Sudan University of Science and Technology

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Sharp Hardy-Lorentz Spaces for B-Valued Martingales and Inequalities on Solid Torus with Limited Ranges of Muckenhoupt Weights

فضاءات هاردي-لورينتز القاطعة لمارتينجاليس القيمة – B والمتباينات علي النتوء المستدير الصلب مع المداءات المنهية لمرجحات ميكينهوبت

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Dedication

To my Family

Acknowledgements

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Abstract

The Hardy-Lorentz spaces for B-valued martingales space are stablished. We study the atomic decompositions, duality of the spaces .the interpolations and John-Nirenberg inequalities of martingale Hardy-Lorentz-Karamata spaces. The critical and the equivalence between pointwise Hardy inequalities presented. We deal with the characterization of the Triebel-Lizorkin spaces and weighted estimates for Littlewood - Paley functions with radial multipliers and bounds of singular integrals and discrete Littlewood-Palev analysis the realized Hardy inequalities under non-convexity measures ,for functions vanishing on boundary on the solid torus estimates for Littlewood-Paley Stein square functions. Calderon-Zygmund operators and limited ranges of Muckenhoupt weights are investigated.

الخلاصة

تم تأسيس فضاءات هاردي-لوينتز لأجل القيمة –B . قمنا بدراسة التفكيكات الذرية والثنائية للفضاءات والاستكمالات ومتباينات جون-بيرينرج لقضاءات هاردي-لوينتز-كارمات المارتيجاليس. تم تقديم المتيابنات الحرجة والمتكافئة بين متباينات هاردي النقطية. تعاملنا مع التشخيصات لفضاءات- تريبيل – ليزوركين والتقديرات المرجحة لأجل دوال ليتليوو-بالي مع المضاعفات نصف القطرية والحديات للتكاملات الشاذة وتحليل ليتايوود-بالي المتقطع . قمنا بدراسة متباينات هاردي المعروفة وتحت والقياسات غير المحدبة ولأجل الدوال المتلاشية علىالدورية وعلى النتوء المستدير الصلب وتقديرات الدوال المربعة لأجل ليتليوود-بالي-ستين ومؤثرات كالديرون-زيجموند والمداءات المنتهية لمرجح ميكينهوبت.

Introduction

We consider the Hardy-Lorentz spaces $\mathrm{H}^{\mathrm{p},q}(\mathrm{R}^{\mathrm{n}})$, with $0 < \mathrm{p} \leq 1, 0 < \mathrm{q} \leq \infty$. We present three atomic decomposition theorems of Lorentz martingale spaces. With the help of atomic decomposition we obtain a sufficient condition for sublinear operator defined on Lorentz martingale spaces to be bounded.

We study an equivalence result between the validity of a pointwise Hardy inequality in a domain and uniform capacity density of the complement. This result is new even in Euclidean spaces, but our methods apply in general metric spaces as well. We give sharp homogeneous improvements weighted Hardy inequalities involving distance from the boundary. In the case of a smooth domain, we obtain lower and upper estimates for the best constant of the remainder term.

We establish the characterization of the weighted Triebel-Lizorkin spaces for $p = \infty$ by means of a "generalized" Littlewood-Paley function which is based on a kernel satisfying "minimal" moment and Tauberian conditions. This characterization completes earlier work by Bui et al. We show some weighted estimates for certain Littlewood–Paley operators on the weighted Hardy spaces H_w^p $(0 < p \leq 1)$ and on the weighted L^p spaces.

We introduce the martingale Hardy–Lorentz–Karamata spaces. The atomic decompositions of these martingale function spaces are established. We study the Hardy–Lorentz spaces for Banach space valued mar-tingales. The dual spaces are characterized and several martingale inequalities are established.

Considering two different non-convexity measures, we obtain some new Hardy-type inequalities for non-convex domains $\Omega \subset R^n$. We establish the classical Hardy inequality in the solid torus and some variants of it. The general idea is to use the fact that Sobolev embeddings can be improved in the presence of symmetries.

We give new sufficient conditions for Littlewood-Paley-Stein square function

and necessary and sufficient conditions for a Calder´on-Zygmund operator to be bounded on Hardy spaces H^p with indices smaller than 1. New Carleson measure type conditions are defined for Littlewood-Paley-Stein operators, and the authors show that they are sufficient for the associated square function to be bounded from H^p into L^p. We give a full necessary and sufficient set of conditions for a Calderón–Zygmund operator to be bounded on weighted Hardy spaces H_w^p where w is an Muckenhoupt weight and $0 < p < \infty$. In fact, this result is new even when $1 < p < \infty$ since it allows for H_w^p boundedness of an operator when $1 <$

$p < q < \infty$ and $w \in A_q$, where it is possible that $H_w^p \neq L_w^p$. These singular integral results are achieved by proving Littlewood–Paley–Stein square function type estimates from H_w^p into L_w^p for $0 < p < \infty$ and a Muckenhoupt weight w, which are interesting results in their own right. New techniques involving A∞ weight invariant spaces are also used to prove the weighted estimates for Calderón–Zygmund operators.

The Contents

Chapter 1
Hardy and Lorentz Martingale Spaces

We discuss the atomic decomposition of the elements in spaces, their interpolation properties, and the behavior of singular integrals and other operators acting on them. We investigate some inequalities on Lorentz martingale spaces. We discuss the restricted weak-type interpolation, and show the classical Marcinkiewicz interpolation theorem in the martingale setting.

Section (1.1): The Hardy-Lorentz Spaces

The real variable theory of the Hardy spaces represents a fruitful setting for the study of maximal functions and singular integral operators. It is because of the failure of these operators to preserve L^1 that the Hardy space H^1 assumes its prominent role in harmonic analysis. Now, for many of these operators, the role of L^1 can just as well be played by $H^{1,\infty}$, or Weak H^1. However, although these operators are amenable to $H^1 - L^1$ and $H^{1,\infty} - L^{1,\infty}$ estimates, interpolation between H^1 and $H^{1,\infty}$ has not been available. Similar considerations apply to H^p and Weak H^p for $0 < p < 1$.

We provide an interpolation result for the Hardy-Lorentz spaces $H^{p,q}, 0 < p \leq 1, 0 < q \leq \infty$, including the case of Weak H^p as and end point for real interpolation. The atomic decomposition is the key ingredient in dealing with interpolation since neither truncations are available, nor reiteration applies. The Lorentz spaces, including criteria that assure membership in $L^{p,q}, 0 < p < \infty, 0 < q \leq \infty$, are discussed. We show that distributions in $H^{p,q}$ have an atomic decomposition in terms of H^p atoms with coefficients in an appropriate mixed norm space. An interesting application of this decomposition is to $H^{p,q} - L^{p,\infty}$ estimates for Calder' on-Zygmund singular integral operators, $p < q \leq \infty$. Also, by manipulating the different levels of the atomic decomposition, we show that, for $0 < q_1 < q < q_2 \leq \infty, H^{p,q}$ is an intermediate space between H^{p,q_1} and H^{p,q_2}. This result applies to Calder'on-Zygmund singular integral operators, including those with variable kernels, Marcinkiewicz integrals, and other operators. The Lorentz space $L^{p,q}(R^n) = L^{p,q}, 0 < p < \infty, 0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\| f \|_{p,q}$ given by

$$\| f \|_{p,q} = \left(\frac{q}{p} \int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}, 0 < q < \infty,$$

$$\| f \|_{p,\infty} = \sup_{t>0} \left[t^{\frac{1}{p}} f^*(t) \right]^q, q = \infty.$$

The Lorentz quasinorm may also be given in terms of the distribution function $m(f,\lambda) = |\{x \in R^n : |f(x)| > \lambda\}|$, the inverse of the non-increasing rearrangement f^* of f. Indeed, we have

$$\| f \|_{p,q} \left(\frac{q}{p} \int_0^\infty \lambda^{q-1} m(f,\lambda)^{\frac{q}{p}} d\lambda \right)^{1/q} \sim \left(\sum_k \left[2^k m(f,2^k)^{\frac{1}{p}} \right]^q \right)^{1/q},$$

when $0 < q < \infty$, and

$$\| f \|_{p,\infty} = \sup_k 2^k m(f,2^k)^{\frac{1}{p}}, \qquad q = \infty.$$

Note that, in particular, $L^{p,p} = L^p$, and $L^{p,\infty}$ is weak L^p.
The following two results are useful in verifying that a function is in $L^{p,q}$.

Lemma (1.1.1)[1]: Let $0 < p < \infty$, and $0 < q \le \infty$. Assume that the non-negative sequence $\{\mu_k\}$ satisfies $\{2^k \mu_k\} \in \ell^q$. Further suppose that the non-negative function φ verifies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0, we have $\varphi \le \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} is essentially bounded and satisfies $\|\psi_{k_0}\|_\infty \le c 2^{k_0}$, and

$$2^{k_0 \varepsilon p} m(\eta_{k_0}, 2^{k_0}) \le c \sum_{k_0}^{\infty} [2^{k\varepsilon} \mu_k]^p.$$

Then, $\varphi \in L^{p,q}$, and $\| \varphi \|_{p,q} \le c \|\{2^k \mu_k\}\|_{\ell q}$.

Proof: It clearly suffices to verify that $\left\| \left\{ 2^k |\{\varphi > \gamma 2^k\}|^{1/p} \right\} \right\|_{\ell q} < \infty$, where γ is an arbitrary positive constant. Now, given k_0, let ψ_{k_0} and η_{k_0} be as above, and put $\gamma = c + 1$, where c is the constant in the above inequalities; for this choice of $\gamma, \{\varphi > \gamma 2^{k_0}\} \subseteq \{\eta_{k_0} > 2^{k_0}\}$.

When $q = \infty$, we have

$$2^{k_0 \varepsilon} m(\eta_{k_0}, 2^{k_0})^{1/p} \le c \left(\sum_{k_0}^{\infty} [2^{-k(1-\varepsilon)} 2^k \mu_k]^p \right)^{1/p}$$
$$\le c 2^{-k_0(1-\varepsilon)} \sup_{k \ge k_0} [2^k \mu_k].$$

Thus, $2^{k_0} m(\eta_{k_0}, 2^{k_0})^{1/p} \le \sup_{k \ge k_0} [2^k \mu_k]$, and, consequently,

$$2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p} \le c \|\{2^k \mu_k\}\|_{\ell^q}, \text{ all } k_0.$$

When $0 < q < \infty$, let $1 - \varepsilon = 2\delta$ and rewrite the right-hand side above as

$$\sum_{k_0}^{\infty} \frac{1}{2^{k\delta_p}} [2^{k(1-\delta)} \mu_k]^p.$$

When $p < q$, by Hölder's inequality with exponent $r = q/p$ and its conjugate r', this expression is dominated by

$$\left(\sum_{k_0}^{\infty} \frac{1}{2^{k\delta_{pr'}}} \right)^{\frac{1}{r'}} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^{rp} \right)^{1/r}$$
$$\le c 2^{-k_0 \delta_p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{p/q},$$

and, when $0 < q \le p, r < 1$, and we get a similar bound by simply observing that it does not exceed

$$2^{-k_0 \delta_p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^p \right)^{r/r} \le 2^{-k_0 \delta_p} \left(\sum_{k_0}^{\infty} [2^{k(1-\delta)} \mu_k]^q \right)^{\frac{p}{q}}.$$

Whence, continuing with the estimate, we have

$$2^{k_0\varepsilon p} m(\eta_{k_0}, 2^{k_0}) \le c 2^{-k_0 \delta_p} \left(\sum_{k_0}^{\infty} \left[2^{k(1-\delta)} \mu_k\right]^q \right)^{\frac{p}{q}},$$

which yields, since $1 - \varepsilon = 2\delta$,

$$2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p} \le c 2^{k_0 \delta} \left(\sum_{k_0}^{\infty} \left[2^{k(1-\delta)} \mu_k\right]^q \right)^{\frac{1}{q}}.$$

Thus, raising to the q and summing, we get

$$\sum_{k_0} \left[2^{k_0} m(\varphi, \gamma 2^{k_0})^{1/p}\right]^q \le c \sum_{k_0} 2^{k_0 \delta_q} \sum_{k=k_0}^{\infty} \left[2^{k(1-\delta)} \mu_k\right]^q,$$

which, upon changing the order of summation in the right-hand side of the above inequality, is bounded by

$$\sum_{k} \left[2^{k(1-\delta)} \mu_k\right]^q \left[\sum_{k_0=-\infty}^{k} 2^{k_0 \delta_q} \right] \le c \sum_{k} [2^k \mu_k]^q$$

We will have no difficulty in verifying that, for Lemma (1.1.1) to hold, it suffices that ψ_{x_0} satisfies

$$m(\psi_{x_0}, 2^{k_0})^{1/p} \le c \mu_{k_0}, \text{ all } k_0.$$

This holds, for instance, when $\|\psi_{x_0}\|_r^r \le c 2^{k_0 r} \mu_{k_0}^p 0 < r < \infty$. In fact, the assumptions of Lemma (1.1.1) correspond to the limiting case of this inequality as $r \to \infty$.

Another useful condition is given by our next result.

Lemma (1.1.2) [1]: Let $0 < p < \infty$, and let the non-negative sequence $\{\mu_k\}$ be such that $\{2^k \mu_k\} \in \ell^q, 0 < q \le \infty$. Further, suppose that the non-negative function φ satisfies the following property: there exists $0 < \varepsilon < 1$ such that, given an arbitrary integer k_0, we have $\varphi \le \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} satisfy

$$2^{k_0 p} m(\psi_{k_0}, 2^{k_0})^{\varepsilon} \le c \sum_{-\infty}^{k_0} [2^k \mu_k^{\varepsilon}]^p, 0 < \varepsilon < \min\ (1, q/p)$$

$$2^{k_0 \varepsilon} \left|\{\eta_{k_0} > 2^{k_0}\}\right| \le c \sum_{k_0}^{\infty} [2^{k\varepsilon} \mu_k]^p.$$

Then, $\varphi \in L^{p,q}$, and $\| \varphi \|_{p,q} \le c \|\{2^k \mu_k\}\|_{\ell q}$.

We will also require some basic concepts from the theory of real interpolation.

Let A_0, A_1, be a compatible couple of quasinormed Banach spaces, i.e., both A_0 and A_1 are continuously embedded in a larger topological vector space. The Peetre K functional of $f \in A_0 + A_1$ at $t > 0$ is defined by

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \|f_0\|_0 + t\|f_1\|_1,$$

Where $f = f_0 + f_1, f_0 \in A_0$ and $f_1 \in A_1$.

In the particular case of the L^q spaces, the K functional can be computed by Holmstedt's formula, see [13]. Specifically, for $0 < q_0 < q_1 \le \infty$, let α be given by $1/\alpha = 1/q_0 - 1/q_1$. Then,

$$K(t,f;L^{q_0},L^{q_1}) \sim \left(\int_0^{t^\alpha} f^*(s)^{q_0} ds\right)^{1/q_0} + t\left(\int_\infty^{t^\alpha} f^*(s)^{q_1} ds\right)^{1/q_1}.$$

The intermediate space $(A_0,A_1)_{\eta,q}, 0<\eta<1, 0<q<\infty$, consists of those f 's in A_0+A_1 with

$$\| f \|_{(A_0,A_1)_{\eta,q}} = \left(\int_0^\infty [t^{-\eta}K(t,f;A_0,A_1)]^q \frac{dt}{t}\right)^{1/q} < \infty,$$
$$\| f \|_{(A_0,A_1)_{\eta,\infty}} = \sup_{t>0}[t^{-\eta}K(t,f;A_0,A_1)] < \infty, q=\infty.$$

Finally, for the L^q and $L^{p,q}$ spaces, we have the following result. Let $0<q_1<q<q_2\le\infty$, and suppose that $1/q=(1-\eta)/q_1+\eta/q_2$. Then, $L^q=(L^{q_1},L^{q_2})_{\eta,q}$, and, $L1,^q=(L^{1,q_1},L^{1,q_2})_{\eta,q}$, see [5].

we adopt the atomic characterization of the Hardy spaces $H^p, 0<p\le1$. Recall that a compactly supported function a with $[n(1/p-1)]$ vanishing moments is an H^p atom with defining interval I (of course, I is a cube in R^n), if $\operatorname{supp}(a)\subseteq I$, and $|I|^{1/p}|a(x)|\le 1$. The Hardy space $H^p(R^n)=H^p$ consists of those distributions f that can be written as $f=\sum\lambda_j a_j$, where the a_j are H^p atoms, $\sum|\lambda_j|^p<\infty$, and the convergence is in the sense of distributions as well as in H^p. Furthermore,

$$\| f \|_{H^p} \sim \inf \left(\sum |\lambda_j|^p\right)^{1/p}$$

where the infimum is taken over all possible atomic decompositions of f.

This last expression has traditionally been called the atomic H^p norm of f. C. Fefferman, Rivi`ere and Sagher identified the intermediate spaces between the Hardy space $H^{p_0}, 0<p_0<1$, and L^∞, as

$$(H^{p_0},L^\infty)_{\eta,q} = H^{p,q}, 1/p=(1-\eta)/p_0, 0<q\le\infty,$$

where $H^{p,q}$ consists of those distributions f whose radial maximal function $Mf(x)=\sup_{t>0} | f*\varphi_t)(x) |$ belongs to $L^{p,q}$. Here φ is a compactly supported, smooth function with nonvanishing integral, see [11]. R. Fefferman and Soria studied in detail the space $H^{1,\infty}$, which they called Weak H1, see [12].

Just as in the case of $H^p, H^{p,q}$ can be characterized in a number of different ways, including in terms of non-tangential maximal functions and Lusin functions. In what follows we will calculate the quasinorm of f in $H^{p,q}$ by the means of the expression

$$\left\|\left\{2^k m(Mf,2^k)^{\frac{1}{p}}\right\}\right\|_{\ell^q}, 0<p\le 1, 0<q\le\infty,$$

where Mf is an appropriate maximal function of f.

Passing to the atomic decomposition of $H^{p,q}$, the proof is divided in two parts. First, we construct an essentially optimal atomic decomposition; Parilov has obtained independently this result for $H^{1,q}$ when $1\le q$, see [15].

Also, R. Fefferman and Soria gave the atomic decomposition of Weak H^1, see [12], and Alvarez the atomic decomposition of Weak $H^p, 0<p<1$, see [3].

Theorem (1.1.3) [1]: Let $f\in H^{p,q}, 0<p\le1, 0<q\le\infty$. Then f has an atomic decomposition $f=\sum_{j,k}\lambda_{j,k}a_{j,k}$, where the $a_{j,k}$'s are H^p atoms with defining intervals $I_{j,k}$ that have bounded overlap uniformly for each k, the sequence $\{\lambda_{j,k}\}$ satisfies

$\left(\sum_k \left[\sum_j \left|\lambda_{j,k}\right|^p\right]^{\frac{q}{p}}\right)^{1/q} < \infty$, and the convergence is in the sense of distributions. Furthermore, $\left(\sum_k \left[\sum_j \left|\lambda_{j,k}\right|^p\right]^{\frac{q}{p}}\right)^{1/q} \sim\| f \|_{H^{p,q}}$.

Proof: The idea of constructing an atomic decomposition using Calder'on's reproducing formula is well understood, so we will only sketch it here, for further details, see [6] and [19]. Let $Nf(x) = \sup\{|(f * \psi_t)(y)|: |x-y| < t\}$ denote the non-tangential maximal function of f with respect to a suitable smooth function ψ with nonvanishing integral. One considers the open sets $\mathcal{O}_k = \{Nf > 2^k\}$, all integers k, and builds the atoms with defining interval associated to the intervals, actually cubes, of the Whitney decomposition of $\mathcal{O}_k$, and hence satisfying all the required properties. One constructs a sequence of bounded functions f_k with norm not exceeding $c2^k$ for each k, and such that $f - \sum_{|k|\le n} f_k \to 0$ as $n \to \infty$ in the sense of distributions.

These functions have the further property that $f_k(x) = \sum_j \alpha_{j,k}(x)$, where $\left|\alpha_{j,k}(x)\right| \le c2^k, c$ is a constant, each $\alpha_{j,k}$ has vanishing moments up to order $[n(1/p-1)]$ and is supported in $I_{j,k} -$ roughly one of the Whitney cubes -, where the $I_{j,k}$'s have bounded overlaps for each k, uniformly in k. It only remains now to scale $\alpha_{j,k}$,

$$\alpha_{j,k}(x) = \lambda_{j,k}\alpha_{j,k}(x),$$

and balance the contribution of each term to the sum. Let $\lambda_{j,k} = 2^k\left|I_{j,k}\right|^{1/p}$.

Then, $a_{j,k}(x)$ is essentially an H^p atom with defining interval $I_{j,k}$, and one has $\left(\sum_j \left|\lambda_{j,k}\right|^p\right)^{1/p} \sim 2^k|\mathcal{O}_k|^{1/p}$. Thus,

$$\left\|\left(\sum_j \left|\lambda_{j,k}\right|^p\right)^{\frac{1}{p}}\right\|_{\ell^q} \sim\| \left\{2^k|\mathcal{O}_k|^{\frac{1}{p}}\right\} \|_{\ell^q}\sim\| f \|_{H^{p,q},0}< q \le \infty.$$

As an application of this atomic decomposition, we should have no difficulty in showing directly the C. Fefferman, Rivi`ere, Sagher characterization of $H^{p,q}$, see [11].

Another interesting application of this decomposition is to $H^{p,q} - L^{p,\infty}$ estimates for Calder'on-Zygmund singular integral operators $T, p < q \le \infty$.

This approach combines the concept of p-quasi local operator of Weisz, see [18], with the idea of variable dilations of R. Fefferman and Soria, see [12].

Intuitively, since Hormander's condition implies that T maps H^1 into L^1, say, for T to be defined in $H^{1,s}, 1 < s \le \infty$, some strengthening of this condition is required. This is accomplished by the variable dilations. Moreover, since we will include $p < 1$ in our discussion, as p gets smaller, more regularity of the kernel of T will be required. This justifies the following definition.

Given $0 < p \le 1$, let $N = [n(1/p-1)]$, and, associated to the kernel $k(x,y)$ of a Calder onZygmund singular integral operator T, consider the modulus of continuity ω_p given by

$$\omega_p(\delta) = \sup_I \frac{1}{|I|}\int_{R^n\backslash(2/\delta)} \left[\int_I \left|k(x,y) - \sum_{|\alpha|\le N} (y-y_I)^\alpha k_\alpha(x,y_I)\right| dy\right]^p dx,$$

where $0 < \delta \leq 1$, and the sup is taken over the collection of arbitrary intervals I of R^n centered at y_I. Here, for a multi-index $\alpha(\alpha_1, \dots, \alpha_n)$,

$$k_{\alpha(x,y_I)} = \frac{1}{\alpha!} D^\alpha k(x,y)\Big]_{y=y_I}.$$

$\omega_p(\delta)$ controls the behavior of T on atoms. More precisely, if a is an H^p atom with defining interval I, and $0 < \delta < 1$, observe that

$$T(a)(x) = \int_I \left[k(x,y) - \sum_{|\alpha|\leq N} (y - y_I)^\alpha k_\alpha(x, y_I)\right] a(y)dy$$

and, consequently,

$$\int_{R^n \setminus (2/\delta)I} |T(a)(x)|^p dx \leq \omega_p(\delta).$$

We prove the $H^{p,q} - L^{p,\infty}$ estimate for a Calder onZygmund singular integral operator T with kernel $k(x,y)$.

Theorem (1.1.4) [1]; Let $0 < p \leq 1$, and $p < q \leq \infty$. Assume that a Calder'onZygmund singular integral operator T is of weak-type (r,r) for some $1 < r < \infty$, and that the modulus of continuity ω_p of the kernel k satisfies a Dini condition of order $q/(q-p)$, namely,

$$A_{p,q} = \left[\int_0^1 \omega_p(\delta)^{q/(q-p)} \frac{d\delta}{\delta}\right]^{(q-p)/q} < \infty.$$

Then T maps $H^{p,q}$ continuously into $L^{p,\infty}$, and $\| Tf \|_{p,\infty} \leq c A_{p,q}^{1/p} \| f \|_{H^{p,q}}$.

Proof: We need to show that

$$2^{k_0 p} m(Tf, 2^{k_0}) \leq c \| f \|_{H^{p,q}}^p \text{ , all } k_0.$$

Let $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, be the atomic decomposition of f given in Theorem (1.1.3), and set $f_1 = \sum_{k \leq k_0} \sum_j \lambda_{j,k} a_{j,k}$, and $f_2 = f - f_1$. Further, let $\mu_k = \left(\sum_j |\lambda_{j,k}|^p\right)^{1/p}$, and recall that $\|\{\mu_k\}\|_{\ell q} \sim \| f \|_{H^{p,q}}$.

Since $\|f_1\|_r^r \leq c 2^{k_0(r-p)} \| f \|_{H^{p,\infty}}^p$, we have

$$2^{pk_0} m(Tf_1, 2^{k_0}) \leq c \| f \|_{H^{p,\infty}}^p$$

Next, put $I_{j,k}^* = 2^{\frac{1}{n}} \left(\frac{3}{2}\right)^{p(k-k_0)/n} I_{j,k}$, and let

$$\Omega = \bigcup_{k>k_0} \bigcup_j I_{j,k}^*.$$

Since $|I_{j,k}^*| = 2 \left(\frac{3}{2}\right)^{p(k-k_0)} |I_{j,k}| \sim 2^{-k_0 p} \left(\frac{3}{2}\right)^{p(k-k_0)} |\lambda_{j,k}|^p$, we get

$$|\Omega| \quad \sum_{k>k_0} \sum_j |I_{j,k}^*| \leq c 2^{-k_0 p} \sum_{k>k_0} \left(\frac{3}{2}\right)^{p(k-k_0)} |\lambda_{j,k}|^p$$

$$\leq c 2^{-k_0 p} \left[\sup_{k>k_0} \mu_k\right]^p \leq c 2^{-k_0 p} \| f \|_{H^{p,\infty}}^p$$

Also, since $0 < p \leq 1$, it readily follows that

$$|T(f_2)(x)|^p \leq \sum_{k>k_0} \sum_j |\lambda_{j,k}|^p |T(a_{j,k})(x)|^p,$$

and, by Tonelli and the estimate for $T(a)$, we have

$$\sum_{R^n\setminus\Omega} |T(f_2)(x)|^p dx \quad \leq \sum_{k>k_0} \sum_j |\lambda_{j,k}|^p \int_{R^n\setminus I_{j,k}^*} |T(a_{j,k})(x)|^p dx$$

$$\leq \sum_{k>k_0} \omega_p \left(\left(\frac{2}{3}\right)^{\frac{p(k-k_0)}{n}}\right)_k^p$$

$$\leq \left(\sum_{k>0} \omega_p \left(\left(\frac{2}{3}\right)^{\frac{pk}{n}}\right)^{q/(q-p)}\right)^{(q-p)/q} \quad \|\mu_k\|_\ell^{pq}$$

$$\leq c\left[\omega_p(\delta)^{\frac{q}{q-p}} \frac{d\delta}{\delta}\right]^{\frac{q-p}{q}} \parallel f \parallel_{H^{p,q}}^p.$$

This bound gives at once

$$2^{pk_0}\left|\{x \notin \Omega: |T(f_2)(x)| > 2^{k_0}\}\right| \leq cA_{p,q} \parallel f \parallel_{H^{p,q}}^p,$$

which implies that

$$2^{pk_0} m(Tf_2, 2^{k_0-1}) \quad \leq 2^{pk_0}\left[|\Omega| + \left|\{x \notin \Omega: |T(f_2)(x)| > 2^{k_0-1}\}\right|\right]$$
$$\leq c \parallel f \parallel_H^{pp,\infty} + cA_{p,q} \parallel f \parallel_H^{p,q}.$$

Finally,

$$2^{k_0 p} m(Tf, 2^{k_0}) \leq 2^{pk_0} m(Tf_1, 2^{k_0-1}) + 2^{pk_0} m(Tf_2, 2^{k_0-1})$$
$$\leq c \parallel f \parallel_{H^{p,\infty}}^p + cA_{p,q} \parallel f \parallel_H^p\ p,$$

and, since $\parallel f \parallel_{H^{p,\infty}} \leq c \parallel f \parallel_{H^{p,q}}$ for all q, we have finished.

We pass now to the converse of Theorem (1.1.3). It is apparent that a condition that relates the coefficients λ_j with the corresponding atoms a_j involved in an atomic decomposition of the form $\sum_j \lambda_j a_j(x)$ is relevant here. If I_j denotes the supporting interval of a_j, let

$$I_k = \left\{j: 2^k \leq |\lambda_j|/|I_j|^{\frac{1}{p}} < 2^{k+1}\right\},$$

and, for $\lambda = \{\lambda_j\}$, put

$$\parallel \lambda \parallel_{[p,q]} = \left(\sum_k \left[\sum_{j\in I_k} |\lambda_j|^p\right]^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

We then have,

Theorem (1.1.5) [1]: Let $0 < p \leq 1, 0 < q \leq \infty$, and let f be a distribution given by $f = \sum_j \lambda_j a_j(x)$, where the a_j 's are H^p atoms, and the convergence is in the sense of distributions. Further, assume that the family $\{I_j\}$ consisting of the supports of the a_j 's has bounded overlap at each level I_k uniformly in k, and $\parallel \lambda \parallel_{[p,q]} < \infty$. Then, $f \in H^{p,q}$, and $\parallel f \parallel_{H^{p,q}} \leq c \parallel \lambda \parallel_{[p,q]}$.

Proof: Let $Mf(x) = \sup_{t>0}|(f * \psi_t)(x)|$ denote the radial maximal function of f with respect to a suitable smooth function ψ with support contained in $\{|x| \leq 1\}$ and nonvanishing integral. We will verify that Mf satisfies the conditions of Lemma (1.1.1) and is thus in $L^{p,q}$.

Fix an integer k_0 and let

$$g(x)=\sum_{k<k_0}\sum_{j\in I_k}\lambda_j a_j(x).$$

Since $\| Mg \|_\infty \leq \| g \|_\infty$ it suffices to estimate $|g(x)|$. Let C be the bounded overlap constant for the family of the supports of the a_j 's. Then, for $j\in I_k$,

$$|\lambda_j||a_j(x)|=\frac{1}{|I_j|^{\frac{1}{p}}}|\lambda_j||I_j|^{1/p}|a_j(x)|\leq 2^k\chi_{I_j}(x),$$

and, consequently,

$$|g(x)|\leq\sum_{k<k_0}2^k\sum_j\chi_{I_j}(x)\leq C2^{k_0}.$$

Next, let

$$h(x)=\sum_{k\geq k_0}\sum_{j\in I_k}\lambda_j a_j(x).$$

Since a_j has $N=[n(1/p-1)]$ vanishing moments, it is not hard to see that, if I_j is the defining interval of a_j and I_j is centered at x_j, and $\gamma=(n+N+1)/n>1/p$, then, with c independent of $j,\varphi_j(x)=Ma_j(x)$ satisfies

$$\varphi_j(x)\leq c\frac{|I_j|^{\gamma-\frac{1}{p}}}{\left(|I_j|+|x-x_j|^n\right)^\gamma}.$$

Thus, if $1/\gamma<\varepsilon p<1$,

$$Mh(x)^{\varepsilon p}\leq c\sum_{j\in I_k,k>k_0}\frac{\left(|\lambda_j||I_j|^{\gamma-\frac{1}{p}}\right)^{\varepsilon p}}{\left(|I_j|+|x-x_j|^n\right)^{\gamma\varepsilon p}}$$

which, upon integration, yields

$$\int_{R^n}Mh(x)^{\varepsilon p}dx\leq c\sum_{j\in I_k,k>k_0}\left(|\lambda_j||I_j|^{\gamma-\frac{1}{p}}\right)^{\varepsilon p}\int_{R^n}\frac{1}{\left(|I_j|+|x-x_j|^n\right)^{\gamma\varepsilon}}dx.$$

The integrals in the right-hand side above are of order $|I_j|^{1-\gamma\varepsilon p}$ and, consequently, by Chebychev's inequality,

$$2^{k_0\varepsilon p}|\{Mh>2^{k_0}\}|\leq c\sum_{j\in I_k,k\geq k_0}|\lambda_j|^{\varepsilon p}|I_j|^{1-\varepsilon}\leq c\sum_{k\geq k_0}2^{k\varepsilon p}\sum_{j\in I_k}|I_j|.$$

Thus, Lemma (1.1.1) applies with $\varphi=Mf,\psi_{k_0}=Mg,\eta_{k_0}=Mh,$ and $\mu_k=\left(\sum_{j\in I_k}|I_j|\right)^{1/p}$, and we get

$$\|2^k m(Mf,2^k)^{1/p}\|_{\ell q}\leq c\left\|\left\{2^k\left(\sum_{j\in I_k}|I_j|\right)^{\frac{1}{p}}\right\}\right\|_{\ell q},$$

which, since

$$|I_j|\sim\frac{|\lambda_j|^p}{2^{kp}},j\in I_k,$$

is bounded by $c \parallel \lambda \parallel_{[p,q]}, 0 < q \leq \infty$.

The next result is of interest because it applies to arbitrary decompositions in $H^{p,q}$. The proof relies on Lemma (1.1.2).

Theorem (1.1.6) [1]: Let $0 < p \leq 1, 0 < q \leq \infty$, and let f be a distribution given by $f = \sum_j \lambda_j a_j(x)$, where the a_j 's are H^p atoms, and the convergence is in the sense of distributions. Further, assume that $\parallel \lambda \parallel_{[\eta,q]} < \infty$ for some $0 < \eta < \min(p,q)$. Then, $f \in H^{p,q}$, and $\parallel f \parallel_{H^{p,q}} \leq c \parallel \lambda \parallel_{[\eta,q]}$. We are now ready to identify the intermediate spaces of a couple of Hardy-Lorentz spaces with the same first index $p \leq 1$.

Theorem (1.1.7) [1]: Let $0 < p \leq 1$. Given $0 < q_1 < q < q_2 \leq \infty$, define $0 < \eta < 1$ by the relation $1/q = (1-\eta)/q_1 + \eta/q_2$. Then, with equivalent quasinorms,

$$H^{p,q} = (H^{p,q_1}, H^{p,q_2})_{\eta,q}.$$

Proof: Since the non-tangential maximal function Nf of a distribution f in H^{p,q_1} is in L^{p,q_1}, and that of f in H^{p,q_2} is in L^{p,q_2}, we have

$$K(t, Nf; L^{p,q_1}, L^{p,q_2}) \leq cK(t, f; H^{p,q_1}, H^{p,q_2}).$$

Thus,

$$\|Nf\|_{p,q} \sim \|Nf\|_{(L^{p,q_1},L^{p,q_2})_{\eta,q}} \leq c\|f\|_{(H^{p,q_1},H^{p,q_2})_{\eta,q}},$$

and $(H^{p,q_1}, H^{p,q_2})_{\eta,q} \hookrightarrow H^{p,q}$.

To show the other embedding, with the notation in the proof of Theorem (1.1.3), write $f = \sum_k \sum_j \lambda_{j,k} a_{j,k}$, and recall that for every integer k, the level set $I_k = \left\{j: |\lambda_{j,k}|/|I_{j,k}|^{\frac{1}{p}} \sim 2^k\right\}$ contains exclusively the sequence $\{\lambda_{j,k}\}$.

Let $\mu_k^p = \sum_{j \in I_k} |\lambda_{j,k}|^p$. By construction, $\sum_k \mu_k^q \sim \parallel f \parallel_{H^{p,q}}^q$. Now, rearrange $\{\mu_k\}$ into $\{\mu_l^*\}$, and, for each $l \geq 1$, let k_l be such that $\mu_{k_l} = \mu_l^*$. For $l_0 \geq 1$, let $K_{l_0} = \{k_1, \dots, k_{l_0}\}$, and put $f_{1,l_0} = \sum_{k \in k_0} \sum_j \lambda_{j,k} a_{j,k}$ and $f_{2,l_0} = f - f_{1,l_0}$. Then, by Theorem (1.1.4), $f_{1,l_0} \in H^{p,q_1}, f_{2,l_0} \in H^{p,q_2}$, and, with the usual interpretation for $q_2 = \infty$,

$$\|f_{1,l_0}\|_{H^{p,q_1}} \leq c\left(\sum_{1}^{l_0} \mu_l^{*q_1}\right)^{1/q_1}, \|f_{2,l_0}\|_{H^{p,q_2}} \leq c\left(\sum_{l_0+1}^{\infty} \mu_l^{*q_2}\right)^{1/q_2}$$

So, for $t > 0$ and every positive integer l_0, we have

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \leq c\left[\left(\sum_{1}^{l_0} \mu_l^{*q_1}\right)^{1/q_1} + t\left(\sum_{l_0+1}^{\infty} \mu_l^{*q_2}\right)^{1/q_2}\right].$$

Now, by Homstedt's formula, there is a choice of l_0 such that the right-hand side above $\sim K(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2})$, and, consequently,

Thus,

$$K(t, f; H^{p,q_1}, H^{p,q_2}) \leq cK(t, \{\mu_k\}; \ell^{q_1}, \ell^{q_2}).$$

$$\leq c\|\{\mu_k\}\|_{\ell_q} \leq c \parallel f \parallel_{H^{p,q}},$$

and $H^{p,q} \hookrightarrow (H^{p,q_1}, H^{p,q_2})_{\eta,q}$.

We will have no difficulty in verifying that Theorem (1.1.7) gives that if T is a continuous, sublinear map from H^1 into L^1, and from $H^{1,\infty}$ into $L^{1,\infty}$, then $\parallel Tf \parallel_{1,q} \leq c \parallel f \parallel_{H^{1,q}}$ for $1 < q < \infty$. This observation has numerous applications. Consider the Calder' on-Zygmund singular integral operators with variable kernel defined by

$$T_{\Omega}(f)(x) = \text{ p. v. } \int_{R^n} \frac{\Omega(x, x-y)}{|x-\mathrm{y}|^n} f(y) dy.$$

Under appropriate growth and smoothness assumptions on Ω, T maps H^1 continuously into L^1, see [7], and $H^{1,\infty}$ continuously into $L^{1,\infty}$, see [9]. Thus, if Ω satisfies the assumptions of both of these results, T_{Ω} maps $H^{1,q}$ continuously into $L^{1,q}$ for $1 < q < \infty$. A similar result follows by invoking the characterization of $H^{1,q}$ given by C. Fefferman, Rivi'ere and Sagher. However, in this case the $H^p - L^p$ estimate requires additional smoothness of Ω, as shown, for instance, in [7]. Similar considerations apply to the Marcinkiewicz integral, see [10], and [8].

Finally, when $p < 1$, our results cover, for instance, the $\delta - CZ$ operators satisfying $T^*(1) = 0$ discussed by Alvarez and Milman, see [4]. These operators, as well as a more general related class introduced in [16], preserve H^p and $H^{p,\infty}$ for $n/(n+\delta) < p \leq 1$, and, consequently, by Theorem (1.1.7), they also preserve $H^{p,q}$ for p in that same range, and $q > p$.

Section (1.2): Atomic Decompositions and Applications

The idea of atomic decomposition in martingale theory is derived from harmonic analysis. Just as it does in harmonic analysis, the method is key ingredient in dealing with many problems including martingale inequalities, duality, interpolation and so on, especially for small-index martingale and multi-parameter martingale. As well known, Weisz [27] gave some atomic decompositions on martingale Hardy spaces and proved many important theorems by atomic decompositions; Weisz [28] made a further study of atomic decompositions for weak Hardy spaces consisting of Vilenkin martingale, and proved a weak version of the Hardy-Littlewood inequality; Liu and Hou [24] investigated the atomic decompositions for vector-valued martingale and some geometry properties of Banach spaces were charactered; Hou and Ren [22] considered the vector-valued weak atomic decompositions and weak martingale inequalities; [29], [30], discussed the operator interpolation by atomic decompositions of weighted martingale Hardy spaces.

We present three atomic decomposition theorems for Lorentz martingale spaces $H^s_{p,q}, Q_{p,q}, D_{p,q}$. Applying these theorems, a sufficient condition for a sublinear operator defined on the Lorentz martingale spaces to be bounded is given. And then we obtain some continuous imbedding relationships among Lorentz martingale spaces. These are new versions of the basic inequalities in the classical martingale theory. Finally we also give a restricted weak-type interpolation theorem, and obtain the version of classical Marcinkiewicz interpolation theorem in the martingale setting.

Let (Ω, Σ, P) be complete probability space and f be a measurable function defined on Ω. The decreasing rearrangement of f is the function f^* defined by

$$f^*(t) = \inf \ \{s > 0: P(|f| > s) \leq t\}.$$

We adopt the convention $\inf \emptyset = \infty$. The Lorentz space $L_{p,q}(\Omega) = L_{p,q}, 0 < p < \infty, 0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\| f \|_{p,q}$ given by

$$\| f \|_{p,q} = \left(\frac{q}{p} \int_0^{\infty} \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}, 0 < q < \infty,$$

$$\| f \|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), q = \infty.$$

It will be convenient for us to use an equivalent definition of $\| f \|_{p,q}$, namely

$$\| f \|_{p,q} = q\left(\int_0^\infty \left[tP(|f(x)| > t)^{\frac{1}{p}}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, 0 < q < \infty,$$

$$\| f \|_{p,\infty} = \sup_{t>0} tP(|f(x)| > t)^{\frac{1}{p}}, q = \infty.$$

To check that these two expressions are the same, simply make the substitution $y = P(|f(x)| > t)$ and then integrate by parts.

It is well know that if $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, then $L_{p,q}$ is a Banach space, and $\| f \|_{p,q}$ is equivalent to a norm. However, for other values of p and $q, L_{p,q}$ is only a quasi-Banach spaces. In particular, if $0 < q \leq 1 \leq p$ or $0 < q \leq p < 1$ then $\| f \|_{p,q}$ is equivalent to a q-norm. Recall also that a quasi-norm $\|\cdot\|$ in X is equivalent to a p-norm, $0 < p < 1$, if there exists $c > 0$ such that for any $x_i \in X, i = 1, \ldots, n$

$$\|x_1 + \cdots + x_n\|^p \leq c(\|x_1\|^p + \cdots + \|x_n\|^p).$$

For all these properties, and more on Lorentz spaces, see for example [21], [5], [23].

Hölder's inequality for Lorentz spaces is the following, which first appears in work of O'Neil [26],

$$\| fg \|_{p,q} \leq c \| f \|_{p_1,q_1} \| g \|_{p_2,q_2}$$

for all $0 < p, q, p_1, q_1, p_2, q_2 \leq \infty$ such that $\frac{1}{p} = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$.

Let $\{\Sigma_n\}_{n\geq 0}$ a nondecreasing sequence of sub- $\sigma -$ fields of Σ such that $\Sigma = \vee \Sigma_n$. We denote the expectation operator and the conditional expectation operator relative to Σ_n by E and E_n, respectively. For a martingale $f = (f_n)_{n\geq 0}$, we define $\Delta_n f = f_n - f_{n-1}, n \geq 0$ (with convention $f_{-1} = 0, \Sigma_{-1} = \{\Omega, \Phi\}$)

$$M_n(f) = \sup_{0\leq i\leq n} |f_i|, M(f) = \sup_{n\geq 0} |f_n|,$$

$$S_n(f) = \left(\sum_{i=0}^n |\Delta_i f|^2\right)^{\frac{1}{2}}, S(f) = \left(\sum_{n=0}^\infty |\Delta_n f|^2\right)^{\frac{1}{2}},$$

$$= \left(\sum_{i=0}^n E_{i-1}|\Delta_i f|^2\right)^{\frac{1}{2}}, s(f) = \left(\sum_{n=0}^\infty E_{n-1}|\Delta_n f|^2\right)^{\frac{1}{2}}.$$

Denote by Λ, the set of all non-decreasing, non-negative and adapted r.v. sequences $\rho = (\rho_n)_{n\geq 0}$ with $\rho_\infty = \lim_{n\to\infty} \rho_n$. We shall say that a
martingale $f = (f_n)_{n\geq 0}$ has predictable control in $L_{p,q}$ if there is a sequence $\rho = (\rho_n)_{n\geq 0} \in \Lambda$ such that

$$|f_n| \leq \rho_{n-1}, \rho_\infty \in L_{p,q}.$$

As usually, we define Lorentz martingale spaces(see[21]),

$$H_{p,q}^{*} = \left\{ f = (f_n)_{n\geq 0} : \| f \|_{H_{p,q}^{*}} = \| M(f) \|_{p,q} < \infty \right\},$$
$$H_{p,q}^{s} = \left\{ f = (f_n)_{n\geq 0} : \| f \|_{H_{p,q}^{s}}^{s} = \| s(f) \|_{p,q} < \infty \right\},$$
$$H_{p,q}^{S} = \left\{ f = (f_n)_{n\geq 0} : \| f \|_{H_{p,q}}^{S} = \| S(f) \|_{p,q} < \infty \right\},$$
$$Q_{p,q} = \{ f = (f_n)_{n\geq 0} : \exists (\rho_n)_{n\geq 0} \in \Lambda, s.t. S_n(f) \leq \rho_{n-1}, \rho_\infty \in L_{p,q} \},$$
$$ff \left\| {}_{Q_{p,q}} = \inf_{\rho} \right\| \rho_\infty \|_{p,q}$$
$$D_{p,q} = \{ f = (f_n)_{n\geq 0} : \exists (\rho_n)_{n\geq 0} \in \Lambda, s.t. |f_n| \leq \rho_{n-1}, \rho_\infty \in L_{p,q} \},$$
$$ff \left\| {}_{D_{p,q}} = \inf_{\rho} \right\| \rho_\infty \|_{p,q}.$$
If we change the $L_{p,q}$-norms in above definitions by L_p-norms, we get the usual Hardy martingale spaces (see [25]).

Definition (1.2.1)[20]: A measurable function a is called a $(1, p, \infty)$-atom (or $(2, p, \infty)$-atom or $(3, p, \infty)$-atom,respectively) if there exists a stopping time τ such that

(i) $a_n = E_n a = 0, n \leq \tau,$

(ii) $\| S(a) \|_\infty \leq P(\tau < \infty)^{-1/p}$ (or (ii) $\| S(a) \|_\infty \leq P(\tau < \infty)^{-1/p}$ or (ii) $\| M(a) \|_\infty \leq P(\tau < \infty)^{-1/p}$, respectively).

We denote the set of integers and the set of nonnegative integers by Z and N, respectively. We write $A \leq B$ if $A \leq cB$ for some positive constant c independent of appropriate quantities involved in the expressions A and B.

Now we can present the atomic decompositions for Lorenz martingale spaces.

Theorem (1.2.2)[20]: If the martingale $f \in H_{p,q}^{s}, 0 < p < \infty, 0 < q \leq \infty$ then there exist a sequence a^k of $(1, p, \infty)$-atoms and a positive real number sequence $(\mu_k) \in l_q$ such that
$$f_n = \sum_{k\in Z} \mu_k a_n^k, n \in N$$
and
$$\|(\mu_k)_{k\in Z}\|_{l_q} \leq \| f \|_{H_{p,q}^{s}}.$$
Conversely, if $0 < q \leq 1, q \leq p < \infty$, and the martingale f has the above decomposition, then $f \in H_{p,q}^{s}$ and
$$\| f \|_{H_{p,q}^{s}} \leq \inf \ \|(\mu_k)_{k\in Z}\|_{l_q,}$$
where the inf is taken over all the preceding decompositions of f.

Proof: Assume that $f \in H_{p,q}^{s}, q \neq \infty$. Now consider the following stopping time for all $k \in Z$:
$$\tau_k = \inf \ \{ n \in N : s_{n+1}(f) > 2^k \} (\mathrm{i} \ \ \phi = \infty).$$
The sequence of these stopping times is obviously non-decreasing. It easy to see that
$$\sum_{k\in Z} \left(f_n^{\tau_{k+1}} - f_n^{\tau_k} \right) = \sum_{k\in Z} \left(\sum_{m=0}^{n} \chi_{\{m\leq \tau_{k+1}\}} \Delta_m f - \sum_{m=0}^{n} \chi_{\{m\leq \tau_k\}} \Delta_m f \right)$$
$$= \sum_{k\in Z} \left(\sum_{m=0}^{n} \chi_{\{\tau_k < m \leq \tau_{k+1}\}} \Delta_m f \right) = f_n.$$
Let $\mu_k = 2^k 3P(\tau_k < \infty)^{1/p}$, and

$$a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$ then we assume that $a_n^k = 0$. Then for a fixed $k, (a_n^k)$ is a martingale. Since $s(f_n^{\tau_k}) \leq 2^k, s(f_n^{\tau_{k+1}}) \leq 2^{k+1}$

$$s(a_n^k) \leq \frac{s(f_n^{\tau_{k+1}}) + s(f_n^{\tau_k})}{\mu_k} \leq P(\tau_k < \infty)^{-1/p}, n \in N,$$

which implies that (a_n^k) is a L_2-bounded martingale so that there exists $a^k \in L_2$ such that $E_n a^k = a_n^k$. If $n \leq \tau_k$ then $a_n^k = 0$ and we get that a^k is really a $(1, p, \infty)$ atom and

$$\left(\sum_{k\in Z} |\mu_k|^q\right)^{1/q} = 3\left(\sum_{k\in Z} \left(2^k P(\tau_k < \infty)^{\frac{1}{p}}\right)^q\right)^{1/q} = \left(\sum_{k\in Z} \left(2^k P(s(f) > 2^k)^{\frac{1}{p}}\right)^q\right)^{1/q}$$

$$\leq \left(\sum_{k\in Z_{2^{k-1}}} \int_{2^k}^{q-1} dy P(s(f) > 2^k)^{\frac{q}{p}}\right)^{1/q}$$

$$\leq \left(\sum_{k\in Z} \int_{2^{k-1}} y^{q-1} P(s(f) > y)^{\frac{q}{p}} dy\right)^{1/q}$$

$$\leq \left(\int_0^\infty y^{q-1} P(s(f) > y)^{\frac{q}{p}} dy\right)^{1/q} \leq \| f \|_{H_{p,q}^s}$$

For $q = \infty$, standard rectifications can be made.

Conversely, if f has the above decomposition, then from $\| {}_s(a^k)\|_\infty \leq P(\tau_k < \infty)^{-1/p}$ and

$$P(s(a^k) > y) \leq P(s(a^k) \neq 0) \leq P(\tau_k < \infty),$$

we get

$$\|a^k\|_{H_{p,q}^s}^q = q \int_0^\infty y^{q-1} P(s(a^k) > y)^{\frac{q}{p}} dy$$

$$= q \int_0^{P(\tau_k<\infty)^{-1/p}} y^{q-1} P(s(a^k) > y)^{q/p} dy$$

$$\leq P(\tau_k < \infty)^{\frac{q}{p}} \int_0^{P(\tau_k<\infty)^{-\frac{1}{p}}} y^{q-1} dy \leq \frac{1}{q}.$$

For $0 < q \leq 1, q \leq p < \infty, \|\cdot\|_{p,q}$ is equivalent to a $q-$ norm,

$$\|a^k\|_{H_{p,q}^s}^q \leq \left\|\sum_{k\in Z} \mu_k s(a^k)\right\|_{p,q}^q \leq \sum_{k\in Z} \mu_k \|s(a^k)\|_{p,q}^q \leq \sum_{k\in Z} \mu_{k^q}^q,$$

which gives the desired result.

Theorem (1.2.3) [20]: If the martingale $f \in Q_{p,q}, 0 < p < \infty, 0 < q \leq \infty$, then there exist a sequence (a^k) of $(2, p, \infty)$ atoms and a real number sequence $(\mu_k) \in l_q$ such that

$$f_n = \sum_{k\in Z} \mu_k a_n^k, \forall n \in N$$

and

$$\left(\sum_{k\in Z} |\mu_k|^q\right)^{1/q} \leq \| f \|_{Q_{p,q}}.$$

Conversely, if $0 < q \leq 1, q \leq p < \infty$, and the martingale f has the above decomposition, then $f \in Q_{p,q}$ and

$$\| f \|_{Q_{p,q}} \leq \text{i} \left(\sum_{k\in Z} |\mu_k|^q\right)^{\frac{1}{q}},$$

where the inf is taken over all the above decompositions.

Proof: Suppose that $f \in Q_{p,q}$. Let $\beta = (\beta_n)_{n\geq 0}$ be the optimal control of $S_n(f)$, i.e., $\beta \in \Lambda, S_n(f) \leq \beta_{n-1}, \| f \|_{Q_{p,q}} = \|\beta_\infty\|_{Q_{p,q}}$. The stopping times τ_k are defined in this case by

$$\tau_k = \inf\ \{n \in N: \beta^n > 2^k\}(\inf\ \phi = \infty).$$

Let a^k and $\mu_k (k \in Z)$ be defined as in the proof of Theorem (1.2.2). Then for a fixed $k, (a_n^k)$ is also a martingale. Since $S\left(f_n^{\tau_k}\right) = S_{\tau_k}(f) \leq \beta_{\tau_{k-1}} \leq 2^k, S\left(f_n^{\tau_{k+1}}\right) \leq 2^{k+1}$,

$$S(a_n^k) \leq \frac{S\left(f_n^{\tau_{k+1}}\right) + S\left(f_n^{\tau_k}\right)}{\mu_k} \leq P(\tau_k < \infty)^{-1/p}, n \in N.$$

As in Theorem (1.2.2), we can show that a^k is a $(2, p, \infty)$-atom. Also

$$\left(\sum_{k\in Z} |\mu_k|^q\right)^{1/q} = 3\left(\sum_{k\in Z} \left(2^k P(\tau_k < \infty)^{\frac{1}{p}}\right)^q\right)^{1/q}$$

$$= 3\left(\sum_{k\in Z} \left(2^k P(\beta_\infty > 2^k)^{\frac{1}{p}}\right)^q\right)^{1/q}$$

$$\|\beta_\infty\|_{p,q} = \| f \|_{Q_{p,q}}.$$

Conversely, if a^k is $(2, p, \infty)$-atom, one can show that $\|a^k\|_{H_{p,q}^s}^q \leq \frac{1}{q}$. The rest can be proved similar to Theorem (1.2.2).

Theorem (1.2.4) [20]: If the martingale $f \in D_{p,q}, 0 < p < \infty, 0 < q \leq \infty$, then there exist a sequence (a^k) of $(3, p, \infty)$-atoms and a real number sequence $(\mu_k) \in l_q$ such that

$$f_n = \sum_{k\in Z} \mu_k a_n^k, n \in N$$

and

$$\left(\sum_{k\in Z} |\mu_k|^q\right)^{1/q} \leq \| f \|_{D_{p,q}}.$$

Conversely, if $0 < q \leq 1, q \leq p < \infty$, and the martingale f has the above decomposition, then $f \in D_{p,q}$ and

$$\| f \|_{D_{p,q}} \leq \inf \left(\sum_{k\in Z} |\mu_k|^q\right)^{\frac{1}{q}},$$

where the inf is taken over all the above decompositions.

The proof of Theorem (1.2.4) is similar to that of Theorem (1.2.3). We shall obtain a sufficient condition for a sublinear operator to be bounded from Lorentz martingale spaces to function Lorentz spaces. Applying the condition to Mf, Sf and sf, we deduce a series of inequalities on Lorentz martingale spaces.

An operator $T: X \to Y$ is called a sublinear operator if it satisfies

$$|T(f+g)| \leq |Tf| + |Tg|, |T(\alpha f)| \leq |\alpha||Tf|,$$

where X is a martingale space, Y is a measurable function space.

Theorem (1.2.5) [20]: Let $T: H_r^s \to L_r$ be a bounded sublinear operator for some $1 \leq r < \infty$. If

$$P(|Ta| > 0) \leq P(\tau < \infty)$$

for all $(1, p, \infty)$-atoms a, where τ is the stopping time associated with a, then for $0 < p < r, 0 < q \leq \infty$, we have

$$\| Tf \|_{p,q} \leq \| f \|_{H_{p,q}^s}, \ f \in H_{p,q}^s.$$

Proof: Assume that $f \in H_{p,q}^s$. By Theorem (1.2.2), f can be decomposed into the sum of a sequence of $(1, p, \infty)$-atoms. For any fixed $y > 0$ choose $j \in Z$ such that $2^{j-1} \leq y < 2^j$ and let

$$f = \sum_{k \in Z} \mu_k a^k = \sum_{k=-\infty}^{j-1} \mu_k a^k + \sum_{k=j}^{\infty} \mu_k a^k =: g + h.$$

Recall that $\mu_k = 2^k 3 P^{\frac{1}{p}}(\tau_k < \infty)$ and $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$. we have

$$\| g \|_{H_r^s} \leq \left(\int_\Omega \left(\sum_{k=-\infty}^{j-1} \mu_k s(a^k) \right)^r dP \right)^{1/r} \leq \sum_{k=-\infty}^{j-1} \mu_k \left(\int_\Omega \left(s(a^k) \right)^r dP \right)^{1/r}$$

$$\leq \sum_{k=-\infty}^{j-1} \mu_k \left(\int_{\{\tau_k \leq \infty\}} \| s(a^k) \|_\infty^r dP \right)^{1/r}$$

$$\leq \sum_{k=-\infty}^{j-1} \mu_k P(\tau_k < \infty)^{-\frac{1}{p}} P(\tau_k < \infty)^{1/r}$$

$$= \sum_{k=-\infty}^{j-1} 2^k P(\tau_k < \infty)^{1/r}$$

$$= \sum_{k=-\infty}^{j-1} 2^k P(s(f) > 2^k)^{1/r}$$

It follows from the boundedness of T that

$$P(|Tg| > y) \leq y^{-r} E|Tg|^r \leq y^{-r} \parallel g \parallel_{H_r^s}^r$$
$$\leq y^{-r} \left(\sum_{k=-\infty}^{j-1} 2^k P(s(f) > 2^k)^{\frac{1}{r}} \right)^r$$
$$= y^{-r} \left(\sum_{k=-\infty}^{j-1} 2^{k\left(1-\frac{p}{r}\right)} 2^{k\frac{p}{r}} P(s(f) > 2^k)^{1/r} \right)^r)$$
$$\leq y^{-r} \left(\sum_{k=-\infty}^{j-1} 2^{k\left(1-\frac{p}{r}\right)} \right)^r \parallel sf \parallel_{p,\infty}^p$$
$$\leq y^{-p} \parallel {}_s f \parallel_{p,\infty}^p.$$

On the other hand, since $|Th| \leq \sum_{k=j}^{\infty} \mu_k |Ta^k|$, we get

$$P(|Th| > y) \quad \leq P(|Th| > 0) \leq \sum_{k=j}^{\infty} P(|Ta^k| >$$
$$= \sum_{k=j}^{\infty} P(\tau_k < \infty) = \sum_{k=j}^{\infty} 2^{-kp} 2^{kp} P(sf > 2^k)$$
$$\leq \sum_{k=j}^{\infty} 2^{-kp} \parallel sf \parallel_{p,\infty}^p$$
$$\leq y^{-p} \parallel sf \parallel_{p,\infty}^p.$$

Since T is subliear,

$$P(|Tf| > y) \leq P\left(|Tg| > \frac{y}{2}\right) + P\left(|Th| > \frac{y}{2}\right) \leq y^{-p} \parallel {}_s f \parallel_{p,\infty}^p,$$

and thus for all $0 < p < r, T: H_{p,\infty}^s \to L_{p,\infty}$ is bounded. Now for any fixed $0 < p < r$, we can choose $0 < p_0, p_1 < r, 0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

From interpolation theorem (see Theorem 5.11 [5]) and the boundedness of sublinear is hereditary for the interpolation spaces, we obtain for $0 < q \leq \infty$

$$T: H_{p,q}^s = \left(H_{p_0,\infty}^s, H_{p_1,\infty}^s\right)_{\theta,q} \to \left(L_{p_0,\infty}, L_{p_1,\infty}\right)_{\theta,q} = L_{p,q}$$

is bounded. Hence

$$\parallel Tf \parallel_{p,q} \leq \parallel f \parallel_{H_{p,q}^s}.$$

On the lines of the proof of Theorem (1.2.5), we can prove the following Theorems (1.2.6) and (1.2.7) by using Theorems (1.2.3) and (1.2.4), respectively.

Theorem (1.2.6) [20]: Let $T: Q_r \to L_r$ be a bounded sublinear operator for some $1 \leq r < \infty$. If

$$P(|Ta| > 0) \leq P(\tau < \infty)$$

for all $(2, p, \infty)$-atoms a, where τ is the stopping time associated with a, then for $0 < p < r, 0 < q \leq \infty$, we have

$$\parallel Tf \parallel_{p,q} \preceq \parallel f \parallel_{Q_{p,q}}, \ f \in Q_{p,q}.$$

Theorem (1.2.7) [20]: Let $T: D_r \to L_r$ be a bounded sublinear operator for some $1 \leq r < \infty$. If

$$P(|Ta| > 0) \leq P(\tau < \infty)$$
for all $(3, p, \infty)$-atoms a, where τ is the stopping time associated with a, then for $0 < p < r, 0 < q \leq \infty$, we have
$$\| Tf \|_{p,q} \leq \| f \|_{D_{p,q}}, \ f \in D_{p,q}.$$
Theorem (1.2.8) [20]: For all martingale $f = (f_n)_{n\geq 0}$ the following imbeddings hold:

(i) For $0 < p < 2, 0 < q \leq \infty$,
$$H^s_{p,q} \hookrightarrow H^*_{p,q}, H^s_{p,q} \hookrightarrow H^S_{p,q},$$
for $p > 2, 0 < q \leq \infty$,
$$H^*_{p,q} \hookrightarrow H^s_{p,q},\ H^S_{p,q} \hookrightarrow H^s_{p,q}.$$
(ii) For $0 < p < \infty, 0 < q \leq \infty$,
$$Q_{p,q} \hookrightarrow H^*_{p,q}, \quad Q_{p,q} \hookrightarrow H^S_{p,q}, \quad Q_{p,q} \hookrightarrow H^s_{p,q}$$
$$D_{p,q} \hookrightarrow H^*_{p,q} \quad D_{p,q} \hookrightarrow H^S_{p,q}, \quad D_{p,q} \hookrightarrow H^s_{p,q}.$$
Proof: (i) The maximal operator $Tf = Mf$ is sublinear, and $\| Mf \|_2 \leq \| sf \|_2$. If a is any $(1, p, \infty)$-atom and τ is the corresponding stopping time, then $\{|Ta| > 0\} = \{|Ma| > 0\} \subset \{\tau < \infty\}$ and hence $P(|Ta| > 0) \leq P(\tau < \infty)$. It follows from Theorem (1.2.5) that
$$\| Mf \|_{p,q} \leq \| f \|_{H^s_{p,q}}, \ (0 < p < 2).$$
Similarly, consider the operator $Tf = Sf$. We get $\| Sf \|_{p,q} \leq \| f \|_{H^s_{p,q}}$.

Conversely, we use vector-valued interpolation (see [25]) to obtain the case $p > 2, 0 < q \leq \infty$. In fact, we can regard martingale spaces as the subspaces of sequence spaces. Consider the operator $Q: L_p(l_\infty) \to L_p$ defined by $Q(f_n) = s(f) = (\sum_{n=0}^{\infty} E_{n-1}|\Delta_n f|^2)^{1/2}$. For $p \geq 2$, we know that $\| s(f) \|_p \leq \| M(f) \|_p = \|\sup_{n\geq 0}|f_n|\|_p$ and so $Q: L_p(l_\infty) \to L_p$ is bounded for all $p \geq 2$. For any fixed $p > 2$, we can choose $p_0, p_1 > 2, 0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Consequently $Q: L_{p_i}(l_\infty) \to L_{p_i}$ is bounded, $i = 0,1$. By interpolation, for $0 < q \leq \infty$,
$$Q: L_{p,q}(l_\infty) = \left(L_{p_0}(l_\infty), L_{p_1}(l_\infty)\right)_{\theta,q} \to \left(L_{p_0}, L_{p_1}\right)_{\theta,q} = L_{p,q}$$
is bounded. Hence we obtain
$$\| s(f) \|_{p,q} \leq \left\| \sup_{n\geq 0} |f_n| \right\|_{p,q} = \| M(f) \|_{p,q},$$
which gives $H^*_{p,q} \hookrightarrow H^s_{p,q}$. By considering Q defined on the sequence space $L_p(l_2)$, we can similarly prove $H^S_{p,q} \hookrightarrow H^s_{p,q}$

(ii) For all $0 < r < \infty, \| M(f) \|_r, \| S(f) \|_r, \| s(f) \|_r \leq \| f \|_{Q_r}$ and $\| M(f) \|_r, \| S(f) \|_r, \| s(f) \|_r \leq \| f \|_{D_r}$. Note that $a_n^k = 0$ on the set $\{n \leq \tau_k\}$.

Thus
$$\chi(n \leq \tau_k) E_{n-1} \left|\Delta_n a^k\right|^2 = E_{n-1} \chi(n \leq \tau_k) \left|\Delta_n a^k\right|^2 = 0.$$
Hence $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$.

We say that a sublinear operator T is of Lorentz-s restricted weak-type (p, q) if T maps $H^S_{p,1}$ to $L_{p,\infty}$. For convenience, we call T as restricted weak-type (p, q). Then we have the next interpolation from one restricted weak-type estimate to another.

Theorem (1.2.9) [20]: Let T be of restricted weak-type (p_i, q_i) for $i = 0,1$, and $1 < p_i, q_i < \infty$. Put

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\forall 0\le\theta\le 1.$$

Then T is also of restricted weak-type (p,q).

Proof: Suppose that $f\in H_{p,1}^s$. From Theorem (1.2.2), $f=\sum_{k\in Z}\mu_k a^k, a^k$ is a $(1,p,\infty)$-atom with respect to the stopping time τ_k, and $\sum_{k\in Z}|\mu_k|\le\| f\|_{H_{p,1}^s}$. Now we can estimate $\|Ta^k\|_{q,\infty}\le 1$. In fact

$$\|Ta^k\|_{q,\infty}=\sup_{t>0}t^{\frac{1}{p}}(Ta^k)^*(t)=\sup_{t>0}\left(t^{\frac{1}{q_0}}(Ta^k)^*(t)\right)^{1-\theta}\left(t^{\frac{1}{q_1}}(Ta^k)^*(t)\right)^{\theta}$$
$$\le\|Ta^k\|_{q_0,\infty}^{1-\theta}\|Ta^k\|_{q_1,\infty}^{\theta}$$
$$\le\|sa^k\|_{p_0,1}^{1-\theta}\|sa^k\|_{p_1,1}^{\theta}$$
$$\le\|sa^k\|_{2p_0,2p_0}^{1-\theta}\|\chi_{\{\tau_k<\infty\}}\|_{2p_0,l}^{1-\theta}\|sa^k\|_{2p_1,2p_1}^{\theta}\|\chi_{\{\tau_k<\infty\}}\|_{2p_1,m}^{\theta}$$
$$\le P(\tau_k<\infty)^{-\frac{1}{p}}\left(P(\tau_k<\infty)^{\frac{1-\theta}{2p_0}}P(\tau_k<\infty)^{\frac{\theta}{2p_1}}\right)^2$$
$$\le 1,$$

Where $l=\frac{2p_0}{2p_0-1}$ and $m=\frac{2p_1}{2p_1-1}$. Consequently,

$$\| Tf\|_{q,\infty}\le\sum_{k\in Z}|\mu_k|\|Ta^k\|_{q,\infty}\le\sum_{k\in Z}|\mu_k|\le\| f\|_{H_{p,1}^s}.$$

and the proof is complete.

Now we show how restricted weak-type estimate can be transferred to strong type. It is also the version of the classical Marcinkiewicz interpolation theorem in the martingale setting (see Theorem 4.13 in [21]).

Theorem (1.2.10) [20]: Let T be of restricted weak-type (p_i,q_i) for $i=0,1$, and $1<p_i<\infty, 1<q_i\le\infty, q_0=q_1$. Put

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},0\le\theta\le 1.$$

Then T is of type $\left(H_{p,r}^s,L_{q,r}\right)$, for $0<r<1$ and $r\le q$.

Proof: For $0<r<1$ and $r\le q$, we know that $\|\cdot\|_{q,r}$ is equivalent to a r-norm, so it is enough to prove $\| Ta\|_{q,r}\le 1$, for all $(1,p,\infty)$-atoms.

Once it is proved then from Theorem (1.2.2),

$$\| Tf\|_{q,r}^r\le\sum_{k\in Z}\mu_k^r\| Ta\|_{q,r}^r\le\sum_{k\in Z}\mu_k^r\le\| f\|_{H_{p,r}^s}^r.$$

Now we shall show $\| Ta\|_{q,r}\le 1$. Consider the case $q_1,q_2<\infty$. From the proof of Theorem (1.2.9), it is easy to see that

$$\| a\|_{H_{p_i,1}^s}^{p_i}\le P(\tau<\infty)^{1-\frac{p_i}{p}}, i=0,1.$$

Thus, for $q_0<q<q_1$, we get

$$\frac{1}{q}\parallel Ta\parallel_{,r}^{q}=\int_0^\infty y^{r-1}P(|Ta|>y)^{r/q}dy$$
$$\leq \int_0^\delta y^{r-1}\left(\frac{1}{y}\parallel a\parallel_{H^s_{p_0,1}}\right)^{\frac{q_0 r}{q}}dy+\int_\delta^\infty y^{r-1}\left(\frac{1}{y}\parallel a\parallel_{H^s_{p_1,1}}\right)^{\frac{q_1 r}{q}}dy$$
$$\leq \delta^{\frac{r}{q}(q-q_0)}P(\tau<\infty)^{\frac{rq_0}{q}(1/p_0-1/p)}+\delta^{\frac{r}{q}(q-q_1)}P(\tau<\infty)^{\frac{rq_1}{q}(1/p_1-1/p)}$$
Take $\delta=P(\tau<\infty)^\alpha$ with α satisfying
$$q\alpha=\left(\frac{1}{p_0}-\frac{1}{p}\right)/\left(\frac{1}{q}-\frac{1}{q_0}\right)=\left(\frac{1}{p_1}-\frac{1}{p}\right)/\left(\frac{1}{q}-\frac{1}{q_1}\right)$$
In fact, from $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ we find that $q\alpha=\left(\frac{1}{p_0}-\frac{1}{p_1}\right)/\left(\frac{1}{q_1}-\frac{1}{q_0}\right)$, and
$$\frac{r}{q}\left[\alpha(q-q_0)+q_0\left(\frac{1}{p_0}-\frac{1}{p}\right)\right]=\frac{r}{q}\left[\alpha(q-q_1)+q_1\left(\frac{1}{p_1}-\frac{1}{p}\right)\right]=0.$$
Then $\parallel Ta\parallel_{q,r}^{q}\lesssim 1$.
When one of q_i is ∞, say $q_1=\infty$, the proof is modified. More precisely, we have
$$\parallel Ta\parallel_\infty\leq\parallel a\parallel_{H^s_{p_1,1}}\leq P(\tau<\infty)^{\frac{1}{p_1}-\frac{1}{p}}.$$
Thus, from $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\frac{1}{q}=\frac{1-\theta}{q_0}$
$$\frac{1}{q}\parallel Ta\parallel_{q,r}^{q}=\int_0^{\parallel Ta\parallel_\infty} y^{r-1}P(|Ta|>y)^{r/q}dy$$
$$\leq \int_0^{\parallel Ta\parallel_\infty} y^{r-1}P\left(\frac{1}{y}\parallel a\parallel_{H^s_{p_0,1}}\right)^{q_0 r/q}dy$$
$$\leq P(\tau<\infty)^{\frac{rq_0}{q}\left(\frac{1}{p_0}-\frac{1}{p}\right)}+P(\tau<\infty)^{\frac{r}{q}(q-q_0)(1/p_1-1/p)}$$
$$\leq 1$$
the assertion follows.

Chapter 2
Hardy-Sobolev Inequalities and Uniform Fatness

We present a new transparent proof for the fact that uniform capacity density implies the classical integral version of the Hardy inequality in the setting of metric spaces. In addition, we consider the relations between the above concepts and certain Hausdorff content conditions. We show that estimates are sharp in the sense that they coincide when the domain is a ball or an infinite strip. In the case of a ball, we also obtain further improvements.

Section (2.1): Critical Hardy-Sobolev Inequalities

For $\Omega \subset \mathbb{R}^n$ be a domain and K be a compact, C^2 manifold without boundary embedded in $\mathbb{R}^n$, of co-dimension $k, 1 \leqslant k < n$. When $k = 1$ we assume that $K = \partial\Omega$, whereas for $1 < k < n$ we assume that $K \cap \bar{\Omega} \neq \emptyset$. We set $d(x) = \operatorname{dist}(x, K)$.

We also recall for $1 < p$ and $p \neq k$ the following condition that was introduced in [35],

$$-\Delta_p d^{\frac{p-k}{p-1}} \geqslant 0 \text{ on } \Omega \setminus K, (C)$$

where Δ_p is the p-Laplacian, that is $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We note that for $k = 1$, condition (C) becomes $-\Delta \mathrm{d} \geqslant 0$, which is equivalent to the convexity of the domain Ω for $n = 2$, but it is a much weaker condition than convexity of Ω for $n \geqslant 3$.

Under assumption (C) the following Hardy inequality holds true [35],

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geqslant 0, u \in C_0^\infty(\Omega \setminus K), \tag{1}$$

Where $\left|\frac{p-k}{p}\right|^p$ is the best constant.

We show that inequality (1) can be improved by adding a multiple of a whole range of critical norms that at the extreme case become precisely the critical Sobolev norm.

Theorem (2.1.1)[31]: Let $2 \leqslant p < n, p \neq k < n$ and $p < q \leqslant \frac{np}{n-p}$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K is a compact, C^2 manifold without boundary embedded in $\mathbb{R}^n$, of codimension $k, 1 \leqslant k < n$. When $k = 1$ we assume that $K = \partial\Omega$, whereas for $1 < k < n$ we assume that $K \cap \bar{\Omega} \neq \emptyset$.

(i) If in addition Ω and K satisfy condition (C), then there exists a positive constant $c = c(\Omega, K)$ such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geqslant c\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q dx\right)^{\frac{p}{q}}. \tag{2}$$

(ii) Without assuming condition (C), there exist a positive constant $c = c(n, k, p, q)$ independent of Ω, K and a constant $M = M(\Omega, K)$, such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds:

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx + M\int_\Omega |u|^p dx \geqslant c\left(\int_\Omega d^{-q+\frac{q-p}{p}n} |u|^q dx\right)^{\frac{p}{q}}. \tag{3}$$

We note that the term in the right-hand side of (2) and (3) is optimal and in fact (2) is a scale invariant inequality. In the extreme case where $q = \frac{np}{n-p'}$, the term in the right-hand side is precisely the critical Sobolev term.

The only result previously known, in the spirit of estimate (2), concerns the particular case where $\Omega = \mathbb{R}^n, p = 2$ and K is affine, that is, $K = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_k = 0\}, 1 \leqslant k < n, k \neq 2$ and has been established in [49].

The case $p \neq 2$ was posed as an open question in [49].
On the other hand the nonnegativity of the left-hand side of (3) for $p = 2$ has been shown in [36] for $K = \partial\Omega$.
Other improvements of the plain Hardy inequality involving any arbitrary subcritical L^q term are presented in [42] for the case where Ω is a convex domain and $K = \partial\Omega$. For earlier results involving improvements with some subcritical L^q terms see [39].
We emphasize that in our theorem the case $k = n$, which corresponds to taking distance from an interior point, is excluded. As a matter of fact estimate (2) fails in this case. Indeed in this case, the optimal improvement of the plain Hardy inequality involves the critical Sobolev exponent, but contrary to (2) it also has a logarithmic correction [43].
To establish Theorem(2.1.1) a crucial step is to obtain local estimates in a neighborhood of K, see Theorem (2.1.18).
For other directions in improving Hardy inequalities see [32], [35], [36], [37], [38],[40],[44],[47], [48], [49],[51], [52], [53]. We establish auxiliary weighted Sobolev type inequalities, in the special case where distance is taken from the boundary. We then use these inequalities derive Hardy-Sobolev inequalities when distance is taken from the boundary. We consider more general distance functions, where distance is taken from a surface K of co-dimension k, as well as other critical norms via interpolation.
Some preliminary results have been announced in [41].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary and $d(x) = \mathrm{dist}(x, \partial\Omega)$. We denote by $\Omega\delta : = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leqslant \delta\}$ a tubular neighborhood of $\partial\Omega$, for δ small. Then, for δ small we have that $d(x) \in C^2(\Omega_\delta)$. Also, if $x \in \Omega\delta$ approaches $x_0 \in \partial\Omega \in C^2$ then clearly $d(x) \to 0$, and also

$$\Delta d(x) = (N-1)H(x_0) + O(d(x)),$$

where $H(x_0)$ is the mean curvature of $\partial\Omega$ at x_0; see e.g., [45]. As a consequence of this we have that there exists a δ^* sufficiently small and a positive constant c_0 such that

$$|d\Delta d| \leqslant c_0 d, \text{ in } \Omega_\delta, \text{ for } 0 < \delta \leqslant \delta^*. (R)$$

We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies condition (R) if there exists a c_0 and a δ^* such that (R) holds. In case $d(x)$ is not a C^2 function we interpret the inequality in (R) in the weak sense, that is

$$\left| \int_{\Omega_\delta} d\Delta d\phi dx \right| \leqslant c_0 \int_{\Omega_\delta} d\phi dx, \forall \phi \in C_0^\infty(\Omega), \phi \geqslant 0.$$

In our proofs, instead of assuming that Ω is a bounded domain of class C^2 we will sometimes assume that Ω satisfies condition (R). Thus, some of our results hold true for a larger class of domains. For instance, if Ω is a strip or an infinite cylinder, condition (R) is easily seen to be satisfied even though Ω is not bounded.
We first prove an L^1 estimate.

Lemma (2.1.2) [31]: Let Ω be a bounded domain which satisfies condition (R). For any $S \in \left(0, \frac{1}{2} n \pi^{\frac{1}{2}} \left[\left(1 + \frac{n}{2}\right)\right]^{-\frac{1}{n}}\right)$ and any $a > 0$, there exists $\delta_0 = \delta_0(a/c_0)$ such that for all $\delta \in (0, \delta_0]$ there holds:

$$\int_{\Omega_\delta} d^a |\nabla v| dx + \int_{\partial\Omega_\delta^c} d^a |v| dS_x \geqslant S \|d^a v\|_{L^{\frac{N}{N-1}}(\Omega_\delta)}, \forall v \in C^\infty(\Omega). \quad (4)$$

Proof: We will use the following inequality: If $V \subset \mathbb{R}^n$ is any bounded domain and $u \in C^\infty(V)$, then

$$S_n \parallel u \parallel_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \parallel \nabla u \parallel_{L^1(V)} + \parallel \nabla u \parallel_{L^1(\partial V)}, \tag{5}$$

where $S_n = n\pi^{1/2}\left[\left(1+\frac{n}{2}\right)\right]^{-1/n}$; see [49].

For $V = \Omega_\delta$ we apply (5) to $u = d^a v, v \in C^\infty(\Omega)$ to get:

$$S_n \|d^a v\|_{L^{\frac{N}{N-1}}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^a |\nabla v| dx + a \int_{\Omega_\delta} d^{a-1} |v| dx + \int_{\partial \Omega_\delta^c} d^a |v| dS_x \cdot \tag{6}$$

To estimate the middle term of the right-hand side, noting that $\nabla d \cdot \nabla d = 1$ a.e. and integrating by parts we have:

$$a \int_{\Omega_\delta} d^{a-1}|v| dx = \int_{\Omega_\delta} \nabla d^a \cdot \nabla d |v| dx$$

$$= -\int_{\Omega_\delta} d^a \Delta d |v| dx - \int_{\Omega_\delta} d^a \nabla d \cdot \nabla |v| dx + \int_{\partial \Omega_\delta^c} d^a |v| dS_x.$$

Under our condition (R) for δ small we have $|d\Delta d| < c_0 d$ in Ω_δ. It follows that

$$(a - c_0\delta) \int_{\Omega_\delta} d^{a-1} |v| dx \leqslant \int_{\Omega_\delta} d^{a-1} |\nabla v| dx + \int_{\partial \Omega_\delta^c} d^a |v| dS_x. \tag{7}$$

From (6) and (7) we get:

$$\frac{a - c_0\delta}{2a - c_0\delta} S_n \parallel u \parallel_{L^{n-1}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^a |\nabla v| dx + \int_{\partial \Omega_\delta^c} d^a |v| dS_x.$$

The result then follows by taking

$$\delta_0 = \frac{a(S_n - 2S)}{c_0(S_n - S)}. \tag{8}$$

We similarly have

Lemma (2.1.3) [31]: Let Ω be a domain which satisfies condition (R). For any $S \in \left(0, \frac{1}{2} n v_n^{\frac{1}{n}}\right)$ and $a > 0$ there exists $\delta_0 = \delta_0(a/c_0)$ such that for all $\delta \in (0, \delta_0]$ there holds:

$$\int_{\Omega_\delta} d^a |\nabla v| dx \geqslant S \|d^a v\|_L \frac{n}{n-1} (\Omega_\delta), \forall v \in C_0^\infty(\Omega_\delta). \tag{9}$$

The proof is quite similar to that of the previous lemma. Instead of (5) one uses the $(p = 1) -$ Gagliardo-Nirenberg inequality valid for any $V \subset \mathbb{R}^n$, and any $u \in C_0^\infty(V)$,

$$\tilde{S}_n \parallel u \parallel_{L^{\frac{n}{n-1}}(V)} \leqslant \parallel \nabla u \parallel_{L^1(V)}, \tag{10}$$

where $\tilde{S}_n = n v_n^{\frac{1}{n}}$, and v_n denotes the volume of the unit ball in $\mathbb{R}^n$.

We next prove

Theorem (2.1.4) [31]: Let Ω be a bounded domain of class C^2 and $1 < p < n$. Then there exists a $\delta_0 = \delta_0(\Omega, p, n)$ such that for all $\delta \in (0, \delta_0]$ there holds:

$$\int_{\Omega_\delta} d^{p-1} |\nabla v|^p dx + \int_{\partial \Omega_\delta^c} |v|^p dS_x \geqslant C(n,p) \left\| d^{\frac{p-1}{p}} v \right\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}, \forall v \in C^\infty(\Omega), \tag{11}$$

with a constant $C(n,p)$ depending only on n and p.

Proof: We will denote by $C(p), C(n,p)$, etc. positive constants, not necessarily the same in each occurrence, which depend only on their arguments. As a first step we will prove the following estimate:

$$C(n,p)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)} \leqslant \int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx + \left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta^c)}. \quad (12)$$

To this end we apply estimate (4) to $w = |v|^s, s = \frac{(n-1)p}{n-p}$ with $a = \frac{(n-1)(p-1)}{n-p} > 0$. Then,

$$S(n,p)\left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}}|v|^{\frac{np}{n-p}}dx\right)^{\frac{n-1}{n}}$$

$$\leqslant s\int_{\Omega_\delta} d^{\frac{(n-1)(p-1)}{n-p}}|v|^{\frac{n(p-1)}{n-p}}|\nabla v|dx + \int_{\partial\Omega_\delta^c} d^{\frac{(n-1)(p-1)}{n-p}}|v|^{\frac{(p-1)p}{n-p}}dS_x.$$

We next estimate the middle term,

$$\int_{\Omega_\delta} d^{\frac{(n-1)(p-1)}{n-p}}|v|^{\frac{n(p-1)}{n-p}}|\nabla v|dx \leqslant \left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}}|v|^{\frac{np}{n-p}}dx\right)^{\frac{p-1}{p}}\left(\int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx\right)^{\frac{1}{p}}$$

$$\leqslant \varepsilon\left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}}|v|^{\frac{np}{n-p}}dx\right)^{\frac{n-1}{n}} + c_\varepsilon\left(\int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx\right)^{\frac{n-1}{n-p}},$$

whence,

$$(S(n,p)-\varepsilon s)\quad\left(\int_{\Omega_\delta} d^{\frac{n(p-1)}{n-p}}|v|^{\frac{np}{n-p}}dx\right)^{\frac{n-1}{n}}$$

$$\leqslant sc_\varepsilon\left(\int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx\right)^{\frac{n-1}{n-p}} + \int_{\partial\Omega_\delta^c} d^{\frac{(n-1)(p-1)}{n-p}}|v|^{\frac{(n-1)p}{n-p}}dS_x.$$

Raising the above estimate to the power $\frac{n-p}{n-1}$ we easily obtain (12).
To prove (11) we need to combine (12) with the following estimate:

$$C(n,p)\left\|d^{\frac{p-1}{p}}v\right\|^p_L \frac{(n-1)p}{n-p}(\partial\Omega_\delta) \leqslant \int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx + \int_{\partial\Omega_\delta^c}|v|^p dS_x. \quad (13)$$

In the rest of the proof we will show (13) We note that the norm in the left-hand side is the critical trace norm of the function $d^{\frac{p-1}{p}}v$. To estimate it we will use the critical trace inequality [34],

$$\| u \|^p_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega_\delta)} \leqslant C(n,p) \| \nabla u \|^p_{L^p(\Omega_\delta)} + M \| u \|^p_{L^p(\Omega_\delta)^p}, \quad (14)$$

where $M = M(n,p,\Omega)$ in general depends on the domain Ω as well. For reasons that we will explain later we will apply this estimate not directly to $d^{\frac{p-1}{p}}v$ but to the function $d^{\frac{p-1}{p}+\theta}v$ with $\theta > 0$ instead. More specifically we have:

$$\left\|d^{\frac{p-1}{p}}v\right\|^p_L = \delta^{-\theta p}\left\|d^{\frac{p-1}{p}+\theta}v\right\|^p_{\frac{(n-p}{n-1)p}}(\partial\Omega_\delta)$$

$$\leqslant \delta^{-\theta p}\left(C(n,p)\left\|\nabla\left(d^{\frac{p-1}{p}+\theta}v\right)\right\|^p_{L^p(\Omega_\delta)} + M\left\|d^{\frac{p-1}{p}+\theta}\nabla v\right\|_{L^p(\Omega_\delta)}\right).$$

Now,

$$\left\|\nabla\left(d^{\frac{p-1}{p}+\theta}v\right)\right\|_{L^p(\Omega_\delta)} \leqslant \left(\frac{p-1}{p}+\theta\right)\left\|d^{-\frac{1}{p}+\theta}v\right\|_{L^p(\Omega_\delta)} + \left\|d^{\frac{p-1}{p}+\theta}\nabla v\right\|_{L^p(\Omega_\delta)},$$

and

$$\left\| d^{\frac{p-1}{p}+\theta} v \right\|_{L^p(\Omega_\delta)} \leqslant \delta \left\| d^{-\frac{1}{p}+\theta} v \right\|_{L^p(\Omega_\delta)} .$$

From the above three estimates we conclude that

$$\left\| d^{\frac{p-1}{p}} v \right\|_{L}^{\frac{(n-1)p}{n-p}} (\partial\Omega_\delta)$$

$$\leqslant C(p)\delta^{-\theta p} \int_{\Omega_\delta} d^{p-1+p\theta} |\nabla v|^p dx + [C(n,p,\theta) + M\delta^p]\delta^{-\theta p} \int_{\Omega_\delta} d^{-1+p\theta} |v|^p dx,$$

whence, by choosing δ sufficiently small,

$$\left\| d^{\frac{p-1}{p}} v \right\|_{\frac{L}{n-p}}^{p} \quad (\partial\Omega_\delta)$$

$$\leqslant C(p)\delta^{-\theta p} \int_{\Omega_\delta} d^{p-1+p\theta} |\nabla v|^p dx$$

$$+C(n,p,\theta)\delta^{-\theta p} \int_{\Omega_\delta} d^{-1+p\theta} |v|^p dx \qquad (15)$$

To continue we will estimate the last term of the right-hand side of (15). Consider the identity:

$$\theta p^{d-1+\theta p} = -d^{\theta p} \Delta d + \operatorname{div}\left(d^{\theta p} \nabla d\right). \qquad (16)$$

We multiply it by $|v|^p$ and integrate by parts over Ω_δ to get:

$$\theta p \int_{\Omega_\delta} p^{d-1+\theta p} |v|^p dx$$

$$= -\int_{\Omega_\delta} d^{\theta p} \Delta d |v|^p dx - \int_{\Omega_\delta} d^{\theta p} |v|^{p-1} \Delta d |v|^p \cdot \nabla |v| dx$$

$$+ \int_{\partial\Omega_\delta^c} d^{\theta p} |v|^{p-1} dS_x.$$

By our assumption (R) we have that $\left| d^{\theta p} \Delta d \right| \leqslant c_0 \delta d^{-1+\theta p}$. On the other hand,

$$\left| p \int_{\Omega_\delta} d^{\theta p} \left| v \right|^{p-1} \Delta d |v|^p \cdot \nabla |v| dx \right| \leqslant p \int_{\Omega_\delta} d^{\theta p} \left| v \right|^{p-1} |\nabla v| dx$$

$$\leqslant p\varepsilon \int_{\Omega_\delta} d^{-1+\theta p} |v|^p dx + p c_\varepsilon \int_{\Omega_\delta} d^{p-1+\theta p} |\nabla v|^p dx.$$

Putting together the last estimates we get:

$$(\theta p - c_0 \delta - p\varepsilon) \int_{\Omega_\delta} d^{-1+\theta p} |v|^p dx \leqslant p c_\varepsilon \int_{\Omega_\delta} d^{p-1+\theta p} |\nabla v|^p dx + \int_{\partial\Omega_\delta^c} d^{\theta p} |v|^p dS_x, \quad (17)$$

whence, choosing δ, ε sufficiently small,

$$C(p,\theta) \int_{\Omega_\delta} d^{-1+\theta p} |v|^p dx \leqslant C(p) \int_{\Omega_\delta} d^{p-1+\theta p} |\nabla v|^p dx + \int_{\partial\Omega_\delta^c} d^{p\theta} |v|^p dS_x. \quad (18)$$

Combining (15) and (18) we obtain:

$$C(n,p,\theta)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{n-p}(\Omega)} \leqslant \delta^{-\theta p}\int_{\Omega_\delta} d^{p-1+p\delta}|\nabla v|^p dx + \delta^{-\theta p}\int_{\partial\Omega_\delta^c} d^{p\theta} \mid$$

$$\leqslant \int_{\partial\Omega_\delta^c} d^{p-1}|\nabla v|^p dx + \int_{\partial\Omega_\delta^c} |v|^p dS_x. \quad (19)$$

By choosing a specific value of θ, e.g., $\theta = 1$, we get (13). We note that estimate (18) fails if $\theta = 0$, and this is the reason for introducing this artificial parameter.

We next have:

Theorem (2.1.5) [31]: Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying (R) and $1 < p < n$. Then there exists a $\delta_0 = \delta_0(c_0, p, n)$ such that for all $\delta \in (0, \delta_0]$ there holds

$$\int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx \geqslant C(n,p,)\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega)}, \forall v \in C_0^\infty(\Omega_\delta), \quad (20)$$

with a constant $C(n,p)$ depending only on n and p.

Proof: One works as in the derivation of (12), using however (9) in the place of (4).

We finally establish the following:

Theorem (2.1.6) [31]: Let $1 < p < n$ and $D = \sup_{x\in\Omega} d(x) < \infty$. We assume that Ω is a domain satisfying both conditions (C) and (R). Then there exists a positive constant $C = C(n,p,c_0 D)$ such that for any $v \in C_0^\infty(\Omega)$,

$$\int_{\Omega_\delta} d^{p-1}\left|\nabla v|^p dx + \int_{\Omega_\delta} (-\Delta d)\right| v|^p dx \geqslant C\left\|d^{\frac{p-1}{p}}v\right\|^p_{L^{\frac{np}{n-p}}(\Omega)}. \quad (21)$$

Proof: We first define suitable cutoff functions supported near the boundary. Let $\alpha(t) \in C^\infty([0,\infty))$ be a nondecreasing function such that $\alpha(t) = 1$ for $t \in [0,1/2), \alpha(t) = 0$ for $t \geqslant 1$ and $|\alpha'(t)| \leqslant C_0$. For δ small we define $\phi_\delta(x) := \alpha\left(\frac{d(x)}{\delta}\right) \in C_0^2(\Omega)$. Note that $\phi_\delta = 1$ on $\Omega_{\delta/2}, \phi_\delta = 0$ on Ω_δ^c and $|\nabla\phi_\delta| = \left|\alpha'\left(\frac{d(x)}{\delta}\right)\right|\frac{|\nabla d(x)|}{\delta} \leqslant \frac{C_0}{\delta}$ with C_0 a universal constant. For $v \in C_0^\infty(\Omega)$ we write $v = \phi_\delta v + (1-\phi_\delta)v$. The function $\phi_\delta v$ is compactly supported in Ω_δ, and by Lemma (2.1.3), we have:

$$S\|d^a\phi_\delta v\|_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \int_\Omega d^a|\nabla(\phi_\delta v)|dx. \quad (22)$$

On the other hand $(1-\phi_\delta)v$ is compactly supported in $\Omega_{\delta/2}^c$ and using (10), we have:

$$C(n)\|d^a(1-\phi_\delta)v\|_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \left(\frac{2D}{\delta}\right)^a \int_\Omega d^a|\nabla(1-\phi_\delta)v|dx. \quad (23)$$

Combining (22) and (23) and using elementary estimates, we obtain the following L^1 estimate:

$$C\left(a,n,\frac{\delta}{D}\right)\|d^a v\|_{L^{\frac{n}{n-1}}(\Omega_\delta)} \leqslant \int_\Omega |d^a\nabla v|dx + \int_{\Omega_\delta\backslash\Omega_{\frac{\delta}{2}}} d^{a-1}|v|dx. \quad (24)$$

We next derive the corresponding $L^p, p > 1$ estimate. To this end we replace v by $|v|^s$ with $s = \frac{p(n-1)}{n-p}$ in (24) to obtain:

$$C\left(a,n,p,\frac{\delta}{D}\right)\left(\int_{\Omega} d^{\frac{an}{n-1}}||v\Big|^{\frac{an}{n-1}} dx\right)^{\frac{n-1}{n}}$$
$$\leqslant s\int_{\Omega} d^{a}|v|^{\frac{n(p-1)}{n-p}}|\nabla v|dx + \int_{\Omega_{\delta}\backslash\Omega_{\delta/2}} d^{a-1}|v|^{1+\frac{n(p-1)}{n-p}}dx.$$

Using Hölders inequality in both terms of the right-hand side of this we get after simplifying,

$$C\left(a,n,p,\frac{\delta}{D}\right)\left(\int_{\Omega} d^{\frac{an}{n-1}}||v\Big|^{\frac{an}{n-1}} dx\right)^{\frac{n-p}{np}}$$
$$\leqslant s\left(\int_{\Omega} d^{\frac{n(p-1)}{n-p}}|\nabla v|^{p}\right)^{\frac{1}{p}} + \left(\int_{\Omega_{\delta}\backslash\Omega_{\delta}} d^{\frac{n(p-1)}{n-p}}|v|^{p}\right)^{\frac{1}{p}} \tag{25}$$

For $a = \frac{(n-1)(p-1)}{n-p} > 0$, this yields:

$$\left(Cn,p,\frac{\delta}{D}\right)\left\|d^{\frac{p-1}{p}}v\right\|^{p}_{L^{\frac{np}{n-p}}(\Omega)} \leqslant \int_{\Omega} d^{p-1}|\nabla v|^{p}dx + \int_{\Omega_{\delta}\backslash\Omega_{\frac{\delta}{2}}} d^{-1}|v|^{p}dx \tag{26}$$

We note that condition (C) has not been used so far and therefore all previous estimates are valid even for general domains.

To complete the proof we will estimate the last term in (26). For $\theta > 0$, we clearly have:

$$\left(\frac{\delta}{2}\right)^{p\theta}\int_{\Omega_{\delta}\backslash\Omega_{\delta/2}} d^{-1}|v|^{p}dx \leqslant \int_{\Omega_{\delta}\backslash\Omega_{\delta/2}} d^{-1+p\theta}|v|^{p}dx$$
$$\leqslant \int_{\Omega} d^{-1+p\theta}|v|^{p}dx. \tag{27}$$

To estimate the last term we work as in (16)-(18). Thus, we start from the identity (16), multiply by $|v|^{p}$ and integrate by parts in Ω. Now there are no boundary terms and also the term containing d is not a lower order term anymore and has to be kept. Notice however that because of condition (C) we have that $-d\Delta \geqslant 0$ in the distributional sense. Without reproducing the details we write the analogue of (18) which is:

$$C(p,\theta)\int_{\Omega} d^{-1+p\theta}|v|^{p}dx \leqslant C(p)\int_{\Omega} d^{p-1+p\theta}|\nabla v|^{p}dx + \int_{\Omega} d^{p\theta}(-\Delta d)|v|^{p}dx. \tag{28}$$

Combining (27) and (28) and recalling that $d \leqslant D$, we get:

$$C(p,\theta)\left(\frac{\delta}{D}\right)^{p\theta}\int_{\Omega_{\delta}\backslash\Omega_{\delta/2}} d^{-1}|v|^{p}dx \leqslant \int_{\Omega_{\delta}\backslash\Omega_{\delta/2}} d^{-1+p\theta}|v|^{p}dx$$
$$\leqslant \int_{\Omega} d^{-1+p\theta}|v|^{p}dx. \tag{29}$$

Choosing e.g., $\theta = 1$ and combining (29) and (26) the result follows. The dependence of the constant C in (21) on the domain Ω enters through the ratio δ/D. By Lemma (2.1.3) (cf. (8)) we obtain that the dependence of C on Ω enters through $c_0 D$. We also note that $C(n,p,\infty) = 0$. Here we will prove various Hardy Sobolev inequalities. Let $d(x) = \text{dist}(x,\partial\Omega)$ and $V \subset \Omega$. For $p > 1$, and $u \in C_0^{\infty}(\Omega)$ we set:

$$I_p[u](V) := \int_V |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_V \frac{|u|^p}{d^p} dx. \quad (30)$$

For simplicity we also write $I_p[u]$ instead of $I_p[u](\Omega)$. We next put:

$$u(x) = d^{\frac{p-1}{p}}(x) v(x). \quad (31)$$

We first prove an auxiliary inequality:

Lemma (2.1.7) [31]: For $p \geqslant 2$, there exists positive constant $c = c(p)$ such that

$$I_p[u](V) \geqslant c(p) \int_V d^{p-1} |\nabla v|^p dx + \left(\frac{p-1}{p}\right)^{p-1} \int_V \nabla d \cdot \nabla |v|^p dx. \quad (32)$$

Proof: We have that

$$\nabla u = \frac{p-1}{p} d^{\frac{p-1}{p}-1} v \nabla d + d^{\frac{p-1}{p}} \nabla v =: a + b.$$

For $p \geqslant 2$ we have that for $a, b \in \mathbb{R}^n$,

$$|a+b|^p - |a|^p \geqslant c(p)|b|^p + p|a|^{p-2} a \cdot b.$$

Using this we obtain:

$$I_p[u](V) \geqslant c(p) \int_V d^{p-1} |\nabla v|^p dx + \left(\frac{p-1}{p}\right)^{p-1} \int_V \nabla d \cdot \nabla |v|^p dx. \quad (33)$$

which is the sought for estimate.

We first establish estimates in Ω_δ.

Theorem (2.1.8) [31]: Let $2 \leqslant p < n$. We assume that Ω is a bounded domain of class C^2. Then, there exists a $\delta_0 = \delta_0(p, n, \Omega)$ such that for $0 < \delta \leqslant \delta_0$ and all $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega_\delta} |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_{\Omega_\delta} \frac{|u|^p}{d^p} dx \geqslant C \int_{\Omega_\delta} \left(|u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{n}}, \quad (34)$$

Where $C = C(n, p) > 0$ depends only on n and p.

Proof: Using Lemma (2.1.7) we have that

$$C(p) I_p[u](\Omega_\delta) \int_{\Omega_\delta} d^{p-1} |\nabla v|^p dx + \int_{\Omega_\delta} \nabla d \cdot \nabla |v|^p dx.$$

Integrating by parts the last term, we get:

$$C(p) I_p[u](\Omega_\delta) \geqslant \int_{\Omega_\delta} d^{p-1} |\nabla v|^p dx + \int_{\Omega_\delta} (-d) |v|^p dx + \int_{\partial \Omega_\delta^c} |v|^p dS_x \cdot (35)$$

We next estimate the middle term of the right-hand side. By condition (R), we have:

$$\left| \int_{\Omega_\delta} (-\Delta d) \left| v \right|^p dx \right| \leqslant c_0 \int_{\Omega_\delta} \left| v \right|^p dx. \quad (36)$$

Starting from the identity $1 + d\Delta d = \operatorname{div}(d \nabla d)$, we multiply it by $|v|^p$ and integrate by parts over Ω_δ to get:

$$\int_{\Omega_\delta} |v|^p dx + \int_{\Omega_\delta} d\Delta d |v|^p dx = -p \int_{\Omega_\delta} d |v|^{p-1} \nabla d \cdot \nabla |v| dx + \delta \int_{\partial \Omega_\delta^c} |u|^p dS.$$

Using once more (R) and standard inequalities we get:

$$(1 - \delta c_0 - \varepsilon p) \int_{\Omega_\delta} |v|^p dx \leqslant \delta p C_\varepsilon \int_{\Omega_\delta} d^{p-1} |\nabla v|^p dx + \delta \int_{\partial \Omega_\delta^c} |u|^p dS,$$

Whence for ε, δ sufficiently small,

$$\int_{\Omega_\delta} |v|^p dx \leqslant C(p) \delta \int_{\Omega_\delta} d^{p-1} |\nabla v|^p dx + C(p) \delta \int_{\partial \Omega_\delta^c} |u|^p dS. \quad (37)$$

Combining (35), (36) and (37) we obtain:

$$C(p)I_p[u](\Omega_\delta) \geqslant \int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx + \int_{\partial\Omega_\delta^c} |v|^p dS_x. \quad (38)$$

To complete the proof we now use Theorem (2.1.4), that is,

$$\int_{\Omega_\delta} d^{p-1}|\nabla v|^p dx + \int_{\partial\Omega_\delta^c} |v|^p dS_x \geqslant C(n,p) \left\| d^{\frac{p-1}{p}} v \right\|^p_{L^{\frac{np}{n-p}}(\Omega_\delta)} \quad (39)$$
$$= C(n,p) \parallel u \parallel^p_{L^{p-p}\ (\Omega_\delta)}.$$

The result then follows from (38) and (39).

Next we prove:

Theorem (2.1.9) [31]: Let $2 \leqslant p < n$. We assume that Ω is a bounded domain of class C^2. Then there exist positive constants $M = M(n,p,\Omega)$ and $C = C(n,p)$ such that for all $u \in C_0^\infty(\Omega)$, there holds:

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx + M \int_\Omega |u|^p dx$$
$$\geqslant C \int_{\Omega_\delta} \left(|u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{n}}. \quad (40)$$

We emphasize that $C(n,p)$ is independent of Ω.

Proof: Clearly we have:

$$I_p[u](\Omega) = I_p[u](\Omega_\delta) + I_p[u](\Omega_\delta^c). \quad (41)$$

By Theorem (2.1.8), for δ small, we have:

$$I_p[u](\Omega_\delta) \geqslant C(n,p) \parallel u \parallel^p_{L^{\frac{np}{n-p}}(\Omega_\delta)}. \quad (42)$$

Since $d(x) \geqslant \delta$ in Ω_δ^c,

$$I_p[u](\Omega_\delta^c) \geqslant \int_{\Omega_\delta^c} |\nabla u|^p dx - \left(\frac{p-1}{p\delta}\right)^p \int_{\Omega_\delta^c} |u|^p dx. \quad (43)$$

Using the Sobolev embedding of $L^{\frac{np}{n-p}}(\Omega_\delta^c)$ into $W^{1,p}(\Omega_\delta^c)$, see [46], we get:

$$\parallel u \parallel^p_{L^{\frac{np}{n-p}}(\Omega_\delta^c)} \leqslant C(n,p) \int_{\Omega_\delta^c} |\nabla u|^p dx + C(n,p,\Omega) \int_{\Omega_\delta^c} |u|^p dx.$$

From this and (43), we get:

$$I_p[u](\Omega_\delta^c) \geqslant C(n,p) \parallel u \parallel^p_{L^{\frac{np}{n-p}}(\Omega_\delta^c)} - C(n,p,\Omega) \int_\Omega |u|^p dx. \quad (44)$$

The result follows from (41), (43) and (44).

We finally show:

Theorem (2.1.10) [31]: Let $2 \leqslant p < n$ and $D = \sup_{x\in\Omega} d(x) < \infty$. We assume that Ω is a domain satisfying both conditions (C) and (R). Then there exists a positive constant $C = C(n,p,c_0 D)$ such that for any $u \in C_0^\infty(\Omega)$ there holds:

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx \geqslant C \left(\int_\Omega |u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{n}}. \quad (45)$$

Proof: Working as in the derivation of (35), we get:

$$C(p)I_p[u](\Omega) \geqslant \int_\Omega d^{p-1}|\nabla v|^p dx + \int_\Omega (-\nabla d)|v|^p dx.$$

The result then follows from Theorem (2.1.6).

Here we will extend the previous inequalities in two directions. First by considering different distant functions and secondly by interpolating between the Sobolev $L^{\frac{pn}{n-p}}$ norm and the L^p norm. This way we will obtain new scale invariant inequalities.
We denote by K a surface embedded in $\mathbb{R}^n$, of codimension $k, 1 < k < n$. We also allow for the extreme cases $k = n$ or 1 , with the following convention. In case $k = n, K$ is identified with the origin, that is $K = \{0\}$, assumed to be in the interior of Ω. In case $k = 1, K$ is identified with $\partial\Omega$.
From now on distance is taken from K, that is, $d(x) = \operatorname{dist}(x, K)$. We also set $K_\delta := \{x \in \Omega: \operatorname{dist}(x, K) \leqslant \delta\}$ is a tubular neighborhood of K, for δ small, and $K_\delta^c := \Omega \setminus K_\delta$.
We say that K satisfies condition (R) whenever there exists a δ^* sufficiently small and a positive constant c_0 such that

$$|d\Delta d + 1 - k| \leqslant c_0 d, \text{ in } K_\delta, \text{ for } 0 < \delta \leqslant \delta^*. \text{ (R)}$$

For $k = 1$ this coincides with condition (R). For $k > 1$, if K is a compact, C^2 surface without boundary, then condition (R) is satisfied; see, e.g., [33] or [50].
We next present an interpolation lemma:
Lemma (2.1.11) [31]: Let a, b, p and q be such that

$$1 \leqslant p < n,\ p < q \leqslant \frac{pn}{n-p},\ \text{ and } b = a - 1 + \frac{q-p}{qp}n. \tag{46}$$

Then for any $\eta > 0$, there holds:

$$\|d^b v\|_{L^q(\Omega)} \leqslant \lambda\eta^{-\frac{1-\lambda}{\lambda}}\|d^a v\|_{L^{\frac{pn}{n-p}}(\Omega)} + (1-\lambda)\|d^{a-1}v\|_{L^p(\Omega)}, \forall v \in C^\infty(\Omega), \tag{47}$$

Where

$$0 < \lambda := \frac{n(q-p)}{qp} \leqslant 1. \tag{48}$$

Proof: For $p_s := \frac{pn}{n-p}$ and λ as in (48) we use Hölder's inequality to obtain:

$$\int_\Omega d^{q^b}|v|^q dx = \int_\Omega \left(d^{a\lambda q}|v|^{\lambda q}\right)\left(d^{q(b-a\lambda)}|v|^{q(1-\lambda)}\right)dx$$
$$\leqslant \left(\int_\Omega d^{ap_s}|v|^{p_s}dx\right)^{\frac{\lambda q}{p_s}}\left(\int_\Omega d^{p(1-\lambda)}|v|^p dx\right)^{\frac{(1-\lambda)q}{p}}$$

that is,

$$\|d^b v\|_{L^q(\Omega)} \leqslant \|d^a v\|^\lambda_{L^{\frac{pn}{n-p}}(\Omega)}\|d^{a-1}v\|^{1-\lambda}_{L^p(\Omega)}.$$

Combining this with Young's inequality,

$$X^\lambda Y^{1-\lambda} \leqslant \lambda\eta^{-\frac{1-\lambda}{\lambda}}X + (1-\lambda)\eta Y, \eta > 0, \tag{49}$$

the result follows.
We first prove inequalities in K_δ.
Lemma (2.1.12) [31]: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and K a C^2 surface of codimension k, satisfying condition (R). We also assume that

$$p = 1 < q \leqslant \frac{n}{n-1}, b = a - 1 + \frac{q-1}{q}n, \text{ and } a \neq 1 - k. \tag{50}$$

Then there exists a $\delta_0 = \delta_0\left(\frac{|a+k-1|}{c_0}\right)$ and $C = C(a,q,n,k) > 0$ such that for all $\delta \in (0, \delta_0]$, there holds:

$$\int_{K_\delta} d^a|\nabla v|dx + \int_{\partial K_\delta} d^a|v|dS_x \geqslant C\|d^b v\|_{L^q(K_\delta)}, \forall v \in C_0^\infty(\Omega \setminus K). \, (51)$$

Proof: Using the interpolation inequality (47) in K_δ with $\eta = 1$, we get:

$$\|d^b v\|_{L^q(K_\delta)} \leqslant \frac{n(q-1)}{q}\|d^a v\|_{L^{N-1}(K_\delta)}^{N} + \frac{q-n(q-1)}{q}\|d^{a-1}v\|_{L^1(K_\delta)}$$

$$\leqslant C(n,q)\left(\|d^a v\|_{L^{\frac{N}{N-1}}(K_\delta)} + \int_{K_\delta} d^{a-1}|v|dx\right). \qquad (52)$$

For $V = K_\delta$ we apply (5) to $u = d^a v, v \in C^\infty(\Omega)$ to get,

$$S_n\|d^a v\|_{L^{\frac{n}{n-1}}(K_\delta)} \leqslant \int_{K_\delta} d^a|\nabla v|dx + |a|\int_{K_\delta} d^{a-1}|v|dx + \int_{\partial K_\delta} d^a|v|dS_x. \, (53)$$

Combining (52) and (53) we get the analogue of (6) which is,

$$C(a,q,n)\|d^b v\|_{L^q(K_\delta)} \leqslant \int_{K_\delta} d^a|\nabla v|dx + \int_{K_\delta} d^{a-1}|v|dx + \int_{\partial K_\delta} d^a|v|dS_x. \, (54)$$

It remains to estimate the middle term of the right-hand side. Noting that $\nabla d \cdot \nabla d = 1$ a.e. and integrating by parts in K_δ, we have:

$$a\int_{K_\delta} d^{a-1}|v|dx = \int_{K_\delta} \nabla d^a \cdot \nabla d|v|dx$$

$$= -\int_{K_\delta} d^a \Delta d|v|dx - \int_{K_\delta} d^a \nabla d \cdot \nabla|v|dx + \int_{\partial K_\delta} d^a|v|dS_x,$$

whence,

$$(a+k-1)\int_{K_\delta} d^{a-1}|v|dx - \int_{K_\delta} d^{a-1}(d\Delta d + 1 - k)|v|dx - \int_{K_\delta} d^a \cdot \nabla d|v|dx$$

$$+ \int_{\partial K_\delta} d^a|v|dS_x.$$

Using (R) we easily arrive at the analogue of (7), that is,

$$(|a+k-1| - c_0\delta)\int_{K_\delta} d^{a-1}|v|dx \leqslant \int_{K_\delta} d^a|\nabla v|dx + \int_{\partial K_\delta} d^a|v|dS_x. \, (55)$$

For estimate (55) to be useful we need $|a+k-1| > 0$, whence the restriction $a = 1-k$. The result then follows from (54) and (55), taking e.g., $\delta_0 = \frac{|a+k-1|}{2c_0}$.

We next present the analogue of Lemma (2.1.3):

Lemma (2.1.13) [31]: Let $\Omega \subset \mathbb{R}^n$ be a domain and K a surface of co-dimension k, satisfying condition (R). We also assume

$$p = 1 < q \leqslant \frac{n}{n-1}, b = a - 1 + \frac{q-1}{q}n, \text{ and } a \neq 1-k.$$

Then, there exists a $\delta_0 = \delta_0\left(\frac{|a+k-1|}{c_0}\right)$ and $aC = C(a,q,n,k) > 0$, such that for all $\delta \in (0, \delta_0\,]$ there holds:

$$\int_{K_\delta} d^a|\nabla v|dx \geqslant C\|d^b v\|_{L^q(K_\delta)}, \forall v \in C_0^\infty(K_\delta). \qquad (56)$$

The proof is quite similar to that of the previous lemma. The only difference is that instead of (5) one uses (10).

We next have:

Theorem (2.1.14) [31]: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition (R). We also assume:

$$1 \leqslant p < n, p < q \leqslant \frac{pn}{n-p}, \text{ and } b = a - 1 + \frac{q-p}{qp} n, \tag{57}$$

and set $a = \frac{p-k}{p}$. Then there exists a $\delta_0 = \delta_0(p, q, \Omega, K)$ and $C = C(p, q, n, k) > 0$ such that for all $\delta \in (0, \delta_0]$ and all $v \in C_0^\infty(\Omega \setminus K)$, there holds:

$$\int_{K_\delta} d^{p-k} |\nabla v|^p dx + \int_{\partial K_\delta} d^{1-k} |v|^p dS_x \geqslant C \|d^b v\|_{L^q(K_\delta)}^p; \tag{58}$$

in particular the constant C is independent of Ω, K.

Proof: We will use Lemma (2.1.12). Since in this lemma the parameters a, b, p, q have a different meaning, to avoid confusion, we will use capital letters for the parameters a, b, p, q appearing in the statement of the present theorem.

That is, we suppose that

$$1 \leqslant P < n, p < Q \leqslant \frac{Pn}{n-P}, \text{ and } B = A - 1 + \frac{Q-P}{QP} n, \tag{59}$$

and for $A = \frac{P-k}{P}$, we will prove that the following estimate holds true,

$$\int_{K_\delta} d^{P-k} |\nabla v|^P dx + \int_{\partial K_\delta} d^{1-k} |v|^P dS_x \geqslant C \|d^B v\|_{L^Q\ (K_\delta)}^P. \tag{60}$$

We will argue in a similar way, as in the proof of Theorem (2.1.4). We first prove the following $L^Q - L^P$ estimate:

$$\begin{aligned} C(P, Q, n, k) \quad & \|d^B v\|_{L^Q(K_\delta)}^P \\ & \leqslant \int_{K_\delta} d^{P-k} |\nabla v|^P dx + \int_{\partial K_\delta} d^{1-k} |v|^P dS_x \\ & + \left\| d^{\frac{P-k}{P}} v \right\|_{L}^P \frac{(n-1)P}{n-P} (K_\delta). \end{aligned} \tag{61}$$

To this end we replace in (51) v by $|v|^s$ with

$$s = Q \frac{P-1}{P} + 1. \tag{62}$$

Also, for A, B, P and Q as in (59), we set:

$$q = Q s^{-1}, b = Bs, a = b + 1 - \frac{q-1}{q} N = BQ \frac{P-1}{P} + A. \tag{63}$$

It is easy to check that a, b, q thus defined satisfy (50). Then, from (51), we have:

$$\|d^B v\|_{L^Q Q^{1+\frac{P-1}{P}Q}(K_\delta)} = \|d^b |v|^s\|_{L^q(K_\delta)} \leqslant Cs \int_{K_\delta} d^a |v|^{s-1} |\nabla v| dx + C \int_{\partial K_\delta} d^a |v|^s dx, \tag{64}$$

with $C = C(a, q, n, k) = C(P, Q, A, n, k)$. Using Hölder's inequality in the middle term of the right-hand side, we get:

$$\int_{K_\delta} d^a |v|^{s-1}|\nabla v\Big|dx = \int_{K_\delta} d^A\Big|\nabla v\Big|d^{BQ\frac{P-1}{P}}\Big|v|^{Q\frac{P-1}{P}}dx$$

$$\leqslant \|d^A|\nabla v|\|_{L^P(K_\delta)}\|d^B v\|_{L^Q(K_\delta)}^{\frac{P-1}{P}Q} \tag{65}$$

$$\leqslant c_\varepsilon \|d^A|\nabla v|\|_{L^P(K_\delta)}^{1+\frac{P-1}{P}Q} + \varepsilon\|d^B v\|^1_{L^Q Q^{1+\frac{P-1}{P}Q}(K_\delta)}.$$

From now on we use the specific value of $A = \frac{P-k}{P}$. For this choice of A a straightforward calculation shows that

$$a - 1 + k = \frac{P-1}{P}\frac{Q-P}{P}(n-k) \neq 0, \tag{66}$$

and therefore it corresponds to an acceptable value of a, see (50). Because of (66) the case $k = n$ is excluded.

We next estimate the last term of (64). Using Hölder's inequality (similarly as in Lemma (2.1.11)), we get:

$$\int_{\partial K_\delta} d^a |v|^s dx = \int_{\partial K_\delta} d^\mu |v|^{\lambda\left(Q\frac{P-1}{P}+1\right)} d^{BQ\frac{P-1}{P}+A-\mu}|v|^{(1-\lambda)\left(Q\frac{P-1}{P}+1\right)}dx$$

$$\leqslant \left(\int_{\partial K_\delta} d^{\frac{(P-k)(n-1)}{n-P}}|v|^{\frac{P(n-1)}{n-P}}dx\right)^{\frac{\lambda(n-P)}{(n-1)P}\left(Q\frac{P-1}{P}+1\right)}\left(\int_{\partial K_\delta} d^{1-k}|v|^P dx\right)^{\frac{1-\lambda}{P}\left(Q\frac{P-1}{P}+1\right)},$$

where,

$$\lambda = \frac{(n-1)(Q-P)}{Q(P-1)+P}, \text{ and } \mu = \frac{(n-1)(Q-P)(P-k)}{P^2}.$$

Using then Young's inequality (cf. (49)) we obtain for a positive constant $C = C(P,Q,n)$,

$$C\int_{\partial K_\delta} d^a|v|^s dx \leqslant \left(\left\|d^{\frac{P-k}{P}}v\right\|_L^{\frac{P(n-1)}{n-P}(\partial K_\delta)} + \left\|d^{\frac{1-k}{P}}v\right\|_{L^P(\partial K_\delta)}\right)^{Q\frac{P-1}{P}+1}. \tag{67}$$

From (64), (65) and (67) we easily obtain (61).

complete the proof of the theorem we will show that

$$C\left\|d^{\frac{P-k}{P}}v\right\|_L^{P(n-1)} \leqslant \int_{K_\delta} d^{P-k}|\nabla v|^P dx + \int_{\partial K_\delta} d^{1-k}|v|^P dS_x, \tag{68}$$

for a positive constant $C = C(P,Q,n,k)$. The proof of (68) parallels that of (13). In particular, for $k = 1$ this is precisely estimate (13). We will sketch the proof of (68).

Applying the critical trace inequality (14) to $d^{\frac{P-k}{P}+\theta}v, \theta > 0$, in the domain K_δ we obtain for δ sufficiently small the analogue of (15), that is

$$\left\|d^{\frac{P-k}{P}}v\right\|_L^P$$

$$\leqslant C(P,k)\delta^{-\theta P}\int_{K_\delta} d^{P-k+P\theta}|\nabla v|^P dx$$

$$+C(n,P,k,\theta)\delta^{-\theta P}\int_{K_\delta} d^{-k+P\theta}|v|^P dx. \tag{69}$$

We next estimate the last term of (69). Starting from the identity,

$$(1-k+\theta P)d^{-k+\theta P}=-d^{1-k+\theta P}\Delta d+\operatorname{div}\left(d^{1-k+\theta P}\nabla d\right), \qquad (70)$$

we multiply it by $|v|^P$ and integrate by parts over K_δ to get:

$$(1-k+\theta P)\int_{K_\delta} d^{-k+\theta P}|v|^P dx$$

$$=-\int_{K_\delta} d^{1-k+\theta P}\Delta d|v|^P dx-P\int_{K_\delta} d^{1-k+\theta P}|v|^{P-1}\nabla d\cdot\nabla|v|dx+\int_{\partial K_\delta} d^{1-k+\theta P}|v|^P dS_x,$$

or, equivalently,

$$\theta P\int_{K_\delta} d^{-k+\theta P}|v|^P dx \quad =\int_{K_\delta} d^{k+\theta P}(d\Delta d+1-k)|v|^P dx$$

$$-P\int_{K_\delta} d^{1-k+\theta P}|v|^{P-1}\nabla d\cdot\nabla|v|dx+\int_{\partial K_\delta} d^{1-k+\theta P}|v|^P dS_x.$$

By our condition (R) we have that $|d\Delta d+1-k|\leqslant c_0 d$. On the other hand,

$$\left|P\int_{K_\delta} d^{1-k+\theta P}\left|v\right|^{P-1}\nabla d\cdot\nabla|v|dx\right|\leqslant P\int_{K_\delta} d^{1-k+\theta P}\left|\nabla v\right|^{P-1}dx$$

$$\leqslant P\varepsilon\int_{K_\delta} d^{1-k+\theta P}|v|^P dx+Pc_\varepsilon\int_{K_\delta} d^{P-k+\theta P}|\nabla v|^P dx.$$

Putting together the last estimates we obtain, for ε,δ small the analogue of (18) that is

$$C(P,\theta)\leqslant\int_{K_\delta} d^{-k+\theta P}|v|^P dx\leqslant C(P)\int_{K_\delta} d^{P-k+\theta P}|\nabla v|^P dx+\int_{\partial K_\delta} d^{-k+\theta P}|v|^P dS_x. \qquad (71)$$

Combining (69), (71) and using the fact that $d(x)\leqslant\delta$ when $x\in K_\delta$, we complete the proof of (68) as well as of the theorem.

Remark (2.1.15) [31]: The choice $a=\frac{p-k}{p}$ corresponds to the Hardy-Sobolev inequality as it will become clear in the next. We note that the corresponding estimate for $a\in\mathbb{R}$ and b,p,q as in (57) remains true. Thus, there exists a positive constant $C=C(a,n,p,q,k)$ such that for all $v\in C_0^\infty(\Omega\setminus K)$ there holds:

$$\int_{K_\delta} d^{ap}|\nabla v|^p dx+\int_{\partial K_\delta} d^{(a-1)p+1}|v|^p dS_x\geqslant C\|d^b v\|_{L^q(\Omega)}. \qquad (72)$$

The proof of (72) in case $a\neq=\frac{p-k}{p}$ is much simpler than in the case $a=\frac{p-k}{p}$. We also note that if $a\neq\frac{p-k}{p}$ then (72) is true even if $k=n$.

We will finally prove the analogue of Theorem (2.1.6).

Theorem (2.1.16) [31]: Let $\Omega\subset\mathbb{R}^n$ be a domain and K a surface of codimension $k, 1\leqslant k<n$, satisfying both conditions (R).

In addition we assume that $D=\sup_{x\in\Omega}d(x)<\infty$, condition (C) is satisfied, and

$$1\leqslant p<n, p<q\leqslant\frac{pn}{n-p}, \text{ and } b=a-1+\frac{q-p}{qp}n. \qquad (73)$$

We set $x\in\Omega$. Then there exists a positive constant $C=C(p,n,\Omega,K)$ such that for all $v\in C_0^\infty(\Omega\setminus K)$ there holds:

$$\int_\Omega d^{p-k}|\nabla v|^p dx+\left|\int_\Omega d^{-k}(-d\Delta d-1+k)\left|v\right|^p dx\right|\geqslant C\|d^b v\|_{L^q(\Omega)}^p. \qquad (74)$$

Proof: As before, to avoid confusion in the proof, we will use capital letters for the parameters a,b,p,q appearing in the statement of the Theorem. That is, we suppose that

$$1 \leqslant P < n, p < Q \leqslant \frac{Pn}{n-P}, \text{ and } B = A - 1 + \frac{Q-P}{QP} n,$$

and for $A = \frac{P-k}{P}$, we will prove that

$$\int_\Omega d^{P-k} |\nabla v|^P dx + \left| \iint_\Omega d^{-k}(-d\Delta d - 1 + k) \right| v \Big|^P dx \mid \geqslant C \|d^B v\|_{L^Q(\Omega)}. \tag{75}$$

Let $\alpha(t) \in C^\infty([0,\infty))$ be the nondecreasing function defined at the beginning of the proof of Theorem (2.1.5) and $\phi_\delta(x) := \alpha\left(\frac{d(x)}{\delta}\right) \in C_0^2(\Omega)$, so that $\phi_\delta = 1$ on $K_{\delta/2}, \phi_\delta = 0$ on K_δ^c and $|\nabla \phi_\delta| \leqslant \frac{C_0}{\delta}$ with C_0 a universal constant.

For $v \in C_0^\infty(\Omega)$ we write $v = \phi_\delta v + (1 - \phi_\delta)v$. The function $\phi_\delta v$ is compactly supported in K_δ, and by Lemma (2.1.13), we have:

$$C(a,n,q)\|d^B v\|_{L^q(K_\delta)} \leqslant \int_{K_\delta} d^a |\nabla v| dx. \tag{76}$$

On the other hand $(1 - \phi_\delta)v$ is compactly supported in $K_{\delta/2}^c$ and using (10) we easily get

$$\|d^b(1-\phi_\delta)v\|_{L^q\left(K^c_{\frac{\delta}{2}}\right)} \leqslant C(\Omega) \frac{D^{|b|}}{\delta^{|a|}} \|d^a \mid \nabla\big((1-\phi_\delta)v\big)\|_{L^1\left(K^c_{\frac{\delta}{2}}\right)}. \tag{77}$$

Combining (76) and (77) we obtain the analogue of (24) which is

$$C\|d^a v\|_{L^{\frac{n}{n-1}}(\Omega)} \leqslant \int_\Omega |d^a \nabla v| dx + \int_{K_\delta \backslash K_{\frac{\delta}{2}}} d^{a-1}|v| dx. \tag{78}$$

We next pass to $L^Q - L^P$ estimates. We replace in (78) v by $|v|^s$ with s as in (62). Also, for $A = \frac{P-k}{P}$ and B, P, Q as in (63), we get (cf. (64)):

$$C\|d^B v\|_{L^Q(K_\delta)}^{1+\frac{P-1}{P}Q} \leqslant s \int_{K_\delta} d^a |v|^{s-1} |\nabla v| dx + \int_{K_\delta \backslash K_{\frac{\delta}{2}}} d^{a-1} |v|^s dx. \tag{79}$$

Using Hölder's inequality in both terms of the right-hand side we get

$$\int_{K_\delta} d^a |v|^{S-1} |\nabla v| dx = \int_\Omega d^A |\nabla v| d^{BQ\frac{P-1}{P}} |v|^{Q\frac{P-1}{P}} dx$$

$$\leqslant \|d^A |\nabla v|\|_{L^P(\Omega)} \|d^B v\|_{L^Q(\Omega)}^{\frac{P-1}{P}Q},$$

and

$$\int_{K_\delta \backslash K_{\delta/2}} d^{a-1} |v|^S dx = \int_{K_\delta \backslash K_{\delta/2}} d^A |v| d^{BQ\frac{P-1}{P}} |v|^{Q\frac{P-1}{P}} dx$$

$$\leqslant \|d^{A-1}|v|\|_{L^P(K_\delta \backslash K_{\delta/2})} \|d^B v\|_{L^Q(\Omega)}^{\frac{P-1}{P}Q}.$$

Substituting into (79) we get after simplifying,

$$C\|d^B v\|_{L^Q(\Omega)}^P \leqslant \int_\Omega d^{P-k} |\nabla v|^P dx + \int_{K_\delta \backslash K_{\frac{\delta}{2}}} d^{-k} |v|^P dx. \tag{80}$$

Here we have also used the specific value of $A = \frac{P-k}{P}$. To conclude we need to estimate the last term in (80). For $\theta > 0$, we clearly have:

$$\left(\frac{\delta}{2}\right)^{p\theta}\int_{K_\delta\setminus K_{\delta/2}} d^{-k}|v|^P dx \leqslant \int_{K_\delta\setminus K_{\delta/2}} d^{-k+P\theta}|v|^P dx \leqslant \int_\Omega d^{-k+P\theta}|v|^P dx. \tag{81}$$

To estimate the last term we work as in (27)-(28) (see also (70)–(71)) to finally get

$$\int_\Omega d^{-k+P\theta}|v|^P dx \leqslant C(p)\int_\Omega d^{P-k+P\theta}|\nabla v|^P dx + \left|\int_\Omega d^{-k+P\theta}(-d\Delta d+1-k)\left|v\right|^P dx\right|. \tag{82}$$

We not that we also used the fact that

$$p \neq k, \text{ and } (p-k)(d\Delta d+1-k) \leqslant 0, \text{ on } \Omega\setminus K, \tag{83}$$

which is a direct consequence of condition (C); see [35]. Combining (81) and (82) and recalling that $d \leqslant D$, we get:

$$C\left(P,\theta,\frac{\delta}{D}\right)\int_{K_\delta\setminus K_{\delta/2}} d^{-k}|v|^P dx \leqslant \int_\Omega d^{P-k}|\nabla v|^P dx + \left|\int_\Omega d^{-k}(-d\Delta d+1-k)\left|v\right|^P dx\right|, \tag{84}$$

and the result follows easily.

Remark (2.1.17) [31]: In case $a \neq \frac{p-k}{p}$ the analogue of (74) remains true. That is, for b, p, q as in (73),

$$\int_\Omega d^{ap}|\nabla v|^p dx + \left|\int_\Omega d^{(a-1)p}(-d\Delta d-1+k)\left|v\right|^p dx\right| \geqslant C\|d^b v\|_{L^q(\Omega)}, \tag{85}$$

for a constant $C = C(p,q,n,k,a) > 0$. The case $k = n$ is not excluded. we will use the v-inequalities of the previous to prove new Hardy-Sobolev inequalities. For $V \subset \mathbb{R}^n$ we set:

$$I_{p,k}[u](V) : \int_V |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \frac{|u|^p}{d^p} dx. \tag{86}$$

Then for $u(x) = d^H(x)v(x)$ with

$$H := \frac{p-k}{p},$$

we have for $p \geqslant 2$,

$$I_{p,k}[u](V) \geqslant c(p)\int_V d^{p-k}|\nabla v|^p dx + H|H|^{p-2}\int_V d^{1-k}\nabla d\cdot\nabla|v|^p dx. \tag{87}$$

The proof of (87) is quite similar to the proof of (32).

As in the previous,

$$1 \leqslant p < n,\ p < q \leqslant \frac{pn}{n-p}, \text{ and } b = a-1+\frac{q-p}{qp}n. \tag{88}$$

We will be interested in the specific value $a = \frac{p-k}{p}$ which corresponds to the critical Hardy Sobolev inequalities.

We first present estimates in K_δ.

Theorem (2.1.18) [31]: Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition (R). Then, there exist positive constants $C = C(n,k,p,q)$ and $\delta_0 = \delta_0(p,n,\Omega,K)$ such that for $0 < \delta \leqslant \delta_0$ and $u \in C_0^\infty(\Omega \setminus K)$ we have:
(a) If $p > k$ then

$$\int_{K_\delta} |\nabla u|^p dx - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} dx \geqslant C \left(d^{-q+\frac{q-p}{p}n} |u|^q dx \right)^{\frac{p}{q}}. \tag{89}$$

(b) If $p < k$, the Hardy inequality,

$$\int_{K_\delta} |\nabla u|^p dx - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} dx \geqslant 0, \tag{90}$$

in general fails. However, there exists a positive constant M such that

$$\begin{aligned} &\int_{K_\delta} |\nabla u|^p dx - |H|^p \int_{K_\delta} \frac{|u|^p}{d^p} dx + M \int_{K_\delta} |u|^p dx \\ &\geqslant C \left(d^{-q+\frac{q-p}{p}n} |u|^q dx \right)^{\frac{p}{q}}. \end{aligned} \tag{91}$$

We emphasize that $C = C(n,k,p,q) > 0$ is independent of Ω, K.
(c) If in addition, u is supported in K_δ, that is $u \in C_0^\infty(K_\delta \setminus K)$ then, (89) holds true even for $p < k$.

Proof: Using (86) and integrating by parts once we have that

$$\begin{aligned} I_{p,k}[u](K_\delta) \geqslant C(p) \int_{K_\delta} d^{p-k} |\nabla v|^p dx + H|H|^{p-2} \int_{K_\delta} d^{-k}(-d\Delta d + k - 1)|v|^{\mathrm{p}} dx \\ +H \left| H \right|^{p-2} \int_{\partial K_\delta} d^{1-k} \left| v \right|^p dS_x. \end{aligned} \tag{92}$$

At first we estimate the middle term of the right-hand side. We have that

$$|d\Delta d + 1 - k| \leqslant c_0 d, \text{ for } x \in K_\delta, \tag{93}$$

and therefore

$$\left| \int_{K_\delta} d^{-k}(-d\Delta d + k - 1) \left| v \right|^p dx \right| \leqslant c_0 \int_{K_\delta} d^{1-k} \left| v \right|^p dx. \tag{94}$$

At this point we will derive some general estimates that we will use in the sequel. Our goal is to prove (96) and (97) below. For $a \in \mathbb{R}$ we consider the identity $(1+a)d^a + d^{1+a}\Delta d = \operatorname{div}(d^{1+a}\nabla d)$. Multiply by $|v|^p$ and integrate by parts to get:

$$\begin{aligned} (a+1) \int_{K_\delta} d^a |v|^p dx + \int_{K_\delta} d^{a+1} \Delta d |v|^P dx \\ = -p \int_{K_\delta} d^{a+1} \nabla d \cdot \nabla |v| |v|^{p-1} dx + \int_{\partial K_\delta} d^{a+1} |v|^p dS_x, \end{aligned}$$

or, equivalently,

$$\begin{aligned} (a+1) \int_{K_\delta} d^a |v|^p dx + \int_{K_\delta} d^a (d\Delta d + 1 - k) |v|^p dx \\ = \quad -p \int_{K_\delta} d^{a+1} \nabla d \cdot \nabla |v| |v|^{p-1} dx + \int_{\partial K_\delta} d^{a+1} |v|^p dS_x. \end{aligned} \tag{95}$$

We next estimate the first term of the right-hand side of (95),

$$p\int_{K_\delta} d^{a+1}\nabla d\cdot\nabla|v||v|^{p-1}dx \leqslant \left(\int_{K_\delta} d^a|v|^p dx\right)^{\frac{p-1}{p}}\left(\int_{K_\delta} d^{a+p}|\nabla v|^p dx\right)^{\frac{1}{p}}$$
$$\leqslant \varepsilon(p-1)\int_{K_\delta} d^a|v|^p dx + \varepsilon^{-(p-1)}\int_{K_\delta} d^{a+p}|\nabla v|^p dx.$$

From this, (93) and (95) we easily obtain the following two estimates:

$$(|a+k| - c_0\delta - \varepsilon(p-1))\int_{K_\delta} d^a|v|^p dx$$
$$\leqslant \varepsilon^{-(p-1)}\int_{K_\delta} d^{a+p}|\nabla v|^p dx + \int_{\partial K_\delta} d^{a+1}|v|^p dS_x, \tag{96}$$

and

$$\int_{\partial K_\delta} d^{a+1}|v|^p \quad dS_x$$
$$\leqslant \varepsilon^{-(p-1)}\int_{K_\delta} d^{a+p}|\nabla v|^p dx$$
$$+(|a+k| + c_0\delta + \varepsilon(p-1))\int_{K_\delta} d^a|v|^p dx. \tag{97}$$

From (96) taking $a = 1-k$ we get that

$$\int_{K_\delta} d^{1-k}|v|^p dx \leqslant C(p)^\delta\int_{K_\delta} d^{p-k}|\nabla v|^p dx + C(p)^\delta\int_{\partial K_\delta} d^{1-k}|u|^p dS_x. \tag{98}$$

At this point we distinguish two cases according to whether $p > k$ or $p < k$. Assume first that $p > k$, or equivalently, $H > 0$. Then from (92) and (98) we get that

$$I_{p,k}[u](K_\delta) \geqslant C(p)\int_{K_\delta} d^{p-k}|\nabla v|^p dx + C(p,k)\int_{\partial K_\delta} d^{1-k}|v|^p dS_x. \tag{99}$$

Using Theorem (2.1.14) as well as the fact that

$$\|d^b v\|^p_{L^q(K_\delta)} = \left(\int_{K_\delta} d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}},$$

we easily obtain (89).

If $u \in C_0^\infty(K_\delta \setminus K)$ then the boundary terms in (92) and (98) are absent and the same argument yields (89) even if $p < k$.

Suppose now that $p < k$, that is, $H < 0$. Using again (92) and (98) we get that

$$I_{p,k}[u](K_\delta) \geqslant C(p)\int_{K_\delta} d^{p-k}|\nabla v|^p dx - C(p,k)\int_{\partial K_\delta} d^{1-k}|v|^p dS_x. \tag{100}$$

To estimate the last term of this we will use (97) with $a = p-k$ in the following way,

$$\int_{\partial K_\delta} d^{1-k}|v|^p dS_x = \delta^{-p}\int_{\partial K_\delta} d^{1+p-k}|v|^p dS_x$$
$$\leqslant \varepsilon^{-(p-1)}\int_{K_\delta} d^{p-k}|\nabla v|^p dx \tag{101}$$
$$+C(\varepsilon,p)\delta^{-p}\int_{\partial K_\delta} d^{p-k}|v|^p dx.$$

From (100) and (101) choosing ε big we get:

$$I_{p,k}[u](K_\delta) \geqslant C(p)\int_{K_\delta} d^{p-k}|\nabla v|^p dx - M\int_{K_\delta} d^{p-k}|v|^p dx. \qquad (102)$$

On the other hand from (101) and Theorem (2.1.14) we get that

$$C(p,q,n,k)\|d^b v\|^p_{L^q(K_\delta)} \leqslant C(p)\int_{K_\delta} d^{p-k}|\nabla v|^p dx + M\int_{K_\delta} d^{p-k}|v|^p dx. \qquad (103)$$

From (102) and (103) we easily conclude (91).

It remains to explain why when $p < k$ and $u \in C_0^\infty(\Omega \setminus K)$ the simple Hardy (90) in general fails. Let us consider the case where K and therefore K_δ are strictly contained in Ω. In this case the function $u_\varepsilon = d^{H+\varepsilon}$, for $\varepsilon > 0$ is in $W^{1,p}(K_\delta)$. On the other hand for $p < k$ a simple density argument shows that $W^{1,p}(K_\delta \setminus K) = W^{1,p}(K_\delta)$. An easy calculation shows that

$$\int_{K_\delta} |\nabla u_\varepsilon|^p dx \quad -|H|^p\int_{K_\delta} \frac{|u_\varepsilon|^p}{d^p} dx = (|H+\varepsilon|^p - |H|^p)\int_{K_\delta} d^{-k-p\varepsilon} dx \qquad (104)$$
$$< 0,$$

by taking $\varepsilon > 0$ small and noting that $H < 0$.

Theorem (2.1.19) [31]: Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and K a C^2 surface of co-dimension k, with $1 \leqslant k < n$, satisfying condition (R). Then, there exist positive constants $C = C(n,k,p,q)$ and M such that for all $u \in C_0^\infty(\Omega \setminus K)$, there holds:

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx + M\int_\Omega |u|^p dx$$
$$\geqslant C\left(\int_\Omega d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}}. \qquad (105)$$

We note that $C(n,k,p,q)$ is independent of Ω, K.

Proof: Clearly we have:

$$I_{p,k}[u](\Omega) = I_{p,k}[u](K_\delta) + I_{p,k}[u](K_\delta^c). \qquad (106)$$

By Theorem (2.1.18) for δ small, we have:

$$I_{p,k}[u](K_\delta) \geqslant \quad C(n,k,p,q)\left(\int_{k_\delta} d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}}$$
$$-M\int_{K_\delta} |u|^p dx. \qquad (107)$$

Since $d(x) \geqslant \delta$ in K_δ^c,

$$I_{p,k}[u](K_\delta^c) \geqslant \int_{K_\delta^c} |\nabla u|^p dx - C(p,k,\delta)\int_{K_\delta^c} |u|^p dx. \qquad (108)$$

From the Sobolev embedding of $L^{\frac{np}{n-p}}(K_\delta^c)$ into $W^{1,p}(K_\delta^c)$ we get:

$$\| u \|^p_{L^{\frac{np}{n-p}}(K_\delta^c)} \leqslant C(p,n) \geqslant \int_{K_\delta^c} |\nabla u|^p dx + C(p,n,\Omega,K) \geqslant \int_{K_\delta^c} |u|^p dx.$$

Using the interpolation Lemma (2.1.11) (with $a = 0$) we have:

$$C(n,p,q)\left(\int_{K_\delta^c} d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}} \leqslant \quad \| u \|^p_{L^{\frac{np}{n-p}}(K_\delta^c)} + \|d^{-1}u\|^p_{L^p(K_\delta^c)} \tag{109}$$
$$\leqslant \| u \|^p_{L^{\frac{np}{n-p}}(K_\delta^c)} + \delta^{-p} \| u \|^p_{L^p(K_\delta^c)}.$$

From (108)-(109) we get for $M = M(n,p,q,\Omega,K)$,

$$I_{p,k}[u](K_\delta^c) \geqslant C(n,p,q)\left(\int_{K_\delta^c} d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}} - M\int_{K_\delta^c} |u|^p dx. \tag{110}$$

The result follows from (106), (107) and (110).

Our final result reads:

Theorem (2.1.20) [31]: Let $2 \leqslant p < n$ and $p < q \leqslant \frac{np}{n-p}$. We assume that $\Omega \subset \mathbb{R}^n$ is a domain and K a surface of co-dimension k, $1 \leqslant k < n$, satisfying condition (R). In addition we assume that $D = \sup_{x\in\Omega} d(x) < \infty$ and condition (C) is satisfied. Then for all $u \in C_0^\infty(\Omega)$ there holds

$$\int_\Omega |\nabla u|^p dx - \left|\frac{p-k}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p} dx \geqslant C\left(\int_\Omega d^{-q+\frac{q-p}{p}n}|u|^q dx\right)^{\frac{p}{q}}, \tag{111}$$

for $C = C(n,P,Q,\Omega,K) > 0$.

Proof: Working as in the derivation of (92) we get:

$$C(p,k)I_{p,k}[u](\Omega) \geqslant \int_\Omega d^{p-k}|\nabla v|^p dx + H\int_\Omega d^{-k}(-d\Delta d + 1 - k)|v|^p dx. \tag{112}$$

Because of condition (C) we have that $H(-d\Delta d + 1 - k) \geqslant 0$, see (83). The result then follows from Theorem (2.1.16).

Section (2.2): The Equivalence Between Pointwise Hardy Inequalities

For $\Omega \subsetneq \mathbb{R}^n$ be a domain and let $u \in C_0^\infty(\Omega)$. The inequality

$$|u(x)| \leq C d(x,\partial\Omega)\left(M_{2d(x,\partial\Omega)}|\nabla u|^p(x)\right)^{\frac{1}{p}}, x \in \Omega, \tag{113}$$

Where $1 \leq p < \infty$ and M_R is the restricted Hardy-Littlewood maximal operator, can be viewed as a point wise variant of the classical p-Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{d(x,\partial\Omega)^p} dx \leq C\int_\Omega |\nabla u(x)|^p dx. \tag{114}$$

We say that the domain Ω admits the point wise p-Hardy inequality, if there exists a constant $C > 0$ such that inequality (113) holds for all $u \in C_0^\infty(\Omega)$ at every $x \in \Omega$. As our main result, we prove the following characterization for such domains:

Theorem (2.2.1)[54]: Let $1 \leq p < \infty$. A domain $\Omega \subset \mathbb{R}^n$ admits the point wise p-Hardy inequality if and only if the complement of Ω is uniformly p-fat.

Uniform p-fatness is a density condition for the (variational) p-capacity

$$\mathrm{cap}_p(E,\Omega) = \inf\left\{\int_\Omega |\nabla u|^p dx : u \in C_0^\infty(\Omega), u \geq 1 \text{ on } E\right\}, \tag{115}$$

Where $E \subset \mathbb{R}^n$ is a compact subset of an open set $\Omega \subset \mathbb{R}^n$. More precisely, a closed set $E \subset \mathbb{R}^n$ is said to be uniformly p-fat, if there exists a constant $c_0 > 0$ such that

$$\mathrm{cap}_p(E \cap \bar{B}(x,r), B(x,2r)) \geq c_0 \mathrm{cap}_p(\bar{B}(x,r), B(x,2r))$$

for all $x \in E$ and every $r > 0$.

The origins of Hardy inequalities lie in the one-dimensional considerations by Hardy, see [69]. In $\mathbb{R}^n$, for $n \geq 2$, Hardytype inequalities first appeared of Necas [85] of Lipschitz

domains. However, it has been well-known since the works of Ancona [57] ($p = 2$), Lewis [83], and Wannebo [88], that the regularity of the boundary is not essential for Hardy inequalities, as it was shown that uniform p-fatness of the complement suffices for a domain to admit the integral p-Hardy inequality (114). Uniform n-fatness of the complement is also necessary for the n-Hardy inequality (in $\mathbb{R}^n$), see [57], [83], but this is not true for $p < n$.
Point wise Hardy inequalities were introduced by Hajłasz [66] and Kinnunen and Martio [78]. In these works it was shown that uniform p-fatness of the complement guarantees that the domain admits even the point wise p-Hardy inequality; this is the sufficiency part of Theorem (2.2.1).
Using the boundedness of the Hardy-Littlewood maximal operator it is easy to see that a point wise q-Hardy inequality for some $q < p$ implies the p-Hardy inequality (114). This method does not work if we start with a point wise p-Hardy inequality, as only weak type estimates are available when the exponent is not allowed to increase.
Indeed, it has been an open question since the first appearance of pointwise Hardy inequalities whether the point wise p-Hardy inequality implies the integral p-Hardy inequality with the same exponent.
Now, by a remarkable result of Lewis [83], uniform p-fatness has the following self-improvement property: If $1 < p < \infty$ and a set $E \subset \mathbb{R}^n$ is uniformly $p -$ fat, then E is also uniformly $q -$ fat for some $1 < q < p$. Thus Theorem (2.2.1) has the striking consequence that point wise p-Hardy inequalities, for $1 < p < \infty$, enjoy this same property. In particular, we obtain a positive answer to the above question:
Corollary (2.2.2) [54]: Let $1 < p < \infty$. If a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise $p -$ Hardy inequality, then Ω admits the integral p-Hardy inequality.
In fact, by using the approach of Wannebo, we obtain for Corollary (2.2.2) another proof in which we avoid the use of the rather deep self-improvement of uniform fatness. In addition, we establish a further equivalence between the conditions of Theorem (2.2.1) and certain Poincaré type boundary conditions, see Theorem (2.2.4). Notice also the inclusion of the case $p = 1$ in Theorem (2.2.1). On the contrary, the usual 1-Hardy inequality does not hold even in smooth domains.
We remark that it was recently shown in [82] that a domain $\Omega \subset \mathbb{R}^n$ admits a point wise $q -$ Hardy inequality for some $1 < q < p$ if and only if the complement of Ω is uniformly p-fat (note here the difference between our terminology and that of [82]). This result is nevertheless significantly weaker than Theorem (2.2.1), as the crucial end-point $q = p$ is not reached.
The second purpose is to generalize parts of the existing theory of Euclidean Hardy inequalities to the setting of metric measure spaces. As a part of this scheme we also state and prove Theorem (2.2.1) in this more general setting. The relevant parts of the analysis in metric spaces, as well as the exact formulations of our main results, can be found we prove that uniform p-fatness of the complement implies the point wise p-Hardy inequality also in metric spaces. The necessity part of Theorem (2.2.1) is then obtained 5 contains a transparent proof for the fact that uniform p-fatness of the complement (and thus also the point wise p-Hardy inequality) is sufficient for Ω to admit the usual integral version of the p-Hardy inequality. We give further generalizations of the results from [82] to metric spaces by linking point wise Hardy inequalities and uniform fatness to certain Hausdorff content density conditions. In the special case of Carnot-Carathéodory spaces similar generalizations were recently obtained in [64].

Different aspects of Hardy inequalities in the metric setting have also been studied in [60], [75], [80], [81].
We recall some relevant definitions related to analysis on general metric spaces, see [56] and the survey article [67] of Heinonen for more details.
We assume that $X = (X, d, \mu)$ is a complete metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X: d(y, x) < r\}$. For $0 < t < \infty$, we write $tB = B(x, tr)$, and $\bar{B}$ is the corresponding closed ball. We assume that μ is doubling, which means that there is a constant $c_D \geq 1$, called the doubling constant of μ, such that

$$\mu(2B) \leq c_D \mu(B)$$

for all balls B of X. Note that the doubling condition together with completeness implies that the space is proper, that is, closed balls of X are compact.
The doubling condition gives an upper bound for the dimension of X. By this we mean that there is a constant $C = C(c_D) > 0$ such that, for $s = \log_2 c_D$,

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C \left(\frac{r}{R}\right)^s \tag{116}$$

Whenever $0 < r \leq R < \operatorname{diam} X$ and $y \in B(x, R)$. Inequality (116) may hold for some smaller exponents than $\log_2 c_D$, too. In such cases we let s denote the infimum of the exponents for which (116) holds and say that s is the doubling dimension of X.
When $\Omega \subset \mathbb{R}^n$, we obtain, by the density of smooth functions in the Sobolev space $W_0^{1,p}(\Omega)$, that the Hardy inequality (114) holds for all $u \in W_0^{1,p}(\Omega)$ if it holds for all smooth functions $\varphi \in C_0^\infty(\Omega)$.
General metric spaces lack the notion of smooth functions, but there exists a natural counterpart of Sobolev spaces, defined by Shanmugalingam [86] and based on the use of upper gradients. We say that a Borel function $g \geq 0$ is an upper gradient of a function u on an open set $\Omega \subset X$, if for all curves γ joining points x and y in Ω we have

$$|u(x) - u(y)| \leq \int_\gamma g ds, \tag{117}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma^\gamma g ds = \infty$ otherwise. By a curve we mean a nonconstant, rectifiable, continuous mapping from a compact interval to X.
If $g \geq 0$ is a measurable function and (117) only fails for a curve family with zero p-modulus, then g is a p-weak upper gradient of u on Ω. For the p-modulus on metric measure spaces and the properties of upper gradients, see for example [65],[70],[73],[86], [87]. We use the notation g_u for a p-weak upper gradient of u. The Sobolev space $N^{1,p}(\Omega)$ consists of those functions $u \in L^p(\Omega)$ that have a p-weak upper gradient $g_u \in L^p(\Omega)$ in Ω. The space $N^{1,p}(\Omega)$ is a Banach space with the norm

$$\| u \|_{N^{1,p}(\Omega)} = \left(\int_\Omega |u|^p d\mu + \inf_g \int_\Omega |g|^p d\mu \right)^{1/p},$$

where the infimum is taken over all p-weak upper gradients $g \in L^p(\Omega)$ of u. In the Euclidean space with the Lebesgue measure, $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ for all domains $\Omega \subset \mathbb{R}^n$ and $g_u = |\nabla u|$ is a minimal upper gradient of u.
For a measurable set $E \subset X$, the Sobolev space with zero boundary values is

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ in } X \setminus E\}.$$

By [87], also the space $N_0^{1,p}(E)$, equipped with the norm inherited from $N^{1,p}(X)$, is a Banach space. Note that often the definition of $N_0^{1,p}(\Omega)$ is given so that the functions are only required to vanish in $X \setminus E$ outside a set of zero p-capacity. However, our definition gives the same space because functions in $N^{1,p}(X)$ are p-quasicontinuous by [58].
In order to be able to develop the basic machinery of analysis in the metric space X, we need to assume, in addition to the doubling condition, that the geometry of X is rich enough. In practice, this means that there must exist sufficiently many rectifiable curves everywhere in X. This requirement is in a sense quantified by assuming that the space X supports a (weak) $(1,p)$-Poincaré inequality. That is, we assume that there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B \subset X$, all locally integrable functions u and for all p-weak upper gradients g_u of u, we have

$$\int_B |u - u_B| d\mu \leq c_P r \left(\int_{\tau B} g_u^p d\mu \right)^{\frac{1}{p}},$$

Where

$$u_B = \int_B u d\mu = \mu(B)^{-1} \int_B u d\mu$$

is the integral average of u over B.
Standard examples of doubling metric spaces supporting Poincaré inequalities include (weighted) Euclidean spaces, compact Riemannian manifolds, metric graphs, and Carnot-Carathéodory spaces.
See [65].
Let $\Omega \subset X$ be an open set and let $E \subset \Omega$. The p-capacity of E with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf \int_\Omega g_u^p d\mu,$$

where the infimum is taken over all functions $u \in N_0^{1,p}(\Omega)$ such that $u|_E = 1$.
If there are no such functions u, then $\operatorname{cap}_p(E, \Omega) = \infty$. Since the norm of an upper gradient does not increase under truncation, we may assume that $0 \leq u \leq 1$. Note also that because functions in $N^{1,p}(X)$ are p-quasicontinuous by [58], our definition of p-capacity agrees with the classical definition where admissible functions are required to satisfy $u = 1$ in a neighborhood of E. Furthermore, if $E \subset \mathbb{R}^n$ is a compact set, then the above definition agrees with the definition in (115) as well.
There exists a constant $C > 0$ such that the following comparison between the p-capacity and measure holds for each $1 \leq p < \infty$: For all balls $B = B(x, r)$ with $0 < r < (1/6)$ diam X and for each $E \subset B$

$$\frac{\mu(E)}{Cr^p} \leq \operatorname{cap}_p(E, 2B) \leq \frac{C\mu(B)}{r^p}. \tag{118}$$

The lower bound can be obtained by considering $(1,p)$-Poincaré inequality for all admissible functions $0 \leq u \leq 1$ for the capacity $\operatorname{cap}_p(E, 2B)$ in the ball $3B$. For more details, see for example [59].
We say that a set $E \subset X$ is (uniformly) $p-$fat, $1 \leq p < \infty$, if there exists a constant $c_0 > 0$ such that

$$\operatorname{cap}_p(E \cap \bar{B}(x,r), \bar{B}(x, 2r)) \geq c_0 \operatorname{cap}_p(\bar{B}(x,r), B(x, 2r)) \tag{119}$$

for all $x \in E$ and all $0 < r < (1/6)\operatorname{diam} X$. Notice that by the double inequality (118), cap $_p(B(x,r), B(x, 2r))$ is always comparable to $\mu(B) r^{-p}$. There are many natural examples of uniformly p-fat sets. For instance, all nonempty subsets of X are uniformly p-fat for all $p > s$, where s is the doubling dimension of X. Also complements of simply connected subdomains of $\mathbb{R}^2$ and sets satisfying measure density condition

$$\mu(B(x,r) \cap E) \geq C\mu(B(x,r)) \text{ for all } x \in E, r > 0,$$

are uniformly p-fat for all $1 \leq p < \infty$. The p-fatness condition is stronger than the Wiener criterion and it is important for example in the study of boundary regularity of A-harmonic functions, see [72].

As mentioned, uniform fatness is closely related to pointwise Hardy inequalities.

Definition (2.2.3) [54]: Let $1 \leq p < \infty$. An open set $\Omega \subsetneq X$ admits the pointwise p-Hardy inequality if there exist constants $c_H > 0$ and $L \geq 1$ such that, for all $u \in N_0^{1,p}(\Omega)$,

$$|u(x)| \leq c_H d_\Omega(x) \left(M_{Ld_\Omega(x)} g_u^p(x)\right)^{\frac{1}{p}} \tag{120}$$

holds at almost every $x \in \Omega$.

Above

$$M_R u(x) = \sup_{0<r\leq R} \int_{B(x,r)} |u| d\mu$$

is the restricted Hardy-Littlewood maximal function of a locally integrable function u.

By the maximal theorem [68], M_R is bounded on $L^p(X)$ for each $1 < p \leq \infty$. Contrary to the Euclidean case, here $d_\Omega(x) = d(x, \Omega^c)$ is the distance from $x \in \Omega$ to the complement $\Omega^c = X \setminus \Omega$. We use the this distance because in metric spaces $d(x, \partial\Omega)$ may be larger than $d(x, \Omega^c)$.

We are now ready to give the general formulation of our main result, which shows, even in the metric setting, the equivalence between uniform p-fatness of the complement, validity of the point wise p-Hardy inequality, and two Poincaré type inequalities.

Here $\tau \geq 1$ is the dilatation constant from the $(1, p)$-Poincaré inequality.

Theorem (2.2.4) [54]: Let $1 \leq p < \infty$ and let X be a complete, doubling metric measure space supporting a $(1, p)$-Poincaré inequality. Then, for an open set $\Omega \subsetneq X$, the following assertions are quantitatively equivalent:

(a) The complement Ω^c is uniformly $p -$ fat.

(b) For all $B = B(w, r)$, with $w \in \Omega^c$ and $r > 0$, and every $u \in N_0^{1,p}(\Omega)$

$$\int_B |u|^p d\mu \leq Cr^p \int_{5\tau B} g_u^p d\mu. \tag{121}$$

(c) For all $x \in \Omega$ and every $u \in N_0^{1,p}(\Omega)$

$$\left|u_{B_x}\right|^p \leq C d_\Omega(x)^p \int_{20\tau B_x} g_u^p d\mu, \tag{122}$$

where $B_x = B(x, d_\Omega(x))$.

(d) The open set Ω admits the point wise p-Hardy inequality (120), and we may choose the dilatation constant to be $L = 20\tau$.

Corollary (2.2.5) [54]: For $1 < p < \infty$ each of the assertions in Theorem (2.2.4) possesses a selfimprovement property. More precisely, if one of the assertions (a)-(d) holds for $1 < p < \infty$, then there exists some $1 < q < p$ so that the same assertion (and thus each of them) holds with the exponent q and constants depending only on p and the associated data.

Notice that we only assume that X supports a $(1, p)$-Poincaré inequality, but in the above corollary we actually need that X supports a (1, q)-Poincaré inequality for some $q < p$ as

well. By a result of Keith and Zhong [64], this is in fact always true if X is complete, doubling and supports a weak $(1,p)$-Poincaré inequality.
In the previous concerning point wise Hardy inequalities (see e.g.[66], [82]), a sort of a selfimprovement has actually been an a priori assumption when the passage from pointwise inequalities to the usual Hardy inequality was considered. Now, by Corollary (2.2.5), such an extra assumption becomes unnecessary. Especially, using the maximal theorem for an exponent $1 < q < p$, for which Ω still admits the pointwise inequality, we obtain the following corollary just as in the Euclidean case.
Corollary (2.2.6) [54]: If an open set $\Omega \subset X$ admits the pointwise p-Hardy inequality (120) for some $1 < p < \infty$, then Ω admits the p-Hardy inequality, that is, there exists $C > 0$ such that

$$\int_\Omega \frac{u(x)^p}{d_\Omega(x)^p} d\mu \leq C \int_\Omega g_u(x)^p d\mu$$

for every $u \in N_0^{1,p}(\Omega)$.
However, the result of Corollary (2.2.6), when viewed as a consequence of Theorem (2.2.4), depends on a heavy machinery of non-trivial results already in the Euclidean setting, let alone in general metric spaces, as the self-improvement of uniform fatness is involved. In particular, the theory of Cheeger derivatives is needed in the metric case. The ideas of Wannebo [88] lead to an alternative proof for Corollary (2.2.6), which is based on completely elementary tools and methods, and especially avoids the use of the self-improvement. Using this approach, we give in Theorem (2.2.10) a direct proof for the fact that uniform p-fatness of the complement of Ω implies that Ω admits the p-Hardy inequality. Note that this result was first generalized to metric spaces in [60], but there the proof was based on the selfimprovement. As the pointwise p-Hardy inequality implies the uniform fatness of the complement by Theorem (2.2.4), Corollary (2.2.6) follows.
It would also be interesting to acquire an alternative proof for Corollary (2.2.5) by showing the selfimprovement directly for one of the conditions (b)-(d) in Theorem (2.2.4).
Let us remark here that self-improving properties of integral Hardy inequalities were considered in [81], but these results and methods do not seem apply for pointwise inequalities.
deals with the proofs of the implications (a) ⇒ (b) ⇒ (c) ⇒ (d) of Theorem (2.2.4). The implication (a) ⇒ (d), that uniform p-fatness of the complement implies the point wise p-Hardy inequality, is a generalization of an Euclidean result of Kinnunen and Martio [78] and Hajłasz [66].
Our proof utilizes the following Sobolev type inequality, proved in the classical case by Maz'ya (c.f. [84]) and in the metric setting by Björn [59]. We recall the main ideas of the proof for the sake of completeness.
Lemma (2.2.7) [54]: There is a constant $C > 0$ such that for each $u \in N^{1,p}(X)$ and for all balls $B \subset X$ we have

$$\int_B |u|^p d\mu \leq \frac{C}{\operatorname{cap}_p\left(\frac{1}{2}B \cap \{u = 0\}, B\right)} \int_{5\tau B} g_u^p d\mu, \tag{123}$$

Where τ is from the $(1,p) -$ Poincaré inequality.
Proof: Let $B = B(x,r)$ be a ball and let φ be a $2/r$-Lipschitz function such that $0 \leq \varphi \leq 1, \varphi = 1$ on $(1/2)B$ and $\varphi = 0$ outside B. We may assume that $u \geq 0$ in B. The function

$$v = \varphi(1 - u/\bar{u}),$$

Where $\bar{u} = \left(\int_B u^p d\mu\right)^{1/p}$, is a test function for the capacity in (123). The claim follows by estimating the integral of g_v^p,

$$g_v = \left|1 - \frac{u}{\bar{u}}\right| 2r^{-1} + g_u/\bar{u}.$$

Here one needs a (p,p)-Poincaré inequality, which by [68] follows from the $(1,p)$-Poincaré inequality with dilatation constant 5τ.

We also need the following pointwise inequality for $N^{1,p}$-functions in terms of the maximal function of the p-weak upper gradient: There is a constant $C > 0$, depending only on the doubling constant and the constants of the Poincaré inequality, such that

$$|u(x) - u_B| \le \mathrm{Cr}\left(M_{\tau r} g_u^p(x)\right)^{\frac{1}{p}} \tag{124}$$

Whenever $B = B(x,r)$ is a ball and x is a Lebesgue point of u. Estimate (124) follows easily from a standard telescoping argument, see for example [67]. Note that u has Lebesgue points almost everywhere in the p-capacity sense, see [77], [79].

Proof: (Theorem (2.2.4) (a) ⇒ (b) ⇒ (c) ⇒ (d))

(a) ⇒ (b): Let $u \in N_0^{1,p}(\Omega)$ and let $B = B(w,r)$, where $w \in \Omega^c$. Assume first that $0 < r < (1/6)\operatorname{diam} X$. Since u vanishes outside Ω, we have $\Omega^c \subset \{u = 0\}$.

Using the p-fatness of Ω^c, estimate (118), and the doubling property of μ, we obtain

$$\begin{aligned}\operatorname{cap}_p\left(\frac{1}{2}B \cap \{u = 0\}, B\right) &\ge \operatorname{cap}_p\left(\frac{1}{2}B \cap \Omega^c, B\right) \\ &\ge c_0 \operatorname{cap}_p\left(\frac{1}{2}B, B\right) \ge C\mu(B) r^{-p}.\end{aligned}$$

This, together with Lemma (2.2.7), gives

$$\int_B |u|^p d\mu \le \frac{C\mu(B)}{\operatorname{cap}_p\left(\frac{1}{2}B \cap \{u = 0\}, B\right)} \int_{5\tau B} g_u^p d\mu \le C r^p \int_{5\tau B} g_u^p d\mu.$$

If $(1/6)\operatorname{diam} X \le r \le \operatorname{diam} X$, we take $\tilde{B} = B(w, (1/7)\operatorname{diam} X)$. From the triangle inequality it follows that

$$\int_B |u|^p d\mu \le C\left(\int_B |u - u_B|^p d\mu + \mu(B)|u_{\tilde{B}}|^p + \mu(B)|u_{\tilde{B}} - u_B|^p\right).$$

We can then use the $(1,p)$-Poincaré inequality, the above case for the ball $\tilde{B}$, and the doubling property, and the claim for B follows with simple calculations.

Finally, if $r > \operatorname{diam} X$, the claim is clear by the previous cases.

(b) ⇒ (c): Let $u \in N_0^{1,p}(\Omega), x \in \Omega$, and let $B_x = B(x, d\Omega(x))$. Choose a point $w \in \Omega^c$ so that

$$R = d(x,w) \le 2d_\Omega(x),$$

and let $B_0 = B(w,R)$. Now

$$|u_{B_x}| \le |u_{B_x} - u_{B_0}| + |u|_{B_0},$$

where, by the $(1,p)$-Poincaré inequality, the fact that $B_0 \subset 4B_x$ and $B_x \subset 2B_0$, and the doubling property,

$$|u_{B_x} - u_{B_0}| \le C d_\Omega(x)\left(\int_{4\tau B_x} g_u^p d\mu\right)^{1/p}.$$

Using the Hölder inequality, assumption (b), and the doubling property, we obtain

$$|u|_{B_0} \leq \left(\int_{B_0} |u|^p d\mu\right)^{1/p} \leq CR\left(\int_{5\tau B_0} g_u^p d\mu\right)^{1/p}$$
$$\leq C d_\Omega(x)\left(\int_{20\tau B_x} g_u^p d\mu\right)^{1/p}.$$
The claim follows by combining these two estimates.
(c) ⇒ (d): Let $u \in N_p^{1,p}(\Omega)$ and let $x \in \Omega$ be a Lebesgue point of u. Now
$$|u(x)| \leq |u(x) - u_{B_x}| + |u_{B_x}|,$$
where, by (124)
$$|u(x) - u_{B_x}| \leq C d_\Omega(x)\left(M_{\tau d_\Omega(x)} g_u^p(x)\right)^{1/p},$$
and by (c)
$$|u_{B_x}| \leq C d_\Omega(x)\left(\int_{20\tau B_x} g_u^p d\mu\right)^{\frac{1}{p}} \leq C d_\Omega(x)\left(M_{20\tau d_\Omega(x)} g_u^p(x)\right)^{\frac{1}{p}}.$$
The pointwise p-Hardy inequality follows from the above estimates.
By slightly modifying the proof above or the proof in [78], we obtain a p-Hardy inequality containing a fractional maximal function of the upper gradient.
Corollary (2.2.8) [54]: Let $1 \leq p < \infty$ and let $\Omega \subset X$ be an open set whose complement is uniformly p-fat. Then there is a constant $C > 0$, independent of Ω, such that for all $0 \leq \alpha < p$ and for all $u \in N_p^{1,p}(\Omega)$,
$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\alpha}{p}}\left(M_{\alpha,20\tau d_\Omega(x)} g_u^p(x)\right)^{\frac{1}{p}} \tag{125}$$
whenever $x \in \Omega$ is a Lebesgue point of u.
Here, for $\alpha \geq 0$, the restricted fractional maximal function of a locally integrable function u is
$$M_{\alpha,R} u(x) = \sup_{0<r\leq R} r^\alpha \int_{B(x,r)} |u| d\mu.$$
we prove the following lemma, from which the part (d) ⇒ (a) of Theorem (2.2.4) and the previously unknown necessity part of Theorem (2.2.1) follow.
Lemma (2.2.9) [54]: Let $1 \leq p < \infty$ and let $\Omega \subset X$ be an open set. If Ω admits the point wise p-Hardy inequality (120), then Ω^c is uniformly $p-$ fat. The constant in the uniform fatness condition (119) depends only on p, c_H, and the constants related to X.
Proof : Let $B = B(w, R)$, where $w \in \Omega^c$ and $0 < R < (1/6)$ diam X. By (118), it suffices to find a constant $C > 0$, independent of w and R, such that
$$\mu(B)R^{-p} \leq C \int_{2B} g_v^p d\mu \tag{126}$$
whenever g_v is an upper gradient of a function $v \in N_0^{1,p}(2B)$ satisfying $0 \leq v \leq 1$ and $v = 1$ in $\Omega^c \cap \bar{B}$. By the quasicontinuity of $N^{1,p}$-functions, we may assume that $v = 1$ in an open neighborhood of $\Omega^c \cap \bar{B}$.
Let $l = [2(L+1)]^{-1}$, where L is from the pointwise p-Hardy inequality (120). The doubling condition implies that $\mu(lB) \geq l^s \mu(B)/c_D$. If now $v_B > l^s/2c_D$, we obtain from the Poincaré inequality for $v \in N_0^{1,p}(2B)$ (see for example [59]) that

$$1 \leq C \int_B |v| d\mu \leq CR \left(\int_{2B} g_v^p d\mu \right)^{\frac{1}{p}},$$

and (126) follows by the doubling condition.

We may hence assume that $v_B \leq l^s/2c_D$. Let $\psi \in N_0^{1,p}(B)$ be a cut-off function, defined as

$$\psi(x) = \max \left\{0, 1 - \frac{4}{R} d\left(x, \frac{1}{2}B\right)\right\},$$

and take

$$u = \min \{\psi, 1 - v\}.$$

Since $1 - v = 0$ in an open set containing $\Omega^c \cap B$ and $N^{1,p}(X)$ is a lattice, we have that $u \in N_0^{1,p}(\Omega)$. Moreover, u has an upper gradient g_u such that $g_u = g_v$ in $(1/2)B$.

We define $C_1 = l^s/4c_D$ and

$$E = \{x \in lB: u(x) > C_1 \text{ and (120) holds for } u \text{ at } x\},$$

and claim that

$$\mu(E) \geq C_1 \mu(B). \tag{127}$$

To see this, first notice that $u = 1 - v$ in lB and that $\mu(lB) \geq 4C_1\mu(B)$. As $v_B \leq l^s/2c_D = 2C_1$, we obtain

$$\begin{aligned} \int_{lB} u d\mu &= \int_{lB} (1 - v) d\mu \geq \int_B (1 - v) d\mu - \mu(B \setminus lB) \\ &\geq (1 - 2C_1)\mu(B) - \mu(B) + \mu(lB) \\ &\geq 2C_1\mu(B). \end{aligned} \tag{128}$$

Since the point wise p-Hardy holds for almost every $x \in \Omega$, we have $u \leq C_1$ almost everywhere in $lB \setminus E$. Thus a direct computation using estimate (128) yields (127):

$$\begin{aligned} \mu(E) \geq \int_E u d\mu &= \int_{lB} u d\mu - \int_{lB \setminus E} u d\mu \\ &\geq 2C_1\mu(B) - \int_{lB} C_1 d\mu \\ &\geq 2C_1\mu(B) - C_1\mu(B) = C_1\mu(B). \end{aligned}$$

To continue the proof, we fix for each $x \in E$ a radius $0 < r_x \leq Ld_\Omega(x)$ such that

$$M_{Ld_\Omega(x)} g_u^p(x) \leq 2 \int_{B(x,r_x)} g_u^p d\mu.$$

By the standard $5r$-covering theorem (see e.g. [62]), there are pairwise disjoint balls $B_i = B(x_i, r_i)$, where $x_i \in E$ and $r_i = r_{xi}$ are as above, so that $E \subset \bigcup_{i=1}^{\infty} 5B_i$.

It follows immediately from (127) and the doubling condition that

$$\mu(B) \leq C_1^{-1}\mu(E) \leq C \sum_{i=1}^{\infty} \mu(B_i). \tag{129}$$

As $x_i \in lB$ and $w \notin \Omega$, we have $d_\Omega(x_i) \leq lR$. Hence, by the choice of l, we obtain for each $y \in B_i$ that

$$d(w, y) \leq d(w, x_i) + d(x_i, y) \leq lR + Ld_\Omega(x_i) \leq lR(1 + L) = R/2,$$

and so $B_i \subset (1/2)B$. This means, in particular, that $g_u = g_v$ in each B_i. Since $u(x_i) > C_1$ for each i, the pointwise p-Hardy inequality (120) and the choice of the radii r_i imply that

$$C_1^p \leq |u(x_i)|^p \leq C d_\Omega(x_i)^p M_{Ld_\Omega(x_i)} g_u^p(x) \leq CR^p \mu(B_i)^{-1} \int_{B_i} g_u^p d\mu,$$

and so

$$\mu(B_i) \le CR^p \int_{B_i} g_v^p d\mu.$$

Inserting this into (129) leads us to

$$\mu(B) \le CR^p \sum_{i=1}^{\infty} \int_{B_i} g_v^p d\mu \le CR^p \int_{2B} g_v^p d\mu,$$

where we used the fact that the balls $B_i \subset 2B$ are pairwise disjoint. This proves estimate (126), and the lemma follows.

We give a straight-forward proof for the fact that uniform p-fatness of the complement Ω^c suffices for Ω to admit the p-Hardy inequality. Our proof follows the ideas of Wannebo [88]. A similar method was also used in [61] of Orlicz-Hardy inequalities. As mentioned earlier, the following result first appeared in the metric space setting in [60], where the proof was based on the self-improvement of uniform p-fatness.

Theorem (2.2.10) [54]: Let $1 < p < \infty$ and let $\Omega \subset X$ be an open set. If Ω^c is uniformly $p-$ fat then Ω admits the p-Hardy inequality, quantitatively.

Proof: To make the proof as simple as possible, let us assume that the dilatation constant in the righthand side of Theorem (2.2.4) (b) is 2. The general case follows by obvious modifications. Let

$$\Omega_n = \{x \in \Omega : 2^{-n} \le d_\Omega(x) < 2^{-n+1}\}$$

and

$$\tilde{\Omega}_n = \bigcup_{k=n}^{\infty} \Omega_k.$$

Let F_n be a cover of Ω_n with balls of radius 2^{-n-2} such that their center points are not included in any other ball in F_n. Associate to each ball $B \in F_n$ a bigger ball $\tilde{B} \supset B$, whose radius is 2^{-n+2} and whose center point is on $\partial\Omega$. Note that $2B \cap \Omega \subset \tilde{\Omega}_{n-2}$ and that

$$\sum_{B\in F_n} \chi_B < C \text{ and } \sum_{B\in F_n} \chi_{2\tilde{B}} < C,$$

where the constant $C > 0$ only depends on the doubling constant of μ.

Let $u \in N_0^{1,p}1(\Omega)$. The condition (b) of Theorem (2.2.4) (which follows from the uniform p-fatness of the complement) implies that for every $B \in F_n$ we have

$$\int_B |u|^p d\mu \le \int_{\tilde{B}} |u|^p d\mu \le C2^{-np} \int_{2\tilde{B}} g_u^p d\mu.$$

By summing up the inequalities above, we obtain

$$\begin{aligned} \int_{\Omega_n} |u|^p d\mu \le & \sum_{B\in F_n} \int_B |u|^p d\mu \le C2^{-np} \sum_{B\in F_n} \int_{2\tilde{B}} g_u^p d\mu \\ & \le C2^{-np} \int_{\tilde{\Omega}_{n-2}} g_u^p d\mu. \end{aligned} \tag{130}$$

Let $0 < \beta < 1$ be a small constant to be fixed later. We multiply (130) by $2^{n(p+\beta)}$ and sum the inequalities to obtain

$$\int_{\Omega} |u(x)|^p d_{\Omega}(x)^{-p-\beta} d\mu \leq \sum_{n=-\infty}^{\infty} \int_{\Omega} |u(x)|^p 2^{n(p+\beta)} d\mu$$

$$\leq C \sum_{n=-\infty}^{\infty} 2^{n\beta} \int_{\Omega_{n-2}} g_u(x)^p d\mu$$

$$= C \sum_{n=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} 2^{(k+2-n)\beta} \int_{\Omega_k} g_u(x)^p d\mu \right)$$

$$\leq C \sum_{n=-\infty}^{\infty} \frac{2^{k\beta}}{\beta} \int_{\Omega_k} g_u(x)^p d\mu$$

$$\leq \frac{C}{\beta} \int_{\Omega_k} g_u(x)^p \, \mathrm{d}_{\Omega}(x)^{-\beta} d\mu. \tag{131}$$

In the calculations above, we used the fact that $2^{-k} \leq d_{\Omega}(x) \leq 2 \cdot 2^{-k}$ for every $x \in \Omega_k$. Now let $v \in N_0^{1,p} 1(\Omega)$ be a function with a compact support in Ω and let

$$u(x) = v(x) d_{\Omega}(x)^{\beta/p}.$$

Then the function

$$g_u(x) = g_v(x) d_{\Omega}(x)^{\beta/p} + \frac{\beta}{p} v(x) d_{\Omega}(x)^{\beta/p-1}$$

is a p-weak upper gradient of u. Thus, by (131), we have

$$\int_{\Omega} \frac{v(x)^p}{d_{\Omega}(x)^p} d\mu = \int_{\Omega} \frac{u(x)^p}{d_{\Omega}(x)^{p+\beta}} d\mu \leq \frac{C}{\beta} \int_{\Omega} \frac{g_u(x)^p}{d_{\Omega}(x)^{\beta}} d\mu$$

$$\leq \frac{C}{\beta} \int_{\Omega} g_v(x)^p d\mu + \frac{C}{\beta} \frac{\beta^p}{p^p} \int_{\Omega} \frac{v(x)^p}{d_{\Omega}(x)^p} d\mu.$$

If $\beta > 0$ is small enough, the last term on the right-hand side can be included on the left-hand side and we obtain

$$\int_{\Omega} \frac{v(x)^p}{d_{\Omega}(x)^p} d\mu \leq C \int_{\Omega} g_v(x)^p d\mu.$$

This completes the proof because functions with compact support are dense in $N_0^{1,p}(\Omega)$, see [87].

Notice that the requirement $p > 1$ is essential in Theorem (2.2.10). For instance, smooth domains in $\mathbb{R}^n$ admit the pointwise 1-Hardy inequality but not the integral 1-Hardy.

It is well-known that capacities and Hausdorff contents are closely related both in Euclidean spaces and general metric spaces, see e.g. [72], [73]. In metric spaces we follow [55], [56], [79], and use Hausdorff contents H_R^t, defined by applying the Carathéodory construction to functions

$$h\big(B(x,r)\big) = \frac{\mu\big(B(x,r)\big)}{r^t},$$

where $r \leq R$. Thus the Hausdorff content of codimension t of a set $E \subset X$ is given by

$$H_R^t(E) = \inf \left\{ \sum_{-i \in I} h\big(B(x_i, r_i)\big) : E \subset \bigcup_{i \in I} B(x_i, r_i), r_i \leq R \right\}.$$

Here we may actually assume that $x_i \in E$, as this increases $H_R^t(E)$ at most by a constant factor.

If the space X is Q-regular, then $H_{\infty}^{t}(E)$ is comparable with the usual Hausdorff content $H_{\infty}^{Q-t}(E)$, which is defined by using the gauge function $h(B(x,r)) = r^{Q-t}$.
Recall that Q-regularity means that there are constants $c_1, c_2 > 0$ such that

$$c_1 r^Q \le \mu(B(x,r)) \le c_2 r^Q$$

for all balls $B(x,r)$ in X.
Now, by slightly modifying the argument in [73] (see also [63] and [64]), one can show that if $E \subset X$ is a closed set and there exists some $1 \le q < p$ and a constant $C > 0$ so that for all $w \in E$ and every $R > 0$,

$$H_{\frac{R}{2}}^{q}(E \cap \bar{B}(w,R)) \ge C\mu(\bar{B}(w,R))R^{-q}, \tag{132}$$

Then $E \subset X$ is uniformly p-fat. Conversely, by rewriting the argument of [72] (see also [63]) for the content $H_{\frac{R}{2}}^{q}$, it is not hard to see that uniform p-fatness of E leads to (132), but with q replaced by p. Using the self-improvement of uniform fatness, we then conclude that uniform p-fatness of E implies the existence of an exponent $q < p$ for which (132) holds. Hence (132), with an exponent $1 \le q < p$, is actually equivalent with the uniform p-fatness of E.
We investigate similar density conditions for the boundary of a domain $\Omega \subset X$. To this end, we consider a version of the point wise Hardy inequality where the distance is taken to the boundary instead of the complement. We define

$$\delta_\Omega(x) = d(x, \partial\Omega) \text{ for } x \in \Omega.$$

The following lemma is a metric space generalization of a result from [64], [82].
Lemma (2.2.11) [54]: Let $1 \le p < \infty$ and let $\Omega \subset X$ be an open set. Assume that Ω admits the point wise p-Hardy inequality

$$|u(x)| \le c_H \delta_\Omega(x) \left(M_{L\delta_\Omega(x)} g_u^p(x)\right)^{\frac{1}{p}} \tag{133}$$

for all $u \in N_0^{1,p}(\Omega)$. Then

$$H_{\delta_\Omega(x)}^{p}\big(\partial\Omega \cap \bar{B}(x, 2L\delta_\Omega(x))\big) \ge C\delta_\Omega(x)^{-p}\mu\left(\bar{B}\big(x, \delta_\Omega(x)\big)\right). \tag{134}$$

for all $x \in \Omega$.
Proof: Let $x \in \Omega$. We define $R = \delta_\Omega(x), B = \bar{B}(x,R)$, and

$$E = \partial\Omega \cap 2LB.$$

Let $\{B_i\}_{i=1}^{N}$, where $B_i = B(w_i, r_i)$ with $w_i \in E$ and $0 < r_i \le R$, be a covering of E; we may assume that the covering is finite by the compactness of E.
It is now enough to show that there exists a constant $C > 0$, independent of the particular covering, such that

$$\sum_{i=1}^{N} \mu(B_i) r_i^{-p} \ge C\mu(B)R^{-p}. \tag{135}$$

If $r_i \ge R/4$ for some $1 \le i \le N$, then, by (116) and the fact that $r_i^{-p} \ge R^{-p}$, we have

$$\mu(B_i) r_i^{-p} \ge C\mu(B)\left(\frac{r_i}{R}\right)^s R^{-p} \ge C\mu(B)R^{-p},$$

from which (135) readily follows.
We may hence assume that $r_i < R/4$ for all $1 \le i \le N$. Now, define

$$\varphi(y) = \min_{1 \le i \le N} \{1, r_i^{-1} d(y, B_i)\}$$

and let $\psi \in N_0^{1,p}(2LB)$ be a cut-off function such that $0 \le \psi \le 1$ and $\psi(y) = 1$ for all $y \in LB$. Then the function

$$u = \min\ \{\psi, \varphi\}\chi_\Omega$$

belongs to $N_0^{1,p}(\Omega)$. As $r_i < R/4$ for all $1 \le i \le N$, it follows that $d(x, 2B_i) \ge R/2$ for all $1 \le i \le N$, and so $u(x) = 1$.

In addition, u has an upper gradient g_u such that

$$g_u(y)^p \le \sum_{i=1}^{N} r_i^{-p} \chi_{2B_i}(y) \tag{136}$$

for almost every $y \in LB$. Especially, we must have $r > R/2$ in order to obtain something positive when estimating $M_{LR} g_u^p(x)$. As the point wise inequality (133) holds for the continuous function $u \in N_0^{1,p}(\Omega)$ at every $x \in \Omega$, we have

$$1 = |u(x)|^p \le CR^p M_{LR} g_u^p(x) \le CR^p \sup_{\frac{R}{2} \le r \le LR} \int_{B(x,r)} g_u^p d\mu$$

$$\le CR^p \mu\left(\frac{1}{2}B\right)^{-1} \int_{LB} g_u^p d\mu \le CR^p \mu(B)^{-1} \sum_{i=1}^{N} \mu(2B_i) r_i^{-p},$$

Where the last inequality is a consequence of (136). Estimate (135) then easily follows with the help of the doubling property.

Next we show that the inner boundary density condition (134) is actually almost equivalent to the pointwise p-Hardy inequality. The proof below uses an idea from [73], but is new of Hardy inequalities.

Theorem (2.2.12) [54]: Let $1 < p < \infty$ and let $\Omega \subset X$ be an open set. If estimate (134) holds with an exponent $1 \le q < p$ for all $x \in \Omega$, then Ω admits the pointwise p-Hardy inequality (133), but possibly with a different dilatation constant in the maximal function.

Proof: Let us first assume that $u \in N_0^{1,p}(\Omega)$ has a compact support in Ω. Let $B = B(x, R)$, where $x \in \Omega$ and $R = \delta_\Omega(x)$. We are going to show that

$$|u_B|^p \le C\delta_\Omega(x)^p \int_{3\tau LB} g_u^p d\mu, \tag{137}$$

where $C > 0$ and $\tau \ge 1$ are independent of x, whence the point wise p-Hardy inequality follows for almost every $x \in \Omega$ by Theorem (2.2.4).

If $u_B = 0$, the claim (137) is true, and so we may assume that $|u_B| > 0$, and in fact, by homogeneity, that $|u_B| = 1$. Let $w \in \partial\Omega \cap 2LB$ and let $B_k = B(w, r_k)$, where $r_k = (5\tau 2^k)^{-1} R, k \in \mathbb{N}$. It then follows that

$$1 = |u(w) - u_B| \le |u_{B_0}| + |u_{B_0} - u_B|.$$

Now, if $|u_{B_0}| < 1/2$, we infer, using the $(1, p)$-Poincaré inequality, the facts $B_0 \subset 3LB$ and $B \subset 3LB_0$, and the doubling property, that

$$\frac{1}{2} \le |u_{B_0} - u_B| \le |u_{B_0} - u_{3B}| + |u_B - u_{3B}| \le CR\left(\int_{3\tau LB} g_u^p d\mu\right)^{\frac{1}{p}}.$$

As $|u_B| = 1$, the claim follows.

Thus we may assume that $1/2 \le |u_{B_0}| = |u(w) - u_{B_0}|$ for every $w \in \partial\Omega \cap 2LB$.

A standard chaining argument, using the $(1, p)$-Poincaré inequality (see for example [68]) and the assumption that the support of u is compact, leads us to estimate

$$1 \leq C \sum_{k=0}^{\infty} r_k \left(\int_{\tau B_k} g_u^p d\mu \right)^{\frac{1}{p}}. \tag{138}$$

From (138) it follows that there must be a constant $C_1 > 0$, independent of u and w, and at least one index $k_w \in \mathbb{N}$ such that

$$r_{k_w} \left(\int_{\tau B_{k_w}} g_u^p d\mu \right)^{1/p} \geq C_1 2^{-k_w \left(1-\frac{q}{p}\right)} = C_1 \left(\frac{r_{k_w}}{R}\right)^{1-q/p}.$$

In particular, we obtain for each $w \in \partial\Omega \cap 2LB$ a radius $r_w \leq R/(5\tau)$ and a ball $B_w = B(w, r_w)$ such that

$$\mu(\tau B_w) r_w^{-q} \leq C R^{p-q} \int_{\tau B_w} g_u^p d\mu. \tag{139}$$

The $5r$-covering lemma implies the existence of points $w_1, w_2, \dots, w_N \in \partial\Omega \cap 2LB$ such that if we set $r_i = r_{w_i}$, then the balls $\tau B_i = B(w_i, \tau r_i)$ are pairwise disjoint, but still $\cap\, 2LB \subset \bigcup_{i=1}^{N} 5\tau B_i$. Assumption (134), the doubling property, estimate (139), and the pairwise disjointness of the balls $\tau B_i \subset 3\tau LB$ then yield

$$\begin{aligned} R^{-q}\mu(B) &\leq C H_R^q(\partial\Omega \cap 2LB) \\ &\leq C \sum_{i=1}^{N} \mu(5\tau B_i)(5\tau r_i)^{-q} \leq C \sum_{i=1}^{N} \mu(\tau B_i) r_i^{-q} \\ &\leq C \sum_{i=1}^{N} R^{p-q} \int_{\tau B_i} g_u^p d\mu \leq C R^{p-q} \int_{3\tau LB} g_u^p d\mu. \end{aligned} \tag{140}$$

As we assumed $|u_B| = 1$, estimate (137) now follows from (140) and the doubling condition. For a general $u \in N_0^{1,p}(\Omega)$ estimate (137) follows by a suitable approximation with compactly supported functions.

If there now exists a constant $C \geq 1$ such that

$$d_\Omega(x) \leq \delta_\Omega(x) \leq C d_\Omega(x) \text{ for each } x \in \Omega, \tag{141}$$

then it is clear that point wise inequalities (120) and (133) are quantitatively equivalent.

In particular, if the inner boundary density condition (134) with codimension q holds for all $x \in \Omega$, then Theorems (2.2.12) and (2.2.4) imply that Ω^c is uniformly $p-$ fat for all $p > q$. On the other hand, easy examples show that Ω^c need not be uniformly $q-$ fat, or equivalently, Ω need not admit the point wise q-Hardy inequality, if $q > 1$. Hence some information is inevitably lost once we pass from the point wise p-Hardy inequality or uniform p-fatness (for $1 < p < \infty$) to Hausdorff contents; in the case $p = 1$ there is indeed an equivalence, cf. [79]. However, by the selfimprovement of the assertions of Theorem (2.2.4), we can still have the following equivalent characterization in terms of Hausdorff contents (see also [64], [82]). Note that here we need to use again the fact that X supports a $(1, q)$-Poincaré inequality for some $q < p$.

Corollary (2.2.13) [54]: Assume that $\Omega \subset X$ is such that (141) holds. Then all of the assertions in Theorem (2.2.4), with an exponent $1 < p < \infty$, are (quantitatively) equivalent to the following density condition: There exist some $1 < q < p$ and constants $C > 0$ and $L \geq 1$ such that

$$H^q_{\delta_\Omega(x)}\big(\partial\Omega \cap \bar{B}(x, L\delta_\Omega(x))\big) \geq C \delta_\Omega(x)^{-q} \mu\big(\bar{B}(x, \delta_\Omega(x))\big)$$

for all $x \in \Omega$.

It is worth a mention that uniform p-fatness of the boundary $\partial\Omega$ is of course sufficient for the uniform p-fatness of the complement and the point wise p-Hardy inequality, but not necessary, as cusp-type domains in $\mathbb{R}^n, n \geq 3$, show (cf. [82]).
Thus it really is essential that we consider above the density of the boundary only as seen from within the domain, in the sense of (134).

Section (2.3): L^1 Hardy Inequalities

Hardy's inequality involving distance from the boundary of a convex set $\Omega \subsetneq \mathbb{R}^n, n \geq 1$, asserts that

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} dx, p > 1, \tag{142}$$

for all $u \in C_c^\infty(\Omega)$, where $d \equiv d(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. Due to [102], [105], and [48], the constant appearing in (142) is optimal. After the pioneering results in [84] and [36], a sequence have improved (142) by adding extra terms on its right-hand side; see, [92], [93], [31], [99], and primarily [35] and [41], [42] where it was also noted that (142) remains valid with the sharp constant in more general sets than convex ones, and in particular in sets that satisfy $-\Delta d \geq 0$ in the distributional sense.
In the case $p = 1$, (142) reduces to a trivial inequality, at least for sets having nonpositive distributional Laplacian of the distance function. However, in the one dimensional case, the following L^1 weighted Hardy inequality is well known:

$$\int_0^\infty \frac{|u'(x)|}{x^{s-1}} dx \geq (s-1) \int_0^\infty \frac{|u(x)|}{x^s} dx, s > 1, \tag{143}$$

for all absolutely continuous functions $u: [0, \infty) \to \mathbb{R}$, such that $u(0) = 0$. This is the special case $p = 1$ of Theorem 330 in [102]. Inequality (143) is, in fact, an equality for u increasing, and thus the constant on the right-hand side is sharp.
We are concerned with the higher-dimensional generalizations of (143). Let $\Omega \subsetneq \mathbb{R}^n (n \geq 2)$ be open and let $d \equiv d(x) := \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$. We deal with inequalities of the type

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq B_0 \int_\Omega \frac{|u|}{d^s} dx + B \int_\Omega V(d)|u| dx, s \geq 1, \tag{144}$$

Valid for all $u \in C_c^\infty(\Omega)$. Here V is a potential function, i.e., nonnegative and $V \in L^1_{\text{loc}}(\mathbb{R}^+)$, and $B_0 \geq 0, B \in \mathbb{R}$. Questions concerning sets for which this inequality is valid, sharp constants, possible improvements, and optimal potentials will be studied.
Our first theorem reads as follows.

Theorem (2.3.1)[89]: Let Ω be a domain in $\mathbb{R}^n$ with boundary of class C^2 satisfying a uniform interior sphere condition, and we denote by $\underline{H}$ the infimum of the mean curvature of the boundary. Then there exists $B_1 \geq (n-1)\underline{H}$ such that for all $u \in C_c^\infty(\Omega)$ and all $s \geq 1$,

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1) \int_\Omega \frac{|u|}{d^s} dx + B_1 \int_\Omega \frac{|u|}{d^{s-1}} dx. \tag{145}$$

Let $s \geq 2$. If Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^2 having strictly positive mean curvature, then the constant $s-1$ in the first term, as well as the exponent $s-1$ on the distance function on the remainder term in (145), are optimal.
In addition, we have the following estimates:

$$(n-1)\underline{H} \leq B_1 \leq \frac{n-1}{|\partial\Omega|} \int_{\partial\Omega} H(y) dS_y, \tag{146}$$

Where $H(y)$ is the mean curvature of the boundary at $y \in \partial, \Omega$ and $\underline{H}$ is its minimum value.

The following result, which is of independent interest, played a key role in establishing Theorem (2.3.1).

Theorem (2.3.2) [89]: Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. Then $\mu := (-\Delta d)dx$ is a signed Radon measure on Ω. Let $\mu = \mu_{ac} + \mu_s$ be the Lebesgue decomposition of μ with respect to $\mathcal{L}^n$, i.e., $\mu_{ac} \ll \mathcal{L}^n$ and $\mu_s \perp \mathcal{L}^n$. Then $\mu_s \geq 0$ in Ω, and $\mu_{ac} \geq (n-1)\underline{H}dx$ a.e. in Ω, where $\underline{H} := \inf_{y\in\partial\Omega} H(y)$.

For domains with boundary of class C^2 satisfying a uniform interior sphere condition, $-\Delta d$ is a continuous function in a tubular neighborhood of the boundary and, moreover, $-\Delta d(y) = (n-1)H(y)$ for any $y \in \partial\Omega$. This fact together with Theorem (2.3.2) leads to the following corollary.

Corollary(2.3.3[89]): Let Ω be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. Then Ω is mean convex, i.e., $H(y) \geq 0$ for all $y \in \partial\Omega$, if and only if $-\Delta d \geq 0$ holds in Ω, in the sense of distributions.

We note that a set $\Omega \subsetneq \mathbb{R}^n$ with distance function having nonpositive distributional Laplacian is shown in [92], [93] and [42], [31] to be the natural geometric assumption for the validity of various Hardy inequalities.

In special geometries, we are able to compute the best constant B_1 in (145):

In case Ω is a ball of radius R, then the upper and lower estimates (146) coincide, yielding $B_1 = (n-1)/R$. One then may ask whether (145) can be further improved.

We provide a full answer to this question by showing that for $s \geq 2$ one can add a finite series of $[s]-1$ terms on the right-hand side before adding an optimal logarithmic correction. We prove the following.

Theorem (2.3.4) [89]: Let B_R be a ball of radius R. Then, (i) For all $u \in C_c^\infty(B_R)$, all $s \geq 2, \gamma > 1$, it holds that

$$\begin{aligned}\int_{B_R} \frac{|\nabla u|}{d^{s-1}} dx \quad &\geq (s-1)\int_{B_R} \frac{|u|}{d^s} dx + \sum_{k=1}^{[s]-1} \frac{n-1}{R^k}\int_{B_R} \frac{|u|}{d^{s-k}} dx \\ &+ \frac{C}{R^{s-1}}\int_{B_R} \frac{|u|}{d} X^\gamma\left(\frac{d}{R}\right) dx,\end{aligned} \tag{147}$$

Where $X(t) := (1-\log t)^{-1}, t \in (0,1]$, and $C \geq \gamma - 1$. The exponents s and $s-k, k = 1,2,\dots,[s]-1$, on the distance function, as well as the constants $s-1, (n-1)/R^k, k = 1,2,\dots,[s]-1$, in the first and the summation terms, respectively, are optimal.

The last term in (147) is optimal in the sense that if $\gamma = 1$, there is no positive constant C such that (147) holds.

(ii) For all $u \in C_c^\infty(B_R)$, all $1 \leq s < 2, \gamma > 1$, it holds that

$$\int_{B_R} \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1)\int_{B_R} \frac{|u|}{d^s} dx + \frac{C}{R^{s-1}}\int_{B_R} \frac{|u|}{d} X^\gamma\left(\frac{d}{R}\right) dx, \tag{148}$$

Where $X(t) := (1-\log t)^{-1}, t \in (0,1]$, and $C \geq \gamma - 1$. The last term in (148) is optimal in the sense that if $\gamma = 1$, there is no positive constant C such that (148) holds.

Note that this is in contrast with the results in the case $p > 1$, where an infinite series involving optimal logarithmic terms can be added (see [92] and [93]).

In case Ω is an infinite strip, using a more general upper bound on B_1 (see Theorem (2.3.26)), we prove that $B_1 = 0$. As a matter of fact, the finite series structure of (147) disappears and only the final logarithmic correction term survives.

Theorem (2.3.5) [89]: Let S_R be an infinite strip of inner radius R. For all $u \in C_c^\infty(S_R)$, all $s \geq 1, \gamma > 1$, it holds that

$$\int_{S_R} \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1) \int_{S_R} \frac{|u|}{d^s} dx + \frac{C}{R^{s-1}} \int_{S_R} \frac{|u|}{d} X^\gamma \left(\frac{d}{R}\right) dx, \tag{149}$$

Where $C \geq \gamma - 1$. The last term in (149) is optimal in the sense that if $\gamma = 1$, there is no positive constant C such that (149) holds.

We prove weighted L^1 Hardy inequalities in sets without regularity assumptions on the boundary. General open sets, sets with nonnegative distributional Laplacian of the distance function, as well as sets with positive reach are considered. Remainders for sets having finite inner radius are obtained in the first two cases and extremal domains are given. The results imply in particular inequality (149). After recalling further properties of the distance function for smooth domains, we prove Theorem (2.3.2). Theorem (2.3.1) and the optimality in Theorem (2.3.5), where also an interesting lower bound for the Cheeger constant of smooth, strictly mean convex domains is deduced (see Corollary (2.3.25)). Theorem (2.3.4) is proved, and we discuss L^p analogs of our results.

Since all inequalities will follow by the integration by parts formula, we formalize it as follows: Let Ω be an open set in $\mathbb{R}^n$ and T be a vector field on Ω.

Integrating by parts and using elementary inequalities, we get

$$\int_\Omega |T||\nabla u| dx \geq \int_\Omega \operatorname{div}(T)\, |u| dx, \tag{150}$$

for all $u \in C_c^\infty(\Omega)$, where we have also used the fact that $|\nabla |u|| = |\nabla u|$ a.e. in Ω.

We recall some properties of the distance function to the boundary of a general open set and then prove various weighted L^1 Hardy inequalities.

Let $\Omega \subsetneq \mathbb{R}^n$ be open. We set $d: \mathbb{R}^n \to [0, \infty)$ by $d(x) := \inf\{|x-y| : y \in \mathbb{R}^n \setminus \Omega\}$.

It is well known that d is Lipschitz continuous on $\mathbb{R}^n$, and in particular $|\nabla d(x)| = 1$ a.e. in (see [98], Theorem 4.8). The next property of d can be found, for example, in [95], Propositions 2.2.2.(i) and 1.1.3.(c), (e). We prove it for completeness.

Lemma (2.3.6) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open. It holds that

$$-d\Delta d \geq -(n-1) \text{ in } \Omega \text{ in the sense of distributions.} \tag{151}$$

Proof: Estimate (151) rests on the fact that the function $A: \mathbb{R}^n \to \mathbb{R}$ defined by $A(x) := |x|^2 - d^2(x)$ is convex. To see this, we take $x \in \mathbb{R}^n$ and let $y \in \mathbb{R}^n$ be such that $d(x) = |x-y|$. For any $z \in \mathbb{R}^n$ we get

$$\begin{gathered} A(x+z) + A(x-z) - 2A(x) \\ = 2|z|^2 - (d^2(x-z) + d^2(x+z) - 2d^2(x)) \\ \geq 2|z|^2 - (|x+z-y|^2 + |x-z-y|^2 - 2|x-y|^2) \\ = 0. \end{gathered}$$

Since $A(x)$ is also continuous, we obtain that $A(x)$ is convex (see [95], Proposition A1.2). It follows by [97], that the distributional Laplacian of A is a nonnegative Radon measure on $\mathbb{R}^n$. Since in Ω we have $\Delta A = 2(n-1-d\Delta d)$ in the sense of distributions, the result follows.

The weighted L^1 Hardy inequalities we obtain are deduced from the following basic fact.

Lemma (2.3.7) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open. For all $u \in C_c^\infty(\Omega)$ and all $s \geq 1$,

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1) \int_\Omega \frac{|u|}{d^s} dx + \int_\Omega \frac{|u|}{d^{s-1}} (-\Delta d) dx, \tag{152}$$

where $-\Delta d$ is meant in the distributional sense. If Ω is bounded, then equality holds for $u_\varepsilon(x) = (d(x))^{s-1+\varepsilon} \in W_0^{1,1}(\Omega; d^{-(s-1)}), \varepsilon > 0$.
Proof: Inequality (152) follows from (150) by setting $T(x) = -(d(x))^{1-s}\nabla d(x)$ for a.e. $x \in \Omega$, while the second statement is easily checked.
A covering of Ω by cubes was used in [91] to prove the next theorem. We present an elementary proof.
Theorem (2.3.8) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open. For all $u \in C_c^\infty(\Omega)$ and all $s > n$, it holds that
$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq (s-n) \int_\Omega \frac{|u|}{d^s} dx. \tag{153}$$
Proof: Coupling (151) and (152), we get
$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1) \int_\Omega \frac{|u|}{d^s} dx - (n-1) \int_\Omega \frac{|u|}{d^s} dx$$
$$= (s-n) \int_\Omega \frac{|u|}{d^s} dx.$$
Remark (2.3.9) [89]: The constant appearing on the right-hand side of (153) is just a lower bound for the best constant. The best constant in (153) differs from one open set to another. However, $\mathbb{R}^n \setminus \{0\}$ serves as an extremal domain for Theorem (2.3.8). More precisely, letting $\Omega = \mathbb{R}^n \setminus \{0\}$, we have $d(x) = |x|$, and (153) reads as follows:
$$\int_{\mathbb{R}^n} \frac{|\nabla u|}{|x|^{s-1}} dx \geq (s-1) \int_{\mathbb{R}^n} \frac{|u|}{|x|^s} dx, s > n, \tag{154}$$
for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. To illustrate the optimality of the constant on the right-hand side of (154), we define the following function:
$$u_\delta(x) := \chi_{B_\eta \setminus B_\delta}(x), x \in \mathbb{R}^n, \tag{155}$$
where, for any $r > 0$, by B_r we henceforth denote the open ball of radius r with center at the origin. Here $0 < \delta < \eta$ and η is fixed. The distributional gradient of u_δ is $\nabla u_\delta = \vec{v}_{\partial B_\delta} \delta_{\partial B_\delta} - \vec{v}_{\partial B_\eta} \delta_{\partial B_\eta}$ where, for any $r > 0$, $\vec{v}_{\partial B_r}$ stands for the outward pointing unit normal vector field along $\partial B_r = \{x \in \mathbb{R}^n : |x| = r\}$, and by $\delta_{\partial B_r}$ we denote the Dirac measure on ∂B_r. Moreover, the total variation of ∇u_δ is $|\nabla u_\delta| = \delta_{\partial B_\delta} + \delta_{\partial B_\eta}$. Using the co-area formula, we get
$$\frac{\int_{\mathbb{R}^n} \frac{|\nabla u_\delta|}{|x|^{s-1}} dx}{\int_{\mathbb{R}^n} \frac{|u_\delta|}{|x|^s} dx} = \frac{\delta^{1-s}|\partial B_\delta| + \eta^{1-s}|\partial B_\eta|}{\int_\delta^\eta r^{-s}|\partial B_r| dr}$$
$$= \frac{\delta^{n-s} + \eta^{n-s}}{\int_\delta^\eta r^{n-s-1}|\partial B_r| dr}$$
$$= (s-n) \frac{\delta^{n-s} + \eta^{n-s}}{\delta^{n-s} - \eta^{n-s}}$$
$$\to s-n, as \delta \downarrow 0.$$
Although not smooth, functions like u_δ defined in (155) belong to $BV(\mathbb{R}^n)$ (the space of functions of bounded variation in $\mathbb{R}^n$), and thus we can use a C_c^∞ approximation so that the calculation above holds in the limit (see, [97]).
Theorem (2.3.10) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open and such that $R := \sup_{x\in\Omega} d(x) < \infty$. For all $u \in C_c^\infty(\Omega)$, all $s \geq n, \gamma > 1$, it holds that

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}}dx \geq (s-n)\int_\Omega \frac{|u|}{d^s}dx + \frac{C}{R^{s-n}}\int_\Omega \frac{|u|}{d^n}X^\gamma\left(\frac{d}{R}\right)dx, \tag{156}$$

where $C \geq \gamma - 1$.

Proof: We set $T(x) = -(d(x))^{1-s}[1-(d(x)/R)^{s-n}X^{\gamma-1}(d(x)/R)]\nabla d(x)$ for a.e. $x \in \Omega$. Since $|1-(d(x)/R)^{s-n}X^{\gamma-1}(d(x)/R)| \leq 1$ for all $x \in \Omega$, we have

$$\int_\Omega |T||\nabla u|dx \leq \int_\Omega \frac{|\nabla u|}{d^{s-1}}dx.$$

Using the rule $\nabla X^{\gamma-1}(d(x)/R) = (\gamma-1)X^\gamma(d(x)/R)\frac{\nabla d(x)}{d(x)}$ for a.e. $x \in \Omega$, we compute

$$\operatorname{div}(T) = (s-1)d^{-s}\left[1-\left(\frac{d}{R}\right)^{s-n}X^{\gamma-1}\left(\frac{d}{R}\right)\right] + \frac{s-n}{R^{s-n}}d^{-n}X^{\gamma-1}(d/R)$$
$$+\frac{\gamma-1}{R^{s-n}}d^{-n}X^\gamma(d/R) + d^{1-s}\left[1-\left(\frac{d}{R}\right)^{s-n}X^{\gamma-1}\left(\frac{d}{R}\right)\right](-\Delta d).$$

Since $1-\left(\frac{d(x)}{R}\right)^{s-n}X^{\gamma-1}(d(x)/R) \geq 0$ for all $x \in \Omega$, we use (151) on the last term of the above equality, and a straightforward computation gives

$$\operatorname{div}(T) \geq (s-n)d^{-s} + \frac{\gamma-1}{R^{s-n}}d^{-n}X^\gamma\left(\frac{d}{R}\right).$$

This means that

$$\int_\Omega \operatorname{div}(T)\,|u|dx \geq (s-n)\int_\Omega \frac{|u|}{d^s}dx + \frac{\gamma-1}{R^{s-n}}\int_\Omega \frac{|u|}{d^n}X^\gamma\left(\frac{d}{R}\right)dx,$$

and the result follows from (150).

We assume that

$$-\Delta \mathrm{d} \geq 0 \text{ in } \Omega, \text{ in the sense of distributions. (C)}$$

This condition was first used of Hardy inequalities in [35], [92] and has been used intensively in [42], [31], [105]. As we will prove, domains with sufficiently smooth boundary carrying condition (C) are characterized as domains with nonnegative mean curvature of their boundary. However, we do not impose regularity on the boundary.

Theorem (2.3.11) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open and such that condition (C) holds. For all $u \in C_c^\infty(\Omega)$ and all $s > 1$, it holds that

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}}dx \geq (s-1)\int_\Omega \frac{|u|}{d^s}dx. \tag{157}$$

Moreover, the constant appearing on the right-hand side of (157) is sharp.

Proof: Since (C) holds, we may cancel the last term in (152) and (157) follows.

To prove the sharpness of the constant, we pick $y \in \partial\Omega$ and define the family of $W_0^{1,1}(\Omega; (\mathrm{d}^{-(s-1)})$ functions by $u_\varepsilon(x) := \phi(x)(d(x))^{s-1+\varepsilon}, \varepsilon > 0$, where $\phi \in C_c^\infty(B_\delta(y)), 0 \leq \phi \leq 1$, and $\phi \equiv 1$ in $B_{\delta/2}(y)$, for some small but fixed δ. We have

$$\frac{\int_\Omega \frac{|\nabla u_\varepsilon|}{d^{s-1}}dx}{\int_\Omega \frac{|u_\varepsilon|}{d^s}dx} \leq s-1+\varepsilon + \frac{\int_\Omega |\nabla\phi|d^\varepsilon dx}{\int_\Omega \phi d^{-1+\varepsilon}dx}$$
$$\leq s-1 + \varepsilon + \frac{C}{\int_{\Omega\cap B_{\delta/2}(y)} d^{-1+\varepsilon}dx}$$
$$\leq s-1+o_\varepsilon(1),$$

where C is some universal constant (not depending on ε).

Theorem (2.3.12) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open and such that condition (C) holds. Suppose in addition that $R := \sup_{x\in\Omega} d(x) < \infty$. For all $u \in C_c^\infty(\Omega)$, all $s \geq 1, \gamma > 1$, it holds that

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1)\int_\Omega \frac{|u|}{d^s} dx + \frac{C}{R^{s-1}}\int_\Omega \frac{|u|}{d} X^\gamma\left(\frac{d}{R}\right) dx, \tag{158}$$

Where $C \geq \gamma - 1$.

Proof: We set $T(x) = -(d(x))^{1-s}[1-(d(x)/R)^{s-1}X^{\gamma-1}(d(x)/R)]\nabla d(x)$ for a.e. $x \in \Omega$. Since $|1-(d(x)/R)^{s-1}X^{\gamma-1}(d(x)/R)| \leq 1$ for all $x \in \Omega$, we have

$$\int_\Omega |T||\nabla u| dx \leq \int_\Omega \frac{|\nabla u|}{d^{s-1}} dx.$$

Using the rule $\nabla X^{\gamma-1}(d(x)/R) = (\gamma-1)X^\gamma(d(x)/R)\frac{\nabla d(x)}{d(x)}$ for a.e. $x \in \Omega$, by a straightforward calculation we arrive at

$$\int_\Omega \operatorname{div}(T)|u| dx = (s-1)\int_\Omega \frac{|u|}{d^s} dx + \frac{\gamma-1}{R^{s-1}}\int_\Omega \frac{|u|}{d} X^\gamma(d/R) dx$$
$$+\int_\Omega \frac{|u|}{d^{s-1}}\left[1-\left(\frac{d}{R}\right)^{s-1} X^{\gamma-1}\left(\frac{d}{R}\right)\right](-\Delta d) dx.$$

Since $1-(d(x)/R)^{s-1}X^{\gamma-1}(d(x)/R) \geq 0$ for all $x \in \Omega$ and also (C) holds, we may cancel the last term and the result follows by (150).

We obtain an interpolation inequality between (153) and (157) via sets with positive reach. Let $\emptyset \neq K \subsetneq \mathbb{R}^n$ be closed, and consider the distance function to K, i.e., $d_K: \mathbb{R}^n \to [0,\infty)$ with $d_K(x) = \inf\{|x-y|: y \in K\}$. Denote by K_1 the set of points in $\mathbb{R}^n$ which have a unique closest point on K, namely $K_1 = \{x \in \mathbb{R}^n: \exists!\, y \in K \text{ such that } d_K(x) = |x-y|\}$.

Definition (2.3.13) [89]: The reach of a point $x \in K$ is $\operatorname{reach}(K,x) := \sup\{r \geq 0: B_r(x) \subset K_1\}$. The reach of the set K is reach $(K) := \inf_{x\in K}\operatorname{reach}(K,x)$.

The above definition was introduced in [98], where it was also noted that K is convex if and only if $\operatorname{reach}(K) = \infty$.

Lemma (2.3.14) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open, and set $h := \operatorname{reach}(\bar{\Omega}) \geq 0$. Then

$$(h+d)(-\Delta d) \geq -(n-1) \text{ in } \Omega, \text{ in the sense of distributions,} \tag{159}$$

where $d \equiv d(x) = \inf\{|x-y|: y \in \mathbb{R}^n \setminus \Omega\}$.

Proof: If $h = 0$, this is Lemma (2.3.6). For $h > 0$, we set $\Omega_h = \{x \in \mathbb{R}^n: d_{\bar{\Omega}}(x) < h\}$. As in the proof of Lemma (2.3.6), the continuous function $\bar{A}: \mathbb{R}^n \to \mathbb{R}$ defined by $\bar{A}(x) = |x|^2 - d^2_{\Omega_h^c}(x)$ is convex, and thus the distributional Laplacian of $\bar{A}$ is a nonnegative Radon measure on $\mathbb{R}^n$. The result follows, since for $x \in \Omega$ we have $d^2_{\Omega_h^c}(x) = d(x) + h$ (see also [98], Corollary 4.9), and thus $\Delta\bar{A} = 2(n-1-(h+d)\Delta d) \geq 0$ in Ω, in the sense of distributions.

Theorem (2.3.15) [89]: Let $\Omega \subsetneq \mathbb{R}^n$ be open, and set $h := \operatorname{reach}(\bar{\Omega})$. Suppose in addition that $R := \sup_{x\in\Omega} d(x) < \infty$. For all $u \in C_c^\infty(\Omega)$ and all $s > \frac{h+nR}{h+R}$, it holds that

$$\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx \geq \left((s-1)\frac{h}{h+R} + (s-n)\frac{R}{h+R}\right)\int_\Omega \frac{|u|}{d^s} dx. \tag{160}$$

Proof: Inserting (159) to (152), we obtain

$$\begin{aligned}\int_\Omega \frac{|\nabla u|}{d^{s-1}}dx &\geq (s-1)\int_\Omega \frac{|u|}{d^s}dx-(n-1)\int_\Omega \frac{|u|}{d^s}\frac{d}{h+d}dx\\ &=\int_\Omega \frac{(s-1)h+(s-n)d}{h+d}\frac{|u|}{d^s}dx\\ &\geq \frac{(s-1)h+(s-n)d}{h+d}\int_\Omega \frac{|u|}{d^s}dx,\end{aligned}$$

where the last inequality follows since $R<\infty$ and $\frac{(s-1)h+(s-n)d}{h+d}$ is decreasing in d.

Note that this inequality interpolates between the case of a general open set $\mathbb{R}^n$, where we have $h=0$ and the constant becomes $s-n$, and the case of a convex set Ω, where $h=\infty$ and the constant becomes $s-1$.

Before stating our result (Theorem (2.3.2)), we gather some additional properties of the distance function to the boundary that will be in use.

From now on, Ω will be a domain, i.e., an open and connected subset of $\mathbb{R}^n$. We will denote by Σ the set of points in Ω which have more than one projection on $\partial\Omega$.

If $x\in\Omega\setminus\Sigma$, then $\xi(x)$ will stand for its unique projection on the boundary.

The next lemma follows from Lemmas 14.16 and 14.17 in [45].

Lemma (2.3.16) [89]: Let $\Omega\subset\mathbb{R}^n$ be a domain (possibly unbounded) with boundary of class C^2.

(a) If in addition Ω satisfies a uniform interior sphere condition, then there exists $\delta>0$ such that $\widetilde{\Omega}_\delta:=\{x\in\bar{\Omega}: d(x)<\delta\}\subset\Omega\setminus\Sigma$ and $d\in C^2\big(\widetilde{\Omega}_\delta\big)$.

(b) $d\in C^2(\bar{\Omega}\setminus\bar{\Sigma})$ and for any $x\in\bar{\Omega}\setminus\bar{\Sigma}$, in terms of a principal coordinate system at $\xi(x)\in\partial\Omega$, it holds that

(i) $\nabla d(x)=-\vec{v}(\xi(x))=(0,\dots,0,1)$

(ii) $1-\kappa_i(\xi(x))d(x)>0$ for all $i=1,\dots,n-1$

(iii) $[D^2d(x)]=\operatorname{diag}\left[\frac{-\kappa_1(\xi(x))}{1-\kappa_1(\xi(x))d(x)},\dots,\frac{-\kappa_{n-1}(\xi(x))}{1-\kappa_{n-1}(\xi(x))d(x)},0\right]$,

Where $\vec{v}(\xi(x))$ is the unit outer normal at $\xi(x)\in\partial\Omega$, and $\kappa_1(\xi(x)),\dots,\kappa_{n-1}(\xi(x))$ are the principal curvatures of $\partial\Omega$ at the point $\xi(x)\in\partial\Omega$.

Lemma (2.3.17) [89]: Let $\Omega\subset\mathbb{R}^n$ be open. The function $\tilde{A}:\mathbb{R}^n\to\mathbb{R}$ defined by $\tilde{A}(x)=C|x|^2/2-d(x)$, is convex in any open ball $B\subset\subset\Omega$, for any $C\geq 1/\operatorname{dist}(B,\partial\Omega)$.

Proof First note that for all $a,b\in\mathbb{R}^n$ with $a\neq 0$, we have

$$|a+b|+|a-b|-2|a|\leq\frac{|b|^2}{|a|}. \quad (161)$$

We choose an open ball $B\subset\Omega$ with $r:=\operatorname{dist}(B,\partial\Omega)>0$, and take $x\in B$. Let $y\in\partial\Omega$ be such that $d(x)=|x-y|$. For any $z\in\mathbb{R}^n$ such that $x+z,x-z\in B$, we get

$$\begin{aligned}&\tilde{A}(x+z)+\tilde{A}(x-z)-2\tilde{A}(x)\\ &=C|z|^2-(d(x+z)+d(x-z)-2d(x))\\ &\geq C|z|^2-(|x+z-y|+|x-z-y|-2|x-y|)\\ &\text{(by (161)for } a=x-y \text{ and } b=z)\\ &\geq C|z|^2-\frac{|z|^2}{|x-y|}\\ &\geq (C-1/r)|z|^2.\end{aligned}$$

Since $\tilde{A}(x)$ is also continuous, we obtain that $\tilde{A}(x)$ is convex in B for any $C\geq 1/r$.

We denote by $H(y) := \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i(y)$ the mean curvature of $\partial\Omega$ at the point $y \in \partial\Omega$.

Theorem (2.3.18) [89]: Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. Then $\mu := (-\Delta d)dx$ is a signed Radon measure on Ω. Let $\mu = \mu_{ac} + \mu_s$ be the Lebesgue decomposition of μ with respect to $\mathcal{L}^n$, i.e., $\mu_{ac} \ll \mathcal{L}^n$ and $\mu_s \perp \mathcal{L}^n$. Then $\mu_s \geq 0$ in Ω, and $\mu_{ac} \geq (n-1)\underline{H}dx$ a.e. in Ω, where $\underline{H} := \inf_{y\in\partial\Omega} H(y)$.

Proof: Letting δ be as in Lemma (2.3.16)(a), we set $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$. Then $-\Delta d$ is a continuous function on Ω_δ, and so $\mu^0 := (-\Delta d)dx$ is a signed Radon measure on Ω_δ, absolutely continuous with respect to $\mathcal{L}^n$.

Next, let $\{B_i\}_{i\geq 1}$ be a cover of the set $\Omega \setminus \Omega_\delta$, composed of open balls B_i for which $\operatorname{dist}(B_i, \partial\Omega) > \delta/2$ for all $i \geq 1$. According to Lemma (2.3.17), the function $\tilde{A}(x) := |x|^2/\delta - d(x)$ is convex in each B_i. From [97], , we deduce that there exist nonnegative Radon measures $\{v^i\}_{i\geq 1}$, respectively on $\{B_i\}_{i\geq 1}$, such that

$$\int_{B_i} \phi \Delta \tilde{A} dx = \int_{B_i} \phi dv^i,$$

for all $\phi \in C_c^\infty(B_i)$. Since $\Delta\tilde{A} = 2n/\delta - \Delta d$ in the sense of distributions, we get

$$\int_{B_i} \phi(-\Delta d)dx = \int_{B_i} \phi dv^i - \frac{2n}{\delta}\int_{B_i} \phi dx, \qquad (162)$$

for all $\phi \in C_c^\infty(B_i)$, and thus $\mu^i := (-\Delta d)dx = v^i - \frac{2n}{\delta}dx$ is a signed Radon measure on B_i.

Let $\{\eta_i\}_{i\geq 1}$ be a C^∞ partition of unity subordinated to the open covering $\{B_i\}_{i\geq 1}$ of $\Omega \setminus \Omega_\delta$, i.e.,

$$\eta_i \in C_c^\infty(B_i),\ 0 \leq \eta_i(x) \leq 1 \text{ in } B_i \text{ and } \sum_{i=1}^{\infty} \eta_i(x) = 1 \text{ in } \Omega \setminus \Omega_\delta.$$

Further, for $x \in \Omega$ define $\eta_0(x) = 1 - \sum_{i=1}^{\infty}\eta_i(x)$. We then have

$$\operatorname{sprt}\eta_0 \subset \Omega_\delta,\ \eta_0(x) = 1 \text{ in } \Omega_{\delta/2} \text{ and } \sum_{i=1}^{\infty} \eta_i(x) = 1 \text{ in } \Omega.$$

We will now show that $\mu := \sum_{i=0}^{\infty}\eta_i\mu^i$ is a well-defined signed Radon measure on Ω, and $\mu = (-\Delta d)dx$. To this end, for any $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \phi(-\Delta d)dx = \sum_{i=0}^{\infty} \int_\Omega \phi\eta_i(-\Delta d)dx$$

$$by(162) = \int_\Omega \phi\eta_0 d\mu^0 + \sum_{i=0}^{\infty}\left(\int_\Omega \phi\eta_i dv^i - \frac{2n}{\delta}\int_\Omega \phi\eta_i dx\right)$$

$$= \int_\Omega \phi\eta_0 d\mu^0 + \int_\Omega \phi \sum_{i=1}^{\infty} \eta_i dv^i - \frac{2n}{\delta}\int_\Omega \phi \sum_{i=1}^{\infty} \eta_i dx$$

$$= \int_\Omega \phi\eta_0 d\mu^0 + \int_\Omega \phi \sum_{i=1}^{\infty} \eta_i d\mu^i$$

$$= \int_\Omega \phi d\mu,$$

Where the middle equality follows since v^i are positive Radon measures and thus $\sum_{i=1}^{m} \eta_i v^i$ is increasing in m (see [97]).
Next, by the Lebesgue Decomposition Theorem ([97]), $\mu = \mu_{ac} + \mu_s$ where

$$\mu_s = \sum_{i=0}^{\infty} \eta_i \mu_s^i = \sum_{i=1}^{\infty} \eta_i \mu_s^i = \sum_{i=1}^{\infty} \eta_i v_s^i \geq 0,$$

Since $\mu^i = v^i - \frac{2n}{\delta} dx$ and v^i are nonnegative. Finally, from Lemma (2.3.16)(b) we get

$$\begin{aligned} -\Delta d(x) \quad &= \sum_{i=1}^{n-1} \frac{\kappa_i(\xi(x))}{1 - \kappa_i(\xi(x))d(x)} \\ &\geq \sum_{i=1}^{n-1} \kappa_i(\xi(x)) \\ &= (n-1)H(\xi(x)) \\ \geq \quad &(n-1)\underline{H}, \forall x \in \Omega \setminus \bar{\Sigma}. \end{aligned}$$

Now by Lemma (2.3.16)(b), $-\Delta d$ is a continuous function on $\Omega \setminus \bar{\Sigma}$, and so

$$\mu_{ac} = (-\Delta d)dx \geq (n-1)\underline{H}dx \text{ in } \Omega \setminus \bar{\Sigma}.$$

Recalling that $\mathcal{L}^n(\bar{\Sigma}) = 0$ when $\partial\Omega \in C^2$ and since $\Omega = (\Omega \setminus \bar{\Sigma}) \cup \bar{\Sigma}$, we conclude $\mu_{ac} \geq (n-1)\underline{H}dx$ a.e. in Ω.

Definition (2.3.19) [89]: A domain Ω with boundary of class C^2 is said to be mean convex if $H(y) \geq 0$ for all $y \in \partial\Omega$.

Theorem (2.3.18), along with Lemma (2.3.16), provides us a characterization of mean convexity in terms of the distance function for sufficiently smooth domains. We have the following.

Corollary (2.3.20) [89]: Let Ω be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. Then Ω is mean convex if and only if condition (C) holds, i.e., $-\Delta d \geq 0$ holds in Ω, in the sense of distributions.

Let Ω be a domain satisfying property (C). We define the quotient

$$Q_\beta[u] := \frac{\int_\Omega \frac{|\nabla u|}{d^{s-1}} dx - (s-1) \int_\Omega \frac{|u|}{d^s} dx}{\int_\Omega \frac{|u|}{d^{s-\beta}} dx}; s > 1, \tag{163}$$

and we consider the minimization problem

$$B_\beta(\Omega) := \text{i } \{Q_\beta[u] : u \in C_c^\infty(\Omega) \setminus \{0\}\}; 0 < \beta \leq s-1.$$

The next proposition shows that the essential range for β is smaller.

Proposition (2.3.21) [89]: Let Ω be a domain with boundary of class C^2 satisfying property (C). If $s \geq 2$ then $B_\beta(\Omega) = 0$ for all $0 < \beta < 1$. If $1 < s < 2$ then $B_\beta(\Omega) = 0$ for all $0 < \beta \leq s-1$. Proof: For small $\delta > 0$, let $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$ and $\Omega_\delta^c = \Omega \setminus \Omega_\delta$. We test (163) with $u_\delta(x) = \chi_{\Omega_\delta^c}(x)\phi(x)$, where $\zeta \in C_c^\infty(B_\varepsilon(y_0))$ for a fixed $y_0 \in \partial\Omega$ and sufficiently small ε, satisfying $\varepsilon > 3\delta$. We may suppose in addition that $0 \leq \phi \leq 1$ in $B_\varepsilon(y_0)$, $\phi \equiv 1$ in $B_{\frac{\varepsilon}{2}}(y_0)$, and $|\nabla\phi| \leq 1/\varepsilon$. This function is not in $C_c^\infty(\Omega)$, but since it is in $BV(\Omega)$ we can mollify the characteristic function so that the calculations below hold in the limit. The distributional gradient of u_δ is $\nabla u_\delta = \chi_{\Omega_\delta^c}^c \nabla\phi - \vec{v}\phi\delta_{\partial\Omega_\delta^c}$, where $\vec{v}$ is the outward pointing, unit normal vector field along $\partial\Omega_\delta^c$, and $\delta_{\partial\Omega_\delta^c}$ is the Dirac measure on $\partial\Omega_\delta^c$.

Moreover, the total variation of ∇u_δ is $|\nabla u_\delta| = \chi_{\Omega_\delta^c}|\nabla\phi| + \phi\delta_{\partial\Omega_\delta^c}$. Since $\partial\Omega_\delta^c = \{x \in \Omega: d(x) = \delta\}$, we obtain

$$Q_\beta[u_\delta] = \frac{\int_{\Omega_\delta^c} |\nabla\phi| d^{1-s}dx + \delta^{1-s}\int_{\partial\Omega_\delta^c} \phi dS_x - (s-1)\int_{\Omega_\delta^c} \phi d^{-s}dx}{\int_{\Omega_\delta^c} \phi d^{\beta-s}dx}. \quad (164)$$

Using the fact that $|\nabla d(x)| = 1$ for a.e. $x \in \Omega$, we may perform an integration by parts in the last term of the numerator as follows:

$$(s-1)\int_{\Omega_\delta^c} \phi d^{-s}dx = -\int_{\Omega_\delta^c} \phi\nabla d \cdot \nabla d^{1-s}dx$$

$$= \int_{\Omega_\delta^c} [\nabla\phi \cdot \nabla d]d^{1-s}dx + \int_{\Omega_\delta^c} \phi d^{1-s}\Delta d dx - \delta^{1-s}\int_{\partial\Omega_\delta^c} \phi\nabla d \cdot \vec{v} dS_x.$$

Since ∇d is the inner unit normal to $\partial\Omega$, we have $\nabla d \cdot \vec{v} = -1$, and substituting the above equality in (164), the surface integrals will be canceled to get

$$Q_\beta[u_\delta] = \frac{\int_{\Omega_\delta^c} [|\nabla\phi| - \nabla\phi \cdot \nabla d]d^{1-s}dx + \int_{\Omega_\delta^c} \phi d^{1-s}(-\Delta d)dx}{\int_{\Omega_\delta^c} \phi d^{\beta-s}dx}.$$

By the fact that $-\Delta d(x) \le c$ for all $x \in \Omega_\delta^c \cap B_\varepsilon$, and by the properties we imposed on ϕ, we get

$$Q_\beta[u_\delta] \le \frac{\frac{2}{\varepsilon}\int_{\Omega_\delta^c\cap B_\varepsilon} d^{1-s}dx + c\int_{\Omega_\delta^c\cap B_\varepsilon} d^{1-s}dx}{\int_{\Omega_\delta^c\cap B_{\varepsilon/2}} d^{\beta-s}dx}$$

$$= c(\varepsilon)\frac{\frac{2}{\varepsilon}\int_{\Omega_\delta^c\cap B_\varepsilon} d^{1-s}dx}{\int_{\Omega_\delta^c\cap B_{\varepsilon/2}} d^{\beta-s}dx}$$

$$= \quad : c(\varepsilon)\frac{N(\delta)}{D(\delta)}.$$

Now using the co-area formula, we compute

$$N(\delta) = \int_\delta^\varepsilon r^{1-s}\int_{\{x\in\Omega_\delta^c\cap B_\varepsilon: d(x)=r\}} dS_x dr$$

$$\le c_1(\varepsilon)\int_\delta^\varepsilon r^{1-s}dr,$$

Where $c_1(\varepsilon) = \max_{r\in[0,\varepsilon]}|\{x \in \Omega_\delta^c \cap B_\varepsilon: d(x) = r\}|$. Also,

$$D(\delta) = \int_\delta^{\varepsilon/2} r^{\beta-s}\int_{\{x\in\Omega_\delta^c\cap B_{\varepsilon/2}: d(x)=r\}} dS_x dr$$

$$\ge \int_\delta^{\varepsilon/3}\int_{\{x\in\Omega_\delta^c\cap B_{\varepsilon/2}: d(x)=r\}} dS_x dr$$

$$\le c_2(\varepsilon)\int_\delta^{\varepsilon/3} r^{\beta-s}dr,$$

Where $c_2(\varepsilon) = \min_{r\in[0,\varepsilon/3]}\left|\{x \in \Omega_\delta^c \cap B_{\varepsilon/2}: d(x) = r\}\right|$. A direct computation reveals that if $s \ge 2$ then $Q_\beta[u_\delta] \le o_\delta(1)$ for all $0 < \beta < 1$, and also if $1 < s < 2$ then $Q_\beta[u_\delta] \le o_\delta(1)$ for all $0 < \beta \le s - 1$.

We obtain upper and lower estimates for $B_1(\Omega)$. In particular, we prove Theorem (2.3.1) and the optimality in Theorem (2.3.5).

Theorem (2.3.22) [89]: (Lower Estimate) Let Ω be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. If $s \geq 1$, then

$$B_1(\Omega) \geq (n-1)\underline{H}, \tag{165}$$

Where $\underline{H}$ is the infimum of the mean curvature of $\partial\Omega$.

Proof : The estimate follows directly from (152) using Theorem (2.3.18).

Definition (2.3.23) [89]: The Cheeger constant $h(\Omega)$ of a bounded domain Ω with piecewise C^1 boundary is defined by $h(\Omega) := \inf_\omega \frac{|\partial\omega|}{|\omega|}$, where the infimum is taken over all sub-domains $\omega \subset\subset \Omega$ with piecewise C^1 boundary.

For the existence of minimizers, uniqueness, and regularity results concerning the Cheeger constant, see [100].

Proposition (2.3.24) [89]: Let Ω be a bounded domain with piecewise C^1 boundary such that condition (C) holds. For all $s \geq 1$, we have $B_1(\Omega) \leq h(\Omega)$.

Proof : Take $\omega \subset\subset \Omega$ with piecewise C^1 boundary, and let $u_\omega(x) = (d(x))^{s-1}\chi_\omega(x)$. The distributional gradient and the total variation of this $BV(\Omega)$ function are, respectively, $\nabla u_\omega = (s-1)d^{s-2}\chi_\omega \nabla d - \vec{v} d^{s-1}\delta_{\partial\omega}$ and $|\nabla u_\omega| = (s-1)d^{s-2}\chi_\omega + d^{s-1}\delta_{\partial\omega}$, where $\vec{v}$ is the outward pointing, unit normal vector field along $\partial\omega$, and $\delta_{\partial\omega}$ is the uniform Dirac measure on $\partial\omega$. We test (163) with u_ω to get

$$Q_1[u_\omega] = (s-1)\int_\omega d^{-1}dx + \frac{\int_{\partial\omega} dS_x - (s-1)\int_\omega d^{-1}dx}{\int_\omega dx} = \frac{|\partial\omega|}{|\omega|}.$$

In particular, $h(\Omega) = \inf_\omega Q_1[u_\omega]$. By the standard C_c^∞ approximation of the characteristic function of the domain ω, we obtain $B_1(\Omega) \leq \frac{|\partial\omega|}{|\omega|}$, and thus $B_1(\Omega) \leq h(\Omega)$.

From Theorem (2.3.22) and Proposition (2.3.24) for $s = 1$, we conclude the following.

Corollary (2.3.25) [89]: If Ω is a strictly mean convex, bounded domain with boundary of class C^2, it holds that $h(\Omega) \geq (n-1)\underline{H}$.

Note that in [90] it is proved that if a bounded convex domain Ω is a self-minimizer of $h(\Omega)$, then it belongs to the class $C^{1,1}$, and also the stronger estimate $h(\Omega) \geq (n-1)\bar{H}$ holds. Here $\bar{H}$ is the essential supremum of the mean curvature of the boundary (the last being defined in the almost everywhere sense, since $\partial\Omega \in C^{1,1}$).

The following result states a more useful upper bound for $B_1(\Omega)$. It will be combined with Theorem (2.3.22) to give the best possible constant for special geometries.

Theorem (2.3.26) [89]: Let Ω be a domain with boundary of class C^2 satisfying a uniform interior sphere condition. If $s \geq 2$, then for all $\phi \in C_c^1(\partial\Omega)$,

$$B_1(\Omega) \leq \frac{(n-1)\int_{\partial\Omega} |\phi(y)|H(y)dS}{\int_{\partial\Omega} |\phi(y)|dS} + \frac{\int_{\partial\Omega} |\nabla\phi(y)|dS}{\int_{\partial\Omega} |\phi(y)|dS},$$

Where $H(y)$ is the mean curvature at the point $y \in \partial\Omega$.

Proof: Let $\delta > 0$ such that for all $x \in \widetilde{\Omega}_\delta := \{x \in \bar{\Omega}: d(x) < \delta\}$ there exists a unique point

$$\xi \equiv \xi(x) = x - d(x)\nabla d(x) \in \partial\Omega \tag{166}$$

with $d(x) = |x - \xi|$. For any $t \in [0, \delta]$ the surface area element of $\partial\Omega_t^c = \{x \in \Omega: d(x) = t\}$ is given by

$$dS_t = (1-\kappa_1 t)\cdots(1-\kappa_{n-1}t)dS = \big(1-(n-1)tH + O(t^2)\big)dS, \tag{167}$$

where $\kappa_1, \dots, \kappa_{n-1}$, are the principal curvatures of $\partial\Omega, dS$ is the surface area element of $\partial\Omega$, and H is the mean curvature of $\partial\Omega$, (see [107], Sects. 13.5 and 13.6). Now let $0 < \varepsilon < \delta$ and chose $\phi \in C_c^1(\partial\Omega)$. We test (163) with $u_\varepsilon(x) = \chi_{\Omega_\varepsilon^c \setminus \Omega_\delta^c}(x)\phi(\xi(x)), \xi(x)$ as in (166), and then we will check the limit as $\varepsilon \downarrow 0$. The distributional gradient of u_ε, is $\nabla u_\varepsilon = \left(\vec{v}_\delta \delta_{\partial\Omega_\delta^c} - \vec{v}_\varepsilon \delta_{\partial\Omega_\varepsilon^c}\right)|\phi(\xi)| + \chi_{\Omega_\varepsilon^c \setminus \Omega_\delta^c} \nabla_x \phi(\xi)$, where $\vec{v}_\delta, \vec{v}_\varepsilon$ are, respectively, the outward pointing unit normal vector fields along $\partial\Omega_\delta^c, \partial\Omega_\varepsilon^c$. Its total variation is $|\nabla u_\varepsilon| = \left(\delta_{\partial\Omega_\delta^c} + \delta_{\partial\Omega_\varepsilon^c}\right)|\phi(\xi)| + \chi_{\Omega_\varepsilon^c \setminus \Omega_\delta^c}^c |\nabla_x \phi(\xi)|$. Thus,

$$\begin{aligned}\int_\Omega \frac{|\nabla u_\varepsilon|}{d^{s-1}} dx \quad &= \delta^{1-s} \int_{\partial\Omega_\delta^c} |\phi(\xi)| dS_\delta + \varepsilon^{1-s} \int_{\partial\Omega_\delta^c} |\phi(\xi)| dS_\varepsilon \\ &+ \int_{\Omega_\varepsilon^c \setminus \Omega_\delta^c} \frac{|\nabla_x \phi(\xi)|}{d^{s-1}} dx.\end{aligned} \tag{168}$$

The first integral on the right-hand side of (168) is a constant, since we will keep δ fixed. We perform the change of variables $y = \xi(x)$ in the second integral. Using (467), we have

$$\begin{aligned}\varepsilon^{1-s} \int_{\partial\Omega_\varepsilon^c} |\phi(\xi)| dS_\varepsilon &= \varepsilon^{1-s} \int_{\partial\Omega} |\phi(y)| \big(1 - (n-1)\varepsilon H(y) + O(\varepsilon^2)\big) dS \quad (169) \\ &= \varepsilon^{1-s} M - (n-1)\, \varepsilon^{2-s} M_H + O(\varepsilon^{3-s}),\end{aligned}$$

Where $M := \int_{\partial\Omega} |\phi| dS$ and $M_H := \int_{\partial\Omega} |\phi| H dS$. Using the co-area formula, the third term on the righthand side of (168) is written as follows:

$$\int_{\Omega_\varepsilon^c \setminus \Omega_\delta^c} \frac{|\nabla_x \phi(\xi)|}{d^{s-1}} dx = \int_\varepsilon^\delta t^{1-s} \int_{\partial\Omega_t^c} |\nabla_x \phi(\xi)| dS_t dt. \tag{170}$$

From (166) we have $\xi_i(x) = x_i - d(x) \frac{\partial}{\partial x_i}(d(x))$, and thus by Lemma (2.3.16)(c) we compute

$$\begin{aligned}\nabla_x \phi(\xi) \quad &= \left(\sum_{i=1}^n \phi_{\xi_i}(\xi) \frac{\partial \xi_i}{\partial x_1}, \dots, \sum_{i=1}^n \phi_{\xi_i}(\xi) \frac{\partial \xi_i}{\partial x_n} \right) \\ &= \left(\frac{\phi_{\xi_1}(\xi)}{1 - \kappa_1 d}, \dots, \frac{\phi_{\xi_{n-1}}(\xi)}{1 - \kappa_{n-1} d}, 0 \right).\end{aligned}$$

Thus, (170) becomes

$$\int_{\Omega_\varepsilon^c \Omega_\delta^c} \frac{|\nabla_x \phi(\xi)|}{d^{s-1}} dx = \int_\varepsilon^\delta t^{1-s} \int_{\partial\Omega} \left(\sum_{i=1}^{n-1} \left(\frac{\phi_{y_i}}{1 - \kappa_i t} \right)^2 \right)^{1/2} dS_t dt$$

$$\int_\varepsilon^\delta t^{1-s} \int_{\partial\Omega} \left(\sum_{i=1}^{n-1} \left(\phi_{y_i} \prod_{j=1, j\neq i}^{n-1} (1 - \kappa_j t) \right)^2 \right)^{1/2} dS dt,$$

where we have changed variables by $y = \xi(x)$ in the last inequality. Expanding the product as in (167), we get

$$\int_{\Omega_\varepsilon^c \setminus \Omega_\delta^c} \frac{|\nabla_x \phi(\xi)|}{d^{s-1}} dx$$

$$\leq \int_{\varepsilon}^{\delta} t^{1-s} \int_{\partial\Omega} \left(\sum_{i=1}^{n-1} \phi_{y_i}^2 (1-[(n-1)H-\kappa_i]t + c_1 t^2)^2 \right)^{1/2} dSdt$$

$$\leq K \int_{\varepsilon}^{\delta} t^{1-s} dt + c_2 \int_{\varepsilon}^{\delta} t^{2-s} dt, \quad (171)$$

for some $c_1, c_2 \geq 0$, where $K := \int_{\partial\Omega}^{\varepsilon} |\nabla\phi| dS$. Next, using the co-area formula and the same change of variables, we get

$$(s-1)\int_{\Omega} |u_\varepsilon|/d^s dx = (s-1)\int_{\varepsilon}^{\delta} t^{-s} \int_{\partial\Omega_t^c} |\phi(\xi)| dS_t dt$$

$$\geq (s-1)\int_{\varepsilon}^{\delta} t^{-s} \int_{\partial\Omega} |\phi(y)|[1-(n-1)tH(y)+c_3 t^2] dSd$$

$$= M\varepsilon^{1-s} - (s-1)(n-1)M_H \int_{\varepsilon}^{\delta} t^{1-s} dt$$

$$+ c_4 \int_{\varepsilon}^{\delta} t^{2-s}, \quad (172)$$

for some $c_3, c_4 \in \mathbb{R}$, and similarly,

$$\int_{\Omega} \frac{|u_\varepsilon|}{d^{s-\beta}} dx \geq \quad M\int_{\varepsilon}^{\delta} t^{\beta-s} dt - (n-1)M_H \int_{\varepsilon}^{\delta} t^{1+\beta-s} dt$$
$$+ c_5 \int_{\varepsilon}^{\delta} t^{2+\beta-s} dt, \quad (173)$$

for some $c_5 \in \mathbb{R}$. Thus inserting (169), (171), (172) into (168), and by (173) for $\beta = 1$, we get

$$Q_\beta[u_\varepsilon] \leq \frac{(n-1)M_H\left[(s-1)\int_{\varepsilon}^{\delta} t^{1-s}dt - \varepsilon^{2-s}\right] + K\int_{\varepsilon}^{\delta} t^{1-s}dt + c_6 \int_{\varepsilon}^{\delta} t^{2-s}dt}{M\int_{\varepsilon}^{\delta} t^{\beta-s}dt - (n-1)M_H\int_{\varepsilon}^{\delta} t^{1+\beta-s}dt + c_5\int_{\varepsilon}^{\delta} t^{2+\beta-s}dt}, \quad (174)$$

for some $c_6 \in \mathbb{R}$. If $s = 2$, then

$$Q_1[u_\varepsilon] \leq \frac{\left((n-1)M_H + K\right)\log\left(\frac{\delta}{\varepsilon}\right) + O_\varepsilon(1)}{\mathrm{M}\log\left(\frac{\delta}{\varepsilon}\right) + O_\varepsilon(1)},$$

While if $s > 2$, then

$$Q_1[u_\varepsilon] \leq \frac{\frac{1}{s-2}\left((n-1)M_H + K\right)\varepsilon^{2-s} + c_7\int_{\varepsilon}^{\delta} t^{2-s}dt}{\frac{1}{s-2}M\varepsilon^{2-s} - (n-1)M_H\int_{\varepsilon}^{\delta} t^{2-s}dt + c_8\int_{\varepsilon}^{\delta} t^{3-s}dt},$$

For some $c_7, c_8 \in \mathbb{R}$. In any case, letting $\varepsilon \downarrow 0$ we deduce $B_1(\Omega) \leq \frac{(n-1)M_H + K}{M}$.

An immediate consequence is

Corollary (2.3.27) [89]: (Upper Estimate) Let Ω be a bounded domain with boundary of class C^2. If $s \geq 2$, then

$$B_1(\Omega) \leq \frac{n-1}{|\partial\Omega|}\int_{\partial\Omega} H(y)dS,$$

where $H(y)$ is the mean curvature at the point $y \in \partial\Omega$.

Proof : Since Ω is bounded, we can chose $\varphi \equiv 1$ in the above theorem.

The proof of Theorem (2.3.1) follows from Proposition (2.3.21), Theorem (2.3.22), and Corollary (2.3.27).

Example (2.3.28) [89]: (Ball) Let B_R be a ball of radius R. By Theorem (2.3.22) we have $B_1(B_R) \geq \frac{n-1}{R}$, and by Corollary (2.3.27), $B_1(B_R) \geq \frac{n-1}{R}$.We conclude that if $s \geq 2$, then $B_1(B_R) \geq \frac{n-1}{R}$.

Example (2.3.29) [89]: (Infinite Strip: Proof of the Optimality in Theorem (2.3.5)) Let $S_R = \{x = (x', x_n): x \in \mathbb{R}^{n-1}, 0 < x_n < 2R\}$. If $s \geq 2$, then by combining Theorems (2.3.22) and (2.3.26), we can prove that $B_1(S_R) = 0$. In fact, we have $B_\beta(S_R) = 0$ for any $1 < \beta \leq s - 1$, and in particular, we will prove that if $\gamma = 1$, there is no positive constant C such that (158) holds for $\gamma = 1$. To see this, pick any $\phi \equiv \phi(x') \in C_c^1(\mathbb{R}^{n-1})$ such that $\text{sprt}\{\phi\} \subset B_1 \subset \mathbb{R}^{n-1}$, where B_1 is the open ball in $\mathbb{R}^{n-1}$ with radius 1 , centered at $0'$. Let $\eta > 0$ and set $\phi_\eta \equiv \phi_\eta(x') := \phi(\eta x')$. Note that $\text{sprt}\{\phi_\eta\} \subset B_{1/\eta}$. Also let $0 < \varepsilon < \delta$ for some fixed $\delta \leq R$ (so that $d(x) = x_n$). The quotient corresponding to (158) is

$$Q_\gamma[u] = \frac{\int_{S_R} \frac{|\nabla u|}{d^{s-1}} dx - (s-1) \int_{S_R} \frac{|u|}{d^s} dx}{\int_{S_R} \frac{|u|}{d} X^\gamma \left(\frac{d}{R}\right) dx}. \tag{175}$$

As in the proof of Theorem (2.3.26), we test (175) with $u_{\varepsilon,\eta}(x) := \chi_{(\varepsilon,\delta)}(x_n)\phi_\eta(x')$ to arrive at

$$Q_\gamma[u_{\varepsilon,\eta}] = \frac{K_\eta \int_\varepsilon^\delta x_n^{1-s} dx_n + 2M_\eta \delta^{1-s}}{M_\eta \int_\varepsilon^\delta x_n^{-1} X^\gamma(x_n/R) dx_n},$$

where we have set $M_\eta := \int_{B_{1/\eta}} |\phi_\eta(x')| dx$ and $K_\eta := \int_{B_{1/\eta}} |\nabla_{x'} \phi_\eta(x')| dx$. Changing variables by $y' = \delta x'$, we obtain

$$\frac{K_\eta}{M_\eta} = \frac{K_1 \eta^{-(n-2)}}{M_1 \eta^{-(n-1)}} = \frac{K_1}{M_1} \eta,$$

Where $M_1 = \int_{B_1} |\phi(y')| dy'$ and $K_1 = \int_{B_1} |\nabla_{y'} \phi(y')| dy$. Thus,

$$Q_\gamma[u_{\varepsilon,\eta}] = \frac{\frac{K_1}{M_1} \eta \int_\varepsilon^\delta x_n^{1-s} dx_n + 2\delta^{1-s}}{\int_\varepsilon^\delta x_n^{-1} X^\gamma(x_n/R) dx_n}.$$

Now we select $\eta = \varepsilon^{s-2+\epsilon}$ for some fixed $\epsilon > 0$. We deduce

$$Q_1[u_{\varepsilon,\eta}] = \frac{\frac{K_1}{M_1} \varepsilon^{s-2+\epsilon} \int_\varepsilon^\delta x_n^{1-s} dx_n + 2\delta^{1-s}}{\log\left(\frac{X(\delta/R)}{X(\varepsilon/R)}\right)}.$$

It follows that $Q_1[u_{\varepsilon,\eta}] \to 0$, as $\varepsilon \downarrow 0$. Thus, for $\Omega = S_R$, inequality (158) does not hold when $\gamma = 1$ and the exponent 1 on the distance function in the remainder term in (158) cannot be increased. we assume Ω is a ball of radius R. Without loss of generality, we assume it is centered at the origin, and denote it by B_R. The distance function to the boundary is then $d(x) = R - r$, where $r := |x|$. Moreover,

$$-\Delta d(x) = \frac{n-1}{R - d(x)}, x \in B_R \setminus \{0\}. \tag{176}$$

This is devoted to the proof of the following fact.

Theorem (2.3.30) [89]: (a) For all $u \in C_c^\infty(B_R), s \geq 2$, and $\gamma > 1$, it holds that

$$\int_{B_R} \frac{|\nabla u|}{d^{s-1}} dx \quad \geq (s-1) \int_{B_R} \frac{|u|}{d^s} dx + \sum_{k=1}^{[s]-1} \frac{n-1}{R^k} \int_{B_R} \frac{|u|}{d^{s-k}} dx + \frac{C}{R^{s-1}} \int_{B_R} \frac{|u|}{d} dX^\gamma\left(\frac{d}{R}\right) dx, \tag{177}$$

where $C \geq \gamma - 1$. The exponents $s-k, k = 1,2, \dots, [s]-1$, on the distance function as well as the constants $(n-1)/R^k, k = 1,2, \dots, [s]-1$, in the summation terms are optimal. If $\gamma = 1$, the above inequality fails in the sense of (180).

(b) For all $u \in C_c^\infty(B_R), 1 \leq s < 2$, and $\gamma > 1$, it holds that

$$\int_{B_R} \frac{|\nabla u|}{d^{s-1}} dx \geq (s-1) \int_{B_R} \frac{|u|}{d^s} dx + \frac{C}{R^{s-1}} \int_{B_R} \frac{|u|}{d} dX^\gamma\left(\frac{d}{R}\right) dx, \tag{178}$$

where $C \geq \gamma - 1$. If $\gamma = 1$, the above inequality fails in the sense of (180).

Remark (2.3.31) [89]: The optimality of the exponents and the constants stated in the above theorem is meant in the following sense: For any $s \geq 1$, set

$$I_0[u] := \int_{B_R} \frac{|\nabla u|}{d^{s-1}} dx - (s-1) \int_{B_R} \frac{|u|}{d^s} dx,$$

and also for any $s \geq 2$, set

$$I_m[u] := I_0[u] - \sum_{k=1}^{m} \frac{n-1}{R^k} \int_{B_R} \frac{|u|}{d^{s-k}} dx, m = 1, \dots, [s]-1.$$

Then, for any $s \geq 2$,

$$\inf_{u \in C_c^\infty(B_R) \setminus \{0\}} \frac{I_m[u]}{\int_{B_R} \frac{|u|}{d^\beta} dx} = \begin{cases} \frac{n-1}{R^{m+1}}, & \text{if } \beta = s-m-1, \\ 0, & \text{if } \beta > s-m-1, \end{cases} \tag{179}$$

for all $m \in \{0, \dots, [s]-2\}$. Further, for any $s \geq 1$,

$$\inf_{u \in C_c^\infty(B_R) \setminus \{0\}} \frac{I_{[s]-1}[u]}{\int_{B_R} \frac{|u|}{d} X(d/R) dx} = 0. \tag{180}$$

Proof: Inequality (178) is evident by Theorem (2.3.12). Let $s \geq 2$ and $\gamma > 1$. Since inequality (177) is scale invariant, it suffices to prove it for $R = 1$. Testing (150) with

$$T(x) = -(d(x))^{1-s}[1 - (d(x))^{s-1} X^{\gamma-1}(d(x))] \nabla d(x), x \in B_1 \setminus \{0\},$$

we arrive at

$$\int_{B_1} \operatorname{div}(T)|u| dx = (s-1) \int_{B_1} \frac{|u|}{d^s} dx + \int_{B_1} \frac{|u|}{d^{s-1}} (1 - d^{s-1} X^{\gamma-1}(d))(-\Delta d) dx + (\gamma-1) \int_{B_1} \frac{|u|}{d} X^\gamma(d) dx.$$

Thus, using (176) for $R = 1$, we obtain

$$\int_{B_1} \operatorname{div}(T)|u| dx = (s-1) \int_{B_1} \frac{|u|}{d^s} dx + (n-1) \int_{B_1} \frac{|u|}{d^{s-1}} \frac{1 - d^{s-1} X^{\gamma-1}(}{1-d} + (\gamma-1) \int_{B_1} \frac{|u|}{d} X^\gamma(d) dx. \tag{181}$$

Since $s \geq 2$, we take into account in (181) the fact that

$$\frac{1-d^{s-1}X^{\gamma-1}(d)}{1-d} \geq \frac{1-d^{s-1}}{1-d} \geq \frac{1-d^{[s]-1}}{1-d} = \sum_{k=1}^{[s]-1} d^{k-1}, x \in B_1 \setminus \{0\},$$

and finally arrive at

$$I_0[u] \geq (s-1)\int_{B_1} \frac{|u|}{d^s}dx + (n-1)\sum_{k=1}^{[s]-1} dx + (\gamma-1)\int_{B_1} \frac{|u|}{d^s}X^\gamma(d)dx,$$

which is (177) for $R = 1$.

We next prove (179). Suppose first that $2 \leq s < 3$. In this case, all we have to prove is that

$$\inf_{u\in C_0^1(B_1)\setminus\{0\}} \frac{I_0[u]}{\int_{B_1} \frac{|u|}{d^\beta}dx} = \begin{cases} n-1, & \text{if } \beta = s-1 \\ 0, & \text{if } \beta > s-1 \end{cases}. \tag{182}$$

To this end, we pick $u_\delta(x) = \chi_{B_{1-\delta}}(x)$, where $x \in B_1$ and $0 < \delta < 1$. This function is in $BV(B_1)$, and we can take a C_c^∞ approximation of it, so that the calculations below hold in the limit. The distributional gradient of u_δ is $\nabla u_\delta = -\vec{v}_{\partial B_{1-\delta}}\delta_{\partial B_{1-\delta}}$, and the total variation of ∇u_δ is $|\nabla u_\delta| = \delta_{\partial B_{1-\delta}}$. Using the co-area formula, we get

$$\begin{aligned}\frac{I_0[u]}{\int_{B_1} \frac{|u|}{d^\beta}dx} &= \frac{\delta^{1-s}|\partial B_{1-\delta}| - (s-1)\int_0^{1-\delta} (1-r)^{-s}|\partial B_r|dr}{\int_0^{1-\delta} (1-r)^{-\beta}|\partial B_r|dr} \\ &= \frac{\delta^{1-s}(1-\delta)\int_0^{1-\delta} ((1-r)^{-s})' r^{n-1}dr}{\int_0^{1-\delta} (1-r)^{-\beta}r^{n-1}dr} \\ &= (n-1)\frac{\int_0^{1-\delta} (1-r)^{1-s}r^{n-2}dr}{\int_0^{1-\delta} (1-r)^{-\beta}r^{n-1}dr}.\end{aligned}$$

Thus,

$$\frac{I_0[u_\delta]}{\int_{B_1} \frac{|u_\delta|}{d^\beta}dx} \to \begin{cases} n-1, \text{ if } \beta = s-1 \\ 0, \text{ if } \beta > s-1 \end{cases} \text{ as } \delta \downarrow 0.$$

Assume next that $3 \leq s < 4$. This time, besides (182) we have to prove that

$$\inf_{u\in C_c^\infty(B_1)\setminus\{0\}} \frac{I_1[u]}{\int_{B_1} \frac{|u|}{d^\beta}dx} = \begin{cases} n-1, & if \;\; \beta = s-2 \\ \quad 0, & if \; \beta > s-2 \end{cases}.$$

Picking the same u_δ as before and performing the same integration by parts in the second term of the numerator, we conclude

$$\begin{aligned}\frac{I_1[u_\delta]}{\int_{B_1} \frac{|u_\delta|}{d^\beta}dx} &= \frac{(n-1)\int_0^{1-\delta} (1-r)^{-s}r^{n-1}dr - (n-1)\int_0^{1-\delta} (1-r)^{1-s}r^{n-1}dr}{\int_0^{1-\delta} (1-r)^{-\beta}r^{n-1}dr} \\ &= (n-1)\frac{\int_0^{1-\delta} (1-r)^{2-s}r^{n-2}dr}{\int_0^{1-\delta} (1-r)^{-\beta}r^{n-1}dr}.\end{aligned}$$

Thus,

$$\frac{I_1[u_\delta]}{\int_{B_1}\frac{|u_\delta|}{d^\beta}dx}\to\begin{cases}n-1, \text{ if } \beta=s-2\\ 0, \text{ if } \beta>s-2\end{cases}, \text{ as } \delta\downarrow 0.$$

We continue in the same fashion for $4\le s<5$, then $5\le s<6$, and so on.
Next we prove (180). We pick u_δ as before, and perform the same integration by parts to get

$$\frac{I_{[s]-1}[u_\delta]}{\int_{B_1}\frac{|u_\delta|}{d^\beta}X(d)dx}=\frac{(n-1)\int_0^{1-\delta}(1-r)^{1-s}r^{n-2}dr-(n-1)\sum_{k=1}^{[s]-1}\int_0^{1-\delta}(1-r)^{k-s}r^{n}}{\int_0^{1-\delta}(1-r)^{-1}r^{n-1}X(1-r)dr}$$

$$=(n-1)\frac{\int_0^{1-\delta}(1-r)^{[s]-s}r^{n-2}dr}{\int_0^{1-\log\delta}(1-e^{1-t})^{n-1}dt}$$

$$=:(n-1)\frac{N_\delta}{D_\delta}.$$

Since $[s]-s>-1$, we have $N_\delta=O_\delta(1)$ as $\delta\downarrow 0$. Also, $D_\delta\ge\int_0^{1-\log\delta}t^{-1}dt+O_\delta(1)\to\infty$, as $\delta\downarrow 0$.
we discuss how far our results can go in the L^p setting. We start with the L^p analog of Lemma (2.3.7).

Lemma (2.3.32) [89]: Let $\Omega\subset\mathbb{R}^n$ be open. For all $u\in C_c^\infty(\Omega)$, all $s>1, p\ge 1$, it holds that

$$\int_\Omega\frac{|\nabla u|^p}{d^{s-p}}dx-\left(\frac{s-1}{p}\right)^p\int_\Omega\frac{|u|^p}{d^s}dx+\left(\frac{s-1}{p}\right)^p\int_\Omega\frac{|u|^p}{d^{s-1}}(-\Delta d)dx,\quad(183)$$

where $-\Delta d$ is meant in the distributional sense.

Proof: We substitute u by $|u|^p$ with $p>1$ in (152), to arrive at

$$\frac{p}{s-1}\int_\Omega\frac{|\nabla u||u|^p}{d^{s-1}}dx\ge\int_\Omega\frac{|u|^p}{d^s}dx+\frac{1}{s-1}\int_\Omega\frac{|u|^p}{d^{s-1}}(-\Delta d)dx.\quad(184)$$

The left-hand side in (184) can be written as follows:

$$\frac{p}{s-1}\int_\Omega\frac{|\nabla u||u|^p}{d^{s-1}}dx=\int_\Omega\left\{\frac{p}{s-1}\frac{|\nabla u|}{d^{s/p-1}}\right\}\left\{\frac{|u|^{p-1}}{d^{s-s/p}}\right\}dx$$

$$\le\frac{1}{p}\left(\frac{p}{s-1}\right)^{p-1}\int_\Omega\frac{|\nabla u|^p}{d^{s-p}}dx+\frac{p-1}{p}\int_\Omega\frac{|u|^p}{d^s}dx,$$

by Young's inequality. Thus, (184) becomes

$$\frac{1}{p}\left(\frac{p}{s-1}\right)^{p-1}\int_\Omega\frac{|\nabla u|^p}{d^{s-p}}dx\ge\frac{1}{p}\int_\Omega\frac{|u|^p}{d^s}dx+\frac{1}{s-1}\int_\Omega\frac{|u|^p}{d^{s-1}}(-\Delta d)dx.$$

Rearranging the constants, we arrive at the inequality we sought.

Chapter 3
Characterization and Some Weighted Estimates

We show that the definitions and spaces are extended in a natural way and it is proven that this is the same space as which justifies the standard convention in which the spaces are defined to be equal. As a consequence, we obtain a new characterization of the Hölder-Zygmund space. We also show some weighted estimates for the Bochner-Riesz operators and the spherical means.

Section (3.1): Triebel-Lizorkin Spaces for $p = \infty$

[110], [111] We obtained characterizations of weighted Besov-Lipschitz spaces $\dot{B}_{p,q}^{\alpha,w}(-\infty < \alpha < \infty, 0 < p, q \leq \infty)$ and weighted Triebel-Lizorkin spaces $\dot{F}_{p,q}^{\alpha,w}(-\infty < \alpha < \infty, 0 < p < \infty, 0 < q \leq \infty$) by means of "generalized" Littlewood-Paley functions, where $w \in A_\infty$. In this article, We complete the characterizations in [110], [111] by considering the remaining case of the spaces $\dot{F}_{\infty,q}^{\alpha,w}$

Note that $\dot{F}_{\infty,2}^{0} = BMO$ (see [116]), and hence the characterization we obtain is an extension of the known result for this most important case.

A space of particular interest is $\dot{F}_{\infty,\infty}^{\alpha}$. The definition of $\dot{F}_{\infty,q}^{\alpha}$ introduced by Frazier and Jawerth [116] was a major step, but the extension of that definition to the case " $q = \infty$ ' remained elusive. The problem is that any such definition is expected to make that space coincide with the Besov-Lipschitz space $\dot{B}_{\infty,\infty}^{\alpha}$. The ad hoc solution has been to make the two spaces the same by definition. As a consequence of the maximal inequalities used to prove Theorem (3.1.2)(i) we are able to use the natural definition suggested by the work of Frazier and

Jawerth in [116] and obtain the identification of the two spaces in Theorem (3.1.4). Moreover, Theorem (3.1.8) shows that there is a family of "natural" norms which are equivalent to each other and each of which characterizes $\dot{B}_{\infty,\infty}^{\alpha} = \dot{F}_{\infty,\infty}^{\alpha,w}$.

All functions and distributions are defined on $\boldsymbol{R}^n$We use S to denote the Schwartz space of test functions and S' its dual, the space of tempered distributions.

The known result (see [126]) for the space BMO mentioned above is:

Proposition (3.1.1)[108]:

Let $\Phi \in S$ with

$$\int_{R^n} \Phi(\mathrm{x})dx = 0. \tag{1}$$

(i) Suppose $f \in BMO$, and let

$$d\mu(x,t) = |f * \Phi_t(x)|^2 dx \frac{dt}{t},$$

where $\Phi_t(x) = t^{-n}\Phi(x/t)$. Then $d\mu$ is a Carleson measure, and there is a positive constant C such that

$$\| d\mu \|_* = \sup_{x \in R^n, t > 0} \frac{1}{|B(x,t)|} \int_0^t \int_{B(x,t)} d\mu(y,s) \leq C \| f \|_{BMO}^2.$$

(ii) Suppose that there exists $a\Psi \in S$ with $\int_{R^n}^{B(x,t)} \Psi(x)dx = 0$, so that

$$\int_{R^n} \widehat{\Phi}(t\xi)\widehat{\Psi}(t\xi)\frac{dt}{t} = 1 \tag{2}$$

for all $|\xi| \neq 0$. Let f be a measurable function such that

$$\int_{R^n} \frac{|f(x)|}{1+|x|^{n+1}} dx < \infty.$$

Then there is a positive constant C such that

$$\| f \|_{BMO}^2 \leq C \| d\mu \|_* .$$

The proof of (i) of the above proposition was given in [114] for the case of the Poisson kernel, but it also works for a general Φ (see [126]). The proof of (ii) for the Poisson kernel and the Gaussian kernel was due to Fabes et al. [112], and Fabes and Neri [113], respectively; the general case was proved in the above cited monograph [126] by the use of the theory of tent spaces.

An important consequence of Proposition (3.1.1) is that for every (fixed) function f with

$$\int_{R^n} \frac{|f(x)|}{1+|x|^{n+1}} dx < \infty,$$

the statement that

$$d\mu(x,t) = |\Phi_t * f(x)|^2 dx \frac{dt}{t}$$

is a Carleson measure is independent of the function $\Phi \in S$ satisfying the assumptions, (1) and (2), in Proposition (3.1.1). A natural question one might ask is whether or not a similar result holds for the measure

$$|\Phi_t * f(x)|^q dx \frac{dt}{t},$$

$0 < q < \infty$. Our main result (Theorem (3.1.2)) answers this question in the affirmative and shows that the corresponding statement characterizes a distribution in a certain Triebel-Lizorkin space defined by Frazier and Jawerth [116], who were also motivated [114].

We state our principal results: Theorem (3.1.2), Theorem (3.1.3), and Theorem (3.1.4).

We assume that $-\infty < \alpha < \infty, 0 < q \leq \infty$, and w is a non-trivial weight function in the Muckenhoupt class A_∞. Choose $\psi \in S$ such that

$$\operatorname{supp} \hat{\psi} \subseteq \left\{\frac{1}{2} \leq |\xi| \leq 2\right\} \text{ and } \sum_{j=-\infty}^{\infty} \hat{\psi}\left(2^{-j}\xi\right) = 1 \text{ for } |\xi| \neq 0.$$

For each $j \in \boldsymbol{Z}$, let $\psi_j(x) = \psi_{2^{-j}}(x) = 2^{jn}\psi\left(2^j x\right)$, that is, $\hat{\psi}_j(\xi) = \hat{\psi}\left(2^{-j}\xi\right)$. We define the weighted version of the Triebel-Lizorkin space $\dot{F}_{\infty,q}^{\alpha}$ introduced by Frazier and Jawerth [116] as follows: For $f \in S'$, we let

$$\| f \|_{\dot{F}_{\infty,q}^{\alpha,w}} \sup_Q \left\{\frac{1}{w(Q)} \int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} \left(2^{j\alpha}\left|\psi_j * f(x)\right|\right)^q w(x) dx\right\}^{\frac{1}{q}} \tag{3}$$

with the interpretation that when $q = \infty$,

$$\| f \|_{\dot{F}_{\infty,\infty}^{\alpha,w}} = \sup_Q \sup_{j \geq -\log_2 \ell(Q)} \frac{1}{w(Q)} \int_Q 2^{j\alpha}\left|\psi_j * f(x)\right| w(x) dx. \tag{4}$$

where the supremum is taken over all dyadic cubes Q, and $\ell(Q)$ denotes the length of sides of the cube Q. We then define

$$\dot{F}_{\infty,q}^{\alpha,w} = \left\{f \in \frac{S'}{P} : \| f \|_{\dot{F}_{\infty,q}^{\alpha,w}} < \infty\right\},$$

where S'/P denotes the space of tempered distributions modulo polynominals. As stated above, our definition of the spaces when $q = \infty$ is different from [116] where they set

$$\dot{F}^{\alpha}_{\infty,\infty} = \dot{B}^{\alpha}_{\infty,\infty}$$

by definition. However, we shall prove that the above identity holds for our definition (see Theorem (3.1.4)), and we also see that the spaces defined in (4) are independent of the weight w.

In [116] Frazier and Jawerth proved that $\dot{F}^{\alpha}_{\infty,q}$ is independent of the sequence $\{\psi_j\}$ by showing that $\dot{F}^{\alpha}_{\infty,q}$ is isomorphic to a certain space of sequences.

We prove the following theorem:

Theorem (3.1.2) [108]:

(i) Assume that $\mu \in S$ satisfies a moment condition of order $[\alpha]$;i.e.,

$$\int_{R^n} x^{\kappa}\mu(x)dx = 0 \tag{5}$$

for all multi-indices κ with $|\kappa| \le [\alpha]$. Then there is a positive constant C such that

$$N^{*}_{\alpha,q}(f) = \sup_{x\in R^n, t>0}\left\{\frac{1}{w(B(x,t))}\int_{B(x,t)0} (s^{-\alpha}\mu^{*}_{s}f(y))^{q}w(y)\frac{ds}{s}dy\right\}$$
$$\le C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,q}} \tag{6}$$

for all $f \in S'/P$, where

$$\mu^{*}_{s}f(x) = \sup_{y\in R^n} |\mu_s * f(x-y)|\left(1+\frac{|y|}{s}\right)^{-\lambda}$$

and $\lambda > 0$ is sufficiently large and dependent on n, q, w.

(ii) Assume that $v \in S$ satisfies the (standard) Tauberian condition; i.e.,

$$\forall\xi \neq 0 \exists t > 0 \text{ such that } \hat{v}(t\xi) \neq 0. \tag{7}$$

Then there is a positive constant C such that $\parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,q}}$

$$\le C \sup_{x\in R^n, t>0}\left\{\frac{1}{w(B(x,t))}\int_{B(x,t)}\int_0^t (s^{-\alpha}|v_s * f(y)|)^{q}w(y)\frac{ds}{s}dy\right\}^{\frac{1}{q}}$$
$$= CN_{\alpha,q}(f) \tag{8}$$

for all $f \in S'$.

Theorem (3.1.3) [108]: Let $\phi \in S$ satisfy the moment condition (5). Assume that there exists $\eta \in S$ such that $\hat{\eta}$ is supported in an annulus about the origin and that

$$\sum_{j=-\infty}^{\infty} \hat{\phi}(2^j\xi)\hat{\eta}(2^j\xi) = 1 \ \forall\xi \neq 0. \tag{9}$$

For $j \in \boldsymbol{Z}$, let $\phi_j(x) = \phi_{2^{-j}}(x) = 2^{jn}\phi(2^j x)$. Then both

$$\sup_Q\left\{\frac{1}{w(Q)}\int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} \left(2^{j\alpha}\phi^{*}_{j}f(x)\right)^{q}w(x)dx\right\}^{\frac{1}{q}}$$

and

$$\sup_Q\left\{\frac{1}{w(Q)}\int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} \left(2^{j\alpha}|\phi_j * f(x)|\right)^{q}w(x)dx\right\}^{\frac{1}{q}}$$

are norms equivalent to $\|f\|_{\dot{F}^{\alpha,w}_{\infty,q}}$ for all $f \in S'/P$, and the finiteness of either norm characterizes $\dot{F}^{\alpha,w}_{\infty,q}$, Are norms equivalent to $\|f\|_{\dot{F}^{\alpha,w}_{\infty,q}}$ for all $f \in S'/P$, and the finiteness of either norm characterizes $\dot{F}^{\alpha,w}_{\infty,q}$ where

$$\phi_j^* f(x) = \sup_{y \in \boldsymbol{R}^n} \left|\phi_j * f(x-y)\right| \left(1 + 2^j |y|\right)^{-\lambda}$$

and $\lambda > 0$ depends on n, q, and w.

We do not give a complete proof, but a proof of the most difficult part follows from the argument used to prove Theorem (3.1.5) and Lemma (3.1.6).

Observe that if (9) is satisfied, then ϕ satisfies the Tauberian condition (19149 Also, it is not difficult to show that if for every $|\xi| = 1$ there are numbers a, b depending on ξ such that $0 < 2a \leq b < \infty$ and

$$\hat{\phi}(t\xi) \neq 0 \text{ for all } a \leq t \leq b.$$

then (9) is satisfied (see [127]). Theorem (3.1.3) implies, in particular, that $\dot{F}^{\alpha,w}_{\infty,q}$ does not depend on the sequence $\{\phi_j\}$ and the characterization does not require the condition that $\hat{\phi}$ has compact support. We note that the removal of the restriction on the support of $\hat{\phi}$ is important in some applications, such as the characterization of function spaces on domains where one assumes that ϕ has compact support; this would imply that $\hat{\phi}$ does not have compact support unless ϕ is trivial (see [123] where the methods in [110], [111] are used to study function spaces on domains).

In the course of the proof of the main results, we also establish the following interesting result.

Theorem (3.1.4) [108]:

We have the identities

$$\dot{F}^{\alpha,w}_{\infty,\infty} = \dot{B}^{\alpha}_{\infty,\infty} = \dot{F}^{\alpha}_{\infty,\infty}$$

(with equivalent norms).

That $\dot{F}^{\alpha,w}_{\infty,\infty}$ is independent of the weight function w seems a surprising fact at first. However, this independence is consistent with the definitions of the weighted Besov-Lipschitz spaces $\dot{F}^{\alpha,w}_{p,q}$ in [109] where one has $\dot{B}^{\alpha,w}_{\infty,q} = \dot{B}^{\alpha}_{\infty,q}$ to obtain a satisfactory interpolation theory. The result of the theorem in the unweighted case might be expected from a theorem of Meyer [119] on the minimality of the Besov-Lipschitz space $\dot{B}^{0}_{1,1}$, which is equivalent to the maximality of the space $\dot{B}^{0}_{\infty,\infty}$ (see [117] for a proof of Meyer's result). We are grateful to P. Auscher for drawing our attention to this minimality result. Theorem (3.1.4) gives a new characterization of the Hölder-Zygmund space $\dot{B}^{\alpha}_{\infty,\infty}$ in terms of weighted oscillations over cubes.

The results in [110], [111] were extended recently by Rychkov [124] to other classes of weight functions which can have exponential growth at infinity. We anticipate further developments in this direction.

The bulk of the rest is devoted to the proof of Theorem (3.1.2). We use C to denote a positive constant which may depend on the parameters concerned, such as $, q, w, n$ ..... but not on the variable quantity, usually a distribution $f; C$ may have different values on any two consecutive occurrences.

Assume that v satisfies the Tauberian condition (7). First consider the case $q < \infty$. Fix a dyadic cube Q with $\ell(Q) = 2^{-\ell}$. Choose $r < q$ suchthatw $w \in A_q/r$. For $x \in Q$ and $j = \ell, \ell + 1, ...$, by (17) of [110] we have the inequality

$$\left|\psi_j * f(x)\right|^q \leq C\left\{\int_{I_j} v_t^* f(x)^r \frac{dt}{t}\right\}^{\frac{q}{r}}, \tag{10}$$

Where $I_j = \{2^{-j-A}, 2^{-j+A}\}$ for some $A > 1$, and $\lambda > 0$ in the definition $v_t^* f(x)$ satisfies $\lambda r > 2n$. By the arguments on p. 840 of [111],

$$\begin{aligned} v_t^* f(x)^r \leq \quad & C\int_0^t \int_{R^n} |v_s * f(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\frac{\lambda r}{2}} \left(\frac{s}{t}\right)^{\frac{\lambda r}{2}} s^{-n} dz \frac{d}{s} \\ & + \int_{R^n} |v_t * f(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\frac{\lambda r}{2}} t^{-n} dz \\ & = CJ(x,t) + CH(x,t) \end{aligned} \tag{11}$$

Next, write

$$\begin{aligned} J(x,t) = & \int_0^t \int_{\{|x-z| \leq 2^{-\ell+A}\}} |v_s * f(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\frac{\lambda r}{2}} \left(\frac{s}{t}\right)^{\frac{\lambda r}{2}} s^{-n} dz \frac{ds}{s} \\ & + \int_0^t \int_{\{|x-z| \geq 2^{-\ell+A}\}} |v_s * f(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\frac{\lambda r}{2}} \left(\frac{S}{t}\right)^{\frac{\lambda r}{2}} s^{-n} dz \frac{ds}{s} \\ & = J_1(x,t) + J_2(x,t). \end{aligned}$$

By an argument similar to that on p. 840 of [111] we deduce that

$$J_1(x,t) \leq C\int_0^t M\left(\left(|v_s * f|\chi_{Q^*}\right)^r\right)(x) \left(\frac{s}{t}\right)^{\frac{\lambda r}{2}} \frac{ds}{s},$$

where M denotes the Hardy-Littlewood maximal function operator, and $Q^* = c_{n,A} Q$ is such that $\{|x-z| \leq 2^{-\ell+A}\} \subseteq Q^*$ for every $x \in Q$. We use the convention that aQ is the dilation of the cube Q about the center of Q by a factor of $a > 0$. Choose λ such that $\lambda + 2\alpha > 0$. Using the above estimate, Hölder's inequality, and Hardy's inequality, we obtain

$$\begin{aligned} \sum_{j=\ell}^{\infty} 2^{j\alpha q} \left\{\int_{I_j} J_1(x,t) \frac{dt}{t}\right\}^{q/r} & \leq C\sum_{j=\ell}^{\infty} \int_{I_j} t^{-\alpha q} J_1(x,t)\Bigg]^{q/r} \frac{dt}{t} \\ \leq C(2A+1)\int_0^{2^{-\ell+A}} t^{-\left(\alpha+\frac{\lambda}{2}\right)q} & \left\{\int_0^t M\left(\left(|v_s * f|\chi_{Q^*}\right)^r\right)(x) s^{\frac{\lambda r}{2}} \frac{ds}{s}\right)^{q/r} \\ \leq C\int_0^{2^{-\ell+A}} & M\left(\left(t^{-\alpha}|v_t * f|\chi_{Q^*}\right)^r\right)(x)^{\frac{q}{r}} \frac{dt}{t}. \end{aligned} \tag{12}$$

It follows from the weighted estimate for the Hardy-Littlewood maximal function in [120] that

$$\begin{aligned} & \frac{1}{w(Q)} \int_Q \sum_{j=\ell}^{\infty} 2^{j\alpha q} \left\{\int_{I_j} J_1(x,t) \frac{dt}{t}\right\}^{\frac{q}{r}} w(x) dx \\ & \leq \frac{C}{w(Q^*)} \int_{Q^*} \int_0^{2^{-\ell+A}} (t^{-\alpha}|v_t * f(x)|)^q w(x) \frac{dt}{t} dx \\ & \leq C N_{\alpha,q}(f)^q, \end{aligned} \tag{13}$$

where we also use the doubling property of w and the simple fact that Q^* is contained in a ball of comparable to-measure and whose radius is comparable to $2^{-\ell+A}$.
Before proceeding to the estimate for $J_2(x,t)$, we recall few properties of weight functions. Put $p=q/r$. Then, since $w\in A_p$, to satisfies the A_p-condition:

$$(A_p)\left\{\frac{1}{|B|}\int_B w(x)dx\right\}^{\frac{1}{p}}\left\{\frac{1}{|B|}\int_B w(x)^{-p'/p}dx\right\}^{\frac{1}{p'}}\leq C$$

for all balls B. Since $w^{-p'/p}\in A_{p'}$, it satisfies the $B_{p'}$,-condition:

$$(B_{p'})\int_{R^n}\left(1+\frac{|x-z|}{t}\right)^{-np'}w(z)^{-p'/p}dz\leq C\int_{B(x,t)}w(z)^{-p'/p}dz$$

for all $x\in \boldsymbol{R}^n$ and $t>0$ (see [118]).
The A_p-condition then implies that

$$\left\{\int_{R^n}\left(1+\frac{|x-z|}{t}\right)^{-np'}w(z)^{-p'/p}dz\right\}^{p/p'}\leq C\frac{t^{np}}{w\big(B(x,t)\big)}. \qquad (14)$$

Put $d_\ell=2^{-\ell+A}$. Let $x\in Q$. Using Hölder's inequality and Hardy's inequality as in the estimate for $J_1(x,t)$ we obtain

$$\sum_{j=\ell}^{\infty}2^{j\alpha q}\left\{\int_{I_j}J_2(x,t)\frac{dt}{t}\right\}^{\frac{q}{r}}$$

$$\leq C\int_0^{d_\ell}\left\{\int_{\{|x-z|\geq d_\ell\}}(t^{-\alpha}|v_t*f(z)|)^r\left(1+\frac{|x-z|}{t}\right)^{-\frac{\lambda r}{2}}t^{-n}dz\right\}^{q/r}\frac{dt}{t}.$$

Choose λ so large that $\lambda r>2n\max\{p,p'\}$. By Hölder's inequality and (14),

$$\int_0^{d_\ell}\left\{\int_{\{|x-z|\geq d_\ell\}}(t^{-\alpha}|v_t*f(z)|)^r\left(1+\frac{|x-z|}{t}\right)^{-\frac{\lambda r}{2}}t^{-n}dz\right\}^{q/r}\frac{dt}{t}$$

$$\leq C\int_0^{d_\ell}\left\{\int_{\{|x-z|\geq d_\ell\}}(t^{-\alpha}|v_t*f(z)|)^q\left(1+\frac{|x-z|}{t}\right)^{-\frac{\lambda r}{2}}t^{-np}w(z)dz\right\}.$$

$$\left\{\int_{\{|x-z|\geq d_\ell\}}\left(1+\frac{|x-z|}{t}\right)^{-np'}w(z)^{-\frac{p'}{p}}dz\right\}^{p/p'}\frac{dt}{t}$$

$$\leq C\sum_{k=1}^{\infty}\frac{w\left(B(x,2^kd_\ell)\right)}{w\big(B(x,d_\ell)\big)}\cdot\frac{1}{w\big(B(x,d_\ell)\big)}\int_0^{d_\ell}\int_{\{2^{k-1}d_\ell\leq|x-z|\leq 2^kd_\ell\}}(t^{-\alpha}\mid v_t$$
$$*f(z)\mid)^q(d_\ell)^{np}(2^kd_\ell)^{-\frac{\lambda r}{2}}t^{-np+\frac{(\lambda r)}{2}}w(z)dz\frac{dt}{t}$$

$$\leq C\sum_{k=1}^{\infty}\left(2^{-\frac{\lambda r}{2}}B\right)^k$$

$$\cdot\frac{1}{w\big(B(x,2^kd_\ell)\big)}\int_{\{|x-z|\leq 2^kd_\ell\}}\int_0^{d_\ell}(t^{-\alpha}|v_t*f(z)|)^qw(z)dz\frac{dt}{t}$$

for some $B > 1$, by the doubling condition on the weight w. If we further choose λ so large that $2^{-\frac{\lambda r}{2}} B < 1$, then the last sum in the above is dominated by $CN_{\alpha,q}(f)^q$.
Thus, we have proved that

$$\frac{1}{w(Q)} \int_Q \sum_{j=\ell}^{\infty} 2^{j\alpha q} \left\{ \int_{I_j} J(x,t) \frac{dt}{t} \right\}^{\frac{q}{r}} w(x)dx \le CN_{\alpha,q}(f)^q.$$

Since we have a similar estimate for $H(x,t)$, (8) follows by appealing to (10) and (11).
The proof for the case $q = \infty$ is similar. For example, to estimate $J_1(x,t)$, choose r such that $w \in A_{1/r}$ and use the weighted estimate for the Hardy-Littlewood maximal function for weights in $A_{1/r}$. We have thus completed the proof of Theorem (3.1.2) (ii).
"How big is λ ?"

(i) An examination of the proof for the case $q < \infty$ shows that we need to choose p and r so that to $w \in A_p, p = q/r > 1$ and then select λ such that

(a) $\lambda r > 2n\max(p, p')$

(b) $\lambda + 2\alpha > 0$

(c) $B2^{-\lambda r/2} < 1, B$ the doubling constant for w.

Let $p_0 = \inf\{p: w \in A_p\}$. Then

$$\lambda > 2 \max \left\{ np_0^2/q, 4n/q, -\alpha \left(\frac{p_0}{q}\right) \log_2 B \right\}$$

suffices for a choice of p (and hence r) so that the above conditions hold.
(ii) The proof for the case $q = \infty$ shows that we need to choose p and r so that $w \in A_p, p = 1/r > 1$ and then select λ so it satisfies the same three conditions: (a) through (c) as above. Then

$$\lambda > 2 \text{ m } \{np_0^2, 4n, -\alpha p_0 \log_2 B\}$$

Will suffice. (See the proof of Lemma (3.1.6).)
The proof of Theorem (3.1.2) (i) is done in a similar way to the proof of [110] once we establish the Peetre type maximal inequalities for $\dot{F}_{\infty,q}^{\alpha,w}$, in Theorem (3.1.5), and then show at the end that for $f \in \dot{F}_{\infty,q}^{\alpha,w}$, there is a polynomial P and there is a sequence of polynomials $\{P_m\}$ of degrees less than or equal to $[\alpha]$ for which

$$f - P = \lim_{m\to\infty} \left(\sum_{j=-m}^{\infty} \psi_j * f - P_m \right) \tag{15}$$

in S'.

Theorem (3.1.5) [108]:

Let $\{\phi_j\}_{j=-\infty}^{\infty}$ be a sequence in S which satisfies the following properties:

$$\text{supp}\, \hat{\phi}_j \subseteq \{2^{j-A} \le |\xi| \le 2^{j+A}\} \tag{16}$$

for all j and for some $A > 1$;

$$\left| D^\kappa \hat{\phi}_j(\xi) \right| \le C_\kappa 2^{-j|\kappa|} \tag{17}$$

for all j and all multi-indices κ. Then

$$\sup_Q \left\{ \frac{1}{w(Q)} \int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} \left(2^{j\alpha} \phi_j^* f(x)\right)^q w(x)dx \right\}^{1/q} \le C \parallel f \parallel_{\dot{F}_{\infty,q}^{\alpha,w}} \tag{18}$$

for all $f \in S'$, where

$$\phi_j^* f(x) = \sup_{y\in R^n} \left|\phi_j * f(x-y)\right|\left(1+2^j|y|\right)^{-\lambda}$$

and λ is large and dependent on n, q, and w (see the remarks at the end), and

$$C = C' \max_{|\kappa|\le N_1} C_\kappa$$

for a sufficiently large N_1. Consequently, the space $\dot{F}_{\infty,q}^{\alpha,w}$ defined is independent of the sequence $\{\psi_j\}$ used in its definition.

Proof: By (16) there is $N \ge 1$ such that $\phi_j = \sum_{j-N}^{j+N} \phi_j * \psi_k, j = 0, \pm 1, \pm 2, \ldots$, so that

$$\left|\phi_j * f(x-y)\right| \le C\left(1+2^j|y|\right)^{-\lambda}\left\{\int_{R^n}\left|\phi_j(z)\right|\left(1+2^j|z|\right)^{-\lambda} dz\right\}\left\{\sum_{k=j-N}^{j+N} \psi_k^* f(x)\right\}$$

for all $x, y \in \boldsymbol{R}^n$. Since, for all $\lambda > 0$,

$$\sup_j \int_{R^n} \left|\phi_j(z)\right|\left(1+2^j|z|\right)^{-\lambda} dz < \infty$$

by (17), the above implies that

$$\phi_j^* f(x) \le C \sum_{k=j-N}^{j+N} \psi_k^* f(x)$$

Fix a dyadic cube Q_1 with $\ell(Q_1) = 2^{-\ell}$. For $q < \infty$ the above inequality implies that

$$\begin{aligned}
&\frac{1}{w(Q_1)}\int_{Q_1} \sum_{j=\ell}^{\infty} \left(2^{j\alpha}\phi_j^* f(x)\right)^q w(x)dx \\
&\le C\frac{1}{w(Q_1)}\int_{Q_1} \sum_{k=\ell-N}^{\infty} (2^{k\alpha}\psi_k^* f(x))^q w(x)dx \\
&\le C 2^{Nn} \sup_{\ell(Q)=2^{N-\ell}} \frac{1}{w(Q)}\int_{Q} \sum_{k=\ell-N}^{\infty} (2^{k\alpha}\psi_k^* f(x))^q w(x)dx \\
&\le C \sup_{Q} \frac{1}{w(Q)}\int_{Q} \sum_{k=-\log_2 \ell(Q)}^{\infty} (2^{k\alpha}\psi_k^* f(x))^q w(x)dx \\
&\le C \parallel f \parallel_{\dot{F}_{\infty,q}^{\alpha,w}}^q
\end{aligned}$$

This last inequality follows by a discrete adaptation of the proof of Theorem (3.1.2) (ii) (cf. [111]). See also (11), (12), and (13). This completes the proof for $q < \infty$.

Next we give the proof for the case $q = \infty$ separately as Lemma (3.1.6), since we also need this result in our proof of Theorem (3.1.4).

Lemma (3.1.6)[108]:

With the notation as above:

$$\sup_Q \sup_{j\ge -\log_2 \ell(Q)} \frac{1}{w(Q)}\int_Q 2^{j\alpha}\psi_j^* f(x) w(x)dx \le C \parallel f \parallel_{F_{\infty,\infty}^{\alpha,w}}. \tag{19}$$

Proof: We follow the arguments used in the proof of inequality (8).

Fix a dyadic cube Q with $\ell(Q) = 2^{-\ell}$. For $j \ge \ell$ and $x \in Q$, by a discrete version of the arguments in the proof of (8) (cf. [111]), we have the following chain of inequalities:

$$\begin{aligned}\psi_j^* f(x) &\leq C\left\{\sum_{k=j}^{\infty}\int_{\{|x-z|\leq 2^{-\ell}\}}|\psi_k * f(z)|^r(1+2^k|x-z|)^{-\frac{\lambda r}{2}}2^{(j-k)\frac{\lambda r}{2}}2^{kn}\right.\\ &\left.+C\left\{\sum_{k=j}\int_{\{|x-z|\geq 2^{-\ell}\}}|\psi_k * f(z)|^r(1+2^k|x-z|)^{-\frac{\lambda r}{2}}2^{(j-k)\frac{\lambda r}{2}}2^{kn}dz\right\}\right.\\ &= J_1(x,j)+J_2(x,j), \end{aligned} \tag{20}$$

Where $0 < r < 1, w \in A_{1/r}$, and $\lambda > 2n/r$; and

$$J_1(x,j) \leq C\sum_{k=j}^{\infty} 2^{(j-k)\frac{\lambda r}{2}} M\left(\left(\chi_{Q^*}|\psi_k * f|\right)^r\right)(x)^{1/r},$$

where $Q^* = c_n Q$ is chosen so that $\{|x-z| \leq 2^{-\ell}\} \subseteq Q^*$ for all $x \in Q$. Hence it follows from the weighted estimate for the Hardy-Littlewood maximal function for weights in $A_{1/r}$ that

$$\begin{aligned} &\sup_{j\geq\ell}\frac{1}{w(Q)}\int_Q 2^{j\alpha}J_1(x,j)w(x)dx\\ &\leq \sup_{j\geq\ell}\frac{C}{w(Q^*)}\sum_{k=j}^{\infty}2^{(j-k)\left(\alpha+\frac{\lambda r}{2}\right)}\int_{Q^*}2^{k\alpha}|\psi_k * f(x)|w(x)dx\\ &\leq C\sup_{k\geq\ell}\frac{1}{w(Q^*)}\int_{Q^*}2^{k\alpha}|\psi_k * f(x)|w(x)dx\\ &\leq C\parallel f\parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}. \end{aligned}$$

if we choose $\lambda > 0$ such that $\lambda r + 2\alpha > 0$.

On the other hand, by Hölder's inequality applied to the summation with $p = 1/r$

$$C\sum_{k=j}^{\infty}2^{(j-k)\frac{\lambda r}{2}}\left\{\int_{\{|x-z|\geq 2^{-\ell}\}}|\psi_k * f(z)|^r(1+2^k|x-z|)^{-\frac{\lambda r}{2}}2^{kn}dz\right\}^{1/r},$$

and using an argument similar to the estimate for $J_2(x,t)$ in the proof of Theorem (3.1.2) (ii), we obtain

$$\begin{aligned} &\left\{\int_{\{|x-z|\geq 2^{-\ell}\}}|\psi_k * f(z)|^r(1+2^k|x-z|)^{-\frac{\lambda r}{2}}2^{kn}dz\right\}\Bigg\}^{1/r}\\ &\leq C\left\{\int_{\{|x-z|\geq 2^{-\ell}\}}|\psi_k * f(z)|(1+2^k|x-z|)^{-\frac{\lambda r}{2}}2^{\frac{kn}{r}}w(z)dz\right\}\Bigg\}^{\frac{1}{r}}\left\{\frac{2^{-\ell n/r}}{w\left(B(x,2^{-\ell})\right)}\right\}\\ &\leq C\sum_{m=1}^{\infty}\left(B2^{-\frac{\lambda r}{2}}\right)^m 2^{(k-\ell)(n/r-\lambda r/2)}2^{-k\alpha}\cdot\frac{1}{w\left(B(x,2^{m-\ell})\right)}\cdot\int_{\{|x-z|\leq 2^{m-\ell}\}}2^{k\alpha}|\psi_k * f(z)|w\\ &\leq C2^{-k\alpha}\parallel f\parallel_{F^{\alpha,w}_{\infty,\infty}} \end{aligned}$$

if $\lambda > 0$ is chosen so that

$$B2^{-\lambda r/2} < 1, \text{ and } \lambda r > 2n/(\min\ \{r, 1-r\})$$

It follows that

$$2^{j\alpha}J_2(x,j) \leq C\sum_{k=j}^{\infty} 2^{(j-k)\left(\alpha+\frac{\lambda r}{2}\right)} \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}} \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}$$

This implies that

$$\sup_{j\geq\ell} \frac{1}{w(Q)}\int_Q 2^{j\alpha}J_2(x,j)w(x)dx \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}.$$

Combining this estimate with the estimate for $J_1(x,j)$ and (20) we obtain the desired inequality (19). This completes the proof of Lemma (3.1.6) and so completes the proof of Theorem (3.1.5).

We note that in the proof of this lemma, the compactness of the support of $\hat{\psi}$ is not used; only the Tauberian condition (7) of ψ is used.

We next turn to the proof of Theorem (3.1.4). Note that the embedding

$$\dot{B}^{\alpha}_{\infty,\infty} \subseteq \dot{F}^{\alpha,w}_{\infty,\infty}$$

is obvious. To prove the converse, let $f \in \dot{F}^{\alpha,w}_{\infty,\infty}$ and let $j \in \boldsymbol{Z}$. Fix any dyadic cube Q with $\ell(Q) = 2^{-j}$. Then by Lemma (3.1.6) there is $X_Q \in Q$ such that

$$2^{j\alpha}\psi_j^* f\left(x_Q\right) \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}. \tag{21}$$

For any $x \in Q$, there is $y \in \boldsymbol{R}^n$ such that $|y| \leq \sqrt{n}2^{-j}$ and $x = x_Q - y$. Since

$$\psi_j^* f\left(x_Q\right) \geq \left|\psi_j * f\left(x_Q - y\right)\right|\left(1 + 2^j|y|\right)^{-\lambda} \geq (1+\sqrt{n})^{-\lambda}\left|\psi_j * f(x)\right|,$$

we deduce that

$$2^{j\alpha}\sup_{x\in Q}\left|\psi_j * f(x)\right| \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}. \tag{22}$$

Since the collection of all dyadic cubes Q with $\ell(Q) = 2^{-j}$ is a covering of $\boldsymbol{R}^n$, we conclude that

$$2^{j\alpha}\left\|\psi_j * f\right\|_\infty \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}. \tag{23}$$

for all $j \in \boldsymbol{Z}$, so that

$$\parallel f \parallel_{\dot{B}^{\alpha}_{\infty,\infty}} = \sup_j 2^{j\alpha}\left\|\psi_j * f\right\|_\infty \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,\infty}}. \tag{24}$$

Consequently, $\dot{B}^{\alpha}_{\infty,\infty} = \dot{F}^{\alpha,w}_{\infty,\infty}$ with equivalent norms. It holds, in particular, when $w(x) \equiv 1$ and this completes the proof of Theorem (3.1.4).

Finally we prove (15). It suffices to show that

$$\dot{F}^{\alpha,w}_{\infty,\infty} \subset \dot{F}^{\alpha,w}_{\infty,\infty} = \dot{B}^{\alpha}_{\infty,\infty} \tag{25}$$

for all $0 < q \leq \infty$, and (15) is then just a result by Peetre [122].

For $q \geq 1$ it is routine to check that (25) is valid. For $0 < q < 1$ it follows by an argument similar to that of Theorem (3.1.4), and which, in fact, works for all finite q that $\dot{F}^{\alpha,w}_{\infty,q} \subset \dot{B}^{\alpha}_{\infty,\infty}$. First we use Theorem (3.1.5) to obtain:

$$\left\{\frac{1}{w(Q)}\int_Q \left(2^{j\alpha}\psi_j^* f(x)\right)^q w(x)dx\right\}^{1/q} \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,q}}$$

for $\ell(Q) = 2^{-j}$. We then argue as in the proof of Theorem (3.1.4), (21), (22), (23) and (24), and we obtain:

$$\parallel f \parallel_{\dot{B}^{\alpha}_{\infty,q}} = \sup_j 2^{j\alpha}\left\|\psi_j * f\right\|_\infty \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,q}},$$

and this establishes (25).

Using the continuous embedding (25) we obtain the following embedding theorem.

Corollary (3.1.7) [108]:

We have the continuous embedding

$$\dot{F}^{\alpha,w}_{\infty,q_1} \subseteq \dot{F}^{\alpha,w}_{\infty,q_2}$$

for all $0 < q_1 < q_2 \leq \infty$.

Let $0 < \rho < \infty$. For $f \in S'$, let

$$\| f \|_{\dot{X}^{\alpha}_{\rho,w}} = \sup_{Q} \left\{ \sup_{j \geq -\log_2 \ell(Q)} \frac{1}{w(Q)} \int_{Q} \left(2^{j\alpha} \left|\psi_j * f(x)\right|\right)^{\rho} w(x) dx \right\}^{1/\rho},$$

and define

$$\dot{X}^{\alpha}_{\rho,w} = \left\{ f \in S'/P : \| f \|_{\dot{X}^{\alpha}_{\rho,w}} < \infty \right\}.$$

Note then that $\dot{X}^{\alpha}_{\rho,w} = \dot{F}^{\alpha,w}_{\infty,\infty}$.

By mimicking the proofs given in the previous, with particular reference to the case $\rho = 1$, we obtain the following theorem:

Theorem (3.1.8) [108]:

(i) Assume that $\mu \in S$ satisfies a moment condition (5) of order $[\alpha]$. Then there is a positive constant C such that

$$\sup_{x \in R^n, t>0} \left\{ \sup_{0<s<t} \frac{1}{w(B(x,t))} \int_{B(x,t)} (s^{-\alpha} \mu_s^* f(y))^{\rho} w(y) dy \right\}^{\frac{1}{\rho}} \leq C \| f \|_{\dot{X}^{\alpha}_{\rho,w}}$$

for all $f \in S'/P$, where $\mu_s^* f$ is defined as in Theorem (3.1.2) (i).

(ii) Assume that $v \in S$ satisfies the Tauberian condition (1) Then there is a positive constant C such that

$$\| f \|^{\alpha}_{\dot{X}^{\alpha}_{\rho,w}} \leq C \sup_{x \in R^n, t>0} \left\{ \sup_{0<s<t} \frac{1}{w(B(x,t))} \int_{B(x,t)} (s^{-\alpha} |v_s * f(y)|)^{\rho} w(y) dy \right\}^{1/\rho}$$

for all $f \in S'$.

(iii) We have the identities

$$\dot{X}^{\alpha}_{\rho,w} = \dot{B}^{\alpha}_{\infty,\infty} = \dot{F}^{\alpha}_{\infty,\infty}$$

(with equivalent norms).

There is another version of the weighted Triebel-Lizorkin spaces when $p = \infty$ in which the weight, w, only satisfies the doubling condition, which is weaker than being in A_∞. That is, there is a constant B such that

$$w(B(x, 2r)) < Bw(B(x, r))$$

for all $x \in \boldsymbol{R}^n$ and $0 < r < \infty$; the smallest constant B for which the above inequality holds is called the doubling constant of w.

For $f \in S', -\infty < \alpha < \infty$, and $0 < q < \infty$, we define

$$\| f \|_{\mathcal{F}^{\alpha,w}_{\infty,q}} = \sup_{Q} \left\{ \left[\frac{1}{w(Q)} \int_{Q} \sum_{j=-\log_2 \ell(Q)}^{\infty} \left(2^{j\alpha} \left|\psi_j * f(x)\right|\right)^{q} dx \right]^{1/q} \right\}$$

and

$$\dot{\mathcal{F}}^{\alpha,w}_{\infty,q} = \left\{ f \in S'/P : \| f \|^{\alpha,\infty}_{\dot{\mathcal{F}}^{\alpha,w}_{\infty,q}} < \infty \right\}.$$

We then have versions of Theorems (3.1.2), (3.1.3), and (3.1.5) for these spaces. The proofs are very similar to those given above for the corresponding results, except for the fact that we only need to use the estimate for the Hardy-Littlewood maximal function instead of the weighted version used above. This is the reason why we do not need to use the A_∞-condition to prove the corresponding theorems. However, in the new version of Theorem (3.1.2) (i) we shall require that the function $\mu \in S$ have more vanishing moments, and we shall give the precise version of this part explicitly as:

Theorem (3.1.9) [108]: Let w be a weight function with doubling constant B. If $\mu \in S$ has $[\alpha+(n-\log_2 B)/q)]$ vanishing moments, then there is a positive constant C such that

$$\sup_{x\in \boldsymbol{R}^n, t>0}\left\{\frac{1}{w(x,t)}\int_{B(x,t)}\int_0^t (s^{-\alpha}\mu_t^* f(y))^q \frac{ds}{s} dy\right\}^{\frac{1}{q}} \leq C \parallel f \parallel_{\dot{F}^{\alpha,w}_{\infty,q}}$$

for all $f \in S'/P$.

Note that when $w \equiv 1$ then $B=2^n$ so $\log_2 B - n = 0$, and we require only the same $[\alpha]$ vanishing moments as in Theorem (3.1.2) (i).

The reason behind this change in the number of vanishing moments required for μ lies with the representation (15) which we shall briefly discuss below.

Suppose that $f \in \dot{\mathcal{F}}^{\alpha,w}_{\infty,q}$. It is well known that $\sum_{j=0}^{\infty}\psi_j * f$ converges in S' for any $f\in S'$ (see, for example, [122]) From the "new" version of Theorem (3.1.5) for $\dot{\mathcal{F}}^{\alpha,w}_{\infty,q}$ and an argument similar to the proof of Theorem (3.1.4), we obtain, for each $j \in \boldsymbol{Z}$ and for every dyadic cube Q with $\ell(Q) = 2^{-j}$

$$\sup_{x\in Q}\left(2^{j\alpha}\left|\psi_j * f(x)\right|\right)^q \parallel f \parallel^q_{\mathcal{F}^{\alpha,w}_{\infty,q}}.$$

Using this estimate, one can show that the series $\sum_{j=-\infty}^{1}\psi_j * f$. converges in S' when $\alpha < (n-\log_2 B)/q)$. In the general case, for each multi-index κ with $|\kappa| > \alpha - (n-\log_2 B)/q, D^\kappa f \in \dot{\mathcal{F}}^{\alpha-|\kappa|,w}_{\infty,q}$, so that $\sum_{j=-\infty}^{1}\psi_j * D^\kappa f = \sum_{j=-\infty}^{1} D^\kappa\left(\psi_j * f\right)$ converges in S'. These facts imply that we have the representation (15) in which each P_m has degree less than or equal to $[\alpha+(n-\log_2 B)/q)]$ (see [122]).

In [125] Rychkov proved a version of Theorem (3.1.3) for the unweighted, inhomogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{\infty,q}, \alpha\in \boldsymbol{R}, 0<q<\infty$, under a condition on ϕ that is stronger than the Tauberian condition (9). His method, as is ours, is an adaptation of the techniques in [110], [111].

Section (3.2): Littlewood-Paley Functions and Radial Multipliers

Let $n \geqslant 2$ and $\rho(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be positive and homogeneous of degree 1 . We assume $\nabla\rho \neq 0$ and the hypersurface

$$\Sigma = \{x\in \mathbb{R}^n : \rho(x) = 1\}$$

has non-vanishing Gaussian curvature. We define

$$\sigma_\delta(f)(x) = \left(\int_0^\infty \left|S_R^\delta(f)(x) - S_R^{\delta-1}(f)(x)\right|^2 \frac{dR}{R}\right)^{\frac{1}{2}}, \tag{26}$$

Where

$$S_R^\delta(f)(x) = \int_{\mathbb{R}^n}(1-R^{-2}\rho(\xi)^2)_+^\delta \hat{f}(\xi) e^{2\pi i x\xi} d\xi \tag{27}$$

is the Bochner-Riesz means of order δ on $\mathbb{R}^n$ with respect to ρ. By Sogge [146] we are motivated to consider $S_R^\delta(f)$ with $\rho(\xi)$ in place of the Euclidean norm $|\xi|$. We also define

$$\tau_\delta(f)(x) = \left(\int_0^\infty \left|\tilde{S}_R^{\delta-1}(f)(x)\right|^2 \frac{dR}{R}\right)^{\frac{1}{2}} \tag{28}$$

With

$$\tilde{S}_R^\delta(f)(x)=\int_{\mathbb{R}^n}\eta(\rho(\xi)/R)(1-R^{-2}\rho(\xi)^2)_+^\delta\hat{f}(\xi)e^{2\pi ix\xi}d\xi, \qquad (29)$$

Where $\eta\in C^\infty(\mathbb{R})$ is such that $\eta(t)=1$ if $|t|\geqslant 1/4$ and $\eta(t)=0$ if $|t|\leqslant 1/8$.
Put $\delta(p)=n|1/p-1/2|+1/2$. We first study the behavior of $\tau_\delta,\delta\geqslant\delta(p),\delta>\delta(1)$, on the weighted Hardy space $H_w^p(\mathbb{R}^n), 0<p\leqslant 1$. Under these conditions of δ we can write $\tau_\delta(f)=g_\psi(f)$, where $g_\psi(f)$ is the Littlewood-Paley function defined by

$$g_\psi(f)(x)=\left(\int_0^\infty|\psi_t*f(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}};$$

here $\psi_t(x)=t^{-n}\psi(t^{-1}x)$, and ψ satisfies $|\psi(x)|\leqslant c(1+|x|)^{-n-\epsilon}$ with $\epsilon=n(1/p-1)+\delta-\delta(p)>0$ and $\int_{\mathbb{R}^n}\psi(x)dx=0$. So τ_δ is bounded on the weighted Lebesgue spaces L_w^r for all $r\in(1,\infty)$ and all $w\in A_r$ (see Sato [144] and Ding et al. [136]), where we denote by A_r the weight class of Muckenhoupt.

Theorem (3.2.1)[129]: Let τ_δ be as in (28).

(i) Let $0<p<1$. Suppose $w\in B_1$ and $w\in A_\infty$. Then

$$\|\tau_{\delta(p)}(f)\|_{L_w^{p,\infty}}\leqslant C_{p,w}\ \|f\|_{H_w^p}^{p,f}\in S_0(\mathbb{R}^n).$$

(ii) Let $0<p\leqslant 1$ and $\delta>\delta(p)$. Suppose $w\in B_{1+n^{-1}p(\delta-\delta(p))}$ and $w\in A_\infty$. Then

$$\|\tau_\delta(f)\|_{L_w^p}\leqslant C_{p,\delta,w}\ \|f\|_{H_w^p}^p, f\in S_0(\mathbb{R}^n).$$

When $\rho(\xi)=|\xi|$, these results also hold for σ_δ in place of τ_δ.We note that when $\rho(\xi)=|\xi|$ and $w(x)\equiv 1$, Theorem (3.2.1) (with σ_δ in place of τ_δ) is due to Kaneko and Sunouchi [140]. By part (i) the Littlewood-Paley operator $\tau_{\delta(p)}$, initially defined on S_0, has a unique sublinear extension which is bounded from H_w^p to $L_w^{p,\infty}$; and by part (ii) τ_δ extends likewise to a bounded operator from H_w^p to L_w^p. As for a recent article dealing with the boundedness on the Hardy spaces for the Littlewood-Paley functions, see also Ding et al. [10], where they study the Marcinkiewicz integrals.

Theorem (3.2.2) [129]: If $\delta>1/2$ and $0\leqslant\alpha<1$, then

$$\int_{\mathbb{R}^n}|\sigma_\delta(f)(x)|^2|x|^{-\alpha}dx\leqslant C_{\delta,\alpha}\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}dx.$$

In Carbery et al. [135] this is proved for the case $\rho(\xi)=|\xi|$ (see also Rubio de Francia [142] for another proof).We prove Theorem (3.2.2) for the general $\rho(\xi)$ by applying the method of Rubio de Francia [142]. Let S_R^δ be as in (27) and define

$$S_*^\delta(f)(x)=\sup_{R>0}\left|S_R^\delta(f)(x)\right|. \qquad (30)$$

Then Theorem (3.2.2) implies, as in the case $\rho(\xi)=|\xi|$, the following (see [135], [142]) :

Corollary (3.2.3) [129]: Let $0<\lambda\leqslant(n-1)/2$. If $-2\lambda-1<\alpha<2n\lambda/(n-1)$, then

$$\int_{\mathbb{R}^n}\left|S_*^\lambda(f)(x)\right|^2|x|^\alpha dx\leqslant C_{\lambda,\alpha}\int_{\mathbb{R}^n}|f(x)|^2|x|^\alpha dx.$$

As in [135], by Corollary (3.2.3) we see that $\lim_{R\to\infty}S_R^\lambda(f)(x)=f(x)$ a.e. for all $\lambda>0$ and $f\in L^p(\mathbb{R}^n)$ provided $2\leqslant p<2n/(n-1-2\lambda)$ (for the case $p<2$ see Tao [152]).
We can also consider the spherical means with respect to ρ. For $\beta>0$ let

$$M_t^\beta(f)(x)=c_\beta t^{-n}\int_{\rho(y)<t}(1-t^{-2}\rho(y)^2)^{\beta-1}f(x-y)dy(f\in S), \qquad (31)$$

Where $c_\beta=\Gamma(\beta+n/2)/(\pi^{n/2}\Gamma(\beta))$. we shall prove some weighted estimates for a modified version of $M_t^\beta(f)$.

We assume $\rho(x)=|x|$ in (31) for the rest of this section. By taking the Fourier transform, we can embed these operators in an analytic family of operators in β in such a way that

$$M_t^0(f)(x)=c\int_{S^{n-1}} f(x-ty)d\sigma(y),$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1}. We also define

$$M_*^\beta(f)(x)=\sup_{t>0}\left|M_t^\beta(f)(x)\right|.$$

The operator M_t^β was studied in Stein [147] (see also Stein and Wainger [149] and Kaneko and Sunouchi [140]).

Now we see some applications of Theorems (3.2.1) and (3.2.2) to the spherical means.

Remark (3.2.4) [129]: Define, when $\beta+n/2-1>0$,

$$v_\beta(f)(x)=\left(\int_0^\infty\left|\frac{\partial}{\partial t}M_t^\beta(f)(x)\right|^2 tdt\right)^{1/2}$$

$$=2\left|\beta+\frac{n}{2}-1\right|\left(\int_0^\infty\left|M_t^\beta(f)(x)-M_t^{\beta-1}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2}.$$

If $\delta=\beta+n/2-1>0$, then $\sigma_\delta(f)$ and $v_\beta(f)(f\in S)$ are pointwise equivalent; that is, there are two positive constants A and B such that

$$\sigma_\delta(f)(x)\leqslant Av_\beta(f)(x)\leqslant B\sigma_\delta(f)(x). \tag{32}$$

This was proved by [140]. By (32) we immediately get the $v_\beta(f)$ analogue of Theorem (3.2.1) (see the remark below Theorem (3.2.1)).

Remark (3.2.5) [129]: We write

$$M(f)(x)=\sup_{t>0}\left|\int_{S^{n-1}} f(x-ty)d\sigma(y)\right|.$$

Note that $M(f)(x)=cM_*^0(f)(x)$. Let $n\geqslant 2, n/(n-1)<p$. Then Duoandikoetxea and Vega [137] proved that the inequality

$$\int_{\mathbb{R}^n}|M(f)(x)|^p|x|^{-\alpha}dx\leqslant C\int_{\mathbb{R}^n}|f(x)|^p|x|^{-\alpha}dx \tag{33}$$

holds for $n-p(n-1)<\alpha<n-1$ (this was partly proved in Rubio de Francia [141]) and does not hold for $\alpha>n-1$. Stein [147] proved (33) when $n\geqslant 3,\alpha=0$; the result for $\alpha=0$ and $n=2$ is due to Bourgain [130] (see also [146]). (see [147] and also [149]) we can give another proof of the inequality (33) when $n\geqslant 3, 0\leqslant\alpha<n-1$ and $n/(n-1)<p$.

We shall give the proofs of the theorems and the corollary stated above. To show Theorem (3.2.1) we prove a more general result. For a locally integrable function f, a nonnegative integer m and $\sigma\geqslant 0$, we define

$$|f|_{m,\sigma}=\sup_{z\in\mathbb{R}^n, s\in(0,1]}\inf_{Q\in P_m} s^{-\sigma-n}\int_{B(z,s)}|f(y)-Q(y)|dy,$$

where P_m denotes the collection of polynomials of degree less than or equal to m. We also write $|f|_{m,\sigma}=|f{:}\,m,\sigma|$.

Let $\theta>n$ and let ψ be a measurable function on $\mathbb{R}^n$ satisfying the following properties:

$$|\psi(x)|\leqslant C(1+|x|)^{-\theta}, \tag{34}$$

$$\int_{\mathbb{R}^n}\psi(x)dx=0; \tag{35}$$

furthermore, ψ can be written as

$$\psi(x)=\sum_{k=0}^{\infty} 2^{-k\theta}\eta_k(x), \tag{36}$$

Where $\{\eta_k\}_{k\geqslant 0}$ is a sequence of integrable functions satisfying the following:

$$\operatorname{supp}(\eta_k)\subset\{2^{k-2}\leqslant|x|\leqslant 2^{k+2}\}(k\geqslant 1),$$

$$\operatorname{supp}(\eta_0)\subset\{|x|\leqslant 1\}, \tag{37}$$

$$\sup_{j\geqslant 1}\left|\eta_j:[\theta-n],\theta-n+\kappa\right|<\infty \text{ for some } \kappa>0, \tag{38}$$

$$|\eta_0:[\theta-n],\theta-n|<\infty. \tag{39}$$

Here $[a]$ denotes the greatest integer less than or equal to a. Then we shall prove the following:

Proposition (3.2.6) [129]: Let g_ψ be the Littlewood-Paley operator with ψ satisfying (34) to (39).

(i) Let $0<p<1$. Suppose $\theta=n/p, w\in B_1$ and $w\in A_\infty$. Then

$$\|g_\psi(f)\|_{L_w^{p,\infty}}\leqslant C_{p,w}\parallel f\parallel_{H_w^p}, f\in S_0(\mathbb{R}^n).$$

(ii) Let $0<p\leqslant 1$. Suppose $\theta>n/p, w\in B_{p\theta/n}$ and $w\in A_\infty$. Then

$$\|g_\psi(f)\|_{L_w^p}\leqslant C_{p,\theta,w}\parallel f\parallel_{H_w^p}^p\ f\in S_0(\mathbb{R}^n).$$

To prove Proposition (3.2.6) we use the following result:

Proposition (3.2.7) [129]: Let $\Psi\in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\Psi(x)dx=0$ and let $\theta>n$. Suppose that

$$\left(\int_0^\infty \inf_{P\in P[\theta-n]}\left(\int_{|y|<1}|r^n\Psi(r(x-y))-P(y)|dy\right)\frac{dr}{r}\right)^{1/2}\leqslant C|x|^{-\theta} \tag{40}$$

for $|x|>2$. Then we have the following:

(i) Let $0<p<1$. Suppose $\theta=n/p$ and $w\in B_1$. If the operator g_Ψ is bounded on $L_w^{p_0}$ for some $p_0\in(p,\infty)$, then

$$\|g_\Psi(f)\|_{L_w^{p,\infty}}\leqslant C_{p,w}\parallel f\parallel_{H_w^p}, f\in S_0(\mathbb{R}^n).$$

(ii) Let $0<p\leqslant 1$. Suppose $\theta>n/p$ and $w\in B_{p\theta/n}$. If the operator g_Ψ is bounded on $L_w^{p_0}$ for some $p_0\in(p,\infty)$, then

$$\|g_\Psi(f)\|_{L_w^p}\leqslant C_{p,\theta,w}\parallel f\parallel_{H_w^p}, f\in S_0(\mathbb{R}^n).$$

We use the atomic decomposition to prove Proposition (3.2.7). Let N be a non-negative integer and w be a locally integrable positive function on $\mathbb{R}^n$. Then a measurable function a on $\mathbb{R}^n$ is called a (p,N,w) atom $(0<p\leqslant 1)$ if for some x_0 and s we have

$$\operatorname{supp}(a)\subset B(x_0,s), \tag{41}$$

$$\parallel a\parallel_\infty\leqslant w\big(B(x_0,s)\big)^{-\frac{1}{p}}, \tag{42}$$

And

$$\int_{\mathbb{R}^n} a(x)x^\alpha dx=0 \text{ forall } |\alpha|\leqslant N, \tag{43}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index and $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}, |\alpha|=\alpha_1+\cdots+\alpha_n$.

Lemma (3.2.8) [129]: Let $\Psi\in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\Psi(x)dx=0$ and (40).

(i) Let $0<p<1$. Suppose $\theta=n/p$ and $w\in B_1$. If the operator g_Ψ is bounded on $L_w^{p_0}$ for some $p_0\in(p,\infty)$, then for $a(p,[n/p-n],w)$ atom a we have

$$w(\{x\in\mathbb{R}^n: g_\Psi(a)(x)>\lambda\})\leqslant C\lambda^{-p},$$

where C is independent of a and λ.
(ii) Let $0 < p \leqslant 1$. Suppose $\theta > n/p$ and $w \in B_{p\theta/n}$. If the operator g_Ψ is bounded on $L_w^{p_0}$ for some $p_0 \in (p, \infty)$, then for a $(p, [\theta - n], w)$ atom a we have

$$\|g_\Psi(a)\|_{L_w^p} \leqslant C,$$

where C is independent of a.
This follows from the following result:
Lemma (3.2.9) [129]: Let $\Psi \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Psi(x)dx = 0$ and (40). Let a be a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (42). Then we have

$$g_\Psi(a)(x) \leqslant C\left(|B(x_0, s)|/w\big(B(x_0, s)\big)\right)^{1/p} {}_{S}^{(\theta-n/p)}(s + |x - x_0|)^{-\theta}$$

for x with $|x - x_0| > 2s$.
Proof: We first give a proof for the case $w(x) \equiv 1$. By $(40) - (43)$ with $N = [\theta - n]$ we have, if $|x - x_0| > 2s$,

$$g_\Psi(a)(x)^2 = \int_0^\infty \left| \int_{\mathbb{R}^n} a(y) r^n \Psi(r(x - y)) dy \right|^2 \frac{dr}{r}$$

$$= \int_0^\infty \inf_{P \in P_{[\theta-n]}} \left| \int_{B(x_{0,S})} a(y)(r^n \Psi(r(x - y)) - P(y)) dy \right|^2 \frac{dr}{r}$$

$$\leqslant \| a \|_\infty^2 \int_0^\infty \inf_{P \in P_{[\theta-n]}} \left(\int_{B(x_0,S)} |r^n \Psi(r(x - y)) - P(y)| dy \right)^2 \frac{dr}{r}$$

$$= \| a \|_\infty^2 \int_0^\infty \inf_{P \in P_{[\theta-n]}} \left(\int_{|y|<1} \left|(rs)^n \Psi\big(rs(s^{-1}x - s^{-1}x_0 - y)\big) - P(y)\right| dy \right)^2 \frac{dr}{r}$$

$$\leqslant C \| a \|_\infty^2 (s^{-1}|x - x_0|)^{-2\theta}$$

$$\leqslant Cs^{-\frac{2n}{p}+2\theta} |x - x_0|^{-2\theta} \leqslant Cs^{2\left(\theta - \frac{n}{p}\right)} (s + |x - x_0|)^{-2\theta}.$$

Next, let a be a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (42). Then applying the above estimate to

$$\big(w\big(B(x_0, s)\big)/|B(x_0, s)|\big)^{1/p} a,$$

we get the conclusion.
Now we give the proof of Lemma (3.2.8).
Proof: We first prove part (a). Let a be a $(p, [n/p - n], w)$ atom supported in $B(x_0, s)$ with (42). Then

$$w(\{x \in \mathbb{R}^n : g_\Psi(a)(x) > \lambda\}) \leqslant w(\{x \in B(x_0, 2s) : g_\Psi(a)(x) > \lambda\})$$
$$+ w(\{x \in \mathbb{R}^n / B(x_0, 2s) : g_\Psi(a)(x) > \lambda\})$$
$$= \mathrm{I} + \mathrm{II}, \text{ say}.$$

Since g_Ψ is bounded on $L_w^{p_0}$, by Chebyshev's inequality and Hölder's inequality we have

$$I \leqslant \lambda^{-p} \int_{B(x_0,2s)} |g_\Psi(a)(x)|^p w(x)dx$$
$$\leqslant \lambda^{-p} w\big(B(x_0,2s)\big)^{(p_0-p)/p_0} \left(\int |g_\psi(a)(x)|^{p_0} w(x)dx\right)^{p/p_0}$$
$$\leqslant C\lambda^{-p} w\big(B(x_0,2s)\big)^{(p_0-p)/p_0} \left(\int |a(x)|^{p_0} w(x)dx\right)^{p/p_0}$$
$$\leqslant C\lambda^{-p} w\big(B(x_0,2s)\big)^{(p_0-p)/p_0} w\big(B(x_0,2s)\big)^{-1+p/p_0}$$
$$= C\lambda^{-p}, \tag{44}$$
where to get the last inequality we have used the doubling condition.
Next, by Lemma (3.2.9) we see that
$$\text{II} \leqslant w\left(\left\{x \in \mathbb{R}^n : C\left(|B(x_0,s)|/w\big(B(x_0,s)\big)\right)^{1/p} (s+|x-x_0|)^{-n/p} > \lambda\right\}\right)$$
$$= w\big(\{x \in \mathbb{R}^n : Cs^n(s+|x-x_0|)^{-n} > w\big(B(x_0,s)\big)\lambda^p\}\big)$$
$$= \text{ III, say.}$$
Since $w \in B_1$, recalling that $s^n(s+|x-x_0|)^{-n} \approx M\big(\chi_{B(x_0,s)}\big)(x)$, we have
$$\text{III} \leqslant w\big(\{x \in \mathbb{R}^n : M\big(\chi_{B(x_0,s)}\big)(x) > w\big(B(x_0,s)\big)\lambda^p\}\big) \leqslant C\lambda^{-p}.$$
Combining the estimates for I and II, we conclude the proof of part (1).
Next we turn to the proof of part (b). Let a be a $(p,[\theta-n],w)$ atom supported in $B(x_0,s)$ with (42). Then by Lemma (3.2.9) we have
$$\mathrm{g}_\Psi(a)(x) \leqslant Cw\left(B(x_0,s)^{-\frac{1}{p}} M\big(\chi_{B(x_0,s)}\big)(x)^{\frac{\theta}{n}} \text{ for } |x-x_0| > 2s.\right.$$
Since $w \in B_{p\theta/n}$, we find
$$\int_{\mathbb{R}^n\setminus B(x_0,2s)} \mathrm{g}_\Psi(a)(x)^p w(x)dx \leqslant \mathrm{C}w\big(B(x_0,s)\big)^{-1} \leqslant \int_{\mathbb{R}^n} M\big(\chi_{B(x_0,s)}\big)(x)^{\frac{\theta}{n}} w(x)dx \leqslant C.$$
Combining this with the estimate appearing in (44), we get the conclusion.
To prove Proposition (3.2.7)(a) we need the following result (see [148]):

Lemma (3.2.10) [129]: Let $0 < p < 1$. Suppose $\{f_k\}$ is a sequence of measurable functions on $\mathbb{R}^n$ such that
$$\sup_{\lambda>0} \lambda^p w(\{x : |f_k(x)| > \lambda\}) \leqslant 1 \text{ for all } k,$$
and suppose $\{c_k\}$ is a sequence of complex numbers satisfying $\sum |c_k|^p \leqslant 1$. Then we have
$$\sup_{\lambda>0} \lambda^p w\left(\left\{x \in \mathbb{R}^n : \sum |c_k f_k(x)| > \lambda\right\}\right) \leqslant \frac{2-p}{1-p}.$$
Now we can prove Proposition (3.2.7).

Proof: We note that $f \in S_0(\mathbb{R}^n)$ can be decomposed as $f = \sum_p \lambda_k a_k$ by $(p,[\theta-n],w)$-atoms ($w \in B_{p\theta/n}$), where we have $\lambda_k \geqslant 0, \sum \lambda_k^p \leqslant C \parallel f \parallel_{H_w^P}^p \sum \lambda_k a_k = f$ a.e. and $\sum \lambda_k |a_k| \leqslant Cf^*$, with f^* denoting the grand maximal function (see [127]). Using this decomposition, we first prove part (i). Since f^* is bounded, by the dominated convergence theorem we have $\Psi_t * f = \sum \lambda_k \Psi_t * a_k$ a.e. and so $g_\Psi(f) \sum_k \lambda_k g_\Psi(a_k)$. Thus by Lemmas (3.2.8)(1) and (3.2.10) we see that
$$\sup_{\lambda>0} \lambda^p w(\{x \in \mathbb{R}^n : g_\Psi(f)(x) > \lambda\}) \leqslant C \sum \lambda_k^p \leqslant C \parallel f \parallel_{H_w^P}^p.$$

This completes the proof of Proposition (3.2.7)(i). Part (b) can be proved in the same way by using Lemma (3.2.8)(ii).

Now we turn to the proof of Proposition (3.2.6).

Proof: First we see that if ψ satisfies the conditions (34)-(39), then ψ satisfies the condition (40) of Proposition (3.2.7). Let $|x| > 2$. Then by (34) we have

$$\int_1^\infty \left(\int_{|y|<1} |r^n\psi(r(x-y))|dy\right)^2 \frac{dr}{r}$$

$$\leqslant C\int_1^\infty r^{2n}(1+r|x|)^{-2\theta}\frac{dr}{r} \leqslant C|x|^{-2\theta}\int_1^\infty r^{2n-2\theta}\frac{dr}{r} \leqslant C|x|^{-2\theta}. \quad (45)$$

Let $r \leqslant 1$. Suppose $2^m|x|^{-1} \leqslant r < 2^{m+1}|x|^{-1}$ for $m \leqslant m_x := [(\log 2)^{-1}\log|x|]$. If $|y| \leqslant 1$, then $r|x|/2 \leqslant r|x-y| \leqslant 3r|x|/2$. Therefore, if $m \geqslant 5$, by (36) and (37) we have

$$\psi\big(r(x-y)\big) = \sum_{k=m-3}^{m+5} 2^{-k\theta}\eta_k\big(r(x-y)\big).$$

This expression of ψ and (38) imply that there exists a polynomial $P = P_{r,x} \in P_{[\theta-n]}$ such that

$$\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|dy \leqslant Cr^{\kappa+\theta}2^{-m\theta} \leqslant C|x|^{-\kappa-\theta}2^{m\kappa}. \quad (46)$$

If $m \leqslant 4$, then

$$\psi\big(r(x-y)\big) = \sum_{k=0}^{8} 2^{-k\theta}\eta_k\big(r(x-y)\big).$$

Therefore, by (38) and (39) there exists a polynomial $P = P_{r,x} \in P_{[\theta-n]}$ such that

$$\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|dy \leqslant Cr^\theta \leqslant C|x|^{-\theta}2^{m\theta}. \quad (47)$$

By (46) and (47) we have

$$\int_0^1 \inf_{P\in P_{[\theta-n]}} \left(\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|dy\right)^2 \frac{dr}{r}$$

$$\leqslant \sum_{m\leqslant m_x} \int_{2^m|x|^{-1}}^{2^{m+1}|x|^{-1}} \inf_{P\in P_{[\theta-n]}} \left(\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|dy\right)^2 \frac{dr}{r}$$

$$\leqslant \sum_{m\leqslant 4} C|x|^{-2\theta}2^{2m\theta} + \sum_{5\leqslant m\leqslant m_x} C|x|^{-2(\kappa+\theta)}2^{2m\kappa} \leqslant C|x|^{-2\theta}. \quad (48)$$

Now the condition (40) of Proposition (3.2.7) follows from (45) and (48).

Also by [144] we see that the conditions (34) and (35) imply the L^p_w-boundedness of g_ψ for all $p \in (1,\infty)$ and all $w \in A_p$. So Proposition (3.2.6) follows from Proposition (3.2.7).

Now we give the proof of Theorem (3.2.1).

Proof: Let

$$K^\delta(x) = \int_{\mathbb{R}^n} \eta\big(\rho(\xi)\big)(1-\rho(\xi)^2)_+^\delta e^{2\pi i x\xi}d\xi.$$

Then

$$\left|D^\alpha K^{\delta-1}(x)\right| \leqslant C_\alpha(1+|x|)^{-\delta-\frac{n-1}{2}} \quad (49)$$

for all α, where $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ (see [146]). Therefore, by [143] we see that $K^{\delta-1}$ satisfies the conditions (34)-(39) for ψ with $\theta = \delta + (n-1)/2$ and $0 < \kappa \leqslant [\delta - (n+1)/2] + 1 - \delta + (n+1)/2$ in (38). Thus Theorem (3.2.1) follows from Proposition (3.2.6). The following result can be used to prove Theorem (3.2.2).

Proposition (3.2.11) [129]: Let $0 < \delta < 1$ and suppose that $m_\delta(r) = \chi_{[1-\delta,1]}(r)$ or $m_\delta(r)$ is a continuously differentiable function supported in the interval $[1-\delta, 1]$ and satisfying $\|(d/dr)m_\delta\|_{L^1(\mathbb{R})} \leqslant 1$. Define

$$\left(\widehat{U_t^\delta f}\right)(\xi) = \hat{f}(\xi) m_\delta(t\rho(\xi)).$$

Then for $0 \leqslant \alpha < 1$ we have

$$\int_{\mathbb{R}^n} \int_0^\infty \left|U_t^\delta f(x)\right|^2 |x|^{-\alpha} \frac{dt}{t} dx \leqslant C_\alpha \delta \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx,$$

where C_α is independent of δ.

This was proved in Carbery et al. [135] and Rubio de Francia [142] when $\rho(\xi) = |\xi|$. To prove the general case we use the method of [142], which is based on an application of Hirschman's method in [139] and the weighted estimates for the one-dimensional square functions. To apply that method to our case we only need to observe that $\Lambda(x) = (\| x \| / \rho(x))_x$ is bi-Lipschitz, with $\| x \| = \max(|x_1|, \dots, |x_n|)$, that is

$$A|x-y| \leqslant |\Lambda(x) - \Lambda(y)| \leqslant B|x-y|$$

for some constants $A, B > 0$; but this is an easy consequence of the fact that $\rho(x)$ is positive, homogeneous of degree one and C^∞ in $\mathbb{R}^n \setminus \{0\}$.

We decompose

$$\rho(\xi)^2 (1 - \rho(\xi)^2)_+^{\delta-1} = \sum_{k=0}^\infty 2^{-(\delta-1)k} m_k(\rho(\xi)),$$

where $m_k(t) \in C_0^\infty(\mathbb{R})$, $\text{supp}(m_k) \subset [1-2^{-k}, 1]$ and $\left|\left(\frac{d}{dr}\right) m_k(r)\right| \leqslant C2^k$, for $k \geqslant 1$. Put $\psi_k(x) = \mathcal{F}^{-1}(m_k(\rho(\xi)))(x)$ and $g_k(f) = g_{\psi_k}(f)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We can take $m_0(t)$ so that g_0 is bounded on L_w^2 for any $w \in A_2$. Now by Proposition (3.2.11) for $k \geqslant 1$ we have

$$\|g_k(f)\|_{L^2(|x|^{-\alpha})} \leqslant C 2^{-k/2} \| f \|_{L^2(|x|^{-\alpha})} \text{ for } 0 \leqslant \alpha < 1.$$

Thus if $\delta > 1/2$ we have

$$\| \sigma\delta(f) \|_{L^2(|x|^{-\alpha})} \leqslant \sum_{k=0}^\infty 2^{-(\delta-1)k} \|g_k(f)\|_{L^2(|x|^{-\alpha})} \leqslant \sum_{k=0}^\infty C 2^{-(\delta-1)k} 2^{-\frac{k}{2}} \| f \|_{L^2(|x|^{-\alpha})}$$

$$\leqslant C_\delta \| f \|_{L^2(|x|^{-\alpha})}.$$

This completes the proof.

To apply the result to the maximal operator S_*^δ defined in (30) we use the following, which can be proved as in the case $\rho(\xi) = |\xi|$ (see Stein and Weiss [150]).

Lemma (3.2.12) [129]: Let S_R^δ be as in (27). If $\beta > 0$ and $\delta > -1$, then we have

$$S_R^{\delta+\beta}(f)(x) = \frac{2\Gamma(\delta+\beta+1)}{\Gamma(\delta+1)\Gamma(\beta)} \int_0^1 (1-t^2)^{\beta-1} t^{2\delta+1} S_{Rt}^\delta(f)(x) dt,$$

for a suitable function f.

Here we give the proof of Corollary (3.2.3).

Proof: Using Lemma (3.2.12) and Theorem (3.2.2) and arguing as in the proof of [150] we have

$$\|S_*^{\lambda}(f)\|_{L^2(|x|^{\alpha})} \leqslant C_{\lambda,\alpha} \parallel f \parallel_{L^2(|x|^{\alpha})} \tag{50}$$

for all $\lambda > 0$ and $-1 < \alpha \leqslant 0$. It is known that if $\lambda \geqslant (n-1)/2$, then

$$\|S_*^{\lambda}(f)\|_{L^2(|x|^{\beta})} \leqslant C_{\lambda,\beta} \parallel f \parallel_{L^2(|x|^{\beta})} \tag{51}$$

for $-n < \beta < n$. We extend the estimates (50) and (51) to complex λ and interpolating between them, we get the conclusion.

For a locally integrable function f, a non-negative integer m and $\sigma \geqslant 0$, we define

$$|f|_{m,\sigma}^* = \sup_{z\in\mathbb{R}^n, s>0} \inf_{Q\in P_m} s^{-\sigma-n} \int_{B(z,s)} |f(y) - Q(y)| dy.$$

Let $\psi \in L^1(\mathbb{R}^n)$ and $\theta \geqslant 0$. We say $\psi \in \mathcal{F}(m,\sigma,\theta)$ if ψ can be written as in (36) with $\{\eta_k\}_{k\geqslant 0}$ satisfying (37) and the condition $\sup_{k\geqslant 0} |\eta_k|_{m,\sigma}^* < \infty$. This function class was introduced by Sato [143] to make a unified approach to the studies of maximal Bochner-Riesz means and maximal spherical means in certain problems. By the methods in the proof of Theorem (3.2.1) we can prove the following:

Proposition (3.2.13) [129]: Let $\theta > n$ and $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$. Define $T^*(f)(x) = \sup_{t>0} |L_t * f(x)|$

(i) Let $0 < p < 1$. Suppose $\theta = n/p$ and $w \in B_1$. Then

$$\|T^*(f)\|_{L_w^{p,\infty}} \leqslant C_{p,w} \parallel f \parallel_{H_w^p}, \ f \in S_0(\mathbb{R}^n).$$

(ii) Let $0 < p \leqslant 1$. Suppose $\theta > n/p$ and $w \in B_{p\theta/n}$. Then

$$\|T^*(f)\|_{L_w^p}^p \leqslant C_{p,\theta,w} \parallel f \parallel_{H_w^p}, \ f \in S_0(\mathbb{R}^n).$$

(iii) Let $0 < p \leqslant 1$. Suppose $\theta > n/p, w \in B_{p\theta/n}$ and $w \in A_\infty$. Then

$$\|L_t * f\|_{H_w^p}^p \leqslant C_{p,\theta,w} \parallel f \parallel_{H_w^p}, \ f \in S_0(\mathbb{R}^n),$$

where the constant $C_{p,\theta,w}$ is independent of $t > 0$.

Proof: Since $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$, arguing as in [143] we have

$$\begin{aligned} T^*(a)(x) \leqslant \quad & C\left(\frac{|B(x_0,s)|}{w(B(x_0,s))}\right)^{\frac{1}{p}} s^{\left(\theta - \frac{n}{p}\right)} (s + |x - x_0|)^{-\theta} \\ & \leqslant Cw(B(x_0,s))^{\frac{1}{p}} M(\chi_{B(x_0,s)})(x)^{\frac{\theta}{n}}, \end{aligned} \tag{52}$$

where a is a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (42). As in the case of the proof of Proposition (3.2.7), this implies parts (i) and (ii). Part (iii) follows from this estimate along with the multiplier characterization of the weighted Hardy spaces (see [127]), which requires the condition $w \in A_\infty$. This completes the proof.

When $w \in A_1$, part (a) of Proposition 4 is in [143]. Also, if $0 < p < 1, w \in A_1$ and $\rho(\xi) = |\xi|$, it is known that $S_*^{\delta(p)-1}$ extends to a bounded operator from H_w^p to $L_w^{p,\infty}$ (see[143]). Let $\theta = \delta + (n-1)/2, \delta \geqslant \delta(p), 0 < p \leqslant 1, \delta > \delta(1)$. Then the estimate (49) implies that $K^{\delta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$ (see [143]). Thus by Proposition (3.2.13) we have the following:

Corollary (3.2.14) [129]: Let $\tilde{S}_*^{\delta}(f)(x) = \sup_{R>0} \left|\tilde{S}_R^{\delta}(f)(x)\right|$, where $\tilde{S}_R^{\delta}(f)(x)$ is as in (29).

(i) Let $0 < p < 1$ and $w \in B_1$. Then

$$\left\|\tilde{S}_*^{\delta(p)-1}(f)\right\|_{L_w^{p,\infty}} \leqslant C_{p,w} \parallel f \parallel_{H_w^p}, f \in S_0(\mathbb{R}^n).$$

(ii) Let $0 < p \leqslant 1, \delta > \delta(p)$ and $w \in B_{1+n^{-1}p(\delta - \delta(p))}$. Then

$$\|\tilde{S}_*^{\delta-1}(f)\|_{L_w^p} \leqslant C_{p,\delta,w} \parallel f \parallel_{H_w^p}, f \in S_0(\mathbb{R}^n).$$

(iii) Let $0 < p \leqslant 1, \delta > \delta(p), w \in B_{1+n^{-1}p(\delta-\delta(p))}$ and $w \in A_\infty$. Then

$$\|\tilde{S}_*^{\delta-1}(f)\|_{H_w^p} \leqslant C_{p,\delta,w} \parallel f \parallel_{L_w^p}, f \in S_0(\mathbb{R}^n),$$

where the constant $C_{p,\delta,w}$ is independent of $R > 0$.

Part (iii) of Corollary (3.2.14) extends a result of Sjölin [145] to the weighted Hardy spaces. When $\rho(\xi) = |\xi|$ and $w(x) \equiv 1$, part (i) (with $S_*^\delta(f)$ in place of $S\delta * (f)$) is proved in Stein et al. [148]. The estimate for $\tilde{S}_*^\delta$ similar to [148] immediately follows from (49), as we can see from the proof of [148]. We can also have the estimate (52) for $\tilde{S}_*^{\delta-1}$ in place of T^* as an application of that estimate. If $0 < p < 1, w \in A_1$ and $\rho(x) = |x|$, then it is known that $M_*^{\beta(p)-1/2}$ is bounded from H_w^p to $L_w^{p,\infty}$, where $\beta(p) = n(1/p - 1) + 3/2$ (see [143]). For $\beta > 0$ let

$$\widetilde{M}_t^\beta(f)(x) = c_\beta t^{-n} \int_{\rho(y)<t} \eta(\rho(y)/t)(1 - t^{-2}\rho(y)^2)^{\beta-1} f(x-y)dy,$$

where c_β is as in (31) and η is as in (29). Then $\eta(\rho(y))(1-\rho(y)^2)_+^{\beta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$, where $\beta > 1$ and $\theta = \beta + n - 1$, and hence by Proposition (3.2.13) we also have the following:

Corollary (3.2.15) [129]: Let

$$\widetilde{M}_*^\beta(f)(x) = \sup_{t>0} \left|\widetilde{M}_t^\beta(f)(x)\right|$$

and write $\beta(p) = n(1/p - 1) + 3/2$.

(i) Let $0 < p < 1$ and $w \in B_1$. Then

$$\left\|\widetilde{M}_*^{\beta(p)-1/2}(f)\right\|_{L_w^{p,\infty}} \leqslant C_{p,w} \parallel f \parallel_{H_w^p}, f \in S_0(\mathbb{R}^n).$$

(ii) Let $0 < p \leqslant 1, \beta > \beta(p)$ and $w \in B_{1+n^{-1}p(\beta-\beta(p))}$. Then

$$\left\|\widetilde{M}_*^{\beta-1/2}(f)\right\|_{L_w^p} \leqslant C_{p,\beta,w} \parallel f \parallel_{H_w^p}, f \in S_0(\mathbb{R}^n).$$

(iii) Let $0 < p \leqslant 1, \beta > \beta(p), w \in B_{1+n^{-1}p(\beta-\beta(p))}$ and $w \in A_\infty$. Then

$$\left\|\widetilde{M}_t^{\beta-\frac{1}{2}}(f)\right\|_{H_w^p} \leqslant C_{p,\beta,w} \parallel f \parallel_{H_w^p}, f \in S_0(\mathbb{R}^n),$$

where the constant $C_{p,\beta,w}$ is independent of $t > 0$.

When $\rho(\xi) = |\xi|$ and $w(x) \equiv 1$, part (i) of Corollary (3.2.15) with $M_*^\beta(f)$ in place of $\widetilde{M}_*^\beta(f)$ is proved in Stein et al. [148]. The estimate (52) for $\widetilde{M}_*^{\beta-1/2}(f)$ in place of T^* also follows from an application of the argument in [148].

Chapter 4

Atomic Decompositions with Dual and Hardy-Lorentz Spaces

Dual spaces are identified and some interpolation properties of the martingale Hardy-LorentzKaramata spaces are obtained. The proofs mainly depend on the classical tool of atomic decompositions. As usual, these conclusions are closely related to the geometrical properties of the underlying Banach spaces.

Section (4.1): Dual Spaces and Interpolations of Martingale Hardy-Lorentz-Karamata Spaces

The family of martingale Hardy spaces is one of the important martingale function spaces. The study of the martingale Hardy spaces is extended to the martingale Hardy-Lorentz spaces [20], [27]. This aims to provide a further extension of the martingale Hardy spaces to the martingale Hardy-LorentzKaramata spaces. The family of martingale Hardy-Lorentz-Karamata spaces is defined in terms of Lorentz-Karamata spaces.

The family of Lorentz-Karamata spaces is a generalization of the Lorentz spaces, the Lorentz-Zygmund spaces and the generalized Lorentz-Zygmund spaces [159], [170]. It is defined via the slowly varying functions. Some important theorems for Lorentz-Karamata spaces are presented in [159].

The main theme is the generalization of those important results in martingale Hardy spaces to the martingale Hardy-Lorentz-Karamata spaces.

We introduce five martingale Hardy-Lorentz-Karamata spaces in Definition (4.1.6).

We establish the BurkHölder-Davis-Gundy inequality when the filtration of the underlying probability space is regular.

We obtain the atomic decompositions of the martingale Hardy-Lorentz-Karamata spaces. Using the atomic decompositions, we show that the dual spaces of the martingale Hardy-Lorentz-Karamata spaces are the space of functions of bounded mean oscillation (BMO) and the Lipschitz spaces associated with slowly varying functions.

We establish some interpolation properties of the martingale Hardy-LorentzKaramata spaces. We find that they are interpolation spaces of the martingale Hardy spaces under a new interpolation functor tailor-made for the Lorentz-Karamata spaces introduced in [160]. Finally, by using these interpolation results, we prove the identification of the five martingale Hardy-Lorentz-Karamata spaces when the filtration of the underlying probability space is regular. we recall the definition of Lorentz-Karamata spaces and state some properties of these function spaces. For a more detail account of Lorentz-Karamata spaces, see [159], [170].

For $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. We denote the space of measurable functions on $(\Omega, \Sigma, \mathbb{P})$ by M.

For any $f \in M$ and $s > 0$, write

$$d_f(s) = \mathbb{P}(\{x \in \Omega : |f(x)| > s\})$$

and

$$f^*(t) = \inf \{s > 0 : d_f(s) \leq t\}, 0 \leq t \leq 1.$$

We write $f \approx g$ if

$$Bf \leq g \leq Cf,$$

for some constants $B, C > 0$ independent of appropriate quantities involved in the expressions of f and g.

We recall the definition of slowly varying function in order to define the Lorentz-Karamata spaces.

A function $f:[1,\infty) \to (0,\infty)$ is equivalent to a non-decreasing function (non-increasing function) if there exists a non-decreasing function (non-increasing function) g and constants $B, C > 0$ such that $f \approx$ g on $[1,\infty)$.

Definition (4.1.1)[153]: A Lebesgue measurable function b: $[1,\infty) \to (0,\infty)$ is said to be a slowly varying function if for any given $\epsilon > 0$, the function $t^{\epsilon}b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.

Let b be a slowly varying function on $[1,\infty)$, define γ_b on (0,1] by

$$\gamma_b(t) = b(t^{-1}), 0 < t \leq 1.$$

We restate the definition of slowly varying function from [176, Definition 3.4.32], while we modify the definition of γ_b according to our setting for probability space.

The following presents some remarkable features of slowly varying functions. It is a modification of [159] to slowly varying functions defined on (0,1].

Proposition (4.1.2) [153]: Let b be a slowly varying function.

(i) For any given $r \in \mathbb{R}$, the function b^r is slowly varying and $\gamma_{b^r} = \gamma_b^r$.

(ii) For any given $\epsilon > 0$, the function $t^{\epsilon}\gamma_b$ is equivalent to a non-decreasing function and $t^{-\epsilon}\gamma_b$ is equivalent to a non-increasing function on (0,1].

(iii) If $a > 0$, then for all $t > 0$,

$$\int_0^t s^{a-1}\gamma_b(s)ds \approx \sup_{0<s\leq t} \gamma_b(s) \approx t^a\gamma_b(t)$$

and

$$\int_t^1 s^{-a-1}\gamma_b(s)ds \approx \sup_{0<s<t} s^{-a}\gamma_b(s) \approx t^{-a}\gamma_b(t).$$

(iv) For any $a > 0$, the function b_1 defined on $[1,\infty)$ by $b_1(t) = b(t^a)$ is slowly varying.

For the proofs of the above results, see [159].

We recall the definition of the Lorentz-Karamata space from [159].

Definition (4.1.3) [153]: Let $0 < p, q \leq \infty$ and b be a slowly varying function. The LorentzKaramata space $L_{p,q,b}$ consists of those Lebesgue measurable functions that satisfy $\| f \|_{L_{p,q,b}} < \infty$, where

$$\| f \|_{L_{p,q,b}} = \begin{cases} \left[\int_0^1 \left(t^{1/p}\gamma_b(t)f^*(t)\right)^q \frac{dt}{t}\right]^{1/q} & ,0 < q < \infty, \\ \sup_{0\leq t\leq 1} \left\{t^{1/p}\gamma_b(t)f^*(t)\right\}, & q = \infty. \end{cases}$$

The Lorentz-Karamata space is a rearrangement-invariant (r.-i.) quasi-Banach function space [164]. When $1 \leq p, q < \infty, L_{p,q,b}$ is a Banach space [159]. When $b \equiv 1$, the Lorentz-Karamata space becomes the Lorentz space $L_{p,q}$. When $b(t) = 1 + \log t$, the LorentzKaramata space reduces to the Lorentz-Zygmund space introduced and studied in [154].

Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$. Define the family of positive functions $\{l_i\}_{i=0}^m$ on $(0,\infty)$ by

$$l_0(t) = t^{-1}, l_i(t) = 1 + \log l_{i-1}(t), 1 \leq i \leq m, 0 < t \leq 1$$

Moreover, define

$$\theta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t).$$

The generalized Lorentz-Zygmund space consists of all $f \in M$ such that $\| f \|_{L_{p,q,\alpha}} < \infty$ where

$$\| f \|_{L_{p,q,\alpha}} = \begin{cases} \left[\int_0^1 \left(t^{1/p}\theta_\alpha^m(t) f^*(t)\right)^q \frac{dt}{t}\right]^{1/q} & ,0 < q < \infty, \\ \sup_{0 \le t \le 1} \{t^{1/p}\theta_\alpha^m(t) f^*(t)\}, & q = \infty. \end{cases}$$

Apparently, the generalized Lorentz-Zygmund space is a member of Lorentz-Karamata spaces (see [159]).

The following gives an equivalent quasi-norm for $L_{p,q,b}$. This equivalent quasi-norm is used to establish the atomic decompositions of the martingale Hardy-Lorentz-Karamata spaces.

Lemma (4.1.4) [153]: Let $0 < p < \infty, 0 < q \le \infty$ and b be a slowly varying function. Then $\|\cdot\|_{Lp,q,b}$ and

$$\| f \|_{L_{p,q,b,*}} = \left(\int_0^\infty \left[d_f(u)^{\frac{1}{p}} \gamma_b\left(d_f(u)\right) u\right]^q \frac{du}{u}\right)^{\frac{1}{q}} \tag{1}$$

are equivalent quasi-norms.

Proof : For any $f \in M$, there exists a sequence of non-negative simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \uparrow |f|$ a.e. Moreover, $d_{f_n} \uparrow d_f$ and $f_n^* \uparrow f^*$. Therefore, by using Lebesgue monotone convergence theorem, it suffices to establish that the quasi-norm defined in (1) is equivalent with $\|\cdot\|_{L_{p,q,b}}$ for nonnegative simple functions.

Let $f(x) = \sum_{j=1}^N a_j \chi_{E_j}(x)$ where $\{E_j\}_{j=1}^N$ is a family of finite Lebesgue measurable sets and $\{a_j\}_{j=1}^N \subset \mathbb{R}$ satisfying $a_i \ge a_j \ge 0$ when $i \ge j$.

For any $u > 0$, we have

$$d_f(u) = \sum_{j=0}^N B_j \chi_{[a_{j+1},a_j)}(u),$$

where $B_j = \sum_{i=1}^j |E_j|$. Furthermore, we find that

$$f^*(t) = \sum_{j=1}^N a_j \chi_{[B_{j-1},B_j)}(t),$$

where $B_0 = 0$. Write

$$\Gamma(u) = \int_0^u t^{\frac{q}{p}-1} \gamma_b(t) dt,$$

we obtain

$$\| f \|_{L_{p,q,b}}^q = \sum_{j=1}^N a_j^q \int_{B_{j-1}}^{B_j} t^{q/p-1} \gamma_b(t) dt = \sum_{j=1}^N a_j^q \left(\Gamma(B_j) - \Gamma(B_{j-1})\right)$$
$$= \sum_{j=1}^N (a_{j+1}^q - a_j^q) \Gamma(B_j).$$

Item (3) of Proposition (4.1.2) assures that $\Gamma(t) \approx t^{\frac{q}{p}} \gamma_b(t)$. Thus,

$$A\sum_{j=1}^{N}\left(a_{j+1}^{q}-a_{j}^{q}\right)B_{j}^{q/p}\gamma_{b}(B_{j})\leq\|f\|_{L_{p,q,b}}^{q}\leq C\sum_{j=1}^{N}\left(a_{j+1}^{q}-a_{j}^{q}\right)B_{j}^{\frac{q}{p}}\gamma_{b}(B_{j}),$$

for some $A, C > 0$ independent of f. Since

$$\|f\|_{L_{p,q,b,}^{*}}^{q}=\sum_{j=1}^{N}\left(a_{j+1}^{q}-a_{j}^{q}\right)B_{j}^{q/p}\gamma_{b}(B_{j}),$$

we have $A\|f\|_{L_{p,q,b,}^{*}}\leq\|f\|_{L_{p,q,b}}\leq C\|f\|_{L_{p,q,b,}^{*}}$ for some constants $A, C > 0$ independent of f.

The above lemma is an extension of the corresponding result for Lorentz spaces, see [161].

Let $0 < r < \infty$.We say that a quasi-norm $\|\cdot\|$ is an $r-$ norm if

$$\|x_1+x_2+\cdots+x_n\|^r\leq C(\|x_1\|^r+\|x_2\|^r+\cdots+\|x_n\|^r),$$

for some $C > 0$ independent of $\{x_i\}_{i=1}^{n}$.

Proposition (4.1.5) [153]: Let $0 < p < \infty, 0 < q \leq \infty$ and b be a slowly varying function. If $r = \min(p, q) \leq 1$, then $\|\cdot\|_{L_{p,q,b}}$ is an $r-$ norm.

Proof: In view of Items (i) and (ii) of Proposition (4.1.2), b^r is a slowly varying function and, hence, $L_{p/r,q/r,b^r}$ is a Lorentz-Karamata space. Furthermore, we have

$$\||f|^r\|_{L_{p/r,q/r,b^r}}=\|f\|_{L_{p,q,b}}^{r}.$$

Let $\{f_i\}_{i=1}^{n}\subset L_{p,q,b}$. As $0 < r \leq 1$, the r-inequality asserts that

$$\|f_1+\cdots+f_n\|_{L_{p,q,b}}=\||f_1+\cdots+f_n|^r\|_{L^r_{p/r,q/r,b}}^{\frac{1}{r}}\leq\||f|_1^r+\cdots+|f|_n^r\|_{L^r_{p/r,q/r,b}}^{\frac{1}{r}}.$$

Since $L_{p/r,q/r,b^r}$ is a Banach space, hence, $\|\cdot\|_{L_{p/r,q/r,b^r}}$ is equivalent to a 1-norm [159], we find that

$$\|f_1+\cdots+f_n\|_{L_{p,q,b}}\leq C\left(\||f_1|^r\|_{L_{p/r,q/r,b}}+\cdots+\||f_n|^r\|_{L_{p/r,q/r,b}}\right)^{1/r}.$$

Therefore,

$$\|f_1+\cdots+f_n\|_{L_{p,q,b}}^{r}\leq C\left(\|f_1\|_{L^r_{p/r,q/r,b}}^{r}+\cdots+\|f_n\|_{L^r_{p/r,q/r,b^r}}^{r}\right).$$

Proposition (4.1.5) is used to obtain the atomic decompositions of martingale Hardy-LorentzKaramata spaces.

We introduce the martingale Hardy-Lorentz-Karamata spaces. We begin with some fundamental notions and notation from martingale theory.

Let $\mathcal{F}=(\mathcal{F}_n)_{n\geq 0}$ be a filtration on $(\Omega,\Sigma,\mathbb{P})$. That is, $(\mathcal{F}_n)_{n\geq 0}$ is a non-decreasing sequence of subσ-algebras of Σ with $\Sigma=\sigma(\cup_{n\geq 0}\mathcal{F}_n)$. Let $\mathcal{F}_{-1}=\mathcal{F}_0$.

Let $\mathbb{E}$ denote the expectation operator. The conditional expectation operator related to $\mathcal{F}_n$ is denoted by $\mathbb{E}_n$. For any martingale $f=(f_n)_{n\geq 0}$ on Ω, write $d_i f = f_i - f_{i-1}, i \geq 0$. For any stopping time v, write $f_n^v=\sum_{i=0}^{n}\chi(v\geq i)d_i f$.

The maximal function, the square function (quadratic variation) and the conditional square function (conditional quadratic variation) of f are defined by

$$M_n(f) = \sup_{0\le i\le n} |f_i|, M(f) = \sup_{i\ge 0}|f_i|,$$

$$S_n(f) = \left(\sum_{i=0}^{n} |d_i f|^2\right)^{1/2}, S(f) = \left(\sum_{i=0}^{\infty} |d_i f|^2\right)^{1/2},$$

$$S_n(f) = \left(\sum_{i=0}^{\infty} \mathbb{E}_{i-1}|d_i f|^2\right)^{\frac{1}{2}}, S(f) = \left(\sum_{i=0}^{\infty} \mathbb{E}_{i-1}|d_i f|^2\right)^{\frac{1}{2}},$$

respectively.

Let $0 < p, q \le \infty$ and b be a slowly varying function. Let $\Lambda_{p,q,b}$ denote the class of all nondecreasing, non-negative family of adapted random variables $\rho = (\rho_n)_{n\ge 0}$ with $\rho_\infty = \lim_{n\to\infty}\rho_n \in \mathrm{L}_{p,q,b}$. A martingale $f = (f_n)_{n\ge 0}$ is said to have predictable control in $\mathrm{L}_{p,q,b}$ if there exists a sequence $\rho = (\rho_n)_{n\ge 0}$ such that

$$|f_n| \le \rho_{n-1} \text{ and } \rho \in \Lambda_{p,q,b}, n \ge 1.$$

Definition (4.1.6) [153]: Let $0 < p, q \le \infty$ and b be a slowly varying function. We have the following martingale Hardy-Lorentz-Karamata spaces:

$$H^*_{p,q,b} = \left\{f = (f_n)_{n\ge 0} : \| f \|_{H^*_{p,q,b}} = \| M(f) \|_{L_{p,q,b}} < \infty\right\},$$

$$H^s_{p,q,b} = \left\{f = (f_n)_{n\ge 0} : \| f \|^s_{H^s_{p,q,b}} = \| S(f) \|_{L_{p,q,b}} < \infty\right\},$$

$$H^S_{p,q,b} = \left\{f = (f_n)_{n\ge 0} : \| f \|_{H^S_{p,q,b}} = \| S(f) \|_{L_{p,q,b}} < \infty\right\},$$

$$\mathcal{Q}_{p,q,b} = \{f = (f_n)_{n\ge 0} : \exists\rho = (\rho_n)_{n\ge 0} \in \Lambda_{p,q,b} \text{ s.t. } S_n(f) \le \rho_{n-1}\}$$

$$with \|f\|_{Q_{p,q,b}} = \inf_{\rho} \rho_\infty \|_{L_{p,q,b}}$$

$$P_{p,q,b} = \{f = (f_n)_{n\ge 0} : \exists\rho = (\rho_n)_{n\ge 0} \in \Lambda_{p,q,b} \text{ s.t. } |f| \le \rho_{n-1}\}$$

with $\| f \|_{Q_{p,q,b}} = \inf_\rho \| {}_\infty \|_{\mathrm{L}_{p,q,b}}$.

The family of Lorentz martingale spaces $H^*_{p,q}, H^S_{p,q}, H^S_{p,q} = P_{p,q}$ and $Q_{p,q}$ introduced and studied in [27] are special cases of the family of martingale Hardy-Lorentz-Karamata spaces when $b \equiv 1$. In particular, the martingale Hardy-Lorentz-Karamata spaces cover the Hardy martingale spaces $H^*_p, H^s_p, H^S_p = P_p$ and Q_p[25], [27]. The atomic decomposition of the martingale Hardy-Lorentz spaces is given in [20].

Furthermore, the above definition also generalizes the study of martingale function spaces to martingale Hardy-Lorentz-Zygmund space and martingale Hardy-type generalized-Lorentz-Zygmund space.

For a detailed study of the martingale Hardy spaces, see [163], [168], [171], [27], [28]. The atomic decompositions for H^s_p, P_p and Q_p are presented in [27], respectively. Using the atomic decomposition, the dual spaces of the martingale Hardy spaces are studied in [27]. A detailed account of the interpolation properties of martingale Hardy space is given in [27]. Particularly, the interpolation of function spaces show that, when $\mathcal{F}$ is regular, the martingale Lorentz-Hardy spaces, $H^*_{p,q}, H^S_{p,q}, H^S_{p,q} = P_{p,q}$ and $Q_{p,q}$, are equivalent.

Recall that $\mathcal{F}$ is said to be regular if there exists a number $R > 0$ such that

$$f_n \le R f_{n-1} \ \forall n \in \mathbb{N},$$

for all non-negative martingales $f = (f_n)_{n\in\mathbb{N}}$, see [27].

Note that the regularity of $\mathcal{F}$ is equivalent to the strong good stopping time property.

Proposition (4.1.7) [153]: If $\mathcal{F}$ is regular, then for all non-negative adapted processes $\gamma = (\gamma_n)_{n\geq 0}$ and $\lambda \geq \|\gamma_0\|_{L^\infty}$, there exist a constant $C > 0$ and a stopping time τ_λ such that

$$\{M(\gamma) > \lambda\} \subset \{\tau_\lambda < \infty\}, \tag{2}$$

$$\mathbb{P}(\{\tau_\lambda < \infty\}) \leq C\mathbb{P}(\{M(\gamma) > \lambda\}), \tag{3}$$

$$\sup_{n\leq\tau_\lambda} \gamma_n = M_{\tau_\lambda}(\gamma) \leq \lambda, \tag{4}$$

$$\lambda_2 \geq \lambda_1 \geq \|\gamma_0\|_{L^\infty} \Rightarrow \tau_{\lambda_1} \leq \tau_{\lambda_2}. \tag{5}$$

For the proof of the preceding result, see [169].

Proposition (4.1.8) [153]: Let $0 < p < \infty$. We have

$$P_p = Q_p. \tag{6}$$

When $\mathcal{F}$ is regular, we have

$$H_p^* = H_p^S = H_p^s = P_p = Q_p. \tag{7}$$

For the proof of (6), see [27]. For the proofs of (7), see [27].

We now extend the identification $H_{p,q,b}^* = H_{p,q,b}^S$ when $\mathcal{F}$ is regular in the following. It is also an extension of BurkHölder-Davis-Gundy inequality to $L_{p,q,b}$, $0 < p, q \leq \infty$.

We first recall the good λ-inequality satisfied by the maximal function and the square function [25].

Proposition (4.1.9) [153]: Let $\mathcal{F}$ be regular. For any $\alpha > 1$ and $\beta > 0$, we have $\epsilon_{\alpha,\beta}, k_{\alpha,\beta} > 0$ satisfying $\lim_{\beta\to 0} \epsilon_{\alpha,\beta} = 0$,

$$\mathbb{P}(\{M(f) > \alpha\lambda\}) \leq \epsilon_{\alpha,\beta}\mathbb{P}(\{M(f) > \lambda\}) + k_{\alpha,\beta}\mathbb{P}(\{S(f) > \beta\lambda\}), \lambda > 0, \tag{8}$$

$$\mathbb{P}(\{S(f) > \alpha\lambda\}) \leq \epsilon_{\alpha,\beta}\mathbb{P}(\{S(f) > \lambda\}) + k_{\alpha,\beta}\mathbb{P}(\{M(f) > \beta\lambda\}), \lambda > 0. \tag{9}$$

For the proof of the above inequalities, see [25].

The subsequent supporting lemma gives a special feature of slowly varying functions.

Lemma (4.1.10) [153]: Let $0 < p \leq \infty$ and b be a slowly varying function. We have a constant $C > 0$ such that for any $A, B > 0$

$$(A + B)^{\frac{1}{p}}\gamma_b(A + B) \leq C\left(A^{\frac{1}{p}}\gamma_b(A) + B^{\frac{1}{p}}\gamma_b(B)\right).$$

Proof: For any $\epsilon > 0, t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function. Thus,

$$A^{\frac{1}{p}+\epsilon}(A + B)^{-\epsilon}\gamma_b(A + B) \leq CA^{\frac{1}{p}}\gamma_b(A),$$

$$B^{\frac{1}{p}+\epsilon}(A + B)^{-\epsilon}\gamma_b(A + B) \leq CB^{\frac{1}{p}}\gamma_b(B),$$

for some $C > 0$. Hence, we have

$$\left(A^{\frac{1}{p}+\epsilon} + B^{\frac{1}{p}+\epsilon}\right)(A + B)^{-\epsilon-\frac{1}{p}}(A + B)^{\frac{1}{p}}\gamma_b(A + B) \leq C\left(A^{\frac{1}{p}}\gamma_b(A) + B^{\frac{1}{p}}\gamma_b(B)\right).$$

Furthermore, we have a constant $C > 0$ such that for any $A, B > 0$,

$$C(A + B)^{1/p+\epsilon} \leq A^{1/p+\epsilon} + B^{1/p+\epsilon}.$$

Hence, our desired inequalities follow.

The subsequent proposition asserts that distributional inequality can be transformed to be norm inequality for $L_{p,q,b}$.

Proposition (4.1.11) [153]: Let $\alpha > 1$ and $\beta > 0$. Let $0 < p < \infty, 0 < q \leq \infty$ and b be a slowly varying function. Let F and G be locally integrable functions. If there exist $\epsilon_{\alpha,\beta}, k_{\alpha,\beta} > 0$ satisfying $\lim_{\beta\to 0}\epsilon_{\alpha,\beta} = 0$ and

$$\mathbb{P}(\{F > \alpha\lambda\}) \leq \epsilon_{\alpha,\beta}\mathbb{P}(\{F > \lambda\}) + k_{\alpha,\beta}\mathbb{P}(\{G > \beta\lambda\}), \lambda > 0. \tag{10}$$

Then

$$\| F \|_{L_{p,q,b}} \leq C \| G \|_{L_{p,q,b}},$$

for some $C > 0$ independent of F and G.

Proof : We rewrite (10) in terms of the distribution functions, we obtain

$$d_F(\alpha\lambda) \leq \epsilon_{\alpha,\beta} d_F(\lambda) + k_{\alpha,\beta} d_G(\beta\lambda).$$

Therefore, Lemma (4.1.10) with $A = \epsilon_{\alpha,\beta} d_F(\lambda)$ and $B = k_{\alpha,\beta} d_G(\beta\lambda)$ ensures that

$$\left(\int_0^\infty \left[d_F(\alpha\lambda)^{\frac{1}{p}}\gamma_b(d_F(\alpha\lambda))\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q}$$

$$\leq C\left(\int_0^\infty \left[\left(\epsilon_{\alpha,\beta} d_F(\lambda)\right)^{\frac{1}{p}}\gamma_b\left(\epsilon_{\alpha,\beta} d_F(\lambda)\right)\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q}$$

$$+C\left(\int_0^\infty \left[\left(k_{\alpha,\beta} d_G(\beta\lambda)\right)^{\frac{1}{p}}\gamma_b\left(k_{\alpha,\beta} d_G(\beta\lambda)\right)\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q},$$

for some $C > 0$.

By using Lemma (4.1.4), we have

$$\frac{C_0}{\alpha} \| F \|_{L_{p,q,b}} \leq \left(\int_0^\infty \left[d_F(\alpha\lambda)^{\frac{1}{p}}\gamma_b\left(d_F(\alpha\lambda)\right)^\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} \tag{11}$$

and

$$\left(\int_0^\infty \left[d_G(\beta\lambda)^{\frac{1}{p}}\gamma_b(d_G(\beta\lambda))^\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q} \leq \frac{C_1}{\beta} \| G \|_{L_{p,q,b}}, \tag{12}$$

for some $C_0, C_1 > 0$.

Furthermore, since $t^{1/2p}\gamma_b(t)$ is equivalent to a non-decreasing function. For any $\epsilon < 1$, we have

$$(\epsilon d_F(\lambda))^{\frac{1}{2p}}\gamma_b(\epsilon d_F(\lambda)) \leq C(d_F(\lambda))^{1/2p}\gamma_b(d_F(\lambda)).$$

Therefore,

$$\left(\int_0^\infty \left[(\epsilon d_F(\lambda))^{\frac{1}{p}}\gamma_b(\epsilon d_F(\lambda))\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q}$$

$$\leq C\epsilon^{\frac{1}{2p}}\left(\left[d_F(\lambda)^{\frac{1}{p}}\gamma_b\left(d_F(\lambda)\right)\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}. \tag{13}$$

Similarly, for any $k > 1$, as $t^{-\frac{1}{2p}}\gamma_b(t)$ is equivalent to a non-increasing function, we have

$$\left(\int_0^\infty \left[(k d_G(\lambda))^{\frac{1}{p}}\gamma_b(k d_G(\lambda))\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{1/q}$$

$$\leq C k^{\frac{1}{2p}}\left(\left[d_G(\lambda)^{\frac{1}{p}}\gamma_b\left(d_G(\lambda)\right)\lambda\right]^q \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}}. \tag{14}$$

Then (11)-(14) guarantee that

$$\frac{C_0}{\alpha} \| F \|_{L_{p,q,b}} \leq \epsilon_{\alpha,\beta}^{\frac{1}{2p}} \| F \|_{L_{p,q,b}} + \frac{C_1 k_{\alpha,\beta}^{\frac{1}{2p}}}{\beta} \| G \|_{L_{p,q,b}},$$

for some $C_0, C_1 > 0$. As $\lim_{\beta\to 0} \epsilon_{\alpha,\beta} = 0$, we obtain

$$\| F \|_{L_{p,q,b}} \leq C \| G \|_{L_{p,q,b}},$$

for some $C > 0$ independent of F and G.

The following BurkHölder-Davis-Gundy's inequalities for martingale Hardy-Lorentz-Karamata spaces follow from Propositions (4.1.9), (4.1.11) and Lemma (4.1.10).

Theorem (4.1.12) [153]: Let $0 < p < \infty, 0 < q \leq \infty$ and b be a slowly varying function. If $\mathcal{F}$ is regular, then there exist constants $A, B > 0$ such that

$$B \| f \|_{H^s_{p,q,b}} \leq \| f \|_{H^*_{p,q,b}} \leq A \| f \|_{H^s_{p,q,b}}.$$

Note that the BurkHölder-Davis-Gundy inequalities are not valid for L^p when $0 < p < 1$. see [27] for a counterexample.

On the other hand, if the Boyd indices of a given r.-i. quasi-Banach function space X are located strictly in between one and infinity, then the BurkHölder-Davis-Gundy inequality is valid on X. see [165] for a proof of this result. Even though $L_{p,q,b}$ are r.-i. when $0 < p < 1$, the Boyd indices of $L_{p,q,b}$ are p. Thus, the results in [165] do not apply to $L_{p,q,b}$ when $0 < p < 1$.

Proposition (4.1.11) can be used to establish the Fefferman-Stein inequalities on Lorentz-Karamata spaces $L_{p,q,b}$ even on the range $0 < p \leq 1$ [166].

We present the other main results We generalize the atomic decompositions, the duality theory and the interpolation properties to the martingale Hardy-LorentzKaramata spaces in the rest.

We present and prove one of the remarkable features of martingale function spaces, the atomic decompositions of the martingale Hardy-Lorentz-Karamata spaces in this section.

The atomic decompositions of martingale function spaces have a long history. For the atomic decompositions of P_1, it was established by Herz [162]. The extension of the atomic decompositions to P_p was obtained in [155], [158]. The atomic decompositions of H^s_p were given in [171], [27]. It was extended to martingale Hardy-Lorentz spaces $H^s_{p,q}$ in [20].

We recall the definition of atoms from [27].

Definition (4.1.13) [153]: Let $0 < p < \infty$. A pair (a, v) of Lebesgue measurable function a and stopping time v is a $(1, p, \infty)$ atom if

$$a_n = \mathbb{E}_n a = 0 \text{ if } v \geq n, \tag{15}$$

$$\| s(a) \|_{L^\infty} \leq \mathbb{P}(v \neq \infty)^{-\frac{1}{p}}. \tag{16}$$

Moreover, if we replace (16) by

$$\| s(a) \|_{L^\infty} \leq \mathbb{P}(v \neq \infty)^{-1/p} \text{ and } \| M(a) \|_{L^\infty} \leq \mathbb{P}(v \neq \infty)^{-1/p},$$

then we have the definitions of $(2, p, \infty)$ atom and $(3, p, \infty)$ atom, respectively.

We write $\{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_i, i = 1,2,3$ if $\{(a^k, v_k)\}_{k\in\mathbb{Z}}$ are (i, p, ∞) atom, $i = 1,2,3$, respectively.

The atomic decompositions of the martingale Hardy-Lorentz-Karamata spaces consist of two results, the decomposition theorem and the reconstruction theorem. We first present and prove the decomposition theorem.

Theorem (4.1.14) [153]: Let $0 < p < \infty, 0 < q \leq \infty$ and b be a slowly varying function. For any $f \in H^s_{p,q,b}$, there exist $\{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_1$ and $\{\mu_k\}_{k\in\mathbb{Z}} \subset [0, \infty)$ satisfying

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k \neq \infty)\big)\mu_k^q\right)^{1/q} \leq C \parallel f \parallel_{H_{p,q,b}^s}, \tag{17}$$

such that for all $n \in \mathbb{N}$

$$\sum_{k=-\infty}^{\infty} \mu_k \mathbb{E}_n a^k = f_n. \tag{18}$$

Proof: Let $f \in H_{p,q,b}^s$. For any $k \in \mathbb{Z}$, define

$$v_k = \inf \ \{n \in \mathbb{N}: s_{n+1}(f) > 2^k\}.$$

Apparently, $v_{k+1} \geq v_k$. Therefore,

$$f_n = \sum_{k=-\infty}^{\infty} \left(f_n^{v_{k+1}} - f_n^{v_k}\right).$$

Set $\mu_k = 2^k 3\mathbb{P}(v_k \neq \infty)^{1/p}$. When $\mu_k \neq 0$, define

$$a_n^k = \frac{f_n^{v_{k+1}} - f_n^{v_k}}{\mu_k}.$$

Obviously, $(a_n^k)_{n\geq 0}$ is a martingale. Moreover, in view of the definition of v_k, we have $s\left(f_n^{v_k}\right) \leq 2^k$. Hence, by using the definition of μ_k

$$s(a_n^k) \leq \frac{s\left(f_n^{v_{k+1}}\right) - s\left(f_n^{v_k}\right)}{\mu_k} \leq \mathbb{P}(v_k \neq \infty)^{-1/p}, n \in \mathbb{N}.$$

That is, $(a_n^k)_{n\geq 0}$ is an $L^2 -$ martingale. Thus, there exists an $a^k \in L^2$ such that

$$\mathbb{E}_n a^k = a_n^k,$$
$$\|s(a^k)\|_{L^\infty} \leq \mathbb{P}(v_k \neq \infty)^{-1/p}.$$

Furthermore, $a_n^k = 0$ when $v_k \geq n$, therefore, a^k is a $(1, p, \infty)$ atom.
Since $\mathbb{P}(v_k \neq \infty) = d_{s(f)}(2^k)$, we find that

$$\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k \neq \infty)\big)\mu_k^q = C\sum_{k\in\mathbb{Z}} \left(d_{s(f)}(2^k)^{\frac{1}{p}}\gamma_b\left(d_{s(f)}(2^k)\right)2^k\right)^q.$$

As $u^{\frac{1}{p}}\gamma_b(u)$ is equivalent to a non-decreasing function and for any $2^{k-1} \leq u \leq 2^k$, we have $d_{s(f)}(2^k) \leq d_{s(f)}(u)$, we find that

$$\left(d_{s(f)}(2^k)\right)^{\frac{1}{p}}\gamma_b\left(d_{s(f)}(2^k)\right) \leq \left(d_{s(f)}(u)\right)^{\frac{1}{p}}\gamma_b\big(d_{s(f)}(u)\big)$$

$$\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k \neq \infty)\big)\mu_k^q = C\sum_{k\in\mathbb{Z}} \left(d_{s(f)}(2^k)^{\frac{1}{p}}\gamma_b\left(d_{s(f)}(2^k)\right)2^k\right)^q$$

$$\leq C \sum_{k\in\mathbb{Z}_{2^{k-1}}} \int_{\leq C}^{2^k} \left[d_{s(f)}(u)^{\frac{1}{p}}\gamma_b\big(d_{s(f)}(u)\big)u\right]^q \frac{du}{u}$$

$$\leq C \parallel s(f) \parallel_{Lp,q,b}^q.$$

Now, we state and prove the reconstruction theorem for the atomic decompositions of $H_{p,q,b}^s$.

Theorem (4.1.15) [153]: Let $0 < p \leq 1, 0 < q \leq \infty$ and b be a slowly varying function. Let $r = \min(p, q) \leq 1$ and

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k,$$

where $\{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_1$ satisfying

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big)|\mu_k|^r\right)^{1/r} < \infty.$$

Then, $f \in H_{p,q,b}^s$ and

$$\| f \|_{H_{p,q,b}^s} \leq C\left(\sum_{k\in\mathbb{Z}} \gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big)|\mu_k|^r\right)^{\frac{1}{r}},$$

for some $C > 0$.

Proof : As $\|s(a^k)\|_{L^\infty} \leq \mathbb{P}(v_k \neq \infty)^{-1/p}$, we have

$$\|a^k\|_{H_{p,q,b}^s} \leq C\left(\int_0^\infty \left[d_{s(a^k)}(u)^{\frac{1}{p}}\gamma_b\left(d_{s(a^k)}(u)\right)u\right]^q \frac{du}{u}\right)^{1/q}$$

$$\leq C\left(\int_0^{\mathbb{P}(v_k\neq\infty)^{-\frac{1}{p}}} \left[d_{s(a^k)}(u)^{\frac{1}{p}}\gamma_b\left(d_{s(a^k)}(u)\right)u\right]^q \frac{du}{u}\right)^{\frac{1}{q}}.$$

For any $u > 0$,

$$d_{s(a^k)}(u) \leq \mathbb{P}(v_k \neq \infty).$$

The fact that $t^{1/p}\gamma_b(t)$ is equivalent to a non-decreasing function assures that

$$\|a^k\|_{H_{p,q,b}^s} \leq C\left(\int_0^{\mathbb{P}(v_k\neq\infty)^{-1/p}} \left[d_{s(a^k)}(u)^{\frac{1}{p}}\gamma_b\left(d_{s(a^k)}(u)\right)u\right]^q \frac{du}{u}\right)^{1/q}$$

$$\leq C\mathbb{P}(v_k \neq \infty)^{-\frac{1}{p}}\gamma_b\big(\mathbb{P}(v_k \neq \infty)\big)\left(\int_0^{\mathbb{P}(v_k\neq\infty)^{-1/p}} u^{q-1}du\right)^{1/q}$$

$$\leq C\gamma_b\big(\mathbb{P}(v_k \neq \infty)\big).$$

Finally, since $\|\cdot\|_{L_{p,q,b}}$ is an $r-$ norm, we have

$$\| s(f) \|_{L_{p,q,b}}^r \leq C\sum_{k\in\mathbb{Z}} |\mu_k|^r\|a^k\|_{H_{p,q,b}^s}^r \leq C\sum_{k\in\mathbb{Z}} |\mu_k|^r\gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big),$$

for some $C > 0$. That is, $f \in H_{p,q,b}^s$ and

$$\| f \|_{H_{p,q,b}^s} \leq C\left(\sum_{k\in\mathbb{Z}} \gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big)|\mu_k|^r\right)^{1/r}.$$

The combination of the preceding theorems give us the atomic characterizations of $H_{p,q,b}^s$ when $0 < q \leq p \leq 1$.

Theorem (4.1.16) [153]: Let $0 < q \leq p \leq 1$ and b be a slowly varying function. For any $f \in H_{p,q,b}^s$, we have

$$\| f \|_{H_{p,q,b}^s} \approx \inf\left\{\left(\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k \neq \infty)\big)\mu_k^q\right)^{\frac{1}{q}} : f = \sum_{k\in\mathbb{Z}} \mu_k a^k, \{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_1\right\}.$$

The above atomic decompositions and characterizations are also valid for $Q_{p,q,b}$ and $P_{p,q,b}$ with the $(1, p, \infty)$ atoms being replaced by the $(2, p, \infty)$ atoms and the $(3, p, \infty)$ atoms, respectively.

For brevity, we skip the detail and see [27].

With the assumption that $\mathcal{F}$ is regular, we also have atomic decompositions for $H^*_{p,q,b}$. As $H^*_{p,q,b}$ is defined via the maximal function, we need to use the atoms defined in terms of maximal function.

That is, atoms from A_3.

Theorem (4.1.17) [153]: Let $0 < p < \infty, 0 < q \le \infty$ and b be a slowly varying function. Suppose that $\mathcal{F}$ is regular. Then, for any $f \in H^*_{p,q,b}$, there exist $\{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_3$ and $\{\mu_k\}_{k\in\mathbb{Z}} \subset [0,\infty)$ satisfying

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k \neq \infty)\big)\mu_k^q\right)^{\frac{1}{q}} \le C \parallel f \parallel_{H^*_{p,q,b}},$$

such that for all $n \in \mathbb{N}$

$$\sum_{k=-\infty}^{\infty} \mu_k \mathbb{E}_n a^k = f_n.$$

Proof : We apply Proposition (4.1.7) to the process $(|f_n|)_{n\ge0}$ and $\lambda = 2^k, k \in \mathbb{Z}$. We obtain the stopping time τ_k satisfying $v_k \ge v_l, k \ge l, \lim_{k\to\infty} |\{v_k < \infty\}| = 0, \lim_{k\to\infty} v_k = \infty$ a.e.,

$$\lim_{k\to\infty} f^{v_k} = f \text{ a.e. and } \lim_{k\to-\infty} |f^{v_k}| = 0 \text{ a.e.}$$

Therefore, f can be rewritten as

$$f_n = \sum_{k=-\infty}^{\infty} \left(f_n^{v_k} - f_n^{v_{k-1}}\right).$$

Set $\mu_k = 3 \cdot 2^{k+1} |\{v_{k-1} < \infty\}|^{1/p}$. When $\mu_k \neq 0$, define

$$a_n^k = \frac{f_n^{v_k} - f_n^{v_{k-1}}}{\lambda_k}.$$

Thus, $a^k = (a_n^k)_{n\ge0}$ is a martingale. Since

$$\mathbb{E}_n(a^k) = \frac{f^{\min\,(n,v_k)} - f^{\min\,(n,v_{k-1})}}{\mu_k} = 0,\ v_{k-1} \ge n,$$

$$\|M(a^k)\|_{L^\infty} = \|a^k\|_{L^\infty} \le \mathbb{P}(v_{k-1} \neq \infty)^{-1/p},$$

(a^k, v_{k-1}) is a $(3, p, \infty)$ atom.

In addition, according to Proposition (4.1.7), we have $\mathbb{P}(v_{k-1} \neq \infty) \le C d_{M(f)}(2^{k-1})$. Since $t^{1/p}\gamma_b(t)$ is equivalent to a non-decreasing function,

$$\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_{k-1} \neq \infty)\big)\mu_k^q \le C \sum_{k\in\mathbb{Z}} \left(d_{M(f)}(2^{k-1})^{\frac{1}{p}} \gamma_b\left(d_{M(f)}(2^{k-1})\right) 2^{k+1}\right)^q.$$

Similar to the proof of Theorem (4.1.14), we have

$$\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_{k-1} \neq \infty)\big)\mu_k^q \le C \parallel f \parallel_{H^*_{p,q,b}}.$$

Note that for the reconstruction theorem of $H^*_{p,q,b}$, we do not need the assumption that $\mathcal{F}$ is regular. **Theorem (4.1.18) [153]:** Let $0 < p < \infty, 0 < q \le \infty$ and b be a slowly varying function. Let $r = \min(p,q) \le 1$ and

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k,$$

where $\{(a^k, v_k)\}_{k\in\mathbb{Z}} \in A_3$ satisfying

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big)|\mu_k|^r\right)^{1/r} < \infty.$$

Then, $f \in H^*_{p,q,b}$ and

$$\| f \|_{H^*_{p,q,b}} \leq C\left(\sum_{k\in\mathbb{Z}} \gamma_b^r\big(\mathbb{P}(v_k \neq \infty)\big)|\mu_k|^r\right)^{\frac{1}{r}},$$

for some $C > 0$.

As an application of the above atomic decompositions for martingale Hardy-Lorentz-Karamata spaces, we obtain the embedding

$$P_{p,q,b} \hookrightarrow H^*_{p,q,b},$$

provided that $0 < p < \infty, 0 < q \leq \infty$ and b be a slowly varying function. This embedding is valid without the assumption that $\mathcal{F}$ is regular.

The main result is the identification of the dual space of martingale Hardy-Lorentz-Karamata spaces $H^s_{p,q,b}$. We are particularly interested in the case when $0 < p \leq 1$ as we have the well-known theorem stated that the dual spaces of H^s_p are the BMO space and the Lipschitz spaces $\Lambda_2(\alpha)$ where $\alpha = 1/p - 1$ [27].

We show that the dual spaces of $H^s_{p,q,b}$ are generalizations of $\Lambda_2(\alpha)$, namely, the BMO space and the Lipschitz spaces associated with slowly varying functions. To prove that the dual spaces of $H^s_{p,q,b}$ are the Lipschitz spaces associated with slowly varying functions, we use the atomic decompositions obtained in the previous.

We start with the definition of the Lipschitz spaces associated with slowly varying functions $\Lambda_{2,b}(\alpha)$. Let Γ denote the class of stopping times.

Definition (4.1.19) [153]: Let $\alpha > 0$ and b be a slowly varying function. The space $\Lambda_{2,b}(\alpha)$ consists of those functions $f \in L_2$ such that

$$\| f \|_{\Lambda_{2,b}(\alpha)} = \sup_{v\in\Gamma} \mathbb{P}(v \neq \infty)^{-\frac{1}{2}-\alpha}\gamma_b^{-1}(\mathbb{P}(v \neq \infty))\|f - f^v\|_{L^2} < \infty.$$

Theorem (4.1.20) [153]: Let $0 < p \leq 1$ and $0 < q \leq p$. The dual space of $H^s_{p,q,b}$ is $\Lambda_{2,b}(\alpha)$ where $\alpha = 1/p - 1$.

Proof : According to [159], we have

$$\| f \|_{H^s_{p,q,b}} = \| s(f) \|_{L_{p,q,b}} \leq \| s(f) \|_{L_2} = \| f \|_{L_2}. \tag{19}$$

In addition, and (4.1.15) assure that L_2 is a dense subspace of $H^s_{p,q,b}$.

For any $f \in L_2$, provides the atoms $\{(a^k, v_k)\}_{k\in\mathbb{Z}}$ and the scalars $\{\mu_k\}_{k\in\mathbb{Z}}$ such that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k.$$

Let $\phi \in \Lambda_{2,b}(\alpha)$. We define

$$l_\phi(f) = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}(a^k\phi), f \in L_2.$$

We find that

$$\left|l_\phi(f)\right| = \sum_{k\in\mathbb{Z}} |\mu_k|\mathbb{E}\left(\left|a^k \parallel \phi - \phi^{v_k}\right|\right) \le \sum_{k\in\mathbb{Z}} |\mu_k| \|a^k\|_{L_2} \|\phi - \phi^{v_k}\|_{L_2}$$
$$\le \sum_{k\in\mathbb{Z}} \gamma_b(\mathbb{P}(v\neq\infty))|\mu_k|\mathbb{P}(v_k\neq\infty)^{-\frac{1}{p}+\frac{1}{2}}\gamma_b^{-1}(\mathbb{P}(v\neq\infty))\|\phi-\phi^{v_k}\|_{L_2}$$
$$\le \sum_{k\in\mathbb{Z}} \gamma_b(\mathbb{P}(v\neq\infty))|\mu_k| \parallel \phi \parallel_{\Lambda_{2,b}(\alpha)}.$$
Since $0 < q \le 1$, we obtain
$$\left|l_\phi(f)\right|^q \le \sum_{k\in\mathbb{Z}} \gamma_b(\mathbb{P}(v\neq\infty))|\mu_k|^q \parallel \phi \parallel_{\Lambda_{2,b}(\alpha)}^q,$$
by using the q-inequality. Therefore, ensures that
$$\left|l_\phi(f)\right| \le C \parallel f \parallel_{H_{p,q,b}^s} \parallel \phi \parallel_{\Lambda_{2,b}(\alpha)},$$
for some $C > 0$ independent of $f \in L_2$. As L_2 is a dense subspace of $H_{p,q,b}^s, l_\phi$ can be extended to be a bounded linear functional on $H_{p,q,b}^s$ and, hence, the embedding $\Lambda_{2,b}(\alpha) \hookrightarrow \left(H_{p,q,b}^s\right)^*$ is valid.

Next, let $l \in \left(H_{p,q,b}^s\right)^*$ be a bounded linear functional on $H_{p,q,b}^s$. In view of the embedding (19), we have a $\phi \in L_2$ such that
$$l(f) = \mathbb{E}(f\phi), f \in L_2.$$
For any $v \in \Gamma$, define
$$g = \frac{\phi - \phi^v}{\|\phi-\phi^v\|_{L_2}\mathbb{P}(v\neq\infty)^{1/p-1/2}\gamma_b(\mathbb{P}(v\neq\infty))}.$$
The function g satisfies
$$s(\mathrm{g}) = s(\mathrm{g})\chi_{\{v\neq\infty}\}.$$
The Hölder inequality assures that
$$\parallel g \parallel_{H_{p,q,b}^s}^q = \int_0^{\mathbb{P}(v\neq\infty)} \left(t^{\frac{1}{p}}\gamma_b(t)(s(\mathrm{g}))^*(t)\right)^q \frac{dt}{t}$$
$$\le \left(\int_0^{\mathbb{P}(v\neq\infty)} ((s(\mathrm{g}))^*(t))^2 dt\right)^{q/2} \left(\int_0^{\mathbb{P}(v\neq\infty)} \left(t^{q/p-1}\gamma_b^q(t)\right)^{2/(2-q)} dt\right)^{(2-q)/2}.$$
In view of the fact that $t^{q/2p}\gamma_b(t)$ is equivalent to a non-decreasing function, we obtain
$$\int_0^{\mathbb{P}(v\neq\infty)} \left(t^{q/p-1}\gamma_b^q(t)\right)^{2/(2-q)} dt = \int_{\mathbb{P}(v\neq\infty)} \left(t^{\frac{q}{2p}}\gamma_b^q(t)\right)^{2/(2-q)} t^{(q-2p)/p(2-q)}dt$$
$$\le C\mathbb{P}(v\neq\infty)^{\frac{q}{p(2-q)}}\gamma_b(\mathbb{P}(v\neq\infty))^{2q/(2-q)}$$
$$\times \int_0^{\mathbb{P}(v\neq\infty)} t^{(q-2p)/p(2-q)}dt,$$
for some $C > 0$.

As $(q-2p)/p(2-q)+1 = q(1-p)/p(2-q)$, we have
$$\int_0^{\mathbb{P}(v\neq\infty)} t^{(q-2p)/p(2-q)}dt = \frac{p(2-q)}{q(1-p)}\mathbb{P}(v\neq\infty)^{q(1-p)/p(2-q)}.$$
Moreover, $q/p(2-q)+q(1-p)/p(2-q) = q(2-p)/p(2-q)$, we find that

$$\int_0^{\mathbb{P}(v\neq\infty)} \left(t^{q/p-1}\gamma_b^q(t)\right)^{2/(2-q)} dt \leq C\mathbb{P}(v \neq \infty)^{q(2-p)/p(2-q)}\gamma_b(\mathbb{P}(v \neq \infty))^{2q/(2-q)},$$

for some $C > 0$.

The above inequalities yield

$$\| g \|_{H^s_{p,q,b}} \leq C \| g \|_{L_2} \mathbb{P}(v \neq \infty)^{1/p-1/2}\gamma_b(\mathbb{P}(v \neq \infty)) \leq C,$$

for some $C > 0$ independent of ϕ.

Consequently,

$$\| l \| \geq |l(g)| = \mathbb{E}\big(g(\phi - \phi^v)\big) = \mathbb{P}(v \neq \infty)^{-\frac{1}{p}+\frac{1}{2}}\gamma_b(\mathbb{P}(v \neq \infty))^{-1}\|\phi - \phi^v\|_{L_2}.$$

Therefore, $\phi \in \Lambda_{2,b}(\alpha)$ and, hence, we establish the embedding $\left(H^s_{p,q,b}\right)^* \hookrightarrow \Lambda_{2,b}(\alpha)$.

We show that the martingale Hardy-Lorentz-Karamata spaces are interpolation spaces of martingale Hardy space under the action of a new interpolation functor introduced in [165], where the role of this interpolation functor on Lorentz-Karamata spaces is the same as the role of real interpolation functor for Lorentz spaces.

We recall the definition of K-functional from [27].

Definition (4.1.21) [153]: Let (X_0, X_1) be a compatible couple of quasi-normed spaces. For any $f \in X_0 + X_1$, the K-functional is defined as

$$K(f, t, X_0, X_1) = \inf \left\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\right\},$$

where the infimum is taking over all $f = f_0 + f_1$ for which $f_i \in X_i, i = 1,2$.

We will write $K(f, t, X_0, X_1)$ as $K(f, t)$ if no confusion may occur.

In [160], a new interpolation functor is introduced for the study of the Lorentz-Karamata spaces.

Definition (4.1.22) [153]: Let $0 < \theta < 1, 0 < r \leq \infty$ and b be a slowly varying function. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. The space $(X_0, X_1)_{\theta,r,b}$ consists of all f in $X_0 + X_1$ such that $\| f \|_{(X_0,X_1)_{\theta,r,b}} < \infty$ where

$$\| f \|_{(X_0,X_1)_{\theta,r,b}} = \begin{cases} \left[\int_0^\infty \left(t^{-\theta}\gamma_b(t)K(f,t)\right)^r \frac{dt}{t}\right]^{1/r} & , 0 < \theta < 1, 0 < r < \infty, \\ \sup\limits_{t\in\mathbb{R}}\left\{t^{-\theta}\gamma_b(t)K(f,t)\right\}, & 0 < \theta < 1, r = \infty. \end{cases}$$

It is shown in [160] that Lorentz-Karamata spaces can be generated from Lebesgue spaces by using the interpolation functor $(\cdot,\cdot)_{\theta,r,b}$. We have the following result which is a special case of more general result in [160].

Lemma (4.1.23) [153]: Let $0 < \theta < 1, 0 < r \leq \infty, 0 < p_0 < p_1 \leq \infty$ and b be a slowly varying function.

Then

$$\left(L_{p_0}, L_{p_1}\right)_{\theta,r,b} = L_{p,r,b_{\alpha'}}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1, 1/\alpha = 1/p_0 - 1/p_1$ and $b_\alpha(t) = b\left(t^{1/\alpha}\right)$.

Inspired by the above lemma, we study the interpolation properties of martingale Hardy-LorentzKaramata spaces under the action of the interpolation functor $(\cdot,)_{\theta,r,b}$. We accomplish the interpolation result for martingale Hardy-Lorentz-Karamata spaces by establishing a formula of the K-functional of the martingale Hardy spaces.

We have the following well-known Holmstedt formula for the K-functional of Lebesgue spaces [13].

Proposition (4.1.24) [153]: Let $0 < p_0 < p_1 \leq \infty, 0 < q_1, q_2 \leq \infty$. We have

$$K(f,t,L_{p_0},L_{p_1}) \approx \left(\int_0^{t^\alpha} (f^*(s))^{p_0} ds\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^{\infty} (f^*(s))^{p_1} ds\right)^{\frac{1}{p_1}},$$

where $1/\alpha = 1/p_0 - 1/p_1$.

A similar formula for martingale Hardy spaces is established as one of the first main results of this section.

For the corresponding formula for the real Hardy spaces on Euclidean spaces, see [156].

To obtain the above result, we need several supporting results. We first recall an estimate for the K-functional of martingale Hardy spaces from [27].

Lemma (4.1.25) [153]: Let $0 < p_0 \leq 1$. We have

$$K(f,t,H_{p_0}^s,H_\infty^s) \leq C\left(\int_0^{t^{p_0}} ((s(f))^*)^{p_0}(u)du\right)^{\frac{1}{p_0}} + 4t(s(f))^*(t^{p_0}). \quad (20)$$

The validity of the above estimate is guaranteed by the atomic decompositions of the martingale Hardy spaces. Note that the atomic decompositions for P_p and Q_p are also valid [27]. Therefore, similar estimates for the K-functional of P_p and Q_p are also valid with $s(f)$ replaced by $M(f)$ and $S(f)$, respectively.

Next, we present the Hardy inequality associated with slowly varying functions.

Lemma (4.1.26) [153]: Let $1 \leq q \leq \infty, v \neq 0$ and b be a slowly varying function.

(i) If $v < 0$, then there exists a constant $C > 0$ such that

$$\left(\int_0^\infty \left(t^{v-\frac{1}{q}}\gamma_b(t)\int_0^t g(s)ds\right)^q dt\right)^{\frac{1}{q}}/q \leq C\left(\int_0^\infty \left(t^{v+1/q'}\gamma_b(t)g(t)\right)^q dt\right)^{1/q}.$$

(ii) If $v > 0$, then there exists a constant $C > 0$ such that

$$\left(\int_0^\infty \left(t^{v-\frac{1}{q}}\gamma_b(t)\int_t^\infty g(s)ds\right)^q dt\right)^{\frac{1}{q}}/q \leq C\left(\int_0^\infty \left(t^{v+1/q'}\gamma_b(t)g(t)\right)^q dt\right)^{1/q}.$$

The above lemma is a special case of more general boundedness result of Hardy-type operator on Lorentz-Karamata spaces presented in [167], for brevity, see [159], [167].

The following lemma is a supporting technical result for the establishment of the interpolation theorem. It is an extension of the result given in [157] and it follows from the fact that g is a non-increasing non-negative function.

Lemma (4.1.27) [153]: Let $\kappa \in \mathbb{R}$ and $0 < \beta < 1$. We have a constant $C > 0$ such that for any $a > 0$ and non-increasing non-negative function g on $(0,\infty)$,

$$\left(\int_0^a s^\kappa g(s)ds\right)^\beta \leq C\int_0^a s^{(\kappa+1)\beta-1}(g(s))^\beta ds, \quad (21)$$

$$\left(\int_0^\infty g(s)ds\right)^\beta \leq C\int_{\frac{a}{2}}^\infty s^{(\kappa+1)\beta-1}(g(s))^\beta ds, \quad (22)$$

We recall an real interpolation result for martingale Hardy spaces from [27].

Proposition (4.1.28) [153]: Let $0 < p_0 < 1, 0 < \theta < 1$ and $1/p_1 = (1-\theta)/p_0$. We have

$$\left(H_{p_0}^s, H_\infty^s\right)_{\theta,p_1} = H_{p_1}^s.$$

We need another result from the interpolation of quasi-normed spaces obtained in [13].

Lemma (4.1.29) [153]: Let $0 < \theta < 1, 0 < r \leq \infty$ and A_0, A_1 be a couple of quasi-normed spaces. We have

$$K\big(f, t, A_0, (A_0, A_1)_{\theta,r}\big) \approx t\left(\int_{t^{\frac{1}{\theta}}}^{\infty} \left(s^{-\theta} K(f, s, A_0, A_1)\right)^r \frac{ds}{s}\right)^{\frac{1}{r}}. \qquad (23)$$

We are now ready to prove Theorem (4.1.30).

Theorem (4.1.30) [153]: Let $0 < p_0 < p_1 < \infty$. Then $K\big(f, t, H_{p_0}^s, H_{p_1}^s\big)$

$$\approx \left(\int_0^{t^\alpha} ((s(f))^*(y))^{p_0} dy\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^{\infty} ((s(f))^*(y))^{p_1} dy\right)^{\frac{1}{p_1}}, \qquad (24)$$

$$K\big(f, t, P_{p_0}, P_{p_1}\big) \approx \left(\int_0^{t^\alpha} ((M(f))^*(y))^{p_0} dy\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^{\infty} ((M(f))^*(y))^{p_1} dy\right)^{\frac{1}{p_1}}, \qquad (25)$$

$$K\big(f, t, Q_{p_0}, Q_{p_1}\big) \approx \left(\int_0^{t^\alpha} ((S(f))^*(y))^{p_0} dy\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^{\infty} ((S(f))^*(y))^{p_1} dy\right)^{\frac{1}{p_1}}, \qquad (26)$$

where $1/\alpha = 1/p_0 - 1/p_1$.

Proof: As the proofs of (25) and (26) are similar to the proof of (24), for brevity, we only give the proof of (24).

With $r = p_1, A_0 = H_{p_0}^s$ and $A_1 = H_\infty^s$, ensure that

$$K\big(f, t, H_{p_0}^s, H_{p_1}^s\big) \leq Ct\left(\int_{t^{\frac{1}{\beta}}}^{\infty} s^{-\beta p_1}\left(\int_0^{s_0} ((s(f))^*(v))^{p_0} dv\right)^{\frac{p_1}{p_0}} \frac{ds}{s}\right)^{1/p}$$

$$+Ct\left(\int_{t^{\frac{1}{\beta}}} s^{(1-\beta)p_1}\big((s(f))^*(s^{p_0})\big)^{p_1} \frac{ds}{s}\right)^{1/p_1} = I + II,$$

where $\beta = 1 - p_0/p_1$.

We first estimate I. Applying the change of variable $y = s^{p_0}$ to the integral on I, we have

$$I \le Ct\left(\int_{t^\alpha}^{\infty}\left(y^{-1}\int_0^y ((s(f))^*(v))^{p_0}dv\right)^{\frac{p_1}{p_0}}dy\right)^{1/p_1}$$

$$\le Ct\left(\int_{t^\alpha}^{\infty}\left(y^{-1}\int_0^{t^\alpha} ((s(f))^*(v))^{p_0}dv\right)^{\frac{p_1}{p_0}}dy\right)^{1/p_1}$$

$$\left(\int_{t^\alpha}^{\infty}\left(y^{-1}\int_{t^\alpha}^{y} ((s(f))^*(v))^{p_0}dv\right)^{\frac{p_1}{p_0}}dy\right)^{1/p_1} = III + IV,$$

since $p_0/\beta = 1/(1/p_0 - 1/p_1) = \alpha$.

We estimate III. We find that

$$III \le Ct\left(\int_{t^\alpha}^{\infty} y^{-p_1/p_0}dy\right)^{1/p_1}\left(\int_0^{t^\alpha} ((s(f))^*(v))^{p_0}dv\right)^{1/p_0}$$

$$\le C\left(\int_0^{t^\alpha} ((s(f))^*(v))^{p_0}dv\right)^{1/p_0}.$$

The Hardy inequality on the interval (t^α,∞) [159] guarantees that

$$IV \le Ct\left(\int_0^{\infty} ((s(f))^*(y))^{p_1}dv\right)^{\frac{1}{p_1}},$$

for some $C > 0$ independent of f and t.

Applying the change of variable $y = s^{p_0}$ to the integral on II, we obtain

$$II \le Ct\left(\int_{t^\alpha}^{\infty} y^{(1-\beta)\left(\frac{p_1}{p_0}\right)}((s(f))^*(y))^{p_1}\frac{dy}{y}\right)^{\frac{1}{p_1}} = Ct\left(\int_{t^\alpha}^{\infty} ((s(f))^*(y))^{p_1}dy\right)^{\frac{1}{p_1}},$$

for some $C > 0$.

Therefore, the estimates of I, II, III and IV assure that

$$K\left(f,t,H_{p_0}^s,H_{p_1}^s\right) \le \left(\int_0^{t^\alpha} ((s(f))^*(y))^{p_0}dy\right)^{1/p_0}$$

$$+Ct\left(\int_{t^\alpha}^{\infty} ((s(f))^*(y))^{p_1}dy\right)^{\frac{1}{p_1}}, \qquad (27)$$

for some $C > 0$.

Finally, we deal with the reserve inequality. The boundedness of $s(f)$ from H_p^s to L_p yields

$$K\left(s(f),t,L_{p_0},L_{p_1}\right)$$

for some $C > 0$. Proposition (4.1.24) assures that

$$\left(\int_0^{t^\alpha} ((s(f))^*(y))^{p_0}dy\right)^{1/p_0} + t\left(\int_{t^\alpha}^{\infty} ((s(f))^*(y))^{p_1}dy\right)^{1/p_1} \le CK\left(f,t,H_{p_0}^s,H_{p_1}^s\right),$$

for some $C > 0$. Therefore, the above inequality and (27) yield (24).

We now apply the functor $(\because,)_{\theta,r,b}$ to the K-functional for martingale Hardy space. It shows that the martingale Hardy-Lorentz-Karamata spaces can be generated from the martingale Hardy spaces via the interpolation functor $(\because)_{\theta,r,b}$.

Theorem (4.1.31) [153]: Let $0 < \theta < 1, 0 < r \le \infty, 0 < p_0 < p_1 < \infty$ and b be a slowly varying function.
We have

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,r,b} = H^s_{p,r,b^\alpha},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1} \text{ and } b_\alpha(t) = b\left(t^{1/\alpha}\right).$$

Proof : To simplify the notation, we write $(s(f))^*$ by s^*. The definition of the interpolation functor $(\because,)_{\theta,r,b}$ and (24) yield

$$\| f \|_{\left(H^s_{p_0, Hp_1}\right)_{\theta,r,b}} \le C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r\left(\int_0^{t^\alpha} (s^*(u))^{p_0} du\right)^{r/p_0}\frac{dt}{t}\right)^{1/r}$$

$$+C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r t^r\left(\int_{t^\alpha}^\infty (s^*(u))^{p_1} du\right)^{r/q_1}\frac{dt}{t}\right)^{1/r}$$

$$= I + II.$$

We split the estimate of I into two cases, $r \ge p_0$ and $r < p_0$.
We first estimate I for the case $r \ge p_0$. By using the change of variable $v = t^\alpha$, I can be rewritten as

$$I = \left(\int_0^\infty v^{-\frac{\theta r}{\alpha}}\gamma_b\left(v^{\frac{1}{\alpha}}\right)^r\left(\int_0^v (s^*(u))^{p_0} du\right)^{\frac{r}{p_0}}\frac{dv}{v}\right)^{1/r}$$

$$= \left(\int_0^\infty \left(v^{-\frac{\theta p_0}{\alpha}-p_0/r}\gamma_b\left(v^{\frac{1}{\alpha}}\right)^{p_0}\int_0^v (s^*(u))^{p_0} du\right)^{\frac{r}{q_0}} dv\right)^{1/r}.$$

Item (1) of Lemma (4.1.26), with $q = r/p_0 \ge 1$ and $v = -\theta p_0/\alpha < 0$, asserts that

$$I \le C\left(\int_0^\infty \left(u^{-\theta p_0/\alpha+1-p_0/r}\gamma_b\left(u^{1/\alpha}\right)^{p_0}(s*(u))^{p_0}\right)^{r/p_0} du\right)^{1/r},$$

for some $C > 0$.
As $-\theta/\alpha + 1/p_0 = 1/p$, we have

$$I \le C\left(\int_0^\infty \left(u^{\frac{1}{p}}\gamma_b\left(u^{\frac{1}{\alpha}}\right)s^*(u)\right)^r\frac{du}{u}\right)^{\frac{1}{r}} = C \| f \|_{H_{p,r,b_\alpha}}.$$

Next, we consider the case $r < p_0$. In this case, as $(s^*)^{p_0}$ is a non-increasing function, inequality (21) yields

$$I \le C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r\int_0^{t^\alpha} s^{\frac{r}{p_0}-1}\left(s^*(u)\right)^r du\frac{dt}{t}\right)^{\frac{1}{r}}.$$

By using the change of variable $u = vt^\alpha$ for the integral with respect to du, we obtain

$$I \quad \leq C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r \int_0^1 t^{r\alpha/p_0-\alpha} v^{r/p_0-1}\left(s^*(vt^\alpha)\right)^r t^\alpha dv \frac{dt}{t}\right)^{1/r}$$

$$= C\left(\int_0^\infty \int_0^1 t^{-\theta r + r\alpha/p_0}\gamma_b(t)^r v^{r/p_0-1}\left(s^*(vt^\alpha)\right)^r t^\alpha dv \frac{dt}{t}\right)^{1/r}.$$

Applying the change of variable $y = vt^\alpha$ for the integral with respect to t, we find that

$$I \leq C\left(\int_0^\infty \int_0^1 y^{-\theta r/\alpha + r/p_0} v^{\theta r/\alpha - r/p_0}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r v^{r/p_0-1}(s^*(y))^r dv \frac{dy}{y}\right)^{1/r}$$

$$= C\left(\int_0^\infty y^{r/p}\left(\int_0^1 v^{\theta r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv\right)(s^*(y))^r \frac{dy}{y}\right)^{1/r},$$

because $-\theta/\alpha + 1/p_0 = 1/p$

Then, we estimate the integral with respect to dv. We find that

$$\int_0^1 v^{\theta r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv = \alpha y^{\theta r/\alpha}\int_0^1 t^{-\theta r-1}\gamma_b(t)^r dt.$$

Where we use the change of variable $v = yt^{-\alpha}$.

Item (4) of Proposition (4.1.2) asserts that

$$\int_0^1 v^{\theta r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv \approx \alpha y^{\theta r/\alpha} y^{(1/\alpha)(-\theta r)}\gamma_b\left(y^{1/\alpha}\right)^r \approx \gamma_b\left(y^{1/\alpha}\right)^r.$$

Therefore,

$$I \leq C\left(\int_0^\infty y^{r/p}\gamma_b\left(y^{1/\alpha}\right)^r (s^*(y))^r \frac{dy}{y}\right)^{1/r} \leq C \parallel f \parallel_{H_{p,r,b_\alpha}}.$$

The estimate for II is similar to the estimate for I. Therefore, for brevity, we just outline the major modifications.

For II with $r \geq q_1$, we find that by using the change of variable $v = t^\alpha$,

$$II = \left(\int_0^\infty v^{\frac{(1-\theta)r}{\alpha}}\gamma_b\left(v^{\frac{1}{\alpha}}\right)^r \left(\int_v^\infty u^{q_1/p_1-1}(s^*(u))^{q_1} du\right)^{r/q_1}\frac{dv}{v}\right)^{1/r}$$

$$\left(\int_0^\infty \left(v^{(1-\theta)q_1/\alpha - q_1/r}\gamma_b\left(v^{\frac{1}{\alpha}}\right)^{q_1}\int_v^\infty u^{q_1/p_1-1}(s^*(u))^{q_1} du\right)^{r/q_1} dv\right)^{1/r}.$$

Applying Item (2) of Lemma (4.1.26) with $q = r/q_1 \geq 1$ and $v = (1-\theta)q_0/\alpha > 0$, we have $II \leq C \parallel f \parallel H^s_{p,r,b_\alpha}$ because $(1-\theta)/\alpha + 1/p_1 = 1/p$.

For the case $r < q_1$, inequality (22) assures that

$$II \leq C\left(\int_0^\infty t^{(1-\theta)r}\gamma_b(t)^r \int_{t^\alpha}^\infty u^{r/p_1-1}(s^*(u))^r du \frac{dt}{t}\right)^{1/r}.$$

By applying the same series of change of variables used for the estimate of I, we obtain

$$II \leq C\left(\int_0^\infty y^{\frac{r}{p}}\left(\int_1^\infty v^{-(1-\theta)r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv\right)(s^*(y))^r \frac{dy}{y}\right)^{1/r}.$$

Consequently, the change of variable $v = t^\alpha$ yields

$$\int_1^\infty v^{-(1-\theta)r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv = \alpha y^{-(1-\theta)r/\alpha}\int_0^{cy^{1/\alpha}} t^{(1-\theta)r-1}\gamma_b(t)^r dt,$$

for some constant $c > 0$. Thus, Items (3) and (4) of Proposition (4.1.2) yield

$$\int_1^\infty v^{-(1-\theta)r/\alpha-1}\gamma_b\left(y^{1/\alpha}v^{-1/\alpha}\right)^r dv \approx \alpha y^{-(1-\theta)r/\alpha} y^{(1/\alpha)(1-\theta)r}\gamma_b\left(cy^{1/\alpha}\right) \approx \gamma_b\left(y^{1/\alpha}\right).$$

Hence, the above estimates conclude that $H^s_{p,r,b_\alpha} \hookrightarrow \left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,r,b}$.

To establish the reverse embedding, note that s^* is non-increasing, therefore, we have

$$\| f \|_{\left(H_{p_0,}H^s_{p_1}\right)_{\theta,r,b}} \geq C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r\left(\int_0^{t^\alpha}(s^*(u))^{p_0}du\right)^{r/q_0}\frac{dt}{t}\right)^{1/r}$$

$$\geq C\left(\int_0^\infty t^{-\theta r}\gamma_b(t)^r\left(s^*(t^\alpha)\right)^r\frac{dt}{t}\right)^{1/r}$$

$$\geq C\left(\int_0^\infty t^{-\theta r+\alpha r/p_0}\gamma_b(t)^r\left(s^*(t^\alpha)\right)^r\frac{dt}{t}\right)^{1/r}$$

By using the change of variable $s = t^\alpha$, we obtain

$$\| f \|_{(H^s_{p_0},H^s_{p_1})_{\theta,r,b}} \geq C \| f \|_{H^s_{p,r,b_\alpha}},$$

and, hence, the embedding $\left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,r,b} \hookrightarrow H^{s,}_{p,r,b_\alpha}$ is valid.

The above results are also valid for $P_{p,q,b}$ and $Q_{p,q,b}$.

Theorem (4.1.32) [153]: Let $0 < \theta < 1, 0 < r \leq \infty, 0 < p_0 < p_1 < \infty$ and b be a slowly varying function.

We have

$$\left(P_{p_0}, P_{p_1}\right)_{\theta,r,b} = P_{p,r,b_{\alpha'}}$$

$$\left(Q_{p_0}, Q_{p_1}\right)_{\theta,r,b} = Q_{p,r,b_{\alpha'}}$$

Where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1} \text{ and } b_\alpha(t) = b\left(t^{1/\alpha}\right).$$

The preceding interpolation properties for martingale Hardy-Lorentz-Karamata spaces guarantee the identifications of $H^s_{p,q,b}, P_{p,q,b}$ and $Q_{p,q,b}$.

When $\mathcal{F}$ is regular, $H^*_{p,q,b}$ also have the atomic decompositions. Therefore, we also have

$$K\left(f, t, H^*_{p_0}, H^*_\infty\right) \leq C\left(\int_0^{t_0^{p_0}}((M(f))^*)^{p_0}(u)du\right)^{\frac{1}{p_0}} + 4t(M(f))^*(t^{p_0}).$$

Similarly, we have the following results for $H^*_{p,q,b}$ when $\mathcal{F}$ is regular.

Theorem (4.1.33) [153]: Let $0 < p_0 < p_1 < \infty$ and b be a slowly varying function. If $\mathcal{F}$ is regular, then

$$K\left(f, t, H^*_{p_0}, H^*_{p_1}\right) \approx \left(\int_0^{t^\alpha}((M(f))^*(y))^{p_0}dy\right)^{\frac{1}{p_0}} + t\left(\int_{t^\alpha}^\infty((M(f))^*(y))^{p_1}dy\right)^{\frac{1}{p_1}},$$

$$\left(H^*_{p_0}, H^*_{p_1}\right)_{\theta,r,b} = H^*_{p,r,b^\alpha}.$$

The subsequent corollary is a consequence of Item (4) of Propositions (4.1.2), (4.1.8) and Theorems (4.1.31)-(4.1.33).

Theorem (4.1.34) [153]: Let $0 < p < \infty, 0 < q \leq \infty$ and b is a slowly varying function. We have

$$P_{p,q,b} = Q_{p,q,b}. \tag{28}$$

When $\mathcal{F}$ is regular, we have

$$H_{p,q,b}^{s} = H_{p,q,b}^{*} = H_{p,q,b}^{S} = P_{p,q,b} = Q_{p,q,b}, 0 < p < \infty.$$

Proof : In view of Item (4) of Proposition (4.1.2), the identification

$$P_{p,q,b} = Q_{p,q,b}$$

is obtained by using (6) and Theorem (4.1.32).

Similarly, the assertion

$$H_{p,q,b}^{s} = H_{p,q,b}^{*} = P_{p,q,b} = Q_{p,q,b}, 0 < p < \infty,$$

follows from (7) and Theorems (4.1.31)-(4.1.33). Finally,

$$H_{p,q,b}^{*} = H_{p,q,b}^{S}, 0 < p < \infty$$

is guaranteed by the BurkHölder-Davis-Gundy inequalities given in Theorem (4.1.12).

Section (4.2): John-Nirenberg Inequalities of Martingale Hardy-Lorentz-Karamata Spaces

Lorentz spaces play an important role in classical Harmonic analysis; see, [1], [21], [173], [159] and so on. Lorentz-Karamata spaces, as a new generalization of Lorentz spaces and Lorentz-Zygmund, was studied in [159], [175]. Also Neves studied Lorentz-Karamata spaces $L_{p,q,b}(\Omega, \mathbb{P})$ in [170] where $p, q \in (0, \infty]$ and b is a slowly varying function on $[1, \infty)$ and $(\Omega, \mathbb{P})$ is a measure space. see [174],[177], [179] for more information about Lorentz-Karamata spaces.

The family of martingale Hardy-Lorentz spaces is one of the important martingale function spaces.

See [27]. Some martingale inequalities and atomic decompositions on martingale Hardy-Lorentz spaces were established in [180], [20], [181].

The family of martingale Hardy-Lorentz-Karamata spaces defined in terms of Lorentz-Karamata spaces were studied by Ho [153]. We simply recall one of the main results.

Theorem (4.2.1)[172]: [153] Let $0 < p \leq 1, 0 < q \leq p$ and b be a slowly varying function. Then the dual space of $H_{p,q,b}^{s}$ is $\mathrm{BMO}_{2,b}(\alpha)$ with $\alpha = 1/p - 1$.

This theorem gives the dual space of martingale Lorentz-Karamata space for $0 < p \leq 1, 0 < q \leq p$, which is an important result. But how to characterize the dual for $0 < p \leq 1, 1 < q < \infty$ is still unknown. We prove the following two results.

Theorem (4.2.2) [172]: Let $0 < p, q \leq 1$ and b be a non-decreasing slowly varying function. Then the dual space of $H_{p,q,b}^{s}$ is $\mathrm{BMO}_{2,b}(\alpha)$ with $\alpha = 1/p - 1$.

Definition (4.2.3) [172]: Let $1 \leq r, q < \infty, \alpha \geq 0, b$ be a slowly varying function. The generalized BMO martingale space $\mathrm{BMO}_{r,q,b}(\alpha)$ is defined by

$$BMO_{r,q,b}(\alpha) = \left\{ f \in L_r : \|f\|_{BMO_{r,q,b}(\alpha)} < \infty \right\},$$

Where

$$\| f \|_{\mathrm{BMO}_{r,q,b}(\alpha)} = \sup \frac{\sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(v_k < \infty)^{1-1/r} \|f - f^{v_k}\|_r}{\left(\sum_{k \in \mathbb{Z}} \left(2^k \gamma_b\big(\mathbb{P}(v_k < \infty)\big) \mathbb{P}(v_k < \infty)^{1+\alpha}\right)^q\right)^{1/q}}$$

and the supremum is taken over all stopping time sequences $\{v_k\}_{k \in \mathbb{Z}}$ such that $\{2^k \gamma_b(\mathbb{P}(v_k < \infty)) \mathbb{P}(v_k < \infty)^{1+\alpha}\}_{k \in \mathbb{Z}} \in \ell_q$.

We prove that $\mathrm{BMO}_{r,q,b}(\alpha)$ can be regarded as the dual space of martingale Lorentz-Karamata space for the case $0 < p \leq 1, 1 < q < \infty$.

Theorem (4.2.4) [172]: Let $0 < p \leq 1, 1 < q < \infty$ and b be a non-decreasing slowly varying function. Then the dual space of $H_{p,q,s}^{s}$ is $\mathrm{BMO}_{2,q,b}(\alpha)$ with $\alpha = 1/p - 1$.

In order to prove the theorems above, we also establish atomic decomposition of martingale Lorentz-Karamata space. But our decomposition is less restrictive conditions, which is mainly based on some technical estimates very differently from the method in [153], [20]. An important observation is the key to our approach.

We now turn to the second objective, that is, the John-Nirenberg inequality with respect to generalized BMO martingale space $\mathrm{BMO}_{r,q,b}(\alpha)$. Basing mainly on the duality, John-Nirenberg inequality and something else, the space BMO (see [176]) plays a remarkable role in classical analysis and probability. see [185] for the function space version, respectively, to the books [178], [183], [27] for the martingale version of this theorem. Recall that for $1 \le r < \infty$, the BMO_r martingale spaces are defined as follows:

$$BMO_r = \left\{ f = (f_n)_{n\ge0} \in L_r : \|f\|_{BMO_r} = \sup_n \ \|(\mathbb{E}_n|f - f_n|^r)\|_\infty < \infty \right\}.$$

The John-Nirenberg theorem says that if the stochastic basis $\{\mathcal{F}_n\}_{n\ge0}$ is regular, then

$$\mathrm{BMO}_r = \mathrm{BMO}_2, 1 \le r < \infty \tag{29}$$

with equivalent norms [27]. Recall that the stochastic basis $\{\mathcal{F}_n\}_{n\ge0}$ is said to be regular, if there exists an absolute constant $R > 0$ such that

$$f_n \le R f_{n-1} \forall n > 0$$

holds for all non-negative martingales $f = (f_n)_{n\ge0}$. The new John-Nirenberg theorem is described as follows, which is proved by dualities.

Definition (4.2.5) [172]: [159] A Lebesgue measurable function $b: [1, \infty) \to (0, \infty)$ is said to be a slowly varying function if for any given $\epsilon > 0$, the function $t^{\epsilon} b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on $[0, \infty)$.

Let b be a slowly varying function on $[1, \infty)$, define γ_b on [0,1) by

$$\gamma_b(t) = b(t^{-1}), 0 < t \le 1.$$

This definition is from [159] and modified in [153]. And for any given $\epsilon > 0$, the function $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-decreasing function, and the function $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function on [0,1) (see, [153]).

We prove the generalized John-Nirenberg theorem by duality when the stochastic basis $\{\mathcal{F}^n\}_{n\ge0}$ is regular. Some of the dual results are of independent interest. In order to do this, we need the following lemma and see [182], [27] see [186] for new John-Nirenberg inequalities for martingales.

Lemma (4.2.6) [172]: If the stochastic basis $\{\mathcal{F}^n\}_{n\ge0}$ is regular, then the martingale Hardy-Lorentz spaces $H^*_{p,q}, H^S_{p,q}, H^s_{p,q}, Q_{p,q}$ and $P_{p,q}$ are all equivalent for $0 < p < \infty, 0 < q \le \infty$, and $H^*_{p,q}, H^S_{p,q}, H^s_{p,q}, Q_{p,q}, P_{p,q}$ and $L_{p,q}$ are all equivalent for $1 < p < \infty, 0 < q \le \infty$.

Theorem (4.2.7) [172]: Let $0 < p \le 1, 1 < q, r < \infty$ and b be a non-decreasing slowly varying function. If the stochastic basis $\{\mathcal{F}^n\}_{n\ge0}$ is regular, then

$$\left(H^s_{p,q,b}\right)^* = \mathrm{BMO}_{r,q,b}(\alpha), \alpha = \frac{1}{p} - 1$$

with equivalent norms.

Proof : Denote by r' the conjugate number of r. We first claim that $L_{r'} \subset H^s_{p,q,b}$. According to Lemmas (4.2.6), we have

$$\| f \|_{H^s_{p,q,b}} = \| s(f) \|_{p,q,b} \le \| s(f) \|_{r',r'} \approx \| f \|_{r'}$$

Since $\{\mathcal{F}^n\}_{n\ge0}$ is regular.

For any $f \in L_{r'}$, provides a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1,p,\infty)$-atoms and a sequence of real numbers $(\mu_k)_{k\in\mathbb{Z}}$ satisfying $\mu_k = A \cdot 2^k \mathbb{P}(v_k < \infty)^{1/p}$ (where A is a positive constant and $(v_k)_{k\in\mathbb{Z}}$ is the corresponding stopping time sequence) such that $f = \sum_{k\in\mathbb{Z}} \mu_k a^k$ and

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k < \infty)\big)^{\mu_k^q}\right)^{1/q} \leq C \parallel f \parallel_{H_{p,q,b}^s}.$$

For g $\in \mathrm{BMO}_{r,q,b}(\alpha) \subset L_r$, we define

$$\phi_{\mathrm{g}}(f) = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}(a^k\,\mathrm{g}) \forall f \in L_{r'}.$$

Applying Hölder's inequality, we find that

$$\begin{aligned}\left|\phi_{\mathrm{g}}(f)\right| \leq \sum_{k\in\mathbb{Z}} |\mu_k| \mathbb{E}\left(a^k(\,\mathrm{g} - \mathrm{g}^{v_k})\right) &\leq \sum_{k\in\mathbb{Z}} |\mu_k| \|a^k\|_{r'} \|g - \mathrm{g}^{v_k}\|_r \\ &\leq C \sum_{k\in\mathbb{Z}} |\mu_k| \|s(a^k)\|_{r'} \|\mathrm{g} - \mathrm{g}^{v_k}\|_r \\ &\leq C \sum_{k\in\mathbb{Z}} |\mu_k| \mathbb{P}(v_k < \infty)^{1/r' - 1/p} \|g - \mathrm{g}^{v_k}\|_r \\ &= C \cdot A \sum_{k\in\mathbb{Z}} 2^k \mathbb{P}(v_k < \infty)^{1-\frac{1}{r'}} \|g - \mathrm{g}^{v_k}\|_r.\end{aligned}$$

By the definition of $\|\cdot\|_{\mathrm{BMO}_{r,q,b}(\alpha)}$, we obtain

$$\begin{aligned}\left|\phi_{\mathrm{g}}(f)\right| &\leq C \cdot A \left(\sum_{k\in\mathbb{Z}} \gamma_b^q\big(\mathbb{P}(v_k < \infty)\big)^{\mu_k^q}\right)^{1/q} \|\mathrm{g}\|_{BMO_{r,q,b}(\alpha)} \\ &\leq C \|f\|_{H_{p,q,b}^s} \|\mathrm{g}\|_{BMO_{r,q,b}(\alpha)}.\end{aligned}$$

Thus, ϕ_{g} can be extended to a continuous functional on $H_{p,q,b}^s$.

Conversely, let $\phi \in \left(H_{p,q,b}^s\right)^*$. we have $L_{r'} \subset H_{p,q,b}^s \subset H_{p,q,b'}^{at}$, then

$$\left(H_{p,q,b}^{at_{r'}}\right)^* \subset \left(H_{p,q,b}^s\right)^* \subset L_r.$$

Hence, there exists $g \in L_r$ such that

$$\phi(f) = \phi_{\mathrm{g}}(f) = \mathbb{E}(f\,\mathrm{g}) \forall f \in L_{r'},$$

and ϕ can be extended to a continuous functional $\tilde{\phi}$ on $H_{p,q,b}^{at_{r'}}$ such that $\parallel \phi \parallel = \parallel \tilde{\phi} \parallel$. Let $\{v_k\}_{k\in\mathbb{Z}}$ be an arbitrary stopping time sequence such that

$$\left(\sum_{k\in\mathbb{Z}} \left(2^k \gamma_b\big(\mathbb{P}(v_k < \infty)\big)\mathbb{P}(v_k < \infty)^{\frac{1}{p}}\right)^q\right)^{1/q} < \infty.$$

Set

$$h_k = \frac{(g - \mathrm{g}^{v_k})^{r-1} \mathrm{sign}(\mathrm{g} - \mathrm{g}^{v_k})}{\|\mathrm{g} - \mathrm{g}^{v_k}\|_r^{r-1} \mathbb{P}(v_k < \infty)^{\frac{1}{p} - \frac{1}{r'}}}.$$

We show that h_k/C is a $(1,p,r') -$ atom for some constant C. Since $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then

$$\|s(h_k)\|_{r'} \leq C \|h_k\|_{r'} = C\mathbb{P}(v_k < \infty)^{\frac{1}{r'} - \frac{1}{p}}.$$

By the definition of $H_{p,q,b'}^{at_{r'}}$, we find that

$$f=\sum_{k\in\mathbb{Z}} 2^k\mathbb{P}(v_k<\infty)^{\frac{1}{p}}h_k\in H_{p,q,b}^{at_{r'}}$$

and

$$\| f \|_{H_{p,q,b}^{at}}\le C\left(\sum_{k\in\mathbb{Z}}\left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{1/q}.$$

Let N be an arbitrary non-negative integer and

$$f^N=\sum_{-N}^{N} 2^k\mathbb{P}(v_k<\infty)^{\frac{1}{p}}h_k.$$

Then

$$\sum_{k=-N}^{N} 2^k\mathbb{P}(v_k<\infty)^{1-1/r}\|g-g^{v_k}\|_r=\sum_{k=-N}^{N} 2^k\mathbb{P}(v_k<\infty)^{1/p}\mathbb{E}\big(h_k(g-g^{v_k})\big)$$
$$=\mathbb{E}(f^N\text{ g})=\tilde{\phi}(f^N)$$
$$\le\|f^N\|_{H_{p,q,b}at_{r'}}\|\tilde{\phi}\|\le\| f \|_{H_{p,q,b}}^{at_{q'}}\|\phi\|.$$

Thus, we obtain

$$\frac{\sum_{k=-N}^{N} 2^k\mathbb{P}(v_k<\infty)^{1-1/r}\|g-g^{v_k}\|_r}{\left(\sum_{k\in\mathbb{Z}}\left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{1/q}}\le C\|\phi\|.$$

Taking $N\to\infty$ and the supremum over all of stopping time sequences, we get $\| g \|_{\mathrm{BMO}}^{r,q,b} C\|\phi\|$.

Theorem (4.2.8) [172]: Let $0<p,q\le 1,\alpha=1/p-1$ and b be a non-decreasing slowly varying function.

Then $\left(P_{p,q,b}^{*}\right)_1=\mathrm{BMO}_{1,b}(\alpha)$ with equivalent norms.

Proof : Let $g\in\mathrm{BMO}_{1,b}(\alpha)\subset L_1$. Define $\phi_{\mathrm{g}}(f)=\mathbb{E}(f\text{ g}),(f\in L_\infty)$. there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(3,p,\infty)$-atoms and a sequence of real numbers $(\mu_k)_{k\in\mathbb{Z}}$ satisfying $\mu_k=A\cdot 2^k\mathbb{P}(v_k<\infty)^{1/p}$ such that $f=\sum_{k\in\mathbb{Z}}\mu_k a^k$ and $\left(\sum_{k\in\mathbb{Z}}\gamma_b^q\big(\mathbb{P}(v_k<\infty)\big)\mu_k^q\right)^{1/q}\le C\| f \|_{P,q,b}$. Similarly, to the proof of Theorem (4.2.7),

$$\left|\phi_{\mathrm{g}}(f)\right|\le\sum_{k\in\mathbb{Z}}|\mu_k|\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{-\frac{1}{p}}\gamma_b^{-1}\big(\mathbb{P}(v_k<\infty)\big)\|\mathrm{g}-\mathrm{g}^{v_k}\|_1$$
$$\le\sum_{k\in\mathbb{Z}}|\mu_k|\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\|\text{ g }\|_{\mathrm{BMO}_{1,b}(\alpha)}.$$

Since $0<q\le 1$, then

$$\left|\phi_{\mathrm{g}}(f)\right|\le\left(\sum_{k\in\mathbb{Z}}\gamma_b^q\big(\mathbb{P}(v_k<\infty)\big)\,\mu_k^q\right)^{\frac{1}{q}}\|\mathrm{g}\|_{BMO_{1,b}(\alpha)}\le C\|f\|_{P_{p,q,b}}\|\mathrm{g}\|_{BMO_{1,b}(\alpha)}.$$

Then φ_{g} can be extended to a continuous functional on $P_{p,q,b}$, and $\varphi_{\mathrm{g}}\in\left(P_{p,q,b}^{*}\right)$.

To prove the converse, let $\phi \in \left(\left(P_{p,q,b}^{*}\right)\right)_1$, then there exists $g \in L_1$ such that $\phi(f) = \mathbb{E}(f\, g), (f \in L_\infty)$. Let $h = \operatorname{sign}(g - g^{v}), a = \frac{1}{2}\mathbb{P}(v < \infty)^{-\frac{1}{p}}(h - h^{v})$, where $v \in \mathcal{T}$ is an arbitrary stopping time. Then a is a $(3, p, \infty) -$ atom.
Let $\mu = 2A \cdot \mathbb{P}(v < \infty)^{1/p}$, let $h_0 = \mu a = A(h - h^{v})$. Considering the atomic decomposition of h_0, we have $h_0 \in P_{p,q,b}$ and

$$\|h_0\|_{P_{p,q,b}} \leq C|\mu|\gamma_b(\mathbb{P}(v < \infty)) = 2CA \cdot \mathbb{P}(v < \infty)^{\frac{1}{p}}\gamma_b(\mathbb{P}(v < \infty)),$$

then $\|h - h^{v}\|_{P_{p,q,b}} \leq 2C \cdot \mathbb{P}(v < \infty)^{\frac{1}{p}}\gamma_b(\mathbb{P}(v < \infty))$. Thus, we have

$$\begin{aligned}\mathbb{P}(v < \infty)^{-\frac{1}{p}}\gamma_b^{-1}(\mathbb{P}(v < \infty))\|g - g^{v}\|_1 &= \mathbb{P}(v < \infty)^{-\frac{1}{p}}\gamma_b^{-1}(\mathbb{P}(v < \infty))\phi(h - h^{v}) \\ &\leq \mathbb{P}(v < \infty)^{-\frac{1}{p}}\gamma_b^{-1}(\mathbb{P}(v < \infty))\|h - h^{v}\|_{P_{p,q,b}} \parallel \phi \parallel \\ &= 2C \parallel \phi \parallel.\end{aligned}$$

Taking the supremum over all stopping times, then we obtain $\parallel g \parallel_{\mathrm{BMO}_{1,b}(\alpha)} \leq C \parallel \phi \parallel$. The proof of the theorem is complete.

Theorem (4.2.9) [172]: Let $0 < p \leq 1, 1 < q < \infty, \alpha = 1/p - 1$ and b be a non-decreasing slowly varying function. Then $\left(P_{p,q,b}^{*}\right)_1 = \mathrm{BMO}_{1,q,b}(\alpha)$ with equivalent norms.

Proof : Let $g \in \mathrm{BMO}_{1,q,b}(\alpha) \subset L_1$. Define $\phi_g(f) = \mathbb{E}(f_g), (f \in L_\infty)$. Similarly to the proof of Theorem (4.2.7), we obtain

$$\begin{aligned}|\phi_g(f)| &\leq \sum_{k\in\mathbb{Z}} |\mu_k|\mathbb{E}\left(\left|a^k(g - g^{v_k})\right|\right) \leq \sum_{k\in\mathbb{Z}} |\mu_k|\|a^k\|\|g - g^{v_k}\|_1 \\ &\leq \sum_{k\in\mathbb{Z}} |\mu_k|\mathbb{P}(v_k < \infty)^{-\frac{1}{p}}\|g - g^{v_k}\|_1 \leq \sum_{k\in\mathbb{Z}} 2^k\|g - g^{v_k}\|_1.\end{aligned}$$

By the definition of $\|\cdot\|_{\mathrm{BMO}_{1,q,b}(\alpha)}$, we obtain

$$\begin{aligned}|\phi_g(f)| &\leq A\left(\sum_{k\in\mathbb{Z}} \left(2^k\gamma_b^q\left(\mathbb{P}(v_k < \infty)\right)\mathbb{P}(v_k < \infty)^{\frac{1}{p}}\right)^q\right)^{1/q} \|g\|_{BMO_{1,b}(\alpha)} \\ &\leq C\|f\|_{P_{p,q,b}}\|g\|_{BMO_{1,b}(\alpha)}.\end{aligned}$$

Thus, ϕ_g can be extended to a continuous functional on $P_{p,q,b}$. Moreover, $\phi_g \in \left(P_{p,q,b}^{*}\right)_1$.
Conversely, if $\phi \in \left(P_{p,q,b}^{*}\right)_1$, then there exists $g \in L_1$ such that $\phi(f) = \mathbb{E}(f\, g), (f \in L_\infty)$.
Let $\{v_k\}_{k\in\mathbb{Z}}$ be an arbitrary stopping time sequence such that

$$\left(\sum_{k\in\mathbb{Z}} \gamma_b^q\left(\mathbb{P}(v_k < \infty)\right)\mu_k^q\right)^{1/q} < \infty.$$

Let

$$h_k = \operatorname{sign}(g - g^{v_k}), a^k = \frac{1}{2}\left(h_k - h_k^{v_k}\right)\mathbb{P}(v_k < \infty)^{1/p},$$

then a^k is a $(3, p, \infty)$-atom.
Let $f^N = \sum_{k=-N}^{N} 2^k\mathbb{P}(v_k < \infty)^{1/p}a^k$, where N is an arbitrary non-negative integer. we have $f^N \in P_{p,q,b}$ and

$$\|f^N\|_{P_{p,q,b}} \leq C\left(\sum_{k=-N}^{N} \left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{\frac{1}{q}}$$

$$\leq C\left(\sum_{k\in\mathbb{Z}} \left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{1/q}.$$

Consequently,

$$\sum_{k=-N}^{N} 2^k\|g-\mathrm{g}^{v_k}\|_1 = \mathbb{E}(f^N\,\mathrm{g}) = \phi(f^N) \leq \|f^N\|_{P_{p,q,b}}\ \|\ \phi\ \|$$

$$\leq C\left(\sum_{k\in\mathbb{Z}} \left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{1/q}\ \|\ \phi\ \|.$$

Thus, we have

$$\frac{\sum_{k=-N}^{N} 2^k\|g-g^{v_k}\|_1}{\left(\sum_{k\in\mathbb{Z}} \left(2^k\gamma_b\big(\mathbb{P}(v_k<\infty)\big)\mathbb{P}(v_k<\infty)^{\frac{1}{p}}\right)^q\right)^{1/q}} \leq C\ \|\ \phi\ \|.$$

Taking $N\to\infty$ and the supremum over all of stopping time sequences, we get $\|\ g\ \|_{\mathrm{BMO}_{1,q,b}(\alpha)}\ C\ \|\ \phi\ \|$. The proof is complete.

Proposition (4.2.10) [172]: If the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, for $0<p\leq 1, 0<q<\infty$, then $\left(P^*_{p,q,b}\right)_1 = P^*_{p,q,b}$.

Proof: Since $0<p\leq 1$, L_2 can also be embedded continuously in $P_{p,q,b}$. Then $P^*_{p,q,b}\subset (L_2)^* = L_2$. Let ϕ be an arbitrary element of $P^*_{p,q,b}$, then there exists $g\in L_2\subset L_1$ such that $\phi=\phi_{\mathrm{g}}$. By the definition of $\left(P^*_{p,q,b}\right)_1$, we have $\phi\in\left(P^*_{p,q,b}\right)_1$, then $P^*_{p,q,b}\subset\left(P^*_{p,q,b}\right)_1$. And the inclusion relation $\left(P^*_{p,q,b}\right)_1\subset P^*_{p,q,b}$ is evident. Hence, we obtain

$$\left(P^*_{p,q,b}\right)_1 = P^*_{p,q,b}.$$

The proof of the theorem is complete.

We now are in a position to prove Theorem (4.2.11).

Theorem (4.2.11) [172]: Let b be a non-decreasing slowly varying function. Suppose that the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular and $1<q<\infty$. Then

$$\mathrm{BMO}_{r,q,b}(\alpha) = \mathrm{BMO}_{2,q,b}(\alpha)$$

with equivalent norms for all $1\leq r<\infty$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. We denote by $L_0(\Omega,\mathcal{F},\mathbb{P})$, or simply $L_0(\Omega)$, the space of all measurable functions on $(\Omega,\mathcal{F},\mathbb{P})$.

Proof: It follows from Theorems (4.2.4) and (4.2.7) that

$$\mathrm{BMO}_{r,q,b}(\alpha) = \mathrm{BMO}_{2,q,b}(\alpha), 1<r<\infty.$$

For $r=1$, combining Theorem (4.2.9), Proposition (4.2.10), we get

$$\mathrm{BMO}_{1,q,b}(\alpha) = \mathrm{BMO}_{2,q,b}(\alpha).$$

So the proof of Theorem (4.2.11) is complete.

Section (4.3): BV-alued Martingales

The study of Banach space valued (B-valued) martingales began with Pisier's fundamental [200]. Since then, B valued martingale theory has attracted more attentions in last decades. BurkHölder in [188] and [189] discussed martingale transforms and differential subordinations for $\mathbf{B}-$ valued martingales. Liu [194], [195]introduced the p-variation operator and discussed various **B** valued martingale inequalities. Yu [206], [205]investigated the dual spaces of Orlicz-Hardy spaces and weak Orlicz-Hardy spaces for **B** valued martingales. see [201] by Pisier for more information on martingales and Fourier analysis in Banach spaces.

It is well known that Lorentz spaces are more extensive family than Lebesgue spaces (see e.g. [199], [173]). In [20], the Hardy martingale spaces are extended to Hardy-Lorentz martingale spaces. Very recently, Jiao et al. [183] studied the small-index Hardy-Lorentz martingale spaces and established the predual and John-Nirenberg inequalities for the generalized BMO spaces. Weisz [202] characterized the dual of multi-parameter martingale Hardy-Lorentz spaces.

We study Hardy-Lorentz spaces for **B**-valued martingales and extend the dual results in [183]and martingale inequalities in [20] to the **B**-valued martingale setting. Our proof mainly depends on the establishment of atomic decompositions of Hardy-Lorentz spaces for B valued martingales. Recall that atomic decompositions were first introduced by Herz [162], generalized by Weisz [171], [190]and developed by many other authors (see e.g. [172], [153]) in scalarvalued martingale case. As for **B**-valued martingales, Liu et al. [197], [198] investigated the atomic decompositions and characterized some geometrical properties of Banach spaces; Yu [204] established the dual of B-valued martingale Hardy spaces with the help of atomic decompositions.

The results above, and also many other **B**-valued martingale results (see e.g. [200], [191], [192], [182], [201]), are closely related to the geometrical properties of the underlying spaces. Our conclusions have no exception.

Some preliminary lemmas and **B**-valued martingale Hardy-Lorentz spaces are introduced we present atomic decompositions for **B**-valued martin-gale Hardy-Lorentz spaces $H_{r_1,r_2}^{s^p}(\mathbf{B}), Q_{r_1,r_2}^{S_2^p}(\mathbf{B})$ and $D_{r_1,r_2}(\mathbf{B})$. As usual, these theorems depend on the geometrical properties of the underlying Banach space B. Applying atomic decompositions established We prove two duality results. In, we establish several martingale inequalities among Lorentz spaces. These are new versions of the basic inequalities in **B**-valued martingale setting.

The sets of integers, nonnegative integers and complex numbers are always denoted by $\mathbb{Z}, \mathbb{N}$ and $\mathbb{C}$, respectively. We use C to denote a positive constant which may vary from line to line. The symbol $\subset$ means the continuous embedding and $f \sim g$ stands for $C^{-1} g \leq f \leq Cg$. We call f is equivalent to g if $f \sim g$.

Let $(\Omega, F, \mathbb{P})$ be a complete probability space and $\boldsymbol{B}$ denote a Banach space with norm $\|\cdot\|$. For a measurable function f, we define its distribution function by

$$\lambda_s(f) = \mathbb{P}(\{\omega \in \Omega : \| f(\omega) \| > s\}), s \geq 0.$$

And denote by $f^*(t)$ the decreasing rearrangement of f, defined by

$$f^*(t) = \mathrm{i} \;\; \{s \geq 0 : \lambda_s(f) \leq t\}, t \geq 0,$$

with the convention that $\inf\emptyset = \infty$.

Let $0 < p < \infty, 0 < q \leq \infty$. The $\boldsymbol{B}$-valued Lorentz space $L_{p,q}(\mathbf{B})$ (briefly denoted by $L_{p,q}$ in the sequel) consists of those **B**-valued measurable functions f with finite quasi norm $\| f \|_{p,q}$ given by

$$\| f \|_{p,q}= \left(\frac{q}{p}\int_0^{\infty}\left(t^{\frac{1}{p}}f^*(t)\right)^q\frac{dt}{t}\right)^{1/q},0<q<\infty,$$
$$\| f \|_{p,\infty}= \sup_{t>0} t^{\frac{1}{p}}f^*(t), q=\infty.$$
If $1<p<\infty$ and $1\le q\le\infty$, the quasi norm $\|\cdot\|_{p,q}$ is equivalent to a norm (see [161]). Another equivalent definition of $f_{p,q}$ is given by
$$\| f \|_{p,q}=\left(q\int_0^{\infty}\left(t\mathbb{P}(\| f \|>t)^{\frac{1}{p}}\right)^q\frac{dt}{t}\right)^{1/q},0<q<\infty,$$
$$\| f \|_{p,\infty}= \sup_{t>0} t\mathbb{P}\left(\| f \|>\frac{1}{t}\right)^{\frac{1}{p}}, q=\infty.$$
As another application of the atomic decomposition, we obtain a sufficient condition for a σ-sublinear operator to be bounded from B-valued martingale Hardy-Lorentz spaces to function Lorentz spaces.
An operator $T: X\to Y$ is called a $\sigma-$ sublinear operator if for any $\alpha\in\mathbb{C}$ it satisfies $|T(\sum_{k=1}^{\infty}f_k)|\le\sum_{k=1}^{\infty}|T(f_k)|$ and $|T(\alpha f)|\le|\alpha||T(f)|$, where X is a martingale space and Y is a measurable function space.
Lemma (4.3.1)[187]: Let $1<p\le 2$ and let $T: H_p^{s^p}(B)\to L_p(\Omega)$ be a bounded $\sigma-$ sublinear operator. If B is isomorphic to a p-uniformly smooth space and $\{|Ta|>0\}\subset\{v<\infty\}$ for all $(1,r_1,\infty;p)$-atoms a (where v is the stopping time associated with a), then for $0<r_1<p$ and $0<r_2\le\infty$, we have
$$\| Tf \|_{r_1,r_2}\le C\ \| f \|_{H^s_{r_1,r_2}(\mathbf{B})}.$$
Proof: Let $f\in H^p_{r_1,r_2}(\mathbf{B})$. For $k_0\in\mathbb{Z}$, set
$$f=\sum_k \mu_k a^k=\sum_{k=-\infty}^{k_0-1}\mu_k a^k+\sum_{k=k_0}^{\infty}\mu_k a^k=g+h,$$
Where
$$g=\sum_{k=-\infty}^{k_0-1}\mu_k a^k, h=\sum_{k=k_0}^{\infty}\mu_k a^k.$$
Let $\varepsilon=r_1/p$. By Chebyshev's inequality and the sublinearity, boundedness of T, we get
$$2^{k_0r_1}\mathbb{P}(T(g)>2^{k_0})^{\varepsilon}\le 2^{k_0r_1}\left(\frac{1}{2^{k_0p}}\| T(g) \|_p^p\right)^{\varepsilon}$$
$$\le\left(\sum_{k=-\infty}^{k_0-1}\mu_k\|T(a^k)\|_p\right)^{p\varepsilon}\le\left(\sum_{k=-\infty}^{k_0-1}\mu_k\|s^p(a^k)\|_p\right)^{p\varepsilon}$$
$$\le C\left(\sum_{k=-\infty}^{k_0-1}\mu_k\mathbb{P}(v_k<\infty)^{\frac{1}{p}-\frac{1}{r_1}}\right)^{p\varepsilon}$$
$$=C\sum_{k=-\infty}^{k_0-1}\left(2^k\mathbb{P}(v_k<\infty)^{\frac{\varepsilon}{r_1}}\right)^{r_1}\le C\sum_{k=-\infty}^{k_0-1}\left(2^k\mathbb{P}(v_k<\infty)^{\frac{\varepsilon}{r_1}}\right)^{r_1}.$$

On the other hand, we have $\{T(h) > 0\} \subset \infty \cup_{k=k_0}^{\infty} \{v_k < \infty\}$. Then for each $0 < \varepsilon < 1$, we obtain

$$2^{k_0 \varepsilon r_1} \mathbb{P}(T(h) > 2^{k_0}) \leq 2^{k_0 \varepsilon r_1} \mathbb{P}(T(h) > 0) \leq 2^{k_0 \varepsilon r_1} \sum_{k=k_0}^{\infty} \mathbb{P}(v_k < \infty)$$

$$\leq \sum_{k=k_0}^{\infty} 2^{k_0 \varepsilon r_1} \mathbb{P}(v_k < \infty) = \sum_{k=k_0}^{\infty} \left(2^{k\varepsilon} \mathbb{P}(v_k < \infty)^{\frac{1}{r_1}}\right)^{r_1}$$

$$\leq \sum_{k=k_0}^{\infty} \left(2^{k\varepsilon} \mathbb{P}(v_k < \infty)^{\frac{1}{r_1}}\right)^{r_1}.$$

we get $T(f) \in L_{r_1,r_2}(\boldsymbol{B})$ and

$$\| T(f) \|_{r_1,r_2} \leq C \left\| \left\{ 2^k \mathbb{P}(v_k < \infty)^{\frac{1}{r_1}} \right\}_{k \in \mathbb{Z}} \right\|_{l_{2r}} \leq C \| f \|_{H^{s_2}_{r_1,r_2}(B)}.$$

Similar to the proof of Lemma (4.3.1), respectively.

Lemma (4.3.2) [187]: Let $1 < p \leq 2$ and let $T: H_p^{S^p}(\boldsymbol{B}) \to L_p(\Omega)$ be a bounded $\sigma -$ `$\{v < \infty\}$ for all $(2, r_1, \infty; p)$-atoms a(where v is the stopping time associated with a), then for $0 < r_1 < p$ and $0 < r_2 \leq \infty$, we have

$$\| Tf \|_{r_1,r_2} \leq C \| f \|_{Q^{S^p}_{r_1,r_2}(B)}.$$

Lemma (4.3.3) [187]: Let $0 < q < \infty$ and let $T: H_q(\boldsymbol{B}) \to L_q(\Omega)$ be a bounded $\sigma -$ sublinear operator. If $\boldsymbol{B}$ has the $R - N$ property and $\{|Ta| > 0\} \subset \{v < \infty\}$ for all $(3, r_1, \infty) -$ atoms a (where v is the stopping time associated with a), then for $0 < r_1 < q$ and $0 < r_2 \leq \infty$, we have

$$\| Tf \|_{r_1,r_2} \leq C \| Tf \|_{D_{r_1,r_2}(B)}.$$

Let $([0,1), \mathcal{F}, \mu)$ be a probability space such that μ is the Lebesgue measure and subalgebras $\{\mathcal{F}_n\}_{n \geq 0}$ generated as follows:

$$\mathcal{F}_n = \sigma - \text{ algebra generated by atoms } \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), j = 0, \cdots, 2^n - 1.$$

Recall that all martingales with respect to $\{\mathcal{F}_n\}_{n \geq 0}$ are called dyadic martingales.

Theorem (4.3.4) [187]: Let B be a Banach space, $1 < p \leq 2, 0 < r_1 < p$ and $0 < r_2 \leq \infty$. Then the following statements are equivalent:

(i) $\boldsymbol{B}$ is isomorphic to a p-uniformly smooth space;

(ii) There exists a constant $C > 0$ such that for every $f = (f_n)_{n \geq 0} \in H^{s^p}_{r_1,r_2}(\boldsymbol{B})$,

$$\| Mf \|_{r_1,r_2} \leq C \| f \|_{H^{s^p}_{r_1,r_2}};$$

(iii) There exists a constant $C > 0$ such that for every $f = (f_n)_{n \geq 0} \in Q^{s^p}_{r_1,r_2}(\boldsymbol{B})$,

$$\| Mf \|_{r_1,r_2} \leq C \| f \|_{Q^{s^p}_{r_1,r_2}(\boldsymbol{B})}.$$

Proof:(i) ⇒ (ii). The maximal operator $Tf = Mf$ is σ-sublinear. Since $\boldsymbol{B}$ is isomorphic to a p-uniformly smooth space and [201], we have

$$\| Mf \|_p \leq C \|s^p(f)\|_p = C \| f \|_{H_p^{s^p}(B)}.$$

This means $M: H_p^{s^p}(\boldsymbol{B}) \to L_p(\Omega)$ is bounded. For any $(1, r_1, \infty; p)$-atom aand the corresponding stopping time v, we have $\{|Ta| > 0\} = \{|Ma| > 0\} \subset \{v < \infty\}$. Hence, $\mathbb{P}(|Ta| > 0) \leq \mathbb{P}(v < \infty)$. The desired inequality immediately follows from Lemma (4.3.1).

(i) ⇒ (iii). Similar to (i) ⇒ (ii), it can be proved by Lemma(4.3.2).
(ii) ⇒ (i). Let $f = (f_n)_{n\geq 0}$ be an arbitrary $\boldsymbol{B}$-valued martingale with $\mathbb{E}s^p(f)^p = \mathbb{E}(\sum_{n=1}^{\infty}\|df_n\|^p) < \infty$.
Since $0 < r_1 < p$, we have $\|s^p(f)\|_{r_1,r_2} \leq \|s^p(f)\|_p\infty$. So the martingale $f \in H^{S^q}_{r_1,r_2}(\boldsymbol{B})$. Consider $g^{(n)} = \left(g_m^{(n)}\right)_{m\geq 0}$, where $g_m^{(n)} = f_{m+n} - f_n, (m \geq 0)$. It is obvious that $s^p\left(g^{(n)}\right)^p = s^p(f)^p - s_n^p(f)^p \to 0$ as $n \to \infty$ and $s^p\left(g^{(n)}\right) \leq s^p(f)$. Furthermore, by condition (ii) we have

$$\|f_{m+n} - f_n\|_{r_1,r_2} \leq \sup_{m\geq 0}\|f_{m+n} - f_n\|_{r_1,r_2} \leq \|Mg^{(n)}\|_{r_1,r_2} \leq C\|s^p g^{(n)}\|_{r_1,r_2}.$$

Applying the controlled convergence theorem, we obtain $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $L_{r_1,r_2}(\boldsymbol{B})$. Hence f_n is convergent in probability. $\boldsymbol{B}$ is isomorphic to a p-uniformly smooth space.
(iii) ⇒ (i). Let $f = (f_n)_{n\geq 0}$ be an arbitrary $\boldsymbol{B}$-valued dyadic martingale such that $\mathbb{E}(\sum_{n=1}^{\infty}\|df_n\|^p) < \infty$. Similar to (ii) ⇒ (i), we get $f \in H^p_{r_1,r_2}(\boldsymbol{B})$. For $n \geq 0$, let $\lambda_n = s_{n+1}^p(f)$. Then $(\lambda_n)_{n\geq 0}$ is a nonnegative, nondecreasing and adapted sequence. Since f is a $\boldsymbol{B}$-valued dyadic martingale, we have $S_n^p(f) \leq Cs_n^p(f)$. Thus

$$\| f \|_{Q^{s^p}_{r_1,r_2}(B)} \leq \|s^p(f)\|_{r_1,r_2} < \infty.$$

Namely, $f \in Q^p_{r_1,r_2}(\boldsymbol{B})$. Consider $g^{(n)} = \left(g_m^{(n)}\right)_{m\geq 0}$ as above. By condition (iii), we get

$$\|f_{m+n} - f_n\|_{r_1,r_2} \leq \|Mg^{(n)}\|_{r_1,r_2} \leq C\|g^{(n)}\|_{Q^{s^p}_{r_1,r_2}(B)} \leq C\|s^p g^{(n)}\|_{r_1,r_2}.$$

Using the controlled convergence theorem, we obtain $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $L_{r_1,r_2}(\boldsymbol{B})$. Hence f_n is convergent in probability. B is isomorphic to a p-uniformly smooth space. The proof of the theorem is complete.

Lemma (4.3.5) [187]: ([201])Let $2 \leq q \leq r < \infty$. Then the following statements are equivalent:
(i) $\boldsymbol{B}$ is isomorphic to a q-uniformly convex space;
(ii) There exists a constant C such that for every $f = (f_n)_{n\geq 0} \in H_r(\boldsymbol{B})$,

$$\| f \|_{H_r^{sq}(\boldsymbol{B})} \leq C \| f \|_{H_r(\boldsymbol{B})} ;$$

(iii) There exists a constant C such that for every $f = (f_n)_{n\geq 0} \in H_r(\boldsymbol{B})$,

$$\| f \|_{H_r^{s^q(\boldsymbol{B})}} \leq C \| f \|_{H_r(\boldsymbol{B})} .$$

Theorem (4.3.6) [187]: Let $\boldsymbol{B}$ be a Banach space, $2 \leq q < \infty, 0 < r_1 < q$ and $0 < r_2 \leq \infty$. Then the following statements are equivalent:
(i) $\boldsymbol{B}$ is isomorphic to a q-uniformly convex space;
(ii) There exists a constant $C > 0$ such that for every $\boldsymbol{B}$-valued martingale $f = (f_n)_{n\geq 0}$,

$$\| f \|_{H^{S^q}_{r_1,r_2}(\boldsymbol{B})} \leq C \| f \|_{D_{r_1,r_2}(\boldsymbol{B})}$$

(iii) There exists a constant $C > 0$ such that for every $\boldsymbol{B}$-valued martingale $f = (f_n)_{n\geq 0}$,

$$\| f \|_{H^{s^p}_{r_1,r_2}(\boldsymbol{B})} \leq C \| f \|_{D_{r_1,r_2}(\boldsymbol{B})}.$$

Proof: (i) ⇒ (ii). It is obvious that $\{S^q(a) > 0\} \subset \{v < \infty\}$, where ais a $(3, r_1, \infty) -$ atom and v is the corresponding stopping time. By Lemma (4.3.5), we know that the sublinear operator $S^q(\cdot)$ is bounded from $H_q(\boldsymbol{B})$ to $L_q(\Omega)$. Condition (i) implies that the space $\boldsymbol{B}$ has the $R - N$ property. Then by Lemma (4.3.3), we have

$$\|S^q(f)\|_{r_1,r_2(\boldsymbol{B})} \leq C \| f \|_{D_{r_1,r_2}(\boldsymbol{B})}, \forall f = (f_n)_{n\geq 0} \in D_{r_1,r_2}(\boldsymbol{B}).$$

Namely, $\| f \|^q_{H_{r_1,r_2}(B)} \leq C \| f \|_{D_{r_1,r_2}(B)}$.
(i) ⇒ (iii). It can be similarly proved as above.
(ii) ⇒ (i). Let $(f_n)_{n\geq 0}$ be an arbitrary $\boldsymbol{B}$-valued martingale such that $\sup_{n\geq 0}\|f_n\|_\infty < \infty$. Then $\| f \|_{D_{r_1,r_2}(\boldsymbol{B})} < \infty$. Since $\| f \|_{H^{S_2}_{r_1,r_2}(\boldsymbol{B})} \leq C \| f \|_{D_{r_1,r_2}(\boldsymbol{B})}$, we have $S^q(f) < \infty$. B is isomorphic to a q-convex space.
(iii) ⇒ (i). Consider a $\boldsymbol{B}$-valued dyadic martingale $(f_n)_{n\geq 0}$ with $\sup_{n\geq 0}\|f_n\|_\infty < \infty$. Then $S^q(f) \leq C_s q(f) < \infty$. we get the desired result. The proof of the theorem is complete.

Chapter 5
Sharp Hardy Inequalities

We develop a geometric framework for Hardy's inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary. In all cases, using techniques that exploit the symmetry presented by the solid torus, we calculate the displayed best constants and we prove that they are the same as the standard Hardy best constants which appear in convex domains although the solid torus is not convex.

Section (5.1): Some Non-Convexity Measures

We study high dimension variants of the classical integral Hardy-type inequality ([214])

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \mu \int_0^\infty f^p(x)dx, \tag{1}$$

where $p > 1, f(x) \ge 0$, and $F(x) = \int_0^x f(t)dt$ with constant μ. Inequality (1) with its improvements have played a fundamental role in the development of many mathematical branches such as spectral theory and PDE's, see [208], [209], [210], [211],[213] and [216]. We centre our attention on the multi-dimensional version of (1) for $p = 2$, which takes the following form (see [212]):

$$\mu \int_\Omega \frac{|f(x)|^2}{d(x)^2} dx \le \int_\Omega |\nabla f|^2 dx, f \in C_c^\infty(\Omega), \tag{2}$$

where

$$d(x) := \min \ \{|x - y| : y \notin \Omega\}. \tag{3}$$

For convex domains $\Omega \subset \mathbb{R}^n$, the sharp constant μ in (2) has been shown to equal 14 , see [211] and [216]. However, the sharp constant for non-convex domains is unknown, although for arbitrary planar simply-connected domains $\Omega \subset \mathbb{R}^2$, A. Ancona ([57]) proved, using the Koebe one-quarter Theorem, that the constant μ in (2) is greater than or equal to $\frac{1}{16}$. Later A. Laptev and A. Sobolev ([215]) considered, under certain geometrical conditions, classes of domains for which there is a stronger version of the Koebe Theorem, this implied better estimates for the constant μ. Other specific examples of non-convex domains were presented by E. B. Davies ([212]).

We obtain new Hardy-type inequalities under some non-convexity measures for domains in $\mathbb{R}^n, n \ge 3$, focusing on obtaining upper bounds for μ. We have two different conditions "measures" introduced.

We present two 'non-convexity measures' for domains $\Omega \subset \mathbb{R}^n; n \ge 3$. Let w be a point in $\mathbb{R}^n$ and v be a unit vector. For $\alpha \in \left(0, \frac{\pi}{2}\right)$ define

$$C_0(v, \alpha) = \{x \in \mathbb{R}^n : x \cdot v \ge |x| \cos \alpha\},$$

which is a cone in the Euclidean space $\mathbb{R}^n$ with vertex at 0 and symmetry axis in the v direction. Denote by $C_w(v, \alpha) = C_0(v, \alpha) + w$, the transition of $C_0(v, \alpha)$ by $w \in \mathbb{R}^n$, i.e.

$$C_w(v, \alpha) = \{x \in \mathbb{R}^n : (x - w) \cdot v \ge |x - w| \cos \alpha\},$$

which can be seen as an n-dimensional cone with vertex at w and symmetry axis parallel to the v direction with angle 2α at the vertex.

Now for $h \ge 0$, define the half-space $\mathbf{\Pi}_h(v)$ by

$$\mathbf{\Pi}_h(v) = \{x \in \mathbb{R}^n : x \cdot v \ge h\}.$$

Denote by $\mathbf{\Pi}_{h,w}(v) = \mathbf{\Pi}_h(v) + w$, the transition of $\mathbf{\Pi}_h(v)$ by $w \in \mathbb{R}^n$, i.e.

$$\mathbf{\Pi}_{h,w}(v) = \{x \in \mathbb{R}^n : (x - w) \cdot v \ge h\},$$

which is a half-space of 'height h ' from the point w in the v direction.

Define the region $K_{h,w}(v,\alpha)$ to be

$$K_{h,w}(v,\alpha) = C_w(v,\alpha) \cup \mathbf{\Pi}_{h,w}(v).$$

We now state the conditions or 'non-convexity measures' we use throughout the rest.

Condition (5.1.1)[207]: (Exterior Cone Condition).

We say that $\Omega \subset \mathbb{R}^n$ satisfies the Exterior Cone Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset C_w^c(v,\alpha)$, with $(x - w) \cdot v = -|x|$. Condition (5.1.1) means that for every point $x \in \Omega$ we can always find a cone $C_\omega(v,\alpha)$ such that x lies on its symmetry axis where Ω is completely outside that cone.

As a development of the above condition, we establish the following condition.

Condition (5.1.2) [207]: (Truncated Cone Region (TCR). Condition).

We say that $\Omega \subset \mathbb{R}^n$ satisfies the TCR Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset K_{h,w}^c(v,\alpha)$, for some $h \geq 0$, with $(x - w) \cdot v = -|x|$.

Condition (5.1.2) means that for every point $x \in \Omega$ we can always find a truncated conical region $K_{h,\omega}(v,\alpha)$ such that x lies on its symmetry axis, which is the symmetry axis of $C_\omega(v,\alpha)$ where Ω is completely outside that truncated conical region.

Suppose that the domain Ω satisfies one of Conditions (5.1.1) and (5.1.2). For a fixed $x \in \Omega$, choose w, a mutual point of $\partial\Omega$ and ∂B, to be such that $d(x) = |x - w|$. Denote by B the appropriate test domain, i.e. a cone (Condition (5.1.1)) or truncated conical region (Condition (5.1.1)). Furthermore, by $d_u(x)$ we mean the distance from $x \in \Omega$ to $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\ \{|s| : x + su \notin \Omega\}, \tag{4}$$

and $\tilde{d}_u(x)$ the distance from $x \in \Omega$ to ∂B, in the direction u, i.e.

$$\tilde{d}_u(x) := \min\ \{|s| : x + su \in \partial B\}.$$

Finally, denote by $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ the angle at which the line segment representing $\tilde{d}_u(x)$ leaves ∂B to infinity.

The following two theorems are related to the Exterior Cone Condition.

Theorem (5.1.3) [207]: Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition (5.1.1) with some $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then for any $f \in C_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\mu(\alpha) \int_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 dx, \tag{5}$$

where

$$\mu(\alpha) = \frac{1}{4} \tan^2 \frac{\alpha}{2}. \tag{6}$$

Remark (5.1.4) [207]: For convex domains we have $\alpha = \frac{\pi}{2}$. In this case, the function $\mu(n,\alpha)$, given by (23), becomes

$$\mu\left(n, \frac{\pi}{2}\right) = \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta = \frac{1}{4} \text{ for any } n, \tag{7}$$

as expected for a convex case.

For $n = 3$, the function $\mu(n,\alpha)$, given by (23), becomes

$$\mu(3,\alpha)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}\cdot[(2\cot^2\alpha+1(1-\cos\alpha)-\cos\alpha]$$
$$=\frac{1}{4}\left[\left(\frac{2\cos^2\alpha+1-\cos^2\alpha}{1-\cos^2\alpha}\right)(1-\cos\alpha)-\cos\alpha\right]$$
$$=\frac{1}{4}\left[\frac{\cos^2\alpha+1-\cos^2\alpha-\cos^2\alpha}{1+\cos\alpha}\right]=\frac{1}{4}\left[\frac{1-\cos\alpha}{1+\cos\alpha}\right]$$
$$=\frac{1}{4}\tan^2\frac{\alpha}{2},$$

exactly as obtained in (6).

For the advantage of 'measuring how deep the dent' inside the domain is, let us consider domains $\Omega\subset\mathbb{R}^n$ that satisfy Condition(5.1.2).

Theorem (5.1.5) [207]: Suppose that the domain $\Omega\subset\mathbb{R}^3$ satisfies Condition (5.1.2). Then for any $f\in C_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(x,\alpha,h)\frac{|f(x)|^2}{(h+d(x))^2}dx+\int_\Omega \mu_2(x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f(x)|^2dx, \tag{8}$$

where

$$\mu_1(x;\alpha,h)=\frac{1}{4}\cos^3(\tan^{-1}(a(x)\tan\alpha)), \tag{9}$$

and

$$\mu_2(x,\alpha,h)=\frac{1}{4\sin^2\alpha}\left[3-\cos 2\left(\alpha-(\tan^{-1}(a(x)\tan\alpha))\right)-2\cos\left(2\alpha-(\tan^{-1}(a(x)\tan\alpha))\right)\right]\sin^2\frac{\tan^{-1}(a(x)\tan\alpha)}{2}, \tag{10}$$

with $a(x)=\frac{1}{1+\frac{d(x)}{h}}$.

Remark (5.1.6) [207]:

(i) If Ω is a convex domain then $\alpha=\frac{\pi}{2}$. Therefore, for convex domains with $a(x)\neq 0$, i.e. $h\to 0$, we have $\mu_1\left(x,\frac{\pi}{2},h\right)=0$ and $\mu_2\left(x,\frac{\pi}{2},h\right)=\frac{1}{4}$, thus the Hardy-type inequality (8) reproduces the well-known bound (see for instance [211]):

$$\frac{1}{4}\int_\Omega\frac{|f(x)|^2}{d(x)^2}dx\leq\int_\Omega|\nabla f(x)|^2dx. \tag{11}$$

(ii) As $\alpha\nearrow\frac{\pi}{2}$, the domain Ω approaches the convexity case, and hence it is natural to compare $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ given by (9) and (10) respectively, with their values for the convex case. To this end we use the Taylor expansion to expand $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ in powers of $\left(\frac{\pi}{2}-\alpha\right)$. Keeping in mind that for fixed h we have $\theta_0=\tan^{-1}(a(x)\tan\alpha)=\frac{\pi}{2}$ where $\alpha=\frac{\pi}{2}$. Consequently, for $\mu_1(x,\alpha,h)$, we have

$$\mu_1\left(x,\frac{\pi}{2},h\right)=0,\frac{\partial}{\partial\alpha}\mu_1\left(x,\frac{\pi}{2},h\right)=0,\frac{\partial^2}{\partial\alpha^2}\mu_1\left(x,\frac{\pi}{2},h\right)=0.$$

However,

$$\frac{\partial^3}{\partial\alpha^3}\mu_1\left(x,\frac{\pi}{2},h\right)=-\frac{3}{2a(x)^3}=-\frac{3(h+d(x))^3}{2h^3},\cdots \text{ and so on .}$$

Thus $\mu_1(x,\alpha,h)$ can be written as follows:

$$\mu_1(x,\alpha,h)=\frac{(h+d(x))^3}{4h^3}\left(\frac{\pi}{2}-\alpha\right)^3+O\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{12}$$

Similarly, $\mu_2(x,\alpha,h)$ can be written as follows:

$$\mu_2(x,\alpha,h)=\frac{1}{4}+\frac{1}{2}\left(\alpha-\frac{\pi}{2}\right)+O\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{13}$$

For $\alpha=\frac{\pi}{2}$, we have $\mu_1(x,\alpha,h)=0$ and $\mu_2(x,\alpha,h)=\frac{1}{4}$, thus we obtain the same bound as in (11). Relations (12) and (13) show that the second term in inequality (8) is the effective term when talking about the convex case, since $\mu_1(x,\alpha,h)$ decays rapidly to zero while $\mu_2(x,\alpha,h)$ tends to $\frac{1}{4}$, when α tends to $\frac{\pi}{2}$.

Remark (5.1.7) [207]:

(i) If Ω is a convex domain then $\alpha=\theta_0=\frac{\pi}{2}$. Consequently, the Hardy-type inequality (33) reproduces the well-known bound (11) for any convex domain $\Omega\subset\mathbb{R}^n$.

(ii) When $\alpha\nearrow\frac{\pi}{2}$, the domain Ω approaches the convexity case. Therefore, it is natural to compare $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$, given by (34) and (35) respectively, with their values for the convex case. Keeping in mind that when $\alpha=\frac{\pi}{2}$ we set $\theta_0=\theta_0(x,\alpha)=\tan^{-1}(a(x)\tan\alpha)=\frac{\pi}{2}$, and for fixed h, expressions for $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$ can be written as powers of $\left(\alpha-\frac{\pi}{2}\right)$.

We find that, the function $\mu_1(n,x,\alpha,h)$ can be written as follows:

$$\mu_1(n,x,\alpha,h)=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\frac{(h+d(x))^3}{6h^3}\left(\frac{\pi}{2}-\alpha\right)^3+O\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{14}$$

On the other hand, the function $\mu_2\left(n,x,\frac{\pi}{2},h\right)$ can be written as follows:

$$\mu_2(n,x,\alpha,h)=\frac{1}{4}+\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(\alpha-\frac{\pi}{2}\right)+O\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{15}$$

Relations (14) and (15) show that the second term in Hardy-type inequality (33) is the effective term when talking about the convex case, since $\mu_1(n,x,\alpha,h)$ tends to zero while $\mu_2(n,x,\alpha,h)$ tends to $\frac{1}{4}$ as tends to $\frac{\pi}{2}$.

(iii) For fixed α, as h tends to $\infty, a(x)$ tends to 1 , which means implicitly that θ_0 tends to α. Therefore, we obtain the following limit for $\mu_2(n,x,\alpha,h)$ as h tends to ∞ :

$$\lim_{h\to\infty}\mu_2(n,x,\alpha,h) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(((n-1)\cot^2\alpha+1)\int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos(\alpha)\right) = \mu(n,\alpha). \quad (16)$$

Since all functions (f,μ_1,μ_2) are uniformly bounded, we can pass to the limit under the integral, thus the first term in Hardy-type inequality (33) tends to zero and we obtain the same result as in Theorem (5.1.9). On the other hand, as h tends to $0, a(x)$ tends to 0 , which leads to the tendency of θ_0 to 0 as well. This implies that $\mu_1(n,x,\alpha,h)\to\frac{1}{4}$ and $\mu_2(n,x,\alpha,h)\to 0$.

The key ingredient in proving Theorems (5.1.3), (5.1.9), (5.1.5) and (5.1.10) is the following proposition.

Proposition (5.1.8) [207]: (E. B. Davies, [210], [213]). Let Ω be a domain in $\mathbb{R}^n$ and let $f\in C_c^\infty(\Omega)$. Then

$$\frac{n}{4}\int_\Omega \frac{|f(x)|^2}{m(x)^2}dx \le \int_\Omega |\nabla f(x)|^2 dx,$$

where $m(x)$ is given by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{1}{d_u(x)^2}dS(u), \quad (17)$$

and

$$d_u(x) := \min\ \{|t| : x+tu\notin\Omega\},$$

for every unit vector $u\in\mathbb{S}^{n-1}$ and $x\in\Omega$. Here $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Our strategy to prove Theorems (5.1.3), (5.1.9), (5.1.5) and (5.1.10) is to obtain lower bounds for the function $\frac{1}{m(x)^2}$ given by (17), containing $d(x)$, then apply Proposition (5.1.8).

By (17) and the fact that $\tilde{d}_u(x)\ge d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int_{\mathbb{S}^n}\frac{1}{d_u(x)^2}dS(u) \ge \frac{1}{4\pi}\int_{\mathbb{S}^n}\frac{1}{\tilde{d}_u(x)^2}dS(u). \quad (18)$$

Since $\tilde{d}_u(x)$ is a symmetric function, with respect to the rotation about the symmetry axis of the cone $C_\omega(v,\alpha)$, then using spherical coordinates, (r,θ,ϕ) where $r\ge 0, 0\le\theta\le\pi$ and $0\le\phi\le 2\pi$, leads to $u=u(\theta,\phi)$, and $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$.

Therefore, inequality (18) becomes

$$\frac{1}{m(x)^2} \ge \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{\tilde{d}(x,\theta)^2}\sin\theta d\theta d\phi = \int_0^{2\pi}\frac{1}{\tilde{d}(x,\theta)^2}\sin\theta d\theta. \quad (19)$$

However, the angle θ can not exceed α, thus inequality (19) takes the following form:

$$\frac{1}{m(x)^2} \ge \int_0^{\alpha}\frac{1}{\tilde{d}(x,\theta)^2}\sin\theta d\theta\,. \quad (20)$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition (5.1.1) and if we consider the two-dimensional cross that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x,\theta)$, we conclude that

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)}.$$

Thus, the lower bound (20), on the function $\frac{1}{m(x)^2}$, can be written as follows:

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{\int_0^\alpha \sin^2(\alpha-\theta)\sin\theta d\theta}{d(x)^2\sin^2\alpha} \\ &= \frac{\int_0^\alpha (\sin\theta - \sin\theta\cos 2(\alpha-\theta))d\theta}{d(x)^2\sin^2\alpha} \\ &= \frac{(1-\cos\alpha)^2}{3d(x)^2(1-\cos^2\alpha)} = \frac{1}{3d(x)^2}\cdot\frac{1-\cos\alpha}{1+\cos\alpha} \\ &= \frac{\tan^2\frac{\alpha}{2}}{3d(x)^2}. \end{aligned} \tag{21}$$

Apply Proposition (5.1.8) to this lower bound in (21) to obtain the Hardy-type inequality (5) with $\mu(\alpha)$ as given in (6), this completes the proof.

Theorem (5.1.9) [207]: Suppose that the domain $\Omega \subset \mathbb{R}^n; n \geq 3$, satisfies Condition (5.1.1). Then for any function $f \in C_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\mu(n,\alpha)\int_\Omega \frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f(x)|^2 dx, \tag{22}$$

where

$$\mu(n,\alpha) = \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left(((n-1)\cot^2\alpha+1)\int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos\alpha\right). \tag{23}$$

Proof: By (17) and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u) \geq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{24}$$

Because of the definition of $\tilde{d}_u(x)$ and by using spherical coordinates, (r,θ,ϕ) where $r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, we have $u = (\theta,\phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (24) becomes

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|}\int_0^\pi \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} dw \\ &= 2\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\int_0^{\pi/2} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta. \end{aligned}$$

However, the angle θ can not exceed the value $\alpha < \frac{\pi}{2}$, hence

$$\frac{1}{m(x)^2} \geq \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\int_0^\alpha \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta. \tag{25}$$

Since Ω satisfies Condition (5.1.1), i.e., we have symmetry with respect to the axis of $C_\omega(v,\alpha)$, we consider the two-dimensional that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x,\theta)$, so we have

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)}.$$

Thus inequality (25) can be rewritten as follows:

$$\begin{aligned}\frac{1}{m(x)^2} &\geq \frac{2|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|d(x)^2\sin^2\alpha}\int_0^\alpha \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\ &= \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|d(x)^2\sin^2\alpha}\left(\int_0^\alpha \sin^{n-2}\theta d\theta - I_1(\alpha)\right),\end{aligned} \tag{26}$$

Where

$$I_1(\alpha) = \int_0^\alpha \sin^{n-2}\theta\cos 2(\alpha-\theta) d\theta.$$

There are many ways to evaluate $I_1(\alpha)$. Rewrite $I_1(\alpha)$ as follows

$$\begin{aligned}I_1(\alpha) &= \cos 2\alpha\left[\int_0^\alpha \sin^{n-2}\theta d\theta - 2\int_0^\alpha \sin^n\theta d\theta\right] \\ &+\frac{2}{n}\sin 2\alpha\sin^n\alpha \\ &= \cos 2\alpha\left[\int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha - \frac{2n-2}{n}\int_0^\alpha \sin^{n-2}\theta d\theta\right] + \frac{4}{n}\sin^{n+1}\alpha\cos\alpha \\ &= \frac{2-n}{n}\cos 2\alpha\int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha.\end{aligned} \tag{27}$$

Thus using (27), inequality (26) produces the following lower bound on $\frac{1}{m(x)^2}$:

$$\begin{aligned}\frac{1}{m(x)^2} &\geq \frac{|\mathbb{S}^{n-2}|}{nd(x)^2|\mathbb{S}^{n-1}|\sin^2\alpha}\Big[n\int_0^\alpha \sin^{n-2}\theta d\theta + (n-2)\cos 2\alpha\int_0^\alpha \sin^{n-2}\theta d\theta \\ &\quad -2\sin^{n-1}\alpha\cos\alpha] \\ &= \frac{|\mathbb{S}^{n-2}|}{nd(x)^2|\mathbb{S}^{n-1}|\sin^2\alpha}\left[(n+(n-2)\cos 2\alpha)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha\right] \\ &\frac{|\mathbb{S}^{n-2}|}{nd(x)^2|\mathbb{S}^{n-1}|\sin^2\alpha}\left[2\big((n-1)\cos^2\alpha+\sin^2\alpha\big)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha\right] \\ &= \frac{2|\mathbb{S}^{n-2}|}{d(x)^2 n|\mathbb{S}^{n-1}|}\left[\big((n-1)\cot^2\alpha+1\big)\int_0^\alpha \sin^{n-2}\theta d\theta \sin^{n-3}\alpha\cos\alpha\right].\end{aligned} \tag{28}$$

Applying Proposition (5.1.8) to the lower bound (28) putting into account that

$$\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}, \tag{29}$$

returns the Hardy-type inequality (22) with $\mu(n,\alpha)$ as in (23), this completes the proof.
By (17) and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{30}$$

Since the function $\tilde{d}_u(x)$ is symmetric, with respect to the rotation about the symmetry axis of the domain $K_{h,\omega}(v,\alpha)$, then using spherical coordinates, (r,θ,ϕ) where $r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta,\phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (30) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta d\theta d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta d\theta.$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition (5.1.2) and if we consider the two-dimensional cross that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}_u(x)$, we can divide the above interval into two intervals considering the relation between $\tilde{d}(x,\theta)$ and $d(x)$. Thus, for $\theta \in (0,\theta_0)$, the function $\tilde{d}(x,\theta)$ can be expressed in the following form:

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)}.$$

Besides, for $\theta \in \left(\theta_0, \frac{\pi}{2}\right)$, the function $\tilde{d}(x,\theta)$ can be written as follows

$$\tilde{d}(x,\theta) = \frac{h + d(x)}{\cos\theta},$$

where θ_0 satisfies

$$\tan\theta_0 = \frac{1}{1 + \frac{d(x)}{h}} \tan\alpha. \tag{31}$$

Moreover, for $\alpha = \frac{\pi}{2}$ (for which Ω attains the convex case) we have

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Thus, the function $\frac{1}{m(x)^2}$ is bounded from below as follows:

$$\frac{1}{m(x)^2} \geq \frac{\int_0^{\theta_0} \sin^2(\alpha-\theta)\sin\theta d\theta}{d(x)^2\sin^2\alpha} + \frac{\int_{\theta_0}^{\frac{\pi}{2}} \cos^2\theta \sin\theta d\theta}{(h + d(x))^2}.$$

Using the substitution $u = \cos\theta$ in the second integral produces

$$\frac{1}{m(x)^2} \geq -\cos\theta\big|_0^{\theta_0} - \frac{\frac{1}{2}\int_0^{\theta_0} (\sin(2\alpha-\theta) + \sin(3\theta - 2\alpha))d\theta}{2d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3(h+d(x))^2} \tag{32}$$

$$= 1 - \cos\theta_0 - \frac{\frac{\cos(2\alpha-\theta_0)}{2} + \frac{\cos(3\theta_0 - 2\alpha)}{6} + \frac{\cos(2\alpha)}{3}}{2d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3(h+d(x))^2}$$

$$= \frac{\left(3 - \cos 2(\alpha-\theta_0) - 2\cos(2\alpha-\theta_0)\right)\sin^2\frac{\theta_0}{2}}{3d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3(h+d(x))^2}.$$

Applying Proposition (5.1.8) to this lower bound in (32) leads to

$$\int_\Omega \mu_1^*(\theta_0)\frac{|f(x)|^2}{(h+d(x))^2}dx+\int_\Omega \mu_2^*(\theta_0,\alpha)\frac{|f(x)|^2}{d(x)^2}dx\le\int_\Omega |\nabla f(x)|^2dx,$$

Where

$$\mu_1^*(\theta_0)=\frac{\cos^3\theta_0}{4},$$

and

$$\mu_2^*(\theta_0,\alpha)=\frac{\big(3-\cos 2(\alpha-\theta_0)-2\cos(2\alpha-\theta_0)\big)\sin^2\frac{\theta_0}{2}}{4\sin^2\alpha}.$$

Now using (31), the relation between θ_0 and α, enables us to write $\mu_1^*(\theta_0)$ and $\mu_2^*(\theta_0,\alpha)$ as functions of x,α, and h as in (9) and (10) respectively. This completes the proof.

Theorem (5.1.10) [207]: Suppose that the domain $\Omega\subset\mathbb{R}^n; n\ge 3$, satisfies Condition (5.1.2). Then for any function $f\in C_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(n,x,\alpha,h)\frac{|f(x)|^2}{(h+d(x))^2}dx+\int_\Omega \mu_2(n,x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx \le\int_\Omega |\nabla f(x)|^2dx, \tag{33}$$

Where

$$\mu_1(n,x,\alpha,h) =\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}-(\sin^{n-1}\theta_0\cos\theta_0 +\int_{\theta_0}\sin^{n-2}\theta d\theta, \text{ and} \tag{34}$$

$$\mu_2(n,x,\alpha,h) =\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{1}{\sin^2\alpha}\left(((n-1)\cot^2\alpha+1)\int_0^{\theta_0}\sin^{n-2}\theta d\theta -\sin^{n-1}\theta_0\cos(2\alpha-\theta_0)\right), \tag{35}$$

with θ_0 satisfies $\tan\theta_0=\frac{h}{h+d(x)}\tan\alpha$. In particular, when $\alpha=\frac{\pi}{2}$, we have $\mu_1(n,x,\alpha,h)=0$ and $\mu_2(n,x,\alpha,h)=\frac{1}{4}$.

Proof: As have been illustrated before, the function $\frac{1}{m(x)^2}$, has the following lower bound

$$\frac{1}{m(x)^2}\ge 2\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\int_0^{\frac{\pi}{2}}\frac{1}{\tilde d(x,\theta)^2}\sin^{n-2}\theta d\theta\,. \tag{36}$$

Since Ω satisfies Condition (5.1.2), we consider containing the point $x\in\Omega$ and the line segments representing $d(x)$ and $\tilde d(x,\theta)$, then according to the relation between $\tilde d(x,\theta)$ and $d(x)$, we can rewrite inequality (36) as follows:

$$\frac{1}{m(x)^2} \geq 2b[I_1(n,\theta_0) + I_2(n,\theta_0)]; b = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}, \quad (37)$$

Where

$$I_1(n,\theta_0) = \int_0^{\theta_0} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta,$$

$$I_2(n,\theta_0) = \int_{\theta_0}^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta,$$

and $0 \leq \theta_0 < \frac{\pi}{2}$ satisfies

$$\tan\theta_0 = \frac{h}{h+d(x)} \tan\alpha.$$

However, for all angles $\alpha < \frac{\pi}{2}$, we can easily find that: For $\theta \in [0,\theta_0)$, the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)},$$

and for $\theta \in \left[\theta_0, \frac{\pi}{2}\right)$, we have

$$\tilde{d}(x,\theta) = \frac{h+d(x)}{\cos\theta}.$$

On the other hand, for $\alpha = \frac{\pi}{2}$ the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Therefore, we can evaluate the first integral $I_1(n,\theta_0)$ as follows:

$$\begin{aligned} I_1(n,\theta_0) &= \frac{1}{d(x)^2\sin^2\alpha} \int_0^{\theta_0} \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\ &= \frac{1}{d(x)^2\sin^2\alpha}\left(\int_0^{\theta_0} \sin^{n-2}\theta d\theta - I_3(n,\theta_0)\right), \end{aligned} \quad (38)$$

where

$$I_3(n,\theta_0) = \int_0^{\theta_0} \sin^{n-2}\theta\cos^2(\alpha-\theta)\, d\theta.$$

On the other hand, we can rewrite $I_3(n,\theta_0)$ as

$$\begin{aligned} I_3(n,\theta_0) &= \cos 2\alpha \int_0^{\theta_0} \sin^{n-2}\theta\cos 2\theta d\theta \\ &+ \sin 2\alpha \int_0^{\theta_0} \sin^{n-2}\theta\sin 2\theta d\theta \\ &= \cos 2\alpha\left[\int_0^{\theta_0} \sin^{n-2}\theta\cos^2\theta d\theta - \int_0^{\theta_0}\sin^n\theta d\theta\right] + 2\sin 2\alpha\int_0^{\theta_0}\sin^{n-1}\theta\cos\theta d\theta \\ &= \cos 2\alpha\left[\frac{2}{n}\sin^{n-1}\theta_0\cos\theta_0 + \frac{2-n}{n}\int_0^{\theta_0}\sin^{n-2}\theta d\theta\right] + \frac{2}{n}\sin 2\alpha\sin^n\theta_0 \end{aligned} \quad (39)$$

Substituting (39) into (38) produces

$$I_1(n,\theta_0)=\frac{1}{nd(x)^2\sin^2\alpha}\left(\frac{1}{2}(n(1+\cos 2\alpha)-2\cos 2\alpha)\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\cos 2\alpha\sin^{n-1}\theta_0\cos\theta_0\right.$$
$$\left.-\sin 2\alpha\sin^n\theta_0\right) \qquad (40)$$
$$=\frac{1}{nd(x)^2\sin^2\alpha}\left((n\cos 2\alpha-\cos 2\alpha)\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0(\cos\theta_0+\sin 2\alpha\sin\theta_0)\right)$$
$$=\frac{1}{nd(x)^2}\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\frac{\cos(2\alpha-\theta_0)}{\sin^2\alpha}\Big).$$

Concerning $I_2(n,\theta_0)$, we have

$$I_2(n,\theta_0)=\frac{1}{(h+d(x))^2}\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta\cos^2\theta d\theta$$
$$=\frac{1}{(h+d(x))^2}\left[\frac{\sin^{n-1}\theta\cos\theta}{n}\Bigg|_{\theta_0}^{\frac{\pi}{2}}+\frac{1}{n}\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right] \qquad (41)$$
$$=\frac{1}{(h+d(x))^2}\left[-\sin^{n-1}\theta_0\cos\theta_0+\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right].$$

Therefore, substituting (41) and (40) into (37) gives the following lower bound on the function $\frac{1}{m(x)^2}$:

$$\frac{1}{m(x)^2}\geq\frac{2b}{n}\left[\frac{1}{d(x)^2}\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\frac{\cos(2\alpha-\theta_0)}{\sin^2\alpha}\Big)\right.$$
$$\left.+\frac{1}{(h+d(x))^2}\left(\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\cos\theta_0\right)\right]\cdot \qquad (42)$$

Apply Proposition (5.1.8) to the lower bound (42) to obtain the Hardy-type inequality (33) where $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$ as stated in (34) and (35) respectively. On the other hand, when $\alpha=\frac{\pi}{2}$, we have $\theta_0=\frac{\pi}{2}$ as well, this implies

$$\mu_1(n,x,\alpha,h)=0, \text{ and}$$
$$\mu_2(n,x,\alpha,h)=\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta$$
$$=\frac{1}{4}\text{ for any }n.$$

This completes the proof.

Section (5.2): Functions Vanishing on a Part of the Boundary

Hardy's inequality is one of the classical items in analysis [240], [167]. Two milestones among many others in the development of the theory seem to be the result of Necas [85] that Hardy's inequality holds on strongly Lipschitz domains and the insight of Maz'ya [249], [250] that its validity depends on measure theoretic conditions on the domain. The geometric framework in which Hardy's inequality remains valid was that enlarged up to the frontiers of what is possible - as long as the boundary condition is purely Dirichlet, see [54], [82], compare also [57], [83], [88]. over the last years it became manifest Hardy's inequality plays an eminent role in modern PDE theory, see e.g. [219], [223], [225], [229], [44], [237], [243], [245], [252], [255].

What has not been treated systematically is the case where only a part D of the boundary of the underlying domain Ω is involved, reflecting the Dirichlet condition of the equation on this part while on $\partial\Omega \setminus D$ other boundary conditions may be imposed, compare [219], [224], [227], [238], [239] including references therein. The aim is to set up a geometric framework for the domain Ω and the Dirichlet boundary part D that allow to deduce the corresponding Hardy inequality.

$$\int_{\Omega} \left|\frac{u}{\mathrm{dist}_D}\right|^p \mathrm{dx} \leq c \int_{\Omega} |\nabla u|^p \mathrm{dx}.$$

As in the well established case $D = \partial\Omega$ we in essence only require that D is l-thick in the sense of [82]. This condition can be understood as an extremely weak compatibility condition between D and $\partial\Omega \setminus D$.

We reduce to the case $D = \partial\Omega$ by purely topological means, provided two major tools are applicable: An extension operator $\mathfrak{E}: W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$, the subscript D indicating the subspace of those Sobolev functions which vanish on D in an appropriate sense, and a Poincare inequality on $W_D^{1,p}(\Omega)$. This abstract result is established. In a second step these partly implicit conditions are substantiated by more geometric assumptions that can be checked - more or less - by appearance. In particular, we prove that under the mere assumption that D is closed, every linear continuous extension operator $W_D^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ that is constructed by the usual procedure of gluing together local extension operators preserves the Dirichlet condition on D. This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained in [220].

Whether Hardy's inequality also characterizes the space $W_D^{1,p}(\Omega)$, i.e. whether the latter is precisely the space of those functions $u \in W^{1,p}(\Omega)$ for which $u/\ \mathrm{dist}\ _D$ belongs to $L^p(\Omega)$. Under very mild geometric assumptions we answer this question to the affirmative.

Finally, we attend to the naive intuition that the part of $\partial\Omega$ that is far away from D should only be circumstantial for the validity of Hardy's inequality and in fact we succeed to weaken the previously discussed geometric assumptions considerably.

We work in Euclidean space $\mathbb{R}^d, d \geq 1$. We use x, y, etc. for vectors in $\mathbb{R}^d$ and denote the open ball in $\mathbb{R}^d$ around x with radius r by $B(\mathrm{x}, r)$. The letter c is reserved for generic constants that may change their value from occurrence to occurrence. Given $F \subset \mathbb{R}^d$ we write dist wor the sin function that measures the distance to F and $\mathrm{diam}(F)$ for the diameter of F.

In our main results on Hardy's inequality we denote the underlying domain and its Dirichlet part by Ω and D. The various side results that are interesting in themselves and drop off on the way are identified by the use of Λ and E instead.

We introduce the common first-order Sobolev spaces of functions 'vanishing' on a part of the closure of the underlying domain that are most essential for the formulation of Hardy's inequality.

Definition (5.2.1)[217]: If Λ is an open subset of $\mathbb{R}^d$ and E is a closed subset of $\bar{\Lambda}$, then for $p \in [1, \infty[$ the space $W_E^{1,p}(\Lambda)$ is defined as the completion of

$$C_E^\infty(\Lambda) := \{v|_\Lambda : v \in C_0^\infty(\mathbb{R}^d), \mathrm{supp}(v) \cap E = \emptyset\}$$

with respect to the norm $v \mapsto \left(\int_\Lambda |\nabla v|^p + |v|^p \mathrm{dx}\right)^{1/p}$. More generally, for $k \in \mathbb{N}$ we define $W_E^{k,p}(\Lambda)$ as the closure of $C_E^\infty(\Lambda)$ with respect to the norm $v \mapsto \left(\int_\Lambda \sum_{j=0}^k \left|D^j v\right|^p \mathrm{dx}\right)^{1/p}$.

The situation we have in mind is of course when $\Lambda = \Omega$ and $E = D$ is the Dirichlet part D of the boundary $\partial\Omega$.
As usual, the Sobolev spaces $W^{k,p}(\Lambda)$ are defined as the space of those $L^p(\Lambda)$ functions whose distributional derivatives up to order k are in $L^p(\Lambda)$, equipped with the natural norm. Note that by definition $W_0^{k,p}(\Lambda) = W_{\partial\Omega}^{k,p}(\Lambda)$ but in general $W_\emptyset^{k,p}(\Lambda) \subsetneq W^{k,p}(\Lambda)$, cf. [259, Sec. 1.1.6].
The following version of Hardy's inequality for functions vanishing on a part of the boundary is our main result.
Theorem(5.2.2) [217]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $D \subset \partial\Omega$ be a closed part of the boundary and $p \in]1,\infty[$. Suppose that the following three conditions are satisfied.
(i) The set D is l-thick for some $l \in]d-p, d]$.
(ii) The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\| \nabla \cdot \|_{L\ (\Omega)}$.
(iii) There is a linear continuous extension operator $\mathfrak{E}: W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$. Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \left|\frac{u}{\mathrm{dist}_D}\right|^p \mathrm{dx} \leq c \int_\Omega |\nabla u|^p \mathrm{dx} \tag{43}$$

holds for all $u \in W_D^{1,p}(\Omega)$.
Of course the conditions (ii) and (iii) in Theorem (5.2.2) are rather abstract and should be supported by more geometrical ones. This will be the content where we shall give an extensive kit of such conditions. In particular, we will obtain the following version of Hardy's inequality.
Theorem (5.2.3) [217]: (A special Hardy inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1,\infty[$. Let $D \subset \partial\Omega$ be l-thick for some $l \in]d-p, d]$ and assume that for every $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $W^{1,p}$-extension domain. Then there is a constant $c > 0$ such that

$$\int_\Omega \left|\frac{u}{\mathrm{dist}_D}\right|^p \mathrm{dx} \leq c \int_\Omega |\nabla u|^p \mathrm{dx},\ u \in W_D^{1,p}(\Omega).$$

Theorem (5.2.4) [217]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1,\infty[$. Let $D \subset \partial\Omega$ be porous and l-thick for some $l \in]d-p, d]$. Finally assume that for every $x \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $W^{1,p}$-extension domain. If $u \in W^{1,p}$ is such that $u/\mathrm{dist}_D \in L^p(\Omega)$, then already $u \in W_D^{1,p}(\Omega)$.
For convenience we recall the notions from geometric measure theory that are used to describe the regularity of the Dirichlet part D in Hardy's inequality. For $l \in]0,\infty[$ the l-dimensional Hausdorff measure of $F \subset \mathbb{R}^d$ is

$$\mathcal{H}_l(F) := \liminf_{\delta\to 0}\left\{\sum_{j=1}^{\infty} \mathrm{diam}(F_j)^l : F_j \subset \mathbb{R}^d, \mathrm{diam}(F_j) \leq \delta, F \subset \bigcup_{j=1}^{\infty} F_j\right\}$$

and its centered Hausdorff content is defined by

$$\mathcal{H}_l^\infty(F) := \inf \left\{\sum_{j=1}^{\infty} r_j^l : \mathrm{x}_j \in F, r_j > 0, F \subset \bigcup_{j=1}^{\infty} B(\mathrm{x}_j, r_j)\right\}.$$

Definition (5.2.5) [217]: Let $l \in]0,\infty[$. A non-empty compact set $F \subset \mathbb{R}^d$ is called l-thick if there exist $R > 0$ and $\gamma > 0$ such that

$$\mathcal{H}_l^\infty(F \cap B(\mathrm{x}, r)) \geq \gamma r^l \tag{44}$$

holds for all $x \in F$ and all $r \in]0, R]$. It is called l-set if there are two constants $c_0, c_1 > 0$ such that

$$c_0 r^l \leq \mathcal{H}_l(F \cap B(\mathrm{x}, r)) \leq c_1 r^l$$

holds for all $\mathrm{x} \in F$ and all $r \in]0,1]$.

Definition (5.2.6) [217]: A set $F \subset \mathbb{R}^d$ is porous if for some $\kappa \leq 1$ the following statement is true: For every ball $B(\mathrm{x}, r)$ with $\mathrm{x} \in \mathbb{R}^d$ and $0 < r \leq 1$ there is $y \in B(\mathrm{x}, r)$ such that $B(y, \kappa r) \cap F = \emptyset$.

Lemma (5.2.7) [217]: Let $l \in]0, \infty[$. If $F \subset \mathbb{R}^d$ is a compact l-set, then there are constants $c_0, c_1 > 0$ such that

$$c_0 r^l \leq \mathcal{H}_l^\infty(F \cap B(\mathrm{x}, r)) \leq c_1 r^l$$

holds for all $r \in]0,1[$ and all $\mathrm{x} \in F$. In particular, F is l-thick.

Proof: We prove $\mathcal{H}_l^\infty(A) \leq c\mathcal{H}_l(A) \leq c\mathcal{H}_l^\infty(A)$ for all non-empty Borel subsets $A \subset F$. First, fix $\varepsilon > 0$ and let $\{A_j\}_{j \in \mathbb{N}}$ be a covering of A by sets with diameter at most ε. If $A_j \cap A \neq \emptyset$, then A_j is contained in an open ball B_j centered in A and radius such that $r_j^l = \operatorname{diam}(A_j)^l + \varepsilon 2^{-j}$. The so-obtained countable covering $\{B_j\}$ of A satisfies

$$\sum_{\substack{j \in \mathbb{N} \\ A_j \cap A \neq \emptyset}} \operatorname{diam}(A_j)^l \geq \sum_{\substack{j \in \mathbb{N} \\ A_j \cap A \neq \emptyset}} \left(r_j^l - \varepsilon 2^{-j}\right) \geq \mathcal{H}_l^\infty(A) - \varepsilon.$$

Taking the infimum over all such coverings $\{A_j\}_{j \in \mathbb{N}}$ and passing to the limit $\varepsilon \to 0$ after wards, $\mathcal{H}_l^\infty(A) \leq \mathcal{H}_l(A)$ follows. Conversely, let $\{B_j\}_{j \in \mathbb{N}}$ be a covering of A by open balls with radii r_j centered in A. If $r_j \leq 1$, then $\mathcal{H}_l(F \cap B_j) \leq c r_j^l$ since by assumption F is an l-set, and if $r_j > 1$, then certainly $\mathcal{H}_l(F \cap B_j) \leq \mathcal{H}_l(F) r_j^l$. Note carefully that $0 < \mathcal{H}_l(F) < \infty$ holds for F can be covered by finitely many balls with radius 1 centered in F. Altogether,

$$\sum_{j=1}^{\infty} r_j^l \geq c \sum_{j=1}^{\infty} \mathcal{H}_l(F \cap B_j) \geq c\mathcal{H}_l\left(F \cap \bigcup_{j=1}^{\infty} B_j\right) \geq c\mathcal{H}_l(A).$$

Passing to the infimum, $\mathcal{H}_l^\infty(A) \geq c\mathcal{H}_l(A)$ follows.

Lemma (5.2.8) [217]: If $F \subset \mathbb{R}^d$ is l-thick, then it is m-thick for every $m \in]0, l[$.

Proof: Inspecting the definition of thick sets, the claim turns out to be a direct consequence of the inequality

$$\sum_{j=1}^{N} r_j^m \geq \left(\sum_{j=1}^{N} r_j^l\right)^{m/l}$$

for positive real numbers $r_1, \dots, r_N$.

The results rely on deep insights from potential theory and we shall recall the necessary notions beforehand. For further background see [218].

Definition (5.2.9) [217]: Let $\alpha > 0, p \in]1, \infty[$ and let $F \subset \mathbb{R}^d$. Denote by $G_\alpha := \mathcal{F}^{-1}((1 + |\xi|^2)^{-\alpha/2})$ the Bessel kernel of order α. Then

$$C_{\alpha,p}(F) := \inf \left\{ \int_{\mathbb{R}^d} |f|^p : f \geq 0 \text{ on } \mathbb{R}^d \text{ and } G_\alpha * f \geq 1 \text{ on } F \right\}$$

is called (α, p)-capacity of F. The corresponding Bessel potential space is

$$H^{\alpha,p}(\mathbb{R}^d) := \{G_\alpha * f : f \in L^p(\mathbb{R}^d)\} \text{ with norm } \|G_\alpha * f\|_{H^{\alpha,p}(\mathbb{R}^d)} = \| f \|_p.$$

It is well-known that for $k \in \mathbb{N}$ the spaces $H^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$ coincide up to equivalent norms [254]. The capacities $C_{\alpha,p}$ are outer measures on $\mathbb{R}^d$ [218]. A property that holds true for all x in some set $E \subset \mathbb{R}^d$ but those belonging to an exceptional set $F \subset E$ with $C_{\alpha,p}(F) = 0$ is said to be true (α,p)-quasieverywhere on E, abbreviated (α,p)-q.e. A property that holds true (α,p)-q.e. also holds true (β,p)-q.e. if $\beta < \alpha$. This is an easy consequence of [218]. A more involved result in this direction is the following [218]

Lemma (5.2.10) [217]: Let $\alpha,\beta > 0$ and $1 < p,q < \infty$ be such that $\beta q < \alpha p < d$. Then each $C_{\alpha,p}$- nullset also is a $C_{\alpha,p}$-nullset

There is also a close connection between capacities and Hausdorff measures, see [218] for an exhaustive discussion. Most important for us is the following comparison theorem. In the case $p \in]1,d]$ this is proved in [218] and if $p \in]d,\infty[$, then the result follows directly from [218].

Theorem (5.2.11) [217]: (Comparison Theorem) Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p < \infty$. Then every $C_{\alpha,p}$-nullset is also a $\mathcal{H}_l$ - and thus a $\mathcal{H}_l^\infty$-nullset.

Bessel capacities naturally occur when studying convergence of average integrals for Sobolev functions. In fact, if $\alpha > 0, p \in \left]1,\frac{d}{\alpha}\right]$ and $u \in H^{\alpha,p}(\mathbb{R}^d)$, then (α,p)-quasievery $y \in \mathbb{R}^d$ is a Lebesgue point for u in the L^p-sense, that is

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u(\mathrm{x})\mathrm{dx} =: \mathrm{u}(y) \tag{45}$$

and

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} |u(\mathrm{x}) - u(\mathrm{y})|^p \mathrm{dx} = 0 \tag{46}$$

hold [218]. The (α,p)-quasieverywhere defined function u reproduces u within its $H^{\alpha,p}$ class. It gives rise to a meaningful (α,p)-quasieverywhere defined restriction $u|_E := u|_E$ of u to E whenever E has non-vanishing (α,p)-capacity. For convenience we agree upon that $u|_E = 0$ is true for all $u \in H^{\alpha,p}(\mathbb{R}^d)$ if E has zero (α,p)-capacity. Note also that these results remain true if $p \in \left]\frac{d}{\alpha},\infty\right[$, since in this case u has a Hölder continuous representative u which then satisfies (45) and (46) for every $y \in \mathbb{R}^d$.

We obtain an alternate definition for Sobolev spaces with partially vanishing traces.

Definition (5.2.12) [217]: Let $k \in \mathbb{N}, p \in]1,\infty[$ and $E \subseteq \mathbb{R}^d$ be closed. Define

$$\mathcal{W}_E^{k,p}(\mathbb{R}^d) := \left\{u \in W^{k,p}(\mathbb{R}^d) : D^\beta u\big|_E = 0 \text{ holds } (k - |\beta|, p) - q.e. \text{ on } E \text{ for all multiindices } \leq |\beta| \leq k-1\right\}$$

and equip it with the $W^{k,p}(\mathbb{R}^d)$-norm.

The following theorem of Hedberg and Wolff is also called (k,p)-synthesis.

Theorem (5.2.13) [217]: ([218]) The spaces $W_E^{k,p}(\mathbb{R}^d)$ and $\mathcal{W}_E^{k,p}(\mathbb{R}^d)$ coincide whenever $k \in \mathbb{N}, p \in]1,\infty[$ and $E \subset \mathbb{R}^d$ is closed.

Hedberg and Wolff's theorem manifests the use of capacities in the study of traces of Sobolev functions. However, if one invests more on the geometry of E, e.g. if one assumes that it is an l-set, then by the subsequent recent result of Brewster, Mitrea, Mitrea and Mitrea capacities can be replaced by the l-dimensional Hausdorff measure at each occurrence.

Theorem (5.2.14) [217]: ([220]) Let $k \in \mathbb{N}, p \in]1,\infty[$ and let $E \subset \mathbb{R}^d$ be closed and additionally an l-set for some $l \in]d-p,d]$. Then

$$W_E^{k,p}(\mathbb{R}^d) = \mathcal{W}_E^{k,p}(\mathbb{R}^d) = \left\{u \in W^{k,p}(\mathbb{R}^d): D^\beta u\big|_E = 0 \text{ holds } \mathcal{H}_{d-1} - \text{ a.e. on } E \text{ for all multiindices } \beta, 0 \le |\beta| \le k-1\right\},$$

where on the right-hand side $D^\beta u\big|_E = 0$ means, as before, that for $\mathcal{H}_{d-1}$-almost every $y \in E$ the average integrals $\frac{1}{|B(y,r)|}\int_{B(y,r)} D^\beta u(\mathrm{x})\mathrm{dx}$ vanish in the limit $r \to 0$.
We will deduce Theorem (5.2.2) from the following proposition that states the assertion in the case $D = \partial\Omega$.

Proposition (5.2.15) [217]: ([82], see also [54]) Let $\Omega_* \subseteq \mathbb{R}^d$ be a bounded domain and let $p \in]1,\infty[$. If $\partial\Omega_*$ is l-thick for some $l \in]d-p, d]$, then Hardy's inequality is satisfied for all $u \in W_0^{1,p}(\Omega_*)$, i.e. (43) holds with Ω replaced by Ω_* and D by $\partial\Omega_*$.

Below we will reduce to the case $D = \partial\Omega$ by purely topological means, so that we can apply Proposition (5.2.15) afterwards. We will repeatedly use the following topological fact.

(■) Let $\{M_\lambda\}_\lambda$ be a family of connected subsets of a topological space. If $\cap_\lambda M_\lambda \neq \emptyset$, then $U_\lambda M_\lambda$ is again connected.

As required in Theorem (5.2.2) let now $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let D be a closed part of $\partial\Omega$. Then choose an open ball $B \supseteq \bar{\Omega}$ that, in what follows, will be considered as the relevant topological space. Consider

$$\mathcal{C} := \{M \subset B \setminus D: M \text{open, connected and } \Omega \subset M\}$$

and for the rest of the proof put

$$\Omega_. := \bigcup_{M \in \mathcal{C}} M.$$

In the subsequent lemma we collect some properties of Ω_0. Our proof here is not the shortest possible, cf. [5, Lem. 6.4] but it has, however, the advantage to give a description of $\Omega_.$ as the union of Ω, the boundary part $\partial\Omega \setminus D$ and those connected components of $B \setminus \bar{\Omega}$ whose boundary does not consist only of points from D. This completely reflects the naïve geometric intuition.

Lemma (5.2.16) [217]: It holds $\Omega \subseteq \Omega_. \subseteq B$. Moreover, $\Omega_.$ is open and connected and $\partial\Omega_{\mathbf{0}} = D$ in B

Proof: The first assertion is obvious. By construction $\Omega_.$ is open. Since all elements from $\mathcal{C}$ contain Ω the connectedness of $\Omega_{\mathbf{0}}$ follows by (□). It remains to show $\partial\Omega_. = D$.

Let $\mathrm{x} \in D$. Then x is an accumulation point of Ω and, since $\Omega \subseteq \Omega_{\mathbf{o}}$, also of $\Omega_\circ$. On the other hand, $x \notin \Omega_*$ by construction. This implies $x \in \partial\Omega_*$ and so $D \subseteq \partial\Omega_*$.

In order to show the inverse inclusion, we first show that points from $\partial\Omega \setminus D$ cannot belong to $\partial\Omega$. Indeed, since D is closed, for $\mathrm{x} \in \partial\Omega \setminus D$ there is a ball $B_\mathrm{x} \subseteq B$ around x that does not intersect D. Since x is a boundary point of Ω, we have $B_\mathrm{x} \cap \Omega \neq \emptyset$. Both Ω and B_x are connected, so (□) yields that $\Omega \cup B_\mathrm{x}$ is connected. Moreover, this set is open, contains Ω and avoids D, so it belongs to $\mathcal{C}$ and we obtain $\Omega \cup B_\mathrm{x} \subseteq \Omega_0$. This in particular yields $x \in \Omega_.$, so $x \notin \partial\Omega_.$ since $\Omega_.$ is open.

Summing up, we already know that $x \in \bar{\Omega}$ belongs to $\partial\Omega_.$ if and only if $x \in D$. So, it remains to make sure that no point from $B \setminus \bar{\Omega}$ belongs to $\partial\Omega_.$.

As $B \setminus \bar{\Omega}$ is open, it splits up into its open connected components $Z_0, Z_1, Z_2, \ldots$ There are possibly only finitely many such components but at least one. We will show in a first step that for all these components it holds $\partial Z_j \subseteq \partial\Omega$. This allows to distinguish the two cases

$\partial Z_j \subseteq D$ and $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$. In Steps 2 and 3 we will then complete the proof by showing that in both cases Z_j does not intersect $\partial\Omega_*$.

Step 1: $\partial Z_j \subseteq \partial\Omega$ for all

First note that $\partial Z_j \cap \Omega \neq \emptyset$ for all . Indeed, assuming this set to be non-empty and investing that Ω is open, we find that the set $Z_j \cap \Omega$ cannot be empty either and this contradicts the definition of Z_j. Now, to prove the claim of Step 1, assume by contradiction that, for some j, there is a point $x \in \partial Z_j$ that does not belong to $\partial\Omega$. By the observation above we then have $x \notin \bar{\Omega}$ and consequently there is a ball B_x around x that does not intersect $\bar{\Omega}$. Now, the set $B_x \cup Z_j$ is connected thanks to (5), avoids $\bar{\Omega}$ and includes Z_j properly. However, this contradicts the property of Z_j to be a connected component of $B \setminus \bar{\Omega}$

Step 2: If $\partial Z_j \subseteq D$, then $\bar{\Omega}. \cap Z_j = \emptyset$.

We first note that it suffices to show $\Omega. \cap Z_j = \emptyset$. In fact, due to $\bar{\Omega}. = \partial\Omega. \cup \Omega.$ we then get $\bar{\Omega}. \cap Z_j = \emptyset$ since Z_j is open.

So, let us assume there is some $x \in \Omega. \cap Z_j$. Then $\Omega. \cup Z_j$ is connected due to (□). By assumption we have $\partial Z_j \subseteq D$ and by construction the sets Z_j and $\Omega_.$ are both disjoint to D. So we can infer that $\partial Z_j \cap (\Omega_* \cup Z_j) = \emptyset$ and this allows us to write

$$\Omega_* \cup Z_j = (\Omega_* \cup Z_j) \cap \left(Z_j \cup (B \backslash \bar{Z}_j)\right) = Z_j \cup \left(\Omega_* \cap (B \backslash \bar{Z}_j)\right).$$

This is a decomposition of $\Omega_* \cup Z_j$ into two open and mutually disjoint sets, so if we can show that both are nonempty then this yields a contradiction to the connectedness of $\Omega_* \cup Z_j$ and the claim of Step 2 follows. Indeed, we even find

$$\Omega_* \cap (B \setminus \bar{Z}_j) = \Omega. \setminus \bar{Z}_j = \Omega_* \setminus (\partial Z_j \cup Z_j) \supset \Omega \setminus (D \cup Z_j) = \Omega \neq \emptyset,$$

since both D and Z_j do not intersect Ω.

Step 3: If $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$, then $Z_j \subseteq \Omega_*$.

Let $x \in \partial Z_j \cap (\partial\Omega \setminus D)$, and let B_x be a ball around x that does not intersect D. The point x is a boundary point of Z_j, so $B_x \cap Z_j \neq \emptyset$ and we obtain that $B_x \cup Z_j$ is connected by (□). By the same argument, also the set $B_x \cup \Omega$ is connected and putting these two together a third reiteration of the argument yields that $(B_x \cup \Omega) \cup (B_x \cup Z_j) = \Omega \cup B_x \cup Z_j$ is again connected. This last set is open and does not intersect D, so it belongs to $\mathcal{C}$ and we end up with $\Omega \cup B_x \cup Z_j \subseteq \Omega_*$. In particular we have $Z_j \subseteq \Omega_*$.

Remark (5.2.17) [217]: Conversely, it can be shown that the asserted properties characterize Ω • uniquely in the sense that if an open, connected subset $\Xi \supset \Omega$ of B additionally satisfies $\partial\Xi = D$, then necessarily $\Xi = \Omega_*$. In fact, since $\Xi \cap D = \emptyset$ one has $\Xi \subset \Omega_*$, due to the definition of Ω_*. In order to obtain the inverse inclusion we write

$$\Omega_* = (\Omega_* \cap \Xi) \cup (\Omega_* \cap \partial\Sigma) \cup \left(\Omega_* \cap (B \setminus \bar{\Xi})\right) = \Xi \cup \left(\Omega_* \cap (B \setminus \bar{\Xi})\right), \quad (47)$$

since $\Omega_* \cap \partial\Xi = \Omega_* \cap D = \emptyset$. Both $\Xi = \Xi \cap \Omega_*$ and $\Omega_* \cap (B \setminus \Xi)$ are open in $\Omega.$, and $\Xi \supset \Omega$ is nonempty.

Since Ω_* is connected and $\Xi = \cap\, \Omega_*$ is clearly disjoint to $\Omega_* \cap (B \setminus \bar{\Xi})$, this latter set must be empty. Thus, (47) gives $\Xi = \Omega_*$.

Corollary (5.2.18) [217]: Consider Ω_* as a subset of $\mathbb{R}^d$. Then Ω_* is open and connected. Moreover, either $\partial\Omega_* = D$ or $\partial\Omega_* = D \cup \partial B$.

Proof: It is clear that Ω_* remains open. Assume that Ω_* is not connected. Then there are disjoint open sets $U, V \subseteq \mathbb{R}^d$ such that $\Omega_* = U \cup V$. However, the property $\Omega_* \subseteq B$ then

gives $\Omega_* = \Omega_* \cap B = (U \cap B) \cup (V \cap B)$, where $U \cap B$ and $V \cap B$ are open in B and disjoint to each other. This contradicts Lemma (5.2.16).
For the last assertion consider an annulus $A \subseteq B$ that is adjacent to ∂B and does not intersect $\bar{\Omega}$. Let Z_j be the connected component of $B \setminus \bar{\Omega}$ that contains A. We distinguish again the two cases of Step 2 and Step 3 in the proof of Lemma (5.2.16): If $\partial Z_j \subseteq D$, we have shown in Step 2 that Z_j is disjoint to Ω_* and this implies $\partial\Omega_* = \partial\Omega_* \cap B = D$. In the second case, we infer from Step 3 in the above proof that $A \subseteq Z_j \subseteq \Omega_*$ and this implies
$\partial\Omega_* = D \cup \partial B$
We conclude the proof of Theorem(5.2.2).We first observe that in both cases appearing in Corollary (5.2.18) the set $\partial\Omega.$ is m-thick for some $m \in]d-p, d-1]$. In fact, D is l-thick for some $l \in]d-p, d]$ by assumption and using its local representation as the graph of a Lipschitz function, it can easily be checked that ∂B is a $(d-1)$-set, hence $(d-1)$-thick owing to Lemma (5.2.7). The claim follows from Lemma (5.2.8). Altogether, Proposition (5.2.15)
applies to our special choice of Ω_*.
Now, let $\mathfrak{G}$ be the extension operator provided by Assumption (iii) of Theorem (5.2.2). In view of Corollary (5.2.18) we can define an extension operator $\mathfrak{G}.: W_D^{1,p}(\Omega) \to W_0^{1,p}(\Omega_*)$ as follows: If $\partial\Omega_* = D$, then we put $\mathfrak{G}, v := \big(\mathfrak{b}v|_{\Omega.}$ and if $\partial\Omega. = D \cup \partial B$, then we choose $\eta \in C_0^\infty(B)$ with the property $\eta \equiv 1$ on $\bar{\Omega}$ and put $(\mathfrak{v}.v := (\eta\mathfrak{G}v)|_{\Omega.}$. This allows us to apply Proposition (5.2.15) to the functions $5.u \in W_0^{1,p}(\Omega.)$, where u is taken from $W_D^{1,p}(\Omega)$.
With a final help of Assumption (ii) in Theorem (5.2.2) this gives

$$\int_\Omega \left|\frac{u}{d_D}\right|^p \mathrm{dx} \le \int_\Omega \left|\frac{u}{d_{\partial\Omega_*}}\right|^p \mathrm{dx} \le \int_{\Omega_0} \left|\frac{\mathfrak{G}.u}{d_{\partial\Omega_*}}\right|^p \mathrm{dx} \le c\int_{\Omega_*} |\nabla(\mathfrak{G}.u)|^p \mathrm{dx} \le c \parallel \mathfrak{G}.u \parallel_{W_0^{1,p}(\Omega_*)}^p$$

$$\le c \parallel u \parallel_{W_D^{1,p}(\Omega)}^p \le c\int_{\Omega_*} |\nabla u|^p \mathrm{dx}$$

for all $u \in W_D^{1,p}(\Omega)$ and the proof is complete.
Corollary (5.2.19) [217]: The assertion of Theorem (5.2.2) remains valid if instead of its l-thickness we require that D is an l-set.
we discuss the second condition in our main result Theorem (5.2.2), that is the extendability for $W_D^{1,p}(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension.

- Dirichlet cracks can be removed:We open the possibility of passing from Ω to another domain $\Omega_\star$ with a reduced Dirichlet boundary part, while $\Gamma = \partial\Omega \setminus D$ remains part of $\partial\Omega_\star$. In most cases this improves the boundary geometry in the sense of Sobolev extendability, see the example in the following Figure.
- Sobolev extendability is a local property: We show that only the local geometry of the domain around the boundary part Γ plays a role for the existence of an extension operator.
- Preservation of traces: We prove under very general geometric assumptions that the extended functions do have the adequate trace behavior on D for every extension operator.

We believe that these results are of independent interest and therefore decided to directly present them for higher-order Sobolev spaces $W_E^{k,p}$. In the end we review some feasible commonly used geometric conditions which together with our abstract principles really imply the corresponding extendability.

As [vi] there may be boundary parts which carry a Dirichlet condition and belong to the inner of the closure of the domain under consideration. Then one can extend the functions on Λ by 0 to such a boundary part, thereby enlarging the domain and simplifying the boundary geometry.

Lemma (5.2.20) [217]: Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $E \subset \partial\Lambda$ be closed. Define $\Lambda_\star$ as the interior of the set $\Lambda \cup E$. Then the following hold true.

(i) The set $\Lambda_\star$ is again a domain, $\Xi := \partial\Lambda \setminus E$ is a (relatively) open subset of and $\partial\Lambda_\star = \Xi \cup (E \cap \partial\Lambda_\star)$.

(ii) Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Extending functions from $W_E^{k,p}(\Lambda)$ by 0 to $\Lambda_\star$, one obtains an isometric extension operator $\mathrm{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$.

Proof: (i) Due to the connectedness of Λ and the set inclusion $\Lambda \subset \Lambda_\star \subset \bar{\Lambda}$, the set $\Lambda_\star$ is also connected, and, hence a domain. Obviously, one has $\overline{\Lambda_\star} = \bar{\Lambda}$. This, together with the inclusion $\Lambda \subset \Lambda_\star$ leads to $\partial\Lambda_\star \subset \partial\Lambda$. Since $\Xi \cap \Lambda_\star = \emptyset$, one gets $\Xi \subset \partial\Lambda_\star$. Furthermore, Ξ was relatively open in $\partial\Lambda$, so it is relatively open also in $\partial\Lambda_\star$.

The last asserted equality follows from $\partial\Lambda_\star = (E \cap \partial\Lambda_\star) \cup (E \cap \partial\Lambda_\star)$ and $\Xi \subset \partial\Lambda_\star$

(ii) Consider any $\psi \in C_E^\infty(\mathbb{R}^d)$ and its restriction $\psi|_\Lambda$ to Λ. Since the support of ψ has a positive distance to E, one may extend $\psi|_\Lambda$ by 0 to the whole of $\Lambda_\star$ without destroying the C^∞-property. Thus, this extension operator provides a linear isometry from $C_E^\infty(\Lambda)$ onto $C_E^\infty(\Lambda_\star)$ (if both are equipped with the $W^{k,p}$-norm). This extends to a linear extension operator $\mathrm{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$, see the two following commutative diagrams:

Lemma (5.2.21) [217]: Let $k \in \mathbb{N}$ and $p \in]1, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain, let $E \subset \partial\Lambda$ be closed and as before define $\Lambda_\star$ as the interior of the set $\Lambda \cup E$. Every linear, continuous extension operator $\mathfrak{F}: W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$

factorizes as $\mathfrak{F} = \mathfrak{F}_\star \mathrm{Ext}(\Lambda, \Lambda_\star)$ through a linear, continuous extension operator $\mathfrak{F}_\star: W_E^{k,p}(\Lambda_\star) \to W_E^{k,p}(\mathbb{R}^d)$

Proof: Let S be the restriction operator from $W_E^{k,p}(\Lambda_\star)$ to $W_E^{k,p}(\Lambda)$. Then we define, for every $f \in W_E^{k,p}(\Lambda_\star), \mathfrak{F}_\star f := \mathfrak{F} \subseteq f$. We obtain $\mathfrak{F}_\star \mathrm{Ext}(\Lambda, \Lambda_\star) = \mathfrak{F} \subseteq \mathrm{Ext}(\Lambda, \Lambda_\star) = \mathfrak{F}$.

This shows that the factorization holds algebraically. However, one also has

$$\|\mathfrak{F}_\star \mathrm{Ext}(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\mathbb{R}^d)} = \| \mathfrak{F} f \|_{W_E^{k,p}(\mathbb{R}^d)} \leq \| \mathfrak{F} \|_{L\left(W_E^{k,p}(\Lambda); W_E^{k,p}(\mathbb{R}^d)\right)} \| f \|_{W_E^{k,p}(\Lambda)}$$

$$= \| \mathfrak{F} \|_{\mathcal{L}\left(W_E^{k,p}(\Lambda); W_E^{k,p}(\mathbb{R}^d)\right)} \|\mathrm{Ext}(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\Lambda_\star)} .$$

Having extended the functions already to $\Lambda_\star$ one may proceed as follows: Since E is closed, so is $E_\star := E \cap \partial\Lambda_\star$. So, one can now consider the space $W_{E_\star}^{k,p}(\Lambda_\star)$ and has the task to establish an extension operator for this space - while afterwards one has to take into account that the original functions were 0 also on the set $E \cap \Lambda_\star$ and have not been altered by the extension operator thereon. However, note carefully that $E_\star := E \cap \partial\Lambda_\star$ may have a worse geometry than E. For example, take Suppose that this time only Σ forms the whole Dirichlet part of the boundary. Then E is a $(d-1)$-set whereas even $\mathcal{H}_{d-1}(E_\star) = 0$ holds.

To sum up, if one aims at an extension operator $\mathfrak{E}: W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$, one is free to modify the domain Λ to $\Lambda_\star$. In most cases this improves the local geometry concerning Sobolev extensions and we do not have examples where the situation gets worse.

Definition (5.2.22) [217]: A domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$-extension domain for given $k \in \mathbb{N}$ and $p \in [1, \infty[$ if there exists a continuous extension operator $\mathfrak{E}_{k,p}: W^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$. If Λ is a $W^{k,p}$ extension domain for all $k \in \mathbb{N}$ and all $p \in [1, \infty[$ in virtue of the same extension operator, then Λ is a universal Sobolev extension domain.

Proposition (5.2.23) [217]: Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a $W^{k,p}$-extension domain. Then there is a continuous extension operator

$$\mathfrak{E}_{k,p}: W_E^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d).$$

Moreover, if each local extension operator $\mathfrak{E}_\mathrm{x}$ maps the space $W_{E_\mathrm{x}}^{k,p}(\Lambda \cap U_\mathrm{x})$ into $W_{E_\mathrm{x}}^{k,p}(\mathbb{R}^d)$, where $E_\mathrm{x} := \overline{E \cap E_\mathrm{x}} \subset \partial(\Lambda \cap U_\mathrm{x})$, then also

$$\mathfrak{E}_{k,p}: W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d).$$

Proof: For the construction of the extension operator let for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ denote U_x the open neighborhood of x from the assumption. Let $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$ be a finite subcovering of $\overline{\partial\Lambda \setminus E}$. Since the compact set $\overline{\partial\Lambda \setminus E}$ is contained in the open set $U_j U_{\mathrm{x}_j}$, there is an $\varepsilon > 0$, such that the sets $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$, together with the open set $U := \{y \in \mathbb{R}^d : \operatorname{dist}(y, \overline{\partial\Lambda \setminus E}) > \varepsilon\}$, form an open covering of $\bar{\Lambda}$. Hence, on $\bar{\Lambda}$ there is a C_0^∞-partition of unity $\eta, \eta_1, \dots, \eta_n$, with the properties $\operatorname{supp}(\eta) \subset U$, $\operatorname{supp}(\eta_j) \subset U_{\mathrm{x}_j}$

Assume $\psi \in C_0^\infty(\Lambda)$. Then $\eta\psi \in C_0^\infty(\Lambda)$. If one extends this function by 0 outside of Λ, then one obtains a function $\varphi \in C_{\partial\Lambda}^\infty(\mathbb{R}^d) \subset C_E^\infty(\mathbb{R}^d) \subset W_E^{k,p}(\mathbb{R}^d)$ with the property $\| \varphi \|_{W^{k,p}(\mathbb{R}^d)} = \| \eta\psi \|_{W^{k,p}(\Lambda)}$.

Now, for every fixed $j \in \{1, \dots, n\}$, consider the function $\psi_j := \eta_j \psi \in W^{k,p}\left(\Lambda \cap U_{\mathrm{x}_j}\right)$. Since $\Lambda \cap U_{\mathrm{x}_j}$ is a $W^{k,p}$-extension domain by assumption, there is an extension of ψ_j to a $W^{k,p}(\mathbb{R}^d)$-function φ_j together with an estimate $\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c \|\psi_j\|_{W^{k,p}\left(\Lambda \cap U_{\mathrm{x}_j}\right)}$, where c is independent from ψ. Clearly, one has a priori no control on the behavior of φ_j on the set $\Lambda \setminus U_{\mathrm{x}_j}$. In particular φ_j may there be nonzero and, hence, cannot be expected to coincide with $\eta_j \psi$ on the whole of Λ. In order to correct this, let ζ_j be a $C_0^\infty(\mathbb{R}^d)$-function which is identically 1 on $\operatorname{supp}(\eta_j)$ and has its support in U_{x_j}. Then $\eta_j \psi$ equals $\zeta_j \varphi_j$ on all of Λ. Consequently, $\zeta_j \varphi_j$ really is an extension of $\eta_j \psi$ to the whole of Rd which, additionally, satisfies the estimate

$$\|\zeta_j \varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c \|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c \|\eta_j \psi\|_{W^{k,p}\left(\Lambda \cap U_{\mathrm{x}_j}\right)} \le c \| \psi \|_{W^{k,p}(\Lambda)},$$

where c is independent from ψ. Thus, defining $\mathfrak{E}_{k,p}(\psi) = \varphi + \sum_j \zeta_j \varphi_j$ one gets a linear, continuous extension operator from $C_E^\infty(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$. By density, $\mathfrak{E}_{k,p}$ uniquely extends to a linear, continuous operator

$$\mathfrak{E}_{k,p}: W_E^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d).$$

Finally, assume that the local extension operators map $W^{k,p}_{E_{x_j}}\left(\Lambda \cap U_{x_j}\right)$ into $W^{k,p}_{E_{x_j}}(\mathbb{R}^d)$. Using the notation above, this means that φ_j can be approximated in $W^{k,p}(\mathbb{R}^d)$ by a sequence from $C^\infty_{E_{x_j}}(\mathbb{R}^d)$. Since ζ_j is supported in U_{x_j}, multiplication by $\zeta_j \in C_0^\infty(\mathbb{R}^d)$ maps $C^\infty_{E_{x_j}}(\mathbb{R}^d)$ into $C^\infty_E(\mathbb{R}^d)$ boundedly with respect to the $W^{k,p}(\mathbb{R}^d)$-topology. Hence, $\zeta_j\varphi_j \in W^{k,p}_E(\mathbb{R}^d)$. Since in any case $\varphi \in W^{k,p}_E(\mathbb{R}^d)$, the conclusion follows.

Proposition (5.2.23) allows to construct Sobolev extension operators from $W^{k,p}_D(\Omega)$ into $W^{k,p}(\mathbb{R}^d)$ and gives a sufficient condition for preservation of the Dirichlet condition. In this we prove that in fact every such extension operator has this feature. Recall that this is the crux of the matter in Assumption (iii) of Theorem (5.2.2). The key lemma is the following.

Lemma (5.2.24) [217]: Let $k \in \mathbb{N}$ and $p \in]1,\infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain, let $E \subset \partial\Lambda$ be closed and let $\mathfrak{E}_{k,p}: W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d)$ be a bounded extension operator. Any of the following conditions guarantees that $\mathfrak{E}_{k,p}$ in fact maps into $W^{k,p}_E(\mathbb{R}^d)$.

(i) For (k,p)-quasievery $y \in E$ balls around y in Λ have asymptotically nonvanishing relative volume, i.e.

$$\liminf_{r\to 0} \frac{|\, B(y,r) \cap \Lambda) \,|}{r^d} > 0. \tag{48}$$

(ii) The set E is an l-set for some $l \in]d-p,d]$ and (48) holds for $\mathcal{H}_l$-almost every $y \in E$.

(iii) There exists $q > d$ such that $\mathfrak{E}_{k,p}$ maps $C^\infty_E(\Lambda)$ into $W^{k,q}(\mathbb{R}^d)$.

Proof : As $C^\infty_E(\Omega)$ is dense in $W^{k,p}_E(\Lambda)$ and since $\mathfrak{E}_{k,p}$ is bounded, it suffices to prove that given $v \in C^\infty_E$ the function $u := \mathfrak{E}_{k,p}v$ belongs to $W^{k,p}_E(\mathbb{R}^d)$. The proof of (i) is inspired by [258]. Easy modifications of the argument will yield (ii) and (iii).

(i) Fix an arbitrary multiindex β with $|\beta| \le k-1$. Let $\mathcal{D}^\beta u$ be the representative of the distributional derivative $D^\beta u$ of u defined $(k-|\beta|,p)$-q.e. on $\mathbb{R}^d$ via

$$\mathcal{D}^\beta u(y) := \lim_{r\to 0} \frac{1}{|B(y,r)|}\int_{B(y,r)} D^\beta u(\mathrm{x})\mathrm{dx}.$$

Recall from (46) that then

$$\lim_{r\to 0} \frac{1}{|B(y,r)|}\int_{B(y,r)} \left|\mathcal{D}^\beta u(\mathrm{x}) - \mathcal{D}^\beta u(\mathrm{y})\right|\mathrm{dx}$$

$$\le \lim_{r\to 0}\left(\frac{1}{|B(y,r)|}\int_{B(y,r)} \left|\mathcal{D}^\beta u(\mathrm{x}) - \mathcal{D}^\beta u(\mathrm{y})\right|^p\mathrm{dx}\right)^{\frac{1}{p}} = 0. \tag{49}$$

holds for $(k-|\beta|,p)$-q.e. $y \in \mathbb{R}^d$. Since (48) holds for (k,p)-quasievery $y \in E$, it a fortiori holds for $(k-|\beta|,p)$-quasievery such y. Let now $N \subset \mathbb{R}^d$ be the exceptional set such that on $\mathbb{R}^d \setminus N$ the function $\mathcal{D}^\beta u$ is defined and satisfies (49) and such that (48) holds for every $y \in E \setminus N$. Owing to Theorem (5.2.13) the claim follows once we have shown $\mathcal{D}^\beta u(y)$ for all $y \in E \setminus N$.

For the rest of the proof we fix $y \in E \setminus N$. For $r > 0$ we abbreviate $B(r) := B(y,r)$ and define

$$W_j := \left\{\mathrm{x} \in \mathbb{R}^d \setminus N: \left|\mathcal{D}^\beta u(\mathrm{x}) - \mathcal{D}^\beta u(y)\right| > \frac{1}{j}\right\}. \tag{50}$$

Thanks to (49) for each $j \in \mathbb{N}$ we can choose some $r_j > 0$ such that $|B(r) \cap W_j| < 2^{-j}|B(r)|$ holds for all $r \in]0, r_j]$. Clearly, we can arrange that the sequence $\{r_j\}_j$ is decreasing. Now,

$$W := \bigcup_{j \in \mathbb{N}} \left\{ \left(B(r_j) \setminus B(r_{j+1}) \right) \cap W_j \right\} \tag{51}$$

has vanishing Lebesgue density at y, i.e. $r^{-d}|B(r) \cap W|$ vanishes as r tends to 0 : Indeed, if $r \in]r_{l+1}, rl]$, then

$$\begin{aligned} |B(r) \cap W| \leq & | \ B(r) \cap W_l) \cup \bigcup_{j \geq l+1} \left(B(r_j) \cap W_j \right) | \\ & \leq 2^{-l}|B(r)| + \sum_{j \geq l+1} 2^{-j}|B(r_j)| \leq 2^{-l+1}|B(r)|. \end{aligned}$$

Now, (48) allows to conclude

$$\liminf_{r \to 0} \frac{| \ B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W) \big) \ |}{r^d} > 0.$$

Since u is an extension of $v \in C_E^\infty(\Lambda)$ and y is an element of E it holds $\mathcal{D}^\beta u = 0$ a.e. on $B(r) \cap \Lambda$ with respect to the d-dimensional Lebesgue measure if $r > 0$ is small enough. The previous inequality gives $| \ B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W) \big) \ | > 0$ if $r > 0$ is small enough. In particular, there exists a sequence $\{\mathrm{x}_j\}_j$ in $\mathbb{R}^d \setminus W$ approximating y such that $\mathcal{D}^\beta u(\mathrm{x}_j) = 0$ for all $j \in \mathbb{N}$. Now, the upshot is that the restriction of $\mathcal{D}^\beta u$ to $\mathbb{R}^d \setminus W$ is continuous at y since if $\mathrm{x} \in \mathbb{R}^d \setminus W$ satisfies $|\mathrm{x} - y| \leq r_j$ then by construction $|\mathcal{D}^\beta u(\mathrm{x}) - \mathcal{D}^\beta u(\mathrm{y})|$. Hence, $\mathcal{D}^\beta u(\mathrm{y}) = 0$ and the proof is complete.

(ii) If E is an l-set for some $l \in]d - p, d]$, then we can appeal to Theorem (5.2.14) rather than Theorem (5.2.13) and the same argument as in (i) applies.

(iii) By assumption $u \in W_E^{k,p}(\mathbb{R}^d)$, where $q > d$. By Sobolev embeddings each distributional derivative $\mathcal{D}^\beta u, |\beta| \leq k - 1$, has a continuous representative $\mathcal{D}^\alpha u$. As each $y \in E \subset \partial\Lambda$ is an accumulation point of $\Lambda \setminus E$ and since $\mathcal{D}^\alpha u = \mathcal{D}^\alpha v$ holds almost everywhere on Λ, the representative $\mathcal{D}^\alpha u$ must vanish everywhere on E and Theorem (5.2.13) yields $u \in W_E^{k,p}(\mathbb{R}^d)$ as required.

Proposition (5.2.25) [217]: ([232]) If a domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$-extension domain for some $k \in \mathbb{N}$ and $p \in [1, \infty[$, then it is a d-set.

Theorem (5.2.26) [217]: Let $k \in \mathbb{N}$ and $p \in [1, \infty [$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a $W^{k,p}$-extension domain. Then there exists a continuous extension operator

$$\mathfrak{E}_{k,p}: W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d).$$

For the proof we recall the following result of Haiłasz, Koskela and Tuominen.

Proof. According to Proposition (5.2.23) it suffices to check that each local extension operator $\mathfrak{E}_\mathrm{x}$ maps $W_{E_\mathrm{x}}^{k,p}(\Lambda \cap U_\mathrm{x})$ into $W_{E_\mathrm{x}}^{k,p}(\mathbb{R}^d)$, where $E_\mathrm{x} := \overline{E \cap U_\mathrm{x}} \subset \partial(\Lambda \cap U_\mathrm{x})$. Owing to Proposition (5.2.25) the $W^{k,p}$-extension domain $\Lambda \cap U_\mathrm{x}$ is a d-set and as such satisfies (48) around every of its boundary points. So, Lemma (5.2.24).(i) yields the claim.

We finally review common geometric conditions on the boundary part $\overline{\partial\Lambda \setminus E}$ such that the local sets $\Lambda \cap U_{\mathrm{x}}$ really admit the Sobolev extension property required in Proposition (5.2.23).
A first condition, completely sufficient for the treatment of most real world problems, is the following Lipschitz condition.
Definition (5.2.27) [254]: A bounded domain $\Lambda \subset \mathbb{R}^d$ is called bounded Lipschitz domain if for each $\mathrm{x} \in \partial\Lambda$ there is an open neighborhood U_{x} of x and a bi-Lipschitz mapping ϕ_{x} from U_{x} onto a cube, such that $\phi_{\mathrm{x}}(\Lambda \cap U_{\mathrm{x}})$ is the (lower) half cube and $\partial\Lambda \cap U_{\mathrm{x}}$ is mapped onto the top surface of this half cube.
It can be proved by elementary means that bounded Lipschitz domains are $W^{1,p}$-extension domains for every $p \in [1, \infty[$, cf. e.g. [231] for the case $p = 2$. In fact, already the following (ε, δ)-condition of Jones [235] assures the existence of a universal Sobolev extension operator.
Definition (5.2.28) [217]: Let $\Lambda \subset \mathbb{R}^d$ be a domain and $\varepsilon, \delta > 0$. Assume that any two points $\mathrm{x}, y \in \Lambda$, with distance not larger than δ, can be connected within Λ by a rectifiable arc γ with length $l(\gamma)$, such that the following two conditions are satisfied for all points z from the curve γ :

$$l(\gamma) \leq \frac{1}{\varepsilon} \| \mathrm{x} - \mathrm{y} \|, \text{ and } \frac{\| \mathrm{x} - \mathrm{z} \| \| \mathrm{y} - \mathrm{z} \|}{\| \mathrm{x} - \mathrm{y} \|} \leq \frac{1}{\varepsilon} \mathrm{dist}(\mathrm{z}, \Lambda^c)$$

Then Λ is called (ε, δ)-domain .
Theorem (5.2.29) [217]: (Rogers) Each (ε, δ)-domain is a universal Sobolev extension domain.
Theorem (5.2.30) [217]: Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_{x} of x such that $\Lambda \cap U_{\mathrm{x}}$ is a bounded Lipschitz or, more generally, an (ε, δ)-domain for some values $\varepsilon, \delta > 0$. Then there exists a universal operator $\mathfrak{E}$ that restricts to a bounded extension operator $W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in]1, \infty[$.
we will discuss sufficient conditions for Poincare's inequality, thereby unwinding Assumption (ii) of Theorem (5.2.2). Our aim is not greatest generality as e.g. in [250] for functions defined on the whole of $\mathbb{R}^d$, but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense geometric conditions, which may be checked more or less 'by appearance' - at least for problems arising from applications.
The next proposition gives a condition that assures that a closed subspace of $W^{1,p}$ may be equivalently normed by the L^p-norm of the gradient of the corresponding functions only. We believe that this might also be of independent interest, compare also [258]. Throughout 1 denotes the function that is identically one.
Proposition (5.2.31) [217]: Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and suppose $p \in]1, \infty[$. Assume that X is a closed subspace of $W^{1,p}(\Lambda)$ that does not contain 1 and for which the restriction of the canonical embedding $W^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ to X is compact. Then X may be equivalently normed by $v \mapsto \left(\int_\Lambda |\nabla v|^p \mathrm{dx}\right)^{1/p}$
Proof First recall that both X and $L^p(\Lambda)$ are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence $\{v_k\}_k$ from X such that

$$\frac{1}{k} \|v_k\|_{L^p(\Lambda)} \geq \|\nabla v_k\|_{L^p(\Lambda)}.$$

After normalization we may assume $\|v_k\|_{L^p(\Lambda)} = 1$ for every $k \in \mathbb{N}$. Hence, $\{\nabla v_k\}_k$ converges to 0 strongly in $L^p(\Lambda)$. On the other hand, $\{v_k\}_k$ is a bounded sequence in X and hence contains a subsequence $\{v_{k_l}\}_l$, that converges weakly in X to an element $v \in X$. Since the gradient operator $\nabla : X \to L^p(\Lambda)$ is continuous, $\{\nabla v_{k_l}\}_l$ converges to ∇v weakly in $L^p(\Lambda)$. As the same sequence converges to 0 strongly in $L^p(\Lambda)$, the function ∇v must be zero and hence v is constant. But by assumption X does not contain constant functions except for $v = 0$. So, $\{v_{k_l}\}_l$ tends to 0 weakly in X. Owing to the compactness of the embedding $X \hookrightarrow L^p(\Lambda)$, a subsequence of $\{v_{k_l}\}_l$ tends to 0 strongly in $L^p(\Lambda)$. This contradicts the normalization condition $\|v_{k_l}\|_{L^p(\Lambda)} = 1$.

Lemma (5.2.32) [217]: Let $p \in]1, \infty[$, let Λ be a bounded domain and let $E \subset \partial\Lambda$ be l-thick for some $l \in]d - p, d]$. Both of the following conditions assure $\mathbb{1} \notin W_E^{1,p}(\Lambda)$.

(i) The set E admits at least one relatively inner point x. Here, 'relatively inner' is with respect to $\partial\Lambda$ as ambient topological space.

(ii) For every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is $aW^{1,p}$– extension domain.

Proof: We treat both cases separately.

(i) Assume the assertion was false and $\mathbb{1} \in W_E^{1,p}(\Lambda)$. Let x be the inner point of E from the hypotheses and let $B := B(\mathrm{x}, r)$ be a ball that does not intersect $\partial\Lambda \setminus E$. Put $\frac{1}{2}B := B\left(\mathrm{x}, \frac{r}{2}\right)$ and let $\eta \in C_0^\infty(B)$ be such that $\eta \equiv 1$ on $\frac{1}{2}B$.We distinguish whether or not x is an interior point of $\bar{\Lambda}$.

First, assume it is not. For every $\psi \in C_E^\infty(\Lambda)$ the function $\eta\psi$ belongs to $W_0^{1,p}(\Lambda \cap B)$ and as such admits a $W^{1,p}$-extension $\widehat{\eta\psi}$ by zero to the whole of $\mathbb{R}^d$. In particular,

$$\widehat{\eta\psi}(\mathrm{y}) = \begin{cases} \psi(y), & \text{if} \quad \mathrm{y} \in \frac{1}{2}B \cap \Lambda \\ 0, & \text{if} \quad \mathrm{y} \in \frac{1}{2}B \setminus \Lambda \end{cases}$$

and consequently,

$$\| \nabla\widehat{\eta\psi} \|_{L^p\left(\frac{1}{2}B\right)} = \| \nabla\psi \|_{L^p\left(\frac{1}{2}B \cap \Lambda\right)}.$$

Since by assumption $\mathbb{1}$ is in the $W^{1,p}(\Lambda)$-closure of $C_E^\infty(\Lambda)$ and since the mappings $W_E^{1,p}(\Lambda) \ni \psi \mapsto \nabla\widehat{\eta\psi} \in L^p\left(\frac{1}{2}B\right)$ and $W_E^{1,p}(\Lambda) \ni \psi \mapsto \nabla\psi \in L^p\left(\Lambda \cap \frac{1}{2}B\right)$ are continuous, the previous equation extends to $\psi = \mathbb{1}$:

$$\| \nabla\widehat{\eta\mathbb{1}} \|_{L^p\left(\frac{1}{2}B\right)} = \| \nabla\mathbb{1} \|_{L^p\left(\frac{1}{2}B \cap \Lambda\right)} = 0.$$

On the other hand x is not an inner point of $\bar{\Lambda}$ so that in particular $\frac{1}{2}B \setminus \bar{\Lambda}$ is nonempty. Since this set is open, $\left|\frac{1}{2}B \setminus \bar{\Lambda}\right| > 0$. Recall that by construction $\widehat{\eta\mathbb{1}} \in W^{1,p}(B)$ vanishes a.e. on $\frac{1}{2}B \setminus \bar{\Lambda}$. Hence, for some $c > 0$ the Poincare inequality

$$\| \widehat{\eta\mathbb{1}} \|_{L^p\left(\frac{1}{2}B\right)} \leq c \| \nabla\widehat{\eta\mathbb{1}} \|_{L^p\left(\frac{1}{2}B\right)},$$

holds, cf. [258]. However, we already know that the right hand side is zero, whereas the left hand side equals $\left|\frac{1}{2}B\cap\Lambda\right|^{1/p}$, which is nonzero since $\frac{1}{2}B\cap\Lambda$ is nonempty and open −a contradiction.
Now, assume x is contained in the interior of $\bar{\Lambda}$. Upon diminishing B we may assume $B\subset\bar{\Lambda}$. For every $\psi\in C_E^\infty(\mathbb{R}^d)$ we have $\eta\psi\in C_E^\infty(\mathbb{R}^d)$ with an estimate

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)}\leq c\,\|\psi\|_{W^{1,p}(B)}=c\left(\int_B|\psi|^p+|\nabla\psi|^p\mathrm{dx}\right)^{1/p}$$

for some constant $c>0$ depending only on η and p. By our choice of B split

$$B=B\cap\bar{\Lambda}=(B\cap\Lambda)\cup(B\cap\partial\Lambda)=(B\cap\Lambda)\cup(B\cap E).$$

Since ψ vanishes in a neighborhood of E,

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)}\leq c\left(\int_{B\cap\Lambda}|\psi|^p+|\nabla\psi|^p\mathrm{dx}\right)^{\frac{1}{p}}\leq c\,\|\psi\|_{W^{1,p}(\Lambda)}. \qquad (52)$$

Taking into account $\eta\equiv 1$ on $\frac{1}{2}B$, the same reasoning gives

$$\int_{\frac{1}{2}B}|\nabla(\eta\psi)|^p\mathrm{dx}=\int_{\frac{1}{2}B}|\nabla\psi|^p\mathrm{dx}\leq\int_\Lambda|\nabla\psi|^p\mathrm{dx}. \qquad (53)$$

By assumption there is a sequence $\{\psi_j\}_j\subset C_E^\infty(\Lambda)$ tending to $\mathbb{1}$ in the $W^{1,p}(\Lambda)$-topology.Due to (52) and the choice of η, the sequence $\{\eta\psi_j\}_j\subset C_E^\infty(\mathbb{R}^d)$ then tends to some $u\in CW_E^{1,p}(\mathbb{R}^d)$ satisfying $u=1$ a.e. on $\frac{1}{2}B\cap\Lambda$. Due to (53), $\nabla u=0$ a.e. on $\frac{1}{2}B$, meaning that u is constant on this set. Since $\frac{1}{2}B\cap\Lambda$ as a non-empty open set has positive Lebesgue measure, all this can only happen if $u=1$ a.e. on $\frac{1}{2}B$. Hence,

$$\lim_{r\to 0}\frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)}u\mathrm{dx}=1$$

for every $y\in\frac{1}{3}B\cap E$, which by Theorem (5.2.13) is only possible if $C_{1,p}\left(\frac{1}{3}B\cap E\right)=0$. By Theorem (5.2.13) this in turn implies $\mathcal{H}_l^\infty\left(\frac{1}{3}B\cap E\right)=0$ in contradiction to the l-thickness of E.
(ii) Again assume the assertion was false. Then by (i) there exists some $x\in E$ that is not an inner point of E with respect to $\partial\Lambda$. Hence x is an accumulation point of $\partial\Lambda\setminus E$ and by assumption there is a neighborhood $U=U_\mathrm{x}$ of x such that $\Lambda\cap U$ is a $W^{1,p}$ extension domain. We denote the corresponding extension operator by E. We shall localize the assumption $\mathbb{1}\in W_E^{1,p}(\Lambda)$ within U to arrive at a contradiction.
To this end, let $r_0>0$ be such that $\overline{B(\mathrm{x},r_0)}\subset U$ and let $\eta\in C_0^\infty(U)$ be such that $\eta\equiv 1$ on $B(\mathrm{x},r_0)$. Then also $\eta=\eta\mathbb{1}\in W_E^{1,p}(\Lambda)$ and in particular $\eta|_{\Lambda\cap U}$ belongs to $W_F^{1,p}(\Lambda\cap U)$, where $F:=\overline{B(\mathrm{x},r_0/2)\cap E}\subset\partial(\Lambda\cap\mathrm{U})$. Recall from That the $W^{1,p}$-extension domain $\Lambda\cap U$ satisfies in particular

$$\liminf_{r\to 0}\frac{|B(y,r)\cap\Lambda\cap U|}{r^d}>0.$$

around every $y\in\partial(\Lambda\cap U)$. Thus, Lemma (5.2.24)(i) yields $u:=\mathfrak{E}(\eta|_{\Lambda\cap U})\in W_F^{1,p}(\mathbb{R}^d)$. On the other hand, similar to the proof of Lemma (5.2.24) let u be the representative of u that is defined by limits of integral means on the complement of some exceptional set N

with $C_{1,p}(N) = 0$ and fix $y \in F \setminus N$. Take W as in (50) and (51). Repeating the arguments in the proof of Lemma (5.2.24) reveals that the restriction of u to $\mathbb{R}^d \setminus W$ is continuous at y and that $| B(y,r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W) | > 0$ if $r > 0$ is small enough. By construction $u = 1$ a.e. on $B(y,r) \cap \Lambda \cap U \cap (\mathbb{R}^d \setminus W)$ if $r < r_0$. Hence, there is a sequence $\{x_j\}_j$ approximating y such that $u(x_j) = 1$ for every $j \in \mathbb{N}$. By continuity $u(y) = 1$ follows. This proves that $u = 1$ holds $(1,p)$-quasieverywhere on F.
By Theorem (5.2.13) this can only happen if $C_{1,p}(F) = 0$, which as in (i) contradicts the l-thickness of E.
Proposition (5.2.33) [217]: Let $p \in]1,\infty[$ and let Λ be a bounded domain. Suppose that $E \subset \partial\Lambda$ is l-thick for some $l \in]d-p,d]$ and that for each $\mathrm{x} \in \partial\Lambda \setminus E$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a $W^{1,p}$-extension domain. Then $W_E^{1,p}(\Lambda)$ may equivalently be normed by $v \mapsto \left(\int_\Lambda |\nabla v|^p \mathrm{dx}\right)^{1/p}$
Now, also Theorem (5.2.3) follows. In fact, this result is just the synthesis of the above proposition with Theorems (5.2.2).
The strategy of proof is to write u as the sum of $v \in W^{1,p}(\Omega)$ with $v/\mathrm{dist}_{\partial\Omega} \in L^p(\Omega)$ and $w \in W^{1,p}$ with support within a neighborhood of $\overline{\partial\Omega \setminus D}$. Then v can be handled by the following classical result.
Proposition (5.2.34) [217]: ([226]) Let $\emptyset \subsetneq \Lambda \subsetneq \mathbb{R}^d$ be open and let $p \in]1,\infty[$. Then if $u \in W^{1,p}(\Lambda)$ and $u/\mathrm{dist}_{\partial\Lambda} \in L^p(\Omega)$, it follows $u \in W_0^{1,p}(\Lambda)$.
For w we can - since local extension operators are available - rely on the techniques developed. A key observation is an intrinsic relation between the property $\frac{u}{\mathrm{dist}_D} \in L^p(\Omega)$ and Sobolev regularity of the function $\log(\mathrm{dist}_D)$. In fact, a formal computation gives
$$\nabla\big(u\log(\mathrm{dist}_D)\big) = \log(\mathrm{dist}_D)\nabla u + \frac{u}{\mathrm{dist}_D}\nabla \mathrm{dist}_D.$$
Details are carried out in the following five consecutive steps.
Step 1: Splitting u and handling the easy term
As in the proof of Proposition (5.2.23) for every $\mathrm{x} \in \overline{\partial\Omega \setminus D}$, let U_x be the open neighborhood of x from the assumption, let $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$ be a finite subcovering of $\overline{\partial\Omega \setminus D}$ and let $\varepsilon > 0$ be such that the sets $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$, together with $U := \{y \in \mathbb{R}^d : \mathrm{dist}(y, \overline{\partial\Omega \setminus D}) > \varepsilon\}$, form an open covering of $\bar{\Omega}$. Finally, let $\eta, \eta_1, \dots, \eta_n$ be a subordinated C_0^∞-partition of unity.
The described splitting is $u = v + w$, where $v := \eta u$ and $w := \sum_{j=1}^n \eta_j u = (1-\eta)u$. Since
$$\mathrm{dist}_{\partial\Omega}(\mathrm{x}) \geq \min\ \{\varepsilon, \mathrm{dist}_D(\mathrm{x})\} \geq \min\ \{\varepsilon \mathrm{diam}(\Omega)^{-1}, 1\} \cdot \mathrm{dist}_D(\mathrm{x})$$
holds for every $\mathrm{x} \in \mathrm{supp}(\eta) \cap \Omega$, the function $v \in W^{1,p}(\Omega)$ satisfies
$$\int_\Omega \left|\frac{v}{\mathrm{dist}_{\partial\Omega}}\right|^p \mathrm{dx} \leq \mathrm{c}\int_\Omega \left|\frac{v}{\mathrm{dist}_D}\right|^p \mathrm{dx} \leq \mathrm{c}\int_\Omega \left|\frac{u}{\mathrm{dist}_D}\right|^p \mathrm{dx} < \infty$$
by assumption on u. Now, Proposition (5.2.34) yields $v \in W_0^{1,p}(\Omega) \subset W_D^{1,p}(\Omega)$. By assumption the sets $\Omega \cap U_{\mathrm{x}_j}$, $1 \leq j \leq n$, are $W^{1,p}$-extension domains. Since $w = (1-\eta)u$, where $(1-\eta)$ has compact support in the union of these domains, an extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ of $w \in W^{1,p}(\Omega)$ with compact support within $\bigcup_{j=1}^n U_{\mathrm{x}_j}$ can be constructed just as in the proof of Proposition (5.2.23). Now, if we can show $w \in W_D^{1,p}(\Omega)$, then by Step 1 also $u = v + w$ belongs to this space.

Step 3: Estimating the trace of $\widehat{w}$

To prove $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$ we rely once more on the techniques used in the proof of Lemma (5.2.24). So, let $\widehat{\omega}$ be the representative of $\widehat{w}$ defined on $\mathbb{R}^d \setminus N$ via

$$\widehat{\omega}(\mathrm{y}) := \lim_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} \widehat{w}\,\mathrm{dx},$$

where the exceptional set N is of vanishing $(1,p)$-capacity. Put

$$U_\star := \bigcup_{j=1}^{n} U_{\mathrm{x}_j},\ \Omega_\star := \Omega \cap U_\star,\ \text{ and }\ D_\star = \overline{D \cap U_\star} \subseteq \partial\Omega_\star.$$

Since $\widehat{w}$ has support in $U_\star$ it holds $\widehat{\omega}(\mathrm{y}) = 0$ for every $y \in D \setminus D_\star$. For the rest of the step let $y \in D_\star \setminus N$.

By Proposition (5.2.25) each set $\Omega \cap U_{\mathrm{x}_j}$ is a d-set and it can readily be checked that this property inherits to their union $\Omega_\star$. Hence, $\Omega_\star$ satisfies the asymptotically nonvanishing relative volume condition (48) around y with a lower bound $c > 0$ on the limes inferior that is independent of y and - just as in the proof of Lemma (5.2.24) - a set $W \subset \mathbb{R}^d$ can be constructed such that the restriction of $\widehat{\omega}$w to $\mathbb{R}^d \setminus W$ is continuous at y and such that $\left|B(y,r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)\right| \geq cr^d/2$ if $r > 0$ is small enough. By these properties of W :

$$\begin{aligned}|\widehat{\omega}(\mathrm{y})| &= \left|\lim_{r\to 0} \frac{1}{|B(y,r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)|}\int_{B(y,r)\cap\Omega_\star\cap(\mathbb{R}^d\setminus W)} \widehat{\omega}\,\mathrm{dx}\right| \\ &\leq \limsup_{r\to 0} \frac{2}{cr^d}\int_{B(y,r)\cap\Omega_\star} |\widehat{w}|\,\mathrm{dx} \\ &= \limsup_{r\to 0} \frac{2}{cr^d}\int_{B(y,r)\cap\Omega_\star} |w|\,\mathrm{dx}.\end{aligned}$$

In order to force these mean-value integral to vanish in the limit $r \to 0$, introduce the function $\log(\,\mathrm{dist}_{\,D})^{-1}$, which is bounded above in absolute value by $|\log r|^{-1}$ on $B(y,r)$ if $r < 1$. It follows

$$|\widehat{\omega}(\mathrm{y})| \leq c\limsup_{r\to 0} |\log r|^{-1}\left(\frac{1}{r^d}\int_{B(y,r)\cap\Omega_\star} |\mathrm{w}\log(\mathrm{dist}_D)|\mathrm{dx}\right). \qquad (54)$$

So, since $|\log r|^{-1} \to 0$ as $r \to 0$ the function $\widehat{\omega}$ vanishes at every $y \in D_\star \setminus N$ for which the mean value integrals on the right-hand side remain bounded as r tends to zero.

Step 4: Intermezzo on $w\log(dist_D)$

In this step we prove the following result.

Lemma (5.2.35) [217]: Let $p \in]1,\infty[$, let $\Lambda \subset \mathbb{R}^d$ be a bounded d-set, and let $E \subset \partial\Lambda$ be closed and porous. Suppose $u \in W^{1,p}(\Lambda)$ has an extension $v \in W^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{\mathrm{dist}_{\,E}} \in L^p(\Lambda)$. Ifr $\in$] $1,p[$ and $s \in]0,1[$, then the function $|u\log(\mathrm{dist}_E)|$ defined on Λ has an extension in the Bessel potential space $H^{s,p}(\mathbb{R}^d)$ that is positive almost everywhere.

For the proof we need the following extension result of Jonsson and Wallin.

Proposition (5.2.36) [217]: ([236]) Let $s \in]0,1[, p \in]1,\infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a d set. Then there exists a linear operator $\mathfrak{E}$ that extends every measurable function f on Λ that satisfies

$$\| f \|_{L^p(\Lambda)} + \left(\iint_{\substack{\mathrm{x},\mathrm{y}\in\Lambda\\ |\mathrm{y}|<1}} \frac{|f(\mathrm{x}) - f(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{d+sp}}\mathrm{dxdy}\right)^{1/p} < \infty$$

to a function $\mathfrak{E}f$ in the Besov space $B_s^{p,p}(\mathbb{R}^d)$ of all measurable functions g on $(\mathbb{R}^d)$ such that

$$\| g \|_{L^p(\mathbb{R}^d)} + \left(\iint_{x,y\in\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x-y|^{d+sp}} dxdy \right)^{1/p} < \infty.$$

Remark (5.2.37) [217]: The Besov spaces are nested with the Bessel potential spaces in the sense that $B_s^{p,p}(\mathbb{R}^d) \subset H^{s-\varepsilon,p}(\mathbb{R}^d)$ for each $s > 0$ and every $\varepsilon \in]0, s[$. Moreover, $W^{1,p}(\mathbb{R}^d) \subset B_s^{p,p}(\mathbb{R}^d)$. Proofs of these results can be found e.g. in [254].

Using Remark (5.2.37) it suffices to construct an extension in $B_s^{p,p}(\mathbb{R}^d)$ with the respective properties. Moreover, by the reverse triangle inequality it is enough to construct any extension $f \in B_s^{p,p}(\mathbb{R}^d)$ of $u\log \text{dist}_E$ - then $|f|$ can be used as the required extension of | u log dist |. These considerations and Proposition (5.2.36) show that the claim follows provided

$$\|u\log(\text{dist}_D)\|_{L^r(\Lambda)} + \left(\iint_{x,y\in\Lambda} \frac{|u(x)\log(\text{dist}_E(x)) - u(y)\log(\text{dist}_E(x))|^r}{|x-y|^{d+sr}} dxdy \right)^{\frac{1}{r}} \quad (55)$$

is finite.

To bound the L^r norm on the left-hand side of (55) choose $q \in]1, \infty[$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder's inequality

$$\|u\log(\text{dist}_E)\|_{L^r(\Lambda)} \leq \| u \|_{L^p(\Lambda)} \|\log(\text{dist}_D)\|_{L^q(\Lambda)}.$$

For the second term on the right-hand we use that the Aikawa dimension of the porous set E is strictly smaller than d. This entails for some $\alpha < d$ and some $x \in E$ the estimate

$$\int_\Lambda \text{dist}_E(x)^{\alpha-d} dx \leq \int_{B(x,2\,\text{diam}\,\Lambda)} \text{dist}_E(x)^{\alpha-d} dx \leq c_\alpha (2\,\text{diam}\,\Lambda)^\alpha < \infty.$$

Hence, some negative power of dist_E is integrable on Λ and by subordination of logarithmic growth $\log(\text{dist}_E) \in L^q(\Lambda)$ follows. Altogether, $u\log(\text{dist}_E) \in L^r(\Lambda)$ taking care of the first term in (55). By symmetry the domain of integration for the second term on the left-hand side of (55) can be restricted $\text{dist}_E(x) \geq \text{dist}_E(y)$. By adding and subtracting the term $u(y)\log(\text{dist}_E(x))$ it in fact suffices to prove that

$$\left(\int_\Lambda \int_\Lambda \frac{|u(x) - u(y)|^r}{|x-y|^{d+sr}} |\log(\text{dist}_E(x))|^r dxdy \right)^{\frac{1}{r}} \quad (56)$$

and

$$\left(\int_\Lambda |u(y)|^r \int_{\substack{x\in\Lambda \\ \text{dist}_E(x)\geq\text{dist}_E(y)}} \frac{|\log(\text{dist}_E(x)) - \log(\text{dist}_E(y))|^r}{|x-y|^{d+sr}} dxdy \right)^{\frac{1}{r}} \quad (57)$$

are finite. Fix $s < t < 1$, write (56) in the form

$$\left(\int_\Lambda \int_\Lambda \frac{|u(x) - u(y)|^r}{|x-y|^{dr/p+tr}} \frac{|\log(\text{dist}_E(x))|^r}{|x-y|^{dr/q+sr-tr}} dxdy \right)^{1/r}$$

and apply Hölder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ to bound it by

$$\leq \left(\int_\Lambda\int_\Lambda \frac{|u(\mathrm{x})-u(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{d+tp}}\mathrm{dxdy}\right)^{1/p}\left(\int_\Lambda\int_\Lambda \frac{|\log(\mathrm{dist}_E(\mathrm{x}))|^q}{|\mathrm{x}-\mathrm{y}|^{d+(s-t)q}}\mathrm{dydx}\right)^{1/q}$$

$$\leq \|\log(\mathrm{dist}_E)\|_{L^q(\Lambda)}\left(\int_\Lambda\int_\Lambda \frac{|u(\mathrm{x})-u(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{d+tp}}\mathrm{dxdy}\right)^{1/p}\left(\int_{|\mathrm{y}|\leq\mathrm{diam}(\Lambda)} \frac{1}{|\mathrm{y}|^{d+(s-t)q}}\mathrm{dy}\right)^{1/q}$$

Now, $\log(\mathrm{dist}_E)\in L^q(\Lambda)$ has been proved above and the third integral is absolutely convergent since $d+(s-t)q<d$. Finally note that by assumption u has an extension $v\in W^{1,p}(\mathbb{R}^d)$. Since $W^{1,p}(\mathbb{R}^d)\subset B_s^{p,p}(\mathbb{R}^d)$ the middle term above is finite as well, see Remark (5.2.37).

It remains to show that the most interesting term (57) is finite. Here, the additional assumptions on u, s, and r enter the game. By the mean value theorem for the logarithm and since dist_E is a contraction, the r-th power of this term is bounded above by

$$\int_\Lambda |u(\mathrm{y})|^r \int_{\substack{\mathrm{x}\in\Lambda \\ \mathrm{dist}_E(\mathrm{x})\geq\mathrm{dist}_E(\mathrm{y})}} \frac{|\mathrm{dist}_E(\mathrm{x})-\mathrm{dist}_E(\mathrm{y})|^r}{\mathrm{dist}_E(\mathrm{y})^r|\mathrm{x}-\mathrm{y}|^{d+sr}}\mathrm{dxdy}$$

$$\leq \int_\Lambda \left|\frac{u(\mathrm{y})}{\mathrm{dist}_E(\mathrm{y})}\right|^r\int_\Lambda \frac{|\mathrm{x}-\mathrm{y}|^r}{|\mathrm{x}-\mathrm{y}|^{d+sr}}\mathrm{dxdy}$$

$$\leq \int_\Lambda \left|\frac{u(\mathrm{y})}{\mathrm{dist}_E(\mathrm{y})}\right|^r \mathrm{dy}\int_{|\mathrm{x}|\leq\mathrm{diam}(\Lambda)} \frac{1}{|\mathrm{x}|^{d+r(s-1)}}\mathrm{dx}.$$

Now, the integral with respect to x is finite since $r(s-1),0$. The integral with respect to y is finite since by assumption $\frac{u}{\mathrm{dist}_E}$ is p-integrable on the bounded domain Λ and thus r-integrable for every $r<p$.

On noting that by Definition (5.2.6) a subset of a porous set is again porous, Lemma (5.2.35) applies to the bounded d-set $\Omega_\star$ and the porous set $D_\star\subset D$. Moreover, $w=(1-\eta)u\in W^{1,p}(\Omega_\star)$ has the extension $\widehat{W}\in W^{1,p}(\mathbb{R}^d)$ and satisfies

$$\int_{\Omega_*}\left|\frac{u(\mathrm{x})}{\mathrm{dist}_{D_*}(\mathrm{x})}\right|^p \mathrm{dx}\leq\| 1-\eta \|_\infty \int_\Omega \left|\frac{u(\mathrm{x})}{\mathrm{dist}_D(\mathrm{x})}\right|^p<\infty.$$

Hence we can record:

Corollary (5.2.38) [217]: For every $r\in]1,p[$ and every $s\in]0,1[$ the function $\left|w\log\left(\mathrm{dist}_{D_\star}\right)\right|$ defined on $\Omega_\star$ has an extension $f_{s,r}\in H^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.

Step 5: Re-inspecting the right-hand side of (54)

We return to (54). Given $r\in]1,p[$ and $s\in]0,1$ [let $f_{s,r}\in H^{s,r}(\mathbb{R}^d)$ be as in Corollary (5.2.38). By (46) we can infer

$$\limsup_{r\to 0}\frac{1}{r^d}\int_{B(y,r)\cap\Omega_\star}|w\log(\mathrm{dist}_D)|\mathrm{dx}\leq \limsup_{r\to 0}\frac{1}{r^d}\int_{B(y,r)} f_{s,r}\mathrm{dx}<\infty$$

for (s,r)-quasievery $y\in D_\star\setminus N$. By the conclusion of Step 3 this implies $\widehat{\omega}(y)=0$ for (s,r) quasievery $y\in D_\star\setminus N$. To proceed further, we distinguish two cases:

(i) It holds $p\leq d$. Since the product $sr<p\leq d$ can get arbitrarily close to p, Lemma (5.2.10) yields for every $r\in]1,p$ [that $\widehat{\omega}=0$ holds $(1,r)$-quasieverywhere on $D_\star\setminus N$. Moreover, since $C_{1,p}(N)=0$ by definition, $\widehat{\omega}=0$ holds even $(1,r)$-quasieverywhere on $D_\star$.

(ii) It holds $p > d$. Then $\hat{\omega}$ is the continuous representative of $\hat{\omega} \in W^{1,p}(\mathbb{R}^d)$ and N is empty, see the beginning of Step 3. Moreover, we can choose s and r such that $d - l < sr$ and conclude from the comparison theorem, Theorem (5.2.11), that $\hat{\omega}$ vanishes $\mathcal{H}_l^\infty$-a.e. on $D_\star$. Since D is l-thick and $U_\star$ is open, for each $y \in D \cap U_\star$ the set $B(y,r) \cap D \cap U_\star$ coincides with $B(y,r) \cap D$ provided $r > 0$ is small enough and thus has strictly positive $\mathcal{H}_l^\infty -$ measure. So, the continuous function $\hat{\omega}$ has to vanish every where on $D \cap U_\star$ as well as on the closure of the latter set - which by definition is $D_\star$.
Summing up, $\hat{\omega} = 0$ has been shown to hold $(1,r)$-quasieverywhere on $D_\star$ for every $r \in]1,p[$. From the beginning of Step 3 we also know that $\hat{\omega}$ vanishes everywhere on $D \setminus D_\star$ and as $\hat{\omega} \in W^{1,p}(\mathbb{R}^d)$ has compact support, Hölder's inequality yields $\hat{\omega} \in W^{1,p}(\mathbb{R}^d)$. Combining these two observations with Theorem (5.2.13) we are eventually led to

$$\hat{\omega} \in W^{1,p}(\mathbb{R}^d) \cap \bigcap_{1<r<p} W_D^{1,r}(\mathbb{R}^d). \tag{58}$$

We continue by quoting the following result of Hedberg and Kilpelainen.
Proposition (5.2.39) [217]: ([234]) Let $p \in]1,\infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a bounded domain whose boundary is l-thick for some $l \in]d-p,d]$. Then

$$W^{1,p}(\Lambda) \cap \bigcap_{1<r<p} W_0^{1,r}(\Lambda) \subset W_0^{1,p}(\Lambda).$$

If one asks: 'What is the most restricting condition in Theorem (5.2.2)?', the answer doubtlessly is the assumption that a global extension operator shall exist. Certainly, this excludes all geometries that include cracks not belonging completely to the Dirichlet boundary part as in Fig. [2].
Since the distance function dist $_D$ measures only the distance to the Dirichlet boundary part D, points in $\partial\Omega$ that are far from D should not be of great
relevance in view of the Hardy inequality (43). In the following considerations we intend to make this precise. Let $U, V \subset \mathbb{R}^d$ be two open, bounded sets with the properties

$$D \subset U,\ \bar{V} \cap D = \emptyset,\ \bar{\Omega} \subset U \cup V. \tag{59}$$

We take U as a small neighborhood of D which - desirably- excludes the 'nasty parts' of $\partial\Omega \setminus D$. More properties of U, V will be specified below.
Let $\eta_U \in C_0^\infty(U), \eta_V \in C_0^\infty(V)$ be two functions with $\eta_U + \eta_V = 1$ on $\bar{\Omega}$. Then one can estimate

$$\left(\int_\Omega |u|^p \mathrm{dist}_D^{-p}\,\mathrm{dx}\right)^{1/p} \le \left(\int_{U\cap\Omega} |\eta_U u|^p \mathrm{dist}_D^{-p}\,\mathrm{dx}\right)^{1/p} + \left(\int_{V\cap\Omega} |\eta_V u|^p \mathrm{dist}_D^{-p}\,\mathrm{dx}\right)^{1/p}.$$

Since $dist_D$ is larger than some $\varepsilon > 0$ on $\mathrm{supp}(\eta_V) \subset V$, the second term can be estimated by $\frac{1}{\varepsilon}\left(\int_\Omega |u|^p \mathrm{dx}\right)^{1/p}$. If one assumes, as above, Poincar'e's inequality, then this term may be estimated as required. In order to provide an adequate estimate also for the first term, we introduce the following assumption.
Assumption (5.2.40) [217]: The set U from above can be chosen in such a way that $\Lambda := \Omega \cap U$ is again a domain and if one puts $\Gamma := (\partial\Omega \setminus D) \cap U$ and $E := \partial\Lambda \setminus \Gamma$, then there is a linear, continuous extension operator $\mathfrak{F}: W_E^{1,p}(\Lambda) \to W_E^{1,p}(\mathbb{R}^d)$.
Clearly, this assumption is weaker than Condition (iii) in Theorem (5.2.2); in other words: Condition (iii) in Theorem (5.2.2) requires Assumption (5.2.40) to hold for an open set $U \supset \bar{\Omega}$.

We discuss the sense of Assumption (5.2.40) in extenso. Philosophically spoken, it allows to focus on the extension not of the functions u but the functions $\eta_U u$, which live on a set whose boundary does (possibly) not include the 'nasty parts' of $\partial\Omega \setminus D$ that are an obstruction against a global extension operator. In detail: one first observes that, for $\eta = \eta_U \in C_0^\infty(U)$ and $v \in W_D^{1,p}(\Omega)$, the function $\eta v|_\Lambda$ even belongs to $W_E^{1,p}(\Lambda)$, see [233]. Secondly, we have by the definition of E

$$\partial U \cap \Omega = (\partial U \cap \Omega) \setminus \Gamma \subset \partial\Lambda \setminus \Gamma = \mathrm{E}.$$

This shows that the 'new' boundary part $\partial U \cap \Omega$ of Λ belongs to E and is, therefore, uncritical in view of extension. Thirdly, one has $D = D \cap U \subseteq \partial\Omega \cap U \subset \partial\Lambda$, and it is clear that the 'new Dirichlet boundary part' E includes the 'old' one D. Hence, the extension operator $\mathfrak{F}$ may be viewed also as a continuous one between $W_E^{1,p}(\Lambda)$ and $W_D^{1,p}(\mathbb{R}^d)$. Thus, concerning $v := \eta u = \eta_U u$ one is - mutatis mutandis - again in the situation of Theorem (5.2.2): $\eta u \in W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$ admits an extension $\mathfrak{F}(\eta u) \in W_E^{1,p}(\mathbb{R}^d) \subseteq W_D^{1,p}(\mathbb{R}^d)$, which satisfies the estimate $\| \mathfrak{F}(\eta u) \|_{W_D^{1,p}(\mathbb{R}^d)} \leq c \| \eta u \|_{W_D^{1,p}(\Lambda)}$, the constant c being independent from u. This leads, as above, to a corresponding (continuous) extension operato $\mathfrak{F}_. : W_E^{1,p}(\Lambda) \to W_0^{1,p}(\Lambda_.)$. Here, of course, $\Lambda_.$ has again to be defined as the connected component of $B \setminus D$ that contains Λ. Thus onemay proceed again as in the proof of Theorem (5.2.2), and gets, for $u \in W_D^{1,p}(\Omega)$,

$$\int_\Omega \left(\frac{|\eta u|}{\mathrm{dist}_D}\right)^p \mathrm{dx} = \int_\Lambda \left(\frac{|\eta u|}{\mathrm{dist}_D}\right)^p \mathrm{dx} \leq \int_{\Lambda_.} \left(\frac{|\mathfrak{F}_.(\eta u)|}{\mathrm{dist}_{\partial\Lambda}}\right)^p \mathrm{dx} \leq c \|\nabla(\mathfrak{F}_.(\eta u))\|_{L^p(\Lambda_.)}^p$$
$$\leq c \|\mathfrak{F}_.(\eta u)\|_{W^{1,p}(\Lambda_.)}^p \leq c \| \eta u \|_{W^{1,p}(\Lambda)}^p \leq c \left(\| u \|_{L^p(\Omega)}^p + \| \nabla u \|_{L^p(\Omega)}^p \right),$$

since ηu belongs to $W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$. Exploiting a last time Poincar'e's inequality, whose validity will be discussed in a moment, one gets the desired estimate.

When aiming at Poincare's inequality, it seems convenient to follow again the argument in the proof of Proposition (5.2.31): as pointed out above, the property $\mathbb{1} \notin W_D^{1,p}(\Omega)$ has to do only with the local behavior of Ω around the points of D, cf. Lemma (5.2.32). Hence, this will not be discussed further here.

Concerning the compactness of the embedding $W_D^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, one does not need the existence of a global extension operator $\mathfrak{E}: W_D^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$. In fact, writing for every $v \in W_D^{1,p}(\Omega)$ again $v = \eta_U v + \eta_V v$ and supposing Assumption (5.2.40), one gets the following:

If $\{v_k\}_k$ is a bounded sequence in $W_D^{1,p}(\Omega)$, then the sequence $\{\eta_U v_k|_\Lambda\}_k$ is bounded in $W_E^{1,p}(\Lambda)$. Due to the extendability property, this sequence contains a subsequence $\left\{\eta_U v_{k_l}\big|_\Lambda\right\}_l$ that converges in $L^p(\Lambda)$ to an element η_U. Thus, $\left\{\eta_U v_{k_l}\right\}_l$ converges to the function on Ω that equals v_U on Λ and 0 on $\Omega \setminus \Lambda$. The elements $\eta_V v_k$ in fact live on the set II := $\Omega \cap V$ and are zero on $\Omega \setminus \Lambda$. In particular they are zero in a neighborhood of D. Moreover, they form a bounded subset of $W^{1,p}$ (II). Therefore it makes sense to require that II is again a domain, and, secondly that II meets one of the well-known compactness criteria $W^{1,p}$ (II) $\hookrightarrow L^p$ (II), cf. [250]. Keep in mind that such requirements are much weaker than the global $W^{1,p}$-extendability, and in particular include the example in Fig. 2, as long as the triangle Σ has a positive distance to the six outer sides of the cube. Resting on these criteria, one obtains again the convergence of a subsequence $\left\{\eta_V v_{k_l}\big|_{\mathrm{II}}\right\}_l$, that converges in L^p (II) towards a function v_V. The sequence $\{\eta_V v_{k_l}\}_l$, then converges in $L^p(\Omega)$ to a function that equals v_V on II and zero on $\Omega \setminus V$. Altogether, we have extracted a subsequence of $\{v_k\}_k$ that converges in $L^p(\Omega)$.

Theorem (5.2.41) [217]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $D \subset \partial\Omega$ be a closed part of the boundary. Suppose that the following three conditions are satisfied:

(i) The set D is l-thick for some $l \in]d-p, d]$.

(ii) The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\| \nabla \cdot \|_{L^p(\Omega)}$.

(iii) There are two open sets $U, V \subset \mathbb{R}^d$ that satisfy (59) and U satisfies Assumption (5.2.40).

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \left|\frac{u}{\mathrm{dist}_D}\right|^p \mathrm{dx} \leq c \int_\Omega |\nabla u|^p \mathrm{dx}$$

holds for all $u \in W_D^{1,p}(\Omega)$.

Section (5.3): The Solid Torus

The classical Hardy inequality was established by Hardy in 1920's and in the continuous form it informs us that:

If $1 < p < \infty$ and f is a non-negative p-integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive xand

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{60}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in (60)is sharp, i.e. it cannot be replaced by a smaller number so that (60)remains true for all relevant functions, respectively, and equality holds only if $f = 0$.

The inequality (60) was established by Hardy in [270] and was first highlighted in the famous book [214]of Hardy, Littlewood, and Polya or in the original article of Hardy [271], who also showed that the constant is not attained, i.e. the variational problem has no minimizer. As known, inequality (60) is the standard form of the large family of the Hardy and Hardy-type inequalities which constitute an essential tool in Analysis, in the study of PDE's, and in the Calculus of variations. In addition, we can find various applications in Geometry, in Mathematical Physics and in Astrophysics.

A proof of the above inequality was given by Landau, in a letter to Hardy, which officially was published in [274]. For a short but very informative presentation of prehistory of Hardy's inequality, see in [273].

Coming back to the inequality (60), if we set $u(x) = \int_0^x f(t)dt$, we obtain the inequality

$$\int_0^\infty \left(\frac{u(x)}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(u'(x)\right)^p dx, \tag{61}$$

which is the most popular form of the classical Hardy inequality.

The following Hardy inequality is the classical generalization of Hardy inequality (60)to higher dimensions and according to which for $n > 1, 1 \leq p < \infty$ with $p \neq n$ and for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}$, it holds

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leqslant \left|\frac{p}{n-p}\right|^p_{\mathbb{R}^n} |\nabla u(x)|^p dx, \tag{62}$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ is the gradient of u (see [214]or [167]). The constant $\left|\frac{p}{p-1}\right|^p$ is sharp and is not attained in the corresponding Sobolev space, which is $W^{1,2}(\mathbb{R}^n)$ when $1 \leq p < n$ and $W^{1,2}(\mathbb{R}^n \setminus \{0\})$ when $n < p < \infty$.

For $p = 2$ and $n > 2$, this inequality is also called the uncertainty principle. For $p = 2$ and $n = 2$, obviously, is trivial. However, in this case, if we weaken the singularity a bit by adding a logarithmic term or/and some extra conditions to the functions, for all $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, we obtain the following inequalities (see [280]):

$$C \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2(1+\ln^2|x|)} dx \leqslant \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx, \text{ if } \int_{|x|=1} u(x) dx = 0 \quad (63)$$

and

$$C \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} dx \leqslant \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx, \text{ if } \int_{|x|=r} u(x) dx = 0 \text{ for all } r > 0. \quad (64)$$

We note here that in the one-dimensional case, it was proved by Hardy in 1925 that for all p-integrable, $p > 1$ on (0,1), functions u, it holds

$$\int_0^1 \frac{|u(x)|^p}{d_{(0,1)}^p(x)} dx \leqslant \left(\frac{p}{n-p}\right)^p \int_0^1 |u'(x)|^p dx, \quad (65)$$

where $d_{(0,1)}(x) = \min(x, 1-x)$ (see in [270], [271] and [36]).

In addition, Hardy showed that the constant is not attained, i.e. the variational problem has no minimizer. Furthermore, inequality (65) confirms that in the one-dimensional case no geometrical assumption is required on the domain.

It is quite natural to ask: Does an inequality of the form (65) continue to exist in the case of $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$? The answer to this question is positive, however, in regard to Hardy inequalities for domains in $\mathbb{R}^n, n \geqslant 2$ the situation is far more complicated than in the one-dimensional case and in general the best constant in (65) depends on the domain.

If we denote by $x = (x', x_n)a$ point in $\mathbb{R}^n$, where $x' = (x_1, \ldots, x_{n-1})$, the Hardy inequality in the half space $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times (0, \infty)$ asserts that if $p > 1$, then for all $u \in C_0^\infty(\mathbb{R}_+^n)$

$$\left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}_+^n} \frac{|u|^p}{x_n^p} dx \leqslant \int_{\mathbb{R}_+^n} |\nabla u|^p dx, \quad (66)$$

where the constant $\left(\frac{p-1}{p}\right)^p$ is sharp and is not attained in $W_0^{1,2}(\mathbb{R}_+^n)$.

As a direct generalization of inequality (66) on domains in $\mathbb{R}^n, n \geq 2$ we can take the following: Let Ω be a domain in $\mathbb{R}^n, n \geq 2$ with non empty boundary and $1 \leq p < \infty$. Given Ω, let $d_\Omega(x)$ be the distance from x to the boundary $\partial\Omega$, that is

$$d_\Omega(x) = \min \ \{|x-y| : y \notin \Omega\}.$$

Then, the Hardy inequality in higher dimensions should be of the type

$$\mu \int_\Omega \frac{|u|^p}{d_\Omega^p} dx \leqslant \int_\Omega |\nabla u|^p dx, \quad (67)$$

Which means that a positive constant μ exists there so that the inequality (67) is valid for all u belonging to some suitable space. And if that is so, it valid unconditionally on Ω, or some prerequisites are necessary, but which ones?

Maz'ya showed in 1960 that the validity of the Hardy inequality depends on measuring theoretical conditions on the domain [249], [49]. Additionally, Hardy type inequalities in $\mathbb{R}^n, n \geqslant 2$, appeared by N← ecas in 1962 [85] of Lipschitz domains. The result of N← ecas that Hardy inequality holds on strongly Lipschitz domains constitutes a milestone in the study of inequalities and it will later become quite understandable. In 1986, Ancona in [57] proved that for a simply connected domain $\Omega \subset \mathbb{R}^2$ and for all $u \in C_0^\infty(\Omega)$, it holds

$$\frac{1}{16}\int_\Omega \frac{u^2}{d_\Omega^2}dx \leqslant \int_\Omega |\nabla u|^2 dx, \tag{68}$$

which means that the Hardy constant is always at least equal to 116. It is still an open question whether this constant is optimal or not. In 1988, Lewis proved the inequality (66) in domains whose complements are uniformly fat [83]. Two years later, the question discussed by Wannebo [88] is, how general Ω can be in order to allow the inequality to be the same as in the case for bounded Lipschitz domains? It has become very clear due to the works of Ancona, Lewis and Wannebo, that the regularity of the boundary is not essential for Hardy inequalities. In addition, as it was proved in 1995 by Davies [211], in the n-dimensional case the best constant

$$\mu_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \left|\frac{u}{d_\Omega}\right|^p dx},$$

called Hardy constant, varies according the domain, and the first results to this direction certify that it strongly depends on the geometrical properties of $\partial\Omega$. It is now well known that if Ω is an arbitrary open convex domain in $\mathbb{R}^n, n \geq 2$, with boundary $\partial\Omega$, belonging to C^1-class of smoothness in the neighborhood of at least one point $x_0 \in \partial\Omega$, then for arbitrary $1 < p < \infty, \mu_p(\Omega) = \left(\frac{p-1}{p}\right)^p$. A proof for $p = n = 2$ was presented in 1995 by Davies [211]and for $n = 2$ and $1 < p < \infty$ by Matskewich and Sobolevskii in 1997 [276]. A simple proof of the result in the general case was provided in 1998 by Markus, Mizel and Pinchover (see [48], Appendix A). It is shown that for all smooth n-dimensional domains, $\mu_p(\Omega) \leqslant \left(\frac{p-1}{p}\right)^p$. Moreover, for all those domains, it is shown that a minimizer for $\mu_p(\Omega)$ is not possessed. Especially, for $p = 2$, it is proved that $\mu_2(\Omega) < \frac{1}{4}$ if and only if the Rayleigh quotient possesses a minimizer. In addition, the given examples show that strict inequality may occur even for bounded smooth domains, but $\mu_p(\Omega) = \left(\frac{p-1}{p}\right)^p$ for convex domains.

It is clear that Hardy inequality holds in an open domain with the best constant $\mu_p(\Omega) = \left(\frac{p-1}{p}\right)^p$, if Ω is a convex domain. However, it is not clear if Hardy inequality with best constant $\mu_p(\Omega) = \left(\frac{p-1}{p}\right)^p$ is valid only for convex domains, even in the most simple case where $p = 2$. We mention on here that in this direction significant progress has been made since Barbatis, Filippas and Tertikas [92] relaxed the assumption of convexity for the domain by introducing the global geometric condition $-\Delta d_\Omega \geqslant 0$ on Ω (in the distributional sense in Ω). They showed that if Ω satisfies the above condition, then Hardy inequality is valid for $\mu_2(\Omega) = \frac{1}{4}$. We note that this condition is equivalent to the convexity of the domain for $n = 2$; but for $n \geq 3$ it is a much weaker condition than the convexity. it has been proved that it is equivalent to the fact that the mean curvature of the boundary is non-negative (see [278], [103]). Also, we note that no smoothness assumption on the boundary is imposed.

As a first conclusion, to all dimensions for convex domains the value of $\mu_p(\Omega)$ is the same as in the classical one-dimensional case, but there are smooth domains so that $\mu_p(\Omega)$ is

smaller than $\left(\frac{p-1}{p}\right)^p$ (see [276], [48], [269], [261]). Davies, also, has constructed non convex domains in $\mathbb{R}^n, n \geq 3$ with Hardy constant as small as one wishes, thereby proving that for simply connected domains no such positive constant μ exists, not depending on the dimension [212]. Furthermore, Barbatis and Tertikas determined the Hardy constant of an arbitrary quadrilateral in the plane [262]. In addition, in [263] the same computed the Hardy constant for other non-convex planar domains. In both aforementioned articles it is confirmed that the Hardy constant is related to that of a certain infinite sectorial region which has been studied by Davies. On the other hand in 2015, Egert, Haller-Dintelmann, and Rehberg in [217]developed a geometric framework for Hardy inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary.
For someone looking for sufficient conditions on Ω in aim to find the best constant in Hardy inequality, it is logical that they seek for some convexity on Ω because of the strong influence of the spirit of the most articles which are devoted to this problem. However, there are the N`ecas's [85] and Maz'ya's [249] articles where the studied cases do not require some kind of convexity. Concerning this subject, we present at this point an excerpt from the book of Opic and Kufner (see [167]p. 235):
...The conditions derived which guarantee the validity of the N-dimensional Hardy inequality are mostly sufficient. V.G. MAZ'JA [49] has derived necessary and sufficient conditions under which (8)holds for every $u \in C_0^\infty(\Omega)$. His conditions are expressed in terms of capacities and are difficult to verify... In our opinion, the advantage of MAZ'JA's results lies in the possibility of obtaining some information about the capacity of a set, once we have derived some information about the validity of the corresponding Hardy inequality by another method...
In order to derive Hardy-type inequalities for non-convex domains $\mathbb{R}^n, n \geqslant 2$ some geometrical conditions must be considered on the domain which act as 'non-convexity measures'. The objective in this direction is to obtain the Hardy inequality for simply-connected non-convex domains and to investigate how the constant $\mu_p(\Omega)$ depends on the non-convexity parameters. Therefore, in contrast to defining the convexity of a domain, 'measuring of non-convexity' can be done in many ways (see for example [207], [215]). In addition, Korte, Lehrbäck, and Tuominen in [54] proved an equivalent result between the validity of a pointwise Hardy inequality in a domain and uniform capacity density of its complement.
According to all mentioned above in regard to Hardy inequalities for domains Ω in $\mathbb{R}^n, n \geqslant 2$, the best constant in (67) depends on the domain Ω and no universal Hardy constant exists (see [261], [262], [263], [212]).

Based on the above, although it does not seem to be possible to determine a general criterion on the basis of which we can classify the domains on the value of their best Hardy constant, we can extend some results that hold on convex domains and prove that they hold in some cases of non-convex domains, too. We establish the classical Hardy inequality and some variants of it in the solid torus, we calculate their best constants and we prove that they are the same with the standard Hardy best constants which appear in convex domains although the solid torus has no convex boundary but it has all kinds of curvature; i.e. there are points where the curvature is positive, negative or zero. This result confirms that the convexity of the boundary is a sufficient condition to obtain the optimal constants in the Hardy inequalities but not a necessary condition.

we make use of the fact that Sobolev embeddings can be improved in the presence of symmetries (see Theorem 9.2 in [46]). Similar results have been obtained in specific contexts by Strauss [281], by Lions [275] and by Cotsiolis and Iliopoulos [264].
We devoted to the study of the classical Hardy inequality and of a weighted Hardy inequality in the solid torus in detail., some improved Hardy-type and $L^p, p > 1$ Hardy inequalities with weights are considered. Finally, , some results concerning the Hardy-Morrey inequality are presented.
Let T be the solid ring torus in $\mathbb{R}^3$ with minor radius rand major radius R. This is the "doughnutshaped" domain generated by rotating a disk of radius rabout a co-planar axis at a distance R from the center of the disk, and it is represented by

$$T = \left\{(x, y, z) \in \mathbb{R}^3: \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 < r^2, R > r > 0\right\}.$$

Let $W^{1,p}(T), p \geq 1$, be the classical Sobolev space, that is the space of all $u \in L^p(T)$ with $\nabla u \in L^p(T)$ and $W_0^{1,p}(T)$ be the closure of $C_0^\infty(T)$ in $W^{1,p}(T)$. Here, $L^p(T)$ is the usual Lebesque space of order p, and ∇ stands for the gradient operator, acting on the distribution space $D'(T)$. Since the solid torus T is an open bounded domain in $\mathbb{R}^3$ and its boundary is smooth, in order to study the Hardy inequality and some of its variants, it seems that the suitable functional space to be used is $W^{1,p}(T)$. Meanwhile, by Meyers-Serrin's Theorem [277], $W^{1,p}(T) = H^{1,p}(T)$, where $H^{1,p}(T)$ is the completion of $C^\infty(T)$ with respect to the norm

$$\| u \|_{H^{1,p}(T)} = \left(\int_T |\nabla u|^p dx\right)^{1/p} + \left(\int_T |u|^p dx\right)^{1/p}.$$

Therefore, it would be natural to work in $H^{1,p}(T)$. However, the solid torus $\bar{T} = T \cup \partial T$ is invariant under the action of the subgroup $G = O(2) \times I$ of the isometry group O (3), so we will rely on spaces containing functions which are invariant under the action of G.
To exploit the symmetry presented by the torus we are working as follows: Consider an arbitrary plane Π containing the axis zz' which forms with the positive semi-axis Ox angle $\theta, \theta \in \mathbb{R}$, and the interval $I = [\theta, \theta + 2\pi)$ (or $I = (\theta, \theta + 2\pi]$). We, also, denote D the unit disk centered on the beginning of the axes, i.e.

$$D = \{(t, s) \in \mathbb{R}^2: t^2 + s^2 < 1\}.$$

Let now the transformation

$$\xi: T \setminus \{T \cap \Pi\} \to I \times D,$$

defined to be $\xi(x, y, z) = (\omega, t, s)$, with $\omega \in I$ and such that

$$\cos\omega = xx2 + y2, \sin\omega = yx2 + y2,$$

$$\omega = \begin{cases} \arctan\frac{y}{x}, x \neq 0 \\ \frac{\pi}{2}, x = 0, y > 0 \\ 3\frac{\pi}{2}, x = 0, y < 0 \end{cases}, \text{or } \omega = \begin{cases} \arctan\frac{y}{x}, x \neq 0 \\ \frac{\pi}{2}, x = 0, y > 0, \ , x = 0, y < 0, \\ 3\frac{\pi}{2}, x = 0, y < 0 \end{cases}$$

and $(t, s) \in D$ such that

$$t = \frac{\sqrt{x^2 + y^2} - R}{r}, s = \frac{z}{r}, 0 \leq t, s < 1. \quad (69)$$

The Euclidean metric g on (T, ξ) can be expressed as

$$\left(\sqrt{g} \circ \xi^{-1}\right)(\omega, t, s) = r^2(R + rt). \quad (70)$$

Under to above considerations, if for any function u defined on T we define the function $\phi = u \circ \xi^{-1}$, this function does not depend on the variable ω, i.e. it holds that:

$$\phi(t,s) = (u \circ \xi^{-1})(\omega,t,s). \tag{71}$$

Thus, we need to use functions whose the values do not depend of the orientation in the $xy-$ plane and therefore must be of the form

$$u(x,y,z) = u\left(\sqrt{x^2+y^2},0,z\right) = u\left(0,\sqrt{x^2+y^2},z\right).$$

For a better understanding, we mention that these functions play for the torus the same role as the radial functions for the sphere (see in [265] and [46]).

We consider now the following spaces:

$$C_0^\infty,G(T) = \{u \in C_0^\infty(T) : u \circ \tau = u, \forall \tau \in G\},$$

$$C_G^\infty(T) = \{u \in C^\infty(T) : u \circ \tau = u, \forall \tau \in G\},$$

And

$$L_G^p(T) = \{u \in L^p(T) : u \circ \tau = u, \forall \tau \in G\}.$$

We define, also, the Sobolev space $H_G^{1,p}(T)$ for any $p \geq 1$, as the completion of $C_G^\infty(T)$ with respect to the norm $\|\cdot\|_{H^{1,p}}$, and the Sobolev space $H_{0,G}^{1,p}(T)$ as the closure of $C_{0,G}^\infty(T)$ in $H_G^{1,p}(T)$.

Due to (71), for any function $u \in H_G^{1,p}(T)$, the following equalities hold:

$$\int_T |u|^p dx = 2\pi r^2 \int_D |\phi|^p (R+rt) dtds \tag{72}$$

$$\int_T |\nabla u|^p dx = 2\pi r^{2-p} \int_D |\nabla\phi|^p (R+rt) dtds. \tag{73}$$

The following lemma gives the relation between the distance function in the solid torus and in the unit disk which is the key to prove the main theorem, since it allows us to transfer the problem from the solid torus to the unit disk. This fact is essential, since it enables us to show that the classical Hardy inequality which holds in convex domains remains true and in non-convex domains.

Lemma (5.3.1)[259]: The distance function d_T in the solid torus T can be expressed via the distance function d_D in the unit disk D as

$$d_T(\cdot) = r d_D(\cdot). \tag{74}$$

Proof: Let $P(x,y,z) \in T$ an arbitrary point of it and $Q(\omega,t,s) = \xi(P(x,y,z))$ its imagine of P in $I \times D$. Since $\bar{T}$ is invariant under the action of the group $G = O(2) \times I$, the distance $d(P,\partial T)$ of P from the boundary ∂T of the torus T remains invariant for any point which belongs to the orbit of P. We consider the plane defined by the point P and the z-axis and let C be the center of the circle which is defined as the intersection of this plane and the torus $\bar{T}$. Then, considering the relations between the (x,y,z) and (ω,t,s) which are given above, we obtain consecutively

$$d(P,\partial T) = r - (CP) = r - \sqrt{\left(\sqrt{x^2+y^2}-R\right)^2 + z^2}$$

$$= r - \sqrt{(rt)^2+(rs)^2} = r\left(1-\sqrt{t^2+s^2}\right)$$

$$= rd(Q,\partial D).$$

Let $P_j(x_j,y_j,z_j)$ be a point in T and O_{P_j} its orbit under the action of the subgroup $G = O(2) \times I$ of the group $O(3)$ of the type

$$(x,y,z) \to (A(x,y),z), A \in O(2), (x,y,z) \in \mathbb{R}^3.$$

Consider the open small solid torus (a tubular neighborhood of the orbit O_{P_j})

$$T_{\delta_j} = \left\{(x,y,z) \in \bar{T} : \left(\sqrt{x^2+y^2} - R_j\right)^2 + \left(z - z_j\right)^2 < \delta_j^2, \delta_j = \varepsilon_j R_j\right\},$$

where $R_j = \sqrt{x_j^2 + y_j^2}$ is the horizontal distance of the orbit O_{P_j} from the axis $z'z$ and $\varepsilon_j > 0$ given. Then, the following lemma is valid.

Lemma (5.3.2) [259]: For any $\varepsilon_j > 0, j = 1,2, \dots N$ and for all $p > 1$, there exists $\delta_j = \varepsilon_j R_j$ so that for all $u \in C_0^\infty, G\left(T_{\delta_j}\right)$,

$$\int_{T_{\delta_j}} \frac{|u|^p}{d_{T_{\delta_j}}^p} dx \leqslant \frac{1+\varepsilon_j}{1-\varepsilon_j}\left(\frac{p}{p-1}\right)^p \int_{T_{\delta_j}} |\nabla u|^p dx. \quad (75)$$

Proof: On every T_{δ_j} we define the transformation

$$\xi_j : T_{\delta_j} \setminus \left\{T_{\delta_j} \cap \Pi\right\} \to I \times D,$$

in the same way as we defined the transformation $\xi : T \setminus \{T \cap \Pi\} \to I \times D$ defined above. Then the Euclidean metric g on $\left(T_{\delta_j}, \xi_j\right)$ due of (70) can be expressed as

$$\left(\sqrt{g} \circ \xi_j^{-1}\right)(\omega, t, s) = \delta_j^2\left(R_j + \delta_j t\right).$$

For $u \in C_0^\infty, G\left(T_{\delta_j}\right)$ by (71) arises that $\phi_j = u \circ \xi_j^{-1}$ and by (72) because of Lemma (5.3.1)we obtain

$$\int_{T_{\delta_j}} \frac{|u|^p}{d_{T_{\delta_j}}^p} dx \leqslant \left(1+\varepsilon_j\right) 2\pi R_j \delta_j^{2-p} \int_D \frac{\left|\phi_j\right|^p}{d_D^p} dt ds. \quad (76)$$

Because $\phi_j \in C_0^\infty(D)$ and since the space $C_0^\infty(D)$ is dense in $H_0^{1,p}(D)$ with respect to the norm $\|\cdot\|_{H^{1,p}}$ in the unit disk D, according to Theorem 1 of [276], yields

$$\int_D \frac{\left|\phi_j\right|^p}{d_D^p} dt ds \leqslant \left(\frac{p}{p-1}\right)^p \int_D \left|\nabla \phi_j\right|^p dt ds. \quad (77)$$

Moreover, from (73) arises

$$\int_D \left|\nabla \phi_j\right|^p dt ds \leq \frac{1}{\left(1-\varepsilon_j\right) 2\pi R_j \delta_j^{2-p}} \int_{T_{\delta_j}} |\nabla u|^p dx. \quad (78)$$

Therefore, combining the inequalities (76), (77) and (78) we obtain the inequality (75). Our first result concerns the classical Hardy inequality in the solid torus.

Theorem (5.3.3) [259]: Let T be the 3-dimensional solid torus. Then, for all $u \in H_G^{1,p}(T)$,

$$\left(\frac{p}{p-1}\right)^p \int_T \frac{|u|^p}{d_T^p} dx \leqslant \int_T |\nabla u|^p dx, p > 1. \quad (79)$$

In addition, the constant $\left(\frac{p}{p-1}\right)^p$ is the best constant for this inequality.

Proof: We carry through the proof of the theorem in two steps.

Step 1. This first step is devoted to prove that $\left(\frac{p}{p-1}\right)^p$ is the best constant for the Hardy inequality (79) considered for all $u \in H_G^{1,p}\left(T_{\delta_j}\right)$, where T_{δ_j} is any one of the tori $T_{\delta_j}' s$ defined above.

Let $P_j(x_j, y_j, z_j)$ be a point in T, O_{P_j} its orbit under the action of the subgroup $G = O(2) \times I$ and $R_j = \sqrt{x_j^2 + y_j^2}$ as defined above. Let, also, $\varepsilon > 0$ given. Then, we can choose an ε_j depending on ε and P_j so that the open small solid torus T_{δ_j} having the following properties: (i) $\bar{T}_{\delta_j}$ is a submanifold of $\bar{T}$ with boundary, (ii) $d^2\left(\cdot, O_{P_j}\right)$, the distance to the orbit O_{P_j}), is a C^∞ function on $\bar{T}_j$, and (iii) $\bar{T}$ is covered by $\left(T_{\delta_j}\right)_{j\in J}$. Once more denote by $\left(T_{\delta_j}\right)_{j=1,\dots,N}$ a finite covering of $\bar{T}$ consisting of $T'_{\delta_j}S$ sets.

We denote now by $P(x_P, y_P, z_P)$ any point of $P_j's$, by O_P its orbit and by $R_P = \sqrt{x_P^2 + y_P^2}$ its horizontal distance of the orbit O_P from the axis $z'z$, and consider the 'smallish' torus

$$T_\delta = \{Q \in \bar{T} : d(Q, O_P) < \delta, \delta = \varepsilon_0 R_P\},$$

where $\varepsilon_0 = \min_{j\in J} \varepsilon_j$.

For any $\Omega \in \mathbb{R}^n, n \geq 2$, we denote

$$J_\Omega(u) \equiv \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \left|\frac{u}{d_\Omega}\right|^p dx} \tag{80}$$

and

$$\mu_p(\Omega) \equiv \inf_{u\in C_{0,G}^\infty(\Omega)\setminus\{0\}} J_\Omega(u). \tag{81}$$

We will prove that

$$\mu_p(T_\delta) = \left(\frac{p-1}{p}\right)^p. \tag{82}$$

By (80) due to Lemma (5.3.2) arises that for all $p > 1$ and for all $u \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$, it holds

$$J_{T_\delta}(u) \geq \frac{1-\varepsilon_0}{1+\varepsilon_0}\left(\frac{p-1}{p}\right)^p, \tag{83}$$

which yields

$$\mu_p(T_\delta) \geq \frac{1-\varepsilon_0}{1+\varepsilon_0}\left(\frac{p-1}{p}\right)^p. \tag{84}$$

If, for given ε, we choose $\varepsilon_0 = \frac{\varepsilon}{2-\varepsilon'}$ by (84) we obtain that $\mu_p(T_\delta) \geq (1-\varepsilon)\left(\frac{p-1}{p}\right)^p$, which means that perhaps $\mu_p(T_\delta) < \left(\frac{p-1}{p}\right)^p$. We will prove that this can not be the case. Suppose, by contradiction, that there exists a non-constant function $v \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$, so that

$$J_{T_\delta}(v) < \left(\frac{p-1}{p}\right)^p$$

or, equivalently

$$\frac{\int_{T_\delta} |\nabla v|^p dx}{\int_{T_\delta} \left|v/d_{T_\delta}\right|^p dx} < \left(\frac{p-1}{p}\right)^p. \tag{85}$$

Now, on D, the unit disk of $\mathbb{R}^2$, we define the function φ (see equality (71)), to be

$$\varphi(t,s) \equiv (v \circ \xi^{-1})(\omega, t, s). \tag{86}$$

For any $\lambda > 0$ we define the function $\varphi_\lambda(t,s) \equiv \varphi(\lambda t, \lambda s)$ and the function $v_\lambda \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$ to be $v_\lambda \equiv \varphi_\lambda \circ \xi$. By (72) and (73), we obtain respectively

$$\int_{T_\delta} |v|^p dx = 2\pi \left(\frac{\delta}{\lambda}\right)^2 \int_D |\varphi|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds \tag{87}$$

and

$$\int_{T_\delta} |\nabla v|^p dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D |\nabla \varphi|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds. \tag{88}$$

In addition, by (72) due to Lemma (5.3.1), we obtain

$$\int_{T_\delta} \frac{|v|^p}{T_\delta} dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D \frac{|\varphi|^p}{d_D^p} \left(R_P + \frac{\delta}{\lambda} t\right) dtds. \tag{89}$$

By (85), because of (88) and (89), and a direct computation, we obtain successively

$$\frac{2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D |\nabla \varphi|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds}{2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D |\varphi/d_D|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds} < \left(\frac{p-1}{p}\right)^p$$

or

$$\frac{\int_D |\nabla \varphi|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds}{\int_D |\varphi/d_D|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds} < \left(\frac{p-1}{p}\right)^p. \tag{90}$$

By (90), sending $\lambda \to \infty$, arises

$$\frac{\int_D |\nabla \varphi|^p dtds}{\int_D |\varphi/d_D|^p dtds} < \left(\frac{p-1}{p}\right)^p,$$

from which it follows that $\mu_p(D) < \left(\frac{p-1}{p}\right)^p$. This last inequality is a contradiction (see [48]or [276]) and thus, we conclude

$$\mu_p(T_\delta) \geq \left(\frac{p-1}{p}\right)^p. \tag{91}$$

To complete the proof of this part of the theorem it remains to be ruled out the case

$$\mu_p(T_\delta) > \left(\frac{p-1}{p}\right)^p.$$

Assume, by contradiction, that there exists a small but fixed positive ε so that

$$\mu_p(T_\delta) = \left(\frac{p-1}{p}\right)^p + \varepsilon.$$

Then for all $u \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$ we will have

$$J_{T_\delta}(u) \geq \left(\frac{p-1}{p}\right)^p + \varepsilon$$

or, equivalently

$$\frac{\int_{T_\delta} |\nabla u|^p dx}{\int_{T_\delta} |u d_{T_\delta}|^p dx} \geq \left(\frac{p-1}{p}\right)^p + \varepsilon. \tag{92}$$

We set

$$J_D(\phi) \equiv \frac{\int_D |\nabla \phi|^p dtds}{\int_D |\phi/d_D|^p dtds'}$$

and

$$\mu_p(D) \equiv \inf_{\phi \in C_0^\infty(D)\setminus\{0\}} J_D(\phi),$$

where D is the unit disk of $\mathbb{R}^2$.
By Theorem 11 in [48], arises that

$$\mu_p(D) = \left(\frac{p-1}{p}\right)^p. \qquad (93)$$

Consider now a minimizing sequence $(\phi_j) \in C_0^\infty(D) \setminus \{0\}$ of $J_D(\phi)$ and for any $\lambda > 0$ we define the λ-parametric sequence ϕ_{j_λ} to be $\phi_{j_\lambda}(t,s) \equiv \phi_j(\lambda t, \lambda s)$. For any ϕ_j we define, also, the function $u_j \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$ to be $u_j \equiv \phi_j \circ \xi$ and the λ-parametric sequence u_{j_λ} to be $u_{j_\lambda} \equiv \phi_{j_\lambda} \circ \xi$.
By (88) and (89), we obtain respectively

$$\int_{T_\delta} \left|\nabla u_{j_\lambda}\right|^p dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D \left|\nabla \phi_j\right|^p \left(R_P + \frac{\delta}{\lambda} t\right) dtds, \qquad (94)$$

$$\int_{T_\delta} \frac{\left|u_{j_\lambda}\right|^p}{d_{T_\delta}^p} dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-p} \int_D \frac{\left|\phi_j\right|^p}{d_D^p} \left(R_P + \frac{\delta}{\lambda} t\right) dtds. \qquad (95)$$

By (92), because of (94) and (95), we obtain

$$J_{T_\delta}(u_{j_\lambda}) \geq \left(\frac{p-1}{p}\right)^p + \varepsilon. \qquad (96)$$

Letting $\lambda \to \infty$ the inequality, (96) yields

$$J_D(\phi_j) \geq \left(\frac{p-1}{p}\right)^p + \varepsilon. \qquad (97)$$

Now by (97), since (ϕ_j) is a minimizing sequence of $J_D(\phi)$, sending $j \to \infty$, we obtain

$$\mu_p(D) \geq \left(\frac{p-1}{p}\right)^p + \varepsilon. \qquad (98)$$

Because of (93), the inequality (98) is false, and this part of the theorem is proved.
Step 2. In this second step, we will prove that $\mu_p(T) = \mu_p(T_\delta)$. For this purpose, it suffices to find the best constant for the Hardy inequality (79)considered for all $u \in H_G^{1,p}(T)$, where T is the solid torus T defined above. However, regardless of the radius r of the circle rotating around the axis $z'z$ and producing the torus, from Lemma (5.3.1) arises that, ultimately, the problem is reduced to the unit disk of the plane. Therefore, combining the conclusion of Lemma(5.3.2)with the equations (88)and (89)considered on the torus T and repeating the procedure followed above, we find that the best constant for the Hardy inequality in the smallish torus is the same for each torus.
Let C_R be the circle of range R on the xy plane, i.e.

$$C_R = \{x = (x, y, 0) \in T: x^2 + y^2 = R^2\}.$$

Additionally, we denote

$$T^* = T \setminus C_R \text{ and } D^* = D \setminus (0,0)$$

the punctured torus and the unit disk, respectively.
Then, the following corollary holds.
Corollary (5.3.4) [259]: $\mu_p(T^*) < \mu_p(T)$.
The following theorem concerns to a weighted Hardy inequality, which consists an extension from the convex to the non-convex domains.
Theorem (5.3.5) [259]: Let T be the 3-dimensional solid torus. Then, for all $u \in H_G^{1,p}(T)$,

$$\left(\frac{s-1}{p}\right)^p \int_T \frac{|u|^p}{d_T^s} dx \leqslant \int_T \frac{|\nabla u|^p}{d_T^{s-p}} dx, p > 1, s > 1. \tag{99}$$

In addition, the constant $\left(\frac{s-1}{p}\right)^{pT}$ is the best constant for this inequality.

Proof: The proof of this theorem is carried by following the same steps as in that of the first theorem. However, the appropriate functional in this case is

$$I_{T_\delta}(u) = \frac{\int_{T_\delta} | \frac{\nabla u|^p}{d_{T_\delta}^{s-p} dx}}{\int_{T_\delta} | \frac{u|^p}{d_{T_\delta}^S dx}}, \tag{100}$$

and in aim to reduce the problem from the torus to the unit disk, we need to use the two formulas

$$\int_{T_\delta} \frac{|\nabla u|^p}{d_{T_\delta}^{s-p}} dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-s} \int_D \frac{|\nabla \phi|^p}{d_D^{s-p}} \left(R_P + \frac{\delta}{\lambda} t\right) dt ds \tag{101}$$

$$\int_{T_\delta} \frac{|u|^p}{d_{T_\delta}^s} dx = 2\pi \left(\frac{\delta}{\lambda}\right)^{2-s} \int_D \frac{|\phi|^p}{d_D^s} \left(R_P + \frac{\delta}{\lambda} t\right) dt ds, \tag{102}$$

arising by (72) and (73)combining to Lemma (5.3.1)after some simple calculations.

Improved Hardy inequality, in general, means having extra terms on the left hand side of (67)that either contain integrals of $|u|^p$ with weights depending on $|x|$ or integrals of $|\nabla u|^q$ with $p < q$ (see [35], [42], [31], [52]).

This need came from the fact that the best constant in the inequality (67) is not obtained suggesting that perhaps a correction term to be added. Brezis and Marcus in [36] moving in that direction and by staying in the case $p = 2$, improved the inequality (67) by adding a positive term on the left-hand side. In particular, the above mentioned authors proved the following result (see [36]):

For every smooth domain Ω, there exists a constant $\lambda(\Omega) \in \mathbb{R}$ so that for all $u \in W_0^{1,2}(\Omega)$,

$$\frac{1}{4} \int_\Omega \frac{u^2}{d_\Omega^2} dx + \lambda(\Omega) \int_\Omega u^2 dx \leqslant \int_\Omega |\nabla u|^2 dx. \tag{103}$$

The largest such constant is precisely $\lambda^*(\Omega)$, i.e.

$$\lambda^*(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4} \int_\Omega \frac{u^2}{d_\Omega^2} dx}{\int_\Omega u^2 dx},$$

and in view of Theorem I in [36], this infimum is not achieved.

In addition, as mentioned before, there are domains for which $\mu(\Omega) < \frac{1}{4}$ and then $\lambda^*(\Omega) < 0$ (see [48]and [276]). On the other hand, if Ω is convex, then $\mu(\Omega) = \frac{1}{4}$ so that $\lambda^*(\Omega) \geq 0$. Furthermore, it is proved that (see Theorem II)

$$\lambda(\Omega) \geq \frac{1}{4 \operatorname{diam}^2(\Omega)}, \tag{104}$$

where by $\operatorname{diam}(\Omega)$ is denoted the diameter of Ω, i.e. $\operatorname{diam}(\Omega) = \sup\{|x - y| : x, y \in \Omega\}$. Brezis and Marcus asked whether the $\operatorname{diam}(\Omega)$ in (104) can be replaced by an expression depending on the volume of Ω, namely, whether $\lambda(\Omega) \geq \alpha |\Omega|^{2/n}$ ($|\Omega|$ stands for the volume

of Ω), for some universal constant $\alpha > 0$. This question was later answered in affirmative as the following result states (see [47]):
For every convex smooth domain Ω, there exists a constant $\lambda(\Omega) \in \mathbb{R}$ such that for all $u \in W_0^{1,2}(\Omega)$,

$$\frac{1}{4}\int_\Omega \frac{u^2}{d_\Omega^2}dx + \lambda(\Omega)\int_\Omega u^2 dx \leqslant \int_\Omega |\nabla u|^2 dx, \tag{105}$$

where

$$\lambda(\Omega) \geq \frac{c(n)}{|\Omega|^{\frac{2}{n}}},$$

$$c(n) = \frac{n^{\frac{n-2}{n}}|\mathbb{S}^{n-1}|^{\frac{2}{n}}}{4}, \tag{106}$$

and, where by $|\mathbb{S}^{n-1}|$ is denoted the volume of the standard unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$.
We extend this last result which relates to convex domains and we prove a corresponding theorem in the torus. In the following theorem, we prove the inequality (105)in the case of the torus noting that the torus behaves exactly like the unit disk of $\mathbb{R}^2$ without inheriting anything from its dimensional texture. This result is certainly surprising considering that neither in the Sobolev inequalities (see [265], [266], [267]), nor in the Nash inequalities exhibit such a behavior (see [268]).
Theorem (5.3.6) [259]: Let T be the 3-dimensional solid torus. Then, for all $u \in H_G^{1,2}(T)$, there exists a constant $\lambda \in \mathbb{R}$ so that

$$\frac{1}{4}\int_T \frac{u^2}{d_T^2}dx + \lambda\int_T u^2 dx \leq \int_T |\nabla u|^2 dx, \tag{107}$$

Where

$$\lambda = \lambda(D) \geq \frac{c(2)}{|D|} = \frac{1}{4'}$$

and D is the unit disk in $\mathbb{R}^2$.
Proof: Consider the small torus T_δ (defined in Theorem(5.3.3)) and an arbitrary function $u \in H_G^{1,p}(T_\delta)$ and $\phi = u \circ \xi^{-1}$ (see (71)). Then by (73), we obtain

$$\int_{T_\delta} |\nabla u|^2 dx = 2\pi R_P \int_D |\nabla\phi|^2 \left(1 + \frac{\delta}{R_P}t\right) dtds$$

$$\geqslant 2\pi R_P \int_D |\nabla\phi|^2 \left(1 - \frac{\delta}{R_P}\right) dtds$$

$$= 2\pi R_P(1-\varepsilon_0)\int_D |\nabla\phi|^2 dtds. \tag{108}$$

By (108), because of Theorem II in [36]combined with the main result in [47], yields

$$\int_D |\nabla\phi|^2 dtds \geqslant \frac{1}{4}\int_D \frac{\phi^2}{d_D^2}dtds + \lambda(D)\int_D \phi^2 dtds, \tag{109}$$

Where

$$\lambda(D) \geq \frac{c(2)}{|D|} = \frac{\pi}{\pi} = \frac{1}{4} \tag{110}$$

and, where the value $\frac{\pi}{2}$ for $c(2)$ is obtained by (106) after a simple calculation.
By (110), due to (109), arises

$$\int_D |\nabla u|^2 dx \geqslant 2\pi R_P(1-\varepsilon_0)\left(\frac{1}{4}\int_D \frac{\phi^2}{d_D^2} dtds + \lambda(D)\int_D \phi^2 dtds\right). \quad (111)$$

Coming back to (72) and under the considerations of the Lemma(5.3.2), we obtain

$$\int_D \phi^2 dtds \geqslant \frac{1}{2\pi R_P(1+\varepsilon_0)}\int_T u^2 dx. \quad (112)$$

Again by (72), because of Lemma (5.3.1), arises

$$\int_D \frac{\phi^2}{d_D^2} dtds \geqslant \frac{1}{2\pi R_P(1+\varepsilon_0)}\int_T \frac{u^2}{d_T^2} dx. \quad (113)$$

By (111), due to (112) and (113), we obtain

$$\int_{T_\delta} |\nabla u|^2 dx \geqslant \frac{1-\varepsilon_0}{1+\varepsilon_0}\left(\frac{1}{4}\int_{T_\delta} \frac{u^2}{d_T^2} dx + \lambda(D)\int_{T_\delta} u^2 dx\right).$$

Now, for given ε, we can choose $\varepsilon_0 = \frac{\varepsilon}{2-\varepsilon'}$ and then by the last inequality we obtain

$$\int_{T_\delta} |\nabla u|^2 dx \geqslant (1-\varepsilon_0)\left(\frac{1}{4}\int_{T_\delta} \frac{u^2}{d_T^2} dx + \lambda(D)\int_{T_\delta} u^2 dx\right). \quad (114)$$

From this last inequality we conclude that the first best constant for this is maybe smaller than $\frac{1}{4}$. For this aim, we need to borrow ideas from the first part of the proof of the Theorem (5.3.3). Let us present a brief proof of this part. Suppose, by contradiction, that there exists a non-constant function $v \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$, such that

$$\frac{\int_{T_\delta} |\nabla v|^2 dx}{\int_{T_\delta} \left(v/d_{T_\delta}\right)^2 dx + \lambda(D)\int_{T_\delta} v^2 dx} < \frac{1}{4}. \quad (115)$$

On D, the unit disk of $\mathbb{R}^2$, we define the function φ to be $\varphi \equiv v \circ \xi^{-1}$, as in (86). Now for any $\alpha > 0$, we define the function $\varphi_\alpha(t,s) \equiv \varphi(\alpha t, \alpha s)$ and the function $v_\alpha \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$ to be $v_\alpha \equiv \varphi_\alpha \circ \xi$. By (89), (91) and (91), we obtain

$$\int_{T_\delta} v^2 dx = 2\pi\left(\frac{\delta}{\alpha}\right)^2 \int_D \varphi^2\left(R_P + \frac{\delta}{\alpha}t\right) dtds, \quad (116)$$

$$\int_{T_\delta} |\nabla v|^2 dx = 2\pi \int_D |\nabla \varphi|^2\left(R_P + \frac{\delta}{\alpha}t\right) dtds \quad (117)$$

and

$$\int_{T_\delta} \frac{v^2}{d_{T_\delta}^2} dx = 2\pi \int_D \frac{\varphi^2}{d_D^2}\left(R_P + \frac{\delta}{\alpha}t\right) dtds. \quad (118)$$

By (116), (117) and (118) substituting in the (115) and sending $\alpha \to \infty$, yields

$$\frac{\int_D |\nabla\varphi|^2 dtds}{\int_D (\varphi/d_D)^2 dtds} < \frac{1}{4},$$

which is a contradiction (see [48]or [276]).

Thus, we have prove that for all $u \in C_{0,G}^\infty(T_\delta) \setminus \{0\}$,

$$\int_{T_\delta} |\nabla u|^2 dx \geqslant \frac{1}{4}\int_{T_\delta} \frac{u^2}{d_T^2} dx + \lambda(D)\int_{T_\delta} u^2 dx \quad (119)$$

where, due to (110), $\lambda(D) \geq \frac{1}{4}$.

In aim to pass from the "smallish" torus T_δ to the torus T, so as to complete the proof, we only have to use the same arguments as in the second step in Theorem(5.3.3).

As mentioned, there has been much discussion in the direction of asking some 'measure' of convexity. We present some interesting results concerning Hardy inequalities with weights in the solid torus, in which no assumption on the convexity is needed. Firstly, we need some definitions from Geometry (i.e. see [261]).

Definition (5.3.7) [259]: Let Ω be a domain in $\mathbb{R}^n, n \geq 2$ with a C^2 boundary. The mean curvature of $\partial\Omega$ at y is defined to be

$$H(y) = \frac{1}{n-1}\sum_{j=1}^{n-1} k_j(y), y \in \partial\Omega,$$

where $k_j(y), j = 1,2,\dots,n-1$ are the principal curvatures at $y \in \partial\Omega$ with respect to the unit inward normal (i.e. the standard unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ has mean curvature -1everywhere).

Definition (5.3.8) [259]: A domain $\Omega \subset \mathbb{R}^n, n \geq 2$ with C^2 boundary is said to be mean convex if $H(y) < 0, y \in \partial\Omega$, and weakly mean convex if $H(y) \leq 0, y \in \partial\Omega$.

Let $y = (y_1, y_2, y_3) \in \partial T$ have the parametric coordinates

$$y_1 = (R + r\cos s^2)\cos s^1$$
$$y_2 = (R + r\cos s^2)\sin s^1$$
$$y_3 = r\sin s^2$$

where $s^1, s^2 \in (-\pi, \pi]$.

Then, the principal curvatures at $y \in \partial\Omega$ are (i.e. see [272])

$$k_1 = -\frac{1}{r}, k_2 = -\frac{\cos s^2}{R + r\cos s^2}$$

and therefore

$$H(y) = -\frac{R + 2r\cos s^2}{2r(R + r\cos s^2)} \leqslant -\frac{R - 2r}{2r(R - r)}.$$

Hence, the torus T is mean convex if $R > 2r$ and weakly mean convex if $R = 2r$. Thus, a first conclusion is that the torus is a classic example of a domain which in some cases can only be mean (or weakly mean) convex, but in general is not convex. In these cases, namely when $R \geq 2r$, the 'relaxed convexity condition'- $\Delta d_T \geq 0$ of Barbatis, Filippas and Tertikas [35] is true (i.e. see [260]). In addition, Balinsky, Desmond Evans and Lewis in [261](see Corollary 3.7.6), proved that for all $u \in C_0^\infty(T)$, the following inequality holds

$$\int_T |\nabla d \cdot \nabla u|^p dx \geqslant \left(\frac{p-1}{p}\right)^p \int_T \frac{|u|^p}{d_T^p} dx + \left(\frac{p-1}{p}\right)^{p-1} \int_T \left(\frac{1}{r - d_T} - \frac{1}{\sqrt{x_1^2 + x_2^2}}\right) \frac{|u|^p}{d_T^{p-1}} dx,$$

Where $x \in T$ has coordinates (x_1, x_2, x_3).

In our case, we have not put any constrain on R and r since we do not need any convexity on T and we will prove a Hardy inequality with weights in an arbitrary torus. We extend a theorem proved in convex domains, by Psaradakis (see Theorem 2.11 in [278]), to the case of the torus.

We now need to define the function

$$X(\tau) = (1 - \ln\tau)^{-1}, \tau \in (0,1].$$

Theorem (5.3.9) [259]: Let T be the 3-dimensional solid torus $p > 1$ and $s > 1$. Then, there exists a constant $B = B(p,s) \geq 1$ so that for all $u \in H_G^{1,p}(T)$,

$$\int_T \frac{|\nabla u|^p}{d_T^{s-p}} dx - \left(\frac{s-1}{p}\right)^p \int_T \frac{|u|^p}{d_T^s} dx \geqslant C(p,s) \int_T \frac{|u|^p}{d_T^s} X^2\left(\frac{d_T}{C_T}\right) dx, \quad (120)$$

where $C(p,s)=\frac{p-1}{2p}\left(\frac{s-1}{p}\right)^{p-2}$ and $C_T=B\sup_{x\in T}d_T(x)$.
The weight function X^2 is optimal, in the sense that the power 2 cannot be decreased, and the constant on the right-hand side is the best possible.
Proof: By repeating the same procedure as in the beginning of the Theorem(5.3.3), we work in the small torus T_δ (defined in Theorem(5.3.3)). Let an arbitrary function $u\in H_G^{1,p}(T_\delta)$ and $\phi=u\circ\xi^{-1}$ (see (71)). Then, by (101) and (102), we obtain successively

$$\int_{T_\delta}\frac{|\nabla u|^p}{d_{T_\delta}^{s-p}}dx-\left(\frac{s-1}{p}\right)^p\int_{T_\delta}\frac{|u|^p}{d_{T_\delta}^{s}}dx=$$
$$2\pi\delta^{2-s}R_p\int_D\frac{|\nabla\phi|^p}{d_D^{s-p}}\left(1+\frac{\delta}{R_p}t\right)dtds-\left(\frac{s-1}{p}\right)^p 2\pi\delta^{2-s}R_p\int_D\frac{|\phi|^p}{d_D^{s}}\left(1+\frac{\delta}{R_p}t\right)dtds\geqslant$$
$$2\pi\delta^{2-s}R_p\int_D\frac{|\nabla\phi|^p}{d_D^{s-p}}(1-\varepsilon_0)dtds-\left(\frac{s-1}{p}\right)^p 2\pi\delta^{2-s}R_p\int_D\frac{|\phi|^p}{d_D^{s}}(1-\varepsilon_0)dtds=$$
$$(1-\varepsilon_0)2\pi\delta^{2-s}R_p\left(\int_D\frac{|\nabla\phi|^p}{d_D^{s-p}}dtds-\left(\frac{s-1}{p}\right)^p\int_D\frac{|\phi|^p}{d_D^{s}}dtds\right),$$

and due to Theorem 2.11(ii) in [278], we obtain

$$\int_{T_\delta}\frac{|\nabla u|^p}{d_{T_\delta}^{s-p}}dx\quad-\left(\frac{s-1}{p}\right)^p\int_{T_\delta}\frac{|u|^p}{d_{T_\delta}^{s}}dx$$
$$\geqslant(1-\varepsilon_0)2\pi\delta^{2-s}R_pC(p,s)\int_D\frac{|\phi|^p}{d_D^{s}}X^2\left(\frac{d_D}{C_D}\right)dtds,\tag{121}$$

Where $C_D=\sup_{x\in D}d_DB(p,s)=B(p,s)$.
If we define $C_{T_\delta}=\sup_{x\in T_\delta}d_{T_\delta}B(p,s)$, we observe that

$$\frac{d_D}{C_D}=\frac{d_{T_\delta}}{rB(p,s)}=\frac{d_{T_\delta}}{r\sup_{x\in D}d_DB(p,s)}=\frac{d_{T_\delta}}{r\sup_{x\in T_\delta}d_{T_\delta}B(p,s)}=\frac{d_{T_\delta}}{C_{T_\delta}}.\tag{122}$$

Combining (121) with (76), by (122), we get

$$\int_{T_\delta}\frac{|\nabla u|^p}{d_\delta^{s-p}}dx-\left(\frac{s-1}{p}\right)^p\int_{T_\delta}\frac{|u|^p}{d_{T_\delta}^{s}}dx\geqslant\frac{1-\varepsilon_0}{1+\varepsilon_0}C(p,s)\int_{T_\delta}\frac{|u|^p}{d_T^{s}}X^2\left(d_{T_\delta}/C_{T_\delta}\right)dx$$

and, if for given ε we choose $\varepsilon_0=\frac{\varepsilon}{1-\varepsilon}$ then

$$\int_{T_\delta}\frac{|\nabla u|^p}{d_\delta^{s-p}}dx-\left(\frac{s-1}{p}\right)^p\int_{T_\delta}\frac{|u|^p}{d_{T_\delta}^{s}}dx\geqslant(1-\varepsilon)C(p,s)\int_{T_\delta}\frac{|u|^p}{d_T^{s}}X^2\left(\frac{d_{T_\delta}}{C_{T_\delta}}\right)dx.\tag{123}$$

We now conclude that the best constant on the right-hand side in (123) can be less or greater than $C(p,s)$, and these possibilities should be excluded.
Suppose, by contradiction, that the best constant in the right-hand side in this last inequality can be less than $C(p,s)$. Then, we can find a function $u\in H_G^{1,p}(T_\delta)$ such that

$$\int_{T_\delta}\frac{|\nabla u|^p}{d_\delta^{s-p}}dx-\left(\frac{s-1}{p}\right)^p\int_{T_\delta}\frac{|u|^p}{d_{T_\delta}^{s}}dx<C(p,s)\int_{T_\delta}\frac{|u|^p}{d_T^{s}}X^2\left(d_{T_\delta}/C_{T_\delta}\right)dx$$

or, equivalently,

$$\frac{\int_{T_\delta} \frac{|\nabla u|^p}{d_\delta^{s-p}} dx - \left(\frac{s-1}{p}\right)^p \int_{T_\delta} \frac{|u|^p}{d_{T_\delta}^s} dx}{\int_{T_\delta} \frac{|u|^p}{d_T^s} X^2(d_{T_\delta}/C_{T_\delta}) dx} < C(p,s). \qquad (124)$$

The inequality (124) due to (101), (102) and since $\frac{d_D}{C_D} = \frac{d_{T_\delta}}{C_{T_\delta}}$, yields

$$\frac{2\pi\left(\frac{\delta}{\lambda}\right)^{2-s} \int_D \frac{|\nabla \phi|^p}{d_D^{s-p}} \left(R_p + \frac{\delta}{\lambda} t\right) dtds - \left(\frac{s-1}{p}\right)^p 2\pi\left(\frac{\delta}{\lambda}\right)^{2-s} \int_D \frac{|\phi|^p}{d_D^s} \left(R_p + \frac{\delta}{\lambda} t\right) dtds}{2\pi\left(\frac{\delta}{\lambda}\right)^{2-s} \int_D \frac{|\phi|^p}{d_D^s} X^2(d_D/C_D) \left(R_p + \frac{\delta}{\lambda} t\right) dtds}$$
$$< C(p,s),$$

and, sending $\lambda \to \infty$, arises

$$\frac{\int_D \frac{|\nabla \phi|^p}{d_D^{s-p}} dtds - \left(\frac{s-1}{p}\right)^p \int_D \frac{|\phi|^p}{d_D^s} dtds}{\int_D \frac{|\phi|^p}{d_D^s} X^2(d_D/C_D) dtds} < C(p,s). \qquad (125)$$

The inequality (125) is false due to Theorem 2.11(ii) in [278]and this part of the theorem is proved. In aim to prove that the best constant can not be greater than $C(p,s)$, we need to follow an analogous procedure.

To complete the proof of this theorem, it is sufficient to observe that the best constant in the inequality does not depend on the radius of the rotating circle (see Step 2 in Theorem(5.3.3)). Coming back to the inequality (120), we observe that as $p \to 1^+$the right hand side of it vanishes. Therefore, in this case we are looking for a remaining term and having regard to Theorem A in [89], we conclude that such a term should be in the form $B\int_T \frac{|u|}{d_T^{s-1}} dx, B \in \mathbb{R}$.

The question is what the best constant B so that the inequality

$$\int_T \frac{|\nabla u|}{d_T^{s-1}} dx - (s-1)\int_T \frac{|u|}{d_T^s} dx \geqslant B \int_T \frac{|u|}{d_T^{s-1}} dx$$

is valid for all $u \in H_G^{1,p}(T)$.

Answer to this question gives the following corollary.

Corollary (5.3.10) [259]: $B = 1$.

Proof: Following the steps of Theorem(5.3.3), we reduce the problem to the unit disk D of $\mathbb{R}^2$, namely we obtain the inequality

$$\int_T \frac{|\nabla \phi|}{d_D^{s-1}} dx - (s-1)\int_T \frac{|\phi|}{d_D^s} dx \geqslant B \int_T \frac{|\phi|}{d_D^{s-1}} dx$$

and by Theorem A in [89], arises immediately that $B = 1$.

The following theorem consists of a different kind of improvement than the addition of reminders terms of integrals of u^2 and constitutes the natural extension of the Theorem (5.3.9) of Brezis and Marcus in [36] from convex to non-convex domains.

Theorem (5.3.11) [259]: Let T be the 3-dimensional solid torus. Then, for any $u \in H_G^{1,2}(T)$,

$$\frac{1}{4}\int_T \frac{u^2}{d_T^2}\left(1 + X^2\left(\frac{d_T}{2r}\right)\right) dx \leq \int_T |\nabla u|^2 dx, \qquad (125)$$

where r is the range of the rotating circle.

Proof: Let $u \in H_G^{1,p}(T\delta)$ and $\phi = u \circ \xi^{-1}$ (see (71)). Then, by (108), arises

$$\int_{T_\delta} |\nabla u|^2 dV \geq 2\pi R_P(1-\varepsilon_0)\int_D |\nabla \phi|^2 dtds. \qquad (127)$$

Applying the Theorem 5.1 of [36] in the case of the unit disk D, we obtain

$$\begin{aligned}\int_D |\nabla \phi|^2 dtds \quad &\geqslant \frac{1}{4}\int_D \frac{\phi^2}{d_D^2}\left(1+X^2\left(\frac{d_D}{\operatorname{diam}(D)}\right)dtds\right. \\ &= \frac{1}{4}\int_D \frac{\phi^2}{d_D^2}\left(1+X^2\left(\frac{d_D}{2}\right)dtds.\right.\end{aligned}$$

Combining (127), (128)with (112)and due to (74), we have

$$\int_{T_\delta} |\nabla u|^2 dV \geq \frac{1-\varepsilon_0}{1+\varepsilon_0}\left(\frac{1}{4}\int_{T_\delta}\frac{u^2}{d_T^2}\left(1+X^2\left(\frac{d_T}{2\mathrm{r}}\right)dtds\right)\right.,$$

and if for given ε, we choose $\varepsilon_0 = \frac{\varepsilon}{2-\varepsilon'}$ by the last inequality we obtain

$$\int_{T_\delta} |\nabla u|^2 dV \geq (1-\varepsilon)\left(\frac{1}{4}\int_{T_\delta}\frac{u^2}{d_T^2}\left(1+X^2\left(\frac{d_T}{2\mathrm{r}}\right)dtds\right)\right.. \qquad (129)$$

To complete the proof it is sufficient to prove that the best constant in the inequality (129) can be neither greater nor smaller than the $\frac{1}{4}$ and that this inequality is true in the large torus T. For the first one we can, by contradiction, follow the same steps as in the proof of the first part of Theorem(5.3.3). For the second, the answer lies in Step 2 of that theorem.

We recall here that we denoted by $C_R = \{x = (x,y,0) \in T: x^2 + y^2 = R^2\}$ the circle of range R on the xy plane, and by T^*, D^* the punctured torus $T \setminus CR$ and the punctured unit disk $D \setminus \{(0,0)\}$, respectively. Then, the following theorem holds.

Theorem (5.3.12) [259]: Let T be the 3-dimensional solid torus and $p > 2$. Then, there exist constants $B = B(p) \geq 1$ and $C = C(p) > 0$ so that for all $u \in H_G^{1,p}(T^*)$,

$$\begin{aligned}&\sup_{\substack{x,x'\in T\\ x\neq x'}}\left\{\frac{|u(x)-u(x')|}{|x-x'|^{1-2/p}}X^{1/p}\left(\frac{|x-x'|}{2rB}\right)\right\} \\ \leqslant \qquad & -C\frac{1}{2\pi R}\left(\int_T |\nabla u|^p dx - \left(\frac{p-2}{p}\right)^p \int_T \frac{|u|^p}{|x|^p}dx\right)^{\frac{1}{p}}. \qquad (130)\end{aligned}$$

Moreover, the modulus of continuity $1-2/p$ is optimal and the weight function $X^{1/p}$ is optimal, in the sense that the power $1/p$ cannot be decreased.

Proof: Consider an arbitrary plane Π containing the axis zz' and let the disk

$$D_r = \{(\mu, v) \in \mathbb{R}^2: \mu^2 + v^2 < r^2\}$$

be its intersection with the torus T. Since the torus T is invariant under the action of the group $G = O(2) \times I$ (i.e. the group of the rotations around the zz' axis), we may identify each $x \in T$ with its image on the disk D_r. So, when we refer to points of the torus T, we can assume that they belong to the disk D_r. Then, for any $x, x' \in T$, we have

$$\begin{aligned}|x| = \sqrt{(x^2+y^2-R)^2+z^2} = \sqrt{(rt)^2+(rs)^2} = r\sqrt{t^2+s^2} \\ = r|\tau|\end{aligned} \qquad (131)$$

and

$$|x - x'| = \sqrt{\left(\left(\sqrt{x^2+y^2} - R\right) - \left(\sqrt{x'^2+y'^2} - R\right)\right)^2 + (z - z')}$$
$$= \sqrt{(rt - rt')^2 + (rs - rs')^2} = r\sqrt{(t-t')^2 + (s-s')^2}$$
$$= r|\tau - \tau'|, \tag{132}$$

Where $\tau = (t, s), \tau' = (t', s') \in D$, the unit disk on $\mathbb{R}^2$ centering in the origin of the axes. By Theorem B in [279], for all $\phi \in C_0^\infty(D^*)$, we obtain

$$\sup_{\substack{\tau,\tau' \in D \\ \tau \neq \tau'}} \left\{ \frac{|\phi(\tau) - \phi(\tau')|}{|\tau - \tau'|^{1-\frac{2}{p}}} X^{1/p}\left(\frac{|\tau - \tau'|}{2B}\right)\right\}$$
$$\leqslant -C\left(\int_D |\nabla\phi|^p dtds - \left(\frac{p-2}{p}\right)^p \int_D \frac{|\phi|^p}{|\tau|^p}\right)^{\frac{1}{p}}. \tag{133}$$

Due to (71), (72), (73), (131) and (132), the inequality (133) can be written, sequentially

$$\sup_{\substack{x,x' \in T \\ x \neq x'}} \left\{ \frac{|u(x) - u(x')|}{|x - x'|^{1-2/p}} X^{1/p}\left(\frac{|x - x'|}{2rB}\right)\right\}$$
$$= \sup_{\substack{\tau\tau' \in D \\ \tau \neq \tau'}} \left\{ \frac{|\phi(\tau) - \phi(\tau')|}{(r|\tau - \tau'|)^{1-\frac{2}{p}}} X^{1/p}\left(\frac{r|\tau - \tau'|}{2rB}\right)\right\}$$
$$= \frac{1}{r^{1-2/p}} \sup_{\tau,\tau' \in D} \left\{ \frac{|\phi(\tau) - \phi(\tau')|}{(|\tau - \tau'|)^{1-\frac{2}{p}}} X^{1/p}\left(\frac{|\tau - \tau'|}{2B}\right)\right\}$$
$$\leqslant \frac{1}{r^{1-2/p}} C\left(\int_D |\nabla\phi|^p dtds - \left(\frac{p-2}{p}\right)^p \int_D \frac{|\phi|^p}{|\tau|^p} dtds\right)^{1/p}$$
$$\leqslant \frac{1}{r^{1-\frac{2}{p}}} C\left(\frac{1}{(1-\varepsilon_0)2\pi R r^{2-p}} \int_T |\nabla u|^p dx - \left(\frac{p-2}{p}\right)^p \frac{1}{(1-\varepsilon_0)2\pi R r^{2-p}} \int_T \frac{|u|^p}{|x|^p} dx\right)^{\frac{1}{p}}$$
$$\leq C \frac{1}{(1-\varepsilon_0)2\pi R r^{2-p}} \left(\int_T |\nabla u|^p dx - \left(\frac{p-2}{p}\right)^p \int_T \frac{|u|^p}{|x|^p} dx\right)^{\frac{1}{p}}$$
$$= C' \frac{1}{2\pi R} \left(\int_T |\nabla u|^p dx - \left(\frac{p-2}{p}\right)^p \int_T \frac{|u|^p}{|x|^p} dx\right)^{1/p},$$

where $C' = C/(1-\varepsilon_0)$, or

$$\sup_{\substack{x,x' \in T \\ x \neq x'}} \left\{ \frac{|u(x) - u(x')|}{|x - x'|^{1-2/p}} X^{1/p}\left(\frac{|x - x'|}{2rB}\right)\right\}$$
$$\leq -\frac{C'}{2\pi R}\left(\int_T |\nabla u|^p dx - \left(\frac{p-2}{p}\right)^p \int_T \frac{|u|^p}{|x|^p} dx\right)^{\frac{1}{p}}, \tag{134}$$

for all $u \in C_{0,G}^\infty(T^*)$.

We now need some verification concerning the optimality of the modulus of continuity $1 - 2/p$. Because of the invariance with regard to rotation around the z axis presenting the torus, it behaves as a two-dimensional domain and more precisely as a disk. Actually, if $x, x' \in T, (x \neq x)$ and $\tau, \tau' \in D$ their 'images' through the transformation ξ, then for any positive parameter γ holds

$$\frac{|u(x)-u(x')|}{|x-x'|^{\gamma}} = \frac{|\phi(\tau)-\phi(\tau')|}{(r|\tau-\tau'|)^{\gamma}} = \frac{1}{r^{\gamma}} \frac{|\phi(\tau)-\phi(\tau')|}{|\tau-\tau'|^{\gamma}}.$$

Thus, the exponent $1-2/p$ in the optimal seminorm in the unit disk Dremains the same and optimal in the corresponding seminorm in the torus, too.

Theorem (5.3.13) [259]: Let T be the 3-dimensional solid torus and $p > 2$. Then, there exist constants $b = b(p) \geq 1$ and $c = c(p) > 0$ so that for all $u \in H_G^{1,p}(T)$,

$$\sup_{\substack{x,x' \in T \\ x \neq x'}} \left\{ \frac{|u(x)-u(x')|}{|x-x'|^{1-2/p}} X^{1/p} \left(\frac{|x-x'|}{2rb} \right) \right\}$$

$$\leqslant c - \frac{1}{2\pi R} \left(\int_T |\nabla u|^p dx - \left(\frac{p-1}{p} \right)^p \int_T \frac{|u|^p}{|x|^p} dx \right)^{\frac{1}{p}}. \qquad (135)$$

Moreover, the modulus of continuity $1-2/p$ is optimal.

The proof of the Theorem (5.3.13) is omitted.

Chapter 6
Bounds of Singular Integrals and Hardy Space Estimates

We introdnce new polynomial growth BMO conditions for Calder'on-Zygmund operators. These results are applied to prove that Bony paraproducts can be constructed such that they are bounded on Hardy spaces with exponents ranging all the way down to zero. We show the following BMO type weight invariance properties: for a fixed $s \geq 0$, the weighted Sobolev- BMO spaces $I_s(BMO_w)$ coincide for all $w \in A_\infty$, the weighted $p = \infty$ type Triebel-Lizorkin spaces $\dot{F}^{s,2}_{\infty,w}$ coincide for all $w \in A_\infty$, and these two classes of spaces coincide with each other as well, all of which have comparable norms up to constants depending on an A_p character of the weight $w \in A_\infty$

Section (6.1): Weighted Hardy Spaces and Discrete Littlewood-Paley Analysis

Weighted Hardy spaces have been studied extensively in the last fifty years (see, for example, GarciaGuerva [288], Strömberg-Torchinsky [310], [127]), where the weighted Hardy space was defined by using the nontangential maximal functions and atomic decompositions were derived. The relationship between H^p_w and L^p_w for $p > 1$ was considered in both one and multiparameter cases (e.g., Strömberg and Wheeden in [311]).

We consider the weighted Hardy space estimates for singular integrals using the discrete version of Calderón's identity and Littlewood-Paley theory developed in the work of Han with [292]. [292], deal with the multiparameter Hardy spaces $H^p(0 < p \leq 1)$ associated with the flag singular integrals. The H^p to H^p and H^p to L^p boundedness are proved for flag singular integrals in [292] for all $0 < p \leq 1$ which extend the L^p theory for $1 < p < \infty$ developed in NagelRicci-Stein [304]. We derive some explicit bounds, in terms of the A_q constant $[w]_q$ of theMuckenhoupt weight $w \in A_q$ (see the definition of Muckenhoupt weight below) if $q > q_w = \inf\{s: w \in A_s\}$, for the H^p_w to L^p_w mapping norms for all $0 < p \leq 1$ and H^p_w to H^p_w mapping norms for all $0 < p < \infty$ on weighted Hardy spaces for a class of singular integral operators.

In other words, we only assume that the weight w is in the class A_∞.

We recall the definition of A_p weight. For $1 < p < \infty$, a locally integrable nonnegative function w on $\mathbf{R}^n$ is said to be in A_p if

$$[w]_{A_p} = \sup_I \left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty,$$

where for every cube $I \in \mathbf{R}^n, |I|$ denotes its Lebesgue measure, and $[w]_{A_p}$ is called the A_p characteristic constant of w. For the case $p = 1, w$ is said to be in A_1 if

$$Mw(x) \leq C_1 w(x) \text{ for almost all } x \in \mathbf{R}^n$$

and for some constant C_1. If $w \in A_1$, then the quantity

$$[w]_{A_p} = \sup_{I \subset \mathbf{R}^n} \left(\frac{1}{|I|}\int_I w(x)dx\right) \|w^{-1}\|_{L^\infty(I)}$$

is called the A_1 characteristic constant of w. Finally, we define

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p.$$

For $w \in A_\infty$, we denote by $q_w = \inf\{q: w \in A_q\}$ the critical index of w.

It is well known that if $w \notin A_p$, then T may not be bounded on L^p_w. However, it does not contradict with our results since in general $H^p_w \neq L^p_w$

when $w \notin A_p$ for $p > 1$. see Strömberg and Wheeden [311] where the relations between L_u^p and H_u^p of the real line are studied in the case when $p > 1$ and $u(x) = |Q(x)|^p w(x)$, where $Q(x)$ is a polynomial and $w(x)$ satisfies the Muckenhoupt A_p condition. It turns out that H_u^p and L_u^p can be identified when all the zeros of Q are real, and that otherwise H_u^p can be identified with a certain proper subspace of L_u^p.
The growth of the A_p constants on classical weighted estimates in L^p spaces for $1 < p < \infty$ for the Hardy-Littlewood maximal function, singular integrals, and fractional integrals has been investigated extensively. See Buckley [285], Petermichl and Volberg [308], Petermichl [306], [307], Lacey, Moen, Pérez, and Torres [297], Lerner [300], [301], Lerner, Ombrosi, and Pérez [302], [303], Lacey, Petermichl, and Reguera [296] and Hytonen, Lacey, Reguera, and Vagharshakyan [295], etc.
Buckley [285] showed that for $1 < p < \infty, w \in A_p$, the Hardy-Littlewood maximal operator M satisfies

$$\| M \|_{L^p(w) \to L^p(w)} \leq c[w]_{A_p}^{1/(p-1)}, \quad \| M \|_{L^p(w) \to L^{p,\infty}(w)} \leq c[w]_{A_p}^{1/p}$$

and the exponent $1/(p-1)$ is the best possible. A new and rather simple proof of both Muckenhoupt's and Buckley's results were recently given by Lerner [301]. It is shown in [300] that the $L^p(w)(1 < p < \infty)$ operator norms of Littlewood-Paley operators are bounded by a multiple of $[w]_{A_p}^{\gamma_p}$, where $\gamma_p = \max\left\{1, \frac{p}{2}\right\} \frac{1}{p-1}$.
For the singular integrals, Petermichl and Volberg [308] proved for the Ahlfors Beurling transform and Petermichl [306], [307] proved for the Hilbert transform and the Riesz transforms the following estimates:

$$\| T \|_{L_w^p} \leq c_{p,n} [w]_p^{\max\left\{1, \frac{1}{p-1}\right\}}, \; 1 < p < \infty,$$

when the operator T is any one of the aforementioned operators and the exponent $\max\left\{1, \frac{1}{p-1}\right\}$ is the best possible. Very recently, Lacey, Petermichl, and Reguera [296] and Hytonen, Lacey, Reguera, and Vagharshakyan [295] proved sharp bounds in terms of linear $[w]_{A_2}$ constant on weighted L^2 space and sharp bounds in terms of $[w]_{A_p}$ constant on weighted L^p spaces for Haar Shift Operators, respectively. As a corollary to their main result they deduced sharp A_p inequalities for T being either the Hilbert transform in dimension $d = 1$, the Beurling transform in dimension $d = 2$, or a Riesz transform in any dimension $d \geq 2$. Let T_* denote the maximal truncations of these operators. They proved weighted weak and strong-type L_w^p inequalities:

$$\|T_*\|_{L_w^{p,\infty}} \leq [w]_{A_p} \| f \|_{L_w^p}, \; 1 < p < 2,$$

and

$$\|T_*\|_{L_w^p} \leq [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \| f \|_{L_w^p}, \; 1 < p < \infty.$$

These estimates are sharp in the power of the A_p characteristic of the weight w, and are consistent with the best possible bounds without the truncations.
In the work of Dragicevic, Grafakos, Pereyra, and Petermichl [287], sharp L_w^p estimates in terms of $[w]_{A_p}$ in the Rubio de Francia extrapolation theorem [290] have been established. In particular, the main result of [287] shows that if a sublinear operator T is bounded on L_w^2 with the linear bound for $\| T \|_{L_w^2}$ in terms of $[w]_{A_p}$, then T is bounded on L_w^2 for $1 < p <$

∞, and $\| T \|_{L_w^p}$ is at most a multiple of $[w]_{A_p}^{\alpha_p}$ with $\alpha_p = \max\left\{1, \frac{1}{p-1}\right\}$. Therefore, the sharp L_w^2 bound for the Hilbert and Riesz transforms along with extrapolation shows that for these operators the best possible exponent α_p can be achieved for all $p > 1$. For more general singular integrals, the question about the best power of $[w]_{A_p}$ in the operator norm on L_w^p is still open.

In [302] and [303], Lerner, Ombrosi, and Pérez derived some results related to the weak Muckenhoupt and Wheeden conjecture for the Calderón-Zygmund operator , they proved that

$$\| T \|_{L^p(w)\to L^p(w)} \le Cpp'[w]_{A_1} \qquad (1 < p < \infty),$$
$$\| T \|_{L^1(w)\to L^{1,\infty}(w)} \le C[w]_{A_1} \qquad \left(1 + \log[w]_{A_1}\right).$$

Motivated by these results and recent works on discrete Littlewood-Paley theory and Calderón's identity in multiparameter settings [292] and [293], in the present we will describe the explicit dependence of the corresponding $H_w^p \to L_w^p (0 < p \le 1)$ and $H_w^p \to H_w^p (0 < p < \infty)$ operator norms of singular integrals in terms of the A_q characteristic constant of $w \in A_q$ for arbitrary $q > q_w = \inf\{s: w \in A_s\}$ A singular integral operator is defined as follows.

Definition (6.1.1)[282]: A one-parameter kernel on $\mathbf{R}^n$ is a distribution K on $\mathbf{R}^n$ which coincides with a C^∞ function away from the origin and satisfies

(i) (Differential Inequalities) For all multi-indices α, and $\forall x \neq 0$,

$$|\partial^\alpha K(x)| \le C_\alpha |x|^{-n-|\alpha|}. \qquad (1)$$

(ii) (Cancellation Condition) For any normalized bump function ϕ on $\mathbf{R}^n$ and any $R > 0$,

$$\left|\int_{\mathbf{R}^n} K(x)\phi(Rx)dx\right| \le C, \qquad (2)$$

where C is a constant independent of ϕ and $R > 0$. An operator with a oneparameter kernel is called a (one-parameter) singular integral operator.

Remark (6.1.2) [282]: There is another way to describe the cancellation condition (ii), that is

$$\left|\int_{\varepsilon<|x|<N} K(x)dx\right| \le C, \text{ for any } 0 < \varepsilon < \infty. \qquad (3)$$

Under the hypothesis of condition (i), the L^2 boundedness of T holds if and only if any one of the cancellation conditions (ii) or (iii) holds (see [126]).

Fefferman and Stein [114] first obtained the H^p boundedness of these operators for $0 < p \le 1$. In the weighted case, when $w \in A_1, n/(n+1) < p \le 1$, Lin and Lee [298] applied the weighted molecular theory and atomic decomposition to obtain the H_w^p boundedness of these operators.

We obtain H_w^p boundedness of T by only assuming $w \in A_\infty$ and derive the explicit operator norm bounds of the singular integrals on weighted Hardy spaces. This is accomplished by using discrete Littlewood-Paley theory similar to that developed earlier in [292]. Indeed, boundedness of singular integrals on weighted multiparameter Hardy spaces $H_w^p(\mathbb{R}^n \times \mathbb{R}^m)$ has been established in [286] by only assuming $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Generalization of such results to weighted Hardy spaces of arbitrary number of parameters has been done in [309]. However, no explicit constants for the bounds of singular integrals are given in [286], [309]. We begin by recalling some properties of weight functions.

Proposition (6.1.3) [282]: [291] Let $w \in A_p$ for some $1 \le p < \infty$. Then

(i) $\left[\delta^{\lambda}(w)\right]_{A_p} = [w]_{A_p}$, where $\delta^{\lambda}(w)(x) = w(\lambda x_1, \dots, \lambda x_n), \lambda \in \mathbf{R}$.
(ii) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, where $\tau^z(w)(x) = w(x - z), z \in \mathbf{R}^n$.
(iii) $[\lambda w]_{A_p} = [w]_{A_p}$ for all $\lambda > 0$.
(iv) When $1 < p < \infty, \sigma = w^{-1/(p-1)} \in A_p$ with characteristic constant $[\sigma]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$.
(v) $[w]_{A_p} \geq 1$ for all $w \in A_p$. Equality holds if and only if w is a constant.
(vi) For $1 \leq p < q < \infty$, we have $[w]_{A_q} \leq [w]_{A_p}$. And $\lim_{q \to 1^+} [w]_{A_q} = [w]_{A_1}$.
(vii) The measure $w(x)dx$ is doubling: precisely, for all $\lambda > 1$ and all cubes Q we have

$$w(\lambda Q) \leq \lambda^{np} [w]_{A_p} w(Q).$$

Let ψ be a Schwartz function on $\mathbf{R}^n$ which satisfies

$$\int_{\mathbf{R}^n} \psi(x) x^{\alpha} dx = 0, \text{ for all multi - indices } \alpha \tag{4}$$

and

$$\sum_{j \in \mathrm{Z}} \left|\hat{\psi}\left(2^{-j\xi}\right)\right|^2 = 1, \text{ for all } \xi \neq 0. \tag{5}$$

Strictly speaking, the classical Hardy spaces H^p should be defined by using bounded distributions or distributions modulus polynomials; see [291] and [126]. For our purpose here, we need to introduce some new class which is similar to distributions modulus polynomials.

Definition (6.1.4) [282]: A function $f(x)$ defined on $\mathbf{R}^n$ is said to be in $\mathcal{S}_M(\mathbf{R}^n)$ where M is a positive integer, if $f(x)$ satisfies the following conditions:
(i) For $|\alpha| \leq M - 1$,

$$|D^{\alpha} f(x)| \leq C \frac{1}{(1 + |x|)^{n+M+|\alpha|}};$$

(ii) For $|x - x'| \leq \frac{1}{2}(1 + |x|)$ and $|v| = M$,

$$|D^v f(x) - D^v f(x')| \leq C \frac{|x - x'|}{(1 + |x|)^{n+2M}};$$

(iii) For $|\alpha| \leq M - 1$,

$$\int_{\mathbf{R}^n} f(x) x^{\alpha} dx = 0.$$

If $f \in \mathcal{S}_M(\mathbf{R}^n)$ the norm of f in $\mathcal{S}_M(\mathbf{R}^n)$ is then defined by

$$\| f \|_{\mathcal{S}_M(\mathbf{R}^n)} = \inf \{C: \text{(i) and (ii) hold}\}.$$

It is easy to check that $\mathcal{S}_M(\mathbf{R}^n)$ with this norm is a Banach space. Denote by $\left(\mathcal{S}_M(\mathbf{R}^n)\right)'$ the dual of $\mathcal{S}_M(\mathbf{R}^n)$.

For $f \in \left(\mathcal{S}_M(\mathbf{R}^n)\right)'$, define the Littlewood-Paley square function of f by

$$\mathrm{g}(f)(x) = \left\{\sum_{j \in Z} \left|\psi_j * f(x)\right|^2\right\}^{\frac{1}{2}}, \tag{6}$$

where $\psi_j(x) = 2^{-jn} \psi\left(2^{-j}\right)$.

Now we give the definition of one-parameter weighted Hardy spaces H_w^p on $\mathbf{R}^n$.

Definition (6.1.5) [282]: Let $0 < p < \infty, w \in A_\infty$. Let $M = [(2q_w/p - 1)n] + 1$, where [.] denotes the integer function. The one-parameter weighted Hardy spaces are defined by

$$H_w^p(\mathbf{R}^n) = \{f \in (\mathcal{S}_M)' : f \in L_w^p(\mathbf{R}^n)\}$$

and the norm of f in $H_w^p(\mathbf{R}^n)$ is defined by

$$\| f \|_{H_w^p(\mathbf{R}^n)} = \| g(f) \|_{L_w{}^p(\mathbf{R}^n)}.$$

The main result is the following theorem.

[286], showed that the weighted Hardy spaces defined by discrete Littlewood-Paley operators are the same as the classical ones defined by a smooth maximal function (see [288] and [127]). Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \varphi(x)dx = 1$ and the maximal function defined as follows

$$f^*(x) = \sup_{t>0} |\varphi_t * f(x)|$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then weighted Hardy space $\mathcal{H}_w^p(\mathbf{R}^n)$ consists of those tempered distributions for which $f^* \in L_w^p$ with $\| f \|_{\mathcal{H}_w^p} = \|f^*\|_{L_w}^p$.

We end with the following remarks. First of all, a sharp contrast with the weighted L^p boundedness results (where $w \in A_p$ was often required) is that we establish the weighted boundedness of singular integrals on Hardy spaces $H_w^p(\mathbf{R}^n)$ by only assuming $w \in A_\infty$. This also significantly improves the earlier known results on weighted Hardy spaces (see, e.g., [298]). This is accomplished by employing the discrete Littlewood- Paley analysis. We mention in passing that consideration of weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ with $w \in A_\infty(\mathbb{R}^n)$ was given earlier; see [289], [33], and also the more recent work [284], [286], [299]. Second, we are not aware if our results of the operator norm bounds for the singular integrals are sharp or not. In particular, unlike in the case of L^p bounds $(1 < p < \infty)$ the definition of the weighted Hardy spaces depends on the choice of the Schwartz functions we use. We are able to determine a nice bound when the definition is given in terms of the discrete Littlewood-Paley square functions. As a consequence, we are also deriving the bounds when an equivalent definition is taken into account using the discrete Littlewood-Paley analysis.
We first establish the discrete Calderón identity.
Then we prove that the weighted Hardy spaces are well defined by proving a Min-Max comparison principle with an explicit bound. Next, we obtain the bound control of the weighted L_w^p norms of a function in a dense class of H_w^p by their weighted H_w^p norms. To do this, we need to establish an alternative discrete Calderón identity with Schwartz function with compact support. Finally, we prove Theorem (6.1.14) to conclude.
We shall prove the boundedness of singular integrals on weighted Hardy Spaces $H_w^p(\mathbf{R}^n)$.
We introduce some new Littlewood-Paley g function. Let ϕ be a C_0^∞ function on $\mathbf{R}^n$ supported in the unit ball and satisfying condition (5) and

$$\int_{\mathbf{R}^n} \phi(x)x^\alpha dx = 0, \text{ for } |\alpha| \le M_0, \tag{9}$$

where $M_0 \ge M$ and M is the same as in the definition of H_w^p.
We introduce discrete Littlewood-Paley g function and its maximal analogue by

$$\mathrm{g}_d(f)(x) = \left\{\sum_j \sum_Q |\phi_j * f(x_Q)|^2 \chi_Q(x)\right\}^{\frac{1}{2}}$$

and

$$g_d^{\sup}(f)(x) = \left\{\sum_j \sum_Q \sup_{u\in Q}\left|\phi_j * f(u)\right|^2 \chi_Q(x)\right\}^{\frac{1}{2}}$$

respectively, where x_Q is any point in $Q, \phi_j(x) = 2^{-jn}\phi(2^j x)$ and the summation of Q is taken over all dyadic cubes Q with side length 2^{-j-N} in $\mathbf{R}^n$ for each $j \in \boldsymbol{Z}$ and a fixed large integer N.

We need the following weighted Fefferman-Stein vector-valued inequality, for every $1 < p, r < \infty, w \in A_p$ (see [291], and also [283] for an earlier version of such an inequality without the explicit bounds):

$$\left\|\left(\sum_j \left|M(f_j)\right|^r\right)^{1/r}\right\|_{L_w^p} \le K\left(n, r, p, [w]_{A_p}\right)\left\|\left(\sum_j \left|f_j\right|^r\right)^{\frac{1}{r}}\right\|_{L_w^p} \quad (10)$$

for all sequence of functions $\{f_j\}$ in L_w^p, where if we set $N(t) = t^{\frac{1}{r-1}}$,

$$K\left(n, r, p, [w]_{A_p}\right) = \begin{cases} 2N\left(K_1(n,p,r)[w]_{A_p}^{\frac{r-1}{p-1}}\right), & \text{if } p < r, \\ 2^{\frac{pr}{r(p-1)}} N\left(K_2(n,p,r)[w]_{A_p}\right), & \text{if } p \ge r \end{cases}$$

and $K_1(n,p,r), K_2(n,p,r)$ are constants that depend only on n, p, r.

Proposition (6.1.6) [282]: If $M_0 \ge M$ in (9), then weighted Hardy spaces can be characterized by these discrete square functions. That is, for any $0 < p < \infty$,

$$\| f \|_{H_w^p} \approx \|g_d(f)\|_{L_w^p}.$$

It was pointed out in [300] that if $1 < r < \infty$ and $\in A_r$, we have the following weighted version of the Littlewood-Paley-Stein inequality:

$$\| g \|_{L_w^r \to L_w^r} \approx \|g_d\|_{L_w^r \to L_w^r} \le C_n [w]_{A_r}^{\max\left\{1, \frac{r}{2}\right\}\frac{1}{r-1}}.$$

By the duality argument together with Calderón's identity, we also have

$$\| f \|_{L_w^r} \le C_n' [w]_{A_r}^{\max\left\{1, \frac{r'}{2}\right\}} \| g(f) \|_{L_w^r}.$$

In fact, let $(x) = w(x)^{-r'/r}$, then $\sigma \in A_{r'}$.

$$\| f \|_{L_w^r} = \sup_{\|h\|_{L_\sigma}^{r'} \le 1}\left|\int f(x)h(x)dx\right| = \sup_{\|h\|_{L_\sigma^{r'}} \le 1}\left|\int \left(\sum_j \psi_j * \psi_j * f\right)(x)h(x)dx\right|$$

$$\le \sup_{\|h\|_{L_\sigma^{r'}}} \int g(f)(x)g(h)(x)dx \le \sup_{\|h\|_{L_\sigma^{r'}}} \| g(f) \|_{L_w^r} \| g(h) \|_{L_w^{r'}}$$

$$\le C_n' [\sigma]_{A_{r'}}^{max\left\{1, \frac{r'}{2}\right\}\frac{1}{r'-1}} \|g(f)\|_{L_w^r} = C_n' [w]_{A_r}^{\max\left\{1, \frac{r'}{2}\right\}\frac{1}{r'-1}} \|g(f)\|_{L_w^r}.$$

Lemma (6.1.7) [282]: [292] If ψ and ϕ are in the class $\mathcal{S}_M(\mathbf{R}^n)$, then for any given positive integers L, K, there exists a constant $C = C(L, K)$ depending only on L, K such that

$$|\psi_t * \phi_{t'}(x)| \le C\left(\frac{t}{t'} \wedge \frac{t'}{t}\right)^L \frac{(t \vee t')^K}{(t \vee t' + |x|)^{n+K}}.$$

Lemma (6.1.8) [282]: Let I, I' be dyadic cubes in $\mathbf{R}^n$ such that $l(I) = 2^{-j-N}, l(I') = 2^{-j'-N}$. Then for any $u, u^* \in I$ and any r satisfying $\frac{n}{n+K} < r \leq 1$, we have

$$\sum_{I'} \frac{2^{-|j-j'|L}|I'|2^{-(j\wedge j')K}}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+K}} |\psi_{j'} * f(x_{I'})|$$

$$\leq C 2^{-|j-j'|2_2\left(\frac{1}{r}-1\right)Nn} 2^{\left(\frac{1}{r}-1\right)n(j'-j)} + \left(M\left(\sum_{I'} |\psi_{j'} * f(x_{I'})|\chi_{I'}\right)^r (u^*)\right)^{\frac{1}{r}},$$

where $(j'-j)_+ = \max\{j'-j, 0\}, x_{I'} \in I'$ and C is a constant depending on dimension n.

Proof: We set

$$A_0 = \left\{I' : l(I') = 2^{-j'-N}, \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \leq 1\right\},$$

and for $l \geq 1$,

$$A_l = \left\{I' : l(I') = 2^{-j'-N}, 2^{l-1} < \frac{|u - x_{I'}|}{2^{-j\wedge j'}} \leq 2^l\right\}.$$

Then

$$\sum_{I'} \frac{2^{-|j-j'|L}|I'|2^{-(j\wedge j')K}}{\left(2^{-j\wedge j'} + |u - x_{I'}|\right)^{n+K}} |\psi_{j'} * f(x_{I'})|$$

$$\leq \sum_{l\geq 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n} \sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|$$

$$\leq \sum_{l\geq 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n} \left(\sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r\right)^{1/r}$$

$$= \sum_{l\geq 0} 2^{-l(n+K)} 2^{-|j-j'|L} 2^{-n(j'+N)} 2^{(j\wedge j')n}$$

$$\times \left(\int_{I'} |I'|^{-1} \sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r \chi_{I'}\right)^{1/r}$$

$$\leq \sum_{l\geq 0} 2^{-|j-j'|L} 2^{-l\left(n+K-\frac{n}{r}\right)} 2^{\left(\frac{1}{r}-1\right)Nn} 2^{\left(\frac{1}{r}-1\right)n(j'-j)_+}$$

$$\times \left(M\left(\sum_{I'\in A_l} |\psi_{j'} * f(x_{I'})|^r \chi_{I'}\right)(u^*)\right)^{1/r}$$

$$2^{-|j-j'|L} 2^{\left(\frac{1}{r}-1\right)Nn} 2^{\left(\frac{1}{r}-1\right)n(j'-j)_+} \left(M\left(\sum_{I'} |\psi_{j'} * (x_{I'})|\chi_{I'}\right)^r (u^*)\right)^{1/r}$$

the last inequality follows from the assumption that $r > \frac{n}{n+k}$ which can be done by choosing K big enough.

With the almost orthogonality estimate (Lemma (6.1.7)) and Lemma (6.1.8), we nowgive the following discrete Calderón reproducing formula.

Theorem (6.1.9) [282]: Suppose that ψ_j is the same as in (6). Then for any $M \geq 1$, we can choose a large N depending on M and ψ such that the following discrete Calderón reproducing identity:

$$f(x) = \sum_j \sum_I |I| \hat{\psi}_j(x, x_I) \psi_j * f(x_I) \tag{11}$$

holds in $\mathcal{S}_M(\mathbf{R}^n)$ and in the dual space $(\mathcal{S}_M)'$, where $\hat{\psi}_j(x, x_I) \in \mathcal{S}_M(\mathbf{R}^n), I'$ s are dyadic cubes with side-length $l(I) = 2^{-j-N}$, and x_I is a fixed point in I.

Proof : For $f \in \mathcal{S}_M$, we use the discrete Calderón identity $f(x) = \sum_j \psi_j * \psi_j * f(x)$ as follows. We rewrite

$$f(x) = \sum_j \sum_I \int_I \psi_j(x-u)\big(\psi_j * f\big)(u)du$$

$$= \sum_j \sum_I \left[\int_I \psi_j(x-u)du\right]\big(\psi_j * f\big)(x_I) + \mathcal{R}(f)(x).$$

We shall show that $\mathcal{R}$ is bounded on $\mathcal{S}_M$ with a small operator norm when the 's are dyadic cubes with side-length 2^{-j-N} for a large N, and $x_I \in I$.

$$\mathcal{R}(f)(x) = \sum_{j,I} \int_I \psi_j(x-u)\big[\big(\psi_j * f\big)(u) - \big(\psi_j * f\big)(x_I)\big]du$$

$$= \sum_{j,I} \int_I \psi_j(x-u)\left(\int \psi_j(u-u')f(u')du' - \int \psi_j(x_I-u')f(u')du'\right)du$$

$$= \int \left(\sum_{j,I} \int_I \psi_j(x-u)\big[\psi_j(u-u') - \psi_j(x_I-u')\big]du f(u')\right)du'$$

$$= \int \mathcal{R}(x, u', x_I)f(u')du'$$

where $\mathcal{R}(x, u', x_I)$ is the kernel of $\mathcal{R}$. It is not difficult to check that

$$\sum_I \int_I \psi_j(x-u)\big[\psi_j(u-u') - \psi_j(x_I-u')\big]du$$

satisfies all conditions of $\psi_j(x-x_I)$ but with the constant of $\mathcal{S}_M(\mathbf{R}^n)$ norm replaced by $C2^{-N}$. This follows from the smooth conditions of ψ_j and the fact that $u, x_I \in I, l(I) = 2^{-j-N}$. Then $\mathcal{R}(f)(x) \in \mathcal{S}_M(\mathbf{R}^n)$ and

$$\| \mathcal{R}(f) \|_{\mathcal{S}_M(\mathbf{R}^n)} \leq C2^{-N} \| f \|_{\mathcal{S}_M(\mathbf{R}^n)}. \tag{12}$$

Thus if we set

$$T(f)(x) = \sum_j \sum_I \left[\int_I \psi(x-u)du\right]\big(\psi_j * f\big)(x_I),$$

then $T^{-1} = (1-\mathcal{R})^{-1}$ exists and

$$\begin{aligned} T(x) &= T^{-1}T(f)(x) = \sum_{i=0}^{\infty} \mathcal{R}^i T(f)(x) \\ &= \sum_{j,I} \left[\sum_{i=0}^{\infty} \mathcal{R}^i \int_I \psi_j(\cdot - u)du\right](x)\big(\psi_j * f\big)(x_I). \end{aligned}$$

Set $\left[\sum_{i=0}^{\infty}\mathcal{R}^i\int_I\psi_j(\cdot - u)du\right](x) = \hat{\psi}_j(x, x_I)$. Then it follows from (12) that $\hat{\psi}_j \in \mathcal{S}_M(\mathbf{R}^n)$. Thus, the discrete Calderón identity on $\mathcal{S}_M(\mathbf{R}^n)$ is obtained. The proof of Theorem (6.1.9) is completed from the duality argument.

We give the following Plancherel-Pôlya-type inequality, i.e., the Min-Max inequality.

Theorem (6.1.10) [282]: Let $\psi, \varphi \in \mathcal{S}_M(\mathbf{R}^n)$. Suppose ψ_j and φ_j satisfy the same conditions as in (6). Then for $0 < p < \infty, w \in A_\infty, f \in (S_M)'(\mathbf{R}^n)$, and for any r satisfying $\frac{n}{n+K} < r < \min\left\{\frac{p}{q_w}, 1\right\}$,

$$\left\|\left\{\sum_{j,I} \inf_{u\in I}\left|\varphi_j * f(u)\right|^2 \chi_I(x)\right\}^{1/2}\right\| \|_{L_w^p} \quad \left\|\left\{\sum_{j,I} \sup_{u\in I}\left|\psi_j * f(u)\right|^2 \chi_I(x)\right\}^{1/2}\right\|_{L_w^p}$$

$$\leq C(n,p,r)K_2\left([w]_{A_{p/r}}\right) \quad \left\|\left\{\sum_{j,I} \inf_{u\in I}\left|\varphi_j * f(u)\right|^2 \chi_I(x)\right\}^{1/2}\right\|_{L_w^p}$$

where $I \in \mathbf{R}^n$ are dyadic cubes with side-length $l(I) = 2^{-j-N}$ for a fixed large integer N, and $K_2\left([w]_{A_{p/r}}\right)$ is as given in (80).

Proof : The discrete Calderón reproducing formula (11) on $\mathcal{S}_M(\mathbf{R}^n)$ implies that

$$\big(\psi_j * f\big)(u) = \sum_{j'I'} |I'|\big(\psi_j * \tilde{\varphi}_j\big)(u, x_{I'})\big(\varphi_j * f\big)(x_{I'}).$$

From the almost orthogonality estimates in Lemma (6.1.7) by choosing $t = 2^{-j}, t' = 2^{-j'}$ and from Lemma (6.1.8), we have that for any given positive integers L, K and for any $u, u^* \in I$,

$$\begin{aligned} \left|\psi_j * f(u)\right| &\leq C\sum_{j'I'} \frac{2^{-|j-j'|L}2^{-(j\wedge j')K}|I'|}{(2-j\wedge j'|u-x_I|)^{n+K}}\left|\varphi_{j'} * f(x_{I'})\right| \\ &\leq C\sum_{j'I'} 2^{-|j-j'|L}\left(M\left[\left(\sum_{I'} \left|\varphi_{j'} * f(x_{I'})\right|\chi_{I'}\right)^r\right]\right)^{1/r} (u^*). \end{aligned}$$

Summing over j, I yields that

$$\left(\sum_{j,I} \sup_{u\in I}\left|\psi_j * f(u)\right|^2\chi_I\right)^{1/2} \leq \left(\sum_{j'} \left\{M\left[\sum_{I'} \left|\varphi_{j'} * f(x_{I'})\right|\chi_{I'}\right]^r\right\}^{\frac{2}{r}}\right)^{\frac{1}{2}}.$$

Since $\frac{n}{n+K} < r < \min\left\{\frac{p}{q_w}, 1\right\}$, it means that $q_w < p/r$, we have $w \in A_{p/r}$.

The Hölder inequality and the $L_w^{p/r}\left(l^{2/r}\right)$ boundedness of M, i.e., the weighted Fefferman-Stein vector-valued inequality (10), yield

$$\left\|\left(\sum_{j,I}\sup_{u\in I}\left|\psi_j * f(u)\right|^2\chi_I\right)^{1/2}\right\|_{L_w^p} \le C\left\|\sum_{j'}\left\{M\left[\sum_{I'}\sup_{u\in I'}\left|\varphi_{j'} * f(u)\right|^2\chi_I\right]^r\right\}^{2/r}\right)^{1/2}\|_{L_w^p}$$

$$\le C(n,p,r)K_2\left([w]_{A_{\frac{p}{r}}}\right)\left\|\left(\sum_{j'I'}\inf_{u\in I'}\left|\varphi_{j'} * f(u)\right|^2\chi_{I'}\right)^{\frac{1}{2}}\right\|_{L_w^p},$$

where we use the fact that $\chi_{I'}$ is arbitrary in I'.

From this theorem, we know that the definition of weighted Hardy spaces is independent of the particular choice of ψ_j. Moreover, it can be characterized by the discrete Littlewood-Paley square function defined by

$$\mathcal{G}^d(f)(x) = \left\{\sum_{j,I}\left|\psi_j * f(x_I)\right|^2\chi_I(x)\right\}^{\frac{1}{2}},\ x_I \in I.$$

That is, a distribution f belongs to $H_w^p(\mathbf{R}^n)$ if and only if $\mathcal{G}^d(f) \in L_w^p(\mathbf{R}^n)$, and

$$\| f \|_{H_w^p(\mathbf{R}^n)} \approx \|\mathcal{G}^d(f)\|_{L_w^p(\mathbf{R}^n)}.$$

Proposition 3.2 in [364] tell us that when $w \in A_\infty, \mathcal{S}_M(\mathbf{R}^n)$ is dense in $H_w^p(\mathbf{R}^n)$ for $0 < p < \infty$.

To prove this theorem, we need a new version of Calderón-type identity. To be more precise, take $\phi \in C_0^\infty$ with

$$\int_{\mathbf{R}^n}\phi(x)x^\alpha dx = 0,\ \text{for all } \alpha \text{ satisfying } 0 \le |\alpha| \le M_0$$

where M_0 is a large positive integer which will be determined later (indeed $M_0 > (2q_w/p - 1)n$ suffices), and

$$\sum_j\left|\hat{\phi}\left(2^{-j}\xi\right)\right|^2 = 1,\ \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}.$$

Moreover, we may assume that ϕ is supported in the unit ball of $\mathbf{R}^n$. We need a discrete Calderón reproducing formula in terms of ϕ.

Lemma (6.1.11) [353]: There exists an operator T_N^{-1} such that

$$f(x) = \sum_{j,I}|I|\hat{\phi}_j(x - x_I)\phi_j * \left(T_N^{-1}(f)\right)(x_I) \tag{13}$$

where T_N^{-1} is bounded on $L^2(\mathbf{R}^n)$ and $H_w^p(\mathbf{R}^n), 0 < p < \infty$, and the series converges in $L^2(\mathbf{R}^n)$.

Proof : As in the proof of Theorem (6.1.9), for $f \in L^2(\mathbf{R}^n)$, the operator R is defined by the following:

$$f(x) = \sum_j\sum_I\int_I\psi(x-u)\left(\psi_j - f\right)(u)du$$

$$= \sum_j\sum_I\left[\int_I\psi_j(x-u)du\right]\left(\psi_j * f\right)(x_I) + \mathcal{R}(f)(x).$$

We claim that for $0 < p < \infty$, there is a constant $C > 0$ such that

$$\| \mathcal{R}(f) \|_{L^2(\mathbf{R}^n)} \le C2^{-N} \| f \|_{L^2(\mathbf{R}^n)},$$

and

$$\| \mathcal{R}(f) \|_{H_w^p(\mathbf{R}^n)} \leq C2^{-N} K_2\left([w]_{A_{\frac{p}{r}}}\right) \| f \|_{H_w^p(\mathbf{R}^n)}.$$

Assume the claim for the moment. Set

$$T_N(f)(x) = \sum_{j,I} \left[\int_I \phi_j(x-u)du\right] (\phi_j * f)(x_I).$$

The proof in Theorem (6.1.9) shows that if N is large enough, then both T_N and $(T_N)^{-1}$ are bounded on $H_w^p(\mathbf{R}^n) \cap L^2$. Thus,

$$f(x) = \sum_{j,I} |I| \hat{\phi}_j(x - x_I)\big(\psi * T_N^{-1}(f)\big)(x_I),$$

where $\hat{\phi}_j \in \mathcal{S}_M(\mathbf{R}^n)$ and the series converges in $L^2(\mathbf{R}^n)$.

Now we prove the claim. Suppose $f \in L^2(\mathbf{R}^n)$. By Theorem (6.1.9),

$$\| \mathcal{G}(\mathcal{R}(f)) \|_{L_w}^p \leq C \left\| \left\{ \sum_{j,I} \left|\psi_j * \mathcal{R}(f)\right|^2 \chi_I \right\}^{1/2} \right\|$$

$$= C \left\| \left\{ \sum_{j,I} \sum_{j'I'} \left| |I'| \psi_j * \mathcal{R}\left(\tilde{\psi}_{j'}(\cdot, x_{I'}) \cdot \left(\psi_{j'}^p * f\right)(x_{I'})\right)\right|^2 \chi_I \right\}^{\frac{1}{2}} \right\|_{L_w^p}.$$

By the almost orthogonality estimate

$$\left|\left(\psi_j * \mathcal{R}\left(\tilde{\psi}_{j'}(\cdot, x_{I'})\right)\right)(x)\right| \leq C2^{-N} 2^{-|j-j'|M} \frac{2^{-(j\wedge j')M}}{\left(2^{-j\wedge j'} + |x - x_{I'}|\right)^{n+M}}.$$

Then from Lemma (6.1.8), Hölder's inequality, and the $L_w^{p/r}\left(l^{2/r}\right)$ boundedness of the maximal operator $\left(w \in A_{p/r}\right)$, we have

$$\| \mathcal{R}(f) \|_{H_w^p}^p$$

$$\leq 2^{-N} \left\| \left(\sum_{j'} \left[M \left(\sum_{I'} \left|\psi_j * f(x_{I'})\right| \chi_{I'} \right)^r \right]^{2/r} \right)^{1/2} \right\|_{L_w^p}$$

$$\leq C2^{-N} K_2\left([w]_{A_{\frac{p}{r}}}\right) \left\| \left(\sum_{j'I'} \left|\psi_{j'} * f(x_{I'})\right|^2 \chi_{I'} \right)^{\frac{1}{2}} \right\|_{L_w^p} \leq C2^{-N} K_2\left([w]_{A_{p/r}}\right) \| f \|_{H_w^p}.$$

Another inequality in the claim follows immediately by taking $w = 1$ and $p = 2$ in the above inequality. Then the proof of Lemma (6.1.11) is completed.

Using a similar argument as in the proof of Theorem (6.1.10), we can get

Corollary (6.1.12) [282]: Suppose $w \in A_\infty$. If $f \in L^2 \cap H_w^p(\mathbf{R}^n), 0 < p < \infty$, then

$$\| f \|_{H_w^p(\mathbf{R}^n)} \approx \left\| \left\{ \sum_{j,I} \left|\left(\phi_j * T_N^{-1}(f)\right)(x_I)\right|^2 \chi_I \right\}^{1/2} \right\|_{L_w^p}.$$

Now we prove Theorem (6.1.13).

Theorem (6.1.13) [282]: If $f \in L^2(\mathbf{R}^n) \cap H_w^p(\mathbf{R}^n), 0 < p \le 1$, then $f \in L_w^p(\mathbf{R}^n)$, and there exists a constant $C(n,p,q) > 0$ such that

$$\| f \|_{L_w(\mathbf{R}^n)} \le C(n,p,q) K_1\left([w]_{A_q}\right)^2 [w]_{A_q}^{\frac{1}{p}+\mathrm{mqx}\left\{1+\frac{q'}{2}\right\}} \| f \|_{H_w^p(\mathbf{R}^n)'}$$

where q is fixed such that $q > q_w, K_1\left([w]_{A_q}\right)$ is given as in (70).

Proof: We may assume $w \in A_q$ for some $2 < q < \infty$. Define a square function by

$$\tilde{\mathrm{g}}(f)(x) = \left\{\sum_{j,I} \left|\phi_j * \left(T_N^{-1}(f)\right)(x_I)\right|^2 \chi_I(x)\right\}^{\frac{1}{2}}.$$

By Corollary (6.1.12), for $f \in L^2 \cap H_w^p$, we have

$$\| \tilde{g}(f) \|_{L_w^p} \le C \| f \|_{H_w^p}.$$

Let $f \in L^2 \cap H_w^p$, set

$$\Omega_i = \left\{x \in \mathbf{R}^n : \tilde{g}(f)(x) > 2^i\right\}.$$

Denote

$$B_i = \left\{(j,I) : w(I \cap \Omega_i) > \frac{1}{2} w(I), w(I \cap \Omega_{i+1}) \le \frac{1}{2} w(I)\right\}$$

where I are dyadic cubes with side-length $l(I) = 2^{-j-N}$.

We use ϕ_I to denote ϕ_j when $l(I) = 2^{-j-N}$. By the discrete Calderón reproducing formula (13), we can write

$$f(x) = \sum_i \sum_{(j,I)\in B_i} |I| \tilde{\phi}_I(x - x_I) \phi_I * (T_N^{-1}(f))(x_I)$$

where the series converges in L^2 norm and hence almost everywhere and also w almost everywhere.

We claim

$$\left\| \sum_{(j,I)\in B_i} |I| \tilde{\phi}_I(x - x_I) \phi_I * (T_N^{-1}(f))(x_I) \right\|_{L_w^p}^p$$

$$\le C(n,p,q) K_1\left([w]_{A_q}\right)^{2p} [w]_{A_q}^{1+\max\left\{1,\frac{q'}{2}\right\}p} 2^{pi} w(\Omega_i) \tag{14}$$

which together with the fact that $0 < p \le 1$ yields

$$\begin{aligned}
\| f \|_{L_w^p}^p &\le \sum_i \left\| \sum_{(j,I)\in B_i} |I| \tilde{\phi}_I(x - x_I) \phi_I * (T_N^{-1}(f))(x_I) \right\|_{L_w^p}^p \\
&\le C K_1\left([w]_{A_q}\right)^{2p} [w]_{A_q}^{1+\max\{1,q'/2\}p} \sum_i 2^{pi} w(\Omega_i) \\
&\le C K_1\left([w]_{A_q}\right)^{2p} [w]_{A_q}^{1+\max\{1,q'/2\}p} \| \tilde{g}(f) \|_{L_w^p}^p \\
&\le C K_1\left([w]_{A_q}\right)^{2p} [w]_{A_q}^{1+\max\{1,q'/2\}p} \| f \|_{H_w^p}^p.
\end{aligned}$$

To finish the proof, it remains to show the claim (14). Note that for $(j,I)\in B_i$, if $x\in I$, then $M(\chi_{\Omega_i})(x)\geq 1/2$. And note that if ϕ is supported in the unit ball, then $\phi_j(x-x_I)$ is supported in $\widetilde{\Omega}_i=\left\{x: M(\chi_{\Omega_i})(x)>\frac{1}{100}\right\}$. Thus for any fixed $q>q_w$, by Hölder's inequality,

$$\left\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I)\right\|_{L_w^p}^p$$

$$\leq Cw(\widetilde{\Omega}_i)^{1-p/q}\left\|\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I)\right\|_{L_w^p}^p.$$

By the duality argument, for all $h\in L^{q'}(w^{1-q'})$ with $\| f \|_{L^{q'}(w^{1-q'})}\leq 1$.

$$\left|\left\langle\sum_{(j,I)\in B_i}|I|\tilde{\phi}_I(x-x_I)\phi_I*(T_N^{-1}(f))(x_I),h\right\rangle\right|$$

$$=\left|\left\langle\sum_{(j,I)\in B_i}|I|\left(\tilde{\phi}_I*h(x_I)\right),\left(\phi_I*(T_N^{-1}(f))(x_I)\right)\right\rangle\right|$$

$$=\left|\sum_{(j,I)\in B_i}\int_I\left(\tilde{\phi}_I*h(x_I)\right)\left(\phi_I*(T_N^{-1}(f))(x_I)\right)\chi_I(x)dx\right|$$

$$\leq\left(\int\left(\sum_{(j,I)\in B_i}|\phi_I*(T_N^{-1}(f))(x_I)|^2\chi_I(x)\right)^{q/2}w(x)dx\right)^{1/q}$$

$$\times\left(\int\left(\sum_{(j,I)\in B_i}\left|\tilde{\phi}_I*h(x_I)\right|^2\chi_I(x)\right)^{q'/2}w(x)^{1-q'}dx\right)^{1/q'}=\Lambda_1\cdot\Lambda_2.$$

We first estimate Λ_2. Since $w\in A_q$ implies $w^{1-q'}\in A_{q'}$, by the weighted Fefferman-Stein inequality, we have the following estimate:

$$\Lambda_2\leq\left\|\left\{\sum_{(j,I)\in B_i}\left(M(\tilde{\phi}_I*h)\right)^2\chi_I(x)\right\}^{\frac{1}{2}}\right\|_{L^{q'}(w^{1-q'})}$$

$$\leq C_1K_1\left([w]_{A_q}\right)\| g(h) \|_{L^{q'}(w^{1-q'})}\leq C_1K_1\left([w]_{A_q}\right)[w]_{A_q}^{\max\left\{1,\frac{q'}{2}\right\}}. \quad (15)$$

As for Λ_1, since $\chi_I(x)\leq 2M\left(\chi_{\mathrm{In}(\tilde{\Omega}_i\setminus\Omega_{i+1})}\right)(x)$, then using the weighted Fefferman-Stein inequality (10) again, we have

$$\Lambda_1^q = \left\| \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(f)(x_I)|^2 \chi_I(x) \right\}^{\frac{1}{2}} \right\|_{L^q(w)}^q$$

$$= \int \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(f)(x_I)|^2 \chi_I(x) \right\}^{1/2} w(x)dx$$

$$\leq C \int \left\{ \sum_{(j,I)\in B_i} \left|\phi_I * T_N^{-1}(x_I) M\left(\chi_{I\cap\tilde{\Omega}_i\setminus\Omega_{i+1}}\right)(x)\right|^2 \right\}^{q/2} w(x)dx$$

$$\leq C_2 K_1 \left([w]_{A_q}\right)^q \int_{\tilde{\Omega}_i\setminus\Omega_{i+1}} \left\{ \sum_{(j,I)\in B_i} |\phi_I * T_N^{-1}(x_I)|^2 \chi_I(x) \right\}$$

$$\leq C_2 K_1 \left([w]_{A_q}\right)^q 2^{iq}\left(\tilde{\Omega}_i\right). \qquad (16)$$

Note that $\Omega_i \subset \tilde{\Omega}_i$, and by the weak $L^q(w)$ boundedness of the maximal operator, $w\left(\tilde{\Omega}_i\right) \leq C[w]_{A_q} w(\Omega_i)$. Combining these estimates for Λ_1 and Λ_2 proves claim (14). Thus we complete the proof of Theorem (6.1.13).

Theorem (6.1.14) [282]: Let $w \in A_\infty$. The one-parameter singular integral operator T is bounded on H_w^p for $0 < p < \infty$ and bounded from H_w^p to L_w^p for $0 < p \leq 1$. Namely, if r satisfies $\frac{n}{n+M} < r < \min\left\{\frac{p}{q_w}, 1\right\}$ and $q > q_w$ (where q_w is the critical index of the weight w defined above), then

$$\| T(f) \|_{H_w^p} \leq C(n,p,r) K_1\left([w]_{A_{\frac{p}{r}}}\right) \| f \|_{H_w^p}^p \quad 0 < p < \infty,$$

$$\| T(f) \|_{H_w^p(R^n)} \leq C(n,p,q,r) K_1\left([w]_{A_q}\right)^2 K_2\left([w]_{A_{p/r}}\right) [w]_{A_q}^{\frac{1}{p}+\max\left\{1,\frac{q'}{2}\right\}} \| f \|_{H_w^p(\mathbf{R}^n)},\ 0 < p \leq 1,$$

where M is the constant in Definition (6.1.4) and constants $K_1\left([w]_{A_q}\right)$ and $K_2\left([w]_{A_{p/r}}\right)$ are defined as follows:

$$K_1\left([w]_{A_q}\right) = \begin{cases} [w]_{A_q}^{\frac{1}{q-1}}, & \text{if } q \leq 2, \\ [w]_{A_q} & \text{if } q > 2, \end{cases} \qquad (70)$$

and

$$K_2\left([w]_{A_{p/r}}\right) = \begin{cases} \left([w]_{A_{\frac{p}{r}}}\right)^{\frac{r}{p-r}}, & \text{if } p \leq 2, \\ \left([w]_{A_{\frac{p}{r}}}\right)^{\frac{r}{2-r}}, & \text{if } p > 2. \end{cases} \qquad (80)$$

Proof: Since $L^2 \cap H_w^p$ is dense in H_w^p, by the standard density argument, we assume $f \in L^2 \cap H_w^p(\mathbf{R}^n)$. Using Lemma(6.1.11) and Corollary (6.1.12), we have for $0 < p < \infty$

$$\| T(f) \|_{H^p_w} \leq \left\| \left\{ \sum_{j,I} |\phi_I * K * f(x)|^2 \chi_I(x) \right\}^{\frac{1}{2}} \right\|_{L^p_w}$$

$$= C \left\| \left\{ \sum_{j,I} \left[\sum_{j'I'} |I'| \left(\phi_j * K * \tilde{\phi}_{j'}(\cdot - x_{I'}) \right)(x) \right. \right. \right.$$

$$\left. \left. \times \left(\phi_{j'} * T_N^{-1}(f) \right)(x_{I'}) \right]^2 \chi_I(x) \right\}^{1/2} \|_{L6p_w}$$

$$\leq C \left\| \sum_{j'} \left\{ M \left[\sum_{I'}^{p} \left| \phi_{j'} * (T_N^{-1}(f))(x_{I'}) \right| \chi_{I'} \right]^r \right\}^{2/r} \right)^{1/2} \|_{L^p_w}$$

$$\leq CK_2 \left([w]_{A_{p/r}} \right) \left\| \left\{ \sum_{j'I'} \left| \phi_{j'} * (T_N^{-1}(f))(x_{I'}) \right|^2 \chi_{I'} \right\}^{1/2} \right\|_{L^p_w}$$

$$\leq CK_2 \left([w]_{A_{p/r}} \right) \| f \|^p_{H^p_w}$$

where in the second-to-the-last inequality, we use the following almost orthogonality estimate:

$$\left| \left(\phi_j * \tilde{\phi}_{j'}(\cdot - x_{I'}) \right)(x) \right| \leq C 2^{-|j-j'|M} \frac{2^{-(j-j')M}}{\left(2^{-(j \wedge j')} + |x - x_{I'}| \right)^{n+M}}.$$

When $0 < p \leq 1$, since T is bounded on $L^2 \cap H^p_w(\mathbf{R}^n)$, we have $T(f) \in L^2 \cap H^p_w(\mathbf{R}^n)$ whenever $f \in L^2 \cap H^p_w(\mathbf{R}^n)$. Thus from Theorem (6.1.13),

$$\| T(f) \|_{L^p_w(\mathbf{R}^n)} \leq CK_1 \left([w]_{A_q} \right)^2 [w]_{A_q}^{\frac{1}{p} + \max\left\{1, \frac{q'}{2}\right\}} \| T(f) \|_{H^p_w(\mathbf{R}^n)}$$

$$\leq CK_1 \left([w]_{A_q} \right)^2 K_2 \left([w]_{A_{\frac{p}{r}}} \right) [w]_{A_q}^{\frac{1}{p} + \max\left\{1 \frac{q'}{2}\right\}} \| f \|_{H^p_w(\mathbf{R}^n)}.$$

By a density argument again, we complete the proof.

Section (6.2): Littlewood-Paley-Stein Square Functions and Calderon-Zygmund Operators

We prove new Hardy space $H^p(\mathbb{R}^n)$ bounds for Littlewood-Paley- Stein square functions and Calderon-Zygmund integral operators where the index p is allowed to be small.

We draw an explicit connection between Calder'onZygmund operators and Littlewood-Paley-Stein square functions.

It is well known by now that one way to define the real Hardy spaces H^p for $0 < p < \infty$ is by using certain convolution-type Littlewood-Paley-Stein square functions. This has been explored by many mathematicians; some of the fundamental developments of this idea can be found in the work of Stein [328], [329] and Fefferman and Stein [114]. In particular, Fefferman and Stein proved that one can define $H^p = H^p(\mathbb{R}^n)$ using square functions of the form

$$S_Q f(x) = \left(\sum_{k \in \mathbb{Z}} |Q_k f(x)|^2 \right)^{\frac{1}{2}},$$

associated to integral operators $Q_k f = \psi_k * f$ for an appropriate choice of Schwartz function $\psi \in S$, where $\psi_k(x) = 2^{kn}\psi(2^k x)$. There are also results in the direction of determining the most general classes of such convolution operators that can be used to define Hardy spaces, or more generally Triebel-Lizorkin spaces; see for example the work of Bui, Paluszy'nski, and Taibelson [316], [111]. Generalized classes of non-convolution type Littlewood-Paley-Stein square function operators were studied, for example, in [319], [320], [327]. Although all of the bounds in these articles are relegated to Lebesgue spaces with index $p \in (1, \infty)$, which for this range of indices coincide with Hardy spaces.

We consider a general class of non-convolution type Littlewood-Paley-Stein square function operators acting on Hardy spaces with indices smaller than 1.

Before we state our Hardy space estimates for Littlewood-Paley-Stein square functions, we define our classes of Littlewood-Paley-Stein square function operators. Given kernel functions $\lambda_k: \mathbb{R}^{2n} \to \mathbb{C}$ for $k \in \mathbb{Z}$, define

$$\Lambda_k f(x) = \int_{\mathbb{R}^n} \lambda_k(x,y) f(y) dy$$

for appropriate functions $f: \mathbb{R}^{2n} \to \mathbb{C}$. Define the square function associated to $\{\Lambda_k\}$ by

$$S_\Lambda f(x) = \left(\sum_{k \in \mathbb{Z}} |\Lambda_k f(x)|^2 \right)^{1/2}.$$

We say that a collection of operators Λ_k for $k \in \mathbb{Z}$ is a collection of Littlewood-Paley-Stein operators with decay N and smoothness $L + \delta$, written $\{\Lambda_k\} \in \mathrm{LPSO}(N, L+\delta)$, for $N > 0$, an integer $L \geq 0$ and $0 < \delta \leq 1$, if there exists a constant C such that

$$|\lambda_k(x,y)| \leq C\Phi_k^N(x-y) \tag{17}$$

$$|D_1^\alpha \lambda_k(x,y)| \leq C 2^{|\alpha|k} \Phi_k^N(x-y) \text{ for all } |\alpha| = \alpha_1 + \cdots + \alpha_n \leq L \tag{18}$$

$$|D_1^\alpha \lambda_k(x,y) - D_1^\alpha \lambda_k(x,y')| \leq C|y-y'|^\delta 2^{k(L+\delta)} \left(\Phi_k^N(x-y) + \Phi_k^N(x-y') \right) \text{ for all } |\alpha| = L. \quad (19)$$

Here we use the notation $\varphi_k^N(x) = 2^{kn}(1 + 2^k|x|)^{-N}$ for $N > 0, x \in \mathbb{R}^n$, and $k \in \mathbb{Z}$. We also use the notation $D_0^\alpha F(x,y) = \partial_x^\alpha F(x,y)$ and $D_1^\alpha F(x,y) = \partial_y^\alpha F(x,y)$ for $F: \mathbb{R}^{2n} \to \mathbb{C}$ and $\alpha \in \mathbb{N}_0^n$. It can easily be shown that $\mathrm{LPSO}(N, L+\delta) \subset \mathrm{LPSO}(N', L+\delta')$ for all $0 < \delta' \leq \delta \leq 1$ and $0 < N' \leq N$.

We study square functions of the form S_Λ is to prove boundedness properties from H^p into L^p. Note that it is not reasonable to expect S_Λ to be bounded from H^p into H^p when $0 < p \leq 1$ since $S_\Lambda f \geq 0$. It is also not hard to see that the condition $\{\Lambda_k\} \in LPSO(N, L+\delta)$ alone, for any $N > 0, L \geq 0$, and $0 < \delta \leq 1$, is not sufficient to guarantee that S_Λ to be bounded from H^p into L^p for any $0 < p < \infty$. In fact, this is not true even in the convolution setting. This can be seen by taking $\lambda_k(x,y) = \varphi_k(x-y)$ for some $\varphi \in S$ with non-zero integral, where $\varphi_k(x) = 2^{kn}\varphi(2^k x)$.

The square function S_Λ associated to this convolution operator is not bounded from H^p into L^p for any $0 < p < \infty$. Hence some additional conditions are required for Λ_k in order to assure H^p to L^p bounds. For $1 < p < \infty$, this problem was solved in terms of Carleson measure conditions on $\Lambda_k 1(x)$; see for example [317], [318], [326], [327]. We give sufficient conditions for such bounds when the index p is allowed to range smaller than 1 .

The additional cancellation conditions we impose on Λ_k involve generalized moments for non-concolution operators Λ_k. Define the moment function $[[\Lambda_k]]_\beta(x)$ by the following. Given $\{\Lambda_k\} \in \mathrm{LPSO}(N, L+\delta)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < N - n$

$$[[\Lambda_k]]_\alpha(x) = 2^{k|\alpha|} \int_{\mathbb{R}^n} \lambda_k(x,y)(x-y)^\alpha dy$$

for $k \in \mathbb{Z}$ and $x \in \mathbb{R}^{2n}$. It is worth noting that $[[\Lambda_k]]_0(x) = \Lambda_k 1(x)$, which is a quantity that is closely related to L^2 bounds for S_Λ, see for example [319], [320], [327]. We use these moment functions to provide sufficient conditions of H^p to L^p bounds for S_Λ in the following theorem.

Theorem (6.2.1)[312]: Let $\{\Lambda_k\} \in LPSO(N, L+\delta)$, where $N = n + 2L + 2\delta$ for some integer $L \geq 0$ and $0 < \delta \leq 1$. If

$$d\mu_\alpha(x,t) = \sum_{k\in\mathbb{Z}} \left|[[\Lambda_k]]_\alpha(x)\right|^2 \delta_{t=2^{-k}} dx \qquad (20)$$

is a Carleson measure for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq L$, then Λ_k can be extended to a bounded operator from H^p into L^p for all $\frac{n}{n+L+\delta} p \leq 1$.

Here we say that a non-negative measure $d\mu(x,t)$ on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$ is a Carleson measure if there exists $C > 0$ such that $d\mu(Q \times (0, \ell(Q))) \leq C|Q|$ for all cubes $Q \subset \mathbb{R}^n$, where $\ell(Q)$ denotes the sidelength of Q. We only prove a sufficient condition here for boundedness of S_Λ from H^p into L^p, but it is reasonable to expect that the Carleson measure conditions in (20) are also necessary.

We also provide a quick corollary of Theorem (6.2.1) to the type of operators studied in [319], [320], [327], among others.

Corollary (6.2.2) [312]: Let $\{\Lambda_k\} \in \mathrm{LPSO}(n+2\delta, \delta)$ and $0 < \delta \leq 1$. If S_Λ is bounded on L^2, then S_Λ extends to a bounded operator from H^p into L^p for all $\frac{n}{n+\delta} < p \leq 1$.

Corollary (6.2.2) easily follows from Theorem (6.2.1) and the following observation. If S_Λ is bounded on L^2, then $d\mu_0(x,t)$, as defined in (20) for $\alpha = 0$, is a Carleson measure; see [317], [326] for proof of this observation.

We prove a characterization of Hardy space bounds for Calder'onZygmund operators. Some of the earliest development of singular integral operators on Hardy spaces is due to Stein and Weiss [330], Stein [329], and Feffermand and Stein [114]. It was proved by Fefferman and Stein [114] that if T is a convolution-type singular integral operator that is bounded on L^2, then T is bounded on H^p for $p_0 < p < \infty$ where $0 \leq p_0 < 1$ depends on the regularity of the kernel of T. This situation is considerably more complicated in the non-convolution setting, which can be observed in the $T1$ type theorems in [319], [331], [322], [321], [313]. In the 1980's David and Journ'e proved the celebrated $T1$ theorem that provided necessary and sufficient conditions for Lebesgue space L^p bounds for non-convolution Calder'on-Zygmund operators when $1 < p < \infty$, which coincides with the Hardy space bounds for this range of indices. [331], [322], [321], give sufficient $T1$ type conditions for a Calder' on-Zygmund operator to be bounded on H^p for $0 < p \leq 1$. The conditions in [331], [322], [321] are too strong though, in the sense that they are not necessary for Hardy space bounds. The fact that the conditions in [331], [322], [321] are not necessary can be seen by the full necessary and sufficient conditions provided in [313] when $p_0 < p \leq 1$, where $p_0 = \frac{n}{n+\gamma}$ and γ is a regularity parameter for the kernel of T. This can also be seen by considering the Bony paraproduct, which we prove is bounded on H^p for $p_0 < p \leq 1$ and p_0 can be taken arbitrarily close to zero. One We prove at full necessary and sufficient $T1$ type theorem for Calder'on-Zygmund operators on Hardy spaces, thereby generalizing results pertaining to H^p bounds from [313], [114], [331]. [321], [322].

We say that a continuous linear operator T from S into S' is a Calderon-Zygmund operator with smoothness $M+\gamma$, for any integer $M \geq 0$ and $0 < \gamma \leq 1$, if T has function kernel $K: \mathbb{R}^{2n} \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ such that

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{2n}} K(x,y) f(y) g(x) dy dx$$

whenever f, g $\in C_0^\infty = C_0^\infty(\mathbb{R}^n)$ have disjoint support, and there is a constant $C > 0$ such that the kernel function K satisfies

$$\left|D_0^\alpha D_1^\beta K(x,y)\right| \leq \frac{C}{|x-y|^{n+|\alpha|+|\beta|}} \text{ for all } |\alpha|, |\beta| \leq M,$$

$$\left|D_0^\alpha D_1^\beta K(x,y) - D_0^\alpha D_1^\beta K(x',y)\right| \leq \frac{C|x-x'|^\gamma}{|x-y|^{n+M+|\beta|+\gamma}} \text{ for } |\beta| \leq |\alpha| = M, |x-x'| < |x-y|/2,$$

$$\left|D_0^\alpha D_1^\beta K(x,y) - D_0^\alpha D_1^\beta K(x,y')\right| \leq \frac{C|y-y'|^\gamma}{|x-y|^{n+|\alpha|+M+\gamma}} \text{ for } |\alpha| \leq |\beta| = M, |y-y'| < |x-y|/2.$$

We will also define moment distributions for an operator $T \in CZO(M+\gamma)$, but we require some notation first. For an integer $M \geq 0$, define the collections of smooth functions of polynomial growth $O_M = O_M(\mathbb{R}^n)$ and of smooth compactly supported function with vanishing moments $\mathcal{D}_M = \mathcal{D}_M(\mathbb{R}^n)$ by

$$O_M = \left\{f \in C^\infty(\mathbb{R}^n) \colon \sup_{x\in\mathbb{R}^n} |f(x)| \cdot (1+|x|)^{-M} < \infty\right\} \text{ and}$$

$$\mathcal{D}_M = \left\{f \in C_0^\infty(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} f(x) x^\alpha dx = 0 \text{ for all } |\alpha| \leq M\right\}.$$

Let $h \in C_0^\infty(\mathbb{R}^n)$ be supported in $B(0,2), \eta(x) = 1$ for $x \in B(0,1)$, and $0 \leq \eta \leq 1$. Define for $R > 0, \eta_R(x) = \eta(x/R)$. We reserve this notation for η and η_R throughout. [331], [322], [321], define Tf for $f \in O_M$ where T is a linear singular integral operator. We give an equivalent definition to the ones in [331], [322], [321]. Let T be a CZO$(M+\gamma)$ and $f \in O_M$ for some integer $M \geq 0$ and $0 < \gamma \leq 1$. For $\psi \in C_0^\infty(\mathbb{R}^n)$, choose $R_0 \geq 1$ minimal so that $\text{supp}(\psi) \subset \overline{B(0,R_0/4)}$, and define

$$\langle Tf, \psi\rangle = \lim_{R\to\infty} \langle T(\eta_R f), \psi\rangle - \sum_{|\beta|\leq M} \int_{\mathbb{R}^{2n}} \frac{D_0^\beta K(0,y)}{\beta!} x^\beta \big(\eta_R(y) - \eta_{R_0}(y)\big) f(y) \psi(x) dy dx$$

This limit exists based on the kernel representation and kernel properties for $T \in CZO(M+\gamma)$ and is independent of the choice of η, see [331], [322], [321] for proof of this fact. The choice of R_0 here is not of consequence as long as R_0 is large enough so that $\text{supp}(\psi) \subset \overline{B(0,R_0/4)}$; we choose it minimal to make this definition precise. The definition of $\langle Tf, \psi\rangle$ depends on ψ here through the support properties of $\psi \in C_0^\infty$, but for $\psi \in \mathcal{D}_M$, it follows that $\langle Tf, \psi\rangle = \lim_{R\to\infty} \langle T(\eta_R f), \psi\rangle$ since the integral term above vanishes for such ψ. Now we define the moment distribution $[[T]]_\alpha \in \mathcal{D}_M'$ for $T \in CZO(M+\gamma)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ by

$$\langle [[T]]_{\alpha'} \psi\rangle = \lim_{R\to\infty} \int_{\mathbb{R}^{2n}} K(u,y) \psi(u) \eta_R(y) (u-y)^\alpha dy du$$

for $\psi \in \mathcal{D}_{|\alpha|}$, where $K \in S'(\mathbb{R}^{2n})$ is the distribution kernel of T. We abuse notation here in that the integral in this definition is not necessarily a measure theoretic integral; rather, it is the dual pairing between elements of $S(\mathbb{R}^{2n})$ and $S'(\mathbb{R}^{2n})$. We will use K to denote distributional kernels and K to denote function kernels for Calderon-Zygmund operators. When we write K in an integral over $\mathbb{R}^{2n}$, the integral is understood to be a the pairing of $K \in S'(\mathbb{R}^{2n})$ with an element of $S(\mathbb{R}^{2n})$. It is not hard to show that this definition is well-defined by techniques from [331], [322], [321]. This distributional moment associated to T generalizes the notion of $T1$ as used in [319] in the sense that $\langle [[T]]_0, \psi \rangle = \langle T1, \psi \rangle$ for all $\psi \in \mathcal{D}_0$ and hence $[[T]]_0 = T1$. We will also use a generalized notion of BMO here to extend the cancellation conditions $T1, T^*1 \in BMO$, which were used in the $T1$ theorem from [319]. Let $M \geq 0$ be an integer and $F \in \mathcal{D}'_M/P$, that is $\mathcal{D}'_M$ modulo polynomials. We say that $F \in BMO_M$ if

$$\sum_{k\in\mathbb{Z}} 2^{2Mk}|Q_k F(x)|^2 dx\delta_{t=2^{-k}}$$

is a Carleson measure for any $\psi \in \mathcal{D}_M$, where $\psi_k f = \psi_k * f$ and $\psi_k(x) = 2^{kn}\psi(2^k x)$. This definition agrees with the classical definition of BMO. That is, for $F \in BMO_0$,

$$\sum_{k\in\mathbb{Z}} 2^{2Mk}|Q_k F(x)|^2 dx\delta_{t=2^{-k}}$$

is a Carleson measure, and hence $F \in BMO$ by the BMO characterization in terms of Carleson measures in [317], [326]. A similar polynomial growth BMO_M was defined by Youssfi [332]. We use this polynomial growth BMO_M to quantify our cancellation conditions for operators $T \in CZO(M+\gamma)$ in the following result.

Theorem (6.2.3) [312]: Let $T \in CZO(M+\gamma)$ be bounded on L^2 and define $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + \gamma)/2$. If $T^*(x^\alpha) = 0$ in $\mathcal{D}'_M$ for all $|\alpha| \leq L$ and $[[T]]_\alpha \in BMO_{|\alpha|}$ for all $|\alpha| \leq L$, then T extends to a bounded operator on H^p for $\frac{n}{n+L+\delta} < p \leq 1$.

Recall here that the operator T^* is defined from S into S' via $\langle T^* f, g \rangle = \langle T^* g, f \rangle$, and the definition of T^* is extended to an operator from O_M to $\mathcal{D}'_M$ by the methods discussed above. Note also that this is not a full necessary and sufficient theorem for Hardy space bounds as described above. This theorem will be used to prove the boundedness of certain paraproduct operators, which in turn allow us to prove the full necessary and sufficient theorem, which is stated in Theorem (6.2.7).

The choice of L and δ here are such that $L \geq 0$ is an integer, $0 < \delta \leq 1$, and $2(L+\delta) = M + \gamma$. It is also not hard to see that $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$ if and only if $\left[[T^*]\right]_\alpha = 0$ for all $|\alpha| \leq L$. We prove Theorem (6.2.7) by decomposing an operator $T \in CZO(M+\gamma)$ into a collection of operators $\{\Lambda_k\} \in \mathrm{LPSO}(n + 2L + 2\delta, L + \delta')$ for $0 < \delta' < \delta$ and applying Theorem (6.2.1). This decomposition of T into a collection of Littlewood-Paley-Stein operators is stated precisely in the next theorem.

Theorem (6.2.4) [312]: Let $T \in CZO(M+\gamma)$ for some integer $M \geq 1$ and $0 < \gamma \leq 1$ be bounded on L^2, and fix $\psi \in \mathcal{D}_M$. Also let $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + \gamma)/2$. If $T^*(x^\alpha) = 0$ in $\mathcal{D}'_M$ for all $|\alpha| \leq L$, then $\{\Lambda_k\} \in \mathrm{LPSO}(n + 2L + 2\delta, L + \delta')$ for all $0 < \delta' < \delta$, where $\Lambda_k = Q_k T$ and $Q_k f(x) = \psi_k * f(x)$

Furthermore, for $\frac{n}{n+L+\delta} < p \leq 1, T$ extends to a bounded operator on H^p if and only if S_Λ extends to a bounded operator from H^p into L^p.

Throughout, we write $L^p = L^p(\mathbb{R}^n)$ and $H^p = H^p(\mathbb{R}^n)$ for $0 < p < \infty$. We will also apply Theorem (6.2.7) to Bony paraproducts operator, which were originally defined in [315] and famously applied in the $T1$ theorem [319] (see also [314]). Let $\psi \in \mathcal{D}_{L+1}$ for some $L \geq 0$ and $\varphi \in C_0^\infty$. Define $Q_k f = \psi_k * f$ and $P_k f = \varphi_k * f$. For $\beta \in BMO$, define

$$\Pi_\beta f(x) = \sum_{j\in\mathbb{Z}} Q_j\big(Q_j\beta \cdot P_j f\big)(x). \tag{21}$$

It easily follows that $\Pi_\beta \in$ CZO $O(M+\gamma)$ for all $M \geq 0$ and $0 < \gamma \leq 1$. It is well known that $\Pi_\beta^*(1) = 0$, and if one selects ψ and φ appropriately, it also follows that $\Pi_\beta(1) = \beta$ in BMO as well. We are not interested in an exact identification of $\Pi_\beta(1)$ in this work, so we don't worry about the extra conditions that should be imposed on ψ and φ to assure that $\Pi_\beta(1) = \beta$.

Theorem (6.2.5) [312]: Let Π_β be as in (21) for $\beta \in BMO, \psi \in \mathcal{D}_{L+1}$, and $\varphi \in C_0^\infty$. Then Π_β is bounded on H^p for all $\frac{n}{n+L+1} < p \leq 1$.

By Theorem (6.2.5) it is possible to construct Π_β so that it is bounded on H^p for $p > 0$ arbitrarily small by choosing $\psi \in \mathcal{D}_{L+1}$ for L sufficiently large. It should be noted that some Hardy space estimates for a variant of the Bony paraproduct in (21) were proved in [324]. Although we use a different construction of the paproduct, so we will prove Theorem (6.2.5)here as well. Finally, we state the first necessary and sufficient boundedness theorem for Calder' on-Zygmund operators on Hardy spaces.

We dedicated to Littlewood-Paley-Stein square functions and proving Theorem (6.2.1). we prove the singular integral operator results in Theorems (6.2.3) and (6.2.4).

We apply Theorem (6.2.7) to the Bony paraproducts to prove Theorem (6.2.5). we use Theorem (6.2.5) and a result from [321], [322], [331] to prove Theorem (6.2.7).

Applying the first estimate above, we finish the proof.

$$\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}} |P_k f(x)|^2\mu_k(x)\right)^{p/2} dx \leq \int_{\mathbb{R}^n} sp_k|P_k f(x)|^{(2-p)p/2}\left(\sum_{k\in\mathbb{Z}} |P_k f(x)|^p\mu_k(x)\right)^{p/2}$$

$$\leq \left\|(\mathcal{N}^\varphi f)^{\frac{(2-p)p}{2}}\right\|_{L^{r'}}\left\|\left(\sum_{k\in\mathbb{Z}} |P_k f(x)|^p\mu_k(x)\right)^{p/2}\right\|_{L^{r'}}$$

$$= \|\mathcal{N}^\varphi f\|_{L^p}^{\frac{p(2-p)}{2}}\left(\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}} |P_k f(x)|^p\mu_k(x)dx\right)^{p/2}$$

$$\lesssim \|f\|_{H^p}^{\frac{p(2-p)}{2}} \| f \|_{H^p}^{\frac{p^2}{2}} = \| f \|_{H^p}^p.$$

we prove Theorem (6.2.1). To do this, we first prove a reduced version of the theorem.

We apply Theorem (6.2.7) to show that the Bony paraproduct operators from [315] are bounded on H^p, which was stated in Theorem (6.2.5). Let $\psi \in \mathcal{D}_{L+1}$ for some $L \geq 0$ and $\varphi \in C_0^\infty$. Note that $|\widehat{W}(\xi)| \lesssim \min(|\xi|, |\xi|^{-1})$ as well, and since $T_{V(\mu)}\beta \in BMO$ with $\|T_{V(\mu)}\beta\| \lesssim \| \beta \|_{BMO}$, it also follows that

$$\frac{1}{|Q|}\int_Q \sum_{2^{-k}\leq \ell(Q)} 2^{2|\alpha|k}\left|\left\langle\left[\left[\Pi_\beta\right]\right]_\alpha, \psi_k^x\right\rangle\right|^2 \lesssim \sum_{\mu\leq a}\left|c_{\alpha,\mu}C_{\alpha-\mu}\right|^2\int_Q \sum_{2^{-k}\leq\ell(Q)}\left|W_k * \big(T_{V(\mu)}\beta\big)(x)\right|^2$$

$$\lesssim \left\|T_{V(\mu)}\beta\right\|_{BMO}^2 \lesssim \| \beta \|_{BMO}^2.$$

Therefore $\left[\left[\Pi_\beta\right]\right]_\alpha \in BMO_{|\alpha|}$ for $|\alpha| \le L$, and by Theorem (6.2.7) it follows that Π_β is bounded on H^p for all $\frac{n}{n+L+\delta} < p \le 1$, where $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + 1)/2$.
Finally, we return to the proof of Theorem (6.2.7). We have waited to this point to do so since we will need both Theorem (6.2.3) and the Bony paraproduct construction in Theorem (6.2.5).
We need one other result from [396]; we state Theorem 3.13 from [321], [322], [331] adapted to our notation and restricted to the Hardy space setting.
Theorem(6.2.6) [312]: ([322]). Let $T \in CZO(M+\gamma)$ be bounded on L^2 and define $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + \gamma)/2$. If $T^*(x^\alpha) = 0$ in $\mathcal{D}'_M$ for all $|\alpha| \le L$ and $T_1 = 0$ in $\mathcal{D}_0$, then T is bounded on H^p for all $\frac{n}{n+L+\delta} < p \le 1$.
In the notation of [322], this theorem is stated with $q = 2, 0 < p \le 1, J = n/p, L = U - n\rfloor = \lfloor n/p - n \rfloor, \alpha = 0$, and $H^p = \dot{F}_P^{0,2}$.
Theorem (6.2.7) [312]: Let $T \in CZO(M+\gamma)$ be bounded on L^2 and define $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + \gamma)/2$. Then $T^*(x^\alpha) = 0$ in $\mathcal{D}'_M$ for all $|\alpha| \le L$ if and only if T extends to a bounded operator on H^p for $\frac{n}{n+L+\delta} < p \le 1$.
Proof: Let $T \in CZO(M+\gamma)$ be bounded on L^2 and define $L = \lfloor M/2 \rfloor$ and $\delta = (M - 2L + \gamma)/2$. Assume that $T^*(x^\alpha) = 0$ in $\mathcal{D}'_M$ for all $|\alpha| \le L$. Then $T1 \in BMO$, and by Theorem (6.2.5) there exists $\Pi \in CZO(M+1)$ such that $\Pi(1) = T(1), \Pi^*(y^\alpha) = 0$ for $|\alpha| \le M$, and Π is bounded on H^p for all $\frac{n}{n+L+\delta} < p \le 1$. Then $T = S + \Pi$, where $S = T - \Pi$. Noting that $S^*(y^\alpha) = 0$ for all $|\alpha| \le L$ and $S1 = 0$, by Theorem (6.2.8) it follows that S is bounded on H^p for all $\frac{n}{n+L+\delta}$. Therefore T is bounded on H^p for all $\frac{n}{n+L+\delta} < p \le 1$.
Now assume that T is bounded on H^p for all $\frac{n}{n+L+\delta} < p \le 1$. For $\psi \in \mathcal{D}_L$, it follows that $T_\psi \in H^p \cap L^2$ for all $\frac{n}{n+L+\delta} < p \le 1$. It is not hard to show that

$$\int_{\mathbb{R}} T\psi(x) x^\alpha dx$$

is an absolutely convergent integral for any $|\alpha| < \sup\left\{n/p - n : \frac{n}{n+L+\delta} < p \le 1\right\} = L + \delta$.
By Theorem 7 in [323], it follows that

$$\int_{\mathbb{R}} T\psi(x) x^\alpha dx = 0$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < L + \delta$. Since $\delta > 0$, this verifies that $T^*(y^\alpha) = 0$ for all $|\alpha| \le L$.
Corollary (6.2.8) [356]: Let $f_{\mathrm{r}}: R^n \to$ Ca non-negative continuous function, $\varepsilon \ge 0$, and $\frac{n}{n+v} < \varepsilon \le 1$. Then

$$\sum_{\ell(Q)=2^{-(j+(1+\varepsilon))_0}} \sum |Q| \Phi^{n+1+\varepsilon}_{\min(j,k)}\left(x - c_Q\right) f_{\mathrm{r}}\left(c_Q\right) \lesssim 2^{\max(0,j-k)(1+\varepsilon)} \sum \mathcal{M}_j^{1+2\varepsilon} f_{\mathrm{r}}(x)$$

for all $x \in R^n$, where $\mathcal{M}_j^{1+\varepsilon}$ is defined and the summation indexed by $\ell(Q) = 2^{-(j+(1+\varepsilon)_0)}$ is the sum over all dyadic cubes with side length $2^{-(j+(1+\varepsilon)_0)})$ and c_Q denotes the center of cube Q.
Proof Define

$$A_0 = \left\{Q \text{ dyadic} : \ell(Q) = 2^{-(j+(1+\varepsilon)_0)} \text{ and } \left|x - c_Q\right| \le 2^{-(j+(1+\varepsilon)_0)}\right\}$$

$A_\ell = \{Q \text{ dyadic} : \ell(Q) = 2^{-(j+(1+\varepsilon)_0)} \text{ and } 2^{\ell-1-(j+(1+\varepsilon)_0)} < |x - c_Q| \le 2^{\ell-(j+(1+\varepsilon)_0)}\}$ for $\ell \ge 1$. Now for each $Q \in A_0$

$$\Phi^{n+1+\varepsilon}_{\min (j,k)}(x - c_Q) = \frac{2^{\min (j,k)n}}{\left(1 + 2^{\min (j,k)}|x - c_Q|\right)^{n+1+\varepsilon}} \le 2^{\min (j,k)n} \le 2^{jn}$$

and for each $Q \in A_\ell$ when $\ell \ge 1$

$$\Phi^{n+1+\varepsilon}_{\min (j,k)}(x - \quad c_Q) = \frac{2^{\min (j,k)n}}{\left(1 + 2^{\min (j,k)}|x - c_Q|\right)^{n+1+\varepsilon}} \le \frac{2^{\min (j,k)n}}{(1 + 2^{\min (j,k)}2^{\ell-1-(j)})^{n+1+\varepsilon}}$$

$$\le 2^{\min (j,k)n}2^{-(n+1+\varepsilon)\min (j,k)}2^{-(n+1+\varepsilon)\ell+n+1+\varepsilon+(n+1+\varepsilon)(j-(1+\varepsilon)_0)}$$

$$\lesssim 2^{\max (0,j-k)(1+\varepsilon)}2^{-(n+1+\varepsilon)\ell}2^{jn}$$

Since $U_\ell A_\ell$ makes up the collection of all dyadic cubes with side length $2^{-(j+(1+\varepsilon)_0)}$, it follows that

$$\sum_{\ell(Q)=2^{-(j+(1+\varepsilon)_0)}} \quad \sum_{\ell=0} |Q|\Phi^{n+1+\varepsilon}_{\min (j,k)}(x - c_Q)f_{\mathrm{r}}(c_Q)$$

$$= \sum_{Q\in A_\ell} \sum_{\ell} 2^{-(j+(1+\varepsilon)_0)n}\Phi^{n+1+\varepsilon}_{\min (j,k)}(x - c_Q)f_{\mathrm{r}}(c_Q)$$

$$\lesssim \sum_{Q\in A_0} \sum_{\mathrm{r}} (c_Q)2^{\max (0,j-k)1+\varepsilon} \sum_{\ell=1}^{\infty} 2^{\ell(n+1+\varepsilon)} \sum_{Q\in A_\ell} \sum_{\mathrm{r}} (c_Q)$$

$$\le 2^{\max (0,j-k)1+\varepsilon} \sum_{\ell=0}^{\infty} 2^{\ell(n+1+\varepsilon)} \left(\sum_{Q\in A_\ell} \sum_{\mathrm{r}} (c_Q)^{1+2\varepsilon}\right)^{\frac{1}{1+2\varepsilon}}$$

For $Q \in A_\ell$ and $x + \varepsilon \in Q$ it follows that

$$|x - x + \varepsilon| \le |x - c_Q| + |x + \varepsilon - c_Q| \le 2^{-(j+(1+\varepsilon)_0)} + 2^{\ell-(j+(1+\varepsilon)_0)} \le 2^{\ell+1-(j+(1+\varepsilon)_0)}$$

Hence $\cup_{Q\in A_\ell} Q \subset B\left(x, 2^{\ell+1-(j+(1+\varepsilon)_0)}\right)$. We also have that $|A_\ell| \ge 2^{n(\ell-2)}$; so

$$\left|\bigcup_{Q\in A_\ell} Q\right| \ge 2^{-(j+(1+\varepsilon)_0)n}2^{n(\ell-2)} = 2^{-2n}2^{(\ell-(j+(1+\varepsilon)_0))n}$$

$$\ge |B(0,1)|^{-1}2^{-2n}\left|B\left(0, 2^{\ell-(j+(1+\varepsilon)_0)}\right)\right|.$$

Now we estimate the sum in Q above:

$$\sum_{Q\in A_\ell}\sum f_{\mathrm{r}}(c_Q)^{1-2\varepsilon} \leq \frac{1}{|U_{Q\in A_\ell}Q|}\int_{U_{Q\in A_\ell}Q} \chi_{\cup_{Q\in A_\ell}Q}(y)\sum_{Q\in A_\ell}\sum f_{\mathrm{r}}(c_Q)^{1+2\varepsilon} d(x+\varepsilon)$$

$$\leq \frac{1}{|U_{Q\in A_\ell}Q|}\int_{U_{Q\in A_\ell}Q} 2^{(\ell-1)n}\sum_{Q\in A_\ell}\sum f_{\mathrm{r}}(c_Q)^{1+2\varepsilon}\chi_Q(y)d(x+\varepsilon)$$

$$\lesssim \frac{2^{\ell n}}{|B(x,2^{\ell-(j+(1+\varepsilon)_0)})|}\int_{B(x,2^{\ell-(j+(1+\varepsilon)_0)})}\sum_{Q\in A_\ell}\sum f_r(c_Q)^{1+2\varepsilon}\chi_Q(x+\varepsilon)d(x+\varepsilon)$$

$$= \frac{2^{\ell n}}{|B(x,2^{\ell-(j+(1+\varepsilon)_0)})|}\int_{B(x,2^{\ell-(j+(1+\varepsilon)_0)})}\left(\sum_{Q\in A_\ell}\sum f_{\mathrm{r}}(c_Q)\chi_Q(x+\varepsilon)\right)^{1+2\varepsilon} d(x+\varepsilon)$$

$$\lesssim 2^{\ell n}\mathcal{M}\left[\left(\sum_{Q\in A_l}\sum f_{\mathrm{r}}(c_Q)\chi_Q\right)^{1+2\varepsilon}\right](x).$$

Then we have that

$$\sum_{\ell(Q)=2^{-(j+(1+\varepsilon)_0)}}\sum |Q|\Phi^{n+1+\varepsilon}_{\min\,(j,k)}(x-c_Q)\sum f_{\mathrm{r}}(c_Q)$$

$$\lesssim 2^{\max\,(0,j-k)1+\varepsilon}\sum_{\ell=0}^{\infty} 2^{-\ell(n+1+\varepsilon-n/r)}\left\{\mathcal{M}\left[\left(\sum_{Q\in A_\ell}\sum f_{\mathrm{r}}(c_Q)\chi_Q\right)^{1+2\varepsilon}\right](x)\right\}^{\frac{1}{1+2\varepsilon}}$$

$$\lesssim 2^{\max\,(0,j-k)1+\varepsilon}\left\{\mathcal{M}\left[\left(\sum_{\ell(Q)=2^{-(j+N)}}\sum f_{\mathrm{r}}(c_Q)\chi_Q\right)^{1+2\varepsilon}\right](x)\right\}^{\frac{1}{1+2\varepsilon}}$$

Corollary (6.2.9) [356]: Suppose

$$d\mu(x,t) = \sum_{k\in\mathbb{Z}} \mu_k(x)\delta_{t=2^{-k}}dx$$

is a Carleson measure, where μ_k is a non-negative, locally integrable function for all $k\in\mathbb{Z}$. Also let $\varphi_{\mathrm{r}}\in\mathcal{Y}$, and define $P_k(\sum f_{\mathrm{r}}) = \sum(\varphi_{\mathrm{r}})_k * f_{\mathrm{r}}$, where $(\varphi_{\mathrm{r}})_k(x) = 2^{kn}\varphi_{\mathrm{r}}(2^k x)$ for $k\in\mathbb{Z}$. Then

$$\left\|\left(\sum_{k\in\mathbb{Z}}\sum |P_k f_{\mathrm{r}}|^{1+\varepsilon}\mu_k(x)\right)^{\frac{1}{1+\varepsilon}}\right\|_{(1+\varepsilon)^{1+\varepsilon}} \lesssim \sum \|f_{\mathrm{r}}\|_{H^{1+\varepsilon}} \text{ for all } 0<\varepsilon\leq\infty$$

and

$$\left\|\left(\sum_{k\in\mathbb{Z}}\sum |P_k f_{\mathrm{r}}|^2\mu_k(x)\right)^{\frac{1}{2}}\right\|_{L^{1+\varepsilon}} \lesssim \sum \|f_{\mathrm{r}}\|_{H^{1+\varepsilon}} \text{ for all } 0<\varepsilon\leq 2$$

Proof : Let $f_{\mathrm{r}}\in H^{1+\varepsilon}$, and we begin the proof of the the first estimate above by looking at

$$\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\sum |P_k f_{\rm r}(x)|^{1+\varepsilon}\mu_k(x)dx$$
$$= P\int_0^\infty d\mu\left(\left\{(x,t):\left|\int_{\mathbb{R}^n}\sum t^{-n}\varphi_{\rm r}(t^{-1}(x-x+\varepsilon))f_{\rm r}(x+\varepsilon)d(x+\varepsilon)\right| > \lambda\right\}\right)\lambda^{1+\varepsilon}\frac{d\lambda}{\lambda}.$$

Define $E_\lambda = \{x: |\mathcal{N}^{\varphi_r} f_{\rm r}(x)| > \lambda\}$, and it follows that
$$\left\{(x,t):\left|\int_{\mathbb{R}^n}\sum t^{-n}\varphi_{\rm r}(t^{-1}(x-x+\varepsilon))f_{\rm r}(x+\varepsilon)d(x+\varepsilon)\right| > \lambda\right\} \subset \widehat{E_\lambda}$$
where $\hat{E} = \{(x,t): B(x,t) \subset E\}$. Therefore
$$\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\sum |P_k f_{\rm r}(x)|^{1+\varepsilon}\mu_k(x)dx = P\int_0^\infty d\mu(\widehat{E_\lambda})\lambda^{1+\varepsilon}\frac{d\lambda}{\lambda} \lesssim P\int_0^\infty |E_\lambda|\lambda^{1+\varepsilon}\frac{d\lambda}{\lambda}$$
$$= \sum \|\mathcal{N}^{\varphi_r} f_{\rm r}\|_{(1+\varepsilon)^{1+\varepsilon}}^{1+\varepsilon}$$
$$= \sum \|f_{\rm r}\|_{H^{1+\varepsilon}}^{1+\varepsilon}.$$

Here we use that $d\mu(\widehat{E_\lambda}) \lesssim |E|$ for any open set $E \subset \mathbb{R}^n$, which is a well known estimate for Carleson measures. In the case $\varepsilon = 2$, the second estimate coincides with the first and hence there is no more to prove.When $0 < \varepsilon < 2$, we set $1 + 2\varepsilon = \frac{2}{1+\varepsilon} > 1$ and then the Hölder conjugate of $1+\varepsilon$ is $1 + 2\varepsilon = \frac{2}{2-(1+\varepsilon)}$. Now applying the first estimate above, we finish the proof.
$$\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}\sum |P_k f_{\rm r}(x)|^2\mu_k(x)\right)^{(1+\varepsilon)/2} dx$$
$$\le \int_{\mathbb{R}^n} sP_k\sum |P_k f_{\rm r}(x)|^{(2-(1+\varepsilon))(1+\varepsilon)/2}\left(\sum_{k\in\mathbb{Z}}\sum |P_k f_{\rm r}(x)|^{1+\varepsilon}\mu_k(x)\right)^{(1+\varepsilon)/2}$$
$$\le \sum \left\|(\mathcal{N}^{\varphi} f_{\rm r})^{\frac{(2-(1+\varepsilon))(1+\varepsilon)}{2}}\right\|_{\frac{2}{2-(1+\varepsilon)}}\left\|\left(\sum_{k\in\mathbb{Z}} |P_k f_{\rm r}(x)|^{(1+\varepsilon)}\mu_k(x)\right)^{\frac{1+\varepsilon}{2}}\right\|_{\frac{2}{2-(1+\varepsilon)}}$$
$$= \sum \|\mathcal{N}^{\varphi_r} f_{\rm r}\|_{(1+\varepsilon)^{1+\varepsilon}}^{\frac{(1+\varepsilon)(2-(1+\varepsilon))}{2}}\left(\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\sum |P_k f_{\rm r}(x)|^{(1+\varepsilon)}\mu_k(x)dx\right)^{(1+\varepsilon)/2}$$
$$\lesssim \sum \|f_{\rm r}\|_{H^{1+\varepsilon}}^{\frac{(1+\varepsilon)(2-(1+\varepsilon))}{2}}\sum \|f_{\rm r}\|_{H^{1+\varepsilon}}^{\frac{(1+\varepsilon)^2}{2}} = \sum \|f_{\rm r}\|_{H^{1+\varepsilon}}^{1+\varepsilon}$$

Corollary (6.2.10) [356]: Assume $\{\Lambda_k\} \in \mathrm{LPSO}(n + 2(1+\varepsilon) + 2\delta, (1+\varepsilon)+\delta)$ for some integer $\varepsilon \ge 0$ and $0 < \delta \le 1$. If $\Lambda_k(y^\alpha) = 0$ for all $k \in Z$ and $|\alpha| \le 1+\varepsilon$, then $\sum\|S_\Lambda f_{\rm r}\|_{(1+\varepsilon)^{1+\varepsilon}} \lesssim \sum\|f_{\rm r}\|_{H^{1+\varepsilon}}$ for all $f_r \in H^{1+\varepsilon} \cap (1+\varepsilon)^2$ and $\frac{n}{n+1+\varepsilon+\delta} < 1+\varepsilon \le 1$.

We call this a reduced version because we have strengthened the assumptions of from the Carleson measure estimates to the vanishingmoment type assumption above; $\Lambda_k(y^\alpha) = 0$ for $|\alpha| \le 1+\varepsilon$.

Proof: Fix $1+\varepsilon \in (n/(1+\varepsilon)-n, 1+\varepsilon+\delta)$, which is possible since our assumption on $1+\varepsilon$ implies that $\frac{n}{1+\varepsilon}-n < 1+\varepsilon+\delta$. Also fix $0<\varepsilon<1$)such that $\frac{n}{n+1+\varepsilon} < 1+2\varepsilon < 1+\varepsilon$. Let $f_{\mathrm{r}} \in H^{1+\varepsilon} \cap (1+\varepsilon)^2$, and we decompose

$$\Lambda_k\left(\sum f_{\mathrm{r}}(x)\right) = \sum_{j\in\mathbb{Z}} \sum_{Q} |Q|\tilde{\phi}_j * f_{\mathrm{r}}(c_Q)\Lambda_k(\psi_{\mathrm{r}})_j^{c_Q}(x)$$

$$= \sum_{j\in\mathbb{Z}} \sum_{Q} \sum |Q|\left(\widetilde{\phi_{\mathrm{r}}}\right)_j * \sum f_{\mathrm{r}}(c_Q) \int_{\mathbb{R}^n} \lambda_k(x, x+\varepsilon)(\psi_{\mathrm{r}})_j^{c_Q}(x+\varepsilon)d(x+\varepsilon).$$

The summation in Q is over all dyadic cubes with side lengths $\ell(Q) = 2^{-(j+N_0)}$. Then we have the following almost orthogonality estimates

$$\left|\int_{\mathbb{R}^n} \sum \lambda_k(x, x+\varepsilon)(\psi_{\mathrm{r}})_j^{c_Q}(x+\varepsilon)d(x+\varepsilon)\right|$$

$$= \left|\int_{\mathbb{R}^n} \sum \lambda_k(x, x+\varepsilon)\left(\phi_j^{c_Q}(x+\varepsilon) - \sum_{|\alpha|\le 1+\varepsilon} \frac{D^\alpha(\psi_{\mathrm{r}})_j^{c_Q}(x)}{\alpha!}(x+\varepsilon-x)^\alpha\right)d(x+\varepsilon)\right|$$

$$\lesssim \int_{\mathbb{R}^n} \Phi_k^{n+2(1+\varepsilon)+2\delta}(x-x+\varepsilon)\left(2^j|x-x+\varepsilon|\right)^{1+\varepsilon+\delta}\left(\Phi_j^{n+1+\varepsilon+\delta}(x+\varepsilon-c_Q)\right.$$
$$\left.+\Phi_j^{j+1+\varepsilon+\delta}(x-c_Q)\right)d(x+\varepsilon)$$

$$\lesssim 2^{(1+\varepsilon+\delta)(j-k)} \int_{\mathbb{R}^n} \Phi_k^{n+2(1+\varepsilon)+2\delta}(x-x+\varepsilon)\left(\Phi_j^{n+1+\varepsilon+\delta}(x+\varepsilon-c_Q)\right.$$
$$\left.+\Phi_j^{n+1+\varepsilon+\delta}(x-c_Q)\right)d(x+\varepsilon) \lesssim 2^{(1+\varepsilon+\delta)(j-k)}\Phi_{\min\,(j,k)}^{n+1+\varepsilon+\delta}(x-c_Q)$$

Also, using the vanishing moment properties of ϕ_j, we have the following estimate,

$$\left|\int_{\mathbb{R}^n} \lambda_k(x, x+\varepsilon)\phi_j^{c_Q}(x+\varepsilon)d(x+\varepsilon)\right|$$

$$= \left|\int_{\mathbb{R}^n} \left(\sum \lambda_k(x, x+\varepsilon) - \sum_{|\alpha|\le L} \frac{D^\alpha(\psi_{\mathrm{r}})_j^{c_Q}(x)}{\alpha!}(x-x+\varepsilon)^\alpha\right)\phi_j^{c_Q}(x+\varepsilon)d(x+\varepsilon)\right|$$

$$\lesssim \int_{\mathbb{R}^n} \Phi_j^{n+1+\varepsilon+\delta}(x-x+\varepsilon)\left(2^k|x+\varepsilon-c_Q|\right)^{1+\varepsilon+\delta}\Phi_j^{j+2(1+\varepsilon)+2\delta}(x+\varepsilon-c_Q)d(x$$
$$+\varepsilon^2$$
$$+\int_{\mathbb{R}^n} \Phi_j^{n+1+\varepsilon+\delta}(x-c_Q)\left(2^k|x+\varepsilon-c_Q|\right)^{1+\varepsilon+\delta}\Phi_j^{j+2(1+\varepsilon)+2\delta}(x+\varepsilon-c_Q)d(x+\varepsilon)$$

$$\lesssim 2^{(1+\varepsilon+\delta)(k-j)} \int_{\mathbb{R}^n} \Phi_k^{n+1+\varepsilon+\delta}(x-x+\varepsilon)\Phi_j^{n+1+\varepsilon 1+\varepsilon+\delta}(x+\varepsilon-c_Q)d(x+\varepsilon)$$
$$+2^{(1+\varepsilon+\delta)(k-j)} \int_{\mathbb{R}^n} \Phi_k^{n+1+\varepsilon+\delta}(x-c_Q)\Phi_j^{n+1+\varepsilon+\delta}(x+\varepsilon-c_Q)d(x+\varepsilon)$$
$$\lesssim 2^{(1+\varepsilon+\delta)(k-j)}\Phi_{\min\,(j,k)}^{n++\delta}(x-c_Q)$$

Therefore

$$\left|\int_{\mathbb{R}^n} \lambda_k(x, x+\varepsilon)\phi_j^{c_Q}(x+\varepsilon)d(x+\varepsilon)\right| \lesssim 2^{-((1+\varepsilon)+\delta)|j-k|}\Phi_{\min\,(j,k)}^{n+1+\varepsilon}(x-c_Q)$$

$$\left|\sum \Lambda_k f_{\mathrm{r}}(x)\right| \lesssim \sum_{j\in\mathbb{Z}} \sum_{Q} |Q|\tilde{\phi}_j * f_{\mathrm{r}}(c_Q) 2^{-(1+\varepsilon+\delta)|j-k|} \Phi^{n+1+\varepsilon}_{\min\,(j,k)}(x-c_Q)$$

$$\lesssim \sum_{j\in\mathbb{Z}} \sum^{-(1+\varepsilon+\delta)|j-k|} 2^{1+\varepsilon \max\,(0,k-j)} \mathcal{M}_j^{1+2\varepsilon}(\tilde{\phi}_j * f_{\mathrm{r}})(x)$$

$$\leq \sum_{j\in\mathbb{Z}} 2^{-\varepsilon|j-k|} \mathcal{M}_j^{1+2\varepsilon}(\tilde{\phi}_j * f_{\mathrm{r}})(x).$$

where $\varepsilon = 1+\varepsilon+\delta-1+\varepsilon > 0$; recall that these parameter are chosen such that $1+\varepsilon < 1+\varepsilon+\delta$. to $M_j^{1+2\varepsilon}(\tilde{\varphi}_j * f_r)$(recall that $1+\varepsilon$ was chosen such that $\frac{n}{n+1+\varepsilon} < 1+2\varepsilon < 1+\varepsilon$) yields the appropriate estimate below,

$$\sum \|S_\Lambda f_{\mathrm{r}}\|_{(1+\varepsilon)^{1+\varepsilon}}$$

$$\lesssim \left\| \left(\sum_{k\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}} \sum 2^{-\varepsilon|j-k|} \mathcal{M}_j^{1+\varepsilon}(\tilde{\phi}_j * f_{\mathrm{r}}) \right]^2 \right)^{\frac{1}{2}} \right\|_{(1+\varepsilon)^{1+\varepsilon}}$$

$$\lesssim \left\| \left(\sum_{j,k\in\mathbb{Z}} \sum 2^{-\varepsilon|j-k|} \left[\mathcal{M}_j^{1+2\varepsilon}(\tilde{\phi}_j * f_{\mathrm{r}})\right]^2 \right)^{\frac{1}{2}} \right\|_{(1+\varepsilon)^{1+\varepsilon}} \lesssim \sum \|f_{\mathrm{r}}\|_{H^{1+\varepsilon}}$$

Next we construct paraproducts to decompose Λ_k. Fix an approximation to identity operator $P_k\left(\sum f_r\right) = \sum \phi_k * f_r$, where $\phi_k(x) = 2^{kn}\phi(2^k x)$ and $\varphi_r \in \mathcal{Y}$ with integral 1. Define for $\alpha, \alpha+\varepsilon \in \mathbb{N}_0^n$

$$M_{\alpha,\beta} = \begin{cases} (-1)^{|\alpha+\varepsilon|-|\alpha|} \dfrac{\alpha+\varepsilon!}{(\alpha+\varepsilon-\alpha)!} \displaystyle\int_{\mathbb{R}^n} \varphi(x+\varepsilon)(x+\varepsilon)^{\alpha+\varepsilon-\alpha} d(x+\varepsilon)\varepsilon \geq 0 \\ \qquad\qquad\qquad\qquad 0 \end{cases}$$

Here we say $\varepsilon \leq 0$ for $\alpha = (\alpha_1,\dots,\alpha_n), \alpha+\varepsilon = (\alpha_1+\varepsilon,\dots,\alpha_n+\varepsilon) \in \mathbb{N}_0^n$ if $\alpha_i \leq \alpha_i + \varepsilon_i$ for all $i = 1,\dots,n$. It is clear that $\left|(1+\varepsilon)_{\alpha,\alpha+\varepsilon}\right| < \infty$ for all $\alpha, \alpha+\varepsilon \in \mathbb{N}_0^n$ since $\phi_r \in S$. Also note that when $|\alpha| = |\alpha+\varepsilon|$

$$(1+\varepsilon)_{\alpha,\alpha+\varepsilon} = \begin{cases} \alpha+\varepsilon! & \varepsilon = 0 \\ \quad 0 & \alpha \neq \alpha+\varepsilon \text{ and } |\alpha| = |\alpha+\varepsilon| \end{cases}.$$

We consider the operators $P_k D^\alpha$ defined on $\mathcal{Y}'$, where D^α is taken to be the distributional derivative acting on $\mathcal{Y}'$. Hence $P_k D^\alpha f_{\mathrm{r}}(x)$ is well defined for $f_{\mathrm{r}} \in \mathcal{Y}'$ since $\sum P_k D^\alpha f_{\mathrm{r}}(x) = \sum \langle \varphi_k^x, D^\alpha f_{\mathrm{r}} \rangle = (-1)^{|\alpha|} \sum \langle D^\alpha((\varphi_r)_k^x), f_{\mathrm{r}} \rangle$ and $D^\alpha((\varphi_{\mathrm{r}})_k^x) \in \mathcal{Y}$. In fact, this gives a kernel representation for $P_k D^\alpha$; estimates for this kernel are addressed in the proof of We also have

$$\left[[P_k D^\alpha]\right]_{\alpha+\varepsilon}(x) = 2^{|\alpha+\varepsilon|k} \int_{\mathbb{R}^n} \sum (\varphi_r)_k (x-x+\varepsilon)^{\alpha+\varepsilon}\, \partial^\alpha_{x+\varepsilon}((x-x+\varepsilon)^{\alpha+\varepsilon}) d(x+\varepsilon)$$

$$= 2^{k|\alpha|}(\alpha+\varepsilon)_{\alpha,\alpha+\varepsilon}.$$

For $k \in \mathbb{Z}$, define

$$\Lambda_k^{(0)} \sum f_{\mathrm{r}}(x) = \Lambda_k \sum f_{\mathrm{r}}(x) - \sum \left[[\Lambda_k]\right]_0(x) \cdot P_k f_{\mathrm{r}}(x),$$

$$\Lambda_k^{(m)} \sum f_{\mathrm{r}}(x)$$

$$= \Lambda_k^{(m-1)} \sum f_{\mathrm{r}}(x) - \sum_{|\alpha|=m} \sum (-1)^{|\alpha|} \frac{\left[\left[\Lambda_k^{(m-1)}\right]\right]_\alpha (x)}{\alpha!} \cdot 2^{-k|\alpha|} P_k D^\alpha f_{\mathrm{r}}(x).$$

for $\varepsilon \geq 0$.

Corollary (6.2.11) [356]: Let $\{\Lambda_k\} \in \mathrm{LPSO}(N, L+\delta)$, where $1+\varepsilon = n + 2(1+\varepsilon) + 2\delta$ for some integer $L \geq 0$ and $0 < \delta \leq 1$, and assume that

$$d\mu_\alpha(x,t) = \sum_{k\in\mathbb{Z}} \left|\left[\left[\Lambda_k^{(m)}\right]\right]_\alpha (x)\right|^2 \delta_{t=2^{-k}} dx.$$

is a Carleson measure for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq 1+\varepsilon$. Also let $\Lambda_k^{(m)}$ be as in as for $\varepsilon \geq -1$. Then $\Lambda_k^{(m)} \in \mathrm{LPSO}(1+\varepsilon, 1+\varepsilon+\delta)$ for the same $1+\varepsilon, 1+2\varepsilon$, and δ, and satisfy the following:

(i) $\left[\left[\Lambda_k^{(m)}\right]\right]_\alpha = o$for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 1+\varepsilon \leq 1+2\varepsilon$.

(ii) $d\mu_m(x,t)$ is a Carleson measure for all $0 \leq 1+\varepsilon \leq 1+2\varepsilon$, where $d\mu_m$ is defined

$$d\mu_m(x,t) = \sum_{k\in\mathbb{Z}} \sum_{|\alpha|\leq 1+\varepsilon} \left|\left[\left[\Lambda_k^{(m)}\right]\right]_\alpha (x)\right|^2 \delta_{t=2^{-k}} dx.$$

Proof : Since $\{\Lambda_k\} \in \mathrm{LPSO}(n + 2(1+\varepsilon) + 2\delta, 1+\varepsilon+\delta)$, we know that $\left|[[\Lambda_k]]_\alpha(x)\right| \lesssim 1$ for all $|\alpha| \leq 1+\varepsilon$. Then to verify that $\left\{\Lambda_k^{(m)}\right\} \in \mathrm{LPSO}(n + 2(1+\varepsilon) + 2\delta, 1+\varepsilon+\delta)$ for $0 \leq 1+\varepsilon \leq 1+\varepsilon$, it is sufficient to showthat $\left\{2^{-k|\alpha|} P_k D^\alpha\right\} \in \mathrm{LPSO}(n + 2(1+\varepsilon) + 2\delta, 1+\varepsilon+\delta)$ for all $\alpha \in \mathbb{N}_0^n$. For $f_{\mathrm{r}} \in \mathcal{Y}$, we have the following integral representation for $2^{-k|\alpha|} P_k D^\alpha f_{\mathrm{r}}$, which was alluded to above,

$$2^{-k|\alpha|} \sum P_k D^\alpha f_{\mathrm{r}}(x) = (-1)^{|\alpha|} 2^{-k|\alpha|} \sum \langle D^\alpha((\varphi_{\mathrm{r}})_k^x), f_{\mathrm{r}}\rangle$$

$$= (-1)^{|\alpha|} \sum_{\mathcal{Y}} (D^\alpha \varphi_{\mathrm{r}}) f_k * f_{\mathrm{r}}(x)$$

Since $\varphi_{\mathrm{r}} \in \mathcal{Y}$, it easily follows that $D^\alpha \varphi_{\mathrm{r}} \in \mathcal{Y}$ for all $\alpha \in \mathbb{N}_0^n$ and that $\left\{2^{-k|\alpha|} P_k D^\alpha\right\} \in LPSO(n + 2(1+\varepsilon) + 2\delta, 1+\varepsilon+\delta)$. Now we prove (1) by induction: the $m=0$ case for (1) is not hard to verify

$$\left[\left[\Lambda_k^{(0)}\right]\right]_0 = \Lambda_k 1 - [[\Lambda_k]]_0 \cdot P_k 1 = [[\Lambda_k]]_0 - [[\Lambda_k]]_0 = 0$$

Now assume that (1) holds for $m-1$, that is, assume $\left[\left[\Lambda_k^{(m-1)}\right]\right]_\alpha = 0$ for all $|\alpha| \leq m-1$.

Then for $|\alpha+\varepsilon| \leq m-1$

$$\left[\left[\Lambda_k^{(m)}\right]\right]_{\alpha+\varepsilon} = \left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon} - \sum_{|\alpha|=m} \frac{\left[\left[\Lambda_k^{(m-1)}\right]\right]_\alpha (x)}{\alpha!} (-1)^{|\alpha|} M_{\alpha,\alpha+\varepsilon} = 0$$

The first term here vanished by the inductive hypothesis. The second term is zero since $|\beta| < m = |\alpha|$ and hence $(\alpha+\varepsilon)_{\alpha,\alpha+\varepsilon} = 0$. For $|\alpha+\varepsilon| = m$,

$$\left[\left[\Lambda_k^{(m)}\right]\right]_{\alpha+\varepsilon} = \left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon} - \sum_{|\alpha|=m} \frac{\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha}(x)}{\alpha!}(-1)^{|\alpha|}(\alpha+\varepsilon)_{\alpha,\alpha+\varepsilon}$$
$$= \left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon} - \left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon} = 0$$

where the sum collapses. By induction, this verifies (1) for all $m \le L$. Given the Carleson measure assumption for $d\mu_\alpha(x,t)$ one can easily prove (2) if the following statement holds: for all $0 \le \varepsilon \le 1$

$$\sum_{|\alpha|\le L} \left|\left[\left[\Lambda_k^{(m)}\right]\right]_{\alpha}(x)\right| \le (1+C_0)^{m+1} \sum_{|\alpha|\le 1+\varepsilon} \left|[[\Lambda_k]]_\alpha(x)\right|, \text{ where } C_0$$
$$= \sum_{|\alpha|,|\alpha+\varepsilon|\le 1+\varepsilon} |M_{\alpha,\beta}|$$

We verify by induction. For $m=0$, let $|\alpha+\varepsilon| \le 1+\varepsilon$, and it follows that

$$\left[\left[\Lambda_k^{(0)}\right]\right]_{1+\varepsilon} = [[\Lambda_k]]_{1+\varepsilon} - [[\Lambda_k]]_0 \cdot [[\mathrm{P}_k]]_{1+\varepsilon} = [[\Lambda_k]]_{1+\varepsilon} - [[\Lambda_k]]_0 \cdot (1+\varepsilon)_{0,1+\varepsilon}$$

Then

$$\sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|\left[\left[\Lambda_k^{(0)}\right]\right]_{\alpha+\varepsilon}\right| \le \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|[[\Lambda_k]]_{\alpha+\varepsilon}\right| + \sum_{|\beta|\le 1+\varepsilon} \left|[[\Lambda_k]]_0\right| (1+\varepsilon)_{0,\alpha+\varepsilon}$$
$$\le (1+C_0) \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|[[\Lambda_k]]_{\alpha+\varepsilon}\right|$$

Now assume holds for $m-1$, and consider

$$\sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|\left[\left[\Lambda_k^{(m)}\right]\right]_{\alpha+\varepsilon}\right|$$
$$\le \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon}\right| + \sum_{|\alpha+\varepsilon|\le 1+\varepsilon |\alpha|=m} \left|\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha}\right| (1+\varepsilon)_{\alpha,\alpha+\varepsilon}$$
$$\le \left(1 + \sum_{|\alpha|\le m, |\alpha+\varepsilon|\le 1+\varepsilon} \left|(1+\varepsilon)_{\alpha,\alpha+\varepsilon}\right|\right) \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon}\right|$$
$$\le (1+C_0) \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon}\right| \le (1+C_0)^{m+1} \sum_{|\alpha+\varepsilon|\le 1+\varepsilon} \left|[[\Lambda_k]]_{\alpha+\varepsilon}\right|.$$

We use the inductive hypothesis in the last inequality here to bound the $\left[\left[\Lambda_k^{(m-1)}\right]\right]_{\alpha+\varepsilon}$. Then by induction, the estimate holds for all $0 \le m \le L$, and completes the proof.

Corollary (6.2.12) [356]: Let $T \in CZO(1+\varepsilon+\gamma)$ be bounded on $(1+\varepsilon)^2$ and satisfy $T^*(x^\alpha) = 0$ for all $|\alpha| \le 1+\varepsilon = [1+\varepsilon/2]$. For $\psi_r \in \mathcal{D}_{1+\varepsilon}$, define

$$d\mu_{\psi_r}(x,t) = \sum_{|\alpha|\le 1+\varepsilon} \sum_{k\in\mathbb{Z}} \left|[[Q_k T]]_\alpha\right|^2 \delta_{t=2^{-k}} dx$$

where $Q_k f_r = (\psi_r)_k * f_r$ and $(\psi_r)_k(x) = 2^{kn}\psi_r(2^k x)$. If $[[T]]_\alpha \in BMO_{|\alpha|}$ for all $|\alpha| \le 1+\varepsilon$, then $d\mu\psi$ is a Carleson measure for any $\psi_r \in \mathcal{D}_{M+1+\varepsilon}$.

Proof : Assume that $[[T]]_\alpha \in BMO|\alpha|$ for all $|\alpha| \le 1+\varepsilon$. Let $\in \mathcal{D}_{1+\varepsilon}$, and it follows that $\{Q_k T\} \in \mathrm{LPSO}(L,\delta')$ for all $\delta' < \delta$, where $Q_k f_r$ is defined as above and $1+\varepsilon = [1+\varepsilon/2$

and $\delta = (1+\varepsilon - \ 2(1+\varepsilon)+\gamma)/2$. We also define $Q_k^{\alpha+\varepsilon} f_r = (\psi_r)_k^{\alpha+\varepsilon} * f_r$, where $(\psi_r)^{\alpha+\varepsilon}(x) = (-1)^{|\alpha+\varepsilon|}\psi_r(x)x^{\alpha+\varepsilon}$. It follows that $(\psi_r)^{\alpha+\varepsilon} \in \mathcal{D}_{1+\varepsilon+1+\varepsilon-|\alpha+\varepsilon|}$. Now let $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le 1+\varepsilon$. Note that for $\alpha+\varepsilon \le \alpha$, it follows that $(\psi_r)^{\beta} \in \mathcal{D}_{\alpha+\varepsilon}$, and hence $\{Q_k^{\alpha+\varepsilon}T\} \in \mathrm{LPSO}(n+2(1+\varepsilon)+2\delta, 1+\varepsilon+\delta')$ for all $0<\delta'<\delta$ as well. Then it follows that

$$[[Q_k T]]_\alpha(x)$$

$$= \int_{\mathbb{R}^n} \sum T^*(\psi_r)_k^x(x+\varepsilon)(x-x+\varepsilon)^\alpha d(x+\varepsilon)$$

$$= \lim_{R\to\infty} 2^{k|\alpha|} \int_{\mathbb{R}^{2n}} \mathcal{K}(u,x+\varepsilon)(\psi_r)_k^x(u)\eta_R(x+\varepsilon)(x-x+\varepsilon)^\alpha dud(x+\varepsilon)$$

$$= \lim_{R\to\infty} \sum_{\alpha+\varepsilon\le\alpha} c_{\alpha,\beta} 2^{k|\alpha|} \int_{\mathbb{R}^{2n}} \sum \mathcal{K}(u,x+\varepsilon)(\psi_r)_k^x(u)(x-u)^{\alpha+\varepsilon}(x-x+\varepsilon)^\varepsilon dud(x +\varepsilon)$$

$$= \lim_{R\to\infty} \sum_{\alpha+\varepsilon\le\alpha} c_{\alpha,\beta} 2^{(|\alpha|-|\alpha+\varepsilon|)k} \int_{\mathbb{R}^{2n}} \sum \mathcal{K}(u,x+\varepsilon)((\psi_r)_k^{\alpha+\varepsilon})^x(u)\eta_R(x +\varepsilon)(x-x+\varepsilon)^{\alpha+\varepsilon} dud(x+\varepsilon)$$

$$= \sum_{\alpha+\varepsilon\le\alpha} c_{\alpha,\alpha+\varepsilon} 2^{(|\alpha|-|\alpha+\varepsilon|)k} \sum_{\mathbb{R}^n} \langle [[T]]_{\varepsilon'}((\psi_r)_k^{\alpha+\varepsilon})^x \rangle.$$

Let $Q \subset \mathbb{R}^n$ be a cube with side length $\ell(Q)$. It follows that

$$\sum_{2^{-k}\le\ell(Q)} \int_Q \left|[[Q_k T]]_\alpha\right|^2 dx$$

$$\le \sum_{2^{-k}\le\ell(Q)} \int_Q \sum_\alpha \left(\sum_{\alpha+\varepsilon\le\alpha} c_{\alpha,\alpha+\varepsilon} 2^{(|\alpha|-|\alpha+\varepsilon|)k} \langle [[T]]_{\varepsilon'}((\psi_r)_k^{\alpha+\varepsilon})^x \rangle \right)^2 dx$$

$$\lesssim \sum_{\alpha+\varepsilon\le\alpha} \sum_{2^{-k}\le\ell(Q)} \int_Q \sum 2^{(|\alpha|-|\alpha+\varepsilon|)k} |\langle [[T]]_{\alpha+\varepsilon'}((\psi_r)_k^{\alpha+\varepsilon})^x\rangle|^2 dx \lesssim |Q|$$

The last inequality holds since $[[T]]_{\alpha+\varepsilon} \in BMO_{|\alpha|-|\alpha+\varepsilon|}$ and $(\psi_r)_k^{\alpha+\varepsilon} \in 1+\varepsilon \subset \mathcal{D}_{|\alpha|-|1+\varepsilon|}$ for all $\alpha+\varepsilon \le \alpha$.

Section (6.3): Limited Ranges of Muckenhoupt Weights

We are concerned with boundedness properties of Calderón-Zygmund singular integral operators and Littlewood-Paley-Stein square function operators on weighted Hardy spaces. The primary issue for singular integral operators for us is the continuity of a Calderón-Zygmund operator T from H_w^p into H_w^p for $0<p<\infty$ and $w \in A_\infty$, for which we give necessary and sufficient conditions; see Theorem (6.3.10). We also prove new results for square function operators from H_w^p into L_w^p for $w \in A_q$ where $p<q$; see Theorems (6.3.7) and (6.3.8). Our approach to these problems uses Muckenhoupt weight invariance properties of BMO and Sobolev-BMO spaces; see Theorem (6.3.11). In fact, the use of BMO weight invariant properties in this way provides a new way to prove L_w^p type estimates for operators.

There is a lot known about Hardy space H^p estimates for Calderón-Zygmund operators, going back to the groundbreaking work of Stein and Weiss [330], Stein [329], and Fefferman and Stein [114], among others. When T is a convolution type operator, things are simplified

considerably. It was shown in [114] that if T is a convolution type Calderón- Zygmund operator and is bounded on L^2, then T is also bounded on H^p for $p_0 < p \leq 1$ where $p_0 < 1$ depends on the regularity of the kernel of T. In particular, if the convolution kernel of T is smooth away from the origin, then T is bounded on H^p for all $0 < p \leq 1$. There are also a number of situations where the boundedness properties of a singular integral operator T on weighted Hardy spaces are already known. Still working in the convolution setting, some weighted Hardy space estimates were proved by Lu and Zhu [282]. In that work, they prove that if a convolution operator T has a smooth convolution kernel away from the origin and is bounded on L^2, then T is also bounded on H_w^p for all $0 < p < \infty$ and $w \in A_\infty$. One should note here that when $1 < p < \infty$ this result does not collapse to the well-known Lebesgue space theory for singular integral operators. Since the result in [282] allows w to be in any A_q class regardless of the p, it does not follow that $L_w^p = H_w^p$; in particular, when $1 < p < q < \infty$ and $w \in A_q \setminus A_p$ the spaces L_w^p and H_w^p do not coincide. Hence one can conclude from the work in [282] the initially surprising fact that there are convolution operators that are bounded on H_w^p, but not bounded on L_w^p for appropriate selections of $1 < p < \infty$ and $w \in A_\infty$.
In the non-convolution setting, Hardy space estimates are considerably more difficult to prove. Sufficiency results for a non-convolution operator T to be bounded on H^p for $0 < p \leq 1$ were given by Torres [331], Frazier, Torres, and Weiss [339], and Frazier, Han, Jawerth, and Weiss [338]. Full necessary and sufficient theorems for the H^p boundedness of non-convolution Calderón-Zygmund operators were achieved by Alvarez and Milman [334] and the first and Lu [343]. In [334], the authors give necessary and sufficient conditions for T to be bounded on H^p when p is close to 1 (more precisely when $\frac{n}{n+\gamma} < p \leq 1$ where $0 < \gamma \leq 1$ is the Hölder regularity parameter for the kernel of T), and the full characterization for any $0 < p \leq 1$ was established in [343].
In the current work, We show a full necessary and sufficient theorem for non convolution type singular integral operator bounds on weighted Hardy spaces H_w^p. The conditions of our theorem are the same as the necessary and sufficient conditions for unweighted H^p bounds found in [343], but as may be expected from the discussion, some intriguing properties are exhibited of weighted Hardy spaces. In the ideal situation of [282]- where T is a convolution operator with smooth kernel away from the origin - one can conclude boundedness on H_w^p for the full range $0 < p < \infty$ and $w \in A_\infty$. This cannot be expected in the non-convolution setting. Indeed, it has already been shown that even in the un weighted situation for non-convolution operators, a Calderón-Zygmund operator T is bounded on H^p for only a limited range of exponents p depending on the kernel regularity and cancellation of T. We find that one must limit the range of q for $w \in A_q$ as well. This is a new result, and an interesting one. It seems that increased kernel regularity and cancellation properties for T allow for H_w^p boundedness where w can "move up the scale" of A_q classes where $q > p$, but cannot be taken to be just any weight in A_∞ as in [282]. In particular, given a Calderón-Zygmund operator T we prove that it is bounded on H_w^p for ranges of p and w of the form $\frac{n}{n+L+\delta} < p < \infty$ and $w \in A_q$ where $q = p\frac{n+L+\delta}{n}$; here $L \in \mathbb{N}_0$ and $0 < \delta \leq 1$ are determined by the kernel regularity and cancellation properties of T.
It should be noted that the complications of working in the non-convolution setting can be directly observed in terms of cancellation conditions for T. If T is a convolution operator

that is bounded on L^2, then $T(x^\alpha) = T^*(x^\alpha) = 0$ for any $\alpha \in \mathbb{N}_0^n$ such that $T(x^\alpha)$ and $T^*(x^\alpha)$ can be defined (the multi-indices α for which these are defined depend on the kernel regularity of T). This collapses all of the cancellation conditions used to trivial ones when T is a convolution operator. So studying convolution operators reduces immediately to the situation where $T(x^\alpha) = T^*(x^\alpha) = 0$ are satisfied for all α, which simplify things considerably. In the non-convolution setting, the conditions $T(x^\alpha) = 0$ are not necessary for H^p or H_w^p boundedness. This in large part, is why it is more difficult to work in the non-convolution setting than the convolution setting.

We prove weighted Hardy space estimates for Littlewood-PaleyStein square function operators. These operators have been studied intensely over the past halfcentury, but the majority of the attention to these operators has been along two lines of research (at least pertaining to boundedness on L_w^p and H_w^p spaces): (i) Proving estimates for convolution type square function operators mapping $L^p \to L^p, L_w^p \to L_w^p, H^p \to L^p$, and $H_w^p \to L_w^p$, and (ii) proving estimates for non-convolution type square function operators mapping $L^p \to L^p$ and $L_w^p \to L_w^p$. The unweighted estimates along the lines of (i) have been studied extensively, and by now are often considered classical. The subject of weighted estimates for square functions is also a well-studied problem, being attacked since the seminal works of Wilson [353]-[355]. Other weighted estimates for convolution-type square functions can be found in the work of Sato [144], [129], Lanzhe [345], Duoandikoetxea [337], and Ding, Fan, and Pan [136], to name a few. For results in the direction of (2) in the unweighted setting, see for example the work of David and Journé [319], David, Journé, and Semmes [335], and Semmes [327]. However, there has been relatively little work done to prove H^p to L^p estimates for square functions defined in terms of non-convolution operators. One place where such estimates are proved is in [343], show that a class of nonconvolution type square function operators are bounded from H^p into L^p where p is allowed to range all the way down to 0 . We prove that a similar class of square function operators are bounded from H_w^p into L_w^p for $p_0 < p < \infty$ and $w \in A_q$, where in some situations $q > p$ (specifically, we take $q = p/p_0$ where $0 < p0 \leq 1$).

We devoted to developing weight invariant properties of BMO and related spaces. The origins of the notion of weight invariance can be traced back to an article by Muckenhoupt and Wheeden [346] about weighted bounded mean oscillation (although they did not use the term weight invariant). In short, they prove that if one defines BMO_w analogous to BMO with the Lebesgue measure dx replaced by $w(x)dx$ where $w \in A_\infty$, then $BMO_w = BMO$ with comparable norms. This property $BMO_w = BMO$ is what is meant by weight invariance of BMO. There has been limited investigation of this property of BMO since the work of Muckenhoupt and Wheeden; we have found a few related results by Harboure, Salinas, and Viviani [342], Hytönen and Pérez [344], and Tsutsui [352] (although they do not use the term weight invariance either). We further develop these weight invariant ideas in the setting of Carleson measure conditions, $p = \infty$ type Triebel-Lizorkin spaces, and Sobolev-BMO spaces, which are a crucial component in the proofs of our singular integral operator results. We prove that weighted Sobolev-BMO spaces and weighted $p = \infty$ type Triebel-Lizorkin spaces coincide with each other and coincide for all $w \in A_\infty$, all having comparable norms. For more information on $p = \infty$ type Triebel-Lizorkin spaces see for example the work of Frazier and Jawerth [116] and Bui and Taibleson [108], and for more on Sobolev-BMO spaces see for example the work of Neri [347], Strichartz [350], [351], and Garnett, Jones, Le, and Vese [340].

The role that the weight invariant properties of Sobolev-BMO spaces play highlights a new method for proving L^p_w type estimates by "passing through" weight invariant spaces. Very briefly, we use conditions of the form $T1 \in BMO$ to conclude that $T1 \in BMO_w$ and $\| T1 \|_{BMO} \approx \| T1 \|_{BMO_w}$, since $BMO_w = BMO$ with comparable norms. Then, after reproducing some Carleson measure theory with a weight $w \in A_2$ attached, we prove that $T1 \in BMO_w$ implies that the Bony paraproduct operator Π_{T1} associated to $T1$ is bounded on L^2_w for all $w \in A_2$. By the weight extrapolation theory of Rubio de Francia [348], [349] (we use the version proved by Duoandikoetxea [336]), it follows that T is bounded on L^p_w for all $1 < p < \infty$ and $w \in A_p$. The argument described here is not exactly the one we use since we must adapt it to the H^p_w setting for $0 < p < \infty$ in place of L^2_w, but in principle it is the approach that we take. In the proofs below, we modify this approach to work with higher order moments of T in addition to $T1$ and with Sobolev-BMO conditions in addition to traditional BMO conditions. In the end, we find that this new technique of "passing through" a weight invariant space to prove L^p_w type estimates is very effective here, and has potential to be useful in many applications to weighted estimates. The topic is the Carleson measure and BMO type spaces in the weighted setting, and we prove weight invariant properties of Sobolev-BMO. are used to prove estimates for square functions and singular integral, respectively.

Definition (6.3.1)[333]: Let $w \in L^1_{loc}(\mathbb{R}^n)$ be a non-negative function. We say w is a Muckenhoupt A_p weight, written $w \in A_p(\mathbb{R}^n)$, for $1 < p < \infty$ if

$$[w]_{A_p} = \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} dx \right)^{p-} 1 < \infty$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Let $S = S(\mathbb{R}^n)$ be the Schwartz class of smooth, rapidly decreasing functions with the typical Schwartz semi-norm topology. Define the Fourier transform $\hat{\varphi}$ of a function $\varphi: \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi} dx.$$

Let $S_\infty = S_\infty(\mathbb{R}^n)$ be the subspace of Schwartz functions ψ such that $|\hat{\psi}(\xi)| \lesssim |\xi|^M$ for all $M \in \mathbb{N}_0$, i.e. the subspace of Schwartz functions with vanishing moments of all orders. Let $S' = S'(\mathbb{R}^n)$ be the dual space of $S(\mathbb{R}^n)$, which as usual we call the class of tempered distributions. We will also work with the class of tempered distributions modulo polynomials S'/P, which we interpret as equivalence classes of S' of the form $f(x) + p(x)$ where $f \in S'$ and p is a polynomial. Let $D_M(\mathbb{R}^n)$ be the subspace of $C_0^\infty(\mathbb{R}^n)$ made up of functions $f \in C_0^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x) x^\alpha dx = 0$$

for all $|\alpha| \le M$. Define the Riesz potential I_s for $s \in \mathbb{R}$ and $f \in S_\infty$ by $\widehat{I_s\psi}(\xi) = |\xi|^{-s}\hat{\psi}(\xi)$ for $\xi \in \mathbb{R}^n$. It follows that $I_s\psi \in S_\infty(\mathbb{R}^n)$ for all $\psi \in S_\infty(\mathbb{R}^n)$. For $f \in S'/P$, we define $I_s f \in S'/P$ by $\langle I_s f, \psi \rangle = \langle f, I_s\psi \rangle$ for $\psi \in S_\infty(\mathbb{R}^n)$. This is well-defined since given $f \in S'$, if $\langle f, \psi \rangle = 0$ for all $\psi \in S_\infty(\mathbb{R}^n)$, then $\mathrm{supp}(\hat{f}) \subset \{0\}$ (the support of $\hat{f}$ as a tempered distribution) and hence f is a polynomial. So I_s is well-defined on tempered distributions modulo polynomials.

Definition (6.3.2)[333]: Let $w \in A_\infty(\mathbb{R}^n)$. Define BMO_w to be the collection of all $f \in L^1_{loc,w}(\mathbb{R}^n)$ such that

$$\| f \|_{BMO_w} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q |f(x) - f_{w,Q}| w(x) dx < \infty,$$

$$\text{where } f_{w,Q} = \frac{1}{w(Q)} \int_Q f(x) w(x) dx.$$

It follows from Theorem 5 in [346] that $BMO_w = BMO$ with comparable norms for every $w \in A_\infty$. Despite the fact that these are, for all intents and purposes, the same space, we keep the notation BMO_w in addition to BMO to help clarify what properties of BMO_w and BMO we are using at any particular point.

Definition (6.3.3) [333]: Let $w \in A_\infty(\mathbb{R}^n)b$ and $s \geq 0$. Define $I_S(BMO_w)$ to be the collection of $f \in S'/P$ such that $I_{-s}f \in BMO_w$ equipped with the norm

$$\| f \|_{I_S(BMO_w)} = \|I_{-s} f\|_{BMO_w}$$

Note that $I_{-s}f$ is well defined as an element of S'/P for $f \in S'/P$ by the discussion above. It also follows that $I_S(BMO_w)$ is a Banach space for $w \in A_\infty$ since $BMO_w = BMO$ with comparable norms, and it was established in [350] that $I_S(BMO)$ is a Banach space.

Definition (6.3.4)[333]: Let $w \in A_\infty(\mathbb{R}^n)$. A collection of non-negative locally integrable functions $F_k(x)$ for $k \in \mathbb{Z}$ is a w-Carleson collection if

$$\|\{F_k\}\|^2_{C_w} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q \sum_{k:2^{-k} \leq \ell(Q)} F_k(x) w(x) dx < \infty.$$

We say that the collection $F_k(x)$ for $k \in \mathbb{Z}$ is an A_∞-Carleson collection if there exists an increasing function $N: [1, \infty) \to [0, \infty)$ such that $\|\{F_k\}\|_{C_w} \leq N\left([w]_{A_p}\right)$ for any $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$.

Now we formulate the construction of the weighted $p = \infty$ type Triebel-Lizorkin spaces $\dot{F}^{s,2}_{\infty,w}$ for $s \geq 0$ and $w \in A_\infty(\mathbb{R}^n)$ from [108] (see [116]).

Definition (6.3.5)[333]: Let $w \in A_\infty(\mathbb{R}^n)$ and $\psi \in S(\mathbb{R}^n)$ such that $\hat{\psi}(\xi)$ is supported in the annulus $1/2 \leq |\xi| \leq 2$ and such that $\sum_{k \in \mathbb{Z}} \hat{\psi}(2^{-k}\xi) = 1$ for $\xi = 0$. Define $\dot{F}^{s,2}_{\infty,w}$ to be the collection of all $f \in S'/P$ such that

$$\| f \|^2_{\dot{F}^{s,2}_{\infty,w}} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q \sum_{k:2^{-k} \leq \ell(Q)} 2^{2ks} |\psi_k * f(x)|^2 w(x) dx < \infty.$$

It was shown in [108] that this is a norm, that $\dot{F}^{s,2}_{\infty,w}$ is a Banach space, and the norm is independent of the choice of ψ satisfying the properties above.

Definition (6.3.6) [333]: Define the non-tangential maximal function

$$N^\varphi f(x) = \sup_{t>0} \sup_{|x-y| \leq t} \left| \int_{\mathbb{R}^n} t^{-n} \varphi(t^{-1}(y-u)) * f(u) du \right|,$$

where $\varphi \in S$ with non-zero integral. For a weight w, define H^p_w to be the collection of $f \in S'$ such that $\| f \|_{H^p_w} = \|N^\varphi f\|_{L^p_w} < \infty$. It follows that this space is a (quasi-)Banach space (for $0 < p < 1$) where $\|N^\varphi(\cdot)\|_{L^p_w(\mathbb{R}^n)}$ for different elements $\varphi \in S(\mathbb{R}^n)$ with non-zero integral define comparable (quasi-)norms. When $\varphi \in S$ has nonzero integral, we will use the notation $N = N^\varphi$ for convenience. Given kernel functions $\lambda_k: \mathbb{R}^{2n} \to \mathbb{C}$ for $k \in \mathbb{Z}$, define

$$\Lambda_k f(x) = \int_{\mathbb{R}^n} \lambda_k(x,y) f(y) dy$$

for appropriate functions $f: \mathbb{R}^{2n} \to \mathbb{C}$. Define the square function associated to $\{\Lambda_k\}$ by

$$S_\Lambda f(x) = \left(\sum_{k \in \mathbb{Z}} |\Lambda_k f(x)|^2 \right)^{\frac{1}{2}}.$$

We say that a collection of operators Λ_k for $k \in \mathbb{Z}$ is a collection of Littlewood-Paley- Stein operators with decay N and smoothness $L + \delta$, written $\{\Lambda_k\} \in \mathrm{LPSO}(N, L + \delta)$, for $N > 0$, an integer $L \geq 0$ and $0 < \delta \leq 1$, if there exists a constant C such that

$$|\lambda_k(x,y)| \leq C \Phi_k^N(x - y) \tag{22}$$

$$|D_1^\alpha \lambda_k(x,y)| \leq C^{2|\alpha|k} \Phi_k^N(x - y) \text{ for all } |\alpha| = \alpha_1 + \cdots + \alpha_n \leq L \tag{23}$$

$$|D_1^\alpha \lambda_k(x,y) - D_1^\alpha \lambda_k(x,y')| \leq C|y - y'|^\delta 2^{k(L+\delta)} \left(\Phi_k^N(x - y) + \Phi_k^N(x - y') \right)$$
$$\text{for all } |\alpha| = L. \tag{24}$$

Here we use the notation $\Phi_k^N(x) = 2^{kn}(1 + 2^k|x|)^{-N}$ for $N > 0, x \in \mathbb{R}^n$, and $k \in \mathbb{Z}$. We also use the notation $D_0^\alpha F(x,y) = \partial_x^\alpha F(x,y)$ and $D_0^\alpha F(x,y) = \partial_y^\alpha F(x,y)$ for $F: \mathbb{R}^{2n} \to \mathbb{C}$ and $\alpha \in \mathbb{N}_0^n$. It should be noted that we only need to impose regularity in the second variable of $\lambda_k(x,y)$.

We will impose additional cancellation conditions on Λ_k that involve a generalized notion of moments for non-convolution operators Λ_k. Define the α moment function $\left[[\Lambda_k]\right]_\alpha(x)$ by the following. Given $\{\Lambda_k\} \in \mathrm{LPSO}(N, L + \delta)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < N - n$,

$$\left[[\Lambda_k]\right]_\alpha(x) = 2^{k|\alpha|} \int_{\mathbb{R}^n} \lambda_k(x,y)(x - y)^\alpha dy$$

for $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. It is worth noting that $\left[[\Lambda_k]\right]_0(x) = \Lambda_k 1(x)$. Further discussion of the class $\mathrm{LPSO}(N, L + \delta)$ and $\left[[\Lambda_k]\right]_\alpha$ can be found in [343]. Our main results for square function operators are the next two theorems.

Theorem (6.3.7) [333]: Let $\{\Lambda_k\} \in \mathrm{LPSO}(N, L + \delta)$, where $N = n + 2L + 2\delta$ for some integer $L \geq 0$ and $0 < \delta \leq 1$. Let $\frac{n}{n+L+\delta} < p \leq 2$ and $w \in A_{p\frac{n+L+\delta}{n}}(\mathbb{R}^n)$. If $\left|\left[[\Lambda_k]\right]_\alpha(x)\right|^2$ is a w-Carleson collection for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq L$, then S_Λ extends to a bounded operator from $H_w^p(\mathbb{R}^n)$ into $L_w^p(\mathbb{R}^n)$. Furthermore, the operator norm of S_Λ depends on $[w]_{A_{p\frac{n+L+\delta}{n}}}$, not on w itself.

Theorem (6.3.8) [333]: Let $\{\Lambda_k\} \in \mathrm{LPSO}(N, L + \delta)$, where $N = n + 2L + 2\delta$ for some integer $L \geq 0$ and $0 < \delta \leq 1$. If $\left|\left[[\Lambda_k]\right]_\alpha(x)\right|^2$ is an A_∞-Carleson collection for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq L$, then S_Λ extends to a bounded operator from $H_w^p(\mathbb{R}^n)$ into $L_w^p(\mathbb{R}^n)$ for all $\frac{n}{n+L+\delta} < p < \infty$ and $w \in A_{p\frac{n+L+\delta}{n}}$.

The main difference between the conclusions of these two theorems is that Theorem (6.3.8) allows for p to exceed 2, whereas Theorem (6.3.7) imposes $p \leq 2$. There are some technical issues that arise in working with these square functions when $p > 2$ that are rooted in the Carleson measure estimates we use. In principle, it seems reasonable to expect that Theorem (6.3.7) holds for $p > 2$ as well, but we cannot conclude these estimates here. The extension from Theorem (6.3.7) to Theorem (6.3.8) is an application of weight extrapolation, for which we must have a uniform estimate over all Muckenhoupt weights given in the A_∞-Carleson condition.

We say that a continuous linear operator T from S into S' is a Calderón-Zygmund operator with smoothness $M+\gamma$, for an integer $M \geq 0$ and $0 < \gamma \leq 1$, if T has function kernel $K: \mathbb{R}^{2n} \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ such that

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{2n}} K(x,y) f(y) g(x) dy dx$$

whenever $f, g \in C_0^\infty = C_0^\infty(\mathbb{R}^n)$ have disjoint support, and there is a constant $C > 0$ such that the kernel function K satisfies

$$\left|D_0^\alpha D_1^\beta K(x,y)\right| \leq \frac{C}{|x-y|^{n+|\alpha|+|\beta|}} \text{ for all } x \neq y \text{ and } |\alpha|, |\beta| \leq M,$$

$$\left|D_0^\alpha D_1^\beta K(x,y) - D_0^\alpha D_1^\beta K(x+h,y)\right| \leq \frac{C|h|^\gamma}{|x-y|^{n+M+|\beta|+\gamma}} \text{ for } |\beta| \leq |\alpha| = M,$$

$$\left|D_0^\alpha D_1^\beta K(x,y) - D_0^\alpha D_1^\beta K(x,y+h)\right| \leq \frac{C|h|^\gamma}{|x-y|^{n+|\alpha|+M+\gamma}} \text{ for } |\alpha| \leq |\beta| = M$$

whenever $|h| < |x-y|/2$. We will also define moment distributions for an operator $T \in \mathrm{CZO}(M+\gamma)$, but we require some notation first. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be supported in $B(0,2)$ and $\eta(x) = 1$ for $x \in B(0,1)$. Define for $R > 0, \eta_R(x) = \eta(x/R)$. Let $f \in C^\infty(\mathbb{R}^n)$ with $|f(x)| \leq C(1+|x|)^M, \psi \in C_0^\infty(\mathbb{R}^n)$, and choose $R_0 \geq 1$ minimal so that $\mathrm{supp}(\psi) \subset B(0, R_0/4)$. Define

$$\langle Tf, \psi\rangle = \lim_{R\to\infty} \langle T(\eta_R f), \psi\rangle - \sum_{|\beta|\leq M} \int_{\mathbb{R}^{2n}} \frac{D_0^\beta K(0,y)}{\beta!} x^\beta \left(\eta_R(y) - \eta_{R_0}(y)\right) f(y) \psi(x) dy dx.$$

This limit exists based on the kernel representation and kernel properties for $T \in CZO(M+\gamma)$. Furthermore, using different functions η_R, constructed as above, in this definition causes Tf to differ only by a polynomial. Hence the definition of Tf is independent of the particular selection of η_R as long as we work modulo polynomials. Now we define the moment distribution $[[T]]_\alpha \in S'(\mathbb{R}^n)$ for $T \in \mathrm{CZO}(M+\gamma)$ and $\alpha \in \mathbb{N}_0^n$ with $0 < |\alpha| \leq M$ by

$$\langle [[T]]_\alpha, \psi\rangle = \lim_{R\to\infty} \int_{\mathbb{R}^{2n}} K(u,y) \psi(u) \eta_R(y) (u-y)^\alpha dy du$$

$$- \sum_{|\beta|\leq|\alpha|} \int_{\mathbb{R}^{2n}} \frac{D_0^\beta K(0,y)}{\beta!} u^\beta (\eta_R(y) - \eta_1(y)) f(y) \psi(u) dy du$$

for $\psi \in S$, where $K \in S'(\mathbb{R}^{2n})$ is the distributional kernel of T and $K_\alpha(x,y) = K(x,y)(x-y)^\alpha$. We abuse notation here in that the integral in this definition is not necessarily a measure theoretic integral; rather, it is the dual pairing between elements of $S(\mathbb{R}^{2n})$ and $S'(\mathbb{R}^{2n})$. Throughout this work, we will use K to denote distributional kernels and K to denote function kernels for Calderón-Zygmund operators. When we write K in an integral over $\mathbb{R}^{2n}$, the integral is understood to be the pairing of $K \in S'(\mathbb{R}^{2n})$ with an element of $S(\mathbb{R}^{2n})$. It is not hard to show that this definition is well-defined by slightly modifying the techniques from [331], [339], [338], [343]. These details are provided later (in the proof of Proposition (6.3.9)), in which we prove that $[[T]]_\alpha$ can actually be realized as an element of S' when $\alpha \neq 0$, even though it is initially defined by pairing with elements of C_0^∞.

Proposition (6. 3. 9) [333]: Let $T \in CZO(M+\gamma)$ be bounded on L^2. Then $[[T]]_\alpha \in S'$ for $\alpha \in \mathbb{N}_0^n$ with $0 < |\alpha| \le M$. Furthermore, $[[T]]_\alpha \in S'/P$ coincides (modulo polynomials) with the function

$$[[T]]_\alpha(x) = \int_{\mathbb{R}^n} K_\alpha(x,y)\eta_1(y)dy + \int_{\mathbb{R}^n} \left(K_\alpha(x,y) - \sum_{|\beta|\le|\alpha|} \frac{D_0^\beta K_\alpha(0,y)}{\beta!} x^\beta \right)(1-\eta_1(y))dy,$$

where $K_\alpha(x,y) = K(x,y)(x-y)^\alpha$ and $|[[T]]_\alpha(x)| \lesssim 1 + |x|^{|\alpha|}$ as long as $0 < |\alpha| \le M$.
Proposition (6.3.9) is very useful since it gives an actual function representation for $[[T]]_\alpha$ rather than just a distribution. Also note that we require $\alpha \ne 0$ in Proposition (6.3.9). We only consider $[[T]]_0 = T1$ as an element of $\left(C_0^\infty(\mathbb{R}^n)\right)'$, not necessarily $S'(\mathbb{R}^n)$ here. Formally, it is reasonable to expect $[[T]]_0 = T1 \in S'(\mathbb{R}^n)$ using the definition of $[[T]]_\alpha$ when $\alpha = 0$, but as will be seen in the proof of Proposition (6.3.9), there is a technical issue that arises in the $\alpha = 0$ case. So we leave $[[T]]_0$ to be $T1$ as a distribution in $\left(C_0^\infty(\mathbb{R}^n)\right)'$ modulo polynomials.
This work is the following $T1$ type theorem for Calderón-Zygmund operators, which extends results from [114], [334], [331], [339], [338], [282], [343].
Theorem (6.3.10) [333]: Let $T \in CZO(M+\gamma)$ be bounded on L^2, and define $L = M/2$ and $\delta = (M+\gamma)/2 - L$. If $T^*(y^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0$ such that $|\alpha| \le L$, then T extends to a bounded operator on $H_w^p(\mathbb{R}^n)$ for any $\frac{n}{n+L+\delta} < p < \infty$ and $w \in A_{p\frac{n+L+\delta}{n}}(\mathbb{R}^n)$. Furthermore, the $H_w^p(\mathbb{R}^n)$ operator norm of T depends on $[w]_{A_{p\frac{n+L+\delta}{n}}}$, not on w itself.
Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. The choice of L and δ here depend on M and γ; it is the choice of L and δ such that $L \ge 0$ is an integer and $2(L+\delta) = M+\gamma$ where $0 < \delta \le 1$. Above we claimed that our main theorem is necessary and sufficient for H_w^p bounds, but its statement only includes one direction of that equivalence. The other direction follows essentially for free. By Theorem 7 in [341], it follows that the condition $T^*(y^\alpha) = 0$ for $|\alpha| \le L$ is necessary for T to be bounded on H^p for $\frac{n}{n+L+\delta} < p \le 1$. Hence the cancellation conditions in Theorem (6.3.10) are necessary and sufficient for H_w^p bounds.
We consider the weight invariance of a number of different spaces. For $s \ge 0$, we show that the spaces $I_s(BMO_w)$ for $w \in A_\infty(\mathbb{R}^n)$ all coincide with $I_s(BMO)$. Furthermore all of these spaces also coincide with the $p = \infty$ type Triebel- Lizorkin space $\dot{F}_{\infty,w}^{s,2}$. In our next theorem, we extend the results in Theorem 5 of [346] and Proposition 4 of [342].
Theorem (6.3.11) [333]: Let $s \ge 0$ and $w \in A_\infty(\mathbb{R}^n)$. Then $\dot{F}_{\infty,w}^{s,2} = \dot{F}_\infty^{s,2} = I_s(BMO_w) = I_S(BMO)$. Moreover, for $f \in S'/P$

$$\| f \|_{\dot{F}_{\infty,,w}^{s,2}} \approx \| f \|_{\dot{F}_\infty^{s,2}} \approx \|I_{-s}f\|_{BMO_w} \approx \|I_{-s}f\|_{BMO},$$

where the implicit constants depend on $[w]_{A_p}$ for some $1 \le p < \infty$, but not on w itself.
We will use the following Frazier and Jawerth type discrete Calderón reproducing formula [115](see also [293] for a multi parameter formulation of this reproducing formula): there exist $\phi_j, \tilde{\phi}_j \in S_\infty$ for $j \in \mathbb{Z}$ such that

$$f(x) = \sum_{j\in\mathbb{Z}_\ell} \sum_{\ell(Q)=2-(j+N_0)} |Q|\phi_j(x-cQ)\tilde{\phi}_j * f(cQ) \text{ in } L^2 \qquad (25)$$

for $f \in L^2$. The summation in Q here is over all dyadic cubes with side length $\ell(Q) = 2^{-(j+N_0)}$, where N_0 is some large constant, and cQ denotes the center of cube Q. Throughout we reserve the notation ϕ_j and $\tilde{\phi}_j$ for the operators constructed in this discrete Calderón decomposition.

We will also use a more traditional formulation of Calderón's reproducing formula: fix $\varphi \in C_0^\infty(B(0,1))$ with integral 1 such that

$$\sum_{k\in\mathbb{Z}} Q_k f = f \text{ in } L^2 \tag{26}$$

for $f \in L^2$, where $\psi(x) = 2^n\varphi(2x) - \varphi(x), \psi_k(x) = 2^{kn}\psi(2^k x)$, and $Q_k f = \psi_k * f$. Furthermore, we can assume that ψ has an arbitrarily large, but fixed, number of vanishing moments. There are many equivalent definitions of the real Hardy spaces $H_w^p = H_w^p(\mathbb{R}^n)$ for $0 < p < \infty$ and $w \in A_\infty$. We use the one given on in terms of the non-tangential maximal function Nf. It follows that

$$\left\| \sup_{k\in\mathbb{Z}} |\varphi_k * f| \right\|_{L_w^p} \lesssim \| f \|_{H_w^p}.$$

Let $\psi \in D_M$ for some integer $M > n(1/p - 1)$, and let ψ_k and Q_k be as above, satisfying (26). For $f \in S'/P, f \in H_w^p$ if and only if

$$\left\| \left(\sum_{k\in\mathbb{Z}} |Q_k f|^2 \right)^{\frac{1}{2}} \right\|_{L_w^p} < \infty,$$

and this quantity is comparable to $\| f \|_{H_w^p}$; see Theorem 1.4 in [109]. The space H_w^p can also be characterized by the operators ϕ_j and $\tilde{\phi}_j$ from the discrete Littlewood-Paley-Stein decomposition in (25). This characterization is given by the following, which can be found in Proposition 2.1 of [282]. Given $0 < p < \infty$

$$\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{\ell(Q)=2^{-(j+N_0)}} \left|\tilde{\phi}_j * f(cQ)\right|^2 \chi_Q \right)^{\frac{1}{2}} \right\|_{L_w^p} \approx \|f\|_{H_w^p}, \tag{27}$$

where $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$ for a subset $E \subset \mathbb{R}^n$. The summation again is indexed by all dyadic cubes Q with side length
$\ell(Q) = 2^{-(j+N_0)}$. For a continuous function $f: \mathbb{R}^n \to \mathbb{C}$ and $0 < r < \infty$, define

$$M_j^r f(x) = \left\{ M\left[\left(\sum_{\ell(Q)=2^{-(j+N_0)}} f(cQ)\chi_Q \right)^r \right](x) \right\}^{\frac{1}{r}}, \tag{28}$$

where M is the Hardy-Littlewood maximal operator. The following estimate was proved for the unweighted situation in [293]; we give a quick proof of it here for the weighted situation.

Proposition (6.3.12) [333]: For any $v > 0, \frac{n}{n+v} < r < \min(2,p) < \infty, w \in A_{\frac{p}{r}}$, and $f \in H_w^p$

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(M_j^r(\tilde{\phi}_j * f) \right)^2 \right)^{\frac{1}{2}} \right\|_{L_w^p} \lesssim \| f \|_{H_w^p},$$

where M_j^r is defined as in (28).

Proof: Let $v > 0, \frac{n}{n+v} < r < \min(2, p) \le p < \infty, w \in A_{\frac{p}{r}}$, and $f \in H_w^p$. Then using the definition of M_j^r in (28) and applying the Fefferman-Stein vector-valued maximal inequality from [283] with $p/r > 1$ and $2/r > 1$ gives

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(M_j^r(\tilde{\phi}_j * f) \right)^2 \right)^{\frac{1}{2}} \right\|_{L_w^p} = \left\| \left(\sum_{j\in\mathbb{Z}} \left\{ M \left[\left(\sum_{\ell(Q)=2^{-(j+N)}} |\tilde{\phi}_j * f(cQ)|\chi_Q \right)^r \right] \right\}^{\frac{2}{r}} \right)^{\frac{r}{2}} \right\|_{L_w^{\frac{p}{r}}}^{\frac{1}{2}}$$

$$\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{Q} |\tilde{\phi}_j * f(cQ)|^2 \chi_Q \right)^{\frac{1}{2}} \right\|_{L_w^p} \lesssim \| f \|_{H_w^p}$$

The next result was proved in [293] in a slightly different form and in [343] as it is stated here.

Proposition (6.3.13)[333]: Let $f: \mathbb{R}^n \to \mathbb{C}$ a non-negative continuous function, $v > 0$, and $\frac{n}{n+v} < r \le 1$. Then

$$\sum_{\ell(Q)=2^{-(j+N_0)}} |Q| \Phi^{n+v}_{\min\,(j,k)}(x - cQ) f(cQ) \lesssim 2^{\max\,(0,j-k)^v} M_j^r f(x)$$

for all $x \in \mathbb{R}^n$, where M_j^r is defined in (28) and the summation indexed by $\ell(Q) = 2^{-(j+N_0)}$ is the sum over all dyadic cubes with side length $2^{-(j+N_0)}$ and cQ denotes the center of cube Q.

Corollary(6.3.14)[356]: . For any $\varepsilon \ge 0, \frac{n}{n+1+\varepsilon} < r < \min(2, \varepsilon+1) < \infty, w_n \in A_{\frac{\varepsilon+1}{r}}$, and $f_{m_0} \in H_{w_n}^{\varepsilon+1}$

$$\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{m_0} \left(M_j^r(\tilde{\phi}_j * f_{m_0}) \right)^2 \right)^{\frac{1}{2}} \right\|_{L_{w_n}^{\varepsilon+1}} \lesssim \sum_{m_0} \|f_{m_0}\|_{H_{w_n}^{\varepsilon+1}}$$

where M_j^r is defined as .

Proof. Let $\varepsilon \ge 0, \frac{n}{n+v} < r < \min(2, \varepsilon+1) \le \varepsilon+1 < \infty, w_n \in A_{\frac{\varepsilon+1}{r}}$, and $f_{m_0} \in H_{w_n}^{\varepsilon+1}$. Then using the definition of M_j^r in (3.4) and applying the Fefferman-Stein vector-valued maximal inequality from Type equation here.with $\varepsilon + 1/r > 1$ and $2/r > 1$ gives

$$\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{m_0}\left(M_j^r\left(\tilde{\phi}_j * f_{m_0}\right)\right)^2\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon+1}_{w_n}}$$

$$=\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{m_0}\left\{M\left[\left(\sum_{\ell(Q)=2^{-(j+N)}}\left|\tilde{\phi}_j * f_{m_0}(cQ)\right|\chi_Q\right)^r\right]\right\}^{\frac{2}{r}}\right)^{\frac{r}{2}}\right\|^{\frac{1}{2}}_{L^{\frac{\varepsilon+1}{r}}_{w_n}}$$

$$\lesssim\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{Q}\sum_{m_0}\left|\tilde{\phi}_j * f_{m_0}(cQ)\right|^2\chi_Q\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon+1}_{w_n}}\lesssim\| f_{m_0}\|_{H^{\varepsilon+1}_{w_n}}$$

Corollary(6.3.15)[356]: If $w_n\in A_\infty(\mathbb{R}^n), b\in BMO_{w_n}$, and $\psi_{m_0}\in S(\mathbb{R}^n)$ with mean zero, then $|Q_kb(x)|^2$ is a w_n-Carleson collection where $Q_kb=\left(\psi_{m_0}\right)_k * b$ and $\left(\psi_{m_0}\right)_k(x)=2^{kn}\psi_{m_0}(2^kx)$. In particular, $BMO_{w_n}\subset\dot{F}^{0,2}_{\infty,w_n}$ with $\|f_{m_0}\|_{\dot{F}^{0,2}_{\infty,w_n}}\lesssim\|f_{m_0}\|_{BMO_{w_n}}$.

Proof. Let $w_n\in A_\infty$ and $b\in BMO_{w_n}$. Then for any cube $Q\subset\mathbb{R}^n$

$$\int_{Q_{2^-k_{\le\ell(Q)}}}|Q_kb(x)|^2w_n(x)dx\le 2\int_{Q^{-k}\le\ell(Q)}\sum_k\left|Q_k\left(\left[b-b_{Q,w_n}\right]\chi_{2Q}\right)(x)\right|^2w_n(x)dx$$

$$+2\int_{Q_{2^-k_{\le}(Q)}}\sum_{w_k}\left|\left[b-b_{Q,w_n}\right]\chi_{(2Q)^c)}(x)\right|^2w_n(x)dx$$

$$=I+II.$$

$$=I+II$$

Since $w_n\in A_\infty$ it follows that $w_n\in A_{\varepsilon+1}$ for some $0\le\varepsilon<\infty$. Since the square function associated to the collection Q_k is bounded on $L^{\varepsilon+1}_{w_n}(\mathbb{R}^n)$, it follows that

$$I=2w_n(Q)^{\frac{\varepsilon-1}{\varepsilon+1}}\left\|S_{\psi_{m_0}}\left(\left[b-b_{Q,w_n}\right]\chi_{2Q}\right)\right\|^2_{L^{\varepsilon+1}_{w_n}}$$

$$\lesssim w_n(Q)^{\frac{\varepsilon-1}{\varepsilon+1}}\left(\int_{2Q}\left|b(x)-b_{Q,w_n}\right|^{\varepsilon+1}w_n(x)dx\right)^{\frac{2}{\varepsilon+1}}\lesssim\| b\|^2_{BMO^2_{w_n}}w_n(Q).$$

Before we estimate II, we first note that for any $\ell\in\mathbb{N}$ we have

$$\left(\frac{1}{w_n(2^{\ell+1}Q)}\int_{2^{\ell+1}Q}\left|b(x+\varepsilon)-b_{Q,w_n}\right|^{\varepsilon+1}w_n(x)d(x+\varepsilon)\right)^{\frac{1}{\varepsilon+1}}$$

$$\leq\left(\frac{1}{w_n(2^{\ell+1}Q)}\int_{2^{\ell+1}Q}\left|b(x+\varepsilon)-b_{2^{\ell+1}Q,w_n}\right|^{\varepsilon+1}w_n(x)d(x+\varepsilon)\right)^{\frac{1}{\varepsilon+1}}+\sum_{m=0}^{\ell}\left|b_{2^{m+1}Q,w_n}-b_{2^mQ,w_n}\right|$$

$$\leq\| b\|_{BMO_{w_n}^{\varepsilon+1}}+\sum_{m=0}^{\ell}\frac{1}{w_n(2^mQ)}\int_{2^mQ}\left|b(x+\varepsilon)-b_{2^{m+1}Q,w_n}\right|w_n(x+\varepsilon)d(x+\varepsilon)$$

$$\leq\| b\|_{BMO_{w_n}^{\varepsilon+1}}+\sum_{m=0}^{\ell}\frac{1}{w_n(2^{m+1}Q)}\int_{2^{m+1}Q}\left|b(x+\varepsilon)-b_{2^{m+1}Q,w_n}\right|w_n(x+\varepsilon)d(x+\varepsilon)$$

$$\leq\left(\| b\|_{BMO_{wn}^{\varepsilon+1}}+\| b\|_{BMO_{w_n}}\right).$$

Now to bound II, we consider for $x\in Q$

$$\lesssim 2^{-k}\sum_{\ell=1}^{\infty}\frac{1}{2^\ell\ell(Q)}\frac{1}{|2^{\ell+1}Q|}\int_{2^{\ell+1}Q}\left|b(x+\varepsilon)-b_{Q,w_n}\right|w_n(x+\varepsilon)^{\frac{1}{\varepsilon}+1}w_n(x+\varepsilon)^{-\frac{1}{\varepsilon}+1}d(x+\varepsilon)$$

$$\lesssim 2^{-k}\sum_{\ell=1}^{\infty}\frac{1}{2^\ell\ell(Q)}\left(\frac{1}{|2^{\ell+1}Q|}\int_{2^{\ell+1}Q}\left|b(x+\varepsilon)-b_{Q,w_n}\right|^{\varepsilon+1}w_n(x+\varepsilon)d(x+\varepsilon)\right)^{\frac{1}{\varepsilon+1}}$$

$$\times\left(\frac{1}{|2^{\ell+1}Q|}\int_{2^{\ell+1}Q}w_n(x+\varepsilon)^{-\varepsilon+1}d(x+\varepsilon)\right)^{\frac{1}{\varepsilon+1}}$$

$$\lesssim 2^{-k}\ell(Q)^{-1}\left(\| b\|_{BMO_{wn}^{\varepsilon+1}}+\| b\|_{BMO_{w_n}}\right)\sum_{\ell=1}^{\infty}\ell\cdot 2^{-\ell}\frac{w_n\left(2^{\ell+1}Q\right)^{\frac{1}{\varepsilon+1}}}{|2^{\ell+1}Q|^{\frac{1}{\varepsilon+1}}}\left(\frac{\left|2^{\ell+1}Q\right|}{w_n(2^{\ell+1}Q)}\right)^{\frac{1}{\varepsilon+1}}$$

$$2^{-k}\ell(Q)^{-1}\| b\|_{BMO_{w_n}}.$$

Here we have used that $\| b\|_{BMO_{w_n}^{\varepsilon+1}}\approx\| b\|_{BMO_{w_n}}$ for $w_n\in A_\infty(\mathbb{R}^n)$, where the implicit constants depend on $[w_n]_{A_{1+2\varepsilon}}$ for some $1\leq\varepsilon\leq\infty$. It easily follows that $II\leq\| b\|_{BMO_{w_n}}^2(Q)$. For appropriate choices of ψ_{m_0}, it easily follows from this estimate that $\|f_{m_0}\|_{\dot F^{0,2}_{\infty,w_n}}\lesssim\|f_{m_0}\|_{BMO_{w_n}}$.

Corollary(6.3.16)[356]: . Let $w_n\in A_\infty$. If $f_{m_0}\in S'/P$, then

$$\sup_{P\text{ dyadic}}\frac{1}{w_n(P)}\int_P\sum_{Q\subset P}\sum_{m_0}\left(\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right||Q|^{-1/2}\chi_Q(x)\right)^2w_n(x)dx\lesssim\|f_{m_0}\|^2_{(f_\infty^{0,2})_{m_0}}.$$

Proof. Let $w_n\in A_\infty$ and $\varphi_{m_0}\in S$ be as above. Following the work in [108], define

$$\left(\varphi_{m_0}\right)_k^*f_{m_0}(x)=\sup_{x+\varepsilon\in\mathbb{R}^n}\left|\left(\varphi_{m_0}\right)_k*f_{m_0}(\varepsilon)\right|(1+2^k|x+\varepsilon|)^{-\lambda}.$$

for λ fixed sufficiently large, we have

$$\sup_Q\frac{1}{w_n(Q)}\int_Q\sum_{k:2^{-k}\leq\ell(Q)}\sum_{m_0}\left(\varphi_{m_0}\right)_k^*f_{m_0}(x)^2w_n(x)dx\lesssim\|f_{m_0}\|^2_{(f_\infty^{0,2})_{m_0}}$$

for all $f_{m_0}\in S'$. Also for any dyadic cube Q such that $\ell(Q)=2^{-k}$ and $2^{-k}\vec{m}$ is the lower left hand corner of Q, it follows that for $x\in Q$

$$|Q|^{-1/2}\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right| = \left|\left(\varphi_{m_0}\right)_k * f_{m_0}(\vec{m})\right| \lesssim \left|\left(\varphi_{m_0}\right)_k * f_{m_0}(\vec{m})\right|(1+2^k|x-\vec{m}|)^{-\lambda}$$
$$\le \left(\varphi_{m_0}\right)_k^* f_{m_0}(x).$$

Then for any dyadic cube P, it follows that

$$\int_P \sum_{Q\subset P}\sum_{m_0}\left(\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right||Q|^{-1/2}\chi_Q(x)\right)^2 w_n(x)dx$$
$$\lesssim \int_P \sum_{k:2^{-k}\le\ell(P)}\sum_{m_0}\left(\varphi_{m_0}\right)_k^* f_{m_0}(x)^2 \sum_{Q:\ell(Q)=2^{-k}}\chi_Q(x)w_n(x)dx$$
$$\le \int_P \sum_{k:2^{-k}\le\ell(P)}\sum_{m_0}\left(\varphi_{m_0}\right)_k^* f_{m_0}(x)^2 w_n(x)dx \lesssim w_n(P)\|f_{m_0}\|^2_{(f_\infty^{0,2})_{m_0}}.$$

The first inequality in the last line holds since the dyadic cubes Q with $\ell(Q)=2^{-k}$ are disjoint, and so we have $\sum_{Q:\ell(Q)=2^{-k}}\chi_Q(x)\le 1$. Dividing by $w_n(P)$ and taking the supremum over P completes the proof.

Corollary(6.3.17)[356]: If $w_n\in A_\infty(\mathbb{R}^n)$, then $\|f_{m_0}\|_{\dot{F}_\infty^{0,2}} \lesssim \|f_{m_0}\|_{(f_\infty^{0,2})_{m_0}}$ for all $f_{m_0}\in \dot{F}^{0,2}_{\infty,w_n}$. Hence $\dot{F}^{0,2}_{\infty,w_n}\subset \dot{F}^{0,2}_\infty$. More precisely, if $w_n\in A_{\varepsilon+1}(\mathbb{R}^n)$ for some $0\le\varepsilon\le\infty$, then $\|f_{m_0}\|^2_{\dot{F}_\infty^{0,2}} \lesssim [w_n]_{A_{\varepsilon+1}}\|f_{m_0}\|^2_{\dot{F}^{0,2}_{\infty,w_n}}$ for all $f_{m_0}\in S'/P$

Proof. Let $w_n\in A_{\varepsilon+1}$ for some $0\le\varepsilon\le\infty$ and $f_{m_0}\in\dot{F}^{0,2}_{\infty,w_n}$. To complete the proof, it is sufficient to show that $\left\|S_{\varphi_{m_0}}f_{m_0}\right\|^2_{f_\infty^{0,2}} \lesssim [w_n]_{A_{\varepsilon+1}}\|f_{m_0}\|^2_{\dot{F}^{0,2}_{\infty,w_n}}$.

$$\left\|S_{\varphi_{m_0}}f_{m_0}\right\|^2_{\dot{f}_\infty^{0,2}} \approx \sup_{P\text{ dyadic}}\frac{1}{|P|}\int_P\sum_{Q\subset P}\sum_{m_0}\left(|Q|^{-1/2}\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right|\right)^2 dx$$
$$\approx \sup_{P\text{ dyadic}}\left[\frac{1}{|P|}\int_P\left(\sum_{Q\subset P}\sum_{m_0}\left(\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right||Q|^{-1/2}\chi_Q(x)\right)^2\right)^{1/\varepsilon+1}dx\right]$$
$$\le \sup_{P\text{ dyadic}}\left[\frac{1}{|P|}\int_P\sum_{Q\subset P}\sum_{m_0}\left(\left|\left(S_{\varphi_{m_0}}f_{m_0}\right)_Q\right||Q|^{-1/2}\chi_Q(x)\right)^2 w_n(x)dx\right]$$
$$\times\left[\frac{1}{|P|}\int_P w_n(x)^{\varepsilon-1}dx\right]^{1/\varepsilon+1}$$
$$\lesssim \|f_{m_0}\|^2_{\dot{F}^{0,2}_{\infty,w_n}}\sup_{\text{P dyadic}}\frac{w_n(P)}{|P|}\left[\frac{1}{|P|}\int_P w_n(x)^{-1/\varepsilon+1}dx\right]^{1/\varepsilon+1}\le [w_n]_{A_{\varepsilon+1}}\|f_{m_0}\|^2_{\dot{F}^{0,2}_{\infty,w_n}}.$$

Corollary(6.3.18)[356]: Let $w_n\in A_\infty$. If a non-negative measure $d\mu_{w_n}$ on $\mathbb{R}^{n+1}_+$ satisfies $d\mu_{w_n}(Q\times(0,\ell(Q)))\le Aw_n(Q)$ for all cubes $Q\subset\mathbb{R}^n$, then $d\mu_{w_n}(\hat{E})\lesssim Aw_n(E)$ for any open set $E\subset\mathbb{R}^n$ where $\hat{E}=\{(x,t): B(x,t)\subset E\}$. Here the suppressed constant does not depend on the constant A or the set E.

This proof is a standard argument that can be found in many articles and books on harmonic analysis, only modified by replacing the Lebesgue measure dx with the weighted Lebesgue measure $w_n(x)dx$. We include the proof for the sake of completeness.

Proof. Let $w_n \in A_\infty$, and assume that $d\mu_{w_n}$ satisfies $d\mu_{w_n}(Q \times (0, \ell(Q))) \le Aw_n(Q)$ for all cubes $Q \subset \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be an open set. Define $E_N = E \cap Q_N$ where Q_N is the cube centered at the origin with side length N., there exist a constant b depending on the doubling constant of w_n and a collection of disjoint dyadic sub-cubes $\{Q_j\}$ of Q_N such that

$$\frac{1}{2} < \frac{1}{w_n(Q_j)} \int_{Q_j} \chi_{E_N}(x) w_n(x) dx \le \frac{1}{2} b$$

and $\chi_{E_N}(x) \le \frac{1}{2}$ almost everywhere on $Q_N \setminus U_j Q_j$. Since $\chi_{E_N} = 1$ on E_N, it follows that $E_N \subset \cup_j Q_j$. Then it also follows that $\hat{E}_N \subset \cup_j Q_j \times \left(0, 2\sqrt{n}\ell(Q_j)\right)$. Therefore

$$\begin{aligned} d\mu_{w_n}(\hat{E}_N) &\le \sum_j d\mu_{w_n}\left(Q_j \times \left(0, 2\sqrt{n}\ell(Q_j)\right)\right) \le A \sum_j w_n(2\sqrt{n}Q_j) \lesssim A \sum_j w_n(Q_j) \\ &\lesssim A \sum_j \int_{Q_j} \chi_{E_N}(x) w_n(x) dx \le A \int_{E_N} w_n(x) dx = Aw_n(E_N). \end{aligned}$$

Note that since $w_n \in A_\infty$, it follows that w_n is a doubling measure with doubling constant depending on $A_{\varepsilon+1}$ character of w_n for some $0 \le \varepsilon \le \infty$, which we used in the previous estimate. Therefore $d\mu_{w_n}(\hat{E}_N) \lesssim w_n(E_N)$ for all $N \ge 1$ where the constant is independent of N, E, and E_N. Since $\hat{E}_N$ and E_N are increasing exhaustions of $\hat{E}$ and E respectively, it follows that

$$d\mu_{w_n}(E) = \lim_{N\to\infty} d\mu_{w_n}(E_N) \lesssim A \lim_{N\to\infty} w_n(E_N) = Aw_n(E).$$

Corollary(6.3.19)[356]: Let $w_n \in A_\infty(\mathbb{R}^n)$. Also let $\varphi_{m_0} \in S$, and define $P_k f_{m_0} = (\varphi_{m_0})_k * f_{m_0}$, where $(\varphi_{m_0})_k(x) = 2^{kn}\varphi_{m_0}(2^k x)$ for $k \in \mathbb{Z}$. If $\mu_k(x)$ is a w_n-Carleson collection, then

$$\left\| \sum_{m_0} \left(\sum_{k\in\mathbb{Z}} \left|P_k f_{m_0}\right|^2 \cdot \mu_k(x) \right)^{\frac{1}{2}} \right\|_{L^{\varepsilon+1}_{w_n}} \lesssim \|\{\mu_k\}\|_{C_{w_n}} \sum_{m_0} \|f_{m_0}\|_{H^{\varepsilon+1}_{w_n}} \text{ for all } 0 < \varepsilon \le 1.$$

Proof. Let $f_{m_0} \in H^{\varepsilon+1}_{w_n}(\mathbb{R}^n)$, and we begin the proof by looking at

$$\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \sum_{m_0} \left|P_k f_{m_0}(x)\right|^{\varepsilon+1} \mu_k(x) w_n(x) dx$$

$$(\varepsilon+1) \int_0^\infty d\mu_{w_n}\left(\left\{(x,t): \left|\int_{\mathbb{R}^n} \sum_{m_0} t^{-n} \varphi_{m_0}(t^{-1}(\varepsilon)) f_{m_0}(x+\varepsilon) d(x+\varepsilon)\right| > \lambda\right\}\right) \lambda^{\varepsilon+1} \frac{d\lambda}{\lambda},$$

where $d\mu_{w_n}$ is a non-negative measure on $\mathbb{R}^{n+1}_+$ defined by

$$d\mu_{w_n}(x,t) = \sum_{k\in\mathbb{Z}} \mu_k(x) \delta_{t=2^{-k}} w_n(x) dx.$$

Define $E_\lambda = \left\{x: \left|N^{\varphi_{m_0}} f_{m_0}(x)\right| > \lambda\right\}$, and it follows that

$$\left\{(x,t):\left|\int_{\mathbb{R}^n}\sum_{m_0} t^{-n}\varphi_{m_0}(t^{-1}(\varepsilon))f_{m_0}(x+\varepsilon)d(x+\varepsilon)\right|>\lambda\right\}\subset E_\lambda,$$

where $\hat{E}=\{(x,t):B(x,t)\subset E\}$. Note that It follows that $d\mu_{w_n}(x,t)$ satisfies the estimate $d\mu_{w_n}(\hat{E})\lesssim\|\{\mu_k\}\|^2_{c_{w_n}}w_n(E)$. Therefore

$$\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\sum_{m_0}\left|P_kf_{m_0}(x)\right|^{\varepsilon+1}\mu_k(x)w_n(x)dx\leq\varepsilon+1\int_0^\infty d\mu_{w_n}(\hat{E}_\lambda)\lambda^{\varepsilon+1}\frac{d\lambda}{\lambda}$$

$$\lesssim(\varepsilon+1)\|\{\mu_k\}\|^2_{C_{w_n}}\int_0^\infty w_n(E_\lambda)\lambda^{\varepsilon+1}\frac{d\lambda}{\lambda}$$

$$=\|\{\mu_k\}\|^2_{C_{w_n}}\left\|N^{\varphi_{m_0}}f_{m_0}\right\|^{\varepsilon+1}_{L^{\varepsilon+1}_{w_n}}\lesssim\|\{\mu_k\}\|^2_{C_{w_n}}\left\|f_{m_0}\right\|^{\varepsilon+1}_{H^{\varepsilon+1}_{w_n}}.$$

In the case $\varepsilon=3$, this completes the proof. When $0<\varepsilon<1$, we set $\frac{2}{\varepsilon+1}>1$ and then the Hölder conjugate is $\frac{2}{\varepsilon-1}$. Now applying the estimate above, we obtain

$$\int_{\mathbb{R}^n}\sum_{m_0}\left(\sum_{k\in\mathbb{Z}}\left|P_kf_{m_0}(x)\right|^2\mu_k(x)\right)^{\frac{\varepsilon+1}{2}}w_n(x)dx$$

$$\leq\int_{\mathbb{R}^n}\sup_k\sum_{m_0}\left|P_kf_{m_0}(x)\right|^{(\varepsilon-1)\varepsilon+1/2}\left(\sum_{k\in\mathbb{Z}}\left|P_kf_{m_0}(x)\right|^{\varepsilon+1}\mu_k(x)\right)^{\frac{\varepsilon+1}{2}}w_n(x)dx$$

$$\leq\sum_{m_0}\left\|\left(N^{\varphi_{m_0}}f_{m_0}\right)^{(\varepsilon-1)(\varepsilon+1)/2}\right\|_{L'_{w_n}}\left\|\left(\sum_{k\in\mathbb{Z}}\left|P_kf_{m_0}\right|^{\varepsilon+1}\mu_k\right)^{\frac{\varepsilon+1}{2}}\right\|_{L^r_{w_n}}$$

$$=\sum_{m_0}\left\|N^{\varphi_{m_0}}f_{m_0}\right\|^{\frac{(\varepsilon+1)(\varepsilon-1)}{2}}_{L^{\varepsilon+1}_{w_n}}\left(\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\left|P_kf_{m_0}(x)\right|^{\varepsilon+1}\mu_k(x)w_n(x)dx\right)^{\frac{\varepsilon+1}{2}}$$

$$\lesssim\|\{\mu_k\}\|^{\varepsilon+1}_{C_{w_n}}\sum_{m_0}\left\|f_{m_0}\right\|^{(\varepsilon-1)/2}_{H^{\varepsilon+1}_{w_n}}\left\|f_{m_0}\right\|^{(\varepsilon+1)^2/2}_{H^{\varepsilon+1}_{w_n}}=\|\{\mu_k\}\|^{\varepsilon+1}_{C_{w_n}}\sum_{m_0}\left\|f_{m_0}\right\|^{\varepsilon+1}_{H^{\varepsilon+1}_{w_n}}.$$

Corollary(6.3.20)[356]: Let $\varphi_{m_0}\in S$, and define $P_kf_{m_0}=\left(\varphi_{m_0}\right)_k*f_{m_0}$, where $\left(\varphi_{m_0}\right)_k(x)=2^{kn}\varphi_{m_0}(2^kx)$ for $k\in\mathbb{Z}$. If $\mu_k(x)$ is an A_∞-Carleson collection, then

$$\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{m_0}\left|P_kf_{m_0}\right|^2\cdot\mu_k(x)\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon+1}_{w_n}}\lesssim\left\|\{\mu_k\}\right\|_{c_{w_n}}\sum_{m_0}\left\|f_{m_0}\right\|_{H^{\varepsilon+1}_{w_n}}\text{ for all }0\leq\varepsilon\leq\infty\text{ and }w_n\in A_\infty(\mathbb{R}^n).$$

Let $\psi_{m_0}\in S$ such that ψ_{m_0} has mean zero, and define $Q_kf_{m_0}=\left(\psi_{m_0}\right)_k*f_{m_0}$, where $\left(\psi_{m_0}\right)_k(x)=2^{kn}\psi_{m_0}(2^kx)$ for $k\in\mathbb{Z}$. If $b\in BMO$, then $|Q_kb(x)|^2$ is an A_∞-Carleson collection and

$$\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{m_0}\left|P_kf_{m_0}\right|^2\cdot|Q_kb|^2\right)^{\frac{1}{2}}\right\|_{L^{\varepsilon+1}_{w_n}} \lesssim \| b \|_{BMO}\sum_{m_0}\| f_{m_0}\|_{H^{\varepsilon+1}_{w_n}} \text{ for all } 0\le\varepsilon \le\infty \text{ and } w_n \in A_\infty(\mathbb{R}^n).$$

Proof. Fix $1<\varepsilon<\infty$ to be specified later and $w_n\in A_{\varepsilon+1}$. Now we consider the collection

$$C_{\varepsilon+1}=\left\{\left[Nf_{m_0}\right]^{2/r},\left(\sum_{k\in\mathbb{Z}}\left|P_kf_{m_0}\right|^2\cdot\mu_k(x)\right)^{1/\varepsilon+1}:f_{m_0}\in S\right\}.$$

With $\varepsilon=1$, for any $\left(g_{m_0},h_{m_0}\right)\in C_{\varepsilon+1}$, we have

$$\left\|h_{m_0}\right\|_{L^{\varepsilon+1}_{w_n}}=\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{m_0}\left|P_kf_{m_0}\right|^2\cdot\mu_k\right)^{1/2}\right\|^{2/\varepsilon+1}_{L^2_{w_n}}\le CN\left([w_n]_{A_{\varepsilon+1}}\right)^{2/r}\sum_{m_0}\left\|Nf_{m_0}\right\|^{2/r}_{L^2_{w_n}}$$
$$=\mathrm{CN}\left([w_n]_{A_r}\right)^{\frac{2}{r}}\sum_{m_0}\left\|g_{m_0}\right\|_{L^2_{w_n}}.$$

Therefore by extrapolation (see the version) $\left\|h_{m_0}\right\|_{L^{1+2\varepsilon}_w}\lesssim\left\|g_{m_0}\right\|_{L^{1+2\varepsilon}_{w_n}}$ for all $1\le\varepsilon\le\infty, w_n\in A_{1+2\varepsilon}$ and $\left(g_{m_0},h_{m_0}\right)\in C_{\varepsilon+1}$. That is, for any $1<r<\infty$ and C_r defined as above, we have $\left\|h_{m_0}\right\|_{L^1_w{1+2\varepsilon}}\lesssim\left\|g_{m_0}\right\|^{1+2\varepsilon}_{L_w}$ for all $1\le\varepsilon\le\infty, w_n\in A_{1+2\varepsilon}$, and $\left(g_{m_0},h_{m_0}\right)\in C_r$. Fix $1\le\varepsilon\le\infty$ and $w_n\in A_\infty$. Then there is a $\varepsilon\ge0$ large enough so that $w_n\in A_{1+2\varepsilon}$. Let $2(1+2\varepsilon)/\varepsilon+1$, which is larger than 1 , and it follows that

$$\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{m_0}\left|P_kf_{m_0}\right|^2\cdot\mu_k\right)^{1/2}\right\|_{L^{\varepsilon+1}_{w_n}}=\left\|h_{m_0}\right\|^{r/2}_{L^{1+2\varepsilon}_{w_n}}\sum_{m_0}\left\|g_{m_0}\right\|^{r/2}_{L^{1+2\varepsilon}_{w_n}}=\sum_{m_0}\left\|Nf_{m_0}\right\|_{L^{\varepsilon+1}_{W_n}}$$

for all $f_{m_0}\in S$, where we use the identification for $\left(g_{m_0},h_{m_0}\right)$ given by $C_{\varepsilon+1}$. It was shown in Theorem 2.4 of [109] that S is dense in $H^{\varepsilon+1}_{w_n}$ for $1<\varepsilon<\infty$ and $w_n\in A_\infty$. So by density, this completes the proof.

It is easy now to note that if $b\in BMO$, then $\mu_k(x)=|Q_kb(x)|^2$ is a A_∞-Carleson collection satisfying $\|\{\mu_k\}\|_{C_{w_n}}\le N\left([w_n]_{A_{\varepsilon+1}}\right)\| b \|_{BMO}$ for some increasing function N and for all $0\le\varepsilon\le\infty$ and $w_n\in A_{\varepsilon+1}$. Hence the second inequality follows as well.

List of Symbols

Symbol		Page
L^1:	Lebesgue space on the real line	1
H^1:	Hardy space	1
$H^{1,\infty}$:	Hardy space	1
$L^{1,\infty}$:	Lebesgue space	1
H^p:	Hardy space	1
$H^{p,q}$:	Hardy – Lorentz space	1
$L^{p,q}$:	Lorentz space	1
sup:	supremum	1
ℓ^q:	Dual of Lebesgue space	2
L^q:	Dual of Lebesgue space	3
min:	minimum	3
inf:	infimum	3
supp:	support	4
L^∞:	essential Lebesgue space	4
$B_s^{p,q}, Q_{p,q}, D_{p,q}$:	Lorentz martingale space	12
ℓ_∞:	Essential Hilbert space of sequences	17
ℓ_2:	Hilbert space of sequences	17
dist:	distance	20
$W^{1,p}$:	Sobolev space	40
cop_p:	p-capacity	42
$N^{1,p}$:	Sobolev space	44
diam:	diameter	45
max:	maximum	49
loc:	local	56
BV:	Bounded variation	60
$\dot{F}_{p,q}^{\alpha,\omega}$	Weighted Triebel – Lizorkin space	75
BMO:	Bounded mean oscillation	75
$\dot{B}_{\infty,\infty}^{\alpha}$:	Besov – Lipschitz space	75
H_ω^p:	Weighted Hardy space	88
$L_{p,q,b}$:	Lorentz – Karamata	99
$H_{p,q,b}^s$:	Hardy space	119
$P_{p,q,b}$:	Lorentz space	119
$Q_{p,q,b}$:	Karamata space	119
H_q:	Dual Hardy space	128
TCR:	Truncated cone Region	132
ext:	extension	153
$a.e$:	almost every where	157
$H^{s,p}(\mathbb{R}^d)$:	Bessel potential space	164
$B_s^{p,q}(\mathbb{R}^d)$:	Besove space	164
$\dot{F}_{\infty,w}^{s,2}$:	Triebel – Lizorkin space	180
$LPSO$:	Littlewood – Paley	187
CZO:	Calderon – Zygmund Operator	186

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