). Since then, some related results have appeared; for example [24], which has some overlap with or [29], [32], where our results for finite factors are applied.

ForA be a C^*-algebra. A_{sa} denotes the real vector space of selfadjoint elements of A, A^+ the cone of positive elements, G1(A) the group of invertible elements of A and $U(A)$ its unitary group. We also assume that all the C^*-algebras in consideration are unital. Given a Hilbert space $\mathcal{H}$, we denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The range of $c \in L(\mathcal{H})$ will be denoted by (c) , its null space by ker$c$ and its spectrum by $\sigma(c)$. If p, $q \in L(\mathcal{H})$ are orthogonal projections, we denote by $p \wedge q$ (respectively $p \vee q$) the orthogonal projection onto of their ranges (respectively the closed subspace generated by their ranges).

In what follows $E_a[I]$ denotes the spectral projection of a self-adjoint operator a in a von Neumann algebra A, corresponding to a (Borel) subset $I \subseteq \mathbb{R}$. Let us recall the notion of spectral preorder.

Definition (1.2.1)[22]: Let A be a von Neumann algebra. Given a, $b \in A_{sa}$, we say that $a < b \sim$ if $E_a[(\alpha, +\infty)]$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $E_b[(\alpha, +\infty)]$ for every real number α.
In finite factors the following result can be proved (see [4], [32]).

Proposition (1.2.2)[22]: *Let A be a finite factor, with normalized trace* tr. *Given* $a, b \in \mathcal{A}_{sa}$, *the following conditions are equivalent*:
(i) $a \precsim b$.
(ii) $\mathrm{tr}(f(a)) \leq \mathrm{tr}(f(b))$ *for every continuous increasing function* f *defined on an interval containing both* $\sigma(a)$ *and* $\sigma(b)$.
(iii) *There exists a sequence* $\{u_n\}$ *in* $\mathcal{U}(\mathcal{A})$ *such that* $u_n b u_n^* \overrightarrow{n \to \infty} c$ *and* $c \geq a$.

Finally, Eizaburo Kamei defined the notion of majorization and submajorization in finite factors (see [31]):

Definition (1.2.3)[22]: Let $\mathcal{A}$ be a finite factor with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, we say that a is submajorized by b, and denote $a \prec_w b$, if the inequality

$$\int_0^\alpha e_a(t)dt \leq l^\alpha b(t)dt0^e$$

holds for every real number α, where $e_c(t) = \inf\{\gamma: \mathrm{tr}(E_c[(\gamma, \infty)]) \leq t\}$ for $c \in \mathcal{A}_{sa}$. An equivalent condition (see [30]) is that

$$\sup\{\mathrm{tr}(ap): p \in \mathcal{P}_k\} \leq \sup\{\mathrm{tr}(bp): p \in \mathcal{P}_k\}, 0 \leq k \leq 1,$$

where $\mathcal{P}_k = \{p \in \mathcal{A}_{sa}: p^2 = p$ and $\mathrm{tr}\, p = k\}$. We say that a is majorized by b (and denote $a \prec b$) if $a \prec_w b$ and $\mathrm{tr}(a) = \mathrm{tr}(b)$.

The following characterization of submajorization also appears in [31].

Proposition (1.2.4)[22]: *Let A be a finite factor with normalized trace* tr. *Given* , $b \in \mathcal{A}_{sa}$, *the following conditions are equivalent*:

(i) $a \prec_w b$.

(ii) $\mathrm{tr}(g(a)) \leq \mathrm{tr}(g(b))$ *for every non-decreasing convexfunction g defined on an interval containing both $\sigma(a)$ and $\sigma(b)$* .

Throughout φ is a positive unital map from a C^*-algebra A to another C^*-algebra $\mathcal{B}$, $f: I \to \mathbb{R}$ is a convex function defined on an open interval I and $a \in \mathcal{A}_{sa}$ whose spectrum lies in I. Note that the spectrum of $\varphi(a)$ is also contained in I. As we mentioned we cannot expect a Jensen's type inequality of the form

$$f(\varphi(a)) \leq \varphi(f(a)) \tag{69}$$

for an arbitrary convex function f without other assumptions. This is the reason why, in order to study inequalities similar to (ii) for different subsets of convex functions, we shall use the spectral and submajorization (pre)orders, or we shall change the hypothesis made over $\mathcal{B}$. Although most of the inequalities considered involve unital positive maps, similar results can be obtained for contractive positive maps by adding some extra hypothesis on f.

We shall consider monotone convex and concave functions. The following result, due to Brown and Kosaki [4], indicates that the appropriate order relation for this class of functions is the spectral preorder. Let $\mathcal{A}$ be a semi-finite von Neumann algebra, and let $v \in \mathcal{A}$ be a contraction; then, for every positive operator $a \in \mathcal{A}$ and every continuous monotone convex function f defined in $[0, +\infty)$ such that $f(0) = 0$, it holds that

$$v^* f(a) v \precsim f(v^* a v) .$$

The following statement is an analogue of Brown and Kosaki's result, in terms of positive unital maps and monotone convex functions. The proof we give below follows essentially the same lines as that in [4].

Theorem (1.2.5)[22]: *If $\mathcal{B}$ is a von Neumann algebra, and f is monotone convex, then*

$$f(\varphi(a)) \precsim \varphi(f(a)) . \tag{70}$$

Proof. Given $\alpha \in \mathbb{R}$, denote by $\{f > \alpha\} = \{t \in \mathbb{R}: f(t) > \alpha\}$. We shall prove that there exists a projection $q_\alpha \in A$ such that

$$E_{f(\varphi(a))}[(\alpha, +\infty)] = E_{\varphi(a)}[\{f > \alpha\}] \sim q_\alpha \leq E_{\varphi(f(a))}[(\alpha, +\infty)].$$

We claim that $E_{\varphi(a)}[\{f > \alpha\}] \wedge E_{\varphi(f(a))}[(-\infty, \alpha]] = 0$. Consider a unit vector $\overline{\eta} \in R(E_{\varphi(a)}[\{f > \alpha\}])$. Since f is monotone we have that $\alpha < f(\langle\varphi(a)\overline{\eta}, \overline{\eta}\rangle)$ and, using Jensen's inequality for states, Eq. (68), we get $\alpha < \langle\varphi(f(a))\overline{\eta}, \overline{\eta}\rangle$. On the other hand, if $\overline{\xi} \in R(E_{\varphi(f(a))}[(-\infty, \alpha]])$ is a unit vector, then $\geq \langle\varphi(f(a))\overline{\xi}\ ,\ \overline{\xi}\rangle$. So, using Kaplansky's formula, we have

$$E_{f(\varphi(a))}[(\alpha, +\infty)] = E_{f(\varphi(a))}[(\alpha, +\infty)] - (E_{f(\varphi(a))}[(\alpha, +\infty)] \wedge E_{\varphi(f(a))}[(-\infty, \alpha]])$$
$$\sim (E_{f(\varphi(a))}[(\alpha, +\infty)] \vee E_{\varphi(f(a))}[(-\infty, \alpha]]) - E_{\varphi(f(a))}[(-\infty, \alpha]]$$
$$\leq I - E_{\varphi(f(a))}[(-\infty, \alpha]] = E_{\varphi(f(a))}[(\alpha, +\infty)].$$

If $\varphi(a)$ and $\varphi(f(a))$ are compact operators and $f(0) = 0$, Theorem (1.2.5) can be rephrased in terms of the following interpretation of the spectral order:

Proposition (1.2.6)[22]: *Let $\mathcal{H}$ be a Hilbert space. Let $a, b \in L(\mathcal{H})^+$ be compact operators, and $\{\lambda_n\}_{n\leq N}$ (respectively $\{\mu_n\}_{n\leq M}$) the decreasing sequence of positive eigenvalues of a (respectively b), counted with multiplicity $(N, M \in \mathbb{N}\text{U}\{\infty\})$. Suppose that $a \precsim b$. Then*
(i) $\lambda_m \leq \mu_m$ for every $m \leq \min\{N, M\}$.
(ii) *There exists a partial isometry $u \in L(\mathcal{H})$ with initial space $\overline{R(a)}$ such that, if $c = uau^*, c \leq b$ and $cb = bc$.*
Moreover, if $\dim(\mathcal{H}) < \infty$, *then* (ii) *holds for $a, b \in L(\mathcal{H})_{sa}$, for some $u \in u(\mathcal{H})$.*

Proof. Given $\alpha > 0$, let $n_\alpha = \max\{m \leq N: \lambda_m > \alpha\}$ and $m_\alpha = \max\{m \leq M: \mu_m > \alpha\}$ (we take $n_\alpha = 0$ if $\alpha \geq \|a\|$ and similarly for m_α). By hypothesis

$$n_\alpha = \text{tr}E_a[(\alpha, +\infty)] \leq \text{tr}E_b[(\alpha, +\infty)] = m_\alpha.$$

Taking $\alpha = \lambda_m - \varepsilon$ (for every $0 < \varepsilon < \lambda_m$), one deduces that $\lambda_m \leq \mu_m$. Let $\{\xi_n\}_{n\leq N}$ be an orthonormal basis of $\overline{R(a)}$ of eigenvectors associated to the sequence $\{\lambda_n\}_{n\leq N}$ of a. Define in a similar way, $\{\eta_m\}_{m\leq M}$ associated to $\{\mu_m\}_{m\leq M}$ for b. Consider the isometry $u: \overline{R(a)} \to \overline{R(b)}$, given by $(\xi_n) = \eta_n$, $n \leq N$, and extend u to a partial isometry with ker$u$ =ker$a$. Let $c = uau^*$. Then $cb = bc$ and $c \leq b$, since $\{\eta_m\}_{m\leq N}$ is an orthonormal basis of $\overline{R(c)}$ of eigenvectors associated to $\{\lambda_n\}_{n\leq N}$. If $\dim \mathcal{H} = n < \infty$ and $, b \in L(\mathcal{H})_{sa}$, we can include the eventual nonpositive eigenvalues to the previous argument, getting orthonormal basis of $\mathcal{H}$, so that u becomes unitary.

Example (1.2.7)[22]: Let $B = \{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of the separable Hilbert space $\mathcal{H}$ and let $b \in L(\mathcal{H})^+$ be the diagonal operator (w.r. t. B) with diagonal $\{2^{-n}\}_{n\in\mathbb{N}}$. If s is the backward shift (w.r. t. B), let $a = s^*bs$, i.e. $a\xi_1 = 0$ and $a\xi_n = 2^{n-1}\xi_n$ for $n \geq 2$. Then, $a \prec b \sim$ because $\text{tr}[E_a(\lambda, \infty)] = \text{tr}[E_b(\lambda, \infty)]$ for every $\lambda \geq 0$. Suppose that there exists an isometry $v \in L(\mathcal{H})$ such that $v^*bv \geq a$. Then

$$\big(bv(\xi_2), v(\xi_2)\big) \geq \{a(\xi_2), a(\xi_2)) = \frac{1}{2} \Rightarrow \text{v}(\xi_2) = \xi_1.$$

Similarly, using that v is an isometry, it can proved that $v(\xi_n) = \xi_{n-1}$ for every $n \geq 2$. Therefore, $v = s$ which is not an isometry.

The following example, due to J.S. Aujla and Γ.C. Silva [25], shows that Theorem (1.2.5) may be false if the function f is not monotone.

Example (1.2.8)[22]: Consider the positive map $\varphi: \mathcal{M}_4 \to \mathcal{M}_2$ given by

$$\varphi\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{A_{11}+A_{22}}{2}.$$

Take $f(t) = |t|$ and let A be the following matrix

$$A = \left(\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Then

$$\varphi(f(A)) = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \text{ and } f(\varphi(A)) = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

Therefore

$$rank\ \left(E_{\varphi(f(A))}[(0.5, +\infty)]\right) = 1 < 2 = \ rank\ \left(E_{f(\varphi(A))}[(0.5, +\infty)]\right).$$

A Jensen's type inequality holds with respect to the usual order for every convex function, if the map φ takes values in a commutative algebra $\mathcal{B}$.

Proposition (1.2.9)[22]: *If $\mathcal{B}$ is abelian, then $(\varphi(a)) \leq \varphi(f(a))$.*

Proof. For every character Γ of the algebra B, $\Gamma \circ \varphi$ is a state over the C^*-algebra A. Thus, using Eq. (i) (Jensen's inequality for states), $\left(f(\varphi(a))\right) = f\left(\Gamma(\varphi(a))\right) \leq \Gamma\left(\varphi(f(a))\right)$.

Now, we shall prove a Jensen inequality for arbitrary convex functions, with respect to the submajorization (pre)order.

Lemma (1.2.10)[22]: *Let* tr *be a trace defined on $\mathcal{B}$, and let $b \in \mathcal{B}$. Then, there exist a Borel measure μ defined on $\alpha(b)$ and a positive unital linear map $\Psi: \mathcal{B} \to L^\infty(\sigma(b), \mu)$ such that*:

(i) $(f(b)) = f$ for *every* $f \in C(\sigma(b))$.

(ii) $\mathrm{tr}(x) = \int_{\sigma(b)} \Psi\,(x)(t)d\mu(t)\ for\ every\ x \in \mathcal{B}.$

Proof. First of all, note that for every continuous function g defined on the spectrum of b, the map

$$g \to \mathrm{tr}(g(b))$$

is a bounded linear functional on $C(\sigma(b))$. Therefore, by the Riesz's representation theorem, there exists a Borel measure μ defined on the Borel subsets of (b) , such that for every continuous function g on $\sigma(b)$,

$$\mathrm{tr}(g(b)) = \int_{\sigma(b)} g\,(t)d\mu(t)\,.$$

Now, given $x \in \mathcal{B}^+$, define the following functional on $C(\sigma(b))$:

$$g \to \mathrm{tr}(xg(b))\,.$$

Since for every $y \in \mathcal{B}^+$, $\mathrm{tr}(xy) = \mathrm{tr}(y^{1/2}xy^{1/2}) \leq \|x\|\mathrm{tr}(y)$, this functional is not only bounded but also dominated by the functional defined before. So, there exists an element h_X of $L^\infty(\sigma(b), \mu)$ such that, for every $\in C(\sigma(b))$,

$$\mathrm{tr}(xg(b)) = \int_{\sigma(b)} g\,(t)h_x(t)d\mu(t)\,.$$

The map $\mapsto h_x$, extended by linearization, defines a positive unital linear map $\Psi: \mathcal{B} \to L^\infty(\sigma(b), \mu)$ which satisfies conditions (i) and (ii) because

$$\operatorname{tr}\big(f(b)g(b)\big) = \operatorname{tr}\big(fg(b)\big) = \int_{\alpha(b)} g\,(t)f(t)d\mu(t)\ and$$

$$\operatorname{tr}(x) = \operatorname{tr}\big(x1(b)\big) = \int_{\sigma(b)} 1\,\Psi(x)(t)d\mu(t) = \int_{\sigma(b)} \Psi\,(x)(t)d\mu(t)\,.$$

Theorem (1.2.11)[22]: *Suppose that $\mathcal{B}$ is a finite factor. Then*
In order to prove this theorem, we need the following lemma.

$$f\big(\varphi(a)\big) \prec_w \varphi\big(f(a)\big)\,. \tag{71}$$

Proof. By Proposition (1.2.4), it is enough to prove that

$$\operatorname{tr}\left[g\left(f\big(\varphi(a)\big)\right)\right] \le \operatorname{tr}\left[g\left(\varphi\big(f(a)\big)\right)\right]$$

for every non-decreasing convex function g such that $g\left(f\big(\varphi(a)\big)\right)$ and $g\left(\varphi\big(f(a)\big)\right)$ are well defined. Fix such a function g. Let $\Psi{:}\,\mathcal{B} \to L^\infty(\sigma(b),\mu)$ be the positive unital linear map given by Lemma (1.2.10) for $= \varphi(a)$. Define the map

$$\Phi{:}\,C\big(\sigma(a)\big) \to L^\infty(\sigma(b),\mu)\ by\ \Phi(h) = \Psi\left(\varphi\big(h(a)\big)\right).$$

Then, Φ is bounded, unital and positive. Moreover, given $h \in C\big(\sigma(a)\big)$,

$$\operatorname{tr}\left(\varphi\big(h(a)\big)\right) = \int_{\sigma(b)} \Psi\left(\varphi\big(h(a)\big)\right)(t)d\mu(t) = \int_{\sigma(b)} \Phi\,(h)(t)d\mu(t)\,.$$

Then, using Proposition (1.2.9) and the fact that g is non-decreasing,

$$\begin{aligned}\operatorname{tr}\big(g\big[f\big(\varphi(a)\big)\big]\big) &= \operatorname{tr}\big(g\mathrm{o}f(b)\big) = \int_{\sigma(b)} g\,\mathrm{o}f(t)d\mu(t) = \int_{\sigma(b)} g\,\big[f\big(\Phi(Id)\big)\big]\,d\mu(t)\\ &\le \int_{\sigma(b)} g\,[\Phi(f)(t)]d\mu(t) = \int_{\sigma(b)} g\left[\Psi\left(\varphi\big(f(a)\big)\right)(t)\right]d\mu(t)\\ &\le \int_{\sigma(b)} \Psi\,\big(g\big[\varphi\big(f(a)\big)\big]\big)(t)d\mu(t) = \operatorname{tr}\big(g\big[\varphi\big(f(a)\big)\big]\big)\,,\end{aligned}$$

which completes the proof.

Remark (1.2.12)[22]: Let $C \subseteq B$ be C^*-algebras. A conditional expectation $8{:}\,B \to C$ is a positive C-linear projection from $\mathcal{B}$ onto C of norm 1. The centralizer of 8 is the C^*-subalgebra of $\mathcal{B}$ defined by $\mathcal{B}^{\mathcal{E}} = \{b \in B{:}\,\mathcal{E}(ba) = \mathcal{E}(ab), \forall a \in \mathcal{B}\}$. Following similar ideas as those in Lemma (1.2.10) and the proof of Theorem (1.2.11) it can be proved that, if $\varphi(a)$ belongs to $\mathcal{B}^{\mathcal{E}}$, then

$$\mathcal{E}\big(g\big[f\big(\varphi(a)\big)\big]\big) \le \mathcal{E}\big(g\big[\varphi\big(f(a)\big)\big]\big) \tag{72}$$

for every non-decreasing convex function $g{:}\,J \to \mathbb{R}$ defined on some open interval J such that $f(I) \subseteq J$. This fact is similar to Hansen and Pedersen's results obtained in [1].

We shall be concetned with the restatement, in the multi-variable context, of several results obtained for related results, see [30], [33].

Let $\mathcal{A}$ be a unital C^*-algebra and let $a_1, \dots, a_n$ be mutually commuting elements of $\mathcal{A}_{sa}$. If $\mathcal{B} = C^*(a_1, \dots, a_n)$ denotes the unital C^*-subalgebra of $\mathcal{A}$ generated by these elements, then B is abelian. So there exists a compact Hausdorff space X such that $B$is $*$-isomorphic to (X) . Actually X is (up to homeomorphism) the space of characters of $\mathcal{B}$.

Recall that in the case of one operator $\in \mathcal{A}_{sa}$, X is homeomorphic to $\sigma(a)$. In general, characters of the algebra $\mathcal{B}$ are associated in a continuous and injective way to n-tuples $(\lambda_1 , \lambda_n) \in \prod_{i=1}^{n} \sigma(a_i)$ by the correspondence $\mapsto (\gamma(a_1), \dots, \gamma(a_n))$. Thus X is homeomorphic to its image under this map, which we calljoint spectrum and denote $(a_1, \dots, a_n)$.

Let $\in C(\sigma(a_1, \dots, a_n))$. Denote by $(a_1, \dots, a_n) \in C^*(a_1, \dots, a_n)$, the element that corresponds to f by the above $*$-isomorphism. Note that by Tietze's extension theorem we can consider functions defined on $\prod_{i=1}^{n} \sigma(a_i) \subseteq \mathbb{R}^n$ without loss of generality. Therefore the association $f \mapsto f(a1, a_n)$ is a $*$-homomorphism from $C(\prod_{i=1}^{n} \sigma(a_i))$ onto $\mathcal{B}$, which generalizes the functional calculus of one variable. C^*-algebras and $\varphi: A \to B$ is a positive unital map. We fix U, an open convex subset of $\mathbb{R}^n$, a convex function $f: U \to \mathbb{R}$ and a mutually commuting n-tuple $\mathrm{a} = (a_1, \dots, a_n) \in A_{sa}^n$ such that $\prod_{i=1}^{n} \sigma(a_i) \subseteq U$. We denote $\vec{\varphi}(\mathrm{a}) = (\varphi(a_1), \dots, \varphi(a_n))$.

Theorem (1.2.13)[22]: *Suppose that $\varphi(f(\mathrm{a})), \varphi(a_1) , \varphi(a_n)$ are also mutually commuting. Then*

$$f(\varphi(a_1), \dots, \varphi(a_n)) = f(\vec{\varphi}(\mathrm{a})) \le \varphi(f(\mathrm{a})) = \varphi(f(a_1, \dots, a_n)) . \quad (73)$$

Moreover, if $\vec{0} = (0, \dots, 0) \in U$ and $f(\vec{0}) \le 0$ then Eq. (73) *holds even if φ is positive contractive.*

Proof. Denote by $\widehat{\mathcal{B}}$the abelian C^*-subalgebra of B generated by $\varphi(a_1) , \dots, \varphi(a_n)$ and $(f(\mathrm{a}))$. On the other hand, let $\{f_i\}_{i\ge1}$ be the linear functions given Since $f \ge f_i (i \ge 1)$ we have that $f(\mathrm{a}) \ge f_i(\mathrm{a})$ and therefore

$$\varphi(f(\mathrm{a})) \ge \varphi(f_i(\mathrm{a})) = f_i(\vec{\varphi}(\mathrm{a})) , \quad (74)$$

where the last equality holds because f_i is linear. As $f_i(\vec{\varphi}(\mathrm{a})) \in \widehat{\mathcal{B}}$ for every $i \ge 1$ and also $\varphi(f(\mathrm{a})) \in \widehat{\mathcal{B}}$, which is abelian,

$$\varphi(f(\mathrm{a})) \ge \max_{\mathrm{l\le i\le n}} \ f_i(\vec{\varphi}(\mathrm{a})) = \left(\max_{\mathrm{l\le i\le n}} \ f_i\right) (\vec{\varphi}(\mathrm{a})) .$$

Now, since $\max_{1\le i\le n} f_i \to \overrightarrow{n \to \infty} f$ uniformly on compact sets, we can deduce from Dini's theorem that $(f(\mathrm{a})) \ge f(\varphi(\mathrm{a}))$. If φ is contractive and $f(0) \le 0$, the functions f_i also satisfy that $f_i(0) \le 0$ and we can replace Eq. (74) by $\varphi(f(\mathrm{a})) \ge \varphi(f_i(a)) \ge f_i(\varphi(a))$. Then we can repeat the same argument to get the desired inequality.

The following results are the multi-variable versions of Lemma (1.2.10), Remark (1.2.12) and Theorem (1.2.11). The proofs of those results were chosen in such a way that they still hold in the multi-variable case without substantial differences.

Lemma (1.2.14)[22]: *Let ϕ be a state defined on $\mathcal{B}$, and let* $\mathrm{b} = (b_1, \dots, b_n) \in (\mathcal{B}^{\phi})^n$ *be an n-tuple with mutually commuting entries. Then, there exist a Borel measure μ defined on K* $:= \sigma(\mathrm{b})$ *and a positive unital linear map $\Psi: \mathcal{B} \to L^\infty(K, \mu)$ such that*:

(i) $\Psi(f(\mathrm{b})) = f$ for every $f \in C(K)$.

(ii) $\phi(x) = \int_K \Psi(x)(t) d\mu(t)$ for every $x \in \mathcal{B}$.

Theorem (1.2.15)[22]: *Let $\mathcal{E}: B \to C$ be a conditional expectation onto the C^*-subalgebra C. If* $(\mathrm{a}_1) , \dots, \varphi(a_n) \in \mathcal{B}^{\mathcal{E}}$ *and are mutually commuting, then*

$$\mathcal{E}\left(g\left[f\left(\vec{\varphi}(\mathrm{a})\right)\right]\right) \leq \mathcal{E}\left(g\left[\varphi\left(f(\mathrm{a})\right)\right]\right) \tag{75}$$

for every convex increasing map $g: J \to \mathbb{R}$ such that $f(I) \subseteq J$.

Theorem (1.2.16)[22]: *If $\mathcal{B}$ is a finite factor and (a_1) , ... , $\varphi(a_n)$ are mutually commuting, then $f\left(\vec{\varphi}(\mathrm{a})\right) <_w \varphi\left(f(\mathrm{a})\right)$.*

we rewrite the already obtained Jensen's inequalities in the finite dimensional case. We use the notations $\mathcal{M}_n = L(\mathbb{C}^n)$, $\mathcal{M}_n^{sa}$ for the space of selfadjoint matrices, $\mathcal{M}_n^+$ for the cone of positive matrices and $\mathcal{U}(n)$ the unitary group of $\mathcal{M}_n$. Given $\in \mathcal{M}_n^{sa}$, by means of $\lambda(A) = \left(\lambda_1(A), \dots, \lambda_n(A)\right)$ we denote the eigenvalues of A counted with multiplicity and arranged in non-increasing order. Now, we recall the aspect of the spectral preorder and majorization in $\mathcal{M}_n^{sa}$

(i) By Proposition (1.2.6), the following conditions are equivalent

(a) $A \precsim B$.

(b) There is $U \in u(n)$ such that $(UAU^*)B = B(UAU^*)$ and $UAU^* \leq B$.

(c) $\lambda_j(A) \leq \lambda_i(B)(1 \leq i \leq n)$.

(ii) Straightforward calculations showthat,given aselfadjoint matrix *C*, the functions $e_C(t)$ considered in the definition of majorization satisfy that $e_C(t) = \lambda_k(C)$ for $\frac{k-1}{n} \leq t < \frac{k}{n}$, $1 \leq k \leq n$. Therefore, $A <_w B$ if and only if $\lambda(A) <_w \lambda(B)$ (as vectors).

We summarize the different versions, in this setting, of Jensen's inequality obtained

Proposition (1.2.17)[22]: *Let A be an unital C^*-algebra and $\varphi: \mathcal{A} \to \mathcal{M}_n$ a positive unital map. Suppose that $a \in \mathcal{A}_{sa}$ and $: I \to \mathbb{R}$ is afunction with $\sigma(a) \subseteq I$.*

(i) *If f is monotone convex,* $\lambda_i\left(f\left(\varphi(a)\right)\right) \leq \lambda_i\left(\varphi\left(f(a)\right)\right)$, for $1 \leq i \leq n$.

(ii) *If f is convex,* $\sum_{i=1}^{k} \lambda_j\left(f\left(\varphi(a)\right)\right) \leq \sum_{i=1}^{k} \lambda_i\left(\varphi\left(f(a)\right)\right), 1 \leq k \leq n.$

If $0 \in I$ and $f(0) \leq 0$ the above inequalities also hold for contractive positive maps.

Example (1.2.18)[22]: Given two $n \times n$ matrices $A = \left(a_{ij}\right)$ and $= \left(b_{ij}\right)$, we denote $A \circ B = \left(a_{ij} b_{ij}\right)$ their Schur's product. It is a well-known fact that the map $A \mapsto A \circ B$ is completely positive for each positive matrix B, and if we further assume that $I \circ B = I$ then it is also unital. So the above inequalities can be rewritten taking $\varphi: \mathcal{M}_n \to \mathcal{M}_n$ given by $\varphi(A) = A \circ B$ where B satisfies the mentioned properties (see also [24]).

Example (1.2.19)[22]: Let $A \subseteq L(\mathcal{H})$ be a C^*-algebra. Take $\varphi: A^r \to \mathcal{M}_n$ given by $(A_1, \dots, A_r) = \sum_{i=1}^{r} W_i^* A_i W_i$, where $W_1 , \dots , W_r \in L(\mathbb{C}^n, \mathcal{H})$ are bounded operators such that $\sum_{i=1}^{r} W_i^* W_i = I$. Since this map is positive, one can apply Proposition (1.2.17) to get new versions of the inequalities appearing in [27], [1].

Chapter 2
Transformations and Orthogonality of Subspaces

We extend Wigner result from the 1-dimensional case to the case of n-dimensional subspaces of H with $n \in \mathbb{N}$ fixed. We show that this relation on the set of all non-isotropic k-subspaces can be used as a single primitive notion for metric-projective geometry provided that the polarity is not symplectic and $n \neq 2k + 1$.

Section (2.1): Set of all n-Dimensional Subspaces of a Hilbert Space Preserving Principal Angles

For H be a (real or complex) Hilbert space and denote $B(H)$ the algebra of all bounded linear operators on H. By a projection we mean a self-adjoint idempotent in $B(H)$. For any $n \in \mathbb{N}, P_n(H)$ denotes the set of all rank-n projections on H, and $P_\infty(H)$ stands for the set of all infinite rank projections. Clearly, $P_n(H)$ can be identified with the set of all n-dimensional subspaces of H. Wigner's theorem describes the bijective trans-formations on the set $\mathcal{L}$ of all 1-dimensional subspaces of H which preserve the angle between the elements of $\mathcal{L}$. It seems to be a very natural problem to try to extend this result from the 1-dimensional case to the case of higher dimensional subspaces (in [44], [45], [46] we have presented several other generalizations of Wigner's theorem for different structures). But what about the angle between two higher dimensional subspaces of H? The most adequate concept of angles is that of the so-called principal angles (or canonical angles, in a different terminology). This concept is a generalization of the usual notion of angles between 1-dimensional subspaces and reads as follows. If P, Q are finite dimensional projections, then the principal angles between them (or, equivalently, between their ranges as subspaces) is defined as the arccos of the square root of the eigenvalues (counted according multiplicity) of the positive (self-adjoint) finite rank operator $QP\,Q$ (see, for example, [26] or [40]). We remark that this concept of angles was motivated by [36] of Jordan and it has serious applications in statistics, for example (see the canonical correlation theory of Hotelling [37], and also see the introduction of [42]). The system of all principal angles between P and Q is denoted by $\angle(P, Q)$. Thus, we have the desired concept of angles between finite rank projections. But in what follows we would like to extend Wigner's theorem also for the case of infinite rank projections. Therefore, we need the concept of principal angles also between infinite rank projections. Using deep concepts of operator theory (like scalar-valued spectral measure and multiplicity function) this could be carried out, but in order to formulate a Wigner-type result we need only the equality of angles. Hence, we can avoid these complications saying that for arbitrary projections P, Q, P', Q' on H we have $\angle(P, Q) = \angle(P', Q')$ if and only if the positive operators QP Q and $Q'P'Q'$ are unitarily equivalent. This obviously generalizes the equality of principal angles between pairs of finite rank projection.

Keeping in mind the formulation of Wigner's theorem given we are now in a position to formulate the main result which, we believe, also has physical interpretation.

Theorem (2.1.1)[34]: Let $n \in \mathbb{N}$. Let H be a real or complex Hilbert space with dim $H \geq n$. Suppose that $\phi: P_n(H) \to P_n(H)$ is a transformation with the property that

$$\angle\big(\phi(P), \phi(Q)\big) = \angle(P, Q) \qquad \big(P, Q \in P_n(H)\big).$$

If n=1 or $n \neq dim\, H/2$, then there exists a linear or conjugate-linear isometry V on H such that

$$\phi(P) = V\, P\, V^* \qquad \big(P \in P_n(H)\big).$$

If H is infinite dimensional, the transformation $\phi: P_\infty(H) \to P_\infty(H)$ satisfies

$$\angle(\phi(P), \phi(Q)) = \angle(P, Q) \quad (P, Q \in P_\infty(H)),$$

and ϕ is surjective, then there exists a unitary or antiunitary operator U on H such that

$$\phi(P) = UP\,U^* \qquad (P \in P_\infty(H)).$$

As one can suspect from the formulation of our main result, there is a system of exceptional cases, namely, when we have $\dim H = 2n, n > 1$. we show that in those cases there do exist transformations on $P_n(H)$ which preserve the principal angles but cannot be written in the form appearing in Theorem (2.1.1) above.

We devoted to the proof of Theorem (2.1.1). In fact, this will follow from the statements below.

The idea of the proof can be summarized in a single sentence as follows. We extend our transformation from $P_n(H)$ to a Jordan homomorphism of the algebra F(H) of all finite rank operators on H which preserves the rank-1 operators. Fortunately, those maps turn to have a form and using this we can achieve the desired conclusion. On the other hand, quite unfortunately, we have to work hard to carry out all the details of the proof that we are just going to begin.

Let H be a real or complex Hilbert space and let $n \in \mathbb{N}$. Since our statement obviously holds when dim H=n, hence we suppose that $dim\,H > n$. Let tr be the usual trace functional on operators. The ideal of all finite rank operators in B(H) is denoted by F(H). Clearly, every element of $F(H)$ has a finite trace. We denote by $F_s(H)$ the set of all self-adjoint elements of F(H).

We begin with two key lemmas. In order to understand why we consider the property (1) in Lemma (2.1.2), we note that if $\angle(P, Q) = \angle(P', Q')$ for some finite rank projections P, Q, P', Q', then, by definition, the positive operators $QP\,Q$ and $Q'P'Q'$ are unitarily equivalent. This implies that $\mathrm{tr}\,QP\,Q = \mathrm{tr}\,Q'P'Q'$. But, by the properties of the trace, we have $\mathrm{tr}\,QP\,Q = \mathrm{tr}\,P\,QQ = \mathrm{tr}\,P\,Q$ and, similarly, $\mathrm{tr}\,Q'P\,'Q' = \mathrm{tr}\,P'Q'$. So, if our transformation preserves the principal angles between projections, then it necessarily preserves the trace of the product of the projections in question. This justifies the condition (1) in the next lemma.

Lemma (2.1.2)[34]: Let $\mathcal{P}$ be any set of finite rank projections on H. If $\phi: \mathcal{P} \to \mathcal{P}$ is a transformation with the property that

$$tr\,\phi(P)\phi(Q) = tr\,P\,Q \qquad (P, Q \in \mathcal{P}), \tag{1}$$

then ϕ has a unique real-linear extension Φ onto the real-linear span $span_{\mathbb{R}}\mathcal{P}$ of $\mathcal{P}$. The transformation Φ is injective, preserves the trace and satisfies

$$tr\,\Phi(A)\Phi(B) = tr\,AB \qquad (A, B \in span_{\mathbb{R}}\mathcal{P}). \tag{2}$$

Proof. For any finite sets $\{\lambda_i\} \subset \mathbb{R}$ and $\{P_i\} \subset \mathcal{P}$ we define

$$\Phi\left(\sum_i \lambda_i P_i\right) = \sum_i \lambda_i \phi(P_i).$$

We have to show that Φ is well-defined. If $\sum_i \lambda_i P_i = \sum_k \mu_k Q_k$, where $\{\mu_k\} \subset \mathbb{R}$ and $\{Q_k\} \subset \mathcal{P}$ are finite subsets, then for any $R \in \mathcal{P}$ we compute

$$tr\left(\sum_i \lambda_i\phi(P_i)\phi(R)\right)=\sum_i \lambda_i tr\big(\phi(P_i)\phi(R)\big)=\sum_i \lambda_i\, tr(P_iR)=tr\left(\sum_i \lambda_iP_iR\right)$$
$$=tr\left(\sum_k \mu_kQ_kR\right)=\sum_k \mu_k tr(Q_kR)=\sum_k \mu_k\, tr\big(\phi(Q_k)\phi(R)\big)$$
$$=tr\left(\sum_k \mu_k\phi(Q_k)\phi(R)\right).$$

Therefore, we have

$$tr\left(\left(\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)\right)\phi(R)\right)=0$$

for every $R\in\mathcal{P}$. By the linearity of the trace functional it follows that we have similar equality if we replace $\phi(R)$ by any finite linear combination of $\phi(R)$'s. This gives us that

$$tr\left(\left(\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)\right)\left(\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)\right)\right)=0.$$

The operator $\big(\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)\big)^2$, being the square of a self-adjoint operator, is positive. Since its trace is zero, we obtain that

$$\left(\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)\right)^2=0$$

which plainly implies that

$$\sum_i \lambda_i\phi(P_i)-\sum_k \mu_k\phi(Q_k)=0.$$

This shows that Φ is well-defined. The real-linearity of Φ now follows from the definition. The uniqueness of Φ is also trivial to see. From (1) we immediately obtain (2). One can introduce an inner product on $F_s(H)$ by the formula

$$\langle A,B\rangle=tr\,AB \qquad \big(A,B\in F_s(H)\big)$$

(the norm induced by this inner product is called the Hilbert-Schmidt norm). The equality (2) shows that Φ is an isometry with respect to this norm. Thus, Φ is injective. It follows from (1) that

$$tr\,\phi(P)=tr\,\phi(P)^2=tr\,P^2=tr\,P \qquad (P\in\mathcal{P})$$

which clearly implies that

$$tr\,\Phi(A)=tr\,A \qquad (A\in span_{\mathbb{R}}\mathcal{P}).$$

This completes the proof of the lemma.

In what follows we need the concept of Jordan homomorphisms. If $\mathcal{A}$ and $\mathcal{B}$ are algebras, then a linear transformation $\Psi:\mathcal{A}\to\mathcal{R}$ is called a Jordan homomorphism if it satisfies

$$\Psi(A^2)=\Psi(A)^2 \qquad (A\in\mathcal{A}),$$

or, equivalently, if

$$\Psi(AB+BA)=\Psi(A)\Psi(B)+\Psi(B)\Psi(A) \qquad (A,B\in\mathcal{A}).$$

Two projections P, Q on H are said to be orthogonal if $P\,Q = QP = 0$ (this means that the ranges of P and Q are orthogonal to each other). In this case we write $P \perp Q$. We denote $P \leq Q$ if $P\,Q = QP = P$ (this means that the range of P is included in the range of Q). In what follows, we shall use the following useful notation. If $x, y \in H$, then $x \otimes y$ stands for the operator defined by

$$(x \otimes y)z = \langle z, y\rangle x \qquad (z \in H).$$

Lemma (2.1.3)[34]: Let $\Phi: F_s(H) \to F_s(H)$ be a real-linear transformation which preserves the rank-1 projections and the orthogonality between them. Then there is an either linear or conjugate-linear isometry V on H such that

$$\Phi(A) = V\,AV^{*} \qquad \big(A \in F_s(H)\big).$$

Proof. Since every finite-rank projection is the finite sum of pairwise orthogonal rank-1 projections, it is obvious that Φ preserves the finite-rank projections. It follows from [35] and the spectral theorem that Φ is a Jordan homomorphism (we note that [35] is about self-adjoint operators on finite dimensional complex Hilbert spaces, but the same argument applies for $F_s(H)$ even if it is infinite dimensional and/or real).

We next prove that Φ can be extended to a Jordan homomorphism of F(H). To see this, first suppose that H is complex and consider the transformation $\widetilde{\Phi}: F(H) \to F(H)$ defined by

$$\widetilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B) \qquad \big(A, B \in F_s(H)\big).$$

It is easy to see that Φ is a linear transformation which satisfies $\widetilde{\Phi}(T^2) = \widetilde{\Phi}(T)^2\big(T \in F(H)\big)$. This shows that $\widetilde{\Phi}$ is a Jordan homomorphism.

If H is real, then the situation is not so simple, but we can apply a deep algebraic result of Martindale as follows (cf. the proof of [43]). Consider the unitalized algebra $F(H) \oplus \mathbb{R}I$ (We have to add the identity only when H is infinite dimensional). Defining $\Phi(I) = I$, we can extend Φ to the set of all symmetric elements of the enlarged algebra in an obvious way. Now we are in a position to apply the results in [41] on the extendability of Jordan homomorphisms defined on the set of symmetric elements of a ring with involution. in [41] Jordan homomorphism means an additive map Ψ which, besides $(s^2) = \Psi(s)^2$, also satisfies $\Psi(sts) = \Psi(s)\Psi(t)\Psi(s)$. But if the ring in question is 2-torsion free (in particular, if it is an algebra), this second equality follows from the first one (see, for example, the proof of [48]). The statements [41] in the case when dim H≥3 and [41] if dim H=2 imply that Φ can be uniquely extended to an associative homomorphism of $F(H) \oplus \mathbb{R}I$ into itself. To be honest, since the results of Martindale concern rings and hence linearity does not appear, we could guarantee only the additivity of the extension of Φ. However, the construction in [41] shows that in the case of algebras, linear Jordan homomorphisms have linear extensions.

In every case we have a Jordan homomorphism of F(H) extending Φ. We use the same symbol Φ for the extension as well.

As F(H) is a locally matrix ring (every finite subset of F(H) can be included in a subalgebra of F(H) which is isomorphic to a full matrix algebra), it follows from a classical result of Jacobson and Rickart [39] that Φ can be written as $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is a homomorphism and Φ_2 is an antihomomorphism. Let P be a rank-1 projection on H. Since $\Phi(P)$ is also rank-1, we obtain that one of the idempotents $\Phi_1(P), \Phi_2(P)$ is zero. Since $F(H)$ is a simple ring, it is easy to see that this implies that either Φ_1 or Φ_2 is identically zero, that is, Φ is either a homomorphism or an antihomomorphism of F(H). We can assume that Φ is a homomorphism. Since the kernel of Φ is an ideal in F(H) and F(H) is simple, we obtain that Φ is injective.

We show that Φ preserves the rank-1 operators. Let $A \in F(H)$ be of rank 1. Then there is a rank-1 projection P such that P A=A. We have Φ(A)=Φ(P A)=Φ(P)Φ(A) which proves that Φ(A) is of rank at most 1. Since Φ is injective, we obtain that the rank of Φ(A) is exactly 1. From the conditions of the lemma it follows that ϕ sends rank-2 projections to rank-2 projections. Therefore, the range of Φ contains an operator with rank greater than 1. We now refer to Hou's work [38] on the form of linear rank preservers on operator algebras. It follows from the argument leading to [38] that either there are linear operators T, S on H such that Φ is of the form

$$\Phi(x \otimes y) = (Tx) \otimes (Sy) \quad (x, y \in H)$$

or there are conjugate-linear operators T', S' on H such that Φ is of the form

$$\Phi(x \otimes y) = (S'y) \otimes (T'x) \qquad (x, y \in H). \tag{3}$$

Suppose that we have the first possibility. By the multiplicativity of Φ we obtain that

$$\begin{aligned}\langle u, y\rangle Tx \otimes Sv &= \langle u, y\rangle \Phi(x \otimes v) = \Phi(x \otimes y \cdot u \otimes v) \\ &= \Phi(x \otimes y)\Phi(u \otimes v) = \langle T u, Sy\rangle Tx \otimes Sv.\end{aligned} \tag{4}$$

This gives us that $\langle T u, Sy\rangle = \langle u, y\rangle$ for every $u, y \in H$. On the other hand, since Φ sends rank-1 projections to rank-1 projections, we obtain that for every unit vector $x \in H$ we have $Tx = Sx$. These imply that T=S is an isometry and with the notation V=T=S we have

$$\Phi(A) = V AV^*$$

for every $A \in F_s(H)$.

We show that the possibility (3) cannot occur. In fact, similarly to (4) we have

$$\langle u, y\rangle S'v \otimes T'x = \langle S'v, T'x\rangle S'y \otimes T'u \qquad (x, y, u, v \in H).$$

Fixing unit vectors $x = y = u$ in H and considering the operators above at $T'x$, we find that

$$S'v = \langle S'v, T'x\rangle\langle T'x, T'u\rangle S'y$$

giving us that S' is of rank 1. Since Φ sends rank-2 projections to rank-2 projections, we arrive at a contradiction. This completes the proof of the lemma.

We present a new proof of the nonsurjective version of Wigner's theorem which is equivalent to the statement of Theorem (2.1.1) in the case when n=1. For another proof see [49].

To begin, observe that if P, Q are finite rank projections such that $tr\ P\ Q = 0$, then we have $tr(P\ Q)^*PQ = tr\ QPQ = tr\ P\ QQ = tr\ P\ Q = 0$ which implies that $(P\ Q)^*(P\ Q) = 0$. This gives us that $P\ Q = 0 = QP$. Therefore, P is orthogonal to Q if and only if $tr\ P\ Q = 0$.

Theorem (2.1.4)[34]: Let $\phi: P_1(H) \to P_1(H)$ be a transformation with the property that

$$tr\ \phi(P)\phi(Q) = tr\ P\ Q\ \big(P, Q \in P_1(H)\big). \tag{5}$$

Then there is an either linear or conjugate-linear isometry V on H such that

$$\phi(P) = V\ P\ V^* \qquad \big(P \in P_1(H)\big).$$

As for the cases when n>1 we need the following lemma. Recall that we have previously supposed that dim H>n.

Lemma (2.1.5)[34]: Let $1 < n \in \mathbb{N}$. Then $span_{\mathbb{R}}P_n(H)$ coincides with $F_s(H)$.

Proof. Since the real-linear span of $P_1(H)$ is $F_s(H)$, it is sufficient to show that every rank-1 projection is a real-linear combination of rank-n projections. To see this, choose orthonormal vectors $e_1, \dots, e_{n+1}$ in H. Let $E = e_1 \otimes e_1 + \dots + e_n + 1 \otimes e_{n+1}$ and define

$$P_k = E - e_k \otimes e_k \qquad (k = 1, \dots, n + 1).$$

Clearly, every P_k can be represented by $a(n + 1) \times (n + 1)$ diagonal matrix whose diagonal entries are all 1's with the exception of the k^{th} one which is 0. The equation

$$\lambda_1 P_1 + \dots + \lambda_{n+1} P_{n+1} = e_1 \otimes e_1$$

gives rise to a system of linear equations with unknown scalars $\lambda_1,\dots,\lambda_{n+1}$. The matrix of this system of equations is an $(n+1)\times(n+1)$ matrix whose diagonal consists of 0's and its off-diagonal entries are all 1's. It is easy to see that this matrix is nonsingular, and hence $e_1\otimes e_1$ (and, similarly, every other $e_k\otimes e_k$) is a real-linear combination of $P_1,\dots,P_{n+1}$. This completes the proof.

We continue with a technical lemma.

Lemma (2.1.6)[34]: Let P, Q be projections on H. If QP Q is a projection, then there are pairwise orthogonal projections R, R', R'' such that $= R + R', Q = R + R''$. In particular, we obtain that QP Q is a projection if and only if $PQ = QP$.

Proof. Let R=QP Q. Since R is a projection whose range is contained in the range of Q, it follows that $R'' = Q - R$ is a projection which is orthogonal to R.

If x is a unit vector in the range of R, then we have $\|QP\,Qx\| = 1$. Since $P\,Qx$ is a vector whose norm is at most 1 and its image under the projection Q has norm 1, we obtain that $P\,Qx$ is a unit vector in the range of Q. Similarly, we obtain that Qx is a unit vector in the range of P and, finally, that x is a unit vector in the range of Q. Therefore, x belongs to the range of P and Q. Since x was arbitrary, we can infer that the range of R is included in the range of P. Thus, we obtain that $R' = P - R$ is a projection which is orthogonal to R.

Next, using the obvious relations

$$P\,R = RP = R, QR = RQ = R$$

we deduce

$$\begin{aligned}(Q-R)(P-R)(Q-R) &= QP\,Q - QP\,R - QRQ + QR - RP\,Q + RP\,R + RQ - R\\ &= R - R - R + R - R + R + R - R = 0. \qquad (6)\end{aligned}$$

Since $A^*A = 0$ implies $A = 0$ for any $A \in B(H)$, we obtain from (6) that $R'R'' = (P-R)(Q-R) = 0$.

The second part of the assertion is now easy to check.

We next prove the assertion of Theorem (2.1.1) in the case when $1 < n \in \mathbb{N}$ and H is infinite dimensional.

Theorem (2.1.7)[34]: Suppose $1 < n \in \mathbb{N}$ and H is infinite dimensional. If $\phi: P_n(H) \to P_n(H)$ is a transformation such that

$$\angle\big(\phi(P),\phi(Q)\big) = \angle(P,Q)\big(P,Q \in P_n(H)\big),$$

then there exists a linear or conjugate-linear isometry V on H such that

$$\phi(P) = V\,P\,V^* \qquad \big(P \in P_n(H)\big).$$

Proof. By Lemma (2.1.2) and Lemma (2.1.5), ϕ can be uniquely extended to an injective real-linear transformation Φ on $F_s(H)$. The main point of the proof is to show that Φ preserves the rank-1 projections. In order to verify this, just as in the proof of Lemma (2.1.5), we consider orthonormal vectors $e_1,\dots,e_{n+1}$ in H, define $E = e_1\otimes e_1 + \dots + e_{n+1}\otimes e_{n+1}$ and set

$$P_k = E - e_k \otimes e_k \qquad (k = 1,\dots,n+1).$$

We show that the ranges of all $P'_k = \phi(P_k)'s$ can be jointly included in an (n+1)-dimensional subspace of H. To see this, we first recall that Φ has the property that

$$tr\,\Phi(A)\Phi(B) = tr\,AB\,\big(A,B \in F_s(H)\big)$$

(see Lemma (2.1.2)). Next we have the following property of Φ: if P, Q are orthogonal rank-1 projections, then $\Phi(P)\Phi(Q) = 0$. Indeed, if P, Q are orthogonal, then we can include them into two orthogonal rank-(n+ 1) projections. Now, referring to the construction given in Lemma (2.1.5) and having in mind that Φ preserves the orthogonality between rank-n projections, we obtain that $\Phi(P)\Phi(Q) = 0$. (Clearly, the same argument works if dim $H \geq$

$2(n+1)$.) Since the rank-n projections P_k are commuting, by the preserving property of ϕ and Lemma (2.1.6), it follows that the projections $\Phi(P_k)$ are also commuting. It is well-known that any finite commuting family of operators in $F_s(H)$ can be diagonalized by the same unitary transformation (or, in the real case, by the same orthogonal transformation). Therefore, if we resctrict Φ onto the real-linear subspace in $F_s(H)$ generated by $P_1, \dots, P_{n+1}$, then it can be identified with a real-linear operator from $\mathbb{R}^{n+1}$ to $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Clearly, this restriction of Φ can be represented by an $m \times (n+1)$ real matrix $T = (t_{ij})$. We examine how the properties of Φ are reflected in those of the matrix T. First, Φ is trace preserving. This gives us that for every $\underline{\lambda} \in \mathbb{R}^{n+1}$ the sums of the coordinates of the vectors $T\,\underline{\lambda}$ and $\underline{\lambda}$ are the same. This easily implies that the sum of the entries of T lying in a fixed column is always 1. As we have already noted, $\Phi(e_i \otimes e_i)\Phi(e_j \otimes e_j) = 0$ holds for every $i \neq j$. For the matrix T this means that the coordinatewise product of any two columns of T is zero. Consequently in every row of T there is at most one nonzero entry. Since Φ sends rank-n projections to rank-n projections, we see that this possibly nonzero entry is necessarily 1. So, every row contains at most one 1 and all the other entries in that row are 0's. Since the sum of the elements in every column is 1, we have that in every column there is exactly one 1 and all the other entries are 0's in that column. These now easily imply that if $\underline{\lambda} \in \mathbb{R}^{n+1}$ is such that its coordinates are all 0's with the exception of one which is 1, then $T\,\underline{\lambda}$ is of the same kind. What concerns Φ, this means that Φ sends every $e_k \otimes e_k (k = 1, \dots, n+1)$ to a rank-1 projection.
So, we obtain that Φ preserves the rank-1 projections and the orthogonality between them. Now, by Lemma (2.1.3) we conclude the proof.
We turn to the case when H is finite dimensional.

Theorem (2.1.8)[34]: Suppose $1 < n \in \mathbb{N}$, H is finite dimensional and $n \neq dim\, H/2$. If $\phi: P_n(H) \to P_n(H)$ satisfies

$$\angle(\phi(P), \phi(Q)) = \angle(P, Q) \qquad (P, Q \in P_n(H)),$$

then there exists a unitary or antiunitary operator U on H such that

$$\phi(P) = UP\,U^* \qquad (P \in P_\infty(H)). \tag{7}$$

Proof. First suppose that $dim\, H = 2d, 1 < d \in \mathbb{N}$. If $n = 1, \dots, d-1$, then we can apply the method followed in the proof of Theorem (2.1.7) concerning the infinite dimensional case. If $n = d+1, \dots, 2d-1$, then consider the transformation $\psi: P \;\longmapsto\; I - \phi(I-P)$ on $P_{2d-n}(H)$. We learn from [40] that if $\angle(P, Q) = \angle(P', Q')$, then there exists a unitary operator U such that $UP\,U^* = P'$ and $UQU^* = Q'$. It follows from the preserving property of ϕ that for any P, $Q \in P_{2d-n}(H)$ we have

$$\phi(I-P) = U(I-P)U^*, \qquad \phi(I-Q) = U(I-Q)U^*$$

for some unitary operator U on H. This gives us that

$$\angle(\psi(P), \psi(Q)) = \angle(UP\,U^*, UQU^*) = \angle(P, Q).$$

In that way we can reduce the problem to the previous case. So, there is an either unitary or antiunitary operator U on H such that

$$\psi(P) = UP\,U^* \qquad (P \in P_{2d-n}(H)).$$

It follows that $\phi(I-P) = I - \psi(P) = I - UP\,U^* = U(I-P)U^*$, and hence we have the result for the considered case.
Next suppose that $dim\, H = 2d+1, d \in \mathbb{N}$. If $n = 1, \dots, d-1$, then once again we can apply the method followed in the proof of Theorem (2.1.7). If $n = d+\ 2, \dots, 2d+\ 1$, then using the 'dual method' that we have applied right above we can reduce the problem to the

previous case. If $n = d$, consider a fixed rank-d projection P_0. Clearly, if P is any rank-d projection orthogonal to P_0, then the rank-d projection $\phi(P)$ is orthogonal to $\phi(P_0)$. Therefore, ϕ induces a transformation ϕ_0 between d+1-dimensional spaces (namely, between the orthogonal complement of the range of P_0 and that of the range of $\phi(P_0)$) which preserves the principal angles between the rank-d projections. Our 'dual method' and the result concerning 1-dimensional subspaces lead us to the conslusion that the linear extension of ϕ_0 maps rank-1 projections to rank-1 projections and preserves the orthogonality between them. This implies that the same holds true for our original transformation ϕ. Just as before, using Lemma (2.1.2) and Lemma (2.1.3) we can conclude the proof. In the remaining case n=d+1 we apply the 'dual method' once again.

We now show that the case when $1 < n \in \mathbb{N}, n = dim\, H/2$ is really exceptional. To see this, consider the transformation $\phi: P \longmapsto I - P$ on $P_n(H)$. This maps $P_n(H)$ into itself and preserves the principal angles. As for the complex case, the preserving property follows from [26] while in the real case it was proved already by Jordan in [36] (see [47], p. 310). We suppose that the transformation ϕ can be written in the form (7). Pick a rank-1 projection Q on H. We know that it is a real linear combination of some $P_1, \ldots, P_{n+1} \in P_n(H)$. It would follow from (7) that considering the same linear combination of $\phi(P_1), \ldots, \phi(P_{n+1})$, it is a rank-1 projection as well. But due to the definition of ϕ, we get that this linear combination is a constant minus Q. By the trace preserving property we obtain that this constant is 1/n. Since n>1, the operator $(1/n)I - Q$ is obviously not a projection. Therefore, we have arrived at a contradiction. This shows that the transformation above can not be written in the form (7).

It would be a nice result if one could prove that in the present case (i.e., when $1 < n, n = dim\, H/2$) up to unitary-antiunitary equivalence, there are exactly two transformations on $P_n(H)$ preserving principal angles, namely, $P \longmapsto P$ and $P \longmapsto I - P$.

We now turn to our statement concerning infinite rank projections. In the proof we shall use the following simple lemma. If $A \in B(H)$, then denote by rng A the range of A.

Lemma (2.1.9)[34]: Let H be an infinite dimensional Hilbert space. Suppose P, Q are projections on H with the property that for any projection R with finite corank we have RP=P R if and only if RQ=QR. Then either P=Q or P=I-Q.

Proof. Let R be any projection on H commuting with P. By Lemma (2.1.6), it is easy to see that we can choose a monotone decreasing net (R_α) of projections with finite corank such that (R_α) converges weakly to R and R_α commutes with P for every α. Since R_α commutes with Q for every α, we obtain that R commutes with Q. Interchanging the role of P and Q, we obtain that any projection commutes with P if and only if it commutes with Q.

Let x be any unit vector from the range of P. Consider $R = x \otimes x$. Since R commutes with P, it must commute with Q as well. By Lemma (2.1.6) we obtain that x belongs either to the range of Q or to its orthogonal complement. It follows that either $d(x, rng\ Q) = 0$, or $d(x, rng\ Q) = 1$. Since the set of all unit vectors in the range of P is connected and the distance function is continuous, we get that either every unit vector in rng P belongs to rng Q or every unit vector in rng P belongs to $(rng\ Q)^\perp$. Interchanging the role of P and Q, we find that either $rng\ P = rng\ Q$ or $rng\ P = (rng\ Q)^\perp$. This gives us that either P=Q or P=I-Q.

Theorem (2.1.10)[34]: Let H be an infinite dimensional Hilbert space. Suppose that $\phi: P_\infty(H) \rightarrow P_\infty(H)$ is a surjective transformation with the property that

$$\angle\big(\phi(P), \phi(Q)\big) = \angle(P, Q) \qquad \big(P, Q \in P_\infty(H)\big).$$

Then there exists a unitary or antiunitary operator U on H such that

$$\phi(P) = UP\,U^* \qquad \big(P \in P_\infty(H)\big).$$

Proof. We first prove that ϕ is injective. If $P, P' \in P_\infty(H)$ and $\phi(P) = \phi(P')$, then by the preserving property of ϕ we have $\angle(P, Q) = \angle(P', Q)\big(Q \in P_\infty(H)\big)$. (8) Putting $Q = I$, we see that P is unitarily equivalent to P'. We distiguish two cases. First, let P be of infinite corank. By (8), we deduce that for every $Q \in P_\infty(H)$ we have $Q \perp P$ if and only if $Q \perp P'$. This gives us that $P = P'$. As the second possibility, let P be of finite corank. Then P, P' can be written in the form $P = I - P_0$ and $P' = I - P_0'$, where, by the equivalence of P, P', the projections P_0 and P_0' have finite and equal rank. Let Q_0 be any finite rank projection on H. It follows from

$$\angle(I - P_0, I - Q_0) = \angle(I - P_0', I - Q_0)$$

that there is a unitary operator W on H such that

$$W(I - Q_0)(I - P_0)(I - Q_0)W^* = (I - Q_0)(I - P_0')(I - Q_0).$$

This implies that

$$W(-Q_0 - P_0 + P_0Q_0 + Q_0P_0 - Q_0P_0Q_0)W^* = -Q_0 - P_0' + P_0'Q_0 + Q_0P_0' - Q_0P_0'Q_0.$$

Taking traces, by the equality of the rank of P_0 and P_0', we obtain that

$$tr\; P_0Q_0 = tr\; P_0'Q_0. \qquad (9)$$

Since this holds for every finite rank projection Q_0 on H, it follows that $P_0 = P_0'$ and hence we have $= P'$. This proves the injectivity of ϕ.

Let $P \in P_\infty(H)$ be of infinite corank. Then there is a projection $Q \in P_\infty(H)$ such that $Q \perp P$. By the preserving property of ϕ, this implies that $\phi(Q) \perp \phi(P)$ which means that $\phi(P)$ is of infinite corank. One can similarly prove that if $\phi(P)$ is of infinite corank, then the same must hold for P. This yields that $P \in P_\infty(H)$ is of finite corank if and only if so is ϕ(P).

Denote by $P_f(H)$ the set of all finite rank projections on H. It follows that the transformation $\psi: P_f(H) \to P_f(H)$ defined by

$$\psi(P) = I - \phi(I - P) \qquad \big(P \in P_f(H)\big)$$

is well-defined and bijective. Since $\phi(I - P)$ is unitarily equivalent to $I - P$ for every $P \in P_f(H)$ (this is because $\angle\big(\phi(I - P), \phi(I - P)\big) = \angle(I - P, I - P))$, it follows that ψ is rank preserving.

We next show that

$$tr\; \psi(P)\psi(Q) = tr\; P\,Q \qquad \big(P, Q \in P_f(H)\big). \qquad (10)$$

This can be done following the argument leading to (9). In fact, by the preserving property of ϕ there is a unitary operator W on H such that

$$W\big(I - \psi(Q)\big)\big(I - \psi(P)\big)\big(I - \psi(Q)\big)W^* = (I - Q)(I - P)(I - Q).$$

This gives us that

$$W\big(-\psi(Q) - \psi(P) + \psi(P)\psi(Q) + \psi(Q)\psi(P) - \psi(Q)\psi(P)\psi(Q)\big)W^*$$
$$= -Q - P + P\,Q + QP - QP\,Q.$$

Taking traces on both sides and referring to the rank preserving property of ψ, we obtain (10). According to Lemma (2.1.2), let $\Psi: F_s(H) \to F_s(H)$ denote the unique real-linear extension of ψ onto $span_{\mathbb{R}} P_f(H) = F_s(H)$. We know that Ψ is injective. Since $P_f\,(H)$ is in the range of Ψ, we obtain that Ψ is surjective as well. It is easy to see that Lemma (2.1.3) can be applied and we infer that there exists an either unitary or antiunitary operator U on H such that

$$\Psi(A) = UAU^* \qquad \big(A \in F_s(H)\big).$$

Therefore, we have

$$\phi(P) = UP\,U^*$$

for every projection $P \in P_\infty(H)$ with finite corank. It remains to prove that the same holds true for every $P \in P_\infty(H)$ with infinite corank as well. This could be quite easy to show if we know that ϕ preserves the order between the elements of $P_\infty(H)$. But this property is far away from being easy to verify. So we choose a different approach to attack the problem.

Let $P \in P_\infty(H)$ be a projection of infinite corank. By the preserving property of ϕ we see that for every $Q \in P_\infty(H)$ the operator $\phi(Q)\phi(P)\phi(Q)$ is a projection if and if QP Q is a projection. By Lemma (2.1.6), this means that $\phi(Q)$ commutes with ϕ(P) if and only if Q commutes with P. Therefore, for any $Q \in P_\infty(H)$ of finite corank, we obtain that Q commutes with $U^*\phi(P)U$ (this is equivalent to that $\phi(Q) = UQU^*$ commutes with $\phi(P)$) if and only if Q commutes with P.

By Lemma (2.1.9) we have two possibilities, namely, either $U^*\phi(P)U = P$ or $U^*\phi(P)U = I - P$. Suppose that $U^*\phi(P)U = I - P$. Consider a complete orthonormal basis $e_0, e_\gamma (\gamma \in \Gamma)$ in the range of P and, similarly, choose a complete orthonormal basis $f_0, f_\delta (\delta \in \Delta)$ in the range of $I - P$. Pick nonzero scalars λ, μ with the property that $|\lambda|^2 + |\mu|^2 = 1$ and $|\lambda| \neq |\mu|$. Define

$$Q = (\lambda e_0 + \mu f_0) \otimes (\lambda e_0 + \mu f_0) + \sum_\gamma e_\gamma \otimes e_\gamma + \sum_\delta f_\delta \otimes f_\delta.$$

Clearly, Q is of finite corank (in fact, its corank is 1). Since $\phi(Q)\phi(P)\phi(Q) = UQU^*\phi(P)UQU^*$ is unitarily equivalent to QPQ, it follows that the spectrum of $QU^*\phi(P)UQ$ is equal to the spectrum of QPQ. This gives us that the spectrum of $Q(I - P)Q$ is equal to the spectrum of QPQ. By the construction of Q this means that

$$\{0, 1, |\mu|^2\} = \{0, 1, |\lambda|^2\}$$

which is an obvious contradiction. Consequently, we have $U^*\phi(P)U = P$, that is, $\phi(P) = UP\,U^*$. Thus, we have proved that this latter equality holds for every $P \in P_\infty(H)$ and the proof is complete.

Section (2.2): Metric-Projective Geometry

The idea of ortho-adjacency relation is taken from [68] and [58]. Given a linear space with an orthogonality relation defined on its line-set, two lines are called ortho-adjacent if they are concurrent and orthogonal. In [58] Havlicek proves that ortho-adjacency preserving transformations of elliptic spaces with dimensions other than 3 are induced by orthogonality-preserving collineations. Later, in [60], he completes his result for 3-dimensional spaces. A similar result for symplectic spaces is also due to Havlicek in [59] and for hyperbolic spaces is due to List in [64]. Orthogonality-preserving transformations on lines of Euclidean spaces are extensively investigated in [51], [52], [53], [54] as well as in [63], [69]. For hyperbolic spaces see [65], [66], [67].

In [68] ortho-adjacency is treated more generally as a relation on all k-subspaces, not only on lines, of an Euclidean space. It is proved there that ortho-adjacency-preserving transformations on k-subspaces are induced by orthogonality-preserving collineations of the underlying n-dimensional Euclidean space where $k + 2 \leq n$. In other words orthoadjacency on k-subspaces can be used as a single primitive notion for at least $(k + 2)$-dimensional Euclidean geometry.

We generalize results of [58], [59] and [64] by taking ortho-adjacency on k-subspaces as it was done in [68] and by unified reasoning for elliptic, symplectic and hyperbolic geometry. An elliptic space is similar to an Euclidean space in that there are no isotropic subspaces.

This is false in general and when we deal with a metric-projective space we restrict our ortho-adjacency to non-isotropic k-subspaces. The methods we use are similar to those used in [68].
We prove the following theorem:
Theorem (2.2.1)[50]: Let $\langle S, \mathcal{L}\rangle$ be an abstract desarguesian projective space of finite dimension n equipped with a polarity π which is neither symplectic nor a pseudo-polarity and let k be such that $0 \le k \le n-1$.
If $n \neq 2k+1$, then the ortho-adjacency relation on the set of all non-isotropic ksubspaces can be used as a single primitive notion for the n-dimensional metric-projective geometry $\langle S, \mathcal{L}, \perp\rangle$ where ⊥ is the perpendicularity of lines given by π.
In Chow style (cf. [55], [57]) this theorem would read as follows:
Theorem (2.2.2)[50]: Under the assumptions of (2.2.1), every bijective transformation of non-isotropic k-subspaces preserving ortho-adjacency in both directions is induced by a collineation of $\langle S, \mathcal{L}\rangle$ that preserves perpendicularity of lines given by π.

For $\mathfrak{P} = \langle S, \mathcal{L}\rangle$ be an abstract desarguesian projective space of finite dimension n and let π be a polarity on $\mathfrak{P}$. In terms of linear algebra $\mathfrak{P}$ corresponds to a projective space over a left vector space of dimension $n+1$ over a division ring and the polarity π corresponds to a non-degenerate reflexive sesqui-linear form.
The polarity π maps any point $u \in S$ to a hyperplane $U = u^\pi$ of $\mathfrak{P}$, it also maps U to the same point u. We call u the pole of U, and we call U the polar hyperplane of u. If u, w are points such that $\in w^\pi$, then we say that u and w are conjugate points. The points $u \in S$ with $u \in u^\pi$ are called absolute (or self-conjugate) and they form the absolute space with respect to the polarity π (absolute space is sometimes called quadric); the other points are called regular. A subspace U of $\mathfrak{P}$ is absolute (or totally isotropic) if $u \in w^\pi$ for all $u, w \in U$. If U is a subspace of $\mathfrak{P}$ we denote by U^π the $\cap_{u\in U} u^\pi$; so U is absolute iff $U \subseteq U^\pi$.
The incidence structure whose points are the absolute points of $\mathfrak{P}$ and whose lines are absolute lines of $\mathfrak{P}$ is a polar space. For a complete survey on polarities, their connections with polar spaces and reflexive sesqui-linear forms see [56].
Now, we fix a natural number k such that $0 \le k \le n-1$. By $\wp_k$ we denote the set of all k-dimensional subspaces of $\mathfrak{P}$. The meet of subspaces U, W of $\mathfrak{P}$ is $U \cap W$ and the join, i.e. the meet of all subspaces containing them, is $U \sqcup W$. So $\langle \wp, \cap, \sqcup\rangle$ is a projective lattice, where $\wp$ is the set of all subspaces of $\mathfrak{P}$.
In geometries whose lattices of subspaces are not modular (e.g. affine lattices) it makes sense to distinguish two adjacency relations of k-subspaces (like in [68]) as two k-subspaces can have a common (k+1)-subspace that covers them and they need not to share $a\ (k-1)$-subspace. In a projective space, whose lattice is modular, for $U, W \in \wp_k$ we have $dim(U \cap W) = k - m$ iff $dim(U \sqcup W) = k + m$. So, we simply call U, W adjacent and write

$$U \sim W \quad \text{iff} \quad dim(U \cap W) = k - 1. \tag{11}$$

Note that adjacent implies distinct, so our adjacency relation is not reflexive.
It is well known that for $1 \le k \le n-2$ maximal ∼-cliques fall into two classes: stars and tops. A star of k-subspaces is simply the set of all k-subspaces containing some (k-1)-subspace — the vertex of the star, and a top is the set of all k-subspaces in some (k+1)-subspace — the base of the top.
Let $U, W \in \wp_k$. We say that U intersects orthogonally W, or they are ortho-adjacent, and write

$$U \underset{\sim}{\perp} W \quad iff \quad U \sim W \quad \text{and} \quad U \cap W^\pi \neq \emptyset. \tag{12}$$

Again, let us stress that ortho-adjacent subspaces must be distinct, as adjacency is involved. From the above definition it is not obvious that ortho-adjacency is symmetric and the next proposition addresses that problem.

Proposition (2.2.3)[50]: The relation $\underset{\sim}{\perp}$ is symmetric.

Proof. Let $U, W \in \wp_k$. Assume that $U \underset{\sim}{\perp} W$. By (12) there is $\in U \cap W^\pi$. Note that $W \subseteq x^\pi$ and $U^\pi \subseteq x^\pi$. Hence $W \sqcup U^\pi \subseteq x^\pi$, but $dim(x^\pi) = n - 1$ and $dim(W \sqcup U^\pi) = n - k - 1 + k - dim(W \cap U^\pi)$ which yields that $0 \leq dim(W \cap U^\pi)$. This means that $W \cap U^\pi \neq \emptyset$ and by (12) we get $W \underset{\sim}{\perp} U$.

The radical of a subspace U is the subspace $ad(U) = U \cap U^\pi$. We call a subspace U non-isotropic if $Rad(U) = \emptyset$ and it is isotropic if $Rad(U) \neq \emptyset$ (cf. [20]). Let

$$S = \{U \in \wp : Rad(U) = \emptyset\},$$

and let S_k be the subset of S of k-subspaces. Recall that S_0 is the set of all regular points of $\mathfrak{P}$, i.e. those off the quadric induced by π. Note that if π is elliptic, then $S_k = \wp_k$, while if π is symplectic, then $S_k = \emptyset$ for $k = 0, 2, 4, \ldots$ An isotropic line L can be either a tangent line to the quadric or a generator of the quadric. Then, respectively Rad(L) is the point of tangency or L is absolute. The symplectic case is exceptional in that $Rad(L) \neq \emptyset$ implies that the line L is already totally isotropic (cf. [59]).

Lemma (2.2.4)[50]: If the polarity π is symplectic and $U, W \in \wp_k$ with $U \underset{\sim}{\perp} W$, then $Rad(U) \neq \emptyset$ or $Rad(W) \neq \emptyset$.

Proof. Let $x \in U \cap W^\pi$ by (12). If $x \in W$, then already $Rad(W) \neq \emptyset$. So, assume that $x \notin W$. Trivially, $U \cap W \subseteq W$ and thus $W^\pi \subseteq (U \cap W)^\pi$ which means that $\in (U \cap W)^\pi$. Since $U \cap W$ is a hyperplane in U we have $U = (U \cap W) \sqcup x$ and $U^\pi = (U \cap W)^\pi \cap x^\pi$. The polarity is symplectic, so $x \in x^\pi$, and finally $x \in U^\pi$ which means that $Rad(U) \neq \emptyset$.

A subspace U is non-isotropic iff U^π is non-isotropic. This follows immediately from the fact that $Rad(U^\pi) = U^\pi \cap (U^\pi)^\pi = U^\pi \cap U = Rad(U)$. Let us write down more properties of the radical and non-isotropic subspaces that will be used later.

Lemma (2.2.5)[50]: Let $U \in \wp_k$ and $m = dim(Rad(U))$.

(i) If Z is a maximal non-isotropic subspace with $Z \subseteq U$, then $dim(Z) = k - m - 1$.

(ii) If Y is a minimal non-isotropic subspace with $U \subseteq Y$, then $dim(Y) = k + m + 1$.

(iii) There are $Z \in S_{k-m-1}$ and $Y \in S_{k+m+1}$ such that $Z \subseteq U \subseteq Y$.

Proof. (i) Obviously $U^\pi \subseteq Z^\pi$ as $Z \subseteq U$. Hence

$$Z \cap Rad(U) = Z \cap U \cap U^\pi = Z \cap U^\pi \subseteq Rad(Z) = \emptyset.$$

Since Z is maximal in U, it is a complement of $Rad(U)$ in U, and thus its dimension is $k - m - 1$.

(ii) Note that Y^π is a maximal non-isotropic subspace with $Y^\pi \subseteq U^\pi$. So, by (i) we get $dim(Y) = n - dim(Y^\pi) - 1 = n - (dim(U^\pi) - m - 1) - 1 = k + m + 1$.

(iii) Take Z a complement of Rad(U) in U. Then $U = Z \sqcup Rad(U)$ and hence $U^\pi = Z^\pi \cap (U^\pi \sqcup U) \supseteq Rad(Z)$. This yields that $Rad(Z) \subseteq Z \cap U^\pi = Z \cap U \cap U^\pi = Z \cap Rad(U) = \emptyset$.

By what we have just proved there is $Z \in S_{n-k-m-2}$ such that $\subseteq U^\pi$. Now, let us take $Y = Z^\pi$. Clearly $U \subseteq Y$, Y is non-isotropic and $dim(Y) = k + m + 1$.

We say that a line K is perpendicular to a line L and write

$$K \perp L \qquad iff\ K \cap L \neq \emptyset \qquad and \qquad K \cap L^\pi \neq \emptyset. \tag{13}$$

That way we get the metric-projective space $\langle S, \mathcal{L}, \perp\rangle$ (cf. [62] for dimensions 2, 3). We reconstruct $\langle S, \mathcal{L}, \perp\rangle$ in the structure $\langle S_k, \underset{\sim}{\perp}\rangle$. For k=1 and a hyperbolic polarity π the result is already known (cf. [64], [65], [66], [67]) as well as for an elliptic polarity (cf. [58]) and for a symplectic polarity (cf. [59]).

The first question that we are asking is: what are the cliques of $\sim \perp$ on non-isotropic subspaces?

We assume that $\underset{\sim}{\perp} \subseteq S_k \times S_k$. According to (2.2.4) this relation is empty for symplectic polarities, so the claims of (2.2.1) and (2.2.2) are false for them. If π is a pseudopolarity, i.e. π corresponds to a symmetric bilinear form and the ground field is of characteristic 2, then the quadric induced by π is a hyperplane (cf. [61]). To avoid these inconveniences, in particular, to ensure that each non-isotropic line contains a regular point and each regular point is on a non-isotropic line, we assume further that the polarity π is not symplectic and is not a pseudo-polarity

When we deal with non-isotropic subspaces the adjacency relation $\sim$ says nothing but the dimension of the meet and join of subspaces. For this reason we need two more specific relations. Let $U, W \in S_k$. Then U, W are meet-adjacent, in symbols

$$U \sim^- W \qquad iff \qquad U \cap W \in S_{k-1}, \tag{14}$$

and they are join-adjacent,

$$U \sim^+ W \qquad iff \qquad U \sqcup W \in S_{k+1}. \tag{15}$$

Trivially

$$if \quad U \sim^- W \quad or \quad U \sim^+ W, \quad then \quad U \sim W. \tag{16}$$

Lemma (2.2.6)[50]: Let $U_1, U_2 \in S_k$.

(i) If $U_1 \sim^- U_2$, then there is $Y \in S_m$ such that $U_1, U_2 \subset Y$ and $m \leq k+2$.

(ii)If $U_1 \sim^+ U_2$, then there is $Z \in S_m$ such that $Z \subset U_1, U_2$ and $k-2 \leq m$.

Proof. (i) Set $B = U_1 \sqcup U_2$. By (16) B is a (k+1)-subspace but we do not know if it is non-isotropic. If it is take Y=B. Otherwise, note that U1 is a maximal nonisotropic subspace in B which by (2.2.5)(i) gives that $dim(Rad(B)) = 0$. By (2.2.5)(iii) there is $Y \in S_{k+1+1}$ such that $B \subset Y$.

(ii) If $U_1 \sim^+ U_2$, then $U_1^\pi \sim^- U_2^\pi$ and we are through by (i).

The connection between the two new adjacency relations and ortho-adjacency is as follows.

Lemma (2.2.7)[50]: If $U, W \in S_k$ and $U \underset{\sim}{\perp} W$, then $U \sim^- W$ and $U \sim^+ W$.

Proof. (i) Set $H := U \cap W$. By (14) assume to the contrary that there is $x \in H \cap H^\pi$.

Let $u \in U \cap W^\pi$ by (12). Note that $u \notin H$ as otherwise we would have $u \in W$ and thus $\in W \cap W^\pi$. Hence $U = H \sqcup u$ which implies that $U^\pi = H^\pi \cap u^\pi$. Since $W \subseteq u^\pi$ we have $x \in H \subseteq W \subseteq u^\pi$. In consequence $x \in U^\pi$ which gives that $x \in U \cap U^\pi$ as $x \in H \subseteq U$. A contradiction arises because $U \in S$.

(ii) Now, set $B := U \sqcup W$ and by (15) assume to the contrary that there is $x \in B \cap B^\pi$.

Let $u \in U \cap W^\pi$ and $w \in W \cap U^\pi$ by (12). Hence note that $u^\pi \cap B = W$ and $w^\pi \cap B = U$ as both left hand sides are hyperplanes in B. Since $B \subseteq x^\pi$ we get $u, w \in x^\pi$ and thus $x \in u^\pi \cap w^\pi \cap B = U \cap W$. This contradicts that both U and W are non-isotropic as $x \in B^\pi = U^\pi \cap W^\pi$.

When we deal with a binary relation it is convenient to know the cliques of that relation. As in graph theory a subset K of S_k is called an ortho-clique, or an $\underset{\sim}{\perp}$-clique, if $U \sim\perp$ W for all distinct $U, W \in K$. Before we give an account on the cliques we need to put down two technical facts.

Lemma (2.2.8)[50]: If U is a non-isotropic subspace and u is a regular point in U^{π}, then $U \sqcup u$ is non-isotropic.

Proof. Given $W := U \sqcup u$ we obtain $W^{\pi} = U^{\pi} \cap u^{\pi}$ and by modularity $W \cap U^{\pi} = (u \sqcup U) \cap U^{\pi} = u \sqcup (U \cap U^{\pi}) = u$. Therefore $W \cap W^{\pi} = W \cap U^{\pi} \cap u^{\pi} = u \cap u^{\pi} = \emptyset$.

Lemma (2.2.9)[50]: If U is a non-isotropic subspace and u is a regular point in U, then $u^{\pi} \cap U$ is non-isotropic.

Proof. By modularity $(u^{\pi} \cap U) \cap (u \sqcup U^{\pi}) = u^{\pi} \cap [u \sqcup (U \cap U^{\pi})] = u^{\pi} \cap u = \emptyset$.

According to (12) maximal $\underset{\sim}{\perp}$-cliques are subsets of maximal $\sim$-cliques, i.e. stars or tops. This suggests that we also have two types of ortho-cliques.

Proposition (2.2.10)[50]: A set K is a maximal ortho-clique iff

(i) $k \leq n-2$ and there is $H \in S_{k-1}$ such that

$$K = \{H \sqcup u : u \in a \text{ maximal self} - \text{polar simplex in } H^{\pi}\},$$

(ii)$1 \leq k$ and there is $B \in S_{k+1}$ such that

$$K = \{u^{\pi} \cap B : u \in a \text{ maximal self} - \text{polar simplex in } B\}.$$

Proof. Verification that the set of the form (i) or (ii) is a maximal ortho-clique is fairly easy with the use of (2.2.7), (2.2.8), and (2.2.9). So, assume that K is a maximal ortho-clique. If K is a subset of a star then we get the Case (i), otherwise we get the Case (ii).

Case (i). For $k = n-1$ the clique K is not maximal as it is contained in the clique of the form (ii) with $B = S$. Hence the assumption $k \leq n-2$. By (2.2.7), we have some $H \in S_{k-1}$ contained in all the elements of K. Let $U \in K$. We will show that $|U \cap H^{\pi}| = 1$. So, assume to the contrary that $a, b \in U \cap H^{\pi}$ and $a \neq b$. The line $l := a, b$ lies in $U \cap H^{\pi}$ and thus l meets H as H is a hyperplane in U. This is however impossible as $H \cap H^{\pi} = \emptyset$ by (2.2.7). In consequence, for every $U \in K$ there is a unique $u \in H^{\pi}$ with $U = H \sqcup u$. Let E be the set of all such points u, i.e.

$$E := \{u \in H^{\pi} : H \sqcup u \in K\}.$$

We will show that E is the simplex we are looking for. First of all, note that E is maximal in that $\langle E \rangle = H^{\pi}$ since K is maximal.

Let us fix some $u \in E$ for a moment and let $U = H \sqcup u$. Note that $U \in K$. Now take $w \in E \backslash \{u\}$. Set $W := H \sqcup w$. It is clear that $W \in K$, so $W \underset{\sim}{\perp} U$. By (12) there is $x \in W \cap U^{\pi}$. Since $U^{\pi} \subset H^{\pi}$ we have $x \in W \cap H^{\pi}$ and thus $x = w$. Therefore $w \in U^{\pi}$. We have actually shown that

$$E \backslash \{u\} \subseteq u^{\pi} \quad \text{for all} \quad u \in E,$$

which means that E is self-polar.

Suppose that $u \in u^{\pi}$. Then we have $E \subseteq u^{\pi}$ and consequently $H^{\pi} = \langle E \rangle \subseteq u^{\pi}$ which means that $u \in H$. Hence $u \in H \cap H^{\pi}$ which contradicts the fact that H is non-isotropic.

Now, observe that $\langle E \backslash \{u\} \rangle \subseteq u^{\pi}$ which implies that $\langle u \notin \backslash \{u\} \rangle$ as otherwise we would have $u \in u^{\pi}$ which is impossible. Hence E is a simplex.

Case (ii). If k=0, then the clique K is not maximal as it is contained in the clique of the form (i) with $H = \emptyset$. This justifies the assumption $1 \leq k$. The polarity π is a duality in that it reverses the inclusion $\subseteq$ and thus it determines a bijection from S_k onto S_{n-k-1} which preserves ortho-adjacency in both directions and which maps stars onto tops and vice versa. Therefore the proof in this case runs dually to the Case (i).

We will call ortho-cliques of type (i) ortho-stars and those of type (ii) ortho-tops. As it is seen in (2.2.10), to each ortho-star K we can uniquely assign its vertex (bottom) $H = b(K) \in S_{k-1}$. Similarly, to each ortho-top K we can uniquely assign its base (top) $B = t(K) \in S_{k+1}$.

If K is a maximal ortho-star, then $|K| = dim(H^{\pi}) + 1$ for some $H \in S_{k-1}$, and if K is a maximal ortho-top, then $|K| = dim(B) + 1$ for some $B \in S_{k+1}$. Finally

$$K = \begin{cases} n - k + 1, & if\ K\ is\ a\ maximal\ ortho - star, \\ k + 2, & if\ K\ is\ a\ maximal\ ortho - top. \end{cases} \quad (17)$$

That way we can distinguish the type of a maximal ortho-clique in $\langle S_k, \underset{\sim}{\perp} \rangle$ provided that $n - k + 1 \neq k + 2$, that is when

$$n \neq 2k + 1.$$

Fact (2.2.11)[50]: Three non-isotropic and pairwise orthogonally intersecting k-subspaces determine the type of a maximal ortho-clique containing them.

With (2.2.11) in mind, for $U_1, U_2, U_3 \in S_k$ we introduce two relations:

$\Delta^-(U_1, U_2, U_3) :\Leftrightarrow U_1, U_2, U_3$ are pairwise distinct in an ortho-star, (18)

$\Delta^+(U_1, U_2, U_3) :\Leftrightarrow U_1, U_2, U_3$ are pairwise distinct in an ortho-top. (19)

We are now able to express meet-adjacency and join-adjacency on non-isotropic subspaces strictly in terms of ortho-adjacency.

Proposition (2.2.12)[50]: Let $U_1, U_2 \in S_k$. If $k \leq n - 4$, then $U_1 \sim^- U_2 \Leftrightarrow$

$$U_1 \neq U_2 \wedge \exists W_1, W_2 \in S_k[\Delta^-(W_1, W_2, U_1) \wedge \Delta^-(W_1, W_2, U_2)]. \quad (20)$$

If $3 \leq k$, then $U_1 \sim^+ U_2 \Leftrightarrow$

$$U_1 \neq U_2 \wedge \exists W_1, W_2 \in S_k[\Delta^+(W_1, W_2, U_1) \wedge \Delta^+(W_1, W_2, U_2)]. \quad (21)$$

Proof. Assume that $k \leq n - 4$. The right-to-left implication is evident. So take $H = U_1 \cap U_2$. By definition of meet-adjacency H is non-isotropic. Set $B = U_1 \sqcup U_2$. In view of (2.2.10)(i) we are looking for two conjugate regular points w_1, w_2 in H^{π} such that $w_1, w_2 \in U_1^{\pi} \cap U_2^{\pi} = B^{\pi}$. By (2.2.6)(i) consider $Y \in S_m$ such that $B \subseteq Y$ and $m = k + 1$ or $m = k + 2$. Note that $dim(Y^{\pi}) = n - m - 1$. By our assumption $1 \leq n - k - 3$ which means that $1 \leq n - m - 1$ for both possible values of m. So, Y^{π} is non-isotropic and it is at least a line. Since $Y^{\pi} \subseteq B^{\pi}$ we can always find required points w_1, w_2 in Y^{π}.

The other part of the proof runs dually.

We will try to reconstruct $\langle S_{k-1}, \underset{\sim}{\perp} \rangle$ in $\langle S_k, \underset{\sim}{\perp} \rangle$. First we define the incidence relation $\subset \subseteq S_{k-1} \times S_k$.

Proposition (2.2.13)[50]: Let K be a maximal ortho-star and $H = b(K)$. Then

$$H \subset U \ \Leftrightarrow \forall W \in K[W = U \vee W \sim^{-U}]. \quad (22)$$

Proof. ⇒: If $W \in K$, then $H \subset U$, W and hence $U = W$ or $U \sim^- W$.

⇐: If $U \in K$, then our claim is evident. So, assume that $U \notin K$. Note that $|K| \geq 3$ by (2.2.10). This let us take pairwise distinct $U_1, U_2, U_3 \in K$. They are pairwise adjacent and thus they belong to a bundle of subspaces, not a pencil in the Grassmann space associated with P. Since $U \sim U_1, U_2, U_3$, we are done.

Consider the following relation. Two ortho-stars K_1, K_2 are said to be related, which is written $K_1 \approx K_2$, iff $b(K_1) = b(K_2)$. We will show that this relation can be expressed within $\langle S_k, \underset{\sim}{\perp} \rangle$.

Proposition (2.2.14)[50]: Let K_i be a maximal ortho-star and $H_i = b(K_i)$ for $i = 1, 2$. Then

$$H_1 = H_2 \Leftrightarrow \forall U \in S_k[H_1 \subset U \ \Leftrightarrow H_2 \subset U]. \quad (23)$$

We can identify the elements of S_{k-1} with the equivalence classes of ortho-stars under the relation ≈. Moreover, two distinct $H_1, H_2 \in S_{k-1}$ are adjacent (or collinear in the sense of incidence ⊂) if there is $U \in S_k$ such that $H_1, H_2 \subset U$. By (2.2.13) this can be worded purely in terms of $\langle S_k, \underset{\sim}{\perp} \rangle$. That way we have the incidence structure

$$\langle S_{k-1}, S_k, \subset \rangle$$

defined in $\langle S_k, \underset{\sim}{\perp}\rangle$. What is still missing is ortho-adjacency on S_{k-1}. To solve the issue we need some technical fact.

Lemma (2.2.15)[50]: For $U, W \in S_k$, if $U \underset{\sim}{\perp} W$, then $|U \cap W^\pi| = 1$.

Proof. In view of (22) assume on the contrary that there are distinct $a, b \in U \cap W^\pi$, so we have a line $\overline{a,b}$ that entirely lies in both U and W^π. By (22) and (21) $H := U \cap W$ is a hyperplane in U. Therefore the line $\overline{a,b}$ shares at least a point, say c, with H. So $c \in W$ as $H \subset W$, and $c \in W^\pi$ as $\overline{a,b} \subseteq W^\pi$, which contradicts the fact that W is non-isotropic.

Proposition (2.2.16)[50]: If $1 \le k$ and $H_1, H_2 \in S_{k-1}$, then $H_1 \underset{\sim}{\perp} H_2 \Leftrightarrow$

$$\exists U_1, U_2, U_3 \in S_k[\Delta^+(U_1, U_2, U_3) \wedge H_1 \subset U_2, U_3 \wedge H_2 \subset U_1, U_3]. \qquad (24)$$

Proof. $\Rightarrow$: Let $U_3 := H_1 \sqcup H_2$. By (2.2.7) we have $U_3 \in S_k$. Take a regular point $x \in U_3^\pi$. In view of (2.2.8) the subspaces $U_1 := H_2 \sqcup x, U_2 := H_1 \sqcup x$ are non-isotropic. It is easy to verify that they satisfy the right-hand side of (24).

$\Leftarrow$:We have $H_1 = U_2 \cap U_3$ and $H_2 = U_1 \cap U_3$. Set $B = U_1 \sqcup U_2 \sqcup U_3$. By assumptions U_1, U_2, U_3 form an ortho-top, so $B \in S_{k+1}$. It is clear that $H_1 \cap H_2 = U_1 \cap U_2 \cap U_3$, and thus $H_1 \sim H_2$. By (2.2.15) we have two points $a \in U_1^\pi \cap U_2$ and $b \in U_3^\pi \cap U_2$. Observe that $a \neq b$ as otherwise we would have $a \in B \cap B^\pi = \emptyset$. Take the line $L := \overline{a,b}$. Note that $L \subseteq U_1^\pi \sqcup U_3^\pi = H_2^\pi$ and $L \subseteq U_2$. Since H_1 is a hyperplane in U_2, the line L meets H_1, and thus $H_1 \cap H_2^\pi \neq \emptyset$ which completes the proof.

To summarize what we have done so far observe that in (2.2.13) and (2.2.14) maximal orthostars are involved, so by (2.2.10) we have to assume that $k \le n-2$. Considering (2.2.16) we need $1 \le k$. By (2.2.12) meet-adjacency $\sim^-$ can be defined only if $k \le n-4$.

Proposition (2.2.17)[50]: If $1 \le k \le n-4$, then the structure

$$\langle S_{k-1}, S_k, \subset, \underset{\sim}{\perp}_{k-1}\rangle$$

can be defined in $\langle S_k, \underset{\sim}{\perp}\rangle$.

With a dual reasoning we get

Proposition (2.2.18)[50]: If $3 \le k \le n-2$, then the structure

$$\langle S_{k+1}, S_k, \supset, \underset{\sim}{\perp}_{k-1}\rangle$$

can be defined in $\langle S_k, \underset{\sim}{\perp}\rangle$.

If $1 \le k \le n-4$, then applying (2.2.17) we get the structure $\langle S_0, S_1, \subset, \underset{\sim}{\perp}\rangle$. If $3 \le k \le n-2$, then applying (2.2.18) we get $\langle S_{n-1}, S_{n-2}, \supset, \underset{\sim}{\perp}\rangle$ which is the dual to the previous one. In the elliptic case this is actually what we need, that is $\langle S, \mathcal{L}, \perp\rangle$ the underlying metric-projective space. Otherwise, we need to go through to get the same result.

What is left are the following three cases:

(i) $k = 0$,

(ii) $k = n-1$,

(iii) $k < 3$ and $n < k+4$.

Considering that $\mathfrak{V}$ is self-dual via polarity π and that $2k \neq n-1$ we can restrict ourselves to $2k < n-1$. Accordingly, there are two specific cases left to investigate:

(i) $k = 0$,

(ii) $k = 1$ and $n = 4$.

Here we deal with the structure of conjugacy on regular points. Every non-isotropic line $L \in S_1$ contains two distinct conjugate regular points u_1, u_2. These two points can be completed to a maximal self-polar simplex of regular points $u_1, \dots u_{n+1}$ in $\mathfrak{V}$. Note that u_i^{π} is a non-isotropic hyperplane and hence

$$L = \bigcap_{i=3,\dots,n+1} u_i^{\pi}.$$

This way we have defined $\langle S_0, S_1, \subset, \underset{\sim}{\perp} \rangle$ and can move

For a triangle Δ in $\langle S_0, S_1, \subset, \underset{\sim}{\perp} \rangle$ with the sides L_1, L_2, L_3 we define the set

$$\Pi_0(\Delta) = \{u \in S_0: (\exists L \in S_1)(\exists w_1, w_2 \in S_0) \\ [u, w_1, w_2 \in L \wedge w_1 \neq w_2 \wedge \vee_{i,j=1,2,3, i\neq j} (w_1 \in L_i \wedge w_2 \in L_j)]. \quad (25)$$

Clearly $\Pi_0(\Delta)$ is contained in a projective plane $\Pi \in \wp_2$ which contains the vertices of Δ, and $Rad(\Pi)$ is at most a point (i.e. it is a point or the plane Π is non-isotropic).
It can be computed1 that for every projective plane $\Pi \in \wp_2$ with $dim(Rad(\Pi)) \leq 0$ there is a triangle Δ such that $\Pi_0(\Delta)$ is the set $S_0(\Pi)$ of all regular points on Π. Then the set

$$\Pi_1(\Delta) = \{L \in S_1: |L \cap \Pi_0(\Delta)| \geq 2\} \quad (26)$$

is the set $S_1(\Pi)$ of all non-isotropic lines on Π. Two non-isotropic lines are coplanar if they both are in one set of the form $\Pi_1(\Delta)$. That way we get the following lemma.
Lemma (2.2.19)[50]: The family

$$\bar{S}_2^0 = \{S_0(\Pi): \Pi \in \wp_2, dim(Rad(\Pi)) \leq 0\}$$

of point-subplanes of all projective planes whose radicals are at most a point, the family

$$\bar{S}_2^1 = \{S_1(\Pi): \Pi \in \wp_2, dim(Rad(\Pi)) \leq 0\}$$

of analogous line-subplanes as well as the adjacency $\sim$ on the set S_1 is definable in terms of $\langle S_0, S_1, \subset, \underset{\sim}{\perp} \rangle$.
Next is a star of non-isotropic lines through a regular point $a \in S_0$, i.e. the set

$$S(a) = \{L \in S_1: a \in L\}. \quad (27)$$

Now consider the following relation. Let $L_1, L_2, L_3 \in S_1$, then $\nabla(L_1, L_2, L_3) :\Leftrightarrow$

$$\sim (L_1, L_2, L_3) \wedge \nexists a \in S_0[L_1, L_2, L_3 \in S(a)] \wedge \nexists U \in \bar{S}_2^1[L_1, L_2, L_3 \in U] \quad (28)$$

The relation $\nabla(L_1, L_2, L_3)$ means that the lines L_1, L_2, L_3 go through some absolute point p and they are non-coplanar. Note by definition (28) that the lines L_1, L_2, L_3 are pairwise adjacent, so they form a triangle or a pencil (possibly flat). Suppose that L_1, L_2, L_3 lie on a plane $\Pi \in \wp_2$. Then $dim(Rad(\Pi)) \leq 0$, and hence $L_1, L_2, L_2 \in S_1(\Pi) \in \bar{S}_2^1$, which contradicts (28). So, the lines L_1, L_2, L_3 go through some point p, which by (28) is absolute.
Every star S(a) of non-isotropic lines, as defined in (27), can be identified with a regular point a and every set of non-isotropic lines L such that $L \sim^{-} L_1, L_2, L_3$ for some $L_1, L_2, L_3 \in S_1$ with $\nabla(L_1, L_2, L_3)$ can be identified with the absolute point of lines L_1, L_2, L_3. That way we have reinterpreted the point-set S of the underlying projective space $\mathfrak{V}$.
The idea we use to reinterpret isotropic lines is that each line in $\mathcal{L}$, no matter if nonisotropic or isotropic, lies on at least two non-isotropic planes. Such a non-isotropic plane can be identified with a non-isotropic triangle i.e. with three pairwise adjacent nonisotropic lines that do not go through a point, formally

$$M(L_1, L_2, L_3) :\Leftrightarrow \sim (L_1, L_2, L_3) \wedge \nexists p \in S[p \in L_1, L_2, L_3]. \quad (29)$$

Two such triangles determine two distinct planes if two of the all six of their sides are skew. This allows to define the collinearity relation L for all the points in S, namely for $p_1, p_2, p_3 \in S$

$$L(p_1, p_2, p_3) \Leftrightarrow \neq (p_1, p_2, p_3) \wedge \exists L_1, K_1, L_2, K_2, L_3, K_3 \\ \in S_1[p_1 \in L_1, K_1 \wedge p_2 \in L_2, K_2 \wedge p_3 \in L_3, K_3 \wedge \Delta(L_1, L_2, L_3) \wedge \Delta(K_1, K_2, K_3) \wedge L_1 \nsim K_2]. \quad (30)$$

That way we have all the underlying metric-projective space $\langle S, \mathcal{L}, \perp\rangle$ reconstructed in the structure $\langle S_0, S_1, \subset, \underset{\sim}{\perp}\rangle$.

This is a structure of ortho-adjacency on lines in a 4-dimensional projective space. From (2.2.10) maximal ortho-stars are 4-element sets and maximal ortho-tops are 3-element sets.

Let U_1, U_2, W_1, W_2 be non-isotropic lines such that $U_1 \neq U_2, \Delta^-(W_1, W_2, U_1)$ and $\Delta^-(W_1, W_2, U_2)$ (i.e. the right hand side of (30) holds). Trivially $U_1 \sim^- U_2$. But by (2.2.10) there is a regular point H and conjugate regular points $w_1, w_2 \in H^\pi$ such that $W_i = H \sqcup w_i$. The key observation is that $U_1 \sqcup U_2 = w_1^\pi \cap w_2^\pi$ which means that $U_1 \sim^+ U^2$ as w_1^π, w_2^π are non-isotropic meet-adjacent hyperplanes. Therefore, all non-isotropic lines U with $\Delta^-(W_1, W_2, U)$ for fixed non-isotropic lines W_1, W_2 such that $W_1 \underset{\sim}{\perp} W_2$ form a regular pencil, i.e.

$$p(H, B) := \{U \in S_1 : H \subset U \subset B\} = \{U \in S_1 : \Delta^-(W_1, W_2, U)\},$$

where $H = W_1 \cap W_2$ and $B = (W_1 \cap W_2) \sqcup (W_1 \sqcup W_2)^\pi$. Indeed, $W_1 \sqcup W_2$ is a nonisotropic plane and thus $L := (W_1 \sqcup W_2)^\pi$ is a non-isotropic line with the property that for all $u \in L$ we have on the one hand $\Delta^-(W_1, W_2, H \sqcup u)$ and on the other $H \subset H \sqcup u \subset B$.

Let us write $L(U_1, U_2, U_3)$ iff U_1, U_2, U_3 are non-isotropic lines in one regular pencil. Using this new relation we define incidence between regular points and non-isotropic lines. Let $\mathcal{R}$ be the set of so defined regular pencils and let $\mathfrak{B}$ be the partial linear space $\mathfrak{B} = \langle S_1, \mathcal{R}\rangle$. If lines L_1, L_2, L_3 are the vertices of a triangle in $\mathfrak{B}$ then either

(a) they lie on a non-isotropic plane Π, or

(b) they are in a star S(a) of non-isotropic lines through a regular point a.

Both cases there are non-isotropic lines M_1, M_2, M_3 which are the vertices of a triangle Δ in $\mathfrak{B}$ such that $L_1, L_2, L_3 \in \Pi_0(\Delta)$; here $\Pi_0(\Delta)$ is defined in $\mathfrak{B}$ by the formula analogous to (25) with points interpreted as the points of $\mathfrak{B}$ and lines interpreted as the lines of $\mathfrak{B}$ i.e. as regular pencils. Moreover, in case (a), $\Pi_0(\Delta) = S_1(\Pi)$ and in case (b), the set of the elements of $\Pi_0(\Delta)$ with a π is $S_0(\Gamma)$, where Γ is a plane in a^π such that $dim(Rad(\Gamma)) \leq 0$. We call such a triangle Δ a spanning triangle; this property can be characterized in the language of $\mathfrak{B}$. In case (a), however, there is a triple of lines $K_1, K_2, K_3 \in \Pi_0(\Delta)$ with $\Delta^+(K_1, K_2, K_3)$ and there is no such a triple in case (b).

Assume (b); then every $X \in \Pi_0(\Delta)$ goes through a. Let $q \in \mathcal{R}$ be arbitrary and let a be the vertex of q. Next, let $L_1, L_2 \in q$ with $L_1 \perp L_2$. Consider arbitrary $L \in S(a)$ such that $L \notin q$. Then the points $L \cap a^\pi, L_1 \cap a^\pi$, and $L_2 \cap a^\pi$ are on a plane Γ in a π with $dim(Rad(\Gamma)) \leq 0$. Thus there are $M_1, M_2, M_3 \in S(a)$ which are the vertices of a triangle Δ in $\mathfrak{B}$ such that $L, L_1, L_2 \in \Pi_0(\Delta)$ and the elements of $\Pi_0(\Delta)$ and a^π is $S_0(\Gamma)$. In symbols, our considerations can be summarized as follows:

$$a \in L \Leftrightarrow L \in q \vee (\exists\Delta)(\exists L_1, L_2 \in q)\ [\Delta \text{ is a spanning triangle in } \mathfrak{B} \wedge L_1 \neq L_2 \wedge L, L_1, L_2 \in \Pi_0\ (\Delta) \wedge \nexists K_1, K_2, K_3 [K_1, K_2, K_3 \in \Pi_0(\Delta) \wedge \Delta^+(K_1, K_2, K_3)]. \quad (21)$$

That way we have defined the stars S(a) of non-isotropic lines, where a is a regular point, and thus we have reinterpreted $\langle S_0, S_1, \subset, \underset{\sim}{\perp}\rangle$, so we can move

The case considered was the missing one to have the underlying metric-projective space $\langle S, \mathcal{L}, \perp\rangle$ reconstructed in the structure $\langle S_k, \underset{\sim}{\perp}\rangle$ and to prove our theorem.

In this case not all automorphisms of $\langle S_k, \underset{\sim}{\perp}\rangle$ are induced by automorphisms of $\langle S, \mathcal{L}, \pi i\rangle$ (or equivalently of $\langle S, \mathcal{L}, \underset{\sim}{\perp}\rangle$).

Proposition (2.2.20)[50]: If **n=2k +1**, then the polarity π is an automorphism of $\langle S_k, \underset{\sim}{\perp}\rangle$ which maps ortho-stars onto ortho-tops and vice versa.

Proof. The polarity π reverses the inclusion, i.e. for $U, W \in \wp$ such that $U \subseteq W$ we have $W^\pi \subseteq U^\pi$. Let $U, W \in S_k$ such that $U \underset{\sim}{\perp} W$. Then $U \cap W$ is a hyperplane in both U and W, so U^π and W^π are hyperplanes in $(U \cap W)^\pi$. Moreover, $U \cap W^\pi \neq \emptyset$, as in (22), implies that $U^\pi \cap (W^\pi)^\pi = U^\pi \cap W \neq \emptyset$ which suffices to state that π preserves ortho-adjacency $\underset{\sim}{\perp}$ and thus, it is an automorphism of $\langle S_k, \underset{\sim}{\perp}\rangle$.

The polarity π also maps S_{k-1} onto S_{k+1} and vice versa. By (2.2.10) we are done.

What it all means is that the assertions of (2.2.1) and (2.2.2) are false when $n = 2k + 1$. If that would be the case, the notion of a point would be definable and thus preserved by automorphisms of $\underset{\sim}{\perp}$. As in the case of polar geometry one can expect that a weaker condition holds: given an automorphism f of $\langle S_k, \underset{\sim}{\perp}\rangle$ either f or fπ is determined by an automorphism of $\langle S, \mathcal{L}, \perp\rangle$. In the hyperbolic and elliptic case $\langle S_k, \underset{\sim}{\perp}\rangle$ is a Plücker space, i.e. it is connected. If this is also true in general then we can use the argument that involves chains of intersecting ortho-cliques as bijective transformations of S_k which preserve the ortho-adjacency map intersecting ortho-cliques of the same type to ortho-cliques of the same type. We leave it as an open question whether this form of the Chow theorem holds for $n = 2k + 1$.

Chapter 3
Complete Positivity and Nuclear

We show that the construction is explicit and involves a convolution operator with a particular Gauß function. We show that the completion contains many interesting functions like exponentials. The star product is shown to converge absolutely and provides an entire deformation. We show that the completion has an absolute Schauder basis whenever V has an absolute Schauder basis. Moreover, the Weyl algebra is nuclear iff V is nuclear. We discuss functoriality, translational symmetries, and equivalences of the construction. We show how the Peierls bracket in classical field theory on a globally hyperbolic spacetime can be used to obtain a local net of Weyl algebras.

Section (3.1): Rieffel Deformation Quantization by Actions of $\mathbb{R}^d$

In deformation quantization [72] the transition from classical mechanics to quantum mechanics is obtained as an associative deformation of the classical observable algebra, modelled by a certain class of functions on the classical phase space. In *formal* deformation quantization this is accomplished by constructing a new associative product, the star product, as a formal power series with formal parameter h. While this theory is by now very well understood, see [74], [84], [85], [87], [89], [95] for existence and classification results, and [100] for a gentle introduction, from a physicist's perspective the formal character of the star products is still not satisfying: h is not a formal parameter after all, whence at the end of the day, some sort of "convergence" in h is needed.

Attacking the convergence problem of the formal series seems to be complicated though in examples this can be done [73]. More successful are approaches that are intrinsically non-formal like the Berezin-Toeplitz inspired quantizations [75], [80], [81], [82], [83] or Rieffel's approach using oscillatory integrals based on group actions of R^d In this version [96], the starting point is a C^*-algebra $\mathfrak{A}$ endowed with an isometric, strongly continuous action b y*-automorphisms by some finite-dimensional vector space V. Out of this and the choice of a symplectic form on V, Rieffel constructs a deformation of $\mathfrak{A}$ in the sense of a continuous field of C^*-algebras, the field parameter being h. While the construction is very general, there are yet many examples of Poisson manifolds which can be deformation quantized this way. In this framework of strict deformations many results have been obtained, most notably [92], [94].

While the above constructions deal with the observable algebra, for a physically complete description of quantization also the states have to be taken into account. In both approaches, the appropriate notion of states is that of positive linear functionals on the observable algebras. While for C^*-algebras this is of course a well-known concept, also in the formal deformation quantization this leads to a physically reasonable definition incorporating a reasonable representation theory; see, e.g., [76], [78], [99].

A fundamental question is whether a given classical state arises as the classical limit of a quantum state. In formal deformation quantization there is a general and affirmative answer to this question [77], [79]. In the strict approaches, Landsman discussed this in [91] for a certain class of examples: the appropriate notion of classical limit and deformation of states is that of a continuous field of states with respect to a given continuous field of C^*-algebras.

His construction is based on particular $*$-representations and certain coherent states and their Wigner functions. Landsman uses continuous fields of states in his discussion of the Born rule [93].

We consider Rieffel's deformation by actions of R^d in general and prove that every state of the undeformed algebra can be deformed into a continuous field of states for the field of deformed algebras. We give an explicit construction including a detailed study of the asymptotics of the deformed states for $h \to 0$; see also [88]. It turns out that the asymptotic expansion coincides in a very precise sense with the formal deformations obtained in [77]. We recall Rieffel's deformation in the Fréchet algebraic framework and define an operator S_h being the "convolution" with a GauB function. The precise form of S_h resembles the Wigner functions Landsman used, however now S_h is defined directly on the algebra. The asymptotics of S_h for $h \to 0^+$ is studied in detail. We show that S_h maps squares $a^* \star_h a$ of the deformed algebra to positive elements of the undeformed algebra. This allows to define a positive functional $\omega_h = \omega \circ S_h$ of the deformed algebra for every positive functional ω of the undeformed algebra. A detailed asymptotic expansion is obtained as well. We devoted to the more particular case of a C^*-algebra deformation. Here we show that the operator S_h is also continuous in the C^*-topology of the deformed algebra whence it extends to the C^*-algebraic completion. Finally, We show that the positive functionals $\{\omega_h\}_{h\geq 0}$ indeed form a continuous field of states.

Denotes a Fréchet$*$-algebra endowed with a strongly continuous action α by $*$-homomorphisms of a finite-dimensional vector space V which we assume without restriction to be even dimensional. Moreover, one requires that there is a system of seminorms $\|\cdot\|_k$ defining the topology of $\mathcal{A}$ such that with respect to these seminorms the action is isomemc. By $\mathcal{A}^\infty \subseteq \mathcal{A}$ we denote the subspace of smooth vectors in $\mathcal{A}$ with respect to α. It is well known that $\mathcal{A}^\infty$ is a dense $*$-subalgebra of $\mathcal{A}$. Moreover, $\mathcal{A}^\infty$ carries a finer topology making it into a Fréchet algebra, too. A system of seminorms defining the topology is explicitly given by

$$\|a\|_{k,\mu} = \sup_{|\beta|\leq\mu} \|\partial^\beta a\|_k,$$

where using multi-index notation $\partial^\beta\, a$ denotes the derivative of $\alpha_u(a)$ with respect to u at $u = 0$; see, e.g., [98] for more background on smooth vectors.

In a next step one chooses a non-degenerate bilinear anti-symmetric form θ on V and $h > 0$. Then Rieffel showed in [96] that

$$a \star_h b = \frac{1}{(\pi h)^{2n}} \int_{V\times V} \alpha_u(a)\alpha_v(b) \mathrm{e}^{\frac{2\mathrm{i}}{h}\theta(u,v)} \mathrm{d}(u,v)\,, \quad (1)$$

which is defined for a , $b \in A^\infty$, yields a well-defined associative product such that $\star_h$ is still continuous with respect to the $\mathcal{A}^\infty$-topology. Moreover, the original $*$-involution of $\mathcal{A}^\infty$ is still a*-involution with respect to $\star_h$. The precise definition of the integral in an oscillatory sense is sophisticated and can be found in Rieffel's booklet [96]. Note that we have to choose a normalization for the Haar measure on V in (1). We shall also make use of linear coordinates denoted by $v = u_j e_j$ in the sequel.

Definition (3.1.1)[71]: Let $g: V \times V \to \mathrm{R}$ be a positive definite inner product on V. Then the linear operator $S_g: \mathcal{A} \to \mathcal{A}$ is defined by

$$S_g(a) = \int_V \mathrm{e}^{-g(u,u)}\, \alpha_u(a)\mathrm{d}u. \tag{2}$$

Thanks to the fast decay of the Gau13 function and the fact that the action α is isometric, the definition of S_g in (2) as an improper Riemann integral is possible. More general, we need the following construction: let $B(V,\mathcal{A})$ denote the $\mathcal{A}$-valued functions on V such that $\sup_{v\in V}\|f(v)\|_k < \infty$ for all k, i.e., the bounded functions with respect to the seminorms $\|\cdot\|_k$ of $\vartheta\{$. Then we define

$$\tilde{S}_g f = \int_V \mathrm{e}^{-g(u,u)} f(u)\mathrm{d}u$$

for $\in B(V,\mathcal{A})$. Again, a naive definition of the integral is possible. Finally, let $C_u^0(V,\mathcal{A})$ be the uniformly continuous functions in $B(V,A)$ and let $C_u^\infty(V,A)$ be the smooth functions with all partial derivatives in $C_u^0(V,\mathcal{A})$. Clearly, the spaces $C_u^0(V,\mathcal{A})$ as well as $C_u^\infty(V,A)$ are equipped with a natural Fréchet topology by taking the sup -norm over V of seminorms of the values of the (derivatives of the) functions. Then the following Proposition lists some properties of S_g and $\tilde{S}_g$:

Proposition (3.1.2) [71]:

(i) *The operator $S_g\colon \mathcal{A} \to \mathcal{A}$ is continuous.*

(ii) *We have $S_g(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ and $S_g\colon \mathcal{A}^\infty \to \mathcal{A}^\infty$ is continuous, too.*

(iii) *The restriction of $\tilde{S}_g$ to $C_u^0(V,\mathcal{A})$ and $C_u^\infty(\mathrm{V},\mathcal{A})$ is continuous in the respective topologies.*

(iv) *The restriction of $\tilde{S}_g$ to $C_u^0(V,A^\infty)$ and $C_u^\infty(V,\mathcal{A}^\infty)$ takes values in A^∞ and is again continuous.*

Proof. The first two statements can be recovered from the third and fourth by considering the function $f(u) = \alpha_u(a)$ for $a \in A$ or $a \in \mathcal{A}^\infty$, respectively: as the action is isometric we have $f \in C_u^0(V,A)$ and $C_\mathrm{u}^\infty(V,A^\infty)$, respectively. The continuity statements in the third and fourth part are then a straightforward estimate.

In a next step we want to understand the asymptotics of the operator S_g. To this end we rescale the inner product by $h > 0$ and consider the normalized GauB function

$$G_h(u) = \frac{\sqrt{\det G}}{(\pi h)^n}\mathrm{e}^{-\frac{g(u,u)}{h}} \tag{3}$$

where $\det G > 0$ is the determinant of g with respect to the Haar measure on V and $2n = \dim V$. The normalization constant is chosen such that the integral of G_h is 1. For a fixed choice of g we consider the operator

$$S_h(a) = \int_V G_h(u)\alpha_u(a)\mathrm{d}u. \tag{4}$$

Lemma (3.1.3) [71]: *For every $a \in \mathcal{A}$ we have $\lim_{h\searrow 0} S_h(a) = a$ in the topology of A. Moreover, for every $a \in \mathcal{A}^\infty$ we have*

$$\lim_{h\searrow 0} S_h(a) = a \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}h}(S_h a) = S_h\left(\frac{1}{4}\triangle_g a\right), \tag{6}$$

both with respect to the topology of $\mathcal{A}^\infty$ where

$$\triangle_g a = \sum_{i,j}(G^{-1})^{ij}\frac{\partial^2}{\partial u_i \partial u_j}\alpha_u(a)|_{u=0}$$

is the Laplacian with respect to the inner product g and the action α viewed as continuous operator on $\mathcal{A}^\infty$. The operator $\triangle_g$ does not depend on the choice of linear coordinates.

Proof. By substitution $u \to \sqrt{h}u$ we have

$$S_h(a) = \frac{\sqrt{\det G}}{\pi^n}\int_V \mathrm{e}^{-g(u,u)}\alpha_{\sqrt{h}u}(a)\mathrm{d}u.$$

To exchange the order of integration and $\lim_{h\searrow 0}$ we consider

$$\|\int_V \mathrm{e}^{-g(u,u)}\left(\alpha_{\sqrt{h}u}(a) - a\right)\mathrm{d}u\|_{k,\mu}$$

$$\leq \int_K \mathrm{e}^{-g(u,u)}\|\alpha_{\sqrt{h}u}(a) - a\|_{k,\mu}\mathrm{d}u + \int_{V\setminus K}\mathrm{e}^{-g(\mathrm{u},u)}\|\alpha_{\sqrt{h}\mathrm{u}}(a) - a\|_{k,\mu}\mathrm{d}u,$$

where K denotes a compact set in V. For $h \searrow 0$ the function $\alpha_h(a): u \mapsto \alpha_{\sqrt{h}u}(a)$ converges uniformly to the constant function $u \mapsto a$ on every compact set in V. Furthermore, since α is isometric, the estimate $\|\alpha_{\sqrt{h}u}(a) - a\|_{k,\mu} \leq 2\|a\|_{k,\mu}$ holds for all $u \in V$. Thus, choosing K large enough makes the second term small, independently of h. Afterwards, choosing h small brings the first term for the fixed K below every positive bound. By the choice of the normalization constant in front of the GauB function this shows (5). The case for $a \in \mathcal{A}$ is analogous. For the last statement we first note that for a fixed a the differentiation in V-directions is a limit in $C_u^\infty(V_\vartheta \mathrm{t}^\infty)$. By the linearity and continuity of $\tilde{S}_g$ as in Proposition (3.1.2) we can thus exchange differentiation and the integral. This gives

$$\frac{\mathrm{d}}{\mathrm{d}h}S_h a = \frac{\mathrm{d}}{\mathrm{d}h}\int_V \frac{\sqrt{\det G}}{\pi^n}\mathrm{e}^{-g(u,u)}\alpha_{\sqrt{h}u}(a)\mathrm{d}u$$

$$= \frac{\sqrt{\det G}}{\pi^n}\int_V \mathrm{e}^{-g(u,\mathrm{u})}\sum_i \frac{u_i}{2\sqrt{h}}\frac{\partial}{\partial u_i}\alpha_u(a)|_{\sqrt{h}u}\mathrm{d}u$$

$$= -\frac{1}{4\sqrt{h}}\frac{\sqrt{\det G}}{\pi^n}\int_V \sum_{i,j}(G^{-1})^{ij}\frac{\partial}{\partial u_j}\mathrm{e}^{-g(u,u)}\frac{\partial}{\partial u_i}\alpha_u(a)|_{\sqrt{h}u}\mathrm{d}u$$

$$= \frac{1}{4}\frac{\sqrt{\det G}}{\pi^n}\mathrm{e}^{-g(u,\mathrm{u})}(G^{-1})^{ij}\frac{\partial^2}{\partial u_i \partial u_j}\alpha_u(a)|_{\sqrt{h}u}\mathrm{d}u$$

$$= \frac{1}{4}\frac{\sqrt{\det G}}{\pi^n}\mathrm{e}^{-g(u,u)}\alpha_{\sqrt{h}u}\left((G^{-1})^{ij}\frac{\partial^2}{\partial v_i \partial v_j}\alpha_v(a)|_{v=0}\right)\mathrm{d}u,$$

where we have used an integration by parts as well as the fact that α is an action. Note that the operator $\triangle_g$ is well defined on $\mathcal{A}^\infty$. This completes the proof.

Since with $a \in \mathcal{A}^\infty$ we also have $\triangle_g\, a \in \mathcal{A}^\infty$, the iteration of (6) immediately yields the following statement:

Theorem (3.1.4)[71] *The operator $S_h: \mathcal{A}^\infty \to \mathcal{A}^\infty$ has the formal asymptotic expansion*

$$S_h \simeq h \searrow 0 \mathrm{e}^{\frac{h}{4}\Delta_g}$$

with respect to the topology of $\mathcal{A}^\infty$.

This means that the asymptotic expansion of S_h corresponds to the formal equivalence transformation leading from the Weyl star product to the Wick product; see, e.g., [100], eq. (3).

Recall that a functional $\omega: \mathcal{A} \to \mathbb{C}$ is called positive if for all $a \in \mathcal{A}$ we have

$$\omega(a^* a) \geq 0.$$

While this is a purely algebraic definition, for a topological algebra $\mathcal{A}$ we require furthermore that *co* is *continuous*. An algebra element $a \in \mathcal{A}$ is called *positive* if $\omega(a) \geq 0$ for all (continuous) positive functionals ω. The positive algebra elements will be denoted by $\mathcal{A}^+$. Note that for general $*$-algebras a definition of positivity like $a = b^* b$ will not lead to a reasonable notion of positive elements due to the lack of a functional calculus. Note also that the above definition coincides with the usual definition of positive elements in the case of a C^*-algebra. There are even more sophisticated notions of positivity, e.g., for O^*-algebras; see the discussion in [97]. However, for our purposes the above definition will be sufficient as for C^*-algebras positive functionals are always continuous.

Now we can use the operator S_h to deform a positive functional of $\mathcal{A}$ into a positive functional with respect to $\star_h$. To this end we observe the following lemma:

Lemma (3.1.5) [71]: *For $a \in \mathcal{A}^\infty$ we have*

$$S_h(a^* \star_h a)$$
$$= \frac{1}{(\pi h)^{2n}} \int_{V\times V} \mathrm{e}^{-\frac{1}{h} g(v,v)}\, \alpha_v(a^*) \mathrm{e}^{-\frac{1}{h} g(w,w)} \alpha_w(a) \mathrm{e}^{\frac{2}{h}(g(v,w)+\mathrm{i}\theta(v,w))} \mathrm{d}v\, dw.$$

Proof. The proof is a straightforward computation using the fact that α is an action as well as a linear change of coordinates and a Fourier transform of the GauB function.

In the particular case that g and θ are *compatible*, i.e., $g\,(u, v) = \theta(u\,, Jv)$ with a complex structure J, the combination $h\,(u, v) = g(u, v) + \mathrm{i}\theta(u, v)$ is known to be a Hermitian metric on the complex vector space (V, J) . In this case there exists a symplectic basis $\{e_1, \dots, e_n, f_1, , f_n\}$ of V with coordinates q^i and p^i and there exist complex coordinates $z^i = q^i + \mathrm{i}p^i$ and $\overline{z}^j = q^i - \mathrm{i}p^i$ such that

$$g(u, u) = \sum_i z_u^i\, \overline{z}_u^i = \|z_u\|^2 \ and\ h(v, w) = \sum_i \overline{z}_v^i\, z_w^i\,.$$

From now on we assume that g is compatible with θ. Using these coordinates, the above integral can be rewritten as

$$S_h(a^* \star_h a) = \frac{1}{(\pi h)^{2n}} \int_{V\times V} \mathrm{e}^{-\frac{1}{h}\|z_v\|^2}\, \alpha_v(a^*) \mathrm{e}^{-\frac{1}{h}\|z_w\|^2} \alpha_w(a) \mathrm{e}^{\frac{2}{h}\overline{z}_v \cdot z_w} \mathrm{d}v\, dw. \quad (7)$$

Lemma (3.1.6) [71]: *For $a \in \mathcal{A}^\infty$ we have*

$$S_h(a^* \star_h a) = \sum_{L\geq 0} \frac{2^{|L|}}{L!} a_L^* a_L \tag{8}$$

with respect to the $\mathcal{A}^\infty$*-topology, where for a multi-index* $L = (l_1, l_n)$ *one defines*

$$a_L = \frac{1}{\pi^n} \int_V \mathrm{e}^{-\|z_v\|^2} z_v^L \alpha_{\sqrt{\hbar}v}(a) \mathrm{d}v.$$

Proof. First note that rescaling the variables in (7) by $\sqrt{h}$ allows to get rid of the negative powers of h. Then (8) is obtained from expanding the exponential function $\mathrm{e}^{2\bar{z}_v \cdot z_w}$ into the Taylor series and exchanging summation and integration. The fact that the latter exchange of limits is allowed follows from a similar argument as in the proof of Lemma (3.1.3): First we split the integration into two parts, one over a compact subset $K \subseteq V$ and the other over $V \backslash K$. On K the Taylor expansion converges uniformly including all derivatives. Outside K, the GauB function decays fast enough to over-compensate the exponential increase. Thus first choosing K large enough to make the second integral small then using the uniform convergence gives the result. Note that the convergence is in the sense of $\mathcal{A}^\infty$.

Theorem (3.1.7) [71]: *Let* g *be a compatible positive definite inner product and* S_h *the corresponding operator as in* (4).

(i) *For every continuous positive linearfunctional* $\omega: \mathcal{A} \to \mathbb{C}$ *the functional*

$$\omega_h = \omega \circ S_h: \mathcal{A}^\infty \to \mathbb{C}$$

is positive and continuous in the $\mathcal{A}^\infty$*-topology.*

(ii) *For every* $a \in \mathcal{A}^\infty$ *we have*

$$S_h(a^* \star_h a) \in \mathcal{A}^+.$$

Proof. Let $\omega: \mathcal{A} \to \mathbb{C}$ be positive and continuous. Since the topology of $\mathcal{A}^\infty$ is finer than the original one, it follows that $\omega: \mathcal{A}^\infty \to \mathbb{C}$ is still continuous. Therefore $\omega\big(S_h(a^* \star_h a)\big) \geq 0$ follows immediately from (8) and the continuity of *co*. Moreover, since S_h is continuous the first part follows. Thus the second part is clear.

Corollary (3.1.8) [71]: *Let* $\omega: \mathcal{A} \to \mathbb{C}$ *be a positive and continuous linear functional. Then on* $\mathcal{A}^\infty$, $\omega_h = \omega \circ S_h$ *has the asymptotic expansion*

$$\omega_h \simeq h \searrow 0 \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{h}{4}\right)^r \omega \circ \Delta_g^r$$

in the $\mathcal{A}^\infty$*-topology.*

In a next step we want to apply Theorem (3.1.7) to the more particular case of a C^*-algebraic deformation. Let $\mathfrak{A}$ be a unital C^*-algebra endowed with an isometric and strongly continuous action of V b y*-automorphisms. Then Rieffel has shown how to construct a C^*-norm on the Fréchet$*$-algebra $(h) = (\mathfrak{A}^\infty, \star_h, *)$. In general, $A(h)$ is not complete. The norm completion of $\mathcal{A}(h)$ will then be denoted by (h) . We briefly recall the construction of the C^*-norm on (h) . Let $S(V, \mathfrak{A}) \subseteq C_u^\infty(V, \mathfrak{A})$ be the subset of functions which are still in $C_u^\infty(V, 2\mathrm{I})$ when multiplied by arbitrary polynomials on V. For f, $g \in S(V, \mathfrak{A})$ one defines the?J-valued inner product

$$\langle f, g \rangle = \int_V f(v)^* g(v) \mathrm{d}v,$$

which makes $S(V,\mathfrak{A})$ into a pre-Hilbert right?J-module; see, e.g., [90] for details on Hilbert modules. In particular, by

$$\|f\|_S = \sqrt{\|\langle f, f\rangle\|}$$

one obtains a norm on $S(V,\mathfrak{A})$, where the norm on the right-hand side is the C^*-norm of $\mathfrak{A}$. Using this norm, Rieffel showed that for every $F \in C_u^\infty(V,\mathfrak{A})$ the operator

$$F \star_h\colon\ S(V,\mathfrak{A}) \ni f \mapsto F \star_h f \in S(V,\mathfrak{A})$$

is continuous with respect to $\|\cdot\|_S$ and adjointable with adjoint given by $F^* \star_h$ Since for $a \in \mathfrak{A}^\infty$ the function $u \mapsto \alpha_u(a)$ is in $C_{\mathrm{u}}^\infty(V,\mathfrak{A})$ we obtain an induced operator on the pre-Hilbert module $\alpha(a) \star_h$. which is continuous and adjointable. A final computation then shows that $a \mapsto \alpha(a) \star_h$ is a*-homomorphism with respect to the deformed product $\star_h$ of $\mathfrak{A}^\infty$. This allows to define

$$\|a\|_h = \|\alpha(a) \star_h .\|,$$

where on the right-hand side we use the operator norm. Since it is well known that the continuous and adjointable operators on a (pre-)Hilbert module constitute a C^*-algebra, Rieffel arrives at a C^*-norm $\|\cdot\|_h$ for $\mathcal{A}(h)$.

We want to show that the operator S_h being defined only on $\mathcal{A}(h)$ is also continuous in the C^*-norm and thus extends to (h) . To show the continuity of S_h we will need the following lemma that shows that there is a star root of the Gau13 function.

Lemma (3.1.9) [71]: *Let G_h be the normalized Gaufl function as in* (3) *used to define the operator S_h. Then we have*

$$G_h \star G_h = \frac{1}{(2\pi h)^n}\frac{1}{\sqrt{\det G}} G_h.$$

Proof. The proof is a straightforward and well-known computation; see, e.g., [87], □ From equation (8) and the trivial fact that $\sqrt{\det G} a_{L=0} = S_h(a)$ we obtain the following statement:

Lemma (3.1.10) [71]. *For $a \in \mathcal{A}^\infty$ we have*

$$S_h(a^* \star_h a) = \frac{1}{\det G} S_h(a^*) S_h(a) + b,$$

where $b \in \mathcal{A}^+$ is positive.

Theorem (3.1.11) [71]: *Let $(\mathfrak{A},\cdot,\|\cdot\|)$ be a C^*-algebra with isometric and strongly continuous action α of V and let $\mathcal{A}(h) = (\mathfrak{A}^\infty, \star_h, \|\cdot\|_h)$ be the Rieffel deformed pre $- C^*-$ algebra. Then the operator*

$$S_h\colon sA\ (h) \to \mathfrak{A}$$

is a continuous operator in the C^*-norms of $\mathcal{A}(h)$ and $\mathfrak{A}$.

Proof. Since is a C^*-algebra, we have $\|S_h a\|^2 = \|(S_h a)^*(S_h a)\|$. From Lemma (3.1.10) it follows that $(S_h a)^*(S_h a) \leq \det(G) S_h(a^* \star_h a)$ in the sense of positive elements in 2I. From this it follows that the same holds for the norms, i.e., $\|(S_h a)^*(S_h a)\| \leq \det(G)\|S_h(a^* \star_h a)\|$. In order to compute the last norm we need the following fact that

$$\int_V f \star_h g = \int_V f\, g \qquad (9)$$

for all $f, g \in S(V, \mathfrak{A})$; see [96], Lemma 3.8. Moreover, due to the fast decay of functions in $(V, \mathfrak{A})$, eq. (9) still holds if one of them is in $C_u^\infty(V, \mathfrak{A})$. Using this and Lemma (3.1.9) we find

$$\begin{aligned}\|S_h(a^* \star_h a)\| &= \det(G)\| \int_V \left(G_h \star_h \alpha(a^* \star_h a)\right)(u)\mathrm{d}u\| \\ &= (2\pi h)^n\left(\det(G)\right)^{\frac{3}{2}}\| \int_V \left(G_h \star_h G_h \star_h \alpha(a^* \star_h a)\right)(u)\mathrm{d}u\| \\ &= (2\pi h)^n\left(\det(G)\right)^{\frac{3}{2}}\| \int_V \left(G_h \star_h \alpha(a)^* \star_h \alpha(a) \star_h G_h\right)(u)\mathrm{d}u\| \\ &= (2\pi h)^n\left(\det(G)\right)^{\frac{3}{2}}\|\{\alpha(a) \star_h G_h , \alpha(a) \star_h G_h)\| \\ &\leq (2\pi h)^n\left(\det(G)\right)^{\frac{3}{2}}\|G_h\|_S^2\|a\|_h^2,\end{aligned}$$

by observing that G_h is central for the *undeformed* pointwise product of $C_u^\infty(V, \mathfrak{A})$. Thus we have the desired continuity

$$\|S_h a\|^2 \leq (2\pi h)^n\left(\det(G)\right)^{\frac{3}{2}}\|G_h\|_S^2\|a\|_h^2. \tag{10}$$

Corollary (3.1.12) [71]: *Let $\omega: \mathfrak{A} \to \mathbb{C}$ be a positive linear functional of the undeformed C^*-algebra. Then $\omega_h = \omega \circ S_h: \mathcal{A}(h) \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_h$ and extends to a positive linearfunctional $\omega_h: \mathfrak{A}(h) \to \mathbb{C}$.*

Thus we have constructed for every classical state *co* a corresponding quantum state using the operator S_h. We shall also use the symbol

$$S_h: \mathfrak{A}(h) \to \mathfrak{A}$$

for the extension of the operator S_h to the completions in the corresponding C*-topologies.

In a last step we want to discuss in which sense ω_h can be considered as a deformation of ω: clearly we have $\omega(a) = \lim_{h \searrow 0} \omega_h(a)$ pointwise for every $a \in \mathcal{A}^\infty$ but we want to show some continuity properties beyond that trivial observation.

One of the main results in Rieffel's work [96] is that the deformed C*-algebras $\{\mathfrak{A}(h)\}_{h\geq 0}$ actually yield a *continuous field* in the sense of Dixmier [86]: Recall that a continuous field structure on a collection $\{\mathfrak{A}(h)\}_{h\geq 0}$ of C*-algebras consists in the choice of *continuous s* $\Gamma \subseteq \prod_{h\geq 0} \mathfrak{A}(h)$ subject to the following technical conditions: Γ is a*-algebra with respect to the pointwise product and for each fixed h the set of possible values $\{a(h)\}_{a\in\Gamma} \subseteq \mathfrak{A}(h)$ is dense. For unital C*-algebras, we require that the unit $h \mapsto \mathrm{I}(h) = \mathrm{I}_{\mathfrak{A}(h)}$ be always in Γ. Moreover, the function $h \mapsto \|a(h)\|_h$ is continuous for all $a \in \Gamma$. Finally, if an arbitrary $b \in \prod_{h\geq 0} \mathfrak{A}(h)$ can locally be approximated uniformly by continuous it is already continuous itself, i.e., if b is a such that for all $\varepsilon > 0$ and all h_0 there exists an open neighborhood $U \subseteq [0, \infty)$ of h_0 and a continuous $a \in \Gamma$ such that $\|a(h) - b(h)\|_h \leq \varepsilon$ uniformly for all $h \in U$, then $b \in \Gamma$. It follows that Γ necessanly contains $C^0(\mathrm{R}_0^+)$.

In the case of the Rieffel deformation the $*$-algebra of continuous sections Γ can be obtained from the "constant" sections $a(h) = a \in \mathfrak{A}^\infty$. In detail, one has the following (technical) characterization:

Proposition (3.1.13) [71]: *Let $A(h) = (\mathfrak{A}^{\infty}, \star_h, \|\cdot\|_h)$ be the Rieffel deformed pre $- C^*-$ algebras and let $\{\mathfrak{A}(h)\}_{h\geq 0}$ be the corresponding field of C^*-algebras. Moreover, let*

$$\Gamma = \{b \in \prod_{h\geq 0} \mathfrak{A}(h) | \forall \varepsilon > 0 \forall h_0 \geq 0 \exists U(h_0) \exists a \in \Gamma_0 \forall h \in U(h_0) : \|b(h) - a(h)\|_h \leq \varepsilon\}$$

be the set of sections generated by the set Γ_0 of sections. Then for all three choices, (i) $\Gamma_0 = \mathfrak{A}^{\infty}$,
(ii) $\Gamma_0 = C^0(\mathrm{R}_0^+) \otimes \mathfrak{A}^{\infty}$,
(iii) Γ_0 *is the $*$-algebra generated by the vector space $C^0(\mathrm{R}_0^+) \otimes \mathfrak{A}^{\infty}$ with respect to $\star_h$, the set Γ is the same and defines the structure of a continuous field.*

In other words, the $*$-algebra Γ of continuous sections yields the smallest continuous field built on the collection $\{\mathfrak{A}(h)\}_{h\geq 0}$ which contains the constant sections $a: h \mapsto a(h) = a \in \mathfrak{A}^{\infty}$. The second choice of Γ_0 is the smallest $C^0(\mathrm{R}_0^+)$-module, while the last choice corresponds to the smallest $*$-algebra containing?J^{∞} and $C^0(\mathrm{R}_0^+)$. In the following we shall always refer to this continuous field structure Γ.

We want to show that the set of states $\omega_h = \omega \circ S_h$, where $\omega: \mathfrak{A} \to \mathbb{C}$ is a classical state, form a continuous field of states in the following sense,

Definition (3.1.14) [71]: A continuous field of states on a continuous field of C*-algebras $(\{\mathfrak{A}(h)\}_{h\geq 0}, \Gamma)$ is a family of states ω_h on $\mathfrak{A}(h)$ such that

$$h \mapsto \omega_h\big(a(h)\big)$$

is continuous for every continuous $a \in \Gamma$.

Lemma (3.1.15) [71]: *If $a \in \Gamma$ is a continuous section, then the map $\mathbb{R}_0^+ \ni h \mapsto S_h a(h) \in \mathfrak{A}$ is continuous in the* (*undeformed*) *C^*-norm of $\mathfrak{A}$.*

Proof. Note that here we use the extension of S_h to the completion? $\mathfrak{A}$. Moreover, by Proposition (3.1.13) we can approximate a by sections in $\Gamma_0 = C^0(\mathbb{R}_0^+) \otimes \mathfrak{A}^{\infty}$. First, we show the continuity at $h \neq 0$:

$$\begin{aligned}\|S_h a(h) - S_{h'} a(h')\| &\leq \|S_h a(h) - S_h a(h')\| + \|S_h a(h') - S_{h'} a(h')\| \\ &= \|S_h\big(a(h) - a_{\Delta h}(h)\big)\| + \|(S_h - S_{h'})\big(a(h')\big)\| \\ &\leq c(h)\|a(h) - a_{\Delta h}(h)\|_h + \|(S_h - S_{h'})\big(a(h')\big)\|.\end{aligned}$$

Here $a_{\Delta h}(h) = a(h + \triangle h)$ with $\triangle h = h' - h$ and $c(h)$ is the constant from the estimate (10). It is now easy to see that the $a_{\Delta h}$ is approximated by sections of the form $\sum_n \tau_{\Delta h} f_n a_n$, where $(\tau_{\Delta h} f_n)(h) = f_n(h + \triangle h)$. Thus $a_{\Delta h}$ is still in Γ and approximates a for $\triangle h \to 0$. Hence the first term becomes small for $h' \to h$. The second term requires more attention. We can approximate a by sections of the form $\sum_n f_n a_n \in \Gamma_0$ with a finite sum and $f_n \in C^0(\mathbb{R}_0^+)$ and $a_n \in \mathfrak{A}^{\infty}$. Then we have

$$\begin{aligned}\|(S_h - S_{h'})\big(a(h')\big)\| &\leq \|S_h\left(a(h') - \sum f_n(h') a_n\right)\| \\ &+ \|(S_h - S_{h'})\left(\sum f_n(h') a_n\right)\| + \|S_{h'}\left(a(h') - \sum f_n(h') a_n\right)\|\end{aligned}$$

$$\leq c(h)\|a_{\Delta h}(h)-\sum \tau_{\Delta h} f_n(h)a_n\|_h$$

$$+\|\sum f_n(h')a_n\|\int |G_h(u)-G_{h'}(u)|\mathrm{d}u$$

$$+c(h')\|a(h')-\sum f_n(h')a_n\|_{h'}.$$

The constants $c(h)$ and $c(h')$ are bounded in a small neighborhood of $h \neq 0$. Since the functions f_n are continuous, $\|\sum f_n(h')a_n\|$ is bounded on a neighborhood. The other factors become smaller than any $\varepsilon > 0$ for $h' \to h$. This shows the continuity at $h \neq 0$. For the continuity at 0 we have with $S_0 = \mathrm{id}$:

$$\|S_h(a(h)) - S_0(a(0))\| \leq \|S_h\left(a(h) - S_h(a(0))\right)\| + \|S_h(a(0)) - a(0)\|.$$

The first term gives

$$\|\int G_h(u)\alpha_u(a(h)-a(0))\mathrm{d}u\| \leq \|a(h)-a(0)\|\int G_h(u)\mathrm{d}u$$
$$= \|a(h)-a(0)\|,$$

since the Gau13 function is normalized and α is isometric. Now $a(h) = a_h(0)$ approximates $a(0)$ in a neighborhood of zero whence this contribution becomes small for $h \searrow 0$. The second term becomes small thanks to the asymptotics from Lemma (3.1.3) in the topology of $\mathfrak{A}$. This shows the continuity at 0, too.

From this lemma we immediately obtain the main result:

Theorem (3.1.16) [71]: *For every classical state $\omega: \mathfrak{A} \to \mathbb{C}$ andfor every continuous $a \in \Gamma$ the map*

$$h \mapsto \omega\left(S_h(a(h))\right) = \omega_h(a(h))$$

is continuous. Hence $\{\omega_h\}_{h\geq 0}$ is a continuous field of states with $\omega_0 = \omega$.

Section (3.2): A Nuclear Weyl Algebra

The Weyl algebra as the mathematical habitat of the canonical commutation relations has many incarnations and variants: in a purely algebraic definition it is the universal unital associative algebra generated by a vector space V subject to the commutation relations $vw - wv = \Lambda(v, w)1$ where Λ is a symplectic form on V and v, $w \in V$. For $V = \mathbb{K}^2$ with basis q, p and the standard symplectic form, the canonical commutation relations take the familiar form

$$\mathrm{qp} - \mathrm{pq} = 1. \tag{11}$$

Typically, some scalar prefactor in front of]⌊ is incorporated. When working over the complex numbers, a C^*-algebraic version of the canonical commutation relations is defined by formally exponentiating the generators from V and using the resulting commutation relations from (11) with iħ in front of]⌊ for the exponentials. It results in a universal C^*-algebra generated by the exponentials subject to the commutation relations. This version of the Weyl algebra is most common in the axiomatic approaches to quantum field theory and quantum mechanics. An alternative construction of a Weyl algebra in a C^*-algebraic

framework has been proposed and studied in [102] based on resolvents instead of exponentials. In [103] the C^*-algebraic Weyl algebra was shown to be a strict deformation quantization of a certain Poisson algebra which consists of certain "bounded" elements in contrast to the "unbounded" generators from V itself, ultimately leading to a continuous field of C^*-algebras.
The two principle versions ofthe Weyl algebra differ very much in their behavior. While the C^*-algebraic formulation has a strong analytic structure, the algebraic version based on the canonical commutation relations does not allow for an obvious topology: in fact, it can easily be proven that any submultiplicative seminorm on the Weyl algebra generated by q and p with commutation relations (11) necessarily vanishes. In particular, there will be no structure of a normed algebra possible.
We provide a reasonable topology for the algebraic Weyl algebra making the product continuous. Starting point will be a general locally convex topology on V, where we also allow for a graded vector space and a graded version of the canonical commutation relations, i.e. we treat the Weyl algebra and the Clifford algebra on the same footing. The algebraic version of the Weyl algebra will be realized by means of a deformation quantization [72] of the symmetric algebra $S^{\cdot}(V)$ encoded in a star product. The idea is to treat the Poisson bracket arising from the bilinear form Λ as a *constant* Poisson bracket and use the Weyl-Moyal star product quantizing it. From a deformation quantization point ofview this is a very trivial situation, though we of course allow for an infinite-dimensional vector space V, see [100] for a gentle introduction to deformation quantization. The bilinear form Λ will not be required to be antisymmetric or non-degenerate. However, we need some analytic properties. For convenience, we require Λ to be *continuous*, a quite strong assumption in infinite dimensions. Many interesting examples fulfill this requirement. In particular, for a finite-dimensional space V this is always the case.
On the tensor algebra $T^{\cdot}(V)$ and hence on the symmetric algebra $S^{\cdot}(V)$ there is of course an abundance of locally convex topologies which all induce the projective topology on each $V^{\otimes n}$. The two extreme cases are the direct sum topology and the Cartesian product topology. The direct sum topology for the Weyl algebra with finitely many generators was used in [104] to study bivariant K-theory. In [105] a slightly coarser topology on $S^{\cdot}\left(\mathcal{S}(\mathbb{R}^d)\right)$ than the direct sum topology was studied in quantum field theories, where $\mathcal{S}(\mathbb{R}^d)$ is the usual Schwartz space. It turns out that this topology makes $S^{\cdot}\left(\mathcal{S}(\mathbb{R}^d)\right)$ a topological algebra, too. However, for our purposes, this topology is still too fine. Interesting new phenomena are found in [106] for formal star products in the case the underlying locally convex space V is a Hilbert space. For the class of functions considered, the classification program shows much richer behavior than in the well-known finite-dimensional case. However, the required Hilbert-Schmidt property will differ from the requirements we state.
The first main result is that we can define a new locally convex topology on the tensor algebra $T^{\cdot}(V)$ and hence also on the symmetric algebra $S^{\cdot}(V)$, quite explicitly by means of seminorms controlling the growth of the coefficients $a_n \in S^n(V)$, in such a way that the star product is continuous. The completion of $S^{\cdot}(V)$ with respect to this locally convex topology will contain many interesting entire functions like exponentials of elements in V.

It turns out that even more is true: the star product converges absolutely and provides an entire deformation in the sense of [107]. In fact, we have two versions of this construction depending on a real parameter $R \geq \frac{1}{2}$ leading to a Weyl algebra $W_R(V)$ and a projective limit $W_{R-}(V)$ for $R > \frac{1}{2}$. Both share many properties but differ in others.

If the underlying vector space V has an absolute Schauder basis we prove that the corresponding Weyl algebra also has an absolute Schauder basis. The second main result is that the Weyl algebra $\mathrm{W}_R(V)$ is nuclear whenever we started with a nuclear V. This is of course a very desirable property and shows that the Weyl algebra enjoys some good properties. If V is even strongly nuclear then the second version $\mathrm{W}_{R-}(V)$ gives a strongly nuclear Weyl algebra. In the case where V is finite-dimensional, we have both for trivial reasons: an absolute Schauder basis of V and strong nuclearity. Thus in this case the corresponding Weyl algebra turns out to be a (strongly) nuclear algebra with an absolute Schauder basis. In fact, we can show even more: the underlying locally convex space is a particular Köthe space which can explicitly be described.

The construction depends functorially on the data V and Λ as well as on a parameter R which controls the coarseness of the topology. The particular value $R = \frac{1}{2}$ seems to be distinguished as it is the limit case for which the product is continuous. For the second variant of our construction, the case $R = 1$ is distinguished as this is the limit case where the exponentials are part of the completion.

We show that the topological dual V' acts on the Weyl algebra by translations. These automorphisms are even *inner* if the element in V' is in the image of the canonical map $V \to V'$ induced by the antisymmetric part of Λ: here we show that the exponential series of elements in V are contained in the Weyl algebra $W_R(V)$, provided $R < 1$, and in $\mathrm{W}_{R-}(V)$ for $R \leq 1$. Since the Weyl algebra does not allow for a general holomorphic calculus, this is a nontrivial statement and puts heuristic formulas for the star-exponential on a solid ground. In particular, these exponentials are also the generators of the C^*-algebraic version of the Weyl algebra, showing that there is still a close relation. However, it does not seem to be easy to make the transition to the C^*-algebraic world more explicitly.

For a finite-dimensional even vector space V, we relate our general construction to the following two earlier versions of the Weyl algebra: first we show that for a suitable choice of the parameters and the Poisson structure the Weyl algebra discussed in [108] coincides with $W_R(V)$. Second, we show that the results from [109], [73] yield the second version $\mathrm{W}_{R-}(V)$ for the particular value $R = 1$. This way, we have now a clear picture on the relation between the two approaches.

We apply our general construction to an example from (quantum) field theory. We consider a linear field equation on a globally hyperbolic spacetime manifold. The Green operators of the normally hyperbolic differential operator encoding the field equation define a Poisson bracket, the so-called Peierls bracket. We show the relevant continuity properties in order to apply the construction of the Weyl algebra to this particular Poisson bracket. It is shown that the resulting Poisson algebra and Weyl algebra relate to the canonical Poisson algebra and Weyl algebra on the initial data of the field equation. The result will be a local net of Poisson algebras or Weyl algebras obeying a version of the Haag-Kastler axioms including the time-slice axiom. On one hand this is a very particular case ofthe Peierls bracket discussed in [110], on the other hand, we provide a simple quantum theory with honestly converging star product in this situation thereby going beyond the formal star products as discussed in [111], [112]. It would be very interesting to see how the much more general (and non-constant) Poisson structures in [110] can be deformation quantized with a convergent star product.

In finite dimensions it is always possible to choose a compatible almost complex structure for a given symplectic Poisson structure. Such a choice gives a star product ofWick type where the symmetric part of Λ now consists of a suitable multiple of the compatible positive definite inner product. The Wick product enjoys the additional feature of being a *positive* deformation [77]. In particular, the evaluation functionals at the points of the dual will become positive linear functionals on the Wick algebra. In [73] the corresponding GNS construction was investigated in detail and yields the usual Bargmann-Fock space representation for the canonical commutation relations. The case of a Hilbert space of arbitrary dimension will be the natural generalization for this. In general, the existence of a compatible almost complex structure having good continuity properties is far from being obvious. · Closely related will be the question what the states of the locally convex Weyl algebra will be in general. While this question might be quite hard to attack in full generality, the more particular case of the Weyl algebra arising from the Peierls bracket will be already very interesting: here one has certain candidates of the so-called Hadamard states from (quantum) field theory. It would be interesting to see whether and how they can be matched with compatible almost complex structures and evaluations at points in the dual.

· In infinite dimensions there are important examples of bilinear forms which are not continuous but only separately continuous. It would be interesting to extend our analysis to this situation as well in such a way that one obtains a separately continuous star product. Yet another scenario would be to investigate a *bornological* version of the Weyl algebra construction: many bilinear forms turn out to be compatible with naturally defined bornologies rather than locally convex topologies. Thus a bornological star product would be very desirable and has the potential to cover many more examples not yet available with our present construction. A good starting point might be [113], [114].

· Finally, already in finite dimensions it will be very challenging to go beyond the geometrically trivial case of constant Poisson structures. One possible strategy is to use the completed nuclear Weyl algebra build on each tangent space of a symplectic manifold. This leads to a Weyl algebra bundle, now in our convergent setting. In a second step one should try to understand how the Fedosov construction [87] of a formal star product can be transferred to this convergent setting provided the curvature and its covariant derivatives of a suitably chosen symplectic connection satisfy certain (still to be found) bounds.

We recall some well-known algebraic facts on constant Poisson structures and their deformation quantizations. Contains the core results. We first construct several systems of seminorms on the tensor algebra and investigate the continuity properties ofthe tensor product with respect to them. The continuity ofthe Poisson bracket is then established but the continuity of the star product requires a suitable projective limit construction in addition. The resulting systems of seminorms are still described explicitly. This way, we arrive at our definition of the Weyl algebra and show that it yields a locally convex algebra We prove that the star product converges absolutely, provides an entire deformation, and enjoys good reality properties we show two main results: first that if V has an absolute Schauder basis the Weyl algebra also has an absolute Schauder basis. Second, we prove that the Weyl algebra is (strongly) nuclear iff V is (strongly) nuclear. devoted to various symmetries and equivalences. We prove that the algebraic symmetries can be cast into the realm of the locally convex Weyl algebra, too, and yield a good functoriality of the construction. If the convergence parameter R is less than 1 then translations are shown to act by inner automorphisms. We show that the isomorphism class of the Weyl algebra only depends on the antisymmetric part ofthe bilinear form Λ. Finally, we relate our construction to the one from [108] in Proposition (3.2.58) as well as to the version from [73], [109] The final and quite large contains a first nontrivial example: the canonical and covariant Poisson structures arising in noninteracting field theories on globally hyperbolic spacetimes. We recall the necessary preliminaries to define and compare the two Poisson structures in detail. The continuity properties of both allow to apply our general construction of the Weyl algebra, leading to a detailed description As a first application we show that both on the classical side as well as on the quantum side the construction leads to a local net of observables satisfying the time-slice axiom.

Let $\mathbb{K}$ be a field of characteristic 0 and let V be a $\mathbb{K}$ -vector space.

In order to treat the symmetric and the Grassmann algebra on the same footing, we assume that $V = V_0 \oplus V_1$ is $\mathbb{Z}_2$- graded. In many applications, V is even Z-graded and the induced $\mathbb{Z}_2$-grading is then given by the even and the odd part of V. A vector $v \in V_0$ is called homogeneous of parity 0 while a vector in V_1 is called homogeneous of parity 1.

We denote the parity of the vector v with the same symbol $v \in \mathbb{Z}_2$ and we also shall refer to even and odd parity. We will make use of the *Koszul sign rule*, i.e. if two things with parities $a,\ b \in \mathbb{Z}_2$ are exchanged this gives an extra sign $(-1)^{ab}$. The homogeneous components of $v \in V$ will be denoted by $v = v_0 + v_1$.

We will need the following signs for *symmetrization*. For homogeneous vectors v_1 , , $v_n \in V$ and a permutation $\sigma \in \mathrm{S}_n$ one defines the sign

$$\operatorname{sign}(v_1,, v_n; \sigma) = \prod_{i<j} \frac{\sigma(i) + (-1)^{v_{\sigma(i)} v_{\sigma(j)}} \sigma(i)}{i + (-1)^{v_i v_j} j}. \tag{12}$$

Then $\operatorname{sign}(v_1,, v_n; \sigma) = 1$ if all the v_1 , , v_n are even and $\operatorname{sign}(v_1,, v_n; \sigma) = \operatorname{sign}(\sigma)$ is the usual signum of the permutation for all v_1 , , v_n odd. It is then straightforward to check that

$$(v_1 \otimes \cdots \otimes v_n) \lhd \sigma = \operatorname{sign}(v_1,, v_n; \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \tag{13}$$

extends to a well-defined right action of S_n on $V^{\otimes n}$. We use this right action to define the symmetrization operator

$$\mathcal{S}_n : V^{\otimes n} \ni v \mapsto \mathcal{S}_n(v) = \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} v \lhd \sigma \in V^{\otimes n}. \tag{14}$$

One has $\mathrm{s}_{t_\iota} \circ \mathrm{s}_{l1} = \mathcal{S}_n$ since (13) is an action. In the case where $V = V_0$, the operator $\mathcal{S}_n$ is the usual total symmetrization, if $V = V_1$ we get the total antisymmetrization operator.

For later use it will be advantageous to define the symmetric algebra not as a quotient algebra of the tensor algebra but as a subspace with a new product. Thus we set

$$\mathrm{S}^n(V) = \mathrm{im}\mathcal{S}_n \subseteq V^{\otimes n} \; and \; \mathrm{S}^0(V) = k. \tag{15}$$

The elements in $\mathrm{S}^n(V)$ consist of the symmetric, i.e. invariant tensors with respect to the action (13). Moreover, we set

$$\mathrm{S}^{\cdot}(V) = \bigoplus_{n=0}^{\infty} \mathrm{S}^{\mathrm{n}}(V) \subseteq \mathrm{T}^{\cdot}(V) = \bigoplus_{\uparrow l=0}^{\infty} V^{\otimes \mathrm{n}}. \tag{16}$$

Alternatively, we can use the idempotent $s = \bigoplus_{\mathrm{n}=0}^{\infty} \mathrm{s}_n$ where $\mathrm{s}_0 = \mathrm{id}$ and get $\mathrm{S}^{\cdot}(V) = im\mathcal{S}_n$. Next we define the symmetric tensor product $\mu: \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \to \mathrm{S}^{\cdot}(V)$ of $v, w \in \mathrm{S}^{\cdot}(V)$ as usual by

Most of the time we shall omit the symbol μ for products. Then the following statement is well-known:

$$vw = \mu(v \otimes w) = \mathcal{S}(v \otimes w)\,. \tag{17}$$

Lemma (3.2.1) [101]: *The symmemc tensor product turns* $\mathrm{S}^{\cdot}(V)$ *into an associative commutative unital algebra freely generated by* V.

Moreover, it is z-graded with respect to the tensor degree. Note however, that we do not use this degree for sign purposes at all. Freely generated means that a homogeneous map $\varphi: V \to A$ of parity 0 into another associative $\mathbb{Z}_2$-graded commutative unital algebra A has a unique extension $\Phi: \mathrm{S}^{\cdot}(V) \to A$ as unital algebra homomorphism.

Beside the symmetric algebra we will also need tensor products of algebras. Thus let A and $'B$ be two associative $\mathbb{Z}_2$-graded algebras. On their tensor product $A \otimes {}'\mathrm{B}$ a new product is defined by linear extension of

$$(a \otimes b)(a' \otimes b') = (-1)^{ba'} aa' \otimes bb', \tag{18}$$

where $0, a' \in A$ and $b, b' \in {}'\mathrm{B}$ are homogeneous elements. This turns $A \otimes {}'B$ again into an associative $\mathbb{Z}_2$-graded algebra. Finally, we recall that the I⟨oszul sign rule also applies to tensor products of maps and evaluations, i.e. for homogeneous maps $\varphi: V \to W$ and $\psi: \tilde{V} \to \widetilde{W}$ we define their tensor product $\varphi \otimes \psi: V \otimes \tilde{V} \to W \otimes \widetilde{W}$ by

is called a *multiderivation* if for each argument it satisfies the Leibniz rule

$$(\varphi \otimes \psi)(v \otimes w) = (-1)^{\psi v} \varphi(v) \otimes \psi(w) \tag{19}$$

on homogeneous vectors and extend linearly.

$$P: \mathrm{S}^{\cdot}(V) \times \times \mathrm{S}^{\cdot}(V) \to \mathrm{S}^{\cdot}(V) \tag{20}$$

Recall that a homogeneous multilinear map of parity 0

$$\begin{aligned} &P(v_1, \ldots, v_{k-1}, v_k v_k', v_{k+1}, v_{\mathrm{t}1}) \\ &= (-1)^{(v_1+\cdots+v_{k-1})vk} U_k P(v_1,, v_k',, v_n) \\ &+(-1)^{v_k'(v_{k+1}+\cdots+v_{\mathrm{n}})} P(v_1, \ldots\ldots, v_k, \ldots\ldots, v_n) v_k'\,, \end{aligned} \tag{21}$$

where we follow again the Koszul sign rule. Note that if one v_i is a constant, i.e. $v_i \in \mathrm{S}^0(V)$, then $P(v_1,, v_{\mathrm{n}}) = 0$. Since V generates $\mathrm{S}^{\cdot}(V)$, a multiderivation is uniquely determined by its values on $V \times \times V$. Conversely, any multilinear homogeneous map $V \times \times V \to \mathrm{S}^{\cdot}(V)$ of parity 0 extends to a multiderivation since V generates $\mathrm{S}^{\cdot}(V)$ *freely*. Though one can also consider odd multiderivations,

For later estimates, we will need the following more explicit form of this extension for the particular case of a *bilinear* homogeneous map

$$\Lambda: V \times V \to \mathrm{k} = \mathrm{S}^0(V) \tag{22}$$

of parity 0. To this end we first define the linear map

$$P_\Lambda: \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \to \mathrm{S}^{*-1}(V) \otimes \mathrm{S}^{*-1}(V) \tag{23}$$

to be the linear extension of

$$P_\Lambda(v_1 \ldots v_n \otimes w_1 \cdots w_m)$$
$$= \sum_{k=1}^{ll} \sum_{\ell=1}^{m} k - (1)^{(\mathrm{v}_{\mathrm{k}+1}+\cdots+v_n)+w_\ell(w_\ell+\cdots+w_{\ell-1})} \Lambda(v_k, w_\ell) v_1 \ldots \wedge^k \ldots v_n \otimes w_1 \ldots \wedge \ell \ldots w_m. \tag{24}$$

The requirement on the Z-grading implies that P_Λ vanishes on tensors of the form $]\lfloor\otimes w$ or $v \otimes]\lfloor$.

First note that Λ is of parity 0 and thus $\Lambda(v_k, w_\ell)$ is only nontrivial if v_k and w_ℓ have the same parity. Hence we can exchange the parities v_k and w_ℓ in the above sign. Second, note that the map P_Λ is indeed well-defined since the right hand side is totally symmetric in v_1 , , v_n and in w_1 , , w_m. With respect to the algebra structure (18) of $S^{\cdot}(V) \otimes S^{\cdot}(V)$ we can characterize the map P_Λ now as follows:

Lemma (3.2.2) [101]: *The map P_Λ is the unique map with*

(i) $P_\Lambda(v \otimes w) = \Lambda(v, w) 1 \otimes \mathrm{n}$ *for all* $v, w \in V$,

(ii) $P_\Lambda(v \otimes wu) = P_\Lambda(v \otimes w)(1 \otimes u) + (-1)^{wv}(\mathrm{I} \otimes w) P_\Lambda(v \otimes u)$ *for all* v, w, $u \in \mathrm{S}^{\cdot}(V)$,

(iii) $P_\Lambda(vw \otimes u) = (v \otimes]\lfloor) P_\Lambda(w \otimes u) + (-1)^{wu} P_\Lambda(v \otimes u)(1 \otimes w)$ *for all* v, w, $u \in \mathrm{S}^{\cdot}(V)$.

Note that the two Leibniz rules imply $P_\Lambda(v \otimes \mathrm{n}) = 0 = P_\Lambda(]\lfloor \otimes v)$ for all $\in \mathrm{S}^{\cdot}(V)$.

In order to rewrite the Leibniz rules for P_Λ in a more conceptual way, we have to introduce the canonical flip operator $\tau_{VW}: V \otimes W \to W \otimes V$ for $\mathbb{Z}_2$-graded vector spaces V and W by

$$\tau_{VN}(v \otimes w) = (-1)^{vw} w \otimes v \tag{25}$$

on homogeneous elements and linear extension to all tensors. We usually write τ if the reference to the underlying vector spaces is clear. Using τ we define the operators

$$P_\Lambda^{12}, P_\Lambda^{23}, P_\Lambda^{13}: \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \to \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \tag{26}$$

on the triple tensor product by

$$P_\Lambda^{12} = P_\Lambda \otimes \mathrm{id}, P_\Lambda^{23} = \mathrm{id} \otimes P_\Lambda \text{, and } P_\Lambda^{13}$$
$$= (\mathrm{id} \otimes \tau) \mathrm{o} (P_\Lambda \otimes \mathrm{id}) \mathrm{o} (\mathrm{id} \otimes \tau) . \tag{27}$$

These operators have again parity 0 and change the tensor degrees by $(-1, -1, 0)$, $(0, -1, -1)$, and by $(-1, 0, -1)$, respectively.

Lemma (3.2.3) [101]: *The Leibniz rules for P_Λ can be written as*

$$P_\Lambda \mathrm{o} (\mu \otimes \mathrm{id}) = (\mu \otimes \mathrm{id}) \mathrm{o} (P_\Lambda^{13} + P_\Lambda^{23}) \tag{28}$$

and

$$P_\Lambda \mathrm{o} (\mathrm{id} \otimes \mu) = (\mathrm{id} \otimes \mu) \mathrm{o} (P_\Lambda^{12} + P_\Lambda^{13}) . \tag{29}$$

Analogously, we have similar Leibniz rules for the operators P_Λ^{12}, P_Λ^{23}, and P_Λ^{13} which show that they will be uniquely determined by their values on generators of $\mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V) \otimes \mathrm{S}^{\cdot}(V)$. Hence the products $P_\Lambda^{12} \mathrm{o} P_\Lambda^{23}$ etc. will be uniquely determined by their values on quadratic expressions in the generators. This will allow for a rather straightforward computation leading to the following obsen ation:

Lemma (3.2.4) [101]: *The operators* P_Λ^{12}, P_Λ^{23}, *and* P_Λ^{13} *commute pairwise*.
This lemma together with the Leibniz rule in the form of Lemma (3.2.3) gives immediately the following result, see e.g. [100] for a detailed proof:
Proposition (3.2.5) [101]: *On* $\mathrm{S}^\cdot(V)[[v]]$ *one obrains a* $\mathbb{Z}_2$*-graded associarive* $\mathrm{k}[[v]]$*-bilinear mulriplicarion by*

$$v \star_{v\Lambda} w = \mu\mathrm{o}\mathrm{e}^{vP_\Lambda}(v \otimes w)\,, \tag{30}$$

where all $\mathbb{K}$*-multilinear maps are extended to be* $\mathrm{k}[[v]]$*-mulrilinear as usual*.
Proposition (3.2.6) [101]: *Let* A *be an associative* $\mathbb{Z}_2$*-graded commutative algebra and let* $\star$ *be a formal associative deformation of it such that* $(A[[v]],\star)$ *is still* $\mathbb{Z}_2$*-graded. Then the first order of the* $\star$*-commutator defmes a Poisson bracket on* A.
We always take the $\mathbb{Z}_2$-graded commutators and Poisson brackets. In our example, this leads to the following Poisson bracket:
Corollary (3.2.7) [101]: *Let* Λ *be as above and set* $P_\Lambda^{\mathrm{opp}} = \tau\mathrm{o}P_\Lambda\mathrm{o}\tau$. *Then*

$$\{v, w\}_\Lambda = \mu\mathrm{o}\left(P_\Lambda - P_\Lambda^{\mathrm{opp}}\right)(a \otimes b) \tag{31}$$

defines a Poisson bracket for $\mathrm{S}^\cdot(V)$.
Alternatively, we can also consider the *symmetric* and *antisymmetr c part*

$$\Lambda_\pm = \frac{1}{2}(\Lambda \pm \Lambda 0\tau) \tag{32}$$

of Λ such that $\Lambda = \Lambda_+ + \Lambda_-$. Then we note that with $\Lambda^{\mathrm{opp}} = \Lambda 0\tau$ we have

$$P_{\Lambda^{\mathrm{opp}}} = P_\Lambda^{\mathrm{opp}} \tag{33}$$

and thus

$$P_\Lambda - P_\Lambda^{\mathrm{opp}} = 2P_{\Lambda -}. \tag{34}$$

Thus $\{v, w\}_\Lambda = 2\mu\mathrm{o}P_{\Lambda-}(v \otimes w)$ depends only on the antisymmetric part. The star product $\star$ in (30) depends on Λ and not just on Λ_-. It is this Poisson bracket for which $\star_{v\Lambda}$ provides a formal deformation quantization.
In general, one requires only a formal star product but since our Poisson bracket is rather particular, we can sharpen the deformation result as follows:
Corollary (3.2.8) [101]: *The product* $\star_{v\Lambda}\, resmc[s$ *to* $\mathrm{S}^\cdot(V)[v]$ *which becomes an associarive* $\mathbb{Z}_2$*-graded algebra over* $\mathrm{k}[v]$.
For $v,\, w \in \mathrm{S}^\cdot(V)$ we have $P_\Lambda^n(v \otimes w) = 0$ as soon as $n \in \mathrm{N}_0$ is larger than the maximal symmetric degree in v or w. It follows that in $\mathrm{S}^\cdot(V)[v]$ we can replace the formal parameter v by any element of k and get a well-defined associative multiplication from $\star_{v\Lambda}$.
Also the following result is well-known and obtained from an easy induction: the elements of V generate $\mathrm{S}^\cdot(V)[v]$ with respect to $\star_{v\Lambda}$:
Corollary (3.2.9) [101]: *The* $\mathrm{k}[v]$*-algebra* $\mathrm{S}^\cdot(V)[v]$ *is generated by* V.

The symmetric algebra $\mathrm{S}^\cdot(V)$ can be interpreted as the polynomials on the “predual” of V, which, of course, needs not to exist in in finite dimensions. Alternatively, $S^\cdot(V)$ injects as a subalgebra into the polynomials $\mathrm{Pol}^\cdot(V^*)$ on the dual of V. We use this heuristic point of view now to establish some symmetries of $\{.\ ,\ \}_\Lambda$ and $\star_{v\Lambda}$ which justify the term “constant” Poisson structure.
Let $\phi \in V^*$ be an even linear functional, i.e. $\phi \in V_0^*$, then the linear map $v \mapsto v + \phi(v)1$ is even, too, and thus it extends uniquely to a unital algebra homomorphism $\tau_\phi^*: \mathrm{T}^\cdot(V) \to$

$T^{\cdot}(V)$. Clearly, the symmetry properties of the tensors in $T^{\cdot}(V)$ are preserved by τ_ϕ^* and thus it restricts to a unital algebra homomorphism

$$\tau_\phi^*: S^{\cdot}(V) \to S^{\cdot}(V), \qquad (35)$$

now with respect to the symmetric tensor product. One has $\tau_0^* = \mathrm{id}$ and $\tau_\phi^* \tau_\psi^* = \tau_{\phi+\psi}^*$ for all , $\psi \in V_0^*$. Thus we get an action of the abelian group V_0^* on $S^{\cdot}(V)$ by automorphisms. In the interpretation of polynomials these automorphisms correspond to pull-backs with *translations* via ϕ, hence the above notation.

The other important symmetry emerges from the endomorphisms of V itself. Let $A: V \to V$ be an even linear map and denote the extension as unital algebra homomorphism again by $: S^{\cdot}(V) \to S^{\cdot}(V)$. This yields an embedding of $\mathrm{End}_0(V)$ into the unital algebra endomorphisms of $S^{\cdot}(V)$. In particular, we get a group homomorphism of $\mathrm{GL}_0(V)$ into $\mathrm{Aut}_0(S^{\cdot}(V))$. For $A \in \mathrm{GL}_0(V)$ and $\phi \in V_0^*$ we have the relation $A^{-1}\tau_\phi^* A v = \tau_{A^*\phi}^* v$ for the generators $v \in V$ and hence also in general

$$A^{-1}\tau_\phi^* A = \tau_{A^*\phi}. \qquad (36)$$

This gives an action of the semidirect product $\mathrm{GL}_0(V) \ltimes V^*$ on $S^{\cdot}(V)$ via unital algebra automorphisms.

For the bilinear map Λ we consider the group of invertible even endomorphisms of V preserving it and denote this group by

$$\mathrm{Aut}(V, \Lambda) = \{A \in \mathrm{GL}_0(V) | \Lambda(Av, Aw) = \Lambda(v, w) \; for \; all \; v, w \in V\}. \qquad (37)$$

Note that such an automorphism preserves Λ_+ and Λ-separately. However, Λ_- and Λ_+ might have a larger invariance group than Aut(V, Λ).

Lemma (3.2.10) [101]: *The subgroup* $\mathrm{Aut}(V, \Lambda) \ltimes V^*$ *acts on* $S^{\cdot}(V)$ *as automorphisms of* $\{\cdot, \cdot\}_\Lambda$ *and* $\star$.

Proof. First consider $\in \mathrm{Aut}(V, \Lambda)$. Then on generators one sees that $P_\Lambda \mathrm{o}(A \otimes A) = (A \otimes A)\mathrm{o}P_\Lambda$, which therefore holds in general. From this we see that A is an automorphism of both, the Poisson bracket and the star product. Analogously, for $\phi \in V_0^*$ one checks first on generators and then in general that $P_\Lambda \mathrm{o}(\tau_\phi^* \otimes \tau_\phi^*) = (\tau_\phi^* \otimes \tau_\phi^*)\mathrm{o}P_\Lambda$. □

In this sense, both the Poisson bracket and the star product are *constant*, i.e. translation-invariant.

In the next step we discuss to what extend the automorphisms are inner. We consider only the infinitesimal picture as the integrated version will require analytic tools. The bilinear form Λ induces a linear map into the dual V^*. We need the antisymmetric part Λ_- of Λ as it appears also in the Poisson bracket (31). This defines an even linear map

$$\#: V \ni v \mapsto v^\# = \Lambda_-(v, \cdot) \in V^* \qquad (38)$$

Lemma (3.2.11) [101]: *Let* $\phi \in V^*$ *be homogeneous and denote by* $X_\phi: S^{\cdot}(V) \to S^{\cdot}(V)$ *the homogeneous derivation extending* $\phi: V \to \mathrm{k}$.

(i) X_ϕ *is a Poisson denvation of parity* ϕ, *i.e. we have*

$$X_\phi\{a, b\}_\Lambda = \{X_\phi(a), b\}_\Lambda + (-1)^{\phi a}\{a, X_\phi(b)\}_\Lambda \qquad (39)$$

for all homogeneous a, $b \in S^{\cdot}(V)$.

(ii) X_ϕ *is inner iff* $\phi \in \mathrm{im}\#$. *In this case* $X_\phi = \{v, \cdot\}_\Lambda$ *for any* $v \in V$ *with* $2v^\# = \phi$.

(iii) X_ϕ *is a derivation of* $\star_{v\Lambda}$, *i.e. we have*

$$X_\phi(a \star_{v\Lambda} b) = X_\phi(a) \star_{v\Lambda} b + (-1)^{\phi a} a \star_{v\Lambda} X_\phi(b) \quad (40)$$

for all homogeneous a, $b \in S^\cdot(V)$.

(iv) *X_ϕ is a quasi-inner derivation of $\star_{v\Lambda}$, i.e. $X_\phi = \frac{1}{v}[a,\cdot]_{\star_{v\Lambda}}$ for some $a \in S^\cdot(V)[[v]]$, iff* $\phi \in \mathrm{im}\#$. *In this case $a = v \in V$ with $2v^\# = \phi$ will do thejob.*

Proof. Consider an even linear map $P: S^\cdot(V) \otimes S^\cdot(V) \to S^\cdot(V) \otimes S^\cdot(V)$ satisfying the Leibniz rules from Lemma (3.2.2)(ii) and (iii), and let X be any homogeneous derivation of either even or odd parity. Then we claim that the operator

$$D = P\mathrm{o}(X \otimes \mathrm{id} + \mathrm{id} \otimes X) - (X \otimes \mathrm{id} + \mathrm{id} \otimes X)\mathrm{o}P$$

satisfies the Leibniz rules

$$D(ab \otimes c) = (-1)^{bc} D(a \otimes c)(b \otimes]\lfloor) + (-1)^{Xa}(a \otimes]\lfloor) D(b \otimes c)$$

and

$$D(a \otimes bc) = D(a \otimes b)(1 \otimes c) + (-1)^{(X+a)b}(1 \otimes b) D(a \otimes c)$$

for all homogeneous a, b, $c \in S^\cdot(V)$. This is a simple verification and does not use that P is (anti-) symmetric. In our case, we conclude that D is uniquely determined by its values on the generators of $S^\cdot(V) \otimes S^\cdot(V)$. For $P = P_\Lambda$ and $X = X_\phi$ it is easy to check that $D = 0$ on generators and thus P_Λ and $(X_\phi \otimes \mathrm{id} + \mathrm{id} \otimes X_\phi)$ commute. But this implies the first as well as the third part. Now consider $\phi \in \mathrm{im}\#$, i.e. there is a $v \in V$ with $\phi = 2\Lambda_-(v,\cdot)$. In this case we get for $w \in V$

$$\{v, w\}_\Lambda = \Lambda(v, w)1 - (-1)^{vw}\Lambda(w, v)1 = 2\Lambda_-(v, w)1 = \phi(w)1 = X_\phi(w) .$$

Since the derivation X_ϕ is determined by its values on generators this implies $X_\phi = \{v,\cdot\}_\Lambda$. For the converse, assume that $X_{(\ell} = \{v,\cdot\}_\Lambda$ for some $v \in S^\cdot(V)$ which we write as $v = \sum_n v_\mathrm{n}$ with $v_n \in S^n(V)$. Then for $w \in V$ we have $\{v_\mathrm{n}, w\}_\Lambda \in S^{n-1}(V)$ while $X_\ell(w) \in S^0(V)$. Thus we necessarily have $X_\phi(w) = \{v_1, w\}_\Lambda$, i.e. the higher order terms in v are not necessary. But then $X_\phi = \{v_1, \}_\Lambda$ follows, proving $\phi = 2v_1^\#$. The fourth part is similar, since for $v \in V$ we have $v \star_{v\Lambda} a = va + v\mu 0 P_\Lambda(v \otimes a)$ without higher order terms. Thus $[v, a]_\star = v\{v, a\}_\Lambda$ and we can argue as in the second part. □

We can extend the results of Lemma (3.2.10) in the following way: suppose we have two vector spaces V and W with two bilinear forms Λ_V and Λ_W on them. Then an even linear map $A: V \to W$ is called a *Poisson map* if

$$\Lambda_W(A(v), A(v')) = \Lambda_V(v, v') \quad (41)$$

for all v, $v' \in V$. The induced map $A: S^\cdot(V) \to S^\cdot(W)$ is then easily shown to satisfy $P_{\Lambda_W}\mathrm{o}(A \otimes A) = (A \otimes A)\mathrm{o}P_{\Lambda_V}$, generalizing the computation in the proof of Lemma (3.2.10) slightly. From this we see that A is a homomorphism of Poisson algebras and star product algebras. Thus we arrive at the following simple functoriality statement:

Proposition (3.2.12) [101]: *The constnuction of $\{.,\}_\Lambda$ and $\star_{v\Lambda}$ is funcronal with respect to Poisson maps.*

We now discuss how we can change the star product by changing the *symmetr c* part Λ_+ of Λ as in (32). Symmetry means that $\Lambda_+(v, w) = (-1)^{vw}\Lambda_+(w, v)$ for homogeneous elements in V.

Let $g: V \times V \to \mathrm{k}$ be another symmetric and even bilinear form, which we can think ofas a $\mathbb{Z}_2$-graded version ofan inner product. We define now a second order Laplacian" associated to g as follows. For homogeneous vectors v_1 , , $v_n \in V$ we set

$$\Delta_g(v_1 \cdots v_n) = \sum_{i<j} \quad (-1)^{\mathrm{v_i}\,(\mathrm{v_1}+\cdots+\mathrm{v_{i-1}})}$$

$$(-1)^{\mathrm{v_j}\,(\mathrm{v_1}+\cdots+\mathrm{v_{i-1}}+\mathrm{v_{i+1}}+\cdots\mathrm{v_{j-1}})}\, \mathrm{g}(\mathrm{v_i}, \mathrm{v_j})\mathrm{v_1} \ldots .\wedge^{i} \ldots \wedge j \ldots v_n, \quad (42)$$

and extend this again by linearity to an operator

$$\Delta_g : \mathrm{S}^*(V) \to \mathrm{S}^{*-2}(V)\,. \quad (43)$$

Note that Δ_g has even parity since g vanishes on vectors of different parities. This is no longer a derivation but a second order differential operator. We have the following "Leibniz rule" for Δ_g:

Lemma (3.2.13) [101]: *The operator Δ_g sarisfies*

$$\Delta_g 0\mu = \mu \mathrm{o}\big(\Delta_g \otimes \mathrm{id} + P_g + \mathrm{id} \otimes \Delta_g\big)\,. \quad (44)$$

Lemma (3.2.14) [101]: *Let Λ, Λ', $g: V \times V \to \mathbb{K}$ be even bilinear maps and let g be symmetric. Then the operators $\Delta_g \otimes \mathrm{id}$, $\mathrm{id} \otimes \Delta_g$, P_Λ, and $P_{\Lambda'}$ commute pairwise.*

Proof. Again, onejust checks this on $v_1 \cdots v_n \otimes w_1 \cdots w_m$ for homogeneous vectors v_1 ,⋯, v_n, w_1 , ⋯, $w_m \in V$ which is a lengthy but straightforward computation.

We use these commutation relations now to prove the following equivalence statement: the isomorphism class of the deformation depends only on the *antisymmetric* part of Λ.

Proposition (3.2.15) [101]: *Let Λ, Λ' : $V \times V \to \mathbb{K}$ be two even bilinear forms on V such that their antisymmetric parts $\Lambda_- = \Lambda'_-$ coincide. Then the corresponding star products $\star_{v\Lambda}$ and $\star_{v\Lambda'}$ are equivalent via the equivalence transformation*

$$\mathrm{e}^{v\Delta_g}(a \star_{v\Lambda} b) = (\mathrm{e}^{v\Delta_g} a) \star_{v\Lambda'} (\mathrm{e}^{v\Delta_g} b) \quad (45)$$

for all $a, b \in \mathrm{S}^{\cdot}(V)[[v]]$ where $g = \Lambda' - \Lambda = \Lambda'_+ - \Lambda_+$.

Proof. The proof is now fairly easy. Analogously to [100] we have

$$\begin{aligned}
\mathrm{e}^{v\Delta_g}(a \star_{v\Lambda} b) &= \mathrm{e}^{v\Delta_g} \mathrm{o}\mu \mathrm{o} \mathrm{e}^{vP_\Lambda}(a \otimes b) \\
&= \mu \mathrm{o} \mathrm{e}^{v(\Delta_g \otimes \mathrm{id} + P_g + \mathrm{id} \otimes \Delta_g)} \mathrm{o} \mathrm{e}^{vP_\Lambda}(a \otimes b) \\
&= \mu \mathrm{o} \mathrm{e}^{v(P_\Lambda + P_g)} \mathrm{o}(\mathrm{e}^{v\Delta_g} \otimes \mathrm{e}^{v\Delta_g})(a \otimes b) \\
&= \mu \mathrm{o} \mathrm{e}^{vP_{\Lambda'}} \mathrm{o}(\mathrm{e}^{v\Delta_g} a \otimes \mathrm{e}^{v\Delta_g} b)\,,
\end{aligned}$$

since $P_\Lambda + P_g = P_{\Lambda+g}$ and since $\Lambda + g = \Lambda'$. Note that g is indeed symmetric. □

We establish a locally convex topology on $\mathrm{S}^{\cdot}(V)$ for which the formal star product, after substituting the formal parameter by a real or complex number z, will be continuous. Starting point is a locally convex topology on V, which we will assume to be Hausdorff, and a continuity assumption on Λ. From now on the field of scalars K is either $\mathbb{R}$ or $\mathbb{C}$.

For V be now a real or complex $\mathbb{Z}_2$-graded Hausdorff locally convex vector space. We require that the grading is *comparible* with the topological structure, i.e. the projections onto the even and odd parts in $V = V_0 \oplus V_1$ are continuous. Thus we have for every continuous seminorm p on V another continuous seminorm q with $\mathrm{p}(v_0)$, $\mathrm{p}(v_1) \leq \mathrm{q}(v)$ for all $v \in V$. This implies that the even and odd parts of V constitute complementary closed subspaces.

In principle, there are many interesting locally convex topologies on $S^{\cdot}(V)$ induced by the one on V. We shall construct now a rather particular one.

First we will endow the tensor products $V^{\otimes n}$ with the π-topology. Recall that for seminorms $p_1, , p_n$ on V one defines the seminorm $p_1 \otimes \cdots \otimes p_n$ on $V^{\otimes n}$ by

$$(p_1 \otimes \cdots \otimes p_n)(v) = \inf \left\{ \sum_i p_1\left(v_i^{(1)}\right) \cdots p_n\left(v_i^{(n)}\right) | v = \sum_i v_i^{(1)} \otimes \cdots \otimes v_i^{(n)} \right\}, \tag{46}$$

where the infimum is taken over all possibilities to write the tensor v as a linear combination of elementary (i.e. factorizing) tensors. One has $(p_1 \otimes \cdots \otimes p_n) \otimes (q_1 \otimes \cdots \otimes q_m) = p_1 \otimes \cdots \otimes p_n \otimes q_1 \otimes \cdots \otimes q_m$ and on factorizing tensors one gets $(p_1 \otimes \cdots \otimes p_n)(v_1 \otimes \cdots \otimes v_n) = p_1(v_1) \cdots p_n(v_n)$. We shall use the abbreviation $p^n = p \otimes \cdots \otimes p$ for n copies ofthe same seminorm p, where by definition p^0 is the usual absolute value on $\mathbb{K}$.

The π-topology on $V^{\otimes n}$ is obtained by taking all seminorms of the form $p_1 \otimes \cdots \otimes p_n$ with $p_1, , p_n$ being continuous seminorms on V. Equivalently, one can take all p^n with p being a continuous seminorm on V. Analogously, one defines the π-topology for the tensor products of different locally convex spaces. We denote the tensor product endowed with the π-topology by $\otimes_\pi$. It is clear that the induced $\mathbb{Z}_2$-grading of $V^{\otimes n}$ is again compatible with the π-topology. More explicitly, we have the following statement:

Lemma (3.2.16) [101]: *Let* p *be a continuous seminorm on* V *and choose a continuous seminorm* q *such that* $p(v_0)$, $p(v_1) \leq q(v)$. *Then for all* $v \in V^{\otimes n}$ *one has*

$$p^n(v_0), p^n(v_1) \leq q^n(v) . \tag{47}$$

For concrete estimates the following simple lemma is useful: it suffices to check estimates on factorizing tensors only:

Lemma (3.2.17) [101]: *Let* $V_1, , V_n, W$ *be vectorspaces and let* $\varphi: V_1 \times \cdots \times V_n \to W$ *be an* n*-linear map, identified with a linear map* $\varphi: V_1 \otimes \cdots \otimes V_n \to W$ *as usual. If* $p_1, , p_n$, q *are seminorms on* $V_1, , V_n, W$, *respecrively, such [hatfor all* $v_1 \in V_1, , v_n \in V_n$ *one has*

$$q\left(\varphi(v_1, , v_n)\right) \leq p_1(v_1) \cdots p_n(v_n) , \tag{48}$$

then one has for all $v \in V_1 \otimes \cdots \otimes V_n$

$$q(\varphi(v)) \leq (p_1 \otimes \cdots \otimes p_n)(v) . \tag{49}$$

Since we view the symmetric powers $S^n(V)$ as subspace of $V^{\otimes n}$ we can inherit the π-topology also for $S^n(V)$, indicated by $S_\pi^n(V)$. Then we get the following simple properties of the symmetric tensor product:

Lemma (3.2.18) [101]: *Let* $n, m \in \mathbb{N}_0$ *and let* p *be a continuous seminorm on* V.

(i) *The symmetr zer* $S_n: V^{\otimes_\pi 1} \to V^{\otimes_\pi n}$ *is continuous and for all* $v \in V^{\otimes_\pi n}$ *one has*

$$p^n(\mathcal{S}_n(v)) \leq p^n(v) . \tag{50}$$

(ii) $S_\pi^n(V) \subseteq V^{\otimes_\pi n}$ *is a closed subspace.*

(iii) *For* $v \in S^n(V)$ *and* $w \in S^m(V)$ *one has*

$$p^{n+m}(vw) \leq p^n(v) p^m(w) . \tag{51}$$

Proof. The first part is clear for factorizing tensors and hence Lemma (3.2.17) applies. The second follows as $S^n(V) = \ker(\mathrm{id} - \mathcal{S}_n)$ by definition. The third is clear from the definition and from (50).

On the tensor algebra $\mathrm{T}^{\cdot}(V)$ there are at least two canonical locally convex topologies: the Cartesian product topology inherited from $\prod_{n=0}^{\infty} \quad V^{\otimes_\pi n}$ and the direct sum topology which is the inductive limit topology of the finite direct sums. While the first is very coarse, the second is very fine. Both of them induce the π-topology on each subspace $V^{\otimes n}$. We are now searching for something in between.
We fix a parameter $R \in \mathbb{R}$ and consider for a given continuous seminorm p on V the new seminorm

$$\mathrm{p}_R(v) = \sum_{n=0}^{\infty} \mathrm{p}^n\,(v_n) n!^R \tag{52}$$

on the tensor algebra $\mathrm{T}^{\cdot}(V)$, where we write $v = \sum_{n=0}^{\infty} v_\mathrm{n}$ as the sum of its components with fixed tensor degree $v_n \in V^{\otimes \iota}$. Analogously, we define

$$\mathrm{p}_{R,\infty}(v) = \sup_{n \in \mathbb{N}_0} \quad \{\mathrm{p}^n(v_n) n!^R\}. \tag{53}$$

In principle, we have also ℓ^{p}-versions for all $\mathrm{p} \in [1, \infty)$, but the above two extreme cases will suffice for the following.
The seminorms control the growth of the contributions $\mathrm{p}^n(v_n)$ for $n \to \infty$ compared to a power of $n!$ which we can view as weights from a weighted counting measure. The choice of the factorials as weights will become clear later. We list some first elementary properties of these seminorms.
Proof. The parts (ii), (iii), and (iv) are clear. Also the first estimate in (i) is obvious. For the second, we note
Lemma (3.2.19) [101]: *Let* p *and* q *be seminorms on* V *and* $R, R' \in$ R.
(i) *One has for all* $v \in \mathrm{T}^{\cdot}(V)$

$$\mathrm{p}_{R,\infty}(v) \leq \mathrm{p}_R(v) \leq 2(2\mathrm{p})_{R,\infty}(v)\,. \tag{54}$$

(ii) *Both seminorms* p_R *and* $\mathrm{p}_{R,\infty} restr \cdot ct\ to\ n!^R$ p^n *on* $V^{\otimes n}$.
(iii) *If* $\mathrm{q} \leq \mathrm{p}$ *then* $\mathrm{q}_R \leq \mathrm{p}_R$.
(iv) *If* $R' > R$ *then* $\mathrm{p}_R(v) \leq \mathrm{p}_{R'}(v)$ *for all* $\in \mathrm{T}^*(V)$.

$$\mathrm{p}_R(v) = \sum_{n=0}^{\infty} \mathrm{p}^n\,(v_n)_{7l!^R} = \sum_{n=0}^{\infty} 2^n\,\mathrm{p}^n(v_n) \uparrow \iota!^R \frac{1}{2^{ll}} \leq \sup_{n \in N_0} \quad 2^n \mathrm{p}^n(v_n) n!^{R\prime} \sum_{n=0}^{\infty} \frac{1}{2^n}$$

which is the second estimate in (54).
The seemingly trivial first part will have an important consequence later when we discuss the nuclearity properties of the Weyl algebra.
We can use now all the seminorms p_R for a fixed R to define a new locally convex topology on the tensor algebra. In particular, the lemma shows that we can safely restrict to the seminorms of the type p_R as long as we take *all* continuous seminorms on V. It is clear from an analogous estimate that also the ℓ^{p}-versions would not yield anything new.
Definition (3.2.20) [101]: Let $R \in \mathbb{R}$. Then $\mathrm{T}_R^*(V)$ is the tensor algebra of V equipped with the locally convex topology determined by all the seminorms p_R with p running through all continuous seminorms on V.
In the following, we will mainly be interested in the case of positive R where we have a *decay* of the numbers $\mathrm{p}^n(v_n)$.
(i) *The tensor product is continuous on* $\mathrm{T}_R^*(V)$. *More precisely, one has*

Lemma (3.2.21) [101]: (i) *Let* $R' > R \geq 0$.

$$\mathrm{p}_R(v \otimes w) \leq (2^R\mathrm{p})_R(v)(2^R\mathrm{p})_R(w) \tag{55}$$

for all $v, w \in \mathrm{T}_R^*(V)$.

(ii) *For all* $n \in \mathbb{N}_0$ *the projections and the inclusions*

$$\mathrm{T}_R^*(V) \to V^{\otimes_\pi n} \to \mathrm{T}_R^*(V) \tag{56}$$

are continuous.

(iii) *The completion* $\widehat{\mathrm{T}}_R^*(V)$ *of* $\mathrm{T}_R^*(V)$ *can explicitly be descnbed by*

$$\widehat{\mathrm{T}}_R^*(V) = \{v = \sum_{n=0}^{\infty} v_n \,|\mathrm{p}_R(v) < \infty \, for\ all\ p\} \subseteq \coprod_{n=0}^{\infty} V^{\otimes_\pi n} \wedge, \tag{57}$$

where p *runs through all continuous seminorms of* V *and we extend* p_R *to the Cartesian product by allowing the value* $+\infty$ *as usual*.

(iv) *We have a continuous inclusion*

$$\widehat{\mathrm{T}^*}_{R'}(V) \to \widehat{\mathrm{T}}^*{}_R(V)\,. \tag{58}$$

(v) *The* $\mathbb{Z}_2$*-grading of* $\mathrm{T}_R^*(V)$ *is conrinuous. More explicirly, if* p *and* q *are conrinuous seminorms with* $\mathrm{p}(v_0)$, $\mathrm{p}(v_1) \leq \mathrm{q}(v)$ *for all* $v \in V$ *then we have*

$$\mathrm{p}_R(v_0), \mathrm{p}_R(v_1) \leq \mathrm{q}_R(v) \tag{59}$$

for all $v \in \mathrm{T}_R^*(V)$.

Proof. The first part is a simple estimate; we have

$$\begin{aligned}
\mathrm{p}_R(v \otimes w) &= \sum_{k=0}^{\infty} \mathrm{p}^k\left(\sum_{n+m=k} v_n \otimes w_m\right) k!^R \\
&\leq \sum_{k=0}^{\infty} \sum_{\mathrm{n}+m=k} \mathrm{p}^n\,(v_n)\mathrm{p}^m(w_m)(n+m)!^R \\
&\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathrm{p}^{\mathrm{n}}\,(v_n)\mathrm{p}^m(w_m)2^{\mathbb{R}\mathrm{n}}2^{\mathbb{R}m}n!^{\mathbb{R}}\,\mathrm{m}!^{\mathbb{R}} \\
&= (2^{\mathbb{R}}\mathrm{p})_{\mathbb{R}}(v)\;(2^{\mathbb{R}}\mathrm{p})_{\mathbb{R}}(w)\,,
\end{aligned}$$

where we used $(n+m)\,! \leq 2^{n+m} n!\, m!$. Since with p also 2^Rp is a continuous seminorm on V, the continuity of$\otimes$follows. The second and third parts are standard, here we use the completed π-tensor product $V^{\otimes_\pi n} \wedge$ to achieve completeness at every fixed $n \in \mathbb{N}_0$. The fourth part is a consequence of Lemma (3.2.19), (iv). The last part follows from Lemma (3.2.16).

Remark (3.2.22) [101]: The case $R = 0$ gives a well-known topology on $\mathrm{T}^{\cdot}(V)$ which becomes the *free locally multiplicatively convex* unital algebra generated by V as discussed e.g. by Cuntz in [116]. For $R > 0$ the completion $\widehat{\mathrm{T}^*}_R(V)$ behaves differently: it does not even have an entire holomorphic calculus. To see this take the entire function

$$f_\varepsilon(z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!^\varepsilon} \tag{60}$$

for a parameter $\varepsilon > 0$. If $R > \varepsilon$ then for every nonzero $v \in V$ the series $f_\varepsilon(v)$ does not converge in $\widehat{\mathrm{T}}^*{}_R(V)$. In particular, the tensor algebra $\mathrm{T}_R^*(V)$ can *not* be locally multiplicatively convex unless $R = 0$.
We have an analogous statement for the symmetric algebra. We equip $S^\cdot(V)$ with the induced topology from $\mathrm{T}_R^*(V)$ and denote it by $\mathrm{S}_R^*(V)$. From (50) we get immediately

$$p_R(\mathcal{S}(v)) \leq \mathrm{p}_R(v)\,, \tag{61}$$

which implies the continuity statement

$$\mathrm{p}_R(vw) \leq (2^R\mathrm{p})_R(v)(2^R\mathrm{p})_R(w) \tag{62}$$

for all $v, w \in \mathrm{S}^\cdot(V)$. This shows that $\mathrm{S}_R^*(V)$ becomes a locally convex algebra, too. Again, it will be locally multiplicatively convex only for $R = 0$ in which case it is the *free locally multiplicatively convex commutative* unital algebra generated by V. The completion of $\mathrm{S}_R^*(V)$ can be described analogously to (57). We also have the continuous inclusions

$$\hat{\mathrm{S}}^*{}_{R'}(V) \to \hat{\mathrm{S}}^*{}_R(V) \tag{63}$$

for $R' > R$. Finally, we have the continuous projections and inclusions

$$\mathrm{S}_R^*\,(V) \to \mathrm{S}_\pi^n(V) \to \mathrm{S}_R^*(V) \tag{64}$$

for all $n \in \mathbb{N}_0$. Thus it makes sense to speak of the n-th component v_n of a vector $v \in \widehat{\mathrm{S}^*}_R(V)$ even after the completion. In fact, it is easy to see that the series of components

$$v = \sum_{n=0}^{\infty} v_n \tag{65}$$

converges to $v \in \widehat{S^*}_R(V)$, even absolutely.
To get rid of the somehow arbitrary parameter R we can pass to the projective limit $R \to \infty$. The resulting locally convex algebras will be denoted by

$$\widehat{\mathrm{T}^*}_\infty(V) = proj\ \lim_{R\to\infty} \widehat{\mathrm{T}^*}_R\ (V)\ and\ \hat{\mathrm{S}}_\infty^*(V) = proj\ \lim_{R\to\infty} \hat{\mathrm{S}}_R{}^*\,(V) \tag{66}$$

in the symmetric case. We have a more explicit description of $\hat{\mathrm{T}}_\infty^*\,(V)$ and $\hat{\mathrm{S}}_\infty^*(V)$ as consisting of those formal series $v = \sum_{n=0}^{\infty} v_n$ with $v_n \in V^{\otimes_\pi^* n} \wedge$ or $v_n \in \hat{\mathrm{S}}_\pi^n(V)$, respectively, such that $\mathrm{p}_R(v) < \infty$ for *all* $R \geq 0$ and for all continuous seminorms p on V. Note that these completions will be rather small as we require a rather strong decay of the coefficients v_n.

We consider an even bilinear form $\Lambda: V \times V \to \mathbb{K}$ which we require to be continuous. Thus there exists a continuous seminorm p on V such that

$$|\Lambda(v,w)| \leq \mathrm{p}(v)\mathrm{p}(w) \tag{67}$$

for all $v, w \in V$. Note that we require continuity and not just separate continuity. Note also, that if p satisfies (67) then we also have the estimates

$$|\Lambda_\pm(v,w)| \leq \mathrm{p}(v)\mathrm{p}(w)\,, \tag{68}$$

showing the continuity of the antisymmetric and symmetric parts of Λ. The continuity of Λ implies the continuity of the operator P_Λ when restricted to fixed symmetric degrees:
Lemma (3.2.23) [101]: *Let* p *be a continuous seminorm of* V satisfying (67) . *Then for all* $u \in \mathrm{S}^n(V) \otimes \mathrm{S}^m(V)$ *one has*

$$(\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})(P_\Lambda(u)) \leq nm\mathrm{p}^{n+m}(u)\,. \tag{69}$$

The same estimate holds for $P_{\Lambda\pm}$.

Proof. We work on the whole tensor algebra first. Thus consider homogeneous vectors v_1 , , v_n and w_1 , , $w_m \in V$ and define $\tilde{P}_\Lambda: \mathrm{T}^{\cdot}(V) \otimes \mathrm{T}^{\cdot}(V) \to \mathrm{T}^{\cdot}(V) \otimes \mathrm{T}^{\cdot}(V)$ by the linear extension of

$$\tilde{P}_\Lambda(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m)$$

$$= \sum_{k=1}^{n}\sum_{\ell=1}^{m} (-1)^{\mathrm{v_k}\,(\mathrm{v_k}+1+\cdots+\mathrm{v_n})\,(-1)\,\mathrm{w}\ell\,(\mathrm{w_1}+\cdots+\mathrm{w}_{\ell-1})} \Lambda(v_k, w_\ell) v_1 \otimes \cdots \wedge \ldots \otimes v_n \otimes w_1 \otimes \cdots \wedge \ldots \otimes w_m.$$

Then we have $P_\Lambda \mathrm{o}(\mathcal{S}_n \otimes \mathcal{S}_m) = (\mathcal{S}_{n-1} \otimes \mathcal{S}_{m-1})^{\circ}\tilde{P}_\Lambda$. For general tensors of arbitrary degree this yields

$$P_\Lambda \mathrm{o}\,(\mathcal{S} \otimes \mathcal{S}) = (\mathcal{S} \otimes \mathcal{S})\,{}^{\circ}\tilde{P}_\Lambda. \qquad (*)$$

For homogeneous vectors we get now the estimate

$$(\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\left(\tilde{P}_\Lambda(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m)\right)$$
$$\leq \sum_{k=1}^{n}\sum_{\ell=1}^{m} |\Lambda(v_k, w_\ell)| \mathrm{p}(v_1) \wedge \cdots \mathrm{p}(v_n)\mathrm{p}(w_1)k \ldots \wedge \cdots \mathrm{p}(w_m)\ell$$
$$\leq nm\mathrm{p}(v_1)\cdots\mathrm{p}(v_\mathrm{n})\mathrm{p}(w_1)\cdots\mathrm{p}(w_m)$$
$$= nm(\mathrm{p}^n \otimes \mathrm{p}^m)(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m)\,.$$

By Lemma (3.2.17) we conclude that for all $u \in \mathrm{T}^n(V) \otimes \mathrm{T}^m(V)$ we have

$$(\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\left(\tilde{P}_\Lambda(u)\right) \leq nm(\mathrm{p}^n \otimes \mathrm{p}^m)(u)\,.$$

Finally, for $u \in S^n(V) \otimes S^m(V)$ we have $(\mathcal{S}\mathrm{s}_n \otimes \mathcal{S}_m)(u) = u$ and thus by $(*)$ and (50)

$$(\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\big(P_\Lambda(u)\big) = (\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\big(P_\Lambda(@_n \otimes \mathcal{S}_m)(u)\big)$$
$$= (\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\left((\mathcal{S}_{n-1} \otimes \mathcal{S}_{m-1})\tilde{P}_\Lambda(u)\right)$$
$$\leq (\mathrm{p}^{n-1} \otimes \mathrm{p}^{m-1})\left(\tilde{P}_\Lambda(u)\right)$$
$$\leq nm(\mathrm{p}^n \otimes \mathrm{p}^m)(u)\,.$$

The last statement follows from (68).

In fact, this estimate just reflects the fact that P_Λ is a biderivation. If Λ is nontrivial then it cannot be improved in general. It implies immediately the continuity of the Poisson bracket $\{.\,,\}_\Lambda$:

Proposition (3.2.24) [101]: *Let Λ be continuous. Then the Poisson bracket $\{.,\}_\Lambda$ is continuous on $\mathrm{S}_R^*(V)$ for every $R \geq 0$. More precisely, for $v, w \in \mathrm{S}_R^*(V)$ and any continuous seminorm* p *on V with* (67) *we have a constant $c > 0$ such that*

$$\mathrm{p}_R(\{v, w\}_\Lambda) \leq (2^{R+1}\mathrm{p})_R\,(v)(2^{R+1}\mathrm{p})_R(w)\,. \qquad (70)$$

Proof. Let v, $w \in \mathrm{S}_R^*\,(V)$ with components $v = \sum_n v_\mathrm{n}$ and $w = \sum_m w_m$ as usual. Then we have for a seminorm p satisfying (67)

$$\mathrm{p}_R(\{v, w\}_\Lambda) = \sum_{k=0}^{\infty} \mathrm{p}^k\left(\sum_{n+m-2=k} \{v_n, w_m\}_\Lambda\right) k!^R$$

$$\leq \sum_{k=0}^{\infty}\sum_{n+m-2=k} \mathrm{p}^{n+m-2}\,\big(2\mu\mathrm{o}p_{\Lambda-}(v_n \otimes w_m)\big)k!^R$$

$$\leq \sum_{k=0}^{\infty} \sum_{n+m-2=k} 2\, nm \mathrm{p}^n(v_n)\mathrm{p}^m(w_m)(n+m-2)!^R$$

$$\leq \sum_{n,m=0}^{\infty} 2\, nm 2^{(\mathrm{n}+m)R} \mathrm{p}^n(v_n)\mathrm{p}^m(w_m) n!^R\, m!^R$$

$$\leq c \sum_{n,m=0}^{\infty} 2^{(n+m)(R+1)}\, \mathrm{p}^n(v_n)\mathrm{p}^m(w_m) n!^R\, m!^R,$$

where we have used Lemma (3.2.23) for Λ_- and the standard estimate $(ri+m)! \leq 2^{n+m} n!\, m!$.

In this sense, $\mathrm{S}_R^*(V)$ becomes a *locally convex Poisson algebra* for every $R \geq 0$. In particular, the Poisson bracket extends to the completion $\hat{\mathrm{S}}_R^*(V)$ and still obeys the continuity estimate (70) as well as the algebraic properties of a Poisson bracket. However, for the star product the situation is more complicated: first we note that thanks to Corollary (3.2.8) we can use the formal star product to get a well-defined *non-formal* star product $\star_{z\Lambda}$ by replacing v with some real or complex number $z \in \mathbb{K}$, depending on our choice of the underlying field. We fix z in the following and consider the dependence of $\star_{z\Lambda}$ on z later in The next lemma provides the key estimate for all continuity properties of $\star_{z\Lambda}$: the main point is that we have to limit the range of the possible values of R:

Lemma (3.2.25) [101]: *Let $R \geq \frac{1}{2}$. Then there exist constants $c, c' > 0$ such rhat for all a, $b \in S^{\cdot}(V)$ and all seminorms* p *with* (67) *we have*

$$\mathrm{p}_R(a \star_{z\Lambda} b) \leq c'(c\mathrm{p})_R(a)(c\mathrm{p})_R(b)\,. \tag{71}$$

Proof. Let $a, b \in S^{\cdot}(V)$ be given and denote by a_n, b_m their homogeneous parts with respect to the tensor degree as usual. We have to distinguish several cases ofthe parameters. The non-trivial case is for $\frac{1}{2} \leq R \leq 1$ and $|z| \geq 1$, where we estimate

$$\mathrm{p}_R(a \star_{z\Lambda} b) \leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \mathrm{p}_R\left(\mu \mathrm{o} P_\Lambda^k(a \otimes b)\right)$$

$$\leq \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \sum_{k \leq n,m} (n+m-2k)\,!^R\, \mathrm{p}^{n+m-2k}\left(\mu \mathrm{o} P_\Lambda^k(a_n \otimes b_m)\right)$$

$$\overset{(a)}{\leq} \sum_{k=0}^{\infty} \frac{|z|^k}{k!} \sum_{k \leq n,m} (n+m-2k)\,!^R\, \frac{n!}{(n-k)!} \frac{m!}{(m-k)!} \mathrm{p}^n(a_n)\mathrm{p}^m(b_m)$$

$$\overset{(b)}{\leq} \sum_{k=0}^{\infty} \sum_{k \leq n,m} \frac{|z|^k 2^{R(n+m-2k)}}{k!} \frac{n!^{1-R}}{(n-k)!^{1-R}} \frac{m!^{1-R}}{(m-k)!^{1-R}} n!^R\, \mathrm{p}^n(a_n) m!^R\, \mathrm{p}^m(b_m)$$

$$\overset{(c)}{\leq} \sum_{k=0}^{\infty} \sum_{k \leq n,m} \frac{|z|^k 2^{R(n+m-2k)}}{k!} 2^{(1-R)n} 2^{(1-R)m} k!^{2(1-R)}\, n!^R\, \mathrm{p}^n(a_n) m!^R\, \mathrm{p}^m(b_m)$$

$$\overset{(d)}{\leq} \sum_{k=0}^{\infty} \sum_{k\leq n,m} \frac{1}{|z|^k k!^{2R-1}\, 2^{2Rk}} n!^R\, (2|z|)^n \mathrm{p}^n(a_n) m!^R\, (2|z|)^m \mathrm{p}^m(b_m)$$

$$\overset{(e)}{\leq} \left(\sum_{k=0}^{\infty} \frac{1}{|z|^k k!^{2R-1}\, 2^{2Rk}}\right)\left(\sum_{\mathrm{n}=0}^{\infty} n\,!^R\, (2|z|)^n \mathrm{p}^{l1}(a_n)\right)\left(\sum_{m=0}^{\infty} m\,!^R\, (2|z|)^m \mathrm{p}^m(b_m)\right)$$

$$= c'(2|z|\mathrm{p})_R(a)(2|z|\mathrm{p})_R(b)$$

where in (a) we used k - times the estimate from Lemma (3.2.25), in (b) we used $(n+m-2k)\ ! \leq 2^{n+m-2k}(n-k)!\,(m-k)\ !$, in (c) we used $n! \leq 2^n(n-k)!\,k!$ and $m! \leq 2^m(m-k)!\,k!$ together with the assumption $R \leq 1$, in (d) we use $|z| \geq 1$ as well as $k \leq n, m,$

and finally, in (e) we note that the series over k converges to a constant thanks to $R \geq \frac{1}{2}$ while the remaining series over n and m give the seminorm $(2|z|p)_R$ for the rescaled seminorm $2|z|p$. If instead $|z| < 1$ then we continue instead of (d) by

$$\mathrm{p}_R(a \star_{z\Lambda} b) \leq \cdots . \overset{(d')}{\leq} \left(\sum_{k=0}^{\infty} \frac{|z|^k 2^{-2Rk}}{k!^{2R-1}}\right)(2\mathrm{p})_R(a)(2\mathrm{p})_R(b)\,,$$

where the series over k always converges since $R \geq \frac{1}{2}$ and $|z| < 1$. Finally, if $R > 1$ then we continue instead of (c) with

$$\mathrm{p}_R(a \star_{Z\Lambda} b) \leq \cdots . \overset{(c')}{\leq} \sum_{k=0}^{\infty} \sum_{k\leq n,m} \frac{|z|^k 2^{-2Rk}}{k!} n!^R\, 2^{Rn} \mathrm{p}^n(a_n) m!^R\, 2^{Rm} \mathrm{p}^m(b_m)$$

$$\leq \left(\sum_{k=0}^{\infty} \frac{|z|^k 2^{-2Rk}}{k!}\right)(2^R\mathrm{p})_R(a)(2^R\mathrm{p})_R(b)\,,$$

where again the series over k converges. In total, we always get an estimate for all $R \geq \frac{1}{2}$ and all z as claimed.

Remark (3.2.26) [101]: We also note that the limiting case $R = \frac{1}{2}$ is sharp in the following sense: consider the most simple nontrivial situation $V = \mathbb{R}^2$ with basis vectors q and p as well as the bilinear form $\Lambda_{\mathrm{std}}(p,q) = 1$ and zero for the other combinations. The corresponding Poisson bracket is the canonical Poisson bracket and the star product is the standard-ordered star product if we take $z = \frac{h}{i}$, see e.g. [100]. Identifying elements in $S^*(\mathbb{R}^2)$ with polynomials in q and p we have the more explicit formula

$$f \star_{\mathrm{std}} g = \sum_{k=0}^{\infty} \frac{(-\mathrm{i}\hbar)^k}{k!} \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial q^k}. \qquad (72)$$

Using again the function f_ε from (60) we see that $f_\varepsilon(q)$ and $f_\varepsilon(p)$ belong to $\hat{\mathrm{S}}_R^*(\mathbb{R}^2)$ as soon as $R < \varepsilon$. However, for the star product we get (formally)

$$f_\varepsilon(p) \star_{\mathrm{std}} f_\varepsilon(q) = \sum_{n,m,k\leq m,n} \frac{(-\mathrm{i}\hbar)^k}{k!} \frac{n!^{1-\varepsilon}}{(n-k)!} \frac{n!^{1-\varepsilon}}{(n-k)!} q^{n-k} p^{n-k}. \qquad (73)$$

Since the projection $\hat{S}_R^*(V) \to S_\pi^n(V)$ is continuous for all $R \geq 0$ we consider the coefficient of (73) in $S^0(V)$ which is obtained for $n = m = k$, i.e.

This clearly diverges for $\varepsilon < \frac{1}{2}$ unless $\hbar = 0$. Thus for $R < \frac{1}{2}$ we cannot expect a continuous star product.

$$\sum_{\ell=0}^{\infty} \frac{(-\mathrm{i}\hbar)^\ell}{\ell!} \ell!^{1-\varepsilon} \ell!^{1-\varepsilon} = \sum_{\ell=0}^{\infty} (-\mathrm{i}\hbar)^\ell \ell!^{1-2\varepsilon} \tag{74}$$

The estimate from Lemma (3.2.25) shows that the star product will be continuous for the topology of $S_R^*(V)$ *provided* the parameter $\mathbb{R}$ satisfies $R \geq \frac{1}{2t}$.This will motivate the following definition of the Weyl algebra.. However, we will also give an alternative definition for later use, where we want to compare with the results from [73], [109].

Definition (3.2.27) [101]: For $R \in \mathbb{R}$ we endow $S_R^*(V)$ with the product $\star_{z\Lambda}$ and call the resulting algebra the Weyl algebra $W_R(V, \star_{z\Lambda})$. Moreover, for $R > \frac{1}{2}$ we set

$$W_{R-}(V) = proj \lim_{\varepsilon \to 0} S_{R-\varepsilon}^*(V) \tag{75}$$

and endow $W_{R-}(V)$ with the Weyl product $\star_{z\Lambda}$, too.

This way, we arrive at two possible definitions of the Weyl algebra. The projective limit can be made more explicitly, since the underlying vector space is always the same: we use *all* seminorms $p_{R-\varepsilon}$ for $\varepsilon > 0$ and p a continuous seminorm on V for $W_{R-}(V)$ and have $W_{R-}(V) = S^{\cdot}(V)$ as a linear space as before. It will turn out that this projective limit enjoys some more interesting properties when it comes to strong nuclearity.

The completion $\widehat{W}_R(V)$ will be given as those formal series $v = \sum_{n=0}^{\infty} v_n$ with $v_n \in \hat{S}_\pi^n(V)$ such that *all* seminorms $p_R(v)$ are finite for all continuous seminorms p on V. Correspondingly, for the completion of the projective limit we have to have finite seminorms $p_{R-\varepsilon}(v)$ for all $\varepsilon > 0$ and all continuous seminorms p on V.

A last option is to take the projective limit $R \to \infty$. Most of the following statements will therefore also be available for the case $R = \infty$. However, we will not be too much interested in this case as the completion $\widehat{W}_{\infty}(V) = \hat{S}_\infty^*(V)$ is rather small.

We start now to collect some basic features of $W_{R-}(V)$. From Lemma (3.2.19) we get immediately the following statement:

Lemma (3.2.28) [101]: *Let* $R' \geq R \geq 0$.

(i) *For all* $n \in \mathbb{N}_0$ *the induced topology on* $S^n(V) \subseteq W_{R-}(V)$ *is the* π*-topology.*

(ii) *The projection and the inclusion maps*

$$W_{R-}(V) \to S_\pi^n(V) \to W_{R-}(V) \tag{76}$$

are continuous for all $n \in \mathbb{N}_0$.

(iii) *The inclusion map*$W_{R-}(V) \to W_{R-}(V)$ *is continuous.*

(iv) *The* $\mathbb{Z}_2$*-grading is continuous for* $W_{R-}(V)$.

The analogous statements for $W_R(V)$ hold for trivial reasons: we have discussed them already for the symmetric algebra $S_R^*(V)$.

This lemma has the important consequence that also after completion of $W_{R-}(V)$ to $\hat{w}_{R-}(V)$ we can speak of the n-th component $a_n \in \hat{S}_\pi^n(V)$ of an element $a \in \widehat{W}_{R-}(V)$ in a meaningful way. a can be expressed as a convergent series in its components of fixed tensor degree:

Lemma (3.2.29) [101]: *Let $R \in \mathbb{R}$ and let $a \in \widehat{\mathrm{W}}_{R-}(V)$ with components $a_n \in \hat{\mathrm{S}}^{\mathrm{n}}_{\pi}(V)$ for $n \in \mathrm{N}_0$. Then*

$$a = \sum_{\mathrm{n}=0}^{\infty} a_{\mathrm{n}} \tag{77}$$

converges absolutely.

Proof. Identifying $a_{\mathrm{n}} \in \hat{\mathrm{S}}^{l1}_{\pi}(V)$ with its image in $\widehat{W}_{R-}(V)$ we get for every continuous seminorm p on V the equation

$$\mathrm{p}_{R-\varepsilon}(a) = \sum_{n=0}^{\infty} \mathrm{p}^{n}\,(a_n) n!^{R-\varepsilon} = \sum_{n=0}^{\infty} \mathrm{p}_{R-\varepsilon}\,(a_n)\,,$$

from which the statement follows immediately.

In particular, the direct sum $\bigoplus_{n=0}^{\infty} \hat{\mathrm{S}}^{n}_{\pi}(V)$ of the completed symmetric π-tensor powers of V is *sequentially* dense in $\widehat{\mathrm{W}}_{R-}(V)$. Again, this statement is also true for the case $\widehat{W}_{R}(V)$, see (65).

The first main result is now that $\mathrm{W}_{R}(V,\star_{z\Lambda})$ as well as $W_{R-}(V,\star_{z\Lambda})$ are indeed locally convex algebras provided R is suitably chosen:

Theorem (3.2.30) [101]: *Let $R \geq \frac{1}{2}$. The Weyl algebra $\mathrm{W}_{R}(V,\star_{z\Lambda})$ is a locally convex algebra. Moreover, $\mathrm{W}_{R-}(V,\star_{z\Lambda})$ is a locally convex algebra for $R > \frac{1}{2}$. Jn both cases, the Weyl algebra is first countable iff V is first countable.*

Proof. The continuity of the product for $\mathrm{W}_{R}(V)$ is just Lemma (3.2.29).This gives also the continuity in the case of $\mathrm{W}_{R-.}(V)$. Note that we need $R > \frac{1}{2,)}$ for the second case A locally convex space V is the fist countable iff we can choose sequence of continuous semi norms $\mathrm{p}^{(1)} \leq \mathrm{p}^{(2)} \leq \cdots$ such that for every other continuous seminorm q on$V$ we have some $n \in \mathbb{N}$ with $\mathrm{q} \leq \mathrm{p}^{(n)}$. Then Lemma (3.2.19) shows that the seminorms $\mathrm{p}^{(\mathrm{n})}_{R}$ will do the job for $W_{R}(V)$ while for $\mathrm{W}_{R-}(V)$ we can take the seminorms $\left(\mathrm{p}^{(n)}\right)_{R-\frac{1}{n}}$ to determine the topology. The converse is obvious from Lemma (3.2.28)(i).

As a first application we show that in the completion $\widehat{\mathrm{W}}_{R}(V)$ we have exponentials of every vector in V, provided R is small enough:

Proposition (3.2.31) [101]: *Assume $V_0 \neq \{0\}$.*

(i) *One has* $\exp\,(v) \in \widehat{\mathrm{W}}_{R}(V)$ *for every non-zero $v \in V$ iff $R < 1$.*

(ii) *Let $R < 1$ and $v \in V$. The map $\mathbb{K} \ni r \mapsto$* $\exp\,([v) \in \widehat{\mathrm{W}}_{R}(V)$ *is real-analytic in the case $\mathbb{K} = \mathbb{R}$ with radius of convergence ∞ and entire in the case $\mathbb{K} = \mathbb{C}$. The Taylor sense converges absolutely.*

Proof. Let $v \in V_0$ be non-zero and choose a seminorm p with $\mathrm{p}(v) > 1$ which is possible thanks to the Hausdorff property and by an appropriate rescaling of v. Then $\mathrm{p}^{n}(v^{n}) = \left(\mathrm{p}(v)\right)^{n}$ since $v \otimes \cdots \otimes v = v \cdots v$ in this case. The exponential series therefore gives

$$\mathrm{p}_R\big(\exp(v)\big) = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n(v^n)}{n!} n!^R = \sum_{n=0}^{\infty} \big(\mathrm{p}(v)\big)^n n!^{R-1},$$

which converges iff $R < 1$, showing the first part. The second part is clear from Lemma (3.2.29) since the homogeneous components of $\exp(tv)$ are given by $\frac{t^n v^v}{n!}$. □

We have established the continuity of $\star_{z\Lambda}$ on $\mathrm{W}_R(V)$ and thus we can conclude that $\star_{z\Lambda}$ has a unique extension to a continuous product $\star_{z\Lambda}$ on the completion $\widehat{\mathrm{W}}_R(V)$. We shall now re-interpret the proof of Lemma (3.2.25) to get the more specific statement that also the formula for $\star_{z\Lambda}$ stays valid:

Proposition (3.2.32) [101]: *Let $R \geq \frac{1}{2}$ and let $a, b \in \widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$. Then*

$$a \star_{z\Lambda} b = \mu \mathrm{o} \mathrm{e}^{zP_\Lambda}(a \otimes b) = \sum_{k=0}^{\infty} \frac{Z^k}{k!} \mu \mathrm{o} P_\Lambda^k (a \otimes b) \tag{78}$$

converges absolutely in $\widehat{\mathrm{W}}_R(V)$. The same statement holds for the projective limit version $\widehat{\mathrm{W}}_{R-}(V,\star_{z\Lambda})$ and $R > \frac{1}{2}$.

Proof. We have to show that for all seminorms p_R of the defining system of se minorms the series$\sum_{k=0}^{\infty} \frac{|z|^k}{a^k i^1} p_R\left(\mu o P_\Lambda^k (a \otimes b)\right)$ converges. But this was exactly what we did in the proof of Lemma (3.2.25). The projective limit case is analogous.

This proposition also allows us to discuss the dependence on the deformation parameter z: here we have the best possible scenario. In the real case, $z \in \mathbb{R}$, we have a real-analytic dependence on z in $a \star_{z\Lambda} b$ with an explicit Taylor expansion around $z = 0$ given by the absolutely convergent series (78). In the complex case, $z \in \mathbb{C}$, we have an entire dependence, again by (78). Note that it is important for such statements that the topology of the Weyl algebra is actually independent of z. Holomorphic deformations were introduced and studied in detail in [107], mainly in the context of Hopfalgebra deformations.

Proposition (3.2.33) [101]: *Lef$R \geq \frac{1}{2}$.*

(i) *If* $\mathrm{K} = \mathbb{R}$ *then for every $a, b \in \widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ the map*

$$\mathbb{R} \ni z \mapsto a \star_{z\Lambda} b \in \widehat{W}_R(V,\star_{z\Lambda}) \tag{79}$$

is real-analytic with Taylor expansion around $z = 0$ given by (78).

(ii) *If* $\mathrm{K} =$ (D *then for every $a, b \in \widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ the map*

$$\mathbb{C} \ni z \mapsto a \star_{z\Lambda} b \in \widehat{W}_R(V,\star_{z\Lambda}) \tag{80}$$

is holomorphic (even entire) with Taylor expansion around $z = 0$ given by (78). *The collection of Weyl algebras $\{\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})\}_{z\in c}$ provides a holomorphic (even entire) deformation of $\widehat{W}_R(V,\mu)$, where $\mu = \star_{z\Lambda}|_{z=0}$ is the symmemc tensor product.*

We treated the real and complex cases on equal footing. However, for applications in physics one typically needs an additional structure, both for the classical Poisson algebra as well as for the quantum algebra: a reality structure in the form of a*-involution.

The following two structures are well-known to be equivalent. We recall their relation in order to establish some notation: either we can start with a real vector space $V_\mathbb{R}$ and complexify it to $V_\mathbb{C} = V_\mathbb{R} \otimes \mathbb{C}$. This gives us an antilinear

involutive automorphism of $V_{\mathbb{C}}$, the complex conjugation, denoted by $v_{\mathbb{R}} \otimes z \mapsto \overline{v_{\mathbb{R}} \otimes z} = v_{\mathbb{R}} \otimes \overline{z}$ for $v_{\mathbb{R}} \in V_{\mathbb{R}}$ and $z \in \mathbb{C}$. We can recover $V_{\mathbb{R}}$ as the real subspace of $V_{\mathbb{C}}$ of those vectors $v \in V_{\mathbb{C}}$ with $\overline{v} = v$. Or, equivalently, we can start with a complex vector space $V_{\mathbb{C}}$ and an antilinear involutive automorphism, still denoted by $v \mapsto \overline{v}$. Then $V_{\mathbb{C}} \cong V_{\mathbb{R}} \otimes \mathbb{C}$ with $V_{1\mathrm{R}}$ consisting again of the real vectors in $V_{(\mathrm{D}}$. In this situation the symmetric algebra $\mathrm{S}^{\cdot}(V_{\mathbb{C}})$ is a*-algebra with respect to the complex conjugation, i.e. we have for homogeneous $a, b \in \mathrm{S}^{\cdot}(V_{\mathbb{C}})$
where in the second equation we use the commutativity of the symmetric tensor product.

$$\overline{ab} = (-1)^{ab}\overline{b}\overline{a} = \overline{a}\overline{b}, \tag{81}$$

If in addition $V_{\mathbb{R}}$ is locally convex we can extend a continuous seminorm p_{lR} on $V_{\mathbb{R}}$ to a seminorm $\mathrm{P}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by setting $\mathrm{p}_{\mathbb{C}}(v \otimes z) = |z|\mathrm{p}_{\mathbb{R}}(v)$. This makes $V_{\mathbb{C}}$ is a locally convex space such that the complex conjugation is continuous. In fact, $\mathrm{p}_{\mathbb{C}}(v) = \mathrm{p}_{\mathbb{C}}(\overline{v}).(v) = \frac{1}{2}\left(\mathrm{q}(v) + \mathrm{q}(\overline{v})\right)$for the seminorms of the form $p_{\mathbb{C}}$. Conversely, if $V_{\mathbb{C}}$ is a locally convex complex vector space with a continuous complex conjugation then for every continuous seminorm q also $\mathrm{p(v)} = \frac{1}{2}\ (q(v)\ +\ q(\overline{v}))$ is continuous, now satisfying is continuous,now satisfyin $p(\mathrm{v}) = \mathrm{P}(\overline{v})$ Clearly, these seminorms still determine the topology of $V_{\mathbb{C}}$. Finally, for $p_{\mathbb{R}} = p|_{V_{\mathbb{R}}}$ we get $(p_{\mathbb{R}})_{\mathbb{C}} = p$. Thus also in the locally convex situation the two structures are equivalent.

Now let $V_{\mathbb{R}}$ be a real locally convex vector space and set $V = V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$, always endowed with the above locally convex topology and the canonical complex conjugation.

Lemma (3.2.34) [101]: *Let $R \in \mathbb{R}$ and let* p *be a continuous seminorm on V with* $\mathrm{p}(v) = \mathrm{p}(\overline{v})$. *Then*

$$\mathrm{p}_R(a) = \mathrm{p}_R(\overline{a}) \tag{82}$$

for all $a \in \mathrm{S}^{\cdot}(V)$. Thus the complex conjugation extends to a continuous antilinear involutive endomorphism of $\widehat{\mathrm{W}}_R(V)$.

For a continuous C-bilinear form $\Lambda: V \times V \to$ (D we define $\overline{\Lambda}: V \times V \to \mathbb{C}$ by

$$\overline{\Lambda}(v, w) = \overline{\Lambda(\overline{v}, \overline{w})} \tag{83}$$

as usual, and set $\mathrm{Re}\,(\Lambda) = \frac{1}{2}\left(\Lambda + \overline{\Lambda}\right)$ as well as $1\mathrm{m}(\Lambda) = \frac{1}{2\mathrm{i}}\left(\Lambda - \overline{\Lambda}\right)$. Then $\overline{\Lambda}$, $\mathrm{Re}\,(\Lambda)$, and $1\mathrm{m}(\Lambda)$ are again continuous C-bilinear forms.

In view of applications in quantum physics we rescale our deformation parameter $z \in \mathbb{C}$ to $z = \frac{\mathrm{i}\hbar}{2}$ and consider only *real* (or even positive) values for ħ. Thus the star product becomes The next result clarifies under which conditions the complex conjugation is a*-involution for $\star_{\frac{\mathrm{i}\hbar}{2}\Lambda}$:

Proposition (3.2.35) [101]: *Let $R \geq \frac{1}{2}$ and $\hbar \in \mathbb{R}\backslash\{0\}$. Then the following statements are equivalent*:

(i) *The complex conjugation is a*-involution with respect to $\star_{\frac{\mathrm{i}\hbar}{2}\Lambda}$, i.e. we have for homogeneous $a, b \in \widehat{\mathrm{W}}_R\left(V, \star_{\frac{\mathrm{i}\hbar}{2}\Lambda}\right)$*

$$a \star_{\frac{i\hbar}{2}\Lambda} b = \mu \circ e^{\frac{i\hbar}{2}\Lambda}(a \otimes b) \ . \tag{84}$$

$$\overline{a \star_{\frac{i\hbar}{2}\Lambda} b} = (-1)^{ab}\overline{b} \star_{\frac{i\hbar}{2}\Lambda} \overline{a}. \tag{85}$$

(ii) $\overline{\Lambda}_+ = -\Lambda_+ and\ \overline{\Lambda}_- = \Lambda_-$.

(iii) $\overline{\Lambda}_+ = -\Lambda_+ and$ *$\widehat{W}_R(V)$ is a Poisson$*$-algebra in the sense that for all $a, b \in \widehat{W}_R(V)$*

$$\overline{\{a, b\}_\Lambda} = \{\overline{a}, \overline{b}\}_\Lambda. \tag{86}$$

Proof. First we note that by continuity it suffices to work on $W_R\left(V, \star_{\frac{i\hbar}{2}\Lambda}\right)$ instead ofthe completion. Suppose (i) and consider homogeneous , $v, w \in V$. Then

$$0 = \overline{v \star_{\frac{i\hbar}{2}\Lambda} w} - (-1)^{vw}\overline{w} \star_{\frac{i\hbar}{2}\Lambda} \overline{v}$$

$$= -\frac{i\hbar}{2}\left(\overline{\Lambda}(\overline{v}, \overline{w}) + (-1)^{vw}\Lambda(\overline{w}, \overline{v})\right)$$

$$= -\frac{i\hbar}{2}\left(\overline{\Lambda}_+(\overline{v}, \overline{w}) + \Lambda_+(\overline{w}, \overline{v}) + \overline{\Lambda}_-(\overline{v}, \overline{w}) - \Lambda_-(\overline{w}, \overline{v})\right),$$

since the star product gives only the zeroth and first order terms and Λ_+ is symmetric while Λ_- is antisymmetric. Now $\overline{\Lambda}_+ + \Lambda_+$ is still symmetric and $\overline{\Lambda}_- - \Lambda_-$ is still antisymmetric. Hence their contributions have to vanish separately which implies (ii). Next, assume (ii). Then

$$\overline{P_{\Lambda\pm}(a \otimes b)} = \mp P_{\Lambda\pm}\left(\overline{a} \otimes \overline{b}\right) \tag{**}$$

follows immediately. For the symmetric tensor product μ we have $\overline{\mu(a \otimes b)} = \mu\left(\overline{a} \otimes \overline{b}\right)$ which combines to give (iii) at once. Conversely, (iii) implies (ii) by evaluating on , $w \in V$. Finally, assume (ii). In general the (anti-)symmetiv of $\Lambda \pm$ implies

$$P_{\Lambda\pm} = \pm\tau \circ P_{\Lambda\pm} \circ \tau, \tag{**}$$

as this follows either by a direct computation or by verifying the Leibniz rules for $\tau \circ P_{\Lambda\pm} \circ \tau$ and then applying the uniqueness result from Lemma (3.2.2). Combining now $(*)$ and $(**)$ with the commutativity $\mu = \mu 0 \tau$ of the symmetric tensor product gives (i) by a computation analogously to [100].

Thus we need a real Poisson bracket to start the deformation and an imaginary symmetric part Λ_+ in the star product $\star_{\frac{i\hbar}{2}\Lambda}$ to have the complex conjugation a s^*-involution. In this case $\widehat{W}_R\left(V, \star_{\frac{i\hbar}{2}\Lambda}\right)$ is a locally convex $*$-algebra.

We collect some additional properties of the Weyl algebra $W_R(V, \star_{z\Lambda})$ which are inherited from V.

Suppose that V has an absolute Schauder basis, i.e. there exists a linearly independent set $\{e_i\}_{i \in I}$, ofvectors in V together with continuous coefficient functionals $\{\phi^i\}_{i \in I}$ in V' such that $\phi^i(e_j) = \delta^i_j$ for $i, j \in I$ and

$$v = \sum_{i \in I} \phi^i\,(v) e_i \tag{87}$$

converges. For an *absolute* Schauder basis one requires that for all continuous seminorms p on V there exists a continuous seminorm q such that

$$\sum_{i \in \mathrm{I}} |\,\phi^i(v)|\mathrm{p}(e_i) \leq \mathrm{q}(v) \tag{88}$$

for all $v \in V$, i.e. the series in (87) converges absolutely for all continuous seminorms and can be estimated by a continuous seminorm. In particular, at most countably many contributions $\phi^i(v)\mathrm{p}(e_i)$ can be different from 0 for a given $v \in V$. Typically, J will be countable itself. In the following we assume to have such an absolute Schauder basis $\{e_i\}_{i\in I}$, for V with coefficient functionals $\{\phi^i\}_{i\in I}$.

The projective topology on $V^{\otimes n}$ is known to be compatible with absolute Schauder bases. We have the following lemma:

Lemma (3.2.36) [101]: *The vecrors* $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{i_1,\ldots,i_n \in I}$ *form an absolute Schauder basis for* $V^{\otimes_\pi n}$ *with coefficienr functionals* $\{\phi^{i_1} \otimes\otimes \phi^{i_n}\}_{i_1,\ldots,i_n\in I}$. *If* p *and* q *are continuous seminorms on* V *with* (88) *then one has for all* $v \in V^{\otimes_\pi n}$

$$\sum_{i_1,\ldots,i_n \in \mathrm{I}} |\left(\phi^{i_1} \otimes \cdots \otimes \phi^{i_n}\right)(v)|\mathrm{p}^n\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) \leq \mathrm{q}^{\dagger l}(v)\,. \tag{89}$$

In a next step we consider the whole tensor algebra $\mathrm{T}_R^*(V)$ endowed with the topology from Definition (3.2.20) for some fixed $R \geq 0$. We claim that the collection $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{\uparrow\iota\in \mathrm{N}_0, i_1,\ldots,i_n\in I}$, provides an absolute Schauder basis for $\mathrm{T}_R^*(V)$ with corresponding coefficient functionals $\{\phi^{i_1} \otimes \cdots \otimes \phi^{i_n}\}_{n\in \mathrm{N}_0, i_1,\ldots,i_n\in I}$ Here for $n = 0$ we take the standard basis vector $1 \in \mathrm{K}$ with the corresponding coefficient functional. First we note that the linear functionals $\phi^{i_1} \otimes \cdots \otimes \phi^{i_n}$ are continuous on $T_R^{\cdot}(V)$ when they are extended by 0 to the tensor degrees different from n. Moreover, we have

$$\mathrm{p}_R\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) = \mathrm{p}^n\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) n!^R \tag{90}$$

by Lemma (3.2.19)(ii). This results in the following statement:

Lemma (3.2.37) [101]: *The vectors* $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{n\in \mathrm{N}_0, i_1,\ldots,i_n\in}$, *form an absolute Schauder basis for* $\mathrm{T}_R^*(V)$ *with coefficient functionals* $\{\phi^{i_1} \otimes \cdots \otimes \phi^{i_n}\}_{n\in \mathrm{N}_0, i_1,\ldots,i_n\in I}$. *If* p *and* q *are continuous seminorms on* V *with* (88) *then one has for all* $v \in \mathrm{T}_R^*(V)$

$$\sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n \in \mathrm{f}} |\left(\phi^{j_1} \otimes \cdots \otimes \phi^{i_n}\right)(v)|\mathrm{p}_R\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) \leq \mathrm{q}_R(v)\,. \tag{91}$$

Remark (3.2.38) [101]: We note that an *absolute* Schauder basis stays an absolute Schauder basis after completion, see [117].

The absolute Schauder basis descends now to the symmetric algebra by symmetrizing. If $v \in \mathrm{S}^{\cdot}(V)$ then we have $v = \mathcal{S}v$ with the continuous symmetrization map from (61). Applying$\mathcal{S}$ twice, this shows the convergence

$$v = \sum_{n=0}^{\infty} \sum_{i_1,\dots,i_n \in 1} \left(\phi^{i_1} \cdots \phi^{i_n}\right)(v) e_{i_1} \cdots e_{i_n} \tag{92}$$

for all $v \in S_R(V)$, where $\phi^{i_1} \cdots \phi^{i_n} = \left(\phi^{i_1} \otimes \cdots \otimes \phi^{i_n}\right)$. Moreover, using again $v = \mathcal{S}v$ we get from the estimate (61) and (91) the estimate

$$\sum_{n=0}^{\infty} \sum_{i_1,\dots,i_n \in \mathrm{f}} |\left(\phi^{i_1} \cdots \phi^{i_\mathrm{n}}\right)(v)| \mathrm{p}_R\left(e_{i_1} \cdots e_{i_n}\right) \leq \mathrm{q}_R(v)\,. \tag{93}$$

So we have all we need for an absolute Schauder basis *except* that the symmetrizations $e_{i_1} \cdots e_{i_\mathrm{n}}$ will no longer be linearly independent: some of them will be zero if they contain twice the same odd vector and some of them will differ by signs. So we only have to single out a maximal linearly independent subset, the choice of which might be personal taste:

Proposition (3.2.39) [101]: Let $R \geq 0$ *and* let $\{e_i\}_{i \in I}$, *be an absolute Schauder basis of* V *of homogeneous vectors with coefficient functionals* $\{\phi^i\}_{i \in \mathrm{J}}$. *Then any choice of a maximal linearly independent subset of* $\{e_{i_1} \cdots e_{i_n}\}_{n \in]\mathrm{N}_0, i_1,\dots,i_n \in \mathrm{J}}$ *will give an absolute Schauder basis of* $\mathrm{S}_R^*(V)$ *and of* $W_{R-}(V)$ *with coefficient functionals given by the corresponding subset of* $\{\phi^{i_1} \cdots \phi^{i_\mathrm{n}}\}_{n \in]\mathrm{N}_0, i_1,\dots,i_\mathrm{n} \in}$*. One has the estimate* (93) *whenever* p *and* q *satisfy* (88).

Having an absolute Schauder basis is a very strong property for a locally convex space. In fact, they are completely known: after completion one obtains a Köthe (sequence) space where the index set for the "sequences" is I, see e.g. [117] for a detailed description. The Köthe matrix K_V of V is obtained from $K_V = \left(\lambda_{i,\mathrm{p}}\right)$ with $\lambda_{i,\mathrm{p}} = \mathrm{p}(e_i)$ where $i \in I$ and p ranges over a defining system of continuous seminorms of V. Thus the corresponding Köthe matrix of the tensor algebra $\mathrm{T}_R^*(V)$ is given by $K_{\mathrm{T}_R^*(V)} = \left(\lambda_{(\mathrm{n}, i_1,\dots,i_n),\mathrm{p}}\right)$ with

$$\lambda_{(n, i_1,\dots,i_n),\mathrm{p}} = n!^R \, \lambda_{i_1,\mathrm{p}} \cdots \lambda_{i_n,\mathrm{p}}. \tag{94}$$

Thus we have an *explicit* description in terms of the Köthe matrix of V. Analogously, one can proceed for the Weyl algebra $\mathrm{W}_R(V)$. For Köthe spaces many properties are (easily) encoded in their Köthe matrix, so we see here that the appearance of $n!^R$ will play a prominent role when passing from V to $\mathrm{T}_R^*(V)$ or $W_R(V)$.

We discuss nuclearity properties of the Weyl algebra $W_R(V)$ originating from those of V: since $V \subseteq \mathrm{W}_R(V)$ is a closed subspace inheriting the original topology from the one of $W_R(V)$, we see that nuclearity of $W_R(V)$ implies the nuclearity of V. The aim of this to show the converse.

To this end, it will be convenient to work with the tensor algebra $\mathrm{T}_R^*(V)$ instead of the symmetric algebra $\mathrm{S}_R^*(V)$ since we do not have to take care of the combinatorics of symmetrization.

Let $U \subseteq V$ be a subspace and denote by $\{U\rangle \subseteq \mathrm{T}^\cdot(V)$ the two-sided ideal generated by U. Then the quotient algebra $\mathrm{T}^\cdot(V)/\{U\rangle$ is still $\mathbb{Z}$-graded by the tensor degree since U has homogeneous generators of tensor degree one. The map

$$\iota\colon \mathrm{T}^\cdot(V)/\langle U) \to \mathrm{T}^\cdot(V/U) \tag{95}$$

determined by $\iota([v_1 \otimes \cdots \otimes v_n]) = [v_1] \otimes \cdots \otimes [v_n]$ turns out to be an isomorphism ofgraded algebras. We shall now show that l also respects the seminorms p_R. First recall that for a seminorm p on V one defines a seminorm [p] on V/U by

$$[\mathrm{p}]([v]) = \inf\{\mathrm{p}(v+u) | u \in U\}. \tag{96}$$

Then the locally convex quotient topology on V/U is obtained from all the seminorms [p] where p runs through all the continuous seminorms of V.

Lemma (3.2.40) [101]: *Let $u \subseteq V$ be a subspace and let* p *be a seminorm on V. Then for all $R \in \mathbb{R}$ one has*

$$[\mathrm{p}_R] = [\mathrm{p}]_R 0\iota. \tag{97}$$

Proof. Let $\{U\}_n = \sum_{\ell=1}^{n} V \otimes \cdots \otimes U \otimes \cdots \otimes V \subseteq V^{\otimes n}$ with U being at the ℓ -th position. This is the n-th homogeneous part of $\langle U \rangle$. Then $V^{\otimes n}/\{U\rangle_n$ gives the n-th homogeneous part of the graded algebra $\mathrm{T}^{\cdot}(V)/\{U\rangle$. For a seminorm p on V and the induced isomorphism ι restricted to $V^{\otimes n}/\langle U \rangle_n$ a simple argument shows

$$[\mathrm{p}^n] = [\mathrm{p}]^n 0\iota.$$

From this, (97) follows at once.

This simple lemma has an important consequence which we formulate in two ways:

Corollary (3.2.41) [101]: *Let $U \subseteq V$ be a subspace and $R \in \mathbb{R}$.*

(i) *The isomorphism* (95) *induces an isomorphism*

$$\iota\colon \mathrm{T}_R^*(V)/\langle U \rangle \to \mathrm{T}_R^*(V/U) \tag{98}$$

of locally convex algebras if the left hand side as well as V/U carry the locally convex quotient topologies.

(ii) *The isomorphism* (95) *induces an isomorphism*

$$\iota\colon \mathrm{T}_{R-}^*(V)/(U\rangle \to \mathrm{T}_{R-}^*(V/U) \tag{99}$$

of locally convex algebras.

Proof. For the second part, we note that the seminorms $\left[\mathrm{p}_{R-\varepsilon,1}\right]$ and $[\mathrm{p}]_{R-\varepsilon,1}$ for $\varepsilon > 0$ and p a continuous seminorm on V constitute a defining system of seminorms for the projective limit topologies.

We now assume that V is nuclear. There are many equivalent ways to characterize nuclearity, see e.g. [117], we shall use the following very basic one: for a given continuous seminorm p on V we consider $V/\ker\mathrm{p}$ with the quotient seminorm [p]. This is now a normed space as we have divided by kerp. Thus we can complete $V/\ker\mathrm{p}$ to a Banach space denoted by V_{p}. Then V is called nuclear if for everv continuous seminorm p there is a another continuous seminorm $\mathrm{q} \geq \mathrm{p}$ such that the canonical map $i_{\mathrm{qp}}\colon V_{\mathrm{q}} \to V_{\mathrm{p}}$ is a nuclear map. This means that there are vectors $e_i \in V_{\mathrm{p}}$ and continuous linear functionals $\varepsilon^i \in V_{\mathrm{q}}'$ such that

$$\iota_{\mathrm{qp}}(v) = \sum_{i=1}^{\infty} \varepsilon^i(v) e_i \text{ with } \sum_{i=1}^{\infty} \| \varepsilon^i \|_{\mathrm{q}} \| e_i \|_{\mathrm{p}} < \infty, \tag{100}$$

where we use the notation $\| \cdot \|_{\mathrm{p}} = [\mathrm{p}]$ for the Banach norms on V_{p} and

$$\|\varepsilon^j\|_{\mathrm{q}} = \sup_{v \neq 0} \frac{\varepsilon^i(v)}{\|v\|_{\mathrm{q}}} \tag{101}$$

denotes the functional norm as usual. The following lemma is well-known:

Lemma (3.2.42) [101]: *Let* $(V, \|\cdot\|)$ *be a Banach space and let* $\phi_1\ ,\ ,\ \phi_n \in V'$. *Then* $\phi_1 \otimes \cdots \otimes \phi_n \in \left(V^{\otimes_\pi n}\right)'$ *with*

$$\left\|\varphi_1 \bigotimes \ \cdots \bigotimes \ \varphi_n\right\| = \|\varphi_1\| .. \|\varphi_n\|, \tag{102}$$

where on $V^{\otimes_\pi n}$ *we use the norm* $\|\cdot\| \otimes \cdots \otimes \|\cdot\|$ *as usual.*

The next lemma shows how $\ker \mathrm{p} \subseteq V$ is related to icer $\mathrm{p}_R \subseteq T_R^{\cdot}(V)$.

Lemma (3.2.43) [101]: *Let* $R \in \mathbb{R}$ *and let* p *be a continuous seminorm on V. Then*

$$\langle \ker \mathrm{p}) = \ker \mathrm{p}_R. \tag{103}$$

Proof. Since [p] is a norm on $V/\ker \mathrm{p}$, also $[\mathrm{p}]^n$ is a norm on $(V/\ker p)^{\otimes n}$. It follows that $[\mathrm{p}]_{R,1}$ is a norm on $\mathrm{T}^{\cdot}(V/\ker \mathrm{p})$ as well, implying that $[\mathrm{p}_R]$ is a norm on $\mathrm{T}(V)/\langle \ker \mathrm{p})$ according to Lemma (3.2.40). Thus $[v] \in \ker \mathrm{p}_R$ iff $[\mathrm{p}_R]([v]) = 0$ iff $\in \{\ker \mathrm{p})$. □

The following lemma is the key to understand nuclearity:

Lemma (3.2.44) [101]: *Let* $\mathrm{p} \leq \mathrm{q}$ *be continuous seminorms such that the canonical map* $\iota_{\mathrm{qp}}: V_{\mathrm{q}} \to V_{\mathrm{p}}$ *has a nuclear representation*

$$\iota_{\mathrm{q,p}} = \sum_{i=1}^{\infty} \varepsilon^i \otimes e_i \tag{104}$$

with $\varepsilon^i \in V'_{\mathrm{q}}$ *and* $e_j \in V_{\mathrm{p}}$. *Then there is a constant* $c > 0$ *such that the canonical map*

$$\iota_{(c\mathrm{q})_R, \mathrm{p}_R}: \mathrm{T}_R^*(V)_{(c\mathrm{q})_R} \to \mathrm{T}_R^*(V)_{\mathrm{p}_R} \tag{105}$$

has the nuclear representation

$$l_{(c\mathrm{q})_R, \mathrm{p}_R} = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{\infty} \left(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_n}\right) \otimes \left(e_{i_1} \otimes \cdots \otimes e_{i_{\mathrm{n}}}\right) \tag{106}$$

Proof. By rescaling the seminorm q we can achieve that the numerical value

$$x = \sum_{i=1}^{\infty} \|\varepsilon^i\|_{\mathrm{q}} \|e_i\|_{\mathrm{p}} < 1 \tag{$*$}$$

of the convergence condition in (100) is not only finite but actually as small as we need: rescaling of q by c shrinks the value of $(*)$ by $\frac{1}{c}$ since we need the dual norm $\|\cdot\|_q$. Thus we may assume $(*)$ without restriction defining the possibly necessary rescaling factor c. Next we note that $\left(\mathrm{T}_R^*(V)\right)_{\mathrm{q}_R}$ is the Banach space completion of $\mathrm{T}_R^*(V)/\ker \mathrm{q}_R \cong \mathrm{T}_R^*(V/\ker \mathrm{q})$ with respect to the norm $[\mathrm{q}_R] = [\mathrm{q_R}]\,{}^\circ\iota$, according to Lemma (3.2.40), and analogously for $\left(\mathrm{T}_R^*(V)\right)_{\mathrm{p}_R}$. In this sense we have $e_{i_1} \otimes \cdots \otimes e_{i_n} \in \left(\mathrm{T}_R^*(V)\right)_{\mathrm{p}_R}$ as well as $\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_n} \in \left(\left(\mathrm{T}_R^*(V)\right)_{\mathrm{q}_R}\right)'$. For the norms of these vectors and linear functionals we have

$$\mathrm{p}_R\left(e_{i_1} \otimes \cdots \otimes e_{i_n}\right) = n!^R\, [\mathrm{p}]^n\left(e_{i_1} \otimes \cdots \otimes e_{i_{\mathrm{n}}}\right) = n!^R\, \|e_{i_1}\|_{\mathrm{p}} \ldots \ldots \|e_{i_n}\|_{\mathrm{p}}$$

and

$$\|\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_n}\|_{\mathrm{q}_R} = \frac{1}{n!^R} \|\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_n}\|_{\mathrm{q}^n} = \frac{1}{n!^R} \|\varepsilon^{i_1}\|_{\mathrm{q}}\, \|\varepsilon^{i_n}\|_{\mathrm{q}}.$$

Again, due to dualizing, the prefactor $n!^R$ appears now in the denominator. Combining these results we have

$$\sum_{n=0}^{\infty}\sum_{i_1,\dots,i_n=0}^{\infty}\| \varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_n}\|_{q_R}p_R\left(e_{i_1}\otimes\cdots\otimes e_{i_n}\right)=\sum_{n=0}^{\infty}x^n<\infty,$$

since we arranged $x < 1$. Finally, it is clear that(106) holds before the completion and thus also afterwards by continuity.

Theorem (3.2.45) [101]: *Let* $R \geq 0$. *Then the following statements are equivalent*:

(i) V *is nuclear*.

(ii) $T_R^*(V)$ *is nuclear*.

(iii) $W_R(V)$ *is nuclear*.

Proof. We consider the tensor algebra $T_R^*(V)$. Then $W_R(V) = S_R^*(V)$ is a closed subspace of $T_R^*(V)$ and V is a closed subspace of $W_R(V)$. Hence it will sumce to show that $T_R^*(V)$ is nuclear whenever V is nuclear. Since the topology of$T_R^*(V)$ is determined by all the seminorms p_R, Lemma (3.2.44) gives us the nuclear representation of $l_{(cq)_R,p_R}$ whenever we have one for $i_{q,p}$. Hence $T_{R,1}^*(V)$ is nuclear.

For the projective limit version $W_{R-}(V)$ we can argue either along the same line of proof as above or use the above result and rely on the general fact that projective limits of nuclear spaces are again nuclear. However, the following statement is less obvious and shines some new light on the projective version of the Weyl algebra:

Theorem (3.2.46) [101]: *Let* $R \geq 0$. *Then the following statements are equivalent*:

(i) V *is strongly nuclear*.

(ii) $T_{R-}(V)$ *is strongly nuclear*.

(iii) $W_{R-}(V)$ *is strongly nuclear*.

Proof. Again, since closed subspaces inherit strong nuclearity, we only have to show the implication (i) ⇒ (ii) . Thus let p be a continuous seminorm on V with a matching continuous seminorm q such that (100) holds with

for all $\alpha > 0$. Now we have to take $q_{R-\varepsilon'}$ for $p_{R-\varepsilon}$ with some $0 < \varepsilon' < \varepsilon$. Then the series

$$x(\alpha)=\sum_{i=1}^{\infty}\| \varepsilon^{i}\|_{q}^{\alpha}\|e_i\|_{p}^{\alpha}<\infty$$

$$\sum_{n=0}^{\infty}\sum_{i_1,\dots,i_n=0}^{\infty}\left(\|\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_n}\|_{q_{R-\varepsilon'}}p_{R-\varepsilon}\left(e_{i_1}\otimes\cdots\otimes e_{i_n}\right)\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{1}{n!^{\alpha(\varepsilon-\varepsilon')}}x(\alpha)^{\alpha}$$

still converges for all $\alpha > 0$ showing the strong nuclearity of $T_{R-}(V)$. □

Note that it is crucial to have some (small) inverse power of $n!$ at hand: without this option we cannot succeed in showing the strong nuclearity as we need the α-summability for *all* $\alpha > 0$. Thus, concerning strong nuclearity, the projective limit version $W_{R-}(V)$ turns out to behave nicer than the more direct version $W_R(V)$.

Example (3.2.47). For V finite-dimensional we get a strongly nuclear Weyl algebra $W_{R-}(V,\star_{z\Lambda})$. Here we can either use the above theorem since V being finite-dimensional is strongly nuclear, or we can rely on the explicit description of V and hence of $W_{R-}(V,\star_{z\Lambda})$ as a Köthe space: since for a finite-dimensional vector space it suffices to take a single norm, the Köthe matrix is finite and hence its entries are bounded, say by $c > 0$. Thus the Köthe matrix (94) has entries bounded by $c^n n!^{R-\varepsilon}$ where $\varepsilon > 0$. To this result one can apply the

Grothendieck-Pietsch criterion and conclude strong nuclearity directly, see [117]. For a finite-dimensional V the Weyl algebra $\mathrm{Y'}_R(V,\star_{z\Lambda})$ is still a nuclear space.

We discuss how the algebraic symmetries and equivalences translate into our locally convex framework.

Suppose that V and W are two $\mathbb{Z}_2$ - graded locally convex Hausdorff spaces and let Λ_V and Λ_W be continuous even bilinear forms on V and W, respectively. We want to extend the functoriality statement from Proposition (3.2.12). The following estimates are obvious:

Lemma (3.2.48) [101]: *Let $A: V \to W$ be an even linear map and let* p *and* q *be seminorms on V and W such that* $\mathrm{q}(A(v)) \leq \mathrm{p}(v)$ *for all $v \in V$. Then*

$$\mathrm{q}_R(A(v)) \leq \mathrm{p}_R(v) \tag{107}$$

for all $v \in \mathrm{T}^{\cdot}(V)$.

Proof. We clearly have $\mathrm{q}^n\left(A^{\otimes n}(v)\right) \leq \mathrm{p}^n(v)$ for all $\in \mathrm{T}^n(V)$. From this, (107) is clear by the definition of p_R.

Proposition (3.2.49) [101]: *Let $R \geq \frac{1}{2}$ and $z \in \mathbb{C}$. Then the Weyl algebra $\mathrm{Y'}_R(V,\star_{z\Lambda})$ as well as its completion $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ depend functonally on (V,Λ) with respect to continuous Poisson maps.*

In particular, the *continuous* Poisson automorphisms in $Aut\ (V,\Lambda)$ act on the Weyl algebra $\mathrm{W}_R(V,\star_{z\Lambda})$ as well as on its completion $\widehat{\mathrm{W}}_{R^{\cdot}}(V,\star_{z\Lambda})$ by continuous automorphisms. The analogous statement holds for the projective version $\mathrm{W}_{R-}(V,\star_{z\Lambda})$

We investigate the action of the translations by linear forms on V as done algebraically in (35). We discuss the continuity of the translations τ_ϕ^* in two ways: first directly for a general even continuous $\phi \in V'$ and second for a ϕ in the image of # from (38): since Λ is continuous, an element $\phi \in \mathrm{im}\# \subseteq V^*$ is clearly continuous as well. In this more special situation we show a much stronger statement, namely that τ_ϕ^* is an inner automorphism.

We start with the following basic estimate for the continuity of the translation operators τ_ϕ^*:

Lemma (3.2.50) [101]: *Let $\phi \in V'$ be even and let* p *be a continuous seminorm on V such that $|\phi(v)| \leq \mathrm{p}(v)$ for all $v \in V$. Then for $R \geq 0$ we have for all $v \in \mathrm{T}_R(V)$*

$$\mathrm{p}_R(\tau_\phi^* v) \leq (2\mathrm{p})_R(v)\,. \tag{108}$$

Proof. We write $v = \sum_{n=0}^{\infty} v_n \in \mathrm{T}^{\cdot}(V)$ with its homogeneous components $v_n \in V^{\otimes n}$, all of which are zero except finitely many. Moreover, we write

$$v_n = \sum_i v_i^{(1)} \otimes \cdots \otimes v_i^{(n)} \tag{$*$}$$

as usual. Then the homomorphism property of τ_ϕ^* gives

$$\tau_\phi^* v = \sum_{n=0}^{\infty} \sum_i \left(v_i^{(1)} + \phi\left(v_i^{(1)}\right)\mathrm{I}\right) \otimes \cdots \otimes (v_i^{(n)} + \phi\left(v_i^{(n)}\right) 1)\,.$$

For every n we get now various contributions in all tensor degrees $k \leq n$. The contributions in the tensor degree k consist of linear combinations ofa choice of k vectors among the $v_i^{(1)}$

, , $v_i^{(n)}$, taking their tensor product, applying ϕ to the remaining $n-k$ vectors, and multiplying everything together in the end. For a fixed index i there are $\binom{n}{k}$ possibilities to distribute $n-k$ copies of ϕ to the n vectors $v_i^{(1)}$, , $v_i^{(n)}$. Finally, using the estimate $|\phi(w)| \leq \mathrm{p}(w)$ for all $w \in V$ we obtain that the contributions to p_R from these terms can be estimated by

$$\mathrm{p}_R\left(\left(v_i^{(1)} + \phi\left(v_i^{(1)}\right)1\right) \otimes \cdots \otimes \left(v_i^{(n)} + \phi\left(v_i^{(n)}\right)1\right)\right)$$
$$\leq \sum_{k=0}^{\mathrm{n}} \binom{n}{k} k!^R \, \mathrm{p}\left(v_i^{(1)}\right) \cdots \mathrm{p}\left(v_i^{(n)}\right).$$

In total, we get the estimate

$$\mathrm{p}_R\left(\tau_\phi^* v\right) \leq \sum_{n=0}^{\infty} \sum_i \sum_{k=0}^{n} \binom{n}{k} k!^R \, \mathrm{p}\left(v_i^{(1)}\right) \cdots \mathrm{p}\left(v_i^{(\mathrm{n})}\right)$$
$$\leq \sum_{n=0}^{\infty} \sum_i 2^n n!^R \, \mathrm{p}\left(v_i^{(1)}\right) \cdots \mathrm{p}\left(v_i^{(\mathrm{n})}\right).$$

Since the decomposition ($*$) was arbitrary, we can take the infimum over all such decompositions resulting in (108).

From this estimate we get immediately the following continuity statements:

Proposition (3.2.51) [101]: *Let $\phi \in V'$ be an even continuous linear functional and let $R \geq 0$.*

(i) *The algebra automorphism $\tau_\phi^*: \mathrm{T}_R^*(V) \to \mathrm{T}_R^*(V)$ is continuous.*

(ii)For $R \geq \frac{1}{2}$, the algebra automorphism$\tau_\phi^*: \mathrm{W}_R(V, \star_{z\Lambda}) \to \mathrm{W}_R(V, \star_{z\Lambda})$*is continuous.*

(iii)For $R > \frac{1}{2}$, the algebra automorphism $\tau_\phi^*: \mathrm{W}_R(V, \star_{z\Lambda}) \to \mathrm{W}_R(V, \star_{z\Lambda})$ is continuous.

In particular, τ_ϕ^* extends in all three cases to the corresponding completions and yields a continuous automorphism for the completions, too.

In a next step we want to understand which of the τ_ϕ^* are inner automorphisms. This is a well-known statement: if the linear functional ϕ is in the image of # then τ^* is inner via the star-exponential of a pre-image of ϕ with respect to #. Also the heuristic formula for the star-exponential is foliose. Our main point here is that we have an analytic framework where the star-exponential actually makes sense: this is in so far nontrivial as we know that the canonical commutation relations do not allow for a general entire calculus. Thus the existence of an exponential has to be shown by hand.

It will be crucial to have $R \leq 1$ in view of Proposition (3.2.31). We start with some basic properties of the exponential series:

Lemma (3.2.52) [101]: *Let $R < 1$ and $w \in V_0$ be an even vector.*

(i) *For all $v \in V$ we have*

$$\exp(w) \star_{z\Lambda} v = \exp(w)\big(v + z\Lambda(w, v)\big), \tag{109}$$

and

$$v \star_{z\Lambda} \exp(w) = \exp(w)\big(v + z\Lambda(v, w)\big). \tag{110}$$

(ii) *For all* t ∈ K *one has*

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}=\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}\star_{z\Lambda}w=w\star_{z\Lambda}\mathrm{e}^{tw+\frac{\mathrm{t}^2z}{2}\Lambda(w,w)1} \quad (111)$$

(iii) *The star-exponential series for* $w\in V$ *converges absolutely in* $W_R(V,\star_{z\Lambda})$ *and*

$$\mathrm{Exp}_{\star_{z\Lambda}}(\mathrm{tw})=\sum_{n=0}^{\infty}\frac{r^n}{n!}w\star_{z\Lambda}\cdots\star_{z\Lambda}w=\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1} \quad (112)$$

Proof. We use the continuity of $\star_{z\Lambda}$ and the (absolute) convergence of the exponential series to get

$$\exp(w)\star_{z\Lambda}v=\sum_{n=0}^{\infty}\frac{1}{n!}w^n\star_{z\Lambda}v$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}(w^{\mathrm{n}}v+z\mu\mathrm{o}P_\Lambda(w^{\mathrm{n}}\otimes v)+0)$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(w^{\mathrm{n}}v+nzw^{n-1}\Lambda(w,v)\right)$$
$$=\exp(w)\left(v+z\Lambda(w,v)\right),$$

since $\mu\mathrm{o}P_\Lambda(.\,,v)$ is a derivation of the undeformed symmetric tensor product and w is even. The second equation is analogous .For the second part.we first note that $t\mapsto\exp\left(\mathrm{t}w+\frac{\mathrm{r}^2z}{2\mathrm{P}}\Lambda(w,w)1\right)$is real –analytic (entire in the case $\mathbb{K}=\mathbb{C}$) with convergent Taylor expansion around 0 for all $t\in\mathbb{K}$ thanks to Proposition (3.2.31)(ii). We compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}=\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}\left(w+z\Lambda(tw,w)\right)=\mathrm{e}^{fw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}\star_{z\Lambda}w,$$

using the first part and the fact that $\Lambda(w,w)]\lfloor$ is central. Analogously, we can write $w\star_{z\Lambda}$ in front. This shows the second part. Together, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{e}^{fw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}=w\star_{z\Lambda}\cdots\star_{z\Lambda}w$$

for the Taylor coefficients of the real-analytic (entire) function f $\mapsto\mathrm{e}^{tw+\frac{\mathrm{r}^2z}{2}\Lambda(w,w)1}$. Since its Taylor series converges absolutely, the last part follows. □

The following statement is now an easy computation:

Proposition (3.2.53) [101]: *Let* $R<1$ *and let* $\phi\in V'$ *be even. If* ϕ *is in the image of* J *then* τ_ϕ^**is an inner automorphism of* $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ *for all* $z\neq 0$. *In fact,*

$$\tau_\phi^*(a)=\mathrm{Exp}_{\star_{z\Lambda}}(w)\star_{z\Lambda}a\star_{z\Lambda}\mathrm{Exp}(-w) \quad (113)$$

for all $a\in\widehat{W}_R(V,\star_{z\Lambda})$ *where* $w\in V_0$ *is such that* $2zw^\#=\phi$.

Proof. First we note that the star-exponential function gives a one-parameter group of invertible elements in $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ with respect to the star product $\star_{z\Lambda}$. This is clear from the absolute convergence ofthe star-exponential series. Thus the right hand side of(113) defines an inner automorphism of the Weyl algebra. Now consider $v\in V$. Using Lemma (3.2.52) we compute for $v\in V$

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{Exp}_{\star_{z\Lambda}}(\mathrm{t}w) \star_{z\Lambda} v \star_{z\Lambda} \mathrm{Exp}_{\star_{z\Lambda}}(-tw)$$
$$= \mathrm{Exp}_{\star_{z\Lambda}}(\mathrm{t}w) \star_{z\Lambda} (w \star_{z\Lambda} v - v \star_{z\Lambda} w) \star_{z\Lambda} \mathrm{Exp}_{\star_{z\Lambda}}(-\mathrm{t}w)$$
$$= \mathrm{Exp}_{\star_{z\Lambda}}(\mathrm{t}w) \star_{z\Lambda} (z\Lambda(w,v)1 - z\Lambda(v,W)1)) \star_{z\Lambda} \mathrm{Exp}_{\star_{z\Lambda}}(-\mathrm{t}w)$$
$$= 2z\Lambda_{-}(w,v)1,$$
where we use that $\mathrm{Exp}_{\star_{z\Lambda}}(-\mathrm{f}w)$ is the $\star_{z\Lambda}$-inverse of $\mathrm{Exp}_{\star_{z\Lambda}}$ (tw). On the other hand, $r \mapsto \tau^{*}_{\mathrm{f}\phi}(v) = v + \mathrm{f}\phi(v)]|$ has the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau^{*}_{\mathrm{f}\phi}(v) = \phi(v)1.$$

Thus taking w such that $2w\Lambda_{-}(w,) = \phi$, i.e. $2zw^{\#} = \phi$, shows (113) for $a = v$. Now both sides are automorphisms and hence both sides coincide on al1 $\star_{z\Lambda}$-polynomials in elements from V. But V together with 1 generates $W_R(V,\star_{z\Lambda})$ according to Corollary (3.2.9). Thus the two automorphisms coincide on $\mathrm{W}_R(V,\star_{z\Lambda})$. Since both are continuous, they also coincide on the completion $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$.

We have seen that the same antisymmetric part Λ_{-} yields equivalent deformations, no matter what the symmetric part Λ_{+} of Λ is. We extend this now to the analytic framework. The following lemma shows the continuity of the equivalence transformation from Proposition (3.2.15):

Lemma (3.2.54) [101]: *Let $g: V \times V \to \mathbb{K}$ be an even symmetr c bilinearform. Let $R \geq \frac{1}{2}$ and let* p *be a seminorrrg on V with*

$$|g(v,w)| \leq \mathrm{p}(v)\mathrm{p}(w) \tag{114}$$

for all $v, w \in V$. Then we have for all $a \in \mathrm{S}^{\cdot}(V)$

$$\mathrm{p}_R\left(\Delta_g a\right) \leq (2\mathrm{p})_R(a)\,. \tag{115}$$

Moreover, there are constants $c, c' > 0$ with

$$\mathrm{p}(\mathrm{e}^{t\Delta_g}a) \leq c'(c\mathrm{p})_R(a)\,. \tag{116}$$

Proof. First we extend the operator Δ_g to the whole tensor algebra $\mathrm{T}^{\cdot}(V)$ as usual by setting

$$\tilde{\Delta}_g(v_1 \otimes \cdots \otimes v_n)$$
$$= \sum_{i<j} (1)^{v_1(+v+\cdots v_{i-1})} (-1)^{v_j(v_1+\cdots+v_{i-1}+\cdots+v_{i-1})} g(v_j,v_j) v_1 \otimes \wedge^i \ldots \wedge^j \ldots \otimes v_n$$

on factorizing homogeneous tensor and extending linearly. Then we have

$$\mathcal{S}\circ\tilde{\Delta}_g = \Delta_g \circ \mathcal{S} \tag{$*$}$$

as already for P_Λ. With an analogous estimate as for the Poisson bracket we get

$$\mathrm{p}^{n-2}\left(\tilde{\Delta}_g(v_1 \otimes \cdots \otimes v_n)\right) \leq \sum_{i<j} |\, g(v_i,v_j)| \mathrm{p}(v_1) \wedge i \ldots \wedge \cdots \mathrm{p}(v_n) j$$
$$\leq \frac{n(n-1)}{2}\mathrm{p}(v_1)\cdots\mathrm{p}(v_n)\,.$$

This implies for all $a_n \in \mathrm{T}^n(V)$ the estimate

$$\mathrm{p}^{n-2}(\tilde{\Delta}_g a_n) \leq \frac{n(n-1)}{2}\mathrm{p}^{\mathrm{n}}(a_n)\,.$$

Thanks to (∗) we get the same estimate for $a_{\mathrm{n}} \in S^n(V)$ and Δ_g in place of $\tilde{\Delta}_g$. By induction this results in

$$\mathrm{p}^{n-2k}\left(\Delta_g^k a_n\right) \leq \frac{n!}{2^k(n-2k)!}\mathrm{p}^n(a_n) \qquad (**)$$

as long as $n-2k \geq 0$ and $\Delta_g^{\mathrm{k}} a_n = 0$ for $2k > n$. This gives

$$\mathrm{p}_R\left(\Delta_g a\right) = \sum_{n=2}^{\infty} (n-2)\,!^R\, \mathrm{p}^{n-2}\left(\Delta_g a_n\right)$$
$$\leq \sum_{n=2}^{\infty} (n-2)\,!^R \frac{n(n-1)}{2}\mathrm{p}^n(a_n) \leq \sum_{n=0}^{\infty} n\,!^R\, 2^n \mathrm{p}^n(a_n)\,,$$

which is the first estimate (115). For the second we have to be slightly more efficient with the estimates as a simple iteration of(115) would not suffice. We have for $|\mathrm{f}| \geq 1$

$$\mathrm{p}_R(\mathrm{e}^{\mathrm{t}\Delta_g} a) \;\leq\; \sum_{k=0}^{\infty} \frac{|\mathrm{r}|^k}{k!}\mathrm{p}_R\left(\Delta_g^k a\right)$$
$$\leq \; \sum_{n,k=0}^{\infty} \frac{|r|^{\mathrm{k}}}{k!}\mathrm{p}_R\left(\Delta_{\mathrm{g}}^k a_n\right)$$
$$\overset{(**)}{\leq} n,k=0 \sum_{n\geq 2k}^{\infty} \frac{|t|^k}{k!}(n-2k)!^R \frac{n!}{2^k(n-2k)!}\mathrm{p}^n(a_{1l})$$
$$= \; n,k=0 \sum_{n\geq 2k}^{\infty} \frac{|r|^k}{2^k k!}\left(\frac{n!}{(\uparrow 1-2k)!}\right)^{1-R} n!^R\, \mathrm{p}^n(a_n)$$
$$\overset{(a\,)}{\leq} n,k=0 \sum_{n\geq 2k}^{\infty} \frac{2^{2k(1-R)}|\mathrm{f}|^k}{2^k} \frac{1}{k!^{1-2(1-R)}} n!^R\, 2^{\mathrm{n}}\mathrm{p}^n(a_{\mathrm{n}})$$
$$\overset{(b)}{\leq} n,k=0 \sum_{n\geq 2k}^{\infty} \frac{1}{|r|^k k!^{2R-1}} n!^R\,(2|\mathrm{f}|)^n \mathrm{p}^{\mathrm{n}}(a_n)$$
$$\leq \left(\sum_{k=0}^{\infty} \frac{1}{|t|^{\mathrm{k}} k!^{2R-1}}\right)(2|t|\mathrm{p})_R(a_n)\,,$$

where in (a) we use $n! \leq 2^n(n-2k)!\,(2k)\,!$ and $(2k)\,! \leq k!^2\, 2^{2k}$, and in ($b$) we use the assumption $|\mathrm{f}| \geq 1$ as well as $k \leq 2k \leq n$ and $R \geq \frac{1}{2}$. If instead $|t| < 1$ then we proceed in (b) by

$$\mathrm{p}_R\left(\mathrm{e}^{\mathrm{f}\Delta_g} a\right) \leq \cdots \overset{(b')}{\leq} \left(\sum_{k=0}^{\infty} \frac{|r|^k}{k!^{2R-1}}\right)(2\mathrm{p})_R(a_n)\,.$$

In both cases the series over k converges as long as $R \;\geq \frac{1}{2}$ and yield constants c ′ as required for the estimate (116). The constant c can be taken as the largest of the numbers $2|t|$ and 2. We see that the idea of this estimate is rather similar to the one in Lemma (3.2.25). These estimates provide now the key to establish the equivalences also in the analytic framework:

Proposition (3.2.55) [101]: *Let* $R \geq \frac{1}{2}$ *and let* Λ, Λ' : $V \times V \to \mathrm{K}$ *be even continuous bilinearforms such that their antisymmetr c parts coincide. Then the Weyl algebras* $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ *and* $\widehat{\mathrm{W}}_R(V,\star_{z\Lambda'})$ *are isomorphic via the continuous equivalence rransformation*

$$\mathrm{e}^{z\Delta_g}(a \star_{z\Lambda} b) = (\mathrm{e}^{z\Delta_g}a) \star_{z\Lambda'} (\mathrm{e}^{z\Delta_g}b)\,, \qquad (117)$$

where $g = \Lambda' - \Lambda = \Lambda'_+ - \Lambda_+$ *and* $a, b \in \widehat{W}_R(V)$. .

Proof. First we note that the continuity of Λ and Λ' implies the continuity of g. Moreover, to test the continuity of the map $\mathrm{e}^{z\Delta_g}$ it clearly suffices to consider only those seminorms p_R of $\widehat{W}_R(V,\star_{z\Lambda})$ with p being a seminorm such that (114) holds. Thus we can apply Lemma (3.2.54) to conclude that $\mathrm{e}^{z\Delta_g}$ is continuous on $W_R(V,\star_{z\Lambda})$ and hence extends to a continuous endomorphism of $\widehat{W}_R(V,\star_{z\Lambda})$ as well. Then the relation (117) holds for all a, $b \in \mathrm{W}_R(V,\star_{z\Lambda})$ by Proposition (3.2.15) and hence for all 0, $b \in \widehat{\mathrm{W}}_R(V,\star_{z\Lambda})$ by continuity. Finally, for a fixed $a \in \widehat{W}_R(V,\star_{z\Lambda})$ the exponential series

$$\mathrm{e}^{t\Delta_g}a = \sum_{k=0}^{\infty} \frac{r^k}{k!} \Delta_g^k a$$

converges absolutely in the topology of $\mathrm{t}\widehat{Reject}_R(V,\star_{z\Lambda})$. Indeed, this follows from the estimate in the proof of Lemma (3.2.54). Thus for $z, w \in \mathbb{K}$ we get $\mathrm{e}^{z\Delta_g}\mathrm{o}\mathrm{e}^{w\Delta_g} = \mathrm{e}^{(z+w)\Delta_g}$ at once. This shows that $\mathrm{e}^{z\Delta_g}$ is indeed invertible and hence a continuous isomorphism with continuous inverse $\mathrm{e}^{-z\Delta_g}$.

In the finite-dimensional case the situation is very simple: first we note that there is only one Hausdorff locally convex topology on V and all bilinear maps are continuous. In this situation we get a defining system of continuous seminorms for the topology of $W_R(V)$ and $\mathrm{W}_{R-}(V)$ very easily:

Lemma (3.2.56) [101]: *Let* V *befinite-dimensional cm*d *let* p *be a norm on* V. *Then the nonns* $\{(c\mathrm{p})_R\}_{c>0}$ *yield a defining system of seminorms for* $\mathrm{W}_R(V)$.

Proof. Let q be an arbitrary seminorm on V. Then there is a constant $c > 0$ with $\mathrm{q} \leq c\mathrm{p}$ since p is a norm and we are in finite dimensions. Then we have $\mathrm{q}^n \leq c^n\mathrm{p}^\mathrm{n}$. Hence we also get $\mathrm{q}_R \leq (c\mathrm{p})_\mathbb{R}$ from which the claim follows.

Lemma (3.2.57) [101]: *Let* V *be finite-dimensional and let* p *be a norm on* V. *Then the norms* $\{\mathrm{p}_{R-\varepsilon}\}_{\varepsilon>0}$ *yield a defining system of seminorms for* $W_{R-}(V)$.

Proof. Here we do not even need the multiples of p. As before, for a given seminorm q on V there is a constant $c > 0$ with $\mathrm{q} \leq \mathrm{cp}$ and hence $\mathrm{q}^n \leq c^n\mathrm{p}^\mathrm{n}$. Now fix $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$ and let $\mathrm{C} > 0$ be a constant such that $c^{l1} \leq \mathrm{C}n!^{\varepsilon-\varepsilon'}$ for all $n \in \mathbb{N}$. Then we have

$$\mathrm{q}_{R-\varepsilon}(a) = \sum_{n=0}^{\infty} n\,!^{R-\varepsilon}\,\mathrm{q}^n(a_n) \leq \sum_{n=0}^{\infty} n\,!^{R-\varepsilon}\,c^\mathrm{n}\mathrm{p}^n(a_n) \leq \mathrm{C}\sum_{\mathrm{n}=0}^{\infty} n\,!^{R-\varepsilon'}\,\mathrm{p}^n(a_n)$$
$$= \mathrm{C}\mathrm{p}_{R-\varepsilon'}(a)\,.$$

This shows that we can estimate every seminorm of the form $\mathrm{q}_{R-\varepsilon}$ by a suitable $\mathrm{p}_{R-\varepsilon'}$.

Let $V = V_0 \oplus V_1$ be finite-dimensional and real. Moreover, let $\Lambda: V \times V \to \mathbb{R}$ be antisymmetric and even. Then $\Lambda = \Lambda_0 + \Lambda_1$ with

$$\Lambda_0: V_0 \times V_0 \to \mathbb{R} \; and \; \Lambda_1: V_1 \times V_1 \to \mathbb{R}, \qquad (118)$$

such that Λ_0 is an antisymmetric bilinear form on V_0 and Λ_1 is a symmetric bilinear form on V_1. By the linear Darboux Theorem we can find a basis q_1 , , q_d, p_1 , , p_d, c_1 , , c_k of V_0 such that the only nontrivial pairing is

$$\Lambda_0(\mathrm{q}_i, \mathrm{p}_j) = \delta_{ij} = -\Lambda_0(\mathrm{p}_j, \mathrm{q}_i)\,. \tag{119}$$

For the odd part, we can find a basis e_1 , , e_r , f_1 , , f_s, x_1 , , x_r with the only nontrivial pairings

$$\Lambda_1(e_i, e_j) = \delta_{ij} \; and \; \Lambda_1(f_\mathrm{i}, f_j) = -\delta_{ij}. \tag{120}$$

Here $\dim V_0 = 2d + k$ and $\dim V_1 = r + s + r$. Then Λ_0 is symplectic iff $k = 0$ and Λ_1 is an (indefinite) inner product iff f = 0, its signature is then given by (r, s) . If we use Λ directly for building the star product $\star_{z\Lambda}$ in this case then we obtain the usual Weyl-Moyal star product for the even part and a Clifford multiplication for the odd part. Thus the numbers $d, k, r, s,$ f encode the isomorphism class of $W_R(V, \star_{z\Lambda})$ as well as those of $W_{R-}(V, \star_{z\Lambda})$ in the finite-dimensional case. The complex case is analogous.

We now use this simple classification to compare our general construction with a previous construction in finite dimensions: first we want to relate our construction to the one of [108], where the Weyl-Moyal type star product was considered, i.e. no symmetric contribution to the symplectic antisymmetric part Λ.

In the approach of [108], the relevant topology on the complexified symmetric algebra $\mathrm{S}^{\cdot}(V_{\mathbb{R}}) \otimes \mathbb{C} = \mathbb{C}[z^1 , , z^d, \overline{z}^1 , , \overline{z}^d]$ over $V_{\mathbb{R}} = \mathbb{R}^{2d}$ is obtained as follows. For a parameter $0 < p \leq 2$ the topology is defined by the seminorms

$$\|a\|_{p,s} = \sup_{X \in \mathbb{C}^{2d}} \{|a(x)|\mathrm{e}^{-s|x|^p}\}, \tag{121}$$

where we denote by $x \in \mathbb{C}^{2d}$ a point in the complexified vector space and use the obvious extension of $a \in \mathrm{S}^{\cdot}(V_{\mathbb{R}}) \otimes \mathbb{C}$ as a function (polynomial) on $\mathbb{C}^{2d}$. Moreover, $|x|$ denotes the euclidean norm ofx. Then the locally convex topology used is the one determined by all the seminorms $\| \cdot \|_{p,s}$ for all $s > 0$. In fact, in [108] only the case of $d = 1$ is considered, but it is clear that everything can be done in higher (finite) dimensions as well. The following proposition clarifies the relation between the two approaches, the proof of which is implicitly contained already in [108].

Proposition (3.2.58) [101]: *Let* $0 < p \leq 2$. *Then the locally convex topology on* $\mathrm{S}^{\cdot}(V_{\mathrm{JR}}) \otimes \mathbb{C}$ *induced by the seminorms* $\| \cdot \|_{p,s}$ *for* $s > 0$ *coincides with the topology of* $\mathrm{S}_R^*(V_{\mathbb{R}}) \otimes \mathbb{C}$ *if we set* $R = \frac{1}{p}$.

Proof. We have to find mutual estimates for the two families of seminorms. We begin with some preparatory material. In view of Lemma (3.2.56) we are free to chose the following ℓ^1-like norm

$$\mathrm{p}(a) = \sum_\alpha |\, a_\alpha| \tag{122}$$

with respect to the canonical basis e_1 , , e_{2d}on $\mathbb{R}^{21} \otimes \mathbb{C}$. The reason to chose this P^1-norm is that it behaves most nicely for the tensor product. We write a polynomial as

$$a=\sum_{n=0}^{\infty}\sum_{\alpha_1,\dots,\alpha_n=1}^{2d} a_{\alpha_1\dots\alpha_n}\,x_{\alpha_1}\cdots x_{\alpha_n}, \tag{123}$$

where x_α are the coordinate functions in $x=\sum_\alpha x_\alpha\, e_\alpha$. The components $a_{\alpha_1\dots\alpha_n}$ are totally symmetric. For the seminorm p^n of the homogeneous part a_n of a of degree n we get

$$\mathrm{p}^n(a_n)=\sum_{\alpha_1,\dots,\alpha_n=1}^{2d}|\,a_{\alpha_1\dots\alpha_n}|. \tag{$*$}$$

For a homogeneous monomial we estimate the seminorm $\|\cdot\|_{p,s}$, yielding

$$\sup_{X\in\mathbb{C}^{2d}}\left\{||x_{\alpha_1}\cdots \mathrm{x}_{\alpha_n}|\mathrm{e}^{-s|x|^p}\right\}\le |x|^{tl}\mathrm{e}^{-s|x|^p}\le\left(\frac{n}{sp}\right)^{\frac{n}{p}}\mathrm{e}^{-\frac{n}{p}},. \tag{$**$}$$

by explicitly computing the maximum value of the scalar function $g(r)=r^n\mathrm{e}^{-sr^p}$ for $r\ge 0$, see also [108]. With this preparation and noting $n^n\le\frac{1}{\mathrm{e}}n!\,\mathrm{e}^n$ we get

$$\begin{aligned}\|a\|_{p,s}&\le\sum_{n=0}^{\infty}\sum_{\alpha_1,\dots,\alpha_n=0}^{2d}|\,a_{\alpha_1\dots\alpha_n}|\|x_{\alpha_1}\cdots x_{\alpha_n}\|_{p,s}^{(*),(**)}\\&\le\sum_{n=0}^{\infty}\mathrm{p}^n\,(a_n)\left(\frac{n}{sp}\right)^{\frac{n}{p}}\mathrm{e}^{-\frac{n}{p}}\le\sum_{n=0}^{\infty}\mathrm{p}^n\,(a_n)n!^{\frac{1}{p}}\,c^n\end{aligned}$$

where $c=\left(\frac{p}{s}\right)^{\frac{1}{p}}$. This gives the estimate $\|a\|_{p,s}\le(c\mathrm{p})_{\frac{1}{p}}(a)$. For the converse estimate we first note that we can apply the (multi-variable) Cauchy formula for the polynomial a. This gives an estimate for the Taylor coefficients following [108]: first we have for a fixed $r>0$ the estimate

$$\sum_{\alpha_1,\dots,\alpha_n=1}^{2d}|\,a_{\alpha_1\dots\alpha_n}|\le\frac{1}{r^n}\mathrm{e}^{sr^p}\|a\|_{p,s}.$$

Using the minimum value $(sp)^n n^{-\frac{n}{p}}\mathrm{e}^{\frac{n}{p}}$ of the scalar function $r^{-n}\mathrm{e}^{sr^p}$ this gives for a fixed $s>0$ and $R=\frac{1}{p}$ the estimate

$$\begin{aligned}(c\mathrm{p})_R(a)&=\sum_{n=0}^{\infty}\sum_{\alpha_1,\dots,\alpha_n=1}^{2d}|\,a_{\alpha_1\dots\alpha_n}|n!^R\,c^n\\&\le\sum_{n=0}^{\infty}n\,!^R\,\frac{1}{r^n}\mathrm{e}^{sr^p}\|a\|_{p,s}c^n\\&\le\sum_{n=0}^{\infty}n\,!^R\,(sp)^n n^{-\frac{n}{p}}\mathrm{e}^{\frac{\mathrm{n}}{p}}\|a\|_{p,s}c^n\\&\le\sum_{n=0}^{\infty}\mathrm{e}^R\,n^R n^{Rn}n^{-Rn}(sp)^n\mathrm{e}^{nR}\|a\|_{p,s}c^n\end{aligned}$$

$$= \sum_{n=0}^{\infty} \mathrm{e}^R \, n^R \left(spc\sqrt[p]{E}\right)^n \|a\|_{p,s}.$$

Now choosing $s > 0$ sufficiently small such that $spc\sqrt[p]{\mathrm{e}} < 1$ gives a converging series over n and hence a suitable constant $c' > 0$ with $(c\mathrm{p})_R(a) \leq c'\|a\|_{p,s}$ for $R = \frac{1}{p}$ and the above chosen s. Thus the two topologies coincide.

The second situation which we shall relate our general Weyl algebra $W_R(V)$ to is the convergent Wick star product as in [73], [109]. Here we are again in the real symplectic situation with $V_{\mathbb{R}} = \mathbb{R}^{2d}$ and its canonical symplectic form. Using the same notation as above, the Wick star product

$$f \star_{\mathrm{Wick}} g = \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{|N|} f}{\partial z^N} \frac{\partial^{|N|} g}{\partial \overline{z}^N} \tag{124}$$

from [73], [109] can then be written as $\star_{\mathrm{Wick}} = \star_{\frac{\mathrm{i}\hbar}{2}\Lambda}$ with Λ given by

$$\Lambda\left(z^k, \overline{z}^{\ell}\right) = \frac{4}{\mathrm{i}} \delta_{kp}, \tag{125}$$

and all other pairings trivial. In [109] it was shown that the previously constructed locally convex topology for the Wick star product from [73] can be described as follows: write $0 \in \mathrm{S}^{\cdot}(V)$ as a Taylor polynomial

$$a = \sum_{.IJ=0}^{\infty} a_{IJ} \frac{z^{\mathrm{I}} \, z^{-J}}{\mathrm{I}!J!}. \tag{126}$$

Then the defining system of seminorms is given by

$$\|a\|_{\varepsilon} = \sup_{I.J} \frac{|a_{\mathrm{I}J}|}{|\mathrm{I}+J|!^{\varepsilon}}, \tag{127}$$

where $\varepsilon > 0$. With other words, the Taylor coefficients a_J have *sub-factorial* growth with respect to the multiindices I and J. Note that in [109] an addition factor $(2\hbar)^{|\mathrm{f}|+\mathrm{N}|}$ is present in the denominator in (126). But clearly such an exponential contribution will not change the sub-factorial growth properties at all. Therefore the seminorms (127) give the same topology as the one in [109].

Proposition (3.2.59) [101]: *The locally convex topology on* $\mathrm{S}^{\cdot}(V)$ *induced by rhe seminorms* $\{\|\cdot\|_{\varepsilon}\}_{\varepsilon>0}$ *coincides with the topology of the Weyl algebra* $\mathrm{W}_{R-}(V)$ *for* $R = 1$.

Proof. To get the combinatorics of the Taylor and tensor coefficients right, we note that for a homogeneous $a_n \in \mathrm{S}^n(V) \subseteq \mathrm{T}^n(V)$ written as in (123) we have

$$\frac{a_{\mathrm{f}J}}{I!J!} = \sum_{\alpha \in (\mathrm{f}j)} a_{\alpha_1 \ldots \alpha_n},$$

where the summation runs over all those n-tuples $\alpha = (\alpha_1, , \alpha_n)$ containing i_1 times the index 1, , i_d times the index d, j_1 times the index $\overline{1}$, , and j_d times the index $\overline{d}$. Since the coefficients are totally symmetric we get

$$\frac{|a_{1J}|}{\mathrm{I}!J!} = \sum_{\alpha \in (\mathrm{J}J)} |\, a_{\alpha_1 \dots \alpha_n}|. \qquad (*)$$

The first estimate we need is now

$$\|a\|_\varepsilon = \sup_{I.J} \frac{|a_{\mathrm{J}J}|}{|I+J|!^\varepsilon}$$

$$= \sup_n \frac{1}{n!^\varepsilon} += \sup_{\substack{I.J \\ |I+j|=n}} \sum_{\alpha \in (\mathrm{f}J)} |\, a_{\alpha_1 \dots \alpha_n}| \mathrm{I}! J!$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!^\varepsilon} \sum_{\alpha_1, \dots, \alpha_n} |\, a_{\alpha_1 \dots \alpha_n}| n!$$

$$= \mathrm{p}_{1-\varepsilon}(a)\,,$$

where p is again the norm (122). For the other direction we take the same norm p and choose again a $0 < \varepsilon' < \varepsilon$. Then

$$\mathrm{p}_{1-\varepsilon}(a) = \sum_{n=0}^{\infty} n\,!^{1-\varepsilon} \sum_{\alpha_1, \dots, \alpha_\mathrm{n}} |\, a_{\alpha_1 \dots \alpha_n}|$$

$$= \sum_{n=0}^{\infty} n\,!^{1-\varepsilon} \sum_{|\mathrm{J}+J\mathrm{I}=n} \frac{|a_{\mathrm{J}J}|}{IvJ!}$$

$$\leq \sum_{n=0}^{\infty} n\,!^{1-\varepsilon} \sum_{|1+J\mathrm{I}=n} \frac{\|a\|_{\varepsilon'} n!^{\varepsilon'}}{\mathrm{I}!J!}$$

$$= \|a\|_{\varepsilon'} \sum_{n=0}^{\infty} \frac{(2d)^{\mathrm{t}l}}{n!^{\varepsilon - \varepsilon}},$$

which gives an estimate $\mathrm{p}_{R-\varepsilon}(a) \leq c\|a\|_{\varepsilon'}$ with c being the above convergent series. Since by Lemma (3.2.57) it suffices to consider one norm p for V, this finishes the proof.

We discuss a first example in infinite dimensions: the Poisson bracket and the corresponding Weyl algebra underlying a free, i.e. linear, field theory. Our main focus here is the precise definition of the relevant locally convex topologies as well as the global aspects of the construction. One has essentially two possibilities for the Poisson structure: the canonical Poisson structure built on a Hamiltonian formulation using the initial value problem and the covariant Poisson structure, also called the Peierls bracket, built on a Lagrangian approach. In the following, we will exclusively work in the smooth category, all manifolds and bundles will be $\mathcal{C}^\infty$.

The material on the Cauchy problem on globally hyperbolic spacetimes is standard and can be found e.g. in [120]. The comparison of the canonical and the covariant Poisson brackets is was taken from [121]. For a much more far-reaching discussion of the Peierls bracket including also non-linear field equations see [110]: in fact, it would be a very interesting project to combine the results from [110] on the classical side with the nuclear Weyl algebra quantization to obtain the corresponding quantum side.

We consider an n-dimensional connected Lorentz manifold (M, g) with a Lorentz metric g of signature $(+,-,,-)$. The important concept we need is the causal structure: first we require that (M, g) is time-orientable and time-oriented. Then one defines the causal future $J_M^+(p)$ of a point $p \in M$ to be the set of those points which can be reached by a future directed causal curve. Analogously, $J_M^-(p)$ denotes the causal past of p. For two points p, $q \in M$ one defines the diamond $JM\ (p, q) = J_M^+(p) \mathrm{n} J_M^-(q)$. For an arbitrary subset $A \subseteq M$ we set

$$J_M^{\pm}(A) = \bigcup_{p \in A} \quad J_M^{\pm}(A)\ and\ J_M(A) = J_M^+(A) \mathrm{U} J_M^-(A)\ . \tag{128}$$

A time-oriented Lorentz manifold is called *globally hyperbolic* if it is causal, i.e. there are no closed causal loops, and if all diamonds are compact. A first consequence is that$J_M^{\pm}(p)$ is always a closed subset of M. In the following we always assume that (M, g) is globally hyperbolic. A celebrated theorem of Bernal and Sánchez, refining a topological statement of Geroch, states that this is equivalent to the existence of a smooth spacelike Cauchy surface

$$\iota: \Sigma \to M \tag{129}$$

together with a smooth Cauchy temporal function $\mathrm{f} \in \mathcal{C}^\infty(M)$, i.e. the gradient of f is future directed and time-like everywhere and the level sets of f are smooth spacelike Cauchy surfaces for all times. Moreover, M is diffeomorphic to the product manifold $\Gamma \mathrm{t} \times \Sigma$ with the metric being

$$g = \beta \mathrm{d}t^2 - g_t\ , \tag{130}$$

where $\beta \in \mathcal{C}^\infty(\Gamma \mathrm{t} \times \Sigma)$ is positive and g_{f} is a Riemannian metric on Σ smoothly depending on r. Finally, the Cauchy temporal function f can be chosen in such a way that Σ is the $\mathrm{f} = 0$ level set. For a detailed discussion see the review [122].

Since M is diffeomorphic to]$\mathbb{R} \times \Sigma$ we get a global vector field $\frac{\partial}{\partial t}$ on M. Normalizing this to a unit vector field gives

$$\mathrm{n} = \frac{1}{\sqrt{\beta}} \frac{\partial}{\partial r} \in \Gamma^\infty(TM)\ , \tag{131}$$

which is a future-directed time-like unit vector field such that it is normal to every level surface of the Cauchy temporal function. In particular $l^\# \mathrm{tt} \in \Gamma^\infty(TM_\Sigma)$ will be normal to the Cauchy surface Σ. Here $TM_\Sigma = \iota^\# TM$ is the restriction (pullback via ι) of the tangent bundle to Σ and $\iota^\# \mathrm{n} = n|_\Sigma$ is the pull-back of a to Σ.

The fields we are interested in will be modeled by of a vector bundle over M: we require the vector bundle $E \to M$ to be real and equipped with a fiber metric h, not necessarily positive definite but non-degenerate. The dynamics of the field is now governed by a second order differential operator $D \in \mathrm{DiffOp}^2(E)$ with the following property: there is a metric connection ∇^E for (E, h) and a zeroth order differential operator $B \in \mathrm{DiffOp}^0(E) = \Gamma^\infty\big(\mathrm{End}(E)\big)$ such that

$$D = \square^\nabla + B, \tag{132}$$

where $\square^\nabla$ denotes the d'Alembert operator obtained from ∇^E and pairing with the metric g. In particular, D is *normally hyperbolic*. Conversely, note that for a normally hyperbolic differential operator there is a unique connection ∇^E and a unique $B \in \Gamma^\infty\big(\mathrm{End}(E)\big)$ such that (132) holds, see e.g. [120]. Thus the only additional requirement we need is that ∇^E is also metric with respect to h.

Let $\mu_g \in \Gamma^\infty(|\Lambda^n|T^*M)$ be the canonical metric density induced by g which we shall use for various integrations over M. First we can use μ_g to define the *rranspose* of a differential operator $D \in DiffOp^*(E)$ to be the unique differential operator $D^{\mathrm{T}} \in DiffOp^*(E^*)$ such that

$$\int_M (D^{\mathrm{T}}\phi)\cdot u\mu_g = \int_M \phi\,.(Du)\,\mu_g \tag{133}$$

for all $\phi \in \Gamma^\infty(E^*)$ and $u \in \Gamma^\infty(E)$, at least one of them having compact support. Here. means pointwise natural pairing.

Note that D^{T} depends on the choice of μ_g and has the same order as D. Taking into account also the fiber metric h we can define the *adjoint* of D to be the unique differential operator $D^* \in \mathrm{DiffOp}\cdot(E)$, again of the same order as D, such that

$$\int_M h\,(D^*u, v)\mu_g = \int_M h\,(u, Dv)\mu_g \tag{134}$$

for u, $v \in \Gamma^\infty(E)$, at least one of them having compact support. Denoting the musical isomorphisms induced by h by $\#: E^* \to E$ and $\mathrm{b}: E \to E^*$ as usual, we get $D^*u = (D^{\mathrm{T}}u^{\triangleright})^\#$ for $\in \Gamma^\infty(E)$.

This allows us now to formulate the last requirement on $D = \Box^\nabla + B$, namely we need D to be a *symmetr c* operator, i.e.

$$D^* = D. \tag{135}$$

Since the connection ∇^E is required to be metric, it is easy to see that (135) is equivalent to $B^* = B$.

Let D be a normally hyperbolic differential operator as before. Then the *wave equation* we are interested in is simply given by

$$Du = 0 \tag{136}$$

for a u of E. Depending on the regularity of u we can interpret (136) as a pointwise equation or as an equation in a distributional sense. Dualizing, we have the corresponding wave equation

$$D^{\mathrm{T}}\phi = 0 \tag{137}$$

for a ϕ of the dual bundle E^*.

Under our general assumption that (M, g) is globally hyperbolic we have the existence and uniqueness of advanced and retarded *Green operators*

$$G_M^\pm : \Gamma_0^\infty(E) \to \Gamma^\infty(E) \tag{138}$$

for D. This means that there are unique, linear, and continuous maps $G_M^\pm$ such that

$$DG_M^\pm = \mathrm{id}_{\Gamma_0^\infty(E)} = G_M^\pm D|_{\Gamma_0^\infty(E)} \tag{139}$$

and

$$supp\ G_M^\pm \subseteq J_M^\pm(supp\ u) \tag{140}$$

for all $u \in \Gamma_0^\infty(E)$. The continuity refers to the usual LF and Fréchet topologies of $\Gamma_0^\infty(E)$ and $\Gamma^\infty(E)$, respectively. Using the volume density μ_g we can identify the distributional $\Gamma^{-\infty}(E^*)$ with the dual $\Gamma_0^\infty(E)'$ and $\Gamma_0^{-\infty}(E^*)$ becomes identified with the dual $\Gamma^\infty(E)'$.

We need the following space: Let $K \subseteq M$ be compact. Then denote by $\Gamma^{\infty}_{J_{M(K)}}(E)$ those in $\Gamma^{\infty}(E)$ with $supp\ u \subseteq J_M(K)$. Since on a globally hyperbolic spacetime $J_M(K)$ is a closed subset, $\Gamma^{\infty}_{J_{M(K)}}(E) \subseteq \Gamma^{\infty}(E)$ is a closed subspace and hence a Fréchet space itself. For $K \subseteq K'$ we have $\Gamma^{\infty}_{J_{M(K)}}(E) \subseteq \Gamma^{\infty}_{J_{M(K)'}}(E)$ and the thereby induced topology on $\Gamma^{\infty}_{J_{M(K)}}(E)$ coincides with the original. Hence we can consider the inductive limit

$$\Gamma^{\infty}_{\mathrm{sc}}(E) = \bigcup_{\substack{k\subseteq M \\ K\text{compact}}} \Gamma^{\infty}_{J_{M(K)}}(E) \tag{141}$$

of those smooth of E which have compact support in spacelike directions. It is a strict inductive limit, and since we can exhaust M with a sequence of compact subsets, it is a countable strict inductive limit, endowing $\Gamma^{\infty}_{\mathrm{sc}}(E)$ with a LF topology. The continuity statement (138) can then be sharpened to the statement that

$$G_M^{\pm}\colon \Gamma_0^{\infty}(E) \to \Gamma^{\infty}_{\mathrm{sc}}(E) \tag{142}$$

is continuous. In fact, this follows in a straightforward manner from the continuity of (138) and the causality condition (140).

We consider now the *propagator* which is defined by

$$G_M = G_M^{+} - G_M^{-}\colon \Gamma_0^{\infty}(E) \to \Gamma^{\infty}_{\mathrm{sc}}(E)\,, \tag{143}$$

for which one has the following crucial properties: the sequence

$$0 \to \Gamma_0^{\infty}(E) \to^{D} \Gamma_0^{\infty}(E) \to^{G_M} \Gamma^{\infty}_{\mathrm{sc}}(E) \to^{D} \Gamma^{\infty}_{\mathrm{sc}}(E) \tag{144}$$

of continuous linear maps is *exact*. Note that the exactness relies crucially on the assumption that (M, g) is globally hyperbolic.

The Green operators can now be used to give a solution to the Cauchy problem of the wave equation (136). For the time $r = 0$ level surface Σ we want to specify initial conditions u_0, $u_0 \in \Gamma_0^{\infty}(E_\Sigma)$. Then we want to find a section $u \in \Gamma^{\infty}(E)$ with

$$Du = 0, \iota^{\#}u = u_0, and\ \iota^{\#}\nabla^{E}_{\mathrm{t}\downarrow}u = u_0. \tag{145}$$

A core result in the globally hyperbolic case is that this is indeed a well-posed Cauchy problem: for any (u_0, u_0) we have a unique solution u of the Cauchy problem (145) such that the map

$$\Gamma_0^{\infty}(E_\Sigma) \oplus \Gamma_0^{\infty}(E_\Sigma) \ni (u_0, \dot{u}_0) \mapsto u \in \Gamma^{\infty}_{\mathrm{sc}}(E) \tag{146}$$

is continuous and $supp\ u \subseteq J_M(supp\ u_0 supp \sqcup \dot{u}_0)$. Moreover, this solution u can be characterized by the formula

$$\int_M \phi \cdot u \mu_g = \int_\Sigma \left(\iota^{?\mathrm{f}}\left(\nabla^{E}_{\mathrm{tt}} F_M(\phi)\right) \cdot u_0 - \iota^{\#}\left(F_M(\phi)\right) \cdot \dot{u}_0\right) \mu_\Sigma, \tag{147}$$

where F_M is the propagator of D^{T} and $\phi \in \Gamma_0^{\infty}(E^*)$. The density $\mu\Sigma$ is the one induced by μ_g.

Since we assume that $D^* = D$ we have a last property of the Green operators, namely

$$\left(G_M^{\pm}\right)^* = G_M^{\mp}\ and\ G_M^* = -G_M. \tag{148}$$

This antisymmetry of the propagator will play a crucial role in the definition of the covariant Poisson bracket. Moreover, for all $\phi \in \Gamma_0^{\infty}(E^*)$ we have

$$G_M^{\pm}(\phi^{\#}) = \left(F_M^{\pm}(\phi)\right)^{\#} \tag{149}$$

The results and their proofs as well as many more additional features of the Cauchy problem of the wave equation on a globally hyperbolic spacetime can be found in [120], [121].

The canonical i.e. Hamiltonian approach uses an algebra of functions on the initial data, which constitute the classical phase space

$$\mathcal{P}_\Sigma) = \Gamma_0^\infty(E_\Sigma) \oplus \Gamma_0^\infty(E_\Sigma)\,. \tag{150}$$

We view $'y_\Sigma$ as symplectic vector space via the symplectic form

$$\omega_\Sigma\big((u_0, \dot{u}_0), (v_0, \dot{v}_0)\big) = \int_\Sigma \big(h_\Sigma(u_0, \dot{v}_0) - h_\Sigma(\dot{u}_0, v_0)\big)\,\mu_\Sigma, \tag{151}$$

where h_Σ is the restriction of h to $E|_\Sigma$. We have the following basic result:

Lemma (3.2.60) [101]: *The two-form ω_Σ on $\mathcal{P}_\Sigma^2$ is antisymmetr c, non-degenerate, and continuous.*

Proof. The non-degeneracy and the antisymmetry are clear. For the continuity we can rely on several standard arguments: first we note that every vector bundle can be written as a subbundle of a suitable trivial vector bundle $\Sigma\ \times]\mathbb{R}^N$. This gives an identification $\Gamma_0^\infty(E_\Sigma) \subseteq \Gamma_0^\infty(\Sigma \times \mathbb{R}^N)$ as a *closed embedded* subspace. We can extend h_Σ in some way to a smooth fiber metric on the trivial bundle and this way, ω_Σ is just the restriction of the corresponding symplectic form on $\Gamma^\infty(\Sigma \times \Gamma \mathrm{t}^N)$. Thus it suffices to consider a trivial bundle from the beginning. There we have $\Gamma_0^\infty(\Sigma \times \Gamma \mathrm{t}^N) \cong \mathcal{C}_0^\infty(\Sigma)^N$. Thus we have to show the continuity of a bilinear map of the form

$$\mathcal{C}_0^\infty(\Sigma)^N \times \mathcal{C}_0^\infty(\Sigma)^N \ni \big((u_i), (v_i)\big) \mapsto \sum_{i_j=1}^{N} u_i\, H_{ij} v_j \in \mathcal{C}_0^\infty(\Sigma)\,, \tag{$*$}$$

where $H_{ij} \in \mathcal{C}^\infty(\Sigma)$. But since the multiplication of compactly supported smooth functions is continuous (and not just separately continuous) the continuity of $(*)$ follows. The final integration needed for (151) is continuous as well. □

It is obvious that ω_Σ is separately continuous, however, we are interested in continuity. In the case where Σ is compact, this would follow directly from separate continuity as then $\Gamma_0^\infty(E_\Sigma) = \Gamma^\infty(E_\Sigma)$ is a Fréchet space. The case of a non-compact Σ is of interest, too.

We look at certain polynomial functions on $\mathcal{P}_\Sigma$ and endow them with the Poisson bracket originating from $\mathcal{P}'_\Sigma$. It turns out that the symmetric algebra over the dual $\mathcal{P}_E$ will be too big and problematic when it comes to the comparison with the covariant Poisson structure. Hence we decide here for a rather small piece of all polynomials, namely for the symmetric algebra over

$$V_\Sigma = \Gamma_0^\infty(E_\Sigma^*) \oplus \Gamma_0^\infty(E_\Sigma^*)\,. \tag{152}$$

Using the density μ_Σ we can indeed pair elements from V_Σ with points in: $\mathcal{P}_\Sigma$:

Lemma (3.2.61) [101]: *The integration*

$$(\phi_0, \dot{\phi}_0)(u_0, \dot{u}_0) = \int_\Sigma \big(\phi_0 \cdot u_0 + \dot{\phi}_0 \cdot \dot{u}_0\big)\,\mu_\Sigma \tag{153}$$

provides a continuous bilinear pairing beDweeV_Σ *and* P_Σ.

The proof of the continuity is analogous to the one in Lemma (3.2.60). In particular, we can view points in P_Σ as elements of the dual of V_Σ and vice versa.

Lemma (3.2.62) [101]: *The symplectic form ω_Σ induces a non-degenerate antisymmetr c continuous bilinear form*

$$\Lambda_\Sigma : V_\Sigma \times V_\Sigma \to \mathbb{R} \tag{154}$$

explicitly given by

$$\Lambda_\Sigma\left((\phi_0, \dot\phi_0), (\psi_0, \dot\psi_0)\right) = \int_\Sigma \left(h_\Sigma^{-1}(\phi_0, \dot\psi_0) - h_\Sigma^{-1}(\dot\phi_0, \psi_0)\right)\mu\Sigma. \tag{155}$$

Here h_Σ^{-1} stands for the induced fiber metric on E_Σ^* and the Poisson bracket is determined by ω_Σ in the sense that the Hamiltonian vector field of the linear function $(\phi_0, \dot\phi_0)$ on 'J) Σ is determined via ω_Σ and the Poisson bracket is determined by the Hamiltonian vector field as usual.

We can now use the Poisson bracket $\{.\,,\}_{\Lambda_\Sigma}$ for the symmetric algebra $\mathrm{S}^{\cdot}(V_\Sigma)$ as described together with its quantization given by the star product $\star\,\Sigma = \star_{\frac{i\hbar}{2}\Lambda_\Sigma}$ for the corresponding nuclear Weyl algebra $\mathrm{W}_R(V_\Sigma \otimes (\mathrm{D})$. This will be the canonically quantized model of our Hamiltonian picture of the field theory. Since we started with a real vector bundle, the resulting Weyl algebra carries the complex conjugation a s^*-involution. We only described the kinematic part, the field equation did not yet enter at all.

As the covariant "phase space" we take simply all possible field configurations on the spacetime, i.e.

$$\wp_{\text{ cov}}) = \Gamma_{\text{sc}}^\infty(E)\,, \tag{156}$$

whether or not they satisfy the wave equation. This will not be a symplectic vector space in any reasonable sense as $\wp_{\text{ cov}})$ contains all the unwanted field configurations as well. This is perhaps the surprising observation, the symmetric algebra over its dual allows for a Poisson bracket: again, we take only a small part of the dual, namely $\Gamma_0^\infty(E^*)$, where we evaluate $\phi \in \Gamma_0^\infty(E^*)$ on $u \in \Gamma_{\text{sc}}^\infty(E)$ by means ofthe integration with respect to μ_g as usual. As before, we denote this integration simply by (u) .

The Poisson bracket will then be determined by a bilinear form on $\Gamma_0^\infty(E^*)$ as before. Using the propagator F_M of D^{T} we define

$$\Lambda_{\text{cov}}(\phi, \psi) = \int_M h^{-1}\,(F_M(\phi), \psi)\mu_g. \tag{157}$$

Note that the compact support of ψ makes this integration well-defined. We have the following property:

Lemma (3.2.63) [101]: *The bilinear form $\Lambda_{\text{cov}} : \Gamma_0^\infty(E^*) \times \Gamma_0^\infty(E^*) \to$ R is antisymmetric and continuous.*

Proof. The antisymmetry is clear since D^{T} is symmetric and hence (148) applies also to F_M. The continuity is slightly more involved: first we note that $F_M : \Gamma_0^\infty(E^*) \to \Gamma^\infty(E^*)$ is continuous by the continuity of the Green operators $F_M^{\pm}$. Next, we use the fact that the inclusion $\Gamma^\infty(E^*) \to \Gamma_0^\infty(E^*)'$ given by the integration with respect to μ_g using h^{-1} is also continuous where we equip the dual $\Gamma_0^\infty(E^*)'$ with the *strong* topology. This shows that the corresponding "musical" homomorphism

$$\#_{\text{cov}} : \Gamma_0^\infty(E^*) \ni \phi \mapsto \Lambda_{\text{cov}}(\phi, \cdot) \in \Gamma_0^\infty(E^*)'$$

is continuous with respect to the LF and the strong topology, respectively. Hence the Kernel Theorem for the nuclear space $\Gamma_0^\infty(E^*)$ states that $\Lambda_{\text{cov}}(.\,,)$ is a distribution on the Cartesian product, or, equivalently, a continuous bilinear map, see e.g. [117].

Definition (3.2.64)[101]: The covariant Poisson algebra for D is the symmetric algebra $S^{\cdot}(V_{\text{cov}})$, where $V_{\text{cov}} = \Gamma_0^\infty(E^*)$, with the constant Poisson bracket $\{.\,,\}_{\text{cov}}$ coming from Λ_{cov}.

This is indeed a Poisson algebra with a continuous Poisson bracket ifwe endow it with one of the topologies discussed see Proposition (3.2.24). A first and heuristic appearance of this Poisson bracket in a very particular case seems to be [123].

From our general theory we know that the corresponding covenant *Weyl algebra* $W_R(V_{\text{cov}} \otimes \mathbb{C}, \star_{\text{cov}})$ with the *covariant star product* $\star_{\text{cov}} = \star_{\frac{i\hbar}{2}\Lambda_{\text{cov}}}$ is a nuclear $*$-algebra with respect to the complex conjugation, where as usual $R \geq \frac{1}{2}$.

We note that Λ_{cov} is now degenerate. We can determine its degeneracy space explicitly [121]:

Lemma (3.2.65) [101]: *Let* $\in \Gamma_0^\infty(E^*)$. *Then the following statements are equivalent*: (i) ϕ *is a Casimir element of* $S^{\cdot}(V_{\text{cov}})$, *i.e.* $\{\phi,\cdot\}_{\text{cov}} = 0$.

(ii) ϕ *vanishes on solutions* $u \in \Gamma_{\text{sc}}^\infty(E)$, *i.e. we hove*

$$\int_M \phi \cdot u\mu_g = 0 \textit{ whenever } Du = 0. \tag{158}$$

(iii) $\phi \in \ker F_M$.

Proof. Assume $\{\phi,\cdot\}_{\text{cov}} = 0$ then we have $0 = \{\phi,\psi\}_{\text{cov}} = \int_M h^{-1}(F_M(\phi),\psi)\mu_g$ for all $\in \Gamma_0^\infty(E^*)$. Since h^{-1} is nondegenerate this implies $F_M(\phi) = 0$. Next, assume $F_M(\phi) = 0$. Then we icnow $\phi = D^{\mathrm{T}}\chi$ for some $X \in \Gamma_0^\infty(E^*)$ by (144) applied to D^{T}. Thus (158) follows by the definition of D^{T} as in (133). Finally, assume (158) and let $\psi \in \Gamma_0^\infty(E^*)$ be arbitraiv. Then $\left(F_M(\psi)\right)^{\#} = G_M(\psi^{\mathrm{Q}})$ by (149) and it solves the wave equation $DG_M(\psi^{\mathrm{Q}}) = 0$ by (144). Thus $\{\phi,\psi\}_{\text{cov}} = 0$ follows. Since $S^{\cdot}(V_{\text{cov}})$ is generated by V_{cov} and $\{\phi,\cdot\}_{\text{cov}}$ is a derivation, $\{\phi,\cdot\}_{\text{cov}} = 0$ follows. □

Since the elements of $\ker F_M \subseteq \Gamma_0^\infty(E^*)$ are Casimir elements, the ideal they generate inside $S^{\cdot}(V_{\text{cov}})$ is a Poisson ideal. It turns out that it is even a two-sided ideal with respect to $\star_{\text{cov}}$:

Lemma (3.2.66) [101]: *Let* $\langle \ker F_M) \subseteq S^{\cdot}(V_{\text{cov}})$ *be the ideal generated by* $\ker F_M$ *with respect to the symmetr c tensor product. Then we have*:

(i) $\{\ker F_M\rangle$ *is a Poisson ideal with respect to* $\{.\,,\}_{\text{cov}}$.

(ii) $\{\ker F_M\rangle \otimes \mathbb{C} \subseteq W_R(V_{\text{cov}} \otimes \mathbb{C})$ *is a***-ideal for* $\star_{\text{cov}}$, *in fact generated by* $\ker F_M$.

Proof. The first part is clear by Lemma (3.2.65)(i). Now let $\Phi \in W_R(V_{\text{cov}} \otimes \mathbb{C})$ be an arbitrary tensor and let $\in \Gamma_0^\infty(E^*)$. Then we have

$$\Phi \star_{\text{cov}} \phi = \Phi\phi + \frac{i\hbar}{2}\{\Phi,\phi\}_{\text{cov}} \textit{ and } \phi \star_{\text{cov}} \Phi = \Phi\phi + \frac{i\hbar}{2}\{\phi,\Phi\}_{\text{cov}}\,,$$

since ϕ has tensor degree 1 and hence the higher order contributions in $\star_{\text{cov}}$ all vanish. Thus for $\phi \in 1 < \text{er}F_M$ we get $\Phi \star_{\text{cov}} \phi = \Phi\phi = \phi \star_{\text{cov}} \Phi$. But this shows that

$$\{\ker F_M) \otimes \mathbb{C} = W_R(V_{\text{cov}} \otimes \mathbb{C}) \star_{\text{cov}} \ker F_M \star_{\text{cov}} W_R(V_{\text{COV}} \otimes \mathbb{C})\,.$$

Since $\ker F_M$ consists of real sections, it is clear that $\langle \ker F_M\rangle \otimes \mathbb{C}$ is a*-ideal.

We relate the two Poisson algebras $S^{\cdot}(V_{\Sigma})$ and $S^{\cdot}(V_{cov})$. In view of (147) it is reasonable to relate a section $\phi \in \Gamma_0^{\infty}(E^*)$ to sections $\phi_0, \dot{\phi}_0 \in \Gamma_0^{\infty}(E_{\Sigma}^*)$ by defining

$$\phi_0 = \iota^{\#}\left(\nabla_{\mathrm{n}}^{E^*} F_M(\phi)\right) \textit{ and } \dot{\phi}_0 = -\iota^{\#}\left(F_M(\phi)\right), \qquad (159)$$

thereby defining a linear map

$$\varrho_{\Sigma}: \Gamma_0^{\infty}(E^*) \ni \phi \mapsto \left(\phi_0, \dot{\phi}_0\right) \in \Gamma_0^{\infty}(E_{\Sigma}^*) \oplus \Gamma_0^{\infty}(E_{\Sigma}^*), \qquad (160)$$

with $\phi_0, \dot{\phi}_0$ given as in (159). This map Q_{Σ} has the following property:

Lemma (3.2.67) [101]: *The map Q_{Σ} is conrinuous and for all solurions $u \in \Gamma_{sc}^{\infty}(E)$ of rhe wave equarion wirh inirial condirions $u_0, \dot{u}_0$ we have*

$$\phi(u) = \varrho_{\Sigma}(\phi)(u_0, \dot{u}_0). \qquad (161)$$

Proof. The propagator F_M gives a continuous map into $\Gamma_{sc}^{\infty}(E^*)$ and the covariant derivative is clearly continuous, too, mapping $\Gamma_{sc}^{\infty}(E^*)$ into $\Gamma_{sc}^{\infty}(E^*)$. For every compact subset $K \subseteq \Sigma$ the restriction

$$l^{\#}: \Gamma_{J_{M(K)}}^{\infty}(E^*) \to \Gamma_K^{\infty}(E_{\Sigma}^*)$$

is a continuous map between Fréchet spaces. But then also $l^{\#}: \Gamma_{sc}^{\infty}(E^*) \to \Gamma_0^{\infty}(E_{\Sigma}^*)$ is continuous by the universal property of LF topologies. This shows the continuity of Q_{Σ}, the equality in (161) isjust (147). □

Since $S^{\cdot}(V_{cov})$ is freely generated by V_{cov} we get a unique unital algebra homomorphism extending Q_{Σ} which we still denote by the same symbol

$$\varrho_{\Sigma}: S^{\cdot}(V_{cov}) \to S^{\cdot}(V_{\Sigma}). \qquad (162)$$

Since (160) is continuous, also (162) is continuous as a linear map from $S_R^*(V_{cov})$ to $S_R(V_{\Sigma})$ by Lemma (3.2.48). Moreover, we have

$$\Phi(u) = \varrho_{\Sigma}(\Phi)(u_0, \dot{u}_0) \qquad (163)$$

for all $\Phi \in S^{\cdot}(V_{cov})$ and all solutions $u \in \Gamma_{sc}^{\infty}(E)$ of the wave equation $Du = 0$ with initial conditions $(u_0, \dot{u}_0)$. This is clear since evaluation of an element in the symmetric algebra on a point is a homomorphism and Q_{Σ} is a homomorphism as well. Since we only have to check the equality of two homomorphisms on generators, (161) is all we need to conclude (163).

Lemma (3.2.68) [101]: *The algebra homomorphism Q_{Σ} is a Poisson morphism as well as a continuous∗-algebra homomorphism*

$$\varrho_{\Sigma}: W_R(V_{cov} \otimes \mathbb{C}, \star_{cov}) \to W_R(V_{\Sigma} \otimes \mathbb{C}, \star\, \Sigma). \qquad (164)$$

Proof. Thanks to Propositions (3.2.12) and (3.2.49) we only have to show that (160) is a Poisson map. Thus let $\phi,\ \psi \in \Gamma_0^{\infty}(E^*)$ be given and let $\left(\phi_0, \dot{\phi}_0\right) = \varrho_{\Sigma}(\phi)$ as well as $\left(\psi_0, \dot{\psi}_0\right) = \varrho_{\Sigma}(\psi)$ be their images in V_{Σ} under Q_{Σ}. Consider now $u = (F_M\psi)^{\#} = G_M(\psi^{\#}) \in \Gamma_{sc}^{\infty}(E)$ which is a solution of the wave equation $Du = 0$ with initial conditions $u_0 = \iota^{\#}u = \iota^{\#}\left(F_M(\psi)\right)^{\#}$ and $\dot{u}_0 = \iota^{\#}\left(\nabla_{t\downarrow}^{E} u\right) = \iota^{\#}\left(\nabla_{tt}^{E^*} F_M(\psi)\right)^{\#}$. Here we use that ∇^E is metric and hence compatible with the musical isomorphism # induced by h. Now we have

$$\Lambda_{\Sigma}\left(\left(\phi_0, \dot{\phi}_0\right), \left(\psi_0, \dot{\psi}_0\right)\right)$$

$$= -\int_{\Sigma}\left(\left(\iota^{\#}\nabla_{\mathrm{n}}^{E^*} F_M(\phi)\right) \cdot -\underbrace{\left(\iota^{\#} F_M(\phi)\right)^{\#}}_{u_0} \left(\iota^{\#} F_M(\phi)\right) \cdot \underbrace{\left(\iota^{\#}\nabla_{\mathrm{n}}^{E^*} F_M(\phi)\right)^{\#}}_{u_0}\right)\mu\Sigma$$

$$= -\int_M \phi \cdot u\mu_g$$
$$= -\int_M h^{-1}\left(\phi, F_M(\psi)\right)\mu_g$$
$$= \Lambda_{\text{cov}}(\phi,\psi)\,.$$
Clearly, Q_Σ is real and hence commutes with the complex conjugation. □

Lemma (3.2.69) [101]: *The kernel of Q_Σ coincides with the Poisson ideal generated by* $\ker F_M$ *which consists of those elements in* $\mathrm{S}^{\cdot}(V_{\text{cov}})$ *which vanish on all solutions.*

Proof. Clearly, the kernel of $Q_\Sigma|_{V_{\text{cov}}}$ is given by $1 < \text{er}F_M$ by Lemma (3.2.65). This implies $\ker\varrho_\Sigma = \{\ker F_M)$ in general. The second statement then follows from (163) at once. □

This statement has a very natural physical interpretation: ifwe view , $\Psi \in \mathrm{S}^{\cdot}(V_{\text{cov}})$ as observables ofthe field theory, their expectation values for a given field configuration are just the evaluations $\Phi(u)$, $\Psi(u) \in \mathbb{R}$, where $u \in \Gamma_{\text{sc}}^\infty(E)$. But since physically only those $u \in \Gamma_{\text{sc}}^\infty(E)$ occur which also satisfy the wave equation $Du = 0$, we have to identify the observables Φ and Ψ as soon as they coincide on the solutions. This is the case iff $\Phi - \Psi \in \langle ker\ F_M\rangle$.

Lemma (3.2.70) [101]: *The homomorphism Q_Σ is surjective.*

Proof. Let $\phi_0, \dot\phi_0 \in \Gamma_0^\infty(E_\Sigma^*)$ be given. Then there is a (unique) solution $\Phi \in \Gamma_{\text{sc}}^\infty(E^*)$ of the wave equation $D^{\mathrm{T}}\Phi = 0$ with the initial conditions
$$l^\#\Phi = -\dot\phi_0 \ and\ \iota^\#\nabla_{\text{n}}^{E^*}\Phi = \phi_0,$$
since D^{T} is normally hyperbolic as well. By (144) for D^{T} and F_M we know that $\Phi = F_M\phi$ for some $\in \Gamma_0^\infty(E^*)$. Then $\varrho\Sigma(\phi) = \left(\phi_0, \dot\phi_0\right)$ follows.

We can collect now the above results in the following statement leading to the comparison between the covariant and the canonical Poisson bracket and their Weyl algebras, see [121] for the classical part:

Theorem (3.2.71) [101]:$Fix R \geq \frac{1}{2}. Let (M, g)$ be globally hyperbolic space time and let $E \to M$ be a real vector bundle with fiber metric h and metric connection ∇^E. Moreover, let $D = \square^\nabla + B$ wirh $B = B^* \in \Gamma^\infty(\text{End}(E))$ be a symmetric normally hyperbolic differential operator on E and denote by F_M the propagator of its adjoint D^*. Finally, let $\iota\colon \Sigma \to M$ be a smooth spacelike Cauchy surface.

(i) *The following subspaces of* $\mathrm{S}^{\cdot}(V_{\text{cov}})$ *coincide*:

· *The vanishing ideal of the solutions of the wave function* $Du = 0$.

· *The Poisson ideal generated by the Casimir elements* $\phi \in V_{\text{cov}}$.

· *The ideal* $\langle \text{icer}F_M)$.

· *The kernel of the Poisson homomorphism* $\varrho_\Sigma\colon \mathrm{S}^{\cdot}(V_{\text{cov}}) \to \mathrm{S}^{\cdot}(V_\Sigma)$.

(ii) *The locally convex quotient Poisson algebra* $\mathrm{S}_R^*(V_{\text{cov}})/\{\text{i} < \text{er}F_M\rangle$ *is canonically isomorphic to the Poisson algebra* $\mathrm{S}_R^*(V_{\text{cov}}/\ker F_M)$, *with the Poisson bracket coming from* (157) *defined on classes*.

(iii) *The Poisson homomorphism $Q\Sigma$ induces a continuous Poisson isomorphism*
$$Q_\Sigma\colon \mathrm{S}_R^*(V_{\text{cov}})/\{\ker F_M) \to \mathrm{S}_R^*(V_\Sigma)\,. \qquad (165)$$

(iv) *The locally convex quotient* $*$*-algebra* $\mathrm{W}_R(V_{\mathrm{cov}} \otimes \mathbb{C}, \star_{\mathrm{cov}})/(\langle \ker F_M \rangle \otimes \mathbb{C})$ *is***-isomorphic to the Weyl algebra* $\mathrm{W}_R((V_{\mathrm{cov}}/\ker F_M) \otimes \mathbb{C}, \star_{\mathrm{cov}})$ *with* $\star_{\mathrm{cov}}$ *coming from the Poisson bracket* (157) *of* $V_{\mathrm{cov}}/1 < \mathrm{er}F_M$.
(v) *The*$*$*-homomorphism* $Q\Sigma$ *induces a continuous*$*$*-isomorphism*

$$Q_\Sigma: W_R(V_{\mathrm{cov}} \otimes \mathbb{C}, \star_{\mathrm{cov}})/(\{\ker F_M\rangle \otimes \mathbb{C}) \to \mathrm{W}_R(V_\Sigma \otimes (\mathrm{D}, \star_\Sigma)\,. \quad (166)$$

Proof. The only things left to prove are the continuity statements with respect to the quotient topologies. But these follow from the general situation discussed in Lemma (3.2.40) and Corollary (3.2.41).

We collect some further properties of the covariant Poisson bracket and its Weyl algebra quantization as required by the Haag-Kastler approach to (quantum) field theory [124]: locality and the time-slice axiom.

Let $U \subseteq M$ be a non-empty open subset. Then we denote by $\tilde{A}_{\mathrm{c1}}(U) \subseteq \mathrm{S}^{\cdot}(V_{\mathrm{cov}})$ the unital Poisson subalgebra generated by those $\phi \in \Gamma_0^\infty(E^*)$ with supp $\phi \subseteq U$. Analogously, we define $\tilde{\mathrm{A}}(U) \subseteq W_R(V_{\mathrm{cov}} \otimes \mathbb{C}, \star_{\mathrm{cov}})$ to be the unital$*$-subalgebra generated by those $\phi \in \Gamma_0^\infty(E^*)$ with $supp\ \phi s \subseteq U$. For $U = \emptyset$ we set $\tilde{A}_{\mathrm{c1}}(\emptyset) = \mathbb{C}]|{=}\tilde{A}\,(\emptyset)$. Finally, we set $A_{\mathrm{c1}}(U)$ and $A(U)$ for the images of $\tilde{A}_{\mathrm{c1}}(U)$ and $\tilde{A}(U)$ in the quotients $\mathrm{S}^{\cdot}(V_{\mathrm{cov}})/\langle \ker F_M \rangle$ and $\mathrm{W}_R(V_{\mathrm{cov}} \otimes \mathbb{C}, \star_{\mathrm{cov}})/(\{\ker F_M\rangle \otimes \mathbb{C})$, respectively. Clearly, $A_{\mathrm{c1}}(M)$ and $A(M)$ yield again everything. Then the following properties are obvious from the causal properties ofF_M:

Proposition (3.2.72)[101]: *Let* $u, u' \subseteq M$ *be open subsets of* M.

$$\{A_{\mathrm{c1}}(U), A_{\mathrm{c1}}(U')\}_{\mathrm{cov}} = 0 \text{ and } [A(U), A(U')]_{\star\mathrm{cov}} = 0 \qquad (167)$$

whenever U *and* U' *are spacelike.*
(ii) *For* $U \subseteq U'$ *we have*

$$A_{cl}(U) \subseteq A_{\mathrm{cl}}(U') \text{ and } A(U) \subseteq A(U')\,. \qquad (168)$$

(iii) *We have*

$$\bigcup_{U \subseteq M\ open} \mathrm{A_{cl}} = \mathrm{A_{cl}(M)} \text{ and } \bigcup_{U \subseteq M\ open} \mathrm{A} = \mathrm{A\,(M)}. \qquad (169)$$

Proof. For $\varphi, \psi \in \Gamma_0^\infty(E^*)$ we clearly have $\{\varphi, \psi\}_{\mathrm{cov}} = 0 = [\varphi, \psi]_{\star\mathrm{cov}}$ whenever U and U' are spacelike and $supp\ F_M(\phi) \subseteq J_M(supp\phi) \subseteq J_M(U)$ Which does not intersect U'.*Then the Leibniz rule shows (167) in general.* Then t.The remaining statements are clear .

The time-slice axiom requires that a small neighborhood of a Cauchy surface contains already all the information about the observables. In our framework, this can be formulated as follows:

Proposition (3.2.73) [101]: *Let* $\iota: \Sigma \to M \cong \mathbb{R} \times \Sigma$ *be a smooth Cauchy surface and let* $\varepsilon > 0$. *Then*

$$A_{\mathrm{c1}}(\Sigma_\varepsilon) = A_{\mathrm{c1}}(M) \text{ and } A(\Sigma_\varepsilon) = A(M)\,, \qquad (170)$$

where $\Sigma_\varepsilon = (-\varepsilon, \varepsilon) \times \Sigma$ *is the* ε*-time slice around* Σ.

Proof. First we note that Σ_ε is a globally hyperbolic spacetime by its own and the inclusion $\Sigma_\varepsilon \subseteq M$ is causally compatible, i.e. we have$J^{\pm}_{\Sigma_\varepsilon}(p) = J^{\pm}_M(p)\mathrm{n}\Sigma_\varepsilon$ for all $p \in \Sigma_\varepsilon$. Restricting D and D^{T} to Σ_ε gives globally hyperbolic differential operators with Green operators $G^{\pm}_{\Sigma_\varepsilon}$ and $F^{\pm}_{\Sigma_\varepsilon}$, respectively. By the uniqueness ofthe Green operators we have for $\phi \in \Gamma_0^\infty\left(E^*|_{\Sigma_\varepsilon}\right)$ the equality

$$F_{\Sigma_\varepsilon}^{\pm}(\phi) = F_M^{\pm}(\phi)|_{\Sigma_\varepsilon}.$$

Thus the covariant Poisson bracket for $S^{\cdot}\left(\Gamma_0^{\infty}\left(E^*|_{\Sigma_\varepsilon}\right)\right)$ is the restriction of the one from $S^{\cdot}\left(\Gamma_0^{\infty}(E^*)\right)$ to the subalgebra $S^{\cdot}\left(\Gamma_0^{\infty}\left(E^*|_{\Sigma_\varepsilon}\right)\right)$. Next, let $\phi \in \Gamma_0^{\infty}\left(E^*|_{\Sigma_\varepsilon}\right)$ be given. Then in the condition $\phi(u) = 0$ only $u|_{\Sigma_\varepsilon}$ enters. This shows that also the kernels ofF_M and F_{Σ_ε} correspond, i.e. we have

$$ker\, F_{\Sigma_\varepsilon} = \, ker\left(F_M|_{\Gamma_0^{\infty}(E^*|_{\Sigma_\varepsilon})}\right).$$

Putting this together we conclude that the two isomorphisms Q_Σ with respect to the Cauchy surface Σ, once sitting inside M and the other time sitting inside $(-\varepsilon, \varepsilon) \times \Sigma$, give the desired isomorphism needed for (170).

For more information on the locality properties and the time-slice axiom of quantum field theories on globally hyperbolic space times see [125], [110], [126] as well as [120], where the C*-algebraic version and the functorial aspects of the above construction are discussed in detail.

Chapter 4
Inequalities and Extensions to Operators on a Hilbert Space

We show that after some discussion of K-theory and Whitehead torsion, we indicate the relevance of these determinants to the study of L^2-torsion in topology. We study the perturbations that happen when positive matrices are added to diagonal blocks of the original matrix. We show that the perturbations are added to the inverses of the matrices. We show that a conceptual framework is established for viewing these inequalities as manifestations of Jensen's inequality in conjunction with the theory of operator monotone and operator convex functions on $[0, \infty)$. We place emphasis on documenting necessary and sufficient conditions for equality to hold.

Section (4.1): Fuglede–Kadison Determinant

For $\mathcal{R}$ be a ring with unit. For an integer $n \geq 1$, denote by $\mathrm{M}_n(\mathcal{R})$ the ring of n-by- n matrices over $\mathcal{R}$ and by $\mathrm{GL}_n(\mathcal{R})$ its group of units. $\mathcal{R}^*$ stands for $\mathrm{GL}_1(\mathcal{R})$.

Suppose $\mathcal{R}$ is commutative. The determinant

$$\det: \mathrm{M}_n(\mathcal{R}) \to \mathcal{R} \tag{1}$$

is defined by a well-known explicit formula, a polynomial in the matrix entries. It is alternate multilinear in the columns of the matrix and normalized by $\det(1_n) = 1$; when $\mathcal{R}$ is a field, these properties constitute an equivalent definition, as was lectured on by Weierstrass and Kronecker (probably) in the 1860s and published much later ([128]).

For $x, y \in \mathrm{M}_n(\mathcal{R})$, we have $\det(xy) = \det(x)\det(y)$. For $x \in \mathrm{M}_n(\mathcal{R})$ with $\det(x)$ invertible, an explicit formula shows that x itself is invertible, so that $\det(x) \in \mathcal{R}^*$ if and only if $x \in \mathrm{GL}_n(\mathcal{R})$. The restriction

$$\mathrm{GL}_n(\mathcal{R}) \to \mathcal{R}^*,\ x \mapsto \det x \tag{2}$$

is a group homomorphism.

Suppose that $\mathcal{R}$ is the field C of complex numbers. The basic property of determinants that we wish to point out is the relation

$$\det(\exp y) = \exp(\mathrm{trace}(y)) \text{ for all } y \in \mathrm{M}_n(\mathbf{C}). \tag{3}$$

Some expository books present this as a very basic formula ([129]); it reappears below as (20). It can also be written as

$$\det(x) = \exp(\mathrm{trace}(\log x)) \text{ for appropriate } x \in \mathrm{GL}_n(\mathbf{C}). \tag{4}$$

"Appropriate" can mean several things. If $\| x - 1 \| < 1$, then $\log x$ can be defined by the convergent series

$$\log x = \log(1 + (x - 1)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x - 1)^k.$$

If x is conjugate to a diagonal matrix, then $\log x$ can be defined component-wise (in pedantic terms, this is functional calculus, justified by the spectral theorem). In [131], note that the indeterminacy in the choice of the logarithm of a complex number is swallowed by the exponential, because $\exp 2\pi i = 1$

Let $x \in \mathrm{GL}_n(\mathbf{C})$. Because the group is connected, we can choose a piecewise smooth path $\xi: [0,1] \to \mathrm{GL}_n(\mathbf{C})$ from 1 to x. Because $\log \xi(\alpha)$ is a primitive of $\dot{\xi}(\alpha)\xi(\alpha)^{-1} d\alpha$, it follows from [131] that

$$\det(x) \stackrel{!}{=} \exp \int_0^1 \mathrm{trace}\big(\dot{\xi}(\alpha)\xi(\alpha)^{-1}\big) d\alpha. \tag{5}$$

This is our motivating formula, and in particular for (18).

The sign $\overset{!}{=}$ stands for a genuine equality, but indicates that some comment is in order. A priori, the integral depends on the choice of ξ, and we have also to worry about the determination of $\log \xi(\alpha)$. As there is locally no obstruction to choosing a continuous determination of the primitive $\log \xi(\alpha)$ of $\dot{\xi}(\alpha)\xi(\alpha)^{-1}d\alpha$, the integral is invariant under small changes of the path (with fixed ends) and therefore depends only on the homotopy class of ξ, so that it is defined modulo its values on (homotopy classes of) closed loops. The fundamental group $\pi_1(\mathrm{GL}_n(\mathbf{C}))$ is infinite cyclic, generated by the homotopy class of

$$\xi_0 \colon [0,1] \to \mathrm{GL}_n(\mathbf{C}),\ \alpha \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & 1_{n-1} \end{pmatrix},$$

and we have $\int_0^1 \mathrm{trace}\big(\dot{\xi}_0(\alpha)\xi_0(\alpha)^{-1}\big)d\alpha = 2\pi i$. Consequently, the integral in the right-hand side of [132] is defined modulo $2\pi i\mathbf{Z}$, so that the right-hand side itself is well defined. (This is repeated in the proof of Lemma (4.1.10).)

Because a connected group is generated by each neighborhood of the identity, there exist $x_1, \dots, x_k \in \mathrm{GL}_n(\mathbf{C})$ such that $x = x_1 \cdots x_k$ and $\|x_j - 1\| < 1$ for $j = 1, \dots, k$, and one can choose

$$\xi(\alpha) = \exp\big(\alpha(\log x_1)\big) \cdots \exp\big(\alpha(\log x_k)\big).$$

A short computation with this ξ gives

$$\exp\int_0^1 \mathrm{trace}\big(\dot{\xi}(\alpha)\xi(\alpha)^{-1}\big)d\alpha = \exp\big(\mathrm{trace}(\log x_1)\big) \cdots \exp\big(\mathrm{trace}(\log x_k)\big)$$

and it is now obvious that [131] implies [132].

Determinants arise naturally with linear systems of equations, first with $\mathcal{R} = \mathbf{R}$ and more recently also with $\mathcal{R} = \mathbf{C}$. They have a prehistory in Chinese mathematics from the second century B.C. (3). In modern Europe, there has been an early contribution by Leibniz in $1693^{\dagger}$, unpublished until 1850. Gabriel Cramer wrote an influential book, published in 1750 . Major mathematicians who have written about determinants include Bézout, Vandermonde, Laplace, Lagrange, Gauss, Cauchy, Jacobi, Sylvester, Cayley, and others. The connection between determinants of matrices in $M_3(\mathbf{R})$ and volumes of parallelepipeds is often attribued to Lagrange (1773). We mention an amazing book on the history of determinants, [131]: four volumes, altogether more than 2,000 pages, an ancestor of the Mathematical Reviews, for one subject, covering the period 1693-1900.

There is an extension of [128] to a skew-field k by Dieudonné, where the range of the mapping det defined on $\mathrm{M}_n(k)$ is $(k^*/Dk^*) \sqcup \{0\}$, where the notation "$D\Gamma$" denotes the group of commutators of a group Γ (see [132] and [133], and also [134] for a discussion of when k is the skew field of Hamilton quaternions). The theory of determinants, in the case of a noncommutative ring $\mathcal{R}$, has motivated a lot of work, in particular by Gelfand and coauthors since the early 1990s (8). We also mention a version for supermathematics due to Berezin ([136] and [137]), as well as "quantum determinants", of interest in low-dimensional topology (see, for example, [138]).

The notion of determinants extends to matrices over a ring without unit (by "adjoining a unit to the ring"). In particular, in functional analysis, there is a standard notion of determinants that appears in the theory of Fredholm integral equations, for example for operators on a Hilbert space of the form $1 + x$, where x is the "trace class" (12,13).

The oldest occurrence I know of $\exp y$ or $\log x$, including the notation, defined by the familiar power series in the matrix y or $x - 1$, in [141]; see also [142]. However,

exponentials of linear differential operators appear also early in Lie theory, see for example [143] and [144], even if Lie never uses a notation like $\exp X$ (unlike Poincaré) (see his $e^{\alpha X}$ in [145]).
There is a related and rather old formula known as the "Abel-Liouville-Jacobi-Ostrogradskii identity". Consider a homogeneous linear differential equation of the first order $y'(t) = A(t)y(t)$, for an unknown function $y: [t_0, t_1] \to \mathbf{R}^n$. The columns of a set of n linearly independent solutions constitute the Wronskian matrix $W(t)$. It is quite elementary (at least nowadays!) to show that $W'(t) = A(t)W(t)$, hence $\frac{d}{dt}\det W(t) = \mathrm{trace}(A(t))\det W(t)$, and therefore

$$\det W(t) = \det W(t_0) \exp\left(\int_{t_0}^{t} \mathrm{trace}\big(A(s)\big)\, ds\right),$$

a close cousin of (5). The name of this identity refers to Abel (1827, case $n = 2$), Liouville (19), Ostrogradskii (1838), and Jacobi (1845). This was pointed out to me by Gerhard Wanner ([147]); also, Philippe Henry showed me this identity on the last five lines of [148] (which does not contain references to previous work).

Finally, a few words are necessary about the authors of the 1952 paper alluded to in our title. Bent Fuglede is a Danish mathematician born in 1925. He has been working on mathematical analysis; he is also known for a book on Harmonic Maps Between Riemannian Polyhedra (coauthor Jim Eells, preface by Misha Gromov). Richard Kadison is an American mathematician, born in this same year, 1925. He is known for his many contributions to operator algebras; his "global vision of the field was certainly essential for my own development" (words of Alain Connes, when Kadison was awarded the Steele Prize in 1999 for Lifetime Achievement, [149])

We a review of von Neumann algebras based on three types of examples, an exposition of the original Fuglede– Kadison idea, we stress the difference between the complex-valued standard determinant and the real-valued Fuglede-Kadison determinant, and a review of some notions of K-theory. We expose the main variations of our title: determinants defined for connected groups of invertible elements in complex Banach algebras. We end by recalling a few facts about Whitehead torsion, with values in $\mathrm{Wh}(\Gamma)$, which is a quotient of the group K_1 of a group algebra $\mathbf{Z}[\Gamma]$, and by alluding to L^2-torsion, which is defined in terms of (a variant of) the Fuglede-Kadison determinant.

In a series of papers from 1936 to 1949 , Francis Joseph Murray and John von Neumann founded the theory of von Neumann (in their terminology "rings of operators"), which are complex $*$-algebras representable by unital weakly closed $*$-subalgebras of some $\mathcal{L}(\mathcal{H})$, the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$.
We first give three examples of pairs $(\mathcal{N}, \tau)$, with $\mathcal{N}$ a finite von Neumann algebra and τ a finite trace on it. We then recall some general facts and define a few terms, such as "finite von Neumann algebra", "finite trace", and "factor of type II_1 ".
Example (4.1.1)[127]: (factors of type $\mathbf{I}_n$)[127]: For any $n \geq 1$, the matrix algebra $\mathrm{M}_n(\mathbf{C})$ is a finite von Neumann algebra known as a factor of type I_n. The involution is given by $(x^*)_{j,k} = \overline{x_{k,j}}$. The linear form $x \mapsto \frac{1}{n}\sum_{j=1}^{n} x_{j,j}$ is the (unique) normalized trace on $\mathrm{M}_n(\mathbf{C})$.
Example (4.1.2)[127]: (Abelian von Neumann Algebras)[127]: Let Z be a locally compact space and ν be a positive Radon measure on Z. The space $L^\infty(Z, \nu)$ of complex-valued functions on Z that are measurable and ν-essentially bounded (modulo equality

locally ν-almost everywhere) is an abelian von Neumann algebra. The involution is given by $f^*(z) = \overline{f(z)}$. Any abelian von Neumann algebra is of this form.
If ν is a probability measure, the linear form $\tau_\nu: f \mapsto \int_Z f(z) d\nu(z)$ is a trace on $L^\infty(Z, \nu)$, normalized in the sense $\tau_\nu(1) = 1$.

Example (4.1.3)[127]: (Group von Neumann Algebra)[127]: Let Γ be a group. The Hilbert space $\ell^2(\Gamma)$ has a scalar product, denoted by $(\cdot|\cdot)$, and a canonical orthonormal basis $(\delta_\gamma)_{\gamma\in\Gamma}$, where $\delta_\gamma(x)$ is 1 if $x = \gamma$ and 0 otherwise. The left-regular representation λ of Γ on $\ell^2(\Gamma)$ is defined by $(\lambda(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $\gamma, x \in \Gamma$ and $\xi \in \ell^2(\Gamma)$.
The von Neumann algebra $\mathcal{N}(\Gamma)$ of Γ is the weak closure in $\mathcal{L}(\ell^2(\Gamma))$ of the set of **C**-linear combinations $\sum_{\gamma\in\Gamma}^{\text{finite}} z_\gamma \lambda(\gamma)$; it is a finite von Neumann algebra. The involution is given by $\left(z_\gamma \lambda(\gamma)\right)^* = \bar{z_\gamma}\lambda(\gamma^{-1})$. There is a canonical trace, given by $x \mapsto \langle x\delta_1 \mid \delta_1\rangle$, which extends $\sum_{\gamma\in\Gamma}^{\text{finite}} z_\gamma \lambda(\gamma) \mapsto z_1$.
Moreover, $\mathcal{N}(\Gamma)$ is a factor of type II_1 if and only if Γ is icc ‡ (this is lemma 5.3.4 in [150]; see also [151]).

In 1952, Fuglede and Kadison defined their determinant

$$\det{}_\tau^{FK}: \begin{cases} \mathrm{GL}_1(\mathcal{N}) & \to & \mathbf{R}_+^* \\ x & \mapsto & \exp\left(\tau\left(\log\left((x^*x)^{\frac{1}{2}}\right)\right)\right) \end{cases}, \tag{6}$$

which is a partial analog of [129]. The number $\det_\tau^{FK}(x)$ is well defined by functional calculus, and most of the work in [160] is to show that $\det_\tau^{FK}$ is a homomorphism of groups. For the definition given below, it will be the opposite: some work to show that the definition makes sense, but a very short proof to show it defines a group homomorphism.
In the original paper, $\mathcal{N}$ is a factor of type II_1, and τ is its unique trace with $\tau(1) = 1$; but everything carries over to the case of a von Neumann algebra and a normalized trace ([151]).
Besides being a group homomorphism, $\det_\tau^{FK}$ has the following properties:

- $\det_\tau^{FK}(e^y) = \left|e^{\tau(y)}\right| = e^{\mathrm{Re}(\tau(y))}$ for all $y \in \mathcal{N}$ and in particular $\det_\tau^{FK}(\lambda 1) = |\lambda|$ for all $\lambda \in \mathbf{C}$,
- $\det_\tau^{FK}(x) = \det_\tau^{FK}\left((x^*x)^{\frac{1}{2}}\right)$ for all $x \in \mathrm{GL}_1(\mathcal{N})$ and in particular $\det_\tau^{FK}(x) = 1$ for all $x \in \mathrm{U}_1(\mathcal{N})$.

For a $*$-ring $\mathcal{R}$ with unit, $\mathrm{U}_1(\mathcal{R})$ denotes its unitary group, defined to be $\{x \in R \mid x^*x = xx^* = 1\}$.
Instead of [**6**], we could equally view $\det_\tau^{FK}$ as a family of homomorphisms $\mathrm{GL}_n(\mathcal{N}) \to \mathbf{R}_+^*$, one for each $n \geq 1$; if the traces on the $\mathrm{M}_n(\mathcal{N})$s are normalized by $\tau(1_n) = n$, we have $\det_\tau^{FK}(\lambda 1_n) = |\lambda|^n$. More generally, for any projection $e \in \mathrm{M}_n(\mathcal{N})$, we have a von Neumann algebra $\mathrm{M}_e(\mathcal{N}) := e\mathrm{M}_n(\mathcal{N})e$ and a Fuglede-Kadison determinant $\det_\tau^{FK}: \mathrm{GL}_e(\mathcal{N}) \to \mathbf{R}_+^*$ defined on its group of units.
There are extensions of $\det_\tau^{FK}$ to noninvertible elements, but this raises some problems and technical difficulties. Two extensions are discussed in [160]: the "algebraic extension" for which the determinant vanishes on singular elements (this is not mentioned again) and the "analytic extension" that relies on (6), in which one should understand

$$\det{}_\tau^{FK}(x) = \exp\left(\tau\left(\log\left((x^*x)^{\frac{1}{2}}\right)\right)\right) = \exp\int_{\mathrm{sp}\left((x+x)^{\frac{1}{2}}\right)} \ln\lambda\, d\tau(E_\lambda), \tag{7}$$

where $(E_\lambda)_{\lambda\in\mathrm{sp}((x*x)^{1/2})}$ denotes the spectral resolution of $(x^*x)^{1/2}$; of course $\exp(-\infty) = 0$. (Note that we write "log" for logarithms of matrices and operators and "ln" for logarithms of numbers.) For example, if x is such that there exists a projection e with $x = x(1-e)$ and $\tau(e) > 0$, we have $\det_\tau^{FK}(x) = 0$. For all $x, y \in \mathcal{N}$, we have

$$\det_\tau^{FK}\big((x^*x)^{1/2}\big) = \lim_{\epsilon\to 0+} \det_\tau^{FK}\big((x^*x)^{1/2} + \epsilon 1\big)$$

$$\det_\tau^{FK}(x)\det_\tau^{FK}(y) = \det_\tau^{FK}(xy)$$

(see [160]). However, an element x with $\det_\tau^{FK}(x) \neq 0$ need not be invertible, and no extension $\mathcal{N} \to \mathbf{R}_+$of the mapping $\det_\tau^{FK}$ of [**6**] is norm continuous ([160]).
We discuss another extension $\det_\tau^{FKL}$ to singular elements.
More generally, $\det_\tau^{FK}(x)$ can be defined for x as an operator "affiliated" to $\mathcal{N}$, and also for traces that are semifinite rather than finite as above. See [161]-[166], among others. We do not comment further on this part of the theory.
Example (4.1.4)[127]: [Fuglede-Kadison determinant for M_n(C)]. Let $\mathcal{N} = \mathrm{M}_n(\mathbf{C})$ be the factor of type I_n, as in Example (4.1.1) , let det be the usual determinant, and let $\tau: x \mapsto \frac{1}{n}\sum_{j=1}^n x_{j,j}$ be the trace normalized by $\tau(1_n) = 1$. Then

$$\det_\tau^{FK}(x) = |\det(x)|^{\frac{1}{n}} = \left(\det\left((x * x)^{\frac{1}{2}}\right)\right)^{\frac{1}{n}} \tag{8}$$

for all $x \in \mathrm{M}_n(\mathbf{C})$.
Example (4.1.5)[127]: (Fuglede-Kadison Determinant for Abelian von Neumann Algebras). Let $L^\infty(Z,\nu)$ and τ_ν be as in Example (4.1.2) , with ν a probability measure. The corresponding Fuglede-Kadison determinant is given by

$$\det_\tau^{FK}(f) = \exp\int_Z \ln|f(z)|d\mu(z) \in \mathbf{R}_+. \tag{9}$$

In [136], observe that $\ln|f(z)|$ is bounded above on Z, because $|f(z)| \leq \| f \|_\infty < \infty$ for ν-almost all z. However, $|f(z)|$ need not be bounded away from 0 , so that $\ln|f(z)| = -\infty$ occurs. If the value of the integral is $-\infty$, then $\det_\tau^{FK}(f) = \exp(-\infty) = 0$.
Consider an integer $d \geq 1$ and the von Neumann algebra $\mathcal{N}(\mathbf{Z}^d)$ of the free abelian group of rank d. Fourier transform provides an isomorphism of von Neumann algebras

$$\mathcal{N}(\mathbf{Z}^d) \xrightarrow{\approx} \mathrm{L}^\infty(T^d,\nu), x \mapsto \hat{x},$$

where ν denotes the normalized Haar measure on the d-dimensional torus T^d. Moreover, the composition of this isomorphism with the trace τ_ν of Example (4.1.2) is the canonical trace on $\mathcal{N}(\mathbf{Z}^d)$, in the sense of Example (4.1.3).
Example (4.1.6)[127]: (Fuglede-Kadison Determinant and Mahler Measure)[127]: Let x be a finite linear combination $\sum_{n\in\mathbb{Z}^d}^{\text{finite}} z_n\lambda(n) \in \mathcal{N}(\mathbf{Z}^d)$, so that $\hat{x} \in L^\infty(T^d,\nu)$ is a trigonometric polynomial. Then the τ_ν-Fuglede-Kadison determinant of x is given by the exponential Mahler measure of $\hat{x}$:

$$\det_{\tau_\nu}^{FK}(x) = M(\hat{x}) := \exp\int_{T^d} \ln|\hat{x}(z)|d\nu(z).$$

In the one-dimensional case ($d = 1$), if

$$\hat{x}(z) = a_0 + a_1 z + \cdots + a_s z^s = a_s \prod_{j=1}^{s} \big(z - \xi_j\big), \text{ with } a_0 a_s \neq 0,$$

a computation shows that

$$\int_T \ln|\hat{x}(z)|d\nu(z) = \int_0^1 \ln\left|\hat{x}\left(e^{2\pi i\alpha}\right)\right|d\alpha = \ln|a_s| + \sum_{j=1}^{s} \max\ \{1, |\xi_j|\}$$

([167]).

Mahler measures occur in particular as entropies of $\mathbf{Z}^d$-actions by automorphisms of compact groups. More precisely, for $x \in \mathbf{Z}[\mathbf{Z}^d]$, which can be viewed as the inverse Fourier transform of a trigonometric polynomial, the group $\mathbf{Z}^d$ acts naturally on the quotient $\mathbf{Z}[\mathbf{Z}^d]/(x)$ of the group ring by the principal ideal (x) and hence on the Pontryagin dual $(\mathbf{Z}[\mathbf{Z}^d]/(x))^\wedge$ of this countable abelian group, which is a compact abelian group. For example, if $x(z) = 1 + z - z^2 \in \mathbf{Z}[z, z^{-1}] \approx \mathbf{Z}[\mathbf{Z}]$, then $(\mathbf{Z}[\mathbf{Z}]/(x)) \approx T^2$, and the corresponding action of the generator of $\mathbf{Z}$ on T^2 is described by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ([167]).

Every action of $\mathbf{Z}^d$ by automorphisms of a compact abelian group arises as above from some $x \in \mathbf{Z}[\mathbf{Z}^d]$. More on this is found in [167], [169], and [170].

The logarithm of the Fuglede-Kadison determinant occurs also in the definition of a "tree entropy", namely in the asymptotics of the number of spanning trees in large graphs (44,45).

It is natural to ask why $\mathbb{R}_+^*$ appears on the right-hand side of [133], even though $\mathcal{N}$ is a complex algebra, for example a II_1-factor, whereas C^* appears on the right-hand side of [132] when $\mathcal{N} = \mathrm{M}_n(\mathbf{C})$.

This is not due to some shortsightedness of Fuglede and Kadison. Indeed, for $\mathcal{N}$ a factor of type II_1, it has been shown that the FugledeKadison determinant provides an isomorphism from the abelianized group $\mathrm{GL}_1(\mathcal{N})/D\mathrm{GL}_1(\mathcal{N})$ onto $\mathbf{R}_+^*$. We have the following:

Proposition (4.1.7)[127]: (Properties of Operators with Trivial Fuglede-Kadison Determinant in a Factor of Type II_1). Let $\mathcal{N}$ be a factor of type II_1.

i) Any element in $\mathrm{U}_1(\mathcal{N})$ is a product of finitely many multiplicative commutators of unitary elements.

ii) The kernel $\mathrm{SL}_1(\mathcal{N})$ of the homomorphism [133] coincides with the group of commutators of $\mathrm{GL}_1(\mathcal{N})$.

Property i is due to Broise (46). It is moreover known that any proper normal subgroup of $\mathrm{U}_1(\mathcal{N})$ is contained in its center, which is $\{\lambda \text{ id } | \lambda \in \mathbf{C}^*, |\lambda| = 1\} \approx \mathbf{R}/\mathbf{Z}$, ([174]); this sharpens an earlier result on the classification of norm-closed normal subgroups of $U_1(\mathcal{N})$ ([175]).

Property ii is from [159]. It follows that the quotient of $\mathrm{SL}_1(\mathcal{N})$ by its center [which is the same as the center of $U_1(\mathcal{N})$] is simple, as an abstract group ([176]).

As a kind of answer to our motivating question, we see below that, when the Fuglede-Kadison definition is adapted to a separable Banach algebra, the right-hand side of the homomorphism analogous to [133] is necessarily a quotient of the additive group C by a countable subgroup. For example, when $A = \mathrm{M}_n(\mathbf{C})$, this quotient is $\mathbf{C}/2i\pi\mathbf{Z}^{\exp(\cdot)} \approx \mathbf{C}^*$; see Corollary (4.1.13). On the contrary, when A is a I_1-factor (not separable as a Banach algebra), this quotient is $\mathbf{C}/2i\pi\mathbf{R}^{\exp(\mathrm{Re}(\cdot))}\mathbf{R}_+^*$; see Corollary (4.1.14). The case of a separable Banach algebra can sometimes be seen as providing an interpolation between the two previous cases;.

Let $\mathcal{R}$ be a ring, say with unit to simplify several small technical points. We first recall one definition of the abelian group $K_0(\mathcal{R})$ of K-theory.

We have a nested sequence of rings of matrices and (nonunital) ring homomorphisms

$$\mathcal{R} = \mathrm{M}_1(\mathcal{R}) \subset \cdots \subset \mathrm{M}_n(\mathcal{R}) \subset \mathrm{M}_{n+1}(\mathcal{R}) \subset \cdots \subset \mathrm{M}_\infty(\mathcal{R}) := \bigcup_{n\geq 1} \mathrm{M}_n(\mathcal{R}), \quad (10)$$

where the inclusions at finite stages are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$.
An idempotent in $\mathrm{M}_\infty(\mathcal{R})$ is an element e such that $e^2 = e$. Two idempotents $e, f \in M_\infty(\mathcal{R})$ are equivalent if there exist $n \geq 1$ and $u \in \mathrm{GL}_n(\mathcal{R})$ such that $e, f \in \mathrm{M}_n(\mathcal{R})$ and $f = u^{-1}eu$. Define an addition on equivalence classes of idempotents, by

$$(\text{class of } e \in \mathrm{M}_k(\mathcal{R})) + \big(\text{ class of } f \in \mathrm{M}_\ell(\mathcal{R})\big) = \text{ class of } e \oplus f \in \mathrm{M}_{k+\ell}(\mathcal{R}), \quad (11)$$

where $e \oplus f$ denotes the matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. Two idempotents $e, f \in \mathrm{M}_\infty(\mathcal{R})$ are stably equivalent if there exists an idempotent g such that the classes of $e \oplus g$ and $f \oplus g$ are equivalent; we denote by $[e]$ the stable equivalence class of an idempotent e. The set of stable equivalence classes of idempotents, with the addition defined by $[e] + [f] := [e \oplus f]$, is a semigroup. The Grothendieck group $K_0(\mathcal{R})$ of this semigroup is the set of formal differences $[e] - [e']$, up to the equivalence defined by $[e] - [e'] \sim [f] - [f']$ if $[e] + [f'] = [e'] + [f]$.
Note that K_0 is a functor: To any (unital) ring homomorphism $\mathcal{R} \to \mathcal{R}'$ corresponds a natural homomorphism $K_0(\mathcal{R}) \to K_0(\mathcal{R}')$ of abelian groups. Note also the isomorphism $K_0(\mathrm{M}_n(\mathcal{R})) \approx K_0(\mathcal{R})$, which is a straightforward consequence of the definition and of the isomorphisms $\mathrm{M}_k(\mathrm{M}_n(\mathcal{R})) \approx \mathrm{M}_{kn}(\mathcal{R})$.
[To an idempotent $e \in \mathrm{M}_\infty(\mathcal{R})$ is associated an $\mathcal{R}$-linear mapping $\mathcal{R}^n \to \mathcal{R}^n$ for n large enough, of which the image is a projective $\mathcal{R}$-module of finite rank. From this it can be checked that the definition of $K_0(\mathcal{R})$ given above coincides with another standard definition, in terms of projective modules of finite rank. Details are in [177].

Rather than a general ring $\mathcal{R}$, consider now the case of a complex Banach algebra A with unit. For each $n \geq 1$, the matrix algebra $\mathrm{M}_n(A)$ is again a Banach algebra, for some appropriate norm, and we can furnish $\mathrm{M}_\infty(A)$ with the inductive limit topology. The following is rather easy to check (e.g., [178]): Two idempotents $e, f \in \mathrm{M}_\infty(A)$ are equivalent if and only if there exists a continuous path

$$[0,1] \to \big\{ \text{ idempotents of } \mathrm{M}_\infty(A)\big\}, \alpha \mapsto e_\alpha$$

such that $e_0 = e$ and $e_1 = f$. This has the following consequence:
Proposition (4.1.8)[127]: If the Banach algebra A is separable, the abelian group $K_0(A)$ is countable.
Proposition (4.1.9)[127]: If $\mathcal{N}$ is a factor of type II_1, then $K_0(\mathcal{N}) \approx \mathbf{R}$ is uncountable.
Indeed, if τ denotes the canonical trace on $\mathcal{N}$, the mapping that associates to the class of a self-adjoint idempotent e in $\mathcal{N}$ its von Neumann dimension $\tau(e) \in [0,1]$ extends to an isomorphism $K_0(\mathcal{N}) \xrightarrow{\approx} \mathbf{R}$.
On the proof: This follows from the "comparison of projections" in von Neumann algebras ([151]).
For historical indications on the early connections between K-theory and operator algebras, which go back to the mid-1960s, see [179].

For any ring $\mathcal{R}$ with unit, we have a nested sequence of group homomorphisms

$$\mathcal{R}^* = \mathrm{GL}_1(\mathcal{R}) \subset \cdots \subset \mathrm{GL}_n(\mathcal{R}) \subset \mathrm{GL}_{n+1}(\mathcal{R}) \subset \cdots \subset \mathrm{GL}_\infty(\mathcal{R}) := \bigcup_{n\geq 1} \mathrm{GL}_n(\mathcal{R}), \quad (12)$$

where the inclusions at finite stages are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$.

By definition,

$$K_1(\mathcal{R}) = \mathrm{GL}_\infty(\mathcal{R})/D\mathrm{GL}_\infty(\mathcal{R}) \tag{13}$$

is an abelian group, usually written additively. Note that K_1 is a functor from rings to abelian groups.

For a commutative ring $\mathcal{R}$, the classical determinant provides a homomorphism $K_1(\mathcal{R}) \to \mathcal{R}^*$; it is an isomorphism in several important cases, for example when $\mathcal{R}$ is a field or the ring of integers in a finite extension of $\mathbf{Q}$ ([180]). In general ($\mathcal{R}$ commutative or not), the association of an element in $K_1(\mathcal{R})$ to a matrix in $\mathrm{GL}_\infty(\mathcal{R})$ can be viewed as a kind of determinant or rather of a log of a determinant because $K_1(\mathcal{R})$ is written additively. Accordingly, the torsion defined in [151] below can be viewed as an alternating sum of logs of determinants; we recall this when defining the L^2-torsion in (29).

Let $\mathcal{R}$ be, again, an arbitrary ring with unit. The reduced K_1-group is the quotient $\bar{K}_1(\mathcal{R})$ of $K_1(\mathcal{R})$ by the image of the natural homomorphism $\{1,-1\} \subset \mathrm{GL}_1(\mathcal{R}) \subset \mathrm{GL}_\infty(\mathcal{R}) \to K_1(\mathcal{R})$.

In the case that $\mathcal{R} = \mathbf{Z}[\Gamma]$ is the integral group ring of a group Γ, the Whitehead group $Wh(\Gamma)$ is the cokernel $K_1(\mathbf{Z}[\Gamma])/\langle \pm 1, \Gamma \rangle$ of the natural homomorphism $\Gamma \subset \mathrm{GL}_1(\mathbf{Z}[\Gamma]) \to K_1(\mathbf{Z}[\Gamma]) \to \bar{K}_1(\mathbf{Z}[\Gamma])$.

When Γ is finitely presented, there is a different (but equivalent) definition of $\mathrm{Wh}(\Gamma)$, with geometric content. In short, let L be a connected finite CW complex with $\pi_1(L) = \Gamma$. One defines a group $\mathrm{Wh}(L)$ of appropriate equivalence classes of pairs (K, L), with K a finite CW complex containing L in such a way that the inclusion $L \subset K$ is a homotopy equivalence. The unit is represented by pairs $L \subset K$ for which the inclusion is a simple homotopy equivalence. It can be shown that the functors $L \to \mathrm{Wh}(L)$ and $L \to \mathrm{Wh}(\pi_1(L))$ are naturally equivalent ([181]).

Examples are $\mathrm{Wh}(\mathrm{Z}^d) = 0$ for free abelian groups Z^d and $\mathrm{Wh}(F_d) = 0$ for free groups F_d. For finite cyclic groups, Wh $(\mathrm{Z}/q\mathrm{Z})$ is a free abelian group of finite rank for all $q \geq 1$ and is the group $\{0\}$ if and only if $q \in \{1,2,3,4,6\}$.

See [180]-[184].

For A be a Banach algebra with unit. For each $n \geq 1$, the group $\mathrm{GL}_n(A)$ is an open subset of the Banach space $\mathrm{M}_n(A)$, and the induced topology makes it a topological group. The group $\mathrm{GL}_\infty(A)$ of [139] is also a topological group, for the inductive limit topology; we denote by $\mathrm{GL}_\infty^0(A)$ its connected component.

It is a simple consequence of the classical "Whitehead lemma" that, for any Banach algebra, the group $D\mathrm{GL}_\infty(A)$ is perfect and coincides with $D\mathrm{GL}_\infty^0(A)$; see, for example, [185], appendix. In particular, $D\mathrm{GL}_\infty(A) \subset \mathrm{GL}_\infty^0(A)$, so that the quotient group

$$K_1^{\mathrm{top}}(A) := \pi_0(\mathrm{GL}_\infty(A)) = \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty^0(A) \tag{14}$$

is commutative. Note that $\mathrm{GL}_1(A)/\mathrm{GL}_1^0(A)$ need not be commutative (59), even if its image in $\mathrm{GL}_\infty(A)/\mathrm{GL}_\infty^0(A)$ is always commutative.

Moreover, we have a natural quotient homomorphism

$$\mathrm{GL}_\infty(A)/D\mathrm{GL}_\infty^0(A) = K_1(A) \to K_1^{\mathrm{top}}(A) = \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty^0(A), \tag{15}$$

which is surjective. It is an isomorphism if and only if the group $\mathrm{GL}_\infty^0(A)$ is perfect; this is the case if A is an infinite simple C^*-algebra, for example if A is one of the Cuntz algebras O_n briefly mentioned below.

If the Banach algebra A is separable, the group $K_1^{\mathrm{top}}(A)$ is countable (compare with Proposition (4.1.8)).

To an idempotent $e \in \mathrm{M}_n(A)$, we can associate the loop

$$\xi_e : \begin{cases} [0,1] & \to \mathrm{GL}_n(A) \subset \mathrm{GL}_\infty(A) \\ \quad \alpha & \mapsto \exp(2\pi i \alpha e) = \exp(2\pi i \alpha)\, e + (1-e); \end{cases} \tag{16}$$

note that $\xi_e(0) = \xi_e(1) = 1$. If two idempotents e and f have the same image in $K_0(A)$, it is easy to check that ξ_e and ξ_f are homotopic loops. It is a fundamental fact, which is a form of Bott periodicity, that the assignment $e \mapsto \xi_e$ extends to a group isomorphism

$$K_0(A) \xrightarrow{\approx} \pi_1\big(\mathrm{GL}^0_\infty(A)\big) \tag{17}$$

([187]). The terminology is due to a generalization of [144]: $K_i^{\mathrm{top}}(A) \approx K_{i+2}^{\mathrm{top}}(A)$ for any integer $i \geq 0$; by definition, $K_i^{\mathrm{top}}(A) = \pi_{i-1}(\mathrm{GL}_\infty(A))$, for all $i \geq 1$, and $K_0^{\mathrm{top}}(A) = K_0(A)$.

Let $A = \mathcal{C}(T)$ be the Banach algebra of continuous functions on a compact space T. Then $K_0(A) = K^0(T)$ and $K_1^{\mathrm{top}}(A) = K^1(T)$, where $K^0(T)$ and $K^1(T)$ stand for the (Grothendieck)-Atiyah-Hirzebruch-Bott K-theory groups of the topological space T, defined in terms of complex vector bundles. For example, if T is a sphere, we have

$$\begin{aligned} K_0\big(\mathcal{C}(\mathbf{S}^{2m})\big) &\approx \mathbf{Z}^2,\ K_1^{\mathrm{top}}\big(\mathcal{C}(\mathbf{S}^{2m})\big) = 0, \\ K_0\big(\mathcal{C}(\mathbf{S}^{2m+1})\big) &\approx \mathbf{Z}, K_1^{\mathrm{top}}\big(\mathcal{C}(\mathbf{S}^{2m+1})\big) \approx \mathbf{Z}, \end{aligned}$$

for all $m \geq 0$. If T is a compact CW complex without cells of odd dimension, then $K_1^{\mathrm{top}}(\mathcal{C}(T)) = 0$.

Let A be an AF algebra, namely a C^*-algebra that contains a nested sequence $A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ of finite-dimensional sub-C algebras with $\bigcup_{n>1} A_n$ dense in A. Then $K_0(A)$ is rather well understood, and $K_1^{\mathrm{top}}(A) = 0$. The group $K_0(A)$ is the basic ingredient in Elliott's classification of AF algebras, from the 1970s; this was the beginning of a long and rich story, with numerous offspring ([178], [188] and [189]). A particular case is the so-called CAR algebra, or C*-algebra of the canonical anticommutation relations, or UHF algebra of type (2^i) in [170]: It is the C*-closure of the limit of the inductive system of finite matrix algebras

$$\mathbf{C} \subset \cdots \subset \mathrm{M}_{2^n}(\mathbf{C}) \subset \mathrm{M}_{2^{n+1}}(\mathbf{C}) \subset \cdots,$$

where the inclusions are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. For this,

$$K_0(CAR) = \mathbf{Z}[1/2] \text{ and } K_1^{\mathrm{top}}(CAR) = 0$$

(for K_1 of CAR and a few other AF algebras).

The Jiang-Su algebra $\mathcal{Z}$ is a simple infinite-dimensional C*-algebra with unit that plays an important role in Elliott's classification program of C^*-algebras. It has the same K-theory as $\mathbf{C}$(64).

The reduced C*-algebra of a group Γ is the norm-closure $C_\lambda^*(\Gamma)$ of the algebra $\left\{\sum_{\gamma\in\Gamma}^{\text{finite}} z_\gamma \lambda(\gamma)\right\}$, see Example (4.1.3), in the algebra of all bounded operators on $\ell^2(\Gamma)$. For the free groups F_d (nonabelian free groups if ≥ 2), we have (65)

$$K_0\big(C_\lambda^*(F_d)\big) \approx \mathbf{Z} \text{ and } K_1^{\mathrm{top}}\big(C_\lambda^*(F_d)\big) \approx \mathbf{Z}^d.$$

For a so-called irrational rotation C*-algebra A_θ, generated by two unitaries u, v satisfying the relation $uv = e^{2\pi i\theta} vu$, where $\theta \in [0,1]$ with $\theta \notin \mathbf{Q}$, we have (66)

$$K_0(A_\theta) \approx \mathbf{Z}^2 \text{ and } K_1^{\mathrm{top}}(A_\theta) \approx \mathbf{Z}^2.$$

For the infinite Cuntz algebras O_n, generated by $n \geq 2$ elements $s_1, \dots, s_n$ satisfying $s_j^* s_k = \delta_{j,k}$ and $\sum_{j=1}^n s_j s_j^* = 1$, we have (67)

$$K_0(O_n) \approx \mathbf{Z}/(n-1)\mathbf{Z} \text{ and } K_1^{\text{top}}(O_n) = 0.$$

For $\mathcal{N}$ a factor of type II_1, we have

$$K_0(\mathcal{N}) \approx \mathbf{R} \text{ and } K_1(\mathcal{N}) = \mathbf{R}_+^*.$$

For K_0, see Proposition (4.1.9); for K_1, see [159], already cited for Proposition (4.1.7).ii. More generally, for $\mathcal{N}$ a von Neumann algebra of type II_1, with center denoted by $\mathcal{Z}$, we have

$$K_0(\mathcal{N}) \approx \{z \in \mathcal{Z} \mid z^* = z\},$$

where the right-hand side is viewed as a group for the addition, and

$$K_1(\mathcal{N}) \approx \{z \in \mathcal{Z} \mid z \geq \epsilon > 0\} \ (\epsilon \text{ depends on } z),$$

where the right-hand side is viewed as a group for the multiplication; see [175] or [168]. For any von Neumann algebra $\mathcal{N}$

$$K_1^{\text{top}}(\mathcal{N}) = 0,$$

because $\text{GL}_n(\mathcal{N})$ is connected for all $n \geq 1$; indeed, by polar decomposition and functional calculus, any $x \in \text{GL}_n(\mathcal{N})$ is of the form $\exp(a)\exp(ib)$, with a, b self-adjoint in $\text{M}_n(\mathcal{N})$, so that x is connected to 1 by the path $\alpha \mapsto \exp(\alpha a)\exp(i\alpha b)$.

If $\mathcal{N}$ is a factor of type II_1, the isomorphism [144] of Bott periodicity shows that

$$\pi_1(\text{GL}_\infty(\mathcal{N})) \approx K_0(\mathcal{N}) \approx \mathbf{R}.$$

Thus, by Bott periodicity,

$$\pi_{2j}(\text{GL}_\infty(\mathcal{N})) = 0 \text{ and } \pi_{2j+1}(\text{GL}_\infty(\mathcal{N})) \approx \mathbf{R}$$

for all $j \geq 0$.

For π_1, it is known more precisely that $\pi_1(\text{GL}_n(\mathcal{N})) \approx \mathbf{R}$ and that the embedding of $\text{GL}_n(\mathcal{N})$ into $\text{GL}_{n+1}(\mathcal{N})$ induces the identity on π_1, for all $n \geq 1$(69,70). Note that, still for the norm topology, polar decomposition shows that the unitary group $\text{U}_1(\mathcal{N})$ is a deformation retract of $\text{GL}_1(\mathcal{N})$; in particular, we have also $\pi_1(\text{U}_1(\mathcal{N})) \approx \mathbf{R}$.

For the strong topology, the situation is quite different; indeed, for "many" I_1-factors, for example for those associated to infinite amenable icc groups or to nonabelian free groups, it is known that the group $U_1(\mathcal{N})^{\text{strong topology}}$ is contractible (71).

Most can be found in [179]. For other expositions of part of what follows, see [180]. Let A be a complex Banach algebra (with unit, again for reasons of simplicity), E be a Banach space, and $\tau: A \to E$ be a continuous linear map that is tracial, namely such that $\tau(yx) = \tau(xy)$ for all $x, y \in A$. Then τ extends to a continuous linear map $\text{M}_\infty(A) \to E$, defined by $x \mapsto \sum_{j \geq 1} \tau(x_{j,j})$, and again denoted by τ. If $e, f \in \text{M}_\infty(A)$ are equivalent idempotents, we have $\tau(e) = \tau(f)$; it follows that τ induces a homomorphism of abelian groups

$$\underline{\tau}: K_0(A) \to E,\ [e] \mapsto \tau(e).$$

For example, if $A = \mathbf{C}$ and $\tau: \mathbf{C} \to \mathbf{C}$ is the identity, the stable equivalence class of an idempotent $e \in \text{M}_n(\mathbf{C})$ is precisely described by the dimension of the image $\text{Im}(e) \subset \mathbf{C}^n$, so that $K_0(\mathbf{C}) \approx \mathbf{Z}$, and the image of $\underline{\tau}$ is the subgroup $\mathbf{Z}$ of the additive group $\mathbf{C}$.

For a piecewise differentiable path $\xi: [\alpha_1, \alpha_2] \to \text{GL}_\infty^0(A)$, we define

$$\tilde{\Delta}_\tau(\xi) = \frac{1}{2\pi i}\tau\left(\int_{\alpha_1}^{\alpha_2} \dot{\xi}(\alpha)\xi(\alpha)^{-1} d\alpha\right) = \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\big(\dot{\xi}(\alpha)\xi(\alpha)^{-1}\big) d\alpha. \quad (18)$$

(If X is a compact space, for example if $X = [\alpha_1, \alpha_2] \subset \mathbf{R}$, the image of a continuous map $X \to \mathrm{GL}_\infty^0(A)$ is inside $\mathrm{GL}_n(A)$, and therefore in the Banach space $\mathrm{M}_n(A)$, for n large enough; the integral can therefore be defined naively as a limit of Riemann sums.)
The normalization in [145] is such that, if $A = \mathbf{C}$ and $\tau = \mathrm{id}$, the loop defined by $\xi_0(\alpha) = \exp(2\pi i\alpha)$ for $\alpha \in [0,1]$ gives rise to $\tilde{\Delta}_\tau(\xi_0) = 1$.
Lemma (4.1.10)[127]: Let A be a complex Banach algebra with unit, E be a Banach space, $\tau: A \to E$ be a tracial continuous linear map, and

$$\tilde{\Delta}_\tau: \left\{ \text{paths in } \mathrm{GL}_\infty^0(A) \text{ as above} \right\} \to E$$

be the mapping defined by [145].
i) If ξ is the pointwise product of two paths ξ_1, ξ_2 from $[\alpha_1, \alpha_2]$ to $\mathrm{GL}_\infty^0(A)$, then $\tilde{\Delta}_\tau(\xi) = \tilde{\Delta}_\tau(\xi_1) + \tilde{\Delta}_\tau(\xi_2)$.
ii) If $\| \xi(\alpha) - 1 \| < 1$ for all $\alpha \in [\alpha_1, \alpha_2]$, then $\tau\left(\dot{\xi}(\alpha)\xi(\alpha)^{-1}\right) \mathrm{d}\,\alpha$ has a primitive $\tau(\log \xi(\alpha))$, so that

$$2\pi i \tilde{\Delta}_\tau(\xi) = \tau\left(\log \xi(\alpha_2)\right) - \tau\left(\log \xi(\alpha_1)\right).$$

iii) $\tilde{\Delta}_\tau(\xi)$ depends only on the homotopy class of ξ.
iv) Let $e \in \mathrm{M}_\infty(A)$ be an idempotent and let ξ_e be the loop defined as in [143]; then

$$\tilde{\Delta}_\tau(\xi_e) = \tau(e) \in E.$$

Sketch of proof: Claim i follows from the computation

$$\begin{aligned}
\tilde{\Delta}_\tau(\xi_1\xi_2) &= \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\left(\left(\dot{\xi}_1(\alpha)\xi_2(\alpha) + \xi_1(\alpha)\dot{\xi}_2(\alpha)\right)\xi_2(\alpha)^{-1}\xi_1(\alpha)^{-1}\right) d\alpha \\
&= \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\left(\dot{\xi}_1(\alpha)\xi_1(\alpha)^{-1}\right) d\alpha + \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\left(\xi_1(\alpha)\dot{\xi}_2(\alpha)\xi_2(\alpha)^{-1}\xi_1(\alpha)^{-1}\right) d\alpha \\
&= \tilde{\Delta}_\tau(\xi_1) + \tilde{\Delta}_\tau(\xi_2).
\end{aligned}$$

Claims ii and iii are straightforward. Claim iv follows again from an easy computation.
Definition (4.1.11)[127]: Let A be a complex Banach algebra with unit, E be a Banach space, and $\tau: A \to E$ be a tracial continuous linear map. Define

$$\Delta_\tau: \mathrm{GL}_\infty^0(A) \to E/\underline{\tau}\left(K_0(A)\right) \tag{19}$$

to be the mapping that associates to an element x in the domain the class modulo $\underline{\tau}(K_0(A))$ of $\tilde{\Delta}_\tau(\xi)$, where ξ is any piecewise differentiable path in $\mathrm{GL}_\infty^0(A)$ from 1 to x.
Theorem (4.1.12)[127]: Let the notation be as above.
i) The mapping Δ_τ of [146] is a homomorphism of groups, with image $\tau(A)/\underline{\tau}(K_0(A))$; in particular Δ_τ is surjective if τ is surjective.
ii) $\Delta_\tau(e^y)$ is the class of $\tau(y)$ modulo $\underline{\tau}(K_0(A))$ for all $y \in \mathrm{M}_\infty(A)$.
Corollary (4.1.13)[127]: If $\tau: A \to \mathbf{C}$ is a trace such that $\underline{\tau}(K_0(A)) = \mathbf{Z}$, then

$$\exp(2\pi i \Delta_\tau): \mathrm{GL}_\infty^0(A) \to \mathbf{C}^*$$

is a homomorphism of groups, and

$$\exp(2\pi i \Delta_\tau)(e^y) = e^{\tau(y)} \tag{20}$$

for all $y \in \mathrm{M}_\infty(A)$. Compare with [130].
In particular, if $A = \mathbf{C}$ and if τ is the identity, then $\exp(2i\pi\Delta_\tau)$ is the usual determinant on $\mathrm{GL}_\infty(\mathbf{C})$.
Corollary (4.1.14)[127]: If $\mathcal{N}$ is a factor of type II_1 and τ its canonical trace, then $\underline{\tau}(K_0(\mathcal{N})) = \mathbf{R}$,

$$\exp(\mathrm{Re}(2\pi i \Delta_\tau)): \mathrm{GL}_\infty(\mathcal{N}) \to \mathbf{R}_+^* \tag{21}$$

is a surjective homomorphism of groups, and its restriction to $\mathrm{GL}_1(\mathcal{N})$ is the Fuglede-Kadison determinant.
If A is a separable Banach algebra given with a trace τ[††] then the range of Δ_τ is the quotient of C by a countable group, by Proposition (4.1.8) . Suppose that A is a C*-algebra with unit, that τ is a faithful tracial continuous linear form on A that is factorial, and that the GNS representation associated to τ provides an embedding $A \to \mathcal{N}$ into a factor of type II_1, where τ on A is the restriction of the canonical trace on $\mathcal{N}$.

Let A be a complex Banach algebra. Denote by E_u the Banach space quotient of A by the closed linear span of the commutators $[x, y] = xy - xy, x, y \in A$; thus $E_u = A/\overline{[A, A]}$. The canonical projection $\tau_u: A \to E_u$ is the universal tracial continuous linear map on A. In some cases, the space E_u has been characterized: For a finite von Neumann algebra $\mathcal{N}$ with center $\mathcal{Z}$, the universal trace (as defined in [151]) induces an isomorphism $E_u \approx \mathcal{Z}$ ([159]). Information on E_u for stable C*-algebras and simple AFC*-algebras can be found in [181] and [182].
To the universal τ_u corresponds the universal determinant

$$\Delta_u: \mathrm{GL}_\infty^0(A) \to E_u/\underline{\tau_u}(K_0(A)).$$

Observe that any tracial linear map $\tau: A \to \mathrm{C}$ is the composition $\sigma\tau_u$ of the universal τ_u with a continuous linear form σ on E_u. We have

$$D\mathrm{GL}_\infty^0(A) \overset{(1)}{\subset} \ker(\Delta_u) \overset{(2)}{\subset} \bigcap_{\sigma \in (E_u)^*} \ker\left(\Delta_{\sigma\tau_u}\right) \subset \mathrm{GL}_\infty^0(A). \qquad (22)$$

Both $\overset{(1)}{\subset}$ and $\overset{(2)}{\subset}$ can be strict inclusions, but $\overset{(2)}{\subset}$ is always an equality if A is separable. The last but one term on the right need not be closed in $\mathrm{GL}_\infty^0(A)$. For all this, see [179].

We agree that the universal determinant is sharp if the inclusions $\overset{(1)}{\subset}$ and $\overset{(2)}{\subset}$ are equalities, equivalently if the natural mapping from the kernel $\mathrm{GL}_\infty^0(A)/D\mathrm{GL}_\infty^0(A)$ of [142] to $E_u/\underline{\tau_u}(K_0(A))$ is an isomorphism.
If A is a simple AFC*-algebra with unit, its universal determinant is sharp. More precisely, if A is an AF$^{\text{algebra with unit,}}$ $\mathrm{GL}(A)$ is connected for all $n \geq 1$ and a fortiori so is $\mathrm{GL}_\infty(A)$. If A is moreover simple, then

$$D\mathrm{GL}_n(A) = \ker\left(\Delta_u: \mathrm{GL}_n(A) \to E_u/\underline{\tau_u}(K_0(A))\right)$$

for all $n \geq 1$, and a similar equality holds for $\mathrm{U}_n(A)$ and $D\mathrm{U}_n(A)$ ([183], theorem I and proposition 6.7). If A is a simple C*-algebra with unit that is infinite, there are no traces on A (75), and therefore no Δ_τ, and $\mathrm{GL}_n^0(A)$ is a perfect group ([183]).
Moreover, if G is one of these groups, the quotient of DG by its center is a simple group (58).

We follow [183].

Let $\mathcal{R}$ be a ring; we assume that free $\mathcal{R}$-modules of different finite ranks are not isomorphic. Let F be a free $\mathcal{R}$-module of finite rank, say n; let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two bases of F. There is a matrix $x \in \mathrm{GL}n(\mathcal{R})$ such that $a_j = \sum_{k=1}^{n} x_{j,k} b_k$, and therefore a class of x in $\bar{K}_1(\mathcal{R})$, denoted by $[b/a]$
Let

$$C: 0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0 \qquad (23)$$

be a chain complex of free $\mathcal{R}$-modules of finite ranks such that the homology groups H_i are also free $\mathcal{R}$-modules (the latter is automatic if $H_i = 0$, a case of interest in topology). Suppose that, for each i, there is given a basis c_i of C_i and a basis h_i of H_i (the latter is automatic if $H_i = 0$).

Assume, first, that each boundary submodule B_i is also free, with a basis b_i. Using the inclusions $0 \subset B_i \subset Z_i \subset C_i$ and the isomorphisms $Z_i/B_i \approx H_i, C_i/Z_i \approx B_{i-1}$, there is a natural way to define (up to some choices) a second basis of C_i, denoted by $b_i h_i b_{i-1}$. By definition, the torsion of C, given together with the basis c_i and h_i, is the element ##

$$\tau(C) = \sum_{i=0}^{n} (-1)^i \left[\frac{b_i h_i b_{i-1}}{c_i}\right] \in \bar{K}_1(\mathcal{R}). \tag{24}$$

It can be shown to be independent of the other choices made to define $b_i h_i b_{i-1}$; in particular, the signs $(-1)^i$ are crucial for $\tau(C)$ to be independent of the choices of the basis b_i s.

In the case that the hypothesis on B_i being free is not fulfilled, it is easy to check that the B_i s are stably free, and there is a natural way to extend the definition of $\tau(C)$. This can be read in [183]. (An $\mathcal{R}$-module A is stably free if there exists a free $\mathcal{R}$-module F such that $A \oplus F$ is free.)

Suppose now that C is acyclic, namely that $H_*(C) = 0$. There exists a chain contraction, namely a degree-one morphism $\delta: C \to C$ such that $\delta d + d\delta = 1$, and therefore an isomorphism

$$d + \delta|_{\text{odd}} : C_{\text{odd}} = C_1 \oplus C_3 \oplus \cdots \to C_{\text{even}} = C_0 \oplus C_2 \oplus \cdots. \tag{25}$$

Because C_{odd} and C_{even} have bases (from the c_i s), this isomorphism defines an element in $\bar{K}_1(\mathcal{R})$; we have

$$\tau(C) = \text{ class of } d + \delta|_{\text{odd}} \text{ in } \bar{K}_1(\mathcal{R}) \tag{26}$$

([181]). Formula [153] is sometimes better suited than [151].

Consider a pair (K, L) consisting of a finite connected CW complex K and a subcomplex L that is a deformation retract of K; set $\Gamma = \pi_1(L) \approx \pi_1(K)$. For a CW pair (X, Y), consider the complex that defines cellular homology theory, with groups $C_i^{\text{CW}}(X, Y) = H_i^{\text{sing}}\left(|X^i \cup Y|, |X^{i-1} \cup Y|\right)$; here, H_i^{sing} denotes singular homology with trivial coefficients $\mathbf{Z}$, and $|X^i \cup Y|$ denotes the space underlying the union of the ith skeleton of X with Y. If $\tilde{K}$ and $\tilde{L}$ denote the universal covers of L and K, the groups $C_i^{CW}(\tilde{K}, \tilde{L})$ are naturally free $\mathbf{Z}[\Gamma]$-modules; moreover, they have free bases as soon as a choice has been made of one oriented cell in $\tilde{K}$ above each oriented cell in K. For each of these choices, and the corresponding basis, we have a torsion element $\tau\left(C^{\text{CW}}(K, L)^{+\text{choices}}\right) \in \bar{K}_1(\mathbf{Z}[\Gamma])$. To obtain an element independent of these choices, it suffices to consider the quotient $\text{Wh}(\Gamma) = K_1(\mathbf{Z}[\Gamma])/\langle\{1, -1\}, \Gamma\rangle$. The class

$$\tau(K, L) \in \text{Wh}(\Gamma)$$

of $\tau\left(C^{\text{CW}}(K, L)^{+\text{choices}}\right)$ is the Whitehead torsion of the pair (K, L). In 1966, it was known to be combinatorially invariant (namely invariant by subdivision of CW pairs); more on this is in [183]. Since then, it has been shown to be a topological invariant of the pair (|K|,|L|) (77); this was a spectacular success of infinite-dimensional topology (manifolds modeled on the Hilbert cube and all that).

An h-cobordism is a triad $(W; M, M')$ where W is a smooth manifold whose boundary is the disjoint union $M \sqcup M'$ of two closed submanifolds, such that both M and M' are deformation retracts of W. Products $W = M \times [0,1]$ provide trivial examples; in [185], there is a nontrivial example of an h-cobordism $(W, L \times \mathbf{S}^4, L' \times \mathbf{S}^4)$, with L and L' two three-dimensional lens manifolds that are homotopically equivalent but not homeomorphic. By a 1965 result of Stallings ([183]):

If $\dim M \geq 5$, any $\tau \in \mathrm{Wh}(\pi_1(M))$ is of the form $\tau(W, M)$ for some h-cobordism $(W; M, M')$.

Together with the s-cobordism theorem (below), this implies:

For two h-cobordisms $(W_1; M, M_1), (W_2; M, M_2)$ such that $\tau(W_1, M) = \tau(W_2, M)$, there exists a diffeomorphism $W_1 \to W_2$ that preserves M.

An h-cobordism gives rise to a chain complex and a torsion invariant $\tau(W, M) \in \mathrm{Wh}(\pi_1(M))$. Here is the basic s-cobordism theorem of Barden, Mazur, and Stallings (79):

If $\dim W \geq 6$, then W is diffeomorphic to the product $M \times [0,1]$ if and only if $\tau(W, M) = 0 \in \mathrm{Wh}(\pi_1(M))$.

In particular, if M is simply connected, then W is always diffeomorphic to $M \times [0,1]$; this is the h-cobordism theorem of [187].

For example, if Σ is a homotopy sphere of dimension $n \geq 6$, if W is the complement in Σ of two open discs with disjoint closures, and if S_0, S_1 are the boundaries of these discs (they are standard spheres), then $(W; S_0, S_1)$ is an h-cobordism, and W is diffeomorphic to $S^{n-1} \times [0,1]$. It follows that Σ is diffeomorphic to a manifold obtained by gluing together the boundaries of two closed n-balls under a suitable diffeomorphism and that Σ is homeomorphic to the standard n-sphere; the last conclusion is still true in dimension $n = 5$. This is the generalized Poincaré conjecture in large dimensions, established in the early 1960s. The first proof was that of Smale ([188] and [187]); very soon after, there were other proofs of other formulations of the Poincaré conjecture, logically independent of Smale's proof but inspired by his work, by Stallings (for $n \geq 7$) and Zeeman (for $n \geq 5$). The other dimensions were settled much later: by Freedman in 1982 for $n = 4$ and by Perelman in 2003 for $n = 3$.

Because K_1 is a functor, any linear representation $h: \Gamma \to \mathrm{GL}_k(\mathbf{R})$ provides a ring homomorphism $\mathbf{Z}[\Gamma] \to \mathrm{M}_k(\mathbf{R})$ and therefore a morphism of abelian groups

$$K_1(\mathbf{Z}[\Gamma]) \to K_1(\mathrm{M}_k(\mathbf{R})) = K_1(\mathbf{R}) \approx \mathbf{R}^*,$$

where $\approx$ is induced by the determinant $\mathrm{GL}_\infty(\mathbf{R}) \xrightarrow{\det} \mathbf{R}^*$, and also a morphism $\bar{K}_1(\mathbf{Z}[\Gamma]) \to \bar{K}_1(\mathbf{R}) \approx \mathbf{R}^*_+$. When the representation is orthogonal, $h: \Gamma \to O(k)$, this induces a morphism of abelian groups $\mathrm{Wh}(\Gamma) \to \mathbf{R}^*_+$.

For a complex of $\mathbf{Z}[\Gamma]$-modules C with torsion $\tau(C) \in \bar{K}_1(\mathbf{Z}[\Gamma])$, the image of $\tau(C)$ is the Reidemeister torsion $\tau_h(C) \in \mathbf{R}^*_+$, which is a real number [in fact, $\tau_h(C)$ may be well defined even in cases where $\tau(C)$ is not]. This is the basic invariant in important work by Reidemeister, Franz, and de Rham (earliest papers published in 1935).

Given a Riemannian manifold M and an orthogonal representation $h: \pi_1(M) \to O(k)$ of its fundamental group, one defines a complex C of differential forms with values in a bundle associated with h. Under appropriate hypotheses, one has a famous analytical expression of the Reidemeister-Franz-de Rham torsion and an equality

$$\tau_h(C) = \frac{1}{2}\sum_{k=0}^{n} (-1)^k \ln\det(d_k^* d_k) = \frac{1}{2}\sum_{k=0}^{n} (-1)^k k\ln\det(d_k^* d_k + d_{k+1}d_{k+1}^*) \quad (27)$$

Let $\mathcal{N}$ be a finite von Neumann algebra and let $\tau: \mathcal{N} \to \mathbf{C}$ be a finite trace. For $x \in \mathcal{N}$, let $(E_\lambda)_{\lambda\geq 0}$ denote the spectral resolution of $(x^*x)^{\frac{1}{2}}$. Define

$$\det{}_\tau^{FKL}(x) = \begin{cases} \exp \lim_{\epsilon\to 0+} \int_\epsilon^\infty \ln(\lambda)\, d\tau(E_\lambda) & \text{if } \lim_{\epsilon\to 0+} \int_\epsilon^\infty \cdots > -\infty, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

It is immediate that $\det_\tau^{FKL}(x) = \det_\tau^{FK}(x)$ when x is invertible, but the equality does not hold in general (det $_\tau^{FK}(x)$ is as in [134]). For example, if $x \in \mathrm{GL}_1(\mathcal{N})$, and $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{N})$, we have

$$0 = \det{}_\tau^{FK}(X) \neq \det{}_\tau^{FKL}(X) = \det{}_\tau^{FKL}(x) = \det{}_\tau^{FK}(x) > 0.$$

The main properties of $\det_\tau^{FKL}$, including

$$\det{}_\tau^{FKL}(xy) = \det{}_\tau^{FKL}(x)\det{}_\tau^{FKL}(y) \text{ for } x, y \in \mathcal{N} \text{ such that } x \text{ is injective and } y \text{ has dense image,}$$

are given in [168].

Since Atiyah's work on the L^2-index theorem (83), we know that (complexes of) $\mathcal{N}$-modules are relevant in topology, say for $\mathcal{N} = \mathcal{N}(\Gamma)$ and for Γ the fundamental group of the relevant space. Let $\mathcal{N}$ and τ be as above. Let

$$C: 0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

be a finite complex of $\mathcal{N}$-modules, with appropriate finiteness conditions on the modules (they should be projective of finite type), with a condition of acyclicity on the homology (the image of d_j should be dense in the kernel of d_{j-1} for all j), and with a nondegeneracy condition on the differentials d_j (which should be of "determinant class", namely $\det_\tau^{FKL}(d_j^* d_j)$ should be as in the first case of [155]). The L^2-torsion of C is defined to be

$$\rho^{(2)}(C) = \sum_{k=0}^{n} (-1)^k \ln\det{}_\tau^{FKL}\left(\left(d_j^* d_j\right)^{\frac{1}{2}}\right) \in \{-\infty\} \sqcup \mathbf{R} \quad (29)$$

(compare with [154]). There is an L^2-analog of [153].
L^2-torsion, and related notions, have properties that parallel those of classical torsions, in particular of Whitehead torsion, and seem to be relevant for geometric problems, e.g., for understanding volumes of hyperbolic manifolds of odd dimensions. We refer (once more) to [168].

It is tempting to ask whether (or even speculate that!) modules over reduced C*-algebras $A = C^*_{\text{red}}(\Gamma)$ and refinements $\Delta_\tau^{(A)}$ will be relevant one time or another, rather than modules over $\mathcal{N}(\Gamma)$ and Fuglede-Kadison determinants det de$_\tau^{FK}(\cdot)$.

Section (4.2): Determinants of Perturbed Positive Matrices and Linear Algebra

Given k complex square matrices $B_1, \ldots, B_k$ of format $n_1 \times n_1, n_2 \times n_2, \ldots, n_k \times n_k$, let us denote by diag $(B_1, \ldots, B_k)$ the matrix of the format $(n_1 + \cdots + n_k) \times (n_1 + \cdots + n_k)$ whose main diagonal blocks are $B_1, \ldots, B_k$ and all other entries are 0. In other words:

$$diag\ (\mathrm{B}_1, B_k) = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & B_k \end{bmatrix}.$$

Given two vectors u, $v \in \mathbb{C}^k$ such that $u = \langle u_1, u_k \rangle$ and $= \langle v_1, v_k \rangle$, we define their inner product $\langle u, v \rangle = \sum_{i=1}^{k} u_i \overline{v_i}$. For a complex $n \times m$ matrix R, we use R^* to denote its adjoint matrix. In other words, R^* is the transpose of the complex conjugate of R, and for $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$ the following is satisfied: $\langle Ru, v \rangle = \langle u, R^* v \rangle$. A square matrix A is self-adjoint if $A^* = A$.

The self-adjoint matrix A of format $n \times n$ is called positive (or positive definite) if $\langle Ax, x \rangle > 0$ for each non-zero vector $x \in \mathbb{C}^n$. If the strict inequality is replaced by $\geq$, the matrix is called non-negative (or positive semi-definite). If A and B are two square matrices of the same format, we will write $A \geq B$ (resp. $A > B$) if $A - B \geq 0$ (resp. $A - B > 0$) .

For $n \in \mathbb{N}$ we will denote by I_n the $n \times n$ identity matrix. The subscript n will be omitted when there is no danger of ambiguity.

We will prove the following two inequalities regarding positive matrices with complex entries.

The two inequalities presented have the flavor of Fischer's determinantal inequality, although in (30) the sign is reversed. An inequality related to our results, which features quotients of perturbed matrices, has been established previously [219]. For refinements of Fischer-type inequalities with singular values, see [215] and [216]. After taking the logarithms of left and right sides of the inequality (31), one obtains

$$\phi\big(C, \mathrm{diag}(D_1, D_2)\big) \leq \phi(C_1, D_1) + \phi(C_2, D_2)\ ,$$

where $(X, Y) = \log\ \det(X + Y) - \log\ \det(X)$. Similar inequalities are known to hold for concave functions ϕ, and such results can be found in [213].

The proof of the first theorem relies on Lemma (4.2.7) which is established using Grothendieck's determinantal inequality. The lemma implies that the function $U \mapsto \det(U + D)/\det(U)$ is operator-decreasing. Several results about operator-monotone functions are available in [218]. Generalizations and improvements of Grothendieck's inequality have been established in [214] and [222] and they have been used in the past to prove results regarding block matrices.

The inequality (31) can be used to establish super-additivity for functions of diffusions in random environments. We consider one-dimensional Brownian motion Z, and let W be another Brownian motion independent on Z. Define

$$f(t) = \log \mathbb{E}\left[\exp\left(-\int_0^t |W\big(Z(s)\big)|^2 ds\right)\right], \qquad (30)$$

where E denotes the expected value with respect to the Brownian motion W. We will now illustrate that $f(t_1 + t_2) \geq f(t_1) + f(t_2)$ is a special case of the inequality (31). Assume that $0 = s_0 < s_1 < \cdots < s_n = t_1$ is the partition of the interval $[0, t_1]$ into n sub-intervals of length λ_1. Similarly, let $t_1 = s_n < s_{n+1} < \cdots < s_{n+m} = t_1 + t_2$ be the partition of the interval $[t_1, t_1 + t_2]$ into m intervals of length λ_2. Let us denote

$$W_i = W\big(Z(s_i)\big)\ ,$$
$$\vec{w}_1 = \langle W_1, W_n \rangle,$$

$$\vec{w}_2 = \langle W_{n+1}, \dots, W_{n+m}\rangle, and$$
$$\vec{w} = \langle W_1, \dots, W_{n+m}\rangle.$$
Then $\sum_{i=1}^{n} |W_i|^2\lambda_1 = \langle \lambda_1 I\vec{w}_1, \vec{w}_1\rangle$, $\sum_{i=n+1}^{n+m} |W_i|^2\lambda_2 = \langle \lambda_2 I\vec{w}_2, \vec{w}_2\rangle$, and
$$\sum_{i=1}^{n} |W_i|^2\lambda_1 + \sum_{i=n+1}^{n+m} |W_i|^2\lambda_2 = \langle \mathrm{diag}(\lambda_1 I_n, \lambda_2 I_m)\vec{w}, \vec{w}\,\rangle.$$
If we fix the Brownian motion Z, then $[W(Z(s_1)), \dots, W(Z(s_2))]$ is a multivariate Gaussian random variable and as such it has a covariance matrix C. Denote by C_1 and C_2 the covariance matrices of $[W(Z(s_1)), \dots, W(Z(s_n))]$ and $[W(Z(s_{n+1})), \dots, W(Z(s_{n+m}))]$. Then C_1 and C_2 are the diagonal blocks of C. Moreover,
$$\mathbb{E}[\exp(-\langle \lambda_1 I\,\vec{w}_1, \vec{w}_1\rangle)] = \frac{1}{M\sqrt{\det C_1}}\int e^{-\langle \vec{w}_1\vec{w}_1\rangle - \frac{1}{2}\langle C_1\vec{w}_1, \vec{w}_1\rangle}\,\lambda_1 d\vec{w}_1$$
$$= \frac{1}{M\sqrt{\det C_1}}\int e^{-\langle (C_1^{-1}+\lambda_1 I)\overrightarrow{w_1}, \overrightarrow{w_1}\rangle}\,d\vec{w}_1$$
$$= \frac{\sqrt{\det\ C_1^{-1}}}{\sqrt{\det(C_1^{-1}+\lambda_1 I)}},$$
where M is a normalizing constant. We obtain similar equalities for the quantities $\mathrm{E}[\exp(-\langle \lambda_2 I\vec{w}_2, \vec{w}_2\rangle)]$ and $\mathrm{E}[\exp(-\langle \mathrm{diag}(\lambda_1 I_n, \lambda_2 I_m)\vec{w}, \vec{w}\rangle)]$. The inequality (31) implies that
$$\log \mathbb{E}[\exp(-\langle \mathrm{diag}(\lambda_1 I_n, \lambda_2 I_m)\vec{w}, \vec{w}\rangle)]$$
$$\geq \log \mathbb{E}[\exp(-\langle \lambda_1 \mathrm{I}\vec{w}_1, \vec{w}_1\rangle)] + \log \mathbb{E}[\exp(-\langle \lambda_2 \mathrm{I}\vec{w}_2, \vec{w}_2\rangle)].$$
Taking the limit as $\lambda_1, \lambda_2 \to 0$ we obtain $(t_1 + t_2) \geq f(t_1) + f(t_2)$.
This technique is potentially useful for establishing large deviations for random processes with drifts. Sub-additive properties are known to hold for killed Brownian motions in random environments [223], [224]. However, in the case of drifts introduced to random diffusions, no analogous results have yet been established. If a drift is assumed to be a multivariate Gaussian process, a possible approach is to express the large deviation probabilities in terms of determinants. However, there is still work to be done to transform the general case of sub-additive inequalities into the language of their covariance matrices [221].

We will start with listing the known theorems that we will use to establish the inequalities. For the derivations of the results presented, the reader is referred to [220].

Theorem (4.2.1) [212]: *If A and B are non-negative matrices such that $A \geq B$ then* $\det A \geq \det\ B$. *If both of them are invertible then $A^{-1} \leq B^{-1}$*

Theorem (4.2.2) [212]: *Let A and D be square $n \times n$ and $m \times m$ matrices respectively. Assume that B and C are matrices of the formats $n \times m$ and $m \times n$ and assume that A and $S_A = D - CA^{-1}B$ are invertible. Then*
$$\begin{bmatrix} \mathrm{A} & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1} \\ -S_A^{-1}CA^{-1} & S_A^{-1} \end{bmatrix}.$$
Similarly, if D and $S_D = A - BD^{-1}C$ are invertible, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_D^{-1} & -S_D^{-1}BD^{-1} \\ -D^{-1}CS_D^{-1} & D^{-1}+D^{-1}CS_D^{-1}BD^{-1} \end{bmatrix}.$$

The matrices S_A and S_D are called Schur complements of A and D.

The following two consequences of the previous result are known as Woodbury's matrix identity and Fischer's inequality.

Theorem (4.2.3) [212]: *Let A, B, C, D, S_A, S_D be as in Theorem* (4.2.2). *Then*

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}BS_A^{-1}CA^{-1}$$

provided that the inverses are defined. Moreover if the matrix $M = \begin{Bmatrix} AB \\ CD \end{Bmatrix}$ is positive, then S_A and S_D are positive.

Theorem (4.2.4) [212]: *Let A, B, C, D, S_A, S_D be as in Theorem* (4.2.2). *Let $M = \begin{Bmatrix} A & B \\ C & D \end{Bmatrix}$. Then*

$$\det M = \det A \cdot \det S_A.$$

If the matrix M is positive then $\det S_A \leq \det D$, and $\det M \leq \det A \cdot \det D$.

Theorem (4.2.5) [212]: *If A and B are non-negative symmetric matrices of the format $n \times n$ and I the $n \times n$ identity matrix then*

$$\det(I + A + B) \leq \det(I + A)\det(I + B). \qquad (31)$$

We will start by proving Theorem (4.2.6) since it is easier to prove than Theorem (4.2.8).

Theorem (4.2.6) [212]: *Assume that $k \in \mathbb{N}$ and that n_1, n_k are positive integers. Assume that $(C_i)_{i=1}^k$ and $(D_i)_{i=1}^k$ are two sequences of positive matrices such that for each $i \in \{1,2,,k\}$ the matrices C_i and D_i are of format $n_i \times n_i$. Assume that C is a positive matrix such that the diagonal blocks of C^{-1} are C_1^{-1}, C_k^{-1} The following inequality holds*:

$$\frac{\det\left(C + \mathrm{diag}(D_1, \dots, D_k)\right)}{\det C} \leq \frac{\det(C_1 + D_1)}{\det C_1} \dots \frac{\det(C_k + D_k)}{\det C_k}. \qquad (31)$$

Proof. Let us denote $B_i = C_i^{-1}$ for $i \in \{1,2,k\}$ and $B = C^{-1}$. Using the multiplicative property of determinants we transform the inequality (31) into equivalent one:

$$\det(I + B\, diag(D_1, D_k)) \leq det(I + B_1D_1) \cdots \det(I + B_kD_k). \quad (32)$$

Since the matrices $D_1, \dots, D_k$ are positive we have that each of them has a square root. In other words, for each i, there exists a unique positive matrix $\sqrt{D_i}$ that commutes with D_i and satisfies $D_i = \sqrt{D_i} \cdot \sqrt{D_i}$. Clearly, the matrix diag $\left(\sqrt{D_1}, \dots, \sqrt{D_k}\right)$ is the square root of diag $(D_1, \dots, D_k)$.

Applying Sylvester's determinant identity $\det(I + XY) = \det(I + YX)$ to the matrices $X = B\sqrt{\mathrm{diag}(D_1, \dots, D_k)}$ and $= \sqrt{\mathrm{diag}(D_1, \dots, D_k)}$ we transform the left-hand side of (32) into:

$$\det\left(I + B\mathrm{diag}(D_1, \dots, D_k)\right)$$
$$= \det\left(I + \mathrm{diag}\left(\sqrt{D_1}, \dots, \sqrt{D_k}\right) \cdot B \cdot \mathrm{diag}\left(\sqrt{D_1}, \dots, \sqrt{D_k}\right)\right).$$

Similarly, the right-hand side of (32) is:

$$\det(I + B_1D_1) \cdots \det(I + B_kD_k)$$
$$= \det\left(I + \sqrt{D_1}B_1\sqrt{D_1}\right) \dots \left(I + \sqrt{D_k}B_k\sqrt{D_k}\right).$$

We will use the induction on k to prove the inequality (32). Let us start with $k = 2$ and assume that $B = \begin{Bmatrix} B_1 & R \\ R^* & B_2 \end{Bmatrix}$ for an $n_1 \times n_2$ matrix R. Elementary calculations imply:

$$\det (I + \ diag\left(\sqrt{D_1}, \sqrt{D_2}\right) \cdot B \cdot \ diag\left(\sqrt{D_1}, \sqrt{D_2}\right))$$
$$= \ \det \begin{bmatrix} I + \sqrt{D_1} B_1 \sqrt{D_1} \sqrt{D_1} R \sqrt{D_2} \\ \sqrt{D_1} R^* \sqrt{D_1} I + \sqrt{D_2} B_2 \sqrt{D_2} \end{bmatrix}.$$

We can now use Theorem (4.2.4) to conclude that

$$\det \left(I + \mathrm{diag}\left(\sqrt{D_1}, \sqrt{D_2}\right) \cdot B \cdot \mathrm{diag}\left(\sqrt{D_1}, \sqrt{D_2}\right)\right)$$
$$\leq \ \det \left(I + \sqrt{D_1} B_1 \sqrt{D_1}\right) \cdot \ \det \left(I + \sqrt{D_2} B_2 \sqrt{D_2}\right)$$
$$= \ \det (I + D_1 B_1) \cdot \ \det (I + D_2 B_2) \ .$$

Therefore the inequality (32) is established for $k = 2$.

Assume now that $k \geq 3$ and that the inequality (32) is true for $k - 1$. Assume that $D_1, \ldots, D_k$, and $B_1, \ldots, B_k$ are positive matrices of formats $n_1 \times n_1, \ldots, n_k \times n_k$. Assume that B_2' is the sub-matrix of the matrix B obtained by removing the first n_1 rows and first n_1 columns. According to the induction hypothesis we have

$$\det (I + B_2' \ diag(D_2 \ , \ldots, D_k)) \ \leq \ det \ (I + B_2 D_2) \cdots \ \det (I + B_k D_k) \ . \quad (33)$$

We denote $D_2' = \mathrm{diag} \ (D_2, \ldots, D_k)$. Then we can write $\mathrm{diag} \ (\mathrm{D}_1, \ldots, D_k) = \mathrm{diag}(D_1, D_2')$. Applying the inequality (32) with $k = 2$ we obtain

$$\det \left(I + B\mathrm{diag}(D_1, D_2')\right) \leq \ \det (I + B_1 D_1) \cdot \ \det (I + B_2' D_2') \ . \quad (34)$$

The inequalities (33) and (34) together imply the inequality (32). This completes the proof of Theorem (4.2.6).

In order to prove Theorem (4.2.8) we will need the following lemma.

Lemma (4.2.7) [212]: *Assume that $U \geq V$ and D are $n \times n$ non-negative matrices such that U and V are invertible. Then the following inequality holds*:

$$\frac{\det (V + D)}{\det V} \geq \frac{\det (U + D)}{\det U}.$$

Proof. The matrix V^{-1} is positive and as such it has a positive square root. Let us denote it by $V^{-\frac{1}{2}}$. Assume that $U = V + W$ for some non-negative matrix W. The required inequality is equivalent to

$$\det (V + W) \det (V + D) \geq \ \det V \cdot \ \det (V + W + D) \ .$$

In our next step we multiply both left and right side of the previous inequality by $\left[\det \left(V^{-\frac{1}{2}}\right)\right]^4$:

$$\det \left(V^{-\frac{1}{2}}\right) \det (V + W) \det \left(V^{-\frac{1}{2}}\right) \cdot \ \det \left(V^{-\frac{1}{2}}\right) \det (V + D) \det \left(V^{-\frac{1}{2}}\right)$$
$$\geq \ \det \left(V^{-\frac{1}{2}}\right) \det (V + W + D) \det \left(V^{-\frac{1}{2}}\right).$$

The last inequality is equivalent to:

$$\det \left(I + V^{-\frac{1}{2}} W V^{-\frac{1}{2}}\right) \cdot \ \det \left(I + V^{-\frac{1}{2}} D V^{-\frac{1}{2}}\right)$$
$$\geq \ \det \left(I + V^{-\frac{1}{2}} W V^{-\frac{1}{2}} + V^{-\frac{1}{2}} D V^{-\frac{1}{2}}\right).$$

The last inequality follows when we apply (31) to the positive definite matrices $A = V^{-\frac{1}{2}}WV^{-\frac{1}{2}}$ and $B = V^{-\frac{1}{2}}DV^{-\frac{1}{2}}$.

Theorem (4.2.8) [212]: *Assume that $k \in \mathbb{N}$ and that n_1, n_k are positive integers. Assume that $(C_i)_{i=1}^{k}$ and $(D_i)_{i=1}^{k}$ are two sequences of positive matrices such that for each $i \in \{1,2,\dots,k\}$ the matrices C_i and D_i are of format $n_i \times n_i$. Assume that C is a positive matrix whose diagonal blocks are $C_1,\dots,C_k$. The following inequality holds*:

$$\frac{\det\left(C + \operatorname{diag}(D_1,\dots,D_k)\right)}{\det C} \geq \frac{\det(C_1 + D_1)}{\det C_1} \cdots \frac{\det(C_k + D_k)}{\det C_k}. \qquad (35)$$

Proof. We will first prove the theorem for the case $k = 2$. Assume that $C = \begin{Bmatrix} C_1 & R \\ R^* & C_2 \end{Bmatrix}$ for some matrix R of the format $n_1 \times n_2$. From Theorem (4.2.4) we conclude that the required inequality is equivalent to

$$\frac{\det(C_1 + D_1)\det[(C_2 + D_2) - R^*(C_1 + D_1)^{-1}R]}{\det C_1 \det(C_2 - R^*C_1^{-1}R)} \geq \frac{\det(C_1 + D_1)\cdot\det(C_2 + D_2)}{\det C_1 \cdot \det C_2}.$$

This inequality can be re-written as

$$\frac{\det[(C_2 + D_2) - R^*(C_1 + D_1)^{-1}R]}{\det(C_2 + D_2)} \geq \frac{\det(C_2 - R^*C_1^{-1}R)}{\det C_2}.$$

Let us denote $V = C_2 - R^*(C_1 + D_1)^{-1}R$. We can prove that $V > 0$ by applying Theorem (4.2.3) to the matrix $\tilde{C} = \begin{Bmatrix} C_1 + D_1 & R \\ R^* & C_2 \end{Bmatrix}$. The theorem requires the positivity of $\tilde{C}$, and this is satisfied since $\tilde{C} = C + \operatorname{diag}(D_1, 0)$. Let $U = C_2$. Moreover, $U - V = R^*(C_1 + D_1)^{-1}R \geq 0$, therefore we can apply Lemma (4.2.7) to matrices U and V to obtain:

$$\frac{\det[(C_2 + D_2) - R^*(C_1 + D_1)^{-1}R]}{\det(C_2 + D_2)} \geq \frac{\det[C_2 - R^*(C_1 + D_1)^{-1}R]}{\det C_2}.$$

From $C_1 + D_1 \geq C_1$ we have $(C_1 + D_1)^{-1} \leq C_1^{-1}$. Therefore

$$R^*(C_1 + D_1)^{-1}R \leq R^*C_1^{-1}R, and$$
$$C_2 - R^*(C_1 + D_1)^{-1}R \geq C_2 - R^*C_1^{-1}R.$$
$$\det(C_2 - R^*(C_1 + D_1)^{-1}R) \geq \det(C_2 - R^*C_1^{-1}R)$$

Theorem (4.2.1) now implies

which completes the proof of Theorem (4.2.8) when $k = 2$.

We will use induction to finish the proof for general $k \in \mathbb{N}$. Assume that $k \geq 3$ and that the statement is true for $k - 1$. We will now prove the inequality for matrices $C_1,\dots,C_k$, $D_1,\dots,D_k$. Let us denote by C_2' the sub-matrix of the matrix C obtained by removing its first n_1 rows and first n_1 columns. The matrix C can be regarded as a block matrix with diagonal blocks C_1 and C_2'. Similarly, for $D_2' = \operatorname{diag}(D_2,\dots,D_k)$ we have that diag $(\mathrm{D}_1,\dots,D_k) = \operatorname{diag}(D_1,\dots,D_2')$. Using the induction hypothesis we obtain

$$\frac{\det\left(C_2' + \operatorname{diag}(D_2,\dots,D_k)\right)}{\det C_2'} \geq \frac{\det(C_2 + D_2)}{\det C_2} \cdots \frac{\det(C_k + D_k)}{\det C_k}. \qquad (36)$$

Using the inequality established for $k = 2$ we conclude

$$\frac{\det\left(C+\operatorname{diag}(D_1,\dots,D_k)\right)}{\det C}=\frac{\det\left(C+\operatorname{diag}(D_1,D_2')\right)}{\det C}$$
$$\geq\frac{\det(C_1+D_1)}{\det C_1}\cdot\frac{\det(C_2'+D_2')}{\det C_2'}.\quad(37)$$

The inequalities (36) and (37) imply the desired result.

we will show that neither of the above inequalities can be generalized to allow for diag $(\mathrm{D}_1,\dots,D_k)$ to be replaced by an arbitrary positive matrix whose diagonal blocks are $D_1,\dots,D_k$. We first present an example that illustrates the case in which the reverse inequality occurs in (35) under previously mentioned generalization. Take

$$C=\begin{bmatrix}10 & 2\\ 2 & 5\end{bmatrix}\ and\ D=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}.$$

Then $C_1=10$, $C_2=5$, $D_1=2$, $D_2=1$ and a simple calculation shows that the left-hand side of (35) corresponds to

$$\frac{\det(C+D)}{\det C}=\frac{63}{46},$$

while the right-hand side is

$$\frac{\det(C_1+D_1)}{\det C_1}.\frac{\det(C_2+D_2)}{\det C_2}=\frac{72}{50}=\frac{36}{25}>\frac{63}{46}.$$

To see a counter-example to (31) if we allow for diag $(\mathrm{D}_1,\dots,D_k)$ to be replaced by general D, we consider

$$C=\begin{bmatrix}2 & -2\\ 2 & 4\end{bmatrix}\ and\ D=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}.$$

We now have $D_1=1$ and $D_2=2$. In order to determine C_1 and C_2 we first find

$$C^{-1}=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

which implies that $C_1=1$ and $C_2=2$. It is now easy to find that the left-hand side of (31) corresponds to

$$\frac{\det(C+D)}{\det C}=\frac{17}{4}>4=\frac{\det(C_1+D_1)}{\det C_1}\cdot\frac{\det(C_2+D_2)}{\det C_2}.$$

Thus, the inequality (31) does not always hold if the matrix diag $(\mathrm{D}_1,\dots,D_k)$ is replaced with a positive matrix D of a more general form.

Section (4.3): The Hadamard Determinant Inequality on Hilbert Space

Many applications of determinants in mathematical analysis are based on the geometric interpretation of the determinant of a square matrix as the (signed) volume of an n-parallelepiped with sides as the column vectors of the matrix. The change-of-variables formula in multidimensional integration involves the determinant of the Jacobian matrix. The study of estimates for the determinant of a matrix in terms of determinants of its principal submatrices is often useful as information about compressions of a matrix A to certain subspaces (*i.e.* PAP for a projection P) may be more readily available. An element of $M_n(\mathbb{C})$, the set of complex $n\times n$ matrices, is said to be *positive semidefinite* if it is Hermitian with non-negative eigenvalues, and *positive-definite* if it is positive-semidefinite with strictly positive eigenvalues. Let A be a positive-definite matrix with $(i,j)^{\text{th}}$ entry denoted by a_{ij}. Hadamard's inequality ([229]) states that the determinant of a positive-definite matrix is less than or equal to the product of the diagonal entries of the matrix *i.e.* $\det A \leq \prod_{i=1}^{n} a_{ii}$. Further, equality holds if and only if A is a diagonal matrix. As a corollary, which is usually referred to by the same name, we get that the absolute value of the determinant of a square matrix is less than or equal to the product of the Euclidean norm of its column vectors (or alternatively, row vectors). In the case of real matrices, the inequality conveys the geometrically intuitive idea that an n-parallelepiped with prescribed lengths of sides has largest volume if and only if the sides are mutually orthogonal. An important application of this inequality to the theory of integral equations is in proving convergence results in classical Fredholm theory [233]. More generally, a similar inequality, known as Fischer's inequality ([228]), holds if one considers the principal diagonal blocks of a positive-definite matrix in block form. Hadamard's inequality is a corollary of Fischer's inequality by considering blocks of size 1×1.

For $n \in \mathbb{N}$, we denote the indexing set $\{1, 2, , n\}$ by $\langle n \rangle$. In Fischer's inequality below, for an $n \times n$ matrix A and $\alpha \subseteq \langle n \rangle$, the principal submatrix of A from rows and columns indexed by α is denoted by $A[\alpha]$.

Theorem (4.3.1) [225]: *Let A be a positive-definite matrix in* $M_n(\mathbb{C})$. *Let* $\alpha_i \subseteq \langle n \rangle$ *for* $i \in \langle k \rangle$ *such that* $\alpha_i \mathrm{n} \alpha_j = \otimes$ *for* $i, j \in \langle k \rangle$, $i \neq j$. *Then*

$$\det\left(A\left[\mathrm{U}_{i=1}^{k} \alpha_i\right]\right) \leq \prod_{i=1}^{k} \det\ (A[\alpha_i])$$

with equality if and only if $A\left[\mathrm{U}_{i=1}^{k} \alpha_i\right] = \mathrm{Pdiag}\ (A[\alpha_1], \cdots, A[\alpha_k])P^{-1}$ *for some permutation matrix* P.

We have that the determinant of a positive-definite matrix (in block form) is less than or equal to the product of the determinants of its principal diagonal blocks, with equality if and only if the entries outside the principal diagonal blocks are all 0. We state an application of this result to information theory. For a multivariate normal random variable $(X_1, X_2, \cdots, X_n)$ with mean 0, covariance matrix Σ and hence density

$$f(\mathrm{x}) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}\mathrm{x}^T \Sigma^{-1} \mathrm{x}\right), \mathrm{x} \in \mathbb{R}^n,$$

the Shannon entropy is given by,

$$h\ (X_1, \cdots, X_n) = -\int_{\mathbb{R}^n} f\ \log f d\mathrm{x} = \frac{1}{2} \ln\left((2\pi e)^n \det(\Sigma)\right).$$

Fischer's inequality conveys the sub-additivity of entropy in the case of normal random variables *i.e.* if the collection of normal random variables $X_1, \cdots, X_n$ is partitioned into disjoint subcollections, the sum of the entropies of the subcollections is bigger than the entropy of the whole collection, with equality if and only if the subcollections are mutually independent.

In [162], Arveson obtains a generalized version of Hadamard's inequality for von Neumann algebras with a tracial state τ, involving the Fuglede-Kadison determinant denoted by $\triangle$, which we paraphrase below.

Theorem (4.3.2) [225]: *Let* Φ *be a* τ*-preserving conditional expectation on a von Neumann subalgebra* $\mathcal{S}$ *of a von Neumann algebra* $\mathcal{R}$. *Then*

$$\triangle(A) \leq \triangle(\Phi(A)),$$

for every positive A in $\mathcal{R}$.

Theorem (4.3.22) along gives us a proof of the above theorem which is different from the one in [162]. We make the additional assumptions of faithfulness of the tracial state τ and regularity of the positive operator A unless stated otherwise. Note that the first assumption necessitates the finiteness of the von Neumann algebra $\mathcal{R}$. The new proof has the added advantage of directly yielding the conditions under which equality holds, given by $\triangle(A) = \triangle(\Phi(A)) \Leftrightarrow \Phi(A) = A$. In Theorem (4.3.24), using this equality condition we are able to prove that $\Phi(A^{-1}) = \Phi(A)^{-1}$ if and only if $\Phi(A) = A$ which is somewhat surprising as the statement itself has no direct reference to $\triangle$. Our investigation reveals that this offers a small glimpse of a bigger picture. For instance, in Theorem (4.3.28), we prove that if f is a non-constant positive-valued operator monotone function on $(0,\infty)$, $\triangle(f(A)) \leq \triangle(f(\Phi(A)))$ with equality if and only if $\Phi(A) = A$. In fact, the result still holds for singular positive operators A and positive-valued operator monotone functions on $[0,\infty)$ but the simple form of the equality condition is rendered ineffective in this scenario. Further in Theorem (4.3.34), for a trace-preserving unital positive map $\Phi: \mathcal{R} \to \mathcal{R}$ and a continuous log -convex function f, we see that $\triangle(f(\Phi(A))) \leq \triangle(f(A))$. As a corollary [Corollary (4.3.35)], we obtain a version of Theorem (4.3.2) for tracepreserving unital positive maps. In [201], Matic proves two inequalities, in the vein of Fischer's inequality, for the ratio $\frac{\det(A+D)}{\det A}$ a to study the change in the determinant of a positive-definite matrix A perturbed by positive-definite block diagonal matrix D. For $k \in \mathbb{N}$, let $n, n_1, \cdots, n_k$ be positive integers such that $n = n_1 + \cdots + n_k$. For $i \in \langle k \rangle$, if A_i is in $M_{n_i}(\mathbb{C})$, we define an $n \times n$ matrix diag $(A_1, \cdots, A_k)$ by,

$$diag\,(A_1, \cdots, A_k) := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix}$$

We recall the two main inequalities from [201].

Theorem (4.3.3) [225]: *For each $i \in \langle k \rangle$, let C_i, D_i be positive-definite matrices in $M_{n_i}(\mathbb{C})$. Let C be a positive-definite matrix in block form in $M_n(\mathbb{C})$ with principal diagonal blocks given by $C_1, C_2, \cdots, C_k$. Then the following inequality holds,*

$$\frac{\det\left(C + \operatorname{diag}(D_1, \cdots, D_k)\right)}{\det(C)} \geq \frac{\det(C_1 + D_1)}{\det(C_1)} \cdots \frac{\det(C_k + D_k)}{\det(C_k)}. \quad (38)$$

As an application of the general framework developed, we view the above inequalities as manifestations of Jensen's inequality in the context of conditional expectations on a finite von Neumann algebra for the choice of the operator monotone function $\left(1 + \frac{1}{x}\right)^{-1}$ on $(0,\infty)$. Theorem (4.3.3) and Theorem (4.3.38) may be considered as specific cases of Theorem (4.3.4) and Corollary (4.3.5) respectively, which we state below. Not only does this provide us more insight but it also helps us directly identify the conditions under which equality holds. These equality conditions were not considered in [201]. In the two results mentioned below, $\mathcal{R}$ is a finite von Neumann algebra with a faithful normal tracial state τ and Φ is a τ-preserving conditional expectation onto the von Neumann subalgebra $\mathcal{S}$ of $\mathcal{R}$.

Theorem (4.3.4) [225]: *For* a regular *positive operator* A *in* $\mathcal{R}$, and a *positive operator* B in $\mathcal{S}$, the *following inequality* $10olds$:

$$\frac{\triangle\,(\Phi(A)+B))}{\triangle\,(\Phi(A))} \leq \frac{\triangle\,(A+B)}{\triangle\,(A)}. \tag{39}$$

If B is regular, equality holds if *and only if* $\Phi(A) = A$ *i.e.* $A \in \mathcal{S}$.

Corollary (4.3.5) [225]: *For* a regular *positive operator* A *in* $\mathcal{R}$, *and* a *positive operator* B in $\mathcal{S}$, *the following inequality* Holds :

$$\frac{\triangle\,(A+B)}{\triangle\,(A)} \leq \frac{\triangle\,(\Phi(A^{-1})^{-1}+B)}{\triangle\,(\Phi(A^{-1})^{-1})}, \tag{40}$$

with equality if and only if $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \in \mathcal{S}$. In particular, if B is regular, equality holds in (40) if and only if $A \in \mathcal{S}$.

We note some improvements to Theorem (4.3.3) and Theorem (4.3.38) obtained from these generalizations. The inequalities (38) and (60) still hold if the matrices $D_1, \cdots, D_k$ are positive semidefinite. If the matrices $D_1, \cdots, D_k$ are positive-definite, equality holds in (38) and (60) if and only if $C = \operatorname{diag}(C_1, \cdots, C_k)$ *i.e.* C is in block diagonal form. If the matrices $D_1, \cdots, D_k$ are positive-semidefinite and $D :=\operatorname{diag}\,(D_1, \cdots, D_k)$, equality holds in inequality (60) if and only if $D^{\frac{1}{2}}C^{-1}D^{\frac{1}{2}}$ is in block diagonal form. Note that when D is positive-definite, $D^{\frac{1}{2}}C^{-1}D^{\frac{1}{2}}$ is in block diagonal form if and only if C is in block diagonal form. Substantially more effort goes into proving Theorem (4.3.4) as compared to Corollary (4.3.5).

A lot has been said in the literature about operator theoretic versions of Jensen's inequality. For instance, using results by Davis ([5]), Choi ([227]), we obtain a general operator theoretic version of Jensen's inequality involving operator convex functions and unital positive maps between C^*-algebras. For a continuous function f on an interval $I \subseteq \mathbb{R}$, if $f\left(\sum_{i=1}^{k} A_i\, T_i A_i^*\right) \leq \sum_{i=1}^{k} A_i\, f(T_i) A_i^*$, for self-adjoint operators $T_1, \cdots, T_k$ on an infinite dimensional Hilbert space $\mathcal{H}$ with spectra in I, and operators $A_1, \cdots A_k$ on $\mathcal{H}$ such that $\sum_{i=1}^{k} A_i\, A_i^* = I$, we say that f is operator C^*-convex. In [1], Hansen and Pedersen prove the equivalence of the class of operator convex functions on an interval I and the class of operator C^*-convex functions on I.

We setup the background for our discussion. We offer a primer on the theory of von Neumann algebras, the Fuglede-Kadison determinant, conditional expectations on von Neumann algebras, and the theory of operator convex and operator monotone functions on $[0, \infty)$. We collect some technical lemmas. The crux of the discussion where extensions of Hadamard's inequality involving the Fuglede-Kadison determinant are proved. we discuss applications of the results derived and obtain the aforementioned inequalities as special cases.

The space $B(\mathcal{H})$ may be considered as a Banach algebra with the operator norm, and multiplication given by composition of operators. In addition, with the adjoint operation as involution $(T \to T^*)$, it is also a C^*-algebra. All norm-closed $*$-subalgebras of $B(\mathcal{H})$ are also C^*-algebras. A positive linear functional β on a unital C^*-algebra is said to be a *state* if

$\rho(I) = 1$. A state ρ is said to be faithful if $\rho(A^*A) = 0$ if and only if $A = 0$. There are several interesting topologies on $B(\mathcal{H})$ coarser than the norm topology. Two important ones are *the weak-operator topology*, which is the coarsest topology such that for any vectors x, y in $\mathcal{H}$ the functional $\rho_{x,y} : \mathcal{R}(\mathcal{H}) \to \mathbb{C}$ defined by $\rho_{x,y}(T) = \langle Tx, y\rangle$ is continuous, and the *strong-operator topology*, which is the coarsest topology such that for any vector x in $\mathcal{R}(\mathcal{H})$ the map $w_x : \mathcal{R}(\mathcal{H}) \to \mathbb{C}$ defined by $w_x(T) = Tx$ is norm-continuous. The commutant of a non-empty subset $\mathcal{F}$ of $\mathcal{R}(\mathcal{H})$ is defined as $\mathcal{F}' := \{T : AT = TA, \forall A \in \mathcal{F}\}$. Before we define von Neumann algebras, we recall the von Neumann double commutant theorem ([238]) to put the algebraic and analytic aspects of von Neumann algebras into perspective, the commutant being an algebraic object, and the topologies considered determining the analytic aspect.

Theorem (4.3.6) [225]: *Let $\mathfrak{A}$ be a self-adjoint subalgebra of $B(\mathcal{H})$ containing the identity operator. Then the following are equivalent*:

(*i*) $\mathfrak{A}$ is weak-operator closed,

(*ii*) $\mathfrak{A}$ is strong-operator closed,

(*iii*) $(\mathfrak{A}')' = \mathfrak{A}$.

Definition (4.3.7) [225]: A von Neumann algebra is a self-adjoint subalgebra of $B(\mathcal{H})$ containing the identity operator which is closed under the weak operator topology.

The inspiration to study von Neumann algebras comes from, amongst other things, the study of group representations on infinite-dimensional Hilbert spaces. They were introduced in [231] by Murray and von Neumann, as rings of operators. Von Neumann algebras have plenty of projections, in the sense that, the set of linear combinations of projections in a von Neumann algebra $\mathcal{R}$ is norm-dense in $\mathcal{R}$. This is clear from the spectral theorem for self-adjoint operators and the observation that any operator T can be written as a linear combination of self-adjoint operators $\left(T = \frac{T+T^*}{2} + \mathrm{i}\frac{T-T^*}{2\mathrm{i}}.\right)$

In a bid to classify von Neumann algebras, Murray and von Neumann developed the comparison theory of projections. Two projections E, F in $\mathcal{R}$ are said to be equivalent if there is an operator $V \in \mathcal{R}$ such that $VV^* = E$ and $V^*V = F$, and such a V is called a partial isometry. In general, it is possible to have equivalent projections E, F such that $E \leq F$ and $E \neq F$. If $\mathcal{R}$ does not allow for such occurrences, it is said to be a finite von Neumann algebra. Alternatively, the characterizing property for a finite von Neumann algebra is that every isometry is a unitary operator *i.e.* for $V \in \mathcal{R}$ if $V^*V = I$, then $VV^* = I$. A major part of their study involved classifying the so-called *factors*, which are von Neumann algebras with trivial center, the set of scalar multiples of the identity. In a nutshell, factors may be thought of as building blocks of von Neumann algebras, and finite factors may be thought of as building blocks of finite von Neumann algebras. Finite factors come in two flavors: the finite-dimensional kind, given by $M_n(\mathbb{C})$, $n \in \mathbb{N}$, and the infinite-dimensional kind, the II_1 factors. A characterizing property of a finite factor is the existence of a unique faithful normal tracial state *i.e.* a linear functional τ satisfying the following conditions : (i) $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{R}$, (ii) $\tau(I) = 1$, (iii) (faithfulness) $\tau(A^*A) = 0$ if and only if $A = 0$, and (iv) (normality) for an increasing sequence of projections $\{E_n\}$ we have $\tau(\sup_{n\in\mathbb{N}} E_n) =$

$\sup_{n\in\mathbb{N}}\tau(E_n)$. A von Neumann algebra $\mathcal{R}$ has a faithful normal tracial state if and only if $\mathcal{R}$ is finite. See [13] for a fuller discussion of this topic.

In [160], for a finite factor $\mathcal{M}$ with the unique faithful normal tracial state τ, Fuglede and Kadison define the notion of a determinant on $GL_1(\mathcal{M})$, the set of regular operators in $\mathcal{M}$, in the following manner:

$$\triangle: GL_1(\mathcal{M}) \to \mathbb{R}_+,$$
$$\triangle(A) = \exp\left(\tau\left(\log (A^*A)^{\frac{1}{2}}\right)\right).$$

That $\triangle$ makes sense is a consequence of the continuous functional calculus. Several properties of $\triangle$ are studied in [160] and a major portion of the effort goes in proving that it is a group homomorphism. Further, it is proved that for any 'reasonable' determinant theory, the determinant of an operator with a non-trivial nullspace must vanish. But in general, there can be other extensions of the definition to singular operators with trivial nullspace. Two such extensions are mentioned; the *algebraic extension* defines the determinant to be zero for all singular operators in $\mathcal{M}$, whereas the *analytic extension* uses the spectral decomposition of $(A^*A)^{\frac{1}{2}}\left(=\int \lambda\, dE_\lambda\right)$ to define $\triangle(A) := \exp\left(\int \log \lambda d\tau(E_\lambda)\right)$ with the understanding that $\triangle(A) = 0$ if $\int \log \lambda d\tau(E_\lambda) = -\infty$.

Although originally in [160], the Fuglede-Kadison determinant is defined for finite factors, the discussion carries over to the case of a von Neumann algebra with a tracial state τ. Our primary interest is in the case when the tracial state τ is faithful and thus, $\mathcal{R}$ is finite. As our results are true for any choice of a faithful normal tracial state, the dependence of $\triangle$ on τ will be suppressed in the notation. See [127] for a masterful account of the Fuglede-Kadison determinant (and its variants) by de la Harpe.

Note that (v) follows from the fact that log is an operator monotone function on $(0,\infty)$, (thus, $\log A \le \log B$) and by the faithfulness of the tracial state, we have that $\triangle(A) = \triangle(B) \Leftrightarrow \tau(\log A) = \tau(\log B) \Leftrightarrow \tau(\log B - \log A) = 0 \Leftrightarrow \log A = \log B \Leftrightarrow A = B$. We will have more to say about operator monotone functions.

Example (4.3.8) [225]: Let us denote the usual determinant function on $M_n(\mathbb{C})$ by det, and the normalized trace on $M_n(\mathbb{C})$ by tr, defined as the average of the diagonal entries of the matrix. Later when we want to emphasize n, we will denote the determinant, normalized trace on $M_n(\mathbb{C})$ by $\det_n$, tr_n, respectively. For a matrix A in $M_n(\mathbb{C})$, if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the positive-definite matrix $(A^*A)^{\frac{1}{2}}$ (counted with multiplicity), we have that,

$$\mathrm{tr}\left(\log (A^*A)^{\frac{1}{2}}\right) = \frac{\log \lambda_1 + \cdots + \log \lambda_n}{n} = \log\left(\sqrt[n]{\lambda_1 \lambda_n}\right) = \log\left(\sqrt[n]{\left(\det (A^*A)^{\frac{1}{2}}\right)}\right)$$

Thus we see that for the type I_n factor $M_n(\mathbb{C})(n \in \mathbb{N})$, we have the following relationship between $\triangle$ and det :

$$\triangle(A) = \sqrt[n]{\left(\det (A^*A)^{\frac{1}{2}}\right)} = \sqrt[n]{|\det A|}, \quad (41)$$

for any A in $M_n(\mathbb{C})$.

Let $\mathcal{R}$ denote a von Neumann algebra with identity I. Let $\mathcal{S}$ denote a von Neumann subalgebra of $\mathcal{R}$. Then a map $\Phi: \mathcal{R} \to \mathcal{S}$ is said to be a *conditional expectation* from $\mathcal{R}$ onto $\mathcal{S}$ if it satisfies the following :
(i) Φ is linear, positive and $\Phi(I) = I$,
(ii) $\Phi(S_1 R S_2) = S_1 \Phi(R) S_2$ for R in $\mathcal{R}$, and S_1, S_2 in $\mathcal{S}$.
From (ii), we have that $\Phi(T) = T$ if and only if T is in $\mathcal{S}$.
For a finite von Neumann algebra $\mathcal{R}$ with a faithful normal tracial state τ, a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be *τ-preserving* or *trace-preserving* if $\tau(\Phi(A)) = \tau(A)$ for A in $\mathcal{R}$. We are primarily interested in trace-preserving conditional expectations on finite von Neumann algebras.

Example (4.3.9) [225]: Let $D_n(\mathbb{C})$ denote the subalgebra of $M_n(\mathbb{C})$ consisting of diagonal matrices. Define a map $: M_n(\mathbb{C}) \to D_n(\mathbb{C})$ by $\Phi(A) := \operatorname{diag}(a_{11}, \cdots, a_{nn})$. The map Φ is a trace preserving conditional expectation from the finite von Neumann algebra $M_n(\mathbb{C})$ onto $D_n(\mathbb{C})$.

Example (4.3.10) [225] Let $n = n_1 + \cdots + n_k$, where $n, n_1, \cdots, n_k \in \mathbb{N}$. Note that for matrices $A_1, \cdots, A_k$ in $M_{n_1}(\mathbb{C}), \cdots, M_{n_k}(\mathbb{C})$ respectively, we may construct a matrix A in $M_n(\mathbb{C})$ with these matrices as the principal diagonal blocks and 0's elsewhere, *i.e.* $A :=$diag $(A_1, \cdots, A_k)$. In this manner, one may consider the matrix algebra $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ as a subalgebra of $M_n(\mathbb{C})$ with the same identity. Consider the map $: M_n(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, defined by $\Phi(A) = A_{11} \oplus \cdots \oplus A_{kk}$ where A_{ii}'s are the principal $n_i \times n_i$ diagonal blocks of A. It is left to the reader to check that Φ is a trace-preserving conditional expectation from $M_n(\mathbb{C})$ onto $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.

A positive linear map $\Psi: \mathcal{R} \to \mathcal{R}$ is said to be *n-positive* (for $n \in \mathbb{N}$) if $\Psi \otimes I_n : \mathcal{R} \otimes M_n(\mathbb{C}) \to \mathcal{R} \otimes M_n(\mathbb{C})$ is positive. Further if Ψ is n-positive for all n in $\mathbb{N}$, we say that Ψ is *completely positive*. We mention without proof the following two theorems about conditional expectations on finite von Neumann algebras that play a fundamental role

Theorem (4.3.11) [225]: *Let $\mathcal{R}$ be a finite von Neumann algebra with a faithful normal tracial state τ and $\mathcal{S}$ be a von Neumann subalgebra of $\mathcal{R}$. Then there is a unique map $: \mathcal{R} \to \mathcal{S}$ such that $\tau(\Phi(R)S) = \tau(RS)$ for $R \in \mathcal{R}$, $S \in \mathcal{S}$, and such a map Φ is a trace-preserving normal conditional expectation from $\mathcal{R}$ onto $\mathcal{S}$.*

Theorem (4.3.12) ([232]) [225]: *Let $\mathcal{R}$ be a finite von Neumann algebra with a faithful normal tracial state τ and $\mathcal{S}$ be a von Neumann subalgebra of $\mathcal{R}$. Then the trace-preserving normal conditional expectation from $\mathcal{R}$ onto $\mathcal{S}$ is a completely positive map.*

A continuous function f defined on the interval $\Gamma \subseteq \mathbb{R}$ is said to be *operator monotone* if for self-adjoint operators A, B on an infinite-dimensional Hilbert space $\mathcal{H}$, with spectra in Γ, such that $A \leq B$, we have that $f(A) \leq f(B)$. For positive operators A, B with $A \leq B$ it is not necessarily true that $A^2 \leq B^2$ But for $0 < r \leq 1$, the Löwner-Heinz inequality states that $A^r \leq B^r$ for $0 < r \leq 1$. Thus x^r is an operator monotone function on $[0, \infty)$ for $0 < r \leq 1$. Other examples include $\log(1 + x)$ on $[0, \infty)$, $\log x$ on $(0, \infty)$. In his seminal [14], Löwner studied operator monotone functions in detail establishing their relationship with a class of analytic functions called Pick functions. We state an integral representation of operator monotone functions on $[0, \infty)$ (cf. [238]).

Theorem (4.3.13) [225]: *A continuous real-valued function f on $[0,\infty)$ is operator monotone if and only if there is a finite positive measure μ on $(0,\infty)$ such that*

$$f(t) = a + bt + \int_0^\infty \frac{(\lambda+1)t}{\lambda+t} d\mu(\lambda), t \in [0,\infty),$$

for some real number a, and $b \geq 0$.

The concept of operator monotonicity is closely related to operator convex functions which were studied by Kraus ([15]). Let f be a continuous function on the interval $\Gamma \subseteq \mathbb{R}$. We say that f is *operator convex* if $(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$, for each λ in [0,1], and self-adjoint operators A, B on an infinite-dimensional Hilbert space $\mathcal{H}$ with spectra in Γ. We state an integral representation of operator convex functions on $[0,\infty)$ for which $f'(0^+)$ exists, which may be derived from Theorem (4.3.13) above and [11, Theorem (4.3.8)].

Theorem (4.3.14) [225]: *A continuous real-valued function f on $[0,\infty)$ such that $f'(0^+)$ exists is operator convex if and only if there is a finite positive measure μ on $(0,\infty)$ such that*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{(\lambda+1)t^2}{\lambda+t} d\mu(\lambda), t \in [0,\infty),$$

for some real numbers a, b, and $c \geq 0$.

we gather a collection of disparate results which serve us well

Definition (4.3.15) [225]: A real-valued convex function on the interval $[a,b] \subseteq \mathbb{R}$ is said to be *strictly convex* if for $x, y \in [a,b]$ such that $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$, we must have $x = y$.

Lemma (4.3.16) [225]: *For* a *convex* function f *on* the real interval $[a,b]$ *and* a *probability* measure μ on a *compact subset S of* $[a,b]$, we have the following inequality,

$$f\left(\int_S \lambda \, d\mu(\lambda)\right) \leq \int_S f(\lambda) d\mu(\lambda).$$

Further if f is strictly convex, then equality holds *if and only if μ is supported on a point i.e. μ is* a *Dirac* measure.

Lemma (4.3.17) [225]: *Let $\mathfrak{A}$ be* a *unital C^**-algebra *and ρ be* a *state* on $\mathfrak{A}$. *For* a *continuous convex* function f *on* a real *interval* $[a,b]$ *and* a *self-adjoint operator A* in $\mathfrak{A}$ *with spectrum* in $[a,b]$, *we have the following inequality,*

$$f(\rho(A)) \leq \rho(f(A)).$$

Further if ρ is faithful and f *is strictly convex, then equality holds if and only if A is a scalar multiple of the identity*.

Proof. The unital C^*-algebra generated by A is $*$-isomorphic to $C(\sigma(A))$ and henceforth referred to interchangeably. Note that the restriction of ρ to $C(\sigma(A))$ is also a state, and faithful if ρ is faithful. By the Riesz representation theorem, there is a probability measure μ on $\sigma(A)$ such that for any continuous function f on (A) , we have that $\rho(f(A)) = \int_{\sigma(A)} f(\lambda) d\mu(\lambda)$. Note that on the compact subset $\sigma(A)$ of $[a,b]$, by Jensen's inequality,

$$f\big(\rho(A)\big) = f\left(\int_{\sigma(A)} \lambda \, d\mu(\lambda)\right) \leq \int_{\sigma(A)} f\,(\lambda) d\mu(\lambda) = \rho\big(f(A)\big)\,.$$

If f is strictly convex, equality holds if and only if μ is a Dirac measure, say, supported on $\lambda' \in [a, b]$. In addition if ρ is a faithful state, then μ is a Dirac measure supported on $\{\lambda'\}$ if and only if $\sigma(A) = \{\lambda'\} \Leftrightarrow A = \lambda' I$.

Lemma (4.3.18) [225]: *For* a *finite von Neumann algebra $\mathcal{R}$ with* a *tracial state τ and* a *positive operator A in $\mathcal{R}$, we have* the *following inequality,*

$$\triangle\,(A) \leq \tau(A)\,. \tag{42}$$

If A is a *regular positive operator* and *τ is faithful, equality* holds *if* and *only if A is* a *positive scalar multiple of the identity I.*

Proof. We first prove the inequality for a regular positive operator A in $\mathcal{R}$. Note that the spectrum of A is contained in $[\|A^{-1}\|^{-1}, \|A\|]$. The function- $\log x$ defined on $[\|A^{-1}\|^{-1}, \|A\|]$ is strictly convex as the second derivative is $\frac{1}{x^2}$ which is strictly positive on $[\|A^{-1}\|^{-1}, \|A\|]$. From Lemma (4.3.17) for $-\log$, we have that - $\log \tau(A) \leq \tau(-\log A) \Rightarrow \tau(\log A) \leq \log \tau(A) \Rightarrow \triangle\,(A) = \exp\big(\tau(\log A)\big) \leq \exp\big(\log \tau(A)\big) = \tau(A)$, and if τ is faithful, equality holds if and only if A is a positive scalar multiple of the identity.

We next prove the inequality for a singular positive operator A. For $\varepsilon > 0$, note that $A + \varepsilon I$ is a regular positive operator. Thus we have that $\triangle\,(A + \varepsilon I) \leq \tau(A + \varepsilon I)$. Using the norm-continuity of τ, and taking the limits as $\varepsilon \to 0^+$, we get that $\triangle\,(A) \leq \tau(A)$.

In this paragraph, we set up the notation to be used in the proof of Lemma (4.3.19). Let $\mathcal{R}$ be a von Neumann algebra, acting on the Hilbert space $\mathcal{H}$, with a tracial state τ. For the von Neumann algebra $M_2(\mathcal{R}) \cong \mathcal{R} \otimes M_2(\mathbb{C})$ (acting on $\mathcal{H} \oplus \mathcal{H}$), we are interested in the tracial state on $M_2(\mathcal{R})$ given by $\tau_2 = \tau \otimes \mathrm{tr}_2$ *i.e.* for an operator A in $M_2(\mathcal{R})$, $\tau_2(A) = \frac{\tau(A_{11}) + \tau(A_{22})}{2}$ where $A_{ij} \in \mathcal{R}(1 \leq i, j \leq 2)$ denotes the $(i, j)^{\text{th}}$ entry of A. We denote the Fuglede-Kadison determinant on $M_2(\mathcal{R})$ corresponding to $\tau \otimes \mathrm{tr}_2$ by $\triangle_2$. For operators A_1, A_2 in $\mathcal{R}$, we define

$$diag\,(A_1, A_2) := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in M_2(\mathcal{R})\,.$$

Lemma (4.3.19) [225]: *Let $\mathcal{R}$ be* a *von Neumann algebra with* a *tracial state τ. For operators A, B in $\mathcal{R}$, we have that $\triangle\,(I + AB) = \triangle\,(I + BA)$.*

Proof. For a unital ring R with multiplicative identity 1 and an element x in $M_2(R)$, we have that

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x & 1_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus the operators

$$\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}, \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \in M_2(\mathcal{R})$$

are regular and their Fuglede-Kadison determinant is strictly positive. Using the multiplicativity of $\triangle_2$, the algebraic identity given below

$$\begin{bmatrix} I & A \\ -B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I + BA \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I+AB & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix},$$

we conclude that $\triangle (I + AB) = \triangle (I + BA)$.

In Lemma (4.3.20) below, we adapt in [226] which involves positive-definite matrices to the context of positive operators on a Hilbert space by mimicking the algebraic trick used therein.

Lemma (4.3.20) [225]: *Let A, C* be *positive operators in $B(\mathcal{H})$ with A* being regular. *Let B be an operator* in $\mathcal{R}(\mathcal{H})$. Then the *self-adjoint operator*,

$$\mathrm{P} := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

in $\mathcal{R}(\mathcal{H} \oplus \mathcal{H})$ *is positive if and only if* the Schur *complement $C - B^*A^{-1}B$ is positive.*

Proof. Consider the operator in $\mathcal{R}(\mathcal{H} \oplus \mathcal{H})$ given by,

$$X := \begin{bmatrix} I & 0 \\ -B^*A^{-1} & I \end{bmatrix}$$

As X is regular, P is positive if and only if XPX^* is positive. A straight forward matrix computation shows that

$$\mathrm{XPX}^* = \begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix}.$$

Thus, P is positive ⇔ XPX^* is positive ⇔ $C - B^*A^{-1}B$ is positive.

We paraphrase Proposition (4.3.21) from [234] below without proof.

Theorem (4.3.21) [225]: *Let $\mathfrak{A}$,* B *be unital C^*-algebras, such that* B *is abelian, and let* $\Phi: \mathfrak{A} \to$ B *be a unital positive map. Let f be a real-valued continuous convex function defined on the interval $[a, b]$. For a self-adjoint operator A of $\mathfrak{A}$ whose spectrum is contained in $[a, b]$, we have the following inequality*:

$$f\big(\Phi(A)\big) \le \Phi\big(f(A)\big). \tag{43}$$

$\mathcal{R}$ will denote a finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$ with identity I and a faithful normal tracial state τ, $\mathcal{S}$ will denote a von Neumann subalgebra of $\mathcal{R}$, and Φ will denote a τ-preserving conditional expectation from $\mathcal{R}$ onto $\mathcal{S}$. Note that for A in $\mathcal{R}$, if $\varepsilon I \le A$, then $\varepsilon I \le \Phi(A)$. As a consequence, if A is a positive regular operator, $\Phi(A)$ is also positive and regular.

Theorem (4.3.22) [225]: *For any* regular *positive operator A in $\mathcal{R}$,* we *have that*

$$\triangle \left(\Phi(A^{-1})^{-1}\right) \le \triangle (A) \le \triangle \big(\Phi(A)\big). \tag{44}$$

with equality on either side if and *only if $\Phi(A) = A$* i.e $A \in \mathcal{S}$. *If A is positive* (*but not necessarily* regular), *one still has the inequality on the right i.e.* $\triangle (A) \le \triangle \big(\Phi(A)\big)$.

Proof. Let A be a regular positive operator in $\mathcal{R}$. As $\Phi(A)^{-1}$ is in $\mathcal{S}$, using Lemma (4.3.18) for the regular positive operator $\Phi(A)^{-\frac{1}{2}} A \Phi(A)^{-\frac{1}{2}}$, and keeping in mind the trace-preserving nature of Φ, we note that,

$$\triangle \left(\Phi(A)^{-\frac{1}{2}} A \Phi(A)^{-\frac{1}{2}}\right) \le \tau \left(\Phi(A)^{-\frac{1}{2}} A \Phi(A)^{-\frac{1}{2}}\right) = \tau(A\Phi(A)^{-1})$$
$$= \tau\big(\Phi(A\Phi(A)^{-1})\big) = \tau(\Phi(A)\Phi(A)^{-1})$$
$$= \tau(I) = 1,$$

with equality if and only if $\Phi(A)^{-\frac{1}{2}} A \Phi(A)^{-\frac{1}{2}} = I \Leftrightarrow \Phi(A) = A$.

Using the multiplicativity of $\triangle$, we prove the desired inequality below.

$$\triangle\left(\Phi(A)^{-\frac{1}{2}}A\Phi(A)^{-\frac{1}{2}}\right) \leq 1$$

$$\Rightarrow \triangle\left(\Phi(A)\right)^{-\frac{1}{2}} \triangle(A) \triangle\left(\Phi(A)\right)^{-\frac{1}{2}} \leq 1$$

$$\Rightarrow \triangle(A) \leq \triangle\left(\Phi(A)\right)^{\frac{1}{2}} \triangle\left(\Phi(A)\right)^{\frac{1}{2}} = \triangle\left(\Phi(A)\right), \quad (45)$$

with equality if and only if $\Phi(A) = A$. Using the inequalityjust proved for the regular positive operator A^{-1}, we have $\triangle(A^{-1}) \leq \triangle\left(\Phi(A^{-1})\right) \Leftrightarrow \triangle(\Phi(A^{-1})^{-1}) \leq \triangle(A)$, with equality if and only if $\Phi(A^{-1}) = A^{-1}$ Note that $\Phi(A^{-1}) = A^{-1} \Leftrightarrow A^{-1} \in \mathcal{S} \Leftrightarrow A \in \mathcal{S} \Leftrightarrow \Phi(A) = A$.

Let $\varepsilon > 0$. If A is positive but not necessarily regular, applying inequality (45) to the regular operator $A + \varepsilon I$ yields the following inequality

$$\triangle(A + \varepsilon I) \leq \triangle\left(\Phi(A + \varepsilon I)\right) = \triangle(\Phi(A) + \varepsilon I).$$

Taking the limit as $\varepsilon \to 0^+$, we see that $\triangle(A) \leq \triangle\left(\Phi(A)\right)$.

Corollary (4.3.23) [225]: *For* a *regular positive operator A in $\mathcal{R}$, and* a *positive operator B in $\mathcal{S}$, the following* inequality *holds* :

$$\frac{\triangle(A+B)}{\triangle(A)} \leq \frac{\triangle(\Phi(A^{-1})^{-1}+B)}{\triangle(\Phi(A^{-1})^{-1})}, \quad (46)$$

with equality if and *only if $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \in \mathcal{S}$. In particular, if B is regular, equality* holds in (46) *if* and *only if $A \in \mathcal{S}$.*

Proof. Using the multiplicativity of $\triangle$ and Lemma (4.3.19), we can rewrite both sides of the inequality in the following manner:

$$\frac{\triangle(A+B)}{\triangle(A)} = \triangle(I + A^{-1}B) = \triangle\left(I + B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right),$$

$$\frac{\triangle(\Phi(A^{-1})^{-1}+B)}{\triangle(\Phi(A^{-1})^{-1})} = \triangle(I + \Phi(A^{-1})B) = \triangle\left(I + B^{\frac{1}{2}}\Phi(A^{-1})B^{\frac{1}{2}}\right).$$

Note that $X := B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$ is a positive operator and thus $I + X$ is a regular positive operator. As $B^{\frac{1}{2}}$ is in $\mathcal{S}$, $\Phi(X) = B^{\frac{1}{2}}\Phi(A^{-1})B^{\frac{1}{2}}$. From Theorem (4.3.22), we have that

$$\triangle(I + X) \leq \triangle\left(\Phi(I + X)\right) = \triangle\left(I + \Phi(X)\right)$$

with equality if and only if $I + X = \Phi(I + X) \Leftrightarrow \Phi(X) = X \Leftrightarrow B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \in \mathcal{S}$. If B is regular, $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \in \mathcal{S} \Leftrightarrow A^{-1} \in \mathcal{S} \Leftrightarrow A \in \mathcal{S}$.

Theorem (4.3.24) [225]: *For a self-adjoint operator A in $\mathcal{R}$, we have the following inequality* :

$$\Phi(A)^2 \leq \Phi(A^2). \quad (47)$$

If A is positive and regular, we have that

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \quad (48)$$

Further, in inequalities (47) *and* (48) *, equality holds if and only if $\Phi(A) = A$ i.e. $A \in \mathcal{S}$ (for the A under consideration*).

Proof. As Φ is a positive map, note that $\Phi(A)$ is self-adjoint and hence, so is $\Phi(A) - A$. We have $(\Phi(A) - A)^2 \geq 0$ and thus,

$$0 \leq \Phi((\Phi(A) - A)^2) = \Phi(\Phi(A)^2 - \Phi(A)A - A\Phi(A) + A^2) = \Phi(A^2) - \Phi(A)^2$$

Above we used the fact that $\left(A\Phi(A)\right) = \Phi(A)\Phi(A) = \Phi(\Phi(A)A)$. This proves the inequality (47). In this case, equality holds if and only if $\Phi((\Phi(A) - A)^2) = 0$. From the faithfulness and positivity of the tracial state, and trace-preserving nature of the conditional expectation, we get that equality holds if and only if $\tau(\Phi(\Phi(A) - A)^2)) = \tau((\Phi(A) - A)^2))$ $= 0 \Leftrightarrow (\Phi(A) - A)^2 = 0 \Leftrightarrow \Phi(A) = A$. This completes the proof of inequality (47).
We next prove inequality (48). From Theorem (4.3.12), as Φ is a completely positive map, the map $\otimes I_2 : \mathcal{R} \otimes M_2(\mathbb{C}) \to \mathcal{S} \otimes M_2(\mathbb{C})$ is a positive map. Applying $\Phi \otimes I_2$ to the positive operator,

$$\begin{bmatrix} A & I \\ I & A^{-1} \end{bmatrix} \in \mathcal{R} \otimes M_2(\mathbb{C}) ,$$

we conclude that

$$\begin{bmatrix} \varphi(A) & I \\ I & \varphi(A^{-1}) \end{bmatrix} \in \mathcal{S} \otimes M_2(\mathbb{C})$$

is a positive operator. Using Lemma (4.3.20), we conclude that $(A)^{-1} \leq \Phi(A^{-1})$.

We then investigate conditions under which equality holds. If $\Phi(A) = A$ for a regular positive operator A in $\mathcal{R}$, we have that $A \in \mathcal{S} \Rightarrow A^{-1} \in \mathcal{S} \Rightarrow \Phi(A^{-1}) = A^{-1} = \Phi(A)^{-1}$ Conversely, if $\Phi(A^{-1}) = \Phi(A)^{-1}$, using (44), we see that,

$$\frac{1}{\triangle\left(\Phi(A)\right)} \leq \frac{1}{\triangle(A)} = \triangle(A^{-1}) \leq \triangle\left(\Phi(A^{-1})\right) = \triangle(\Phi(A)^{-1}) = \frac{1}{\triangle\left(\Phi(A)\right)}.$$

Thus $\triangle(A)^{-1} = \triangle\left(\Phi(A)\right)^{-1} \Rightarrow \triangle\left(\Phi(A)\right) = \triangle(A)$. From the equality condition in Theorem (4.3.22), we conclude that $\Phi(A) = A$.

Corollary (4.3.25) [225]: *For* a *positive operator* A *in* $\mathcal{R}$ and $\lambda > 0$, *the following inequalities hold*:

(*i*) $\Phi(A(\lambda I + A)^{-1}) \leq \Phi(A)\left(\lambda I + \Phi(A)\right)^{-1}$,

(*ii*) $\Phi(A)^2\left(\lambda I + \Phi(A)\right)^{-1} \leq \Phi(A^2(\lambda I + A)^{-1})$

In both inequalities, *equality* holds *if* and *only if* $\Phi(A) = A$ *i.e.* $A \in \mathcal{S}$.

Proof. These inequalities follow from Theorem (4.3.24), after noting the following algebraic identitites :

$$X(\lambda I + X)^{-1} = I - \lambda I(\lambda I + X)^{-1},$$
$$X^2(\lambda I + X)^{-1} = X - \lambda I + \lambda^2 I(\lambda I + X)^{-1}$$

Theorem (4.3.26) [225]: *Let* f *be an operator monotone function defined on* the *interval* $[0, \infty)$. *Then for a regular positive operator* A, *we have the following* ii_2equality :

$$\Phi\left(f(A)\right) \leq f\left(\Phi(A)\right), \qquad (49)$$

with equality if and only if either f *is linear with positive slope or* $\Phi(A) = A$.

Proof. Let f be operator monotone. From Theorem (4.3.13), we have a finite positive measure μ on $(0, \infty)$ and real numbers a, b with $b \geq 0$, such that

$$f(t) = a + bt + \int_0^\infty \frac{(\lambda + 1)t}{\lambda + t} d\mu(\lambda) .$$

Consider the continuous family of operators $\mathrm{H}(\lambda) := \Phi\left((\lambda + 1)A(\lambda I + A)^{-1}\right) - (\lambda + 1)\Phi(A)(\lambda I + \Phi(A))^{-1}$ parametrized by $\in (0, \infty)$. Note that as A is regular, $\mathrm{H}(0)$ is well-defined and equal to 0. By Corollary (4.3.25) (i), we have that the family H consists of positive operators and $\mathrm{H}(\lambda) = 0$ for some $\lambda \in (0, \infty)$ if and only if $\Phi(A) = A$. We get the desired inequality below,

$$\Phi\big(f(A)\big) - f\big(\Phi(A)\big) = \int_0^{\infty} \mathrm{H}\,(\lambda) d\mu(\lambda) \geq 0.$$

The next step is to find necessary and sufficient conditions for equality in (49). Note that for any continuous function f on $[0, \infty)$, if A is in $\mathcal{S}$, then $f(A)$ is also in $\mathcal{S}$. Thus $\Phi(A) = A$ implies that $\big(f(A)\big) = f(A) = f\big(\Phi(A)\big)$. Also if f is linear, clearly $\big(\Phi(A)\big) = \Phi\big(f(A)\big)$. If $f\big(\Phi(A)\big) = \Phi\big(f(A)\big)$ and $\Phi(A) \neq A$, from the equality condition in Corollary (4.3.25), we conclude that $\mathrm{H}(\lambda') \neq 0$ for any $\lambda' \in (0, \infty)$. For a vector x in $\mathcal{H}$, one may define a positive continuous function h_x on $(0, \infty)$ by $h_x(\lambda) = \langle \mathrm{H}(\lambda)x, x\rangle$. Note that for each λ' in $(0, \infty)$, there is a vector $x_{\lambda'}$ such that $h_{x_\lambda}, (\lambda') > 0$ and thus a neighborhood $N_{\lambda'}$ of λ' where h_{x_λ}, is strictly positive. As $\int_0^{\infty} h_x\,(\lambda) d\mu(\lambda) = 0$ for all vectors x in $\mathcal{H}$, in particular, we have that $\int_0^{\infty} h_{x_\lambda}, (\lambda) d\mu(\lambda) = 0$. We conclude that μ is not supported on $N_{\lambda'}$ for any λ' in $(0, \infty)$ and thus $f(t) = a + bt$. Hence if $f\big(\Phi(A)\big) = \Phi\big(f(A)\big)$ and $\Phi(A) \neq A$, we have that f must be a linear function with positive slope.

Theorem (4.3.27) [225]: *Let* f be *an operator convex function* defined on the *interval* $[0, \infty)$. *Then for a regular positive operator* A in $\mathcal{R}$, we *have the following inequality*:

$$f\big(\Phi(A)\big) \leq \Phi\big(f(A)\big)\,, \qquad (50)$$

with equality if and *only if either* f *is* linear, *or* $\Phi(A) = A$ *i.e.* $A \in \mathcal{S}$.

Proof. Let f be operator convex and assume that $f'(0^+)$ exists. From Theorem (4.3.14), we have a finite positive measure μ on $[0, \infty)$ and real numbers a, b, c with c non-negative, such that

$$f(t) = a + bt + ct^2 + \int_0^{\infty} \frac{(\lambda + 1)t^2}{\lambda + t} d\mu(\lambda)\,.$$

Using Corollary (4.3.25) (ii) and inequality (47), we may essentially mimic the proof in Theorem (4.3.26) adapting it to the case of operator convex functions. What deserves mention is the disappearance of the quadratic term cx^2 in the equality case. We start with the assumption that $\Phi(A) \neq A$. Since $\big(f(A)\big) = f\big(\Phi(A)\big)$, we must have $c\Phi(A^2) = c\Phi(A)^2$ as $\langle (c\Phi(A^2) - c\Phi(A)^2)x, x\rangle = 0$ for all x in $\mathcal{H}$. Thus, from the equality condition in Theorem (4.3.24), we see that $c = 0$. The rest of the proof for the equality case is similar to the case of operator monotone functions. We conclude that if $f\big(\Phi(A)\big) = \Phi\big(f(A)\big)$ and $\Phi(A) \neq A$, f must be a linear function.

Finally we get rid of the assumption of existence of $f'(0^+)$. Let $\varepsilon > 0$ be such that $\varepsilon I \leq A$ and define a function g on $[0, \infty)$ by $g(x) := f\left(x + \frac{\varepsilon}{2}\right)$.Note that g is an operator convex function on $[0, \infty)$ and $g'(0) = f'\left(\frac{\varepsilon}{2}\right)$ exists. For the regular positve operator $A_\epsilon := A - \frac{\epsilon}{2} I$, we conclude that $g\big(\Phi(A_\varepsilon)\big) \leq \Phi\big(g(A_\varepsilon)\big) \Rightarrow f\big(\Phi(A)\big) = f\left(\Phi(A_\varepsilon) + \frac{\varepsilon}{2} I\right) \leq \Phi(f\left(A_\varepsilon + \frac{\varepsilon}{2} I\right) = \Phi\big(f(A)\big)$with equality if and only if either g is linear or $\Phi(A_\varepsilon) = A_\varepsilon i.e. f$ is linear or $\Phi(A) = A$.

The inequalities in Theorem (4.3.26), Theorem (4.3.27) involve similar-looking quantities, the direction is reversed.

Theorem (4.3.28) [225]: *Let* $f:(0,\infty)\to(0,\infty)$ be a *non-constant operator monotone function. Then for a regular positive operator* A in $\mathcal{R}$, we *have the following inequality*,

$$\triangle\left(f(A)\right)\leq\triangle\left(f\left(\Phi(A)\right)\right),\qquad(51)$$

with equality if and *only if* $\Phi(A)=A$ *i.e.* $A\in\mathcal{S}$.

Proof. As $\log$ is an operator monotone function on $(0,\infty)$, we observe that $\log f$ is also operator monotone on $(0,\infty)$. As the exponential function is not operator monotone, note that $\log f$ is not linear. Thus from Theorem (4.3.26), we have that

$$\Phi\left(\log f(A)\right)\leq\log f\left(\Phi(A)\right)),$$

with equality if and only if $\Phi(A)=A$. As τ is a faithful state, using the trace-preserving nature of Φ, we have

$$\tau\left(\Phi\left(\log f(A)\right)\right)=\tau\left(\log f(A)\right)\leq\tau\left(\log f\left(\Phi(A)\right)\right),$$

and equality holds if and only if $\Phi(A)=A$. Applying the exponential function, which is strictly increasing, to both sides of the inequality, we get the desired inequality with equality if and only if $\Phi(A)=A$.

Corollary (4.3.29) [225]: *For* a *regular positive operator* A *in* $\mathcal{R}$, *we have* that

$$\triangle\left(I+\Phi(A)^{-1}\right)\leq\triangle\left(I+A^{-1}\right),\qquad(52)$$

with equality if and *only* if $\Phi(A)=A$ *i.e.* $A\in\mathcal{S}$.

Theorem (4.3.30) [225]: *For* a regular *positive operator* A in $\mathcal{R}$, and a *positive operator* B in $\mathcal{S}$, *the following inequality holds* :

$$\frac{\triangle\left(\Phi(A)+B\right))}{\triangle\left(\Phi(A)\right)}\leq\frac{\triangle\left(A+B\right)}{\triangle\left(A\right)}\qquad(53)$$

If B *is* regular, *equality* holds *if and only if* $\Phi(A)=A$ *i.e.* $A\in\mathcal{S}$.

(*Note* that when $B=0$, *equality* holds *for any* regular *positive operator A*. This *illustrates that when* B *is not regular*, the characterizing *conditions for equality may not* be as *simple* in *form*.)

Proof. As A is regular, there is an $\varepsilon>0$ such that $\varepsilon I\leq A$. First we prove the inequality for the regular operators $A_{\varepsilon}:=A-\frac{\varepsilon}{2}I$, $B_{\varepsilon j}:=B+\frac{\varepsilon}{2}I$. We observe that $\frac{\varepsilon:}{2}I\leq\Phi\left(A_{\varepsilon j}\right),\frac{\varepsilon}{2}I\leq B_{\varepsilon}$ as a result of which $\Phi(A_{\varepsilon})$, B_{ε} are also regular. Using the multiplicativity of $\triangle$, we can rewrite both sides of the inequality in the following manner :

$$\frac{\triangle\left(A_{\varepsilon:}+B_{\varepsilon:}\right)}{\triangle\left(A_{6}\right)}=\triangle\left(I+B^{\frac{1}{\varepsilon^{2}}}A_{\varepsilon}^{-1}B^{\frac{1}{\varepsilon^{2}}}\right),$$

$$\frac{\triangle\left(\Phi(A_{\varepsilon})+B_{\varepsilon}\right)}{\triangle\left(\Phi(A_{\varepsilon})\right)}=\triangle\left(I+B^{\frac{1}{\varepsilon^{2}}}\Phi(A_{\varepsilon})^{-1}B^{\frac{1}{\varepsilon^{2}}}\right).$$

Note that $X:=B_{\varepsilon}^{-\frac{1}{2}}A_{\varepsilon}B_{\varepsilon}^{-\frac{1}{2}}$ is a regular positive operator. As $B_{\varepsilon}^{-\frac{1}{2}}$ is in $\mathcal{S}$, $\Phi(X)=B_{\varepsilon}^{-\frac{1}{2}}\Phi(A_{\varepsilon})B_{\varepsilon}^{-\frac{1}{2}}$ From Corollary (4.3.28),

$$\triangle\left(I+\Phi(X)^{-1}\right)\leq\triangle\left(I+X^{-1}\right).$$

which proves the inequality for A_{ε}, B_{ε}. Note that $\Phi(A)+B=\Phi(A_{\varepsilon})+B_{\varepsilon}$ and $A+B=A_{\varepsilon}+B_{\varepsilon}$. Thus we have that

$$\frac{\triangle\left(\Phi(A)+B\right)}{\triangle\left(\Phi(A_\varepsilon)\right)} \leq \frac{\triangle\left(A+B\right)}{\triangle\left(A_\varepsilon\right)}.$$

Taking the limit as $\varepsilon \to 0^+$, we get the required inequality.

If B is regular, then we may directly follow the above steps without performing the perturbative step of defining A_ε, B_ε and instead defining $:= B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Noting that $B^{-\frac{1}{2}}$ is in $\mathcal{S}$ and using the equality condition in Corollary (4.3.29), we see that equality holds if and only if $\Phi(X) = X \Leftrightarrow B^{-\frac{1}{2}}\Phi(A)B^{-\frac{1}{2}} = \Phi\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right) = B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \Leftrightarrow \Phi(A) = A$.

Remark (4.3.31) [225]: The following generalized form of inequality (53),

$$\frac{\triangle\left(\Phi(A)+\Phi(B)\right)}{\triangle\left(\Phi(A)\right)} = \frac{\triangle\left(\Phi(A+B)\right)}{\triangle\left(\Phi(A)\right)} \leq \frac{\triangle\left(A+B\right)}{\triangle\left(A\right)} \qquad (54)$$

does not hold for an arbitrary choice of positive operators A, B in $\mathcal{R}$ with A being regular. From Theorem (4.3.22), we have that $\triangle\left(I+B\right) \leq \triangle\left(\Phi(I+B)\right) = \triangle\left(I+\Phi(B)\right)$ with equality if and only if $\Phi(I+B) = I+B$ *i.e.* $\Phi(B) = B$. So inequality (54) is clearly untrue if $A = I$ and B is not in $\mathcal{S}$ *i.e.* $\Phi(B) \neq B$.

In the remaining portion we establish versions of some of the inequalities proved above, in a more general setting. Broadly speaking, conditional expectations on $\mathcal{R}$ are replaced by the more general unital positive maps on $\mathcal{R}$, again denoted by Φ. For a detailed study of positive linear maps between operator algebras, we direct the reader to [234] (see also [235]). By the Kadison-Schwarz inequality ([230]), inequality (47) still holds in this context. In addition, if Φ were a unital 2-positive map, then the proof for inequality (48) in Theorem (4.3.24) goes through. The map $: \mathcal{R} \to \mathcal{R}$ is said to be τ-preserving or trace-preserving if $\tau(X) = \tau\left(\Phi(X)\right)$ for all X in $\mathcal{R}$. A careful scrutiny would reveal that for Φ a trace-preserving unital 2-positive map, the determinant inequalities (51), (52) are still valid. We mention the appropriate version of the result for unital 2-positive maps below. The trade-off for this level of generality is that we are unable to find straightforward conditions for equality.

Theorem (4.3.32) [225]: *Let* $\Phi: \mathcal{R} \to \mathcal{R}$ *be a unital* 2*-positive map which is* τ*-preserving and* A *be a regular positive operator in* $\mathcal{R}$*. For a positive-valued operator monotone function* f *on* $(0,\infty)$, *we have the following inequality*:

$$\triangle\left(f(A)\right) \leq \triangle\left(f\left(\Phi(A)\right)\right).$$

In addition to the preceding comments, we also explore another direction of generalization below. We first prove a convexity inequality as a form of Jensen's inequality. Although the basic idea is contained in the proof of [234], we adapt the relevant parts to our discussion for the sake of clarity and continuity.

Proposition (4.3.33) [225]: *Let* $\Phi: \mathcal{R} \to \mathcal{R}$ *be* a *unital positive map. Let* f *be* a *real-valued continuous convex function defined on the interval* $[a,b] \subseteq \mathbb{R}$*. Then for every self-adjoint operator* A in $\mathcal{R}$ *with spectrum* in $[a,b]$, $\Phi(A)$ *also* has *spectrum in* $[a,b]$ and *we* have *the following inequality* :

$$\tau\left(f\left(\Phi(A)\right)\right) \leq \tau\left(\Phi\left(f(A)\right)\right).$$

Proof. As the spectrum of A is contained in $[a,b]$, we note that $aI \leq A \leq bI$. Since Φ is a unital positive map, we conclude that $aI = \Phi(aI) \leq \Phi(A) \leq \Phi(bI) = bI$. Thus $\Phi(A)$ also has spectrum in $[a,b]$.

Let $\mathcal{A}$ be a *masa* of $\mathcal{R}$ containing (A) . By Theorem (4.3.12), there is a unique tracepreserving conditional expectation $\Psi: \mathcal{R} \to \mathcal{A}$. Note that $\Psi \circ \Phi$ is a unital positive map into a commutative von Neumann algebra $\mathcal{A}$, and $(\Psi \circ \Phi)(A) = \Phi(A)$. From Theorem (4.3.21), we have that

$$f\big(\Phi(A)\big) = f\big((\Psi \circ \Phi)(A)\big) \leq (\Psi \circ \Phi)\big(f(A)\big).$$

Using the positivity of the trace and trace-preserving nature of Ψ, we conclude that

$$\tau(f\big(\Phi(A)\big) \leq \tau\left(\Psi\left(\Phi\big(f(A)\big)\right)\right) = \tau\left(\Phi\big(f(A)\big)\right).$$

Theorem (4.3.34) [225]: *Let* $\Phi: \mathcal{R} \to \mathcal{R}$ be a trace-preserving unital positive map. Let f be a continuous positive function defined on the interval $[a,b] \subseteq \mathbb{R}$ *such* that $\log f$ is convex (in other words, f is log-convex). Then for every positive operator A in $\mathcal{R}$, we have the following inequality:

$$\triangle\,(f\big(\Phi(A)\big) \leq \triangle\,\big(f(A)\big) \tag{55}$$

Proof. Using Proposition (4.3.33), from the convexity of $\log f$ and trace-preserving nature of Φ, we observe that,

$$\tau\left(\log f\big(\Phi(A)\big)\right) \leq \tau\left(\Phi\big(\log f(A)\big)\right) = \tau\big(\log f(A)\big).$$

Thus $\triangle\left(f\big(\Phi(A)\big)\right) = \exp\left(\tau\left(\log f\big(\Phi(A)\big)\right)\right) \leq \exp\left(\tau\big(\log f(A)\big)\right) = \triangle\left(f(A)\right)$.

Corollary (4.3.35) [225]: *Let* $\Phi: \mathcal{R} \to \mathcal{R}$ *be* a trace-*preserving unital* positive *map*. Then for *every positive operator A i*n $\mathcal{R}$, *w*e have the following *inequality*:

$$\triangle\,(A) \leq \triangle\,\big(\Phi(A)\big) \tag{56}$$

Further, *if A is also regular*, *then we have that*

$$\triangle\,(\Phi(A^{-1})^{-1}) \leq \triangle\,(A) \tag{57}$$

Proof. For $b > \varepsilon > 0$, the function $f: [\varepsilon, b] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, is a $\log$-convex function. From Theorem (4.3.34), for a regular positive operator A with spectrum in $[\varepsilon, b]$, we have that

$$\triangle\,(\Phi(A)^{-1}) \leq \triangle\,(A^{-1}).$$

Thus using the multiplicativity of the Fuglede-Kadison determinant, we conclude that

$$\triangle\,(A) \leq \triangle\,\big(\Phi(A)\big).$$

If A is any positive operator (not necessarily regular), applying the inequality to the regular positive operator $A + \varepsilon I$ (for $\varepsilon > 0$), we note that $\triangle\,(A + \varepsilon I) \leq \triangle\,\big(\Phi(A + \varepsilon I)\big) = \triangle\,(\Phi(A) + \varepsilon I)$. Keeping in mind that $\triangle$ is a continuous function on $\mathcal{R}$, and taking the limit as $\varepsilon \to 0^+$, we note that $\triangle\,(A) \leq \triangle\,\big(\Phi(A)\big)$ for all positive operators A in $\mathcal{R}$.

If A is regular, then by the inequality proved above $\triangle\,(A^{-1}) \leq \triangle\,\big(\Phi(A^{-1})\big)$. Using the multiplicativity of $\triangle$, we conclude that

$$\triangle\,(\Phi(A^{-1})^{-1}) \leq \triangle\,(A).$$

Corollary (4.3.36) [225]: *Let* $\Phi: \mathcal{R} \to \mathcal{R}$ *be* a trace-preserving unital *positive map*. Then *for every* regular *positive operator A in* $\mathcal{R}$, *we have that*

$$\triangle\left(I+\Phi(A)^{-1}\right) \leq \triangle\left(I+A^{-1}\right). \qquad (58)$$

Proof. As A is a regular positive operator, there is an $\varepsilon > 0$ such that $\varepsilon I \leq A$. For $b > \varepsilon > 0$,the function f: $[\varepsilon, b] \to \mathbb{R}$ defined by $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$ is a log -convex function as the second derivative of $\log\left(\frac{x+1}{t^x}\right) = \log(x+.1) - \log$ is $\frac{1}{x^2} - \frac{\frac{1}{x}}{(x+1)^2}$ which is greater than 0.The required inequality is just inequality (55) for f considered above.

Corollary (4.3.37) [225]: *Let* Φ: $\mathcal{R} \to \mathcal{R}$ *be* a $t\tau ace$-preserving *unital positive map*. *For positive operators* A, B *in* $\mathcal{R}$ *with* A *being* regular, *the following inequality holds* :

$$\frac{\triangle(A+B)}{\triangle(A)} \leq \frac{\triangle\left(\Phi(A)+\Phi(B)\right)}{\triangle\left(\Phi(A^{-1})^{-1}\right)} \qquad (59)$$

Proof. Using Corollary (4.3.35), we observe that the following inequalities hold,

$$\triangle(A+B) \leq \triangle\left(\Phi(A+B)\right) = \triangle\left(\Phi(A)+\Phi(B)\right),$$

$$\frac{1}{\triangle(A)} = \triangle(A^{-1}) \leq \triangle\left(\Phi(A^{-1})\right) = \frac{1}{\triangle\left(\Phi(A^{-1})^{-1}\right)}.$$

Multiplying both the inequalities gives us (59).

Here we make the appropriate choices of the finite von Neumann algebra $\mathcal{R}$, the von Neumann subalgebra $\mathcal{S}$ of $\mathcal{R}$, and the trace-preserving conditional expectation Φ, to obtain the inequalities mentioned. We also provide several subtle improvements to Theorem (4.3.3) and Theorem (4.3.38).

We follow the notation in Example (4.3.9). Let $\mathcal{R} = M_n(\mathbb{C})$, $\mathcal{S} = D_n(\mathbb{C})$, and Φ : $\mathcal{R} \to \mathcal{S}$ be given by $\Phi(A) := \operatorname{diag}(a_{11}, \cdots, a_{nn})$. For a positive-definite matrix A, we have from Theorem (4.3.22) that $\sqrt[n]{\det A} = \triangle(A) \leq \triangle\left(\Phi(A)\right) = \sqrt[n]{a_{11} a_{nn}}$. Taking n^{th} powers on both sides, we obtain Hadamard's inequality, and equality holds if and only if $\Phi(A) = A$ *i.e*. A is a diagonal matrix.

For the remaining three inequalities, we are in the setting of Example (4.3.10). Let $\mathcal{R} = M_n(\mathbb{C})$, $\mathcal{S} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, and Φ: $\mathcal{R} \to \mathcal{S}$ be given by $\Phi(A) = \operatorname{diag}(A_{11}, \cdots, A_{kk})$.

For a positive-definite matrix A, we have from Theorem (4.3.22) that $\det A = \triangle(A)^n \leq \triangle\left(\Phi(A)\right)^n = \det\left(\operatorname{diag}(A_{11}, \cdots, A_{kk})\right) = (\det A_{11}) \cdots (\det A_{kk})$. This gives us Fischer's inequality, and equality holds if and only if $\Phi(A) = A$ *i.e*. A is a block diagonal matrix.

Consider a positive-definite matrix C in $M_n(\mathbb{C})$, and a positive-definite matrix $D = \operatorname{diag}(D_1, \cdots, D_k)$ in $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Let the principal diagonal blocks of C be denoted by $C_1, \cdots, C_k$. We observe that,

$$\frac{\det(C_1+D_1)}{\det(C_1)} \cdots \frac{\det(C_k+D_k)}{\det(C_k)} = \frac{\det\left(\operatorname{diag}(C_1+D_1, \cdots.'C_k+D_k)\right)}{\det\left(\operatorname{diag}(C_1, \cdot\cdot, C_k)\right)}$$

$$= \frac{\det(\Phi(C)+D)}{\det\left(\Phi(C)\right)}.$$

Thus, from Theorem (4.3.30), we have that,

$$\frac{\det\left(\Phi(C)+D\right)}{\det\left(\Phi(C)\right)}=\left(\frac{\triangle\left(\Phi(C)+D\right)}{\triangle\left(\Phi(C)\right)}\right)^n\leq\left(\frac{\triangle\left(C+D\right)}{\triangle\left(C\right)}\right)^n=\frac{\det\left(C+D\right)}{\det\left(C\right)}.$$

This proves Theorem (4.3.3) and equality holds if and only if $\Phi(C)=C$ *i.e.* C is in block diagonal form. If D is positive-semidefinite, the inequality still holds but the equality condition may not be applicable as noted in the parenthetical remark following the statement of Theorem (4.3.30).

Theorem (4.3.38) [225]: *For each $i\in\langle k\rangle$, let C_i, D_i be positive-definite matrices in $M_{n_i}(\mathbb{C})$. Let C be a positive-definite matrix in block form in $M_n(\mathbb{C})$ such that the principal diagonal blocks of C^{-1} (in block form) is given by $C_1^{-1}, C_2^{-1}, \cdots, C_k^{-1}$ Then the following inequality holds,*

$$\frac{\det\left(C+\operatorname{diag}(D_1,\cdots,D_k)\right)}{\det\left(C\right)}\leq\frac{\det\left(C_1+D_1\right)}{\det\left(C_1\right)}\cdots\frac{\det\left(C_k+D_k\right)}{\det\left(C_k\right)}. \quad (60)$$

Proof . Consider a positive-definite matrix C in $M_n(\mathbb{C})$, and a positive-semidefinite matrix $D=\operatorname{diag}\left(D_1,\cdots,D_k\right)$ in $M_{n_1}(\mathbb{C})\oplus\cdots\oplus M_{n_k}(\mathbb{C})$. Let $C_1, \cdots, C_k$ be positive-definite matrices such that the principal diagonal blocks of C^{-1} are given by $C_1^{-1}, \cdots, C_k^{-1}$; in other words, $\Phi(C^{-1})=\operatorname{diag}(C_1^{-1},\cdots,C_k^{-1})$ or $\Phi(C^{-1})^{-1}=\operatorname{diag}\left(C_1,\cdots,C_k\right)$. We observe that,

$$\frac{\det\left(C_1+D_1\right)}{\det\left(C_1\right)}\cdots\frac{\det\left(C_k+D_k\right)}{\det\left(C_k\right)}=\frac{\det\left(\operatorname{diag}(C_1+D_1,\dots,C_k+D_k)\right)}{\det\left(\operatorname{diag}(C_1,\dots,C_k)\right)}$$
$$=\frac{\det\left(\Phi(C^{-1})^{-1}+D\right)}{\det\left(\Phi(C^{-1})^{-1}\right)}.$$

Thus, from Corollary (4.3.23), we have that,

$$\frac{\det\left(\Phi(C^{-1})^{-1}+D\right)}{\det\left(\Phi(C^{-1})^{-1}\right)}=\left(\frac{\triangle\left(\Phi(C^{-1})^{-1}+D\right)}{\triangle\left(\Phi(C^{-1})^{-1}\right)}\right)^n\geq\left(\frac{\triangle\left(C+D\right)}{\triangle\left(C\right)}\right)^n=\frac{\det\left(C+D\right)}{\det\left(C\right)}.$$

This proves Theorem (4.3.38) and equality holds if and only if $D^{\frac{1}{2}}C^{-1}D^{\frac{1}{2}}$ is in block diagonal form *i.e.* $D^{\frac{1}{2}}C^{-1}D^{\frac{1}{2}}$ is in $M_{n_1}(\mathbb{C})\oplus\cdots\oplus M_{n_k}(\mathbb{C})$.

Chapter 5
Isometries and Geometric Version

A variety of tools is used, such as topological, geometrical and linear algebra techniques. The famous two projections theorem for two finite rank projections will be re-proven using linear algebraic methods. A theorem of Gyory and on orthogonality preservers on Grassmann spaces will be strengthened as well. This latter result will be obtained by using Chow's fundamental theorem of geometry of Grassmannians. We give a very natural joint generalisation of Wigner's and Molnár's theorems, namely, we show a characterisation of all (not necessarily bijective) transformations on the Grassmann space which fix the quantity Tr P Q (i.e. the sum of the squares of cosines of principal angles) for every pair of rank-n projections P and Q. We get the following: if the dimension of H is finite and greater than $2k$, then every transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator.

Section (5.1): Grassmann Spaces

The form of (surjective) isometries of linear normed spaces is an area of functional analysis. A theorem of Mazur and Ulam states that every surjective isometry between two real normed spaces is automatically an affine map, i.e. a composition of a linear map and a translation by a vector. Therefore if the spaces are isomorphic as metric spaces, then they are also isomorphic as vector spaces. Another classical example of this is the famous Banach-Stone theorem which describes surjective linear isometries between Banach spaces $C(X)$ and $C(Y)$ of continuous functions over compact Hausdorff spaces X and Y. An immediate consequence of this result is that the existence of a metric isomorphism between $C(X)$ and $C(Y)$ implies the topological equivalence of the underlying spaces X and Y. The non-commutative extension of this theorem was provided by Kadison, who in particular showed that surjective linear isometries between C^*-algebras are closely related to algebra isomorphisms. See [244], [244] for more results in this direction.

Isometries of non-linear spaces are also very important in functional analysis. The famous Wigner's theorem, playing an important role in the probabilistic aspects of quantum mechanics, can be interpreted as a structural result for isometries of a certain non-linear space. Let H be a complex (or real) Hilbert space. In quantum physics the Grassmann space $P_1(H)$ of all rank-one (orthogonal) projections is used to represent the set of pure states of the quantum system, and the quantity tr (PQ) is the so-called transition probability between two pure states. Wigner's theorem describes those transformations of $P_1(H)$ which preserve the transition probability. The conclusion is that these transformations are induced by linear or conjugate-linear isometries of H. One can easily obtain the following equation: $\|P - Q\| = \sqrt{1 - \mathrm{tr}PQ}\big(P, Q \in P_1(H)\big)$, where $\|\cdot\|$ denotes the operator norm. The metric on $P_1(H)$ (or on any other subset of projections) which is induced by the operator norm is usually called the gap metric. Therefore, Wigner's theorem characterizes isometries of $P_1(H)$ with respect to the gap metric, and in fact it states that these maps are induced by isometries of the underlying space H. we note that in its original version, Wigner's theorem describes surjective mappings of this kind, but as was shown later in several the above conclusion holds for non-surjective transformations as well. The gap metric was introduced and investigated by Sz.-Nagy and independently by Krein and Krasnoselski under the name “ aperture” It has a wide range of applications from pure mathematics to engineering. One can easily find several references demonstrating this broad applicability, among others, we list the following fields: perturbation theory of linear operators, perturbation analysis of invariant subspaces, optimization, robust control, multi-variable control, system identification and signal processing.

Since Wigner’s theorem quite a lot of attention has been paid to the study of isometries of non-linear spaces. Here, we are interested in the description of surjective isometries on the Grassmann space $P_n(H)$ of all rank n projections with respect to the gap metric $(n \in \mathbb{N})$. In [247], Molnár characterized (not necessarily surjective) transformations of $P_n(H)$ which preserve the complete system of the so-called principal angles. These transformations are implemented by an isometry of H. The notion of principal angles was first investigated by Jordan, and has a wide range of applications such as in mathematical statistics, geometry, etc. We recall that the sines of the non-zero principal angles are exactly the non-zero singular values of the operator $P - Q$, each of them counted twice (see e.g. (ii) of [245]). This further implies that the quantity $\|P - Q\|$ is the sine of the largest principal angle. Recently, Botelho, Jamison, and Molnár have obtained a characterization of surjective isometries of $P_n(H)$ with respect to the gap metric for complex Hilbert spaces H under the dimensionality constraint $\dim H \geq 4n$ ([241]). Their approach was to apply a non-commutative Mazur-Ulam type result on the local algebraic behaviour of surjective isometries between substructures of metric groups. Then they proved that such a mapping preserves orthogonality in both directions, and finally they applied a theorem of Györy and Šemrl ([246], [247]), which contains some dimensionality constraint, too.

We provide a completely different approach to Botelho-Jamison-Molnár’s generalization of Wigner’s theorem. We will remove the dimensionality assumption, and in finite dimensions we are able to drop the surjectivity condition. As a byproduct, we are also able to handle the real case. Furthermore, an additional possibility occurs in the case when $\dim H = 2n$, which was not covered in [241].

We state the main result on isometries of the Grassmannians.

In the case when $\dim H = 2n$, *we have either* (*1*), *or the following additional possibility occurs*:

$$\varphi(P) = U(I - P)U^*\big(P \in P_n(H)\big) . \tag{1}$$

If $\dim H < \infty$, *then we have the above conclusion without assuming surjectivity*.

whenever we say a projection we automatically mean an orthogonal projection. We will consider two arbitrary projections P and Q, and we will investigate the set $M(P, Q)$ which consists of those projections whose distance to both P and Q is less than or if P *and* Q *are or*thogonal

. This will imply that orthogonality is preserved in both directions by φ. In the case when $\dim H > 2n$ the proof is completed by a straightforward application of our second main result stated below. In the case when $\dim H = 2n$, the orthogonality preservers can behave badly. So, in this special case another approach is needed. It is based on a theorem of Blunck and Havlicek on complementarity preservers.

We consider a manifold, we always mean a topological manifold without boundary.

Now, we state our improvement of the theorem of Györy and Šemrl.

We will first prove several lemmas concerning the properties of the above mentioned set (P, Q) . We also include a proof of the well-known two projections theorem in the case when the projections are both from $P_n(H)$. is devoted to the proofs of Theorems (5.1.9) and (5.1.8).

We will often use matrix representation of operators. In all such cases the matrices and the block-matrix forms are written with respect to an orthonormal system or an orthogonal decomposition, respectively. By Diag (. . .) we will denote a (block-)diagonal matrix. In our first lemma we consider the operator norm of certain two by two matrices. *Furthermore, in both cases equality holds if and only if* $\alpha = -\frac{1}{2}$.

Lemma (5.1.1) [239]: *We have*

$$\left\| \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \alpha \end{bmatrix} \right\| \geq \frac{1}{\sqrt{2}} \text{ and } \left\| \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \alpha \end{bmatrix} \right\| \geq \frac{1}{\sqrt{2}} \left(-\frac{1}{2} \leq \alpha \leq \frac{1}{2}\right).$$

Proof. The proof of the two cases are almost identical, so we will only deal with the first one. If we set $A_\alpha = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \alpha \end{bmatrix}$ with $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, then clearly,

$$\frac{1}{2} + \alpha = \mathrm{tr} A_\alpha \geq 0 \text{ and } \frac{\alpha}{2} - \frac{1}{4} = \det A_\alpha \leq 0. \tag{2}$$

Since A_α is hermitian in the complex case and symmetric in the real case, we have $\|A_\alpha\| = \max\{|t_1|, |t_2|\}$, where t_1, t_2 are the (possibly equal) eigenvalues of A_α. Because of (2) we get

$$f(\alpha) = 2\|A_\alpha\| = 2\max\{|t_1|, |t_2|\} = t_1 + t_2 + |t_1 - t_2|$$
$$= \mathrm{tr} A_\alpha + \sqrt{(\mathrm{tr} A_\alpha)^2 - 4\det A_\alpha} = \alpha + \frac{1}{2} + \sqrt{\alpha^2 - \alpha + \frac{5}{4}}.$$

Since $f(-1/2) = \sqrt{2}$, and

$$f'(\alpha) = 1 + \frac{-1 + 2\alpha}{2\sqrt{\frac{5}{4} - \alpha + \alpha^2}} > 0 \left(\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right),$$

we easily complete the proof.

The set of bounded linear operators acting on H is denoted by (H) . For any two projections $P, Q \in P_n(H)$ we define the following set:

$$M(P, Q) = \{R \in P_n(H) : \|R - P\| \leq \frac{1}{\sqrt{2}} \text{ and } \|R - Q\| \leq \frac{1}{\sqrt{2}}\}.$$

This set will play an important role. If $A \subset H$ is a set, then $A^\perp$ and span A denote the set of all vectors which are orthogonal to every element of A, and the (not necessarily closed) linear manifold generated by A, respectively. We will give a useful description of the set $M(P, Q)$ when $\|P - Q\| = 1$.

Lemma (5.1.2) [239]: *Let H be a real or complex Hilbert space, n a positive integer, and P, $Q \in P_n(H)$ such that $\|P - Q\| = 1$. Then for every $R \in M(P, Q)$ there exist an* *orthogonal decomposition* $H = H_1 \oplus H_2$ *with* $\dim H_1 = 2$ *and an orthonormal basis* $\{e_1, e_2\}$ *in* H_1, *such that with respect to this decomposition and this orthonormal basis the projections* P, Q, R *have the following matrix representations*:

$$P = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & P_1 \end{bmatrix} Q = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & Q_1 \end{bmatrix}, \text{and } R = \begin{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} & 0 \\ 0 & R_1 \end{bmatrix}$$

where $P_1, Q_1, R_1 \in P_{n-1}(H_2)$.

Proof. In both, the real and the complex case, we have $1 = \|P - Q\| = \max\{|\langle (P-Q)x, x\rangle| : x \in H, \|x\| = 1\}$. Since P and Q are projections we know that

$$0 \leq \langle Px, x\rangle, \langle Qx, x\rangle \leq 1$$

holds for every unit vector $x \in H$. Thus, after interchanging P and Q, if necessary, we may assume that there exists $e_1 \in H$ such that $\|e_1\| = 1$, $\langle Pe_1, e_1\rangle = 1$ and $\langle Qe_1, e_1\rangle = 0$. It follows that $Pe_1 = e_1$ and $Qe_1 = 0$. Since P and Q are projections, their matrix representations with respect to the orthogonal decomposition $H = \text{span}\{e_1\} \oplus e_1^{\perp}$ are

$$P = \begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}$$

for some projections P_2, Q_2 acting on $e_1^{\perp}$. The corresponding matrix representation of the projection R is

$$R = \begin{bmatrix} r_1 & x^* \\ x & R_2 \end{bmatrix},$$

where r_1 is a real number, x a vector from $e_1^{\perp}$, and $R_2 \in B(e_1^{\perp})$. From $R^2 = R$,

$$\frac{1}{\sqrt{2}} \geq \|R - Q\| = \|\begin{bmatrix} r_1 & x^* \\ x & R_2 - Q_2 \end{bmatrix}\|,$$

and

$$\frac{1}{\sqrt{2}} \geq \|R - P\| = \|\begin{bmatrix} r_1 - 1 & x^* \\ x & R_2 - P_2 \end{bmatrix}\|$$

the following equalities and inequalities can be obtained:

$$r_1^2 + \|x\|^2 = r_1, \sqrt{r_1^2 + \|x\|^2} \leq \frac{1}{\sqrt{2}}, \text{and } \sqrt{(r_1 - 1)^2 + \|x\|^2} \leq \frac{1}{\sqrt{2}}.$$

These readily imply $r_1 = \frac{1}{2} = \|x\|$. Setting $e_2 = 2x$, the matrix representations of P, Q, R with respect to the orthogonal decomposition $H = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \{e_1, e_2\}^{\perp}$ are

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_2 & y^* \\ 0 & y & P_1 \end{bmatrix} Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q_2 & w^* \\ 0 & w & Q_1 \end{bmatrix} \text{ and } R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & r_2 & z^* \\ 0 & z & R_1 \end{bmatrix},$$

for some $p_2, q_2, r_2 \in [0,1]$, $y, w, z \in \{e_1, e_2\}^{\perp}$, and some $P_1, Q_1, R_1 \in B(\{e_1, e_2\}^{\perp})$. It follows from $R^2 = R$ that $r_2 = \frac{1}{2}$ and $z = 0$.

Let $S \in B(H)$ be the projection onto the two-dimensional subspace $\{e_1, e_2\}$. From

$$\|S(R - Q)S\| \leq \frac{1}{\sqrt{2}}$$

we conclude that

$$\| \left[\frac{1}{2} - q_2 \frac{1}{2} \right] \| \leq \frac{1}{\sqrt{2}},$$

and hence, by Lemma (5.1.1), we have $q_2 = 1$. But then $Q \leq I$ yields that $w = 0$. Thus, Q is of the desired form, and in exactly the same way we see that also P is of the form as described in the conclusion of the lemma.

We note that so far we have not proven that $M(P,Q)$ is non-empty. We only showed that if $\|P - Q\| = 1$ and $\in M(P,Q)$, then we have the conclusion of Lemma (5.1.2).

In what follows, u_r will denote either the unitary group on the r-dimensional complex Hilbert space, or the orthogonal group on the r-dimensional real Hilbert space. The symbols I_r and 0_r will denote the r by r identity and zero matrices, respectively.

Corollary (5.1.3) [239]: *Let* $P,\ Q \in P_n(H)$ *such that* $\|P - Q\| = 1$. *Then there exists a number* $1 \leq r \leq n$ *such that*

$$P = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_1 \end{bmatrix} \textit{and}\ Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & Q_1 \end{bmatrix} \qquad (3)$$

with respect to an orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$, $\dim H_1 = \dim H_2 = r$, *and* $P_1,\ Q_1 \in P_{n-r}(H_3)$, $\|P_1 - Q_1\| < 1$ (*in the case when* $r = n$ *we have* $P_1 = Q_1 = 0$). *Moreover, in this case* $M(P,Q)$ *is the set of all projections of the form*

$$\begin{bmatrix} \frac{1}{2} I_r & \frac{1}{2} U & 0 \\ \frac{1}{2} U^* & \frac{1}{2} I_r & 0 \\ 0 & 0 & R_4 \end{bmatrix} \qquad (4)$$

where $U \in u_r$, $R_1 \in P_{n-r}(H_3)$, $\|R_1 - P_1\| \leq \frac{1}{\sqrt{2}}$, *and* $\|R_1 - Q_1\| \leq \frac{1}{\sqrt{2}}$.

Proof. We begin with verifying (3). After interchanging P and Q if necessary, we may use exactly the same arguments as at the beginning of the proof of
Lemma (5.1.2) to conclude that P and Q are unitary (orthogonal) similar to

$$\begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix} \textit{and} \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix},$$

where P_2 and Q_2 are projections of rank $n - 1$ and n, respectively. It follows that there exists a unit vector from the $\text{Im}\ Q_2 \cap \text{Ker} P_2$. In other words, P and Q are unitary (orthogonal) similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \textit{and} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}$$

where P_3 and Q_3 are projections both of rank $n - 1$. Now, we apply the inductive approach to obtain (3).

Next, let R be of the form (4). An easy calculation shows that $\in P_n(H)$. We observe that the upper-left two by two corners of $P - R$ and $Q - R$ are $\frac{1}{\sqrt{2}}$ multiples of unitary (orthogonal) operators. Therefore R is indeed in (P,Q) . We consider a projection $\in M(P,Q)$. Then, by Lemma (5.1.2) there exists a unitary (orthogonal) operator U such that

$$P = U\begin{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} & 0\\ 0 & P'\end{bmatrix}U^*, Q = U\begin{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} & 0\\ 0 & Q'\end{bmatrix}U^*,$$

and

$$R = U\begin{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} & 0\\ 0 & R'\end{bmatrix}U^*,$$

where P', Q', R' are projections of rank $n-1$. We also have $\|R' - P'\| \leq \frac{1}{\sqrt{2}}$ and $\|R' - Q'\| \leq \frac{1}{\sqrt{2}}$. If $\|P' - Q'\| < 1$ we stop here. Otherwise we apply Lemma (5.1.2) once again, this time for projections P', Q', and $R' \in M(P', Q')$. Inductively we arrive at

$$P = V\ Diag\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \dots, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, P''\right)V^*,$$

$$Q = V\ Diag\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \dots, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, Q''\right)V^*,$$

$$R = V\ Diag\left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}, \dots, \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}, R''\right)V^*,$$

for some unitary (orthogonal) operator V and some projections P'', Q'', R'' with $\|P'' - Q''\| < 1$ and $R'' \in M(P'', Q'')$. Let k denote the number of two by two diagonal blocks appearing in the above matrix representations of , Q, and R. After rearranging the first $2k$ elements of the orthonormal basis of H we get

$$P = \begin{bmatrix}I_r & 0 & 0\\0 & 0 & 0\\0 & 0 & P_1\end{bmatrix} = W\begin{bmatrix}I_k & 0 & 0\\0 & 0 & 0\\0 & 0 & P''\end{bmatrix}W^*, \tag{5}$$

$$Q = \begin{bmatrix}0 & 0 & 0\\0 & I_r & 0\\0 & 0 & Q_1\end{bmatrix} = W\begin{bmatrix}0 & 0 & 0\\0 & I_k & 0\\0 & 0 & Q''\end{bmatrix}W^*, \tag{6}$$

and

$$R = W\begin{bmatrix}\frac{1}{2}I_k & \frac{1}{2}I_k & 0\\ \frac{1}{2}I_k & \frac{1}{2}I_k & 0\\ 0 & 0 & R''\end{bmatrix}W^* \tag{7}$$

for some unitary (orthogonal) operator W. One has to be careful when reading the above three equations. Namely, the block matrix representations of operators on the left sides of equations correspond to the direct sum decomposition $H = H_1 \oplus H_2 \oplus H_3$, while the block matrix representations of the same operators on the right sides correspond to some possibly different direct sum decomposition of the underlying space. But already in the next step we will show that $k = r$, and then (after changing W, if necessary) we may, and we will assume that the two decompositions coincide.

From (5) and (6) we infer

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & -I_r & 0 \\ 0 & 0 & P_1 - Q_1 \end{bmatrix} = W \begin{bmatrix} I_k & 0 & 0 \\ 0 & -I_k & 0 \\ 0 & 0 & P'' - Q'' \end{bmatrix} W^*$$

Comparing the eigenspaces of the two sides and taking into account that the right-bottom corners have norm less than 1, we conclude that $k = r$. fur-thermore, the representation of W with respect to the decomposition $H = H_1 \oplus H_2 \oplus H_3$ is

$$W = \begin{bmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & W_3 \end{bmatrix}.$$

Finally, from (7), an easy calculation gives us (4) with $U = W_1 W_2^* \in u_r$ and $R_1 = W_3 R'' W_3^* \in P_{n-r}(H_3)$. This completes the proof.

We still do not know whether the phenomena $M(P, Q) = (l]$ can happen or not. Non-emptiness of (P, Q) , for arbitrary $P,\ Q \in P_n(H)$, is a consequence of the two projections theorem, which we will prove after the following corollary. However, if P and Q are orthogonal projections, then we do know that $M(P, Q) \neq (l]$, which is stated below.

Let $P,\ Q \in P_n(H)$ be orthogonal projections, that is, $PQ = 0$, or equivalently, $QP = 0$, which is equivalent to $\operatorname{Im} P \perp \operatorname{Im} Q$. Then with respect to the orthogonal decomposition $H = \operatorname{Im} P \oplus \operatorname{Im} Q \oplus H_0$ the projections $P,\ Q$ have the following matrix representations:

$$P = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} and\ Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Corollary (5.1.4) [239]: *Let H be a complex or real Hilbert space, n a positive integer, and $P,\ Q \in P_n(H)$ projections given by* (8). *Then*

$$M(P, Q) = \left\{ \begin{bmatrix} \frac{1}{2} I_n & \frac{1}{2} U & 0 \\ \frac{1}{2} U^* & \frac{1}{2} I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} : U \in \mathcal{U}_n \right\}.$$

In particular, $M(P, Q)$ is a compact manifold.

Proof. The first part is a direct consequence of the previous statement, while the second part of the conclusion follows from the well-known facts that both the orthogonal and unitary groups are compact manifolds.

The following lemma is known as the two projections theorem (see [242], [245]) in the special case when $P,\ Q \in P_n(H)$ and $\|P - Q\| < 1$. We give a proof here. The case of the two projections theorem in which the latter inequality is dropped can be obtained by combining the following lemma and Corollary (5.1.3).

Lemma (5.1.5) [239]: *Let P, Q be projections of rank n acting on a Hilbert space H. Assume that $\|P - Q\| < 1$. Denote the dimension of* $\operatorname{Im} P \cap \operatorname{Im} Q$ *by* $(0 \le p \le n)$. *Then P and Q are unitary* (*orthogonal*) *similar to operators*

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix} and \begin{bmatrix} I_p & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where E and F are $2(n-p) \times 2(n-p)$ *matrices given by*

$$E = Diag\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

and

$$F = Diag\left(\begin{bmatrix} \mathrm{d_j} & \sqrt{\mathrm{d_j}(1-d_j)} \\ \sqrt{\mathrm{d_j}(1-d_j)} & 1-\mathrm{d_j} \end{bmatrix} : 1 \le j \le n-p\right),$$

with $0 < d_1, \dots, d_{n-p} < 1$.

Proof. We set $H_1 = \mathrm{Im}\, P \cap \mathrm{Im}\, Q$ and $H_2 = H_1^{\perp}$. With respect to the orthogonal decomposition $H = H_1 \oplus H_2$ we have

$$P = \begin{bmatrix} I_p & 0 \\ 0 & P_1 \end{bmatrix} \textit{ and } Q = \begin{bmatrix} I_p & 0 \\ 0 & Q_1 \end{bmatrix},$$

where P_1 and Q_1 are projections of rank $n-p$. The subspace H_2 is the orthogonal sum of $H_3 = \mathrm{Im}\, P_1$ and $H_4 = \mathrm{Ker} P_1$. With respect to the decomposition $H = H_1 \oplus H_3 \oplus H_4$ the projections P and Q have the following matrix representations:

$$P = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{n-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} \textit{ and } Q = \begin{bmatrix} I_p & 0 & 0 \\ 0 & D_1 & D_2 \\ 0 & D_2^* & D_3 \end{bmatrix}.$$

After applying unitary (orthogonal) similarity, if necessary, we may assume with no loss of generality that D_1 is diagonal, $D_1 = \mathrm{Diag}\,(d_1, \dots, d_{n-p})$. Moreover, the rank of the submatrix $[D_2^* D_3]$ is at most $n-p$, and therefore, the subspace H_4 can be decomposed into an orthogonal sum of two subspaces, the first one being of dimension at most $n-p$, such that the corresponding matrix representations of P and Q are

$$P = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{n-p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \textit{ and } Q = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & D_1 & E_2 & 0 \\ 0 & E_2^* & E_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since Q is a projection, we have $0 \le D_1 \le I_{n-p}$, and because $\|I_{n-p} - D_1\| \le \|P - Q\| < 1$ we conclude that $0 < d_1, \dots, d_{n-p} \le 1$. Actually, we have $0 < d_1, \dots, d_{n-p} < 1$, since otherwise, one of $d_1, \dots, d_{n-p}$, say d_1, would be equal to 1, and then since Q is a projection, the first row of E_2 and the first column of E_2^* would be zero yielding that $\dim(\mathrm{Im}\, P \cap \mathrm{Im}\, Q) \ge p+1$, a contradiction. The size of the matrix E_2 is $(n-p) \times k$ with $k \le n-p$. We claim that actually we have $k = n-p$. For if this was not true, it would follow from $Q^2 = Q$ that

$$D_1^2 + E_2 E_2^* = D_1, \tag{9}$$

and consequently, the diagonal matrix $D_1 - D_1^2$ would not be of full rank, which is a contradiction.

We can now apply the polar decomposition $E_2 = PU$, where U is unitary (orthogonal) and P is positive semidefinite. Applying unitary (orthogonal) similarity once more, we can assume that already E_2 is positive. But then (9) yields that E_2 is the unique positive square root of the diagonal matrix $D_1 - D_1^2$. It follows that

$$E_2 = Diag\left(\sqrt{d_1(1-d_1)}, \dots, \sqrt{d_{n-p}(1-d_{n-p})}\right),$$

And then trivially

$$E_3 = Diag\left(1 - d_1, \dots, 1 - d_{n-p}\right).$$

We complete the proof by rearranging the orthonormal basis of H.

The general case of the two projections theorem, i.e. when we have two finite rank projections with possibly different ranks, could be obtained from the above

Lemma, Corollary (5.1.3), and some elementary facts concerning two projections.We consider the rank one projections

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; and \; T(d) = \begin{bmatrix} \mathrm{d} & \sqrt{\mathrm{d}(1-\mathrm{d})} \\ \sqrt{\mathrm{d}(1-\mathrm{d})} & 1-\mathrm{d} \end{bmatrix} (0 \le d \le 1).$$

Some easy computations give us the following equalities and inequalities:

$$\|S - T(1/2)\| = \frac{1}{\sqrt{2}}, \|T(d) - T(1/2)\| = \frac{\sqrt{1 - 2\sqrt{(1-d)d}}}{\sqrt{2}} \le \frac{1}{\sqrt{2}}, (10)$$

$$\|S - T\left(\left(1 + \sqrt{d}\right)/2\right)\| = \|T(d) - T\left(\left(1 + \sqrt{d}\right)/2\right)\| = \frac{\sqrt{2 - 2\sqrt{d}}}{2} \le \frac{1}{\sqrt{2}}. (11)$$

If we combine (10) (or (11)) with the two projections theorem, then we obtain that $M(P,Q)$ is indeed non-empty for every two $P, Q \in P_n(H)$. We point out that if $0 \le d < 1$, then there exists a positive number ε such that we have

$$\|S - T(1/2 - \tilde{\varepsilon})\| > \frac{1}{\sqrt{2}} \; and \; \|T(d) - T(1/2 - \tilde{\varepsilon})\| < \frac{1}{\sqrt{2}} (0 < \tilde{\varepsilon} < \varepsilon). (12)$$

This could be verified by straightforward calculations.

Next, as a counterpart to Corollary (5.1.4), we have the following statement.

Corollary (5.1.6) [239]: *Let H be a complex or real Hilbert space, n a positive integer, $2n \le \dim H$, and P, $Q \in P_n(H)$. Assume that $\|P - Q\| = 1$ and that P and Q are not orthogonal. Then $M(P,Q)$ is not a compact manifold. Moreover, when H is of infinite dimension, then $M(P,Q)$ is not even a compact set.*

Proof. According to Corollary (5.1.3) we may, and we will assume that P and Q are of the form (3). We need to prove that the set $M(P,Q)$ in (4) is not a compact manifold.

Using Lemma (5.1.5) and (11) it is straightforward to find

$$R = \begin{bmatrix} \frac{1}{2} I_r & \frac{1}{2} U_1 & 0 \\ \frac{1}{2} U_1^* & \frac{1}{2} I_r & 0 \\ 0 & 0 & R_1 \end{bmatrix} \in M(P,Q)$$

with $\|R_1 - P_1\| < \frac{1}{\sqrt{2}}$ and $\|R_1 - Q_1\| < \frac{1}{\sqrt{2}}$.

Hence, there exists a positive real number ε such that the set u_ε consisting of all projections of the form

$$\begin{bmatrix} \frac{1}{2} I_r & \frac{1}{2} V & 0 \\ \frac{1}{2} V^* & \frac{1}{2} I_r & 0 \\ 0 & 0 & S \end{bmatrix},$$

where $V \in u_r$ with $\|V - U_1\| < \varepsilon$ and $S \in P_{n-r}(H_3)$ with $\|S - R_1\| < \varepsilon$, is an open subset of $M(P, Q)$. In particular, if $\dim H = \infty$, then $M(P, Q)$ is not compact at all.
We assume from now on that H is finite-dimensional. Assume also that $M(P, Q)$ is a compact manifold. Having these assumptions we need to arrive at a contradiction.
In both the real and the complex cases, the topological spaces u_r and $P_{n-r}(H_3)$ are compact manifolds. Denote their dimensions by q_1 and q_2, respectively (the exact values of q_1 and q_2 are well-known, but not important here). We set

$$S = \left\{ \begin{bmatrix} \frac{1}{2} I_r & \frac{1}{2} U & 0 \\ \frac{1}{2} U^* & \frac{1}{2} I_r & 0 \\ 0 & 0 & L \end{bmatrix} U \in \mathcal{U}_r, L \in P_{n-r}(H_3) \right\}. \tag{13}$$

Then $M(P, Q) \subset S$ and S is a compact manifold of dimension $q_1 + q_2$. Using the fact that u_ε is an open neighbourhood of R in $M(P, Q)$ as well as in S we conclude that the dimension of $M(P, Q)$ is equal to $q_1 + q_2$.
Using Lemma (5.1.5) and (10), we can find

$$T = \begin{bmatrix} \frac{1}{2} I_r & \frac{1}{2} W & 0 \\ \frac{1}{2} W^* & \frac{1}{2} I_r & 0 \\ 0 & 0 & T_1 \end{bmatrix} \in M(P, Q) T_1$$

such that $\|T_1 - P_1\| < \frac{1}{\sqrt{2}}$ and $\|T_1 - Q_1\| = \frac{1}{\sqrt{2}}$. Moreover, by (12), it is possible to find in an arbitrary neighbourhood of $T_1 \in P_{n-r}(H_3)$ of a projection $T_2 \in P_{n-r}(H_3)$such that $||T_2 - Q_1\| > \frac{1}{\sqrt{2}}$ and $||T_2 - P_1\| < \frac{1}{\sqrt{2}}$.

Finally, the inclusion of $M(P, Q)$ into S is a continuous injective map. The invariance of domain theorem states that any injective and continuous map between manifolds of the same dimensions is automatically an open map. Applying this theorem, we conclude that $M(P, Q)$ must be an open subset of S, contradicting the fact that $T \in M(P, Q)$. Therefore $M(P, Q)$ is not a compact manifold.
In the proof of our main results is Chow's fundamental theorem of geometry of Grassmann spaces [243]. we prefer to speak of (orthogonal) projections rather than of subspaces. But if we apply the obvious identification, where a subspace of dimension n is identified with a projection of rank n whose image is this subspace, then we arrive at the following definition of adjacency of two projections of rank n: projections $P, Q \in P_n(H)$ are said to be adjacent if and only if $\dim(\operatorname{Im} P + \operatorname{Im} Q) = n + 1$ which is equivalent to $\dim(\operatorname{Im} P \cap \operatorname{Im} Q) = n - 1$.
By the two projections theorem we easily conclude that $P, Q \in P_n(H)$ are adjacent if and only if they are unitary (orthogonal) similar to operators of the following form:

$$P = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

and

$$Q = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \begin{bmatrix} d & \sqrt{d(1-d)} \\ \sqrt{d(1-d)} & 1-d \end{bmatrix} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (15)$$

for some real d, $0 \le d < 1$. Equivalently, we can say that P and Q are adjacent if and only if rank $(P - Q) = 2$.

A semi-linear map is an additive map $A: H \to H$ such that there exists a field automorphism $\sigma: \mathbb{C} \to \mathbb{C}$ ($\sigma: \mathbb{R} \to \mathbb{R}$ in the real case) which satisfies $A(\lambda x) = \sigma(\lambda)x$ for every vector $x \in H$ and every number λ. In the case of real numbers, the only automorphism is the identity, therefore every semi-linear map is linear. In the case of complex numbers, two trivial automorphisms are the identity and the conjugation, but there are several other automorphisms. The above mentioned Chow's theorem states that if $2n + 1 \le \dim H < \infty$, and we have a bijective map $\varphi: P_n(H) \to P_n(H)$ which preserves adjacency in both directions, i.e.

$$rank\ (P - Q) = 2 \Leftrightarrow rank\ \big(\varphi(P) - \varphi(Q)\big) = 2\big(P, Q \in P_n(H)\big),$$

then there exists a bijective semi-linear transformation $A: H \to H$ such that we have

$$\operatorname{Im} \varphi(P) = A(\operatorname{Im} P)\ \big(P \in P_n(H)\big). \quad (16)$$

If $\dim H = 2n$, then either (16) holds, or we have

$$\operatorname{Im} \varphi(P) = \big(A(\operatorname{Im} P)\big)^{\perp}\ \big(P \in P_n(H)\big). \quad (17)$$

For a subset $\mathcal{A} \subset P_n(H)$ we define the following set

$$\mathcal{A}^{\mathrm{T}} = \{Q \in P_n(H) :\ QP = 0\ for\ all\ P \in \mathcal{A}\}.$$

The last lemma characterizes adjacency of two n-rank projections with the help of orthogonality.

Lemma (5.1.7) [239]: *Let $n \ge 2$ and $\dim H \ge 2n + 1$. For P, $Q \in P_n(H)$, $P \ne Q$, the following conditions are equivalent*:

P and Q are adjacent;

for every $R \in P_n(H) \backslash \{P, Q\}^{\mathrm{T}}$ the set $(\{R\} \mathrm{U} \{P, Q\}^{\mathrm{T}})^{\mathrm{T}}$ contains at most one projection.

Proof. Assume first that P and Q are adjacent. Then there is no loss of generality in assuming that they are of the form (14) and (15) with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$. It follows that $\{P, Q\}^{\mathrm{T}}$ is the set of all rank n projections of the form

$$\begin{bmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & * \end{bmatrix}.$$

Note that the size of the bottom-right corner is at least $n \times n$, and therefore, $\{P, Q\}^{\mathrm{T}}$ is not empty.

Hence, if $T \in (\{P, Q\}^{\mathrm{T}})^{\mathrm{T}}$, then Ker$T$ contains H_3. We fix an arbitrary $R \in P_n(H) \backslash \{P, Q\}^{\mathrm{T}}$ and assume that $T \in (\{R\} \mathrm{U} \{P, Q\}^{\mathrm{T}})^{\mathrm{T}}$. Clearly, there exist a non-zero vector $x_{12} \in H_1 \oplus H_2$ and another (possibly zero) one $x_3 \in H_3$ such that $x := x_{12} \oplus x_3 \in \operatorname{Im} R \subset$ Ker$T$. Therefore, we obtain Ker$T$ = span $\{x_{12}\} \oplus H_3$, and conclude that either $(\{R\} \mathrm{U} \{P, Q\}^{\mathrm{T}})^{\mathrm{T}}$ is empty, or it contains only one projection, whose range is (span $\{x_{12}\} \oplus H_3)^{\perp}$.

We consider now the case when P and Q are not adjacent. Denote $W = \operatorname{Im} P + \operatorname{Im} Q$. Then $\{P, Q\}^{\mathrm{T}}$ is either empty and in this case it is trivial to complete the proof; or it is the set of all projections of rank n whose matrix representation with respect to the orthogonal decomposition $H = W \oplus W^{\perp}$ is of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}.$$

Choose

$$R = \begin{bmatrix} E_1 & 0 \\ 0 & R_1 \end{bmatrix},$$

with $E_1 \in P_1(W)$ and $R_1 \in P_{n-1}(W^{\perp})$. Using the fact that $\dim W \geq n+2$, we easily conclude that the set $(\{R\} \cup \{P, Q\}^{\mathrm{T}})^{\mathrm{T}}$ contains infinitely many rank n projections.

Now, we have the follow results.

Theorem (5.1.8) [239]: *Let H be a complex (real) Hilbert space and n a positive integer such that $2n < \dim H$ is satisfied. Assume that a surjective map φ: $P_n(H) \to P_n(H)$ preserves orthogonality in both directions. Then there exists either a unitary or an antiunitary operator (orthogonal operator) U on H such that*

$$\varphi(P) = UPU^* \left(P \in P_n(H)\right).$$

In order to prove this result, we will apply Chow's fundamental theorem of geometry of Grassmann spaces.

Proof. The infinite dimensional case was covered in [246], [247]. So we may assume that $2n + 1 \leq \dim H < oo$ is satisfied. We would like to show that φ (which is onto) is a bijective map which preserves adjacency in both directions. Assume first that we have $\varphi(P) = \varphi(Q)$. Then $R \in P_n(H)$ is orthogonal to P if and only if $\varphi(R)$ is orthogonal to $\varphi(P) = \varphi(Q)$ which is equivalent to the orthogonality of R and Q. It follows easily that $P = Q$. Hence, φ is injective, and hence bijective.

Now, by Lemma (5.1.7), we easily conclude that φ preserves adjacency in both directions. Therefore it follows from Chow's theorem that φ has the form of (16) with some semi-linear mapping $A: H \to H$. Let x and y be two non-zero orthogonal vectors in H. We consider two projections P, $Q \in P_n(H)$ such that we have $Px = x$, $Py = 0$, $Qx = 0$, $Qy = y$ and $PQ = 0$. Therefore Ax and Ay are also orthogonal. Similarly, we can conclude that if Ax and Ay are orthogonal, then x and y has to be orthogonal as well. An easy application of Uhlhorn's theorem [249] (or [248] together with Wigner's theorem) gives that A is a non-zero scalar multiple of a unitary or an antiunitary transformation (orthogonal in the real case). Clearly, we can choose A to be unitary or antiunitary (or orthogonal in the real case). Finally, using the fact that $P \in P_n(H)$ implies $UPU^* \in P_n(H)$ with $\operatorname{Im}(UPU^*) = U(\operatorname{Im} P)$, our proof is done.

If $\dim H = 2n$, then we call two projections P, $Q \in P_n(H)$ complementary if $\operatorname{Im} P + \operatorname{Im} Q = H$ is fulfilled.

Theorem (5.1.9) [239]: *Let H be a complex (real) Hilbert space and n a positive integer, $n < \dim H$. Assume that a surjective map φ: $P_n(H) \to P_n(H)$ is an isometry with respect to the gap metric. If $\dim H \neq 2n$, then there exists either a unitary or an antiunitary operator (orthogonal operator) U on H such that φ is of the following form:*

$$\varphi(P) = UPU^* \left(P \in P_n(H)\right). \tag{18}$$

Proof. The case when $n = 1$ is the classical version of Wigner's theorem, so we will assume $n \geq 2$ throughout the proof.

First, assume that $\dim H < \infty$ is satisfied. On one hand, since $P_n(H)$ is a compact manifold, its image is also compact. On the other hand, the domain invariance theorem ensures that $\operatorname{Im} \varphi$ is open as well. Since $P_n(H)$ is connected, we conclude the bijectivity of φ. Therefore the surjectivity assumption is indeed disposable in the finite dimensional cases.
Second, obviously φ is a homeomorphism with respect to the topology induced by the gap metric. We also have

$$\varphi\big(M(P,Q)\big) = M\big(\varphi(P), \varphi(Q)\big)\,. \tag{19}$$

If $\dim H = \infty$, then by Corollaries (5.1.4) and (5.1.6), the projections P and Q are orthogonal if and only if $M(P,Q)$ is compact. Therefore the map φ preserves orthogonality in both directions, and the Györy-Šemrl theorem completes the proof of this case.
Next, we assume $2n \leq \dim H < \infty$, and we show that φ preserves orthogonality in both directions. Let us assume the contrary, i.e. we either have P, Q with $P \perp Q$ but their images are not orthogonal; or P, Q are not orthogonal but $(P) \perp \varphi(Q)$. Since φ^{-1} is also a surjective isometry, it is enough to consider the second possibility. Then $M\big(\varphi(P), \varphi(Q)\big)$ is a compact manifold, but $M(P,Q)$ is not, which contradicts (19) and the fact that both φ and φ^{-1} are continuous.
Clearly, Theorem (5.1.8) completes the proof in the case when $2n < \dim H < \infty$. Next, let us suppose that $\dim H = 2n$. By the two projections theorem we conclude that any two elements P, $Q \in P_n(H)$ are complementary if and only if $\|(I-P) - Q\| < 1$. But this is equivalent to $\|\varphi(I-P) - \varphi(Q)\| = \|\big(I - \varphi(P)\big) - \varphi(Q)\| < 1$, which is satisfied if and only if $\varphi(P)$ and $\varphi(Q)$ are complementary. Hence φ preserves complementarity in both directions, and a straightforward application of [240] completes the proof of this case.
It remains to consider the case when $n < d := \dim H < 2n$ case. Since $\|P - Q\| = \|(I-P) - (I-Q)\| \big(P, Q \in P_n(H)\big)$, the map $\tilde{\varphi}: P_{d-n}(H) \to P_{d-n}(H), \tilde{\varphi}(I-P) = I - \varphi(P)\big(P \in P_n(H)\big)$ is also an isometry, but on the Grassmann space $P_{d-n}(H)$. Because of $1 \leq 2(d-n) < d$, we obtain that φ is of the form (18).

Section (5.2): Wigner Theorem on Grassmann Spaces

For H be a complex Hilbert space and I stand for the identity operator. If n is a positive integer, then we denote the set of all rank-n (self-adjoint) projections by $P_n(H)$.
This space can be naturally identified with the *Grassmann space* of all n-dimensional subspaces of H using the map $P \mapsto \operatorname{Im} P$. In case when $n = 1$, we get the usual projective space that represents the set of all pure states of a quantum system. For P, $Q \in P_n(H)$ let us call the quantity $\operatorname{Tr} PQ$ the *transition probability* between the two projections. If $n = 1$, then this is a commonly used notion in quantum mechanics, furthermore, $\operatorname{Tr} PQ = \cos^2\vartheta$ where ϑ is the angle between $\operatorname{Im} P$ and $\operatorname{Im} Q$. Wigner's theorem characterises symmetry transformations of $P_1(H)$ that respect the transition probability, or equivalently, that leave the angle invariant. However, this theorem can be significantly improved, namely, we can drop the bijectivity assumption and have a similar conclusion.

Theorem (5.2.1)[250]: *Let $\varphi: P_1(H) \to P_1(H)$ be a (not necessarily bijective) transformation which satisfies*

$$Tr\, \varphi(P)\varphi(Q) = TrPQ\ \big(P, Q \in P_1(H)\big)\,.$$

Then φ is induced by either a linear or a conjugate-linear isometry $V: H \to H, i.e.$

$$\varphi(P) = VPV^* \left(P \in P_1(H)\right).$$

The above result is commonly referred to as the *optimal version of Wigner's theorem*. Various generalisations of this essential result have been provided, see [241], [251], [252], [239], [246], [45]–[18], [259], [261]– [262]. This short note is particularly concerned with *Molnár's generalisation* which we explain now. Assume that $n > 1$ and $P, Q \in P_n(H)$, then the *principal angles* between P and Q are the arcuscosines of the n largest singular values of PQ ([249], [40]). The system of all principal angles is denoted by $\angle(P,Q) := (\vartheta_1, \dots \vartheta_n)$ where $\frac{\pi}{2} \geq \vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n \geq 0$. The origin of the notion goes back to Jordan's work [36] and has serious applications, see e.g. [253], [255], [256], [260], [27]. Molnár proved the following.

Theorem (5.2.2)[250]: *Let* $\dim H > n \geq 2$ *and* $\varphi: P_n(H) \to P_n(H)$ *be a* (*not necessarily bijective*) *transformation that satisfies*

$$\angle\left(\varphi(P), \varphi(Q)\right) = \angle(P,Q) \left(P, Q \in P_n(H)\right). \tag{20}$$

Then either φ *is induced by a linear or a conjugate-linear isometry* $V: H \to H$, *i.e.*

$$\varphi(P) = VPV^* \qquad \left(P \in P_n(H)\right),$$

or we have $\dim H = 2n$ *and*

$$\varphi(P) = I - VPV * \left(P \in P_n(H)\right).$$

Molnár's original desire was to prove a more general result. Namely, note that by the two projections theorem ([242], [245], [239]) we have Tr $PQ = \sum_{j=1}^{n} \cos^2 \vartheta_j$, therefore if φ satisfies (20), then it automatically preservers the transition probability as well (see (21) below). Actually, in the first few steps of the proof of Theorem (5.2.2) Molnár used only this weaker property, although, there is a point where the methods start to heavily rely on (20).

We provide this missing result which is stated below, and hence giving a very natural joint generalisation of the Wigner and Molnár theorems.

Theorem (5.2.3)[250]: *Let* $\dim H > n \geq 2$ *and* $\phi: P_n(H) \to P_n(H)$ *be a* (*not necessarily bijective*) *map which preserves the transition probability, that is*

$$Tr\ \phi(P)\phi(Q) = \mathrm{Tr} PQ \quad \left(P, Q \in P_n(H)\right). \tag{21}$$

Then either ϕ *is induced by a linear or conjugate-linear isometry* $V: H \to H$, *i.e.*

$$\phi(P) = VPV^* \quad \left(P \in P_n(H)\right), \tag{22}$$

or we have $\dim H = 2n$ *and*

$$\phi(P) = I - VPV^* \qquad \left(P \in P_n(H)\right). \tag{23}$$

We point out that all the above three theorems hold for real Hilbert spaces as well and their proofs are almost the same, even simpler, as in the complex case. We present the proof of the Main Theorem (5.2.3)

We note that (21) is equivalent to the following property

$$\|\phi(P) - \phi(Q)\|_{HS} = \|P - Q\|_{HS} \left(P, Q \in P_n(H)\right), \tag{24}$$

where $\|\ \|_{HS}$ denotes the Hilbert-Schmidt norm. Therefore our result describes the general form of not necessarily surjective *isometries of the Grassmannian* with respect to this special norm. [241], [239] have been published about the same problem for the case of the operator norm. However, the characterisation of non-bijective isometries of $P_n(H)$ with respect to the operator norm is still an open problem in the case when $\dim H = \infty$. We hope that the proof gives some additional insight into that problem as well.

Let $F_s(H)$ be the set of all finite-rank self-adjoint operators on H. We begin with stating a lemma which is a trivial consequence of [258] and [17], and which was crucial in [17], as well as here.

Lemma (5.2.4)[250]: *If ϕ satisfies the conditions of Main Theorem (5.2.3), then it has a unique real-linear extension $\Phi: F_s(H) \to F_s(H)$ which is injective and satisfies*

$$Tr\, \Phi(A)\Phi(B) = Tr\, AB \left(A, B \in F_s(H)\right). \tag{25}$$

An immediate consequence of Lemma (5.2.4) is that if $\dim H < \infty$, then Φ is a homeomorphism, moreover, by the domain invariance theorem ϕ is a homeomorphism as well. We call two rank-n projections P and Q *adjacent* if dim (Im $P \cap$ Im $Q) = n-1$, or equivalently, if rank $(P - Q) = 2$, and in this case we use the notation $P \sim aQ$. Note that $P \sim aQ$ implies $P \neq Q$. It is apparent by the two projections theorem that $P \sim aQ$ if and only if $\angle(P, Q)$ contains exactly one non-zero angle.

Here we will utilise the following special case of Chow's fundamental theorem of geometry of Grassmann spaces.

Theorem (5.2.5)[250]: *Let* $\dim H = 2n$ *and $\varphi: P_n(H) \to P_n(H)$ be a continuous bijection which preserves adjacency in both directions, $i.e.$*

$$\varphi(P) \sim^a \varphi(Q) \Leftrightarrow P \sim^a Q \left(P, Q \in P_n(H)\right).$$

Then there exists a linear or conjugate-linear bijection $A: H \to H$ such that either

$$\operatorname{Im} \varphi(P) = A(\operatorname{Im} P) \left(P \in P_n(H)\right), \tag{26}$$

or

$$\operatorname{Im} \varphi(P) = \left(A(\operatorname{Im} P)\right)^{\perp} \left(P \in P_n(H)\right). \tag{27}$$

In the general version of Chow's theorem continuity is not assumed, however, then A can be a non-continuous semilinear bijection as well. That version also covers the $2n < \dim H < \infty$ case where the conclusion (27) is of course excluded.

We call P and $Q \in P_n(H)$ *orthogonal adjacent* if $P \sim^a Q$ and $\vartheta_1 = \frac{\pi}{2}$, in notation $P \sim^{\perp a} Q$. Similarly, P, $Q \in P_n(H)$ are said to be *non-orthogonal adjacent* if $P \sim^a Q$ and $\vartheta_1 < \frac{\pi}{2}$, in notation $P^7 \sim Q\mathrm{L}a$. For any $k \in \mathbb{N}$, subspace M, and P, $Q \in P_k(M)$ we define the set

$$A_{P,Q}^{(k)} = \{R \in P_k(M): P + Q - R \in P_k(M)\}.$$

We will show that ϕ preserves non-orthogonal adjacency in both directions in which the following topological characterisation of the relation $\sim^a$ plays a crucial role.

Lemma (5.2.6)[250]: *Suppose that P, $Q \in P_n(H)$. Then $A_{P,Q}^{(n)}$ is a one-dimensional* (*real*) *manifold if and only if $P \sim^a QX$.*

Proof. Clearly, we have $\mathcal{A}_{P,P}^{(n)} = \{P\}$, therefore from now on we may assume that $P \neq Q$. Let us first investigate the case when $P \sim^a Q$. Then P and Q can be represented by the following block-matrices with respect to the orthogonal decomposition $H = M_1 \oplus M_2 \oplus M_3$ where $M_1 = \operatorname{Im} P \cap \operatorname{Im} Q$, $M_1 \oplus M_2 = \operatorname{Im} P + \operatorname{Im} Q$, $\dim M_1 = \dim M_3 = n - 1$, $\dim M_2 = 2$ and p, q $\in P_1(M_2)$:

$$P = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} \quad and \quad Q = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}.$$

Suppose that $R \in \mathcal{A}_{P,Q}^{(n)}$ and set $= P + Q - R \in P_n(H)$. Since we have

$$\|Rx\|^2 + \|Sx\|^2 = \langle (R+S)x, x\rangle = \langle (P+Q)x, x\rangle = 2\|x\|^2 \ (x \in M_1)$$

and

$$\|Rx\|^2 + \|Sx\|^2 = \langle (R+S)x, x\rangle = \langle (P+Q)x, x\rangle = 0 \ (x \in M_3)\ ,$$

we immediately infer $M_1 \subseteq$ Im $R \cap$ Im S and $M_3 \subseteq \ker R \cap \ker S$. Thus the block-matrix representations of R and S in the decomposition $H = M_1 \oplus M_2 \oplus M_3$ are

$$R = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} \quad and \quad S = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix},$$

where $\mathrm{t}, s \in P_1(M_2)$, whence we easily conclude the following:

$$\mathcal{A}_{P,Q}^{(n)} = \left\{ \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} \ : \mathrm{t} \in \mathcal{A}_{\mathrm{p,q}}^{(1)} \right\}$$

In particular, $A_{P,Q}^{(n)}$ and $A_{\mathrm{p,q}}^{(1)}$ are homeomorphic.

Next, we investigate the set $A_{\mathrm{p,q}}^{(1)}$, where $\mathrm{p} \neq \mathrm{q}$. We shall represent elements of $F_S(M_2)$ by 2×2 Hermitian matrices. If $\mathrm{p} + \mathrm{q} = I_2$, i.e. $P^{\perp a} \sim Q$, then obviously $A_{\mathrm{p,q}}^{(1)} = P_1(M2)$, hence $A_{P,Q}^{(n)}$ is a two-dimensional manifold. Suppose that $\mathrm{p} + \mathrm{q} \neq I_2$, i.e. $P \ \sim^{La} Q$, then applying unitary similarity we may assume without loss of generality that $\mathrm{p} + \mathrm{q} = \begin{pmatrix} s & 0 \\ 0 & 2-s \end{pmatrix}$ where $0 < s < 1$. Since for any $\mathrm{t} \in P_1(M_2)$ we have Tr $(\mathrm{p} + \mathrm{q} - \mathrm{c}) = 1$, we infer that $\mathrm{t} \in A_{\mathrm{p,q}}^{(1)}$ if and only if $\mathrm{p} + \mathrm{q} - \mathrm{t}$ is singular. But this holds exactly when $I_2 - (\mathrm{p} + \mathrm{q})^{-1}\mathrm{t}$ is singular, that is equivalent to Tr $(\mathrm{p} + \mathrm{q})^{-1} t = 1$, since $(\mathrm{p} + \mathrm{q})^{-1}\mathrm{t}$ is of rank one. Therefore an Hermitian 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ belongs to $\mathcal{A}_{\mathrm{p,q}}^{(1)}$ if and only if $Tr\,A = a_{11} + a_{22} = 1, Tr\,(p+q)^{-1}A = \frac{a_{11}}{s} + \frac{a_{22}}{2-s} = 1$ and A is of rank 1. Observe that the two equations immediately yield $a_{11} =$ and $a_{22} = \frac{2-s}{2}$, and since A has rank one, we also obtain $a_{12} = \overline{a_{21}} = \frac{\sqrt{(2-s)s}}{2} e^{it}$ with a real number t. This implies that $A_{\mathrm{p,q}}^{(1)}$ is a one-dimensional manifold, and therefore so is $A_{P,Q}^{(n)}$.

Finally, let us suppose that $P \neq Q$ and $P \nsim^{a} Q$. Then there is an orthogonal decomposition $H = H_1 \oplus \oplus H_n$ such that $\dim H_j = 2$ for every j and that we have the following block-diagonal representations where $\mathrm{p}_j, \mathrm{q}_j \in P_1(M_j) (j = 1, \ldots n)$:

$$P = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix} \quad and\ Q = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{pmatrix}$$

Observe that $\mathrm{p}_j \neq \mathrm{q}_j$ holds for at least two indices and that we obviously have

$$\left\{\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} : \mathrm{t}_j \in \mathcal{A}^{(1)}_{\mathrm{p}_j,\mathrm{q}_j}\right\} \subset \mathcal{A}^{(n)}_{P,Q}.$$

Since the left-hand side is a manifold of dimension at least two, the right-hand side cannot be a one-dimensional manifold, which completes the proof.

Utilising Molnár's lemma we easily obtain the following property:

$$\phi\left(\mathcal{A}^{(n)}_{P,Q}\right) = \Phi\left(\mathcal{A}^{(n)}_{P,Q}\right) = \mathcal{A}^{(n)}_{\Phi(P),\Phi(Q)} = \mathcal{A}^{(n)}_{\phi(P),\phi(Q)}\ \left(P, Q \in P_n(H)\right).$$

Since ϕ is a homeomorphism, we infer the following equivalence-chain:

$P\ \sim^{La} Q \Leftrightarrow \mathcal{A}^{(n)}_{P,Q}$ is a one-dimensional manifold

$\Leftrightarrow \mathcal{A}^{(n)}_{\phi(P),\phi(Q)}$ is a one-dimensional manifold $\Leftrightarrow \phi(P) \sim^{a} \phi(Q)$,

i.e. ϕ preserves non-orthogonal adjacency in both directions. The lower semicontinuity of the rank on $F_s(H)$ yields the following for every $\in P_n(H)$:

$$\{R \in P_n(H): PX \sim^{a} R\}^{-} = \{P\} \cup \{R \in P_n(H): P \sim^{a} R\}$$

where. $-$ denotes the closure. Therefore, since ϕ is a homeomorphism, it preserves adjacency in both directions, which implies that ϕ satisfies either (26) or (27). Finally, by (21) the map A preserves orthogonality, and thus A must be a scalar multiple of a unitary or an antiunitary operator which completes the proof of the present case.

We make an important observation here. Clearly, every rank-one projection $\mathrm{p} \in P_1(H)$ can be expressed as a real-linear combination of $n+1$ rank-n projections ([258]), moreover, if this linear combination is $\mathrm{p} = \sum_{j=1}^{n+1} t_j\, P_j$, then taking the trace of both sides gives $\sum_{j=1}^{n+1} t_j = \frac{1}{n}$. Therefore, in case of (23) we have

$$\Phi(\mathrm{p}) = \sum_{j=1}^{n+1} t_j\, \Phi(P_j) = V\left(\sum_{j=1}^{n+1} t_j\left(I_{2n} - P_j\right)\right)V^* = \frac{1}{n} I_{2n} - V\mathrm{p}V^*\left(\mathrm{p} \in P_1(H)\right),$$

and similarly, in case of (22) we obtain $\Phi(\mathrm{p}) = V\mathrm{p}V^*$ for every $\mathrm{p} \in P_1(H)$.

By the following three properties it is apparent that the case of $\dim H < 2n$ follows from the $\dim H > 2n$ case: $P \in P_n(H)$ holds if and only if $-P \in P_{\dim H - n}(H)$, we have $\mathrm{Tr}\,(I-P)(I-Q) = \dim H - 2n + \mathrm{Tr}PQ$ for every $P, Q \in P_n(H)$, and the following map preserves the transition probability:

$$\psi: P_{\dim H-n}(H) \to P_{\dim H-n}(H), \psi(\tilde{P}) = I - \phi(I - \tilde{P})\left(\tilde{P} \in P_{\dim H-n}(H)\right).$$

Next, assume that $\dim H > 2n$ and fix two orthogonal rank-n projections P and Q. By (21) we obtain that $\phi(P)$ and $(\ell(Q)$ are orthogonal as well, and since for any $R \in P_n(H)$ we have $R \le P + Q$ if and only if $R \in A^{(n)}_{P,Q}$, we easily conclude $\phi(R) \le \phi(P) + \phi(Q)$. By the observation following the $2n$-dimensional case we get either $(\mathrm{p}) \in P_1(H)(\mathrm{p} \in P_1(H)$, $\mathrm{p} \le P + Q)$, or $\mathrm{Im}\,\Phi(\mathrm{p}) = \mathrm{Im}\,(\ell(P) \oplus \mathrm{Im}\,\phi(Q)(\mathrm{p} \in P_1(H), \mathrm{p} \le P + Q)$. Assume for a moment that the second possibility holds. If we replace in the above method Q by another $Q' \in P_n(H)$ that is still orthogonal to P, then we easily obtain $\mathrm{Im}\,\Phi(\mathrm{p}) = \mathrm{Im}\,\phi(P) \oplus \mathrm{Im}\,\phi(Q')(\mathrm{p} \in P_1(H), \mathrm{p} \le P + Q')$, since we obviously cannot have $\Phi(\mathrm{p}) \in P_1(H)$ for any $\mathrm{p} \in P_1(H)$, $\mathrm{p} \le P$. In particular, we obtain $\Phi\left(P_1(H)\right)\mathrm{n}P_1(H) = \emptyset$, whence we infer that

$\operatorname{Im}\phi(P) \oplus \operatorname{Im}\phi(Q)$ must be the same subspace for every orthogonal pair P and Q, from which we conclude $\operatorname{Im}\Phi(A) \subseteq \operatorname{Im}\phi(P) \oplus \operatorname{Im}\phi(Q)\big(A \in F_s(H)\big)$ that contradicts to the injectivity of Φ. Therefore we must have $\big(P_1(H)\big) \subset P_1(H)$, and finally, (25), Theorem (5.2.1) and the linearity of Φ imply (22).

Section (5.3): Wigner Theorem for Hilbert Grassmannians

For H be a complex Hilbert space of dimension not less than 3. There is a natural one-to-one correspondence between closed subspaces of H and projections, i.e. self-adjoint idempotents in the algebra of all bounded linear operators on H. Denote by $\mathcal{G}_k(H)$ the Grassmannian formed by all k-dimensional subspaces of H, i.e. all projections of rank k.

In quantum mechanics, projections of rank 1 are identified with so-called *pure states*. The *transition probability* $\operatorname{Tr}(P, P')$ for two pure states $P,\ P' \in \mathcal{G}_1(H)$ is equal to $|\langle x,\ x'\rangle|^2$, where $x \in P$ and $x' \in P'$ are unit vectors. In other words, the transition probability is $\cos^2\theta$, where θ is the angle between P and P'. By classical Wigner's theorem, every bijective transformation of $\mathcal{G}_1(H)$ preserving the transition probability is induced by a unitary or anti-unitary operator on H. This statement plays an important role in the mathematical foundations of quantum mechanics.

It was observed by Uhlhorn in [249] that the same holds for bijective transformations of $g_1(H)$ preserving the orthogonality relations in both directions (this fact is a simple consequence of the Fundamental Theorem of Projective Geometry). Since the transition probability is zero if and only if the corresponding pure states are orthogonal, classical Wigner's theorem is contained in Uhlhorn's result.

This statement was extended to other Grassmannians. Györy [246] and Šemrl [247] (see also [239]) proved independently that every bijective transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator on H under the assumption that $\dim H > 2k$ (if $\dim H = 2k$, then for every $X \in \mathcal{G}_k(H)$ the orthogonal complement $X^\perp$ is the unique element of $g_k(H)$ orthogonal to X and such transformations might be wild). It was noted in [247] that there are non-bijective transformations of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions which cannot be obtained from linear or conjugate-linear isometries.

We show that this happens only in the infinite-dimensional case. In other words, if the dimension of H is finite, then every (not necessarily bijective) transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator on H (as above, we assume that $\dim H > 2k$). The proof of this statement is based on a modification of Molnár's result [34], [259] (see also [258]) which will be described below.

There is a non-bijective version of Wigner's theorem which states that every transformation of $\mathcal{G}_1(H)$ preserving the transition probability is induced by a linear or conjugate-linear isometry on H (see, for example, [254]). In [34], [259] Molnár proposed the following extension of this statement: every transformation of $\mathcal{G}_k(H)$ preserving all principal angles between subspaces is induced by a linear or conjugate-linear isometry on H or $\dim H = 2k$ and it is the composition of the transformation induced by an isometry and the orthocomplementation. Gehér [250] obtained the same result for transformations of $\mathcal{G}_k(H)$

preserving the transition probability (the transition probability is defined as the sum of squares of cosines for all principal angles).

We show that the transformations of $\mathcal{G}_k(H)(\dim H \geq 2k > 2)$ induced by linear or conjugate-linear isometries can be characterized as transformations preserving some of principal angles corresponding to the orthogonality, adjacency and ortho-adjacency relations. To prove this statement we use a modification of methods from [268].

For X be a set and let $R \subset X \times X$ be a relation on X. We write xRy if $(x, y) \in R$. A transformation $: X \to X$ is said to be R *preserving* if for all $x, y \in X$ we have

$$xRy \Rightarrow f(x)Rf(y) ;$$

$$xRy \Leftrightarrow f(x)Rf(y)$$

in the case when

for all $x, y \in X$, we say that f is R *preserving in both directions*.

The *principal angles* $0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \pi/2$ between k-dimensional subspaces $X, Y \subset H$ are defined as follows. Let θ_1 be the minimal value of $\arccos(|\langle x, y\rangle|)$ for unit vectors $\in X$, $y \in Y$, and let $x_1 \in X$, $y_1 \in Y$ be unit vectors realizing this minimum. For $i \geq 2$ the principal angle θ_i and unit vectors $x_i \in X$, $y_i \in Y$ are defined recursively, i.e. θ_i is the minimal value of $\arccos(|\langle x, y\rangle|)$ for unit vectors $x \in X$ and $y \in Y$ orthogonal to x_1, ..., x_{i-1} and $y_1, \dots, y_{i-1}$ (respectively), and $x_i \in X$, $y_i \in Y$ are unit vectors satisfying the latter conditions and realizing this minimum.

Two elements of $\mathcal{G}_k(H)$ are orthogonal if and only if all principal angles between them are equal to $\pi/2$. We will always suppose that $\dim H \geq 2k$ (otherwise, $\mathcal{G}_k(H)$ does not contain orthogonal elements).

Two elements of $\mathcal{G}_k(H)$ are called *adjacent* if their is $(k-1)$-dimensional, in other words, only one of the principal angles between them is non-zero. Two adjacent elements of $\mathcal{G}_k(H)$ are said to be *ortho-adjacent* if the unique non-zero principal angle between them is equal to $\pi/2$.

Theorem (5.3.1)[265]: *Suppose that* $\dim H > 2k > 2$. *Let* f *be an orthogonality preserving transformation of* $\mathcal{G}_k(H)$ *which satisfies one of the following additional conditions*:

Then f *is induced by a linear or conjugate-linear isometry on* H.

(A) f *is adjacency preserving*,

(OA) f *is an ortho-adjacency preserving injection*.

We use Theorem (5.3.1) to prove the following.

Theorem (5.3.2)[265]: *If the dimension of* H *is finite and greater than* $2k$, *then every transformation of* $\mathcal{G}_k(H)$ *preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator on* H.

For the case when $\dim H = 2k$, we can prove only the following weak version of Theorem (5.3.1).

Proposition (5.3.3)[265]: *Suppose that* $\dim H = 2k > 2$. *Let* f *be an orthogonality preserving transformation of* $\mathcal{G}_k(H)$ *which preserves the adjacency relation in both directions*. *Then* f *is a bijection induced by a unitary or anti-unitary operator on* H.

The *Grassmann graph* $\Gamma_k(H)$ is the graph whose vertex set is $\mathcal{G}_k(H)$ and whose edges are pairs of adjacent elements. The case when $k = 1$ is trivial (any two distinct elements of $\mathcal{G}_1(H)$ are adjacent) and we suppose that $k > 1$.
For a subspace S of dimension not greater than k we denote by $[S\rangle_k$ the set of all k-dimensional subspaces containing S; this set is called a *star* if $\dim S = k - 1$. If U is a subspace whose dimension is not less than k, then we write $\langle U]_k$ for the set of all k-dimensional subspaces contained in U; we say that this is a *top* if $\dim U = k + 1$.
A *clique* in a graph is a set formed by mutually adjacent vertices. It is clear that stars and tops are cliques in $\Gamma_k(H)$. Conversely, every maximal clique of $\Gamma_k(H)$ is a star or a top (see, for example, [268]).
Two closed subspaces X, $Y \subset H$ are called *compatible* if there are mutually orthogonal closed subspaces X', Y', Z such that

$$X = X' + Z \text{ and } Y = Y' + Z.$$

Two closed subspaces of H are compatible if and only if there is an orthonormal basis of H such that these subspaces are spanned by subsets of this basis. Two elements of $\mathcal{G}_k(H)$ are ortho-adjacent if they are adjacent and compatible.
A subset of $\mathcal{G}_k(H)$ is called *compatible* if any two distinct elements from this subset are compatible. Every maximal compatible subset of $\mathcal{G}_k(H)$ is an *orthogonal apartment,* i.e. it consists of all k-dimensional subspaces spanned by subsets of a certain orthonormal basis for H[12]. Compatible subsets of cliques are formed by mutually ortho-adjacent elements.
Lemma (5.3.4)[265]: *Every maximal compatible subset of a top contains precisely $k + 1$ elements. Every maximal compatible subset of a star contains precisely $n - k + 1$ elements if* $\dim H = n$ *is finite, and it is infinite if H is infinite-dimensional.*
The distance $d(v, w)$ between two vertices v and w in a connected graph is the smallest number of edges in a path connecting these vertices. Every path from v to w formed by $d(v, w)$ edges is called a *geodesic*. The Grassmann graph $\Gamma_k(H)$ is connected and the distance $d(X, Y)$ between X, $Y \in \mathcal{G}_k(H)$ in this graph is equal to $-\dim(X \cap Y)$. So, we have $d(X, Y) = k$ if and only if $\boldsymbol{X} \cap \boldsymbol{Y} = 0$. In particular, the distance between two orthogonal elements of $\mathcal{G}_k(H)$ is equal to k.
Lemma (5.3.5)[265]: *If $X, X_1, \dots, X_{i-1}, Y$ is a geodesic in the graph $\Gamma_k(H)$, then*

$$X \cap Y \subset X \cap X_{i-1} \subset \cdots \subset X \cap X_1$$

and

$$X \cap Y \subset X \cap X_1 \subset \cdots \subset X \cap X_{\mathrm{i}-1}.$$

Proof. First, we show that $x \cap Y$ is contained in $x \cap x_j$ for every $j \in \{1, \dots, i - 1\}$. Since , X_1,..., X_{i-1}, Y is a geodesic, we have $d(X, Y) = i$ and

$$d(X, X_j) = j, d(X_j, Y) = i - j.$$

Then

$$\dim (X \cap x_j) = k - j \textit{ and } \textit{dim} (Y \cap x_j) = k - i + j.$$

If $X \cap \mathrm{X_j}$ does not contain $X \cap Y$, then the dimension of the of these subspaces is less than $\dim(X \cap Y) = k - i$. This means that $(X \cap X_j)\backslash Y$ contains a collection of $i - j + 1$ linearly independent vectors. Then

$$\dim X_j \geq i - j + 1 + \dim (Y \cap X_j) = (i - j + 1) + (k - i + j) = k + 1$$

which is impossible. So, every $X \cap X_j$ contains $X \cap Y$.
Applying the same arguments to the geodesic $X, X_1, \dots, X_j$ with $j \le i-1$, we establish that $X \cap X_j$ is contained in $X \cap X_l$ for every $l < j$.
To prove the second chain of inclusions, we consider the reversed geodesic Y, X_{i-1}, ..., X_1, X.
We will use the following characterization of the compatibility relation in terms of orthogonality and adjacency.
Lemma (5.3.6)[265]: *Every geodesic in $\Gamma_k(H)$ joining orthogonal elements consists of mutually compatible elements. Any two compatible $X, Y \in \mathcal{G}_k(H)$ are contained in a certain geodesic of $\Gamma_k(H)$ connecting X with an element orthogonal to X.*
Proof. If X and Y are orthogonal elements of $g_k(H)$ and X, X_1, ..., X_{k-1}, Y is a geodesic of $\Gamma_k(H)$, then for every $i \in \{1, \dots, k-1\}$ we have
$$\dim\,(X \cap X_i) = k - i \ and \ \dim\,(Y \cap X_i) = i.$$
This means that X_i is the orthogonal sum of $X \cap X_i$ and $Y \cap X_i$, i.e. X_i is compatible to both X and Y. If $i, j \in \{1, \dots, k-1\}$ and $i < j$, then $X \cap X_j$ is contained in $x \cap x_i$, and $Y \cap X_i$ is contained in $Y \cap X_j$ by Lemma (5.3.5). Therefore, X_i and X_j are compatible.
Consider compatible X, $Y \in \mathcal{G}_k(H)$. We take $Z \in \mathcal{G}_k(H)$ intersecting Y precisely in $(X \ \cap Y)^{\perp} \cap Y$ and orthogonal to X. Then X, Y, Z are mutually compatible and there is an orthogonal apartment containing them. This apartment contains a geodesic joining X with Z and passing through Y.
A mapping $L\colon H \to H$ is said to be a *semilinear operator* if
$$L(x+y) = L(x) + L(y)$$
for all $x, y \in H$ and there is an endomorphism σ of the field $\mathbb{C}$ such that
$$L(ax) = \sigma(a)L(x)$$
for all $a \in \mathbb{C}$ and all $x \in H$. If an endomorphism of the field $\mathbb{C}$ is continuous, then it is the identity or the conjugation. Non-continuous endomorphisms of $\mathbb{C}$ exist. If a semilinear operator is bounded, then the associated endomorphism of $\mathbb{C}$ is continuous, and the operator is linear or conjugate-linear.
Every injective semilinear operator on H induces a transformation of $\mathcal{G}_1(H)$ and every non-zero scalar multiple of this operator defines the same transformation. We will need the following consequence of the Fundamental Theorem of Projective Geometry [266], [267].
Fact (5.3.7)[265]: Let f be an injective transformation of $\mathcal{G}_1(H)$. If for every $U \in \mathcal{G}_2(H)$ there is $U' \in \mathcal{G}_2(H)$ such that
$$f(\langle U]_1) \subset \langle U']_1$$
and there is no 2-dimensional subspace containing all elements from the image of f, then f is induced by an injective semilinear operator on H. Such operator is unique up to a non-zero scalar multiple.
Lemma (5.3.8)[265]: *If an injective semilinear operator on H sends orthogonal vectors to orthogonal vectors, then it is a non-zero scalar multiple of a linear or conjugate-linear isometry.*

Proof. Let L be a semilinear operator on H sending orthogonal vectors to orthogonal vectors. If $x, y \in H$ are orthogonal unit vectors, then $x + y, x - y$ are orthogonal. Since $L(x), L(y)$ and $L(x) + L(y), L(x) - L(y)$ are pairs of orthogonal vectors, we have

$$||L(x)|| = ||L(y)||.$$

If unit vectors $x, y \in H$ are non-orthogonal, then we choose a unit vector z orthogonal to both x, y (this is possible, since $\dim H \geq 3$ by our assumption) and get

$$||L(x)|| = ||L(z)|| = ||L(y)||.$$

So, the function $x \to ||L(x)||$ is constant on the set of unit vectors which means that L is bounded, i.e. L is linear or conjugate-linear.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then there is an orthonormal basis {eí}i∈I of $L(H)$ such that $L(e_i) = a_i e_i'$ for non-zero scalars $a_i \in \mathbb{C}$. It was established above that $|a_i| = |a_j|$ for any pair $i, j \in I$. Then there is a positive real number b such that $a_i = bb_i$ and $|b_i| = 1$ for every $i \in I$. The linear or conjugate-linear operator transferring every e_i to $b_i e_i'$ is an isometry. We have $L = bL'$, where L' is one of these operators.

Let f be an orthogonality preserving transformation of $\mathcal{G}_k(H)$ and $k > 1$.

Suppose that $\dim H > 2k$ and f is adjacency preserving.

Lemma (5.3.9)[265]: *The transformation f is ortho-adjacency preserving.*

Proof. If $X, Y \in \mathcal{G}_k(H)$ are ortho-adjacent, then $f(X), f(Y)$ are adjacent and we need to show that they are compatible. By Lemma (5.3.6), X and Y are contained in a certain geodesic γ of $\Gamma_k(H)$ which connects X with an element $Z \in \mathcal{G}_k(H)$ orthogonal to X. Since f is adjacency preserving, $f(\gamma)$ is a path in $\Gamma_k(H)$. The elements X and Z are orthogonal, and the same holds for $f(X)$ and (Z) . Then

$$d(X,Z) = d\big(f(X), f(Z)\big) = k$$

which implies that $f(\gamma)$ is a geodesic in $\Gamma_k(H)$ connecting $f(X)$ with $f(Z)$ and containing $f(Y)$. Since $f(X)$ and $f(Z)$ are orthogonal, Lemma (5.3.6) gives the claim.

Lemma (5.3.10)[265]: *For every star $S \subset \mathcal{G}_k(H)$ there is the unique star containing (S) .*

Proof. Since f is adjacency preserving, $f(S)$ is a clique in $\Gamma_k(H)$ (not necessarily maximal), and it is contained in a certain maximal clique (a star or a top). Let $\mathcal{X}$ be a maximal compatible subset of S. By Lemma (5.3.9), $f(\mathcal{X})$ is a compatible subset in a star or a top. Note that $\mathcal{X}$ and $f(\mathcal{X})$ are of the same cardinality.

Lemma (5.3.4) shows that $f(\mathcal{X})$ cannot be contained in a top (since $\dim H > 2k$). Therefore, $f(S)$ is a subset in a star. The intersection of two distinct stars contains at most one element. This means that there is the unique star containing (S) .

Therefore, f induces a transformation f_{k-1} of $\mathcal{G}_{k-1}(H)$ such that

$$f([S\rangle_k) \subset [f_{k-1}(S)\rangle_k$$

for every $S \in \mathcal{G}_{k-1}(H)$. Then

$$f_{k-1}(\langle X]_{k-1}) \subset \langle f(X)]_{k-1}$$

for every $X \in \mathcal{G}_k(H)$.

Lemma (5.3.11)[265]: *The transformation f_{k-1} is orthogonality preserving.*

Proof. If X and Y are orthogonal elements of $\mathcal{G}_{k-1}(H)$, then there exist orthogonal $X', Y' \in \mathcal{G}_k(H)$ containing X and Y, respectively. We have

$f_{k-1}(X) \subset f(X'), f_{k-1}(Y) \subset f(Y') \; and \; f(X') \perp f(Y')$

which implies that $f_{k-1}(X)$ and $f_{k-1}(Y)$ are orthogonal.

Lemma (5.3.12)[265]: *The following assertions are fulfilled*:

(i) *If* $X, Y \in \mathcal{G}_{k-1}(H)$ *are adjacent, then* $f_{k-1}(X)$ *and* $f_{k-1}(Y)$ *are adjacent or* $f_{k-1}(X) = f_{k-1}(Y)$.

(ii) f_{k-1} *is ortho-adjacency preserving*.

Proof. The statements are trivial for $k = 2$. Indeed, any two distinct elements of $\mathcal{G}_1(H)$ are adjacent, and elements of $\mathcal{G}_1(H)$ are ortho-adjacent if they are orthogonal. Consider the case when $k > 2$.

(i) If $X, Y \in \mathcal{G}_{k-1}(H)$ are adjacent, then the corresponding stars $[X\rangle_k$ and $[Y\rangle_k$ have a non-empty intersection. The transformation f sends these stars to subsets of the same star or subsets of distinct stars with a non-empty intersection. This gives the claim.

(ii) By (i), f_{k-1} sends every path in $\Gamma_{k-1}(H)$ to a path (possibly of shorter length). So, the proof of this statement is similar to the proof of Lemma (5.3.9).

In the case when $k \geq 3$, we use Lemma (5.3.12) and arguments from the proof of Lemma (5.3.10) to show that for every star $S \subset \mathcal{G}_{k-1}(H)$ there is the unique star containing $f_{k-1}(S)$.

Step by step, we construct a sequence $= f_k, f_{k-1}, \dots, f_1$, where every f_i is an orthogonality and ortho-adjacency preserving transformation of $\mathcal{G}_i(H)$. If $i \geq 2$, then we have

$$f_i([Y\rangle_i) \subset [f_{i-1}(Y)\rangle_i$$

for every $Y \in \mathcal{G}_{i-1}(H)$ and

$$f_{i-1}(\langle X]_{i-1}) \subset \langle f(X)]_{i-1}$$

for every $X \in \mathcal{G}_i(H)$. This implies that

$$f_1(\langle X]_1) \subset \langle f(X)]_1 \ if \ X \in \mathcal{G}_k(H) . \tag{28}$$

Lemma (5.3.13)[265]: *The transformation* f_1 *is injective*.

Proof. For any distinct $P, Q \in \mathcal{G}_1(H)$ there exist mutually orthogonal $P_1, \dots , P_{k-1} \in \mathcal{G}_1(H)$ which are orthogonal to both P, Q. Consider the k-dimensional subspaces

$$X = P_1 + \cdots + P_{k-1} + P \ and \ Y = P_1 + \cdots + P_{k-1} + Q.$$

Since $f_1(P_1)$, $f_1(P_{k-1})$, $f_1(P)$ are mutually orthogonal, (28) implies that

$$f(X) = f_1(P_1) + \cdots + f_1(P_{k-1}) + f_1(P) .$$

Similarly, we establish that

$$f(Y) = f_1(P_1) + \cdots + f_1(P_{k-1}) + f_1(Q) .$$

The equality $f_1(P) = f_1(Q)$ implies that $(X) = f(Y)$. On the other hand, X and Y are adjacent and the same holds for $f(X)$ and (Y) .

So, f_1 is an orthogonality preserving injective transformation of $\mathcal{G}_1(H)$ such that

$$f_1(\langle Y]_1) \subset \langle f_2(Y)]_1$$

for every $Y \in \mathcal{G}_2(H)$. This means that f_1 satisfies the conditions of the Fundamental Theorem of Projective Geometry (Fact (5.3.7)), i.e. f_1 is induced by an injective semilinear operator on H. This operator sends orthogonal vectors to orthogonal vectors and Lemma (5.3.8) implies that it is a non-zero scalar multiple of a linear or conjugate-linear isometry. Using (28), we show that this isometry induces f.

Suppose that $\dim H > 2k$ and f is an ortho-adjacency preserving injection. By the required statement is a direct consequence of the following.

Lemma (5.3.14)[265]: *The transformation* f *is adjacency preserving*.

Proof. If $X, Y \in \mathcal{G}_k(H)$ are adjacent, then $\dim(X+Y)=k+1$ and we have $\dim(X+Y)^{\perp} \geq 2$ (since $\dim H > 2k > 2$). This implies the existence of orthogonal P, $Q \in \mathcal{G}_1(H)$ contained in $(X+Y)^{\perp}$. We take

$$X' = (X \cap Y) + P \text{ and } Y' = (X \cap Y) + Q.$$

Then X, X', Y' are mutually ortho-adjacent and the same holds for Y, X', Y'. Let $\mathcal{X}$ be a maximal compatible subset of the star $[X \cap Y\rangle_k$ containing X, X', Y'. Then $f(\mathcal{X})$ is a compatible subset in a star or a top. Since $\dim H > 2k$, Lemma (5.3.4) implies that $f(\mathcal{X})$ cannot be contained in a top, i.e. it is a subset of a star. This means that $f(X)$ contains the $(k-1)$-dimensional subspace $(X') \cap f(Y')$. Similarly, we show that this subspace is contained in (Y). Since f is injective, $f(X)$ and $f(Y)$ are adjacent.

Suppose that $\dim H = 2k$ and f preserves the adjacency relation in both directions. By [268], for every star $S \subset \mathcal{G}_k(H)$ there is the unique maximal clique (a star or a top) containing (S), and one of the following possibilities is realized:

(S) all stars go to subsets of stars,

(T) all stars go to subsets of tops.

In the case (S), f is induced by a unitary or anti-unitary operator on H (arguments from In the case (T), we consider the composition of f and the orthocomplementation. This transformation satisfies (S), i.e. it is induced by a unitary or anti-unitary operator. This gives the claim.

For f be a transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions. We suppose that $\dim H = n$ is finite and greater than $2k$.

Lemma (5.3.15)[265]: *The transformation f is injective.*

Proof. For distinct $X, Y \in \mathcal{G}_k(H)$ we take $Z \in \mathcal{G}_k(H)$ orthogonal to X and non-orthogonal to Y. Then $f(Z)$ is orthogonal to $f(X)$ and non-orthogonal to (Y). This implies that $f(X)$ and $f(Y)$ are distinct.

Consider the case when $k=1$. For every $U \in \mathcal{G}_2(H)$ we take mutually orthogonal $Q_1, \ldots, Q_{n-2} \in g_1(H)$ contained in $U^{\perp}$. If $P \in \langle U]_1$, then $f(P)$ is contained in the orthogonal complement of the $(n-2)$-dimensional subspace $f(Q_1)+\cdots+f(Q_{n-2})$. Therefore, f satisfies the conditions of the Fundamental Theorem of Projective Geometry (Fact (5.3.7)). Since H is finite-dimensional, Lemma (5.3.8) shows that f is induced by a unitary or anti-unitary operator.

From this moment we will suppose that $k > 1$.

Lemma (5.3.16)[265]: *Let $X_1, \ldots, X_i, Y$ be mutually distinct elements of $\mathcal{G}_k(H)$ such that Y is not contained in the subspace $X_1+\cdots+X_i$ and*

$$\dim(X_1+\cdots+X_i) \leq n-k. \tag{29}$$

Then $f(Y)$ is not contained in $(X_1)+\cdots+f(X_i)$.

Proof. The condition (29) implies the existence of elements of $\mathcal{G}_k(H)$ orthogonal to $X_1+\cdots+X_i$. Since Y is not contained in $X_1+\cdots+X_i$, there is $Z \in \mathcal{G}_k(H)$ orthogonal to $X_1+\cdots+X_i$ and non-orthogonal to Y. Then $f(Z)$ is orthogonal to $f(X_1)+\cdots+f(X_i)$ and non-orthogonal to $f(Y)$ which means that $f(Y)$ is not contained in $(X_1)+\cdots+f(X_i)$.

Lemma (5.3.17)[265]: *The transformation f is adjacency preserving.*

Proof. Let X and Y be adjacent elements of $\mathcal{G}_k(H)$. Consider a sequence $X_0, X_1, \ldots, X_{n-2k}$ of elements from $Q_k(H)$ such that $X_0 = X, X_1 = Y$ and

$$\dim\left(X_0 + X_1 + \cdots + X_j\right) = k + j$$
for every $j \in \{1, \ldots, n - 2k\}$. Then X_j is not contained in $X_0 + \cdots + X_{j-1}$. We have $k + j \leq n - k$ for every $j \in \{1, \ldots, n - 2k\}$ and Lemma (5.3.16) implies that
$$f\left(X_j\right) \not\subset f(X_0) + \cdots + f\left(X_{j-1}\right).$$
Therefore,
$$\dim\left(f(X_0) + f(X_1) + \cdots + f(X_{n-2k})\right) \geq \dim\left(f(X_0) + f(X_1)\right) + n - 2k - 1. \quad (30)$$
If $f(X) = f(X_0)$ and $f(Y) = f(X_1)$ are not adjacent, then
$$\dim\left(f(X_0) + f(X1)\right) > k + 1$$
and (30) shows that
$$\dim\left(f(X_0) + f(X_1) + \cdots + f(X_{n-2k})\right) > n - k.$$
In this case, there is no element of $\mathcal{G}_k(H)$ orthogonal to all $\left(X_j\right)$. On the other hand, the equality
$$\dim(X_0 + X_1 + \cdots + X_{n-2k}) = n - k$$
implies the existence of $Z \in \mathcal{G}_k(H)$ orthogonal to all X_j . Then $f(Z)$ is orthogonal to every $f(X_j)$. This contradiction shows that $f(X)$ and $f(Y)$ are adjacent.
The statement is a consequence of Lemma (5.3.17) and Theorem (5.3.1).

Chapter 6
C^*-Completions and Convergent Star Products

We show when taking the C^*-algebra of continuous sections vanishing at infinity, we arrive at a functor associating a C^*-algebra to any Poisson vector bundle and recover the original DFR-algebra as a particular example. We show that many properties of the resulting locally convex algebra are explained. We compare this approach to various other discussions of convergent star products in finite and infinite dimensions. We pay special attention to the case of a Hilbert space and to nuclear spaces.

Section (6.1): The DFR-Algebra

Doplicher, Fredenhagen, and Roberts (DFR) have introduced a special C^*-algebra to provide a model for spacetime in which localization of events can no longer be performed with arbitrary precision: they refer to it as a model of "quantum spacetime apart from being beautifully motivated, their construction admits a mathematically simple (re)formulation: it starts from a symplectic form on Minkowski space and considers the corresponding canonical commutation relations (CCRs), which can be viewed as a representation of a well-known finite-dimensional nilpotent Lie algebra, the Heisenberg (Lie) algebra. The CCRs appear in Weyl form, i.e., through an irreducible, strongly continuous, unitary representation of the corresponding Heisenberg (Lie) group-which, according to the well-known von Neumann theorem, is unique up to unitary equivalence. That representation is then used to define a C^*-algebra that we propose to call the Heisenberg C^*-algebra, related to the original representation through Weyl quantization, that is, via the Weyl-Moyal star product.

The main novelty in the DFR construction is that the underlying symplectic form is treated as a *variable*. In this way, one is able to reconcile the construction with the principle of relativistic invariance: since Minkowski space $\mathbb{R}^{1,3}$ has no distinguished symplectic structure, the only way out is to consider, simultaneously, *all* possible symplectic structures on Minkowski space that can be obtained from a fixed one, that is, its orbit Σ under the action of the Lorentz group. This orbit turns out to be isomorphic to $TS^2 \times \mathbb{Z}_2$, thus explaining the origin of the extra dimensions that appear in this approach. (In passing, we note that the factor $\mathbb{Z}_2$ comes from the fact that we are dealing with the full Lorentz group; it would be absent if we dropped (separate) invariance under parity P or time reversal T. Also, the generic feature that any deformation quantization of the function algebra over Minkowski space must contain, within its classical limit, some kind of extra factor has been noted and emphasized.)

Assuming the symplectic form to vary over the orbit Σ of some fixed representative produces notjust a single Heisenberg C^*-algebra but an entire C^*-bundle over this orbit, with the Heisenberg C^*-algebra for the chosen representative as typical fiber. The continuous sections of that C^*-bundle vanishing at infinity then define a "section" C^*-algebra, which carries a natural action of the Lorentz group induced from its natural action on the underlying bundle of C^*-algebras (which moves base points as well as fibers). Besides, this "section" C^*-algebra is also a C^*-module over the "scalar" C^*-algebra $C_0(\Sigma)$ of continuous functions on Σ vanishing at infinity. In the special case considered by DFR, the underlying C^*-bundle turns out to be globally trivial, which in view of von Neumann's theorem implies a classification result on irreducible as well as on Lorentz covariant representations of the DFR-algebra.

In retrospect, it is clear that when formulated in this geometrically inspired language, the results yearn for generalization- even if only for purely mathematical reasons.

From a more physical side, one of the original motivations for the present work was an idea of Barata, who proposed to look for a clearer geometrical interpretation of the classical limit of the DFR-algebra in terms of coherent states, as developed by Hepp.15 This led the second to investigate possible generalizations of the DFR construction to other vector spaces than four-dimensional Minkowski space and other Lie groups than the Lorentz group in four dimensions. As it turned out, the crucial mathematical input for the construction of the DFR-algebra is a certain symplectic vector bundle over the orbit Σ, namely, the trivial vector bundle $\Sigma \times \mathbb{R}^{1,3}$ equipped with the "tautological" symplectic structure, which on the fiber over a point $(J^{\cdot} \in \Sigma$ is just $(J^{\cdot}$ itself. Here, we show how, following an approach similar to the one, one can generalize this construction to any Poisson vector bundle, without supposing homogeneity under some group action or nondegeneracy of the Poisson tensor.

The basic idea of the procedure is to use the given Poisson structure to first construct a bundle of Fréchet∗-algebras over the same base space whose fibers are certain function spaces over the corresponding fibers of the original vector bundle, the product in each fiber being the Weyl-Moyal star product given by the Poisson tensor there: the DFR-algebra is then obtained as the C^*-completion algebra of this Fréchet∗-algebra bundle. However, a geometrically more appealing interpretation would be algebra of some C^*-bundle, which should be obtained directly from the underlying Fréchet∗-algebra bundle by a process of C^*-completion. The concept of a C^*-completion at the level of *bundles* is novel, and one of the main goals we achieve is to develop this new theory to the point needed and then apply it to the situation at hand.

We gather a few known facts about the construction of C^*-completions of a given ∗-algebra: provided that such completions exist at all, they can be controlled in terms of the corresponding universal enveloping C^*-algebra, which in particular provides a criterion for deciding whether such a completion is unique.

We notice that when the given ∗-algebra is embedded into some C^*-algebra as a spectrally invariant subalgebra, then that C^*-algebra is in fact its universal enveloping C^*-algebra.

We propose a new definition of "the C^*- algebra of the canonical commutation relations" (for systems with a finite number of degrees of freedom) which we propose to call the Heisenberg C^*-algebra: it comes in two variants, namely, a nonunital one, ε_σ, and a unital one, $\subset H_\sigma$, obtained as the unique C^*-completions of certain Fréchet∗- algebras S_σ and B_σ, respectively. These are simply the usual Fréchet spaces S of rapidly decreasing smooth functions and $\mathcal{B}$ of totally bounded smooth functions (= bounded smooth functions with bounded partial derivatives) on a given finite-dimensional vector space, equipped with the Weyl-Moyal star product induced by a-possibly degenerate-bivector σ, whose definition, in the unital case, requires the use of oscillatory integrals as developed in Rieffel's theory of strict deformation quantization. The main advantage of this definition as compared to others is that the representation theory of these C^*-algebras corresponds precisely to the

representation theory of the Heisenberg group: as a result of uniqueness of the C^*-completion, there is no need to restrict to a subclass of “regular” representations.
We begin by introducing the concept of a bundle of locally convex $*$-algebras, which contains that of a C^*-bundle as a special case, and following the approach of Dixmier, Fell and other we show how the topology of the total space of any such bundle is tied to its algebra of continuous. Next, we pass to the C^* setting, where we explore the notion of a $C_0(X)$-algebra (X being some fixed locally compact topological space). At first sight, this appears to generalize the natural module structure algebra of a C^*-bundle over X, but according representation theorem [33], it actually provides a necessary and sufficient condition for a C^*- algebra to algebra of a C^*-bundle over X. Here, we formulate a somewhat strengthened version of that theorem which establishes a categorical equivalence between C^*-bundles over X and $C_0(X)$-algebras. Finally, we introduce the (apparently novel) concept of C^*-completion of a bundle of locally convex $*$-algebras and show that, using this (essentially fiberwise) definition and imposing appropriate conditions on the behavior of sections at infinity, the two processes of completion and of passing to section algebras commute: the C^*-completion of the algebra of continuous sections with compact support of a bundle of locally convex $*$-algebras is naturally isomorphic to the algebra of continuous sections vanishing at infinity of its C^*-completion.
We combine the methods developed construct, from an arbitrary Poisson vector bundle E over an arbitrary manifold X, with Poisson tensor $\alpha\,\cdot$, two bundles of Fréchet$*$-algebras over X, $\mathcal{E}(E,\sigma)$ and $B(E,\sigma)$, as well as two C^*-bundles over X, $\varepsilon(E,\sigma)$ and $\subset H(E,\sigma)$, the latter being the C^*-completions of the former with respect to the C^* fiber norms induced by the unique C^*-norms on each fiber, according to the prescriptions of We propose to refer to these C^*-bundles as DFR-bundles and to the corresponding algebras as DFR-algebras, since we show that the original DFR-algebra can be recovered as a special case, by an appropriate and natural choice of Poisson vector bundle. Moreover, that construction can be applied fiberwise to the tangent spaces of any Lorentzian manifold to define a functor from the category of Lorentzian manifolds (of fixed dimension) to that of C^*-algebras which might serve as a starting point for a notion of “locally covariant quantum spacetime.
The overall picture that emerges is that the constructions presented establish a systematic method for producing a vast class of examples of C^*-algebras provided with additional ingredients that are tied up with structures from classical differential geometry and/or topology in a functorial manner. To what extent this new class of examples can be put to good use remains to be seen. But we believe that even the original question of how to define the classical limit of the DFR-algebra, or more generally how to handle its space of states, will be deeply influenced by the generalization presented here, which is of independent mathematical interest.

We want to discuss the question of existence and uniqueness of the C^*-completion of a $*$-algebra (possibly equipped with some appropriate locally convex topology of its own), which is closely related to the concept of a spectrally invariant subalgebra, as well as the issue of continuity of the inversion map on the group of invertible elements.
We begin by recalling a general and well-known strategy for producing C^*-norms on $*$-algebras. It starts from the observation that given any $*$-algebra B and any $*$-representation

p of B on a Hilbert space $\mathfrak{H}_\beta$, we can define a C^*-seminorm $\|.\|_p$ on B by taking the operator norm in $B(\mathfrak{H}_\beta)$, i.e., by setting, for any $b \in B$,

$$\|b\|_p = \|p(b)\|. \tag{1}$$

Obviously, this will be a C^*-norm if and only if p is faithful. More generally, given any set R of $*$-representations of B such that, for any $b \in B$, $\{\|p(b)\| | p \in R\}$ is a bounded subset of $\mathbb{R}$, setting

$$\|b\|_R = \sup_{p \in R} \|b\|_p \tag{2}$$

will define a C^*-seminorm on B, which is even a C^*-norm as soon as the set R separates B (i.e., for any $b \in B \backslash \{0\}$, there exists $p \in R$ such that $p(b) \neq 0$). Taking into account that every C^*-seminorm s on B is the operator norm for some $*$-representation of B (this follows from applying the Gelfand-Naimark theorem [25] to the C^*-completion of $B/\ker s$, together with the fact that every faithful C^*-algebra representation is automatically isometric [25]), we can take R to be the set Rep (B) of al1 $*$-representations of B (up to equivalence) to obtain a C^*-seminorm on B which is larger than any other one, provided that, for any $b \in B$,

$$\sup_{p \in \mathrm{Rep(B)}} \|b\|_p < \infty. \tag{3}$$

Moreover, when Rep (B) separates B, we obtain the well-known *maximal* C^*-norm on B, which gives rise to the *minimal* C^*-*completion* of B, also denoted by $C^*(B)$ and called the *universal enveloping* C^*-*algebra* of B because it satisfies the following universal property: for every C^*-algebra C, every $*$-algebra homomorphism from B to C extends uniquely to a C^*-algebra homomorphism from $C^*(B)$ to C.

Next, given a $*$-algebra B embedded in some C^*-algebra A as a dense $*$-subalgebra, one method for guaranteeing existence of the universal enveloping C^*-algebra relies on the concept of spectral invariance, which is defined as follows: B is said to be *spectrally invariant* in A if, for every element b of B, its spectrum in A, $\sigma_A(b)$, is the same as its spectrum in B, $\sigma_B^{\vee}(b)$. Note that, in general, $(\sigma_A(b) \subset \sigma_B(b)$, i.e., the spectrum shrinks under the inclusion of B into A, so only the opposite inclusion is a nontrivial condition. (Actually, the spectrum shrinks under any morphism. To see this, suppose that A and B are any two $*$-algebras and $\varphi: B \to A$ is any $*$-algebra homomorphism. If , B, and φ are unital it suffices to note that $\lambda \notin \sigma_B(b)$ means that $\lambda 1_B - b$ has an inverse in B whose image under φ serves as an inverse of $\lambda 1_A - \varphi(b)$ in A, so $\lambda \notin \sigma_A(\ell(b))$. If A, B, and φ are nonunital, we can apply the same argument, with $\lambda \neq 0$, to their unitizations $\tilde{A}$, $\tilde{B}$, and $\tilde{\varphi}$. At any rate, we conclude that, for any b in B, $\mathrm{c}\sigma_A(\varphi(b)) \subset (\sigma_B(b).)$ Returning to the situation where B is a spectrally invariant dense $*$-subalgebra of a C^*-algebra A, we may conclude that, for any self-adjoint element b of B,

$$\sup_{p \in \mathrm{Rep(B)}} \|b\|_p \leq r(b) \, ,$$

where $r(b)$ denotes the spectral radius of b in B, which by hypothesis coincides with its spectral radius in A and hence (for self-adjoint b) also with its C^*-norm in A. But this means that the C^*-norm in A is in fact the maximal C^*-norm and hence that the C^*-algebra A is precisely the universal enveloping C^*-algebra of : $A = C^*(B)$.

As an example showing the usefulness of this concept, we note the following.

Theorem (6.1.1)[270]: *Let A be a (nonunital) C^*-algebra, equipped with the standard partial ordering induced by the cone $A^+$of positive elements, and let B be a spectrally invariant $*$-subalgebra of A. Then A admits an approximate identity consisting of elements of B, i.e., a directed set $(e_\lambda)_{\lambda\in\Lambda}$ of elements e_λ of B such that, in A, $e_\lambda \geq 0$, $\|e_\lambda\| \leq 1$, $e_\lambda \leq e_\mu$ if $\lambda \leq \mu$ and, for every $a \in A$, $\lim_\lambda e_\lambda\, a = a = \lim_\lambda a\, e_j$.*

The proof is an easy adaptation of that of a similar theorem due to Inoue, of locally C^*-algebras, see [12]; we note here that the version given above can also be generalized to locally C^*-algebras without additional effort. The main difference is that we assume B to bejust a dense $*$-subalgebra, rather than a dense $*$-ideal, and spectral invariance turns out to be the crucial ingredient to make the proof work.

Once the existence of the universal enveloping C^*-algebra $C^*(B)$ of B is settled- usually by realizing it explicitly as a spectrally invariant $*$-subalgebra of a given C^*-algebra $A -$ we can address the question of classifying *all* possible C^*-norms on B. Making use of the fact that, in this situation, any C^*-norm on B can be uniquely extended to a C^*-seminorm on A whose kernel is a closed $*$-ideal in A that has trivial intersection with B, it follows that if we can determine what are the closed $*$-ideals in A and prove that none of them intersects B trivially, then we can conclude that B admits one and only one C^*-norm.

Finally, it is worth noting that in many cases of interest, B will not be merely a $*$-algebra but will come equipped with a (locally convex) topology of its own, with respect to which it is complete. We have the following result.

Proposition (6.1.2)[270]: *Let B be a Fréchet$*$-algebra, i. e., a $*$-algebra which is also a Fréchet space such that multiplication and involution are continuous, and assume that B is continuously embedded in some C^*-algebra A as a spectrally invariant $*$-subalgebra. Then the group G_B of invertible elements of B is open and the inversion map*

$$\begin{aligned} G_B &\to G_B \\ b &\mapsto b^{-1} \end{aligned}$$

is continuous not only in the induced C^*-topology but also in the Fréchet topology.

Proof. The statement of this proposition is well-known for the C^*-topology, but that it also holds for the finer Fréchet topology is far from obvious, as can be inferred from the extensive discussion of concepts related to this question that can be found in the literature, such as that of Q-algebras" and of "topological algebras with inverses see [12] and [271] Spectral invariance guarantees that G_B is equal to $B \cap \quad G_A$, i.e., it is the inverse image of G_A, which is open in A, under the inclusion map $B \leftrightarrow A$, which by hypothesis is continuous. Continuity of the inversion map then follows from the Arens-Banach theorem [271] or from a more general direct argument.

Let V be a Poisson vector space, i.e., a real vector space of dimension n, say, equipped with a fixed bivector σ of rank $2r$; in other words, the dual V^* of V is a presymplectic vector space. (We emphasize that we do *not* require 0^- to be nondegenerate.) It gives rise to an $(n + 1)$-dimensional Lie algebra $\mathfrak{h}_0$– which is a one-dimensional central extension of the abelian Lie algebra V^* defined by the cocycle σ and will be called the *Heisenberg algebra* or, more precisely, *Heisenberg Lie algebra* (associated to V^* and σ) : as a vector space, $\mathfrak{h}_\sigma = V^* \oplus \mathbb{R}$, with commutator given by

$$[(\xi,\lambda),(\eta,\mu)] = \big(0,\sigma(\xi,\eta)\big) \qquad for\ \xi,\eta \in V^*, \lambda,\mu \in \mathbb{R}. \quad (4)$$

Associated with this Lie algebra is the *Heisenberg group* or, more precisely, *Heisenberg Lie group*, H_σ: as a manifold, $H_\sigma \cdot= V^* \times \mathbb{R}$, with product given by

$$(\xi,\lambda)(\eta,\mu) = \left(\xi+\eta, \lambda+\mu-\frac{1}{2}\sigma(\xi,\eta)\right) for\ \xi,\eta \in V^*, \lambda,\mu \in \mathbb{R}. \quad (5)$$

We shall discuss various forms of giving a precise mathematical meaning to the concept of a *representation of the canonical commutation relations* defined by σ. From the very beginning, we shall restrict ourselves to representations that can be brought into *Weyl form*, i.e., that correspond to strongly continuous unitary representations π of the Heisenberg group H_o–: abbreviating $\pi(\xi,0)$ to $\pi(\xi)$, these relations can be written in the form

$$\pi(\xi)\pi(\eta) = e^{-\frac{i}{2}\sigma(\xi,\eta)}\pi(\xi+\eta)\,. \qquad (6)$$

At the infinitesimal level, they correspond to representations $\dot{\pi}$ of the Heisenberg algebra $\mathfrak{h}_\sigma$. which are often called "regular": according to Nelson's theorem these are precisely the representations of $\mathfrak{h}_\sigma$ by essentially skew adjoint operators on a common dense invariant domain of analytic vectors. use these representations of the canonical commutation relations to construct what we shall call the *Heisenberg C^*-algebra*: This algebra comes in two versions, namely, a nonunital one and a unital one, denoted here by ε_σ. and by CH_σ ·, respectively: as it turns out, the latter is simply the multiplier algebra of the former. We emphasize that our construction differs substantially from previous ones that can be found in the literature, such as the Weyl algebra or the resolvent algebra: both of those use the method of constructing a C^*-algebra from an appropriate set of generators and relations. Instead, we focus on certain Fréchet∗-algebras that play a central role in Rieffel's theory of strict deformation quantization and show that each of these admits a unique C^*-norm, so it has a unique C^*-completion.

we given any (finite-dimensional) real vector space W, say, we denote by $S(W)$ the Schwartz space of rapidly decreasing smooth functions on W and by $\mathcal{B}(W)$ the space of totally bounded smooth functions on W. (A smooth function is said to be totally bounded if it is bounded and so are all of its partial derivatives.)

To begin with, we want to briefly recall how one can use the bivector σ to introduce a new product on the space $S(V)$ which is a deformation of the standard pointwise product, commonly known as the *Weyl-Moyal star product* and will then comment on how that deformed product can be extended to the space $\mathcal{B}(V)$.

Given any strongly continuous unitary representation π of the Heisenberg group H_σ on some Hilbert space $\mathfrak{H}_\pi$, we can construct a continuous linear map

$$\begin{aligned} W_\pi &: S(V) \to \mathcal{B}(\mathfrak{H}_\pi) \\ f &\mapsto W_\pi f \end{aligned} \qquad (7)$$

from $S(V)$ to the space of bounded linear operators on $\mathfrak{H}_\pi$, called the *Weyl quantization map*, by setting

$$W_\pi f = \int_{V^*} d\,\xi \check{f}(\xi)\pi(\xi)\,, \qquad (8)$$

which is to be compared with

$$f(x) = \int_{V^*} d\,\xi \check{f}(\xi) e^{i\langle \xi, x\rangle}, \tag{9}$$

where $\check{f}$ is the inverse Fourier transform of f,

$$\check{f}(\xi) \equiv (F^{-1}f)(\xi) = \frac{1}{(2\pi)^n}\int_V \, d\,x f(x) e^{-i\langle \xi, x\rangle}. \tag{10}$$

Note that Equation (8) should be understood as stating that, for every vector ψ in $\mathfrak{H}_\pi$, we have

$$(W_\pi f)\psi = \int_{V^*} d\,\xi \check{f}(\xi)\pi(\xi)\psi,$$

since it is this integral that makes sense as soon as π is strongly continuous; then it is obvious that $W_\pi f \in B(\mathfrak{H}_\pi)$, with $\|W_\pi f\| \le \|\check{f}\|_1$, where $\|.\|_1$ is the L^1-norm on $S(V^*)$ which, as shown in the Appendix (see Equation (A2)), can be estimated in terms of a suitable Schwartz seminorm of f,

$$\|W_\pi f\| \le \|\check{f}\|_1 \le (2\pi)^n \sum_{|\alpha|,|\beta|\le 2n} \sup_{\mathrm{x}\in\mathrm{V}} \; |x^\alpha \partial_\beta f(x)|. \tag{11}$$

Moreover, an explicit calculation shows that, independently of the choice of π, we have, for $f, g \in S(V)$,

$$W_\pi f W_\pi g = W_\pi (f \star_\sigma g)\,, \tag{12}$$

where $\star_\sigma$ denotes the *Weyl-Moyal star product* of f and g, which is given by any one of the following two twisted convolution integrals:

$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi e^{i\langle \xi, x\rangle} \int_{V^*} d\,\eta \check{f}(\eta)\check{g}(\xi - \eta) e^{\frac{i}{2}\sigma(\xi,\eta)}, \tag{13}$$

$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi e^{i\langle \xi, x\rangle} \int_{V^*} d\,\eta \check{f}(\xi - \eta)\check{g}(\eta) e^{-\frac{i}{2}\sigma(\xi,\eta)}. \tag{14}$$

The proof is a simple computation (we omit the ψ),

$$\begin{aligned}
W_\pi f W_\pi g &= \int_{V^*} d\,\eta \int_{V^*} d\,\zeta \check{f}(\eta)\check{g}(\zeta)\pi(\eta)\pi(\zeta) \\
&= \int_{V^*} d\,\eta \int_{V^*} d\,\zeta \check{f}(\eta)\check{g}(\zeta) e^{-\frac{i}{2}o^-(\eta,\zeta)}\pi(\eta+\zeta) \\
&= \int_{V^*} d\,\eta \int_{V^*} d\,\xi \check{f}(\eta)\check{g}(\xi-\eta) e^{-\frac{i}{2}o-(\eta,\xi)}\pi(\xi) \\
&= \int_{V^*} d\,\xi \int_{V^*} d\,\eta \check{f}(\eta)\check{g}(\xi-\eta) e^{\frac{i}{2}o^-(\xi,\eta)}\pi(\xi) \\
&= \int_{V^*} d\,\xi F^{-1}(f \star_\sigma g)(\xi)\pi(\xi) \\
&= W_\pi (f \star_\sigma g)\,.
\end{aligned}$$

For the sake of comparison, we note an alternative form of this product using the “musical homomorphism” $\# : V^* \to V$ induced by $(J^\cdot$ (i.e., $\langle \xi, \sigma^\# \eta\rangle = \sigma(\eta, \xi)$) , we get

$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi e^{i\langle \xi, x\rangle} \int_{V^*} d\,\eta \check{f}(\eta)\check{g}(\xi-\eta) e^{-\frac{i}{2}\langle \xi, \sigma^\# \eta\rangle}$$
$$= \int_{V^*} d\,\eta \int_{V^*} d\,\xi' \check{f}(\eta)\check{g}(\xi') e^{i\left(\xi'+\eta, x-\frac{1}{2}o^{-\#}\eta\right)}$$
$$= \int_{V^*} d\,\eta \check{f}(\eta) g\left(x - \frac{1}{2}\sigma^\# \eta\right) e^{i\langle \eta, x\rangle}$$
$$= \frac{1}{(2\pi)^n} \int_{V^*} d\,\eta \int_V d\,w f(w) g\left(x - \frac{1}{2}\sigma^\# \eta\right) e^{i\langle \eta, x-w\rangle},$$
and similarly
$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi e^{i\langle \xi, x\rangle} \int_{V^*} d\,\eta \check{f}(\xi-\eta)\check{g}(\eta) e^{\frac{i}{2}\langle \xi, \sigma^\# \eta\rangle}$$
$$= \int_{V^*} d\,\eta \int_{V^*} d\,\xi' \check{f}(\xi')\check{g}(\eta) e^{i\langle \xi'+\eta, x+\frac{1}{2}\sigma^\# \eta\rangle}$$
$$= \int_{V^*} d\,\eta f\left(x + \frac{1}{2}\sigma^\# \eta\right) \check{g}(\eta) e^{i\langle \eta, x\rangle}$$
$$= \frac{1}{(2\pi)^n} \int_{V^*} d\,\eta \int_V d\,w f\left(x + \frac{1}{2}\sigma^\# \eta\right) g(w) e^{i\langle \eta, x-w\rangle},$$
i.e., after a change of variables $w \to u = w - x$, $\eta \to \xi = \eta/2\pi$ in the first case and $w \to v = w - x, \eta \to \xi = -\eta/2\pi$ in the second case,
$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi \int_V d\,u f(x+u) g(x - \pi(J^\# \xi) e^{-2\pi i\langle \xi, u\rangle}, \quad (15)$$
$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi \int_V d\,v f(x - \pi\sigma^\# g) g(x+v) e^{2\pi i\langle \xi_U\rangle}. \quad (16)$$
The Weyl-Moyal star product is iointly) continuous with respect to the standard Fréchet topology on $S(V)$ (this is well known and is also an immediate consequence of Proposition (6.1.17) in the Appendix). It follows that, with respect to the Weyl-Moyal star product, together with the standard involution of pointwise complex conjugation and the standard Fréchet topology, the space $S(V)$ becomes a Fréchet$*$-algebra, which we shall denote by S_O- and call the *Heisenberg-Schwartz algebra* (with respect to $\sigma^\#$) .
Dealing with the Weyl-Moyal star product between two functions in (V) , rather than $S(V)$, is substantially more complicated. In this case, its definition is based on Equation (15) or Equation (16), whose rhs has to be interpreted as an oscillatory integral on $V^* \times V$. Fortunately, all of the necessary analytic tools have been provided by Rieffe1 (with the identification $= -\pi\sigma^\#$) , so we may just state, as one of the results, that with respect to the Weyl-Moyal star product, together with the standard involution of pointwise complex conjugation and the standard Fréchet topology, the space $B(V)$ becomes a Fréchet$*$-algebra, which we shall denote by B_σ. and propose to call the *Heisenberg-Rieffel algebra* (with respect to σ).
We note in passing that both algebras are noncommutative when $r \neq 0$, but their deviation from commutativity is explicitly controlled by a simple formula,

$$g \star_\sigma f = f \star_{-\sigma} g. \tag{17}$$

Retutning to explicit integral formulas, we note next that an intermediate situation, which will be of particular interest in what follows, occurs when one factor belongs to $B(V)$ while the other belongs to (V) , since as the above calculation has shown, we have

$$(f \star_\sigma g)(x) = \int_{V^*} d\xi \check{f}(\xi) g\left(x - \frac{1}{2}\sigma^\# \xi\right) e^{i\langle\xi,x\rangle}, \tag{18}$$

and similarly,

$$(f \star_\sigma g)(x) = \int_{V^*} d\,\xi f\left(x + \frac{1}{2}\sigma^\# \xi\right) \check{g}(\xi) e^{i\langle\xi,x\rangle}. \tag{19}$$

Note that the expression in Equation (18) makes sense when $f \in S(V)$, $g \in B(V)$ and similarly, the the expression in Equation (19) makes sense when $f \in B(V), g \in S(V)$: both are then ordinary integrals that become iterated integrals when the expression (10) for the inverse Fourier transform is written out explicitly. (Obviously, the two formulae can be converted into each other by means of Equation (17).) Moreover, it follows from elementary estimates which can be found in the Appendix (see Proposition (6.1.17)) that, in either case, $\star_\sigma \cdot\, g \in S(V)$, and the linear operators

$$\begin{aligned} L_\sigma f \colon S_\sigma &\to S_\sigma \\ h &\mapsto f \star_\sigma h \end{aligned} \tag{20}$$

of left translation by $f \in B_\sigma$ and

$$\begin{aligned} R_\sigma g \colon\ S_\sigma &\to S_\sigma \\ h &\to h \star_\sigma g \end{aligned} \tag{21}$$

of right translation by $g \in B_\sigma$ are continuous in the Schwartz topology. In particular, S_O– is a $*$-ideal in E_σ (but neither closed nor dense); see [31] for more details. Thus we get a $*$-homomorphism

$$\begin{aligned} B_\sigma &\to M(S_\sigma) \\ f &\mapsto (L_\sigma f, R_\sigma f) \end{aligned} \tag{22}$$

which provides an embedding of B_σ into what might be called the multiplier algebra $M(S_\sigma)$ of S_σ. However, we have refrained from using this terminology since there is no established definition of the concept of multiplier algebra beyond the realm of Banach algebras: there are "*a priori*" many possible candidates for its locally convex topology. (This is of course a generic statement: it does not exclude the existence of special cases where the "most natural" ones among these topologies coincide, as happens in the case of $M(S_\sigma)$ when $a.$ is nondegenerate. Of course, this ambiguity will no longer be a problem as soon as we pass to the C^*-completions.

The Fréchet algebras S_σ and B_σ both admit various norms. The naive choice would be the standard sup norm, but this is a C^*-norm for the usual pointwise product, not for the Weyl-Moyal star product. Hence the first question is whether there exist C^*-norms on S_σ and on B_σ at all. Fortunately, the answer is affirmative: it suffices to take the operator norm in the regular representation. Consider the $*$-representation

$$\begin{aligned} L_\sigma \colon S_\sigma &\to B\big(L^2(V)\big) \\ f &\mapsto L_\sigma f \end{aligned} \tag{23}$$

of S_σ, which extends to a $*$-representation

$$L_\sigma : B_\sigma \; \rightarrow \; B\big(L^2(V)\big)$$
$$f \rightarrow L_\sigma f \qquad (24)$$

of B_σ, both defined by taking the operator $L_{(J} \cdot f \; : \; L^2(V) \rightarrow L^2(V)$ to be the unique continuous linear extension of the operator $L_\sigma f : S(V) \rightarrow S(V)$ of Equation (20). Similarly, we may also consider the (anti-)∗-representation

$$R_\sigma : S_\sigma \; \rightarrow \; B\big(L^2(V)\big)$$
$$g \; \mapsto \; R_\sigma g \qquad (25)$$

of S_σ, which extends to an (anti-)∗-representation

$$R_\sigma : B_\sigma \; \rightarrow \; B\big(L^2(V)\big)$$
$$g \rightarrow R_\sigma g \qquad (26)$$

of B_σ, both defined by taking the operator $R_\sigma g : L^2(V) \rightarrow L^2(V)$ to be the unique continuous linear extension of the operator $R_\sigma g : S(V) \rightarrow S(V)$ of Equation (21). Obviously, any $L_{o-}f$ commutes with any $R_\sigma g$: this is nothing but associativity of the star product. This construction presupposes that the operators $L_\sigma f$ of Equation (20) (and analogously, the operators $R_\sigma g$ of Equation (21)) are continuous not only in the Schwartz topology but also in the L^2-norm. We need to show that the linear maps L_σ in Equations (23) and (24) (and analogously, the linear maps R_σ in Equations (25) and (26)) are continuous with respect to the appropriate topologies. And finally, we want these continuity properties to hold locally uniformly when we vary $\sigma\;\cdot$. Fortunately, all these statements can be derived from a single estimate, as we explain in what follows.

First, consider the case when f belongs to (V) : then we can rewrite Equation (18) in the form of Equation (8), since

$$L_\sigma f = \int_{V^*} d\,\xi \check{f}(\xi) \pi^{\mathrm{reg}}(\xi)\,, \qquad (27)$$

where π^{reg} is the regular representation of the Heisenberg group H_σ, that is, the strongly continuous unitary representation of H_σ on the Hilbert space $L^2(V)$ defined by setting

$$(\pi^{\mathrm{reg}}(\xi)\psi)(x) = e^{i\langle \xi, x\rangle} \psi\left(x - \frac{1}{2}\sigma^{\#}\xi\right), \qquad (28)$$

i.e., $\pi^{\mathrm{reg}}(\xi)$ is the operator of translation by $-\frac{1}{2}\sigma^{\#}\xi$ followed by that of multiplication with the phase function $e^{i\langle \xi, .\rangle}$. As before, it follows that $L_\sigma f \in B\big(L^2(V)\big)$, with $\|L_\sigma f\| \leq \|\check{f}\|_1$, where $\|.\|_1$ is the L^1-norm on $S(V^*)$ which, as shown in the Appendix (see Equation (A2)), can be estimated in terms of a suitable Schwartz seminorm of f,

$$\|L_\sigma f\| \leq \|\check{f}\|_1 \leq (2\pi)^n \sum_{|\alpha|,|\beta| \leq 2n} \sup_{\mathrm{x}\in\mathrm{V}} \; |x^\alpha \partial_\beta f(x)|. \qquad (29)$$

To handle the case when f belongs to (V) , we need a better estimate. Fortunately, we can resort to a famous theorem from the theory of pseudo-differential operators, known as the Calderón-Vaillancourt in the version we need here, which deals with a very special symbol class (since the function space B coincides with Hörmander's symbol class space S^0_{00}) but on the other hand requires an improvement of the pertinent estimate, taken from it states that, given any totally bounded smooth function a on $V \times V^*$, setting

$$(Au)(x) = \int_{V^*} d\,\xi a(x,\xi)\check{u}(\xi)e^{i\langle \xi,x\rangle}\, for\ u \in S(V)$$

defines, by continuous linear extension, a bounded linear operator on $L^2(V)$ with operator norm

$$\|A\| \leq C \sum_{|\alpha|,|\beta|\leq n} \sup_{\mathrm{x\in V,\xi\in V^n}} |\partial_{x,\alpha}\partial_{\xi,\beta}a(x,\xi)|,$$

where C is a combinatorial constant depending only on the dimension n of V. Applying this result to the operator $L_\sigma f$ defined by Equations (19) and (20), we see that for every f in $E(V)$, $L_\sigma f$ is a bounded linear operator on $L^2(V)$ whose operator norm satisfies an estimate of the form

$$\|L_\sigma f\| \leq |P(\sigma^\vee)| \sum_{|\alpha|\leq n} \sup_{\mathrm{x\in V}} |\partial_\alpha f(x)|, \tag{30}$$

where $P(\sigma)$ is a polynomial of degree $\leq n$ in σ whose coefficients are combinatorial constants depending only on the dimension n of V.

From these results, it follows that we can define a C^* - norm on S_σ as well as on B_O- by setting

$$\|f\| = \|L_\sigma f\|. \tag{31}$$

That this is really a norm and not just a seminorm is due to the fact that the left regular representation is faithful. Namely, given $\in \mathcal{B}(V)$, $f \neq 0$, and any point x in V such that $f(x) \neq 0$, take $g \in S(V)$ such that $\check{g} \in S(V^*)$ becomes

$$\overline{\check{g}(\xi) = f\left(x + \frac{1}{2}\sigma^\#\xi\right)}\, e^{\,-i\langle \xi,x\rangle} e^{-q(\xi)},$$

where q is any positive definite quadratic form on V^*; then by Equation (19), $(f \star_\sigma g)(x)$ is equal to the L^2 - norm of the function $\xi \mapsto f\left(x + \frac{1}{2}\sigma^\#\xi\right)$ with respect to the Gaussian measure $e^{-q(\xi)}d\xi$ on V^* and hence is > 0, since this function is smooth and $\neq 0$ at $\xi = 0$, so $L_\sigma f \cdot g \neq 0$ and hence $L_\sigma f \neq 0$.

The completions of S_σ and of $E_{C\Gamma}$ with respect to this C^* - norm will be denoted by ε_σ and by $\mathcal{H}_\sigma$ ·, respectively, and will be referred to as *Heisenberg C^* - algebras*: more precisely, ε_σ. is the *nonunital Heisenberg C^* - algebra* while $\mathcal{H}_\sigma$ is the *unital Heisenberg C^* - algebra* (with respect to σ) . (We admit that using the symbol ε with this meaning may be a bit confusing because ε_σ. has nothing to do with the Schwartz space $\varepsilon(V)$ of arbitrary smooth functions on V: after all, when σ is nondegenerate, ε_σ will be isomorphic to the algebra of compact operators on the Hilbert space $L^2(V_L)$, where V_L is some lagrangian subspace of V. Still, we have decided to adopt this notation because of the connection, explained below, with the DFR - algebra, which was called ε and also because the space $\varepsilon(V)$ will play no role except for an intermediate argument in the Appendix.) Obviously, the estimate (29) and the (much better) estimate (30) imply that the natural Fréchet topologies on S_σ and on B_σ. are finer than the C^* - topologies induced by their embeddings into ε_σ and $\mathcal{H}_\sigma$, respectively. Moreover, by construction, the (faithful)$*$ - representations (23) and (24) extend to (faithful) C^* - representations of ε_σ and of $\mathcal{H}_\sigma$, respectively, for which we maintain the same notation, writing

$$L_\sigma \cdot : \varepsilon_\sigma \ \to \ B\big(L^2(V)\big) \qquad (32)$$

$$f \ \mapsto \ L_\sigma f$$

and

$$L_\sigma : \mathcal{H}_\sigma \ \to \ B\big(L^2(V)\big),$$

$$f \ \mapsto \ L_\sigma f \qquad (33)$$

respectively. It is also clear that the embedding of S_σ into B_σ (as a $*$ - ideal) extends canonically to an embedding of ε_σ. into $H_{c\tau}$ (as a $*$ - ideal) and similarly that the embedding of Equation (22) extends canonically to an embedding of ($\mathcal{H}_\sigma$ into the multiplier algebra $M(\varepsilon_\sigma)$ of ε_σ, which we shall write in a form analogous to Equation (22),

$$\mathcal{H}_\sigma \ \to \ M(\varepsilon_\sigma) f \ \mapsto \ (L_\sigma \cdot f, R_\sigma \cdot f)\,. \qquad (34)$$

The (faithful) C^* - representation of ε_σ. in Equation (32) is nondegenerate. (To explain this statement, recall that a $*$ - representation of a $*$ - algebra A by bounded operators on a Hilbert space $\mathfrak{H}$ is called nondegenerate if the subspace generated by vectors of the form $\pi(a)\psi$, where $a \in A$ and $\psi \in \mathfrak{H}$, is dense in $\mathfrak{H}$, or equivalently, if there is no nonzero vector in $\mathfrak{H}$ that is annihilated by all elements of A. Obviously, if A has a unit, every (unital)$*$ - representation of A is nondegenerate. Also, irreducible $*$ - representations and, more generally, cyclic $*$ - representations are always nondegenerate. In the situation of interest here, the statement follows easily from the existence of approximate identities in the Heisenberg - Schwartz algebra $S_\sigma \cdot$, as formulated in Proposition (6.1.18) of the Appendix: given any L^2 - function $\psi \in L^2(V)$, it suffices to approximate it in L^2 - norm by some Schwartz function $f \in S(V)$ and then approximate that, in the Schwartz space topology and hence also in L^2 - norm, by some Schwartz function of the form $X_k \star_\sigma f$, where $\mathrm{X}_k \in S(V)$.) This property of nondegeneracy is important here because it implies that the (faithful) C^* - representation of ε_σ in Equation (32) extends uniquely to a (faithful) C^* - representation

$$L_\sigma : M(\varepsilon_\sigma) \ \to \ B\big(L^2(V)\big) \qquad (35)$$

$$m \ \mapsto \ L_\sigma m$$

of the multiplier algebra $M(\varepsilon_\sigma)$ of ε_σ.: for later use, let us quickly recall how to construct this extension. Writing elements of $M(\varepsilon_\sigma)$ as pairs $m = (m_L, m_R)$, where $m_L \in L(\varepsilon_\sigma)$ is a left multiplier $(m_L(f \star_\sigma g) = m_L(f) \star_\sigma g)$ and $m_R \in L(\varepsilon_\sigma.)$ is a right multiplier $\big(m_R(f \star_\sigma g) = f \star_\sigma m_R(g)\big)$, related by the condition that $f \star_\sigma m_L(g) = m_R(f) \star_\sigma g$, and using the fact that the representation L_σ in Equation (32) is nondegenerate, which means that the subspace of $L^2(V)$ generated by vectors of the form $L_\sigma f \cdot \psi$ with $f \in \varepsilon_\sigma$ and $\psi \in L^2(V)$ (or even $\psi \in S(V)$) is dense in $L^2(V)$, the operator $L_\sigma m \in B\big(L^2(V)\big)$ is defined by

$$L_\sigma \cdot m \cdot (L_\sigma \cdot f \cdot \psi) = L_\sigma \cdot \big(m_L(f)\big) \cdot \psi. \qquad (36)$$

That this is well-defined follows from the fact that ε_σ is an essential $*$-ideal in (ε_σ) , i.e., a $*$-ideal that has nontrivial with any nontrivial $*$-ideal in $M\big(\varepsilon_{(f}\big)$. Moreover, it follows that, just like any $L_\sigma \cdot f$ (originally for $f \in S_\sigma$ but then, by continuity, also for $f \in \varepsilon_\sigma$), any L_O-m also commutes with any $R_\sigma g$ (originally for $g \in S_\sigma$ but then, by continuity, also for $g \in \varepsilon_\sigma$),

$$\begin{aligned}(L_\sigma m R_\sigma g)\cdot(L_\sigma f\cdot\psi) &= (L_\sigma m R_\sigma g)\cdot(f\star_\sigma\psi) = L_\sigma m\cdot\big((f\star_\sigma\psi)\star_\sigma g\big)\\ &= L_\sigma m\cdot\big(f\star_\sigma.(\psi\star_\sigma\cdot g)\big) = L_\sigma\cdot m\cdot\big(L_\sigma f\cdot(\psi\star_\sigma g)\big)\\ &= \big(L_\sigma\cdot\big(m_L(f)\big)R_\sigma\cdot g\big)\cdot\psi = \Big(R_\sigma\cdot gL_\sigma\cdot\big(m_L(f)\big)\Big)\cdot\psi\\ &= (R_\sigma - gL_\sigma m)\cdot(L_\sigma f\cdot\psi)\,.\end{aligned}$$

Finally, we see that with this construction, the representation (33) becomes simply the composition of the representation (35) with the embedding (34).

Having settled the question of existence of a C^*-norm on the Fréchet∗-algebras S_σ and B_σ, we want to address the question of its uniqueness. We follow the script laid which turns out to work perfectly for the Heisenberg-Schwartz and Heisenberg-Rieffel algebras. The first step will be to prove the following fact.

Theorem (6.1.3)[270]: *The Heisenberg-Schwartz and Heisenberg-Rieffel algebras, S_σ and B_σ, are spectrally invariant in their respective C^*-completions, ε_σ* and $\mathcal{H}_\sigma$, *as defined above. Therefore, ε_σ and $\mathcal{H}_\sigma$ are the universal enveloping C^*-algebras of the Heisenberg-Schwartz algebra S_σ and of the Heisenberg-Rieffel algebra E_σ, respectively.*

The assertion of Theorem (6.1.3) is known to hold in the commutative case, i.e., when $\sigma = 0$ [14] and also when σ is nondegenerate [13], but for the general deformed algebras, it does not seem to have been stated explicitly anywhere in the literature: in what follows, we shall give a different and direct proof in which the rank of σ. plays no role.

Proof. The proof will be based on the main theorem which can be formulated as follows. To begin with, let Ω denote the standard symplectic form on the doubled space $V \oplus V^*$, defined by

$$\Omega\,((x,\xi),(y,\eta) \;=\; \xi(y) \;-\; \eta(x) \tag{37}$$

let H_Ω denote the corresponding Heisenberg group (which has nothing to do with the Heisenberg group H_σ considered before), and consider the corresponding strongly continuous unitary representation

$$W_\Omega : H_\Omega \to U\big(L^2(V)\big) \tag{38}$$

of H_Ω on $L^2(V)$, explicitly given by

$$(W_\Omega(x,\xi,\lambda)\psi)(z) = e^{-i\langle\xi,z-\frac{1}{2}x\rangle+i\lambda}\psi(z-x)\,. \tag{39}$$

Next, consider the continuous isometric representation

$$\mathrm{Ad}(W_\Omega) : H_\Omega \to \mathrm{Aut}\Big(B\big(L^2(V)\big)\Big) \tag{40}$$

of H_Ω on $B\big(L^2(V)\big)$ obtained from it by taking the adjoint action (i.e., for $T \in B\big(L^2(V)\big)$, we have $\mathrm{Ad}(W_\Omega)(h)T = W_\Omega(h)TW_\Omega(h)^{-1}$). Then given an operator $\in B\big(L^2(V)\big)$, we say that it is *Heisenberg-smooth* if it is a smooth vector with respect to this representation, i.e., if the function

$$\begin{aligned} H_\Omega \;&\to\; B\big(L^2(V)\big)\\ (x,\xi,\lambda) \;&\mapsto\; W_\Omega(x,\xi,\lambda)TW_\Omega(x,\xi,\lambda)^{-1}\end{aligned}$$

is smooth. Now the main theorem states that an operator $T \in B(L^2(V))$ is of the form $L_\sigma f$ (see Equations (19), (20), and (24)), with $f \in B_\sigma$, if and only if it is Heisenberg-smooth and commutes with all operators of the form $R_{C\Gamma} g$ (see Equations (18), (21), and (26)), where $g \in B_\sigma$. (or equivalently, $g \in S_\sigma$). This fact, applied in both directions, will enable us to complete the proof, as follows.

Suppose first that $f \in \mathcal{B}_\sigma$. is invertible in $\mathcal{H}_\sigma$. Then the operator $L_\sigma \cdot f \in B(L^2(V))$ is Heisenberg smooth and commutes with all operators of the form $R_\sigma \cdot g$, where $g \in S_\sigma$. But this implies that the inverse operator $(L_\sigma f)^{-1} \in B(L^2(V))$ is also Heisenberg-smooth, since

$$W_\Omega(x,\xi,\mu)(L_\sigma f)^{-1} W_\Omega(x,\xi,\mu)^{-1} = (W_\Omega(x,\xi,\mu) L_o\text{-} f W_\Omega(x,\xi,\mu)^{-1})^{-1}$$

and since inversion of bounded linear operators is a smooth map, and that it also commutes with all operators of the form $R_\sigma \cdot g$, where $g \in S_\sigma \cdot$. Thus it follows that $(L_\sigma f)^{-1}$ is of the form $L_\sigma g$ for some $g \in B_\sigma$, showing that B_σ is spectrally invariant in $\mathcal{H}_\sigma$. To prove that, similarly, S_σ is spectrally invariant in ε_σ, consider the unitizations $\tilde{S}_\sigma$ of S_σ (still contained in B_σ) and $\tilde{\varepsilon}_\sigma$ of ε_σ (still contained in $\mathcal{H}_\sigma$), and suppose $f \in S_\sigma$. to be such that $\lambda 1 + f \in \tilde{S}_\sigma$ is invertible in $\tilde{\varepsilon}_\sigma$ (note that this implies $\lambda \neq 0$). Then, as we have already shown, $(\lambda 1 + L_\sigma f)^{-1}$ is of the form $L_\sigma h$ for some $h \in B_\sigma \cdot$, which we can rewrite in the form $h = \lambda^{-1} 1 + g$ with $g \in B_\sigma \cdot$, implying

$$1 = (\lambda 1 + f) \star_\sigma (\lambda^{-1} 1 + g) = 1 + \lambda^{-1} f + \lambda g + f \star_\sigma \cdot g$$
$$g = -\lambda^{-2} f - \lambda^{-1} f \star_\sigma \cdot g.$$

But S_σ is an ideal in $B_\sigma \cdot$, so it follows that $g \in S_\sigma$ and hence $\lambda^{-1} 1 + g \in \tilde{S}_\sigma$.
and thus
The same techniques can be used to prove the following interesting and useful theorem about the relation between ε_{CT} and $\mathcal{H}_\sigma$.

Theorem (6.1.4)[270]: *The C^*-algebra $\mathcal{H}_\sigma$. is the multiplier algebra of the C^*-algebra ε_σ,*

$$\mathcal{H}_\sigma = M(\mathcal{E}_\sigma)\,, \tag{41}$$

and in fact it is a von Neumann algebra.

Proof. What needs to be shown is that the embedding (34) is in fact an isomorphism. To this end, let R be the subspace of $B(L^2(V))$ consisting of right translations by elements of S_σ,

$$R = \{R_\sigma(g) | g \in S_\sigma\}.$$

What will be of interest here is its commutant R', which is a closed subspace (and in fact even a von Neumann subalgebra) of $B(L^2(V))$. As has been shown at the end B, the representation (35) maps $M(\varepsilon_O\text{-})$ into R'. On the other hand, the relation

$$W_\Omega(x,\xi,\lambda) R_\sigma(g) W_\Omega(x,\xi,\lambda)^{-1} = R_\sigma\left(W_\Omega\left(x + \frac{1}{2}\sigma^\# \xi, 0,0\right) g\right) \tag{42}$$

shows that R is an invariant subspace for the representation $\mathrm{Ad}(W_\Omega)$ of H_Ω on $B(L^2(V))$ (see Equation (40)); hence so is R'. Therefore, the main theorem can be reformulated as the statement that the image of B_σ under the representation (24) is precisely the subspace of smooth vectors for the representation $\mathrm{Ad}(W_\Omega)$ of H_Ω on R' obtained by restriction, and hence it is dense in R'. It follows that the image of $cH_o\text{-}$ under the representation (33) is precisely R', a von Neumann algebra.

For the second step, we use a result that is of independent interest, namely, the fact that, as shown in [31], the Heisenberg C^*-algebra ε_σ. is isomorphic to the algebra of continuous functions, vanishing at infinity, on a certain subspace V_0 of V (dual to kerσ) and taking values in the algebra $\mathcal{K}$ of compact linear operators in a separable Hilbert space, which can also be written as a C^*- algebra tensor product,

$$\varepsilon_\sigma \cong C_0(V_0, \mathcal{K}) \cong C_0(V_0) \otimes \mathcal{K}. \tag{43}$$

To see this explicitly, we first recall the "musical homomorphism" ($\sigma^{\#} : V^* \to V$ induced by a. (i.e., $\langle \xi, \sigma^{\#}\eta \rangle = \sigma \cdot (\eta, \xi)$) whose image is a subspace of V that we shall denote by W: it is precisely the annihilator of the kernel of (σ in V^*,

$$W = \mathrm{im}\sigma^{\#} = (\ker\sigma)^{\perp}, \tag{44}$$

and it carries a symplectic form, denoted by oj and defined by $(v((\sigma^{\#}\xi, (\sigma^{\#}\eta) = \sigma(\xi, \eta)$. Now choosing a subspace V_0 of V complementary to W, we get a direct decomposition

$$V = V_0 \oplus W. \tag{45}$$

Taking the corresponding annihilators, we also get a direct decomposition for the dual,

$$V^* = V_0^* \oplus W^*, where\ V_0^* = W^{\perp} = \ker\sigma\ and\ W^* = V_0^{\perp}. \tag{46}$$

Of course, W^* also carries a symplectic form, again denoted by w, which is simply the resmction of σ to this subspace, on which it is nondegenerate. Now according to the Schwartz nuclear theorem, we have

$$S(V) \cong S(V_0) \otimes S(W)\,, \tag{47}$$

and similarly

$$S(V^*) \cong S(V_0^*) \otimes S(W^*)\,, \tag{48}$$

and it is clear that the Fourier transform $\mathcal{F} : S(V) \to S(V^*)$ is the tensor product of the Fourier transforms $\mathcal{F} : S(V_0) \to S(V_0^*)$ and $\mathcal{F} : S(W) \to S(W^*)$. Hence looking at the definition of the Weyl-Moyal star product, we see that the tensor products in Equations (47) and (48) are in fact tensor products of algebras, i. e.,

$$S_\sigma \cong S_0 \otimes S_w, \tag{49}$$

where S_0 is the commutative algebra of Schwartz test functions ($S(V_0)$ with the ordinary pointwise product or $S(V_0^*)$ with the ordinary convolution product) while S_w is the Heisenberg-Schwartz algebra associated with the nondegenerate 2-form w. Taking the universal C^*-completions, we get

$$\varepsilon_\sigma \cong \varepsilon_0 \otimes \varepsilon_w. \tag{50}$$

But obviously, $\varepsilon_0 \cong C_0(V_0) \cong C_0(V_0^*)$, and it is well known that $\varepsilon_w \cong \mathcal{K}$.

In passing, we note that the tensor product in Equations (43) and (50) is the tensor product of C^*-algebras and as such is unique (there is only one C^*-norm on the algebraic tensor product) since one of the factors is nuclear (in fact, both of them are; see [25]).

To complete the argument, we make use of the fact that any ideal in ε_σ is of the form

$$\{(\ell \in C_0(V_0, \mathcal{K}) | \varphi|_F = 0\},$$

or equivalently,

$$\{f \in C_0(V_0) | f|_F = 0\} \otimes \mathcal{K},$$

where F is a closed subset of the space V_0. (That these are in fact all ideals in ε_σ is a special case of a much more general statement, whose formulation and proof can be found in [279], together with the fact that $\mathcal{K}$ is simple.) But obviously, each of these ideals has nontrivial with the Heisenberg-Schwartz algebra.

Finally, we can extend the conclusion from ε_σ. to $'\mathcal{H}_\sigma$: since the latter is the multiplier algebra of the former, any nontrivial ideal of $\mathcal{H}_\sigma$ intersects ε_σ in a nontrivial ideal of ε_σ, which in turn has nontrivial with S_σ. and hence with B_σ ··
Summarizing, we have proved

Theorem (6.1.5)[270]: *The Heisenberg-Schwartz algebra S_σ and the Heisenberg-Rieffel algebra B_σ each admit one and only one C^*-norm, and hence the Heisenberg C^*-algebras ε_σ.* and$\mathcal{H}_\sigma$. *are their unique C^*-completions*.

Returning to the situation discussed at the beginning assume we are given any strongly continuous unitary representation π of the Heisenberg group H_σ. Then Weyl quantization produces a $*$-representation W_π of the Heisenberg-Schwartz algebra S_σ, defined according to Equations (7) and (8), which according to Equation (11) is continuous with respect to the Schwartz topology. But in fact it is also continuous with respect to the C^*-topology since that is defined by the maximal C^*-norm on S_σ which is an upper bound for all C^*-seminorms on S_σ, including the operator seminorm for W_π, and therefore W_π extends uniquely to a C^*-representation of the nonunital Heisenberg C^*-algebra ε_σ which will again be denoted by W_π. Moreover, we have

Lemma (6.1.6)[270]: *Given any strongly continuous unitary representation π of the Heisenberg group H_σ, the resulting $*$-representation W_π of the Heisenberg-Schwartz algebra S_σ, and hence also of the Heisenberg C^*-algebra ε_σ, is nondegenerate*.

Proof. Given any vector ψ in the Hilbert space $\mathfrak{H}$ of the representations π and W_π and any $\varepsilon > 0$, strong continuity of π implies the existence of an open neighborhood U^* of 0 in V^* such that

$$\|\pi(\xi)\psi - \psi\| < \varepsilon \; for \; \xi \in U^*,$$

since $\pi(0) = 1$. Now choose $f \in S(V)$ such that $\check{f} \in S(V^*)$ is nonnegative, with integral normalized to 1, and has compact support contained in U^*. Then

$$\|(W_\pi f)\psi - \psi\| = \| \int_{V^*} d\,\xi \check{f}(\xi)\pi(\xi)\psi - \psi\| \le \int_{V^*} d\,\xi \check{f}(\xi)\|\pi(\xi)\psi - \psi\| < \varepsilon.$$

As a result, these $*$-representations extend to (unital) $*$-representations of the Heisenberg-Rieffel algebra B_σ and of the unital Heisenberg C^*-algebra $\mathcal{H}_\sigma$, respectively, which will again be denoted by W_π.

Conversely, given any nondegenerate C^*-representation W of ε_σ, we can extend it uniquely to a (unital) C^*-representation of H_σ ·, again denoted by W, which restricts to a unitary representation π_W of H_σ defined according to

$$\pi_W(\xi) = W\left(e_\xi\right), \tag{51}$$

where $e_\xi \in B_\sigma$ denotes the phase function given by $e_\xi(u) = e^{i\langle \xi, v\rangle}$. To show that π_W is automatically strongly continuous, we note that, according to Equations (9) and (19), we have, for any $f \in S_\sigma$ ·,

$$\left(e_\xi \star_\sigma f\right)(x) = e^{i\langle \xi, x\rangle} f\left(x - \frac{1}{2}\sigma^{\#}\xi\right),$$

so $e_\xi \star_\sigma f$ converges to f as ξ tends to zero, in the Schwartz topology and hence also in the C^*-topology. Now since W is supposed to be nondegenerate and S_σ is dense in ε_σ, every vector in $\mathfrak{H}_W$ can be approximated by vectors of the form $W(f)\psi$, where $f \in S_\sigma$ and $\psi \in$

$\mathfrak{H}_W$. But on such vectors, we have strong continuity, since for any $f \in S_\sigma$ and any $\psi \in \mathfrak{H}_W$, $\pi_W(\xi)W(f)\psi = W(e_\xi \star_\sigma f)\psi$ tends to $W(f)\psi$ as ξ tends to zero.
Finally, it is easy to see that composing the two operations of passing (a) from a strongly continuous unitary representation π of H_σ. to a nondegenerate C^*-representation W_π of ε_σ and (b) from a nondegenerate C^*-representation W of ε_σ to a strongly continuous unitary representation π_W of H_σ, in any order, reproduces the original representation, so we have proved.

Theorem (6.1.7)[270]: *There is a bijective correspondence between the strongly continuous unitary representations of the Heisenberg group H_σ and the nondegenerate C^*-representations of the nonunital Heisenberg C^*-algebra ε_σ. Moreover, this correspondence takes irreducible representations to irreducible representations*.

As a corollary, we can state a classification theorem for irreducible representations which is based on one of von Neumann's famous theorems, according to which there is a *unique* such representation, generally known as the *Schrödinger representation of the canonical commutation relations*, provided that σ is nondegenerate. To handle the degenerate case, i.e., when σ has a nontrivial null space, denoted by ker σ, we use the same trick as above: choose a subspace W^* of V^* complementary to kerσ (see Equation (46)), so that the restriction w of σ to $W^* \times W^*$ is nondegenerate, and introduce the corresponding Heisenberg algebra $\mathfrak{h}_{\mathrm{w}} = W^* \oplus \mathbb{R}$ and Heisenberg group $H_{\mathrm{w}} = W^* \times \mathbb{R}$ to decompose the original ones into the direct sum $\mathfrak{h}_\sigma = \ker\sigma \oplus \mathfrak{h}_w$ of two commuting ideals and $H_\sigma = \ker\sigma \times H_{\mathrm{w}}$ of two commuting normal subgroups. (As is common practice in the abelian case, we consider the same vector space kerσ as an abelian Lie algebra in the first case and as an additively written abelian Lie group in the second case, so that the exponential map becomes the identity.) It follows that every (strongly continuous unitary) representation of H_w is the tensor product of a (strongly continuous unitary) representation of kerσ and a (strongly continuous unitary) representation of H_w, where the first is irreducible if and only if each of the last two is irreducible. Now since kerσ is abelian, its irreducible representations are one-dimensional and given by their character, which proves the following.

Theorem (6.1.8)[270]: *With the notation above, the strongly continuous, unitary, irreducible representations of the Heisenberg group H_σ, or equivalently, the irreducible representations of the nonunital Heisenberg C^*-algebra ε_σ., are classified by their highest weight, which is a vector u in V, or more precisely, its class $[u]$ in the quotient space $V/(\ker\sigma)^\perp$, such that*

$$\pi_{[v]}(\xi,\eta) = e^{i\langle\xi,v\rangle}\pi_w(\eta) for\ \xi \in \ker\sigma, \eta \in H_w,$$

where π_w is of course the Schrödinger representation of H_ω.

We point out that the correspondence of Theorem (6.1.7) does *not* hold when we replace ε_σ. by $\mathcal{H}_\sigma$, simply because $\mathcal{H}_\sigma$ admits C^*-representations whose restriction to ε_σ. is trivial: just consider any representation of the corona algebra $\mathcal{H}_\sigma \cdot/\varepsilon_\sigma$.. That is why it is important to consider not only $\mathcal{H}_\sigma$ but also ε_σ.

We would like to comment on the difference between our definition of the Heisenberg C^*-algebra and others that can be found - more specifically, the Weyl algebra $\Delta(V^*, (r)$ of Refs

22 and 23 and the resolvent algebra $\mathcal{R}(V^*, \sigma \cdot)$ these are defined as the universal enveloping C^*-algebras of the $*$-algebra $\Delta(V^*, \sigma \cdot)$ generated by the phase functions e_ξ and of the $*$-algebra $\mathcal{R}_0(V^*, \sigma)$ generated by the resolvent functions R_ξ, respectively, where $e_\xi(u) = e^{i\langle \xi u \rangle}$, as before, and similarly, $R_\xi(v) = (i - \langle \xi, u \rangle)^{-1}$.
The main problem with these constructions is that the resulting C^*-algebras are, in a certain sense, "too small as indicated by the fact that they accommodate lots of "purely algebraic" representations and one has to restrict to a suitable class of "regular" representations in order to establish a bijective correspondence with the usual representations of the CCRs: nonregular representations do not even allow to define the "infinitesimal" operators that would be candidates for satisfying the CCRs. Moreover, the choice of the respective generating $*$-subalgebras $\Delta(V^*, \sigma)$ and $\mathcal{R}_0(V^*, \sigma)$ is to a certain extent arbitrary, and even though they admit maximal C^*-norms, they do *not* in general admit a *unique* C^*-norm. What is remarkable about the extensions proposed here, using the larger C^*- algebras ε_σ. or H_σ, together with the larger generating $*$-subalgebras S_σ. or B_σ, is that this procedure eliminates the unwanted representations (whose inclusion would invalidate the analogue of Theorem (6.1.8) classifying the irreducible representations) as well as the ambiguity in the choice of C^*-norm.
On the other hand, it must be emphasized that our approach is restricted to the case of finite-dimensional Poisson vector spaces (quantum mechanics): the question of whether, and how, it is possible to extend it to infinite-dimensional situations (quantum field theory) is presently completely open.

we want to introduce concepts that will allow us to extend the process of C^*-completion of $*$-algebras discussed to bundles of $*$-algebras.
Assume that $(V_X)_{x \in X}$ is a family of sets indexed by the points x of some other set X. Then we may introduce the set V defined as their disjoint union,

$$V = \dot{\bigcup_{x \in X}} V_{x,} \tag{52}$$

together with the surjective map $p: V \to X$ that takes V_X to x: this defines a "*bundle* " with *total space* V, *base space* X, and *projection* p, with $V_x = p^{-1}(x)$ as the *fiber over* the point x. The question is what additional conditions should be imposed on this kind of structure in order to allow us to remove the quotation marks on the expression "bundle." For example, in of topology, it is usually required that both V and X should be topological spaces and that p should be continuous and open. Similarly, of differential geometry, one requires that, in addition, both V and X should be manifolds and that p should be a submersion. Of course, special care must be taken when these manifolds are infinite-dimensional, since dealing with these is a rather touchy business; in particular, the standard theory that works in of Banach spaces and manifolds, for which we may refer to does not apply to more generally locally convex spaces and manifolds, for which one must resort to more sophisticated techniques such as the"convenient calculus" developed.
A central role is played by the condition of *local triviality*, which requires the existence of a fixed topological space or of a fixed manifold V_0, called the *typical fiber*, and of some covering of the base space by open subsets such that for each one of them, say U, the subset

$p^{-1}(U)$ of the total space is homeomorphic (in the case of topological spaces) or diffeomorphic (in the case of manifolds) to the cartesian product $U \times V_0$: in this case, one says that V is a *fiber bundle* over X and refers to the afore-mentioned homeomorphisms or diffeomorphisms as *local trivializations*. When V_0 and each of the fibers $V_X(x \in X)$ come with a certain (fixed) type of additional structure and local trivializations can be found which preserve that structure, an appropriate reference is incorporated into the terminology: for example, one says that V is a *vector bundle* over X when V_0 and each of the fibers $V_x(x \in X)$ are vector spaces and local trivializations can be chosen to be fiberwise linear. Thus the standard terminology used in topology and differential geometry suggests that fiber bundles, vector bundles, etc. - and in particular, C^*-algebra bundles- should be locally trivial.

Definition (6.1.9)[270]: *Abundle of locallyconvex∗-algebras over a locally compact topological space X is a topological space $\mathcal{A}$ together with a surjective continuous and open map $p: \mathcal{A} \to X$, equipped with the following structures*: (*a*) *operations of fiberwise addition, scalar multiplication, multiplication and involution that turn each fiber $\mathcal{A}_x = p^{-1}(x)$ into a ∗-algebra and are such that the corresponding maps*

$$\mathcal{A} \times_x \mathcal{A} \to \mathcal{A}\mathbb{C} \times \mathcal{A} \to \mathcal{A}$$
$$(a_1, a_2) \mapsto a_1 + a_2 \ (\lambda, a) \mapsto \lambda a$$

and

$$\mathcal{A} \times_x \mathcal{A} \to \mathcal{A}\mathcal{A} \to \mathcal{A}$$
$$(a_1, a_2) \mapsto a_1 a_2 \ a \mapsto a^*$$

where $\mathcal{A} \times_X \mathcal{A} = \{(a_1, a_2) \in \mathcal{A} \times \mathcal{A} | p(a_1) = p(a_2)\}$ is the fiber product of $\mathcal{A}$with itself over X, are continuous, and (*b*) *a directed set Σ of nonnegative functions $s : \mathcal{A} \to \mathbb{R}$ which, at every point x in X, provides a directed set $\Sigma_x = \{s|_{\mathcal{A}_x} | s \in \Sigma\}$ of seminorms on the fiber $\mathcal{A}_x = p^{-1}(x)$ turning it into a locally convex ∗-algebra*; *we shall refer to thefunctions s in Σ as fiber seminorms on $\mathcal{A}$. Moreover, when each of these fiber seminorms is either continuous or else just upper semicontinuous, and when taken together they satisfy the additional continuity condition that any net $(a_i)_{i \in I}$ in $\mathcal{A}$such that $s(a_j) \to 0$ for every $s \in \Sigma$ and $p(a_i) \to x$ for some $x \in X$ actually converges to $0_x \in \mathcal{A}_x$, then we say that $\mathcal{A}$is either a continuous or else an upper semicontinuous bundle of locally convex ∗-algebras, respectively. Finally, we shall say that such a bundle $\mathcal{A}$is unital if all of its fibers $\mathcal{A}_x$* are ∗-*algebras with unit and, in addition,*

$$X \to \mathcal{A}$$
$$x \mapsto 1_x$$

is continuous. Special cases are

· *$\mathcal{A}$ is a bundle of Fréchet ∗-algebras if Σ is countable and each fiber is complete in the induced topology*: *in this case, Σ can* (*and will*) *be arranged in the form of an increasing sequence*.

· *$\mathcal{A}$is a bundle of Banach ∗-algebras if Σ is finite and each fiber is complete in the induced topology*: *in this case, Σ can* (*and will*) *be replaced by a singlefunction $\|.\| : \mathcal{A} \to \mathbb{R}$, called the fiber norm, which induces a Banach ∗-algebra norm on each fiber*.

· *$\mathcal{A}$is a bundle of C^*-algebras, or simply C^*-bundle, if it is a bundle of Banach ∗-algebras whose fiber norm induces a C^*-norm on each fiber*.

We remark that, in this definition, the condition on the index set Σ to be directed refers to the natural order on the set of all nonnegative functions on $\mathcal{A}$, defined pointwise. Also, a simple generalization of an argument that can be found in [33] shows that it is sufficient to require that scalar multiplication should be continuous in the second variable, i.e., for each $\lambda \in \mathbb{C}$, the map $\mathcal{A} \to \mathcal{A}, a \to \lambda a$ is continuous: this condition is often easier to check in practice, but it already impliesjoint continuity.
It may be worthwhile to stress that, according to the convention adopted, bundles of $*$-algebras over X need not be locally trivial and hence the property of local triviality- either in the sense of topology when X is a topological space (continuous transition functions) or in the sense of differential geometry when X is a manifold (smooth transition functions)- will have to be stated explicitly when it is satisfied and relevant.
We note that a first version of this definition was formulated by Dixmier, through his notion of a "continuous field of C^*-algebras Somewhat later, Fell introduced the concept of a continuous C^*-bundle (see, e.g., [279]), providing an (ultimately) equivalent but intuitively more appealing approach. Finally, it was observed that most of the important results continue to hold with almost no changes for upper semicontinuous C^*-bundles, the main difference being that in this case, the total space $\mathcal{A}$ may fail to be Hausdorff. The extension proposed here, to bundles whose fibers are more general locally convex $*$-algebras (of various types), seems natural and will be useful for what follows.
The additional continuity condition formulated in the above definition guarantees that the topology on the total space $\mathcal{A}$is uniquely determined by the set of fiber seminorms Σ; this follows directly from the following generalization of a theorem of Fell.
Theorem (6.1.10)[270]: *Assume that $(\mathcal{A}_X)_{x\in X}$ is a family of $*$-algebras indexed by the points x of a locally compact topological space X, and consider the disjoint union*

$$\mathcal{A} = \dot{\bigcup_{x\in X}} \mathcal{A}_X \tag{53}$$

as a "bundle" over X (in the purely set-theoretical sense). Assumefurther that Σ is a directed set of fiber seminorms on $\mathcal{A}$[turning each fiber $\mathcal{A}_X$ of $\mathcal{A}$into a locally convex $*$-algebra (Fréchet
$*$-algebra/ Banach $*-algebra/C^*$-algebra) and that Γ is a $*$-algebra of this "bundle," satisfying the following properties.
(a) For each $\phi \in \Gamma$ and each fiber seminorm $s \in \Sigma$, thefunction $X \to \mathbb{R}_c$, $x \mapsto s\big(\phi(x)\big)$ is upper semicontinuous (or continuous).
(b) For each point x in X, the $*$-subalgebra $\Gamma_x = \{\phi(x)|\phi \in \Gamma\}$ of $\mathcal{A}_x$ is dense in $\mathcal{A}_X$.
Then there is a unique topology on $\mathcal{A}$ turning it into an upper semicontinuous (or continuous) bundle of locally convex $*$-algebras (Fréchet$*$-algebras/Banach$*$-algebras/$C*$-algebras) over X, respectively, such that Γ becomes a $*$-subalgebra of the $*$-algebra $\Gamma(X, \mathcal{A})$ of all continuous of $\mathcal{A}$[.
Similar statements can be found, e.g., in (for continuous bundles of Banach spaces) and in (for upper semicontinuous bundles of C^*-algebras), but the proof is easily adapted to the more general situation considered here; in particular, a basis of the desired topology on $\mathcal{A}$is given by the subsets

$$W(\phi, U, s, \varepsilon) = \{a \in \mathcal{A} | p(a) \in U, s\left(a - \phi\big(p(a)\big)\right) < \varepsilon\},$$

where $p: \mathcal{A} \to X$ is the bundle projection, $\phi \in \Gamma$, U is an open subset of X, $s \in \Sigma$ and $\varepsilon > 0$. Whatever may be the specific class of bundles considered, the notion of morphism between them is the natural one.

Definition (6.1.11)[270]: *Given two bundles of locally convex* $*$*-algebras* $\mathcal{A}$*and* $\mathcal{B}$ *over locally compact topological spaces* X *and* Y*, respectively, a bundle morphism from* $\mathcal{A}$*to* $\mathcal{B}$ *is a continuous map* $\varphi : \mathcal{A} \to \mathcal{B}$ *which is fiber preserving in the sense that there exists a* (*necessarily unique*) *continuous map* $\check{\ell} : X \to$ y *such that the following diagram commutes,*

$$\mathcal{A} \to^{\varphi} B$$
$$\mathcal{A} \downarrow \check{\varphi} \downarrow p_{\mathcal{B}}$$
$$X \to \mathrm{Y}$$

and such that for every point x *in* X*, the restriction* ($\ell_x : \mathcal{A}_X \to E_{(\check{\beta}(x)}$ *of* φ *to the fiber over* x *is a homomorphism of locally convex* $*$*-algebras. When* $\mathrm{Y} = X$ *and* $\check{\varphi}$ *is the identity, we say that* φ *is*
strict(*over* X).

Theorem (6.1.10) above already makes it clear that an important object associated with any upper semicontinuous bundle $\mathcal{A}$of locally convex $*$-algebras over X is the space $\Gamma(X, \mathcal{A})$ of all continuous of $\mathcal{A}$ which, when equipped with the usual pointwise defined operations of addition, scalar multiplication, multiplication and involution, is easily seen to become a $*$-algebra. Moreover, given a directed set Σ of fiber seminorms s on ffl that generates its topology, as explained above, we obtain a directed set of seminorms $\|.\|_{s,K}$ on $\Gamma(X, \mathcal{A})$ by taking the usual sup seminorms over compact subsets K of X,

$$\|\phi\|_{s,K} = \sup_{x \in K} \; s\big(\phi(x)\big) for \; \phi \in \Gamma(X, \mathcal{A}) \,, \qquad (54)$$

turning $\Gamma(X, \mathcal{A})$ into a locally convex $*$-algebra with respect to what we may continue to call the topology of uniform convergence on compact subsets. Over and above that, $\Gamma(X, \mathcal{A})$ carries two important additional structures. The first is that $\Gamma(X, \mathcal{A})$ is a *module* over the locally convex $*$-algebra $C(X)$ of continuous functions on X, as expressed by the compatibility conditions

$$\begin{gathered} f(\phi_1\phi_2) = (f\phi_1)\phi_2 = \phi_1(f\phi_2)\,, (f\phi)^* = \overline{f}\phi^*, \\ \|f\phi\|_{s,K} \leq \|f\|_K \|\phi\|_{s,K} \\ for\; f \in C(X), \phi, \phi_1, \phi_2 \in \Gamma(X, \mathcal{A})\,. \end{gathered} \qquad (55)$$

The second additional structure is that $\Gamma(X, \mathcal{A})$ comes equipped with a family $(\delta_x)_{x \in X}$, indexed by the points x of the base space X, of continuous homomorphisms of locally convex $*$-algebras, the evaluation maps

$$\to \; \mathcal{A}\mathrm{t}_x \; \mapsto \; \phi(x)$$
$$\delta_x : \Gamma(X, v) \; \to \; \mathcal{A}_x \qquad (56)$$
$$\phi \; \mapsto \; \phi(x)$$

Obviously, when X is compact, we can omit the reference to compact subsets since then $C(X)$ comes with the natural sup norm while every fiber seminorm s on $\mathcal{A}$will generate a seminorm $\|.\|_s$ on $\Gamma(X, \mathcal{A})$ by taking the sup over all of X; the resulting topology is simply that of uniform convergence on all of X. On the other hand, when X is locally compact but not compact, the situation is a bit more complicated since we have to worry about the behavior of functions at infinity. One way to deal with this issue consists in restricting to the algebras $C_0(X)$ of continuous functions on X and $\Gamma_0(X,\mathcal{A})$ of continuous sections of $\mathcal{A}$that vanish at infinity (in the usual sense that $f \in C(X)$ belongs to $C_0(X)$ and $\phi \in \Gamma(X, \mathcal{A})$ belongs to $\Gamma_0(X,\mathcal{A})$ if for each $\varepsilon > 0$ and, in the second case, each $s \in \Sigma$, there exists a compact subset K of X such that $|f(x)| < \varepsilon$ and $s(\phi(x)) < \varepsilon$ whenever $x \notin K$): as in the compact case, these are locally convex $*$-algebras with respect to the topology of uniform convergence on all of X and the latter is a module over the former, with the same compatibility conditions and the same evaluation maps as before (see Equations (55) and (56)). Moreover, we have a condition of nondegeneracy, which is necessary since we are now dealing with nonum tal $*$-algebras: it states that the $*$-ideal generated by elements of the form $f\phi$, with $f \in C_0(X)$ and $\phi \in \Gamma_0(X,\mathcal{A})$, should be the entire algebra $\Gamma_0(X,\mathcal{A})$. (this condition can equally well be formulated in the compact case but is then trivially satisfied since the condition of vanishing at infinity is then void and so we can identify $C_0(X)$ with $C(X)$, which has a unit, and $\Gamma_0(X,\mathcal{A})$ with $\Gamma(X,\mathcal{A})$.) An alternative choice would be to consider the (larger) algebras $C_b(X)$ of bounded continuous functions on X and $\Gamma_b(X,\mathcal{A})$ of bounded continuous sections of $\mathcal{A}$ (in the obvious sense that $\phi \in \Gamma(X,\mathcal{A})$ is bounded if and only if its composition with each fiber seminorm $s \in \Sigma$ is bounded), again with the topology of uniform convergence on all of X, which has the advantage that $C_b(X)$ is unital. In fact, both $C_0(X)$ and $C_b(X)$ are C^*-algebras, and the latter is the multiplier algebra of the former,

$$C_b(X) = M(C_0(X))\,. \tag{57}$$

All these constructions of section algebras become particularly useful when we start out from an upper semicontinuous C^*-bundle $\mathcal{A}$over X. In that case, $\Gamma_0(X,\mathcal{A})$ and $\Gamma_b(X,\mathcal{A})$ will both be C^*-algebras (which coincide among themselves and with $\Gamma(X,\mathcal{A})$ when X is compact), and the aforementioned structure of $\Gamma_0(X,\mathcal{A})$ as a $C_0(X)$-module can be reintetpreted as providing a C^*-algebra homomorphism $\Phi : C_0(X) \to Z\left(M(\Gamma_0(X,\mathcal{A}))\right)$, where $M(\Gamma_0(X,\mathcal{A}))$ is the multiplier algebra of $\Gamma_0(X,\mathcal{A})$ and $Z\left(M(\Gamma_0(X,\mathcal{A}))\right)$ its center. algebra $\Gamma_0(X,\mathcal{A})$ is a $C_0(X)$-algebra in the sense of Kasparov.

Definition (6.1.12)[270]: *Given a locally compact topological space X, a $C_0(X)$-algebra is a C^*-algebra A equipped with a C^*-algebra homomorphism*

$$\Phi: C_0(X) \to Z(M(A)) \tag{58}$$

which is nondegenerate, i.e., such that the $*$-ideal generated by elements of the form fa, with $f \in C_0(X)$ and $a \in A$, is the entire algebra A. (*We shall simply write fa, instead of $\Phi(f)(a)$, whenever convenient*.)

Note that the nondegeneracy condition imposed in Definition (6.1.12) above means that Φ extends uniquely to a C^*-algebra homomorphism

$$\Phi: C_b(X) \to Z(M(A)) \tag{59}$$

i.e., $C_0(X)$-algebras are automatically also $C_b(X)$-algebras. However, not every $C_b(X)$-algebra is also a $C_0(X)$-algebra, since the nondegeneracy condition may fail: an obvious example is provided by $C_b(X)$ itself, which is trivially a module over $C_b(X)$ itself and hence also over $C_0(X)$ but, as such, is degenerate; in fact, in this case the $*$-ideal mentioned in

Definition (6.1.12) above is $C_0(X)$ and not all of $C_b(X)$. At any rate, of the present the extension of the module structure from $C_0(X)$ to $C_b(X)$ will not play any significant role.
The notion of a $C_0(X)$-algebra homomorphism is, once again, the natural one: it is a $*$-algebra homomorphism which is also a homomorphism of $C_0(X)$-modules.
With these concepts at our disposal, we can now think of the process of passing from bundles to their section algebras as *afunctor*. The version of interest here is the following: given any locally compact topological space X, we have a corresponding *section algebra functor*

$$\Gamma_0(X,\cdot): \mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(X) \to \mathrm{C}_0(X)\mathrm{Alg} \quad (60)$$

from the category $\mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(X)$ of upper semicontinuous C^*-bundles over X, whose morphisms are the strict bundle morphisms over X, to the category $\mathrm{C}_0(X)\mathrm{Alg}$ of $C_0(X)$-algebras, whose morphisms are the $C_0(X)$-algebra homomorphisms. Indeed, it is clear that given any strict bundle morphism $\varphi : \mathcal{A} \to \mathcal{B}$ between upper semicontinuous C^*-bundles $\mathcal{A}$and $\mathcal{B}$ over X, pushing forward sections with φ provides a corresponding $C_0(X)$-algebra homomorphism $\Gamma_0(X,\varphi) : \Gamma_0(X,\mathcal{A}) \to \Gamma_0(X,\mathcal{B})$.
Conversely, we can construct a *sectional representation functor*

$$\mathrm{SR}(X,\cdot): \mathrm{C}_0(X)\mathrm{Alg} \to \mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(X) \quad (61)$$

as follows. First, given any $C_0(X)$-algebra A, we define SR(X, A), as a “bundle” over X (in the purely set-theoretical sense), by writing

$$\mathrm{SR}(X,A) = \dot{\bigcup_{x\in X}} SR(X,A)_X \quad (62)$$

where the fiber SR $(\mathrm{X},A)_x$ over any point x in X is defined by

$$\mathrm{SR}(X,A)_X = A/\overline{\Phi(I_x)A}, where\ I_x = \{f \in C_0(X) | f(x) = 0\}. \quad (63)$$

The structure of $\mathrm{SR}(X,A)$ as a C^*-bundle is then determined by the construction described in Theorem (6.1.10) above, specialized to the case of C^*-bundles and with

$$\Gamma = \{\phi_a | a \in A\}\ where\ \phi_a(x) = a + \overline{\Phi(I_x)A} \in A/\overline{\Phi(I_x)A}, \quad (64)$$

since this space Γ satisfies the two conditions of Theorem (6.1.10) (condition (b) is obvious and condition (a) is shown in [33]). Second, given any homomorphism $\varphi_X : A \to B$ between $C_0(X)$-algebras A and B, passing to quotients provides a corresponding strict bundle morphism $\mathrm{SR}(X,\varphi_X)$: SR(X, A) $\to \mathrm{SR}(X,B)$.
The construction outlined in the previous paragraph is actually the central point in the proof of a famous theorem in the field, generally known as the sectional representation theorem, which asserts that every $C_0(X)$-algebra A can be obtained as the section algebra $\Gamma_0(X,\mathcal{A})$ of an appropriate upper semicontinuous C^*-bundle $\mathcal{A}$over X; for an explicit statement with a complete proof, see [33]. Here, we state a strengthened version of this theorem, which extends it to an equivalence of categories [21].
Theorem (6.1.13) [270]: *Given a locally compact topological space X, the functors $\Gamma_0(X,\cdot)$ and* SR(X, ·) *establish an equivalence between the categories* $\mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(X)$ *and* $\mathrm{C}_0(X)\mathrm{Alg}$.
Proof. Explicitly, the statement of the theorem means that, for any upper semicontinuous C^*-bundle $\mathcal{A}$over X, there is a strict bundle isomorphism $\mathcal{A} \cong \mathrm{SR}\big(X,\Gamma_0(X,\mathcal{A})\big)$ which behaves naturally under strict bundle morphisms, and similarly that, for any $C_0(X)$-algebra A, there is a $C_0(X)$-algebra isomorphism $A \cong \Gamma_0\big(X,\mathrm{SR}(X,A)\big)$ which behaves naturally under $C_0(X)$-algebra homomorphisms. The existence of the second of these isomorphisms

is precisely the content of the traditional formulation of the sectional representation theorem [33], whereas the first is constructed similarly. Namely, given any upper semicontinuous C^*-bundle $\mathcal{A}$ over X, note that, for any point x in X, we have

$$\overline{\Phi(I_x)\Gamma_0(X,\mathcal{A})} = \{\phi \in \Gamma_0(X,\mathcal{A}) | \phi(x) = 0\}$$

since the inclusion $\subset$ is trivial and the inclusion $\supset$ follows from a standard argument: given $\phi \in \Gamma_0(X,\mathcal{A})$ and any $\varepsilon > 0$, there are an open neighborhood U of x with compact closure $\overline{U}$ and a compact subset K containing it such that the function $x \mapsto \|\phi(x)\|_x$ is $< \varepsilon$ in U (since it vanishes at x and the C^* fiber norm on $\mathcal{A}$is upper semicontinuous) as well as outside of K (since ϕ vanishes
at infinity), so applying Urysohn's lemma we can find a function $f \in C_c(X)$ with $0 \leq f \leq 1$ which is $\equiv 0$ outside of U but satisfies $f(x) = 1$ and combine it with another function $g \in C_0(X)$ with $0 \leq f \leq 1$ which is $\equiv 1$ on K to get a function $(1-f)g \in C_0(X)$ which is $\equiv 1$ on $K\backslash U$ but vanishes at x and from that deduce that the sup norm of $\phi - (1-f)g\phi$ is $< \varepsilon$. Therefore, for any point x in X, we get a C^*-algebra isomorphism

$$\mathrm{SR}\big(X,\Gamma_0(X,\mathcal{A})\big)_x \cong \mathcal{A}_X$$

which provides the desired bundle isomorphism as x varies over the base space X.
An interesting question would be to fully incorporate the notions of pull-back and of change of base ring into this picture. On the one hand, given any proper continuous map $f: X \to \mathrm{Y}$ between locally compact topological spaces X and Y, we can define a corresponding *pull-backfunctor*

$$f^*: \mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(\mathrm{Y}) \to \mathrm{C}^*_{\mathrm{us}}\mathrm{Bun}(X)\,, \qquad (65)$$

associating to each upper semicontinuous C^*-bundle B over Y its pull-back via f, which is an upper semicontinuous C^*- bundle f^*E over X, fiberwise defined by $(f^*\mathcal{B})_X = B_{f(x)}$, and associating to each strict bundle morphism $\ell : B \to B'$ over Y its pull-back via f, which is a strict bundle morphism $f^*\varphi : f^*E \to f^*B'$ over X, fiberwise defined by $f^*\ell|_{(f^*B)_x} = \ell|_{B_{f(x)}}$. On the other hand, given any proper continuous map $f: X \to \mathrm{Y}$ between locally compact topological spaces X and Y, we can define a corresponding *change of base ringfunctor*

$$f_\#: \mathrm{C}_0(X)\mathrm{Alg} \to \mathrm{C}_0(\mathrm{Y})\mathrm{Alg}, \qquad (66)$$

associating to each $C_0(X)$-algebra A a $C_0(\mathrm{Y})$-algebra $f_\# A$ which as a C^*-algebra is equal to A but with a modified module structure, defining multiplication with functions in $C_0(\mathrm{Y})$ to be given by multiplication with the corresponding functions in $C_0(X)$ obtained by pull-back via f, and associating to each $C_0(X)$-algebra homomorphism $\varphi_X: A \to A'$ a $C_0(\mathrm{Y})$-algebra homomorphism $f_\#\varphi_X : f_\# A \to f_\# A'$ which as a C^*-algebra homomorphism is equal to ϕ_X but is now linear with respect to the modified module structure. It should be noted that these two functors do *not* translate into each other under the equivalence established by the sectional representation theorem because, they go in opposite directions and the first preserves the fibers while changing the section algebras whereas the second preserves the section algebras while changing the fibers. Indeed, for any upper semicontinuous C^*-bundle $\mathcal{B}$ over Y, composition of sections with f induces a C^*-algebra homomorphism

$$f^*: \Gamma_0(\mathrm{Y},B) \to \Gamma_0(X,f^*B) \qquad (67)$$

which, in general, is far from being an isomorphism since it may have a nontrivial kernel (consisting of sections of $\mathcal{B}$ over Y that vanish on the image of *f*) as well as a nontrivial image (consisting of sections of $f^*\mathcal{B}$ over X that are constant along the level sets of f). Similarly, given any $C_0(X)$-algebra A and using f to also consider it as a $C_0(Y)$-algebra $f_\# A$, we can apply the respective sectional representation functors to introduce the corresponding C^*-bundles $\mathcal{A} = \mathrm{SR}(X, A)$ over X and $f_\#\mathcal{A} = \mathrm{SR}(\mathrm{Y}, f_\# A)$ over Y, so that $A \cong \Gamma_0(X, \mathcal{A})$ and $f_\# A \cong \Gamma_0(\mathrm{Y}, f_\#\mathcal{A})$: then we find that the fibers of $f_\#\mathcal{A}$are related to the fibers of $\mathcal{A}$by

$$((f_\#\mathcal{A})_y \cong \Gamma(f^{-1}(y), \mathcal{A}. \tag{68}$$

We are now ready to address the central point of this section, namely, the construction of the C^*-completion at the level of bundles and its relation with the C^*-completion at the level of the corresponding section algebras. To this end, suppose that X is a locally compact topological space and $\mathcal{A}$ is a bundle of locally convex $*$-algebras over X, with bundle projection $p: \mathcal{A} \to X$ and with respect to some directed set Σ of fiber seminorms on $\mathcal{A}$, as in Definition (6.1.9) above. Suppose furthermore that we are given a function $\|.\| : \mathcal{A} \to \mathbb{R}$ which is a C^* fiber norm (in the sense of inducing a C^*-norm on each fiber of $\mathcal{A}$). From these data, we can construct a “fiberwise C^*-completion” $\mathcal{A}^-$ of $\mathcal{A}$ by taking, for every point x of X, the completion $\overline{\mathcal{A}_X}$ of $\mathcal{A}_x$ with respect to the C^*-norm $\|.\|_x$ to obtain a family $(\mathcal{A}_x^-)_{x\in X}$ of C^*-algebras and consider the disjoint union

$$\bar{\mathcal{A}} = \dot{\bigcup_{x\in X}} \mathcal{A}_x \tag{69}$$

as a “bundle” of C^*-algebras over X (in the purely set-theoretical sense); obviously, $\mathcal{A} \subset \bar{\mathcal{A}}$ and the original bundle projection $p: \mathcal{A} \to X$ is simply the restriction of the bundle projection $\overline{p}: \bar{\mathcal{A}} \to X$. In order to control the topological aspects involved in this construction, we have to impose additional hypotheses. We shall assume that $\mathcal{A}$is an upper semicontinuous bundle of locally convex $*$-algebras over X, as in Definition (6.1.9) above, and that $\|.\|$ is *locally bounded* by Σ, i.e., for every point x of X there exist a neighborhood U_X of x, a fiber seminorm s belonging to Σ and a constant $C > 0$ such that $\|a\| \leq Cs(a)$ for $a \in p^{-1}(U_x)$. We show that, under these circumstances, $Reject^-$ admits a unique topology turning it into an upper semicontinuous C^*-bundle over X such that the space of its continuous sections vanishing at infinity is the completion of the space of continuous sections of compact support of $Reject$ with respect to the sup norm induced by the C^* fiber norm $\|.\|$: this will provide us with a natural and concrete example of the abstract sectional representation theorem.

To do so, note first that since the fiber norm $\|.\|$ is locally bounded by the seminorms in Σ which are upper semicontinuous, and since X is locally compact, it follows that, given any continuous section ϕ of $Reject$, the function $x \mapsto \|\phi(x)\|_X$ on X is locally bounded and hence bounded on compact subsets of X, so for each compactly supported continuous section ϕ of $\mathcal{A}$, $\|\phi\|_\infty = \sup_{x\in X}\|\phi(x)\|_x$ exists. It is then clear that $\|.\|_\infty$ defines a C^*-norm on $\Gamma_c(X, \mathcal{A})$: let $\Gamma_c(X, \mathcal{A})$ be the corresponding C^*-completion. Next, note that $\Gamma_c(X, \mathcal{A})$ is also a module over $C_0(X)$, and hence so is $\Gamma_c(X, \mathcal{A})$ (since multiplication is obviously a continuous bilinear map with respect to the pertinent C^*-norms); moreover, we have the equality $C_0(X)\Gamma_c(X, \mathcal{A}) = \Gamma_c(X, \mathcal{A})$, since any $\phi \in \Gamma_c(X, \mathcal{A})$ can be written in the form $f\phi$ for some $f \in C_0(X)$ (it suffices to choose f to be equal to 1 on the support of ϕ, using Urysohn's lemma), so $\Gamma_c(X,\mathcal{A})$ is in fact a $C_0(X)$-algebra. Therefore, by the construction of the sectional representation functor, we have

$$\bar{\mathcal{A}} = \mathrm{SR}\big(X, \overline{\Gamma_c(X,\mathcal{A})}\big)\,.$$

In particular, $Reject^{-}$ admits a unique topology turning it into an upper semicontinuous C^*-bundle over X such that continuous sections of compact support of $\mathcal{A}$ become continuous sections of compact support of $\bar{\mathcal{A}}$, since the space $\Gamma_c(X,\mathcal{A})$ satisfies the two conditions of Theorem (6.1.10) (condition (b) is obvious and condition (a) is stated in [33]). In fact, it follows from the construction of the topology on $\bar{\mathcal{A}}$ in the proof of Theorem (6.1.10) that the inclusion $\mathcal{A} \subset \bar{\mathcal{A}}$ is continuous, since if U is a sufficiently small open subset of X such that $\|a\| \le Cs(a)$ for $a \in p^{-1}(U)$ with some fiber seminorm $s \in \Sigma$ and some constant $C > 0$, we have, for any $\phi \in \Gamma_c(X,\mathcal{A})$,

$$W_{\mathcal{A}}(\phi, U, s, \varepsilon/C) \subset W_{\bar{\mathcal{A}}}(\phi, U, \|.\|, \varepsilon) \bigcap \mathcal{A},$$

and $W_{\mathcal{A}}(\phi, U, s, \varepsilon/C)$ is open in $\mathcal{A}$ since ϕ is continuous and s is upper semicontinuous; this fact also implies that the original C^* fiber norm $\|.\|$ on $\mathcal{A}$, just like its extension to the C^* fiber norm (also denoted by $\|.\|$) on $\bar{\mathcal{A}}$, is automatically upper semicontinuous; moreover, $\mathcal{A}$ is dense in $\bar{\mathcal{A}}$(simply because, by construction, every fiber $\mathcal{A}_X$ of $\mathcal{A}$ is dense in the corresponding fiber $\bar{\mathcal{A}}_X$ of $\bar{\mathcal{A}}$). All of this justifies calling $\bar{\mathcal{A}}$ the *fiberwise C^*-completion* of $Reject$ with respect to the given C^* fiber norm. And finally, it is clear that, by construction, the section algebra $\Gamma_0(X,\bar{\mathcal{A}})$ is the C^*-completion of the section algebra $\Gamma_c(X,\mathcal{A})$ with respect to the sup norm $\|.\|_\infty$.

Theorem (6.1.14)[270]: *Given a locally compact topological space X, let $\mathcal{A}$ be an upper semicontinuous bundle of locally convex $*$-algebras over X, with respect to some directed set Σ of fiber seminorms, let $\|.\|$ be a C^*fiber norm on $\mathcal{A}$ which is locally bounded with respect to Σ and let $\bar{\mathcal{A}}$ be the corresponding fiberwise C^*-completion of $\mathcal{A}$. Then there is a unique topology on $\bar{\mathcal{A}}$* ([*turning it into an upper semicontinuous C^*-bundle over X such that the C^*-completion of the section algebra $\Gamma_c(X,\mathcal{A})$ with respect to the sup norm $\|.\|_\infty$ is the section algebra $\Gamma_0(X,\bar{\mathcal{A}})$,*

$$\overline{\Gamma_c(X,\mathcal{A})} = \Gamma_0(X,\bar{\mathcal{A}})\,. \tag{70}$$

Regarding universal properties of such C^*-completions at the level of bundles and of their section algebras, it is now easy to see that these depend essentially on whether the corresponding universal properties hold fiberwise, at the level of algebras, provided we take into account that when dealing with the section algebras, we must work in the category of $C_0(X)$-algebras rather than just C^*-algebras. More specifically, under the same hypotheses as before (namely, that $\mathcal{A}$ is an upper semicontinuous bundle of locally convex $*$-algebras over X and $\|.\|$ is a locally bounded C^* fiber norm on $\mathcal{A}$), we can guarantee the following.

· Universality implies universality. If, for every point x of X, $\|.\|_x$ is the maximal C^*-norm on $\mathcal{A}_x$, then $\|.\|$ is the maximal C^* fiber norm on $\mathcal{A}$ and we can refer to $\bar{\mathcal{A}}$ as the *minimal C^*-completion* or *universal enveloping C^*-bundle* of $\mathcal{A}$. Moreover, $\Gamma_0(X,\bar{\mathcal{A}})$ will under these circumstances be the *universal enveloping $C_0(X)$-algebra* of $\Gamma_c(X,\mathcal{A})$.

· Uniqueness implies uniqueness. If, for every point x of X, $\mathcal{A}_x$ admits a unique C^*-norm and hence a unique C^*-algebra completion, then $\mathcal{A}$ admits a unique C^* fiber norm and hence a unique C^*-bundle completion. Moreover, $\Gamma_0(X, \bar{\mathcal{A}})$ will under these circumstances be the unique $C_0(X)$-completion of $\Gamma_c(X, \mathcal{A})$.

Let $(E, \sigma \cdot)$ be a Poisson vector bundle with base manifold X, i.e., E is a (smooth) real vector bundle of fiber dimension n, say, over a (smooth) manifold X, with typical fiber]E, equipped with a fixed (smooth) bivector field cr; in other words, the dual E^* of E is a (smooth) presymplectic vector bundle. (Again, we emphasize that we do *not* require σ to be nondegenerate or even to have constant rank.) Then it is clear that we can apply all the constructions to each fiber. The question to be addressed is how, using the methods outlined the results can be glued together along the base manifold X and to describe the resulting global objects. Starting with the collection of Heisenberg algebras $\mathfrak{h}_{\sigma(x)}(x \in X)$, we note first of all that these fit together into a (smooth) real vector bundle over X, which is just the direct sum of E^* and the trivial line bundle $X \times \mathbb{R}$ over X. The nontrivial part is the commutator, which is defined by Equation (4) applied to each fiber, turning this vector bundle into a *totally intransitive Lie algebroid* [20] which we shall call the *Heisenberg algebroid* associated to $(E, a \cdot)$ and denote by $\mathfrak{h}(E, \sigma)$: it will even be a *Lie algebra bundle* [20] if and only if (τ has constant rank. spaces of sections (with certain regularity properties) of $\mathfrak{h}(E, \sigma)$ will then form (infinite-dimensional) Lie algebras with respect to the (pointwise defined) commutator, but the correct choice of regularity conditions is a question of functional analytic nature to be dictated by the problem at hand.

Similarly, considering the collection of Heisenberg groups $H_{\sigma(x)}(x \in X)$, we note that these fit together into a (smooth) real fiber bundle over X, which is just the fiber product of E^* and the trivial line bundle $X \times \mathbb{R}$. Again, the nontrivial part is the product, which is defined by Equation (5) applied to each fiber, turning this fiber bundle into a *totally intransitive Lie groupoid* [20] which we shall call the *Heisenberg groupoid* associated to (E, σ) and denote by $H(E, \sigma)$: it will even be a *Lie group bundle* [20] if and only if (τ has constant rank. And again, spaces of sections (with certain regularity properties) of $H(E, \sigma)$ will form (infinite-dimensional) Lie groups with respect to the (pointwise defined) product, but the correct choice of regularity conditions is a question of functional analytic nature to be dictated by the problem at hand.

An analogous strategy can be applied to the collection of Heisenberg C^*-algebras $\varepsilon_{\sigma(x)}$ and ${}^tH_{\sigma(x)}(x \in X)$, but the details are somewhat intricate since the fibers are now (infinite-dimensional) C^*-algebras which may depend on the base point in a discontinuous way, since the rank of σ is allowed to jump. Still, there remains the question whether we can fit the collections of Heisenberg C^*-algebras $\varepsilon_{\sigma(X)}$ an d${}^cH_{\sigma(x)}$ into C^*-bundles over X which are at least upper semicontinuous.

The basic idea that allows us to bypass all these difficulties is to introduce two smooth vector bundles over X, denoted in what follows by $S(E)$ and by $B(E)$, whose fibers are just the Fréchet spaces of Schwartz functions and of totally bounded smooth functions on the fibers of the original vector bundle E, respectively, i.e., $S(E)_X = S(E_x)$ and $B(E)_x = B(E_X)$: note that choosing any system of local trivializations of the original vector bundle E will give rise to induced systems of local trivializations which, together with an adequate partition of unity, can be used to provide appropriate systems of fiber seminorms, both for $S(E)$ and for $B(E)$. Moreover, we use the Poisson bivector field σ. to introduce a fiberwise Weyl-Moyal star product on these vector bundles which, when combined with the standard fiberwise involution, will turn them into continuous bundles of Fréchet∗-algebras, denoted here by $S(E,\sigma)$ and by $B(E,\sigma)$, respectively. (Continuity of the Weyl-Moyal star product again follows from the estimate of Proposition (6.1.17) in the Appendix, in the case of S, and from [31], in the case of B.) We stress that even though both are locally trivial (and smooth) as vector bundles over X, they will fail to be locally trivial as Fréchet∗-algebra bundles-unless σ has constant rank: this is exactly the same situation as for the fiberwise commutator in the Heisenberg algebroid or the fiberwise product in the Heisenberg groupoid.

The next step consists in gathering the C^*-norms on the fibers of these two bundles, as defined, to construct C^*-fiber norms on each of them which, due to the estimate (30), are locally bounded. Therefore, as seen, they admit C^*-completions which we call the *DFR-bundles*, here denoted by $\varepsilon(E,\sigma)$ and by $H(E,\sigma)$, respectively; thus

$$\mathcal{E}(E,\sigma) = \overline{S(E,\sigma)}, \quad \mathcal{H}(\mathrm{E},\sigma) = \overline{B(E,\sigma)}. \tag{71}$$

algebras are then called the *DFR-algebras*.

We stress that this is a canonical construction because the Heisenberg C^*-algebras are the universal enveloping C^*-algebras associated to the Heisenberg-Schwartz and Heisenberg-Rieffel algebras, and even more than that, they are their *only* C^*-completions, so that according to the results obtained the same goes for the corresponding bundles and section algebras: the DFR-bundles are the universal enveloping C^*-bundles of the corresponding Fréchet∗-algebra bundles introduced above, and even more than that, they are their *only* C^*-completions, and an analogous statement holds for the DFR-algebras as "the" C^*-completions of the corresponding.

When σ is nondegenerate, all these constructions can be drastically simplified; in particular, the DFR-bundles $\varepsilon(E,\sigma)$ and $H(E,\sigma)$ can be obtained much more directly from the principal bundle of symplectic frames for E as associated bundles, and the former becomes identical with the Weyl bundle as constructed.

An important special case of the general construction outlined occurs when the underlying manifold X and Poisson vector bundle (E,σ) are homogeneous. More specifically, assume that G is a Lie group which acts properly on X as well as on E and such that σ is G-invariant: this means that writing

$$\begin{array}{c} G \times X \to X \\ (g,x) \mapsto g\cdot x \end{array} \quad \text{and} \quad \begin{array}{c} G \times E \to E \\ (g,u) \to g\cdot u \end{array} \tag{72}$$

for the respective actions, where the latter is linear along the fibers and hence induces an action

$$\begin{array}{c} G \times \wedge^2 E \to \wedge^2 E \\ (g,u) \mapsto g\cdot u \end{array} \tag{73}$$

we should have

$$\sigma(g\cdot x) = g\cdot \sigma(x) \quad \text{for} \quad g\in G, x\in X, \tag{74}$$

Moreover, we shall assume that the action of G on the base manifold X is transitive. Then, choosing a reference point x_0 in X and denoting by H its stability group in G, by E the fiber of E over x_0 and by σ_0 the value of the bivector field σ at x_0, we can identify: X with the homogeneous space G/H, E with the vector bundle $G \times_H \mathbb{E}$ associated to G (viewed as a principal H-bundle over G/H) and to the representation of H on E obtained from the action of G on $\mathbb{E}$ by appropriate restriction, and σ with the bivector field obtained from σ_0 by the association process. Explicitly, for example, we identify the left coset $gH \in G/H$ with the point $g \cdot x_0 \in X$ and, for any $u_0 \in \mathbb{E}$, the equivalence class $[g, u_0] = [gh, h^{-1} \cdot u_0] \in G \times_H \mathbb{E}$ with the vector $g \cdot u_0 \in \mathbb{E}$. As a result, we see that if the representation of Hon $\mathbb{E}$ extends to a representation of G, then the associated bundle $G \times_H \mathbb{E}$ is globally trivial: an explicit trivialization is given by

$$G \times_H \mathbb{E} \to G/H \times \mathbb{E}$$
$$[g, u_0] = [gh, h^{-1}, u_0] \mapsto (gH, g^{-1} \cdot u_0)\,. \qquad (75)$$

G-invariance combined with transitivity implies that σ has constant rank and hence the Heisenberg algebroid becomes a Lie algebra bundle, the Heisenberg groupoid becomes a Lie group bundle and the DFR-bundles $\varepsilon(E, \sigma)$ and $\subset H(E, \sigma\ \cdot)$ become locally trivial (and smooth) C^*-bundles. Moreover, if the representation of Hon]B extends to a representation of G, all these bundles will even be globally trivial.

To recover the original DFR-model, consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$, which has the Lorentz group $O(1, 3)$ as its isometry group, and choose any symplectic form on $\mathbb{R}^{1,3}$, say the one defined by the matrix $\begin{Bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{Bmatrix}$. Let σ_0 be the corresponding Poisson tensor and H be its stability group under the action of $O(1, 3)$. Then we may recover the space Σ from the original as the quotient space $O(1,3)/H$. Moreover, the vector bundle $O(1,3) \times_H \mathbb{R}^{1,3}$ associated to the canonical principal H-bundle $O(1,3)$ over Σ and the defining representation of $H \subset O(1,3)$ on $\mathbb{R}^{1,3}$ carries a canonical Poisson structure defined by using the action of $O(1,3)$ to transport the Poisson tensor σ_0 at the distinguished point $0 \in \Sigma$ (i.e., $[1] \in O(1,3)/H$) to all other points of Σ. According to the previous discussion, the resulting DFR-bundles will be globally trivial, and so we have

$$\varepsilon(O(1,3) \times_H \mathbb{R}^{1,3}, \sigma) \cong \Sigma \times K \; and \; H(O(1,3) \times_H \mathbb{R}^{1,3}, \sigma) \cong \Sigma \times \mathcal{B}. \quad (76)$$

Moreover, the corresponding DFR-algebra

$$\Gamma_0\big(\varepsilon(O(1,3) \times_H \mathbb{R}^{1,3}, \sigma)\big)$$

will then be the same as the one originally defined

We can extend the construction above to obtain a C^*-bundle over an arbitrary spacetime whose fibers are isomorphic to the original DFR-algebra. Let (M, g) be an n-dimensional Lorentz manifold with orthonormal frame bundle (M, g) . Also, let σ_0 be a fixed bivector on $\mathbb{R}^n$ and Σ its orbit under the action of the Lorentz group $(1, n-1)$. Consider the associated fiber bundle

$$\Sigma(M) = O(M, g) \times_{O(1,n-1)} \Sigma$$

over M, whose bundle projection we shall denote by π. Using π to pull back the tangent bundle TM of M to (M) , we obtain a vector bundle π^*TM over $\Sigma(M)$ which carries a canonical bivector field σ defined by the original bivector σ_0. Then the section algebra

$$\Gamma_0\big(\Sigma(M), \varepsilon(\pi^*TM, \sigma)\big)$$

of the resulting DFR-bundle $\varepsilon(\pi^*TM, \sigma)$ is not only a $C_0\big(\Sigma(M)\big)$-algebra but, using the bundle projection π, it also becomes a $C_0(M)$-algebra and hence can be regarded as the section algebra of a C^*-bundle over M. Refining the discussion.

$$\Gamma_0\left(\Sigma(M)_m, \varepsilon\big(O(T_mM, g_m) \times {}_{H_m} T_mM, \sigma_m\big)\right) . \tag{77}$$

In analogy with the term "quantum spacetime" employed by designate the original DFR-algebra, we suggest to refer to the functor that to each Lorentz manifold (M, g) associates the section algebra $\Gamma_0\big(\Sigma(M), \varepsilon(\pi^*TM, \sigma)\big)$ as "locally covariant quantum spacetime.

Our first goal when starting this investigation was to find an appropriate mathematical setting for geometrical generalizations of the DFR-model–a model for "quantum spacetime" which grew out of the attempt to avoid the conflict between the classical idea of sharp localization of events (ideally, at points of spacetime) and the creation of black hole regions and horizons by the concentration of energy and momentum needed to achieve such a sharp localization, according to the Heisenberg uncertainty relations. To begin with, this required translating the Heisenberg uncertainty relations into the realm of C^*-algebra theory in such a way as to maintain complete control over the dependence on the underlying (pre)symplectic form: a problem that we found can be completely solved within Rieffel's theory of strict deformation quantization, leading to a new construction of "the C^*-algebra of the canonical commutation relations" which is an alternative to existing ones such as the Weyl algebra or the resolvent algebra.

The other main ingredient that had to be incorporated and further developed was the general theory of bundles of locally convex $*$-algebras and, in particular, how the process of C^*-completion of $*$-algebras at the level of fibers relates to that at the level of section algebras. The main outcome here is the definition of a novel procedure of C^*-completion, now at the level of bundles, which to each bundle of locally convex $*$-algebras, equipped with a locally bounded C^* fiber seminorm, associates a C^*-bundle over the same base space such that, at the level of $*$-algebras, the fibers of the latter are the C^*-completions of the fibers of the former and, with appropriate falloff conditions at infinity, the C^*-algebra of continuous sections of the latter is the C^*-completion of the $*$-algebra of continuous sections of the former. Combining these two ingredients, we arrive at a generalization of the mathematical construction underlying the DFR-model which, among other things, can be applied in any dimension and in curved spacetime.

It should perhaps be emphasized at this point that it is not clear how much of the original physical motivation behind the DFR-model carries over to our mathematical generalization. However, we believe our construction to be of interest in its own right, as a tool to generate a nontrivial class of C^*-bundles (the DFR- bundles), each of which can be obtained as the (in this case, unique) C^*-completion of a concrete bundle of Fréchet$*$-algebras that is canonically constructed from a given finite-dimensional Poisson vector bundle and, as a bundle of Fréchet spaces, is locally trivial and even smooth. This whole process can be generalized even further by considering other methods to generate C^*-algebras from an appropriate class of vector spaces (to replace the passage from pre-symplectic vector spaces to Heisenberg C^*-algebras) which satisfy continuity conditions in such a way as to allow for

a lift from vector spaces and C^*-algebras to vector bundles and C^*-bundles, in the spirit of the functor lifting.
The construction of the aforementioned bundle of Fréchet$*$-algebras gains additional importance when one considers the necessity of identifying further geometrical structures on the "noncommutative spaces" that the DFR-algebras are supposed to emulate. A first step in this direction is to look at the general definition of smooth subalgebras of C^*-algebras, as discussed. Using the results from it to show that the Heisenberg-Schwartz and Heisenberg-Rieffel algebras are smooth subalgebras of their respective C^*-completions and with a little further effort one can also show that the same holds for the algebras of smooth sections of the corresponding bundles of Fréchet$*$-algebras (with regard to their smooth structure as vector bundles). Another application of our construction of the DFR-bundles is that it provides nontrivial examples of locally C^* $-$algebras namely, by considering the algebras of continuous local sections of our bundles. The concept of locally C^*-algebras is of particular importance for handling noncompact spaces and is encountered naturally when dealing with sheaves of algebras, so prominent in topology and geometry. A collection of new results in this direction, related to what has been done here, can be found.
We are fully aware of the fact that all these questions are predominantly of mathematical nature: the physical interpretation is quite another matter. But to a certain extent this applies even to the original DFR-model, since it is not clear how to extend the interpretation of the commutation relations postulated in terms of uncertainty relations, to other spacetime manifolds, or even to Minkowski space in dimensions $\neq 4$. In addition, it should not be forgotten that, even classically, spacetime coordinates are *not* observables: this means that the basic axiom of algebraic quantum field theory according to which observables should be described by (local) algebras of a certain kind (such as C^*-algebras or von Neumann algebras) does *not at all* imply that in quantum gravity one should replace classical spacetime coordinate functions by noncommuting operators. To us, the basic question seems to be: *How can we formulate spacetime uncertainty relations*, *in the sense of obstructions to the possibility of localizing events with arbitrary precision*, *in terms of observables*? That of course stirs up the question: *How do we actually measure the geometry of spacetime when quantum effects become strong*?

We establish a couple of useful results on the Weyl-Moyal star product, beginning with an estimate for the Schwartz seminorms of the product $f \star_{(J} \cdot g$ of two functions f and g in $B(V)$ when at least one of them belongs to (V) , in terms of the pertinent seminorms of the factors; as shown this implies a corresponding estimate for the C^*-norm on $B(V)$. Such estimates can be found in [31], but we also need some information on how the constants involved in these estimates depend on the Poisson tensor σ, and that part of the required information is not provided there. In a second part, we shall discuss the issue of approximate identities for the Heisenberg-Schwartz algebra (noting that for the Heisenberg Rieffel algebra, this would be a pointless exercise since that already has a unit, namely, the constant function 1).
For simplicity, we shall work in coordinates, so we choose a basis $\{e_1, \dots , e_n\}$ of V and introduce the corresponding dual basis $\{e^1, \dots, e^n\}$ of V^*, expanding vectors x in V and

covectors ξ in V^* according to $x = x^j e_j$, $\xi = \xi_j e^j$ and the bivector 0^- according to $(\tau(\xi,\eta) = \sigma^{kl}\xi_k\xi_l$; then

$$\eta_j = (\sigma^\# \xi) = \langle \eta, \sigma^\# \xi \rangle = \sigma(\xi,\eta) = \sigma^{kj}\xi_k\eta_j$$

implies that $(\sigma^\#\xi)^j = \sigma^{kj}\xi_k$. Moreover, using multiindex notation, we can define the topologies of $S(V)$ and of $B(V)$ in terms of the Schwartz seminorms $s_{p,q}$ (for $S(V)$) and $s_{0,q}$ (for $B(V)$), defined by

$$s_{p,q}(f) = \sum_{|\alpha|\le p, |\beta|\le q} \sup_{x\in V} \left|x^\alpha \partial_\beta f(x)\right|. \tag{78}$$

To begin with, we note the following explicit estimate for the L^1-norm of the (inverse) Fourier transform $\check{f}$ of a Schwartz function f in terms of an appropriate Schwartz seminorm,

$$\|\check{f}\|_1 \le (2\pi)^n s_{2n,2n}(f) \; for\; f \in S(V) . \tag{79}$$

Proof.

$$\|\mathcal{F}'^{-}{}'f\|_1 = \int_{V^*} d\,\xi |(\mathcal{F}^{-1}f)(\xi)|$$

$$= \int \frac{d\xi_1}{1+\xi_1^2}\frac{d\xi_n}{1+\xi_n^2} |(1+\xi_1^2)\dots(1+\xi_n^2)(\mathcal{F}^{-1}f)(\xi)|$$

$$\le \pi^n \sup_{\xi\in V^*} |\left(\mathcal{F}^{-1}\left(\left(1-\partial_{x^1}^2\right)\dots\left(1-\partial_{x^n}^2\right)f\right)\right)(\xi)|$$

$$\le \frac{1}{2^n}\int_V d\,x| \left(\left(1-\partial_{x^1}^2\right)\dots\left(1-\partial_{x^n}^2\right)f\right)(x)|$$

$$= \frac{1}{2^n}\int \frac{dx^1}{1+(x^1)^2}\cdots\frac{dx^n}{1+(x^n)^2}|(1+(x^1)^2)\dots(1+(x^n)^2)$$
$$\left(\left(1-\partial_{x^1}^2\right)\dots\left(1-\partial_{x^n}^2\right)f\right)(x)|$$
$$\le (2\pi)^n s_{2n,2n}(f) .$$

Now from Equation (19), we conclude that

$$\sup_{x\in V} |(f \star_\sigma g)(x)| \le \sup_{x\in V} |f(x)| \int_{V^*} d\,\xi |\check{g}(\xi)| \; for\; f \in \mathcal{B}(V), g \in S(V) ,$$

and hence Equation (A2) gives the following estimate:

$$s_{0,0}(f \star_\sigma g) \le (2\pi)^n s_{0,0}(f) s_{2n,2n}(g) \; for\; f \in \mathcal{B}(V), g \in S(V) . \tag{80}$$

In order to generalize this inequality to higher order Schwartz seminorms, we need the following facts.

Lemma (6.1.15)[270]: *For $f \in \mathcal{B}(V)$ and $\in S(V)$, we have the Leibniz rule*

$$\frac{\partial}{\partial x^j}(f \star_\sigma g) = \frac{\partial f}{\partial x^j} \star_\sigma g + f \star_\sigma \frac{\partial g}{\partial x^j},$$

and therefore the higher order Leibniz rule

$$\partial_\alpha (f \star_\sigma g) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial_\beta f \star_\sigma \partial_{\alpha-\beta} g.$$

Proof. Simply differentiate Equation (19) under the integral sign.

Lemma (6.1.16)[270]: *For $f \in B(V)$ and $\in S(V)$, we have*
$$\mathrm{x}^j(f \star_\sigma g) = f \star_\sigma x^j g + \nabla_\sigma^j f \star_\sigma g,$$
and therefore
$$\mathrm{x}^\alpha(f \star_\sigma g) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla_\sigma^{\alpha-\beta} f \star_{(f} \mathrm{x}^\beta g,$$
where ∇_O– denotes the (pre-)symplectic gradient, defined by
$$\nabla_\sigma^j h = \frac{i}{2} \sigma^{jk} \frac{\partial h}{\partial x^k}$$
and $\nabla_\sigma^\alpha = \prod_{j=1}^n \left(\nabla_\sigma^j\right)^{\alpha_j}$.

Proof. For $f \in \mathcal{B}(V)$ and $\in S(V)$, we have, according to Equation (19),
$$\begin{aligned}\left(f \star_\sigma x^j g\right)(x) &= \int_{V^*} d\,\xi f\left(x + \frac{1}{2}\sigma^\# \xi\right)\left(\mathcal{F}^{-1}\left(x^j g\right)\right)(\xi) e^{i\langle \xi, x\rangle} \\ &= i \int_{V^*} d\,\xi f(x + \frac{1}{2}\sigma^\# \xi) \frac{\partial \check{g}}{\partial \xi_j}(\xi) e^{i\langle \xi, x\rangle} \\ &= -i \int_{V^*} d\,\xi (\left(\frac{\partial}{\partial \xi_j} f\left(x + \frac{1}{2}\sigma^\# \xi\right)\right) \check{g}(\xi) e^{i\langle \xi, x\rangle} \\ &\quad + f\left(x + \frac{1}{2}\sigma^\# \xi\right) \check{g}(\xi) \frac{\partial}{\partial \xi_j} e^{i\langle \xi, x\rangle}) \\ &= \frac{i}{2}\sigma^{kj} \int_{V^*} d\,\xi \frac{\partial f}{\partial x^k}\left(x + \frac{1}{2}\sigma^\# \xi\right) \check{g}(\xi) e^{i\langle \xi, x\rangle} + x^j (f \star_\sigma g)(x) \\ &= -\left(\nabla_\sigma^j f \star_\sigma g\right)(x) + x^j (f \star_\sigma g)(x)\,.\end{aligned}$$
Combining these two lemmas gives the formula
$$x^\alpha \partial_\beta (f \star_\sigma g) = \sum_{\gamma \leq \alpha, \delta \leq \beta} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \left(\nabla_\sigma^{\alpha.-\gamma} \partial_{\beta-\delta} f \star_\sigma \mathrm{x}^\gamma \partial_\delta g\right) \tag{81}$$
$$for\ f \in \mathcal{B}(V), g \in S(V)\,.$$
Taking the sup norm (which is just $s_{0,0}$) and applying the definition of the seminorms $s_{p,q}$ together with the estimate (A3) established above, we arrive at the following.

Proposition (6.1.17)[270]: *For any two natural numbers , q, there exists a polynomial $P_{p,q}$ of degree p in the matrix elements of* σ, *with coefficients that depend only on n, p, and q, such that the following estimate holds*:
$$s_{p,q}\left(f \star_{(T} g\right) \leq |P_{p,q}((\sigma)| s_{0,p+q}(f) s_{p+2n,q+2n}(g)$$
$$for\ f \in \mathcal{B}(V), g \in S(V)\,. \tag{82}$$
With these formulas and estimates at our disposal, we can address the issue of constructing approximate identities for the Heisenberg-Schwartz algebra S_σ. The fact that this is a $*$-subalgebra (and even a $*$-ideal) of the Heisenberg-Rieffel algebra $\mathcal{B}_\sigma$. which does have a unit, namely, the constant function 1, indicates that we should look for sequences $(X_k)_{k\in\mathbb{N}}$ of Schwartz functions $X_k \in S_\sigma$ which converge to 1 in some appropriate sense: without loss of generality, we may assume these functions to be real-valued and to satisfy $0 \leq X_k \leq 1$.

Thus we expect that $X_k \to 1$ and $\partial_\alpha X_k \to 0$ for $\alpha \neq 0$ (or equivalently, $\partial_\alpha(1 - X_k) \to 0$ for all α) as $k \to \infty$, but this convergence can at best hold uniformly on compact subsets of V. (Typically, we may even take $(X_k)_{k\in\mathbb{N}}$ to be a sequence of test functions $X_k \in \mathcal{D}(V)$ that is monotonically increasing and converges to 1 in $\varepsilon(V)$. Note, however, that this sequence does not converge to 1 in the space $S(V)$ and not even in the space $\mathcal{B}(V)$, since the function 1 does not go to 0 at infinity: convergence is only uniform on compact subsets but not on the entire space.) Still, it turns out that any such sequence yields an approximate identity for the Heisenberg-Schwartz algebra—provided we also require the partial derivatives $\partial_\alpha(1 - X_k)$ to be uniformly bounded in k, for all α.

Proposition (6.1.18)[270]: *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of Schwartz functions $X_k \in S(V)$ satisfying $0 \leq X_k \leq 1$ which is bounded in the Fréchet space $\mathcal{B}(V)$ and converges to 1 in the Fréchet space (V) , that is, in the topology of uniform convergence of all derivatives on compact subsets. Then $(X_k)_{k\in\mathbb{N}}$ is an approximate identity in the Heisenberg-Schwartz algebra $S_\sigma, i.e.$, for any $f \in S_\sigma$, we have that $X_k \star_\sigma f \to f$ in $S_\sigma \cdot$, as $k \to \infty$.*

Proof. Fixing $f \in S_\sigma$. and $p, q \in \mathbb{N}$, we have the following estimate:

$$s_{p,q}(X_k \star_\sigma f - f) \leq C_0 \max_{\substack{|\alpha|,|\gamma|\leq p \\ |\beta|,|\delta|\leq q}} \sup_{x\in V} \left|\left(\nabla_\sigma^\alpha \partial_\beta(1 - X_k) \star_\sigma x^\gamma \partial_\delta f\right)(x)\right|,$$

where

$$C_0 = \sum_{|\alpha|\leq p,|\beta|\leq q} \sum_{\gamma\leq\alpha,\delta\leq\beta} \binom{\alpha}{\gamma}\binom{\beta}{\delta},$$

which follows directly from Equation (A4) after some relabeling. Now given $\varepsilon > 0$, we shall split this sup norm into two parts. First, we use that the functions X_k, and hence also the functions

$\nabla_\sigma^\alpha \partial_\beta(1 - X_k)$, form a bounded subset of B_σ, while the $x^\gamma \partial_\delta f$ are fixed functions in S_σ, to conclude that there exists a compact subset K of V such that, for all $|\alpha|$, $|\gamma| \leq p$ and $|\beta|$, $|\delta| \leq q$,

$$\sup_{x\notin K} \left|\left(\nabla_\sigma^\alpha \partial_\beta(1 - x_k) \star_\sigma x^\gamma \partial_\delta f\right)(x)\right| < \frac{\in}{C_0}.$$

Indeed, we may apply Equations (A3) and (A4) to show that the Schwartz functions $(1 + |x|^2)(\nabla_\sigma^\alpha \partial_\beta(1 - X_k) \star_\sigma x^\gamma \partial_\delta f)$ on V are uniformly bounded in k (as well as in all other parameters), so the Schwartz functions $\nabla_\sigma^\alpha \partial_\beta(1 - X_k) \star_{(T} x^\gamma \partial_\delta f$ on V vanish at infinity uniformly in k (as well as in all other parameters). Next, we set

$$C_1 = \max_{|a|\leq p|B|\leq q} s_{0,0}\left(\nabla_\sigma^\alpha \partial_\beta(1 - X_k)\right), C_2 = \max_{|y|\leq p,|\delta|\leq q} \|\mathcal{F}^{-1}(x^\gamma \partial_\delta f)\|_1$$

and introduce a compact subset K^* of V^* such that, for all $|\gamma| \leq p$ and $|\delta| \leq q$,

$$\int_{V^*\setminus K^*} d\,\xi |\mathcal{F}^{-1}(x^\gamma \partial_\delta f)(\xi)| < \frac{\epsilon}{2C_0C_1}.$$

Now let $L = K + \frac{1}{2}\sigma^\# K^*$, which is again a compact subset of V, and finally use the uniform convergence of the functions $1 - Xk$ and their derivatives on L to infer that there exists $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$ and all $|\alpha| \leq p$ and $|\beta| \leq q$,

$$\sup_{y\in L} |\left(\nabla_\sigma^\alpha \partial_\beta(1-X_k)\right)(y)| < \frac{\epsilon}{2C_0C_2}$$

Then it follows from Equation (19) that, for $k \geq k_0$ and all $|\alpha|, |\gamma| \leq p$ and $|\beta|, |\delta| \leq q$,

$$\sup_{x\in K} |\left(\nabla_O^\alpha\text{-}\partial_\beta(1-Xk) \star_\sigma x^\gamma \gamma \partial_\delta f\right)(x)|$$

$$\leq |\int_{V^*\backslash K^*} d\,\xi \left(\nabla_\sigma^\alpha \partial_\beta(1-X_k)\right)\left(x+\frac{1}{2}\sigma^\# \xi\right)\left(\mathcal{F}^{-1}(\mathrm{x}^\gamma \partial_\delta f)\right)(\xi) e^{i\langle \xi, x\rangle}|$$

$$+|\int_{K^*} d\,\xi \left(\nabla_\sigma^\alpha \partial_\beta(1-Xk)\right)\left(x+\frac{1}{2}\sigma^\#\ \xi\right)\left(\mathcal{F}^{-1}(x^\gamma \partial_\delta f)\right)(\xi) e^{i\langle \xi, x\rangle}|$$

$$\leq s_{0,0}\left(\nabla_\sigma^\alpha \partial_\beta(1-X_k)\right)\int_{V^*\backslash K^*} d\,\xi |\left(\mathcal{F}^{-1}(x^\gamma \partial_\delta f)\right)(\xi)|$$

$$+\sup_{y\in L} |\left(\nabla_\sigma^\alpha \partial_\beta(1-Xk)\right)(y)| \int_{V^*} d\,\xi |\mathcal{F}^{-1}(\mathrm{x}^\gamma \partial_\delta f)(\xi)|$$

$$< \frac{\epsilon}{C_0}$$

It may be worthwhile to emphasize that this construction provides an entire class of approximate identities for the Heisenberg-Schwartz algebra but no bounded ones: the $\lambda^\gamma k$ are uniformly bounded in k only in B_o- but not in S_o-. This is unavoidable since it is in fact not difficult to prove that the Heisenberg-Schwartz algebra does not admit any bounded approximate identities, but we shall not pursue the matter any further since we do not need this fact in the present .

Section (6.2): Projective Limits of Hilbert Spaces

The canonical commutation relations

$$QP - PQ\ = \mathrm{i}h$$

are the paradigm of quantum physics. They indicate the transition from formerly commutative algebras of observables in classical mechanics to now non-commutative algebras, those generated by the fundamental variables of position Q and momentum P. While this basic form of the commutation relations is entirely algebraic, the need of physics is to have some more analytic framework. Traditionally, one views Q and P as (necessarily unbounded) self-adjoint operators on a Hilbert space. Then the commutation relation becomes immediately much more touchy as one has to take care of domains. The reasonable way to handle these difficulties is to use the Schrödinger representation which leads to a strongly continuous representation of the Heisenberg group. This way, the commutation relations encode an integration problem, namely from the infinitesimal picture of a Lie algebra representation by unbounded operators to the global picture of a group representation by unitary operators.

While this is all well-understood, things become more interesting in infinite dimensions: here one still has canonical commutation relations now based on a symplectic (or better: Poisson) structure on a vector space V. Physically, infinite dimensions correspond to a field theory with infinitely many degrees of freedom instead of a mechanical system. Then, algebraically, the commutation relations can be realized as a star product for the symmetric algebra over this vector space, see [72] where the basic notions of deformation quantization have been introduced as well as e.g. [108], [100], [287] for introductions. However, beyond the algebraic questions one is again interested in an analytic context: it turns out that now things are much more involved. First, there is no longer an essentially unique way to represent the canonical commutation relations by operators, a classical result which can be stated in many ways. One way to approach this non-uniqueness is now to focus first on the algebraic part and discuss the whole representation theory of this quantum algebra. To make this possible one has to go beyond the symmetric algebra and incorporate suitable completions instead.

Based on a C^*-algebraic formulation there are (at least) two approaches available. The classical one is to take formal exponentials of the unbounded quantities and implement a C^*-norm for the algebra they generate, see [284]. An alternative was proposed by taking formal resolvents and the C^*-algebra they generate [102]. These two approaches can be formulated in arbitrary dimensions and are used extensively in quantum field theory.

Only for finite dimensions there is a third C^*-algebraic way based on (strict) deformation quantization in the framework of Rieffel [96], see also [282], [270] for some more recent development: here one constructs a rather large C^*-algebra by deforming the bounded continuous functions on the underlying symplectic vector space. The deformation is based on certain oscillatory integrals which is the reason that this approach, though extremely appealing and powerful, will be restricted to finite dimensions. In such finite-dimensional situations one has even ways to go beyond the flat situation and include also much more non-trivial geometries of the underlying geometric system, see e.g. [281].

While the C^*-algebraic approaches are very successful in many aspects, some questions seem to be hard to answer within this framework: from a deformation quantization point of view it is not completely obvious in which sense these algebras provide deformations of their classical counterparts, see, [103]. Closely related is the question of how one can get back the analogs of the classically unbounded quantities like polynomials on the symplectic vector space: in the quantum case they can not be elements of any C^*-algebra and thus they have to be recovered in certain well-behaved representations as unbounded operators on the representation space. This raises the question whether they can acquire some intrinsic meaning, independent of a chosen representation. In particular, all the C^*-algebraic constructions completely ignore possible additional structures on the underlying vector space V, like e.g. a given topology. This seems both from the purely mathematical but also from the physical point of view rather unpleasant.

In [101] a first step was taken to overcome some of these difficulties: instead of considering a C^*-algebraic construction, the polynomials, modeled as the symmetric algebra, were kept and quantized by means of a star product directly. Now the additional feature is that a given locally convex topology on the underlying vector space V induces a specific locally convex topology on the symmetric algebra $S^{\cdot}(V)$ in such a way that the star product becomes *continuous*. Necessarily, there will be no nontrivial sub-multiplicative seminorms, making the whole locally convex algebra quite non-trivial. It was then shown that in the completion the star product is a convergent series in the deformation parameter h. This construction has good functorial properties and works for every locally convex space V with continuous constant Poisson structure. The basic feature was that on a fixed symmetric power $S^k(V)$

the *projective* locally convex topology was chosen. In finite dimensions this construction reproduces earlier versions [108], [119] of convergence results for the particular case of the Weyl-Moyal star product.

We want to adapt the construction of [101] to the more particular case of a projective limit of (pre-) Hilbert spaces, i.e. a locally convex space where the topology is determined by Hilbert seminorms coming from (not necessarily non-degenerate) positive inner products. The major difference is now that for each fixed symmetric power $S^k(V)$ we have another choice of the topology, namely the one by extending the inner products first and taking the corresponding Hilbert seminorms afterwards. In general, this is coarser than the projective one and thus yields a larger and hence more interesting completion. We then use a star product coming from an arbitrary continuous bilinear form on V, thereby allowing for various other orderings beside the usual Weyl symmetrization. We are able to determine many features of this new algebra hosting the canonical commutation relations in arbitrary dimensions, including the convergence of the star product and an explicit description of the completion as certain analytic functions on the topological dual.

We outline the construction of the star product and the relevant topology. Since the star product is the usual one of exponential type on a vector space

we can be brief here. The topological properties are discussed in some detail, in particular as they differ at certain points significantly from [101]. After the necessary but technical estimates this results in the construction of the locally convex algebra in Theorem (6.2.13). contains various properties of the star product algebra. First we show that a continuous antilinear involution on V extends to a continuous $*$-involution on the algebra. Then we are able to characterize the topology by some very simple conditions in Theorem (6.2.18), a feature which is absent in the case of [101]. The discussion of equivalences between different star products becomes now more involved as not all continuous symmetric bilinear forms give rise to equivalences as that was the case in [101]. Now in Theorem (6.2.23) we have to add a Hilbert-Schmidt condition similar to the one of Dito in [106]. In Theorem (6.2.39) we are able to characterize the completed star product algebra as certain analytic functions on the topological dual. This will later be used to make contact to the more particular situation considered in [106]. In Theorem (6.2.44) we show the existence of many positive linear functionals provided the Poisson tensor allows for a compatible positive bilinear form of Hilbert-Schmidt type. Since the algebra is (necessarily) not locally multiplicatively convex, we have no general entire calculus. However, we can show that for elements of degree one, i.e. vectors in V, the star exponential series converges absolutely. This is no longer true for quadratic elements, i.e. elements in $S^2(V)$. However, we are able to show that in all GNS representations with respect to continuous positive linear functionals all elements up to quadratic ones yield essentially self-adjoint operators in Theorem (6.2.53). Here our topology is used in an essential way. The statement can be seen as a representation-independent version of Nelson's theorem, as it holds for arbitrary such GNS representations. Finally, contains a discussion of two particular cases of interest: First, we consider the case that V is not just a projective limit of Hilbert spaces but a Hilbert space directly. In this case, Dito discussed formal star products of exponential type and their formal equivalence in [106]. We can show that his algebra of functions contains our algebra, where the star product

converges nicely, as a subalgebra. We extend Dito's results form the formal power series context to a convergent one. In fact, we show a rather strong continuity with respect to the deformation parameter in Theorem (6.2.54).
The second case is a nuclear space V. It is well-known that any (complete) nuclear space can be seen as a projective limit of Hilbert spaces, see e.g. [117]. We prove that in this case our construction coincides with the previous one of [101] as for nuclear spaces the two competing notions of topological tensor products we use coincide. This way we can transfer the abstract characterization of the topology to the case of nuclear spaces in [101], a result which was missing in that approach. The important benefit from the projective Hilbert space point of view is now that we can show the existence of sufficiently many continuous positive linear functionals: an element in the completed $*$-algebra is zero iff all continuous positive functionals on it vanish. It follows that the resulting $*$-algebra has a faithful $*$-representation on a pre-Hilbert space, i.e. it is $*$-semisimple in the sense of [97].
For a set X and $k \in \mathrm{N}_0$ we define X^k as the set of all functions from $\{1, . . . , k\}$ (or the empty set if $k = 0$) to X and usually put the parameter in the index, i.e. $\{1, , k\} \ni i \mapsto f_i \in X$ for $f \in X^k$. Let V be a vector space and $k \in \mathrm{M}_0$, then we write $T^k_{\mathrm{alg}}(V)$ for the space of degree k-tensors over V and $\mathcal{T}^*_{\mathrm{g}}(V) := \bigoplus_{k\in \mathrm{N}_0} T^k_{\mathrm{alg}}(V)$ for the vector space underlying the tensor algebra. For $x \in V^k$ we define the projections on the tensors of degree k by $\langle \cdot \rangle_k : T^*_{\mathrm{alg}}(V) \to T^k_{\mathrm{alg}}(V)$. Let $\mathfrak{S}_k \subseteq \{1, , k\}^k$ be the symmetric group of degree k (in the case $k = 0$ this is $\mathfrak{S}_0 = \{\mathrm{id}_\emptyset\}$), then $\mathfrak{S}_k$ acts linearly on $T^k_{\mathrm{alg}}(V)$ from the right via $(x_1 \otimes \cdots \otimes x_k)^\sigma := x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$. This allows us to define the symmetrisation operators $\mathcal{S}^k : T^k_{\mathrm{alg}}(V) \to T^k_{\mathrm{alg}}(V)$ by $X \mapsto \mathcal{S}^k(X) := \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} X^\sigma$ and $\mathcal{S}^{\cdot} : \mathcal{T}^*_{\mathrm{alg}}(V) \to \mathcal{T}^*_{\mathrm{alg}}(V)$ by $X \mapsto \mathcal{S}^{\cdot}(X) := \sum_{k\in\mathbb{N}_0} \mathcal{S}^k(\langle X\rangle_k)$. These are projectors on subspaces of $T^k_{\mathrm{alg}}(V)$ and $T^*_{\mathrm{alg}}(V)$ which we will denote by $S^k_{\mathrm{alg}}(V)$ and $S^*_{\mathrm{alg}}(V)$.
We will always denote an algebra as a pair (V, o) of a vector space V and a multiplication 0, because we will discuss different products on the same vector space.

As we want to construct a similar algebra like in [101], but by using Hilbert tensor products instead of projective tensor products, we have to restrict our attention to locally convex spaces whose topology is given by Hilbert seminorms.
For V be a locally convex space, then a positive Hermitian form on V is a sesquilinear Hermitian and positive semi-definite form $\langle \cdot \mid \cdot \rangle_\alpha : V \times V \to \mathbb{C}$ (antilinear in the first, linear in the second argument). By $\mathcal{J}_V$ we denote the set of all continuous positive Hermitian forms on V and we will distinguish different positive Hermitian forms by a lowercase greek subscript. Out of $p,\ q \geq 0$ and $\langle \cdot \mid \cdot \rangle_\alpha,\ \langle \cdot \mid \cdot \rangle_\beta \in \mathcal{J}_V$ we get a new continuous positive Hermitian form $\langle \cdot \mid \cdot \rangle_{p\alpha+q\beta} := p\langle \cdot \mid \cdot \rangle_\alpha + q\langle \cdot \mid \cdot \rangle_\beta$.
Every $\langle \cdot \mid \cdot \rangle_{cy} \in \mathcal{J}_V$ yields a continuous Hilbert seminorm on V, defined as $\|v\|_\alpha := \sqrt{\langle v|v\rangle_\alpha}$ for all $v \in V$. The set of all continuous Hilbert seminorms on V will be denoted by $\mathcal{P}_V$. Note that $\|\cdot\|_{p\alpha+q\beta} = \left(q\|\cdot\|^2_\alpha + p\|\cdot\|^2_\beta\right)^{1/2}$ and that $\mathcal{P}_V$ with the usual partial ordering of seminorms (i.e. by pointwise comparison) is an upwards directed poset and that there is a one-to-one correspondence between $\mathcal{J}_V$ and $\mathcal{P}_V$ due to the polarisation identity.

In the following we will always assume that V is a Hausdorff locally convex space whose topology is given by its continuous Hilbert seminorms (hilbertisable" in the language of [117]), i.e. we assume that $\mathcal{P}_V$ is cofinal in the upwards directed set of all continuous seminorms on V. Important examples of such spaces are (pre-) Hilbert spaces and nuclear spaces (see [117]) and, in general, all projective limits of pre-Hilbert spaces in the category of locally convex spaces.

Analogous to [101], we extend all Hilbert seminorms from V to $\mathcal{T}^*_{\text{alg}}(V)$ with the difference that we first extend the $\langle\cdot\mid\cdot\rangle_\alpha$ and reconstruct the seminorms out of these extensions:

Definition (6.2.1)[280]: *For every continuous positive Hermitian form $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_V$ we define the sesquilinear extension $\langle\cdot\mid\cdot\rangle^*_\alpha : \mathcal{T}^*_{\text{alg}}(V) \times \quad \mathcal{T}^*_{\text{alg}}(V) \to \mathbb{C}$*

$$(X,Y) \mapsto \langle X|Y\rangle_\alpha := \sum_{k=0}^{\infty} \langle\, \langle X\rangle_k | \langle Y\rangle_k \rangle_\alpha, \tag{83}$$

where

$$\langle x_1 \otimes \dots \otimes x_k | y_1 \otimes \dots \otimes y_k\rangle^*_\alpha := k! \prod_{m=1}^{k} \langle\, x_m|y_m\rangle_\alpha \tag{84}$$

for all $k \in \mathrm{N}_0$ and all $x, y \in V^k$.

It is well-known that this is a positive Hermitian form on all homogeneous tensor spaces and then it is clear that $\langle\cdot\mid\cdot\rangle^*_\alpha$ is a positive Hermitian form on $\mathcal{T}^*_{\text{alg}}(V)$. We write $\|\cdot\|_\alpha$ for the resulting seminorm on $\mathcal{T}^*_{\text{alg}}\,(V)$ and $T^{\cdot}(V)$ for the locally convex space of $\mathcal{T}^*_{\text{alg}}(V)$ with the topology defined by the extensions of all $\|\cdot\|_\alpha \in \mathcal{P}_V$. Analogously, we write $\mathcal{T}^k(V)$, $S^k(V)$ and $S^{\cdot}(V)$ for the subspaces $\mathcal{T}^k_{\text{alg}}(V)$, $S^k_{\text{alg}}(V)$ and $S_{\text{alg}}(V)$ with the subspace topology. Note that $\|\cdot\|_\alpha \le \|\cdot\|_\beta$ holds if and only if $\|\cdot\|_\alpha \le \|\cdot\|_\beta$. Note that, in general, for a fixed tensor degree the resulting topology on $T^k(V)$ is *not* the projective topology used in
The factor $k!$ in (84) for the extensions of positive Hermitian forms corresponds roughly to the factor $(n!)^R$ for $R = 1/2$ in [101] for the extensions of seminorms (where $R = 1/2$ yields the coarsest topology for which the continuity of the star product could be shown in [101]). We are only interested in this special case because of the characterization .

The following is an easy consequence of the definition of the topology on $T^{\cdot}(V)$:

Proposition (6.2.2)[280]: *$\mathcal{T}^*(V)$ is Hausdorff and is metrizable if and only if V is metrizable.*

For working with these extensions of not necessarily positive definite positive Hermitian forms, the following technical lemma will be helpful:

Lemma (6.2.3)[280]: *Let $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_V$, $k \in \mathbb{N}$ and $X \in T^k(V)$ be given. Then X can be expressed as $X = X_0 + \tilde{X}$ with tensors $X_0, \tilde{X} \in T^k(V)$ that have the following properties*:

$i.$) One has $\|X_0\|^*_\alpha = 0$ and there exists a finite (possibly empty) set A and tuples $x_a \in V^k$ for all $a \in A$ that fulfil $\Gamma\mathrm{I}^k_{n=1}\|x_{a,n}\|^*_\alpha = 0$ and $X_0 = \sum_{a\in A} x_{a,1} \otimes \otimes x_{a,k}$.

$ii.$) There exist a $d \in \mathrm{N}_0$ and a $\langle\cdot\mid\cdot\rangle_\alpha$-orthonormal tuple $e \in V^d$ as well as complex coefficients $X^{a'}$, such that

$$\tilde{X} = \sum_{a\in\{1,\ldots,d\}^k} X^{a'}\, e_{a'_1} \otimes \ldots \otimes e_{a'_k} \; and \; \|X\|_\alpha^{*2}$$

$$= \|\tilde{X}\|_\alpha^{*2} = k! \sum_{a\in\{1,\ldots,d\}^k} |\,X^{a'}|^2 \qquad (85)$$

Proof.: We can express X as a finite sum of simple tensors, $X = \sum_{b\in B} x_{b,1} \otimes \cdots \otimes x_{b,k}$ with a finite set B and vectors $x_{b,i} \in V$. Let

$$V_X := \; span \; \left\{x_{b,i} | b \in B, i \in \{1, \ldots, k\}\right\} \; and \; V_{X_0} := \{v \in V_X | \|v\|_a = 0\}.$$

Construct a complementary linear subspace V_X- of V_{X_0} in V_X, then we can also assume without loss of generality that $x_{b,i} \in V_{X_0} \mathrm{U} V_X-$ for all $b \in B$ and $i \in \{1, k\}$. Note that V_X, V_{X_0} and V_X- are all finite dimensional. Now define $A := \left\{a \in B | \exists_{n\in\{1,\ldots,k\}}: x_{a,n} \in V_{X_0}\right\}$ and $X_0 := \sum_{a\in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$, then $\prod_{n=1}^{k} \|x_{a,n}\|_\alpha = 0$ by construction and so $\|X_0\|_\alpha^* = 0$ and $\|X - X_0\|_\alpha^* = \|X\|_\alpha^*$. Restricted to V_X-, the positive Hermitian form $\langle \cdot \mid \cdot \rangle_\alpha$ is even positive definite, i.e. an inner product. Let $d := \dim (V_X-)$ and $e \in V^d$ be an $\langle \cdot \mid \cdot \rangle_\alpha$-orthonormal base of V_X-. Define $\tilde{X} := X - X_0$, then $\tilde{X} = \sum_{a'\in\{1,\ldots..d\}} X^{a'}\, e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ with complex coefficients $X^{a'}$ and

$$\|X\|_\alpha^{*2} = \|\tilde{X}\|_\alpha^{*2} = \sum_{a'\in\{1,\ldots,d\}^k} |\,X^{a'}|^2 \|e_{a'_1} \otimes\cdots\otimes e_{a'_k}\|_\alpha^{*2} = \sum_{a'\in\{1,\ldots,d\}^k} |\,X^{a'}|^2 k\,!.$$

On the locally convex space $\mathcal{T}^{\cdot}(V)$, the tensor product is indeed continuous and $(T^{\cdot}(V), \otimes)$ is a locally convex algebra. In order to see this, we are going to prove the continuity of the following function:

Definition (6.2.4)[280]: *We define the map* $\mu_\otimes : T^{\cdot}(V) \otimes_\pi T^{\cdot}(V) \to 7^{\cdot}(V)$ *by*

$$X \otimes_\pi Y \mapsto \mu \otimes (X \otimes_\pi Y) := X \otimes Y. \quad (86)$$

Algebraically, $\mu \otimes$ is of course just the product of the tensor algebra. The emphasize lies here on the topologies involved: $\otimes_\pi$ denotes the projective tensor product. We recall that the topology on $T^{\cdot}(V) \otimes_\pi T^{\cdot}(V)$ is described by the seminorms $\|\cdot\|_{\alpha\otimes_\pi\beta}^* : T^{\cdot}(V) \otimes_\pi T^{\cdot}(V) \to [0, \infty[$

$$Z \mapsto \left\|Z\|_{\alpha\otimes_\pi\beta}^* := \inf \sum_{i\in I} \| X_i\|_\alpha^* \right\| Y_i\|_\beta^*, \quad (87)$$

where the infimum runs over all possibilities to express Z as a sum $Z = \sum_{i\in I} X_i \otimes_\pi Y_i$ indexed by a finite set I and $\|\cdot\|_\alpha^*$, $\|\cdot\|_\beta^*$ run over all extensions of continuous Hilbert seminorms on V. The only property of the projective tensor product relevant for our purposes is the following lemma, which is a direct result of the definition of the seminorms $\|\cdot\|_{\alpha\otimes_\pi\beta}^*$:

Lemma (6.2.5)[280]: *Let W be a locally convex space, p a continuous seminorm on W and $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_V$. Let $\Phi: T^{\cdot}(V) \otimes_\pi T^{\cdot}(V) \to W$ be a linear map. Then the two statements*

$i.) p\big(\Phi(X \otimes_\pi Y)\big) \leq \|X\|_\alpha^* \|Y\|_\beta^{*}{}^{\wedge *}$ *for all* $X, Y \in T^{\cdot}(V)$

ii.) $p(\Phi(Z)) \leq \|Z\|^*_{\alpha\otimes_\pi\beta}$ *for all* $Z \in T^{\cdot}(V) \otimes_\pi T^{\cdot}(V)$

are equivalent. Continuity of the bilinear map $T^{\cdot}(V) \times \mathcal{T}(V) \ni (X, Y) \mapsto \Phi(X \otimes_\pi Y) \in W$ *is therefore equivalent to continuity of* Φ.

Proposition (6.2.6)[280]: *The linear map* $\mu \otimes$ *is continuous and the estimate*

$$\|\mu_{\otimes}(Z)\|^*_\gamma \leq \|Z\|^*_{2\gamma\otimes_\pi 2\gamma} \tag{88}$$

holds for all $Z \in T^{\cdot}(V) \otimes_\pi T^{\cdot}(V)$ *and all* $\|\cdot\|_\gamma \in \mathcal{P}_V$. *Moreover, all* $X \in T^k(V)$ *and* $Y \in T^\ell(V)$ *with* $k, \ell \in \mathrm{M}_0$ *fulfil for all* $\|\cdot\|_\gamma \in \mathcal{P}_V$ *the estimate*

$$\|\mu \otimes (X \otimes_\pi Y)\|^*_\gamma \leq \binom{k+\ell}{k}^{\frac{1}{2}} \|X\|_\gamma \|Y\|^*_\gamma. \tag{89}$$

Proof.: Let $X \in T^k(V)$ and $Y \in T^\ell(V)$ with $k, \ell \in \mathrm{M}_0$ be given. Then

$$\left\|X \otimes Y\right\|^*_\gamma = \sqrt{\langle X \otimes Y | X \otimes Y\rangle^*_\gamma} = \binom{k+\ell}{k}^{\frac{1}{2}} \left\| X\right\|^*_\gamma \|Y\|^*_\gamma$$

holds. It now follows for all $X, Y \in T^{\cdot}(V)$ that

$$\begin{aligned}
\|X \otimes Y\|^{*2}_{\gamma'} &= \sum_{m=0}^{\infty} \|\langle X \otimes Y\rangle_m\|^{*2}_{\gamma'} \\
&\leq \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \|\langle X\rangle_{m-n} \otimes \langle Y\rangle_n\|^*_\gamma\right)^2 \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \binom{m}{n}\frac{1}{2} \|\langle X\rangle_{m-n}\|^*_\gamma \|\langle Y\rangle_n\|^*_\gamma\right)^2 \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \left(\binom{m}{n}\frac{1}{2^m}\right)^{\frac{1}{2}} \|\langle X\rangle_{m-n}\|^*_{2\gamma} \|\langle Y\rangle_n\|^*_{2\gamma}\right)^2 \\
\text{cs} &\leq \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \binom{m}{n}\frac{1}{2^m}\right)\left(\sum_{n=0}^{m} \|\langle X\rangle_{m-n}\|^{*2}_{2\gamma} \|\langle Y\rangle_n\|^{*2}_{2\gamma}\right) \\
&= \|X\|^{*2}_{2\gamma}\|Y\|^{*2}_{2\gamma},
\end{aligned}$$

by the Cauchy-Schwarz (CS) inequality.

The star product will be defined on the symmetric tensor algebra with undeformed product $X \vee Y := \mathcal{S}^{\cdot}(X \otimes Y)$ for $X, Y \in S^{\cdot}(V)$, which is indeed continuous:

Proposition (6.2.7)[280]: *The symmetrisation operator is continuous and fulfils* $\|\mathcal{S}^{\cdot}X\|^*_\gamma \leq \|X\|^*_\gamma$ *for all* $X \in T^{\cdot}(V)$ *and* $\|\cdot\|_\gamma \in \mathcal{P}_V$.

Proof.: From Definition (6.2.1) it is clear that $\langle X^\sigma | Y^\sigma\rangle_\gamma = \langle X|Y\rangle_\gamma$ for all $k \in \mathrm{N}_0$, $X, Y \in T^k(V)$ and $\sigma \in \mathfrak{S}_k$, because this holds for all simple tensors and because both sides are (anti-)linear in X and Y. Therefore $\|X^\sigma\|_\gamma = \|X\|_\gamma$ and $\|\mathcal{S}^k X\|_\gamma \leq \|X\|_\gamma$ and we get the desired estimate

$$\|\mathcal{S}^{\cdot}X\|^{*2}_\gamma = \sum_{k=0}^{\infty} \left\|\mathcal{S}^k\langle X\rangle_k\right\|^{*2}_\gamma \leq \sum_{k=0}^{\infty} \|\langle X\rangle_k\|^2_\gamma = \|X\|^{*2}_\gamma$$

on $T^{\cdot}(V)$.

Analogously to $\mu \otimes$ we define the linear map $\mu_{Reject} := \mathcal{S}^{\cdot} 0 \mu_{\otimes} : T^{\cdot}(V) \otimes_{\pi} T^{\cdot}(V) \to \mathcal{T}(V)$. Then the restriction of μ_{V} to $S^{\cdot}(V)$ describes the symmetric tensor product V and Propositions (6.2.6) and (6.2.7) yield:

Corollary (6.2.8)[280]: *The linear map μ_{Reject} is continuous and the estimate $\|\mu_{\mathrm{V}}(Z)\|_{\gamma} \leq \|Z\|_{2\gamma \otimes_{\pi} 2\gamma}$ holds for all $Z \in 7^{\cdot}(V) \otimes_{\pi} T^{\cdot}(V)$ and all $\| \cdot \|_{\gamma} \in \mathcal{P}_{V}$.*

The following star product is based on a bilinear form and generalizes the usual exponential-type star products like the Weyl-Moyal or Wick star product, see e.g. [100], to arbitrary dimensions:

Definition (6.2.9)[280]: *For every continuous bilinear form Λ on V we define the product $\mu_{\star_{\Lambda}} : T^{\cdot}(V) \otimes_{\pi} \mathcal{T}(V) \to T^{\cdot}(V)$ by*

$$X \otimes_{\pi} Y \mapsto \mu_{\star_{\Lambda}}(X \otimes_{\pi} Y) := \sum_{t=0}^{\infty} \frac{1}{t!} \mu_{\mathrm{V}}\big((\mathrm{P}_{\Lambda})^{t}(X \otimes_{\pi} Y)\big), \quad (90)$$

where the linear map $\mathrm{P}_{\Lambda} : T^{\cdot}(V) \otimes_{\pi} T^{\cdot}(V) \to \mathcal{T}^{-1}(V) \otimes_{\pi} 7^{-1}(V)$ is given on factorizing tensors of degree $k, \ell \in \mathrm{M}$ by

$$\mathrm{P}_{\Lambda}\big((x_1 \otimes \cdots \otimes x_k) \otimes_{\pi} (y_1 \otimes \cdots \otimes y_{\ell})\big)$$
$$:= k\ell \Lambda(x_k, y_1)(x_1 \otimes \cdots \otimes x_{k-1}) \otimes_{\pi} (y_2 \otimes \cdots \otimes y_{\ell}) \quad (91)$$

for all $x \in V^{k}$ and $y \in V^{\ell}$. Moreover, we define the product $\star_{\Lambda}$ on $S^{\cdot}(V)$ as the bilinear map described by the restriction of $\mu_{\star_{\Lambda}}$ to $S^{\cdot}(V)$.

Note that these definitions of P_{Λ} and $\star_{\Lambda}$ coincide (algebraically) on $S^{\cdot}(V)$ with the ones in [101], evaluated at a fixed value for $l \nearrow$ in the truely (not graded) symmetric case $V = V_0$. Note that with our convention the deformation parameter h is already part of Λ.

We prove the continuity of $\star_{\Lambda}$. Therefore we note that continuity of Λ means that there exist $\| \cdot \|_{\alpha}, \| \cdot \|_{\beta} \in \mathcal{P}_{V}$ such that $|\Lambda(v, w)| \leq \|v\|_{\alpha} \|w\|_{\beta}$ holds for all , $w \in V$. So the set

$\mathcal{P}_{V,\Lambda} := \{\| \cdot \|_{\gamma} \in \mathcal{P}_{V} | |\Lambda(v, w)| \leq \|v\|_{\gamma} \|w\|_{\gamma}\ for\ all\ v, w \in V\}$ (92)

contains at least all continuous Hilbert seminorms on V that dominate $\| \cdot \|_{\alpha+\beta}$. Thus this set is cofinal in $\mathcal{P}_{V}$.

Lemma (6.2.10)[280]: *Let Λ be a continuous bilinear form on V, let $\| \cdot \|_{\alpha}, \| \cdot \|_{\beta} \in \mathcal{P}_{V,\Lambda}$ as well as $k, \ell \in \mathrm{N}_0$ and $X \in T^{k}(V), Y \in T^{\ell}(V)$ be given. Then*

$$\|\mathrm{P}_{\Lambda}(X \otimes_{\pi} Y)\|_{\alpha \otimes_{\pi} \beta} \leq \sqrt{k\ell} \|X\|_{\alpha} \|Y\|_{\beta}. \quad (93)$$

Proof.: If $k = 0$ or $\ell = 0$ this is clearly true, so assume $k, \ell \in \mathbb{N}$. We use Lemma (6.2.3) to construct $X_0 = \sum_{a \in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ and $\tilde{X} = \sum_{a' \in \{1,\ldots,e\}^{k}} X^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ with respect to $\langle \cdot | \cdot \rangle_{\alpha}$ as well as $Y_0 = \sum_{b \in B} y_{b,1} \otimes \cdots \otimes y_{b,\ell}$ and $\tilde{Y} = \sum_{b' \in \{1,\ldots,d\}^{\ell}} Y^{b'} f_{b'_1} \otimes \cdots \otimes f_{b'_{\ell}}$ with respect to $\langle \cdot | \cdot \rangle_{\beta}$. Then

$$\|\mathrm{P}_{\Lambda}\left(\left(X_0 + \tilde{X}\right) \otimes_{\pi} \left(Y_0 + \tilde{Y}\right)\right)\|_{\alpha \otimes_{\pi} \beta} \leq \|\mathrm{P}_{\Lambda}\big(\tilde{X} \otimes_{\pi} \tilde{Y}\big)\|_{\alpha \otimes_{\pi} \beta},$$

because

$$\|\mathrm{P}_{\Lambda}\left((\xi_1 \otimes \cdots \otimes \xi_k) \otimes_{\pi} (\eta_1 \otimes \cdots \otimes \eta_{\ell})\right)\|^{*}_{\alpha \otimes_{\pi} \beta}$$
$$= k\ell |\Lambda(\xi_k, \eta_1)| \|\xi_1 \otimes \cdots \otimes \xi_{k-1}\|^{*}_{\alpha} \|\eta_2 \otimes \cdots \otimes \eta\ell\|^{*}_{\beta}, = 0$$

for all $\xi \in V^{k}, \eta \in V^{\ell}$ for which there is at least one $m \in \{1, \ldots, k\}$ with $\|\xi_m\|_{\alpha} = 0$ or one $n \in \{1, \ldots, \ell\}$ with $\|\eta_n\|_{\beta} = 0$. On the subspaces $V_{X^{-}} =$ span $\{e_1, \ldots, e_c\}$ and VỸ=span{fl,

$\ldots, f_d\}$ of V, the bilinear form Λ is described by a matrix $\Omega \in \mathbb{C}^{c\times d}$ with entries $\Omega_{gh} = \Lambda(e_g, f_h)$. By using a singular value decomposition we can even assume without loss of generality that all off-diagonal entries of Ω vanish. We also note that $|\Omega_{gg}| = |\Lambda(e_g, f_g)| \le \|e_g\|_\alpha \|f_g\|_\beta \le 1$. This gives the desired estimate

$$\|\mathrm{P}_\Lambda(X \otimes_\pi Y)\|^*_{\alpha\otimes_\pi\beta}$$
$$\le \|\mathrm{P}_\Lambda(\tilde{X} \otimes_\pi \tilde{Y})\|^*_{\alpha\otimes_\pi\beta}$$

$$= \| \sum_{a'\in\{1\ldots e\}^k} \sum_{b'\in\{1\ldots d\}^\ell} X^{a'} Y^{b'} \mathrm{P}_\Lambda \left((e_{a_1'} \otimes \cdots \otimes e_{a_k'}) \otimes_\pi \left(fbí \otimes \cdots \otimes f_{b_p'} \right) \right) \|^*_{\alpha\otimes_\pi\beta}$$

$$= k\ell \| \sum_{r=1}^{\min\{c,d\}} \sum_{\substack{\overline{a}'\in\{1,\ldots,c\}^{k-1} \\ \overline{a}'\in\{1,\ldots,c\}^{\ell-1}}} X^{(\tilde{a}',r)} Y^{(r,\overline{b}')} \Omega_{rr} \tilde{b}'$$
$$\in \{1,\ldots,d\}^{\ell-1} \left(eãí \otimes \cdots \otimes e_{\overline{a}'_{k-1}} \right) \otimes_\pi \left(fb_í \otimes \cdots \otimes f_{\overline{b}'_{\ell-1}} \right) \|^*_{\alpha\otimes_\pi\beta}$$

$$\le k\ell \sum_{r=1}^{\min\{c,d\}} \| \sum_{\overline{a}'\in\{1,\ldots,c\}^{k-1}} X^{(\overline{a}',r)} eãí \otimes \otimes e_{\tilde{a}'_{k-1}} \|^*_\alpha \| \sum_{\tilde{b}'\in\{1,\ldots,d\}^{\ell-1}} Y^{(r,\tilde{b}')} fb^{\sim} í \otimes \cdots$$
$$\otimes f_{\overline{b}'_{\ell-1}} \|^*_\beta$$

$$\overset{\text{cs}}{\le} \sqrt{k\ell} \|X\|^*_\alpha \|Y\|^*_\beta,$$

where we have used in the last line after applying the Cauchy-Schwarz inequality that

$$\sum_{r=1}^{\min\{c,d\}} \| \sum_{\overline{a}'\in\{1,\ldots,c\}^{k-1}} X\,(\text{ã}', r)\; eãí \otimes \otimes e_{\tilde{a}'_{k-1}} \|^{*2}_\alpha = \sum_{r=1}^{\min\{c,d\}} \sum_{\overline{a}'\in\{1,\ldots,c\}^{k-1}} |X^{(\tilde{a}',r)}|^2 (k-1)!$$
$$\le \frac{1}{k} \|X\|^{*2}_\alpha$$

and analogously for Y.

Proposition (6.2.11)[280]: *Let Λ be a continuous bilinear form on V, then the function* P_Λ *is continuous and fulfils the estimate*

$$\left\| (\mathrm{P}_\Lambda)^t(Z) \|^*_{\alpha\otimes_\pi\beta} \le \frac{c}{c-1} \frac{t!}{c^t} \right\| Z \|^*_{2c\alpha\otimes_\pi 2c\beta} \qquad (94)$$

for all $c > 1$, all $t \in \mathrm{M}_0$, all seminorms $\|\cdot\|_\alpha, \|\cdot\|_\beta \in \mathcal{P}_{V,\Lambda}$, and all $Z \in T^\cdot(V) \otimes_\pi T^\cdot(V)$
.

Proof: Let $X, Y \in T^\cdot(V)$ be given, then the previous Lemma (6.2.10) together with Lemma (6.2.5) yields

$$\|(\mathrm{P}_\Lambda)^t(X \otimes_\pi Y)\|^*_{\alpha\otimes_\pi\beta} \le \sum_{k,\ell=0}^{\infty} \| (\mathrm{P}_\Lambda)^t(\langle X\rangle_{k+t} \otimes_\pi \langle Y\rangle_{\ell+t})\|^*_{\alpha\otimes_\pi\beta}$$
$$\le t! \sum_{k,\ell=0}^{\infty} \quad (k|+tt)\ \frac{1}{2}\ (p|+tt)\ \frac{1}{2} \|\langle X\rangle_{k+t}\|^*_\alpha \|\langle Y\rangle_{\ell+t}\|^*_\beta$$

$$\leq t! \sum_{k,\ell=0}^{\infty} \| \langle X \rangle_{k+t} \|_{2\alpha}^{*} \| \langle Y \rangle_{\ell+t} \|_{2\beta}^{*}$$
$$= \frac{t!}{c^t} \sum_{k,\ell=0}^{\infty} \frac{1}{\sqrt{c}^{k+\ell}} \| \langle X \rangle_{k+t} \|_{2c\alpha}^{*} \| \langle Y \rangle_{\ell+t} \|_{2c\beta}^{*}$$
$$\text{cs} \leq \frac{t!}{c^t} \left(\sum_{k,\ell=0}^{\infty} \frac{1}{c^{k+\ell}} \right)^{\frac{1}{2}} \left(\sum_{k,\ell=0}^{\infty} \| \langle X \rangle_{k+t} \|_{2c\alpha}^{*2} \| \langle Y \rangle_{\ell+t} \|_{2c\beta}^{*2} \right)^{\frac{1}{2}}$$
$$\leq \frac{c}{c-1} \frac{t!}{c^t} \|X\|_{2c\alpha}^{*} \|Y\|_{2c\beta}^{*}. \square$$

Lemma (6.2.12)[280]: *Let Λ be a continuous bilinear form on V, then $\mu_{\star_\Lambda}$ is continuous and, given $R > 1/2$, the estimate*

$$\left\| \mu_{\star_{z\Lambda}}(Z) \right\|_{\gamma}^{*} \leq \sum_{t=0}^{\infty} \frac{1}{t!} \left\| \mu_{\vee}\left((\mathrm{P}_{z\Lambda})^t (Z) \right) \right\|_{\gamma}^{*} \leq \frac{4R}{2R-1} \|Z\|_{8R\gamma \otimes_\pi 8R\gamma}^{*} \quad (95)$$

holds for all $\| \cdot \|_\gamma \in \mathcal{P}_{V,\Lambda}$, all $Z \in T^{\cdot}(V) \otimes_\pi 7^{\cdot}(V)$ and all $z \in \mathrm{C}i$ with $|z| \leq R$.

Proof.: The first estimate is just the triangle-inequality. By combining Corollary (6.2.8) and Proposition (6.2.11) with $c = 2R$ we get the second estimate

$$\sum_{t=0}^{\infty} \frac{1}{t!} \| \mu_{\vee}\left((\mathrm{P}_{z\Lambda})^t (Z) \right) \|_{\gamma}^{*} \leq \sum_{t=0}^{\infty} \frac{|z|^t}{t!} \| (\mathrm{P}_{\Lambda})^t (Z) \|_{2\gamma \otimes_\pi 2\gamma}^{\cdot}$$
$$\leq \frac{2R}{2R-1} \sum_{t=0}^{\infty} \frac{1}{2^t} \|Z\|_{8R\gamma \otimes_\pi 8R\gamma}$$
$$= \frac{4R}{2R-1} \|Z\|_{8R\gamma \otimes_\pi 8R\gamma}. \square$$

This estimate immediately leads to:

Theorem (6.2.13)[280]: *Let Λ be a continuous bilinear form on V, then the product $\star_\Lambda$ is continuous and $(S^{\cdot}(V), \star_\Lambda)$ is a locally convex algebra. Moreover, for fixed tensors X, Y from the completion $S^{\cdot}(V)^{\mathrm{cp1}}$, the product $X \star_{z\Lambda} Y$ converges absolutely and locally uniformly in $z \in \mathbb{C}$ and thus depends holomorphically on z.*

Note that the above estimate also shows that $(S^{\cdot}(V), \star_{z\Lambda})$ describes a holomorphic deformation (as in [107]) of the locally convex algebra $(S^{\cdot}(V), \vee)$. However, in the following we will examine the star product for fixed values of both Λ and z and therefore can absorb the deformation parameter z in the bilinear form Λ.

we want to examine some properties of the products $\star_\Lambda$, namely how the topology on $S^{\cdot}(V)$ can be characterized by demanding that certain algebraic operations are continuous, which products are equivalent, how to transform $S^{\cdot}(V)$ to a space of complex functions, the existence of continuous positive linear functionals and whether or not some exponentials of elements in $S^{\cdot}(V)$ exist and which elements are represented by essentially self-adjoint operators via GNS construction. At some points we will also work with the completion $S^{\cdot}(V)^{\mathrm{cp1}}$ of $S^{\cdot}(V)$ and therefore note that the previous constructions and results extend to $S^{\cdot}(V)^{\mathrm{cp1}}$ by continuity.

We show that the topology on $S^{\cdot}(V)$ that was defined in a rather unmotivated way is— under some additional assumptions— the coarsest possible one. We want to express the extensions of positive Hermitian forms with the help of suitable star products. Due to the sesquilinearity of positive Hermitian forms, this is only possible if we also have an antilinear structure on $S^{\cdot}(V)$, so we construct a*-involution.
There is clearly one and only one possibility to extend an antilinear involution $-$ on V to a $*$-involutio n*: $T^{\cdot}(V) \to T^{\cdot}(V)$ on the tensor algebra over V, namely by $(x_1 \otimes \cdots \otimes x_k)^*$ $:= \overline{x}_k \otimes \cdots \otimes \overline{x}_1$ for all $k \in$ M and $x \in V^k$ and antilinear extension. Its restriction to $S^{\cdot}(V)$ gives a*-involution on $(S^{\cdot}(V), \vee)$.

Proposition (6.2.14)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then the induced*$*$*-involution on* $7^{\cdot}(V)$ *is also continuous*.

Proof: For$\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ define the continuous positive Hermitian form $V^2 \ni (v,w) \mapsto \langle v|w\rangle_{\alpha^*} := \overline{\langle \overline{v}|\overline{w}\rangle_\alpha}$. Then $\langle X^*|Y^*\rangle_\alpha = \langle X|Y\rangle_{\alpha^*}$ and in particular $\|X^*\|_\alpha = \|X\|_{\alpha^*}$ for all X, $Y \in T^{\cdot}(V)$ because this is clearly true for simple tensors and because both sides are (anti-)linear in X and Y. □

For certain bilinear forms Λ on V we can also show that $*$ is a $*$-involution of $\star_\Lambda$, which is of course not a new result:

Definition (6.2.15)[280]: Let$\bar{\cdot}$: $V \to V$ *be a continuous antilinear involution on V. For every continuous bilinear form* $\Lambda: V \times V \to \mathbb{C}$ t*ve define its conjugate* Λ^* *by* $\Lambda^*(v,w) := \Lambda(\overline{w}, \overline{v})$ *, which is again a continuous bilinear form on V. We say that* Λ *is Hermitian if* $\Lambda = \Lambda^*$ *holds*.

Note that the bilinear form $(v,w) \mapsto \Lambda(v,w)$ is Hermitian if and only if the sesquilinear form $(v,w) \mapsto \Lambda(\overline{v},w)$ is Hermitian. The typical example of a complex vector space V with antilinear involutio n$^-$ is that $V = W \otimes$ (D is the complexification of a real vector space W with the canonical involution $\overline{w \otimes \lambda} := w \otimes \overline{\lambda}$. In this case, every bilinear form Λ on V is fixed by two bilinear forms Λ_r, Λ_i: $W \times W \to \mathbb{R}$, the restriction of the real- and imaginary part of Λ to the real subspace $W \cong W \otimes 1$ of V, and Λ is Hermitian if and only if Λ_r is symmetric and Λ_i antisymmetric. Similarly to [101] we get:

Proposition (6.2.16)[280]: Let$\bar{\cdot}$: $V \to V$ *be a continuous antilinear involution and* Λ *a continuous bilinear form on V. Then* $(X \star_\Lambda Y)^* = Y^* \star_{\Lambda^*} X^*$ *holds for all* $X, Y \in S^{\cdot}(V)$ *. Consequently, if* Λ *is Hermitian, then* $(S^{\cdot}(V), \star_\Lambda, *)$ *is a locally convex* $*$*-algebra*.

Proof: The identitie s*o$\mathcal{S}^{\cdot} = \mathcal{S}^{\cdot}0^*$ and $*$ $0\mu \otimes = \mu \otimes 0\tau$o$(^*\otimes_\pi^*)$, with $\tau: T^{\cdot}(V) \otimes_\pi T^{\cdot}(V) \to 7^{\cdot}(V) \otimes_\pi T^{\cdot}(V)$ defined as $\tau(X \otimes_\pi Y) := Y \otimes_\pi X$, can easily be checked on simple tensors, so $*$o $\mu_\vee = \mu_{\vee^0}\tau$o$(\otimes_\pi^*)$. Combining this with τo$(^*\otimes_\pi^*)$ oP$_\Lambda$ = P$_\Lambda * 0\tau$o$(^*\otimes_\pi^*)$ on symmetric tensors, which again can easily be checked on simple symmetric tensors, yields the desired result.

Lemma (6.2.17)[280]: Let$\bar{\cdot}$: $V \to V$ *be a continuous antilinear involution. For every* $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ *we define a continuous bilinear form* Λ_α *on* V *by* $\Lambda_\alpha(v,w) := \langle \overline{v}|w\rangle_\alpha$ *for all* , $w \in V$, *then* Λ_α *is Hermitian and the identities*

$$\sum_{t=0}^{\infty} \frac{1}{t!} \mu \otimes \left(\left(\mathrm{P}_{\Lambda_\alpha}\right)^t (\langle X^* \rangle_t \otimes_\pi \langle Y \rangle_t) \right) = \langle X|Y \rangle_\alpha^* \qquad (96)$$

and

$$\langle \mu_{\star_{\Lambda_\alpha}}(X^* \otimes_\pi Y)\rangle_0 = \langle X|Y\rangle_\alpha^* \tag{97}$$

hold for all $X, Y \in T^{\cdot}(V)$.

Proof: Clearly, Λ_α is Hermitian because $\langle\cdot\mid\cdot\rangle_\alpha$ is Hermitian. Then (97) follows directly from (96) because of the grading of μ_V and P_{Λ_α}. For proving (96) it is sufficient to check it for factorizing tensors of the same degree, because both sides are (anti-)linear in X and Y and vanish if X and Y are homogeneous of different degree. If X and Y are of degree 0 then (96) is clearly fulfiled. Otherwise we get

$$\frac{1}{k!}\mu\otimes\left(\left(P_{\Lambda_\alpha}\right)^k\left((x_1\otimes\cdots\otimes x_k)^*\otimes_\pi(y_1\otimes\cdots\otimes y_k)\right)\right)$$
$$=\frac{1}{k!}\mu\otimes\left(\left(P_{\Lambda_\alpha}\right)^k\left((\overline{x}_k\otimes\cdots\otimes\overline{x}_1)\otimes_\pi(y_1\otimes\cdots\otimes y_k)\right)\right)$$
$$=\frac{1}{k!}\mu\otimes\left((1\otimes_\pi 1)(k!)^2\prod_{m=1}^{k}\Lambda_\alpha(\overline{x}_m,y_m)\right)$$
$$=k!\prod_{m=1}^{k}\Lambda_\alpha(\overline{x}_m,y_m)$$
$$=k!\prod_{m=1}^{k}\langle x_m|y_m\rangle_\alpha$$
$$=\langle x_1\otimes\cdots\otimes x_k|y_1\otimes\cdots\otimes y_k\rangle_\alpha^*. \ \square$$

Theorem (6.2.18)[280]: *The topology on* $S^{\cdot}(V)$ *is the coarsest locally convex one that makes all star products* $\star_\Lambda$ *for all continuous and Hermitian bilinear forms* Λ *on* V *as well as the* $*$*-involution and the projection* $\langle\cdot\rangle_0$ *onto the scalars continuous. In addition we have for all* $X, Y \in S^{\cdot}(V)$ *and all* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_V$

$$\langle X^* \star_{\Lambda_\alpha} Y\rangle_0 = \langle X|Y\rangle_\alpha^*, \tag{98}$$

with Λ_α *as in Lemma* (6.2.17).

Proof.: We have already shown the continuity of the star product and of the $*$-involution, the continuity of $\langle\cdot\rangle_0$ is clear. Conversely, if these three functions are continuous, their compositions yield the extensions of all $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_V$ which then have to be continuous. Then (97) gives (98) for symmetric tensors X and Y. □

Next we want to examine the usual equivalence transformations between star products, given by exponentials of a Laplace operator (see [101] for the algebraic background).

Definition (6.2.19)[280]: *Let* $b: V\times V \to \mathbb{C}$ *be a symmetric bilinear form on* V, *i.e.* $b(v,w) = b(w,v)$ *for all* v, $w \in V$. *Then we define the Laplace operator* $\triangle_b: T^{\cdot}(V) \to T^{-2}(V)$ *as the linear map given on simple tensors of degree* $k \in \mathrm{M}\backslash\{1\}$ *by*

$$\triangle_b\ (x_1\otimes\cdots\otimes x_k) := \frac{k(k-1)}{2} b(x_1,x_2)x_3\otimes\cdots\otimes x_k. \tag{99}$$

Note that $\triangle_b$ can be restricted to symmetric tensors on which it coincides with the Laplace operator from [101]. However, there is no need for $\triangle_b$ to be continuous even if b is continuous, because the Hilbert tensor product in general does not allow the extension of all continuous multilinear forms.

Note that this is very different from the approach taken in [101] where the projective tensor product was used: this guaranteed the continuity of the Laplace operator directly for all continuous bilinear forms.

For the restriction of $\triangle_b$ to $S^2(V)$, continuity is equivalent to the existence of a $\|\cdot\|_\alpha \in \mathcal{P}_V$ that fulfils $|\triangle_b X| \leq \|X\|_\alpha$ for all $X \in S^2(V)$. This motivates the following:

Definition (6.2.20)[280]: *A bilinear form of Hilbert-Schmidt type on V is a bilinear form $b: V \times V \to \mathrm{C}i$ for which there is a seminorm $\|\cdot\|_\alpha \in \mathcal{P}_V$ such that the following two conditions are fulfiled*:

$i.$) *If $\|v\|_\alpha = 0$ or $\|w\|_\alpha = 0$ for vectors , $w \in V$, then $b(v,w) = 0$.*

$ii.$) *For every tuple of $\langle\cdot\mid\cdot\rangle_\alpha$-orthonormal vectors $\in V^d$, $d \in \mathrm{N}$, the estimate*

$$\sum_{i,j=1}^{d} |b(e_i, e_j)|^2 \leq 1 \tag{100}$$

holds.

For such a bilinear form of Hilbert-Schmidt type b we define $\mathcal{P}_{V,b,HS}$ as the set of all $\|\cdot\|_\alpha \in \mathcal{P}_V$ that fulfil these two conditions.

We can characterize the bilinear forms of Hilbert-Schmidt type in the following way:

Proposition (6.2.21)[280]: *Let b be a symmetric bilinear form on V and $\|\cdot\|_\alpha \in \mathcal{P}_V$, then the following two statements are equivalent*:

$i.$) *The bilinear form b is of Hilbert-Schmidt type and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.*

$ii.$) *The estimate $|\triangle_b X| \leq 2^{-1/2}\|X\|_\alpha$ holds for all $\in S^2(V)$.*

Moreover, if this holds then $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$ and b is continuous.

Proof.: If the first point holds, let $X \in T^2(V)$ be given. Construct $X_0 = \sum_{a\in A} x_{a,1} \otimes x_{a,2}$ and $\tilde{X} = \sum_{a_1',a_2'=1}^{d} X^{a_1',a_2'}\, e_{a_1'} \otimes e_{a_2'} \in T^2(V)$ like in Lemma (6.2.3). Then $b(x_{a,1}, x_{a,2}) = 0$ for all $a \in A$ because $\|x_{a,1}\|_\alpha = 0$ or $\|x_{a,2}\|_\alpha = 0$. Moreover,

$$|\triangle_b X| \leq \left|\sum_{a_1',a_2'=1}^{d} X^{a_1',a_2'}\, b(e_{a_1'}, e_{a_2'})\right|$$

$$\text{cs} \leq \left(\sum_{a_1',a_2'=1}^{d} |X^{a_1',a_2'}|^2\right)^{\frac{1}{2}} \left(\sum_{a_1',a_2'=1}^{d} |b(e_{a_1'}, e_{a_2'})|^2\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}}\|X\|_\alpha^*$$

shows that the second point holds. Conversely, from the second point we get $|b(v,w)| = |\triangle_b (v \vee w)| \leq 2^{-1/2}\|v \vee w\|_\alpha \leq \|v\|_\alpha\|w\|_\alpha$ for all , $w \in V$. Hence $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$, the bilinear form b is continuous, and $b(v,w) = 0$ if one of v or w is in the kernel of $\|\cdot\|_\alpha$. Moreover, given an $\langle\cdot\mid\cdot\rangle_\alpha$-orthonormal set of vectors $\in V^d$, $d \in \mathrm{M}$, we define $X := \sum_{j=1}^{d} \overline{b(e_i, e_j)}\, e_i \otimes e_j \in S^2(V)$ and get

$$0 \leq \sum_{i,j=1}^{d} |\, b\big(e_i, e_j\big)|^2 = |\, \triangle_b\, X| \leq \frac{1}{\sqrt{2}} \|X\|_\alpha^* = \left(\sum_{i.j=1}^{d} |\, b\big(e_i, e_j\big)|^2 \right)^{\frac{1}{2}}$$

which implies $\sum_{j=1}^{d} |\, b\big(e_i, e_j\big)|^2 \leq 1.$ □

Note that this also implies that for a bilinear form of Hilbert-Schmidt type b, the set $\mathcal{P}_{V,b,HS}$ is cofinal in $\mathcal{P}_V$, because if $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$, $\|\cdot\|_\beta \in \mathcal{P}_V$ and $\|\cdot\|_\beta \geq \|\cdot\|_\alpha$, then $|\, \triangle_b\, X| \leq 2^{-\frac{1}{2}} \left\| X \|_\alpha^* \leq 2^{-\frac{1}{2}} \right\| X\|_\beta^*$ and so $\|\cdot\|_\beta \in \mathcal{P}_{V,b,HS}$.

As a consequence of the above characterization we see that a symmetric bilinear form b on V has to be of Hilbert-Schmidt type if we want $\triangle_b$ to be continuous. We are going to show now that this is also sufficient:

Proposition (6.2.22)[280]: *Let b be a symmetric bilinear form of Hilbert-Schmidt type on V, then the Laplace operator $\triangle_b$ is continuous and fulfils the estimate*

$$\left\|(\triangle_b)^t X\|_\alpha^* \leq \frac{\sqrt{(2t)!}}{(2r)^t}\right\| X\|_{2r\alpha}^* \tag{101}$$

for all $X \in T^*(V)$, $t \in \mathrm{M}_0$, $r \geq 1$, and all $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.

Proof.: First, let $\in 7^k(V)$, $k \geq 2$, and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$ be given. Construct $X_0 = \sum_{a\in A} x_{a,1} \otimes \ldots \otimes x_{a,k}$ and $\tilde{X} = \sum_{a' \in \{1.....d\}^k} X^{a'}\, e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ like in Lemma (6.2.3). Then again

$$\|\, \triangle_b\, X_0\|_\alpha^* \leq \frac{k(k-1)\sqrt{(k-2)!}}{2} \sum_{a\in A} |\, b\big(x_{a_1}, x_{a_2}\big)| \coprod_{m=3}^{k} \|\, x_{a_m}\|_\alpha = 0$$

shows that $\|\, \triangle_b\, X\|_\alpha^* \leq \|\, \triangle_b\, \tilde{X}\|_\alpha^*$. For$\tilde{X}$ we get:

$$\|\, \triangle_b\, \tilde{X}\|_\alpha^{*2} = \| \frac{k(k-1)}{2} \sum_{a' \in \{1,\ldots,d\}^k} X^{a'}\, b\big(e_{a'_1}, e_{a'_2}\big) e_{a'_3} \otimes \cdots \otimes e_{a'_k}\|_\alpha^{*2}$$

$$= \frac{k^2(k-1)^2}{4} \sum_{\overline{a}' \in \{1,\ldots,d\}^{k-2}} \| \sum_{g,h=1}^{d} X^{\left(g,h,\overline{a}'\right)}\, b\ (e_g, eh) e\tilde{a}í \otimes \cdots \otimes e_{\tilde{a}_{k-2}}\|_\alpha^{*2}$$

$$= \frac{k^2(k-1)^2}{4} \sum_{\overline{a}' \in \{1,\ldots,d\}^{k-2}} | \sum_{g,h=1}^{d} X^{(g,h,\tilde{a}')}\, b\big(e_g, e_h\big)|^2 (k-2)!$$

$$\leq \frac{k(k-1)k!}{4}\ \tilde{a}’ \sum_{\in \{1,\ldots,d\}^{k-2}} \left(\sum_{g,h=1}^{d} |\, X^{(g,h,\tilde{a}')}||b\big(e_g, e_h\big)| \right)^2$$

$$cs \leq \frac{k(k-1)k!}{4} \sum_{\overline{a}' \in \{1,\ldots,d\}^{k-2}} \left(\sum_{g,h=1}^{d} |\, X^{(g,h,\tilde{a}')}|^2 \right) \left(\sum_{g,h=1}^{d} |\, b\big(e_g, e_h\big)|^2 \right)$$

$$\leq \frac{k(k-1)k!}{4} \sum_{a' \in \{1,\ldots d\}^k} ,|X^{a'}|^2$$

$$= \frac{k(k-1)}{4} \|X\|_\alpha^{*2}$$

Using this we get

$$\|(\triangle_b)^t X\|_\alpha^{*2} = \sum_{k=2t}^{\infty} \|(\triangle_b)^t \langle X\rangle_k\|_\alpha^{*2}$$
$$\leq \sum_{k=2t}^{\infty} \binom{k}{2t} \frac{(2t)!}{4^t} \|\langle X\rangle_k\|_\alpha^{*2}$$
$$\leq \frac{(2t)!}{4^t} \sum_{k=2t}^{\infty} \frac{1}{r^k} \|\langle X\rangle_k\|_{2r\alpha}^{*2}$$
$$\leq \frac{(2t)!}{(2r)^{2t}} \|X\|_{2r\alpha}^{*2}$$

for arbitrary $X \in T^{\cdot}(V)$ and $t \in$ N. Finally, the estimate (101) also holds in the case $t = 0$.

Theorem (6.2.23)[280]: *Let b be a symmetric bilinear form on V, then the linear operator $e^{\triangle_b} = \sum_{t=0}^{\infty} \frac{1}{t!} (\triangle_b)^t$ as well as its restriction to $S^*(V)$ are continuous if and only if b is of Hilbert-Schmidt type. In this case*

$$e^{\triangle_b}(X \star_\Lambda Y) = \left(e^{\triangle_b} X\right) \star_{\Lambda+b} \left(e^{\triangle_b} Y\right) \qquad (102)$$

holds for all $X, Y \in S^{\cdot}(V)$ and all continuous bilinear forms Λ on V. Hence $e^{\triangle_b}$ describes an isomorphism of the locally convex algebras $(S^{\cdot}(V), \star_\Lambda)$ and $(S^{\cdot}(V), \star_{\Lambda+b})$. Moreover, for fixed $X \in S^{\cdot}(V)^{\text{cp1}}$, the series $e^{z\triangle_b} X$ converges absolutely and locally uniformly in $z \in$ Ci and thus depends holomorphically on z.

Proof.: As $|\triangle_b X| \leq \|e^{\triangle_b} X\|_\alpha^*$ holds for all $\|\cdot\|_\alpha \in \mathcal{P}_V$ and all $\in S^2(V)$, it follows from Proposition (6.2.21) that continuity of the restriction of $e^{\triangle_b}$ to $S^*(V)$ implies that b is of Hilbert-Schmidt type. Conversely, for all $\in T^{\cdot}(V)$, all $\alpha \in \mathcal{P}_{V,b,HS}$, and $r > 1$, the estimate

$$\|e^{z\triangle_b} X\|_\alpha \leq \sum_{t=0}^{\infty} \frac{1}{t!} \|(z\triangle_b)^t (X)\|_\alpha \leq \sum_{t=0}^{\infty} \frac{|z|^t}{(4r)^t} \binom{2t}{t} \frac{1}{2} \left\| X \right\|_{4r\alpha}^* \leq \sum_{t=0}^{\infty} \frac{1}{2^t} \left\| X \right\|_{4r\alpha}^*$$
$$= 2\|X\|_{4r\alpha}^*$$

holds for all $z \in \mathbb{C}$ with $|z| \leq r$ due to the previous Proposition (6.2.22) if b is of Hilbert-Schmidt type, which proves the continuity of $e^{z\triangle_b}$ for all $z \in \mathbb{C}$ as well as the absolute and locally uniform convergence of the series $e^{z\triangle_b} X$. The algebraic relation (102) is well-known, see e.g. [101]. Finally, as $e^{\triangle_b}$ is invertible with inverse $e^{-\triangle_b}$, and because $\triangle_b$ and thus $e^{\triangle_b}$ map symmetric tensors to symmetric ones, we conclude that the restriction of $e^{\triangle_b}$ to $S^*(V)$ is an isomorphism of the locally convex algebras $(S^*(V), \star_\Lambda)$ and $(S^*(V), \star_{\Lambda+b})$.

We construct an isomorphism of the undeformed $*$-algebra $(S^*(V), \vee, *)$ to a $*$-algebra of smooth functions by a construction similar to the Gel'fand transformation of commutative C^*-algebras.

Let $\bar{\cdot}$ be a continuous antilinear involution on V. We write V_h for the real linear subspace of V consisting of Hermitian elements, i.e.

$$V_h := \{v \in V | \overline{v} = v\}. \qquad (103)$$

The inner products compatible with the involution are denoted by

$$\mathcal{I}_{V,h} := \{\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_V | \overline{\langle v|w\rangle_\alpha} = \langle \overline{v}|\overline{w}\rangle_\alpha \ for\ all\ v,w \in V\}. \quad (104)$$

Moreover, we write V' for the topological dual space of V and V_h' again for the real linear subspace of V' consisting of Hermitian elements, i.e.

$$V_h' := \{\rho \in V' | \overline{\rho(v)} = \rho(\overline{v})\ for\ all\ v \in V \quad (105)$$

Finally, recall that a subset $B \subseteq V_h'$ is *bounded* (with respect to the equicontinuous bornology) if there exists a $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ such that $|p(v)| \leq \|v\|_\alpha$ holds for all $v \in V$ and all $p \in B$. This also gives a notion of boundedness of functions from or to V_h': A (multi-)linear function is bounded if it maps bounded sets to bounded ones.

Note that one can identify V_h' with the topological dual of V_h and $\mathcal{I}_{V,h}$ with the set of continuous positive bilinear forms on V_h. Moreover, $\mathcal{I}_{V,h}$ is cofinal in $\mathcal{I}_V$: every $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_V$ is dominated by $V^2 \ni (v,w) \mapsto \langle v|w\rangle_\alpha + \overline{\langle \overline{v}|\overline{w}\rangle}_\alpha \in$ Cj.

Definition (6.2.24)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $\rho \in V_h'$, *then we define the derivative in direction of* ρ *as the linear map* $D_\rho: T^*(V) \to T^{*-1}(V)$ *by*

$$x_1 \otimes \cdots \otimes x_k \mapsto D_\rho\,(x_1 \otimes \cdots \otimes x_k) := k\rho(x_k)x_1 \otimes \cdots \otimes x_{k-1} \quad (106)$$

for all $k \in$ M *and all* $x \in V^k$. *Next, we define the translation by* p *as the linear map*

$$\tau_\rho^* := \sum_{t=0}^{\infty} \frac{1}{t!}\left(D_\rho\right)^t : \mathcal{T}^*(V) \to \mathcal{T}^*(V)\,, \quad (107)$$

and the evaluation at ρ *by*

$$\delta_\rho := \langle\cdot\rangle_0 \circ \tau_\rho^*: T^*(V) \to \mathbb{C}. \quad (108)$$

Finally, for $k \in$ N *and* $\rho_1, \ldots, \rho_k \in V_h$ *we set* $D^{(k)}_{\beta 1 \cdots, \beta k} := D_{\beta 1} \cdots D_{\rho_k}: T^*(V) \to T^{*-k}(V)$.

Note that τ_p^* is well-defined because for every $X \in T^*(V)$ only finitely many terms contribute to the infinite series $\tau_\rho^* X = \sum_{t=0}^{\infty} \frac{1}{t!}\left(D_\rho\right)^{\,t}(X)$. Note also that D_ρ and consequently also τ_ρ^* can be restricted to endomorphisms of $S^{\cdot}(V)$. Moreover, this restriction of D_ρ is a $*$-derivation of all the $*$-algebras $(S^*(V), \star_\Lambda, *)$ for all continuous Hermitian bilinear forms Λ on V (see [101], the compatibility with the $*$-involution is clear), so that τ_ρ^* turns out to be a unital $*$-automorphism of these $*$-algebras.

Lemma (6.2.25)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* , $\sigma \in V_h'$. *Then*

$$\left(D_\rho D_\sigma - D_\sigma D_\beta\right)(X) = \left(\tau_p^* D_\sigma - D_\sigma \tau_\rho^*\right)(X) = \left(\tau_p^* \tau_\sigma^* - \tau_\sigma^* \tau_p^*\right)(X) = 0 \quad (109)$$

holds for all $X \in S^*(V)$.

Proof.: It is sufficient to show that $\left(D_\rho D_\sigma - D_\sigma D_\rho\right)(X) = 0$ for all $\in S^*(V)$, which clearly holds if X is a homogeneous factorizing symmetric tensor and so holds for all $X \in S^*(V)$ by linearity.

Lemma (6.2.26)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $\rho \in V_h'$. *Then* D_ρ, τ_p^* *and* δ_ρ *are all continuous. Moreover, if* $\|\cdot\|_\alpha \in \mathcal{P}_V$ *fulfils* $|\rho(v)| \leq \|v\|_\alpha$, *then the estimates*

$$\left\|\left(D_p\right)^t X\right\|_\alpha^* \leq \sqrt{t!}\left\| X\right\|_{2\alpha}^* \quad (110)$$

and

$$\left\| \tau_p^*(X) \|_\alpha^* \leq \sum_{t=0}^{\infty} \frac{1}{t!} \right\| \left(D_\rho\right)^{t'} X \|_\alpha^* \leq \frac{2}{\sqrt{2}-1} \|X\|_{2\alpha}^* \tag{111}$$

hold for all $X \in 7^*(V)$ *and all* $t \in \mathrm{M}_0$.

Proof Let $\| \cdot \|_\alpha \in \mathcal{P}_V$ be given such that $|\rho(v)| \leq \|v\|_\alpha$ holds for all $v \in V$. For all $d \in \mathrm{M}_0$ and all $\langle \cdot \mid \cdot \rangle_\alpha$-orthonormal $e \in V^d$ we then get

$$\sum_{i=1}^{d} | \rho(e_i)|^2 d = \rho\left(\sum_{i=1}^{d} e_i \overline{p(e_i)}\right) d \leq \| \sum_{i=1}^{d} e_i \overline{\rho(e_i)} \|_\alpha = \left(\sum_{i=1}^{d} | \rho(e_i)|^2\right)^{\frac{1}{2}} d$$

$$i = 1 \; i = 1 \; i = 1$$

hence $\sum_{i=1}^{d} | \rho(e_i)|^2 \leq 1$. Given $k \in \mathrm{M}$ and a tensor $\in T^k(V)$, then we construct $X_0 = \sum_{a \in A} x_{a,1} \otimes \ . \ . \ . \otimes x_{a,k}$ and $\tilde{X} = \sum_{a' \in \{1.....d\}^k} X^{a'} \, e_{a_1'} \otimes \cdots \otimes e_{a_k'}$ like in Lemma (6.2.3). Then we have $\|D_\rho X_0\|_\alpha^* = 0$ because

$$\|D_\rho \left(x_{a,1} \otimes \cdot \cdot \cdot \otimes x_{a,k}\right)\|_\alpha^* = k|p\left(x_{a,k}\right)| \|x_{a,1} \otimes \cdot \cdot \cdot \otimes x_{a,k-1}\|_\alpha^*$$

$$\leq k\sqrt{(k-1)!} \coprod_{m=1}^{k} \| x_{a,m}\|_\alpha = 0$$

holds for all $a \in A$. Consequently $\|D_\rho X\|_\alpha \leq \|D_\rho \tilde{X}\|_\alpha$ and we get

$$\|D_\rho X\|_\alpha^{*2} \leq \|D_\rho \tilde{X}\|_\alpha^{*2} = \| \sum_{a' \in \{1,\ldots,d\}^k} X^{a'} \, D_\rho \left(e_{a_1'} \otimes \cdots \otimes e_{a_k'}\right) \|_\alpha^{*2}$$

$$= k^2 \text{ ã'} \sum_{\in \{1,\ldots,d\}^{k-1}} \| \sum_{g=1}^{d} X\,(\text{ã'}, g)_{\rho(e_g) e_{\overline{a}}} \text{ í } \otimes \cdots \otimes e_{\overline{a}_{k-1}'} \|_\alpha^{*2}.,$$

$$\leq k^2 (k-1)! \text{ ã'} \sum_{\in \{1,\ldots,d\}^{k-1}} \left(\sum_{g=1}^{d} | X(\text{ã''} g)| |\rho\left(e_g\right)|\right)^2$$

$$\overset{CS}{\leq} k^2 (k-1)! \sum_{\overline{a}' \in \{1,\ldots,d\}^{k-1}} \left(\sum_{g=1}^{d} | X^{(\tilde{a}',g)}|^2\right)\left(\sum_{g=1}^{d} | p\left(e_g\right)|^2\right)$$

$$\leq k^2 (k-1)! \sum_{a' \in \{1,\ldots,d\}^k} | X^{a'}|^2$$

$$= k\|X\|_\alpha^{*2}$$

Using this we can derive the estimate (110), which also proves the continuity of D_ρ: If $t = 0$, then this is clearly fulfiled. Otherwise, let $X \in T^\cdot(V)$ be given, then

$$\|\left(D_\rho\right)^t X\|_\alpha^{*2} = \sum_{k=t}^{\infty} \| \left(D_\rho\right)^t \langle X \rangle_k\|_\alpha^{*2} \leq t! \sum_{k=t}^{\infty} \binom{k}{t} \|\langle X \rangle_k\|_\alpha^{*2} \leq t! \sum_{k=t}^{\infty} \| \langle X \rangle_k\|_{2\alpha}^{*2}$$

$$\leq t! \, \|X\|_{2\alpha}^{*2}.$$

From this we can now also deduce the estimate (111), which then shows continuity of τ_β^* and of $\delta_\rho = \langle\cdot\rangle_0 \circ \tau_\rho^*$: The first inequality is just the triangle inequality and for the second we use that $t! \geq 2^{t-1}$ for all $t \in \mathrm{M}_0$, so

$$\sum_{t=0}^{\infty} \frac{1}{t!} \left\| \left(D_\rho\right)^t X \right\|_\alpha^* \leq \sum_{t=0}^{\infty} \frac{1}{\sqrt{t!}} \left\| X \right\|_{2\alpha}^* \leq \sqrt{2} \sum_{t=0}^{\infty} \frac{1}{\sqrt{2}^t} \left\| X \right\|_{2\alpha}^* \leq \frac{2}{\sqrt{2}-1} \left\| X \right\|_{2\alpha}^*.$$

Proposition (6.2.27)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then the set of all continuous unital*$*$*-homomorphisms from* $(S^*(V)^{\mathrm{cp1}}, \vee, *)$ *to* $\mathbb{C}$ *is* $\{\delta_p | \rho \in V_h'\}$ (*strictly speaking, the continuous extensions to* $S^{\cdot}(V)^{\mathrm{cp1}}$ *of the restrictions of* δ_ρ *to* $S^*(V)$).

Proof On the one hand, every such δ_ρ is a continuous unital $*$-homomorphism, because $\langle\cdot\rangle_0$ and τ_ρ^* are. On the other hand, if $\varphi: (S^{\cdot}(V)^{\mathrm{cp1}}, \vee, *) \to \mathbb{C}$ is a continuous unital $*$-homomorphism, then $V \ni v \mapsto p(v) := \varphi(v) \in \mathbb{C}$ is an element of V_h' and fulfils $\delta_\rho = \varphi$ because the unital $*$-algebra $(S^*(V), \vee, *)$ is generated by V and because $S^*(V)$ is dense in its completion. □

Let $\Phi := \{\delta_\beta | \rho \in V_h'\}$ be the set of all continuous unital $*$-homomorphisms from $(S^*(V)^{\mathrm{cp1}}, \vee, *)$ to $\mathbb{C}$ and $\mathbb{C}^\Phi$ the unital $*$-algebra of all functions from Φ to Ci with the pointwise operations, then the Gel'fand-transformation is usually defined as the unital $*-$homomorphism$^-$: $(S^*(V)^{\mathrm{cp1}}, \vee, *) \to \mathbb{C}^\Phi, X \mapsto \overline{X}$ with $\overline{X}(\varphi) := \varphi(X)$ for all $\varphi \in \Phi$. This is a natural way to transform an abstract commutative unital locally convex $*$-algebras to a*-algebra of complex-valued functions. For our purposes, however, it will be more convenient to identify Φ with V_h' like in the previous Proposition (6.2.27):

Definition (6.2.28)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $X \in S^*(V)^{\mathrm{cp1}}$, *then we define the function* $\hat{X}: V_h' \to$ Ci *by*

$$p \mapsto \hat{X}(p) := \delta_\rho(X)\,. \tag{112}$$

In the following we will show that this construction yields an isomorphism between $(S^*(V)^{\mathrm{cp1}}, \vee, *)$ and a unital $*$-algebra of certain functions on V_h':

Definition (6.2.29)[280]: *Let* $f: V_h' \to$ Ci *be a function. For* $\rho, \sigma \in V_h'$ *we denote by*

$$\left(\widehat{D}_p f\right)(\sigma) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\sigma + tp) \tag{113}$$

(*if it exists*) *the directional derivative of* f *at* σ *in direction* ρ. *If the directional derivative of* f *in direction* ρ *exists at all* $\sigma \in V_h$, *then we denote by* $\widehat{D}_p f: V_h' \to$ Ci *the function* $\sigma \mapsto \left(\widehat{D}_\rho f\right)(\sigma)$. *In this case we can also examine directional derivatives of* $\widehat{D}_\rho f$ *and define the iterated directional derivative*

$$\widehat{D}_\rho^{(k)} f := \widehat{D}_{\rho_1} \cdots \widehat{D}_{\rho_k} f \tag{114}$$

(*if it exists*) *for* $k \in \mathrm{N}$ *and* $\rho \in (V_h')^k$. *For* $k = 0$ *we define* $\widehat{D}^{(0)} f := f$. *Moreover, we say that* f *is smooth if all iterated directional derivatives* $\widehat{D}_\rho^{(k)} f$ *exist for all* $k \in \mathrm{M}_0$ *and all* $p \in (V_h')^k$ *and describe a bounded symmetric multilinear form* $(V_h')^k \ni \rho \mapsto \left(\widehat{D}_\rho^{(k)} f\right)(\sigma) \in$ Ci *for all* $\sigma \in V_h'$. *Finally, we write* $\mathcal{C}^\infty(V_h')$ *for the unital* $*$*-algebra of all smooth functions on* V_h'.

Note that this notion of smoothness is rather weak, we do not even demand that a smooth function is continuous (we did not even endow V_h' with a topology). For example, every bounded linear functional on V_h' is smooth.

Proposition (6.2.30)[280]: *Let $\bar{\cdot}$ be a continuous antilinear involution on V and $X \in S^*(V)^{\text{cp1}}$. Then $\hat{X}: V_h' \to \mathbb{C}$ is smooth and*

$$\widehat{D}_p^{(k)}\hat{X} = \overline{D_\beta^{(k)}X} \tag{115}$$

holds for all $k \in \mathrm{M}_0$ and all $\rho \in (V_h')^k$.

Proof. Let $X \in S^*(V)^{\text{cp1}}$ be given. As the exponential series $\tau_{t\rho}^*(X)$ is absolutely convergent by

Lemma (6.2.26), it follows that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\tau_{t\rho}^*(X) = D_\rho(X)$ for all $p \in V_h'$ and so we conclude that

$$\left(\widehat{D}_\rho\hat{X}\right)(\sigma) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\delta_{\sigma+t\rho}(X) = \langle\tau_\sigma^*\left(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\tau_{t\rho}^*(X)\right)\rangle_0 = \langle\tau_\sigma^*\left(D_\rho(X)\right)\rangle_0 = \overline{D_\rho(X)}(\sigma)$$

holds for all $p, \sigma \in V_h'$, which proves (115) in the case $k = 1$. We see that $\widehat{D}_\rho$ for all $\rho \in V_h'$ is an endomorphism of the vector space $\{\hat{X} | X \in S^\cdot(V)^{\text{cp1}}\}$, so all iterated directional derivatives of such an $\hat{X}$ exist. By induction it is now easy to see that (115) holds for arbitrary $k \in \mathrm{M}_0$. Moreover, $D_\rho D_{\rho'}X = D_{\rho'}D_\rho X$ holds for all $\rho, \rho' \in V_h'$ and all $X \in S^*(V)^{\text{cp1}}$ by Lemmas (6.2.25) and (6.2.26). Together with (115) this shows that directional derivatives on $\hat{X}$ commute. Finally, the multilinear form $(V_h')^k \ni p \mapsto \left(\widehat{D}_p^{(k)}\hat{X}\right)(\sigma) \in \mathrm{C}i$ is bounded for all $\sigma \in V_h'$: It is sufficient to show this for $\sigma = 0$, because τ_σ^* is a continuous automorphism of $S^*(V)$ and commutes with $D_\rho^{(k)}$. If $p \in (V_h')^k$ fulfils $|\rho_i(v)| \le \|v\|_\alpha^*$ for all $i \in \{1,,k\}$, all $v \in V$ and one $\|\cdot\|_\alpha \in \mathcal{P}_V$, then we have $\|D_{\beta 1}\cdots D_{\rho_k}X\|_\alpha^* \le \|X\|_{2^k\alpha}$ due to

Lemma (6.2.26), which is an upper bound of $\left(\widehat{D}_\rho^{(k)}\hat{X}\right)(0)$.

Let $-$ be a continuous antilinear involution on V and let $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ be given, then the degeneracy space of the inner product $\langle\cdot\mid\cdot\rangle_\alpha$ is

$$\ker_h\|\cdot\|_\alpha := \{v \in V_h | \|v\|_\alpha = 0\}. \tag{116}$$

Thus we get a well-defined non-degenerate positive bilinear form on the real vector space $V_h/\ker_h\|\cdot\|_\alpha$. We write $V_{h,\alpha}^{\text{cp1}}$ for the completion of this space to a real Hilbert space with inner product $\langle\cdot\mid\cdot\rangle_\alpha$ and define the linear map $.\,\mathrm{b}_\alpha$ from $V_{h,\alpha}^{\text{cp1}}$ to V_h' as

$$v^{\mathrm{b}_\alpha}(w) := \langle v|w\rangle_\alpha \tag{117}$$

for all $v \in V_{h,\alpha}^{\text{cp1}}$ and all $w \in V$. Note that $.\,\mathrm{b}_\alpha: V_{h,\alpha}^{\text{cp1}} \to V_h'$ is a bounded linear map due to the Cauchy-Schwarz inequality. Analogously, we define

$$\ker\|\cdot\|_\alpha^* := \{X \in T^*(V) | \|X\|_\alpha^* = 0\}, \tag{118}$$

and denote by $T^*(V)_\alpha^{\text{cp1}}$ the completion of the complex vector space $\tau_{\mathrm{a}}\mathrm{i}_{\mathrm{g}}(V)/\ker\|\cdot\|_\alpha$ to a complex Hilbert space with inner product $\langle\cdot\mid\cdot\rangle_\alpha$. Then $S^*(V)_\alpha^{\text{cp1}}$ becomes the linear subspace of (equivalence classes of) symmetric tensors, which is closed because $\mathcal{S}^\cdot$ extends to a continuous endomorphism of $T^*(V)_\alpha^{\text{cp1}}$ by Proposition (6.2.7).

Moreover, for all $\langle\cdot\mid\cdot\rangle_\alpha$, $\langle\cdot\mid\cdot\rangle_\beta \in \mathcal{I}_{V,h}$ with $\langle\cdot\mid\cdot\rangle_\beta \leq \langle\cdot\mid\cdot\rangle_\alpha$, the linear map $\mathrm{id}_{7^{\cdot}(V)}: 7^*(V) \to T^*(V)$ extends to continuous linear maps $\iota_{\infty\alpha}: T^*(V)^{\mathrm{cp1}} \to T^*(V)_\alpha^{\mathrm{cp1}}$ and $\iota_{\alpha\beta}: T^*(V)_\alpha^{\mathrm{cp1}} \to T^*(V)_\beta^{\mathrm{cp1}}$, such that $\iota_{\alpha\beta} 0 \iota_{\infty\alpha} = \iota_{\infty\beta}$ and $\iota_{\beta\gamma} 0 \iota_{\alpha\beta} = \iota_{\alpha\gamma}$ hold for all $\langle\cdot\mid\cdot\rangle_\alpha$, $\langle\cdot\mid\cdot\rangle_\beta$, $\langle\cdot\mid\cdot\rangle_\gamma \in \mathcal{I}_{V,h}$ with $\langle\cdot\mid\cdot\rangle_\gamma \leq \langle\cdot\mid\cdot\rangle_\beta \leq \langle\cdot\mid\cdot\rangle_\alpha$. This way, $T^*(V)^{\mathrm{cp1}}$ is realized as the projective limit of the Hilbert spaces $T^*(V)_\alpha^{\mathrm{cp1}}$ and similarly, $S^*(V)^{\mathrm{cp1}}$ as the projective limit of the closed linear subspaces $S^*(V)_\alpha^{\mathrm{cp1}}$

Lemma (6.2.31)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $\in \mathcal{C}^\infty(V_h')$. *Given* $\rho \in V_h'$ *and* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ *such that* $|\rho(v)| \leq \|v\|_\alpha$ *holds for all* $v \in V$, *then*

$$\widehat{D}_\rho f = \sum_{i\in I} \rho\,(e_i) \widehat{D}_{e_i^{\mathrm{b}\alpha}} f \tag{119}$$

holds for every Hilbert basis $e \in \left(V_{h,\alpha}^{\mathrm{cp1}}\right)^I$ *of* $V_{h,\alpha}^{\mathrm{cp1}}$ *indexed by a set* I.

Proof As f is smooth, the function $V_h' \ni \sigma \mapsto \widehat{D}_\sigma f \in \mathbb{C}$ is bounded, which implies that its restriction to the dual space of $V_{h,\alpha}^{\mathrm{cp1}}$ is continuous with respect to the Hilbert space topology on (the dual of) $V_{h,\alpha}^{\mathrm{cp1}}$. As $\rho = \sum_{i\in I} e_i^{\mathrm{b}\alpha} \rho(e_i)$ with respect to this topology, it follows that $\widehat{D}_p f = \sum_{i\in I} \rho\,(e_i) \widehat{D}_{e_i^{\mathrm{b}\alpha}} f$. □

Definition (6.2.32) [280]: *Let* $-$ *be a continuous antilinear involution on* V. *We say that a function* $f: V_h' \to \mathbb{C}$ *is analytic of Hilbert-Schmidt type, if it is smooth and additionally fulfils the condition that for all* $\sigma, \sigma' \in V_h'$ *and all* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ *there exists a* $C_{\sigma,\sigma',\alpha} \in \mathbb{R}$ *such that*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} \left| \left(\widehat{D}^{(k)}_{\left(e_{i_1}^{\mathrm{b}\alpha},\ldots,e_{i_k}^{\mathrm{b}\alpha}\right)} f \right)(\xi) \right|^2 \leq C_{\sigma,\sigma',\alpha} \tag{120}$$

holds for one Hilbert base $e \in \left(V_{h,\alpha}^{\mathrm{cp1}}\right)^I$ *of* $V_{h,\alpha}^{\mathrm{cp1}}$ *indexed by a set I and every* ξ *from the line-segment between* σ *and* σ', *i.e. every* $\xi = \lambda\sigma + (1-\lambda)\sigma'$ *with* $\lambda \in [0,1]$. *We write* $\mathcal{C}^{wHS}(V_h')$ *for the set of all complex functions on* V_h' *that are analytic of Hilbert-Schmidt type*.

Here and elsewhere a sum over an uncountable Hilbert basis is understood in the usual sense: only countably many terms in the sum are non-zero.

This definition is independent of the choice of the Hilbert basis due to Lemma (6.2.31) and $\mathcal{C}^{wHS}(V_h')$ is a complex vector space. It is not too hard to check that $\mathcal{C}^{0\supset HS}(V_h')$ is even a unital $*$-subalgebra of $\mathcal{C}^\infty(V_h')$. However, we will indirectly prove this later on. Calling the functions in $\mathcal{C}^{w_{HS}}(V_h')$ *analytic* is justified thanks to the following statement:

Proposition (6.2.33)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $f: V_h' \to \mathbb{C}$ *analytic of Hilbert-Schmidt type with* $\left(\widehat{D}_\rho^{(k)} f\right)(0) = 0$ *for all* $k \in \mathrm{N}_0$ *and all* $\rho \in (V_h')^k$. *Then* $f = 0$.

Proof: Given $\sigma \in V_h'$, then define the smooth function $g: \mathbb{R} \to \mathbb{C}$ by $t \mapsto g(t) := f(t\sigma)$. We write $g^{(k)}(t)$ for the k-th derivative of g at t. Then there exists a $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ that fulfils $|\sigma(v)| \leq \|v\|_\alpha$ for all $v \in V$, and consequently $\sigma = L/e^{\mathrm{b}\alpha}$ with a normalized $e \in V_{h,\alpha}^{\mathrm{cp1}}$ and $u \in [0,1]$ by the Fréchet-Riesz theorem. Therefore,

$$\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left|g^{(k)}(t)\right|\right)^2 \overset{CS}{\leq} \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}|g^{(\ell)}(t)|^2$$
$$\leq \mathrm{e}\sum_{\ell=0}^{\infty}\frac{\mathrm{v}^{2p}}{\ell!}\left|\left(\widehat{D}^{(\ell)}_{(e^{\mathrm{b}_\alpha},\dots,e^{\mathrm{b}_\alpha})}f\right)(t\sigma)\right|^2 \leq \mathrm{e}C_{-2\sigma,2\sigma,\alpha}$$

holds for all $t \in [-2,2]$ with a constant $C_{-2\sigma,2\sigma,\alpha} \in \mathbb{R}$, which shows that g is an analytic function on $]-2,2[$. As $g^{(k)}(0) = 0$ for all $k \in \mathrm{M}_0$ this implies $f(\sigma) = g(1) = 0$. □

Note that one can derive even better estimates for the derivatives of g. This shows that condition (120) is even stronger than just analyticity. As an example, consider $V = \mathbb{C}$, $V_h' = \mathbb{R}$, then the function $\mathbb{R} \ni x \mapsto \exp(x^2) \in \mathbb{C}$ is not analytic of Hilbert-Schmidt type.

Definition (6.2.34)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and let* $f, g: V_h' \to$ Ci *be analytic of Hilbert-Schmidt type as well as* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$. *Because of the estimate* (120) *we can define a function* $\ll f|g \gg_\alpha^* : V_h' \to \mathbb{C}$ *by*

$$\rho \mapsto \ll f|g \gg_\alpha^*(\rho) := \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\overline{\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}f\right)(p)}\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}g\right)(\rho), \tag{121}$$

where $e \in \left(V_{h,\alpha}^{\mathrm{cp1}}\right)^I$ *is an arbitrary Hilbert base of* $V_{h,\alpha}^{\mathrm{cp1}}$ *indexed by a set* I.

Note that $\ll f|g \gg_\alpha^*$ does not depend on the choice of this Hilbert base due to Lemma (6.2.31). Essentially, $\ll f|g \gg_\alpha^*(\rho)$ is a weighted ℓ^2-inner product (yet not necessarily positive-definite) of all partial derivatives of f and g at ρ in directions described by (the dual of) a$\langle\cdot\mid\cdot\rangle_\alpha$-Hilbert base. Note that the analyticity condition (120) for a function f is equivalent to demanding that $\ll f|f \gg_\alpha^*(\xi)$ exists for all $\xi \in V_h'$ and all $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ and is uniformly bounded on line segments in V_h'.

Lemma (6.2.35)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V. *Let* $k \in \mathrm{N}$ *and* $x \in (V_h)^k$ *as well as* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ *be given. Then*

$$\left(\widehat{D}^{(k)}_{x^{\mathrm{b}_\alpha}}\widehat{Y}\right)(0) = \langle D^{(k)}_{x^{\mathrm{b}_\alpha}}Y\rangle_0 = \langle x_1 \otimes \cdots \otimes x_k|Y\rangle_\alpha^* \tag{122}$$

holds for all $Y \in S^*(V)^{\mathrm{cp1}}$.

Proof The first identity is just Proposition (6.2.30), and for the second one it is sufficient to show that $\langle D^{(k)}_{x^{\mathrm{b}_\alpha}}Y\rangle_0 = \langle x_1 \otimes \cdots \otimes x_k|Y\rangle_\alpha^*$ holds for all factorizing tensors Y of degree k, because both sides of this equation vanish on homogeneous tensors of different degree and are linear and continuous in Y by Lemma (6.2.26). However, it is an immediate consequence of the definitions of D, $\cdot\,\mathrm{b}_\alpha$, and $\langle\cdot\mid\cdot\rangle_\alpha^*$ that

$$\langle D^{(k)}_{\left(x_1^{\mathrm{b}_\alpha},\dots,x_k^{\mathrm{b}_\alpha}\right)}y_1 \otimes \cdots \otimes y_k\rangle_0 = k!\prod_{m=1}^{k}\langle x_m|y_m\rangle_\alpha = \langle x_1 \otimes \dots \otimes x_k|y_1 \otimes \dots\dots \otimes y_k\rangle_\alpha^*$$

holds for all $y_1, y_k \in V$.

Proposition (6.2.36)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then*

$$\ll \widehat{X}|\widehat{Y} \gg_\alpha^*(\rho) = \langle\tau_\rho^* X|\tau_\rho^* Y\rangle_\alpha^* = \widehat{X^* \star_{\Lambda_\alpha}} Y(\rho) \tag{123}$$

holds for all $X, Y \in S^\cdot(V)^{\mathrm{cp1}}$, *all* $\rho \in V_h'$, *and all* $\langle\cdot\mid\cdot\rangle_{0j} \in \mathcal{I}_{V,h}$, *where* $\Lambda_\alpha: V \times V \to$ Ci *is the continuous bilinear form defined by* $\Lambda_\alpha(v,w) := \langle\overline{v}|w\rangle_\alpha$.

Proof: Let $X, Y \in S^*(V)^{\mathrm{cp}1}, p \in V_h'$ and $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ be given. Let $e \in \left(V_{h,\alpha}^{\mathrm{cp}1}\right)^I$ be a Hilbert base of $V_{h,\alpha}^{\mathrm{cp}1}$ indexed by a set I. Then

$$\ll \hat{X}|\hat{Y} \gg_\alpha^* (\rho) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(k)} \hat{X}\right)(p) \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(k)} \hat{Y}\right)(\rho)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} \langle D_{e_i^{\mathrm{b}\alpha}}^{(k)} \tau_\rho^* X\rangle_0 \langle D_{e_i^{\mathrm{b}\alpha}}^{(k)} \tau_\rho^* Y\rangle_0$$

$$= \sum_{k=0}^{\infty} \sum_{i\in I^k} \frac{1}{k!} \langle \tau_\rho^* X | e_{i_1} \otimes \cdots \otimes e_{i_k}\rangle_\alpha^* \langle e_{i_1} \otimes \cdots \otimes e_{i_k} | \tau_\rho^* Y\rangle_\alpha^*$$

$$= \langle \tau_\rho^* X | \tau_p^* Y\rangle_\alpha^*$$

holds by Proposition (6.2.30) and Lemma (6.2.25) as well as the previous Lemma (6.2.35) and the fact that the tensors $(k!)^{1/2} e_{i_1} \otimes \otimes e_{i_k}$ for all $k \in \mathrm{M}_0$ and $i \in I^k$ form a Hilbert base of $\mathcal{T}^*(V)_\alpha^{\mathrm{cp}1}$ The second identity is a direct consequence of Theorem (6.2.18) because τ_ρ^* is a unital $*$-automorphism of $\star_{\Lambda_\alpha}$. Indeed, we have

$$\langle \tau_\rho^* X | \tau_\rho^* Y\rangle_\alpha^* = \langle \left(\tau_\rho^* X\right)^* \star_{\Lambda_\alpha} \left(\tau_\rho^* Y\right)\rangle_0 = \langle \tau_\rho^* \left(X^* \star_{\Lambda_\alpha} Y\right)\rangle_0 = \widehat{X^* \star_{\Lambda_\alpha}} Y(\rho)\,. \square$$

Corollary (6.2.37)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $X \in S^*(V)^{\mathrm{cp}1}$, *then* $\hat{X} \in \mathcal{C}^{('J_{HS}}(V_h')$.

Proof The function $\hat{X}$ is smooth by Proposition (6.2.30). By the previous Proposition (6.2.36), we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} |\left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(k)} \hat{X}\right)(\xi)|^2 = \ll \hat{X}|\hat{X} \gg_\alpha^* (\xi) = \widehat{X^* \star_{\Lambda_\alpha}} X(\xi)$$

for all $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$, which is finite and depends smoothly on $\xi \in V_h'$ by Proposition (6.2.30) again. Therefore it is uniformly bounded on line segments. $\square$

Lemma (6.2.38)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$. *For every* $f \in \mathcal{C}^{0\supset HS}(V_h')$ *there exists an* $X_f \in S^{\cdot}(V)^{\mathrm{cp}1}$ *that fulfils* $\ll f \big| f \gg_\alpha^* (0) = \ll \hat{X}_f \big| \hat{X}_f \gg_\alpha^* (0)$*and* $\ll f \big| \hat{Y} \gg_\alpha^* (0) = \ll \hat{X}_f \big| \hat{Y} \gg_\alpha^* (0)$ *for all* $Y \in S^{\cdot}(V)^{\mathrm{cp}1}$ *and all* $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$.

Proof For every $\alpha \in \mathcal{I}_{V,h}$ construct $X_{f,\alpha} \in S^*(V)_\alpha^{\mathrm{cp}1}$ as

$$X_{f,\alpha} := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i\in I^k} e_{i_1} \otimes \cdots \otimes e_{i_k} \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(k)} f\right)(0) \in S^v(V)_\alpha^{\mathrm{cp}1},$$

where $e \in \left(V_{h,\alpha}^{\mathrm{cp}1}\right)^I$ is a Hilbert base of $V_{h,\alpha}^{\mathrm{cp}1}$ indexed by a set I. This infinite sum $X_{f,\alpha}$ indeed lies in $S^*(V)_\alpha^{\mathrm{cp}1}$ and fulfils $\langle X_{f,\alpha} \big| X_{f,\alpha}\rangle_\alpha^* = \ll f \big| f \gg_\alpha^* (0)$, because $\left(\widehat{D}_{e_i^{\rangle}\alpha}^{(k)} f\right)(0)$ is invariant under permutations of the $e_{i_1}, \ldots, e_{i_k}$ due to the smoothness of f and because

$$\sum_{k,\ell=0}^{\infty} \sum_{i\in I^k, i'\in I^\ell} \frac{1}{k!\,\ell!} \langle e_{i_1} \otimes \cdots \otimes e_{i_k} \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(k)} f\right)(0) |\, e_{i\acute{}} \otimes \cdots \otimes e_{i_p'} \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}}^{(\ell)} f\right)(0)\rangle_\alpha^*$$

$$= \sum_{k=0}^{\infty} \sum_{i \in I^k} \frac{1}{k!} | \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} f \right) (0)|^2$$

$$= \ll f | f \gg_\alpha^* (0) .$$

Moreover, for all $Y \in S^{\cdot}(V)^{\mathrm{cp1}}$ the identity

$$\ll f | \hat{Y} \gg_\alpha^* (0) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} f \right) (0) \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} \hat{Y} \right) (0)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \langle X_{f,\alpha} | e_{i_1} \otimes \otimes e_{i_k} \rangle_\alpha^* \langle e_{i_1} \otimes \otimes e_{i_k} | Y \rangle_\alpha^* = \langle X_{f,0i} | Y \rangle_\alpha^*$$

holds due to the construction of $X_{f,\alpha}$ and Lemma (6.2.35) and because the tensors $(k!)^{1/2} e_{i_1} \otimes \otimes e_{i_k}$ for all $k \in \mathrm{M}_0$ and all $i \in I^k$ are a Hilbert base of $T^*(V)_\alpha^{\mathrm{cp1}}$

Next, let $\langle \cdot | \cdot \rangle_\beta \in \mathcal{I}_{V,h}$ with $\langle \cdot | \cdot \rangle_\beta \leq \langle \cdot | \cdot \rangle_\alpha$ and a Hilbert basis $d \in \left(V_{h,\beta}^{\mathrm{cp1}} \right)^J$ of $V_{h,\beta}^{\mathrm{cp1}}$ indexed by a set J be given. Using the explicit formulas and the identity

$$\left(\widehat{D}^{(k)}_{d_j^{\mathrm{b}_\beta}} f \right) (0) = \frac{1}{k!} \sum_{i \in I^k} \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} f \right) (0) \langle d_{j_1} \otimes \cdots \otimes d_{j_k} | \iota_{\alpha\beta} \left(e_{i_1} \otimes \otimes e_{i_k} \right) \rangle_\beta^*$$

from Lemma (6.2.31) one can now calculate that

$$\iota_{\alpha\beta} \left(X_{f,\alpha} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \iota_{\alpha\beta} \left(e_{i_1} \otimes \cdots \otimes e_{i_k} \right) \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} f \right) (0)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \sum_{i \in I^k} \sum_{j \in J^k} d_{j_1} \otimes \cdots \otimes d_{j_k} \langle d_{j_1} \otimes \cdots \otimes d_{j_k} | \iota_{\alpha\beta} \left(e_{i_1} \otimes \otimes e_{i_k} \right) \rangle_\beta^* \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} f \right) (0)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j \in J^k} d_{j_1} \otimes \otimes d_{j_k} \left(\widehat{D}^{(k)}_{d_j^{\mathrm{b}_\beta}} f \right) (0)$$

$$= X_{f,\beta}.$$

As $S^*(V)^{\mathrm{cp1}}$ is the projective limit of the Hilbert spaces $S^*(V)_\alpha^{\mathrm{cp1}}$, this implies that there exists a unique $X_f \in S^*(V)^{\mathrm{cp1}}$ that fulfils $\iota_{\infty\alpha} \left(X_f \right) = X_{f,\alpha}$ for all $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$. Consequently and with the help of Proposition (6.2.36),

$$\ll \hat{X}_f | \hat{Y} \gg_\alpha^* (0) = \langle X_f | Y \rangle_\alpha^* = \langle \iota_{\infty\alpha} \left(X_f \right) | Y \rangle_\alpha^* = \langle X_{f,\alpha} | Y \rangle_\alpha^* = \ll f | \hat{Y} \gg_\alpha^* (0)$$

holds for all $Y \in S^{\cdot}(V)^{\mathrm{cp1}}$ and all $\langle \cdot | \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$, and similarly,

$$\ll \hat{X}_f | \hat{X}_f \gg_\alpha^* (0) = \langle X_f | X_f \rangle_\alpha^* = \langle \iota_{\infty\alpha} \left(X_f \right) | \iota_{\infty\alpha} \left(X_f \right) \rangle_\alpha^* = \langle X_{f,\alpha} \middle| X_{f,\alpha} \rangle_\alpha^* = \ll f \middle| f \gg_\alpha^* (0) .$$

After this preparation we are now able to identify the image of the Gel'fand transformation explicitly:

Theorem (6.2.39)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then the Gelfand transformation* $\wedge$: $(S^*(V)^{\mathrm{cp1}}, \vee, *) \to \mathcal{C}^{(\prime)} HS(V_h')$ *is an isomorphism of unital∗-algebras*.

Proof Let $X \in S^*(V)^{\mathrm{cp1}}$ be given, then $\hat{X} \in \mathcal{C}^{(}{}_{Hws}(V_h')$ by Corollary (6.2.37). The Gelfand transformation is a unital ∗-homomorphism onto its image by construction and injective because $\hat{X} = 0$ implies $\langle X|X\rangle_\alpha^* = \ll \hat{X}|\hat{X} \gg_\alpha^* (0) = 0$ for all $\langle\cdot | \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ by Proposition (6.2.36), hence $X = 0$. It only remains to show that $\wedge$ is surjective, so let $f \in \mathcal{C}^{('J_{HS}}(V_h')$ be given. Construct $X_f \in S^*(V)^{\text{cp1}}$ like in the previous Lemma (6.2.38), then

$$\ll f - \hat{X}_f|f - \hat{X}_f \gg_\alpha^* (0) = \ll f|f \gg_\alpha^* (0) - \ll f|\hat{X}_f \gg_\alpha^* (0) - \ll \hat{X}_f|f \gg_\alpha^* (0) + \ll \hat{X}_f|\hat{X}_f \gg_\alpha^* (0)$$
$$= \ll f|f \gg_\alpha^* (0) - \ll \hat{X}_f|\hat{X}_f \gg_\alpha^* (0) - \ll \hat{X}_f|\hat{X}_f \gg_\alpha^* (0) + \ll \hat{X}_f|\hat{X}_f \gg_\alpha^* (0)$$
$$= 0$$

holds for all $\langle\cdot | \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$, hence $f = \hat{X}_f$ due to Proposition (6.2.33). □

Remark (6.2.40)[280]: Let $\bar{\cdot}$ be a continuous antilinear involution on V. For a continuous bilinear form Λ on V the identity

$$\mathrm{P}_\Lambda(X \otimes_\pi Y) = \sum_{i,i'\in I} \Lambda(e_i, e_{i'}) \left(D_{e_i^{\mathrm{b}\alpha}} X \otimes_\pi D_{e_i^{\mathrm{b}\alpha}}, Y\right) \tag{124}$$

holds for all $X, Y \in S^*(V)$ and every $\langle\cdot | \cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ for which $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$ and for every Hilbert base $e \in \left(V_{h,\alpha}^{\text{cp1}}\right)^I$ indexed by a set I. Thus

$$\hat{X} \mathbin{\widehat{\star_\Lambda}} \hat{Y} := \widehat{X \star_\Lambda Y} = \mu\left(\sum_{t=0}^{\infty} \frac{1}{t!}\left(\sum_{i,i\in I} \Lambda(e_i, e_{i'}) \left(\widehat{D}_{e_i^{\mathrm{b}\alpha}} \otimes \widehat{D}_{e_i^{\mathrm{b}\alpha}},\right)\right)^t . \left(\hat{X} \otimes \hat{Y}\right)\right)$$

with $\mu: \mathcal{C}^\infty(V_h') \otimes \mathcal{C}^\infty(V_h') \to \mathcal{C}^\infty(V_h')$ the pointwise product is the usual exponential star product on $\mathcal{C}^{\alpha)}HS(V_h')$. Moreover, if $\mathcal{A} \subseteq \mathcal{C}^\infty(V_h')$ is any unital $*$-subalgebra on which all such products $\widehat{\star_\Lambda}$ for all continuous Hermitian bilinear forms Λ on V converge, then $\mathcal{A} \subseteq \mathcal{C}^{('J_{HS}}(V_h')$, because analogous to Proposition (6.2.36), every $f \in \mathcal{A}$ fulfils $\ll f|f \gg_\alpha^* = f^* f \in \mathcal{A} \subseteq \mathcal{C}^\infty(V_h')$ for all $\langle\cdot | \cdot\rangle_\alpha \in \mathcal{I}_{Vh}$ with corresponding continuous Hermitian bilinear form $V^2 \ni (v, w) \mapsto \Lambda_\alpha(v, w) := \langle\overline{v}|w\rangle_\alpha \in \mathbb{C}$. This is of course just our Theorem (6.2.18) again.

Recall that a linear functional $w: S^*(V) \to$ (D is said to be positive for $\star_\Lambda$ if $w(X^* \star_\Lambda X) \geq 0$ holds for all $\in S^*(V)$. Such positive linear functionals yield important information about the representation theory of a*-algebra, e.g. there exists a faithful $*$-representation as adjointable operators on a preHilbert space if and only if the positive linear functionals are point-separating, see [97]. we will determine the obstructions for the existence of continuous positive linear functionals. First, we need the following lemma which allows us to apply an argument similar to the one used in [77] in the formal case:

Lemma (6.2.41)[280]: *Let $\bar{\cdot}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear form on V such that $\Lambda(\overline{v}, v) \geq 0$ holds for all $v \in V$. Then for all $X \in S^*(V)$ and all $t \in \mathrm{M}_0$ there exist $n \in \mathrm{N}$ and $X_1, \ldots, X_n \in S^\cdot(V)$ such that*

$$(\mathrm{P}_\Lambda)^t(X^* \otimes_\pi X) = \sum_{i=1}^{n} X_i^* \otimes_\pi X_i. \tag{125}$$

Proof This is trivial for scalar X as well as for $t = 0$ and for the remaining cases it is sufficient to consider $t = 1$, the others then follow by induction. So let $k \in \mathrm{N}$ and $X \in$

$S^k(V)$ be given. Expand X as $X = \sum_{j=1}^{m} x_{j,1} \vee \cdots \vee x_{j,k}$ with $m \in \mathrm{N}$ and vectors $x_{1,1},, x_{m,k} \in V$. Then

$$\mathrm{P}_{\Lambda}(X^* \otimes_{\pi} X)$$

$$= \sum_{j\prime,j=1}^{m} \sum_{\ell\prime,\ell=1}^{k} \Lambda\left(\overline{x_{j},, \ell'}, x_{j,\ell}\right)\left(x_{j,1} \vee \cdots \widehat{x_{j\prime,\ell\prime}} \ldots \vee x_{j,k}\right)^* \otimes_{\pi} \left(x_{j,1} \vee \cdots \widehat{x_{j\prime,\ell\prime}} \ldots \vee x_{j,k}\right),$$

where $\wedge$ denotes omission of a vector in the product. The complex $mk \times mk$ -matrix with entries $\Lambda\left(\overline{x_{j',\ell}},, x_{j,\ell}\right)$ is positive semi-definite due to the positivity condition on Λ, which implies that it has a Hermitian square root $R \in \mathbb{C}^{mk\times mk}$ that fulfils $\Lambda\left(\overline{x_{j',\ell'}}, x_{j,l}\right) = \sum_{p=1}^{m}\sum_{q=1}^{k} \overline{R_{(p,q),(j',\ell')}}\, R_{(p,q),(j,\ell)}$ for all $j, j' \in \{1,, m\}$ and $\ell, \ell' \in \{1, k\}$. Consequently,

$$\mathrm{P}_{\Lambda}(X^* \otimes_{\pi} X) =$$

$$= \sum_{p,q=1}^{m,k} \left(\sum_{j,\ell\prime=1}^{m,k} \overline{R_{(p,q),(j\prime,\ell)}} \left(x_{j,1} \vee \cdots \widehat{x_{j\prime,\ell\prime}} \ldots \vee x_{j\prime,k}\right)^*\right) \otimes_{\pi} \left(\sum_{j,\ell=1}^{m,k} R_{(p,q),(j,\ell)} \left(x_{j,1} \vee \widehat{x_{j,\ell\prime}} \cdots \vee x_{j,k}\right)\right)$$

holds which proves the lemma.

Proposition (6.2.42)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ, Λ' *as well as* b *three continuous Hermitian bilinear forms on* V *such that* b *is symmetric and of Hilbert-Schmidt type and such that* $\Lambda'(\overline{v}, v) + b(\overline{v}, v) \geq 0$ *holds for all* $v \in V$. *Given a continuous linear functional* w *on* $S^{\cdot}(V)$ *that is positive for* $\star_{\Lambda}$, *define* $(v_{zb}: S^*(V) \to$ (D *as*

$$X \mapsto w_{zb}(X) := w\left(\mathrm{e}^{z\triangle_b} X\right) \tag{126}$$

for all $z \in \mathbb{R}$. *Then* αj_{zb} *is a continuous linear functional and positive for* $\star_{\Lambda+z\Lambda'}$.

Proof: It follows from Theorem (6.2.23) that w_{zb} is continuous, and given $\in S^*(V)$, then

$$w\left(\mathrm{e}^{z\triangle_b}(X^* \star_{\Lambda+z\Lambda'} X)\right) = w\left(\left(\mathrm{e}^{z\triangle_b} X\right)^* \star_{\Lambda+z(\Lambda'+b)} \left(\mathrm{e}^{z\triangle_b} X\right)\right)$$

$$= \sum_{s,t=0}^{\infty} \frac{1}{s!\,t!} (\lambda) \left(\mu_{\vee}\left((\mathrm{P}_{\Lambda})^s \left(\mathrm{P}_{z(\Lambda'+b)}\right)^t \left(\left(\mathrm{e}^{z\triangle_b} X\right)^* \otimes_{\pi} \left(\mathrm{e}^{z\triangle_b} X\right)\right)\right)\right)$$

$$= \sum_{t=0}^{\infty} \frac{1}{t!} w\left(\mu_{\star_{\Lambda}}\left(\left(\mathrm{P}_{z(\Lambda'+b)}\right)^t \left(\left(\mathrm{e}^{z\triangle_b} X\right)^* \otimes_{\pi} \left(\mathrm{e}^{z\triangle_b} X\right)\right)\right)\right) \geq 0$$

holds because P_{Λ} and $\mathrm{P}_{z(\Lambda'+b')}$ commute on symmetric tensors and because of Lemma (6.2.41). □

Note that Theorem (6.2.23) also shows that w_{zb} depends holomorphically on $z \in \mathbb{C}$ in so far as (D $\ni z \mapsto w_{zb}(X) \in$ (D is holomorphic for all $\in S^*(V)$. This is the analog of statements in [91], [283] in the Rieffel setting.

Proposition (6.2.43)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ *a continuous Hermitian bilinear forms on* V. *If there exists a continuous linear functional* (λ) *on* $S^*(V)$ *that is positive for* $\star_{\Lambda}$ *and fulfils* $(v(1) = 1$, *then the bilinear form* $V^2 \ni (v, w) \mapsto$

$b,(v,w) := w(v \vee w) \in \mathbb{C}$ *is symmetric, Hermitian, of Hilbert-Schmidt type and fulfils* $\Lambda(\overline{v},v) + b_w(\overline{v},v) \geq 0$ *for all* $v \in V$.

Proof. It follows immediately from the construction of b_w that this bilinear form is symmetric and it is Hermitian because $\overline{b_w(v,w)} = \overline{w(v \vee w)} = \overline{b_w(\overline{w} \vee \overline{v})} = \overline{b_w(\overline{w},\overline{v})}$ holds for all , $w \in V$. Continuity of (v especially implies that there exists a $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ such that $|(v(X)| \leq 2^{-\frac{1}{2}} \|X\|_\alpha^*$ holds for all $w \in S^2(V)$, hence b_w is of Hilbert-Schmidt type by Proposition3.8and because $\triangle_{b_\omega} X = w(X)$ for $X \in S^2(V)$. Finally, $0 \leq \alpha_j(v^* \star_\Lambda v) = \Lambda(\overline{v},v) + b_w(\overline{v},v)$ holds due to the positivity of w. □

Theorem (6.2.44)[280]: *Let* $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ *a continuous Hermitian bilinear forms on V. Assume* $V \neq \{0\}$. *There exists a non-zero continuous positive linear functional on* $(S^*(V), \star_\Lambda, *)$ *if and only if there exists a symmetric and hermitian bilinear form of HilbertSchmidt type* b *on* V *such that* $\Lambda(\overline{v},v) + b(\overline{v},v) \geq 0$ *holds for all* $v \in V$. *In this case, the continuous positive linear functionals on* $(S^*(V), \star_\Lambda, *)$ *are point-separating, i.e. their common kernel is* $\{0\}$.

Proof If there exists a non-zero continuous positive linear functional w on $(S^*(V), \star_\Lambda, *)$, then $w(1) \neq 0$ due to the Cauchy-Schwarz identity and we can rescale w such that $\alpha_j(1) = 1$. Then the previous Proposition (6.2.43) shows the existence of such a bilinear form b. Conversely, if such a bilinear form b exists, then Proposition (6.2.42) shows that all continuous linear functionals on $S^\cdot(V)$ that are positive for V can be deformed to continuous linear functionals that are positive for $\star_\Lambda$ by taking the pull-back with $e^{\triangle_b}$. As $e^{\triangle_b}$ is invertible, it only remains to show that that the continuous positive linear functionals on $(S^\cdot(V), \vee^*)$ are point-separating. This is an immediate consequence of Theorem (6.2.39), which especially shows that the evaluation functionals δ_β with $\rho \in V_h'$ are point-separating.

Having a topology on the symmetric tensor algebra allows us to ask whether or not some exponentials (with respect to the undeformed or deformed products) exist in the completion, i.e. we want to discuss for which tensors $X \in S^*(V)^{\mathrm{cp1}}$ the series $\exp_{\star_\Lambda}(X) := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\star_\Lambda n}$ converges, where $X^{\star_\Lambda n}$ denotes the n-th power of X with respect to the product $\star_\Lambda$ for a continuous bilinear form Λ on V. Note that since the algebra is (necessarily) *not* locally multiplicatively convex, this is a non-trivial question. This also allows to give a sufficient criterium for a GNS representation of a Hermitian algebra element to be essentially self-adjoint.

Definition (6.2.45)[280]: *For* $k \in \mathrm{N}_0$ *we define*

$$S^{(k)}(V) := \bigoplus_{\ell=0}^{k} S^\ell(V), \tag{127}$$

and write $S^{(k)}(V)^{\mathrm{cp1}}$ *for the closure of* $S^{(k)}(V)$ *in* $S^\cdot(V)^{\mathrm{cp1}}$.

Lemma (6.2.46)[280]: *One has*

$$\binom{m}{\ell}\binom{m-P \quad +t}{t} \leq \binom{\ell+t}{t}\binom{k(n+1)}{k} \tag{128}$$

for all $k, n \in \mathrm{N}_0$, $m \in \{0, kn\}$, $t \in \{0, \ldots, k\}$, *and all* $\ell \in \{0,\ \min\{m, k-t\}\}$.

Lemma (6.2.47)[280]: *Let* Λ *be a continuous bilinear form on V. Let* $k, n \in \mathrm{N}_0$ *and* X_1, $X_n \in S^{(k)}(V)^{\mathrm{cp1}}$ *be given. Then the estimates*

$$\|\langle X_1 \star_\Lambda \ldots \star_\Lambda X_n\rangle_m\|_\alpha^* \le \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn}\|X_1\|_\alpha^* \cdots \|X_n\|_\alpha^* \quad (129)$$

and

$$\left\|X_1 \star_\Lambda \ldots \star_\Lambda X_n\right\|_\alpha^* \le \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} (2\mathrm{e}^3)^{kn}\left\| X_1\right\|_\alpha^* \|X_n\|_\alpha^* \quad (130)$$

hold for all $m \in \{0,\ldots,kn\}$ *and all* $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$.

Proof. The first estimate implies the second, because $\|X_1 \star_\Lambda \star_\Lambda X_n\|_\alpha^*$ has at most $(1+kn)$ nonvanishing homogeneous components, namely those of degree $m \in \{0,\ldots,kn\}$, and $(1+kn) \le \mathrm{e}^{kn}$. We will prove the first estimate by induction over n: If $n=0$ or $n=1$, then the estimate is clearly fulfiled for all possible k and m, and if it holds for one $n \in \mathrm{N}$, then

$$\begin{aligned}
&\|\langle X_1 \star_\Lambda \ldots \star_\Lambda X_{n+1}\rangle_m\|_\alpha^* \\
&\le \sum_{t=0}^{k} \frac{1}{t!}\|\langle \mu_{Reject}\left((\mathrm{P}_\Lambda)^t((X_1 \star_\Lambda \ldots \star_\Lambda X_n) \otimes_\pi X_{n+1})\right)\rangle_m\|_\alpha^* \\
&\le \sum_{t=0}^{k} \sum_{\ell=0}^{\min(m,k-1)} \frac{1}{t!}\|\mu_{\mathrm{V}}\left((\mathrm{P}_\Lambda)^t(\langle X_1 \star_\Lambda \ldots \star_\Lambda X_n\rangle_{m-\ell+t} \otimes_\pi \langle X_{n+1}\rangle_{\ell+t})\right)\|_\alpha^* \\
&\le \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \frac{1}{t!}\binom{m}{\ell}^{\frac{1}{2}} \|(\mathrm{P}_\Lambda)\ ^t(\langle X_1 \star_\Lambda .\star_\Lambda X_n\rangle_{m-\ell+t} \otimes_\pi \langle X_{n+1}\rangle_{\ell+t})\|_{\alpha\otimes_\pi\alpha} \\
&\le \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \binom{m}{\ell}^{\frac{1}{2}} \binom{m-\ell\quad +t}{t}^{\frac{1}{2}} \binom{\ell\quad +t}{t}^{\frac{1}{2}} \|\langle X_1 \star_\Lambda \star_\Lambda X_n\rangle_{m-\ell+t}\|_\alpha \|\langle X_{n+1}\rangle_{\ell+t}\|_\alpha^* \\
&\le \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \binom{\ell\quad +t}{t}\binom{\mathrm{k(n+1)}}{k}^{\frac{1}{2}} \|\langle X_1 \star_\Lambda \star_\Lambda X_n\rangle_{m-\ell+t}\|_\alpha \|\langle X_{n+1}\rangle_{\ell+t}\|_\alpha^* \\
&\le \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \binom{\ell\quad +t}{t}\left(\frac{\mathrm{k(n+1)}}{(K!)^{n+1}}\right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn}\|X_1\|_\alpha \|X_n\|_\alpha \|X_{n+1}\|_\alpha^* \\
&= \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} (p|+tt)\left(\frac{\left(k(n+1)\right)!}{(k!)^{n+1}}\right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn}\|X_1\|_\alpha^* \ldots \|X_{n+1}\|_\alpha^* \\
&\le \left(\frac{\left(k(n+1)\right)!}{(k!)^{n+1}}\right)^{\frac{1}{2}} (2\mathrm{e}^2)^{k(n+1)}\|X_1\|_\alpha^* \ldots \|X_{n+1}\|_\alpha^*
\end{aligned}$$

holds due to the grading of μ_{V} and P_Λ, the estimates from Propositions (6.2.6) as well as (6.2.7) and Lemma (6.2.10) for μ_{V} and P_Λ, and the previous Lemma (6.2.46).

Proposition (6.2.48)[280]: *Let Λ be a continuous bilinear form on V, then* $\exp_{\star_\Lambda}(v)$ *is absolutely convergent and*

$$\exp_{\star_\Lambda}(v) = \sum_{n=0}^{\infty} \frac{v^{\star_\Lambda n}}{n!} = \mathrm{e}^{\frac{1}{2}\Lambda(v,v)} \exp_{\mathrm{V}}(v) \tag{131}$$

holds for all $v \in V$. Moreover,

$$\exp_{\mathrm{V}}(v) \ \star_\Lambda \exp_{\mathrm{V}}(w) = \mathrm{e}^{\Lambda(v,w)} \exp_{\mathrm{V}}(v+w) \tag{132}$$

and

$$\langle \exp_{\mathrm{V}}(v) | \exp_{\mathrm{V}}(w) \rangle_\alpha^* = \mathrm{e}^{\langle v|w \rangle_\alpha} \tag{133}$$

hold for all $v, w \in V$ and all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$. Finally, $\exp_{\mathrm{V}}(v)^* = \exp_{\mathrm{V}}(\overline{v})$ *for all $v \in V$ if V is equipped with a continuous antilinear involutio n^-*

Proof The existence and absolute convergence of $\star_\Lambda$ -exponentials of vectors follows directly from the previous Lemma (6.2.47) with $k = 1$ and $X_1 = \cdots = X_n = v$:

$$\sum_{n=0}^{\infty} \frac{\|v^{\star_\Lambda n}\|_\alpha^*}{n!} \le \sum_{n=0}^{\infty} \frac{(4\mathrm{e}^3 \|v||_\alpha)^n}{\sqrt{n!}} \frac{1}{2^n} \le \mathrm{cs} \left(\sum_{n=0}^{\infty} \frac{(4\mathrm{e}^3 \|v\|_\alpha)^{2n}}{n!} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{4^n} \right)^{\frac{1}{2}} < \infty$$

The explicit formula can then be derived like in [101]. We just note that

$$\begin{aligned} \mathrm{P}_\Lambda\big(\exp_{\mathrm{V}}(v) \otimes_\pi \exp_{\mathrm{V}}(w)\big) &= \sum_{k,\ell=0}^{\infty} \mathrm{P}_\Lambda \left(\frac{v^{\vee k}}{k!} \otimes_\pi \frac{w^{\vee \ell}}{\ell!} \right) \\ &= \Lambda(v,w) \sum_{k,\ell=1}^{\infty} \frac{k v^{\vee(k-1)}}{k!} \otimes_\pi \frac{p_{w^{\vee(\ell-1)}}}{\ell!} \end{aligned}$$

and so

$$\begin{aligned} \exp_{\mathrm{V}}(v) \ \star_\Lambda \exp_{\mathrm{V}}(w) &= \sum_{t=0}^{\infty} \frac{1}{t!} \mu_{\mathrm{V}} \left((\mathrm{P}_\Lambda)^t \big(\exp_{\mathrm{V}}(v) \otimes_\pi \exp_{\mathrm{V}}(w) \big) \right) \\ &= \mathrm{e}^{\Lambda(v,w)} \exp_{\mathrm{V}}(v) \vee \exp_{\mathrm{V}}(w) \,. \end{aligned}$$

The remaining two identities are the results of straightforward calculations.

We show that there exists a dense $*$-subalgebra consisting of uniformly bounded elements:

Definition (6.2.49)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on V. We define the linear subspace*

$$S_{\mathrm{per}}^*(V) := \ span\, \{expV(iv) \ \in S^*(V)^{\mathrm{cp1}} | v \in \mathrm{V} \ and \ \overline{v} = v\} \tag{134}$$

of $S^*(V)^{\mathrm{cp1}}$.

Proposition (6.2.50)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on V. Then* $S_{\mathrm{per}}^*(V)$ *is a dense $*$-subalgebra of* $(S^*(V)^{\mathrm{cp1}*} \star_\Lambda,)$ *with respect to all products $\star_\Lambda$ for all continuous bilinear Hermitian forms Λ on V and*

$$\|X\|_{\infty,\Lambda} := \ \sup \sqrt{w(X^* \star_\Lambda X)} < \infty \tag{135}$$

holds for all $X \in S_{\mathrm{per}}(V)$, where the supremum runs over all continuous positive linear functionals w on $(S^{\cdot}(V), \star_\Lambda, *)$ that are normalized to $w(1) = 1$.

Proof. Proposition (6.2.48) shows that $S_{\mathrm{per}}^*(V)$ is a*-subalgebra of $S^*(V)^{\mathrm{cp1}}$ with respect to all products $\star_\Lambda$ for all continuous bilinear Hermitian forms Λ on V. As $-\mathrm{i} \frac{\mathrm{d}}{\mathrm{dz}} |_{z=0} \exp_{\mathrm{V}}(\mathrm{i}zv) = v$ for all $v \in V$ with $v = \overline{v}$ we see that the closure of the subalgebra

$S_{\text{per}}^*(V)$ contains V, hence $S^*(V)$ which is (as a unital algebra) generated by V, and so the closure of $S_{\text{per}}^*(V)$ coincides with $S^*(V)^{\text{cp1}}$.

As $S_{\text{per}}^*(V)$ is spanned by exponentials and $(v\big(\exp_{\text{V}}(\text{i}v)^* \star_\Lambda \exp_{\text{V}}(\text{i}v)\big) = \text{e}^{\Lambda(v,v)}(v\big(\exp_{\text{V}}(0)\big) = \text{e}^{\Lambda(v,v)}$ holds for all positive linear functionals w on $(S^*(V),\star_\Lambda,*)$ that are normalized to $(v(1) = 1$ by Proposition (6.2.48), it follows that $\|X\|_{\infty,\Lambda} < \infty$ for all $X \in S_{\text{per}}^*(V)$.

Note that one can show that $\|\cdot\|_{\infty,\Lambda}$ is a C^*-norm on $(S^{\cdot}(V),\star_\Lambda,*)$ if the continuous positive linear functionals are point-separating. In contrast to the existence of exponential of vectors, we get strict constraints on the existence of exponentials of quadratic elements:

Proposition (6.2.51)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on V. Then there is no locally convex topology* τ *on* $S_{\text{alg}}^*(V)$ *with the property that any* (*undeformed*) *exponential* $\exp_{\text{V}}(X) = \sum_{n=0}^{\infty} \frac{X^{\vee n}}{n!}$ *of any* $X \in S^2(V)\backslash\{0\}$ *exists in the completion of* $S_{\text{alg}}^*(V)$ *under* τ *and such that all the products* $\star_\Lambda$ *for all continuous Hermitian bilinear forms* Λ *on* V *as well as the* $*$*-involution and the projection* $\langle\cdot\rangle_0$ *on the scalars are continuous*.

Proof. Analogously to the Proof of Theorem (6.2.18) we see that, if all the products $\star_\Lambda$ for all continuous Hermitian bilinear forms Λ on V as well as the $*$-involution and the projection $\langle\cdot\rangle_0$ on the scalars are continuous, then all the extended positive Hermitian forms $\langle\cdot \mid \cdot\rangle_\alpha^*$ for all $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_V$ would have to be continuous and thus extend to the completion of $S_{\text{alg}}^*(V)$.

Now let $X \in S^2(V)\backslash\{0\}$ be given. There exist $k \in$ M and $x \in V^k$ such that $x_1, \ldots, x_k$ are linearly independent and $X = \sum_{i=1}^k \sum_{j=1}^k \tilde{X}^{ij}\, x_i \vee x_j$ with complex coefficients $\tilde{X}^{ij}$. If there exists an $i \in \{1,k\}$ such that $\tilde{X}^{ii} \neq 0$, then we can assume without loss of generality that $i = 1$ and $\tilde{X}^{11} = 1$ and define a continuous positive Hermitian form on V by $\langle v|w\rangle_{(v} := \overline{w(V)a)}\ (w)$, where $c \supset: V \to \mathbb{C}$ is

a continuous linear form on V that satisfies $w(x_1) = 1$ and $w(x_i) = 0$ for $i \in \{2,k\}$. Otherwise we can assume without loss of generality that $\tilde{X}^{11} = \tilde{X}^{22} = 0$ and $\tilde{X}^{12} = 1$ and define a continuous positive Hermitian form on V by $\langle v|w\rangle_{(v} := \overline{(v(v)}^T\ (v(w)$, where $(v: V \to \mathbb{C}^2$ is a continuous linear map that satisfies $w(x_1) = \begin{pmatrix}1\\0\end{pmatrix}$ $\alpha j(x_2) = \begin{pmatrix}0\\1\end{pmatrix}$ and $\alpha j(x_i) = 0$ for $i \in \{3,,k\}$.

In the first case, this results in $\langle X^{\vee n}|X^{\vee n}\rangle_J(j = (2n)!$ and in the second, $\langle X^{\vee n}|X^{\vee n}\rangle_{(v} = (n!)^2$. So $\sum_{n=0}^{\infty} \frac{X^{\vee n}}{n!}$ cannot converge in the completion of $S_{\text{alg}}(V)$ because

$$\langle\sum_{n=0}^{N} \frac{X^{\vee n}}{n!} | \sum_{n=0}^{N} \frac{X^{\vee n}}{n!}\rangle_{\text{w}}^* \geq \sum_{n=0}^{N} 1 \to^{\overset{N\to\infty}{\longrightarrow}} \infty.$$

A similar result has already been obtained by Omori, Maeda, Miyazaki and Yoshioka in the 2-dimensional case in [108], where they show that associativity of the Moyal-product breaks down on exponentials of quadratic functions. Note that the above proposition does not exclude the possibility that exponentials of *some* quadratic functions exist if one only demands that *some* special deformations are continuous.

Even though exponentials of non-trivial tensors of degree 2 are not contained in $S^*(V)^{\mathrm{cp}1}$, the continuous positive linear functionals are in some sense "analytic" for such tensors:

Proposition (6.2.52)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* Λ *a continuous Hermitian bilinear form on V. Let* $w: S^*(V)^{\mathrm{cp}1} \to \mathbb{C}$ *be a continuous linear functional on* $S^*(V)^{\mathrm{cp}1}$ *that is positive with respect to* $\star_\Lambda$. *Then for all* $X \in S^{(2)}(V)^{\mathrm{cp}1}$ *there exists an* $\varepsilon > 0$ *such that*

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n w((X^{\star_\Lambda n})^* \star_\Lambda X^{\star_\Lambda n})^{\frac{1}{2}}}{n!} < \infty \qquad (136)$$

holds.

Proof: The seminorm $S^*(V)^{\mathrm{cp}1} \ni Y \mapsto \alpha J(Y^* \star_\Lambda Y)^{1/2} \in [0, \infty[$ is continuous by construction, so there exist $C > 0$ and $\|\cdot\|_\alpha \in \mathcal{P}_V$ such that $w(Y^* \star_\Lambda Y)^{1/2} \leq C\|Y\|_\alpha^*$ holds for all $Y \in S^*(V)^{\mathrm{cp}1}$. We can even assume without loss of generality that $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$. Now choose $\varepsilon > 0$ with $\varepsilon(8\mathrm{e}^6\|X\|_\alpha^*) \leq 1$, then Lemma (6.2.47) in the case $k = 2$ and $X_1 = \cdots = X_n = X$ shows that

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n (v((X^{\star_\Lambda n})^* \star_\Lambda X^{\star_\Lambda n})^{\frac{1}{2}}}{n!} \leq C \sum_{n=0}^{\infty} \frac{\varepsilon^n \|X^{\star_\Lambda n}\|_\alpha^*}{n!}$$
$$\leq C \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{\sqrt{2}^{3n} n!}$$
$$\leq C \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}^n}$$
$$= \frac{C\sqrt{2}}{\sqrt{2}-1}.$$

It is an immediate consequence of this proposition that Hermitian tensors of grade at most 2 are represented by essentially self-adjoint operators in every GNS representation corresponding to a continuous positive linear functional (λ) . Recall that for a*-algebra $\mathcal{A}$ with a positive linear functional $(J \supset: \mathcal{A} \to \mathbb{C}$, the GNS representation of $\mathcal{A}$ associated to w is the unital $*$-homomorphism $\pi_{(v}: \mathcal{A} \to \mathrm{Adj}(\mathcal{A}/\mathcal{I}_{\alpha J})$ into the adjointable endomorphisms on the pre-Hilbert space $h_{\alpha j} = \mathcal{A}/\mathcal{I}_{(\lambda)}$ with inner product $\langle \cdot \mid \cdot \rangle_{(\lambda)}$, where $\mathcal{I}_{0\supset} = \{a \in \mathcal{A} | w(a^*a) = 0\}$ and $\langle [a] | [b] \rangle_{(v} = w(a^*b)$ for all $[a]$, $[b] \in h_{(v}$ with representatives $a, b \in \mathcal{A}$.

Theorem (6.2.53)[280]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* Λ *a continuous Hermitian bilinear form on V. Let* $w: S^*(V)^{\mathrm{cp}1} \to \mathbb{C}$ *be a continuous linear functional on* $S^\cdot(V)^{\mathrm{cp}1}$ *that is positive with respect to* $\star_\Lambda$. *Then for* $X^* = X \in S^{(2)}(V)^{\mathrm{cp}1}$ *all vectors in the GNS pre-Hilbert space* $\mathcal{H}_{\alpha'}$ *are analytic for* $\pi_{\alpha)}(X)$ *which is therefore essentially self-adjoint.*

Proof It is clear from the construction of the GNS representation that $\pi_{w)}(X)$ is a symmetric operator on $h_\omega = S^*(V)^{\mathrm{cp}1}/\mathcal{I}_\omega$ and by Nelson's theorem, see e.g. [286], it is sufficient to show that all vectors $[Y] \in \mathcal{H}_{(v}$ are analytic for $\pi_{(\prime J}(X)$:From

$$\langle \pi_{(v}(X)^n[Y] | \pi_{(j}J(X)^n[Y]\rangle_{(v} = (J \supset \left((X^{\star_\Lambda n} \star_\Lambda Y)^* \star_\Lambda (X^{\star_\Lambda n} \star_\Lambda Y)\right)$$
$$= (v(Y^* \star_\Lambda (X^{\star_\Lambda n})^* \star_\Lambda X^{\star_\Lambda n} \star_\Lambda Y)$$

it follows that analyticity of the vector $[Y]$ is equivalent to the analyticity of the continuous positive linear functional $S^*(V)^{\mathrm{cp}1} \ni Z \mapsto w_Y(Z) := w(Y^* \star_\Lambda Z \star_\Lambda Y) \in \mathbb{C}$ in the sense of the previous Proposition (6.2.52).

Finally we want to discuss two special cases that have appeared in the literature before, namely that V is a Hilbert space and that V is a nuclear space.

Assume that V is a (complex) Hilbert space with inner product $\langle \cdot \mid \cdot \rangle_1$. We note that in this case $S^*(V)$ is not a pre-Hilbert space but only a countable projective limit of pre-Hilbert spaces, because the extensions $\langle \cdot \mid \cdot \rangle_\alpha^*$ of the (equivalent) inner products $\langle \cdot \mid \cdot \rangle_\alpha := \alpha \langle \cdot \mid \cdot \rangle_1$ for $\alpha \in]\, 0, \infty[$ are not equivalent. If V is a Hilbert space, then its topological dual and, more generally, all spaces of bounded multilinear functionals on V are Banach spaces. This allows a more detailed analysis of the continuity of functions in $\mathcal{C}^{\alpha)}HS(V_h')$ and of the dependence of the product $\star_\Lambda$ on $\in \mathfrak{B}il(V)$.

Theorem (6.2.54)[280]: *Let V be a (complex) Hilbert space with inner product $\langle \cdot \mid \cdot \rangle_1$ and unit ball $U \subseteq V$ and let $\mathrm{B}il(V)$ be the Banach space of all continuous bilinear forms on V with norm $\|\Lambda\| := \sup_{v,w \in U} |\Lambda(v,w)|$. Then the map $\mathrm{B}il(V) \times S^*(V)^{\mathrm{cp}1} \times S^*(V)^{\mathrm{cp}1} \to S^*(V)^{\mathrm{cp}1}$*

$$(\Lambda, X, Y) \mapsto X \star_\Lambda Y \tag{137}$$

is continuous.

Proof: Note that for a Hilbert space V, the continuous inner products $\langle \cdot \mid \cdot \rangle_\lambda$ with $\lambda > 0$ are cofinal in $\mathcal{I}_V$. Now let $\Lambda \in \mathrm{B}il(V)$, $X, Y \in S^*(V)^{\mathrm{cp}1}$ and $\varepsilon > 0$ be given, then

$$\|X' \star_{\Lambda'} Y' - X \star_\Lambda Y\|_\lambda \leq \|X' \star_{\Lambda'} Y' - X \star_{\Lambda'} Y\|_\lambda + \|X \star_{\Lambda'} Y - X \star_\Lambda Y\|_\lambda$$

holds for all $\lambda > 0$ and all $\Lambda' \in \mathrm{B}i1(V)$ as well as all $X', Y' \in S^{\cdot}(V)^{\mathrm{cp}1}$. Moreover,

$$\|X' \star_{\Lambda'} Y' - X \star_{\Lambda'} Y\|_\lambda \leq \|(X' - X) \star_{\Lambda'} Y'\|_\lambda + \|X \star_{\Lambda'} (Y' - Y)\|_\lambda$$
$$\leq 4\|X' - X\|_{8\lambda}^* \|Y'\|_{8\lambda}^* + 4\|X\|_{8\lambda}^* \|Y' - Y\|_{8\lambda}^*$$

holds for all $X', Y' \in S^*(V)^{\mathrm{cp}1}$ as well as all $\lambda > 0$ and all $\Lambda' \in \mathrm{B}i1(V)$ such that $\|\cdot\|_\lambda \in \mathcal{P}_{V,\Lambda'}$ by Lemma (6.2.12). One can check on factorizing symmetric tensors that P_Λ and $\mathrm{P}_{\Lambda'-\Lambda}$ commute and by using that

$$X \star_{\Lambda'} Y = \sum_{t=0}^{\infty} \frac{1}{t!} \mu_{\vee}\left(\left(\mathrm{P}_{\Lambda+(\Lambda'-\Lambda)}\right)^{t'} (X \otimes_\pi Y)\right)$$
$$= \sum_{t,s=0}^{\infty} \frac{1}{t!\,s!} \mu_{\vee}\left((\mathrm{P}_\Lambda)^t (\mathrm{P}_{\Lambda'-\Lambda})^s (X \otimes_\pi Y)\right)$$
$$= \sum_{s=0}^{\infty} \frac{1}{s!} \mu_{\star_\Lambda}\left((\mathrm{P}_{\Lambda'-\Lambda})^s (X \otimes_\pi Y)\right),$$

it follows that

$$\left\| X \star_{\Lambda'} Y - X \star_\Lambda Y \|_\lambda \leq \sum_{s=1}^{\infty} \frac{1}{\rho^s s!} \right\| \mu_{\star_\Lambda}\left(\left(\mathrm{P}_{\rho(\Lambda'-\Lambda)}\right)^s (X \otimes_\pi Y)\right) \|_\lambda^*$$

$$\le 4\sum_{s=1}^{\infty}\frac{1}{p^s s!}\|\left(\mathrm{P}_{\rho(\Lambda'-\Lambda)}\right)^s(X\otimes_\pi Y)\|^*_{8\lambda\otimes_\pi 8\lambda}$$
$$\le 8\sum_{s=1}^{\infty}\frac{1}{(2\rho)^s}\|X\|^*_{32\lambda}\|Y\|^*_{32\lambda}$$
$$=\frac{8}{2p-1}\|X\|^*_{32\lambda}\|Y\|^*_{32\lambda}$$

holds for all $\rho>\frac{1}{2}$, $\lambda>0$, and all $\Lambda'\in \mathrm{Bi}l(V)$ if $\|\cdot\|_\lambda\in\mathcal{P}_{V,\Lambda}\cap\mathcal{P}_{V,\rho(\mathrm{N}-\Lambda)}$ by Lemma (6.2.12) and Proposition (6.2.11) with $c=2$.

Assume that $\lambda\ge 1+||\Lambda||$ andchoose $p>\frac{1}{2}$ such that $\frac{8}{2\rho-1,\mathrm{d}||}\|X||^*_{32\lambda}||Y\|^*_{32\lambda}\le\frac{\varepsilon}{3,\|}$. Then$\|\cdot\|_\lambda\in\ \mathcal{P}_{V,\Lambda}\cap\mathcal{P}_{V,\rho(\Lambda-\Lambda)}$for all $\Lambda'\in Bil(V)$with $||\Lambda'-\Lambda||\le\frac{1}{\rho}$and $\left||X'\star_\Lambda,Y'-X\star_\Lambda Y|\right|_\lambda$ holds for all these Λ' and all X', $Y'\in S^{*\cdot}(V)^{\mathrm{cp1}}$ with $\|X'-X\|^*_{8\lambda}\le\varepsilon/(12+12\|Y\|_{8\lambda})$ and $\|Y'-Y\|_{8\lambda}\le\ \min\{1,\varepsilon/(12+12\|X\|_{8\lambda})\}$. This proves continuity of $\star$ at (Λ,X,Y) . □

Theorem (6.2.55)[280]: *Let V be a (complex) Hilbert space with inner product $\langle\cdot\mid\cdot\rangle_1$ and a continuous antilinear involutio n^- that fulfils $\langle v|w\rangle_1=\langle\overline{v}|\overline{w}\rangle_1$ for all , $w\in V$, then $\widehat{X}\colon V_h'\to\mathbb{C}$ is smooth in the Fréchet sense for all $X\in S^*(V)^{\mathrm{cp1}}$.*

Proof. By the Fréchet-Riesz theorem we can identify V_h' with V_h by means of the antilinear map . b.$V_h\to V_h$. As the translations τ^* are automorphisms of $S^*(V)^{\mathrm{cp1}}$, it is sufficient to show that $\widehat{X}$ is smooth at $0\in V_h'$. So let $K\in\mathrm{M}_0$ and $r\in V_h$ be given with $r\neq 0$ and $\|r\|_1\le 1$. We have already seen in Proposition (6.2.30) that all directional derivatives of $\widehat{X}$ exist and form bounded symmetric multilinear maps $(V_h')^k\ni\rho\mapsto\left(\widehat{D}^{(k)}_\rho\widehat{X}\right)(0)\in$ (D. These maps are indeed the derivatives of $\widehat{X}$ in the Fréchet sense due to the analyticity of $\widehat{X}$: Define $\hat{r}:=r/\|r\|_1$, then due to Proposition (6.2.30) and Lemma (6.2.26) the estimate

$$\frac{1}{\|r\|_1^{K+1}}|\widehat{X}(r^{\mathrm{b}})-\sum_{k=0}^{K}\frac{1}{k!}\left(\widehat{D}^{(k)}_{(r^{\mathrm{b}},\dots,r^{\mathrm{b}})}\widehat{X}\right)(0)|=\frac{1}{\|r\|_1^{K+1}}|\langle\tau^*_{r^{\mathrm{b}}}(X)-\sum_{k=0}^{K}\frac{1}{k!}\left(D_{r^{\mathrm{b}}}\right)^kX\rangle_0|$$
$$=\frac{1}{\|r\|_1^{K+1}}|\langle\sum_{k=K+1}^{\infty}\frac{1}{k!}\left(D_{r^{\mathrm{b}}}\right)^kX\rangle_0|$$
$$\le|\langle\sum_{k=K+1}^{\infty}\frac{1}{k!}\left(D_{\hat{r}^{\mathrm{b}}}\right)^kX\rangle_0|$$
$$\le\sum_{k=K+1}^{\infty}\frac{1}{k!}\left\|\left(D_{\hat{r}^{\mathrm{b}}}\right)^kX\right\|_1^*$$
$$\le\sum_{k=K+1}^{\infty}\frac{1}{\sqrt{k!}}\|X\|_2^*$$

with $C=\sum_{k=K+1}\frac{1}{\sqrt{k!}}<\infty$ holds uniformly for all $r\neq 0$ with $\|r\|_1\le 1$.

The formal deformation quantization of a Hilbert space in a very similar setting has already been examined in [106] by Dito. There the formal deformations of exponential type of a certain algebra $\mathcal{F}_{HS}$ of smooth functions on a Hilbert space $\mathcal{H}$ was constructed. $\mathcal{F}_{HS}$ consists of all smooth (in the Fréchet sense) functions f whose derivatives fulfil the additional condition that for all $\sigma \in \mathcal{H}$

$$k! \ll f|f \gg^k (\sigma) := \sum_{i \in I^k} |\left(\widehat{D}^{(k)}_{(e_{i_1},\ldots,e_{i_k})} f\right)(\sigma)|^2 < \infty \qquad (138)$$

holds and depends continuously on σ for one (hence all) Hilbert base $e \in \mathcal{H}^I$ of $\mathcal{H}$ indexed by a set I. In this case $\ll f|f \gg^k \in \mathcal{F}_{HS}$ holds.
The convergent deformations discussed and the formal deformations discussed by Dito in [106] are very much analogous: In both cases it is necessary to restrict the construction to a subalgebra of all smooth functions, $\mathcal{F}_{HS}$ or $\mathcal{C}^{0\supset HS}(V_h')$, where the additional requirement is that all the derivatives of fixed order (in the formal case) or of all orders (in the convergent case) at every point σ obey a Hilbert-Schmidt condition and that the square of the corresponding Hilbert-Schmidt norms, $\ll f|f \gg^k (\sigma)$ or $\ll f|f \gg^* (\sigma)$, respectively, depend in a sufficiently nice way on σ such that one can prove that $\ll f|f \gg^k$ and $\ll f|f \gg^*$ are again elements of $\mathcal{F}_{HS}$ or $\mathcal{C}^{wns}(V_h')$ (see the Proof of Proposition (6.2.17) in [106] and our Proposition (6.2.36)). Moreover, the results concerning equivalence of the deformations are similar: In [106] it is shown that two (formal) deformations are equivalent if and only if they differ by bilinear forms of Hilbert-Schmidt type, while our Theorem (6.2.23) shows that the corresponding equivalence transformations are continuous if and only if they are generated by bilinear forms of Hilbert-Schmidt type.

We conclude with a short discussion of the case that V is nuclear. It is well known that the topology of a nuclear space can be described by continuous Hilbert seminorms. Moreover, the topology of the Hilbert tensor product on $S^k(V)$ coincides with the topology of the projective tensor product which was examined in [101]. However, for the comparison of the topologies on $S^*(V)$ we have to be more careful: Let $\|\cdot\|_\alpha \in \mathcal{P}_V$ be given. Define the seminorm $\|\cdot\|^*_{\alpha,\mathrm{pr}}$ as

$$\|X\|^*_{\alpha,\mathrm{pr}} := |\langle X\rangle_0| + \sum_{k=1}^{\infty} \sqrt{k!}\ \inf \sum_{i\in I} \prod_{m=1}^{k} \| x_{i,m}\|_\alpha \qquad (139)$$

for all $X \in \tau_{\mathrm{a}}\mathrm{i}_{\mathrm{g}}(V)$, where the infimum runs over all possibilities to express $\langle X\rangle_k$ as a finite sum of factorizing tensors, i.e. as $\langle X\rangle_k = \sum_{i=1}^{d} x_{i,1} \otimes \otimes x_{i,k}$ with $x_i \in V^k$.

Lemma (6.2.56)[280]: *One has the estimate*

$$\|X\|^*_\alpha \le \|X\|^*_{\alpha,\mathrm{pr}} \qquad (140)$$

for all $X \in \tau^*_{\mathrm{ai\ g}}(V)$. Moreover, if there is a $\|\cdot\|_\beta \in \mathcal{P}_V$, $\|\cdot\|_\beta \ge \|\cdot\|_\alpha$, such that for every $\langle\cdot\mid\cdot\rangle_\beta$- orthonormal $e \in V^d$ and all $d \in$ M the estimate $\sum_{i=1}^{d} \| e_i\|^2_\alpha \le 1$ holds, then

$$\|X\|^*_{\alpha,\mathrm{pr}} \le \|X\|^*_\beta \qquad (141)$$

for all $X \in \tau_{\mathrm{a}}\mathrm{i}_{\mathrm{g}}(V)$.

Proof. Let $X \in \tau_{\mathrm{a}}\mathrm{i}_{\mathrm{g}}(V)$ be given, then $\|X\|^*_\alpha \le \sum_{k-0}^{\infty} \| \langle X\rangle_k\|^*_\alpha$ and $\|X\|^*_{\alpha,\mathrm{pr}} = \sum_{k-0}^{\infty} \| \langle X\rangle_k\|^*_{\alpha,\mathrm{pr}}$ Thus it is sufficient for the first estimate to show that $\|\langle X\rangle_k\|^*_\alpha \le$

$\|\langle X\rangle_k\|^*_{\alpha,\mathrm{pr}}$ for all $k \in \mathrm{M}_0$. $Fix k \in \mathrm{N}_0$ and assume that $\langle X\rangle_k = \sum_{i\in I} x_{i,1} \otimes \cdots \otimes x_{i,k}$ with $x_i \in V^k$. Then

$$\|\langle X\rangle_k\|^*_\alpha \leq \sum_{i\in I} \| x_{i,1} \otimes \cdot\cdot\cdot \otimes x_{i,k}\|^*_\alpha = \sqrt{k!}\sum_{i\in I} \Gamma\, \mathrm{I}^k_{m=1}\|x_{i,m}\|_\alpha$$

shows that $\|\langle X\rangle_k\|^*_\alpha \leq \|\langle X\rangle_k\|^*_{\alpha,\mathrm{pr}}$, hence $\|X\|^*_\alpha \leq \|X\|^*_{\alpha,\mathrm{pr}}$. For the second estimate, let $\|\cdot\|_\beta$ with the stated properties and $X \in T^k_{\mathrm{alg}}(V)$ be given. Use Lemma (6.2.3) to construct $X_0 = \sum_{a\in A} x_{a,1} \otimes \cdots \otimes x_{a,k}$ and $\tilde{X} = \sum_{a\prime\in(1,\dots,d)} X^{a'}\, e_{a'_1} \otimes \quad \otimes e_{a'_k}$ with $e \in V^k$ orthonormal with respect to $\langle\cdot \mid \cdot\rangle_\beta$. Clearly $\|X_0\|^*_{\alpha,\mathrm{pr}} = 0$ and so

$$\|X\|^*_{\alpha,\mathrm{pr}} \leq \|\tilde{X}\|^*_{\alpha,\mathrm{pr}}$$

$$\leq \sqrt{k!} \sum_{a'\in\{1,\dots,d\}^k} |X^{a'}| \coprod_{m=1}^{k} \| e_{a'_m}\|_\alpha$$

$$\leq \mathrm{cs}\left(k!\left(\sum_{a'\in\{1,\dots,d\}^k} |X^{a'}|^2\right)\left(\sum_{a'\in\{1,\dots d\}^k}, \coprod_{m=1}^{k} \| e_{a'_m}\|^2_\alpha\right)\right)^{\frac{1}{2}}$$

$$\leq \left(k!\left(\sum_{a'\in\{1,\dots,d\}^k} |X^{a'}|^2\right)\left(\sum_{i=1}^{d} \| e_i\|^2_\alpha\right)^k\right)^{\frac{1}{2}}$$

$$\leq \|X\|^*_\beta.$$

Proposition (6.2.57)[280]: *Let V be a nuclear space, then the topology on $S^{\cdot}(V)$ coincides with the one constructed for $R = \frac{1}{2}$.*

Proof.: This is a direct consequence of the preceeding lemma because the locally convex topology constructed in [101] for $R = \frac{1}{2}$ is the one defined by the seminorms $\|\cdot\|_{\alpha,\mathrm{pr}}$ for all $\|\cdot\|_\alpha \in \mathcal{P}_V$ and because in a nuclear space, such seminorms $\|\cdot\|_\beta$ as required in the lemma exist for all $\|\cdot\|_\alpha \in \mathcal{P}_V$, see e.g. [285] or also [117].

From [101] we get:

Corollary (6.2.58)[280]: *Let V be a nuclear space, then $S^{\cdot}(V)$ is nuclear.*

And conversely, our Theorem (6.2.18) implies:

Corollary (6.2.59)[280]: *Let V be a nuclear space, then the $R = \frac{1}{2}$ topology constructed in* [101] *is the coarsest one possible under the conditions of Theorem* (6.2.18) *in the truely* (*not graded*) *symmetric case.*

As all continuous bilinear forms on a nuclear space V are automatically of Hilbert-Schmidt type (see [117] or use [285]), we also see that the equivalence transformations $\mathrm{e}^{\triangle_b}$ are continuous for all continuous symmetric bilinear forms b on V, which corresponds to [101]. Our discussion of translations and evaluation functionals then shows the existence of point-separating many positive linear functionals on the deformed algebras:

Theorem (6.2.60)[280]: *Let V be a Hausdorff nuclear space and $-$ a continuous antilinear involution of V as well as Λ a continuous Hermitian bilinear form on V, then there exist point-separating many continuous positive linear functionals of $(S^{\cdot}(V),\star_{\Lambda},*)$.*

Proof: Choose some $\langle\cdot\mid\cdot\rangle_{\alpha}\in\mathcal{I}_{V,h}$ such that $\|\cdot\|_{\alpha}\in\mathcal{P}_{V,\Lambda}$ and define a bilinear form b on V by $b(v,w):=\langle\overline{v}|w\rangle_{\alpha}$ for all $,w\in V$. Then b is continuous and Hermitian by construction and symmetric due to the compatibility of $\langle\cdot\mid\cdot\rangle_{\alpha}$with $-$. Moreover, $\Lambda(\overline{v},v)\leq\|\overline{v}\|_{\alpha}\|v\|_{\alpha}=\|v\|_{\alpha}^{2}=\langle v|v\rangle_{\alpha}=b(\overline{v},v)$ holds for all $v\in V$ and b is of Hilbert-Schmidt type because every continuous bilinear form on a nuclear space is of Hilbert-Schmidt type (again, see [117] or use [285]). Because of this, Theorem (6.2.44) applies.

Corollary (6.2.61)[288]: *Let $\langle\cdot\mid\cdot\rangle_{\alpha}\in\mathcal{I}_{V}$, $k\in\mathbb{N}$ and $X_j\in T^{k}(V)$ be given. Then X_j can be expressed as $X_j=(X_j)_0+\tilde{X}_j$ with tensors $(X_j)_0,\tilde{X}_j\in T^{k}(V)$ that have the following properties*:

i.) One has $\sum_j\ \|(X_j)_0\|_{\alpha}^{*}=0$ and there exists a finite (possibly empty) set A and tuples $x_a^r\in V^k$ for all $a\in A$ that fulfil $\sum_r\ \prod_{n=1}^{k}\ \|x_{a,n}^{r}\|_{\alpha}=0$ and $(X_j)_0=\sum_{a\in A}\sum_r\ x_{a,1}^{r}\otimes\otimes x_{a,k}^{r}$.

ii.) There exist a $d\in\mathrm{N}_0$ and a $\langle\cdot\mid\cdot\rangle_{\alpha}$-orthonormal tuple $e\in V^d$ as well as complex coefficients $X_j^{a'}$, such that

$$\tilde{X}_j=\sum_{a\in\{1,\dots,d\}^k}\sum_j\ X_j^{a'}\,e_{a_1'}\otimes\dots\otimes e_{a_k'}\ and\ \sum_j\ \|X_j\|_{\alpha}^{*2}=\sum_j\ \|\tilde{X}_j\|_{\alpha}^{*2}$$

$$=k!\sum_{a\in\{1,\dots,d\}^k}\sum_j\ |X_j^{a'}|^2 \tag{142}$$

Proof: We can express X_j as a finite sum of simple tensors, $X_j=\sum_{b\in B}\sum_r\ x_{b,1}^{r}\otimes\cdots\otimes x_{b,k}^{r}$ with a finite set B and vectors $x_{b,i}^{r}\in V$. Let

$$V_{X_j}:=\ span\ \left\{x_{b,i}^{\mathrm{r}}|b\in B,i\in\{1,\dots,k\}\right\}\ and\ V_{(X_j)_0}:=\left\{v^r\in V_{X_j}|\|v^r\|_{\mathrm{a}}=0\right\}.$$

Construct a complementary linear subspace $V_{X_j^-}$ of $V_{(X_j)_0}$ in V_{X_j}, then we can also assume without loss of generality that $x_{b,i}^{r}\in V_{(X_j)_0}\mathrm{U}V_{X_j^-}$ for all $b\in B$ and $i\in\{1,\cdots,k\}$. Note that V_{X_j}, $V_{(X_j)_0}$ and $V_{X_j^-}$ are all finitedimensional. Now define $A:=\left\{a\in B|\exists_{n\in\{1,\dots,k\}}:x_{a,n}^{r}\in V_{(X_j)_0}\right\}$ and $\left(X_j\right)_0\ :=\sum_{a\in A}\sum_r\ x_{a,1}^{r}\otimes\cdots\otimes x_{a,k}^{r}$, then $\sum_r\prod_{n=1}^{k}\|x_{a,n}^{r}\|_{\alpha}=0$ by construction and so $\sum_j\ \|(X_j)_0\|_{\alpha}^{*}=0$ and $\sum_j\ \|X_j-(X_j)_0\|_{\alpha}^{*}=\sum_j\ \|X_j\|_{\alpha}^{*}$. Restricted to $V_{X_j^-}$, the positive Hermitian form $\langle\cdot\mid\cdot\rangle_{\alpha}$ is even positive definite, i.e. an inner product. Let $d:=\dim\left(V_{X_j^-}\right)$ and $e\in V^d$ be an $\langle\cdot\mid\cdot\rangle_{\alpha}$-orthonormal base of $V_{X_j^-}$. Define $\widetilde{X_j}:=X_j-(X_j)_0$, then $\widetilde{X_j}=\sum_{a'\in\{1,\dots,d\}}\sum_j\ X_j^{a'}\,e_{a_1'}\otimes\cdots\otimes e_{a_k'}$ with complex coefficients $X_j^{a'}$ and

$$\sum_j \|X_j\|_\alpha^{*2} = \sum_j \|\widetilde{X}_j\|_\alpha^{*2} = \sum_{a'\in\{1,\dots,d\}^k} \sum_j |X_j^{a'}|^2 \|e_{a'_1} \otimes \cdots \otimes e_{a'_k}\|_\alpha^{*2}$$

$$= \sum_{a'\in\{1,\dots,d\}^k} \sum_j |X_j^{a'}|^2 k\,!.$$

Corollary (6.2.62)[288]: (see [280]) *The linear map $\mu \otimes$ is continuous and the estimate*

$$\|\sum_j \mu_\otimes(Z_j)\|_\gamma^* \leq \sum_j \|Z_j\|_{2\gamma\otimes_\pi 2\gamma}^* \tag{143}$$

holds for all $Z_j \in T^\cdot(V) \otimes_\pi T^\cdot(V)$ and all $\|\cdot\|_\gamma \in \mathcal{P}_V$. Moreover, all $X_j \in T^k(V)$ and $Y_j \in T^\ell(V)$ with $k, \ell \in \mathrm{M}_0$ fulfil for all $\|\cdot\|_\gamma \in \mathcal{P}_V$ the estimate

$$\|\sum_j \mu \otimes (X_j \otimes_\pi Y_j)\|_\gamma^* \leq \binom{k+\ell}{k}^{\frac{1}{2}} \sum_j \|X_j\|_\gamma \|Y_j\|_\gamma^*. \tag{144}$$

Proof: Let $X_j \in T^k(V)$ and $Y_j \in T^\ell(V)$ with $k, \ell \in \mathrm{M}_0$ be given. Then

$$\sum_j \left\| X_j \otimes Y_j \|_\gamma^* = \sum_j \sqrt{\langle X_j \otimes Y_j | X_j \otimes Y_j \rangle_\gamma^*} = \binom{k+\ell}{k}^{\frac{1}{2}} \right\| \sum_j X_j\|_\gamma^* \|Y_j\|_\gamma^*$$

holds. It now follows for all $X_j, Y_j \in T^\cdot(V)$ that

$$\sum_j \|X_j \otimes Y_j\|_{\gamma'}^{*2} = \sum_{m=0}^\infty \sum_j \|\langle X_j \otimes Y_j\rangle_m\|_{\gamma'}^{*2}$$

$$\leq \sum_{m=0}^\infty \sum_j \left(\sum_{n=0}^m \|\langle X_j\rangle_{m-n} \otimes \langle Y_j\rangle_n\|_\gamma^*\right)^2$$

$$= \sum_{m=0}^\infty \sum_j \left(\sum_{n=0}^m \binom{m}{n} \frac{1}{2} \|\langle X_j\rangle_{m-n}\|_\gamma^* \|\langle Y_j\rangle_n\|_\gamma^*\right)^2$$

$$= \sum_{m=0}^\infty \sum_j \left(\sum_{n=0}^m \left(\binom{m}{n} \frac{1}{2^m}\right)^{\frac{1}{2}} \|\langle X_j\rangle_{m-n}\|_{2\gamma}^* \|\langle Y_j\rangle_n\|_{2\gamma}^*\right)^2$$

$$\overset{CS}{\leq} \sum_{m=0}^\infty \left(\sum_{n=0}^m \binom{m}{n} \frac{1}{2^m}\right) \left(\sum_{n=0}^m \sum_j \|\langle X_j\rangle_{m-n}\|_{2\gamma}^{*2} \|\langle Y_j\rangle_n\|_{2\gamma}^{*2}\right)$$

$$= \sum_j \|X_j\|_{2\gamma}^{*2} \|Y_j\|_{2\gamma}^{*2},$$

by the Cauchy-Schwarz (CS) inequality.

Corollary (6.2.63)[288]: *The symmetrisation operator is continuous and fulfils $\sum_j \|\mathcal{S}^\cdot X_j\|_\gamma^* \leq \sum_j \|X_j\|_\gamma^*$ for all $X_j \in T^\cdot(V)$ and $\|\cdot\|_\gamma \in \mathcal{P}_V$.*

Proof: From Definition (6.2.1) it is clear that $\sum_j \langle X_j^\sigma | Y_j^\sigma \rangle_\gamma = \sum_j \langle X_j | Y_j \rangle_\gamma$ for all $k \in \mathrm{N}_0$, $X_j, Y_j \in T^k(V)$ and $\sigma \in \mathfrak{S}_k$, because this holds for all simple tensors and because both sides are (anti-)linear in X_j and Y_j. Therefore $\|X_j^{\sigma}\|_{\gamma} = \|X_j\|_{\gamma}$ and $\| \sum_j \ \mathcal{S}^k X_j\|_{\gamma} \leq \sum_j \ \|X_j\|_{\gamma}$ and we get the desired estimate

$$\sum_j \ \left\|\mathcal{S}^{\cdot} X_j\right\|_{\gamma}^{*2} = \sum_{k=0}^{\infty} \sum_j \ \left\|\mathcal{S}^k \langle X_j\rangle_k\right\|_{\gamma}^{*2} \leq \sum_{k=0}^{\infty} \sum_j \ \left\|\langle X_j\rangle_k\right\|_{\gamma}^{2} = \sum_j \ \left\|X_j\right\|_{\gamma}^{*2}$$

on $T^{\cdot}(V)$.

Analogously to $\mu \otimes$ we define the linear map $\mu_{Reject} := \mathcal{S}^{\cdot} 0 \mu_{\otimes} : T^{\cdot}(V) \otimes_{\pi} T^{\cdot}(V) \to \mathcal{T}(V)$. Then the restriction of μ_{V} to $S^{\cdot}(V)$ describes the symmetric tensor product V and Corollaries (6.2.62) and (6.2.63) yield:

Corollary (6.2.64)[288]: [280] *Let Λ be a continuous bilinear form on V, let $\|\cdot\|_{\alpha}, \|\cdot\|_{\beta} \in \mathcal{P}_{V,\Lambda}$ as well as $k, \ell \in \mathrm{N}_0$ and $X_j \in T^k(V), Y_j \in T^{\ell}(V)$ be given. Then*

$$\| \sum_j \ \mathrm{P}_{\Lambda}\left(X_j \otimes_{\pi} Y_j\right)\|_{\alpha \otimes_{\pi} \beta} \leq \sqrt{k\ell} \sum_j \ \|X_j\|_{\alpha}\|Y_j\|_{\beta}. \tag{145}$$

Proof: If $k = 0$ or $\ell = 0$ this is clearly true, so assume $k, \ell \in \mathrm{IN}$. We use Corollary (6.2.61) to construct $(X_j)_0 = \sum_{a \in A} \sum_r \ x_{a,1}^r \otimes \cdots \otimes x_{a,k}^r$ and $\widetilde{X}_j = \sum_{a' \in \{1,\ldots,e\}^k} \sum_j \ X_j^{a'} \, e_{a_1'} \otimes \cdots \otimes e_{a_k'}$ with respect to $\langle \cdot \mid \cdot \rangle_{\alpha}$ as well as $(Y_j)_0 = \sum_{b \in B} \sum_r \ y_{b,1}^r \otimes \cdots \otimes y_{b,\ell}^r$ and $\widetilde{Y}_j = \sum_{b' \in \{1,\ldots,d\}^{\ell}} \sum_j \ Y_j^{b'} \ f_{b_1'} \otimes \ldots \otimes f_{b_{\ell}'}$ with respect to $\langle \cdot \mid \cdot \rangle_{\beta}$. Then

$$\| \sum_j \ \mathrm{P}_{\Lambda}\left(\left((X_j)_0 + \widetilde{X}_j\right) \otimes_{\pi} \left((Y_j)_0 + \widetilde{Y}_j\right)\right) \|_{\alpha \otimes_{\pi} \beta} \leq \sum_j \ \|\mathrm{P}_{\Lambda}\left(\widetilde{X}_j \otimes_{\pi} \widetilde{Y}_j\right)\|_{\alpha \otimes_{\pi} \beta},$$

because

$$\sum_r \ \|\mathrm{P}_{\Lambda}\left((\xi_1^r \otimes \cdots \otimes \xi_k^r) \otimes_{\pi} (\eta_1^r \otimes \cdots \otimes \eta_{\ell}^r)\right)\|_{\alpha \otimes_{\pi} \beta}^{*}$$

$$= k\ell \sum_r \ |\Lambda(\xi_k^r, \eta_1^r)| \|\xi_1^r \otimes \cdots \otimes \xi_{k-1}^r\|_{\alpha}^{*} \|\eta_2^r \otimes \cdots \otimes \eta_{\ell}^r\|_{\beta}^{*}, = 0$$

for all $\xi^r \in V^k, \eta^r \in V^{\ell}$ for which there is at least one $m \in \{1, \ldots, k\}$ with $\sum_r \ \|\xi_m^r\|_{\alpha} = 0$ or one $n \in \{1, \ldots, \ell\}$ with $\|\eta_n^r\|_{\beta} = 0$. On the subspaces $V_{X_j^-} =$ span $\{e_1, \ldots, e_c\}$ and VỸ=span{fl, . . . , $f_d\}$ of V, the bilinear form Λ is described by a matrix $\Omega \in \mathbb{C}^{c \times d}$ with entries $\Omega_{gh} = \Lambda\left(e_g, f_h\right)$. By using a singular value decomposition we can even assume without loss of generality that all off-diagonal entries of Ω vanish. We also note that $|\Omega_{gg}| = |\Lambda\left(e_g, f_g\right)| \leq \|e_g\|_{\alpha}\|f_g\|_{\beta} \leq 1$. This gives the desired estimate

$$\| \sum_j \ \mathrm{P}_{\Lambda}\left(X_j \otimes_{\pi} Y_j\right)\|_{\alpha \otimes_{\pi} \beta}^{*} \leq \sum_j \ \|\mathrm{P}_{\Lambda}\left(\widetilde{X}_j \otimes_{\pi} \widetilde{Y}_j\right)\|_{\alpha \otimes_{\pi} \beta}^{*}$$

$$= \| \sum_{a' \in \{1 \ldots e\}^k} \ \sum_{b' \in \{1 \ldots d\}^{\ell}} \sum_j \ X_j^{a'} \, (Y_j)^{b'} \mathrm{P}_{\Lambda} \left(\left(e_{a_1'} \otimes \cdots \otimes e_{a_k'}\right) \otimes_{\pi} \left(f_{b_1'} \otimes \cdots \otimes f_{b_p'}\right)\right) \|_{\alpha \otimes_{\pi} \beta}^{*}$$

$$= k\ell \| \sum_{r=1}^{\min\{c,d\}} \sum_{\substack{\overline{a}' \in \{1,\dots,c\}^{k-1} \\ \overline{a}' \in \{1,\dots,c\}^{\ell-1}}} \sum_j X_j^{(\tilde{a}',r)} Y_j^{(r,\overline{b}')} \Omega_{rr} \left(e_{\tilde{a}'_1} \otimes \cdots \otimes e_{\tilde{a}'_{k-1}} \right) \otimes_\pi (f_{\tilde{b}'_1} \otimes \cdots \otimes f_{\tilde{b}_{\ell -}}) \|^*_{\alpha \otimes_\pi \beta}$$

$$\leq k\ell \sum_{r=1}^{\min\{c,d\}} \| \sum_{\overline{a}' \in \{1,\dots,c\}^{k-1}} \sum_j X_j^{(\overline{a}',r)} e\tilde{a}í \otimes \dots \otimes e_{\tilde{a}'_{k-1}} \|^*_\alpha \| \sum_{\tilde{b}' \in \{1,\dots,d\}^{\ell-1}} Y_j^{(r,\tilde{b}')} f_{\widetilde{b'_1}} \otimes \cdots \otimes f_{\tilde{b}'_{\ell-1}} \|^*_\beta$$

$$\underset{\leq}{cs} \sqrt{k\ell} \sum_j \|X_j\|^*_\alpha \|Y_j\|^*_\beta,$$

where we have used in the last line after applying the Cauchy-Schwarz inequality that

$$\sum_{r=1}^{\min\{c,d\}} \| \sum_{\overline{a}' \in \{1,\dots,c\}^{k-1}} \sum_j X_j\,(\tilde{a}', r) e_{\tilde{a}'_1} \otimes \dots \otimes e_{\tilde{a}'_{k-1}} \|^{*2}_\alpha$$

$$= \sum_{r=1}^{\min\{c,d\}} \sum_{\overline{a}' \in \{1,\dots,c\}^{k-1}} \sum_j |X_j^{(\tilde{a}',r)}|^2 (k-1)!$$

$$\leq \frac{1}{k} \sum_j \|X_j\|^{*2}_\alpha$$

and analogously for Y_j.

Corollary (6.2.65)[288]:. [280] *Let Λ be a continuous bilinear form on V, then the function* P_Λ *is continuous and fulfils the estimate*

$$\left\| \sum_j (\mathrm{P}_\Lambda)^t (Z_j) \right\|^*_{\alpha \otimes_\pi \beta} \leq \frac{c}{c-1} \frac{t!}{c^t} \left\| \sum_j \right. ||Z_j||^*_{2c\alpha \otimes_\pi 2c\beta} \tag{146}$$

for all $c > 1$, all $t \in \mathrm{M}_0$, all seminorms $\| \cdot \|_\alpha, \| \cdot \|_\beta \in \mathcal{P}_{V,\Lambda}$, and all $Z_j \in T^\cdot(V) \otimes_\pi T^\cdot(V)$.

Proof: Let X_j, $Y_j \in T^\cdot(V)$ be given, then the previous Corollary (6.2.64) together with Lemma (6.2.5) yields

$$\| \sum_j (\mathrm{P}_\Lambda)^t (X_j \otimes_\pi Y_j) \|^*_{\alpha \otimes_\pi \beta} \leq \sum_{k,\ell=0}^{\infty} \sum_j \| (\mathrm{P}_\Lambda)^t \left(\langle X_j \rangle_{k+t} \otimes_\pi \langle Y_j \rangle_{\ell+t} \right) \|^*_{\alpha \otimes_\pi \beta}$$

$$\leq t! \sum_{k,\ell=0}^{\infty} \binom{k+t}{t}^{\frac{1}{2}} \binom{\ell+t}{t}^{\frac{1}{2}} \sum_j \| \langle X_j \rangle_{k+t} \|^*_\alpha \| \langle Y_j \rangle_{\ell+t} \|^*_\beta$$

$$\leq t! \sum_{k,\ell=0}^{\infty} \sum_j \| \langle X_j \rangle_{k+t} \|^*_{2\alpha} \| \langle Y_j \rangle_{\ell+t} \|^*_{2\beta}$$

$$= \frac{t!}{c^t} \sum_{k,\ell=0}^{\infty} \frac{1}{\sqrt{c}^{k+\ell}} \sum_j \ \|\langle X_j \rangle_{k+t}\|_{2c\alpha}^{*} \|\langle Y_j \rangle_{\ell+t}\|_{2c\beta}^{*}$$

$$\overset{cs}{\leq} \frac{t!}{c^t} \left(\sum_{k,\ell=0}^{\infty} \frac{1}{c^{k+\ell}} \right)^{\frac{1}{2}} \left(\sum_{k,\ell=0}^{\infty} \sum_j \ \| \langle X_j \rangle_{k+t}\|_{2c\alpha}^{*2} \|\langle Y_j \rangle_{\ell+t}\|_{2c\beta}^{*2} \right)^{\frac{1}{2}}$$

$$\leq \frac{c}{c-1} \frac{t!}{c^t} \sum_j \ \|X_j\|_{2c\alpha}^{*} \|Y_j\|_{2c\beta}^{*}.$$

Corollary (6.2.66)[288]:. [280] *Let Λ be a continuous bilinear form on V, then $\mu_{\star_\Lambda}$ is continuous and, given $R > 1/2$, the estimate*

$$\left\| \sum_j \ \mu_{\star_{z^r\Lambda}}(Z_j) \right\|_{\gamma}^{*} \leq \sum_{t=0}^{\infty} \frac{1}{t!} \left\| \sum_j \ \mu_{\vee}\left((\mathrm{P}_{z^r\Lambda})^t (Z_j) \right) \right\|_{\gamma}^{*}$$

$$\leq \frac{4R}{2R-1} \sum_j \ \|Z_j\|_{8R\gamma \otimes_\pi 8R\gamma}^{*} \qquad (147)$$

holds for all $\| \cdot \|_\gamma \in \mathcal{P}_{V,\Lambda}$, all $Z_j \in T^{\cdot}(V) \otimes_\pi \mathrm{T}^{\cdot}(V)$ and all $z^r \in \mathbb{C}$ with $|z^r| \leq R$.

Proof: The first estimate is just the triangle-inequality. By combining Corollary (6.2.8) and Corollary (6.2.65) with $c = 2R$ we get the second estimate

$$\sum_{t=0}^{\infty} \frac{1}{t!} \left\| \sum_j \ \mu_{\vee}\left((\mathrm{P}_{z^r\Lambda})^t (Z_j) \right) \right\|_{\gamma}^{*} \leq \sum_{t=0}^{\infty} \sum_j \ \left\| (\mathrm{P}_\Lambda)^t (Z_j) \right\|_{2\gamma \otimes_\pi 2\gamma}^{\cdot}$$

$$\leq \frac{2R}{2R-1} \sum_{t=0}^{\infty} \sum_j \ \frac{1}{2^t} \|Z_j\|_{8R\gamma \otimes_\pi 8R\gamma}$$

$$= \frac{4R}{2R-1} \sum_j \ \|Z_j\|_{8R\gamma \otimes_\pi 8R\gamma}.$$

Corollary (6.2.67)[288]:. [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on V, then the induced$*$-involution on $T^{\cdot}(V)$ is also continuous.*

Proof: For$\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ define the continuous positive Hermitian form $V^2 \ni (v^r, w^r) \mapsto \langle v^r | w^r \rangle_{\alpha^*} := \overline{\langle \overline{v^r} | \overline{w^r} \rangle_\alpha}$. Then $\sum_j \ \langle X_j^* | Y_j^{\ *} \rangle_\alpha = \sum_j \ \langle X_j | Y_j \rangle_{\alpha^*}$ and in particular $\|X_j^*\|_\alpha = \|X_j\|_{\alpha^*}$ for all $X_j, Y_j \in T^{\cdot}(V)$ because this is clearly true for simple tensors and because both sides are (anti-)linear in X_j and Y.

Corollary (6.2.68)[288]:. Let$\bar{\cdot}$: $V \to V$ *be a continuous antilinear involution and Λ a continuous bilinear form on V. Then $\sum_j \ \left(X_j \ \star_\Lambda Y_j \right)^* = \sum_j \ Y_j^{\ *} \star_{\Lambda^*} X_j^*$ holds for all $X_j, Y_j \in S^{\cdot}(V)$. Consequently, if Λ is Hermitian, then $(S^{\cdot}(V), \star_\Lambda, *)$ is a locally convex $*$-algebra.*

Proof: The identitie $\mathrm{s}^* \mathrm{o} \mathcal{S}^{\cdot} = \mathcal{S}^{\cdot} 0^*$ and $* \ 0 \mu \otimes = \mu \otimes 0 \tau \mathrm{o} (^* \otimes_\pi^*)$, with $\tau : T^{\cdot}(V) \otimes_\pi T^{\cdot}(V) \to T^{\cdot}(V) \otimes_\pi T^{\cdot}(V)$ defined as $\sum_j \ \tau\left(X_j \otimes_\pi Y_j \right) := \sum_j \ Y_j \otimes_\pi X_j$, can easily be checked on simple tensors, so $* \mathrm{o} \ \mu_\vee = \mu_{\vee} \mathrm{o} \tau \mathrm{o} (\otimes_\pi^*)$. Combining this with $\tau \mathrm{o} (^* \otimes_\pi^*)$ $\mathrm{o} \mathrm{P}_\Lambda = \mathrm{P}_\Lambda * 0 \tau \mathrm{o} (^* \otimes_\pi^*)$ on symmetric tensors, which again can easily be checked on simple symmetric tensors, yields the desired result.

Corollary (6.2.69)[288]:. [280] Let $\bar{\cdot}: V \to V$ *be a continuous antilinear involution. For every* $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ *we define a continuous bilinear form* Λ_α *on* V *by* $\sum_r \Lambda_\alpha(v^r, w^r) := \sum_r \langle \overline{v^r} | w^r \rangle_\alpha$ *for all ,* $v^r, w^r \in V$*, then* Λ_α *is Hermitian and the identities*

$$\sum_{t=0}^{\infty} \frac{1}{t!} \mu \otimes \sum_j \left(\left(\mathrm{P}_{\Lambda_\alpha}\right)^t \left(\langle X_j^* \rangle_t \otimes_\pi \langle Y_j \rangle_t \right) \right) = \sum_j \langle X_j | Y_j \rangle_\alpha^* \tag{148}$$

and

$$\sum_j \langle \mu_{\star_{\Lambda_\alpha}} \left(X_j^* \otimes_\pi Y_j \right) \rangle_0 = \sum_j \langle X_j | Y_j \rangle_\alpha^* \tag{149}$$

hold for all $X_j, Y_j \in T^*(V)$ *.*

Proof: Clearly, Λ_α is Hermitian because $\langle \cdot \mid \cdot \rangle_\alpha$ is Hermitian. Then (149) follows directly from (148) because of the grading of μ_{V} and $\mathrm{P}_{\Lambda_\alpha}$. For proving (148) it is sufficient to check it for factorizing tensors of the same degree, because both sides are (anti-)linear in X_j and Y_j and vanish if X_j and Y_j are homogeneous of different degree. If X_j and Y_j are of degree 0 then (148) is clearly fulfiled. Otherwise we get

$$\begin{aligned}
&\frac{1}{k!} \mu \otimes \sum_r \left(\left(\mathrm{P}_{\Lambda_\alpha}\right)^k \left((x_1^r \otimes \cdots \otimes x_k^r)^* \otimes_\pi (y_1^r \otimes \cdots \otimes y_k^r) \right) \right) \\
&= \sum_r \frac{1}{k!} \mu \otimes \left(\left(\mathrm{P}_{\Lambda_\alpha}\right)^k \left(\left(\overline{x_k^r} \otimes \cdots \otimes \overline{x_1^r} \right) \otimes_\pi (y_1^r \otimes \cdots \otimes y_k^r) \right) \right) \\
&= \sum_r \frac{1}{k!} \mu \otimes \left((1 \otimes_\pi 1)(k!)^2 \prod_{m=1}^{k} \Lambda_\alpha \left(\overline{x_m^r}, y_m^r \right) \right) \\
&= \sum_r k! \prod_{m=1}^{k} \Lambda_\alpha \left(\overline{x_m^r}, y_m^r \right) \\
&= \sum_r k! \prod_{m=1}^{k} \langle x_m^r | y_m^r \rangle_\alpha \\
&= \sum_r \langle x_1^r \otimes \cdots \otimes x_k^r | y_1^r \otimes \cdots \otimes y_k^r \rangle_\alpha^* .
\end{aligned}$$

Corollary (6.2.70)[288]:. [280] *The topology on* $S^*(V)$ *is the coarsest locally convex one that makes all star products* $\star_\Lambda$ *for all continuous and Hermitian bilinear forms* Λ *on* V *as well as the* $*$*-involution and the projection* $\langle \cdot \rangle_0$ *onto the scalars continuous. In addition we have for all* $X_j, Y_j \in S^*(V)$ *and all* $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$

$$\sum_j \langle X_j^* \star_{\Lambda_\alpha} Y_j \rangle_0 = \sum_j \langle X_j | Y_j \rangle_\alpha^*, \tag{150}$$

with Λ_α *as in Corollary (*6.2.69).

Proof: We have already shown the continuity of the series of star product and of the $*$-involution, the continuity of $\langle \cdot \rangle_0$ is clear. Conversely, if these three functions are continuous, their compositions yield the extensions of all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_V$ which then have to be continuous. Then (149) gives (150) for symmetric tensors X_j and Y_j.

Corollary (6.2.71)[288]:. *Let b be a symmetric bilinear form on V and $\|\cdot\|_\alpha \in \mathcal{P}_V$, then the following two statements are equivalent*:

$i.$) The bilinear form b is of Hilbert-Schmidt type and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.

$ii.$) The estimate $|\sum_j \triangle_b X_j| \le 2^{-1/2}\sum_j \|X_j\|_\alpha$ holds for all $X_j \in S^2(V)$.

Moreover, if this holds then $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$ and b is continuous.

Proof: If the first point holds, let $X_j \in T^2(V)$ be given. Construct $(X_j)_0 = \sum_{a\in A}\sum_r x^r_{a,1} \otimes x^r_{a,2}$ and $\widetilde{X_j} = \sum_{a'_1,a'_2=1}^d \sum_j X_j^{a'_1,a'_2} e_{a'_1} \otimes e_{a'_2} \in T^2(V)$ like in Corollary (6.2.61). Then $\sum_r b(x^r_{a,1}, x^r_{a,2}) = 0$ for all $a \in A$ because $\sum_r \|x^r_{a,1}\|_\alpha = 0$ or $\sum_r \|x^r_{a,2}\|_\alpha = 0$. Moreover,

$$|\sum_j \triangle_b X_j| \le |\sum_{a'_1,a'_2=1}^d \sum_j X_j^{a'_1,a'_2}\, b(e_{a'_1}, e_{a'_2})|$$

$$\overset{CS}{\le} \sum_j \left(\sum_{a'_1,a'_2=1}^d |X_j^{a'_1,a'_2}|^2\right)^{\frac{1}{2}} \left(\sum_{a'_1,a'_2=1}^d |b(e_{a'_1}, e_{a'_2})|^2\right)^{\frac{1}{2}}$$

$$\le \frac{1}{\sqrt{2}}\sum_j \|X_j\|^*_\alpha$$

shows that the second point holds. Conversely, from the second point we get $\sum_r |b(v^r, w^r)| = \sum_r |\triangle_b (v^r \vee w^r)| \le 2^{-1/2}\sum_r \|v^r \vee w^r\|_\alpha \le \sum_r \|v^r\|_\alpha \|w^r\|_\alpha$ for all $v^r, w^r \in V$. Hence $\|\cdot\|_\alpha \in \mathcal{P}_{V,b}$, the bilinear form b is continuous, and $\sum_r b(v^r, w^r) = 0$ if one of v^r or w^r is in the kernel of $\|\cdot\|_\alpha$. Moreover, given an $\langle\cdot\mid\cdot\rangle_\alpha$-orthonormal set of vectors $\in V^d$, $d \in \mathrm{M}$, we define $X_j := \sum_{j=1}^d \overline{b(e_i, e_j)}\, e_i \otimes e_j \in S^2(V)$ and get

$$0 \le \sum_{i,j=1}^d |b(e_i, e_j)|^2 = |\sum_j \triangle_b X_j| \le \frac{1}{\sqrt{2}}\sum_j \|X_j\|^*_\alpha = \left(\sum_{i.j=1}^d |b(e_i, e_j)|^2\right)^{\frac{1}{2}}$$

which implies $\sum_{j=1}^d |b(e_i, e_j)|^2 \le 1$.

Note that this also implies that for a bilinear form of Hilbert-Schmidt type b, the set $\mathcal{P}_{V,b,HS}$ is cofinal in $\mathcal{P}_V$, because if $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$, $\|\cdot\|_\beta \in \mathcal{P}_V$ and $\|\cdot\|_\beta \ge \|\cdot\|_\alpha$, then $|\triangle_b X_j| \le 2^{-\frac{1}{2}}\left\|X_j\|^*_\alpha \le 2^{-\frac{1}{2}}\right\|\sum_j ||X_j||^*_\beta$ and so $\|\cdot\|_\beta \in \mathcal{P}_{V,b,HS}$.

Corollary (6.2.72)[288]:. *Let b be a symmetric bilinear form of Hilbert-Schmidt type on V, then the Laplace operator $\triangle_b$ is continuous and fulfils the estimate*

$$\left\|\sum_j (\triangle_b)^t X_j\|^*_\alpha \le \frac{\sqrt{(2t)!}}{(2r)^t}\right\|\sum_j ||X_j||^*_{2r\alpha} \tag{151}$$

for all $X_j \in T^*(V)$, $t \in \mathrm{M}_0$, $r \ge 1$, and all $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$.

Proof: First, let $X_j \in \mathcal{T}^k(V)$, $k \ge 2$, and $\|\cdot\|_\alpha \in \mathcal{P}_{V,b,HS}$ be given. Construct $(X_j)_0 = \sum_{a\in A}\sum_j x^r_{a,1} \otimes \ldots \otimes x^r_{a,k}$ and $\widetilde{X_j} = \sum_{a'\in\{1,\ldots,d\}^k} X_j^{a'} e_{a'_1} \otimes \cdots \otimes e_{a'_k}$ like in Corollary (6.2.61). Then again

$$\left\|\sum_j \triangle_b (X_j)_0\right\|_\alpha^* \leq \frac{k(k-1)\sqrt{(k-2)!}}{2}\sum_{a\in A}\sum_r |b(x_{a_1}^r, x_{a_2}^r)|\prod_{m=3}^k \| x_{a_m}^r\|_\alpha = 0$$

shows that $\| \sum_j \triangle_b X_j\|_\alpha^* \leq \sum_j \| \triangle_b \widetilde{X_j}\|_\alpha^*$. $For \widetilde{X_j}$ we get:

$$\sum_j \| \triangle_b \widetilde{X_j}\|_\alpha^{*2}$$

$$= \| \frac{k(k-1)}{2} \sum_{a'\in\{1,\ldots,d\}^k} \sum_j X_j^{a'} b(e_{a'_1}, e_{a'_2}) e_{a'_3} \otimes \cdots \otimes e_{a'_k}\|_\alpha^{*2}$$

$$= \frac{k^2(k-1)^2}{4} \sum_{\overline{a}'\in\{1,\ldots,d\}^{k-2}} \| \sum_{g,h=1}^d \sum_j X_j^{(g,h,\overline{a}')} b\,(e_g, eh)e\tilde{a}\acute{\imath} \otimes \cdots \otimes e_{\tilde{a}_{k-2}}\|_\alpha^{*2}$$

$$= \frac{k^2(k-1)^2}{4} \sum_{\overline{a}'\in\{1,\ldots,d\}^{k-2}} |\sum_{g,h=1}^d \sum_j X_j^{(g,h,\tilde{a}')} b(e_g, e_h)|^2 (k-2)!$$

$$\leq \frac{k(k-1)k!}{4}\ \tilde{a}' \sum_{\in\{1,\ldots,d\}^{k-2}} \left(\sum_{g,h=1}^d \sum_j |X_{\mathrm{j}}^{(g,h,\tilde{a}')}||b(e_g, e_h)|\right)^2$$

$$\overset{cs}{\leq} \frac{k(k-1)k!}{4} \sum_{\overline{a}'\in\{1,\ldots,d\}^{k-2}} \left(\sum_{g,h=1}^d \sum_j |X_j^{(g,h,\tilde{a}')}|^2\right)\left(\sum_{g,h=1}^d |b(e_g, e_h)|^2\right)$$

$$\leq \frac{k(k-1)k!}{4} \sum_{a'\in\{1,\ldots d\}^k} \sum_j |X_j^{a'}|^2$$

$$= \frac{k(k-1)}{4}\sum_j \|X_j\|_\alpha^{*2}$$

Using this we get

$$\| \sum_j (\triangle_b)^t X_j\|_\alpha^{*2} = \sum_{k=2t}^\infty \sum_j \| (\triangle_b)^t \langle X_j\rangle_k\|_\alpha^{*2}$$

$$\leq \sum_{k=2t}^\infty \sum_j \binom{k}{2t} \frac{(2t)!}{4^t} \|\langle X_j\rangle_k\|_\alpha^{*2}$$

$$\leq \frac{(2t)!}{4^t}\sum_{k=2t}^\infty \sum_j \frac{1}{r^k} \|\langle X_j\rangle_k\|_{2r\alpha}^{*2}$$

$$\leq \frac{(2t)!}{(2r)^{2t}}\sum_j \|X_j\|_{2r\alpha}^{*2}$$

for arbitrary $X_j \in T^{\cdot}(V)$ and $t \in$ N. Finally, the estimate (151) also holds in the case $t = 0$.

Corollary (6.2.73)[288]: [280] *Let b be a symmetric bilinear form on V, then the linear operator* $e^{\triangle_b}=\sum_{t=0}^{\infty}\frac{1}{t!}(\triangle_b)^t$ *as well as its restriction to $S^*(V)$ are continuous if and only if b is of Hilbert-Schmidt type. In this case*

$$\sum_j e^{\triangle_b}\left(X_j\star_\Lambda Y_j\right)=\left(e^{\triangle_b}X_j\right)\star_{\Lambda+b}\left(e^{\triangle_b}Y_j\right)\tag{152}$$

holds for all $X_j, Y_j\in S^{\cdot}(V)$ and all continuous bilinear forms Λ on V. Hence $e^{\triangle_b}$ describes an isomorphism of the locally convex algebras $(S^{\cdot}(V),\star_\Lambda)$ and $(S^{\cdot}(V),\star_{\Lambda+b})$. Moreover, for fixed $X_j\in S^{\cdot}(V)^{\mathrm{cpl}}$, the series $e^{z^r\triangle_b}X_j$ converges absolutely and locally uniformly in $z^r\in$ Ci and thus depends holomorphically on z^r.

Proof: As $|\triangle_b X_j|\le\|e^{\triangle_b}X_j\|_\alpha^*$ holds for all $\|\cdot\|_\alpha\in\mathcal{P}_V$ and all $X_j\in S^2(V)$, it follows from Corollary (6.2.71) that continuity of the restriction of $e^{\triangle_b}$ to $S^*(V)$ implies that b is of Hilbert-Schmidt type. Conversely, for all $X_j\in T^{\cdot}(V)$, all $\alpha\in\mathcal{P}_{V,b,HS}$, and $r>1$, the estimate

$$\begin{aligned}\|\sum_r e^{z^r\triangle_b}X_j\|_\alpha &\le\sum_{t=0}^{\infty}\sum_j\frac{1}{t!}\|(z^r\triangle_b)^t(X_j)\|_\alpha\\ &\le\sum_{t=0}^{\infty}\sum_j\frac{|z^r|^t}{(4r)^t}\binom{2t}{t}\frac{1}{2}\left\|\,||X_j||_{4r\alpha}^*\le\sum_{t=0}^{\infty}\sum_j\frac{1}{2^t}\right\|\,||X_j||_{4r\alpha}^*\\ &=2\sum_j\|X_j\|_{4r\alpha}^*\end{aligned}$$

holds for all $z^r\in\mathbb{C}$ with $|z^r|\le r$ due to the previous Corollary (6.2.72) if b is of Hilbert-Schmidt type, which proves the continuity of $e^{z^r\triangle_b}$ for all $z^r\in\mathbb{C}$ as well as the absolute and locally uniform convergence of the series $e^{z^r\triangle_b}X_j$. The algebraic relation (152) is well-known, see e.g. [101]. Finally, as $e^{\triangle_b}$ is invertible with inverse $e^{-\triangle_b}$, and because $\triangle_b$ and thus $e^{\triangle_b}$ map symmetric tensors to symmetric ones, we conclude that the restriction of $e^{\triangle_b}$ to $S^*(V)$ is an isomorphism of the locally convex algebras $(S^*(V),\star_\Lambda)$ and $(S^*(V),\star_{\Lambda+b})$.

Corollary (6.2.74)[288]:. [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on V and , $\sigma\in V_h'$. Then*

$$\begin{aligned}\sum_j\left(D_\rho D_\sigma-D_\sigma D_\beta\right)\left(X_j\right)&=\sum_j\left(\tau_p^*D_\sigma-D_\sigma\tau_\rho^*\right)\left(X_j\right)\\ &=\sum_j\left(\tau_p^*\tau_\sigma^*-\tau_\sigma^*\tau_p^*\right)\left(X_j\right)=0\end{aligned}\tag{153}$$

holds for all $X_j\in S^*(V)$.

Proof: It is sufficient to show that $\sum_j\left(D_\rho D_\sigma-D_\sigma D_\rho\right)\left(X_j\right)=0$ for all $X_j\in S^*(V)$, which clearly holds if X_j is a homogeneous factorizing symmetric tensor and so holds for all $X_j\in S^*(V)$ by linearity.

Corollary (6.2.75)[288]:. [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on V and $\rho\in V_h'$. Then D_ρ, τ_p^* and δ_ρ are all continuous. Moreover, if $\|\cdot\|_\alpha\in\mathcal{P}_V$ fulfils $\sum_r|\rho(v^r)|\le\sum_r\|v^r\|_\alpha$, then the estimates*

$$\left\|\sum_j \left(D_p\right)^t X_j\|_\alpha^* \le \sqrt{t!}\right\|\sum_j ||X_j||_{2\alpha}^* \tag{154}$$

and

$$\left\|\sum_j \tau_p^*(X_j)\|_\alpha^* \le \sum_{t=0}^{\infty}\sum_j \frac{1}{t'!}\right\| ||\left(D_\rho\right)^{t'} X_j\|_\alpha^* \le \frac{2}{\sqrt{2}-1}\sum_j \|X_j\|_{2\alpha}^* \tag{155}$$

hold for all $X_j \in \mathcal{S}^*(V)$ and all $t \in$ M_0.

Proof: Let $\|\cdot\|_\alpha \in \mathcal{P}_V$ be given such that $\sum_r |\rho(v^r)| \le \sum_r \|v^r\|_\alpha$ holds for all $v^r \in V$. For all $d \in \mathrm{M}_0$ and all $\langle\cdot\mid\cdot\rangle_\alpha$-orthonormal $e \in V^d$ we then get

$$\sum_{i=1}^d |\rho(e_i)|^2 d = \rho\left(\sum_{i=1}^d e_i\,\overline{p(e_i)}\right)d \le \|\sum_{i=1}^d e_i\,\overline{\rho(e_i)}\|_\alpha = \left(\sum_{i=1}^d |\rho(e_i)|^2\right)^{\frac{1}{2}}$$

hence $\sum_{i=1}^d |\rho(e_i)|^2 \le 1$. Given $k \in \mathrm{M}$ and a tensor $X_j \in T^k(V)$, then we construct $(X_j)_0 = \sum_{a\in A}\sum_r x_{a,1}^r \otimes \ldots \otimes x_{a,k}^r$ and $\widetilde{X_J} = \sum_{a'\in\{1,\ldots,d\}^k}\sum_j X_j^{a'}\, e_{a_1'} \otimes \cdots \otimes e_{a_k'}$ like in Corollary (6.2.61). Then we have $\sum_j \|D_\rho(X_j)_0\|_\alpha^* = 0$ because

$$\sum_r \|D_\rho\left(x_{a,1}^r \otimes \cdots \otimes x_{a,k}^r\right)\|_\alpha^* = k\sum_r |p\left(x_{a,k}^r\right)|\|x_{a,1}^r \otimes \cdots \otimes x_{a,k-1}^r\|_\alpha^*$$

$$\le k\sqrt{(k-1)!}\sum_r \prod_{m=1}^k \|x_{a,m}^r\|_\alpha = 0$$

holds for all $a \in A$. Consequently $\|\sum_j D_\rho X_j\|_\alpha \le \sum_j \|D_\rho \widetilde{X_J}\|_\alpha$ and we get

$$\|\sum_j D_\rho X_j\|_\alpha^{*2} \le \sum_j \|D_\rho\widetilde{X_J}\|_\alpha^{*2} = \|\sum_{a'\in\{1,\ldots,d\}^k}\sum_j X_j^{a'} D_\rho\left(e_{a_1'} \otimes \cdots \otimes e_{a_k'}\right)\|_\alpha^{*2}$$

$$= k^2 \text{ ã'} \sum_{\in\{1,\ldots,d\}^{k-1}} \|\sum_{g=1}^d\sum_j X_j\,(\text{ã'},g)_{\rho(e_g)e_{\overline{a}}}\ \text{í} \otimes \cdots \otimes e_{\overline{a}_{k-1}'}\|_\alpha^{*2}.,$$

$$\le k^2(k-1)!\ \text{ã'} \sum_{\in\{1,\ldots,d\}^{k-1}}\left(\sum_{g=1}^d\sum_j |X_j(\text{ã''}g)||\rho\left(e_g\right)|\right)^2$$

$$\overset{cs}{\le} k^2(k-1)! \sum_{\overline{a}'\in\{1,\ldots,d\}^{k-1}}\sum_j \left(\sum_{g=1}^d |X_j^{(\tilde{a}',g)}|^2\right)\left(\sum_{g=1}^d |p\left(e_g\right)|^2\right)$$

$$\le k^2(k-1)! \sum_{a'\in\{1,\ldots,d\}^k}\sum_j |X_j^{a'}|^2$$

$$= k\sum_j \|X_j\|_\alpha^{*2}$$

Using this we can derive the estimate (154), which also proves the continuity of D_ρ: If $t = 0$, then this is clearly fulfiled. Otherwise, let $X_j \in T^\cdot(V)$ be given, then

$$\|\sum_j \left(D_\rho\right)^t X_j\|_\alpha^{*2} = \sum_{k=t}^{\infty}\sum_j \|\left(D_\rho\right)^t \langle X_j\rangle_k\|_\alpha^{*2} \leq t!\sum_{k=t}^{\infty}\sum_j \binom{k}{t} \|\langle X_j\rangle_k\|_\alpha^{*2}$$
$$\leq t!\sum_{k=t}^{\infty}\sum_j \|\langle X_j\rangle_k\|_{2\alpha}^{*2} \leq t!\sum_j \|X_j\|_{2\alpha}^{*2}.$$

From this we can now also deduce the estimate (155), which then shows continuity of τ_β^* and of $\delta_\rho = \langle\cdot\rangle_0 \circ \tau_\rho^*$: The first inequality is just the triangle inequality and for the second we use that $t! \geq 2^{t-1}$ for all $t \in \mathrm{M}_0$, so

$$\sum_{t=0}^{\infty}\frac{1}{t!}\|\sum_j \left(D_\rho\right)^t X_j\|_\alpha^* \leq \sum_{t=0}^{\infty}\sum_j \frac{1}{\sqrt{t!}}\|X_j\|_{2\alpha}^* \leq \sqrt{2}\sum_{t=0}^{\infty}\sum_j \frac{1}{\sqrt{2}^t}\|X_j\|_{2\alpha}^*$$
$$\leq \frac{2}{\sqrt{2}-1}\sum_j \|X_j\|_{2\alpha}^*.$$

Corollary (6.2.76)[288]:. [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then the set of all continuous unital*$*$*-homomorphisms from* $\left(S^*(V)^{\mathrm{cpl}},\vee,*\right)$ *to* $\mathbb{C}$ *is* $\left\{\delta_p | \rho \in V_h'\right\}$ (*strictly speaking, the continuous extensions to* $S^\cdot(V)^{\mathrm{cpl}}$ *of the restrictions of* δ_ρ *to* $S^*(V)$).

Proof: On the one hand, every such δ_ρ is a continuous unital $*$-homomorphism, because $\langle\cdot\rangle_0$ and τ_ρ^* are. On the other hand, if $\varphi: \left(S^\cdot(V)^{\mathrm{cpl}},\vee,*\right) \to \mathbb{C}$ is a continuous unital $*$-homomorphism, then $V \ni v^r \mapsto p(v^r) := \varphi(v^r) \in \mathbb{C}$ is an element of V_h' and fulfils $\delta_\rho = \varphi$ because the unital $*$-algebra $(S^*(V),\vee,*)$ is generated by V and because $S^*(V)$ is dense in its completion.

Let $\Phi := \left\{\delta_\beta | \rho \in V_h'\right\}$ be the set of all continuous unital $*$-homomorphisms from $\left(S^*(V)^{\mathrm{cpl}},\vee,*\right)$ to $\mathbb{C}$ and $\mathbb{C}^\Phi$ the unital $*$-algebra of all functions from Φ to $\mathbb{C}$ with the pointwise operations, then the Gel'fand-transformation is usually defined as the unital $*$ $-$homomorphism$^-$: $\left(S^*(V)^{\mathrm{cpl}},\vee,*\right) \to \mathbb{C}^\Phi, X_j \mapsto \overline{X_j}$ with $\overline{X_j}(\varphi) := \varphi\left(X_j\right)$ for all $\varphi \in \Phi$. This is a natural way to transform an abstract commutative unital locally convex $*$-algebras to a*-algebra of complex-valued functions. For our purposes, however, it will be more convenient to identify Φ with V_h' like in the previous Corollary (6.2.76):

Corollary (6.2.77)[288]:. [280] *Let* $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $X_j \in S^*(V)^{\mathrm{cpl}}$. *Then* $\widehat{X}_j: V_h' \to \mathbb{C}$ *is smooth and*

$$\sum_j \widehat{D}_p^{(k)}\widehat{X}_j = \sum_j \overline{D_\beta^{(k)} X_j} \tag{156}$$

holds for all $k \in \mathrm{M}_0$ *and all* $\rho \in (V_h')^k$.

Proof: Let $X_j \in S^*(V)^{\mathrm{cpl}}$ be given. As the exponential series $\tau_{t\rho}^*\left(X_j\right)$ is absolutely convergent by

Corollary (6.2.75), it follows that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\tau_{t\rho}^*\left(X_j\right) = D_\rho\left(X_j\right)$ for all $p \in V_h'$ and so we conclude that

$$\sum_j (\widehat{D}_\rho \widehat{X}_J)(\sigma) = \sum_j \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\delta_{\sigma+t\rho}(X_j) = \sum_j \langle\tau_\sigma^*\left(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\tau_{t\rho}^*(X_j)\right)\rangle_0$$
$$= \sum_j \langle\tau_\sigma^*\left(D_\rho(X_j)\right)\rangle_0 = \sum_j \widehat{D_\rho(X_J)}(\sigma)$$

holds for all $p, \sigma \in V_h'$, which proves (156) in the case $k = 1$. We see that $\widehat{D}_\rho$ for all $\rho \in V_h'$ is an endomorphism of the vector space $\{\widehat{X}_J | X_j \in S^\cdot(V)^{\mathrm{cpl}}\}$, so all iterated directional derivatives of such an $\widehat{X}_J$ exist. By induction it is now easy to see that (156) holds for arbitrary $k \in \mathrm{M}_0$. Moreover, $\sum_j D_\rho D_{\rho'} X_j = \sum_j D_{\rho'} D_\rho X_j$ holds for all $\rho, \rho' \in V_h'$ and all $X_j \in S^*(V)^{\mathrm{cpl}}$ by Corollaries (6.2.74) and (6.2.75). Together with (156) this shows that directional derivatives on $\widehat{X}_J$ commute. Finally, the multilinear form $(V_h')^k \ni p \mapsto \left(\widehat{D}_p^{(k)}\widehat{X}_J\right)(\sigma) \in \mathbb{C}$ is bounded for all $\sigma \in V_h'$: It is sufficient to show this for $\sigma = 0$, because τ_σ^* is a continuous automorphism of $S^*(V)$ and commutes with $D_\rho^{(k)}$. If $p \in (V_h')^k$ fulfils $|\sum_r \rho_i(v^r)| \leq \sum_r \|v^r\|_\alpha^*$ for all $i \in \{1,,k\}$, all $v^r \in V$ and one $\|\cdot\|_\alpha \in \mathcal{P}_V$, then we have $\|\sum_j D_{\beta 1} \cdots D_{\rho_k} X_j\|_\alpha^* \leq \sum_j \|X_j\|_{2^k\alpha}$ due to Corollary (6.2.75), which is an upper bound of $\left(\widehat{D}_\rho^{(k)}\widehat{X}_J\right)(0)$.

Let $-$ be a continuous antilinear involution on V and let $\langle\cdot|\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ be given, then the degeneracy space of the inner product $\langle\cdot|\cdot\rangle_\alpha$ is

$$\ker_h\|\cdot\|_\alpha := \{v^r \in V_h | \|v^r\|_\alpha = 0\}. \tag{157}$$

Thus we get a well-defined non-degenerate positive bilinear form on the real vector space $V_h/\ker_h\|\cdot\|_\alpha$. We write $V_{h,\alpha}^{\mathrm{cpl}}$ for the completion of this space to a real Hilbert space with inner product $\langle\cdot|\cdot\rangle_\alpha$ and define the linear map $.\,\mathrm{b}_\alpha$ from $V_{h,\alpha}^{\mathrm{cpl}}$ to V_h' as

$$(v^r)^{\mathrm{b}_\alpha}(w^r) := \langle v^r | w^r\rangle_\alpha \tag{158}$$

for all $v^r \in V_{h,\alpha}^{\mathrm{cpl}}$ and all $w^r \in V$. Note that $.\,\mathrm{b}_\alpha: V_{h,\alpha}^{\mathrm{cpl}} \to V_h'$ is a bounded linear map due to the Cauchy-Schwarz inequality. Analogously, we define

$$\ker\|\cdot\|_\alpha^* := \left\{X_j \in T^*(V) | \|X_j\|_\alpha^* = 0\right\}, \tag{159}$$

and denote by $T^*(V)_\alpha^{\mathrm{cpl}}$ the completion of the complex vector space $\tau_{\mathrm{a}}\mathrm{i}_{\mathrm{g}}(V)/\ker\|\cdot\|_\alpha$ to a complex Hilbert space with inner product $\langle\cdot|\cdot\rangle_\alpha$. Then $S^*(V)_\alpha^{\mathrm{cpl}}$ becomes the linear subspace of (equivalence classes of) symmetric tensors, which is closed because $\mathcal{S}^\cdot$ extends to a continuous endomorphism of $T^*(V)_\alpha^{\mathrm{cpl}}$ by Corollary (6.2.63).

Moreover, for all $\langle\cdot|\cdot\rangle_\alpha$, $\langle\cdot|\cdot\rangle_\beta \in \mathcal{I}_{V,h}$ with $\langle\cdot|\cdot\rangle_\beta \leq \langle\cdot|\cdot\rangle_\alpha$, the linear map $\mathrm{id}_{T^\cdot(V)}: T^*(V) \to T^*(V)$ extends to continuous linear maps $\iota_{\infty\alpha}: T^*(V)^{\mathrm{cpl}} \to T^*(V)_\alpha^{\mathrm{cpl}}$ and $\iota_{\alpha\beta}: T^*(V)_\alpha^{\mathrm{cpl}} \to T^*(V)_\beta^{\mathrm{cpl}}$, such that $\iota_{\alpha\beta} 0 \iota_{\infty\alpha} = \iota_{\infty\beta}$ and $\iota_{\beta\gamma} 0 \iota_{\alpha\beta} = \iota_{\alpha\gamma}$ hold for all $\langle\cdot|\cdot\rangle_\alpha$, $\langle\cdot|\cdot\rangle_\beta$, $\langle\cdot|\cdot\rangle_\gamma \in \mathcal{I}_{V,h}$ with $\langle\cdot|\cdot\rangle_\gamma \leq \langle\cdot|\cdot\rangle_\beta \leq \langle\cdot|\cdot\rangle_\alpha$. This way, $T^*(V)^{\mathrm{cpl}}$ is realized as the projective limit of the Hilbert spaces $T^*(V)_\alpha^{\mathrm{cpl}}$ and similarly, $S^*(V)^{\mathrm{cpl}}$ as the projective limit of the closed linear subspaces $S^*(V)_\alpha^{\mathrm{cpl}}$

Corollary (6.2.78)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $f_j \in \mathcal{C}^\infty(V_h')$. *Given* $\rho \in V_h'$ *and* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ *such that* $|\sum_r \rho(v^r)| \le \sum_r \|v^r\|_\alpha$ *holds for all* $v^r \in V$, *then*

$$\widehat{D}_\rho(\sum_j f_j) = \sum_{i\in I}\sum_j \rho(e_i)\widehat{D}_{e_i^{b\alpha}} f_j \tag{160}$$

holds for every Hilbert basis $e \in \left(V_{h,\alpha}^{\mathrm{cpl}}\right)^I$ *of* $V_{h,\alpha}^{\mathrm{cpl}}$ *indexed by a set* I.

Proof: As f_j is smooth, the function $V_h' \ni \sigma \mapsto \widehat{D}_\sigma f_j \in \mathbb{C}$ is bounded, which implies that its restriction to the dual space of $V_{h,\alpha}^{\mathrm{cpl}}$ is continuous with respect to the Hilbert space topology on (the dual of) $V_{h,\alpha}^{\mathrm{cpl}}$. As $\rho = \sum_{i\in I} e_i^{b\alpha}\rho(e_i)$ with respect to this topology, it follows that $\widehat{D}_p f_j = \sum_{i\in I}\sum_j \rho(e_i)\widehat{D}_{e_i^{b\alpha}} f_j$.

Corollary (6.2.79)[288]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $f_j: V_h' \to \mathbb{C}$ *analytic of Hilbert-Schmidt type with* $\sum_j \left(\widehat{D}_\rho^{(k)} f_j\right)(0) = 0$ *for all* $k \in \mathrm{N}_0$ *and all* $\rho \in (V_h')^k$. *Then* $f_j = 0$.

Proof: Given $\sigma \in V_h'$, then define the smooth function $g_j: \mathbb{R} \to \mathbb{C}$ by $t \mapsto g_j(t) := f_j(t\sigma)$. We write $g_j^{(k)}(t)$ for the k-th derivative of g_j at t. Then there exists a $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ that fulfils $\sum_r |\sigma(v^r)| \le \sum_r \|v^r\|_\alpha$ for all $v^r \in V$, and consequently $\sigma = L/e^{b\alpha}$ with a normalized $e \in V_{h,\alpha}^{\mathrm{cpl}}$ and $u \in [0,1]$ by the Fréchet-Riesz theorem. Therefore,

$$\left(\sum_{k=0}^{\infty}\sum_j \frac{1}{k!}|g_j^{(k)}(t)|\right)^2 \overset{CS}{\le} \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\ell=0}^{\infty}\sum_j \frac{1}{\ell!}|g_j^{(\ell)}(t)|^2$$

$$\le \mathrm{e}\sum_{\ell=0}^{\infty}\frac{\mathrm{v}^{2p}}{\ell!}|\sum_j \left(\widehat{D}^{(\ell)}_{(e^{b\alpha},\dots,e^{b\alpha})} f_j\right)(t\sigma)|^2 \le \mathrm{e}C_{-2\sigma,2\sigma,\alpha}$$

holds for all $t \in [-2,2]$ with a constant $C_{-2\sigma,2\sigma,\alpha} \in \mathbb{R}$, which shows that g_j is an analytic function on $]-2,2[$. As $\sum_j g_j^{(k)}(0) = 0$ for all $k \in \mathrm{M}_0$ this implies $f_j(\sigma) = g_j(1) = 0$.

Note that one can derive even better estimates for the derivatives of g_j. This shows that condition (120) is even stronger than just analyticity. As an example, consider $V = \mathbb{C}$, $V_h' = \mathbb{R}$, then the function $\mathbb{R} \ni x^r \mapsto \exp((x^r)^2) \in \mathbb{C}$ is not analytic of Hilbert-Schmidt type.

Corollary (6.2.80)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V. *Let* $k \in \mathrm{N}$ *and* $x^r \in (V_h)^k$ *as well as* $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ *be given. Then*

$$\sum_j \left(\widehat{D}^{(k)}_{(x^r)^{b\alpha}}\widehat{Y}_j\right)(0) = \sum_j \langle D^{(k)}_{(x^r)^{b\alpha}} Y_j\rangle_0 = \langle x_1^r \otimes \cdots \otimes x_k^r | Y_j\rangle_\alpha^* \tag{161}$$

holds for all $Y_j \in S^*(V)^{\mathrm{cpl}}$.

Proof: The first identity is just Corollary (6.2.77), and for the second one it is sufficient to show that $\sum_j \langle D^{(k)}_{(x^r)^{b\alpha}} Y_j\rangle_0 = \sum_r \langle x_1^r \otimes \cdots \otimes x_k^r | Y_j\rangle_\alpha^*$ holds for all factorizing tensors Y_j of degree k, because both sides of this equation vanish on homogeneous tensors of different degree and are linear and continuous in Y_j by Corollary (6.2.75). However, it is an immediate consequence of the definitions of D, $\cdot\, \mathrm{b}_\alpha$, and $\langle\cdot\mid\cdot\rangle_\alpha^*$ that

$$\sum_r \langle D^{(k)}_{\left((x_1^r)^{\mathrm{b}_\alpha},\ldots,(x_k^r)^{\mathrm{b}_\alpha}\right)} y_1^r \otimes \cdots \otimes y_k^r\rangle_0 = k! \sum_r \prod_{m=1}^{k} \langle x_m^{\mathrm{r}} | y_m^r\rangle_\alpha$$
$$= \sum_r \langle x_1^r \otimes \ldots \otimes x_k^{\mathrm{r}} | y_1^r \otimes \ldots \ldots \otimes y_k^r\rangle_\alpha^*$$

holds for all $y_1^r, y_k^r \in V$.

Corollary (6.2.81)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then*

$$\sum_j \ll \widehat{X}_j | \widehat{Y}_j \gg_\alpha^* (\rho) = \sum_j \langle \tau_\rho^* X_j | \tau_\rho^* Y_j\rangle_\alpha^* = \sum_J \widehat{X_J{}^* \star_{\Lambda_\alpha} Y_j}(\rho) \qquad (162)$$

holds for all $X_j, Y_j \in S^{\cdot}(V)^{\mathrm{cpl}}$, *all* $\rho \in V_h'$, *and all* $\langle\cdot\mid\cdot\rangle_{0j} \in \mathcal{I}_{V,h}$, *where* $\Lambda_\alpha: V \times V \to \mathbb{C}$ *is the continuous bilinear form defined by* $\sum_r \Lambda_\alpha(v^r, w^r) := \sum_r \langle \overline{v^r} | w^r\rangle_\alpha$.

Proof: Let X_j, $Y_j \in S^*(V)^{\mathrm{cpl}}$, $p \in V_h'$ and $\langle\cdot\mid\cdot\rangle_\alpha \in \mathcal{I}_{V,h}$ be given. Let $e \in \left(V_{h,\alpha}^{\mathrm{cpl}}\right)^I$ be a Hilbert base of $V_{h,\alpha}^{\mathrm{cpl}}$ indexed by a set I. Then

$$\sum_j \ll \widehat{X}_j | \widehat{Y}_j \gg_\alpha^* (\rho) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \sum_j \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} \widehat{X}_J\right)(p) \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}} \widehat{Y}_J\right)(\rho)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i \in I^k} \sum_j \langle D^{(k)}_{e_i^{\mathrm{b}_\alpha}} \tau_\rho^* X_j\rangle_0 \langle D^{(k)}_{e_i^{\mathrm{b}_\alpha}} \tau_\rho^* Y_j\rangle_0$$
$$= \sum_{k=0}^{\infty} \sum_{i \in I^k} \sum_j \frac{1}{k!} \langle \tau_\rho^* X_j | e_{i_1} \otimes \cdots \otimes e_{i_k}\rangle_\alpha^* \langle e_{i_1} \otimes \cdots \otimes e_{i_k} | \tau_\rho^* Y_j\rangle_\alpha^*$$
$$= \sum_j \langle \tau_\rho^* X_j | \tau_p^* Y_j\rangle_\alpha^*$$

holds by Corollary (6.2.77) and Corollary (6.2.74) as well as the previous Corollary (6.2.80) and the fact that the tensors $(k!)^{1/2} e_{i_1} \otimes \otimes e_{i_k}$ for all $k \in \mathrm{M}_0$ and $i \in I^k$ form a Hilbert base of $\mathcal{T}^*(V)_\alpha^{\mathrm{cpl}}$ The second identity is a direct consequence of Corollary (6.2.70) because τ_ρ^* is a unital $*$-automorphism of $\star_{\Lambda_\alpha}$. Indeed, we have

$$\sum_j \langle \tau_\rho^* X_j | \tau_\rho^* Y_j\rangle_\alpha^* = \sum_j \langle \left(\tau_\rho^* X_j\right)^* \star_{\Lambda_\alpha} \left(\tau_\rho^* Y_j\right)\rangle_0 = \sum_j \langle \tau_\rho^*\left(X_j^* \star_{\Lambda_\alpha} Y_j\right)\rangle_0$$
$$= \sum_j \widehat{X_J{}^* \star_{\Lambda_\alpha}} Y_j(\rho)\,.$$

Corollary (6.2.82)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $X_j \in S^*(V)^{\mathrm{cpl}}$, *then* $\widehat{X}_j \in \mathcal{C}^{({}^\prime J_{HS}}(V_h')$.

Proof: The function $\widehat{X}_j$ is smooth by Corollary (6.2.77). By the previous Corollary (6.2.81), we have

$$\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\sum_{j}\ |\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}\widehat{X}_J\right)(\xi^r)|^2=\sum_{j}\ \ll \widehat{X}_J|\widehat{X}_J\gg_\alpha^*(\xi^r)=\sum_{j}\ \widehat{X_J{}^*\star_{\Lambda_\alpha}}X_j(\xi^r)$$

for all $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$, which is finite and depends smoothly on $\xi^r\in V_h'$ by Corollary (6.2.77) again. Therefore it is uniformly bounded on line segments.

Corollary (6.2.83)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$. *For every* $f_{\mathrm{j}}\in\mathcal{C}^{0\supset HS}(V_h')$ *there exists an* $(X_j)_{f_{\mathrm{j}}}\in S^{\cdot}(V)^{\mathrm{cpl}}$ *that fulfils* $\sum_j\ \ll f_{\mathrm{j}}\Big|f_{\mathrm{j}}\gg_\alpha^*(0)=\sum_j\ \ll \widehat{(X_J)}_{f_{\mathrm{j}}}\Big|\widehat{(X_J)}_{f_{\mathrm{j}}}\gg_\alpha^*(0)$*and* $\sum_j\ \ll f_{\mathrm{j}}\Big|\widehat{Y}_J\gg_\alpha^*(0)=\sum_j\ \ll \widehat{(X_J)}_{f_{\mathrm{j}}}\Big|\widehat{Y}_J\gg_\alpha^*(0)$ *for all* $Y_j\in S^{\cdot}(V)^{\mathrm{cpl}}$ *and all* $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$.

Proof: For every $\alpha\in\mathcal{I}_{V,h}$ construct $(X_j)_{f_{\mathrm{j}},\alpha}\in S^*(V)_\alpha^{\mathrm{cpl}}$ as

$$(X_j)_{f_{\mathrm{j}},\alpha}:=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\sum_{j}\ e_{i_1}\otimes\cdots\otimes e_{i_k}\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}f_{\mathrm{j}}\right)(0)\in S^{v^r}(V)_\alpha^{\mathrm{cpl}},$$

where $e\in\left(V_{h,\alpha}^{\mathrm{cpl}}\right)^I$ is a Hilbert base of $V_{h,\alpha}^{\mathrm{cpl}}$ indexed by a set I. This infinite sum $(X_j)_{f_{\mathrm{j}},\alpha}$ indeed lies in $S^*(V)_\alpha^{\mathrm{cpl}}$ and fulfils $\sum_j\ \langle(X_j)_{f_{\mathrm{j}},\alpha}\Big|(X_j)_{f_{\mathrm{j}},\alpha}\rangle_\alpha^*=\sum_j\ \ll f_{\mathrm{j}}\Big|f_{\mathrm{j}}\gg_\alpha^*(0)$, because $\left(\widehat{D}^{(k)}_{e_i^{\rangle\alpha}}f_{\mathrm{j}}\right)(0)$ is invariant under permutations of the $e_{i_1},\ldots,e_{i_k}$ due to the smoothness of f_{j} and because

$$\sum_{k,\ell=0}^{\infty}\sum_{i\in I^k,i'\in I^\ell}\sum_{j}\frac{1}{k!\,\ell!}\langle e_{i_1}\otimes\cdots\otimes e_{i_k}\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}f_{\mathrm{j}}\right)(0)|\ e_{i'_1}\otimes\cdots\otimes e_{i'_p}\left(\widehat{D}^{(\ell)}_{e_i^{\mathrm{b}_\alpha}}f_{\mathrm{j}}\right)(0)\rangle_\alpha^*$$

$$=\sum_{k=0}^{\infty}\sum_{i\in I^k}\sum_{j}\ \frac{1}{k!}|\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}f_{\mathrm{j}}\right)(0)|^2$$

$$=\sum_{j}\ \ll f_{\mathrm{j}}|f_{\mathrm{j}}\gg_\alpha^*(0)\,.$$

Moreover, for all $Y_j\in S^{\cdot}(V)^{\mathrm{cpl}}$ the identity

$$\sum_{j}\ \ll f_{\mathrm{j}}|\widehat{Y}_J\gg_\alpha^*(0)=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\sum_{j}\ \left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}f_{\mathrm{j}}\right)(0)\left(\widehat{D}^{(k)}_{e_i^{\mathrm{b}_\alpha}}\widehat{Y}_J\right)(0)$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\sum_{j}\ \langle\,(X_j)_{f_{\mathrm{j}},\alpha}|e_{i_1}\otimes\otimes e_{i_k}\rangle_\alpha^*\langle e_{i_1}\otimes\otimes e_{i_k}|Y_j\rangle_\alpha^*$$

$$=\sum_{j}\ \langle(X_j)_{f_{\mathrm{j}},0i}|Y_j\rangle_\alpha^*$$

holds due to the construction of $(X_j)_{f_{\mathrm{j}},\alpha}$ and Corollary (6.2.80) and because the tensors $(k!)^{1/2}e_{i_1}\otimes\otimes e_{i_k}$ for all $k\in\mathrm{M}_0$ and all $i\in I^k$ are a Hilbert base of $T^*(V)_\alpha^{\mathrm{cpl}}$

Next, let $\langle\cdot\mid\cdot\rangle_\beta\in\mathcal{I}_{V,h}$ with $\langle\cdot\mid\cdot\rangle_\beta\le\langle\cdot\mid\cdot\rangle_\alpha$ and a Hilbert basis $d\in\left(V_{h,\beta}^{\mathrm{cpl}}\right)^J$ of $V_{h,\beta}^{\mathrm{cpl}}$ indexed by a set J be given. Using the explicit formulas and the identity

$$\sum_j \left(\widehat{D}^{(k)}_{d_j^{b_\beta}} f_j\right)(0) = \frac{1}{k!}\sum_{i\in I^k}\sum_j \left(\widehat{D}^{(k)}_{e_i^{b_\alpha}} f_j\right)(0)\langle d_{j_1}\otimes\cdots\otimes d_{j_k}|\iota_{\alpha\beta}\left(e_{i_1}\otimes\otimes e_{i_k}\right)\rangle^*_\beta$$

from Corollary (6.2.78) one can now calculate that

$$\sum_j \iota_{\alpha\beta}\left((X_j)_{f_j,\alpha}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i\in I^k}\sum_j \iota_{\alpha\beta}\left(e_{i_1}\otimes\cdots\otimes e_{i_k}\right)\left(\widehat{D}^{(k)}_{e_i^{b_\alpha}} f_j\right)(0)$$

$$= \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\sum_{i\in I^k}\sum_{j\in J^k}\sum_j d_{j_1}\otimes\cdots\otimes d_{j_k}\langle d_{j_1}\otimes\cdots$$

$$\otimes d_{j_k}|\iota_{\alpha\beta}\left(e_{i_1}\otimes\otimes e_{i_k}\right)\rangle^*_\beta\left(\widehat{D}^{(k)}_{e_i^{b_\alpha}} f_j\right)(0)$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j\in J^k}\sum_j d_{j_1}\otimes\otimes d_{j_k}\left(\widehat{D}^{(k)}_{d_j^{b_\beta}} f_j\right)(0)$$

$$= (X_j)_{f_j,\beta}.$$

As $S^*(V)^{\mathrm{cpl}}$ is the projective limit of the Hilbert spaces $S^*(V)^{\mathrm{cpl}}_\alpha$, this implies that there exists a unique $(X_j)_{f_j}\in S^*(V)^{\mathrm{cpl}}$ that fulfils $\iota_{\infty\alpha}\left((X_j)_{f_j}\right) = (X_j)_{f_j,\alpha}$ for all $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$. Consequently and with the help of Corollary (6.2.81),

$$\sum_j \ll \widehat{(X_j)}_{f_j}|\widehat{Y}_j \gg^*_\alpha (0) = \sum_j \langle (X_j)_{f_j}|Y_j\rangle^*_\alpha = \sum_j \langle\iota_{\infty\alpha}\left((X_j)_{f_j}\right)|Y_j\rangle^*_\alpha$$

$$= \sum_j \langle (X_j)_{f_j,\alpha}|Y_j\rangle^*_\alpha = \sum_j \ll f_j|\widehat{Y}_j \gg^*_\alpha (0)$$

holds for all $Y_j\in S^{\cdot}(V)^{\mathrm{cpl}}$ and all $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$, and similarly,

$$\sum_j \ll \widehat{(X_j)}_{f_j}|\widehat{(X_j)}_{f_j} \gg^*_\alpha (0) = \sum_j \langle (X_j)_{f_j}|(X_j)_{f_j}\rangle^*_\alpha$$

$$= \sum_j \langle\iota_{\infty\alpha}\left((X_j)_{f_j}\right)|\iota_{\infty\alpha}\left((X_j)_{f_j}\right)\rangle^*_\alpha = \sum_j \langle (X_j)_{f_j,\alpha}|(X_j)_{f_j,\alpha}\rangle^*_\alpha = \sum_j \ll f_j|f_j \gg^*_\alpha (0)\,.$$

Corollary (6.2.84)[288]: Let $\bar{\cdot}$ *be a continuous antilinear involution on* V, *then the Gel'fand transformation* $\wedge$: $\left(S^*(V)^{\mathrm{cpl}},\vee,*\right)\to\mathcal{C}^{(\prime)}HS(V'_h)$ *is an isomorphism of unital*$*$*-algebras*.

Proof: Let $X_j\in S^*(V)^{\mathrm{cpl}}$ be given, then $\widehat{X}_j\in\mathcal{C}^{(}{}_{HwS}(V'_h)$ by Corollary (6.2.82). The Gel'fand transformation is a unital $*$-homomorphism onto its image by construction and injective because $\widehat{X}_j = 0$ implies $\sum_j \langle X_j|X_j\rangle^*_\alpha = \sum_j \ll \widehat{X}_j|\widehat{X}_j \gg^*_\alpha (0) = 0$ for all $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_{V,h}$ by Corollary (6.2.81), hence $X_j = 0$. It only remains to show that $\wedge$ is surjective, so let $f_j\in\mathcal{C}^{(\prime J_{HS}}(V'_h)$ be given. Construct $(X_j)_{f_j}\in S^*(V)^{\mathrm{cpl}}$ like in the previous Corollary (6.2.83), then

$$\sum_j \ll f_j - \widehat{(X_J)}_{f_j} | f_j - \widehat{(X_J)}_{f_j} \gg_\alpha^* (0) = \sum_j \ll f_j | f_j \gg_\alpha^* (0) - \sum_j \ll f_j | \widehat{(X_J)}_{f_j} \gg_\alpha^* (0) - \sum_j \ll \widehat{(X_J)}_{f_j} | f_j \gg_\alpha^* (0) + \sum_j \ll \widehat{(X_J)}_{f_j} | \widehat{(X_J)}_{f_j} \gg_\alpha^* (0)$$

$$= \sum_j \ll f_j | f_j \gg_\alpha^* (0) - \sum_j \ll \widehat{(X_J)}_{f_j} | \widehat{(X_J)}_{f_j} \gg_\alpha^* (0) - \sum_j \ll \widehat{(X_J)}_{f_j} | \widehat{(X_J)}_{f_j} \gg_\alpha^* (0) + \sum_j \ll \widehat{(X_J)}_{f_j} | \widehat{(X_J)}_{f_j} \gg_\alpha^* (0)$$

$$= 0$$

holds for all $\langle \cdot \mid \cdot \rangle_\alpha \in \mathcal{I}_{V,h}$, hence $f_j = \widehat{(X_J)}_{f_j}$ due to Corollary (6.2.79).

Corollary (6.2.85)[288]: *Let* $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ *a continuous Hermitian bilinear form on* V *such that* $\sum_r \Lambda\left(\overline{v^r}, v^r\right) \geq 0$ *holds for all* $v^r \in V$. *Then for all* $X_j \in S^*(V)$ *and all* $t \in \mathrm{M}_0$ *there exist* $n \in \mathrm{N}$ *and* $(X_j)_1, \ldots, (X_j)_n \in S^{\cdot}(V)$ *such that*

$$\sum_j (\mathrm{P}_\Lambda)^t\left(X_j^* \otimes_\pi X_j\right) = \sum_{i=1}^n \sum_j (X_j)_i^* \otimes_\pi (X_j)_i. \tag{163}$$

Proof: This is trivial for scalar X_j as well as for $t = 0$ and for the remaining cases it is sufficient to consider $t = 1$, the others then follow by induction. So let $k \in \mathrm{N}$ and $X_j \in S^k(V)$ be given. Expand X_j as $X_j = \sum_{j=1}^m \sum_r x_{j,1}^r \vee \cdots \vee x_{j,k}^r$ with $m \in \mathrm{N}$ and vectors $x_{1,1}^r, , x_{m,k}^r \in V$. Then

$$\sum_j \mathrm{P}_\Lambda\left(X_j^* \otimes_\pi X_j\right)$$

$$= \sum_{j',j=1}^m \sum_{\ell',\ell=1}^k \sum_r \Lambda\left(\overline{x_{j}^r,, \ell'}, x_{j,\ell}^r\right)\left(x_{j,1}^r \vee \cdots \widehat{x_{j',\ell'}^r} \ldots \vee x_{j,k}^r\right)^* \otimes_\pi \left(x_{j,1}^r \vee \cdots \widehat{x_{j',\ell'}^r} \ldots \vee x_{j,k}^r\right),$$

where $\wedge$ denotes omission of a vector in the product. The complex $mk \times mk$ -matrix with entries $\Lambda\left(\overline{x_{j',\ell'}^r}, x_{j,\ell}^r\right)$ is positive semi-definite due to the positivity condition on Λ, which implies that it has a Hermitian square root $R \in \mathbb{C}^{mk \times mk}$ that fulfils $\Lambda\left(\overline{x_{j',\ell'}^r}, x_{j,l}^r\right) = \sum_{p=1}^m \sum_{q=1}^k \overline{R_{(p,q),(j',\ell')}} R_{(p,q),(j,\ell)}$ for all $j, j' \in \{1,, m\}$ and $\ell, \ell' \in \{1, k\}$. Consequently,

$$\sum_j \mathrm{P}_\Lambda\left(X_j^* \otimes_\pi X_j\right)$$
$$= \sum_{p,q=1}^{m,k} \sum_r \left(\sum_{j,\ell'=1}^{m,k} \overline{R_{(p,q),(j',\ell)}} \left(x_{j,1}^r \vee \cdots \widehat{x_{j',\ell'}^r} \ldots \vee x_{j',k}^r\right)^* \right) \otimes_\pi \left(\sum_{j,\ell=1}^{m,k} R_{(p,q),(j,\ell)} \left(x_{j,1}^r \vee \widehat{x_{j,\ell'}^r} \cdots \vee x_{j,k}^r\right) \right)$$

holds which proves the lemma.

Corollary (6.2.86)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ, Λ' *as well as* b *three continuous Hermitian bilinear forms on* V *such that* b *is symmetric and of Hilbert-Schmidt type and such that* $\Lambda'\left(\overline{v^r}, v^r\right) + b\left(\overline{v^r}, v^r\right) \geq 0$ *holds for all* $v^r \in V$. *Given a continuous linear functional* w^r *on* $S^{\cdot}(V)$ *that is positive for* $\star_\Lambda$, *define* $(v^{\mathrm{r}}_{z^r b}: S^*(V) \to \mathrm{D}$ *as*

$$X_j \mapsto (w^r)_{z^r b}\left(X_j\right) := w^r\left(\mathrm{e}^{z^r \triangle_b} X_j\right) \tag{164}$$

for all $z^r \in \mathbb{R}$. *Then* $\alpha j_{z^r b}$ *is a continuous linear functional and positive for* $\star_{\Lambda + z^r \Lambda'}$.

Proof: It follows from Corollary (6.2.73) that $(w^r)_{z^r b}$ is continuous, and given $X_j \in S^*(V)$, then

$$\sum_j \sum_r w^r \left(\mathrm{e}^{z^r \triangle_b}\left(X_j^* \star_{\Lambda + z^r \Lambda'} X_j\right)\right) = \sum_j \sum_r w^r \left(\left(\mathrm{e}^{z^r \triangle_b} X_j\right)^* \star_{\Lambda + z^r(\Lambda' + b)} \left(\mathrm{e}^{z^r \triangle_b} X_j\right)\right)$$
$$= \sum_{s,t=0}^{\infty} \sum_j \sum_r \frac{1}{s!\,t!} (\lambda) \left(\mu_{\mathrm{V}} \left((\mathrm{P}_\Lambda)^s \left(\mathrm{P}_{z^r(\Lambda' + b)}\right)^t \left(\left(\mathrm{e}^{z^r \triangle_b} X_j\right)^* \otimes_\pi \left(\mathrm{e}^{z^r \triangle_b} X_j\right)\right)\right)\right)$$
$$= \sum_{t=0}^{\infty} \sum_j \sum_r \frac{1}{t!} w^r \left(\mu_{\star_\Lambda} \left(\left(\mathrm{P}_{z^r(\Lambda' + b)}\right)^t \left(\left(\mathrm{e}^{z^r \triangle_b} X_j\right)^* \otimes_\pi \left(\mathrm{e}^{z^r \triangle_b} X_j\right)\right)\right)\right) \geq 0$$

holds because P_Λ and $\mathrm{P}_{z^r(\Lambda' + b')}$ commute on symmetric tensors and because of Corollary (6.2.85).

Note that Corollary (6.2.73) also shows that $(w^r)_{z^r b}$ depends holomorphically on $z^r \in \mathbb{C}$ in so far as $(\mathrm{D} \ni z^r \mapsto (w^r)_{z^r b}\left(X_j\right) \in (\mathrm{D}$ is holomorphic for all $X_j \in S^*(V)$. This is the analog of statements in [12, 13] in the Rieffel setting.

Corollary (6.2.87)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution of* V *and* Λ *a continuous Hermitian bilinear forms on* V. *If there exists a continuous linear functional* (λ) *on* $S^*(V)$ *that is positive for* $\star_\Lambda$ *and fulfils* $(v^r(1) = 1$, *then the bilinear form* $V^2 \ni (v^r, w^r) \mapsto b, (v^r, w^r) := w^r(v^r \vee w^r) \in \mathbb{C}$ *is symmetric, Hermitian, of Hilbert-Schmidt type and fulfils* $\sum_r \Lambda\left(\overline{v^r}, v^r\right) + \sum_r b_{w^r}\left(\overline{v^r}, v^r\right) \geq 0$ *for all* $v^r \in V$.

Proof: It follows immediately from the construction of b_{w^r} that this bilinear form is symmetric and it is Hermitian because $\sum_r \overline{b_{w^r}(v^r, w^r)} = \sum_r \overline{\mathrm{w}(v^r \vee w^r)} = \sum_r \overline{b_{w^r}\left(\overline{w^r} \vee \overline{v^r}\right)} = \sum_r \overline{b_{w^r}\left(\overline{w^r}, \overline{v^r}\right)}$ holds for all , $w^r \in V$. Continuity of w^r especially implies that there exists a $\langle\cdot \mid \cdot\rangle_\alpha \in \mathcal{I}_V$ such that $\sum_r \left|(w^r\left(X_j\right)\right| \leq 2^{-\frac{1}{2}} \sum_j \|X_j\|_\alpha^*$ holds for all $X_j \in S^2(V)$, hence b_{w^r} is of Hilbert-Schmidt type by Corollary (6.2.87) and because $\sum_j \sum_r \triangle_{b_{\omega^r}} X_j = \sum_j \sum_r w^r(X_j)$ for $X_j \in S^2(V)$. Finally, $0 \leq \sum_r w^r((v^r)^* \star_\Lambda v^r) = \sum_r \Lambda(\overline{v^r}, v^r) + \sum_r b_{w^r}(\overline{v^r}, v^r)$ holds due to the positivity of w^r.

Corollary (6.2.88)[288]: [280] *Let $\bar{\cdot}$ be a continuous antilinear involution of V and Λ a continuous Hermitian bilinear forms on V. Assume $V \neq \{0\}$. There exists a non-zero continuous positive linear functional on $(S^*(V), \star_\Lambda, *)$ if and only if there exists a symmetric and hermitian bilinear form of HilbertSchmidt type b on V such that $\sum_r \Lambda(\overline{v^r}, v^r) + \sum_r b(\overline{v^r}, v^r) \geq 0$ holds for all $v^r \in V$. In this case, the continuous positive linear functionals on $(S^*(V), \star_\Lambda, *)$ are point-separating, i.e. their common kernel is* $\{0\}$.

Proof: If there exists a non-zero continuous positive linear functional w^r on $(S^*(V), \star_\Lambda, *)$, then $w^r(1) \neq 0$ due to the Cauchy-Schwarz identity and we can rescale w^r such that $\alpha j(1) = 1$. Then the previous Corollary (6.2.87) shows the existence of such a bilinear form b. Conversely, if such a bilinear form b exists, then Corollary (6.2.86) shows that all continuous linear functionals on $S^{\cdot}(V)$ that are positive for V can be deformed to continuous linear functionals that are positive for $\star_\Lambda$ by taking the pull-back with $\mathrm{e}^{\triangle_b}$. As $\mathrm{e}^{\triangle_b}$ is invertible, it only remains to show that that the continuous positive linear functionals on $(S^{\cdot}(V), \vee^*)$ are point-separating. This is an immediate consequence of Corollary (6.2.84), which especially shows that the evaluation functionals δ_β with $\rho \in V_h'$ are point-separating.

Corollary (6.2.89)[288]: [280] *Let Λ be a continuous bilinear form on V. Let $k, n \in \mathrm{N}_0$ and $(X_j)_1, (X_j)_n \in S^{(k)}(V)^{\mathrm{cpl}}$ be given. Then the estimates*

$$\|\sum_j \langle (X_j)_1 \star_\Lambda \dots \star_\Lambda (X_j)_n \rangle_m \|_\alpha^* \leq \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn} \sum_j \|(X_j)_1\|_\alpha^* \cdots \|(X_j)_n\|_\alpha^* \quad (165)$$

and

$$||\sum_j (X_j)_1 \star_\Lambda \dots \star_\Lambda (X_j)_n ||_\alpha^* \leq \left(\frac{(kn)!}{(k!)^n}\right)^{\frac{1}{2}} (2\mathrm{e}^3)^{kn} \sum_j || (X_j)_1||_\alpha^* \dots ||(X_j)_n||_\alpha^* \quad (166)$$

hold for all $m \in \{0, \dots, kn\}$ and all $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$.

Proof: The first estimate implies the second, because $\|(X_j)_1 \star_\Lambda \star_\Lambda (X_j)_n\|_\alpha^*$ has at most $(1 + kn)$ nonvanishing homogeneous components, namely those of degree $m \in \{0, \dots, kn\}$, and $(1 + kn) \leq \mathrm{e}^{kn}$. We will prove the first estimate by induction over n: If $n = 0$ or $n = 1$, then the estimate is clearly fulfiled for all possible k and m, and if it holds for one $n \in \mathrm{N}$, then

$$\|\sum_j \langle (X_j)_1 \star_\Lambda \dots \star_\Lambda (X_j)_{n+1} \rangle_m \|_\alpha^*$$

$$\leq \sum_{t=0}^{k} \frac{1}{t!} \sum_j \|\langle \mu_{Reject} \left((\mathrm{P}_\Lambda)^t \left(\left((X_j)_1 \star_\Lambda \dots \star_\Lambda (X_j)_n \right) \otimes_\pi (X_j)_{n+1} \right) \right) \rangle_m \|_\alpha^*$$

$$\leq \sum_{t=0}^{k} \sum_{\ell=0}^{\min(m,k-1)} \sum_j \frac{1}{t!} \|\mu_\vee \left((\mathrm{P}_\Lambda)^t \left(\langle (X_j)_1 \star_\Lambda \dots \star_\Lambda (X_j)_n \rangle_{m-\ell+t} \otimes_\pi \langle (X_j)_{n+1} \rangle_{\ell+t} \right) \right) \|_\alpha^*$$

$$\leq \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \sum_{j} \frac{1}{t!} \binom{m}{\ell}^{\frac{1}{2}} \|(\mathrm{P}_{\Lambda})\ ^{t}(\langle (X_j)_1 \star_{\Lambda} .\star_{\Lambda} (X_j)_n \rangle_{m-\ell+t} \otimes_{\pi} \langle (X_j)_{n+1} \rangle_{\ell+t})\|_{\alpha\otimes_{\pi}\alpha}$$

$$\leq \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \sum_{j} \binom{m}{\ell}^{\frac{1}{2}} \binom{m-\ell+t}{t}^{\frac{1}{2}} \binom{\ell+t}{t}^{\frac{1}{2}} \|\langle (X_j)_1 \star_{\Lambda} \star_{\Lambda} (X_j)_n \rangle_{m-\ell+t}\|_{\alpha} \|\langle (X_j)_{n+1} \rangle_{\ell+t}\|_{\alpha}^{*}$$

$$\leq \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \sum_{j} \binom{\ell+t}{t} \binom{\mathrm{k(n+1)}}{k}^{\frac{1}{2}} \|\langle (X_j)_1 \star_{\Lambda} \star_{\Lambda} (X_j)_n \rangle_{m-\ell+t}\|_{\alpha} \|\langle (X_j)_{n+1} \rangle_{\ell+t}\|_{\alpha}^{*}$$

$$\leq \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \sum_{j} \binom{\ell+t}{t} \left(\frac{\mathrm{k(n+1)}}{(K!)^{n+1}} \right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn} \|(X_j)_1\|_{\alpha} \|(X_j)_n\|_{\alpha} \|(X_j)_{n+1}\|_{\alpha}^{*}$$

$$= \sum_{t=0}^{k} \sum_{\ell=0}^{\min\{m,k-t\}} \sum_{j} \binom{\ell+t}{t} \left(\frac{\big(k(n+1)\big)!}{(k!)^{n+1}} \right)^{\frac{1}{2}} (2\mathrm{e}^2)^{kn} \|(X_j)_1\|_{\alpha}^{*} \cdots \|(X_j)_{n+1}\|_{\alpha}^{*}$$

$$\leq \left(\frac{\big(k(n+1)\big)!}{(k!)^{n+1}} \right)^{\frac{1}{2}} \sum_{j} (2\mathrm{e}^2)^{k(n+1)} \|(X_j)_1\|_{\alpha}^{*} \cdots \|(X_j)_{n+1}\|_{\alpha}^{*}$$

holds due to the grading of μ_{V} and P_{Λ}, the estimates from Corollaries (6.2.62) as well as (6.2.63) and Corollary (6.2.64) for μ_{V} and P_{Λ}, and the previous Lemma (6.2.46).

Corollary (6.2.90)[288]: [280] *Let Λ be a continuous bilinear form on V, then* $\exp_{\star_{\Lambda}}(v^r)$ *is absolutely convergent and*

$$\sum_{r} \exp_{\star_{\Lambda}}(v^r) = \sum_{n=0}^{\infty} \sum_{r} \frac{(v^r)^{\star_{\Lambda} n}}{n!} = \sum_{r} \mathrm{e}^{\frac{1}{2}\Lambda(v^r,v^r)} \exp_{\mathrm{V}}(v^r) \tag{167}$$

holds for all $v^r \in V$. Moreover,

$$\sum_{r} \exp_{\mathrm{V}}(v^r) \star_{\Lambda} \exp_{\mathrm{V}}(w^r) = \sum_{r} \mathrm{e}^{\Lambda(v^r,w^r)} \exp_{\mathrm{V}}(v^r + w^r) \tag{168}$$

and

$$\sum_{r} \langle \exp_{\mathrm{V}}(v^r) | \exp_{\mathrm{V}}(w^r) \rangle_{\alpha}^{*} = \sum_{r} \mathrm{e}^{\langle v^r | w^r \rangle_{\alpha}} \tag{169}$$

hold for all $v^r, w^r \in V$ and all $\langle \cdot | \cdot \rangle_{\alpha} \in \mathcal{I}_V$. Finally, $\sum_r \exp_{\mathrm{V}}(v^r)^* = \sum_r \exp_{\mathrm{V}}(\overline{v^r})$ *for all $v^r \in V$ if V is equipped with a continuous antilinear involutio n$^-$*

Proof: The existence and absolute convergence of $\star_{\Lambda}$ -exponentials of vectors follows directly from the previous Corollary (6.2.89) with $k=1$ and $(X_j)_1 = \cdots = (X_j)_n = v^r$:

$$\sum_{n=0}^{\infty} \sum_{r} \frac{\|(v^r)^{\star_{\Lambda} n}\|_{\alpha}^{*}}{n!}$$

$$\leq \sum_{n=0}^{\infty} \sum_{r} \frac{(4\mathrm{e}^3 \|v^r\|_{\alpha})^n}{\sqrt{n!}} \frac{1}{2^n} \overset{cs}{\leq} \left(\sum_{n=0}^{\infty} \sum_{r} \frac{(4\mathrm{e}^3 \|v^r\|_{\alpha})^{2n}}{n!} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{4^n} \right)^{\frac{1}{2}} < \infty$$

The explicit formula can then be derived like in [23, Lem. 5.5]. For (168) we just note that

$$\sum_r \mathrm{P}_\Lambda\big(\exp_\vee(v^r)\otimes_\pi \exp_\vee(w^r)\big)=\sum_{k,\ell=0}^{\infty}\sum_r \mathrm{P}_\Lambda\left(\frac{(v^r)^{\vee k}}{k!}\otimes_\pi\frac{(w^r)^{\vee \ell}}{\ell!}\right)$$
$$=\sum_r \Lambda(v^r,w^r)\sum_{k,\ell=1}^{\infty}\frac{k(v^r)^{\vee(k-1)}}{k!}\otimes_\pi\frac{\ell(w^r)^{\vee(\ell-1)}}{\ell!}$$
and so
$$\sum_r \exp_\vee(v^r)\ \star_\Lambda \exp_\vee(w^r)=\sum_{t=0}^{\infty}\sum_r \frac{1}{t!}\mu_\vee\left((\mathrm{P}_\Lambda)^t\big(\exp_\vee(v^r)\otimes_\pi \exp_\vee(w^r)\big)\right)$$
$$=\sum_r \mathrm{e}^{\Lambda(v^r,w^r)}\exp_\vee(v^r)\vee\exp_\vee(w^r)\,.$$
The remaining two identities are the results of straightforward calculations.

Corollary (6.2.91)[288]: Let $\bar{\cdot}$ *be a continuous antilinear involution on V. Then* $S^*_{\text{per}}(V)$ *is a dense* $*$*-subalgebra of* $\big(S^*(V)^{\text{cpl}*}\star_\Lambda,\big)$ *with respect to all products* $\star_\Lambda$ *for all continuous bilinear Hermitian forms* Λ *on V and*
$$\|X_j\|_{\infty,\Lambda}:=\ \sup\sum_j \sqrt{w^r\big(X_j^*\star_\Lambda X_j\big)}<\infty \tag{170}$$
holds for all $X_j\in S_{\text{per}}(V)$ *, where the supremum runs over all continuous positive linear functionals* w^r *on* $(S^{\cdot}(V),\star_\Lambda,*)$ *that are normalized to* $w^r(1)=1$.

Proof: Corollary (6.2.90) shows that $S^*_{\text{per}}(V)$ is a*-subalgebra of $S^*(V)^{\text{cpl}}$ with respect to all products $\star_\Lambda$ for all continuous bilinear Hermitian forms Λ on V. As $\sum_r -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z^r}|_{z^r=0}\exp_\vee(\mathrm{i}z^r v^r)=\sum_r\ v^r$ for all $v^r\in V$ with $v^r=\overline{v^r}$ we see that the closure of the subalgebra $S^*_{\text{per}}(V)$ contains V, hence $S^*(V)$ which is (as a unital algebra) generated by V, and so the closure of $S^*_{\text{per}}(V)$ coincides with $S^*(V)^{\text{cpl}}$.

As $S^*_{\text{per}}(V)$ is spanned by exponentials and $\sum_r\ \big(w^r\big(\exp_\vee(\mathrm{i}v^r)^*\star_\Lambda \exp_\vee(\mathrm{i}v^r)\big)=\sum_r\ \mathrm{e}^{\Lambda(v^r,v^r)}\big(w^r\big(\exp_\vee(0)\big)=\sum_r\ \mathrm{e}^{\Lambda(v^r,v^r)}$ holds for all positive linear functionals w on $(S^*(V),\star_\Lambda,*)$ that are normalized to $(v^r(1)=1$ by Corollary (6.2.90), it follows that $\sum_j\ \|X_j\|_{\infty,\Lambda}<\infty$ for all $X_j\in S^*_{\text{per}}(V)$.

Corollary (6.2.92)[288]: Let $\bar{\cdot}$ *be a continuous antilinear involution on V. Then there is no locally convex topology* τ *on* $S^*_{\text{alg}}(V)$ *with the property that any (undeformed) exponential* $\sum_j\ \exp_\vee\big(X_j\big)=\sum_{n=0}^{\infty}\sum_j\ \frac{X_j^{\vee n}}{n!}$ *of any* $X_j\in S^2(V)\backslash\{0\}$ *exists in the completion of* $S^*_{\text{alg}}(V)$ *under* τ *and such that all the products* $\star_\Lambda$ *for all continuous Hermitian bilinear forms* Λ *on* V *as well as the* $*$*-involution and the projection* $\langle\cdot\rangle_0$ *on the scalars are continuous*.

Proof: Analogously to the proof of Corollary (6.2.70) we see that, if all the products $\star_\Lambda$ for all continuous Hermitian bilinear forms Λ on V as well as the $*$-involution and the projection $\langle\cdot\rangle_0$ on the scalars are continuous, then all the extended positive Hermitian forms $\langle\cdot\mid\cdot\rangle^*_\alpha$ for all $\langle\cdot\mid\cdot\rangle_\alpha\in\mathcal{I}_V$ would have to be continuous and thus extend to the completion of $S^*_{\text{alg}}(V)$.

Now let $X_j\in S^2(V)\backslash\{0\}$ be given. There exist $k\in$ M and $x^r\in V^k$ such that $x^r_1,\ldots,x^r_k$ are linearly independent and $X_j=\sum_{i=1}^k\sum_{j=1}^k\sum_j\ \tilde{X}_j^{ij}\,x_i^r\vee x_j^r$ with complex coefficients $\tilde{X}_j^{ij}$. If there exists an $i \in \{1, k\}$ such that $\tilde{X}_j^{ii} \neq 0$, then we can assume without loss of generality that $i = 1$ and $\tilde{X}_{\mathrm{j}}^{11} = 1$ and define a continuous positive Hermitian form on V by $\sum_r \langle v^r | w^r \rangle_{\mathrm{w^r}} := \sum_r \overline{w^r(v^r)} w^r(w^r)$, where $w^r \supset : V \to \mathbb{C}$ is a continuous linear form on V that satisfies $w^r(x_1^r) = 1$ and $w^r(x_i^r) = 0$ for $i \in \{2, k\}$. Otherwise we can assume without loss of generality that $\tilde{X}_{\mathrm{j}}^{11} = \tilde{X}_{\mathrm{j}}^{22} = 0$ and $\tilde{X}_{\mathrm{j}}^{12} = 1$ and define a continuous positive Hermitian form on V by $\sum_r \langle v^r | w^r \rangle_{\mathrm{w^r}} := \sum_r \overline{\mathrm{w^r}(v^r)}^T w^r(w^r)$, where $\mathrm{w^r}: V \to \mathbb{C}^2$ is a continuous linear map that satisfies $\sum_r w^r(x_1^r) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\sum_r w^r(x_2^r) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\sum_r w^r(x_i^r) = 0$ for $i \in \{3, , k\}$.

In the first case, this results in $\langle X_{\mathrm{j}}^{\vee n} | X_j^{\vee n} \rangle_j^* = (2n)!$ and in the second, $\sum_r \sum_j \langle X_{\mathrm{j}}^{\vee n} | X_j^{\vee n} \rangle_{\mathrm{w^r}} = (n!)^2$. So $\sum_{n=0}^{\infty} \frac{X_j^{\vee n}}{n!}$ cannot converge in the completion of $S_{\mathrm{alg}}(V)$ because

$$\sum_j \langle \sum_{n=0}^{N} \frac{X_j^{\vee n}}{n!} | \sum_{n=0}^{N} \frac{X_j^{\vee n}}{n!} \rangle_{\mathrm{w}}^* \geq \sum_{n=0}^{N} 1 \to \overset{N \to \infty}{\longrightarrow} \infty.$$

A similar result has already been obtained by Omori, Maeda, Miyazaki and Yoshioka in the 2-dimensional case in [108], where they show that associativity of the Moyal-product breaks down on exponentials of quadratic functions. Note that the above proposition does not exclude the possibility that exponentials of *some* quadratic functions exist if one only demands that *some* special deformations are continuous.

Even though exponentials of non-trivial tensors of degree 2 are not contained in $S^*(V)^{\mathrm{cpl}}$, the continuous positive linear functionals are in some sense "analytic" for such tensors:

Corollary (6.2.93)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* Λ *a continuous Hermitian bilinear form on* V*. Let* $w^r : S^*(V)^{\mathrm{cpl}} \to \mathbb{C}$ *be a continuous linear functional on* $S^*(V)^{\mathrm{cpl}}$ *that is positive with respect to* $\star_\Lambda$*. Then for all* $X_j \in S^{(2)}(V)^{\mathrm{cpl}}$ *there exists an* $\varepsilon > 0$ *such that*

$$\sum_{n=0}^{\infty} \sum_r \sum_j \frac{\varepsilon^n w^r\left(\left(X_j^{\star_\Lambda n}\right)^* \star_\Lambda X_j^{\star_\Lambda n}\right)^{\frac{1}{2}}}{n!} < \infty \tag{171}$$

holds.

Proof: The seminorm $S^*(V)^{\mathrm{cpl}} \ni Y_j \mapsto w^r\left(Y_j^* \star_\Lambda Y_j\right)^{1/2} \in [0, \infty[$ is continuous by construction, so there exist $C > 0$ and $\|\cdot\|_\alpha \in \mathcal{P}_V$ such that $\sum_r \sum_j w^r\left(Y_j^* \star_\Lambda Y_j\right)^{1/2} \leq C \sum_j \|Y_j\|_\alpha^*$ holds for all $Y_j \in S^*(V)^{\mathrm{cpl}}$. We can even assume without loss of generality that $\|\cdot\|_\alpha \in \mathcal{P}_{V,\Lambda}$. Now choose $\varepsilon > 0$ with $\sum_j \varepsilon\left(8\mathrm{e}^6 \|X_j\|_\alpha^*\right) \leq 1$, then Corollary (6.2.89) in the case $k = 2$ and $(X_j)_1 = \cdots = (X_j)_n = X_j$ shows that

$$\sum_{n=0}^{\infty} \sum_r \sum_j \frac{\varepsilon^n (w^r\left(\left(X_j^{\star_\Lambda n}\right)^* \star_\Lambda X_j^{\star_\Lambda n}\right)^{\frac{1}{2}}}{n!} \leq C \sum_{n=0}^{\infty} \sum_j \frac{\varepsilon^n \|X_j^{\star_\Lambda n}\|_\alpha^*}{n!}$$

$$\leq C\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{\sqrt{2}^{3n}n!}$$
$$\leq C\sum_{n=0}^{\infty}\frac{1}{\sqrt{2}^{n}}$$
$$=\frac{C\sqrt{2}}{\sqrt{2}-1}.$$

It is an immediate consequence of this proposition that Hermitian tensors of grade at most 2 are represented by essentially self-adjoint operators in every GNS representation corresponding to a continuous positive linear functional w^r . Recall that for a*-algebra $\mathcal{A}$ with a positive linear functional $w^r: \mathcal{A} \to \mathbb{C}$, the GNS representation of $\mathcal{A}$ associated to $\mathrm{w^r}$ is the unital $*$-homomorphism $\pi_{\mathrm{w^r}}: \mathcal{A} \to \mathrm{Adj}(\mathcal{A}/\mathcal{J}_{w^r})$ into the adjointable endomorphisms on the pre-Hilbert space $\mathcal{H}_{w^r} = \mathcal{A}/\mathcal{J}_{\mathrm{w^r}}$ with inner product $\langle\cdot \mid \cdot\rangle_{\mathrm{w^r}}$, where $\mathcal{J}_{w^r} = \{a \in \mathcal{A} | w^r(a^*a) = 0\}$ and $\langle[a]|[b]\rangle_{\mathrm{w^r}} = w^r(a^*b)$ for all $[a]$, $[b] \in \mathcal{H}_{w^r}$ with representatives $a, b \in \mathcal{A}$.

Corollary (6.2.94)[288]: [280] Let $\bar{\cdot}$ *be a continuous antilinear involution on* V *and* Λ *a continuous Hermitian bilinear form on* V*. Let* $w^r: S^*(V)^{\mathrm{cpl}} \to \mathbb{C}$ *be a continuous linear functional on* $S^{\cdot}(V)^{\mathrm{cpl}}$ *that is positive with respect to* $\star_\Lambda$*. Then for* $X_{\mathrm{j}}^* = X_j \in S^{(2)}(V)^{\mathrm{cpl}}$ *all vectors in the GNS pre-Hilbert space* $\mathcal{H}_w^r$ *are analytic for* $\pi_{w^r}(X_j)$ *which is therefore essentially self-adjoint.*

Proof: It is clear from the construction of the GNS representation that $\pi_{w^r}(X_j)$ is a symmetric operator on $\mathcal{H}_{w^r} = S^*(V)^{\mathrm{cpl}}/\mathcal{J}_{\omega^r}$ and by Nelson's theorem, see e.g. [286], it is sufficient to show that all vectors $[Y_j] \in \mathcal{H}_{\mathrm{w^r}}$ are analytic for $\pi_{\mathrm{w}}(X_j)$:From

$$\sum_r\sum_j \langle\pi_{\mathrm{w^r}}(X_j)^n[Y_j]|\pi_{\mathrm{w^r}}J(X_j)^n[Y_j]\rangle_{\mathrm{w^r}} = \sum_r\sum_j w^r\left(\left(X_j^{\star_\Lambda n}\star_\Lambda Y_j\right)^*\star_\Lambda\left(X_j^{\star_\Lambda n}\star_\Lambda Y_j\right)\right)$$
$$=\sum_r\sum_j w^r\left(Y_j^*\star_\Lambda\left(X_j^{\star_\Lambda n}\right)^*\star_\Lambda X_j^{\star_\Lambda n}\star_\Lambda Y_j\right)$$

it follows that analyticity of the vector $[Y_j]$ is equivalent to the analyticity of the continuous positive linear functional $S^*(V)^{\mathrm{cpl}} \ni Z_j \mapsto (w^r)_{Y_j}(Z_j) := w^r(Y_j^* \star_\Lambda Z_j \star_\Lambda Y_j) \in \mathbb{C}$ in the sense of the previous Corollary (6.2.93).

Corollary (6.2.95)[288]: [280] *Let* V *be a (complex) Hilbert space with inner product* $\langle\cdot \mid \cdot\rangle_1$ *and unit ball* $U \subseteq V$ *and let* $\mathrm{Bil}(V)$ *be the Banach space of all continuous bilinear forms on* V *with norm* $\|\Lambda\| := \sup_{v^r, w^r \in U}|\Lambda(v^r, w^r)|$*. Then the map* $\mathrm{Bil}(V) \times S^*(V)^{\mathrm{cpl}} \times S^*(V)^{\mathrm{cpl}} \to S^*(V)^{\mathrm{cpl}}$

$$\left(\Lambda, X_j, Y_j\right) \mapsto X_j \star_\Lambda Y_j \tag{172}$$

is continuous.

Proof: Note that for a Hilbert space V, the continuous inner products $\langle\cdot \mid \cdot\rangle_\lambda$ with $\lambda > 0$ are cofinal in $\mathcal{J}_V$. Now let $\Lambda \in \mathrm{Bil}(V)$, X_j, $Y_j \in S^*(V)^{\mathrm{cpl}}$ and $\varepsilon > 0$ be given, then

$$\sum_j \|X_j' \star_{\Lambda'} Y_j{}' - X_j \star_\Lambda Y_j\|_\lambda$$
$$\leq \sum_j \|X_j' \star_{\Lambda'} Y_j{}' - X_j \star_{\Lambda'} Y_j\|_\lambda + \sum_j \|X_j \star_{\Lambda'} Y_j - X_j \star_\Lambda Y_j\|_\lambda$$

holds for all $\lambda > 0$ and all $\Lambda' \in \mathrm{B}i1(V)$ as well as all $X_j', Y_j{}' \in S^{\cdot}(V)^{\mathrm{cpl}}$. Moreover,

$$\sum_j \|X_j' \star_{\Lambda'} Y_j{}' - X_j \star_{\Lambda'} Y_j\|_\lambda \leq \sum_j \|\big(X_j' - X_j\big) \star_{\Lambda'} Y_j{}'\|_\lambda + \sum_j \|X_j \star_{\Lambda'} \big(Y_j{}' - Y_j\big)\|_\lambda$$
$$\leq 4\sum_j \big\|X_j' - X_j\|_{8\lambda}^* \big\|Y_j{}'\|_{8\lambda}^* + 4\sum_j \|X_j\|_{8\lambda}^* \|Y_j{}' - Y_j\|_{8\lambda}^*$$

holds for all $X_j', Y_j{}' \in S^*(V)^{\mathrm{cpl}}$ as well as all $\lambda > 0$ and all $\Lambda' \in \mathrm{B}i1(V)$ such that $\|\cdot\|_\lambda \in \mathcal{P}_{V,\Lambda'}$ by Corollary (6.2.66). One can check on factorizing symmetric tensors that P_Λ and $\mathrm{P}_{\Lambda'-\Lambda}$ commute and by using that

$$\sum_j X_j \star_{\Lambda'} Y_j = \sum_{t=0}^{\infty}\sum_j \frac{1}{t!}\mu_{\mathrm{V}}\left(\big(\mathrm{P}_{\Lambda+(\Lambda'-\Lambda)}\big)^{t'}\big(X_j \otimes_\pi Y_j\big)\right)$$
$$= \sum_{t,s=0}^{\infty}\sum_j \frac{1}{t!\,s!}\mu_{\mathrm{V}}\left((\mathrm{P}_\Lambda)^t(\mathrm{P}_{\Lambda'-\Lambda})^s\big(X_j \otimes_\pi Y_j\big)\right)$$
$$= \sum_{s=0}^{\infty}\sum_j \frac{1}{s!}\mu_{\star_\Lambda}\left((\mathrm{P}_{\Lambda'-\Lambda})^s\big(X_j \otimes_\pi Y_j\big)\right),$$

it follows that

$$\sum_j \left\|X_j \star_{\Lambda'} Y_j - X_j \star_\Lambda Y_j\|_\lambda \leq \sum_{s=1}^{\infty}\sum_j \frac{1}{\rho^s s!}\right\|\mu_{\star_\Lambda}\left(\big(\mathrm{P}_{\rho(\Lambda'-\Lambda)}\big)^s\big(X_j \otimes_\pi Y_j\big)\right)\|_\lambda^*$$
$$\leq 4\sum_{s=1}^{\infty}\sum_j \frac{1}{p^s s!}\|\big(\mathrm{P}_{\rho(\Lambda'-\Lambda)}\big)^s\big(X_j \otimes_\pi Y_j\big)\|_{8\lambda\otimes_\pi 8\lambda}^*$$
$$\leq 8\sum_{s=1}^{\infty}\sum_j \frac{1}{(2\rho)^s}\big\|X_j\|_{32\lambda}^*\big\|Y_j\|_{32\lambda}^*$$
$$= \frac{8}{2p-1}\sum_j \big\|X_j\|_{32\lambda}^*\big\|Y_j\|_{32\lambda}^*$$

holds for all $\rho > \frac{1}{2}$, $\lambda > 0$, and all $\Lambda' \in \mathrm{Bi}l(V)$ if $\|\cdot\|_\lambda \in \mathcal{P}_{V,\Lambda} \cap \mathcal{P}_{V,\rho(\Lambda'-\Lambda)}$ by Corollary (6.2.66) and Corollary (6.2.65) with $c = 2$.

Assume that $\lambda \geq 1 + ||\Lambda||$ and choose $\rho > \frac{1}{2}$ such that $\frac{8}{2\rho-1}\sum_j \big\|X_j\big|\|_{32\lambda}^* ||Y_j\|_{32\lambda}^* \leq \frac{\varepsilon}{3,\|}$. Then $\|\cdot\|_\lambda \in \mathcal{P}_{V,\Lambda} \cap \mathcal{P}_{V,\rho(\Lambda-\Lambda)}$ for all $\Lambda' \in Bil(V)$ with $||\Lambda' - \Lambda|| \leq \frac{1}{\rho}$ and $\sum_j \left\|X_j' \star_{\Lambda},Y_j{}' - X_j \star_\Lambda Y_j\right\|_\lambda \leq \epsilon$ holds for all these Λ' and all $X_j', Y_j{}' \in S^{*\cdot}(V)^{\mathrm{cpl}}$ with $\sum_j \|X_j' - X_j\|_{8\lambda}^* \leq$

$\sum_j \varepsilon/(12+12\|Y_j\|_{8\lambda})$ and $\sum_j \|Y_j' - Y_jY_j\|_{8\lambda} \leq \min\sum_j \{1, \varepsilon/(12+12\|X_j\|_{8\lambda})\}$. This proves continuity of $\star$ at (Λ, X_j, Y_j) .

Corollary (6.2.96)[288]: [280] *Let V be a (complex) Hilbert space with inner product $\langle\cdot\mid\cdot\rangle_1$ and a continuous antilinear involutio n^- that fulfils $\sum_r \langle v^r|w^r\rangle_1 = \sum_r \langle\overline{v^r}|\overline{w^r}\rangle_1$ for all $v^r, w^r \in V$, then $\widehat{X}_j: V_h' \to \mathbb{C}$ is smooth in the Fréchet sense for all $X_j \in S^*(V)^{\mathrm{cpl}}$.*

Proof: By the Fréchet-Riesz theorem we can identify V_h' with V_h by means of the antilinear map . b: $V_h \to V_h$. As the translations τ^* are automorphisms of $S^*(V)^{\mathrm{cpl}}$, it is sufficient to show that $\widehat{X}_J$ is smooth at $0 \in V_h'$. So let $K \in \mathrm{M}_0$ and $r \in V_h$ be given with $r \neq 0$ and $\|r\|_1 \leq 1$. We have already seen in Corollary (6.2.77) that all directional derivatives of $\widehat{X}_J$ exist and form bounded symmetric multilinear maps $(V_h')^k \ni \rho \mapsto \left(\widehat{D}_\rho^{(k)}\widehat{X}_J\right)(0) \in$ (D. These maps are indeed the derivatives of $\widehat{X}_J$ in the Fréchet sense due to the analyticity of $\widehat{X}_J$: Define $\hat{r} := r/\|r\|_1$, then due to Corollary (6.2.77) and Corollary (6.2.75) the estimate

$$\frac{1}{\|r\|_1^{K+1}}\sum_j |\widehat{X}_J(r^{\mathrm{b}}) - \sum_{k=0}^{K}\frac{1}{k!}\left(\widehat{D}^{(k)}_{(r^{\mathrm{b}},\ldots,r^{\mathrm{b}})}\widehat{X}_J\right)(0)|$$

$$= \frac{1}{\|r\|_1^{K+1}}\sum_j |\langle\tau^*_{r^{\mathrm{b}}}(X_j) - \sum_{k=0}^{K}\frac{1}{k!}(D_{r^{\mathrm{b}}})^k X_j\rangle_0|$$

$$= \frac{1}{\|r\|_1^{K+1}}|\sum_j \langle\sum_{k=K+1}^{\infty}\frac{1}{k!}(D_{r^{\mathrm{b}}})^k X_j\rangle_0| \leq \sum_j |\langle\sum_{k=K+1}^{\infty}\frac{1}{k!}(D_{\hat{r}^{\mathrm{b}}})^k X_j\rangle_0|$$

$$\leq \sum_{k=K+1}^{\infty}\sum_j \frac{1}{k!}\left\|(D_{\hat{r}^{\mathrm{b}}})^k X_j\right\|_1^* \leq \sum_{k=K+1}^{\infty}\sum_j \frac{1}{\sqrt{k!}}\|X_j\|_2^* \leq C\sum_j \|X_j\|_2^*$$

with $C = \sum_{k=K+1}\frac{1}{\sqrt{k!}} < \infty$ holds uniformly for all $r \neq 0$ with $\|r\|_1 \leq 1$.

The formal deformation quantization of a Hilbert space in a very similar setting has already been examined in [106] by Dito. There the formal deformations of exponential type of a certain algebra $\mathcal{F}_{HS}$ of smooth functions on a Hilbert space $\mathcal{H}$ was constructed. More precisely, $\mathcal{F}_{HS}$ consists of all smooth (in the Fréchet sense) functions $f_\mathrm{j}f_\mathrm{j}$ whose derivatives fulfil the additional condition that for all $\sigma \in \mathcal{H}$

$$k!\sum_j \ll f_\mathrm{j}|f_\mathrm{j} \gg^k (\sigma) := \sum_{i\in I^k}\sum_j |\left(\widehat{D}^{(k)}_{(e_{i_1},\ldots,e_{i_k})}f_\mathrm{j}\right)(\sigma)|^2 < \infty \qquad (173)$$

holds and depends continuously on σ for one (hence all) Hilbert base $e \in \mathcal{H}^I$ of $\mathcal{H}$ indexed by a set I. In this case $\ll f_\mathrm{j}|f_\mathrm{j} \gg^k \in \mathcal{F}_{HS}$ holds.

Corollary (6.2.97)[288]: [280] *One has the estimate*

$$\|\sum_j X_j\|_\alpha^* \leq \sum_j \|X_j\|_{\alpha,\mathrm{pr}}^* \qquad (174)$$

for all $X_j \in \tau^*_{\mathrm{ai\ g}}(V)$. Moreover, if there is a $\|\cdot\|_\beta \in \mathcal{P}_V$, $\|\cdot\|_\beta \geq \|\cdot\|_\alpha$, such that for every $\langle\cdot\mid\cdot\rangle_\beta$- orthonormal $e \in V^d$ and all $d \in \mathrm{M}$ the estimate $\sum_{i=1}^d \|e_i\|_\alpha^2 \leq 1$ holds, then

$$\| \sum_j X_j \|^*_{\alpha,\mathrm{pr}} \leq \sum_j \|X_j\|^*_\beta \tag{175}$$

for all $X_j \in \mathcal{T}^*_{\mathrm{alg}}(V)$.

Proof: Let $X_j \in \mathcal{T}^*_{\mathrm{alg}}(V)$ be given, then $\| \sum_j X_j \|^*_\alpha \leq \sum_{k-0}^{\infty} \sum_j \| \langle X_j \rangle_k \|^*_\alpha$ and $\sum_j \|X_j\|^*_{\alpha,\mathrm{pr}} = \sum_{k-0}^{\infty} \sum_j \| \langle X_j \rangle_k \|^*_{\alpha,\mathrm{pr}}$ Thus it is sufficient for the first estimate to show that $\| \sum_j \langle X_j \rangle_k \|^*_\alpha \leq \sum_j \|\langle X_j \rangle_k \|^*_{\alpha,\mathrm{pr}}$ for all $k \in \mathrm{M}_0$. Fix $k \in \mathbb{N}_0$ and assume that $\sum_j \langle X_j \rangle_k = \sum_{i \in I} \sum_r x^r_{i,1} \otimes \cdots \otimes x^r_{i,k}$ with $x^r_i \in V^k$. Then

$$\| \sum_j \langle X_j \rangle_k \|^*_\alpha \leq \sum_{i \in I} \sum_r \| x^r_{i,1} \otimes \cdot \cdot \cdot \otimes x^r_{i,k} \|^*_\alpha = \sqrt{k!} \sum_{i \in I} \sum_r \prod_{m=1}^{k} \|x^r_{i,m}\|_\alpha$$

shows that $\| \sum_j \langle X_j \rangle_k \|^*_\alpha \leq \sum_j \|\langle X_j \rangle_k \|^*_{\alpha,\mathrm{pr}}$, hence $\| \sum_j X_j \|^*_\alpha \leq \sum_j \|X_j\|^*_{\alpha,\mathrm{pr}}$. For the second estimate, let $\| \cdot \|_\beta$ with the stated properties and $X_j \in \mathcal{T}^{\mathrm{k}}_{\mathrm{alg}}(V)$ be given. Use Corollary (6.2.61) to construct $\sum_j (X_j)_0 = \sum_{a \in A} \sum_r x^r_{a,1} \otimes \cdots \otimes x^r_{a,k}$ and $\sum_j \widetilde{X_J} = \sum_{a' \in (1,\ldots,d)} \sum_j X_j^{a'} e_{a'_1} \otimes \otimes e_{a'_k}$ with $e \in V^k$ orthonormal with respect to $\langle \cdot \mid \cdot \rangle_\beta$. Clearly $\sum_j \|(X_j)_0\|^*_{\alpha,\mathrm{pr}} = 0$ and so

$$\begin{aligned}
\| \sum_j X_j \|^*_{\alpha,\mathrm{pr}} &\leq \sum_j \|\widetilde{X_J}\|^*_{\alpha,\mathrm{pr}} \\
&\leq \sqrt{k!} \sum_{a' \in \{1,\ldots,d\}^k} \sum_j | X_{\mathrm{j}}^{a'} | \prod_{m=1}^{k} \| e_{a'_m} \|_\alpha \\
&\overset{cs}{\leq} \sum_j \left(k! \left(\sum_{a' \in \{1,\ldots,d\}^k} | X_j^{a'} |^2 \right) \left(\sum_{a' \in \{1,\ldots d\}^k}, \coprod_{m=1}^{k} \| e_{a'_m} \|^2_\alpha \right) \right)^{\frac{1}{2}} \\
&\leq \sum_j \left(k! \left(\sum_{a' \in \{1,\ldots,d\}^k} | X_j^{a'} |^2 \right) \left(\sum_{i=1}^{d} \| e_i \|^2_\alpha \right)^k \right)^{\frac{1}{2}} \\
&\leq \sum_j \|X_j\|^*_\beta .
\end{aligned}$$

Corollary (6.2.98)[288]: [280] *Let V be a nuclear space, then the topology on $S^{\cdot}(V)$ coincides with the one constructed for $R = \frac{1}{2}$.*

Proof: This is a direct consequence of the preceeding lemma because the locally convex topology constructed in [101] for $R = \frac{1}{2}$ is the one defined by the seminorms $\| \cdot \|_{\alpha,\mathrm{pr}}$ for all $\| \cdot \|_\alpha \in \mathcal{P}_V$ and because in a nuclear space, such seminorms $\| \cdot \|_\beta$ as required in the lemma exist for all $\| \cdot \|_\alpha \in \mathcal{P}_V$, see e.g. [285] or also [117].

From[101] we get:

Corollary (6.2.99)[288]: *Let V be a Hausdorff nuclear space and $-$ a continuous antilinear involution of V as well as Λ a continuous Hermitian bilinear form on V, then there exist point-separating many continuous positive linear functionals of $(S^{\cdot}(V),\star_{\Lambda},*)$.*

Proof: Choose some $\langle\cdot\mid\cdot\rangle_{\alpha}\in\mathcal{I}_{V,h}$ such that $\|\cdot\|_{\alpha}\in\mathcal{P}_{V,\Lambda}$ and define a bilinear form b on V by $\sum_r\ b(v^r,w^r)\ :=\sum_r\ \langle\overline{v^r}|w^r\rangle_{\alpha}$ for all $v^r,w^r\in V$. Then b is continuous and Hermitian by construction and symmetric due to the compatibility of $\langle\cdot\mid\cdot\rangle_{\alpha}$with $-$. Moreover, $\sum_r\ \Lambda\left(\overline{v^r},v^r\right)\leq\sum_r\ \|\overline{v^r}\|_{\alpha}\|v^r\|_{\alpha}=\sum_r\ \|v^r\|_{\alpha}^2=\sum_r\ \langle v^r|v^r\rangle_{\alpha}=\sum_r\ b\left(\overline{v^r},v^r\right)$ holds for all $v^r\in V$ and b is of Hilbert-Schmidt type because every continuous bilinear form on a nuclear space is of Hilbert-Schmidt type (again, see [117] or use [285]). Because of this, Corollary (6.2.88) applies.

List of Symbols

Symbol		Page
sa:	self-adjoint	1
Tr:	trace	2
$\oplus$:	Direct sum	3
dim:	dimension	3
ker:	kernel	3
diag:	diagonal	3
$\otimes$:	tensor product	5
sp:	spectrum	7
$L^{\infty}_{\mu(\varphi)}$:	essential Lebesgue space	8
inf:	infimum	17
sup:	supremum	17
max:	maximum	18
L^{∞}:	essential Lebesgue space	20
ran:	range	30
Rad:	Radial	34
det:	determinant	44
$W_R(V)$:	Weyl algebra	53
im:	imaginary	56
Pol:	Polynomial	59
End:	Endomorphism	59
Aut:	Automorphism	59
ℓ^p:	Lebesgue space of sequences	63
proj:	projection	65
Re:	Real	72
supp:	support	91
cov:	covariant	93
ℓ^2:	Hilbert space of sequences	102
FK:	Fuglede and Kadison	103
L^2:	Hilbert space	107
top:	Topology	107
DFR:	Doplicher-Fredenhagen-Roberts	164
CCRs:	Canonical commutation relations	164
Rep:	Repressentation	167
Reg:	Regular	173
US:	Upper semicontmipus	186
Alg:	Algebra	186
SR:	Sectional representation	186
L^1:	Lebesgue on the real line	195
cs:	Cauchy-Schwarz	205

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