Sudan University of Science and Technology
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Bishop-Phelps-Bollobás Theorem and Properties for Operators and Approximate Hyperplane Series

مبرهنات بيشوب – فيلبس – بولوباس والخصائص للمؤثرات ومتسلسلة المستوي المفرط التقريبي

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedications

To my family

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Abstract

The Lindelöf property and Bishop-Phelps-Bollobás moduli in Banach spaces are studied. The Bishop-Phelps-Bollobás theorem for operators from c_0 to uniformly convex spaces, for bilinear forms and for uniform algebras are established. We characterize the Bishop-Phelps-Bollobás property for numerical radius in $\ell_1(C)$ operators on $C(K)$, for certain spaces of operators and for numerical radius of operators on $L_1(\mu)$. Asplund operators, Γ-flatness and Bishop–Phelps–Bollobás type theorems for operators, version of Lindenstrauss properties A and B and approximation hyperplane series properties are considered.

الخلاصة

تمت دراسة خاصية ليندلوف ومقاييس بيشوب – فيلبس – بولوباس في فضاءات باناخ. قمنا بتأسيس مبرهنة بيشوب – فيلبس – بولوباس لاجل المؤثرات من c_0 إلى الفضاءات المحدبة المنتظمة ولاجل الصيغ ثنائية الخطية والجبريات المنتظمة. تم تشخيص خاصية بيشوب – فيلبس – بولوباس لاجل نصف القطر العددي في $\ell_1(C)$ والمؤثرات على $C(K)$ ولاجل فضاءات معينة للمؤثرات ولنصف القطر العددي للمؤثرات على $L_1(\mu)$. قمنا باعتبار تسطيح-Γ والمبرهنات نوع بيشوب – فيلبس – بولوباس لأجل المؤثرات وإصدار خاصية ليندينستراوس A و B وخاصيات متسلسلة المستوى المفرط التقريبي.

Introduction

We show that topological space (T,τ) is said to be fragmented by a metric d on T if each non-empty subset of T has non-empty relatively open subsets of arbitrarily small d-diameter. We devoted to applications of the basic theorem. A compact Hausdorff space K is Radon-Nikodým compact if, and only if, there is a bounded subset D of $C(K)$ separating the points of K such that $(K,\gamma(D))$ is Lindelöf. If X is a Banach space and H is a weak* -compact subset of the dual X^* which is weakly Lindelöf, then $(H, \text{weak})^{\mathbb{N}}$ is Lindelöf. We deal with a strengthening of the Bishop-Phelps property for operators that in the literature is called the Bishop-Phelps-Bollobás property. Let X be a Banach space and L a locally compact Hausdorff space. We prove that if $T: X \to C_0(L)$ is an Asplund operator and $\|T(x_0)\| \approx \| T \|$ for some $\|x_0\| = 1$, then there is a norm attaining Asplund operator $S: X \to C_0(L)$ and $\|u_0\| = 1$ with $\|S(u_0)\| = \| S \| = \| T \|$ such that $u_0 \approx x_0$ and $S \approx T$.

We show that the set of bounded linear operators from X to X admits a Bishop-Phelps-Bollobás type theorem for numerical radius whenever X is $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. Guirao and Kozhushkina introduced the Bishop-Phelps-Bollobás property says that if we have a state and an operator that almost attains it numerical radius at this state, then there exist another state close to the original state and another operator close to the original operator, such that the new operator attains its numerical radius at this new state. We provide a version for operators of the Bishop-Phelps-Bollobás theorem when the domain space is the complex space $\mathcal{C}_0(L)$.

We show that the pair of Banach spaces (c_0, Y) has the Bishop-PhelpsBollobás property when Y is uniformly convex. Further, when Y is strictly convex, if (c_0, Y) has the Bishop-Phelps-Bollobás property then Y is uniformly convex for the case of real Banach spaces. We provide versions of the Bishop-Phelps-Bollobás Theorem for bilinear forms. Indeed we prove the first positive result of this kind by assuming uniform convexity on the Banach spaces. A characterization of the Banach space Y satisfying a version of the Bishop-Phelps-Bollobás Theorem for bilinear forms on $\ell_1 \times Y$ is also obtained. As a consequence of this characterization, we obtain positive results for finite-dimensional normed spaces, uniformly smooth spaces, the space $C(K)$ of continuous functions on a compact Hausdorff topological space K and the space $K(H)$ of compact operators on a Hilbert space H.

We devoted to showing that Asplund operators with range in a uniform Banach algebra have the Bishop-Phelps-Bollobás property, i.e., they are approximated by norm attaining Asplund operators at the same time that a point where the approximated operator almost attains its norm is approximated by a point at which the approximating operator attains it. We characterize the Banach spaces Y for which certain subspaces of operators from $L_1(\mu)$ into Y have the Bishop-Phelps-Bollobás property in terms of a geometric property of Y, namely AHSP. This characterization applies to the spaces of compact and weakly compact operators. The Bishop-Phelps-Bollobás property deals with simultaneous approximation of an operator T and a vector x at which T nearly attains its norm by an operator T_0 and a vector x_0, respectively, such that T_0 attains its norm at x_0. We extend the already known results about the Bishop-Phelps-Bollobás property for Asplund operators to a wider class of Banach spaces and to a wider class of operators. Instead of proving a BPB-type theorem for each space separately we isolate two main notions: Γ-flat operators and Banach spaces with ACK_ρ structure.

We study the Bishop-Phelps-Bollobás property for numerical radius (in short, BPBp-nu) and find sufficient conditions for Banach spaces to ensure the BPBp-nu. Among other results, we show that $L_1(\mu)$-spaces have this property for every measure μ. We introduce two Bishop-Phelps-Bollobás moduli of a Banach space which measure, for a given Banach space, what is the best possible Bishop-Phelps-Bollobás theorem in this space. We show that there is a common upper bound for these moduli for all Banach spaces and we present an example showing that this bound is sharp. We prove the continuity of these moduli and an inequality with respect to duality. We introduce the notion of the Bishop-Phelps-Bollobás property for numerical radius (BPBp-ν) for a subclass of the space of bounded linear operators. Then, we show that certain subspaces of $\mathcal{L}(L_1(\mu))$ have the BPBp-ν for every finite measure μ.

We study a Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B. For domain spaces, we study Banach spaces X such that (X, Y) has the Bishop-Phelps-Bollobás property (BPBp) for every Banach space Y. We show that in this case, there exists a universal function $\eta X(\varepsilon)$ such that for every Y, the pair (X, Y) has the BPBp with this function. This allows us to show some necessary isometric conditions for X to have the property. We also show that if X has this property in every equivalent norm, then X is one-dimensional. We show the Bishop-Phelps-Bollobás theorem for operators from an arbitrary Banach space X into a Banach space Y whenever the range space has property β of Lindenstrauss. We also characterize those Banach spaces Y for which the Bishop-Phelps-Bollobás theorem holds for operators from ℓ_1 into Y. Several examples of classes of such spaces are provided. We study the Bishop-Phelps-Bollobás property for operators between Banach spaces. Sufficient conditions are given for generalized direct sums of Banach spaces with respect to a uniformly monotone Banach sequence lattice to have the approximate hyperplane series property.

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Chapter 1
Banach Spaces and Bishop-Phelps-Bollobás Theorem

We show that under certain condition $\|\overline{\text{span}(H)}\|$ and $\overline{\text{co}(H)}w^*$ are weakly Lindelöf. We answer a question by Talagrand. Finally we apply the basic theorem to certain classes of Banach spaces including weakly compactly generated ones and the duals of Asplund spaces. We obtain: (A) if T is weakly compact, then S can also be taken to be weakly compact; (B) if X is Asplund (for instance, $X = c_0$), the pair $(X, C_0(L))$ has the Bishop-Phelps-Bollobás property for all L; (C) if L is scattered, the pair $(X, C_0(L))$ has the Bishop-Phelps-Bollobás property for all Banach spaces X.

Section (1.1): The Lindelöf Property:

The starting point of the present investigation is a theorem by [26], namely that a Banach space X is an Asplund space if and only if its dual X^* is Lindelöf with respect to the topology of uniform convergence on bounded countable subsets of X, the γ-topology. We show that this result is a special case of a much more general theorem on function spaces and that it has interesting consequences including a solution to a question by Talagrand.

The basic theorem and its important corollary are stated and proved. A new characterization of Radon-Nikodým compact spaces by the Lindelöf property relative to the γ-topology is derived from the basic theorem. It will be shown that Meyer's characterization of compact scattered spaces [24] by the Lindelöf property with respect the G_δ-topology is also a consequence of the basic theorem.

We use the Lindelöf property relative to the γ-topology to study the weakly Lindelöf property of sets in dual Banach spaces. We show, that the weak *-closed convex hull of a weak *-compact subset which is weakly Lindelöf in a dual Banach space is again weakly Lindelöf. This solves a problem of Talagrand in [33].

The theme is further expanded where it is proved, in particular, that the norm-closed linear span of a weak *-compact subset in a dual Banach space that is weakly Lindelöf is a WLD Banach space, as defined. It should be noted here that each WLD Banach space is weakly Lindelöf and more. We approach depends on the existence of "projectional generators" shown. The results on projectional generators also give a unified approach to the existence of projectional resolutions of the identity for both weakly compactly generated Banach spaces and duals of Asplund spaces.

We present several examples that illustrate the results .

For the notation and terminology see Engelking and Kelley, [11] and [21]. Given a topological space Z we let $C(Z)$ (resp. $C_b(Z)$) denote the space of real continuous (resp. real continuous uniformly bounded) functions defined on Z. For a Banach space X, B_X denotes its closed unit ball and X^* denotes its dual space. When F is a subset of X^*, we write $\sigma(X, F)$ to denote the locally convex topology (maybe non-Hausdorff) on X of pointwise convergence on $F; \sigma(X, X^*)$ is the weak topology of X and $\sigma(X^*, X)$ is the weak topology of X^*. We consider $C_b(Z)$ as a Banach space endowed with the supremum norm.

We first gather definitions of the terms and notation necessary for stating the main theorem, Theorem (1.1.2). Recall that a topological space is said to be Lindelöf if each open cover of the space admits a countable subcover. The following definition is due to Jayne and Rogers [20].

Definition (1.1.1)[1]: Let (Z, τ) be a topological space and ϱ a metric on Z. We say that (Z, τ) is fragmented by ϱ (or ϱ-fragmented) if for each non-empty subset C of Z and for each $\varepsilon > 0$ there exists a non-empty τ-open subset U of Z such that $U \cap C \neq \emptyset$ and $\varrho - \operatorname{diam}(U \cap C) \leq \varepsilon$

It is easily checked that for (Z, τ) to be ϱ-fragmented, it is sufficient that each τ-closed non-empty subset of X has non-empty relatively τ-open subsets of arbitrarily small ϱ-diameter.

Fort (M, ϱ) be a metric space and let D be an arbitrary set. We shall write $\tau_p(D)$ to denote the product topology of the space M^D. Assume henceforth that ϱ is bounded, which can always be done without altering the uniformity of M. For any set $S \subset D$ we define the pseudo-metric d_S on M^D by the formula

$$d_S(x, y) = \sup\{\varrho(x(t), y(t)) : t \in S\} \quad \text{for } x, y \in M^D. \tag{1}$$

The metric d_D will be simply denoted by d; the topology associated to d in M^D is the topology of uniform convergence on I. Let $\gamma(I)$ denote the uniform topology on M^D generated by the family of pseudo-metrics $\{d_A : A \subset D, A \text{ countable}\}$, i.e. the topology of uniform convergence on the family of countable subsets of D.

Let $2^{\mathbb{N}}$ be the space of all sequences of 0's and $1'$s and let $2^{(\mathbb{N})}$ be the set of all finite sequences of 0's and 1's. For a given $t \in 2^{(\mathbb{N})}$, let $|t|$ denote the length of t; for $\sigma \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, we write $\sigma \mid n = (\sigma(1), \ldots, \sigma(n)) \in 2^{(\mathbb{N})}$.

Theorem (1.1.2)[1]: Let (M, ϱ) and D be as above, and let K be a compact subset of (M^D, τ_p). Then the following conditions are equivalent:

(a) The space (K, τ_p) is fragmented by d.

(b) For each countable subset A of D, (K, d_A) is separable.

(c) The space $(K, \gamma(D))$ is Lindelöf.

Proof. (a) ⇒ (b). By Lemma 2.1 of [25], $(K|_A, \tau_p(A))$ is fragmented by d_A. Since M^A is metrizable, $(K|_A, \tau_p(A))$ is compact metrizable; hence it has a countable base. If (K, d_A) is not separable, then there is an uncountable subset Q of $K|_A$ and $\varepsilon > 0$ such that $d_A(p, q) > \varepsilon$ whenever $p, q \in Q$ and $p \neq q$. We may assume that no point of Q is τ_p-isolated in Q since $(K|_A, \tau_p(A))$ has a countable base. Since $(K|_A, \tau_p(A))$ is fragmented by d_A, there is a $\tau_p(A)$-open subset U of $K|_A$ such that $U \cap Q \neq \emptyset$ and $d_A - \operatorname{diam}(U \cap Q) < \varepsilon$. Hence $U \cap Q$ is a singleton, contradicting the fact that no point of Q is $\tau_p(A)$-isolated in Q. Hence (K, d_A) is separable.

(b) ⇒ (a). Suppose that (K, τ_p) is not fragmented by d. Then, for some non-empty τ_p-closed subset C of K and $\varepsilon > 0$, each non-empty τ_p-open subset of C has d-diameter greater that ε. By induction on $n = |s|, s \in 2^{(\mathbb{N})}$, we construct a family $\{U_s : s \in 2^{(\mathbb{N})}\}$ of non-empty relatively τ_p-open subsets of C and a family $\{t_s : s \in 2^{(\mathbb{N})}\}$ of points of D, satisfying the following conditions:

(α) $U_\emptyset = C$.

(β) $\bar{U}_{s0}^{\tau_p} \cup \bar{U}_{s1}^{\tau_p} \subset U_s$ for each s.

(γ) $\varrho(x(t_s), y(t_s)) > \varepsilon$ for each $x \in \bar{U}_{s0}^{\tau_p}$ and $y \in \bar{U}_{s1}^{\tau_p}$.

(α) starts the induction from $n = 0$. Next, for some $n > 0$, assume that $\{U_s : |s| < n\}$ and $\{t_s : |s| < n - 1\}$ have been constructed. Fix an $s \in 2^{(\mathbb{N})}$ with $|s| = n - 1$. By hypothesis, there are $x, y \in U_s$ with $d(x,y) > \varepsilon$.Hence for some $t_s \in D, \varrho(x(t_s), y(t_s)) > \varepsilon$. By the τ_p-continuity of the map

$$(x', y') \mapsto \varrho(x'(t_s), y'(t_s))$$

there are relatively τ_p-open neighborhoods U_{s0} and U_{s1} of x and y, respectively, so that (β) and (γ) are satisfied. This completes the construction. Note that (γ) implies that $\bar{U}_{s0}^{\tau_p} \cap \bar{U}_{s1}^{\tau_p} = \emptyset$ for each $s \in 2^{(\mathbb{N})}$.

For each $\sigma \in 2^{\mathbb{N}}$, choose $x_\sigma \in \bigcap_{n=1}^{\infty} \bar{U}_{\sigma|n}^{\tau_p}$. If $\sigma, \sigma' \in 2^{\mathbb{N}}$ are two different sequences, then for some $n \in \{0\} \cup \mathbb{N}, \sigma|n = \sigma'|n$ and $\sigma|(n+1) \neq \sigma'|(n+1)$. Then by (γ) we have $\varrho\left(x_\sigma(t_{\sigma|n}), x_{\sigma'}(t_{\sigma|n})\right) > \varepsilon$. Letting $A = \{t_s : s \in 2^{(\mathbb{N})}\}$ we have $d_A(x_\sigma, x_{\sigma'}) > \varepsilon$. Since $2^{\mathbb{N}}$ is uncountable, (K, d_A) is not separable, and therefore (b) does not hold.

(c) $\Rightarrow$ (b). This is clear because the topology associated to d_A is weaker than $\gamma(D)$ whenever A is a countable subset of D.

(a)&(b) $\Rightarrow$ (c). Let $\mathcal{U} = \{U_j : j \in J\}$ be a $\gamma(D)$-open cover of K and let $\mathcal{C} = \{A : A \subset D$ and A is countable $\}$. Without loss of generality we may assume that each U_j is of the form

$$U_j = U(x_j, A_j, \varepsilon_j) := \left\{y \in K : d_{A_j}(x_j, y) < \varepsilon_j\right\},$$

where $x_j \in K, A_j \in \mathcal{C}$ and $\varepsilon_j > 0$. For each $A \in \mathcal{C}$, define

$$\mathcal{U}(A) = \left\{U_j : j \in J, A_j \subset A\right\} \text{ and } U(A) = \bigcup \left\{U_j : U_j \in \mathcal{U}(A)\right\}$$

Then we have

$$\mathcal{U} = \bigcup \{\mathcal{U}(A) : A \in \mathcal{C}\} \text{ and } K = \bigcup \{U(A) : A \in \mathcal{C}\} \tag{2}$$

Also if $A \subset A'$ then $U(A) \subset U(A')$.

We claim that $K = U(A)$ for some $A \in \mathcal{C}$. Suppose for a moment this is true. Then since each member of $\mathcal{U}(A)$ is d_A-open and since (K, d_A) is separable by (b), there is a countable subfamily of $\mathcal{U}(A)$ (hence of $\mathcal{U}$) that covers K, which completes the proof.

The proof of the claim is by contradiction. So assume that $U(A) \neq K$ for each $A \in \mathcal{C}$. For each $A \in \mathcal{C}$, let

$$C(A) = K \setminus U(A) \text{ and } C = \bigcap \left\{\overline{C(A)}^{\tau_p} : A \in \mathcal{C}\right\}$$

We note that $C(A) \supset C(A')$ whenever $A \subset A'$. By compactness of (K, τ_p), $C \neq \emptyset$, and now (a) tells us that (C, τ_p) is fragmented by d. So by Lemma 1.1 of [25], there is a point $y \in C$ where the identity map $(C, \tau_p) \to (C, d)$ is continuous. The second equality in (2) ensures us that $y \in U(B)$ for some $B \in \mathcal{C}$. Since $U(B)$ is d_B-open, for some $\varepsilon > 0, y \in U(y, B, \varepsilon) \subset U(B)$. Then for each $x \in C(B) = K \setminus U(B), x \notin U(y, B, \varepsilon)$ and so for some $t \in B$, $\varrho(x(t), y(t)) \geq 2\varepsilon/3$. For each $t \in B$, let

$$v_t = \left\{x \in C(B) : \varrho(x(t), y(t)) \geq \frac{2\varepsilon}{3}\right\}. \tag{3}$$

Then from the above, $C(B) = \bigcup\{D_t : t \in B\}$.

Let V be a τ_p-open neighborhood of y in K such that $-\text{diam}(\bar{V}^{\tau_p} \cap C) \leq \varepsilon/2$. Then we claim that, for some $t \in B, D_t \cap V \cap C(A) \neq \emptyset$ for each $A \in \mathcal{C}$. For, otherwise, for each $t \in B$

there is an $A_t \in \mathcal{C}$ such that $D_t \cap V \cap C(A_t) = \emptyset$. Since B is countable, the set $E := B \cup \bigcup\{A_t : t \in B\}$ is also countable, and $D_t \cap V \cap C(E) = \emptyset$ for all $t \in B$. Hence

$$\emptyset = \left(\bigcup \{D_t : t \in B\}\right) \cap V \cap C(E^\top) = C(B) \cap V \cap C(E^\top) = V \cap C(E^\top),$$

contradicting $y \in C \subset \overline{C(E)}\tau_p$.

Now fix a $t \in B$ so that $D_t \cap V \cap C(A) \neq \emptyset$ for each $A \in \mathcal{C}$, and let

$$z \in \bigcap \left\{\overline{D_t \cap V \cap C(A)}^{\tau_p} : A \in \mathcal{C}\right\}.$$

Then $z \in \bar{V}^{\tau_p} \cap C$, and so

$$d(z, y) \leq \frac{\varepsilon}{2}. \tag{4}$$

On the other hand, since $z \in \overline{D_t}^{\tau_p}$, it follows by (3) that $\varrho(z(t), y(t)) \geq 2\varepsilon/3$, which contradicts (4). This completes the proof of both the claim and the theorem.

It is well known that the product of two Lindelöf spaces is not in general Lindelöf again: indeed, let $Z = \mathbb{R}$ and endow it with the topology for which a basis is given by all the intervals $[x, r)$, where $x, r \in \mathbb{R}, x < r$ and r is a rational number; then Z is a separable first-countable space that is Lindelöf and is not second-countable; moreover $Z \times Z$ is not normal and therefore not Lindelöf (see [11, pp. 248-249]).

Fortunately the Lindelöf property for the spaces $(K, \gamma(D))$ in Theorem (1.1.2) is preserved under the countable power.

Corollary (1.1.3)[1]: Let K, M, D be as in Theorem (1.1.2). If K satisfies one of the three conditions of the theorem, then $(K, \gamma(D))^{\mathbb{N}}$ is Lindelöf. In particular, $(K, \gamma(I)))^n$ is Lindelöf for each $n \in \mathbb{N}$.

Proof. We may assume that the metric ϱ of the space M is bounded by 1. Let $\varphi: (M^D)^{\mathbb{N}} \to (M^{\mathbb{N}})^D$ be the map defined by $\varphi(\xi)(t)(j) = \xi(j)(t)$ for all $\xi \in (M^D)^{\mathbb{N}}, t \in D, j \in \mathbb{N}$. Clearly φ is a homeomorphism when the product topology is used throughout. Now the space $M^{\mathbb{N}}$ is metrizable, and we use the metric $\varrho_\infty(mr, rl') := \sum_{j \in \mathbb{N}} 2^{-j} \varrho(ml(j), mn'(j))$ for $m, mn' \in M^{\mathbb{N}}$. Let d_∞ be the metric on $(M^{\mathbb{N}})^D$ given by

$$d_\infty(x, x') := \text{s } \{\varrho_\infty(x(t), x'(t)) : t \in D\} \text{ for } x, x' \in (M^{\mathbb{N}})^D.$$

We now show that if K is fragmented by d then $\varphi(K^{\mathbb{N}})$ is fragmented by d_∞. Let $\varepsilon > 0$, let C be a non-empty subset of $K^{\mathbb{N}}$ and let $\pi_i: K^{\mathbb{N}} \to K$ be the i-th projection. Then by induction we can construct a decreasing sequence $V_1 \supset V_2 \supset \cdots$ of non-empty relatively open subsets of C such that d-diam $\left(\pi_j(V_j)\right) < \varepsilon/2$ for each $j \in \mathbb{N}$. Choose $k \in \mathbb{N}$ so that $2^{-k} < \varepsilon/2$, and let $\xi, \xi' \in V_k$. Then for each $t \in D$,

$$\begin{aligned}\varrho_\infty(\varphi(\xi)(t), \varphi(\xi')(t)) \quad &\leq \sum_{j \leq k} 2^{-j} \varrho(\xi(j)(t), \xi'(j)(t)) + \sum_{j \geq k+1} 2^{-j} \\ &< \sum_{j \leq k} 2^{-j} d\left(\pi_j(\xi), \pi_j(\xi')\right) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus $\varphi(V_k)$ is a non-empty relatively open subset of $\varphi(C)$ with d_∞-diameter not greater than ε.

Hence by Theorem (1.1.2), $\varphi(K^{\mathbb{N}})$ is $\gamma(D)$-Lindelöf. So we finish the proof by showing that φmaps$(M^D, \gamma(D))^{\mathbb{N}}$ homeomorphically onto $((M^{\mathbb{N}})^D, \gamma(D))$. Let τ_1, τ_2 be the topologies of these two spaces respectively. Then a net ξ_α in $(M^D)^{\mathbb{N}}\tau_1$-converges to $\xi \in (M^D)^{\mathbb{N}}$ if and only if: (i) for each $j \in \mathbb{N}$ and for each countable set $A \subset D, \varrho(\xi_\alpha(j)(t), \xi(j)(t)) \to 0$ uniformly in $t \in A$. On the other hand, the net $\varphi(\xi_\alpha)\tau_2$-converges to $\varphi(\xi)$ if and only if: (ii) for each countable $A \subset D$

$$\varrho_\infty(\varphi(\xi_\alpha)(t), \varphi(\xi)(t)) = \sum_{j\in\mathbb{N}} 2^{-j}\varrho(\xi_\alpha(j)(t), \xi(j)(t)) \to 0$$

uniformly in $t \in A$. The equivalence of statements (i) and (ii) can be seen by an easy calculation similar to the one given above. Hence φ is a $\tau_1 - \tau_2$ homeomorphism.

We obtain the following theorem, whose first part was mentioned. It has been stated in [26] as Theorems B and C. The original proof is quite different and depends on the technique of projections in Banach spaces.

Theorem (1.1.4)[1]: ([26]). A Banach space X is an Asplund space if and only if $(X^*, \gamma(B_X))$ is Lindelöf. If this is the case, then $(X^*, \gamma(B_X))^n$ is Lindelöf for each $n \in \mathbb{N}$.

Proof. Note that $(X^*, \gamma(B_X))^n$ is Lindelöf if and only if $(B_{X^*}, \gamma(B_X))^n$ is Lindelöf, and X is an Asplund space if and only if (B_{X^*}, weak*) is frag- mented by the norm. Therefore the theorem follows by regarding (B_{X^*}, weak*) as a compact subspace of $([-1,1]^B, \tau_p)$.

Let K be a compact Hausdorff space and let D be a uniformly bounded subset of $C(K)$ and $A \subset D$. Then we define the pseudo-metric on K by

$$d_A(x, x') = \sup\{|f(x) - f(x')|: f \in A\} \quad \text{for } x, x' \in K$$

We again write $\gamma(D)$ to denote the uniform topology on K generated by the family of pseudo-metrics $\{d_A: A \subset D, A \text{ countable}\}$. Observe that when D separates the points of K, then K embeds in $[-m, m]^D$ for some $m > 0$. Hence the topology $\gamma(D)$ now defined is the one already given through the embedding $K \subset [-m, m]^D$, and $\gamma(D)$ is stronger than the original topology of K. In particular the equivalences we have seen in Theorem (1.1.2) and Corollary (1.1.3) remain true.

Theorem (1.1.5)[1]: Let K be a compact Hausdorff space and let D be a uniformly bounded subset of $C(K)$. Then the following statements are equivalent:

(i) The space (K, d_A) is separable for each countable $A \subset D$.

(ii) The space $(K, \gamma(D))$ is Lindelöf.

(iii) The space $(K, \gamma(D))^{\mathbb{N}}$ is Lindelöf.

Proof. From the remark above, the theorem is clear in case D separates the points of K. The general case can be reduced to this as follows. Let $m = \sup\{\| f \|: f \in D\}$ and let $\varphi: K \to [-m, m]^D$ be the map given by $\varphi(x)(f) = f(x)$ for all $x \in K$ and $f \in D$. Then $K' := \varphi(K)$ is a compact Hausdorff space. For each $f \in D$, let $\hat{f} \in C(K')$ be the map given by $\hat{f}(\varphi(x)) = f(x)$, and, for each $A \subset D$, let $\hat{A} = \{\hat{f}: f \in A\}$. Then clearly $f \mapsto \hat{f}$ is a one-to-one map of D onto $\widehat{D}$ and $d_A(x, y) = d_{\hat{A}}(\varphi(x), \varphi(y))$ for all $x, y \in K$. It follows that (K, d_A) is separable if, and only if, $(K', d_{\hat{A}})$ is separable. The last equality also implies that, for each $\in K, \{y \in K : d_A(x, y) < \varepsilon\} = \varphi^{-1}(\{z \in K': d_{\hat{A}}(\varphi(x), z) < \varepsilon\})$. Hence a subset U of K is $\gamma(D)$-open if and only if $U = \varphi^{-1}(U')$ for some $\gamma(\widehat{D})$-open subset U' of K'. From this it is straightforward to check that $(K, \gamma(D))(\text{resp.}\,(K, \gamma(D))^{\mathbb{N}})$ is Lindelöf if, and only if, $(K', \gamma(\widehat{D}))$ $\left(\text{resp.}\,(K', \gamma(\widehat{D}))^{\mathbb{N}}\right)$ is Lindelöf. Since $\widehat{D}$ separates the points of K', the conclusion of the theorem is true for $\widehat{D}$ and K'. Hence the theorem is proved in general.

A compact Hausdorff space is said to be Radon-Nikodým compact (or RN-compact) if it is homeomorphic to a weak*-compact subset of the dual of an Asplund space, i.e. a dual Banach space with the RNP. It is shown in [25] that a compact Hausdorff space is RN-compact if, and only if, it is fragmented by a lower semicontinuous metric on the space. When (M,ϱ) is a metric space (with ϱ bounded) the metric d in Theorem (1.1.2) is clearly τ_p lower semicontinuous. Therefore, Theorem (1.1.2) provides the following characterization of RN-compact spaces.

Proposition (1.1.6)[1]: A compact Hausdorff space is RN-compact if, and only if, it is homeomorphic to a pointwise compact subset K of $[-1,1]^D$ for some set D such that $(K,\gamma(D))$ is Lindelöf.

Proof. By Theorem 3.6 of [25] a compact space is RN-compact if, and only if, K is homeomorphic to a pointwise compact subset K of $[-1,1]^D$ for some set D such that (K,d_A) is separable for each countable subset A of D. An application of Theorem (1.1.2) finishes the proof of the proposition.

In terms of spaces of continuous functions the proposition above can be restated as follows.

Corollary (1.1.7)[1]: A compact Hausdorff space K is RN-compact if, and only if, there is a bounded subset D of $C(K)$ separating points of K such that $(K,\gamma(D))$ is Lindelöf. If this is the case, then $(K,\gamma(D))^{\mathbb{N}}$ is Lindelöf.

Proof. Assume K is RN-compact. By Proposition (1.1.6), we may assume that K is a subspace of $\left([-1,1]^D,\tau_p\right)$ for a certain set D with $(K,\gamma(D))$ Lindelöf; for every $d\in D$ let $\pi_d:[-1,1]^D\to[-1,1]$ be the projection defined by $\pi_d(x)=x(d), x\in[-1,1]^D$. If we let $\widehat{D}=\{\pi_d: d\in D\}$, then $\widehat{D}$ is a uniformly bounded subset of $C(K)$ separating the points of K and such that $(K,\gamma(\widehat{D}))$ is Lindelöf. The last part follows from Theorem (1.1.5). A similar argument proves the converse.

For weakly compact subsets of $C(K)$, we have the following.

Corollary (1.1.8)[1]: Let K be a compact Hausdorff space and let $H\subset C(K)$ be a weakly compact (i.e. bounded and τ_p-compact) set. Then $(K,\gamma(H))^{\mathbb{N}}$ is Lindelöf.

Proof. For a countable set $A\subset H, \bar{A}^{\tau_p}\subset C(K)$ is $\tau_p(K)$-metrizable and thus the space $(C(\bar{A}^{\tau_p}),d_{\bar{A}^{\tau_p}})$ is separable. Hence, $\left(K|_{\bar{A}^{\tau_p}}, d_{\bar{A}^{\tau_p}}\right)$ is separable and so is (K,d_A). In view of Theorem (1.1.5), the proof is complete.

We need the following easy lemma that appears in [6].

Lemma (1.1.9)[1]: Let Z be a Lindelöf space, and let $H\subset C(Z)$ be equicontinuous. Then $\left(H,\tau_p(Z)\right)$ is metrizable.

Proof. Let d_H be the pseudo-metric on Z given by

$$d_H(z,z')=\min\left\{1,\sup_{h\in H}|h(z)-h(z')|\right\}$$

Since H is equicontinuous, the d_H-topology is weaker than the given one on Z. So (Z,d_H) is Lindelöf and hence separable. Let $I)$ be a countable d_H- dense subset of Z. Then since H is d_H-equicontinuous, on H the topologies of pointwise convergence on D and on Z coincide. Therefore $\left(H,\tau_p(Z)\right)$ is metrizable.

Given a subset D of $\mathbb{R}^K$, let

$$F(D)=\bigcup\ \{\bar{A}^{\tau_p}: A\subset D, A \text{ countable }\}.$$

Note that if B is a countable subset of $F'(D)$ then there is a countable subset A of D such that $\bar{B}^{\tau_p} \subset \bar{A}^{\tau_p} \subset H'(D)$. In particular, $F'(F'(D)) = F(D)$

Recall that a topological space Z is said to be countably tight (resp. to be a Fréchet-Urysohn space) if for each set $S \subset Z$ and each point $x \in \bar{S}$ there is a countable set $A \subset S$ (resp. sequence $(x_n)_n$ in S) such that $x \in \bar{A}$ (resp. $(x_n)_n$ converges to x); see [3, pp. 5 and 7]. In applying the results, the following theorem of Arkhangel'skil ([3, Theorem II.1.1]) is very useful.

Theorem (1.1.10)[1]: Let T be a topological space such that T^n is Lindelöf for each $n \in \mathbb{N}$. Then $\left(C(T), \tau_p(T)\right)$ is countably tight.

Corollary (1.1.11)[1]: Let K be a compact space and let D be a bounded subset of $C(K)$ such that $(K, \gamma(D))$ is Lindelöf. Then the following properties hold:

(a) For any countable set $A \subset D, \bar{A}^{\tau_p}$ (closure taken in $\mathbb{R}^K$) is $\gamma(D)$ equicontinuous and τ_p-metrizable.

(b) $F'(D) = C(K, \gamma(D)) \cap \bar{D}^{\tau_p}$, where the closure is taken in $\mathbb{R}^K$.

(c) $\left(F'(I), \tau_p\right)$ is a Fréchet-Urysohn space.

Proof. (a) easily follows from the previous lemma: if $A \subset D$ is countable then A is $\gamma(D)$-equicontinuous; its τ_p-closure $\bar{A}^{\tau_p}$ in $\mathbb{R}^K$ is again $\gamma(D)$ equicontinuous and therefore τ_p-metrizable by Lemma (1.1.9). This proves (a).

For (b), we first note that (a) implies $F(D) \subset C(K, \gamma(D)) \cap \bar{D}^{\tau_p}$. Next we note that $(K, \gamma(D))^n$ is Lindelöf for each $n \in \mathbb{N}$ by Theorem (1.1.5). This fact implies that $\left(C(K, \gamma(D)), \tau_p\right)$ is countably tight according to Theorem (1.1.10). Therefore if $f \in C(K, \gamma(I)) \cap \bar{D}\tau_p$ then there is a countable subset A of D) such that $f \in \bar{A}^{\tau_p}$. Hence $f \in F'(D)$, which proves (b).

The proof of (c) is similar: Suppose that $S \subset F'(D)$ and $f \in \bar{S}^{\tau_p} \cap H'(D)$. Then by the countable tightness, there is a countable subset B of S such that $f \in \bar{B}^{\tau_p}$. Then as noted above, there is a countable subset A of D such that $\bar{B}^{\tau_p} \subset \bar{A}^{\tau_p}$. In particular $\bar{B}\tau_p$ is τ_p-metrizable by (a). Therefore there is a sequence in B (hence in S) that τ_p-converges to f. This proves (c).
Recall that a topological space T is said to be scattered if each non-empty subset of T has an isolated point, or equivalently T is fragmented by the (necessarily lower semicontinuous) trivial metric ϱ, where $\varrho(t, s) = 0$ for $t = s$ and $\varrho(t, s) = 1$ for $t \neq s$. It can be shown (cf. [30, Theorem 8.5.4]) that a compact Hausdorff space K is scattered if and only if there is no continuous map from K onto [0,1]. We remark that in the corollary above if $B_{C(K)} \subset F'(D)$ then K is scattered. For then, $\left(B_{C(K)}, \tau_p\right)$ is a FréchetUrysohn space; on the other hand, $\left(B_C[0,1], \tau_p\right)$ is not Fréchet-Urysohn (see [3, Lemma II.3.5]), and consequently K cannot be continuously mapped onto [0,1].

Given a topological space $(Z, \mathcal{T})$, the G_δ-topology associated to $\mathcal{T}$ is the topology on Z whose basis is the family of G_δ-sets, $\{\bigcap_n U_n : U_n \in \mathcal{T}\}$; when no confusion is likely we simply write Z for the topological space and then refer to its $G\delta$-topology.

Lemma (1.1.12)[1]: Let K be a compact Hausdorff space. Then the G δ-topology for K is identical with $\gamma\left(B_{C(K)}\right)$ on K.

Proof. Clearly the G_δ-topology is stronger than $\gamma\left(B_{C(K)}\right)$. Let $a \in K$, and let G be a G_δ-set containing a. Then $G = \bigcap_{n=1}^{\infty} U_n$ where each U_n is open in K. For each n, let f_n be a continuous function $f_n: K \to [0,1]$ such that $f_n(a) = 0$, and $f_n|_{K\setminus U_n} \equiv 1$. Write $A = \{f_n: n \in \mathbb{N}\}$. Then A is a countable subset of $B_{C(K)}$, and $x \in G$ whenever $d_A(a,x) < 1$, i.e.

$$a \in \{x \in K: d_A(a,x) < 1\} \subset G.$$

This shows that $\gamma\left(B_{C(K)}\right)$ is stronger than the G_δ-topology and we are done.

Corollary (1.1.13)[1]: (Meyer, [24]). For a compact Hausdorff space K, let τ_δ denote its G_δ-topology. Then the following statements are equivalent:

(a) K is scattered.

(b) (K, τ_δ) is Lindelöf.

(c) $\left(B_{C_{\mathrm{b}}}(K, \tau_\delta), \tau_p\right)$ is a Fréchet-Urysohn space.

Proof. For (a) ⇔ (b), regarding K as a subset of $\left([-1,1]^B C(K), \tau_p\right)$, we apply Theorem (1.1.2). In this case the metric d is twice the trivial metric and the topology $\gamma\left(B_{C(K)}\right)$ is the G_δ-topology for K by the lemma above. (a) ⇔ (b) now follows. Next assume (b), and we apply Corollary (1.1.11) to our K and $D := B_{C(K)}$. The hypotheses are satisfied by (b). Since the τ_p-closure of D is $[-1,1]^K$, (b) of Corollary (1.1.11) says that $F(D) = B_{C_{\mathrm{b}}(K,\gamma(D))} = B_{C_{\mathrm{b}}(K,\tau_\delta)}$ and (c) of the same corollary says that $\left(B_{C_{\mathrm{b}}}(K, \tau_\delta), \tau_p\right)$ is a Fréchet-Urysohn space. This is (c). If (c) holds, then $\left(B_{C(K)}, \tau_p\right)$ is also a Fréchet-Urysohn space. But as remarked above, this implies (a).

We should comment here that topological spaces for which $G\delta$-sets are again open are called P-spaces. It is a very easy exercise to prove that if Z is a Lindelöf P-space then Z^n is Lindelöf for $n \in \mathbb{N}$ and so $\left(C(Z), \tau_p\right)$ has countable tightness; it also follows from Lemma (1.1.9) that for such a Z the separable subsets of $\left(C(Z), \tau_p\right)$ are metrizable, and hence $\left(C(Z), \tau_p\right)$ is Fréchet-Urysohn; see also [3]. Our argument also shows that, for K compact and scattered, the space of all continuous functions on K endowed with its G_δ-topology is $B_1(K)$, the space of τ_p-limits of sequences in $C(K)$, and that all classes of Baire functions on K are the same [23].

Let D be a dense subset of a compact Hausdorff space K and let H be a bounded $\tau_p(D)$-compact subset of $C(K)$. We investigate the $\tau_p(K)$-Lindelöf property of H by means of the $\gamma(D)$-topology of the earlier sections. As application we prove the results mentioned.

The following simple proposition enables us to extract information on $\left(H, \tau_p(K)\right)$ from that on $(H, \gamma(D))$.

Proposition (1.1.14)[1]: Let K be a compact Hausdorff space, D a dense subset of K and H a subset $C(K)$. If H is $\tau_p(K)$-Lindelöf, then $\gamma(D)$ is stronger than $\tau_p(K)$ on H.

Proof. Let $f \in H, \varepsilon > 0, x \in K$, and

$$U = \{g \in H: |g(x) - f(x)| < \varepsilon\}.$$

Then U is a $\tau_p(K)$-open neighborhood of f in H, and it is sufficient to show that U is a $\gamma(I)$-neighborhood of f in H. For each $d \in D$, let

$$D_d = \{g \in H: |g(d) - f(d)| \leq \varepsilon/2\}$$

If $g \in \bigcap\{D_d: d \in D\}$, then $|g(x) - f(x)| \leq \varepsilon/2$ since $x \in \bar{D}$, and therefore $g \in U$. It follows that $\bigcap\{)_d: d \in D\} \subset U$. Since each D_d is $\tau_p(K)$-closed and H is $\tau_p(K)$-Lindelöf, there is a countable subset A of D such that already $\bigcap\{D_d: d \in A\} \subset U$, i.e. $\{g \in H: \sup_{d\in A}|g(d) - f(d)| \le \varepsilon/2\} \subset U$. Hence U is a $\gamma(D)$-neighborhood of f in H and the proof is finished.

Corollary (1.1.15)[1]: Let K be a compact Hausdorff space, D a dense subset of K and H a bounded $\tau_p(D)$-compact subset of $C(K)$. If $\left(H, \tau_p(K)\right)$ is Lindelöf, then $\left(H, \tau_p(K)\right)^{\mathbb{N}}$ is Lindelöf.

Proof. If H is $\tau_p(D)$-compact and $\tau_p(K)$-Lindelöf, then by [5, Theorem B], H is fragmented by the supremum norm of $C(K)$, i.e. as a compact subset H of $[-m, rn]^D$ for a suitable m, H is fragmented by d in the notation of Theorem (1.1.2). According to Theorem (1.1.2) and Corollary (1.1.3), $(H, \gamma(L))^{\mathbb{N}}$ is Lindelöf. By Proposition (1.1.14), $\gamma(D)$ is stronger than $\tau_p(K)$ on H and therefore $\left(H, \tau_p(K)\right)^{\mathbb{N}}$ is Lindelöf because it is a continuous image of the Lindelöf space $(H, \gamma(D))^{\mathbb{N}}$

In [3, Problem IV.11.11] Arkhangel'skiY asks the following question. Let K be a compact Hausdorff space. If there exists a τ_p-Lindelöf subset H of $C(K)$ that separates the points of K, is K countably tight? The next corollary is an answer to this question under a rather strong restriction on H.

Corollary (1.1.16)[1]: Let K be a compact Hausdorff space, and H a $\tau_p(K)$ Lindelöf bounded subset of $C(K)$ separating the points of K. If H is $\tau_p(D)$ compact for some dense subset $D \subset K$, then K is countably tight.

Proof. An application of Corollary (1.1.15) allows us to conclude that $\left(H, \tau_p(K)\right)^n$ is Lindelöf for $n \in \mathbb{N}$. Hence the space $\left(C\left(H, \tau_p(K)\right), \tau_p(H)\right)$ is countably tight by Theorem (1.1.10). The space K is homeomorphic to a subset of $C\left(H, \tau_p(K)\right)$ because H separates the points of K, and so the proof is done.

If X is a Banach space, then $B_{X^{**}}$ is always assumed to have the weak* topology $\left(= \sigma(X^{**}, X^*)\right)$ unless other topology is specified. Also X and B_X are considered as subspace/subset of X^{**} and $B_{X^{**}}$, respectively, by means of the canonical embedding. Thus (X^*, weak) is a subspace of $\left(C_{X^{**}}), \tau_p(B_X)\right)$ and (X^*, weak) is a subspace of $\left(C(B_{X^{**}}), \tau_p(B_{X^{**}})\right)$. For a subset S of X^*, the weak and weak * closures of S are respectively denoted by $\bar{S}^{\mathrm{W}}$ and $\bar{S}^{\mathrm{w}}$. A particular case of Corollary (1.1.15) is the following:

Corollary (1.1.17)[1]: Let X be a Banach space and let H be a weak*-compact subset of X^* which is weakly Lindelöf. Then $(H$, weak $)^{\mathbb{N}}$ is Lindelöf.

The next result gives an affirmative answer to a question posed by Talagrand that appears in [33] as Problème 4.5.

Theorem (1.1.18)[1]: Let X be a Banach space and let H be a weak*-compact subset of X^* which is weakly Lindelöf. Then:

(a) $\overline{\mathrm{co}(H)}\mathrm{w}^* = \overline{\mathrm{co}(H)}\ \|\|$.

(b) $\overline{\mathrm{co}(H)}\mathrm{w}^*$ is weakly Lindelöf.

Proof. If H is a weak -compact subset of X^* which is also weakly Lindelöf, then (H weak *) is fragmented by the dual norm by Corollary E in [5]. The equality in item (a) now follows from Theorem 2.3 in [25].

Let us prove (b). As noted in the proof of (a), (H, weak *) is fragmented by the norm. Therefore if we let $W = \overline{\mathrm{co}(H)}\mathrm{w}^*$, then W is weak *-compact and (W, weak *) is fragmented by the norm by [25, Theorem 2.5]. By embedding W into $[-rr,rr]^{B_X}$ for a suitable $rr>0$, we see that $(W,\gamma(B_X))$ is Lindelöf by Theorem (1.1.2). Therefore the proof is finished once we show that $\gamma(B_X)$ is stronger than the weak topology on W, or equivalently each member x^{**} of $B_{X^{**}}$ is continuous on $(W,\gamma(B_X))$. So fix an element x^{**} in B_{X*k}. By Corollary (1.1.17), $(H,\text{ weak })^{\mathbb{N}}$ is Lindelöf, and therefore, by Theorem $A,(C(H,\text{ weak}),\tau_p(H))$ is countably tight. Since $B_X|_H$ is $\tau_p(H)$-dense in $B_{X^{**}}|_H \subset C(H,\text{weak })$, there is a countable subset $A\subset B_X$ such that $x^{**}|_H$ is in the $\tau_p(H)$-closure of $A|_H$. Let G be the convex hull of H. Then by the linearity, $x^{**}|_W$ is in the $\tau_p(G)$-closure of $A|_W$. By (a), G is norm-dense in W and $B_{X_{**}}\big|_W$ is an equicontinuous family of functions on $(W,\|\|)$. Hence $\tau_p(W)$ and $\tau_p(G)$ coincide on $B_{X^{**}}|_W$, and so $x^{***}|_W$ is in the $\tau_p(W)$-closure of $A|_W$. Finally, $A|_W$ is an equicontinuous family on $(W,\gamma(B_X))$ and hence $x^{**}|_W$, being in the pointwise closure of $A|_W$, is $\gamma(B_X)$-continuous on W.

Corollary (1.1.19)[1]: Let X be a Banach space, H a weak*-compact subset of X^* and W its weak*-closed convex hull. The following statements are equivalent:

(a) $(H,$weak) is Lindelöf.

(b) $(H,\text{ weak})^{\mathbb{N}}$ is Lindelöf.

(c) $(W,$ weak) is Lindelöf.

(d) $(W,\text{ weak})^{\mathbb{N}}$ is Lindelöf.

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) both follow from Corollary (1.1.17). The implications (b) $\Rightarrow$ (a), (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious. And finally, the implication (a) $\Rightarrow$ (c) is Theorem (1.1.18).

If X is either a weakly compactly generated Banach space or the dual of an Asplund space, then X is generated by an RN-compact subset in the weak or the weak * topology. We shall deal in this section with the class of Banach spaces generated by RN-compact subsets with respect to a topology weaker than the weak topology. To be more concrete, our framework is the following: for a Banach space $(X,\|\|)$ we consider a norming subset $F\subset X^*$ (also called 1-norming subset) for X, that is, a Q-linear set H' satisfying

$$\| x \|= \mathrm{s}\ \{|\langle x,f\rangle|: f\in F'\cap B_{X^*}\}. \tag{5}$$

If a bounded set $H\subset X$ is $\sigma(X,F)$-compact and fragmented by the norm, then $(H,\sigma(X,H'))$ is an RN-compact set since the norm is $\sigma(X,F')$-lower semicontinuous, and we will study the space generated by it, that is, the space $Y=\overline{\operatorname{span}(H)}\ \|\|$. The Banach space Y thus obtained will be called a Banach space generated by an RN-compact subset. We exhibit several examples of such Banach spaces. In order to show the main properties of spaces generated this way we shall first see that these spaces admit projectional generators as defined below. See [12]. If A is a non-empty subset of a Banach space X, then $A^{\perp}$ denotes the subset $\{f\in X^*: f(x)=0 \text{ for all } x\in A\}$ of X^*.

Definition (1.1.20)[1]: Let X be a Banach space. A projectional generator on X is a countable-valued map $\varphi: F\to 2^X$ on a norming subset $F\subset X^*$ such that whenever $B\subset F$ is a $\mathbb{Q}$-linear set, we have

$$\varphi(B)^{\perp}\cap \overline{B\cap B_{X^*}}w^*=\{0\} \tag{6}$$

According to the method developed in [28], [26] and [12], the existence of a projectional generator leads to the existence of a projectional resolution of identity (PRI for short) in the sense that follows. Given a Banach space X, the density character of X (denoted by dens X) is defined to be the least cardinality of a dense subset of X. Let μ be the least ordinal such that $|\mu| = \operatorname{dens} X$, where $|\mu|$ denotes the cardinality of the ordinal μ. A PRI on X is a transfinite sequence $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ of linear projections in X satisfying the following conditions, where α and β are arbitrary ordinals in $[\omega_0, \mu]$:

(a) $\|P_\alpha\| = 1$.

(b) dens $P_\alpha(X) \le |\alpha|$

(c) $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min\{\alpha,\beta\}}$

(d) For each $x \in X$ and each limit ordinal $\alpha, P_\beta(x) \to P_\alpha(x)$ in the norm as $\beta \uparrow \alpha$.

The next proposition gathers the main properties of spaces with a projectional generator. In what follows, "LUR norm" stands for "locally uniformly rotund (or convex) norm".

Theorem (1.1.21)[1]: Let X be a Banach space with a projectional generator $\varphi: F \to 2^X$. Then the following statements hold:

(a) X admits a $PRI\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ such that $P_\alpha(X)$ has a projectional generator for each $\omega_0 \le \alpha < \mu$.

(b) X admits an equivalent LUR norm.

(c) There is a linear continuous one-to-one operator $T': X \to c_0(\Gamma)$ for some set Γ. (d) The Banach space X is $\gamma(X,F)$-Lindelöf, where $\gamma(X,F)$ is the topology on X of uniform convergence on bounded countable subsets of F '.

Proof. (a) With the projectional generator φ in X, a PRI $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ can be constructed, based on pairs (A_α, B_α) of $\mathbb{Q}$-linear subsets, $A_\alpha \subset X$ and $B_\alpha \subset F'$ with $\varphi(B_\alpha) \subset A_\alpha$ and B_α norming for A_α (see Proposition 6.1.7 and Remark 6.1.8 of [12]); so, we have $\overline{B_\alpha \cap B_{X^*}}$ w$^* \cap$ $A_\alpha^\perp = \{0\}$ and P_α is the projection from X onto $\bar{A}_\alpha^{\|}$ $\|$with kernel $B_\alpha^\perp$. The space $P_\alpha^*(X^*) = \bar{B}_\alpha^{w^*}$ is identified with the dual of $P_\alpha(X) = \bar{A}_\alpha^{\|\|}$and therefore $P_\alpha(X)$ also has a projectional generator defined on B_α by $\varphi_\alpha(f) = P_\alpha(\varphi(f)), f \in B_\alpha$. These observations complete the proof of (a).

(b) and (c). Here we use the induction argument encapsulated in [8, Theorem VII.1.8]. Let $\mathcal{P}$ be the class of Banach spaces that admit a projectional generator. Then (a) shows that the hypothesis on $\mathcal{P}$ in [8, Theorem VII.1.8] is satisfied. Therefore each member X of $\mathcal{P}$ admits an equivalent LUR norm. If, in the proof of [8, Theorem VII.1.8], one uses [12, Proposition 6.2.2] instead of Proposition VII.1.6 of [8], then one can also conclude that each member X of $\mathcal{P}$ has property (c).

(d) The proof of Theorem A in [26] gives us this result.

What remains is devoted to proving that a Banach space generated by an RN-compact subset has a projectional generator and therefore enjoys the properties listed in Theorem (1.1.21).

First we recall Simons' lemma [31].

Lemma (1.1.22)[1]: Let $(z_n)_n$ be a uniformly bounded sequence in $\ell^\infty(C)$ and let W be its convex hull. If B is a subset of C such that for every sequence $(\lambda_n)_n$ of positive numbers with $\sum_{n=1}^\infty \lambda_n = 1$ there is $b \in B$ such that

$$\sup\left\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in C\right\} = \sum_{n=1}^{\infty} \lambda_n z_n(b), \tag{7}$$

then

$$\sup_{b\in B}\left\{\limsup_{n\to\infty} z_n(b)\right\} \geq \inf\left\{\sup_C w : w \in W\right\}. \tag{8}$$

A subset of X^* is said to be total if its linear span is weak*-dense in X^*. Clearly a norming subset for X is a total subset of X^*.

Definition (1.1.23)[1]: Let X be a normed space, $C \subset X$ a set and F a total subset in X^*. A subset $B \subset C$ is said to be an F-boundary for C if for every f in F there is a $b \in B$ such that $f(b) = \sup\{f(x): x \in C\}$.

In what follows, when F' is a total norm-closed subspace of X^* we consider the norm associated to F given by

$$p_F(x) = \sup\{|\langle x, f\rangle| : f \in F' \cap B_{X^*}\},$$

for $x \in X$. Then the unit ball of $(X, p_F)^*$ is the set $\overline{F \cap B_{X^*}}$ w* and $(X, p_F)^*$ is the subspace $H = \bigcup_{n=1}^{\infty} n\overline{(F \cap B_{X^*})}$ of X^*. Clearly $F \subset H$.

Proposition (1.1.24)[1]: Let X be a normed space and let F be a total normclosed subspace of X^*. Let C be a bounded subset of X and $B \subset C$ an F'boundary for C such that (B, p_F) is separable. Then

$$\overline{\text{co}(B)}p_F = \overline{\text{co}(C)}\sigma(X, F)$$

Proof. The proof is based on the ideas in [14] (see also [13]). As we remarked, the dual of (X, p_F) is the subspace $H = \bigcup_{n=1}^{\infty} n\bar{G}^{\text{w}}$ of X^*, where $G = B_{X^*} \cap F$, and $F \subset H$. Hence

$$\overline{\text{co}(B)}p_F \subset \overline{\text{co}(C)}p_F = \overline{\text{co}(C)}\sigma(X, H) \subset \overline{\text{co}(C)}\sigma(X, F).$$

Assume that the conclusion of the proposition is false. Then there exists an element $x_0 \in \overline{\text{co}(C)}\sigma(X, F) \setminus \overline{\text{co}(B)}p_F$. Then by the separation theorem, there is a functional $f \in H = (X, p_F)^*$ such that

$$f(x_0) > \alpha > \sup\{f(b): b \in B\}.$$

By scaling we may assume that $f \in \bar{G}^{\text{w}^*}$. Let $U = \{g \in X^*: g(x_0) > \alpha\}$. Then U is convex weak -open and $f \in \bar{G}^{\text{w}^*} \cap U \subset \overline{G \cap U}^{\text{w}}$. Now $\bar{G}^{\text{w}^*}$ is equicontinuous on (X, p_F) and B contains a countable p_F-dense subset D. Therefore in $\bar{G}^{\text{w}^*}$ the topology of pointwise convergence on B is identical with the topology of pointwise convergence on D, and the latter is pseudometrizable. It follows that there is a sequence $\{z_n: n \in \mathbb{N}\}$ in $G \cap U$ such that $\lim_n z_n(b) = f(b)$ for each $b \in B$. Our assumption of F being norm-closed and B being an F-boundary of C implies that the sequence $(z_n)_n$ satisfies the hypothesis of Lemma (1.1.22). Hence by (8),

$$\alpha > \sup_{b\in B} f(b) \geq \inf\left\{\sup_{c\in C} w(c): w \in \text{co}(\{z_n\})\right\}$$

It follows that $\alpha > \sup_C w$ for some $w \in \text{co}(\{z_n\}) \subset G \cap U$. In particular, since $w \in U, w(x_0) > \alpha > \sup_C w$. On the other hand, since $x_0 \in \overline{\text{co}(C)}\sigma(X, F)$ and, being in F, w is $\sigma(X, F')$-continuous, $w(x_0) \leq \sup_C w$, contradicting the previous inequality. This proves the proposition.

The pointwise limit of a sequence of real-valued continuous functions is called a function of the first Baire class. More generally a function f from a topological space M into a normed space X is said to be of the first Baire class if there is a sequence of continuous functions $f_n: M \to X$ such that $(f_n)_n$ converges to f in (X^M, τ_p). A multi-valued map φ from the topological space M to the space of subsets of a topological space T is said to be usco if $\varphi(m)$ is a compact non-empty subset of T for each $m \in M$ and if φ is upper semicontinuous in the sense that, whenever U is an open subset of T, $\{m \in M: \varphi(m) \subset U\}$ is open in M.

Ideas in [15] (see also [30]) allow us to modify Jayne-Rogers' selection theorem, [20], to our situation below.

Theorem (1.1.25)[1]: Let M be a metric space, X a normed space and H' a total norm-closed subspace of X^*. Let H be a norm-bounded $\sigma(X,F')$-compact subset of X which is fragmented by the norm p_F. If ψ is an usco map from M to subsets of $(H,\sigma(X,F))$, then ψ has a first Baire class selector f from M into (X,p_F).

Proof. If we identify (X,p_F) with a subspace of $\ell^\infty(F\cap B_{X^*})$ and H with a weak -compact subset there, then we can apply Remark 17 in [19] to obtain a selector f of ψ_H which is σ-discrete and of the first Borel class from F' to $\ell^\infty(B\cap B_{X^*})$ (see Corollary 7 in [19]). Such a selector as a map from F into (X,p_F) is also σ-discrete of the first Borel class, and by Theorems 1 and 2 of [30], f is of the first Baire class from F' into (X,p_F) (see also [15] [30]).

We prove one of the main properties of the selectors obtained above: the result that follows is a counterpart to the one stated as Theorem 26 in [19], and it is in the setting of topologies of pointwise convergence on total sets.

Theorem (1.1.26)[1]: Let X be a normed space and let F' be a total norm-closed subspace of X^*. Let H be a norm-bounded $\sigma(X,F)$-compact subset of X. Let $\psi_H\colon F\to 2^H$ be the multi-valued map given by

$$\psi_H(f)=\left\{x\in H\colon f(x)=\sup_H f\right\}.$$

Then ψ_H has a selector of the first Baire class from $(F,\|\|)$ into (X,p_F) if, and only if, $\big(H,\sigma(X,H')\big)$ is fragmented by p_F. Moreover, if $f\colon H'\to H$ is such a selector of ψ_H, then

$$\overline{\operatorname{co}(H)}\,\sigma(X,F)=\overline{\cos\big(f(F')\big)}\,p_F. \tag{10}$$

Proof. The arguments here are similar to the ones in [19, Theorem 26]. First it is easy to check that ψ_H is an usco map from $(F,\|\|)$ into compact subsets of $\big(H,\sigma(X,H')\big)$. If $\big(H,\sigma(X,F')\big)$ is fragmented by p_F, then, by Theorem (1.1.25), ψ_H has a first Baire class selector $f\colon(F,\|\ \|)\to(X,p_F)$. Conversely assume that such a selector f exists. Let S be a $\|\ \|$-closed and $\|\ \|$-separable subspace of F, and consider the quotient normed space $(X/S^\perp,\|\|_S)$. Recall that the dual of $(X/S^\perp,\|\|_S)$ is isometric with $\bar{S}^*$ and hence S is a norm-closed total subspace of $(X/S^\perp,\|\|_S)^*$. Let $\pi_S\colon X\to X/S^\perp$ be the canonical quotient map and let p_S be the norm on $X/S^\perp$ given by

$$p_S(\pi_S(x))=\bar{p}_S(x):=\sup\{|g(x)|\colon g\in S\cap B_{X^*}\} \tag{11}$$

for each $x\in X$. Then $\pi_S(H)$ is a $\|\|_S$-bounded, $\sigma(X/S^\perp,S)$-compact subset of $X/S^\perp$, and $\pi_S(f(S))$ is an S-boundary for $\pi_S(H)$. Now let $f_k\colon H'\to X$ be a sequence of $\|{-p_F}$ continuous maps such that for each $g\in F, f_k(g)\to f(g)$ in p_F. For each subset A of F, let

$$\Phi(A)=\bigcup_{k=1}^{\infty}f_k(A)$$

Then $f\big(\bar{A}^{\|\ \|}\big)\subset\overline{\Phi(A)}p_F$ and $\Phi(A)$ is countable whenever A is. If D is a $\|\ \|$ dense countable subset of S, then $f(S)=f\big(\bar{D}\ \|^{\|\ \|}\big)\subset\overline{\Phi(D)}p_F$. Hence $f(S)$ is p_F-separable and so $\pi_S(f(S))$ is p_S-separable. It follows from Proposition (1.1.24) that

$$\overline{\operatorname{co}(\pi_S(f(S)))}p_S=\overline{\operatorname{co}(\pi_S(H))}\sigma(X/S^\perp,S). \tag{12}$$

This shows in particular that, whenever S is a $\|\ \|$-separable $\|\ \|$-closed subspace of $H',\pi_S(H)$ is p_S-separable and hence H is $\bar{p}_S$-separable. Regarding H as a τ_p-compact subset of $[-m,m]^{F\cap B_{X^*}}$ with an appropriate $m>0$, we see from Theorem (1.1.2) that $(H,\sigma(X,F))$ is fragmented by p_F.

We show that (10) is a consequence of (12). For this it is sufficient to prove that for each $u \in X$, there is a $\|\ \|$-separable $\|\ \|$-closed subspace S of F such that

$$p_S - \operatorname{dist}\big(\pi_S(u), \operatorname{co}(\pi_S(f(S)))\big) \geq p_F - \operatorname{dist}(u, \operatorname{co}(f(S))) \tag{13}$$

For if $u \in \overline{\operatorname{co}(H)}\sigma(X, F)$ and if S is chosen as above, then since

$$\pi_S(u) \in \overline{\pi_S(\operatorname{co}(H))}\sigma(X/S^{\perp}, S)$$

we have, by (12),$0 = p_S$-dist $(\pi_S(u), \operatorname{co}(\pi_S(f(S))) \geq p_F$-dist $(u, \operatorname{co}(f(S)))$. Hence $u \in \overline{\operatorname{co}(f(S))}p_F \subset \frac{p_S}{\operatorname{co}(f(F^+))}p_F$. This shows that the left side of (10) is contained in the right side. The reverse inclusion is obvious.

To prove (13), let $u \in X$. For each countable subset M of X, let $\alpha(M)$ be a countable subset of $F' \cap B_{X*}$ such that, for each $x \in M$,

$$p_F(u - x) = \sup\{|g(u - x)| : g \in \alpha(M)\}.$$

Inductively we define a sequence $A_1 \subset A_2 \subset \cdots$ of countable subsets of F as follows: let g_0 be an arbitrary non-zero element of F' and let $A_1 = \{qg_0 : q \in \mathbb{Q}\}$. Assuming that A_n has been defined, let

$$A_{n+1} = \operatorname{span}_{\mathbb{Q}}\Big(\alpha\Big(\operatorname{co}\mathbb{Q}\big(\Phi(A_n)\big)\Big) \cup A_n\Big),$$

where $\operatorname{span}_{\mathbb{Q}}(C)$ (resp. $\operatorname{co}_{\mathbb{Q}}(C)\big)$ denotes the set of all linear (resp. convex) combinations of elements of C with rational coefficients. Let $S = \bigcup_{n=1}^{\infty} A_n$ $\|\|$

Before showing this S satisfies (13), we note that if $y \in \operatorname{co}_{\mathbb{Q}}\big(\Phi(A_n)\big)$ then $p_F(u - y) = \sup\Big\{|g(u - y)| : g \in \alpha\Big(\operatorname{co}_{\mathbb{Q}}\big(\Phi(A_n)\big)\Big)\Big\} \leq \bar{p}_S(u - y) \leq p_F(u - y)$. Hence $p_F(u - y) = \bar{p}_S(u - y)$. Now by the definition of Φ.

Let $x \in \operatorname{co}(f(S))$ and $\varepsilon > 0$ be arbitrary. Then there is a $y \in \operatorname{co}\mathbb{Q}\big(\Phi_n(A_n)\big)$ for some n such that $\bar{p}_S(x - y) \leq p_F(x - y) < \varepsilon$. Then

$$\begin{aligned} p_S(\pi_S(u) - \pi_S(x)) &= \bar{p}_S(u - x) \geq \bar{p}_S(u - y) - \varepsilon = p_F(u - y) - \varepsilon \\ &\geq p_F(u - x) - 2\varepsilon \geq p_F - \operatorname{dist}(u, \operatorname{co}(f(S))) - 2\varepsilon \end{aligned}$$

Since $x \in f(S)$ and $\varepsilon > 0$ are arbitrary, we obtain (13).

Theorem (1.1.27)[1]: Let X be a Banach space, F' a norming subset of X^*, and let H be a bounded $\sigma(X, F)$-compact subset of X fragmented by the norm of X. Then the Banach space $Y = \overline{\operatorname{span}(H)}\ \|$ has a projectional generator.

Proof. We first prove the case $X = Y$. Since H is bounded, $\sigma(X, H')$ and $\sigma(X, \overline{F'}\ \|)$ coincide on H. Hence we may assume that F' is a $\|\|$ closed norming subspace. Let $\psi_H : F \to 2^H$ be the set-valued map given by $\psi_H(g) = \{x \in H : g(x) = \sup_H g\}$ for each $g \in H$. Then by Theorem (1.1.26), ψ_H admits a selector $f : F \to H$ of the first Baire class from $(F, \|\|)$ into $(X, \|\|)$. Let $\{f_k\}$ be a sequence of continuous maps $(F, \|\|) \to (X, \|\|)$ such that $f_k(g) \to f(g)$ in the norm for each $g \in F$, and we define the countable-valued map $\varphi : F' \to 2^X$ by $\varphi(g) = \{f_k(g) : k \in \mathbb{N}\}$. We prove that φ is a projectional generator (cf. Definition (1.1.20)). So let B be a $\mathbb{Q}$-linear subset of F, and let $g \in \varphi(B)^{\perp} \cap \overline{B \cap B_{X^*}}$. We must show that $g = 0$.

Let $S = \bar{B}$ $\|\| \subset F$, let $\pi_S : X \to X/S^{\perp}$ be the quotient map and let p_S be the norm defined on $X/S^{\perp}$ by (11). Since $g \in \overline{S \cap B_{X^*}}\mathrm{w}^*$, g defines a p_S-continuous linear functional $\bar{g}$ on $X/S^{\perp}$ by the formula $\bar{g}(\pi_S(x)) = g(x)$ for each $x \in X$. Now by the definition of $\varphi, f(S) = f\big(\bar{B}^{\|}\ \| \subset \overline{\varphi(B)}$ $\|\|$. Since g vanishes on $\varphi(B)$, it also vanishes on $f(S)$, and hence $\bar{g}$ vanishes on $\pi_S(f(S))$. By the remark following the last theorem, (12) is valid for S and hence $\pi_S(H) \subset$

$\overline{\text{co}(\pi_S(f(S'))} p_S$. Therefore by continuity $\bar{g}$ vanishes on $\pi_S(H)$, i.e. g vanishes on H. Since X is the norm-closed span of $H, g = 0$.

$$\text{co}(f(S)) \subset \text{co}\left(\overline{\Phi\left(\bigcup_{n=1}^{\infty} A_n\right)}^{p_F}\right)$$

$$\subset \overline{\text{co}_{\mathbb{Q}}\left(\bigcup_{n=1}^{\infty} \Phi(A_n)\right)}^{p_F} = \overline{\bigcup_{n=1}^{\infty} \text{co}_{\mathbb{Q}}(\Phi(A_n))}^{p_F}.$$

The general case is proved by applying the special case above to the Banach space Y and the norming subspace F_Y for Y. Note that H is a $\sigma(Y, H_Y|_Y)$-compact subset of Y and it is fragmented by the norm of Y.

Corollary (1.1.28)[1]: Let X be a Banach space, F' a norming subset of X^*, H a bounded subset of X which is $\sigma(X, F)$-compact and fragmented by the norm of X, and let $Y = \overline{\text{span}(H)}^{\|\|}$. Then

(a) $(Y, \gamma(X, F))$ is Lindelöf.

(b) Y has a PRI.

(c) Y has an equivalent LUR norm.

Proof. This is a straightforward consequence of Theorems (1.1.23) and (1.1.27).

Another property of spaces generated by RN-compact sets is the following. For this, we need one more definition. Let (Z, τ) be a topological space and ϱ a metric on Z. Then (Z, τ) is said to be σ-fragmented by ϱ if for each $\varepsilon > 0, Z$ can be written as $Z = \bigcup\{Z_n: n \in \mathbb{N}\}$ with each Z_n having the property that, whenever C is a non-empty subset of Z_n, there exists a τ-open subset U of Z such that $U \cap C$ is non-empty and of ϱ-diameter less than ε

Theorem (1.1.29)[1]: Let X be a Banach space, F a norming subset of X^*, H a bounded subset of X which is $\sigma(X, F)$-compact fragmented by the norm of X, and let $Y = \overline{\text{span}(H)}^{\|\|}$. Then $(Y, \sigma(X, F))$ is σ-fragmented by the norm.

Proof. The proof is analogous to the one given for weakly compactly generated spaces in [15], [17]. Indeed, $W := \overline{\text{co}(H)}^{\sigma(X,F)} = \overline{\text{co}(H)}^{\|\|}$ is $\sigma(X, F')$ compact and fragmented by the norm [7, 4.1, 5.2 and 5.3]. Lemmas 2.1 and 2.2 of [25] entail that $W - W$ is again $\sigma(X, F')$-compact and fragmented by the norm. We now have $Y = \overline{\bigcup_{n=1}^{\infty} n(W - W)}^{\|\ \|}$ and because F is norming, the norm in Y is $\sigma(X, F)$-lower semicontinuous and Lemma 2.3 in [16] gives us the conclusion.

We have obtained so far in the following:

Theorem (1.1.30)[1]: Let X be a Banach space, F' a norming subset of X^*, H a bounded subset of X which is $\sigma(X, F)$-compact, and let $Y = \overline{\text{span}(H)}^{\|\|}$. The following statements are equivalent:

(a) $(H, \sigma(X, F'))$ is fragmented by the norm.

(b) $(Y, \sigma(X, F))$ is σ-fragmented by the norm.

(c) $(H, \gamma(X, H'))$ is Lindelöf.

(d) $(Y, \gamma(X, F'))$ is Lindelöf.

Theorem (1.1.31)[1]: Let $K \subset [-1,1]^D \subset \ell^\infty(D)$ be a τ_p-compact set. The following statements are equivalent:

(a) (K, τ_p) is fragmented by the norm.

(b) $\left(\overline{\operatorname{span}(K)}^{\|\|}, \tau_p\right)$ is σ-fragmented by the norm.
(c) $(K, \gamma(D))$ is Lindelöf.
(d) $(\overline{\operatorname{span}(K)}^{\|\|}, \gamma(L))$ is Lindelöf.

We study Banach spaces which are Lindelöf in the weak topology. Main tools are again the projectional generators. Beyond Theorem (1.1.33) below, which gives quite a general way of deciding when a Banach space is weakly Lindelöf, here we take advantage of the scope of the results and the main results in [5] to prove that a Banach space X generated by a weakly Lindelöf subset which is $\sigma(X, F)$-compact with respect to some norming subspace $F \subset X^*$ is weakly Lindelöf. We need the following definition. For each set Γ, let $\Sigma(\Gamma)$ be the subspace of $\ell^\infty(\Gamma)$ consisting of all $u \in \ell^\infty(\Gamma)$ with $\{\gamma: u(\gamma) \neq 0\}$ at most countable. A compact Hausdorff space K is said to be Corson if, for some Γ, K can be embedded in $\Sigma(\Gamma)$ as a pointwise compact subset.

Definition (1.1.32)[1]: ([2]). A Banach space X is said to be weakly Lindelöf determined (WLD for short) if there is a bounded one-to-one linear map $T: X^* \to \ell^\infty(\Gamma)$, for some set Γ, which is $\sigma(X^*, X)$-pointwise continuous and such that $T(X^*) \subset \sum(\Gamma)$

It was established in [27] that a Banach space is WLD if, and only if, its dual unit ball with the weak* topology is Corson compact. Note that WCG Banach spaces and hence separable Banach spaces are WLD. It is known that a WLD Banach space is $\gamma(B_{X^*})$-Lindelöf ([26]) and renormable by a LUR norm ([35] and [22]). A Banach space X, or more generally a convex subset M of X, is said to have property $\mathcal{C}$ (after Corson) if each collection of relatively closed convex subsets of M with empty intersection has a countable subcollection with empty intersection. If (M, weak) is Lindelöf, then M has property $\mathcal{C}$ since closed convex sets in X are also weak-closed. It is shown in [29] that the Banach space X has the property $\mathcal{C}$ if and only if, whenever $A \subset X^*$ and $f \in \bar{A}^{w^*}$, there is a countable subset C of A such that $f \in \overline{\operatorname{co}}(C)^{w^*}$. This fact is crucial in the proof of the next theorem.

Theorem (1.1.33)[1]: Let X be a Banach space with a projectional generator. If X has property $\mathcal{C}$, then X is WLD, i.e. $(B_{X^*}$, weak $^*)$ is Corson compact.

Proof. Let $\varphi: H' \to 2^X$ be a projectional generator on X, where F' is a norming subspace for X. Then X admits a PRI constructed as we have recalled in Theorem (1.1.21). Let $\{P_\alpha: \omega_0 \leq \alpha \leq \mu\}$ be this PRI. Since property $\mathcal{C}$ is stable under taking closed subspaces, each $P_\alpha(X)$ has property $\mathcal{C}$ and a projectional generator. Now, by a standard induction process on the density character of the Banach space, we may assume that X admits a PRI $\{P_\alpha: \omega_0 \leq \alpha \leq \mu\}$, with μ a limit ordinal, such that, for each $\omega_0 \leq \alpha < \mu$, $P_\alpha(X)$ is WLD; that is, there is a one-to-one norm one operator

$$T_\alpha: P_\alpha^*(X^*) \to \ell^\infty(\Gamma_\alpha) \text{ with } T_\alpha\left(P_\alpha^*(X^*)\right) \subset \sum (\Gamma_\alpha)$$

which is weak *-pointwise continuous. Assume that $\{\Gamma_\alpha: \omega_0 \leq \alpha < \mu\}$ is a disjoint family. Then we define

$$\Gamma = \Gamma_{\omega_0} \cup \bigcup \{\Gamma_{\alpha+1}: \omega_0 \leq \alpha < \mu\}$$

and $T: X^* \to \ell^\infty(\Gamma)$ by the formulas

$$(I'f)(n) = l_{\omega_0}\left(P_{\omega_0}^*(f)\right)(n) \qquad \text{if } n \in \Gamma_{\omega_0} = \mathbb{N},$$
$$(Tf)(\gamma) = T_{\alpha+1}(P_{\alpha+1}^*(f) - P_\alpha^*(f))(\gamma) \quad \text{if } \gamma \in \Gamma_{\alpha+1}, \omega_0 \leq \alpha < \mu.$$

Clearly T is bounded linear and weak *-pointwise continuous. We claim that $Y'(X^*) \subset \sum(\Gamma)$. To prove it, we will see that the set $\{\alpha \in [\omega_0, \mu): P^*_{\alpha+1}(f) - P^*_\alpha(f) \neq 0\}$ is at most countable for each $f \in X^*$. Assume on the contrary that this is not the case and take $f \in X^*$ so that this set is uncountable. Recall that the family $\{B_\alpha : \alpha < \mu\}$ is a long sequence of increasing $\mathbb{Q}$ linear subsets of F with $P^*_\alpha(X^*) = \bar{B}^{w^*}_\alpha$ for each $\alpha < \mu$. Also for each limit ordinal $\beta \leq \mu$ and $f \in X^*$, weak - $^*_\alpha \lim_\alpha P^*_\alpha(f) = P^*_\beta(f)$, and $P^*_\mu = \mathrm{Id}$. Let $\Delta = \{\alpha \in [\omega_0, \mu): P^*_{\alpha+1}(f) - P^*_\alpha(f) \neq 0\}$. Then Δ is an uncountable subset of $[\omega_0, \mu)$ which is well-ordered under the inherited ordering. Therefore there is an order-isomorphism φ from $[0, \omega_1)$ onto an initial segment of Δ. Let $\eta = \sup\varphi([0, \omega_1)) \leq \mu$. Then $P^*_\eta(f) = \text{weak } ^* - \lim_{\gamma \uparrow \omega_1} P^*_{\varphi(\gamma)}(f)$ and therefore

$$P^*_\eta(f) \in \overline{\bigcup_{\gamma < \omega_1} P^*_{\varphi(\gamma)}(f)}^{w^*}.$$

Since X has property $\mathcal{C}$, there is a sequence $\gamma_1 < \gamma_2 < \cdots$ in $[0, \omega_1)$ such that

$$P^*_\eta(f) \in \overline{\mathrm{co}\left(\bigcup_{\iota=1}^{\infty} P^*_{\varphi(\gamma_\iota)}(f)\right)}^{w^*}.$$

Let $\xi = \varphi(\sup_i \gamma_i) \in \Delta$. Then $\xi < \eta \leq \mu$. Since for each i, $P_{\varphi(\gamma_i)}(f) \in \bar{B}^{w^*}_{\varphi(\gamma_i)} \subset \bar{B}^{w^*}_\xi$, we have $P^*_\eta(f) \in \bar{B}^{w^*}_\xi$. It follows that $P^*_\eta(f)$ is a fixed point of P^*_α for all $\alpha \geq \xi$. Hence if $\xi \leq \alpha < \eta$, then $P^*_\eta(f) = P^*_\alpha P^*_\eta(f) = P^*_\alpha(f)$ by the property of PRI: $P_\eta P_\alpha = P_{\min}\{\eta, \alpha\}$. In particular, $P^*_{\xi+1}(f) = P^*_\eta(f) = P^*_\xi(f)$, contradicting $\xi \in \Delta$. Hence $I'(X^*) \subset \sum(\Gamma)$.

To see that $'T$ is one-to-one, let $'(f) = 0$ for an $f \in X^*$. Then $P^*_{\omega_0}(f) = 0$, and $P^*_{\alpha+1}(f) = P^*_\alpha(f) = 0$ for all $\alpha \in [\omega_0, \mu)$. Then by a straightforward (transfinite) induction, $P^*_\alpha(f) = 0$ for all $\alpha \in [\omega_0, \mu)$, and hence $f = \text{weak } ^* \lim_{\alpha \uparrow \mu} P^*_\alpha(f) = 0$.

Corollary (1.1.34)[1]: Let X be a Banach space, F' a norming subset of X^*, H a bounded subset of X which is $\sigma(X, F)$-compact fragmented by the norm of X, and let $Y = \overline{\mathrm{span}(H)}\ \|\|$. If Y has property $\mathcal{C}$, then Y is WLD.

As mentioned earlier, a WLD Banach space is weakly Lindelöf, but the converse is not true; cf. [22, p. 514]. In [22, p. 521], Mercourakis and Negrepontis have asked if this converse is true in dual Banach spaces. The affirmative answer to this question is contained in [26] where it is shown that if X is an Asplund space then X^* is weakly Lindelöf if and only if ($B_{X^{**}}$, weak*) is Corson compact, i.e. X^* is WLD. Recall that Edgar had observed earlier [10] that X is an Asplund space whenever X^* is weakly Lindelöf. The next two corollaries are generalizations of the result in [26] just mentioned. The first one is a special case of the previous corollary.

Corollary (1.1.35)[1]: Let X be an Asplund space, H a subset of X^* which is weak*-compact, and let $Y = \overline{\mathrm{span}(H)}\ \|\|$. If Y has property $\mathcal{C}$ then Y is WLD. In particular, if X is an Asplund space, then X^* is WLD if and only if it has property $\mathcal{C}$.

A combination of most of the results and the main result in [5] finally allows us to prove:

Corollary (1.1.36)[1]: Let X be a Banach space, and let H be a subset of X^* which is weak*-compact and weakly Lindelöf. Then the space generated by $H, Y = \overline{\mathrm{span}(H)}\ \|\|$, is WLD. In particular, Y is weakly Lindelöf.

Proof. By the remark following Theorem (1.1.18), we know that the weak* closed absolute convex hull of H, say W, is also weakly Lindelöf. Hence by Corollary E of [4], (W, weak) is fragmented by the norm. Furthermore, $Y = \overline{\text{span}(W)}^{\|\|} = \overline{\bigcup_{n=1}^{\infty} nW}^{\|\|}$ has property $\mathcal{C}$ by Proposition 2 in [29]. Hence by Corollary (1.1.34), Y is WLD, and since a closed subspace of a WLD Banach space is again WLD ([22]), the corollary follows.

We know from the above that $x^{**}|_Z$ is $\gamma(X^*, X)$-continuous. This means that for each $\varepsilon > 0$ there is a $\gamma(X^*, X)$ open neighborhood $U \subset X^*$ of the origin such that

$$|x^{**}(g)| < \varepsilon \text{ for each } g \in U \cap Z \tag{14}$$

Now U is also $\|$ -open and therefore $\overline{U \cap Z}^{\|} = \overline{U \cap Y}^{\|\|} \supset UY$. Therefore the $\|\|$-continuity of x^{**} and (14) imply that $|x^{**}(f)| \leq \varepsilon$ for every $f \in U \cap Y$. This means that x^{**} is $\gamma(X^*, X)$-continuous on Y, which concludes the proof.

As mentioned, we give several examples of Banach spaces generated by an RN-compact subset. By Theorem (1.1.27), these spaces share all the properties stated in Theorem (1.1.21). Also, by Corollary (1.1.34), for these spaces being WLD is equivalent to having property $\mathcal{C}$

Example (1.1.37)[1]: Spaces with 1-norming Markushevich basis. Let us recall that a Markushevich basis, or M-basis, of a Banach space X is a subset $\{(x_i, f_i): i \in I\}$ of $X \times X^*$ such that

(a) $\overline{\text{span}\{x_i: i \in I\}}^{\|\|} = X$.

(b) $\bigcap_{i\in I} \text{Kernel}(f_i) = \{0\}$.

(c) $f_j(x_i) = \delta_{ij}, i, j \in I$.

We consider the subspace $F' := \text{span}\{f_i\}$, which is a total subspace in X^* by condition (b). If $K := \{x_i: i \in I\} \cup \{0\}$ then it is easy to see that K is a $\sigma(X, F)$-compact set fragmented by the norm. Indeed, $\{x_i: i \in I\}$ is a $\sigma(X, F')$-discrete set with 0 as its unique limit point. When F is norming, the M-basis is called a norming M-basis. Therefore any Banach space with a norming M-basis is generated by an RN-compact subset. The σ fragmentability of spaces with a norming M-basis was first proved in [18]; here, it is a consequence of Theorem (1.1.30).

Example (1.1.38)[1]: Spaces of continuous functions. Let K be a compact space and D a dense subset of K. If $H \subset C(K)$ is $\tau_p(L)$-compact, uniformly bounded, fragmented by the supremum norm and separates the points of K, then $C(K)$ is generated by an N-compact set. Indeed, in this case the norming subspace of $C(K)^*$ is $F' = \text{span}\{\delta_x: x \in L)$ and we observe that for every $n = 1,2, \ldots$ the set $H^n := \{f_1 \cdot \ldots \cdot f_n: f_i \in H, i = 1, \ldots, n\}$ is $\sigma(C(K), F)$-compact and fragmented by the norm in view of Lemmas 2.1 and 2.2 in [25]. Now, $W = \bigcup_{n=1}^{\infty}(1/n)H^n \cup \{0\}$ is also $\sigma(C(K), F)$ compact and σ-fragmented by the norm, hence fragmented [17, Theorem 4.1]. On the other hand, the Stone-Weierstrass theorem gives us the equality $\overline{\text{span}(W)}^{\|\|} = C(K)$ and so $C(K)$ is generated by a $\sigma(C(K), F)$-compact subset fragmented by the norm.

Example (1.1.39)[1]: Spaces of continuous functions defined on solid compact spaces and on compact spaces defined through adequate families of sets. Let I be a set and consider the cube $[0,1]^I$ with the product topology. Given $x \in [0,1]^I$ let us write

$$\text{supp}(x) := \{i \in I: x(i) \neq 0\},$$

$$\mathcal{F}(I) := \{x \in [0,1]^I: \text{supp}(x) \text{ is finite}\}.$$

We claim that if $K \subset [0,1]^I$ is a compact subset such that $K \cap \mathcal{F}(I)$ is dense in K (i.e. K is a special type of Valdivia compact space), then $C(K)$ is generated by an RN-compact subset. Indeed, write $D = K \cap \mathcal{F}(I)$ and let $\pi_i: [0,1]^I \to [0,1]$ denote the canonical projection onto the

i-th coordinate, for each $i \in I$. Without loss of generality we can, and do, assume that for each $i \in I$ there is $x \in K$ such that $\pi_i(x) \neq 0$, because otherwise we can remove from the index set I the element i that is not needed for embedding K in $[0,1]^I$. Observe that $\{\pi_i: i \in I\}$ is $\tau_p(D)$-discrete and that each $\tau_p(D)$ neighborhood of 0 must contain all but at most finitely many $\{\pi_i: i \in I\}$; therefore $\{\pi_i: i \in I\} \cup \{0\}$ is $\tau_p(D)$-compact, $\| \|_\infty$-fragmented and separates the points of K. We now use Example (1.1.38) to conclude that $C(K)$ is generated by an RN-compact subset.

A compact space $K \subset [0,1]^I$ is said to be solid if whenever $x \in K$ and $y \in [0,1]^I$ are such that either $y_i = x_i$ or $y_i = 0$, for every $i \in I$, then $y \in K$. Obviously, if $K \subset [0,1]^I$ is solid, then $K \cap \mathcal{F}(I)$ is dense in K and therefore $C(K)$ is generated by an RN-compact set in view of our former reasoning.

A particular situation to which we can apply the above is when we deal with compact spaces defined through adequate families of sets. Following Talagrand [34], if I is a non-empty set, a family $\mathcal{A}$ of subsets of I is called adequate if it has the following properties:

(a) If $A \in \mathcal{A}$ and $B \subset A$, then $B \in \mathcal{A}$.
(b) $\{i\} \in \mathcal{A}$ for every $i \in I$.
(c) If $A \subset I$ and every finite subset of A belongs to $\mathcal{A}$, then $A \in \mathcal{A}$. If $\mathcal{A}$ is an adequate family in I, then $K: = \{\chi_A: A \in \mathcal{A}\}$

is a solid compact space. Then $C(K)$ is also generated by an RN-compact subset. Talagrand produced in [34, Théorème 4.3] an example of a compact space K defined through an adequate family of sets that is not Eberlein compact; the corresponding $C(K)$ then does not contain a $\tau_p(K)$-compact subset separating the points of K, even though it contains a $\tau_p(I)$-compact (for a certain dense $I) \subset K)$ $\|\|_\infty$-fragmented subset separating the points of K.

Example (1.1.40)[1]: Spaces of Bochner integrable functions. Let $(X, \|\|)$ be a Banach space and $H' \subset X^*$ a norming subspace. It was stated in [7, Corollary 4.3] that if $\sigma(X, F)$-separable compact subsets of X are $\| \|$-separable then $\sigma(X, F)$-compact (norm-bounded) subsets H of X are $\| \|$-fragmented. This is indeed a consequence of the equivalence between the first two statements in Theorem (1.1.2): write $D = F' \cap B_{X^*}$ and consider $H \subset [-1,1]^D$; for $A \subset$ D) countable the set $H|_A \subset [-1,1]^A$ is compact and metrizable, therefore separable; then there is a $\sigma(X, F)$-compact and separable $S \subset H$ such that $S|_A = H|_A$; the restriction map $[-1,1]^D \to [-1,1]^A$ is continuous for the corresponding uniform metrics and therefore $H|_A$ is d_A-separable, because S is d_D-separable (S is $\| \|$-separable).

The above observation is useful in finding more compact spaces "living" in Banach spaces and fragmented by the norm without being necessarily weakly compact.

Given a probability space (Ω, Σ, μ) we will denote by $L^p(\mu, X), 1 \leq p < \infty$, the Banach space of μ-strongly measurable X-valued p-Bochner integrable functions $f: \Omega \to X$ normed by

$$\| f \|_p = \left(\int_\Omega \| f \|^p \, d\mu \right)^{1/p}.$$

The dual $L^p(\mu, X)^*$ of $L^p(\mu, X)$ is a space of weak -measurable functions and the space $L^q(\mu, X^*), 1 = 1/p + 1/q$, which can be isometrically identified with a subspace of $L^p(\mu, X)^*$, is a norming subspace. So $\sigma' = \sigma(L^p(\mu, X), L^q(\mu, X^*))$ is a Hausdorff topology which is weaker than the weak topology of $L^p(\mu, X)$; these two topologies coincide if, and only if, X^* has the

RNP [9, IV.1.1]. It was shown in [7, Example E] that every σ'-separable compact subset of $L^p(\mu, X)$ is norm-separable. Therefore, every σ'-compact subset of $L^p(\mu, X)$ is fragmented by the norm. Thus we can apply the results to state for instance that if $H \subset L^p(\mu, X)$ is σ'-compact then the space $Y = \overline{\text{span}(H)}\ \|\|_p$ has a PRI. This result is related to the main result of [4], which asserts the existence of a bounded one-to-one operator from $\overline{\text{span}(H)}\sigma'$ into some $c_0(\Gamma)$ which is σ'-pointwise continuous.

Section (1.2): Asplund Operators:

We are concerned with the study of simultaneously approximating both operators and the points at which they almost attain their norms by norm attaining operators and the points at which they attain their norms. We study what in recent literature has been called the Bishop-Phelps-Bollobás property. This property is defined as:

Definition (1.2.1)[36]: (Acosta, Aron, García and Maestre, [37]). A pair of Banach spaces (X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBP) if for any $\varepsilon > 0$ there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$, such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then there are $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \beta(\varepsilon) \text{ and } \| T - S \| < \varepsilon.$$

The above BPBP was motivated by the following result of Bollobás:

Theorem (1.2.2)[36]: (Bollobás, [41]). Given $\frac{1}{2} > \varepsilon > 0$, if $x_0 \in S_X$ and $x^* \in S_{X^*}$ are such that

$$|1 - x^*(x_0)| < \frac{\varepsilon^2}{2},$$

then there are $u_0 \in S_X$ and $y^* \in S_{X^*}$ such that

$$y^*(u_0) = 1, \|x_0 - u_0\| < \varepsilon + \varepsilon^2 \text{ and } \|x^* - y^*\| < \varepsilon.$$

Bollobás' result is indeed an observation about the classical Bishop-Phelps' theorem, [40], that "sharpen Bishop-Phelps' theorem and is applied to some problems about the numerical range of operators." Using Definition (1.2.1), Bollobás' Theorem (1.2.2) can be rephrased by saying that for every Banach space X the pair $(X, \mathbb{R})$ has BPBP.

[37] described a number of cases of pairs (X, Y) with BPBP. For instance, they proved that if Y has property (β), see [54], then (X, Y) has BPBP for every Banach space X. Also, (ℓ^1, Y) has BPBP for Y in a large class of Banach spaces that includes the finite dimensional Banach spaces, uniformly convex Banach spaces, spaces $L_1(\mu)$ for a σ-finite measure μ and spaces $C(K)$. Although some particular results can be found in [37] for pairs of the form (ℓ_n^∞, Y) (for instance, Y uniformly convex), [37] comment that their methods do not work for pairs of the form (c_0, Y). We devise a method to study the Bishop-Phelps-Bollobás property that in particular addresses this question when $Y = C_0(L)$, L a locally compact Hausdorff space.

We recall the notions of Asplund space and Asplund operator. We also prove a central technical result (Lemma (1.2.4)) that will be used to prove our main result Theorem (1.2.5). Theorem (1.2.5) establishes that if $T: X \to C_0(L)$ is an Asplund operator and $\|T(x_0)\| \approx \| T \|$ for some $\|x_0\| = 1$, then there is a norm attaining Asplund operator $S: X \to C_0(L)$ and $\|u_0\| = 1$ with $\|S(u_0)\| = \| S \| = \| T \|$ such that $u_0 \approx x_0$ and $S \approx T$

Three consequences follow:

(A) If T is weakly compact, then S can also be taken being weakly compact (see Corollary (1.2.6))

(B) If X is Asplund, then the pair $(X, C_0(L))$ has the BPBP for all L (see Corollary (1.2.7)).

(C) If L is scattered, then the pair $(X, C_0(L))$ has the BPBP for all X (see Corollary (1.2.8)).

We note that in Corollary (1.2.6) even the part of the density of norm attaining weakly compact operators from X to $C_0(L)$ in the family of weakly compact operators $\mathcal{W}(X, C_0(L))$ seems to be new. Corollary (1.2.7) strengthens a result in [47] and Corollary (1.2.8) can be alternatively proved using a result in [37].

All vector spaces are assumed to be real. By X and Y we always denote Banach spaces. B_X and S_X are the closed unit ball and the unit sphere of X, respectively. X^* (resp. X^{**}) stands for the topological dual (resp. bidual) of X. The weak topology is denoted w and w^* is the weak * topology in the dual. $L(X, Y)$ denotes the spaces of bounded linear operators from X to Y endowed with its usual norm.

The letters K and L are reserved to denote compact and locally compact Hausdorff spaces respectively. $C(K)$ (resp. $C_0(L)$) denotes the space of real valued continuous functions (resp. continuous functions vanishing at infinity) on K (resp. on L) endowed with the standard sup norm, that is simply denoted by $\| f \| := \sup\{|f(s)| : s \in K\}$. As usual, given $s \in L$ we denote by $\delta_s : C_0(L) \to \mathbb{R}$ the Dirac measure at s given by $\delta_s(f) = f(s), f \in C_0(L)$.

The Banach space X is called an Asplund space if, whenever f is a convex continuous function defined on an open convex subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [39] under the name strong differentiability space. Asplund spaces have been used profusely since they were introduced. The versatility of this concept is in part explained by its multiple characterizations via topology or measure theory, as for instance in the following:

Theorem (1.2.3)[36]: Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space;

(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

For the notion of Radon-Nikodým property see [42],[44]. The equivalence (iii) ⇔ (iv) is due to Stegall [56], (i) ⇔ (ii) ⇒ (iii) can be found by Namioka and Phelps [50] and, (iii) ⇒ (ii) is due again to Stegall [57]. Recall that a subset C of (X^*, w^*) is said to be fragmented by the norm if for each non-empty subset A of C and for each $\varepsilon > 0$ there exists a non-empty w^*-open subset U of X^* such that $U \cap A \neq \emptyset$ and $\|\cdot\| - \operatorname{diam}(U \cap A) \leq \varepsilon$, [49]. We note that if C is w^*-compact convex, then C is fragmented by the norm if, and only if, C has the Radon-Nikodým property, see [42].

An operator $T \in L(X, Y)$ is said to be an Asplund operator if it factors through an Asplund space, i.e., there are an Asplund space Z and operators $T_1 \in L(X, Z), T_2 \in L(Z, Y)$ such that $T = T_2 \circ T_1$, see [46],[58]. Note that every weakly compact operator $T \in \mathcal{W}(X, Y)$ factors through a reflexive Banach space, see [43], and hence T is an Asplund operator.

Lemma (1.2.4) isolates the technicalities that we need to prove our main result, Theorem (1.2.5). In the proof of the lemma, we use that Theorem (1.2.2) easily yields the following result.

Lemma (1.2.4)[36]: Let $T : X \to Y$ be an Asplund operator with $\| T \| = 1$, let $\frac{1}{2} > \varepsilon > 0$ and choose $x_0 \in S_X$ such that

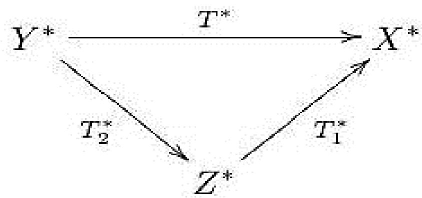
$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$$

For any given 1 -norming set $B \subset B_{Y^*}$ if we write $M := T^*(B)$ then there are:

(a) a w^*-open set $U \subset X^*$ with $U \cap M \neq \emptyset$ and (b) points $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$ such that

$$\|x_0 - u_0\| < \varepsilon \text{ and } \|z^* - y^*\| < 3\varepsilon \text{ for every } z^* \in U \cap M.$$

Proof. Observe first that if T is an Asplund operator, then its adjoint T^* sends the unit ball of Y^* into a w^*-compact subset of (X^*, w^*) that is norm fragmented. Indeed, if $T = T_2 \circ T_1$ is a factorization through the Asplund space Z for T, then its adjoint T^* factors through Z^*



Since T_2^* is $w^* - w^*$ continuous, $T_2^*(B_{Y^*})$ is a w^*-compact subset of Z^*, and we can now appeal to Theorem (1.2.3) to conclude that $T_2^*(B_{Y^*}) \subset (Z^*, w^*)$ is fragmented by the norm of Z^*. On the other hand, $T_1^*: Z^* \to X^*$ is norm-to-norm and $w^* - w^*$ continuous and, therefore it sends the fragmented w^*-compact set $T_2^*(B_{Y^*}) \subset (Z^*, w^*)$ onto the w^*-compact set $T^*(B_{Y^*}) \subset (X^*, w^*)$ that is fragmented by the norm of X^*, see [49], and our observation is proved. (Alternatively, the observation can be proved using [58] and [42].)

Now we really start the proof of the lemma. Use that $B \subset B_{Y^*}$ is 1 -norming and pick $b_0^* \in B$ such that

$$|T^*(b_0^*)(x_0)| = \left|b_0^*\big(T(x_0)\big)\right| > 1 - \frac{\varepsilon^2}{4}$$

Defining $U_1 = \left\{x^* \in X^*: |x^*(x_0)| > 1 - \frac{\varepsilon^2}{4}\right\}$, we have that

$$T^*(b_0^*) \in U_1 \cap M \subset T^*(B_{Y^*}) \subset B_{X^*}.$$

Since $T^*(B_{Y^*})$ is fragmented and $U_1 \cap M$ is non-empty, there exists a w^*-open set $U_2 \subset X^*$ such that $(U_1 \cap M) \cap U_2 \neq \emptyset$ and

$$\|\cdot\| - \operatorname{diam}\big((U_1 \cap M) \cap U_2\big) \leq \varepsilon. \tag{15}$$

Let $U := U_1 \cap U_2$ and fix $x_0^* \in U \cap M$. We have

$$1 \geq \|x_0^*\| \geq |x_0^*(x_0)| > 1 - \frac{\varepsilon^2}{4}.$$

If we normalize we still have

$$1 \geq \frac{|x_0^*(x_0)|}{\|x_0^*\|} \geq |x_0^*(x_0)| \geq 1 - \frac{\varepsilon^2}{4}. \tag{16}$$

Then we obtain $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$ such that

$$\|x_0 - u_0\| < \varepsilon \text{ and } \left\|\frac{x_0^*}{\|x_0^*\|} - y^*\right\| < \varepsilon. \tag{17}$$

Let $z^* \in U \cap M$ be an arbitrary element. Then,

$$\|z^* - y^*\| \quad \leq \|z^* - x_0^*\| + \left\|x_0^* - \frac{x_0^*}{\|x_0^*\|}\right\| + \left\|\frac{x_0^*}{\|x_0^*\|} - y^*\right\|$$

$$\overset{(15),(17)}{\leq} \varepsilon + \|x_0^*\| \left|1 - \frac{1}{\|x_0^*\|}\right| + \varepsilon \overset{(16)}{\leq} 3\varepsilon$$

and the proof is over.

Theorem (1.2.5)[36]: Let $T: X \to C_0(L)$ be an Asplund operator with $\| T \| = 1$. Suppose that $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ are such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X,C_0(L))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \| T - S \| \leq 3\varepsilon.$$

Proof. The natural embedding $\xi: L \to C_0(L)^*$ given by $\xi(s) := \delta_s$, for $s \in L$, is continuous for the topology of L and the w^*-topology in $C_0(L)^*$. Hence the composition $\phi := T^* \circ \xi: L \to X^*$ is continuous for the w^* topology in X^*.

Apply now Lemma (1.2.4) for $Y := C_0(L), B := \{\delta_s : s \in L\} \subset B_{C_0(L)^*}$, our given operator T, and ε. We produce the w^*-open set U and the functional $y^* \in S_{X^*}$ satisfying properties (a) and (b) in the aforesaid lemma. Note that we have $\phi(L) = M$. Since $U \cap M \neq \emptyset$ we can pick $s_0 \in L$ such that $\phi(s_0) \in U$. The w^*-continuity of ϕ ensures that the set $W = \{s \in L: \phi(s) \in U\}$ is an open neighborhood of s_0. By Urysohn's lemma, [53], we can find a continuous function $f: L \to [0,1]$ with compact support, satisfying:

$$f(s_0) = 1 \text{ and } \operatorname{supp}(f) \subset W. \tag{18}$$

Define now the linear operator $S: X \to C_0(L)$ by the formula

$$S(x)(s) = f(s) \cdot y^*(x) + (1 - f(s)) \cdot T(x)(s). \tag{19}$$

It is easily checked that S is well-defined and that $\| S \| \leq 1$. On the other hand, $1 = |y^*(u_0)| = |S(u_0)(s_0)| \leq \|S(u_0)\| \leq 1$ and therefore S attains the norm at the point $u_0 \in S_X$ for which we had $\|u_0 - x_0\| < \varepsilon$.

Now, bearing in mind (18), (19), Lemma (1.2.4) and the definition of W we conclude that

$$\begin{aligned} \| T - S \| &= \sup_{x \in B_X} \| Tx - Sx \| = \sup_{x \in B_X} \sup_{s \in L} f(s) |T(x)(s) - y^*(x)| \\ &= \sup_{x \in B_X} \sup_{s \in W} f(s) |\phi(s)(x) - y^*(x)| \leq \sup_{s \in W} \sup_{x \in B_X} |\phi(s)(x) - y^*(x)| \\ &= \sup_{s \in W} \|\phi(s) - y^*\| \leq 3\varepsilon. \end{aligned}$$

To finish we prove that S is also an Asplund operator. This is based on the fact that the family of Asplund operators between Banach spaces is an operator ideal, see [58]. Observe that S appears as the sum of a rank one operator and the operator $\mapsto (1 - f)T(x)$; the latter is the composition of a bounded operator from $C_0(L)$ into itself with T. Therefore S is an Asplund operator and the proof is over.

Recall that an operator ideal $\mathcal{J}$ is a way of assigning to each pair of Banach spaces (X, Y) a linear subspace $\mathcal{J}(X, Y) \subset L(X, Y)$ that contains all finite rank operators from X to Y and satisfies the following property: $T_2 \circ T \circ T_1 \in \mathcal{J}(Z, V)$ whenever $T \in \mathcal{J}(X, Y)$, $T_1 \in L(Z, X)$, and $T_2 \in L(Y, V)$, see [45],[52].

If we denote by $\mathcal{A}$ the ideal of Asplund operators between Banach spaces, the above theorem applies as well to any sub-ideal $\mathcal{J} \subset \mathcal{A}$.

Corollary (1.2.6)[36]: Let $\mathcal{J} \subset \mathcal{A}$ be an operator ideal. Let $T \in \mathcal{J}(X, C_0(L))$ with $\| T \|= 1$, $\frac{1}{2} > \varepsilon > 0$, and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and $S \in \mathcal{J}(X, C_0(L))$ with $\| S \|= 1$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \| T - S \| \leq 3\varepsilon.$$

We should stress that because $\mathcal{W} \subset \mathcal{A}$, see [43], the above corollary applies in particular to the ideals of finite rank operators $\mathcal{F}$, compact operators $\mathcal{K}$, p-summing operators Π_p and of course to the weakly compact operators $\mathcal{W}$ themselves. Results in this vein can be found for weakly compact operators but, with spaces of continuous functions as domain spaces and only for the so-called Bishop-Phelps property: Schachermayer proved, see [55], that any $T \in \mathcal{W}(C(K), X)$ can be approximated by norm attaining operators. This result was generalized later for operators $T \in \mathcal{W}(C_0(L), X)$, see [38]). With spaces of continuous functions in the range, Johnson and Wolfe, see [47], proved that any $T \in \mathcal{K}(X, C(K))$ can be approximated by finite rank norm attaining operators. Note then, that our Corollary (1.2.6) adds several new versions of the vector-valued Bishop-Phelps theorem. Moreover, these cases provide the Bollobás part of approximation of points at which the norm is attained.

Standard $\varepsilon - \delta$ tricks suffice to prove that for a pair of Banach spaces (X, Y) the following are equivalent:

(i) (X, Y) has $BPBP$ according to Definition (1.2.1);

(ii) there are functions $\eta: (0, +\infty) \to (0,1), \beta, \gamma: (0, +\infty) \to (0, +\infty)$ with $\lim_{t\to 0}\beta(t) = \lim_{t\to 0}\gamma(t) = 0$, such that given $\varepsilon > 0$, for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \beta(\varepsilon) \text{ and } \| T - S \| < \gamma(\varepsilon).$$

Once again, in (ii) above we can always take $\beta(t) = \gamma(t) = t$, but of course changing η if needed!. Consequently we arrive to the following straightforward consequence of Theorem (1.2.5):

Corollary (1.2.7)[36]: For any Asplund space X and any locally compact Hausdorff topological space L the pair $(X, C_0(L))$ has the BPBP.

Note that this corollary extends and strengthens Theorem 2 in [47]; we stress also that we can take as X any $c_0(\Gamma)$ (Γ arbitrary set), or more generally any $C_0(S)$ where S is a scattered locally compact Hausdorff space (see, [51] for scattered or dispersed spaces). Indeed for a locally compact space S, the space $C_0(S)$ is Asplund if, and only if, S is scattered. This can be proved in the following way:

(a) It is known that for K compact, $C(K)$ is Asplund if, and only if, K is scattered, combine [51] with Theorem (1.2.3) or alternatively see [50].

(b) It is easy to check that if S is locally compact, then S is scattered if, and only if, its Alexandroff compactification $S \cup \{\infty\}$ is scattered,

(c) Use now that Asplundness is a three space property, see ([50], Theorems 11,12 and 14), and conclude that $C_0(S)$ is Asplund if, and only, if $C(S \cup \{\infty\})$ is Asplund.

(d) Summarizing, $C_0(S)$ is Asplund if, and only if, S is scattered.

Note that whereas the hypothesis of X being Asplund in the above corollary is an isomorphic property, for the range space we have to use the sup norm in $C_0(L)$. Indeed,

Lindenstrauss [48] established that if $(c_0, \|\cdot\|)$ is a strictly convex renorming of c_0 then $id: c_0 \to (c_0, \|\cdot\|)$ cannot be approximated by norm attaining operators. Notice also, that Corollary (1.2.7) may fail when X is not Asplund: Schachermayer [55] gave an example of an operator $T \in L(L^1[0,1], C[0,1])$ that cannot be approximated by norm attaining operators.
With our comments above together with Theorem (1.2.5) we have:

Corollary (1.2.8)[36]: For any Banach space X and any scattered locally compact Hausdorff topological space L the pair $(X, C_0(L))$ has the BPBP.

An alternative proof for this corollary can be obtained using the fact that for such L the space $Y = C_0(L)$ has property (β), see [54], and for spaces Y with property (β), every pair (X, Y) has BPBP, see [37].

We point out that Lindenstrauss proved in [48] that every operator $T \in L(X, Y)$ can be approximated by operators $S \in L(X, Y)$ such that $S^{**} \in L(X^{**}, Y^{**})$ attains the norm on $B_{X^{**}}$. In [37] it is established that the counterpart of the above Lindenstrauss' result is not longer valid for the corresponding natural Bishop-Phelps-Bollobás with bi-adjoints operators. The example again uses c_0 as a domain space. Replacing Y^{**} by $C(B_{Y^*}, w^*)$, we state our last result.

Corollary (1.2.9)[36]: Let $T: X \to Y$ be an Asplund operator with $\| T \| = 1$, $\frac{1}{2} > \varepsilon > 0$ and $x_0 \in S_X$ be such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

Then there are $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X, C(B_{Y^*}))}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \| i \circ T - S \| \leq 3\varepsilon,$$

where $i: Y \hookrightarrow C(B_{Y^*})$ is the natural embedding.

Chapter 2
The Bishop-Phelps-Bollobás Property

We provide two constructive versions of the classical Bishop-Phelps-Bollobás theorem for $\ell_1(\mathbb{C})$. We study the Bishop-Phelps-Bollobás property for numerical radius of the Banach space of Lebesgue integrable functions over the real line. We show that the pair $(\mathcal{C}_0(L), Y)$ will satisfy the Bishop-Phelps-Bollobás property for operators for every Hausdorff locally compact space L and any $\mathbb{C}$-uniformly convex space.

Section (2.1): Numerical Radius in $\boldsymbol{\ell_1}(\mathbb{C})$:

The Bishop-Phelps theorem states that norm attaining functionals on a Banach space X are dense in its dual space X^*. In 1970, B. Bollobás extended this result in a quantitative way in order to work on problems related to the numerical range of an operator [68]. One of the versions of his extension is presented below:

Theorem (2.1.1)[59]: Let X be a Banach space. Given $\varepsilon > 0$, if $x \in X, x^* \in X^*$ with $\| x \|= \|x^*\| = 1$ and $x^*(x) \geq 1 - \frac{\varepsilon^2}{2}$, then there exist elements $x_0 \in X$ and $x_0^* \in X^*$ such that $\|x_0\| = \|x_0^*\| = x_0^*(x_0) = 1$

$$\|x - x_0\| \leq \varepsilon \text{ and } \|x^* - x_0^*\| \leq \varepsilon$$

However, the known proofs of this fact have an existence nature $-$ they are based on Hahn-Banach extension theorem, the Ekeland variational principle or Brøndsted-Rockafellar principle. We construct, as a necessary tool for our main results, explicit expressions of the approximating pair (x_0, x_0^*) when $X = \ell_1(\mathbb{C})$ $-$ see Theorem (2.1.6) and Theorem (2.1.8).

Paralleling the research of norm attaining operators initiated by Lindenstrauss in [74], B. Sims raised the question of the norm denseness of the set of numerical radius attaining operators $-$ see [76]. Partial positive results have been proved. We emphasize for their importance the results of M. Acosta [62], where a systematic study of the problem was initiated, the renorming result in [63], and joint findings of this with R. Payá [64, 65]. Prior to them, I. Berg and B. Sims in [69] gave a positive answer for uniformly convex spaces and C. S. Cardassi obtained positive answers for $\ell_1, c_0, C(K),\ L_1(\mu)$, and uniformly smooth spaces [70, 71, 72].

Using a renorming of c_0, R. Payá provided an example of a Banach space X such that the set of numerical radius attaining operators on X is not norm dense, answering in the negative Sims' question-see [75]. M. Acosta, F. Aguirre, and R. Payá in [61] gave another counterexample: $X = \ell_2 \oplus_\infty G$, where G is the Gowers space.

M. Acosta et al. studied in [60] a new property, called the Bishop-Phelps-Bollobás property for operators, BPBp for short. A pair of Banach spaces (X, Y) has the BPBp if a "Bishop-Phelps-Bollobás" type theorem can be proved for the set of operators from X to Y. This property implies, in particular, that the norm attaining operators from X to Y are dense in the whole space of continuous linear operators $\mathfrak{L}(X, Y)$. However, as shown in [60], the converse is not true. Consequently, the BPB property is more than a quantitative tool for studying the density of norm attaining operators.

We investigate here an analogue of the Bishop-Phelps-Bollobás property for operators but in relation with numerical radius attaining operators. We call it the Bishop-Phelps-Bollobás property for numerical radius, BPBp-ν for short. The relation between norm attaining and numerical radius attaining operators is far from being clear, although the existence of an interconnection is evident. Accordingly, We define this new property -see Definition (2.1.2) below- and to show that $\ell_1(\mathbb{C})$ and $c_0(\mathbb{C})$ satisfy it -see Theorem (2.1.9) and Theorem (2.1.12). This brings an extension as well as a quantitative version of C. S. Cardassi's results in [71].

Observe that the counterexamples provided in [61] and [75] imply, in particular, that there exist Banach spaces failing the Bishop-Phelps-Bollobás property for numerical radius.

Given a Banach space $(X, \|\cdot\|)$, we denote as usual by S_X and B_X, respectively, the unit sphere and the unit ball of X. By X^* we represent its dual, endowed with its standard norm $\|x^*\| = \sup_{x\in B_X}\{|x^*(x)|\}$ and by $\Pi(X)$ the set

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*}: x^*(x) = 1\}.$$

Given $x \in S_X$ and $x^* \in S_{X^*}$, we set

$$\pi_1(x^*) := \{x \in S_X: x^*(x) = 1\}.$$

By $\mathfrak{L}(X)$ we mean the Banach space of all linear and continuous operators from X into X endowed with its natural norm $\| T \| = \sup_{x\in B_X}\{\| Tx \|\}$. For a given $T \in \mathfrak{L}(X)$, its numerical radius $\nu(T)$ is defined by

$$\nu(T) = \text{s} \ \{|x^*(Tx)|: (x, x^*) \in \Pi(X)\}.$$

It is well known that the numerical radius of a Banach space X is a continuous seminorm on X which is, in fact, an equivalent norm when X is complex. In general, there exists a constant $n(X)$, called the numerical index of X, such that

$$n(X) \| T \| \leq \nu(T) \leq \| T \| \text{, for all } T \in \mathfrak{L}(X).$$

The interest is in spaces of numerical index $1, n(X) = 1$, where the norm and the numerical radius coincide. For background in numerical radius see [66, 67] and in numerical index see [73].

We say that $T \in \mathfrak{L}(X)$ attains its numerical radius if there exists $(x, x^*) \in \Pi(X)$ such that $|x^*(Tx)| = \nu(T)$. The set of numerical radius attaining operators will be denoted by NRA(X) $\subset$ $\mathfrak{L}$(X).

Definition (2.1.2)[59]: (BPBp-ν). A Banach space X is said to have the Bishop-PhelpsBollobás property for numerical radius if for every $0 < \varepsilon < 1$, there exists $\delta > 0$ such that for a given $T \in \mathfrak{L}(X)$ with $\nu(T) = 1$ and a pair $(x, x^*) \in \Pi(X)$ satisfying $|x^*(Tx)| \geq 1 - \delta$, there exist $S \in \mathfrak{L}(X)$ with $\nu(S) = 1$, and a pair $(y, y^*) \in \Pi(X)$ such that

$$\nu(T - S) \leq \varepsilon, \| x - y \| \leq \varepsilon, \|x^* - y^*\| \leq \varepsilon \text{ and } |y^*(Sy)| = 1. \quad (1)$$

Observe that if X is a Banach space with $n(X) = 1$, then the seminorm $\nu(\cdot)$ can be replaced by $\|\cdot\|$ in the definition above. Note that all the spaces studied have numerical index 1.

The $\arg(\cdot)$ stands for the function which sends a non zero complex number z to the unique $\arg(z) \in [0,2\pi)$ such that $z = |z|e^{\arg(z)i}$. For convenience we extend the function to $\mathbb{C}$ by writing $\arg(0) = 0$. Let $\text{Re}(z)$ and $\text{Im}(z)$ be, respectively, the real and imaginary part of the complex number $z \in \mathbb{C}$.

The spaces ℓ_1, ℓ_∞, and c_0 stand respectively for $\ell_1(\mathbb{C}), \ell_\infty(\mathbb{C})$, and $c_0(\mathbb{C})$. The standard basis of ℓ_1 is denoted by $\{e_n\}_{n\in\mathbb{N}}$, and its biorthogonal functionals by $\{e_n^*\}_{n\in\mathbb{N}}$. Given a sequence $\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^\mathbb{N}$ and a complex function $f: \mathbb{C} \to \mathbb{C}$ we write $f(\xi)$ meaning the sequence $\left(f(\xi_j)\right)_{j\in\mathbb{N}}$. The following sets will be of help in the formulation of the results and proofs. Given $x = (x_j)_{j\in\mathbb{N}} \in \ell_1, \varphi = (\varphi_j)_{j\in\mathbb{N}} \in \ell_\infty$ we define

$$\mathcal{N}_{(x,\varphi)} = \{j \in \mathbb{N}: \varphi_j x_j = |x_j|\}, \quad (2)$$

$$\operatorname{supp}(x) = \{j \in \mathbb{N} : |x_j| \neq 0\}.$$

For $r > 0$ we consider

$$\mathcal{A}_\varphi(r) \quad = \{j \in \mathbb{N} : |\varphi_j| \geq 1 - r\}, \tag{3}$$

$$\mathcal{P}_{(x,\varphi)}(r) \quad = \{j \in \operatorname{supp}(x) : \operatorname{Re}(\varphi_j x_j) \geq (1 - r)|x_j|\}. \tag{4}$$

Observe that $\mathcal{P}_{(x,\varphi)}(r) \subset \mathcal{A}_\varphi(r)$ and that if $x_j \geq 0$ for all $j \in \mathbb{N}$-we describe this situation saying that x is positive- then

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \operatorname{supp}(x) : \operatorname{Re}(\varphi_j) \geq (1 - r)\}.$$

For a given set Γ, a subset $A \subset \Gamma$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathbb{1}_A$ the characteristic function of A, that is, the element in $\mathbb{K}^\Gamma$ such that $(\mathbb{1}_A)_\gamma = 1$ if $\gamma \in A$ and $(\mathbb{1}_A)_\gamma = 0$ otherwise.

We present two constructive versions of Theorem (2.1.1), which are the main tools in the proofs of Theorem (2.1.9) and Theorem (2.1.15).

Lemma (2.1.3)[59]: Let $(x, \varphi) \in S_{\ell_1} \times S_{\ell_\infty}$. Then $x \in \pi_1(\varphi)$ if and only if $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$.

Proof. Given a pair $(x, \varphi) \in S_{\ell_1} \times S_{\ell_\infty}$ satisfying $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$, one can compute $\varphi(x) = \sum_{j\in\mathbb{N}} \varphi_j x_j \overset{(2)}{=} \sum_{j\in\mathbb{N}} |x_j| = \| x \| = 1$, which implies that $(x, \varphi) \in \Pi(\ell_1)$

Conversely, let us assume that $(x, \varphi) \in \Pi(\ell_1)$ then,

$$1 = \operatorname{Re}(\varphi(x)) = \sum_{j\in\mathbb{N}} \operatorname{Re}(\varphi_j x_j) \leq \sum_{j\in\mathbb{N}} |\varphi_j x_j| \leq \sum_{j\in\mathbb{N}} |x_j| = 1,$$

which implies that $\operatorname{Re}(\varphi_j x_j) = |\varphi_j x_j| = |x_j|$ for $j \in \mathbb{N}$. Therefore, $\varphi_j x_j = |x_j|$ for every $j \in \mathbb{N}$, which finishes the proof.

Lemma (2.1.3) provides the essential insight into the properties of $\Pi(\ell_1)$ that we need for the proofs of Theorem (2.1.6) and Theorem (2.1.8). A glance at Lemma (2.1.3) gives the following easy result regarding the norm attaining functionals on ℓ_1, $\operatorname{NA}(\ell_1)$.

Corollary (2.1.4)[59]: $\operatorname{NA}(\ell_1) = \{\varphi \in \ell_\infty : \exists \mathrm{n} \in \mathbb{N} \text{ with } |\varphi_\mathrm{n}| = \| \varphi \|\}$.

The following lemma is an adaptation of [60, Lemma 3.3].

Lemma (2.1.5)[59]: Let $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and $0 < \delta < 1$ such that $\operatorname{Re}(\varphi(x)) \geq 1 - \delta$. Then, for every $\delta < r < 1$ we have $\left\|\operatorname{Re}\left(e^{\arg(\varphi)i} x\right) \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1 - (\delta/r)$.

Proof. By assumption, we have that

$$\begin{aligned} 1 - \delta \leq \operatorname{Re}(\varphi(x)) &= \sum_{j\in\mathbb{N}} \operatorname{Re}(\varphi_j x_j) = \sum_{j\in\mathbb{N}} |\varphi_j| \operatorname{Re}\left(e^{\arg(\varphi_j)i} x_j\right. \\ &\leq \sum_{\mathcal{P}_{(x,\varphi)(r)}} \operatorname{Re}\left(e^{\arg(\varphi_j)i} x_j\right) + (1 - r) \sum_{\mathbb{N}\mathcal{P}_{(x,\varphi)}(r)} |x_j| \\ &\leq r \sum_{\mathcal{P}_{(x,\varphi)(r)}} \left|\operatorname{Re}\left(e^{\arg(\varphi_j)i} x_j\right)\right| + (1 - r), \end{aligned}$$

which implies that

$$\left\|\operatorname{Re}\left(e^{\arg(\varphi)i} x\right) \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| = \sum_{j\in\mathcal{P}_{(x,\varphi)}(r)} \left|\operatorname{Re}\left(e^{\arg(\varphi_j)i} x_j\right)\right| \geq 1 - (\delta/r),$$

as we wanted to show.

Observe that the previous lemma implies, in particular, that

$$\left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\right\| \geq 1-(\delta/r).$$

We present next the two constructive versions of the Bishop-Phelps-Bollobás theorem.

Theorem (2.1.6)[59]: Given $(x,\varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and $0<\varepsilon<1$ such that $\operatorname{Re}(\varphi(x)) \geq 1-\frac{\varepsilon^3}{4}$. Then, there exists $(x_0,\varphi_0) \in \Pi(\ell_1)$ such that $\|x-x_0\| \leq \varepsilon, \|\varphi-\varphi_0\| \leq \varepsilon$. Moreover, we can take

$$x_0:=\left\|x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}\right\|^{-1} \cdot x \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}. \tag{5}$$

Proof. Set $P:=\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)$-see (4). Applying Lemma (2.1.5) with $\delta=\varepsilon^2/2$ and $r=\varepsilon$ gives that

$$M:=\|x \cdot \mathbb{1}_P\| \geq 1-(\varepsilon/2). \tag{6}$$

Let us define

$$\varphi_0:=\varphi \cdot \mathbb{1}_{\mathbb{N}\setminus P}+e^{-\arg(x)i} \cdot \mathbb{1}_P \in S_{\ell\infty} \tag{7}$$

and

$$x_0:=M^{-1}x \cdot \mathbb{1}_P \in S_{\ell_1}. \tag{8}$$

On one hand, we can compute

$$\begin{aligned}\|x-x_0\| &\overset{(8)}{=} \|x-M^{-1}x \cdot \mathbb{1}_P\|=(M^{-1}-1)\|x \cdot \mathbb{1}_P\|+\left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \\ &\overset{(6)}{=} (1-M)+\left\|x \cdot \mathbb{1}_{\mathbb{N}\setminus P}\right\| \overset{\|x\|\leq 1}{\leq} 2-2M \overset{(6)}{\leq} \varepsilon,\end{aligned}$$

and, since the support of x_0 is included in P-this is a consequence of (8), we deduce that

$$\varphi_0(x_0)=\sum_{j\in P}(\varphi_0)_j(x_0)_j \overset{(7)}{=} \sum_{j\in P} e^{-\arg(x_j)i}(x_0)_j \overset{(8)}{=} \sum_{j\in P}\left|(x_0)_j\right|=\|x_0\|=1,$$

which is equivalently expressed as $(x_0,\varphi_0) \in \Pi(\ell_1)$.

On the other hand, using that

$$|z-1| \leq \sqrt{2(1-\operatorname{Re}(z))} \text{ for every } z \in \mathbb{C} \text{ such that } |z| \leq 1, \tag{9}$$

we deduce

$$\begin{aligned}\|\varphi-\varphi_0\| &\overset{(7)}{=} \sup_{j\in P}\{|\varphi_j-(\varphi_0)_j|\} \overset{(7)}{=} \sup_{j\in P}\{|\varphi_j-e^{-\arg(x_j)i}|\} \\ &= \sup_{j\in P}\{|e^{\arg(x_j)i}\varphi_j-1|\} \overset{(9)}{\leq} \sup_{j\in P}\left\{\sqrt{2-2\operatorname{Re}\left(e^{\arg(x_j)i}\varphi_j\right)}\right\} \\ &\leq \sqrt{2-2(1-\varepsilon^2/2)}=\varepsilon\end{aligned}$$

which finishes the proof.

An immediate consequence of Theorem (2.1.6) is the following version of the Bishop-Phelps-Bollobás theorem for $\ell_1(\mathbb{C})$.

Corollary (2.1.7)[59]: Let $0<\varepsilon<1$ and $(x,\varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ such that $|\varphi(x)| \geq 1-\frac{\varepsilon^3}{4}$. Then, there exists $(x_0,\varphi_0) \in S_{\ell_1} \times S_{\ell_\infty}$ such that $\|x-x_0\| \leq \varepsilon, \|\varphi-\varphi_0\| \leq \varepsilon$ and $|\varphi_0(x_0)|=1$.

Proof. Apply Theorem (2.1.6) to the pair $\left(e^{-\arg(\varphi(x))i}x,\varphi\right)$ obtaining (z_0,φ_0) belonging to $\Pi(\ell_1)$ such that $\|e^{-\arg(\varphi(x))i}x-z_0\| \leq \varepsilon$ and $\|\varphi-\varphi_0\| \leq \varepsilon$. Therefore, if we set $x_0:=e^{\arg(\varphi(x))i}z_0$, the pair (x_0,φ_0) satisfies the conclusions of the corollary.

Given a pair (x,φ) and $0<\varepsilon<1$, Theorem (2.1.6) ensures the existence of a pair (x_0,φ_0)-defined by (8) and (7)satisfying the conclusions of the Bishop-Phelps-Bollobás theorem. However, φ_0 depends on x, in fact, on $\arg(x)$. In order to prove Theorem (2.1.9) we will need a functional φ_0 depending only on the given ε and φ. So, we present the following result.

Theorem (2.1.8)[59]: Let $(x, \varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and $0 < \varepsilon < 1$ be such that $\mathrm{Re}(\varphi(x)) \geq 1 - \frac{\varepsilon^3}{60}$. Then there exists $(x_0, \varphi_0) \in \Pi(\ell_1)$ such that $\|x - x_0\| \leq \varepsilon, \|\varphi - \varphi_0\| \leq \varepsilon$. Moreover, the functional φ_0 can be defined as

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N} \setminus \mathcal{A}_\varphi(\varepsilon^2/20)} + e^{\arg(\varphi) i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/20)}. \tag{10}$$

Proof. Let us consider the isometry $S: \ell_1 \to \ell_1$ defined by

$$\langle e_j^*, Sy \rangle = e^{\arg(\varphi_j) i} y_j, \text{ for } y \in \ell_1 \text{ and } j \in \mathbb{N}. \tag{11}$$

Set $\tilde{x} = Sx$ and $\tilde{\varphi} = \varphi \circ S^{-1}$. Then, it is clear that the pair $(\tilde{x}, \tilde{\varphi})$ is in $B_{\ell_1} \times B_{\ell_\infty}$, that $\mathrm{Re}(\tilde{\varphi}(\tilde{x})) \geq 1 - \frac{\varepsilon^3}{60}$ and that $\tilde{\varphi} = \left(|\varphi_j|\right)_{j \in \mathbb{N}}$ is positive. Denote by A and P respectively the sets $\mathcal{A}_{\tilde{\varphi}}(r)$ and $\mathcal{P}_{(\tilde{x}, \tilde{\varphi})}(r)$-see definitions (3) and (4), where $r := \frac{\varepsilon^2}{20}$. Let us define

$$\hat{\varphi} := \tilde{\varphi} \cdot \mathbb{1}_{\mathbb{N} \setminus A} + \mathbb{1}_A \in S_{\ell_\infty} \tag{12}$$

and

$$\hat{x} := M^{-1} \mathrm{Re}(\tilde{x}) \cdot \mathbb{1}_P \in S_{\ell_1}, \tag{13}$$

where $M := \|\mathrm{Re}(\tilde{x}) \cdot \mathbb{1}_P\|$. Applying Lemma (2.1.5) with $\delta = \varepsilon^3/60$ and r, gives that $M \geq 1 - \frac{\varepsilon}{3}$. In particular, this means that P, and thus A, are non-empty.

We can compute that

$$\begin{aligned} \| \tilde{\varphi} - \hat{\varphi} \| &\overset{(12)}{=} \sup_{j \in A} \{|\tilde{\varphi}_j - \hat{\varphi}_j|\} \overset{(12)}{=} \sup_{j \in A} \{|\tilde{\varphi}_j - 1|\} \\ &= \sup_{j \in A} \{(1 - \tilde{\varphi}_j)\} \overset{(3)}{\leq} r \leq \varepsilon, \end{aligned} \tag{14}$$

and, since by (4) and (13) the support of $\hat{x}$ is $P \subset A$-which, in particular, implies that $\hat{x}_j > 0$ for $j \in P$, we deduce that

$$\hat{\varphi}(\hat{x}) = \sum_{j \in P} \hat{\varphi}_j \hat{x}_j \overset{(12)}{=} \sum_{j \in P} \hat{x}_j = \sum_{j \in P} |\hat{x}_j| = 1, \tag{15}$$

which is equivalently written as $(\hat{x}, \hat{\varphi}) \in \Pi(\ell_1)$.

In order to show that $\| \tilde{x} - \hat{x} \| \leq \varepsilon$, let us observe first that

$$\|\tilde{x} \cdot \mathbb{1}_P\| = \sum_{j \in P} |\tilde{x}_j| \geq \sum_{j \in P} |\mathrm{Re}(\tilde{x}_j)| = M \geq 1 - \frac{\varepsilon}{3}, \tag{16}$$

from which

$$\begin{aligned} \| \tilde{x} - \hat{x} \| &\overset{(13)}{=} \|\tilde{x} - M^{-1} \mathrm{Re}(\tilde{x}) \cdot \mathbb{1}_P\| = \|\tilde{x} \cdot \mathbb{1}_{\mathbb{N} \setminus P}\| + \|(\tilde{x} - M^{-1} \mathrm{Re}(\tilde{x})) \cdot \mathbb{1}_P\| \\ &\overset{(16)}{\leq} \frac{\varepsilon}{3} + \|(\tilde{x} - M^{-1} \mathrm{Re}(\tilde{x})) \cdot \mathbb{1}_P\|. \end{aligned} \tag{17}$$

We need a bit more care to estimate the last term in (17). From the very definition of P, we know that for every $j \in P$ it holds

$$|\tilde{x}_j| \leq (1 - r)^{-1} \tilde{\varphi}_j \mathrm{Re}(\tilde{x}_j). \tag{18}$$

Therefore,

$$\begin{aligned}\|(\tilde{x}-\mathrm{Re}(\tilde{x}))\cdot \mathbb{1}_P\| &= \sum_{j\in P}\left|\tilde{x}_j-\mathrm{Re}(\tilde{x}_j)\right| = \sum_{j\in P}\left|\mathrm{Im}(\tilde{x}_j)\right| \\ &= \sum_{j\in P}\sqrt{\left|\tilde{x}_j\right|^2-\mathrm{Re}(\tilde{x}_j)^2} \\ &\overset{(18)}{\leq} \sum_{j\in P}\left|\mathrm{Re}(\tilde{x}_j)\right|\sqrt{(1-r)^{-2}-1} \\ &\leq \|\tilde{x}\|\sqrt{(1-r)^{-2}-1} \overset{r=\frac{e^2}{20}}{\leq} \frac{\varepsilon}{3},\end{aligned} \tag{19}$$

which implies that

$$\begin{aligned}\|(\tilde{x}-M^{-1}\mathrm{Re}(\tilde{x}))\cdot \mathbb{1}_P\| &\leq \|(\tilde{x}-\mathrm{Re}(\tilde{x}))\cdot \mathbb{1}_P\| + \|(1-M^{-1})\mathrm{Re}(\tilde{x})\cdot \mathbb{1}_P\| \\ &\overset{(19)}{\leq} \frac{\varepsilon}{3}+(M^{-1}-1)\|\mathrm{Re}(\tilde{x})\cdot \mathbb{1}_P\| \\ &= \frac{\varepsilon}{3}+(1-M)\leq \frac{2\varepsilon}{3}.\end{aligned} \tag{20}$$

Putting together (17) and (20), one obtains

$$\|\tilde{x}-\hat{x}\| \leq \frac{\varepsilon}{3}+\|(\tilde{x}-M^{-1}\mathrm{Re}(\tilde{x}))\cdot \mathbb{1}_P\| \leq \varepsilon, \tag{21}$$

which finishes the core of the proof.

Now, we define

$$x_0 := S^{-1}\hat{x} \text{ and } \varphi_0 = S^*(\hat{\varphi}) = \hat{\varphi}\circ S, \tag{22}$$

which by (15) gives that $\varphi_0(x_0)=\hat{\varphi}(\hat{x})=1$. Since S and S^* are isometries, we deduce from (14), (21), (22) and the definition of $\tilde{x}$ and $\tilde{\varphi}$ that

$$\|x-x_0\|\leq \varepsilon, \|\varphi-\varphi_0\|\leq\varepsilon.$$

Therefore, (x_0,φ_0) is the pair in $\Pi(\ell_1)$ we were looking for.
Bearing in mind (22), one computes

$$(\varphi_0)_j=\varphi_0(e_j)\overset{(22)}{=}\hat{\varphi}(Se_j)\overset{(11)}{=}\hat{\varphi}\left(e^{\arg(\varphi_j)i}e_j\right)=e^{\arg(\varphi_j)i}\hat{\varphi}_j,$$

which together with (12) implies that $\varphi_0=\varphi\cdot\mathbb{1}_{\mathbb{N}\setminus A}+e^{\arg(\varphi)i}\cdot\mathbb{1}_A$. Finally, noting that $A=\mathcal{A}_{\tilde{\varphi}}(r)=\mathcal{A}_{\varphi}(r)$, the validity of (10) has been shown.

As a consequence of Theorem (2.1.6) and Theorem (2.1.8) we show that ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.

Theorem (2.1.9)[59]: Let $T\in S_{\mathfrak{C}(\ell_1)}$, $0<\varepsilon<1$ and $(x,\varphi)\in\Pi(\ell_1)$ such that $\varphi(Tx)\geq 1-(\varepsilon/9)^{9/2}$. Then there exist $T_0\in S_{\mathfrak{C}(\ell_1)}$ and $(x_0,\varphi_0)\in\Pi(\ell_1)$ such that

$$\|T-T_0\|\leq\varepsilon, \|x-x_0\|\leq\varepsilon, \|\varphi-\varphi_0\|\leq\varepsilon \text{ and } \varphi_0(T_0x_0)=1. \tag{23}$$

Proof. First of all, fix $\mu:=\sqrt{\varepsilon^3/240}$. Using a suitable isometry, we can assume that x is positive. In particular, by Lemma (2.1.3) and the definition of $\mathcal{N}_{x,\varphi}$ in (2), we can assume that $\varphi_j=1$ for $j\in\mathrm{supp}(x)$. Since $\mu^3/4\geq(\varepsilon/9)^{9/2}$, Theorem (2.1.6) can be applied to the pair $(x,T^*\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ and μ instead of ε giving $x_0\in\pi_1(\varphi)$ such that $\|x-x_0\|\leq\mu\leq\varepsilon$. Moreover, by (5) we know that

$$x_0=\|x\cdot\mathbb{1}_P\|^{-1}\cdot x\cdot\mathbb{1}_P, \tag{24}$$

where the non-empty set P is defined by

$$P := \mathcal{P}_{(x,T^*\varphi)}(\mu^2/2) = \left\{j \in \operatorname{supp}(x) : \operatorname{Re}\left(T^*\varphi(e_j)\right) \geq 1 - \mu^2/2\right\}. \quad (25)$$

In particular, x_0 is positive.

Since $\mu^2/2 = \frac{(\varepsilon/2)^3}{60}$, for each $j \in P$ we can apply Theorem (2.1.8) to the pair $\left(e^{-\arg(\varphi(Te_j))i}Te_j, \varphi\right)$ and $\varepsilon/2$ to find $(z_j, \varphi_0) \in \Pi(\ell_1)$ such that

$$\|Te_j - a_j z_j\| \leq \varepsilon/2,\ \|\varphi - \varphi_0\| \leq \varepsilon/2$$

and $\Pi_1(\varphi) \subset \Pi_1(\varphi_0)$, where $a_j = e^{\arg(\varphi(Te_j))i}$. Observe that φ_0 can be chosen independently on $j \in P$ and by (10) explicitly written as

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/80)} + e^{\arg(\varphi)i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/80)}. \quad (26)$$

Let us define T_0 as the unique operator in $\mathfrak{L}(\ell_1)$ such that $T_0 e_i = Te_i$ for $i \notin P$ and $T_0 e_j = z_j$ for $j \in P$. Equivalently,

$$T_0 x = \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j\in P} e_j^*(x) z_j, \text{ for } x \in \ell_1. \quad (27)$$

It is clear from (27) that

$$\|T_0\| = \sup_{n\in\mathbb{N}}\{\|T_0 e_n\|\} = \max\left\{\sup_{j\notin P}\{\|Te_j\|\}, \sup_{j\in P}\{\|z_j\|\}\right\} = 1.$$

Given $j \in P$, the identity (25) ensures that $\operatorname{Re}\left(\varphi(Te_j)\right) \geq 1 - \mu^2/2$. Using again the general fact (9), we deduce that $|a_j - 1| \leq \mu \leq \varepsilon/2$.

Therefore,

$$\begin{aligned}
\|T - T_0\| &= \sup_{n\in\mathbb{N}}\{\|Te_n - T_0 e_n\|\} = \sup_{j\in P}\{\|Te_j - z_j\|\} \\
&\leq \sup_{j\in P}\{\|Te_j - a_j z_j\|\} + \sup_{j\in P}\{\|a_j z_j - z_j\|\} \\
&\leq \frac{\varepsilon}{2} + \sup_{j\in P}\{|a_j - 1|\} \leq \varepsilon.
\end{aligned}$$

Since $x_0 \in \pi_1(\varphi)$ and $\pi_1(\varphi) \subset \pi_1(\varphi_0)$, we deduce that (x_0, φ_0) belongs to $\Pi(\ell_1)$. It remains to show that $\varphi_0(T_0 x_0) = 1$ to prove the validity of (23). But, since x_0 is positive, we obtain that

$$\begin{aligned}
\varphi_0(T_0 x_0) &\overset{(27)}{=} \sum_{j\in P} (x_0)_j \varphi_0(z_j) + \sum_{j\notin P} (x_0)_j \varphi_0(Te_j) \\
&\overset{(24)}{=} \sum_{j\in P} (x_0)_j = \sum_{j\in P} |(x_0)_j| = \|x_0\| = 1,
\end{aligned}$$

and the proof is over.

Corollary (2.1.10)[59]: The Banach space ℓ_1 has the Bishop-Phelps-Bollobás property for numerical radius.

Proof. Let us consider $T \in \mathfrak{L}(\ell_1)$ with $\nu(T) = 1$ and $0 < \varepsilon < 1$. Let us take a pair $(x, \varphi) \in \Pi(\ell_1)$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$. In fact, we can assume that $\varphi(Tx) \geq 1 - (\varepsilon/9)^{\frac{9}{2}}$; otherwise, we proceed with $\tilde{T} = e^{-\arg(\varphi(Tx))i}T$. Then Theorem (2.1.9) gives the existence of an operator $T_0 \in S_{\mathfrak{L}(\ell_1)}$ and a pair $(x_0, \varphi_0) \in \Pi(\ell_1)$ that satisfy conditions in (23), which are precisely the requirements (1) in Definition (2.1.2).

Corollary (2.1.11)[59]: ([71]). The set $\text{NRA}(\ell_1)$ is dense in $\mathfrak{L}(\ell_1)$.

Theorem (2.1.9) allows us to show that c_0 has the Bishop-Phelps-Bollobás property for numerical radius as well. Indeed, we rely on the fact that our constructions in ℓ_1 can be dualized.

Theorem (2.1.12)[59]: Let $T \in S_{\mathfrak{E}(c_0)}, 0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(c_0)$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $S \in S_{\mathcal{L}(c_0)}$ and $(x_0, \varphi_0) \in \Pi(c_0)$, such that

$$\| T - S \| \leq \varepsilon, \|x - x_0\| \leq \varepsilon, \|\varphi - \varphi_0\| \leq \varepsilon \text{ and } |\varphi_0(Sx_0)| = 1.$$

Throughout this proof we identify the elements in c_0 with their image in ℓ_∞ through the natural embedding $c_0 \to \ell_\infty$. The adjoint operator of $T, T^*: \ell_1 \to \ell_1$ satisfies

$$|x(T^*\varphi)| = |T^*(\varphi)(x)| = |\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}.$$

Without loss of generality, we can assume that $x(T^*\varphi) \geq 1 - (\varepsilon/9)^{9/2}$. Otherwise, employing techniques from the proof of Corollary (2.1.10), define the operator $\tilde{T} = e^{-\arg(x(T^*\varphi))i} T^*$ and proceed with the proof for $x(\tilde{T}\varphi) = |x(T^*\varphi)|$.

By Theorem (2.1.9), there exists $T_0 \in \mathfrak{L}(\ell_1), \|T_0\| = 1$ and $(\varphi_0, x_0) \in \Pi(\ell_1)$ such that

$$\|T^* - T_0\| \leq \varepsilon,\ \|\varphi - \varphi_0\| \leq \varepsilon,\ \|x - x_0\| \leq \varepsilon$$

and $x_0(T_0\varphi_0) = 1$.

We assert that (x_0, φ_0) is the pair we are looking for. To show this, we will reexamine the proof of Theorem (2.1.9) to establish how x_0, φ_0 and T_0 are defined. Indeed, from (25), (24), (26) and (27) we have respectively

$$\begin{aligned} P &= \mathcal{P}_{(\varphi, T^{**}x)}(\varepsilon^3/480), \\ \varphi_0 &= \|\varphi \cdot \mathbb{1}_P\|^{-1} \cdot \varphi \cdot \mathbb{1}_P, \\ x_0 &= x \cdot \mathbb{1}_{\mathbb{N}\setminus A_x(\varepsilon^2/80)} + e^{\arg(x)i} \cdot \mathbb{1}_{A_x(\varepsilon^2/80)}, \\ T_0 x &= \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j\in P} e_j^*(x) z_j, \text{ for } x \in \ell_1, \end{aligned} \tag{28}$$

where $\{z_j\}_{j\in P} \subset \pi_1(\varphi_0)$.

Note that $\mathcal{A}_x(\varepsilon^2/80) = \{j \in N: |x_j| \geq 1 - \varepsilon^2/80\}$ and that $x \in c_0$. Thus, $A_x(\varepsilon^2/80)$ is finite which, by (28), implies that $x_0 \in c_0$.

We shall show that T_0 is an adjoint operator and thus that there exists $S \in \mathfrak{L}(c_0)$ such that $S^* = T_0$. It will be enough to show that $T_0^*|_{c_0} \subset c_0$. Set $t_{ij} = \langle e_i, T(e_j)\rangle$ for $i, j \in \mathbb{N}$. Fix $i \in \mathbb{N}$, then for $j \in \mathbb{N}$

$$\langle e_j, T_0^*(e_i)\rangle = \begin{cases} t_{ji} & \text{if } j \notin P, \\ (z_j)_i & \text{if } j \in P. \end{cases}$$

Since $x \in c_0, T^{**}x$ belongs to c_0, which implies that P is finite. Accordingly, only finitely many terms of the form $\langle e_j, T_0^*(e_i)\rangle$ differ from the corresponding t_{ji}. On the other hand, since T belongs to $\mathfrak{L}(c_0)$, it holds that $\lim_j |t_{ji}| = 0$. Therefore, we deduce that $|\langle e_j, T_0^*(e_i)\rangle| \to 0$ when $j \to \infty$. This implies that $T_0^* e_i \in c_0$ and, since $i \in \mathbb{N}$ is arbitrarily chosen, we deduce that $T_0^*|_{c_0} \subset c_0$.

Hence we obtain the operator $S = T_0^*|_{c_0} \in \mathfrak{L}(c_0)$ and the pair $(x_0, \varphi_0) \in \Pi(c_0)$ satisfying:

$$\varphi_0(Sx_0) = S^*\varphi_0(x_0) = x_0(S^*\varphi_0) = x_0(T_0\varphi_0) = 1,$$
and
$$\| S - T \| = \|(S - T)^*\| = \|S^* - T^*\| = \|T_0 - T^*\| \leq \varepsilon,$$
which finishes the proof.

Theorem (2.1.12) implies the following two corollaries.

Corollary (2.1.13)[59]: The Banach space c_0 has the Bishop-Phelps-Bollobás property for numerical radius.

Corollary (2.1.14)[59]: ([71]). The set $\mathrm{NRA}(c_0)$ is dense in $\mathfrak{L}(c_0)$.

All the results that have been presented were stated and proved for the Banach spaces $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. However, a glance at their proofs suffices to convince oneself of their validity for $\ell_1(\mathbb{R})$ and $c_0(\mathbb{R})$-shorter proofs and better estimates can be obtained in this case. More generally, given a non-empty set Γ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, these results are, after suitable adjustments, still valid for $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$. The spaces $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$ are, respectively, the ℓ_1-sum and the c_0-sum of Γ copies of the field $\mathbb{K}$. Note that in particular $\ell_1(\mathbb{N}, \mathbb{K}) = \ell_1(\mathbb{K})$.

The Banach space $c_0(\Gamma, \mathbb{K})$ is a predual of $\ell_1(\Gamma, \mathbb{K})$. Observe that both spaces $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$ have numerical index 1 . Previous considerations imply that both of them also have the BPB property for numerical radius. The ω^* topology of $\ell_1(\Gamma, \mathbb{K})$ stands here for the topology induced on $\ell_1(\Gamma, \mathbb{K})$ by pointwise convergence on elements of $c_0(\Gamma, \mathbb{K})$.

On the other hand, the proof of Theorem (2.1.12) shows that in Theorem (2.1.9) we proved more than was stated. Indeed, putting together Theorem (2.1.9), the ideas on duality in the proof of Theorem (2.1.12) and considerations above, one easily proves the following theorem.

Theorem (2.1.15)[59]: Let $T \in S_{\mathfrak{C}(\ell_1(\Gamma,\mathbb{K}))}, 0 < \varepsilon < 1$ and $(x, \varphi) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ such that $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}$. Then there exist $T_0 \in S_{\mathfrak{C}(\ell_1(\Gamma,\mathbb{K}))}$ and $(x_0, \varphi_0) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ such that
$$\|T - T_0\| \leq \varepsilon, \|x - x_0\| \leq \varepsilon, \|\varphi - \varphi_0\| \leq \varepsilon \text{ and } |\varphi_0(T_0x_0)| = 1.$$
Moreover, if T is ω^* - ω^*-continuous and φ is ω^*-continuous, then T_0 and φ_0 will be $\omega^* - \omega^*$-continuous and ω^*-continuous, respectively.

Below are two consequences of Theorem (2.1.15).

Theorem (2.1.16)[59]: The Banach space $\ell_1(\Gamma, \mathbb{K})$ has the BPB property for numerical radius.

Theorem (2.1.17)[59]: The Banach space $c_0(\Gamma, \mathbb{K})$ has the BPB property for numerical radius.

Proof. Fix $0 < \varepsilon < 1, \delta \leq (\varepsilon/9)^{9/2}, T \in S_{\mathfrak{C}(c_0(\Gamma,\mathbb{K}))}$ and $(x, x^*) \in \Pi(c_0(\Gamma, \mathbb{K}))$ such that $|x^*(Tx)| \geq 1 - \delta$. Applying Theorem (2.1.15) to the ω^* - ω^*-continuous operator $T^* \in S_{\mathfrak{C}(\ell_1(\Gamma,\mathbb{K}))}$, the pair (x^*, x) and ε, gives a new $T_0 \in S_{\mathcal{L}(c_0(\Gamma,\mathbb{K}))}$ and a new pair $(x_0^*, x_0^{**}) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ satisfying
$$\|T^* - T_0^*\| \leq \varepsilon, \|x - x_0^{**}\| \leq \varepsilon, \|x^* - x_0^*\| \leq \varepsilon \text{ and } |x_0^{**}(T_0^*x_0^*)| = 1. \tag{29}$$
Moreover, x_0^{**} is ω^*-continuous, so we can identify it with some $x_0 \in S_{c_0(\Gamma,K)}$. Therefore, conditions in (29) become
$$\|T - T_0\| \leq \varepsilon, \|x - x_0\| \leq \varepsilon, \|x^* - x_0^*\| \leq \varepsilon \text{ and } |x_0^*(T_0x_0)| = 1.$$
which are the requirements (1) in Definition (2.1.2). Consequently, $c_0(\Gamma, \mathbb{K})$ has the Bishop-Phelps-Bollobás property for numerical radius.

Section (2.2): Numerical Radius on L_1:

The Bishop-Phelps theorem states that for every Banach space the set of linear and continuous functionals that attain their norm is norm-dense in its dual space. In 1970, B.

Bollobás, motivated by problems related to numerical radius, made a refinement of the Bishop-Phelps theorem, giving a quantitative version of this result as follows:

Theorem (2.2.1)[77]: (See Bollobás [80].) Given $\frac{1}{2} > \epsilon > 0$, if $x \in X$ and $x^* \in X^*$ with $\|x\| = \|x^*\| = 1$ are such that

$$|1 - x^*(x)| < \frac{\epsilon^2}{2}$$

then there are $y \in X$ and $y^* \in X^*$ such that

$$\| y \| = \|y^*\| = y^*(y) = 1, \ \| y - x \| < \epsilon + \epsilon^2 \ \text{ and } \ \|x^* - y^*\| < \epsilon$$

Many extensions of Bishop-Phelps and Bollobás results to more general settings, such as linear operators, multilinear maps or polynomials between Banach spaces have been done (see e.g. [78,79,81,82]). In particular, the space of Lebesgue integrable functions over the real line has received much attention in connection with the extension of Bollobás results to more general cases.

Given a Banach space X, if we denote by $\Pi(X) := \{(x, x^*): x \in X, x^* \in X^*, \| x \| = \|x^*\| = x^*(x) = 1\}$, we can interpret the Bollobás theorem, asserting that any ordered pair that "almost belongs" to $\Pi(X)$ can be approximated in the product norm by elements of $\Pi(X)$. Given an operator $T \in \mathcal{L}(X)$, the numerical radius of T is defined by $\nu(T) = \{|f(T(x))|: (x, f) \in \Pi(X)\}$. Usually every pair of elements (x, f) is called a state. Notice that the numerical radius of a Banach space X is a continuous seminorm on X bounded by the natural norm on $\mathcal{L}(X)$. We say that the Banach space X has numerical index 1 if $\| T \| = \nu(T)$ for all operators T. We say that $T \in \mathcal{L}(X)$ attains its numerical radius if there exists $(x, f) \in \Pi(X)$ such that $|f(T(x))| = \nu(T)$.

In [83] Guirao and Kozhushkina study the Bishop-Phelps-Bollobás property for numerical radius described in Definition (2.2.4) below, having as its main point of interest the natural extension of Bollobás result to the numerical radius on Banach spaces of numerical index 1. The main result is Theorem (2.2.8) where we prove that the Banach space of Lebesgue real-valued integrable functions over the real line, that we will denote by L_1, has the Bishop-Phelps-Bollobás property for numerical radius.

From now on, X will denote a Banach space and X^*, B_X and S_X the strong dual, closed unit ball and unit sphere of X respectively. If X and Y are Banach spaces we will denote by $\mathcal{L}(X, Y)$ the space of all linear and continuous operators from X into Y endowed with its natural norm $\| T \| = \sup_{x \in B}\{\| T(x) \|\}$ and in the particular case of $Y = X$ we will write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

We recall that in 2008, M. Acosta et al. introduced the following property generalizing the Bollobás theorem to operators between Banach spaces, called the Bishop-Phelps-Bollobás property for operators, $BPBp$ for short.

Definition (2.2.2)[77]: ($BPBp$). (See [78].) Let X and Y be real or complex Banach spaces. We say that the pair (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators (or that the Bishop-Phelps-Bollobás theorem holds for all bounded operators from X to Y) if given $\epsilon > 0$, there are $\delta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{t \to 0} \beta(t) = 0$ such that for all $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ with $\| T(x) \| > 1 - \delta(\epsilon)$, then there exist a point $y \in S_X$ and an operator $G \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\| G(y) \| = 1, \ \| y - x \| < \beta(\epsilon) \ \text{ and } \ \| G - T \| < \epsilon$$

[78] showed that a necessary and sufficient condition on Y for the pair (l_1, Y) to satisfy the $BPBp$ is for Y to have the $(AHSP)$.

Definition (2.2.3)[77]: $(AHSP)$. (See [78].) A real Banach space X is said to have the $AHSP$ if for every $\epsilon > 0$ there exists $0 < \gamma < \epsilon$ such that for every sequence $(x_k) \subseteq S_X$ and for every convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \gamma$$

there exist a subset $A \subseteq \mathbb{N}$, a subset $\{y_k : k \in A\} \subseteq S_X$, and a certain $g \in S_{X^*}$ satisfying:

- $\sum_{k \in A} \alpha_k > 1 - \epsilon$
- $\|x_k - y_k\| < \epsilon$ for all $k \in A$.
- $g(y_k) = 1$ for all $k \in A$.

In 2012, Aron et al. showed in [79] that an extension of the Bishop-Phelps-Bollobás theorem holds for all bounded linear operators from $L_1(\mu)$ into $L_\infty[0,1]$, where μ is a σ-finite measure. The same year Choi and Kim [81], motivated by the characterization of the $BPBp$ in terms of the $AHSP$ for ℓ_1, tried to extend this characterization to the space L_1. They showed that if the pair $(L_1(\mu), Y)$ has the $BPBp$ then Y has the $AHSP$, and if Y has the Radon-Nikodým property then the $AHSP$ is also a sufficient condition. However the $AHSP$ on Y is not a sufficient condition for the pair $(L_1(\mu), Y)$ to have the $BPBp$, as Schachermayer showed using the space of continuous functions on the interval $[0,1]$ [84], and the Radon-Nikodým property for Y is not always necessary as can be shown by using the space of L_∞ and the result of Aron et al. [79].

Guirao and Kozhushkina [83] started a new line of research, focusing on approximating operators and states using the numerical radius of the operator instead of the operator norm.

Definition (2.2.4)[77]: $(BPBp\text{-}\nu)$. (See [83].) A Banach space X is said to have the Bishop-Phelps-Bollobás property for numerical radius, $BPBp - \nu$ for short, if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that for all $T \in \mathcal{L}(X)$ of norm one, if $(x, x^*) \in \Pi(X)$ is such that $|x^*(T(x))| > 1 - \delta(\epsilon)$, then there exist $G \in \mathcal{L}(X)$, with $\nu(G) = 1$ and a pair $(y, y^*) \in \Pi(X)$ such that

$$\| T - G \| \leqslant \epsilon, \quad \| x - y \| \leqslant \epsilon, \|x^* - y^*\| \leqslant \epsilon \text{ and } |y^*(G(y))| = 1$$

From now on we will focus on the case where Y is the space L_1. In [78], Acosta et al. proved that L_1 has the $AHSP$. However, we know that in general L_1 does not have the Radon-Nikodým property, and so we cannot apply the techniques of Choi and Kim to obtain that the pair (L_1, L_1) has the $BPBp$. However in [82], Choi et al. have proved that the pair (L_1, L_1) has the $BPBp$. An alternative proof of this result for $L_1(\mathbb{R})$ can be done by modifying the proof presented in Theorem (2.2.8).

Even though Choi et al. have shown that the pair (L_1, L_1) has the $BPBp$, there is no known relation between the pair (X, X) having the $BPBp$ and the space X having the $BPBp$ -ν. Our focus from now on will be to prove that the Banach space of Lebesgue integrable functions over the real line has the Bishop-Phelps-Bollobás property for numerical radius.

For the proof of our main Theorem (2.2.8), we need some necessary technical lemmas. The proofs of Lemma (2.2.5) and Lemma (2.2.6) are omitted.

Lemma (2.2.5)[77]: Let $A \subseteq \mathbb{R}$ be a measurable set. The operator R from $\mathcal{L}(L_1)$ to $\mathcal{L}(L_1)$ defined, for every operator $T \in \mathcal{L}(L_1)$, by $R(T)(f) = T(f\chi_A - f\chi_{A^c})\chi_A - T(f\chi_A - f\chi_{A^c})\chi_{A^c}$ is an isometry, i.e. $\| T \| = \| R(T) \|$ for all operators $T \in \mathcal{L}(L_1)$.

Also for every point $x \in L_1$ and every linear form $f \in L_\infty$, if we denote by $r(f) = f\chi_A - f\chi_{A^c}$ and by $r(x) = x\chi_A - x\chi_{A^c}$, then

$$\langle r(x), r(f)\rangle = \langle x, f\rangle = \int_{\mathbb{R}} x(t)f(t)dt$$

and

$$\langle R(T)(r(x)), r(f)\rangle = \langle T(x), f\rangle = \int_{\mathbb{R}} (T(x)(t))f(t)dt$$

Lemma (2.2.6)[77]: Given a pair $(x, f) \in \Pi(L_1)$ with $f(t) \geqslant 0$ for all $t \in \mathbb{R}$, let $A = \{t \in \mathbb{R} : x(t) > 0\}$. Then $\mu(\{t \in \mathbb{R}: x(t) < 0\}) = 0$ and $\mu(\{t \in A: f(t) < 1\}) = 0$. Also, for every point $y \in L_1$ if $\{t \in \mathbb{R}: y(t) > 0\} \subseteq A$ and $\mu(\{t \in \mathbb{R}: y(t) < 0\}) = 0$, then $\langle y, f\rangle = \| y \|$.

Before presenting our main result we need to prove the last technical lemma that will be used to modify the operator in Theorem (2.2.8).

Lemma (2.2.7)[77]: Given two measurable sets I and S and an operator $T \in \mathcal{L}(L_1)$, for any finite number of pairwise disjoint measurable sets $I_1, \dots, I_j$ of finite measure, with $I = \bigcup_{i=1}^{j} I_i$

$$\left(T(\chi_I)\chi_{\{t\in\mathbb{R}:T(\chi_I)(t)>0\}\cap S}\right)(t) \leqslant \sum_{i=1}^{j} \left(T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}\right)(t)$$

almost everywhere. Also $\left\|\sum_{i=1}^{j} T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}\right\| \leqslant \| T \| \, \|\chi_I\|$

Proof. Since $T(\chi_I) = \sum_{i=1}^{j} T(\chi_{I_i})$ there is no loss of generality in assuming that equality also holds for the measurable functions after taking representatives of the equivalence class, i.e. we assume that $T(\chi_I)(t) = \sum_{i=1}^{j} T(\chi_{I_i})(t)$ for all real number t in S.

Notice that $\left(T(\chi_{I_i})\chi_S\right)(t) \leqslant T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}(t)$ for $i = 1, \dots, j$. Hence, if $T(\chi_I)(t) \leqslant 0$, by the linearity of T the required inequality holds and if $T(\chi_I)(t) > 0$,

$$\begin{aligned} T(\chi_I)\chi_{\{t\in\mathbb{R}:T(\chi_I)(t)>0\}\cap S}(t) &= \sum_{i=1}^{j} \left(T(\chi_{I_i})\chi_S\right)(t) \\ &\leqslant \sum_{i=1}^{j} T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}(t) \end{aligned}$$

To see that $\left\|\sum_{i=1}^{j} T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}\right\| \leqslant \| T \| \, \|\chi_I\|$ we only have to notice that

$$\begin{aligned} &\left\|\sum_{i=1}^{j} T(\chi_{I_i})\chi_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S}\right\| \\ &= \sum_{i=1}^{j} \int_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S} T(\chi_{I_i})(t)dt \\ &\leqslant \sum_{i=1}^{j} \left(\int_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)>0\right\}\cap S} T(\chi_{I_i})(t)dt + \int_{\left\{t\in\mathbb{R}:T\left(\chi_{I_i}\right)(t)<0\right\}\cap S} \left|T(\chi_{I_i})(t)\right|dt\right) \end{aligned}$$

$$= \sum_{i=1}^{j} \|T(\chi_{I_i})\chi_S\| \leqslant \sum_{i=1}^{j} \|T(\chi_{I_i})\|$$

$$\leqslant \sum_{i=1}^{j} \| T \| \|\chi_{I_i}\| = \| T \| \|\chi_I\|.$$

We can now prove the main result.

Theorem (2.2.8)[77]: The Banach space L_1 satisfies the $BPBp-\nu$.

Proof. Consider $x \in B_{L_1}, f \in B_{L_1^*} = B_{L_\infty}$ with $\| x \| = \| f \| = 1$ and $\langle x, f \rangle = 1$. Consider also an operator $T \in \mathcal{L}(L_1), \| T \| = 1$ and assume

$$\langle T(x), f \rangle > 1 - \frac{\delta}{2} \tag{30}$$

with $0 < \delta < 1/4$

From now on, we fix a representative of the equivalence classes of $x \in L_1$ and $f \in L_\infty$. That is, two measurable functions in the equivalence classes and we denote the representatives in the same way as x and f

First we do the proof assuming $f(t) \geqslant 0$ for all real number t.

Since the norm of f is $\| f \| = \text{supess}_{t\in\mathbb{R}} |f(t)|$ we can assume that $0 \leqslant f(t) \leqslant 1$ for all real number t. Denote by

$$B = \{t \in \mathbb{R}: f(t) \geqslant 1 - \delta^{1/4}\}. \tag{31}$$

Consider the measurable set $S := \{t \in \mathbb{R}: x(t) > 0\}$. There exists a function $z \in L_1$ of norm one with $\| x - z \| \leqslant \delta/2$, where $z = \sum_{i=1}^{N} \alpha_i \chi_{D_i \cap S}$ with D_i being mutually disjoint dyadic segments with $\mu(D_i \cap S) > 0$ and $\alpha_i > 0$ for $i = 1, \dots, N$. Therefore

$$\langle T(z), f \rangle \geqslant \langle T(x), f \rangle - \| x - z \| > 1 - \delta.$$

By Lemma (2.2.6) f attains its norm on z, i.e. $(z, f) \in \Pi(L_1)$. Denote by

$$D_0 = \bigcup_{i=1}^{N} D_i \cap S \text{ and } g = \chi_B + f\chi_{B^c}. \tag{32}$$

It is easy to see that

$$\| f - g \| \leqslant \delta^{1/4}. \tag{33}$$

Now, for every natural number $i = 1, \dots, N$ and for $n \in \mathbb{N}$, consider

$$F_{n,i} := \{I \in \Delta_n: \langle T(\chi_{I\cap D_i \cap S}), f \rangle \leqslant (1 - \sqrt{\delta})\|\chi_{I\cap D_i\cap S}\|\}. \tag{34}$$

Put $D^i = (D_i \cap S) \setminus \left(\bigcup_{n\in\mathbb{N}} \bigcup_{I\in F_{n,i}} I\right)$ which is measurable. Define $D = \bigcup_{i=1}^{N} D^i$ and since $\|z\chi_D\| > 0$ as we will see below, define

$$y = \frac{z\chi_D}{\|z\chi_D\|}. \tag{35}$$

Then y has norm one and since $\{t \in \mathbb{R}: y(t) > 0\} \subseteq D_0$ and $\{t \in \mathbb{R} : y(t) < 0\} = \emptyset$. By Lemma (2.2.6) considering the set D_0 and the state $(z, f) \in \Pi(L_1)$ we obtain that $(y, f) \in \Pi(L_1)$. Also, by Lemma (2.2.6), $\mu(\{t \in D_0: f(t) < 1\}) = 0$ so we have $f(t) = 1$ for almost all t in D_0, hence $f(t) = 1$ for almost all t in D. Therefore $1 = \langle y, f \rangle = \int_{\mathbb{R}} y(t)f(t)dt = \int_D y(t)f(t)dt = \int_D y(t) = \int_D y(t)g(t)dt = \int_{\mathbb{R}} y(t)g(t)dt = \langle y, g \rangle$ hence $(y, g) \in \Pi(L_1)$

Let's see that z and y are close and $\|z\chi_D\| > 0$. By the construction of D, $D_0 \setminus D$ is a union of a sequence of disjoint measurable sets $\{I_k\}_{k=1}^{\infty}$ where I_k is of the form $R \cap D_i \cap S$ with R a dyadic set in some Δ_n and D_i one of the sets that appear in the definition of z. By (34) the set I_k is such that $\langle T(\chi_{I_k}), f\rangle \leqslant (1-\sqrt{\delta})\|\chi_{I_k}\|$. Hence using the Monotone Convergence Theorem we obtain

$$\begin{aligned}
1-\delta \quad &< \langle T(z), f\rangle = \left\langle T\left(\sum_{i=1}^{N} \alpha_i \sum_{k=1}^{\infty} \chi_{I_k} + z\chi_D\right), f\right\rangle \\
&= \sum_{i=1}^{N} \alpha_i \sum_{k=1}^{\infty} \langle T(\chi_{I_k}), f\rangle + \langle T(z\chi_D), f\rangle \ \text{(by linearity of } T \text{ and } f\text{)} \\
&\leqslant (1-\sqrt{\delta}) \sum_{i=1}^{N} \alpha_i \sum_{k=1}^{\infty} \|\chi_{I_k}\| + \langle T(z\chi_D), f\rangle \\
&= (1-\sqrt{\delta})\|z\chi_{D_0\setminus D}\| + \langle T(z\chi_D), f\rangle \ \text{(by definition of } z\text{)} \\
&\leqslant (1-\sqrt{\delta})\|z\chi_{D_0\setminus D}\| + \|z\chi_D\| \ \text{(because } T \text{ and } f \text{ are of norm one)} \\
&= 1-\sqrt{\delta}\|z\chi_{D_0\setminus D}\|
\end{aligned}$$

Therefore $\|z\chi_{D_0\setminus D}\| < \sqrt{\delta}$ and $\|z_D\| = \| z \| - \|z\chi_{D_0\setminus D}\| > 1-\sqrt{\delta} > 0$ as required. Also,

$$\begin{aligned}
\| z - y \| \quad &= \left\| z - \frac{z\chi_D}{\|z\chi_D\|} \right\| \\
&\leqslant \|z - z\chi_D\| + \left\| z\chi_D - \frac{z\chi_D}{\|z\chi_D\|} \right\| \\
&= \|z - z\chi_D\| + 1 - \|z\chi_D\| \\
&< 2\sqrt{\delta}
\end{aligned}$$

Hence $\| x - y \| < \delta + 2\sqrt{\delta}$

Now we modify the operator T on the set D.

For simplicity denote by $W_I = \{t \in \mathbb{R}: T(\chi_{I\cap D})(t) > 0\}$ for every dyadic set I in Δ_n for some natural number n. For each such dyadic set I in Δ_n, define the sequence of integrable functions $\{h_k^I\}_k$ by

$$h_k^I = \sum_{i=1}^{2^{k-1}} T(\chi_{I_i\cap D})\chi_{W_{I_i}\cap B} \tag{36}$$

where the sets $\{I_1, \dots, I_{2^{k-1}}\}$ are the disjoint dyadic sets of Δ_{n+k-1} whose union is I.

Then by Lemma (2.2.7) we have a sequence of positive increasing functions almost everywhere, so as a consequence of the Monotone Convergence Theorem the sequence of integrable functions $\{h_k^I\}_{k=1}^{\infty}$ converges to an integrable function h^I. Notice that for every natural number k, $\|h_k^I\| \leqslant \|\chi_I\|$ hence $\|h^I\| \leqslant \|\chi_I\|$, and $\|h_k^I\| = \int_{\mathbb{R}} h_k^I(t)dt = \int_B h_k^I(t)dt = \int_B h_k^I(t)g(t)dt = \langle h_k^I, g\rangle$. Therefore

$$\|h_k^I\| = \langle h_k^I, g\rangle \ \text{and} \ \|h^I\| = \langle h^I, g\rangle. \tag{37}$$

Also, since the sequence of functions $\{h_k^I\}_{k=1}^{\infty}$ is positive and increasing almost everywhere, by (37) we have, for all $k \in \mathbb{N}$,

$$\begin{aligned}\|h^I - h_k^I\| &= \int_{\mathbb{R}} \left|(h^I - h_k^I)(t)\right| dt \\ &= \int_{\mathbb{R}} (h^I - h_k^I)(t) dt \\ &= \int_{\mathbb{R}} h^I(t) dt - \int_{\mathbb{R}} h_k^I(t) dt \\ &= \int_{\mathbb{R}} |h^I(t)| dt - \int_{\mathbb{R}} \left|h_k^I(t)\right| dt \\ &= \|h^I\| - \|h_k^I\| \\ &= \langle h^I, g\rangle - \langle h_k^I, g\rangle \\ &= \langle h^I - h_k^I, g\rangle\end{aligned}$$

Hence

$$\|h^I - h_k^I\| = \langle h^I - h_k^I, g\rangle. \tag{38}$$

By (34) we have that if $\mu(I \cap D) > 0$ then $\langle T(\chi_{I\cap D}), f\rangle > (1-\sqrt{\delta})\|\chi_{I\cap D}\|$, but

$$\begin{aligned}\langle T(\chi_{I\cap D}), f\rangle &= \int_{\mathbb{R}} T(\chi_{I\cap D})(t) f(t) dt \leqslant \int_{W_I} T(\chi_{I\cap D})(t) f(t) dt \\ &= \langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle\end{aligned}$$

So

$$\langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle \geqslant (1-\sqrt{\delta})\|\chi_{I\cap D}\|. \tag{39}$$

Therefore

$$\begin{aligned}(1-\sqrt{\delta})\|\chi_{I\cap D}\| &\leqslant \langle T(\chi_{I\cap D})\chi_{W_I}, f\rangle \\ &= \int_{W_I\cap B} T(\chi_{I\cap D})(t) f(t) dt + \int_{W_I\cap B^c} T(\chi_{I\cap D})(t) f(t) dt \\ &\leqslant \int_{W_I\cap B} T(\chi_{I\cap D})(t) dt + \int_{(W_I\cap B)^c} \left(1-\delta^{1/4}\right)|T(\chi_{I\cap D})(t)| dt \\ &= \|T(\chi_{I\cap D})\chi_{W_I\cap B}\| + \left(1-\delta^{1/4}\right)\|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\| \\ &= \|T(\chi_{I\cap D})\| - \delta^{1/4}\|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\| \\ &\leqslant \|\chi_{I\cap D}\| - \delta^{1/4}\|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\|\end{aligned}$$

Therefore $\delta^{1/4}\|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\| \leqslant \sqrt{\delta}\|\chi_{I\cap D}\|$, and for any dyadic interval I

$$\|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\| \leqslant \delta^{1/4}\|\chi_{I\cap D}\|. \tag{40}$$

Define now the operator G on the simple functions whose measurable sets are dyadic as follows:

$$G\left(\sum_{i=1}^{j} \beta_i \chi_{I_i}\right) = \sum_{i=1}^{j} \beta_i\left(T\left(\chi_{I_i\cap D^c}\right) + h^{I_i} + \left(\|\chi_{I_i\cap D}\| - \|h^{I_i}\|\right)y\right). \tag{41}$$

Notice that $\sum_{i=1}^{j}\beta_i T\left(\chi_{I_i\cap D^c}\right)$ is well defined because T is linear. Also, for every dyadic set $I \in \Delta_n$, if R, Q are two disjoint dyadic sets in Δ_{n+1} whose union is I, by the construction of the sequences $\{h_k^I\}_{k=1}^{\infty}, \{h_k^R\}_{k=1}^{\infty}$ and $\{h_k^Q\}_{k=1}^{\infty}$ we have for all $k > 1, h_k^I = h_{k-1}^R + h_{k-1}^Q$. Hence $h^I = h^R + h^Q$ and since $\|h^I\| = \langle h^I, g\rangle$, $\|h^R\| = \langle h^R, g\rangle$ and $\|h^Q\| = \langle h^Q, g\rangle$

$$\begin{aligned}\|\chi_{I\cap D}\| - \|h^I\| &= \big(\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|\big) - \langle h^I, g\rangle \text{ (by(37))}\\ &= \big(\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|\big) - \langle h^R + h^Q, g\rangle\\ &= \big(\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|\big) - (\langle h^R, g\rangle + \langle h^Q, g\rangle)\\ &= \big(\|\chi_{R\cap D}\| + \|\chi_{Q\cap D}\|\big) - (\|h^R\| + \|h^Q\|) \text{ (by(37))}\\ &= (\|\chi_{R\cap D}\| - \|h^R\|) + \big(\|\chi_{Q\cap D}\| - \|h^Q\|\big)\end{aligned}$$

By using induction, $\sum_{i=1}^{N}\beta_i\big(h^{I_i} + \big(\|\chi_{I_i\cap D}\| - \|h^{I_i}\|\big)y\big)$ is well defined. Hence G is well defined and linear. To finish, by density of the simple functions whose measurable sets are dyadic we can extend the operator G to L_1

Now let's compute the norm of G and the distance between T and G. For this, it enough to compute the norm and the distance over dyadic sets I

$$\begin{aligned}\|G(\chi_I)\| &\leqslant \|T(\chi_{I\cap D^c})\| + \|h^I\| + \|(\|\chi_{I\cap D}\| - \|h^I\|)y\|\\ &\leqslant \|\chi_{I\cap D^c}\| + \|\chi_{I\cap D}\| = \|\chi_I\|\end{aligned}$$

Therefore $\| G \| \leqslant 1$. On the other hand, it is easy to check that $\langle G(f), g\rangle = \langle T(f\chi_{D^c}), g\rangle + \int_D f(t)dt$. Since for any dyadic set I with $\mu(I \cap D) > 0$, we have $\langle G(\chi_{I\cap D}), g\rangle = \|\chi_{I\cap D}\|, G$ has norm one.

Also,

$$\begin{aligned}&\|T(\chi_I) - G(\chi_I)\| \leqslant \|T(\chi_{I\cap D}) - h^I - (\|\chi_{I\cap D}\| - \|h^I\|)y\|\\ &\leqslant \|T(\chi_{I\cap D}) - h_1^I\| + \|h^I - h_1^I\| + (\|\chi_{I\cap D}\| - \|h^I\|)\\ &\leqslant \delta^{1/4}\|\chi_{I\cap D}\| + \|h^I - h_1^I\| + (\|\chi_{I\cap D}\| - \|h^I\|) \text{ (by (40))}\\ &= \delta^{1/4}\|\chi_{I\cap D}\| + \langle h^I - h_1^I, g\rangle + (\|\chi_{I\cap D}\| - \|h^I\|) \text{ (by (38))}\\ &= \delta^{1/4}\|\chi_{I\cap D}\| + \langle h^I, g\rangle - \langle h_1^I, g\rangle + (\|\chi_{I\cap D}\| - \|h^I\|)\\ &= \delta^{1/4}\|\chi_{I\cap D}\| + \|h^I\| - \|h_1^I\| + \|\chi_{I\cap D}\| - \|h^I\| \text{ (by (37))}\\ &= \delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|h_1^I\|\\ &= \delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|T(\chi_{I\cap D})\| + \|T(\chi_{I\cap D})\chi_{(W_I\cap B)^c}\|\\ &\leqslant 2\delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \|T(\chi_{I\cap D})\|\\ &\leqslant 2\delta^{1/4}\|\chi_{I\cap D}\| + \|\chi_{I\cap D}\| - \langle T(\chi_{I\cap D}), f\rangle\\ &\leqslant 2\delta^{1/4}\|\chi_{I\cap D}\| + \sqrt{\delta}\|\chi_{I\cap D}\| \text{ (by (34))}\end{aligned}$$

so $\| T - G \| \leqslant 3\delta^{1/4}$

To conclude it is easy to check that $\langle G(y), g\rangle = \int_D y(t)dt = 1$, which proves the theorem in the case $f(t) \geqslant 0$ for all real number t.

For the general case, consider the measurable set $\langle G(y), g\rangle = \{t \in \mathbb{R} : f(t) < 0\}$. Then by Lemma (2.2.5) applied to the set A, we have $r(f)(t) \geqslant 0$ for all real number t, and $r(f), r(x)$ and $R(T)$ satisfy the conditions of $\langle r(x), r(f)\rangle \in \Pi(L_1), \| R(T) \| = 1$ and $\langle R(T)(r(x)), r(f)\rangle > 1 - \delta/2$. By the previous case we can find $(y, g) \in \Pi(L_1)$ and $G \in \mathcal{L}(L_1)$ such that $\| y - r(x) \| \leqslant \delta + 2\sqrt{\delta}, \| g - r(f) \| \leqslant \delta^{1/4}, \| G - R(T) \| \leqslant 3\delta^{1/4}$ and $\| G \| = \langle G(y), g\rangle = 1$. Now by Lemma (2.2.5) again we obtain that $(r(y), R(G)) \in \Pi(L_1)$ and $R(G) \in \mathcal{L}(L_1)$ are such that $\| r(y)-x \| \leqslant \delta+2\sqrt{\delta}, \| R(G)-f \| \leqslant \delta^{1/4}, \| R(G)-T \| \leqslant 3\delta^{1/4}$ and $\| R(G) \| = \langle R(G)(r(y)), R(G)\rangle = 1$ which concludes the proof.

The results presented can be extended to $\mathcal{L}\big(L_1(\mathbb{R}^n)\big)$, by using the dyadic partitions of the space $\mathbb{R}^n$ on cubes defined by $\Delta_n := \left\{\prod_{i=1}^{n}\left[\frac{z_i}{2^n}, \frac{z_i+1}{2^n}\right) : z_i \in \mathbb{Z}, i = 1, \dots, n\right\}$. In general, for every finite dimensional real Banach space $\mathbb{R}^n$, the space $L_1(\mathbb{R}^n)$ has the Bishop-Phelps-Bollobás property for numerical radius. Clearly the same kind of argument proves that the space $L_1[0,1]$ and actually $L_1(R)$ for any n-interval $R = \prod_{i=1}^{n}[a_i, b_i]$ (of positive measure) in $\mathbb{R}^n$ has the $BPBp-\nu$.

Section (2.3): Operators On $C(K)$:

Bishop-Phelps Theorem states the denseness of the subset of norm attaining functionals in the (topological) dual of a Banach space [96]. Since the Bishop-Phelps Theorem was proved in the sixties, some interesting provided versions of this result for operators. Related to those results, it is worth to mention by Lindenstrauss [114], the somehow surprising result obtained by Bourgain [99] and also results for concrete classical Banach spaces. In full generality there is no parallel version of Bishop-Phelps Theorem for operators even if the domain space is c_0 [114]. Lindenstrauss also provided some results of denseness of the subset of norm attaining operators by assuming some isometric properties either on the domain or on the range space [114]. We mention here two concrete consequences of these results. If the domain space is ℓ_1 or the range space is c_0, every operator can be approximated by norm attaining operators. First Lindenstrauss [114] and later Bourgain [99] proved that certain isomorphic assumptions on the domain space (reflexivity or even Radon-Nikodým property, respectively) implies the denseness of the subset of norm attaining operators in the corresponding space of linear (bounded) operators. For classical Banach spaces (see [108],[106],[117],[118],[92]) and a few containing counterexamples (see [117],[109], [105],[90] and [86]).

Recently [87] dealt with "quantitative" versions of the Bishop-Phelps Theorem for operators. The motivating result is known nowadays as Bishop-Phelps-Bollobás Theorem [97], [98] and has been a very useful tool to study numerical ranges of operators (see [98]). This result can be stated as follows.

Let X be a Banach space and $0<\varepsilon<1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1-x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$such that $y^*(y)=1, \| y-x \| < \varepsilon$ and $\|y^*-x^*\| < \varepsilon$.

Here X^* denotes the (topological) dual of the Banach space X and S_X its unit sphere. We write B_X to denote the closed unit ball of X.

For two Banach spaces X and $Y, L(X,Y)$ is the space of linear bounded operators from X into Y. We recall that the pair (X,Y) has the Bishop-PhelpsBollobás property for operators $(BPBp)$, if for any $\varepsilon>0$ there exists $\eta(\varepsilon)>0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1-\eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ satisfying the following conditions:

$$\|Su_0\| = 1,\ \|u_0-x_0\| < \varepsilon \text{ and } \| S-T \| < \varepsilon.$$

Acosta et all proved that for any space Y satisfying the property β of Lindenstrauss, the pair (X,Y) has the BPBp for operators for every Banach space X [87]. For the domain space, there is no a reasonably general property implying a positive result. However there are some positive results in concrete cases. There is a characterization of the spaces Y such that the pair (ℓ_1, Y) satisfies the BPBp [87] As a consequence of this result, it is known that this condition is satisfied by finite-dimensional spaces, uniformly convex spaces, $C(K)(K$ is some compact topological space) and $L_1(\mu)$ (any measure μ). Aron et all showed that the pair $(L_1(\mu), L_\infty([0,1]))$ has also the BPBp for every σ-finite measure μ [94]. This result has been extended recently by Choi et all (see [102]). Some related results for operators whose domain is $L_1(\mu)$ can be also found in [101], [89] and [102].

Now we point out results stating that the pair (X, Y) has the BPBp in case that the domain space is $C_0(L)$ (space of continuous functions on a locally compact Hausdorff space L vanishing at infinity). Kim proved that in the real case the pair (c_0, Y) has the BPBp for operators whenever Y is uniformly convex [111]. [88] contains also a positive result for the pair $(C(K), C(S))$ in the real case (K and S are compact Hausdorff spaces). Let us point out that in the complex case it is not known yet if the subset of norm attaining operators from $C(K)$ to $C(S)$ is dense in $L(C(K), C(S))$. Very recently Kim, Lee and Lin [113] proved that the pair $(L_\infty(\mu), Y)$ has the BPBp whenever Y is a uniformly convex space and μ is any positive measure. Analogous result in complex case for the pairs (c_0, Y) and $(L_\infty(\mu), Y)(\mu$ is any positive measure) whenever Y is a C-uniformly convex space. It also holds that the pair $(C(K), Y)$ has the BPB in the real case for any uniformly convex space [112].

We show that the subspace of weakly compact operators from $C_0(L)$ into Y satisfies the Bishop-Phelps-Bollobás property for operators in the complex case, for every locally compact Hausdorff space L and for any $\mathbb{C}$-uniformly convex (complex) space. Let us notice that this is an extension of the result in [113] for the complex case in two ways. First we consider any space $C(K)$ instead of $L_\infty(\mu)$ as the domain space and also we consider a strictly more general property on the range space, namely $\mathbb{C}$_ uniform convexity instead of uniform convexity. Our result extends [87] in a satisfactory way and the recent result in [113] for the case that the domain space is $L_\infty(\mu)$. As a consequence, in the complex case the pair $(C(K), L_1(\mu))$ has the BPBp for every compact Hausdorff space K and any measure μ.

The spaces $C(K)$ and $C(S)$ is dense in $L(C(K), C(S))$. However a positive result for real $C(K)$ spaces was proved [108].

We notice that in case that the range space is $C(K)$ or more generally a uniform algebra, [93] and [100] provides positive results for the BPBp for the class of Asplund operators.

For a complex Banach space Y, recall that the $\mathbb{C}$-modulus of convexity δ is defined for every $\varepsilon > 0$ by

$$\delta(\varepsilon) = \mathrm{i}\ \{\mathrm{s}\ \{\| x + \lambda\varepsilon y \| - 1 : \lambda \in \mathbb{C}, |\lambda| = 1\} : x, y \in S_Y\}.$$

Recall that the Banach space Y is $\mathbb{C}$-uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$ [104]. Every uniformly convex complex space is $\mathbb{C}$-uniformly convex and the converse is not true. Globevnik proved that the complex space $L_1(\mu)$ is $\mathbb{C}$-uniformly convex [104].

We will denote by $\bar{D}(0,1)$ the closed unit disc in $\mathbb{C}$. Let us notice that for $0 < s < t$ it is satisfied that $\sup\{\| x + \lambda s y \| : \lambda \in \bar{D}(0,1)\} \leq \sup\{\| x + \lambda t y \| : \lambda \in \bar{D}(0,1)\}$. Hence δ is an increasing function and $\delta(t) \leq t$ for every $t > 0$.

L will be a locally compact Hausdorff topological space and $C_0(L)$ will be the space of continuous complex valued functions on L vanishing at infinity.

We recall the following definition.

Definition (2.2.1)[85]: ([89] Definition 1.3) Let X and Y be both real or complex Banach spaces and M a subspace of $\mathcal{L}(X,Y)$. We say that M satisfies the Bishop-Phelps-Bollobás property if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for any $T \in S_M$, if $x_0 \in S_X$ satisfies that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_M$ satisfying the following conditions:

$$\|Su_0\| = 1,\ \|u_0 - x_0\| < \varepsilon \text{ and } \| S - T \| < \varepsilon.$$

We recall some elementary results.

Lemma (2.2.2)[85]: Assume that $\lambda, w \in \bar{D}(0,1), t \in]0,1[$ and $\operatorname{Re} w\lambda > 1 - t$. Then $|w - \bar{\lambda}| < \sqrt{2t}$.

The proof is straightforward.

For a locally compact Hausdorff topological space L, we denote by $\mathcal{B}(L)$ the space of Borel measurable and bounded complex valued functions defined on L, endowed with the sup norm. If $B \subset L$ is a Borel measurable set, denote by P_B the projection $P_B : \mathcal{B}(L) \longrightarrow \mathcal{B}(L)$ given by $P_B(f) = f\chi_B$ for any $f \in \mathcal{B}(L)$. Of course, in view of Riesz Theorem, the space $\mathcal{B}(L)$ can be identified in a natural way as a subspace of $C_0(L)^{**}$. As a consequence, for an operator $T \in L(C_0(L), Y)$ and a Borel set $B \subset L$, the composition $T^{**}P_B$ makes sense.

The symbol $\mathcal{WC}(X,Y)$ denotes the subspace of weakly compact operators from X to Y for any Banach spaces X and Y.

Lemma (2.2.3)[85]: Let Y be a $\mathbb{C}$-uniformly convex space with modulus of $\mathbb{C}$-convexity δ. Let L be a locally compact Haussdorf topological space and A a Borel set of L. Assume that for some $0 < \varepsilon < 1$ and $T \in S_{\mathcal{WC}(C_0(L),Y)}$ it is satisfied $\|T^{**}P_A\| > 1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$. Then $\|T^{**}(I - P_A)\| \le \varepsilon$

Proof. Assume that T satisfies the assumptions of the result. Since T is a weakly compact operator, then $T^{**}(C_0(L))^{**} \subset Y$ and we consider the subspace $\mathcal{B}(K) \subset C_0(L)^{**}$. We write $\eta = \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$. By the assumption, there exists $f \in S_{\mathcal{B}(L)}$ such that $f = P_A(f)$ and $\|T^{**}(f)\| > 1 - \eta > 0$. For every $g \in B_{\mathcal{B}(L)}$ it is satisfied that $\|f + (I - P_A)(g)\| \le 1$ and so $\|T^{**}(f + \lambda(I - P_A)g)\| \le 1$ for every $\lambda \in \bar{D}(0,1)$. That is, for any $\lambda \in \bar{D}(0,1)$ we have

$$\left\|\frac{T^{**}(f)}{\|T^{**}(f)\|} + \lambda\frac{T^{**}(I - P_A)(g)}{\|T^{**}(f)\|}\right\| \le \frac{1}{\|T^{**}(f)\|} < \frac{1}{1-\eta} = 1 + \delta(\varepsilon).$$

As a consequence $\|T^{**}(I - P_A)(g)\| \le \varepsilon\|T^{**}(f)\| \le \varepsilon$ and so $\|T^{**}(I - P_A)\| \le \varepsilon$.

As we already mentioned, the subset of norm attaining operators between two Banach spaces is not always dense in the corresponding space of operators in case that the domain space is $C_0(L)$. Schachermayer proved a Bishop-Phelps result in the real case for the subspace of weakly compact operators from any space $C_0(L)$ into any Banach space [117]. Alaminos et all extended this result to the complex case [91]. We notice that there are examples of spaces Y for which the subspace of finite-dimensional operators from the space ℓ_∞^2 to Y does not have the Bishop-Phelps-Bollobs property (see [87] Theorem 4.1 and Proposition 3.9 or 10. Corollary 3.3]). For those reasons restrictions are needed both on the class of operators and also on the range space in order to obtain a BPB result in case that the domain space is $C_0(L)$.

Theorem (2.2.4)[85]: The space $\mathcal{WC}(C_0(L),Y)$ satisfies the Bishop-Phelps-Bollobás property for any locally compact Hausdorff topological space L and any $\mathbb{C}$-uniformly convex space Y. Moreover the function η appearing in Definition (2.2.1) depends only on the modulus of convexity of Y.

Proof. Fix $0<\varepsilon<1$ and let $\delta(\varepsilon)$ be the modulus of $\mathbb{C}$-convexity of Y. We denote $\eta=\frac{\varepsilon^2\delta\left(\frac{\varepsilon}{9}\right)^2}{10945\left(1+\delta\left(\frac{\varepsilon}{9}\right)\right)^2}$ and $s=\frac{\eta(2-\varepsilon)\varepsilon^2}{2(\varepsilon^2+2\cdot12^2)}$ Assume that $T\in S_{\mathcal{WC}(C_0(L),Y)}$ and $f_0\in S_{C_0(L)}$ satisfy that

$$\|Tf_0\|>1-s.$$

Our goal is to find an operator $S\in S_{\mathcal{WC}(C_0(L),Y)}$ and $g\in S_{C_0(L)}$ such that

$$\|S(g)\|=1,\ \ \|S-T\|<\varepsilon,\ \text{ and }\ \|g-f_0\|<\varepsilon.$$

We can choose $y_1^*\in S_{Y^*}$ such that

$$\operatorname{Re} y_1^*(Tf_0)=\|Tf_0\|>1-s. \tag{42}$$

We identify $C_0(L)^*$ with the space $M(L)$ of Borel regular complex measures on L in view of Riesz Theorem. We write $\mu_1=T^*(y_1^*)\in M(L)$. Since μ_1 is absolutely continuous with respect to its variation $|\mu_1|$, by the Radon-Nikodým Theorem there is a Borel measurable function $g_1\in\mathcal{B}(L)$ such that $|g_1|=1$ and such that

$$\mu_1(f)=\int_L fg_1d|\mu_1|,\ \forall f\in C_0(L).$$

We write $\beta=\frac{\varepsilon^2}{2\cdot12^2}$ and denote by A the set given by

$$A=\{t\in L:\operatorname{Re} f_0(t)g_1(t)>1-\beta\}.$$

By Lemma (2.2.2) we have that

$$\|(f_0-\overline{g_1})\chi_A\|_\infty\le\sqrt{2\beta}=\frac{\varepsilon}{12}. \tag{43}$$

Clearly A is also Borel measurable and we know that

$$\begin{aligned}1-s\quad &<\operatorname{Re} y_1^*(Tf_0)=\operatorname{Re}\mu_1(f_0)=\operatorname{Re}\int_L f_0g_1d\mid\mu_1\\ &=\operatorname{Re}\int_A f_0g_1d|\mu_1|+\operatorname{Re}\int_{L\setminus A} f_0g_1d|\mu_1|\\ &\le|\mu_1|(A)+(1-\beta)|\mu_1|(L\setminus A)\\ &=|\mu_1|(L)-\beta|\mu_1|(L\setminus A)\\ &\le1-\beta|\mu_1|(L\setminus A).\end{aligned}$$

Hence

$$|\mu_1|(L\setminus A)\le\frac{s}{\beta}=\frac{\eta(2-\varepsilon)12^2}{\varepsilon^2+2\cdot12^2}. \tag{44}$$

By Lusin's Theorem (see for instance [116] Theorem 2.23) and by the inner regularity of μ_1 there is a compact set $B\subset A$ such that the restriction of g_1 to B is continuous, and $|\mu_1|(A\setminus B)\le\frac{\varepsilon\eta}{2}$ and so

$$|\mu_1|(L\setminus B)\le|\mu_1|(L\setminus A)+|\mu_1|(A\setminus B)\le\frac{s}{\beta}+\frac{\varepsilon\eta}{2}. \tag{45}$$

From (42) and the previous estimate we obtain

$$|\mu_1|(B) = |\mu_1|(L) - |\mu_1|(L \setminus B) > 1 - s - \frac{s}{\beta} - \frac{\varepsilon\eta}{2} = 1 - \eta. \tag{46}$$

Hence

$$\begin{aligned}\|T^{**}P_B\| &\geq |\mu_1|(B)\\ &> 1-\eta\\ &> 1 - \frac{\delta\left(\frac{\varepsilon}{9}\right)}{1+\delta\left(\frac{\varepsilon}{9}\right)}\end{aligned}$$

By applying Lemma (2.2.3) we deduce

$$\|T^{**}(I - P_B)\| \leq \frac{\varepsilon}{9}. \tag{47}$$

Since T is a weakly compact operator it is satisfied that $T^{**}(C_0(L)^{**}) \subset Y$. So we can define the operator $\tilde{S} \in \mathcal{WC}(C_0(L), Y)$ by

$$\tilde{S}(f) = T^{**}(f\chi_B) + \varepsilon_1 y_1^*\big(T^{**}(f\chi_B)\big)T^{**}(\overline{g_1}\chi_B)\ (f \in C_0(L)),$$

where $\varepsilon_1 = \frac{1}{6}\frac{\delta\left(\frac{\varepsilon}{9}\right)}{1+\delta\left(\frac{\varepsilon}{9}\right)}$.

Let us notice that $\tilde{S}^{**} = \tilde{S}^{**}P_B$ and we have that

$$\begin{aligned}\| \tilde{S} \| &\geq | y_1^*\big(\tilde{S}^{**}(\overline{g_1}\chi_B) |\\ &= | y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big) + \varepsilon_1\big(y_1^*(T^{**}(\overline{g_1}\chi_B))\big(y_1^*\big(T^{**}(\overline{g_1}\chi_B)\big)\\ &\geq | y_1^*(T^{**}(\overline{g_1}\chi_B)||1 + \varepsilon_1 y_1^*(T^{**}(\overline{g_1}\chi_B) |\\ &\geq |\mu_1|(B)(1 + \varepsilon_1|\mu_1|(B))\\ &> (1-\eta)(1+\varepsilon_1(1-\eta))\ \text{(by (46))}\end{aligned}$$

As a consequence

$$1 \leq 1 - \eta + \varepsilon_1(1-\eta)^2 \leq \| \tilde{S} \| \leq 1 + \varepsilon_1, \tag{48}$$

and so

$$|1 - \| \tilde{S}\|| \leq \varepsilon_1. \tag{49}$$

For every $h \in C(B)$, we will denote by $h\chi_B$ the natural extension of h to L, which is a Borel function on L. Let be S_1 the operator given by

$$S_1(h) = \tilde{S}^{**}(h\chi_B)\ (h \in C(B)),$$

which is clearly a weakly compact operator from $C(B)$ into Y. Since $\tilde{S}^{**} = \tilde{S}^{**}P_B$, it is clear that $\|S_1\| = \| \tilde{S} \|$. We know that B is a compact set and $\tilde{S}$ is weakly compact, by [91] there is weakly compact operator $S_2 \in L(C(B), Y)$ and $h_1 \in S_{C(B)}$ satisfying that

$$\| \tilde{S} \| = \|S_2\| = \|S_2(h_1)\| \ \text{ and } \ \|S_2 - S_1\| < \frac{\varepsilon\eta}{2}. \tag{50}$$

We can choose $y_2^* \in S_{Y^*}$ such that

$$y_2^*\big(S_2(h_1)\big) = \|S_2\|. \tag{51}$$

By rotating the elements h_1 and y_2^* if needed we can also assume that $y_1^*\big(T^{**}(h_1\chi_B)\big) \in \mathbb{R}_0^+$. In view of (50), the choice of y_2^* and by using that $y_1^*\big(T^{**}(h_1\chi_B)\big) \in \mathbb{R}_0^+$we have

$$\| \tilde{S} \| - \frac{\varepsilon\eta}{2} \leq \operatorname{Re} y_2^*\big(S_1(h_1)\big) = \operatorname{Re} y_2^*\left(\tilde{S}^{**}(h_1\chi_B)\right)$$

$$= \operatorname{Re} y_2^*\big(T^{**}(h_1\chi_B)\big) + \varepsilon_1 \operatorname{Re} y_1^*\big(T^{**}(h_1\chi_B)\big) y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big)$$
$$\leq 1 + \varepsilon_1 \operatorname{Re} y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big).$$

Combining this inequality with the estimate (48) we deduce that

$$\operatorname{Re}\left(y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) \geq (1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}. \qquad (52)$$

As a consequence we obtain that

$$\begin{aligned}
\operatorname{Re} y_2^*\left(\tilde{S}^{**}(\overline{g_1}\chi_B)\right) &= \operatorname{Re} y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) + \varepsilon_1 \operatorname{Re} y_1^*\big(T^{**}(\overline{g_1}\chi_B) y_2^*\big(T^{**}(\overline{g_1}\chi_B)\big) \\
&\geq (1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1} + \varepsilon_1|\mu_1|(B)\left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right) \\
&\geq \left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right)\big(1+\varepsilon_1(1-\eta)\big) \text{ (by (46).}
\end{aligned}$$

So

$$\begin{aligned}
\operatorname{Re} y_2^*\left(S_2(\overline{g_{1|B}})\right) &\geq \operatorname{Re} y_2^*\big(S_1(\overline{g_1 \mid B})\big) - \|S_2 - S_1\| \\
&\geq \operatorname{Re} y_2^*\left(\tilde{S}^{**}(\overline{g_1}\chi_B)\right) - \|S_2 - S_1\| \\
&\geq \left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right)\big(1+\varepsilon_1(1-\eta)\big) - \frac{\eta\varepsilon}{2} \text{ (by (50)} \\
&\geq \left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right)\big(1+\varepsilon_1(1-\eta)\big) - \frac{\eta\|S_2\|}{2} \text{ (by (50) and (48)).}
\end{aligned}$$

Let us write $R_2 = \frac{S_2}{\|S_2\|}$ and $\mu_2 = R_2^*(y_2^*) \in M(B)$. Let $g_2 = \frac{d\mu_2}{d|\mu_2|}$ and we can assume that $|g_2| = 1$. From the previous inequality, in view of (50) and (48) we have that

$$\begin{aligned}
\operatorname{Re} y_2^*\left(R_2(\overline{g_{1|B}})\right) &\geq \frac{\left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right)\big(1+\varepsilon_1(1-\eta)\big)}{\|S_2\|} - \frac{\eta\varepsilon}{2} \\
&\geq \frac{\left((1-\eta)^2 - \frac{\eta(2+\varepsilon)}{2\varepsilon_1}\right)\big(1+\varepsilon_1(1-\eta)\big)}{1+\varepsilon_1} - \frac{\eta\varepsilon}{2} \\
&= 1 - \frac{2\eta - 2\eta^2 + \varepsilon_1\big(1-(1-\eta)^3\big) + \frac{2\eta+\eta\varepsilon}{2\varepsilon_1} + \frac{\eta\varepsilon}{2}(2+\varepsilon_1-\eta)}{1+\varepsilon_1} \\
&> 1 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta. \qquad (53)
\end{aligned}$$

We consider the measurable set C of L given by

$$C = \{t \in B : \operatorname{Re}(\overline{g_1}(t) + h_1(t))g_2(t) > 2 - \beta\}.$$

In view of (51) and (53) we have that

$$\begin{aligned}
2 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta &< \operatorname{Re}\mu_2(h_1 + \overline{g_1} \mid B) \\
&= \int_C \operatorname{Re}(h_1 + \overline{g_1})g_2\, d|\mu_2| + \int_{B\setminus C} \operatorname{Re}(h_1 + \overline{g_1})g_2\, d|\mu_2| \\
&\leq 2|\mu_2|(C) + (2-\beta)|\mu_2|(B \setminus C)
\end{aligned}$$

$$= 2|\mu_2|(B) - \beta|\mu_2|(B \setminus C)$$
$$\leq 2 - \beta|\mu_2|(B \setminus C).$$

Hence

$$|\mu_2|(B \setminus C) \leq \frac{6\eta + 2\dfrac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta}. \tag{54}$$

On the other hand, in view of Lemma (2.2.2) we have that

$$\|(g_1 - g_2)\chi_C\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12} \text{ and } \|(h_1 - \overline{g_2})\chi_C\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12}. \tag{55}$$

From the previous inequality and (43) it follows that

$$\|(h_1 - f_0)\chi_C\|_\infty \leq \|(h_1 - \overline{g_2})\chi_C\|_\infty + \|(\overline{g_2} - \overline{g_1})\chi_C\|_\infty + \|(\overline{g_1} - f_0)\chi_C\|_\infty \leq \frac{\varepsilon}{4}. \tag{56}$$

By the inner regularity of μ_2 there is a compact set $K_1 \subset C$ such that

$$|\mu_2|(C \setminus K_1) < \frac{\eta\varepsilon}{2}. \tag{57}$$

Let us notice that

$$\begin{aligned}
\|R_2^{**}P_{K_1}\| &\geq \| y_2^* R_2^{**} P_{K_1} \| = |\mu_2|(K_1) \\
&= |\mu_2|(B) - |\mu_2|(B \setminus C) - |\mu_2|(C \setminus K_1) \\
&\geq \operatorname{Re} y_2^*\big(R_2(\overline{g_1 \mid B})\big) - |\mu_2|(B \setminus C) - |\mu_2|(C \setminus K_1) \\
&\geq \operatorname{Re} y_2^*\big(R_2(\overline{g_1 \mid B})\big) - |\mu_2|(B \setminus C) - \frac{\eta\varepsilon}{2} \text{ (by (57))} \\
&\geq 1 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon\eta - \frac{6\eta + 2\dfrac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta} - \frac{\eta\varepsilon}{2} \text{ (by (57) and (54))} \\
&> 1 - 2\frac{6\eta + 2\dfrac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta} - \frac{\eta\varepsilon}{2} \\
&> 1 - \frac{\delta\left(\frac{\varepsilon}{9}\right)}{1 + \delta\left(\frac{\varepsilon}{9}\right)} > 0
\end{aligned}$$

Hence $K_1 \neq \emptyset$

In view of Lemma (2.2.3) we obtain

$$\|R_2^{**}\big(P_B - P_{K_1}\big)\| \leq \frac{\varepsilon}{9}. \tag{58}$$

We denote by T_2 the element in $L(C_0(L), Y)$ defined by

$$T_2(f) = R_2(f_{|B}) \ (f \in C_0(L)).$$

Clearly it is satisfied that $\|T_2^{**}\big(I - P_{K_1}\big)\| = \|R_2^{**}\big(P_B - P_{K_1}\big)\|$ and since $T_2^{**}\big(P_B - P_{K_1}\big) = T_2^{**}\big(I - P_{K_1}\big)P_B$ in view of (58) we obtain

$$\|T_2^{**}\big(P_B - P_{K_1}\big)\| \leq \frac{\varepsilon}{9}. \tag{59}$$

We also write $R(f) = T^{**}(f\chi_B)$ for every $f \in C(B)$ and so we have

$$\|(T_2^{**} - T^{**})P_B\| = \|R_2 - R\|. \tag{60}$$

By the definition of S_1 we know that

$$\|S_1 - R\| \leq \varepsilon_1. \tag{61}$$

Since $K_1 \neq \emptyset$, let us fix $t_0 \in K_1$. Since $K_1 \subset C$, we have that $|h_1(t_0)| > 1 - \beta > 1 - \frac{\varepsilon}{2}$. So we can choose and open set V in B such that $t_0 \in V \subset \left\{t \in B: |h_1(t)| > 1 - \frac{\varepsilon}{2}\right\}$ and a function $v \in C(B)$ satisfying $v(B) \subset [0,1], v(t_0) = 1$ and supp $v \subset V$. So there are functions $h_i \in C(B)(i = 2,3)$ such that

$$h_2(t) = h_1(t) + v(t)(1 - |h_1(t)|)\frac{h_1(t)}{|h_1(t)|}\ (t \in B). \tag{62}$$

and

$$h_3(t) = h_1(t) - v(t)(1 - |h_1(t)|)\frac{h_1(t)}{|h_1(t)|}\ (t \in B). \tag{63}$$

It is clear that $h_i \in B_{C(B)}$ for $i = 2,3$ and $h_1 = \frac{1}{2}(h_2 + h_3)$. By using that the operator R_2 attains its norm at h_1 we clearly have that

$$\|R_2(h_2)\| = 1 \text{ and } |h_2(t_0)| = 1. \tag{64}$$

Since supp $v \subset V \subset \left\{t \in B: |h_1(t)| > 1 - \frac{\varepsilon}{2}\right\}$ we obtain for $t \in V$ that

$$|h_2(t) - h_1(t)| \leq 1 - |h_1(t)| < \frac{\varepsilon}{2}. \tag{65}$$

For $t \in B \setminus V, h_2(t) = h_1(t)$ so $\|h_2 - h_1\| < \frac{\varepsilon}{2}$. In view of (56) we obtain that

$$\begin{aligned} \|h_2 - f_{0|C}\| &\leq \|h_2 - h_1\| + \|h_1 - f_{0|C}\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\ &= \frac{3\varepsilon}{4}. \end{aligned} \tag{66}$$

Since $B \subset L$ is a compact subset, there is a function $f_2 \in C_0(L)$ such that it extends the function h_2 to L (see for instance [107], Corollary 9.15 and Theorem 12.4 and [110], Theorems 17 and [103]). Since the function $\Phi: \mathbb{C} \longrightarrow \mathbb{C}$ given by $\Phi(z) = z$ if $|z| \leq 1$ and $\Phi(z) = \frac{z}{|z|}$ if $|z| > 1$ is continuous, by using $\Phi \circ f_2$ instead of f_2 if needed, and the fact that $h_2 \in S_{C(B)}$ we can also assume that $f_2 \in S_{C_0(L)}$. Since f_2 is an extension of h_2, by using (66) there is an open set $G \subset L$ such that $K_1 \subset G$ and satisfying also that

$$\|(f_2 - f_0)\chi_G\|_\infty < \frac{7\varepsilon}{8}. \tag{67}$$

By Urysohn's Lemma there is a function $u \in C_0(L)$ such that $u(L) \subset [0,1], u_{|K_1} = 1$ and supp $u \subset G$. We define the function f_3 by

$$f_3 = uf_2 + (1 - u)f_0$$

that clearly belongs to $B_{C_0(L)}$

Notice also that

$$f_3(t) = f_2(t) = h_2(t)\ \forall t \in K_1,\ f_3(t) = f_0(t) \forall t \in L \setminus G \tag{68}$$

and

$$|f_3(t) - f_0(t)| = u(t)|f_2(t) - f_0(t)|,\ \forall t \in G \setminus K_1 \tag{69}$$

In view of (67) we obtain that

$$\|f_3 - f_0\| < \varepsilon. \tag{70}$$

We write $\lambda_0 = \overline{h_2(t_0)}$ and we know that $|\lambda_0| = 1$. Define the operator $S \in L(C_0(L), Y)$ given by

$$S(f) = R_2^{**}\left(\left(f\chi_{K_1}\right)_{|B}\right) + \lambda_0 f(t_0) R_2^{**}\left(h_2\chi_{B\setminus K_1}\right) \ (f \in C_0(L))$$

Since R_2 is weakly compact, S is well-defined and it is also weakly compact. For every $f \in B_{C_0(L)}$ we have that $|\lambda_0 f(t_0)| \leq 1$ and so

$$\left\|\left(f\chi_{K_1}\right)_{|B} + \lambda_0 f(t_0) h_2 \chi_{B\setminus K_1}\right\|_\infty \leq 1.$$

Since $\|R_2\| \leq 1$, then

$$\| S(f) \| = \left\|R_2^{**}\left(\left(f\chi_{K_1}\right)_{|B} + \lambda_0 f(t_0) h_2\chi_{B\setminus K_1}\right)\right\| \leq 1$$

We checked that $S \in B_{\mathcal{WC}(C_0(L),Y)}$. It is also satisfied that

$$\begin{aligned} S(f_3) &= R_2^{**}\left(\left(f_3\chi_{K_1}\right) \mid B\right) + \lambda_0 f_3(t_0) R_2^{**}\left(h_2\chi_{B\setminus K_1}\right) \\ &= R_2^{**}(h_2) \text{ (by (68))} \\ &= R_2(h_2) \end{aligned}$$

and in view of (64) we obtain $\|S(f_3)\| = \|R_2(h_2)\| = 1$. Hence $S \in S_{\mathcal{WC}(C_0(L),Y)}$ and it attains its norm at f_3. We also know that $\|f_3 - f_0\| < \varepsilon$ by inequality (70). It suffices to check that S is close to T. Indeed we obtain the following estimate

$$\begin{aligned} \| S - T \| &\leq \|S^{**} - T^{**}P_B\| + \|T^{**}(I - P_B)\| \\ &\leq \|T_2^{**}P_{K_1} - T^{**}P_B\| + \|R_2^{**}(P_B - P_{K_1})\| + \frac{\varepsilon}{9} \text{ (by (47)} \\ &= \|(T_2^{**} - T^{**})P_B\| + \|T_2^{**}(P_B - P_{K_1})\| + \frac{2\varepsilon}{9} \text{ (by (58)} \\ &\leq \|(T_2^{**} - T^{**})P_B\| + \frac{\varepsilon}{3} \text{ (by (59))} \\ &= \|R_2 - R\| + \frac{\varepsilon}{3} \text{ (by (60))} \\ &\leq \|R_2 - S_2\| + \|S_2 - S_1\| + \|S_1 - R\| + \frac{\varepsilon}{3} \\ &\leq |1 - \|S_2\|| + \frac{\eta\varepsilon}{2} + \varepsilon_1 + \frac{\varepsilon}{3} \text{ (by (50) and (61)} \\ &\leq 2\varepsilon_1 + \frac{\eta\varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon \text{ (by (49) and (50).} \end{aligned}$$

Since any operator from $C_0(L)$ into $L_p(\mu)$ is weakly compact $(1 \leq p < \infty)$ and the complex spaces $L_p(\mu)(1 \leq p < \infty)$ are $\mathbb{C}$-uniformly convex we obtain the following result:

Corollary (2.2.5)[85]: In the complex case the pair $\left(C_0(L), L_p(\mu)\right)$ does have the BishopPhelps-Bollobás property for operators for every Hausdorff locally compact space L, every positive measure μ and $1 \leq p < \infty$ [113].

Chapter 3
The Bishop-Phelps-Bollobás Theorem

We show that the Bishop-Phelps-Bollobás theorem holds for all bounded operators from $L_1(\mu)$ into $L_\infty[0,1]$, where μ is a σ-finite measure. We show that the Bishop-Phelps-Bollobás theorem holds for bilinear forms on $c_0 \times \ell_p (1 < p < \infty)$. On the other hand, the Bishop-Phelps-Bollobás Theorem for bilinear forms on $\ell_1 \times L_1(\mu)$ fails for any infinite-dimensional $L_1(\mu)$, a result that was known only when $L_1(\mu) = \ell_1$.

Section (3.1): $\mathcal{L}(L_1(\mu), L_\infty[0,1])$:

In 1961, Bishop and Phelps [124] proved the celebrated Bishop-Phelps theorem, which shows that for every Banach space X, every element in its dual space X^* can be approximated by ones that attain their norms. Since then, this theorem has been extended to linear operators between Banach spaces [126],[130],[132],[133],[135], and also to nonlinear mappings [120],[123],[121],[127],[131]. On the other hand, Bollobás [125] sharpened it to apply a problem about the numerical range of an operator, now known as Bishop-Phelps-Bollobás theorem. We denote the unit sphere of a Banach space X by S_X, the closed unit ball by B_X, as usual.

Theorem (3.1.1)[119]: (Bishop-Phelps-Bollobás theorem). Suppose $x \in S_X, f \in S_{X^*}$ and $|f(x) - 1| \leqslant \epsilon^2/2 \left(0 < \epsilon < \frac{1}{2}\right)$. Then there exist $y \in S_X$ and $g \in S_{X*}$ such that $g(y) = 1, \| f - g \| < \epsilon$ and $\| x - y \| < \epsilon + \epsilon^2$

Recently, Acosta, Aron, García and Maestre [122] defined the Bishop-Phelps-Bollobás property for a pair of Banach spaces. A pair of Banach spaces (X, Y) is said to have the Bishop-PhelpsBollobás property for operators $(BPBP)$ if for every $\epsilon > 0$ there are $\eta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{\epsilon\to 0}\beta(\epsilon) = 0$ such that for all $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|T(x_0)\| > 1 - \eta(\epsilon)$, there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\| = 1, \ \|u_0 - x_0\| < \beta(\epsilon), \ \text{and} \ \ \| S - T \| < \epsilon$$

This property is a uniform one in nature.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and (I, Σ, m) be the Lebesgue measure space, where $I = [0,1]$. Finet and Payá [129] showed that the set of all norm attaining operators is dense in the space $\mathcal{L}(L_1(\mu), L_\infty(m))$. Further, we will show that the pair $(L_1(\mu), L_\infty(m))$ has the $BPBP$

It is well known that the space $\mathcal{L}(L_1(\mu), L_\infty(m))$ is isometrically isomorphic to the space $L_\infty(\mu \otimes m)$, where $\mu \otimes m$ denotes the product measure on $\Omega \times I$. The operator $\hat{h}$ corresponding to an essentially bounded function h is given by

$$[\hat{h}(f)](t) = \int_\Omega h(\omega, t) f(\omega) d\mu(\omega)$$

for m-almost every $t \in I$ and for all $f \in L_1(\mu)$ (see [128]).

We recall the Lebesgue density theorem: given a measurable set $E \subset \mathbb{R}$, we have $m(E \Delta \delta(E)) = 0$, where $\delta(E)$ is the set of points $y \in \mathbb{R}$ of density of E, that is,

$$\delta(E) = \left\{ y \in \mathbb{R}: \lim_{h\to 0} \frac{m(E \cap [y - h, y + h])}{2h} = 1 \right\}$$

and $E\Delta\delta(E)$ is the symmetric difference of the sets E and $\delta(E)$. In addition, the closed unit ball of $L_1(m)$ is the closed absolutely convex hull of the set $\left\{\frac{\chi_B}{m(B)}: B \in \Sigma, 0 < m(B) < \infty\right\}$, equivalently,

$$\| g \|_\infty = \mathrm{s} \left\{\frac{1}{m(B)}\left|\int_B gdm\right| : B \in \Sigma, 0 < m(B) < \infty\right\}$$

for every $g \in L_\infty(m)$. For a measurable subset M of $\Omega \times I$, let $M_x = \{y \in I : (x,y) \in M\}$ for each $x \in \Omega$ and $M^y = \{x \in \Omega : (x,y) \in M\}$ for each $y \in I$.

Lemma (3.1.2)[119]: Let M be a measurable subset of $\Omega \times I$ with positive measure, $0 < \epsilon < 1$, and $f_0 = \sum_{j=1}^{m} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \in S_{L_1(\mu)}$, where each A_j is a measurable subset of Ω with finite positive measure, $A_k \cap A_l = \emptyset, k \neq l$, and α_j is a positive real number for every $j = 1, \dots, m$ with $\sum_{j=1}^{m} \alpha_j = 1$. If $\|\hat{\chi}_M(f_0)\|_\infty > 1 - \epsilon$, then there exists a simple function $g_0 \in S_{L_1(\mu)}$ such that

$$\|(\hat{\chi}_M + \hat{\varphi})(g_0)\|_\infty = 1 \text{ and } \|f_0 - g_0\|_1 < \frac{4\epsilon}{1-\epsilon},$$

for any simple function φ in $L_\infty(\mu \otimes m)$ such that $\| \varphi \|_\infty \leqslant 1$ and φ vanishes on M.

Proof. Since $\|\hat{\chi}_M(f_0)\|_\infty > 1 - \epsilon$, there is a measurable subset B of I such that $0 < m(B)$ and

$$\left|\left\langle \hat{\chi}_M(f_0), \frac{\chi_B}{m(B)}\right\rangle\right| > 1 - \epsilon.$$

For each $j = 1, \dots, m$ we put $M_j = M \cap (A_j \times B)$ and let

$$H_j = \left\{(x,y): x \in A_j, y \in \delta\left(\left(M_j\right)_x\right)\right\}.$$

As in the proof of Proposition 5 in [134], H_j 's are disjoint measurable subsets of $\Omega \times I$ and $(\mu \otimes m)(H) > 0$, where $H = \bigcup_{j=1}^{m} H_j$. Then there is $y \in I$ such that $\mu(H^y) > 0$. We also note that for each $j = 1, \dots, m$ we have $H_j \subset A_j \times \delta(B)$ and $(\mu \otimes m)(M_j \Delta H_j) = 0$. Let

$$J(y) = \{j: \mu(H_j^y) > 0, 1 \leqslant j \leqslant m\}.$$

For $y \in \delta(B)$ with $J(y) \neq \emptyset$ we define $g_y \in S_{L_1(\mu)}$ by

$$g_y = \sum_{j \in J(y)} \beta_j \frac{\chi_{H_j^y}^{\mu(H_j^y)}}{},$$

where $\beta_j = \alpha_j / \left(\sum_{k \in J(y)} \alpha_k\right)$.

We first claim that $\hat{\chi}_M + \hat{\varphi}$ attains its norm at g_y for every y with $\mu(H^y) > 0$.

Fix such y and let $B_n = [y - \gamma_n, y + \gamma_n]$, where (γ_n) is a sequence of positive numbers converging to 0 . Note that for every $x \in H_j^y$ we have $(x,y) \in H_j$, which implies that

$$\lim_{n\to\infty} \frac{m\left(\left(M_j\right)_x \cap B_n\right)}{m(B_n)} = 1.$$

The Lebesgue dominated convergence and Fubini theorems show that for each $j \in J(y)$

$$1 = \lim_{n\to\infty} \frac{1}{\mu\left(H_j^y\right)} \int_{H_j^y} \frac{m\left(\left(M_j\right)_x \cap B_n\right)}{m(B_n)} d\mu(x) = \lim_{n\to\infty} \frac{(\mu \otimes m)\left(M_j \cap \left(H_j^y \times B_n\right)\right)}{\mu\left(H_j^y\right) m(B_n)}.$$

On the other hand, since the simple function φ is assumed to vanish on M and also $\|\varphi\|_\infty \leqslant 1$, we have

$$\left|\left\langle \hat{\varphi}\left(\frac{\chi_{H_j^y}}{\mu\left(H_j^y\right)}, \frac{\chi_{B_n}}{m(B_n)}\right)\right\rangle\right| = \left|\frac{1}{\mu\left(H_j^y\right) m(B_n)} \int_{H_j^y \times B_n} \varphi d(\mu \otimes m)\right|$$

$$\leqslant \frac{(\mu \otimes m)\left(\left(H_j^y \times B_n\right) \setminus M_j\right)}{\mu\left(H_j^y\right) m(B_n)}$$

$$= 1 - \frac{(\mu \otimes m)\left(M_j \cap \left(H_j^y \times B_n\right)\right)}{\mu\left(H_j^y\right) m(B_n)} \to 0,$$

as $n \to \infty$.

Therefore,

$$1 \geqslant \|(\hat{\chi}_M + \hat{\varphi})(g_y)\|_\infty \geqslant \lim_{n\to\infty} \left|\left\langle (\hat{\chi}_M + \hat{\varphi})\left(\sum_{j\in J(y)} \beta_j \frac{\chi_{H_j^y}}{\mu\left(H_j^y\right)}\right), \frac{\chi_{B_n}}{m(B_n)}\right\rangle\right|$$

$$\geqslant \lim_{n\to\infty} \sum_{j\in J(y)} \beta_j \frac{(\mu \otimes m)\left(M \cap \left(H_j^y \times B_n\right)\right)}{\mu\left(H_j^y\right) m(B_n)}$$

$$- \lim_{n\to\infty} \sum_{j\in J(y)} \beta_j \left|\frac{1}{\mu\left(H_j^y\right) m(B_n)} \int_{H_j^y \times B_n} \varphi d(\mu \otimes m)\right|$$

$$\geqslant \lim_{n\to\infty} \sum_{j\in J(y)} \beta_j \frac{(\mu \otimes m)\left(M_j \cap \left(H_j^y \times B_n\right)\right)}{\mu\left(H_j^y\right) m(B_n)}$$

$$- \lim_{n\to\infty} \sum_{j\in J(y)} \beta_j \left[1 - \frac{(\mu \otimes m)\left(M_j \cap \left(H_j^y \times B_n\right)\right)}{\mu\left(H_j^y\right) m(B_n)}\right] = 1,$$

which shows that $\hat{\chi}_M + \hat{\varphi}$ attains its norm at g_y.

Next we claim that there exists $y \in \delta(B)$ such that $\mu(H^y) > 0$ and

$$\|g_y - f_0\|_1 < \frac{4\epsilon}{1-\epsilon}.$$

For each $j = 1, \dots, m$ we set $B_j^+ = \{y \in \delta(B) : \mu\left(H_j^y\right) > 0\}$, $B_j^0 = \{y \in \delta(B) : \mu\left(H_j^y\right) = 0\}$ and $B^0 = \bigcap_{j=1}^m B_j^0$. By applying Fubini's theorem the sets B_j^+ and B_j^0 are Lebesgue measurable subsets of $[0,1]$.

We note that for each $j = 1, \dots, m$

$$(\mu \otimes m)(M_j) = (\mu \otimes m)\left(\left(A_j \times \delta(B)\right) \cap H_j\right)$$

$$= (\mu \otimes m)\Big(\big(A_j \times \delta(B)\big) \cap \big\{(x,y) \in H_j : \mu\big(H_j^y\big) > 0\big\}\Big).$$

Since

$$\left|\hat{\chi}_M(f_0)\left(\frac{\chi_B}{m(B)}\right)\right| > 1 - \epsilon,$$

we have

$$1 - \epsilon < \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)\big(M_j\big)}{(\mu \otimes m)\big(A_j \times B\big)},$$

which implies that

$$\sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)\Big(\big(A_j \times \delta(B)\big) \setminus \big\{(x,y) \in H_j : \mu\big(H_j^y\big) > 0\big\}\Big)}{(\mu \otimes m)\big(A_j \times B\big)} < \epsilon, \qquad (1)$$

and

$$\sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)\Big(\big(A_j \times B_j^0\big)\Big)}{(\mu \otimes m)\big(A_j \times B\big)}$$

$$\leqslant \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)\Big(\big(A_j \times \delta(B)\big) \setminus \big\{(x,y) \in H_j : \mu\big(H_j^y\big) > 0\big\}\Big)}{(\mu \otimes m)\big(A_j \times B\big)} < \epsilon, \qquad (2)$$

which implies that

$$\sum_{j=1}^{m} \alpha_j m\big(B_j^0\big) < \epsilon m(B). \qquad (3)$$

It follows from this inequality that $m(B^0) < \epsilon m(B)$. For $y \in \delta(B) \setminus B^0$,

$$\begin{aligned} \|g_y - f_0\|_1 &= \sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} \left[\left(\frac{\beta_j}{\mu\left(H_j^y\right)} - \frac{\alpha_j}{\mu\big(A_j\big)}\right)\mu\big(H_j^y\big) + \alpha_j \frac{\mu\big(A_j \setminus H_j^y\big)}{\mu\big(A_j\big)}\right] \\ &= \sum_{j \notin J(y)} \alpha_j + 1 + \sum_{j \in J(y)} \left[-\alpha_j \frac{\mu\big(H_j^y\big)}{\mu\big(A_j\big)} + \alpha_j \frac{\mu\big(A_j \setminus H_j^y\big)}{\mu\big(A_j\big)}\right] \\ &= 2\sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} 2\alpha_j \frac{\mu\big(A_j \setminus H_j^y\big)}{\mu\big(A_j\big)}. \end{aligned}$$

Assume that there is no $y \in \delta(B) \setminus B^0$ such that

Then

$$\begin{aligned} \frac{4\epsilon}{1-\epsilon} m(\delta(B) \setminus B^0) &\leqslant \int_{\delta(B)\setminus B^0} \|g_y - f_0\|_1 dm(y) \\ &= 2\int_{\delta(B)\setminus B^0} \left(\sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} \alpha_j \frac{\mu\big(A_j \setminus H_j^y\big)}{\mu\big(A_j\big)}\right) dm(y). \end{aligned}$$

It follows from the inequalities (1)-(3) that

$$\int_{\delta(B)\backslash B^0} \sum_{j\notin J(y)} \alpha_j \, dm(y) = \int_{\delta(B)\backslash B^0} \sum_{J=1}^{m} \left(\alpha_j \chi_{B_j^0}(y)\right) dm(y)$$
$$\leq \sum_{J=1}^{m} \alpha_j m\left(B_j^0\right) < \epsilon m(B),$$

and

$$\int_{\delta(B)\backslash B^0} \sum_{j\in J(y)} \alpha_j \frac{\mu\left(A_j \setminus H_j^y\right)}{\mu\left(A_j\right)} dm(y)$$
$$= \int_{\delta(B)\backslash B^0} \sum_{j=1}^{m} \left(\alpha_j \frac{\mu\left(A_j \setminus H_j^y\right)}{\mu\left(A_j\right)} \chi_{B_j^+}(y)\right) dm(y)$$
$$= \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)\left(\left(A_j \times B_j^+\right) \setminus \left\{(x,y) \in H_j^y : y \in B_j^y\right\}\right)}{\mu\left(A_j\right)} < \epsilon m(B).$$

Therefore,

$$4\epsilon m(B) \quad < \frac{4\epsilon}{1-\epsilon} m(\delta(B) \setminus B^0)$$
$$\leqslant \int_{\delta(B)\backslash B^0} \|g_y - f_0\|_1 dm(y) < 4\epsilon m(B),$$

which is a contradiction.

Lemma (3.1.3)[119]: (See [122].) Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum_{n=1}^{\infty} \alpha_n$, Re $\sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A = \{i \in \mathbb{N} : \text{Re } c_i > r\}$ satisfies the estimate

$$\sum_{i\in A} \alpha_i \geqslant 1 - \frac{\eta}{1-r}.$$

We recall that the set of simple functions is a dense subspace of $L_\infty(\mu \otimes m)$.

Theorem (3.1.4)[119]: For the complex Banach spaces $L_1(\mu)$ and $L_\infty(m)$, let $T: L_1(\mu) \to L_\infty(m)$ be a bounded operator such that $\| T \| = 1$. Given $0 < \epsilon < 1/5$ and $f_0 \in S_{L_1(\mu)}$ satisfying $\|T(f_0)\|_\infty \rangle\, 1 - \epsilon^8$, there exist $S \in \mathcal{L}(L_1(\mu), L_\infty(m))$, $\| S \| = 1$ and $g_0 \in S_{L_1(\mu)}$ such that

$$\|S(g_0)\|_\infty = 1, \ \ \| T - S \| < \epsilon \text{ and } \|f_0 - g_0\|_1 < 2\epsilon^4 + \frac{4\epsilon}{1-\epsilon}.$$

Proof. Since the set of all simple functions is dense in $L_1(\mu)$, we may assume

$$f_0 = \sum_{j=1}^{m} \alpha_j \frac{\chi_{A_j}}{\mu\left(A_j\right)} \in S_{L_1(\mu)},$$

where each A_j is a measurable subset of Ω with finite positive measure, $A_k \cap A_l = \emptyset, k \neq l$, and every α_j is a nonzero complex number with $\sum_{j=1}^{m} |\alpha_j| = 1$. We may also assume that $0 < \alpha_j \leqslant 1$ for every $j = 1, \dots, m$. Indeed, define $\Psi: L_1(\mu) \to L_1(\mu)$ by

$$\Psi(f)=\sum_{j=1}^{m} e^{-i\theta_j} f\cdot\chi_{A_j}+f\cdot\chi_{(\Omega\setminus\cup_{j=1}^{m}A_j)}$$

where $\theta_j=\arg(\alpha_j)$ for every $j=1,\dots,m$. The operator Ψ is an isometric isomorphism of $L_1(\mu)$ onto $L_1(\mu)$,

$$\|T(f_0)\|_\infty=\left\|(T\circ\Psi^{-1})(\Psi(f_0))\right\|_\infty>1-\epsilon^8$$

and

$$\Psi(f_0)=\sum_{j=1}^{m}|\alpha_j|\frac{\chi_{A_j}}{\mu(A_j)},$$

hence we may replace T and f_0 by $T\circ\Psi^{-1}$ and $\Psi(f_0)$, respectively.

Let h be the element in $L_\infty(\Omega\times I,\mu\otimes m)$, $\| h \|_\infty=1$ corresponding to T, that is, $T=\hat{h}$. We can find a simple function

$$h_0\in L_\infty(\Omega\times I,\mu\otimes m),\ \|h_0\|_\infty=1$$

such that $\|h-h_0\|_\infty<\|T(f_0)\|_\infty-(1-\epsilon^8)$, hence $\|\hat{h}_0(f_0)\|_\infty>1-\epsilon^8$. We can write $h_0=\sum_{l=1}^{p}c_l\chi_{D_l}$, where each D_l is a measurable subset of $\Omega\times I$ with positive measure, $D_k\cap D_l=\emptyset$, $k\neq l$, the complex number $|c_l|\leqslant 1$ for every $l=1,\dots,p$, and $|c_{l_0}|=1$ for some $1\leqslant l_0\leqslant p$.

Let B be a Lebesgue measurable subset of I with $0<m(B)<\infty$ such that

$$\left|\left\langle\hat{h}_0(f_0),\frac{\chi_B}{m(B)}\right\rangle\right|>1-\epsilon^8.$$

Choose $\theta\in\mathbb{R}$ so that

$$\begin{aligned}1-\epsilon^8 &<\left|\left\langle\hat{h}_0(f_0),\frac{\chi_B}{m(B)}\right\rangle\right|\\ &=e^{i\theta}\left\langle\hat{h}_0(f_0),\frac{\chi_B}{m(B)}\right\rangle\\ &=\sum_{j=1}^{m}\alpha_j e^{i\theta}\left\langle\hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right),\frac{\chi_B}{m(B)}\right\rangle.\end{aligned}$$

Let

$$J=\left\{j:1\leqslant j\leqslant m,\operatorname{Re}\left[e^{i\theta}\left\langle\hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right),\frac{\chi_B}{m(B)}\right\rangle\right]>1-\epsilon^4\right\}.$$

By Lemma (3.1.3) we have

$$\alpha_J=\sum_{j\in J}\alpha_j>1-\frac{\epsilon^8}{1-(1-\epsilon^4)}=1-\epsilon^4.$$

We define

$$f_1=\sum_{j\in J}\left(\frac{\alpha_j}{\alpha_J}\right)\frac{\chi_{A_j}}{\mu(A_j)}.$$

Then we can see $\|f_1\|_1=1$,

$$\begin{aligned}\|f_0 - f_1\|_1 &\leqslant \left\|\sum_{j\notin J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)}\right\|_1 + \left(\frac{1}{\alpha_J} - 1\right)\left\|\sum_{j\in J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)}\right\|_1 \\ &= \sum_{j\notin J} \alpha_j + (1-\alpha_J) = 2(1-\alpha_J) < 2\epsilon^4\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle \hat{h}_0(f_1), \frac{\chi_B}{m(B)}\right\rangle\right| &\geqslant \mathrm{Re}\left[e^{i\theta}\left\langle \hat{h}_0(f_1), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &= \frac{1}{\alpha_J}\sum_{j\in J} \alpha_j \mathrm{Re}\left[e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &> \frac{1}{\alpha_J}\sum_{j\in J} \alpha_j(1-\epsilon^4) = 1-\epsilon^4.\end{aligned}$$

On the other hand, for each $j \in J$

$$\begin{aligned}1-\epsilon^4 &< \mathrm{Re}\left[e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &= \mathrm{Re}\left[e^{i\theta}\sum_{l=1}^{p} c_l \frac{(\mu\otimes m)\left(D_l \cap (A_j\times B)\right)}{\mu(A_j)m(B)}\right] \\ &= \mathrm{Re}\left[e^{i\theta}\sum_{l=1}^{p} c_l\gamma_j \frac{\gamma_{j,l}}{\gamma_j}\right]\end{aligned}$$

where

$$\gamma_j = \sum_{l=1}^{p} \frac{(\mu\otimes m)\left(D_l \cap (A_j\times B)\right)}{\mu(A_j)m(B)},$$

and

$$\gamma_{j,l} = \frac{(\mu\otimes m)\left(D_l \cap (A_j\times B)\right)}{\mu(A_j)m(B)}.$$

We define

$$L = \left\{l: 1\leqslant l\leqslant p, \mathrm{Re}\left(e^{i\theta}c_l\right) > 1-\frac{\epsilon^2}{4}\right\},$$

and

$$L_j = \left\{l: 1\leqslant l\leqslant p, \mathrm{Re}\left(e^{i\theta}c_l\gamma_j\right) > 1-\frac{\epsilon^2}{4}\right\}.$$

For each $j \in J$ we can see $\gamma_j > 1-\epsilon^4$, and by Lemma (3.1.3) again

$$\sum_{l\in L_j} \frac{\gamma_{j,l}}{\gamma_j} > 1 - \frac{\epsilon^4}{1-\left(1-\frac{\epsilon^2}{4}\right)} = 1-4\epsilon^2.$$

Hence

$$\sum_{l\in L_j} \gamma_{j,l} > (1-4\epsilon^2)(1-\epsilon^4)$$

For every $j \in J$ we note that $L_j \subset L$ and

$$\begin{aligned}\sum_{l\in L} \frac{(\mu \otimes m)\left(D_l \cap \left(A_j \times B\right)\right)}{\mu\left(A_j\right)m(B)} &\geqslant \sum_{l\in L_j} \frac{(\mu \otimes m)\left(D_l \cap \left(A_j \times B\right)\right)}{\mu\left(A_j\right)m(B)} \\ &= \sum_{l\in L_j} \gamma_{j,l} > (1-4\epsilon^2)(1-\epsilon^4)\end{aligned}$$

Set $D = \bigcup_{l\in L} D_l$.
Therefore

$$\begin{aligned}\left\langle \hat{\chi}_D(f_1), \frac{\chi_B}{m(B)} \right\rangle &= \sum_{j\in J} \left(\frac{\alpha_j}{\alpha_J}\right) \cdot \sum_{l\in L} \frac{\mu \otimes m\left(D_l \cap \left(A_j \times B\right)\right)}{\mu\left(A_j\right)m(B)} \\ &\geqslant \sum_{j\in J} \left(\frac{\alpha_j}{\alpha_J}\right)(1-4\epsilon^2)(1-\epsilon^4) = (1-4\epsilon^2)(1-\epsilon^4) \\ &> 1-5\epsilon^2 > 1-\epsilon\end{aligned}$$

By Lemma (3.1.2) there is $g_0 \in S_{L_1(\mu)}$ such that $\|(\hat{\chi}_D + \hat{\varphi})(g_0)\|_\infty = 1$ and $\|f_1 - g_0\| < \frac{4\epsilon}{1-\epsilon}$, where φ is any simple function in $L_\infty(\mu \otimes m)$ such that $\| \varphi \|_\infty \leqslant 1$ and φ vanishes on D. Therefore, we have

$$\|f_0 - g_0\|_1 \leqslant \|f_0 - f_1\|_1 + \|f_1 - g_0\|_1 \leqslant 2\epsilon^4 + \frac{4\epsilon}{1-\epsilon}$$

Define

$$h_1 = e^{-i\theta}\chi_D + \sum_{l\notin L} c_l \chi_{D_l} \in L_\infty(\mu \otimes m)$$

Let S be the operator in $\mathcal{L}(L_1(\mu), L_\infty(m))$ corresponding to h_1. Then

$$\|S(g_0)\|_\infty = \|\hat{h}_1(g_0)\|_\infty = 1$$

and

$$\|h_0 - h_1\|_\infty = \max_{l\in L}\left|c_l - e^{-i\theta}\right| = \max_{l\in L}\left|e^{i\theta}c_l - 1\right|$$

However, $\mathrm{Re}\left(e^{i\theta}c_l\right) > 1 - \frac{\epsilon^2}{4}$ for every $l \in L$, hence

$$\begin{aligned}\left(\mathrm{Im}\left(e^{i\theta}c_l\right)\right)^2 &\leqslant 1 - \left(\mathrm{Re}\left(e^{i\theta}c_l\right)\right)^2 \\ &< 1 - \left(1 - \frac{\epsilon^2}{4}\right)^2 = \frac{\epsilon^2}{2} - \frac{\epsilon^4}{16}\end{aligned}$$

Since

$$\begin{aligned}\left|e^{i\theta}c_l - 1\right| &= \sqrt{\left(1 - \mathrm{Re}(e^{i\theta}c_l)\right)^2 + \left(\mathrm{Im}(e^{i\theta}c_l)\right)^2} \\ &< \sqrt{\epsilon^4/16 + (\epsilon^2/2 - \epsilon^4/16)} = \frac{\epsilon}{\sqrt{2}},\end{aligned}$$

we conclude

$$\|h_0 - h_1\|_\infty < \frac{\epsilon}{\sqrt{2}},$$

hence

$$\| T - S \|_\infty \leqslant \|h - h_0\|_\infty + \|h_0 - h_1\|_\infty < \epsilon^8 + \frac{\epsilon}{\sqrt{2}} < \epsilon.$$

We observe that for the real Banach spaces $L_1(\mu)$ and $L_\infty(m)$ better estimates could be obtained by inspecting the above proof.

Theorem (3.1.5)[119]: For the real Banach spaces $L_1(\mu)$ and $L_\infty(m)$, let T be a bounded operator from $L_1(\mu)$ into $L_\infty(m)$ such that $\| T \| = 1$. Given $0 < \epsilon < 1/5$ and $f_0 \in S_{L_1(\mu)}$ satisfying $\|T(f_0)\|_\infty > 1 - \epsilon^4$, there exist $S \in \mathcal{L}(L_1(\mu), L_\infty(m))$, $\| S \| = 1$ and $g_0 \in S_{L_1(\mu)}$ such that

$$\|S(g_0)\|_\infty = 1, \;\; \| T - S \| < \epsilon \text{ and } \|f_0 - g_0\|_1 < 2\epsilon^2 + \frac{20\epsilon}{1 - 5\epsilon}.$$

Section (3.2): Operators From c_0 to Uniformly Convex Spaces:

For X be a real or complex Banach space and B_X (resp. S_X) be the closed unit ball (resp. unit sphere) of X. Let $\mathcal{L}(X, Y)$ be the Banach space of all bounded linear operators from X into Y. We say that an operator $T \in \mathcal{L}(X, Y)$ attains its norm if there exists $x_0 \in S_X$ such that

$$\|T(x_0)\| = \| T \| = \mathrm{s}\ \{\| T(x) \| : x \in B_X\}.$$

In 1961, Bishop and Phelps [143] showed that the set of norm-attaining functionals on a Banach space X is dense in its dual space X^*, namely the BishopPhelps Theorem. There have been many efforts to extend this theorem to bounded linear operators between Banach spaces [145],[150],[152],[155],[156],[157], and also to non-linear mappings like multi-linear mappings [142], [148], polynomials and holomorphic mappings [137]. This theorem was sharpened in 1970 by Bollobás [144], and we call it the Bishop-Phelps-Bollobás Theorem.

Theorem (3.2.1)[136]: ([144]). For an arbitrary $\epsilon > 0$, if $x^* \in S_{X^*}$ satisfies $|1 - x^*(x)| < \epsilon^2/4$ for $x \in B_X$, then there are both $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\| y - x \| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.

Afterwards, Acosta et al. [138] began extending this theorem to bounded linear operators between Banach spaces and defined the Bishop-Phelps-Bollobás property ([138]). We say that the pair (X, Y) has the BishopPhelps-Bollobás property for operators (BPBP), if given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon\to 0^+}\beta(\epsilon) = 0$ such that if there exist both $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist both an operator $S \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \ \|x_0 - u_0\| < \beta(\epsilon) \ \text{ and } \ \| T - S \| < \epsilon.$$

They characterized the Banach spaces Y such that the pair (ℓ_1, Y) has the $BPBP$. They also proved that for a uniformly convex space Y the pair (ℓ_∞^n, Y) has the $BPBP$ for every $n \in \mathbb{N}$, but it remained open whether or not the pair (c_0, Y) has the $BPBP$ for a uniformly convex space Y. Our main result states that this problem has a positive answer. It seems worth mentioning that some other results about the $BPBP$ have appeared very recently ([140],[141],[149],[151]).

It was shown that if X is uniformly convex, a pair of Banach spaces (X, Y) has the $BPBP$ for every Banach space Y ([154]). We can ask a natural question that if Y is uniformly convex, then does the pair (X, Y) have the $BPBP$ for every Banach space? Let us note that the above statement does not hold in case that Y is uniformly convex. Indeed there exists a Banach space X such that the set of norm-attaining operators is not dense in $\mathcal{L}(X,\ell_p)$ for $1<p<\infty$ ([153]). However, in order to prove the main result, we use the following perturbation result for operators from a reflexive space X into a uniformly convex space Y.

Theorem (3.2.2)[136]: Let $1>\epsilon>0$ be given. Let X be a reflexive Banach space and Y be a uniformly convex Banach space with modulus of convexity $\delta(\epsilon)>0$. If $T\in S_{\mathcal{L}(X,Y)}$ and $x\in S_X$ satisfy

$$\| Tx \| > 1-\frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right),$$

then there exist $S\in S_{\mathcal{L}(X,Y)}$ and $x_0\in S_X$ such that $\|Sx_0\|=1$, $\| S-T \|<\epsilon$ and $\|Tx-Sx_0\|<\epsilon$.

Proof. Assume $T\in S_{\mathcal{L}(X,Y)}$ and $x\in S_X$ satisfy

$$\| Tx \| > 1-\frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right).$$

Choose $f\in S_{Y^*}$ such that

$$\operatorname{Re} f(Tx) > 1-\frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right).$$

Set $(x_1,f_1,T_1)=(x,f,T)$, and define a sequence

$$(x_i,f_i,T_i)_{i=1}^{\infty}\subset S_X\times S_{Y^*}\times S_{\mathcal{L}(X,Y)},$$

inductively.

If the k-th term was constructed, define

$$\tilde{T}_{k+1}x=T_kx+\frac{\epsilon}{2^{k+2}}f_k(T_kx)T_kx_k,\ \text{ and }\ T_{k+1}=\frac{\tilde{T}_{k+1}}{\|\tilde{T}_{k+1}\|}.$$

Choose x_{k+1} and f_{k+1} satisfying

$$\operatorname{Re} f_{k+1}(\tilde{T}_kx_k)=\left|f_{k+1}(\tilde{T}_kx_k)\right|,$$

$$\operatorname{Re} f_{k+1}(\tilde{T}_{k+1}x_{k+1})>\|\tilde{T}_{k+1}\|-\frac{\epsilon}{2^{k+5}}\delta\left(\frac{\epsilon}{2^{k+1}}\right).$$

This implies that

$$\operatorname{Re} f_{k+1}(T_{k+1}x_{k+1})>\|T_{k+1}\|-\frac{\epsilon}{2^{k+4}}\delta\left(\frac{\epsilon}{2^{k+1}}\right).$$

We can see that

$$\begin{aligned}\|T_k-T_{k+1}\| &\le \|T_k-\tilde{T}_{k+1}\|+\|\tilde{T}_{k+1}-T_{k+1}\|\\ &<\frac{\epsilon}{2^{k+1}}.\end{aligned}$$

Hence, (T_k) is a Cauchy sequence which converges to $S\in S_{\mathcal{L}(X,Y)}$ satisfying $\|T_k-S\|<\frac{\epsilon}{2^k}$.

To see that $(T_ix_i)_{i=1}^{\infty}$ is also a Cauchy sequence, we need to check the following:

$$\begin{aligned}\|\tilde{T}_k\|-\frac{\epsilon}{2^{k+4}}\delta\left(\frac{\epsilon}{2^k}\right) &< \left|f_k(\tilde{T}_kx_k)\right|\\ &=\left|f_k(T_{k-1}x_k)+\frac{\epsilon}{2^{k+1}}f_{k-1}(T_{k-1}x_k)\cdot f_k(T_{k-1}x_{k-1})\right|\\ &\le |f_k(T_{k-1}x_k)|+\frac{\epsilon}{2^{k+1}}|f_{k-1}(T_{k-1}x_k)|\cdot|f_k(T_{k-1}x_{k-1})|\\ &\le \|T_{k-1}\|+\frac{\epsilon}{2^{k+1}}\operatorname{Re} f_k(T_{k-1}x_{k-1}),\end{aligned}$$

$$\begin{aligned}\|\tilde{T}_k\| &\geq \left|f_{k-1}(\tilde{T}_k x_{k-1})\right| \\ &= \left|f_{k-1}(T_{k-1}x_{k-1}) + \frac{\epsilon}{2^{k+1}} f_{k-1}(T_{k-1}x_{k-1}) \cdot f_{k-1}(T_{k-1}x_{k-1})\right| \\ &\geq \left|\left(1 + \frac{\epsilon}{2^{k+1}} f_{k-1}(T_{k-1}x_{k-1})\right) \cdot f_{k-1}(T_{k-1}x_{k-1})\right| \\ &\geq \left(1 + \frac{\epsilon}{2^{k+1}} \operatorname{Re} f_{k-1}(T_{k-1}x_{k-1})\right) \cdot \operatorname{Re} f_{k-1}(T_{k-1}x_{k-1}) \\ &\geq \|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\left(\frac{\epsilon}{2^{k-1}}\right) + \frac{\epsilon}{2^{k+1}}\left(\|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\left(\frac{\epsilon}{2^{k-1}}\right)\right)^2\end{aligned}$$

It follows from the above that

$$\begin{aligned}\operatorname{Re} f_k(T_{k-1}x_{k-1}) &> \left(\|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\left(\frac{\epsilon}{2^{k-1}}\right)\right)^2 - \frac{1}{2}\delta\left(\frac{\epsilon}{2^{k-1}}\right) - \frac{1}{2^3}\delta\left(\frac{\epsilon}{2^k}\right) \\ &\geq 1 - \frac{\epsilon}{2^{k+1}}\delta\left(\frac{\epsilon}{2^{k-1}}\right) - \frac{1}{2}\delta\left(\frac{\epsilon}{2^{k-1}}\right) - \frac{1}{2^3}\delta\left(\frac{\epsilon}{2^k}\right) \\ &\geq 1 - \delta\left(\frac{\epsilon}{2^{k-1}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\frac{T_{k-1}x_{k-1} + T_k x_k}{2}\right\| &\geq \operatorname{Re} f_k\left(\frac{T_{k-1}x_{k-1} + T_k x_k}{2}\right) \\ &> 1 - \frac{\epsilon}{2^{k+4}}\delta\left(\frac{\epsilon}{2^k}\right) - \frac{1}{2}\delta\left(\frac{\epsilon}{2^{k-1}}\right) \\ &\geq 1 - \delta\left(\frac{\epsilon}{2^{k-1}}\right),\end{aligned}$$

which implies that $\|T_{k-1}x_{k-1} - T_k x_k\| < \frac{\epsilon}{2^{k-1}}$. Moreover, $(Sx_k)_{k=1}^{\infty}$ is also a Cauchy sequence and the limits of $(T_k x_k)_{k=1}^{\infty}$ and $(Sx_k)_{k=1}^{\infty}$ are the same. Choose a weakly convergent subsequence $\left(x_{k_i}\right)_{i=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ and let $x_0 \in B_X$ be its weak limit. We can see that the limit of $(T_k x_k)_{k=1}^{\infty}$ is Sx_0. Therefore $\|Tx - Sx_0\| < \epsilon$ and $\|Sx_0\| = 1$. We present here some useful lemmas, see ([138], Lemma 3.3, Lemma 5.1, Lemma 6.1).

Lemma (3.2.3)[136]: ([138], Lemma 3.3): Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for every n, and let $\eta > 0$ be such that for a series of non-negative numbers $\sum_{n=1}^{\infty} \alpha_n \leq 1$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. For every $0 < r < 1$, the set $A = \{i \in \mathbf{N}: \operatorname{Re} c_i > r\}$ satisfies the estimate

$$\sum_{i \in A} \alpha_i \geq 1 - \frac{\eta}{1 - r}.$$

Lemma (3.2.4)[136]: ([138], Lemma 5.1): Let $0 < \epsilon < 1$ be given and Y be a uniformly convex Banach space with modulus of convexity $\delta(\epsilon)$. If $T \in S_{\mathcal{L}(c_0,Y)}$ (resp. $T \in S_{\mathcal{L}(\ell_\infty^n,Y)}$), and $A \subset \mathbb{N}$ (resp. $A \subset \{1, \ldots, n\}$) has the property that $\|TP_A\| > 1 - \delta(\epsilon)$, then we have that $\|T(I - P_A)\| \leq \epsilon$, where P_A is the canonical projection from c_0 (resp. ℓ_∞^n) to ℓ_∞^A.

Lemma (3.2.5)[136]: Let Y be a strictly convex Banach space $T \in L(X, Y)$, where $X = \ell_\infty^n, c_0$, or ℓ_∞. If $\| Tx \| = \| T \|$ for some $x = (x_i) \in S_X$, then $Te_k = 0$ for every $k \in A = \{i: |x_i| < 1\}$.

Theorem (3.2.6)[136]: Let Y be a uniformly convex Banach space. Given $0 < \epsilon < 1$, there exists $0 < \eta(\epsilon) < 1$ such that if $T \in S_{\mathcal{L}(\ell_\infty^n, Y)}$ and $x \in S_{\ell_\infty^n}$ satisfies

$$\| Tx \| > 1 - \eta(\epsilon)^2,$$

then there exist $S \in S_{\mathcal{L}(\ell_\infty^n, Y)}$ and $v \in S_{\ell_\infty^n}$ such that $\| Sv \| = 1, \| S - T \| < \epsilon$ and $\| x - v \| < \sqrt{\epsilon} + \sqrt{\epsilon^2 + 2\epsilon}$ for every $n \in \mathbb{N}$.

Proof. Let the modulus of convexity of Y be $0 < \delta(\epsilon) < 1$. Given $\epsilon > 0$, set $\gamma(\epsilon) = \frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right)$ and $\eta(\epsilon) = \gamma\left(\frac{\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right)\right)$. Assume that $T \in S_{\mathcal{L}(\ell_\infty^n, Y)}$ and $x \in S_{\ell_\infty^n}$ satisfy

$$\| T(x) \| > 1 - \eta(\epsilon)^2.$$

Choose $y^* \in S_{Y^*}$ so that

$$\operatorname{Re} y^* T(x) = \operatorname{Re} T^* y^*(x) > 1 - \eta(\epsilon)^2.$$

Write $T^*y^* = ((T^*y^*)_i)_{i=1}^n \in B_{\ell_1^n}, x = (x_i)_{i=1}^n \in S_{\ell_\infty^n}$, and for any set $D \subset \{1, \dots, n\}$ define $1_D \in \ell_\infty^n$ such that $(1_D)_i = 1$ for $i \in D$ and $(1_D)_i = 0$ for $i \in D^c$. By using an appropriate isometry on ℓ_∞^n, we can assume that $(T^*y^*)_i = \operatorname{Re}(T^*y^*)_i \geq 0$ for all $i = 1, \dots, n$.

Let $A = \{i \in \{1, \dots, n\} : \operatorname{Re} x_i > 1 - \eta(\epsilon)\}$. By Lemma (3.2.3), we can see that

$$\operatorname{Re} \sum_{i \in A} (T^*y^*)_i > 1 - \eta(\epsilon),$$

which implies

$$\|TP_A(1_A)\| > 1 - \eta(\epsilon).$$

From Lemma (3.2.4) and the fact that $\eta(\epsilon) < \frac{\epsilon}{6}\delta\left(\frac{\epsilon}{6}\right)$, it follows that

$$\|T(I - P_A)\| < \frac{\epsilon}{6}\delta\left(\frac{\epsilon}{6}\right).$$

Consider $T_A \in \mathcal{L}(\ell_\infty^A, Y)$ which is the restriction of TP_A on ℓ_∞^A. We clearly have that

$$\left\| \frac{T_A}{\|T_A\|} - T_A \right\| \leq \eta(\epsilon) \ \text{ and } \ \left\| \frac{T_A(1_A)}{\|T_A\|} \right\| > 1 - \eta(\epsilon).$$

By Theorem (3.2.2) there exist $\tilde{T} \in S_{\mathcal{L}(\ell_\infty^A, Y)}$ and $u \in S_{\ell_\infty^A}$ such that

$$\| \tilde{T}(u) \| = 1, \ \left\| \tilde{T} - \frac{T_A}{\|T_A\|} \right\| < \frac{\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right), \ \text{ and } \ \left\| \frac{T_A(1_A)}{\|T_A\|} - \tilde{T}u \right\| < \frac{\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right).$$

This implies that

$$\|\tilde{T}(1_A) - \tilde{T}(u)\| \leq \left\| \frac{T_A(1_A)}{\|T_A\|} - \tilde{T}(1_A) \right\| + \left\| \frac{T_A(1_A)}{\|T_A\|} - \tilde{T}(u) \right\| < \frac{2\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right).$$

Choose $z^* \in S_{Y^*}$ so that $z^*\tilde{T}u = \operatorname{Re} z^*\tilde{T}u = 1$. We may assume that $u = (u_i)_{i \in A}$ is an extreme point of $B_{\ell_\infty^A}$ by Lemma (3.2.5) which implies $|u_i| = 1$ for all $i \in A$. There exist series of non-negative numbers $\sum_{i \in A} \alpha_i$ satisfying $\sum_{i \in A} \alpha_i = 1$ and $(\theta_j)_{j \in A}$ such that $\tilde{T}^* z^* = \left(\alpha_j e^{\theta_j i}\right)_{j \in A}$. Then, we can easily see that $u_j = e^{-\theta_j i}$ for every j such that $\alpha_j \neq 0$.

We get that

$$\operatorname{Re} \sum_{j \in A} \alpha_j e^{\theta_j i} = \operatorname{Re} \tilde{T}^* z^*(1_A) = \operatorname{Re} z^* \tilde{T}(1_A)$$

$$> \operatorname{Re} \tilde{T}^* z^*(u) - \|\tilde{T}(1_A) - \tilde{T}(u)\|$$

$$> 1 - \frac{2\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right)$$

Let us write $= \{j \in A : \operatorname{Re} e^{\theta_j i} > 1 - \epsilon \text{ and } \alpha_j \neq 0\}$. By Lemma (3.2.3) we get

$$\|\tilde{T}P_C\| \geq \sum_{j\in C} \left|\left(\tilde{T}^* z^*\right)_j\right| = \sum_{j\in C} \alpha_j > 1 - \delta\left(\frac{\epsilon}{6}\right).$$

Hence,

$$\|\tilde{T}P_C\| > 1 - \delta\left(\frac{\epsilon}{6}\right) \text{ and } \|\tilde{T}(I - P_C)\| < \frac{\epsilon}{6}.$$

Set $S = \tilde{T}P_C + \tilde{T}(I - P_C)P_u \in S_{\mathcal{L}(\ell_\infty^A, Y)}$ where $P_u(z) = (u_i z_i)_{i\in A}$ for every $z \in \ell_\infty^A$. Then $\| S(\tilde{u}) \| = \| S \| = 1$, where $\tilde{u}_i = u_i$ for all $i \in C$ and $\tilde{u}_i = 1$ for all $i \in A \setminus C$.

We can easily extend S and $\tilde{T}$ to operators from ℓ_∞^n to Y by defining

$$S(e_i) = \tilde{T}(e_i) = 0 \text{ for every } i \in \{1, \ldots, n\} \setminus A.$$

Choose $v \in \ell_\infty^n$ so that $v_i = \tilde{u}_i$ for every $i \in A$ and $v_i = x_i$ for all $i \in A^c$.

We can see that

$$\begin{aligned}
\| S - T \| &\leq \|S - TP_A\| + \|TP_A - T\| \\
&\leq \| S - \tilde{T} \| + \left\|\tilde{T} - \frac{TP_A}{\|TP_A\|}\right\| + \left\|\frac{TP_A}{\|TP_A\|} - TP_A\right\| + \|TP_A - T\| \\
&= \|\tilde{T}(I - P_C)P_u - \tilde{T}(I - P_C)\| + \left\|\tilde{T} - \frac{TP_A}{\|TP_A\|}\right\| \\
&\quad + \left\|\frac{TP_A}{\|TP_A\|} - TP_A\right\| + \|TP_A - T\| \\
&\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \left(\frac{\epsilon}{3}\delta\left(\frac{\epsilon}{6}\right)\right) + \eta(\epsilon) + \left(\frac{\epsilon}{6}\delta\left(\frac{\epsilon}{6}\right)\right) < \epsilon.
\end{aligned}$$

For every $j \in A$, we obtain

$$\begin{aligned}
|x_j - 1| &\leq \sqrt{(1 - \operatorname{Re} x_j)^2 + (\operatorname{Im} x_j)^2} \\
&\leq \sqrt{\eta(\epsilon)^2 + \left(1 - (\operatorname{Re} x_j)^2\right)} \\
&\leq \sqrt{\eta(\epsilon)^2 + 2\eta(\epsilon)}.
\end{aligned}$$

Similarly, for every $j \in C$, from the fact that $\operatorname{Re} u_j = \operatorname{Re} e^{-\theta_j i} = \operatorname{Re} e^{\theta_j i}$ we get

$$|1 - v_j| \leq \sqrt{\epsilon^2 + 2\epsilon}.$$

Therefore,

$$\begin{aligned}
\| x - v \| &= \sup_j |x_j - v_j| \\
&\leq \sup_{j\in A}\left(|x_j - 1| + |1 - v_j|\right) \\
&\leq \sup_{j\in A}|x_j - 1| + \sup_{j\in C}|1 - v_j| \\
&\leq \sqrt{\eta(\epsilon)^2 + 2\eta(\epsilon)} + \sqrt{\epsilon^2 + 2\epsilon} \\
&< \sqrt{\epsilon} + \sqrt{\epsilon^2 + 2\epsilon}.
\end{aligned}$$

Corollary (3.2.7)[136]: Let Y be a uniformly convex Banach space. Then the pair of Banach spaces (c_0, Y) has the BPBP.

Proof. Given $0 < \epsilon < 1$, let $\eta(\epsilon) > 0$ be the positive number in Theorem (3.2.6). Assume that $T \in S_{\mathcal{L}(c_0,Y)}$ and $x \in S_{c_0}$ satisfy $\| Tx \| > 1 - \eta(\epsilon)^2$ and also $\| Tx \| > 1 - \delta(\epsilon)$. Choose $\tilde{x} \in S_{c_0}$ which has a finite support A such that $\| T\tilde{x} \| > 1 - \eta(\epsilon)^2$, $\| T\tilde{x} \| > 1 - \delta(\epsilon)$ and $\| x - \tilde{x} \| < \epsilon$. It follows from Lemma (3.2.4) that $\|TP_A - T\| < \epsilon$.

Consider $T_A \in \mathcal{L}(\ell_\infty^A, Y)$ which is a restriction of TP_A on ℓ_∞^A. It follows from Theorem (3.2.6) that there exist $\tilde{S} \in S_{\mathcal{L}(\ell_\infty^A,Y)}$ and $\tilde{u} \in S_{\ell_\infty^A}$ such that $\| \tilde{S}\tilde{u} \| = 1$, $\left\| \tilde{S} - \frac{T_A}{\|T_A\|} \right\| < 2\epsilon$ and $\| \tilde{x} - \tilde{u} \| < \sqrt{\epsilon} + \sqrt{\epsilon^2 + 2\epsilon}$.

Define a bounded operator $S: c_o \to Y$ by $S(e_i) = \tilde{S}(e_i)$ for every $i \in A$ and $S(e_i) = 0$ for every $i \in A^c$. Choose $u \in c_0$ so that $u_i = \tilde{u}_i$ for all $i \in A$ and $u_i = x_i$ for $i \in A^c$. It is easy to see that $\| Su \| = 1$,

$$\begin{aligned} \| S - T \| &\le \|S - TP_A\| + \|TP_A - T\| \\ &\le \left\| SP_A - \frac{TP_A}{\|TP_A\|} \right\| + \left\| \frac{TP_A}{\|TP_A\|} - TP_A \right\| + \|TP_A - T\| \\ &< 2\epsilon + \eta(\epsilon)^2 + \epsilon, \end{aligned}$$

and

$$\| u - x \| = \|P_A(u - x)\| = \| \tilde{u} - \tilde{x} \| < \sqrt{\epsilon} + \sqrt{\epsilon^2 + 2\epsilon}.$$

In [155], Lindenstrauss proved that if Y is strictly convex and if there is a non-compact operator from c_0 into Y, then the set of norm-attaining operators in $\mathcal{L}(c_0, Y)$ is not dense. In the real case, we show that if Y is strictly convex but not uniformly convex, then (c_0, Y) cannot have the $BPBP$. It is worth mentioning that if Y is strictly convex but not uniformly convex, then (ℓ_1, Y) cannot have the $BPBP$ [138].

Theorem (3.2.8)[136]: Let X be a real Banach space c_0 or $\ell_\infty^n (n \ge 2)$ and let Y be a real strictly convex space. Then (X, Y) has the $BPBP$ if and only if Y is uniformly convex. In particular, if the pair (ℓ_∞, Y) has the $BPBP$, then Y is uniformly convex.

Proof. We prove only the case c_0. The other cases follow from the same argument that we will use. Assume that Y is not uniformly convex. There exist $\epsilon > 0$ and sequences $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \subset S_Y$ such that $\lim_{i\to\infty} \left\| \frac{x_i + y_i}{2} \right\| = 1$ and $\|x_i - y_i\| > \epsilon$.

Let $(e_i)_{i=1}^\infty$ be the canonical basis of c_0. For every $i \in \mathbb{N}$, define $T_i \in S_{\mathcal{L}(c_0,Y)}$ by $T_i(e_1 + e_2) = x_i, T_i(e_1 - e_2) = y_i$ and $T_i(e_k) = 0$ for all $k \in \mathbb{N} \setminus \{1,2\}$. We can see that $\lim_{i\to\infty} \|T_i(e_1)\| = 1$.

Assume that (c_0, Y) has the $BPBP$ with positive numbers $\eta(\epsilon)$ and $\beta(\epsilon)$. Choose $j \in \mathbb{N}$ so that $\|T_j(e_1)\| > 1 - \eta(\epsilon/2)$. Then there exist $\tilde{T} \in S_{\mathcal{L}(c_0,Y)}$ and $x \in S_{c_0}$ satisfying

$$\| \tilde{T}x \| = 1, \ \|\tilde{T} - T_j\| < \epsilon/2, \ \text{and} \ \|x - e_1\| < \beta(\epsilon/2) < 1.$$

By Lemma (3.2.5) and the fact that $\|x - e_1\| < \beta(\epsilon/2) < 1$, we can assume that $x = e_1$. Hence, it follows from the strict convexity of Y that $\tilde{T}(e_1 + e_2) = \tilde{T}(e_1 - e_2) = \tilde{T}(e_1)$, which implies that

$$\begin{aligned} \|x_j - y_j\| &= \|T_j(e_1 + e_2) - T_j(e_1 - e_2)\| \\ &\le \|T_j(e_1 + e_2) - \tilde{T}(e_1 + e_2)\| + \|\tilde{T}(e_1 - e_2) - T_j(e_1 - e_2)\| \\ &< \epsilon. \end{aligned}$$

This is a contradiction.

We say that the Bishop-Phelps-Bollobás theorem holds for bilinear forms on $X \times Y$ if, given $\epsilon > 0$, there exist $\eta(\epsilon)$ and $\beta(\epsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for all $\phi \in S_{\mathcal{L}^2(X\times Y)}$, if $x \in S_X, y \in S_Y$ satisfy $|\phi(x,y)| > 1-\eta(\epsilon)$, then there exist points $x_\epsilon \in S_X, y_\epsilon \in S_Y$ and a bilinear form $\phi_\epsilon \in S_{\mathcal{L}^2(X\times Y)}$ that satisfy

$$|\phi_\epsilon(x_\epsilon, y_\epsilon)| = 1,\ \|x - x_\epsilon\| < \beta(\epsilon),\ \|y - y_\epsilon\| < \beta(\epsilon),\ \|\phi - \phi_\epsilon\| < \epsilon.$$

Theorem (3.2.9)[136]: ([146]). Assume that Y is uniformly convex. Then the BishopPhelps-Bollobás theorem holds for bilinear forms on $X \times Y$ if and only if the pair (X, Y^*) has the $BPBP$.

In general, the Bishop-Phelps-Bollobás theorem does not hold for bilinear forms ([139],[147],[151]). However, Cheng and Dai [146] showed that the BishopPhelps-Bollobás theorem holds for bilinear forms on $\ell_1 \times \ell_p (1 < p < \infty)$ by obtaining Theorem (3.2.9) and they asked whether the Bishop-Phelps-Bollobás theorem holds for bilinear forms on $c_0 \times \ell_p(1 < p < \infty)$. Using Corollary (3.2.7) and Theorem (3.2.9), we get an affirmative answer to this question.

Corollary (3.2.10)[136]: The Bishop-Phelps-Bollobás theorem holds for bilinear forms on $c_0 \times \ell_p$ for $1 < p < \infty$.

Section (3.3): The Bishop-Phelps-Bollobás Theorem for Bilinear Forms:

E. Bishop and R. Phelps in [161] proved that every continuous linear functional x^* on a Banach space X can be approximated, uniformly on the closed unit ball of X, by a continuous linear functional y^* that attains its norm. This result is called the Bishop-Phelps Theorem. Shortly thereafter, B. Bollobás in [162] showed that this approximation can be done in such a way that, moreover, the point at which x^* almost attains its norm is close in norm to a point at which y^* attains its norm. This is a "quantitative version" of the Bishop-Phelps Theorem, known as the Bishop-Phelps-Bollob´as Theorem. As usual, by B_X and S_X we will denote the closed unit ball and the unit sphere of a Banach space X, respectively, and X^* will be the dual of X.

Theorem (3.3.1)[157]: (Bishop-Phelps-Bollobás Theorem, [163], Theorem 16.1). Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1, \|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

[158] proved versions of the Bishop-Phelps-Bollobás Theorem for operators. Amongst them is shown a characterization of the Banach spaces Y satisfying an analogous result of the Bishop-Phelps-Bollobás Theorem for operators from ℓ_1 into Y. There are also positive results for operators from $L_1(\mu)$ to $L_\infty(\nu)$ [160] and for operators from an Asplund space to $C(K)$ [159]. For more results on the subject, also see [164].

Choi and Song initiated the study of versions of the Bishop-Phelps-Bollobás Theorem for bilinear forms [166], showing that this theorem does not hold for $\ell_1 \times \ell_1$. For two Banach spaces X and Y, by using the natural identification of the space of the continuous bilinear forms on $X \times Y$ and the space $L(X, Y^*)$ of linear and continuous operators from X into Y^*, it is clear that the pair (X, Y^*) satisfies the BPBP for operators if the pair (X, Y) has the BPBP for bilinear forms. The converse is not true even for $X = Y = \ell_1$ (see [166] and [158]).

We provide classes of spaces satisfying a version of the Bishop-Phelps-Bollobás Theorem for bilinear forms. All the Banach spaces considered will be over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Except when explicitly stated, all results hold for the real and the complex cases. If A is a subset of a linear space, we will denote by co A and $|co|A$ the convex hull and the absolutely convex hull of A, respectively. For a family of Banach spaces $X_1, \dots, X_n, Y$ we denote by $L^n(X_1 \times \cdots \times X_n, Y)$ the Banach space of all continuous n-linear mappings from $X_1 \times \cdots \times X_n$ to Y. When Y is the scalar field we remove it, i.e. we write $L^n(X_1 \times \cdots \times X_n)$. If $n = 1$ we simply write $L(X, Y)$ and X^* when Y is the scalar field.

In [158] the following property was introduced to study versions of the Bishop-Phelps-Bollobás Theorem for operators. To deal with the bilinear case we need a natural modification of this property.

Definition (3.3.2)[157]: ([158], Definition 1.1). If X and Y are Banach spaces, the pair (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators (for short, BPBP for operators) if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ satisfying the following conditions:

$$\|S(u_0)\| = 1,\ \|u_0 - x_0\| < \beta(\varepsilon), \text{ and } \| S - T \| < \varepsilon.$$

Definition (3.3.3)[157]: ([166]). For two Banach spaces X and Y, the pair (X, Y) satisfies the Bishop-Phelps-Bollobás property for bilinear forms (for short, BPBP for bilinear forms) if for every $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for any $A \in S_{L^2(X\times Y)}$, if $(x_0, y_0) \in S_X \times S_Y$ is such that $|A(x_0, y_0)| > 1 - \eta(\varepsilon)$, then there are $B \in S_{L^2(X\times Y)}$ and $(u_0, v_0) \in S_X \times S_Y$ satisfying the following conditions:

$$|B(u_0, v_0)| = 1,\ \|u_0 - x_0\| < \beta(\varepsilon),\ \|v_0 - y_0\| < \beta(\varepsilon) \text{ and } \| B - A \| < \varepsilon.$$

In this definition we can replace $(x_0, y_0) \in S_X \times S_Y$ by $(x_0, y_0) \in B_X \times B_Y$. Also it is not difficult to check that $\lim_{\varepsilon\to 0^+}\eta(\varepsilon) = 0$.

We will also consider the BPBP for n-linear mappings, which is defined.

We devoted to positive results of spaces satisfying the BPBP for n-linear or bilinear forms. Indeed if $X_i (1 \le i \le n)$ are uniformly convex Banach spaces, then for every Banach space Y, the BPBP for n-linear mappings from $X_1 \times \cdots \times X_n$ to Y is satisfied. Up to now, this is the only sufficient condition known that implies BPBP even for bilinear forms. As a consequence of the previous result, one obtains the corresponding condition for operators whose domain is uniformly convex. We already mentioned that the BPBP for bilinear forms on $X \times Y$ implies the BPBP for operators from X into Y^*, and the converse is no longer true. However, if Y is uniformly convex, then the converse also holds, a result that has also been proved independently by Dai [167]. As a consequence, if Y is a uniformly convex Banach space whose dual satisfies some isometric property (called AHSP), then the pair (ℓ_1, Y) satisfies the BPBP for bilinear forms. This result can be applied for instance to any $L_p(\mu)$ for $1 < p < \infty$.

We obtain a characterization of the Banach spaces Y satisfying that the pair (ℓ_1, Y) has the BPBP for bilinear forms. In order to do this, we introduce a geometrical property and prove that many classical Banach spaces enjoy this property, including the finite-dimensional normed spaces, uniformly smooth spaces, $C(K)$ and $K(H)$. On the other hand, the pair $(\ell_1, L_1(\mu))$ does not satisfy the BPBP for bilinear forms for any infinite-dimensional $L_1(\mu)$, a result that was known only when $L_1(\mu) = \ell_1$. Let us notice that the set of norm attaining bilinear forms is dense in $L^2(\ell_1 \times L_1(\mu))$ (see [172]). For operators, the BPBP in the case $(\ell_1, L_\infty(\mu))$ for every measure μ is also satisfied (see [158]).

We recall that a Banach space X is uniformly convex if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_X, \ \frac{\| u+v \|}{2} > 1-\delta \Rightarrow \| u-v \| < \varepsilon.$$

In such a case, the modulus of convexity of X is given by

$$\delta(\varepsilon) := \mathrm{i} \left\{1 - \frac{\| u+v \|}{2} : u, v \in B_X, \| u-v \| \geq \varepsilon\right\}$$

Given a bounded subset A of X, an element $x^* \in X^*$ and $\alpha > 0$, the slice $S(A, x^*, \alpha)$ is the subset of A given by

$$S(A, x^*, \alpha) := \left\{z \in A : \operatorname{Re} x^*(z) > \sup_{x \in A} \operatorname{Re} x^*(x) - \alpha\right\}$$

The following simple lemma will be useful in the proof of the main result.

Lemma (3.3.4)[157]: If X is uniformly convex, then for every $\varepsilon > 0$,

$$\operatorname{diam} S(B_X, x^*, \delta(\varepsilon)) \leq \varepsilon, \quad \text{for all } x^* \in S_{X^*}.$$

Proof. Indeed if $x^* \in S_{X^*}$ and we choose $x, z \in S(B_X, x^*, \delta(\varepsilon))$, then

$$\| x+z \| \geq |x^*(x+z)| > 2(1-\delta(\varepsilon)).$$

So we deduce $\| x-z \| < \varepsilon$. Since this holds for every pair of elements x, z in the slice, then $\operatorname{diam} S(B_X, x^*, \delta(\varepsilon)) \leq \varepsilon$, as we wanted to show.

We show that the Bishop-Phelps-Bollobás Theorem holds for n linear mappings defined on uniformly convex Banach spaces. Given Banach spaces $X_j (1 \leq j \leq n)$ we will denote by $X = (X_1 \times \cdots \times X_n, \|\cdot\|_\infty)$, $\|\cdot\|_\infty$ being the supremum norm, and by S the set given by

$$S := \left\{x = (x_j) \in \prod_{j=1}^{n} X_j : x_j \in S_{X_j}, \text{ for all } 1 \leq j \leq n\right\}$$

Theorem (3.3.5)[157]: Let $X_1, \dots, X_n, Y$ be Banach spaces and assume that every X_j is uniformly convex with modulus of convexity $\delta_j, 1 \leq j \leq n$. Given $\varepsilon > 0$, if $0 < \eta < \min\{\delta_j(\varepsilon) : 1 \leq j \leq n\} \frac{\varepsilon}{8+2\varepsilon}$, then for every $\bar{A} \in S_{L^n}(X_1 \times \cdots \times X_n, Y)$ and every $x_0 \in S$ such that $\|A(x_0)\| > 1-\eta$, there exist a point $z_0 \in S$ and $B \in S_{L^n(X_1 \times \cdots \times X_n, Y)}$ satisfying the following conditions:

$$\|B(z_0)\| = 1, \ \|z_0 - x_0\|_\infty \leq \varepsilon \text{ and } \| B - A \| < \varepsilon$$

Moreover, if A belongs to some linear subspace of $L^n(X_1 \times \cdots \times X_n, Y)$ containing the finite-type n-linear mappings, then B belongs to the same subspace.

Proof. Let $0 < \varepsilon < 1$. Since every X_j is uniformly convex, by Lemma (3.3.4),

$$\operatorname{diam} S\left(B_{X_j}, f_j, \delta_j(\varepsilon)\right) \leq \varepsilon$$

for all $f_j \in S_{X_j^*}$ and every $1 \leq j \leq n$. We define $\alpha := \min\{\delta_j(\varepsilon) : 1 \leq j \leq n\}$ and choose a real number η such that

$$0 < \eta < \frac{\alpha\varepsilon}{8+2\varepsilon}$$

hence

$$1 + \frac{\varepsilon}{4}\left(1 - \frac{\alpha}{2}\right) < \left(1 + \frac{\varepsilon}{4}\right)(1-\eta).$$

Let $A \in S_{L^n}(X_1 \times \cdots \times X_n, Y)$ and $x_0 = (x_{01}, \ldots, x_{0n}) \in S$ such that $\|A(x_0)\| > 1 - \eta$. For each $1 \le j \le n$ there is a functional $x_j^* \in S_{X_j^*}$ such that $x_j^*(x_{0j}) = 1$, and we know that

$$\operatorname{diam} S\left(B_{X_j}, x_j^*, \alpha\right) \le \varepsilon, \text{ for all } 1 \le j \le n. \tag{4}$$

We define the mapping $C \in L^n(X_1 \times \cdots \times X_n, Y)$ by

$$C(x) := A(x) + \frac{\varepsilon}{4}\left(\prod_{j=1}^{n} x_j^*(x_j)\right) A(x_0) \; \left(x = (x_j) \in X\right)$$

Clearly,

$$C(x_0) = \left(1 + \frac{\varepsilon}{4}\right) A(x_0)$$

and thus

$$\|C(x_0)\| = \left(1 + \frac{\varepsilon}{4}\right) \|A(x_0)\| > \left(1 + \frac{\varepsilon}{4}\right)(1 - \eta). \tag{5}$$

Let $\mathbb{T}$ be the set given by $\mathbb{T} := \{\lambda \in \mathbb{K} : |\lambda| = 1\}$. Thus, if $1 \le j \le n$ and $z_j \in B_{X_j} \setminus \mathbb{T}S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right)$, it is satisfied that

$$\left|x_j^*(z_j)\right| \le 1 - \frac{\alpha}{2}.$$

Hence, if $z \in B_X$ and there exists $1 \le j \le n$ with $z_j \notin \mathbb{T}S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right)$, we have that

$$\| C(z) \| \le 1 + \frac{\varepsilon}{4}\left|x_j^*(z_j)\right| \le 1 + \frac{\varepsilon}{4}\left(1 - \frac{\alpha}{2}\right)$$

From (5) and the previous inequality it follows that

$$\|C(x_0)\| > \| C(z) \| \text{, for all } z \in B_X \setminus \prod_{j=1}^{n} \mathbb{T}S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right)$$

This implies that $\| C \| = \sup\left\{\| C(z) \| : z \in \prod_{j=1}^{n} S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right)\right\}$, and

$$z \in B_X,\ \|C(z)\| > \left(1 + \frac{\varepsilon}{4}\right)(1 - \eta) \Rightarrow z \in \prod_{j=1}^{n} \mathbb{T}S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right). \tag{6}$$

It is also clear that

$$\| C - A \| \le \frac{\varepsilon}{4}. \tag{7}$$

By (5) we can choose $0 < \gamma < \frac{\varepsilon}{4n}$ such that

$$\|C(x_0)\| - \left(1 + \frac{\varepsilon}{4}\right)(1 - \eta) > n\gamma. \tag{8}$$

Let us consider the mapping $\phi : B_X \longrightarrow \mathbb{R}$ given by

$$\phi(x) := \| C(x) \| \;\; (x \in B_X),$$

which is continuous and bounded. Since X_j is uniformly convex for every $1 \le j \le n$, the space X is reflexive, so it has the Radon-Nikodým property. Hence B_X is a Radon-Nikodým set and we can apply [176] to obtain an element $u_0 \in B_X$ and functionals $z_j^* \in X_j^* (1 \le j \le n)$ such that

$$0 < \|z_j^*\| \le \gamma, \quad \text{for all } 1 \le j \le n \tag{9}$$

and the function

$$x = (x_1, \dots, x_n) \mapsto \| C(x) \| + \sum_{j=1}^{n} \operatorname{Re} z_j^*(x_j) \ (x = (x_j) \in B_X)$$

attains its maximum at $u_0 \in B_X$. By using that C is an n-linear mapping and the unit ball of a Banach space is balanced, it is immediate to deduce that

$$u_{0j} \in S_{X_j}, \ \operatorname{Re} z_j^*(u_{0j}) = z_j^*(u_{0j}) = |z_j^*(u_{0j})|, \ \text{ for all } 1 \le j \le n,$$

and indeed it is satisfied that

$$\|C(x)\| \le \| C(x) \| + \sum_{j=1}^{n} |z_j^*(x_j)| \le \|C(u_0)\| + \sum_{j=1}^{n} |z_j^*(u_{0j})|, \text{ for all } x = (x_j) \in B_X.$$

As a consequence $\| C \| \le \|C(u_0)\| + \sum_{j=1}^{n} |z_j^*(u_{0j})|$, and so by using (9) and (8) we have that

$$\|C(u_0)\| \ge \| C \| - n\gamma > \left(1 + \frac{\varepsilon}{4}\right)(1 - \eta) > 0.$$

By now using (6) we obtain

$$u_0 \in \prod_{j=1}^{n} \mathbb{T}S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right). \tag{10}$$

Let us write $y_0 := \frac{C(u_0)}{\|C(u_0)\|}$, and for each $1 \le j \le n$ choose $u_j^* \in S_{X_j^*}$ such that $u_j^*(u_{0j}) = 1$. Now we define

$$D(x) := C(x) + \left(\sum_{j=1}^{n} z_j^*(x_j) \prod_{\substack{i=1 \\ i \ne j}}^{n} u_i^*(x_i) \right) y_0 \ (x = (x_j) \in B_X)$$

It is clear that $D \in L^n(X_1 \times \cdots \times X_n, Y)$ and

$$\| D - C \| \le \sum_{j=1}^{n} \|z_j^*\| \le n\gamma < \frac{\varepsilon}{4}. \tag{11}$$

For any $x = (x_j) \in B_X$ we have that

$$\|D(x)\| \le \| C(x) \| + \sum_{j=1}^{n} |z_j^*(x_j)| \left(\prod_{\substack{i=1 \\ i \ne j}}^{n} \|u_i^*\| \right)$$
$$\le \| C(x) \| + \sum_{j=1}^{n} |z_j^*(x_j)| \le \|C(u_0)\| + \sum_{j=1}^{n} |z_j^*(u_{0j})|$$

Also it is satisfied that

$$\|D(u_0)\| = \left\| C(u_0) + \sum_{j=1}^{n} z_j^*(u_{0j}) y_0 \right\|$$
$$= \left\| C(u_0) + \sum_{j=1}^{n} |z_j^*(u_{0j})| y_0 \right\|$$

$$= \|C(u_0)\| + \sum_{j=1}^{n} |z_j^*(u_{0j})|$$

As a consequence D attains its norm at u_0.

In view of (10), for every $1 \le j \le n$ there is $\lambda_j \in \mathbb{T}$ such that $\lambda_j u_{0j} \in S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right)$. By using (4) we deduce that

$$\|\lambda_j u_{0j} - x_{0j}\| \le \operatorname{diam} S\left(B_{X_j}, x_j^*, \frac{\alpha}{2}\right) \le \varepsilon, \text{ for all } 1 \le j \le n.$$

If we write $z_0 := (\lambda_j u_{0j})$, then $z_0 \in B_X, D$ attains its norm at z_0 and

$$\|z_0 - x_0\|_\infty \le \varepsilon. \tag{12}$$

Let us notice that $D - A$ is an n-linear mapping of finite type. Indeed it is the sum of (at most) $n + 1 n$-linear mappings of the form

$$x \mapsto \prod_{j=1}^{n} x_j^*(x_j) y \; (x = (x_j) \in X),$$

where $x_j^* \in X_j^*$ for every j and $y \in Y$. By taking $B := \frac{D}{|D}$ we have that $B \in S_{L^n(X_1 \times \cdots \times X_n, Y)}, B$ attains its norm at z_0 and

$$\begin{aligned} \|B - A\| &\le \|B - D\| + \|D - A\| \\ &\le |1 - \|D\|| + \|D - A\| \le 2\|D - A\| \\ &\le 2(\|D - C\| + \|C - A\|) < \varepsilon \text{ (by (11) and (7))}. \end{aligned}$$

We have proved that the BPBP for n-linear mappings from $X_1 \times \cdots \times X_n$ to Y is satisfied.

We deduce the following immediate consequence for operators:

Corollary (3.3.6)[157]: Let X and Y be Banach spaces, and assume that X is uniformly convex. Then given $\varepsilon > 0$, there is $\eta > 0$ such that for every $R \in S_{L(X,Y)}$ and $x_0 \in S_X$ such that $\|R(x_0)\| > 1 - \eta$, there exist a point $u_0 \in S_X$ and $T \in S_{L(X,Y)}$ satisfying the following conditions:

$$\|T(u_0)\| = 1, \; \|u_0 - x_0\| \le \varepsilon \text{ and } \|T - R\| < \varepsilon.$$

Actually any $0 < \eta < \frac{\varepsilon\delta(\varepsilon)}{8+2\varepsilon}$ satisfies the above condition.

In addition, if R belongs to some linear space $M \subset L(X, Y)$ containing the finite-rank operators, then T also belongs to M.

The BPBP for operators from X into Y^* is satisfied if the pair (X, Y) satisfies the BPBP for bilinear forms and the converse is not true. We will provide a class of Banach spaces for which the converse holds. The next result has also been proved independently by Dai [167].

Proposition (3.3.7)[157]: Let X and Y be Banach spaces and assume that Y is uniformly convex. If the pair (X, Y^*) has the Bishop-Phelps-Bollobás property for operators, then (X, Y) satisfies the Bishop-Phelps-Bollobás property for bilinear forms.

Proof. Given $\varepsilon_0 > 0$, we can see that the pair (X, Y^*) has the BPBP for operators for $\varepsilon > 0$ small enough such that $\max\{\varepsilon, \beta(\varepsilon)\} < \min\left\{\varepsilon_0, \frac{1}{3}\delta(\varepsilon_0)\right\}$, where $\delta(\varepsilon_0)$ is the modulus of convexity of Y. Let $0 < \eta < \min\left\{\eta(\varepsilon), \frac{1}{3}\delta(\varepsilon_0)\right\}$. Take $A \in S_{L^2(X \times Y)}$ and assume that $(x_0, y_0) \in S_X \times S_Y$ is such that $|A(x_0, y_0)| > 1 - \eta$. By rotating A if necessary, we can assume that $|A(x_0, y_0)| = A(x_0, y_0)$. If $T \in L(X, Y^*)$ is the operator associated to A, then we know that

$$\| T \| = \| A \| = 1, \text{ and } \operatorname{Re} T(x_0)(y_0) = \operatorname{Re} A(x_0, y_0) > 1 - \eta.$$

By assumption, there are an operator $S \in S_{L(X,Y^*)}$ and an element $z_0 \in S_X$ satisfying

$$\|S(z_0)\| = 1, \ \|z_0 - x_0\| < \beta(\varepsilon), \ \| S - T \| < \varepsilon.$$

As a consequence we have

$$\begin{aligned} \operatorname{Re} S(z_0)(y_0) \quad &> \operatorname{Re} T(z_0)(y_0) - \varepsilon > \operatorname{Re} T(x_0)(y_0) - \varepsilon - \|z_0 - x_0\| \\ &> 1 - \eta - \varepsilon - \beta(\varepsilon) > 1 - \delta(\varepsilon_0). \end{aligned}$$

Since Y is uniformly convex, then Y is reflexive, and so there is $u_0 \in S_Y$ such that $S(z_0)(u_0) = 1$

Therefore we have

$$1 - \frac{\delta(\varepsilon_0)}{2} < \operatorname{Re} \frac{S(z_0)(u_0 + y_0)}{2} \leq \|Sz_0\| \frac{\|u_0 + y_0\|}{2} \leq \frac{\|u_0 + y_0\|}{2}$$

By using that Y is uniformly convex we deduce

$$\|u_0 - y_0\| < \varepsilon_0.$$

Hence, if we denote by B the bilinear form associated to S, then we have that

$$B(z_0, u_0) = S(z_0)(u_0) = 1 = \| B \|$$

$$\| B - A \| < \varepsilon_0, \ \|z_0 - x_0\| < \beta(\varepsilon) < \varepsilon_0, \ \|u_0 - y_0\| < \varepsilon_0$$

So the bilinear form B satisfies all the required conditions.

Now we will recall the isometric property that was already used in [158] to describe the Banach spaces Y such that the pair (ℓ_1, Y) satisfies the BPBP for operators.

Definition (3.3.8)[157]: ([158], Remark 3.2). A Banach space X is said to have the Approximate Hyperplane Series property (for short, AHSP) if for every $\varepsilon > 0$ there exist $0 < \eta, \delta < \varepsilon$ such that for every sequence $(x_k) \subset S_X$ and every convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta,$$

there exist a subset $A \subset \mathbb{N}$, a subset $\{z_k : k \in A\} \subset S_X$ and $x^* \in S_{X^*}$ satisfying

(i) $\sum_{k \in A} \alpha_k > 1 - \delta$, and

(ii) (a) $\|z_k - x_k\| < \varepsilon$ for all $k \in A$,

(b) $x^*(z_k) = 1$ for each $k \in A$

Corollary (3.3.9)[157]: If Y is a uniformly convex Banach space and Y^* has the Approximate Hyperplane Series property, then the pair (ℓ_1, Y) has the Bishop-PhelpsBollobás property for bilinear forms.

The previous statement is a consequence of the fact that the pair (ℓ_1, Y^*) has the BPBP for operators only when Y^* has the AHSP (see [158], Theorem 4.1) and Proposition (3.3.7). Examples of classes of spaces having the AHSP can be found in [158] and [164]. For instance, it is known that the finite-dimensional spaces, $C(K)$, $L_1(\mu)$ and uniform convex Banach spaces have this property. Indeed, every almost CL-space satisfies the AHSP (see [170] for the definitions and also [164]). Furthermore every lush space has the AHSP (see [165]). Also, spaces whose dual norm satisfies some uniform condition of smoothness (USSD) at some boundary have this property (see [169] and [164]).

By looking directly at the proof of the BPBP for bilinear forms on the product $\ell_1 \times Y$ (Y a Banach space) we will obtain a more general result than the one appearing in Corollary (3.3.9). The Banach spaces Y such that the pair (ℓ_1, Y) satisfies the BPBP for bilinear forms do have a geometric property that we will characterize.

Definition (3.3.10)[157]: For a Banach space Y we will say that the pair (Y, Y^*) satisfies the Approximate Hyperplane Series property (for short, AHSP) if for every $\varepsilon > 0$ there are $0 < \delta, \eta < \varepsilon$ satisfying that for every convex series $\sum_n \alpha_n$ and for every sequence of functionals $\{y_n^*\}$ in S_{Y*} and $y_0 \in S_Y$ such that $\operatorname{Re} \sum_n \alpha_n y_n^*(y_0) > 1 - \eta$, there are a subset $C \subset \mathbb{N}, \{z_k^*: k \in C\} \subset S_{Y*}$ and $z_0 \in S_Y$ such that

$$\sum_{k \in C} \alpha_k > 1 - \delta, \ \|z_k^* - y_k^*\| < \varepsilon, \ z_k^*(z_0) = 1, \ \text{ for all } k \in C \text{ and } \|z_0 - y_0\| < \varepsilon.$$

It is not difficult to check that by assuming in Definition (3.3.10) that the sequence $\{y_n^*\}$ is contained in B_{Y^*}, an equivalent condition is obtained. It is also clear that if the pair (Y, Y^*) has the AHSP, then Y^* has the AHSP. As we will see later, as a consequence of our results and previous work, both properties are not equivalent (see Proposition (3.3.22)).

The following elementary result is slightly more general than [158], and the proof is almost the same.

Lemma (3.3.11)[157]: Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for every n, let $\eta > 0$ and let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n \leq 1$. Assume also that $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N}: \operatorname{Re} c_i > r\}$ satisfies the estimate

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta}{1 - r}$$

The following result will be useful in order to provide examples of spaces Y satisfying that the pair (Y, Y^*) has the AHSP. It has the advantage that the use of convex series is avoided.

Proposition (3.3.12)[157]: Let Y be a Banach space. If for every $\varepsilon > 0$ there is $\delta > 0$ satisfying that for every finite set F and for every finite sequence of functionals $\{y_i^*: i \in F\} \subset S_{Y^*}$ and $y_0 \in S_Y$ such that $\operatorname{Re} y_i^*(y_0) > 1 - \delta$, for every $i \in F$ there are $\{z_k^{**}: i \in F\} \subset S_{Y*}$ and $z_0 \in S_Y$ such that

$$\|z_i^* - y_i^*\| < \varepsilon, \ z_i^*(z_0) = 1, \text{ for all } i \in F, \ \text{ and } \ \|z_0 - y_0\| < \varepsilon,$$

then the pair (Y, Y^*) satisfies the Approximate Hyperplane Series property.

Proof. Given $\varepsilon > 0$ there is $\delta > 0$ satisfying the assumption and we can clearly assume $\delta < \varepsilon < 1$. Let us consider a convex series $\sum \alpha_n$, a sequence $\{y_n^*\}$ in S_{Y^*} and $y_0 \in S_Y$ such that

$$\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n y_n^*(y_0) > 1 - \delta^2,$$

and choose N large enough such that $\operatorname{Re} \sum_{k=1}^{N} \alpha_k y_k^*(y_0) > 1 - \delta^2$. Let us define

$$C := \{n \in \mathbb{N}: n \leq N, \operatorname{Re} y_n^*(y_0) > 1 - \delta\}$$

By Lemma (3.3.11) applied with $\eta = \delta^2$ and $r = 1 - \delta$, we obtain

$$\sum_{n \in C} \alpha_n > 1 - \delta. \tag{13}$$

By using the assumption for the finite set C we obtain a finite set

$$\{z_i^*: i \in C\} \subset S_{Y^*}, \text{ and an element } z_0 \in S_Y,$$

satisfying

$$\|z_i^* - y_i^*\| < \varepsilon, \ z_i^*(z_0) = 1, \text{ for all } i \in C, \ \|z_0 - y_0\| < \varepsilon,$$

and we also know that $\sum_{n\in C}\alpha_n > 1-\delta$, as we wanted to show. Let us notice that the above condition is a version of the Bishop-Phelps-Bollobás Theorem which uses any finite set of functionals instead of one. We will see later that this change really makes a difference.

For a Banach space X and $x \in S_X$, we will denote

$$D(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}.$$

The set-valued mapping $D: S_X \mapsto S_X *$ is called the duality mapping of X. It is clear that from Proposition (3.3.12) the following result can be deduced.

Corollary (3.3.13)[157]: Let X be a Banach space. If for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in S_X$ there is $y \in S_X$ satisfying

(a) $\| y - x \| < \varepsilon$,

(b) if $x^* \in S_{X*}$ satisfies $\operatorname{Re} x^*(x) > 1-\delta$, then dist $(x^*, D(y)) < \varepsilon$, then the pair (X, X^*) satisfies the Approximate Hyperplane Series property.

Now, we characterize the Banach spaces Y satisfying that the pair (ℓ_1, Y) has the BPBP for bilinear forms. The following elementary fact will be useful for this purpose.

Lemma (3.3.14)[157]: Let z be a complex number with $|z| \le 1$ and $0 < r < 1$. If $\operatorname{Re} z > r$, then $|z-1|^2 < 2(1-r)$.

Proof. It is clear that

$$\begin{aligned}|z-1|^2 &= (\operatorname{Re} z - 1)^2 + \operatorname{Im}^2 z \\ &= \operatorname{Re}^2 z + \operatorname{Im}^2 z + 1 - 2\operatorname{Re} z \le 2(1-\operatorname{Re} z) < 2(1-r)\end{aligned}$$

Theorem (3.3.15)[157]: Let Y be a Banach space. Then the pair (ℓ_1, Y) has the BishopPhelps-Bollobás property for bilinear forms if and only if the pair (Y, Y^*) satisfies the Approximate Hyperplane Series property.

Proof. Assume that the pair (ℓ_1, Y) has the BPBP for bilinear forms. Then for every $\varepsilon > 0$ there are $\beta(\varepsilon)$ and $\eta(\varepsilon)$ satisfying the conditions of Definition (3.3.3).

Given $0 < \varepsilon_0 < 1$, we are going to show that (Y, Y^*) has the AHSP for $\delta = \beta(\varepsilon) + \sqrt{\beta(\varepsilon)}$ and $\eta = \eta(\varepsilon)$, where $\varepsilon > 0$ is so that

$$3\sqrt{2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon)} + (4\beta(\varepsilon))^{\frac{1}{4}} + \varepsilon < \varepsilon_0. \tag{14}$$

Let us take a convex series $\sum_n \alpha_n$, a sequence of functionals $\{y_n^*\}$ in S_{Y^*} and $y_0 \in S_Y$ such that $\operatorname{Re}\sum_n \alpha_n y_n^*(y_0) > 1 - \eta(\varepsilon)$.

Now we define the bilinear form A on $\ell_1 \times Y$ given by

$$A(x,y) = \sum_{n=1}^{\infty} x(n) y_n^*(y) \ ((x,y) \in \ell_1 \times Y, x = (x(n)))$$

It is clear that A is well defined and, moreover, $A \in S_{L^2(\ell_1 \times Y)}$ since for every $(x,y) \in \ell_1 \times Y$ it holds that

$$|A(x,y)| \le \sum_{n=1}^{\infty} |x(n)||y_n^*(y)| \le \sum_{n=1}^{\infty} |x(n)|\|y\| = \|x\|_1 \|y\|$$

and also $A(e_1, y) = y_1^*(y)$, so $\|A\| \ge \sup_{y\in S_Y} |y_1^*(y)| = \|y_1^*\| = 1$. Here $\|\cdot\|_1$ denotes the usual norm and (e_n) the canonical basis in ℓ_1.

The condition $\operatorname{Re}\sum_n \alpha_n y_n^*(y_0) > 1 - \eta(\varepsilon)$ implies that the element $x_0 = (\alpha_n)$ satisfies $|A(x_0, y_0)| > 1 - \eta(\varepsilon)$. By assumption we can find elements $(u_0, v_0) \in S_{\ell_1} \times S_Y$ and a bilinear form $B \in S_{L^2(\ell_1 \times Y)}$ such that

$$\|B-A\|<\varepsilon,\ \|u_0-x_0\|_1<\beta(\varepsilon),\ \|v_0-y_0\|<\beta(\varepsilon),\ |B(u_0,v_0)|=1. \tag{15}$$

Since $|B(u_0,v_0)|=1$, there is a real number $\theta\in\mathbb{R}$ satisfying $e^{i\theta}=B(u_0,v_0)$. We clearly have that

$$\begin{aligned}1=|B(u_0,v_0)|&=e^{-i\theta}B(u_0,v_0)=\operatorname{Re}e^{-i\theta}B(u_0,v_0)\\&=\operatorname{Re}\left(\sum_{u_0(n)\neq 0}|u_0(n)|e^{-i\theta}\frac{u_0(n)}{|u_0(n)|}B(e_n,v_0)\right)\\&=\sum_{u_0(n)\neq 0}|u_0(n)|\operatorname{Re}\left(e^{-i\theta}\frac{u_0(n)}{|u_0(n)|}B(e_n,v_0)\right)\\&\leq\sum_{u_0(n)\neq 0}|u_0(n)|=1.\end{aligned}$$

It follows that

$$\begin{aligned}n\in\mathbb{N},u_0(n)\neq 0\quad&\Rightarrow\operatorname{Re}\left(e^{-i\theta}\frac{u_0(n)}{|u_0(n)|}B(e_n,v_0)\right)=1\\&\Rightarrow\frac{u_0(n)}{|u_0(n)|}B(e_n,v_0)=e^{i\theta}.\end{aligned}\tag{16}$$

By using (15) we also have that

$$\begin{aligned}1-\beta(\varepsilon)&=e^{-i\theta}B(u_0,v_0)-\beta(\varepsilon)\\&<\operatorname{Re}e^{-i\theta}B(x_0,v_0)=\operatorname{Re}\sum_{n=1}^{\infty}\alpha_n e^{-i\theta}B(e_n,v_0).\end{aligned}$$

Since we fixed ε at the beginning, we write $r:=1-\sqrt{\beta(\varepsilon)}$ and define

$$H:=\{n\in\mathbb{N}:\operatorname{Re}e^{-i\theta}B(e_n,v_0)>r\}.$$

In view of Lemma (3.3.11) we obtain that

$$\sum_{n\in H}\alpha_n>1-\sqrt{\beta(\varepsilon)}. \tag{17}$$

Now we define $C:=H\cap\{n\in\mathbb{N}:u_0(n)\neq 0\}$. Then we deduce that

$$\begin{aligned}\sum_{n\in C}\alpha_n=\sum_{n\in H}\alpha_n-\sum_{\substack{n\in H\\u_0(n)=0}}\alpha_n&\geq\sum_{n\in H}\alpha_n-\|u_0-x_0\|_1\\&>1-\sqrt{\beta(\varepsilon)}-\beta(\varepsilon)=1-\delta.\end{aligned}\tag{18}$$

Now it suffices to give the functionals $\{z_n^*:n\in C\}$ that will satisfy the condition needed in order to prove that the pair (Y,Y^*) has the AHSP. For this purpose we will consider appropriate small perturbations of the functionals $y\mapsto B(e_n,y)$.

Let us fix $n\in C$. Since $\|B\|=1$ and $\operatorname{Re}e^{-i\theta}B(e_n,v_0)>r=1-\sqrt{\beta(\varepsilon)}$, by Lemma (3.3.14) we deduce that

$$\left|e^{-i\theta}B(e_n,v_0)-1\right|^2<2\sqrt{\beta(\varepsilon)}. \tag{19}$$

On the other hand, by using (15) it follows that

$$\operatorname{Re}e^{i\theta}=\operatorname{Re}B(u_0,v_0)>\operatorname{Re}A(u_0,v_0)-\|A-B\|$$

$$
\begin{aligned}
&> \operatorname{Re} A(x_0, v_0) - \|x_0 - u_0\|_1 - \|A - B\| \\
&\quad > \operatorname{Re} A(x_0, y_0) - \|y_0 - v_0\| - \|x_0 - u_0\|_1 - \|A - B\| \\
&> 1 - \eta(\varepsilon) - 2\beta(\varepsilon) - \varepsilon.
\end{aligned}
$$

Now we can apply Lemma (3.3.14) to obtain that

$$
\left|e^{i\theta} - 1\right|^2 < 2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon). \tag{20}
$$

By again applying Lemma (3.3.14) we also deduce that

$$
n \in H \;\Rightarrow\; \left|e^{-i\theta} B(e_n, v_0) - 1\right|^2 < 2\sqrt{\beta(\varepsilon)}. \tag{21}
$$

Hence for every $n \in C$, in view of (16) we obtain that

$$
\begin{aligned}
\left|\frac{u_0(n)}{|u_0(n)|} - 1\right| &\le \left|\frac{u_0(n)}{|u_0(n)|} - \frac{u_0(n)}{|u_0(n)|} B(e_n, v_0)\right| + \left|\frac{u_0(n)}{|u_0(n)|} B(e_n, v_0) - 1\right| \\
&= |1 - B(e_n, v_0)| + \left|e^{i\theta} - 1\right| \\
&\le \left|1 - e^{i\theta}\right| + \left|e^{i\theta} - B(e_n, v_0)\right| + \left|1 - e^{i\theta}\right| \\
&< 2\sqrt{2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon)} + \left|1 - e^{-i\theta} B(e_n, v_0)\right| \text{ (by (20))} \\
&< 2\sqrt{2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon)} + (4\beta(\varepsilon))^{\frac{1}{4}} \text{ (by (21))}
\end{aligned}
$$

That is,

$$
\left|\frac{u_0(n)}{|u_0(n)|} - 1\right| \le 2\sqrt{2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon)} + (4\beta(\varepsilon))^{\frac{1}{4}}, \quad \text{for all } n \in C. \tag{22}
$$

Finally we define the functionals z_n^* by

$$
z_n^*(y) := \frac{u_0(n)}{|u_0(n)|} e^{-i\theta} B(e_n, y) \text{ and } v_n^*(y) := B(e_n, y) \; (n \in C).
$$

Clearly it is satisfied that $\{z_n^* : n \in C\} \subset S_{Y^*}$, and in view of (16) we know that

$$
z_n^*(v_0) = 1, \quad \text{for all } n \in C. \tag{23}
$$

Also we have that

$$
\begin{aligned}
\|z_n^* - y_n^*\| &= \left\|\frac{u_0(n)}{|u_0(n)|} e^{-i\theta} v_n^* - y_n^*\right\| \\
&\le \left\|\frac{u_0(n)}{|u_0(n)|} e^{-i\theta} v_n^* - e^{-i\theta} v_n^*\right\| + \|e^{-i\theta} v_n^* - v_n^*\| + \|v_n^* - y_n^*\| \\
&\le \left|\frac{u_0(n)}{|u_0(n)|} - 1\right| + \left|e^{-i\theta} - 1\right| + \|B - A\| \\
&\le 3\sqrt{2(\eta(\varepsilon) + 2\beta(\varepsilon) + \varepsilon)} + (4\beta(\varepsilon))^{\frac{1}{4}} + \varepsilon < \varepsilon_0 \text{ (by (22), (20) and (15))}.
\end{aligned}
$$

In view of (18), the above inequality, (15) and (23) we have proved that the pair (Y, Y^*) has the AHSP.

Conversely, assume that (Y, Y^*) satisfies the AHSP. Given $0 < \varepsilon < 1$, there are $0 < \delta, \eta < \frac{\varepsilon}{2}$ satisfying the conditions in Definition (3.3.10) for $\frac{\varepsilon}{2}$. Suppose that $A \in S_{L^2(\ell_1 \times Y)}$ and for some pair $(x_0, y_0) \in S_{\ell_1} \times S_Y$ it holds that $|A(x_0, y_0)| > 1 - \eta$. By rotating A if necessary we can assume that $A(x_0, y_0) > 0$. Also, by applying a convenient isometry on ℓ_1 we may assume that $x_0(n) \ge 0$ for every natural number n. Hence we have that

$$
\operatorname{Re} \sum_{n=1}^{\infty} x_0(n) A(e_n, y_0) > 1 - \eta.
$$

Since $\| A \|= 1$, the sequence of functionals $\{y_n^*\}$ given by $y_n^*(y) := A(e_n, y), n \in \mathbb{N}$, is a subset of B_{Y^*}. By using the AHSP for the pair (Y, Y^*) we can find a subset $C \subset \mathbb{N}$ such that

$$\sum_{n \in C} x_0(n) > 1 - \delta,$$

and a subset of functionals $\{z_k^* : k \in C\} \subset S_{Y^*}$ and $z_0 \in S_Y$ such that

$$\|z_k^* - y_k^*\| < \frac{\varepsilon}{2},\ z_k^*(z_0) = 1, \text{ for all } k \in C \text{ and } \|z_0 - y_0\| < \frac{\varepsilon}{2}$$

Now we define the bilinear form B on $\ell_1 \times Y$ given by

$$B(x, y) = \sum_{n \in C} x(n) z_n^*(y) + \sum_{n \notin C} x(n) A(e_n, y) \ ((x, y) \in \ell_1 \times Y, x = (x(n))).$$

It is immediate to check that B is bounded since $\|z_n^*\| = 1$ for every $n \in C$ and $\| A \|= 1$. Since C is nonempty, it is easy to deduce that $B \in S_{L^2(\ell_1 \times Y)}$. Also,

$$\begin{aligned} \|B - A\| &\le \sup_n \sup_{y \in B_Y} |(B - A)(e_n, y)| \\ &= \sup_{n \in C} \sup_{y \in B_Y} |(B - A)(e_n, y)| \\ &= \sup_{n \in C} \|z_n^* - y_n^*\| \le \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

Let us take $u_0 := \frac{1}{\sum_{n \in C} x_0(n)} \sum_{n \in C} x_0(n) e_n$ that satisfies $u_0 \in S_{\ell_1}$. Also,

$$\begin{aligned} \|u_0 - x_0\|_1 &= \left\| \frac{1}{\sum_{n \in C} x_0(n)} \sum_{n \in C} x_0(n) e_n - x_0 \right\|_1 \\ &\le \left\| \frac{1}{\sum_{n \in C} x_0(n)} \sum_{n \in C} x_0(n) e_n - \sum_{n \in C} x_0(n) e_n \right\|_1 + \left\| \sum_{n \notin C} x_0(n) e_n \right\| \\ &= 1 - \sum_{n \in C} x_0(n) + \sum_{n \notin C} x_0(n) < 2\delta < \varepsilon. \end{aligned}$$

Finally we have that

$$\begin{aligned} B(u_0, z_0) &= \sum_{n \in C} u_0(n) B(e_n, z_0) \\ &= \frac{1}{\sum_{n \in C} x_0(n)} \sum_{n \in C} x_0(n) z_n^*(z_0) = 1. \end{aligned}$$

Therefore (ℓ_1, Y) has the BPBP for bilinear forms.

We are going to provide many classes of Banach spaces X such that (X, X^*) has the AHSP, and so by Theorem (3.3.15) the pair (ℓ_1, X) has the BPBP for bilinear forms. We begin with the class of uniformly smooth Banach spaces. We recall that a Banach space is uniformly smooth if its norm is uniformly Fréchet differentiable at the points of the unit sphere.

Proposition (3.3.16)[157]: If X is a uniformly smooth Banach space, then the pair (X, X^*) has the Approximate Hyperplane Series property.

Proof. If X is uniformly smooth, then X^* is uniformly convex (see for instance [171]). We will check that the assumption of Proposition (3.3.12) is satisfied.

Given $\varepsilon > 0$, we take $\delta := 2\delta_X * (\varepsilon)$, where $\delta_{X^*}(\varepsilon)$ is the modulus of convexity of X^*, which is a positive real number. Now assume that $x^* \in S_{X^*}$ and $x_0 \in S_X$ satisfy $\operatorname{Re} x^*(x_0) > 1 - \delta = 1 - 2\delta_{X^*}(\varepsilon)$. If $x_0^* \in S_{X^*}$ satisfies $x_0^*(x_0) = 1$, then

$$1 - \delta_{X^*}(\varepsilon) < \operatorname{Re}\frac{x^* + x_0^*}{2}(x_0) \leq \frac{\|x^* + x_0^*\|}{2}$$

Hence $\|x^* - x_0^*\| < \varepsilon$. By applying Proposition (3.3.12) we obtain that the pair (X, X^*) has the AHSP.

We recall that a Banach space X is smooth if $D(x)$ is a singleton for every $x \in S_X$. Not every finite-dimensional space is smooth. However, we will show that every finite-dimensional normed space X also satisfies that (X, X^*) has the AHSP. This fact is a consequence of a refinement of [158] that we will now show. We include the proof of this result for the sake of completeness.

Proposition (3.3.17)[157]: Every finite-dimensional normed space X satisfies that (X, X^*) has the Approximate Hyperplane Series property.

Proof. We are going to check that the hypotheses of Corollary (3.3.13) hold. We argue by contradiction. Assume that there is some positive real number ε_0 not satisfying those hypotheses. Thus, for every positive $\delta > 0$ there exists $x_\delta \in S_X$ satisfying the following condition:

$$y \in S_X, \|y - x_\delta\| < \varepsilon_0 \Rightarrow \exists x^* \in S_{X^*}: \operatorname{Re} x^*(x_\delta) > 1 - \delta \text{ and } \operatorname{dist}\big(x^*, D(y)\big) \geq \varepsilon_0. \quad (24)$$

Hence, for every $n \in \mathbb{N}$, there is an element $x_n \in S_X$ satisfying (24) for $\delta = \frac{1}{n}$. Since $\dim X < \infty$ we can also assume that $(x_n) \to x \in S_X$ and $\|x - x_n\| < \varepsilon_0$ for every natural number n.

So by using (24), for every n there is $x_n^* \in S_{X^*}$ satisfying

$$\operatorname{Re} x_n^*(x_n) > 1 - \frac{1}{n} \text{ and } \operatorname{dist}\big(x_n^*, D(x)\big) \geq \varepsilon_0. \quad (25)$$

By passing to a subsequence, if needed, we can assume that $(x_n^*) \to x^* \in S_{X^*}$. In view of (25) we have that

$$x^*(x) = 1 \text{ and } \operatorname{dist}\,(x^*, D(x)) \geq \varepsilon_0$$

which is a contradiction.

Now we will provide an important class of classical Banach spaces that are very far from the uniformly smooth spaces but still satisfy the fact that there is a version of the Bishop-Phelps-Bollobás Theorem for bilinear forms on the product of ℓ_1 and any space in this class.

Proposition (3.3.18)[157]: For every locally compact Hausdorff topological space Ω, the space $Y = \mathcal{C}_0(\Omega)$, of real or complex-valued and continuous functions on Ω vanishing at infinity, satisfies that (Y, Y^*) has the Approximate Hyperplane Series property.

Proof. We prove both the real and the complex cases. It suffices to check that $\mathcal{C}_0(\Omega)$ satisfies the assumptions of Proposition (3.3.12). In order to do that we will use the Riesz Theorem to identify the topological dual of $\mathcal{C}_0(\Omega)$ with the space $\mathcal{M}(\Omega)$ of real or complex Radon measures on Ω, endowed with the norm given by the total variation, i.e. given $x^* \in \mathcal{C}_0(\Omega)^*$ there exists a unique $\mu \in \mathcal{M}(\Omega)$, such that

$$x^*(f) = \int_\Omega f\, d\mu, \text{ for all } f \in \mathcal{C}_0(\Omega)$$

and $\| x^* \| = |\mu|(\Omega)$, where $|\mu|$ denotes the positive measure called the total variation of μ (see e.g. [173], 6.19 Theorem). It is well known that $|\mu|$ is a finite positive regular measure on Ω and there exists a Borel measurable function $h: \Omega \to \mathbb{C}$ with $|h(t)| = 1$ for all t in Ω so that

$$x^*(f) = \int_\Omega f h d|\mu|, \text{ for all } f \in \mathcal{C}_0(\Omega)$$

(see e.g. [173]). Given $0 < \varepsilon < 1$, we choose $0 < \eta < 1$ such that $2\eta + \sqrt{2\eta} < \varepsilon$. Let $\{y_j^*: j \in F\} \subset S_{\mathcal{C}_0(\Omega)^*}$ be a finite set and $f_0 \in S_{\mathcal{C}_0(\Omega)}$ such that $\operatorname{Re} y_j^*(f_0) > 1 - \eta^2$, for every $j \in F$. Let $\{\mu_j: j \in F\} \subset S_{\mathcal{M}(\Omega)}$ be such that

$$y_j^*(f) = \int_\Omega f d\mu_j, \text{ for all } f \in \mathcal{C}_0(\Omega) \text{ and all } j \in F.$$

Let $\{h_j: j \in F\}$ be the Borel measurable functions on Ω such that

$$|h_j| = 1 \text{ and } y_j^*(f) = \int_\Omega f h_j d|\mu_j|, \text{ for all } j \in F \text{ and all } f \in \mathcal{C}_0(\Omega).$$

We are assuming that

$$\int_\Omega \operatorname{Re}(f_0 h_j) d|\mu_j| = \operatorname{Re}\left(\left(\int_\Omega f_0 h_j d|\mu_j|\right)\right) = \operatorname{Re} y_j^*(f_0) > 1 - \eta^2, \text{ for all } j \in F$$

Now for each $j \in F$ let us consider the Borel set

$$B_j = \{t \in \Omega: \operatorname{Re}(f_0(t) h_j(t)) > 1 - \eta\}.$$

For every $j \in F$ we also have

$$\begin{aligned} 1 - \eta^2 < \int_\Omega \operatorname{Re}(f_0 h_j) d|\mu_j| &= \int_{B_j} \operatorname{Re}(f_0 h_j) d|\mu_j| + \int_{\Omega \setminus B_j} \operatorname{Re}(f_0 h_j) d|\mu_j| \\ &\leq \int_{B_j} d|\mu_j| + (1 - \eta) \int_{\Omega \setminus B_j} d|\mu_j| \\ &= |\mu_j|(B_j) + (1 - \eta)|\mu_j|(\Omega \setminus B_j) = \eta|\mu_j|(B_j) + 1 - \eta \end{aligned}$$

Hence

$$|\mu_j|(B_j) > 1 - \eta.$$

Since each $|\mu_j|$ is regular, then for each $j \in F$ there is a compact set $K_j \subset B_j$, such that

$$|\mu_j|(K_j) > 1 - \eta > 0, \text{ for all } j \in F. \tag{26}$$

As a consequence,

$$|\mu_j|(\Omega \setminus K_j) < \eta, \text{ for every } j \in F. \tag{27}$$

Let us take $K := \bigcup_{j \in F} K_j$, which is a compact subset of Ω satisfying

$$K = \bigcup_{j \in F} K_j \subset \bigcup_{j \in F} B_j \subset \{t \in \Omega: |f_0(t)| > 1 - \eta\}.$$

The set $U := \{t \in \Omega: |f_0(t)| > 1 - \eta\}$ is open. Since every locally compact space is completely regular, there is a function $m \in \mathcal{C}_0(\Omega)$ that separates the closed set $\Omega \setminus U$ and the compact set K, i.e., such that $0 \leq m \leq 1, m(K) = \{1\}$ and $m(\Omega \setminus U) = \{0\}$. So the function h_0 defined on Ω by

$$h_0(t) := \begin{cases} \dfrac{f_0(t)}{|f_0(t)|} m(t) & \text{if } t \in U, \\ 0 & \text{if } t \in \Omega \setminus U \end{cases}$$

is continuous, and since it vanishes outside the relatively compact set U, it belongs to $\mathcal{C}_0(\Omega)$. We take $g_0 := h_0 + (1 - m) f_0$, which is also a continuous function on Ω vanishing at infinity.

It is also clear that g_0 satisfies

$$|g_0(t)| \leq m(t) + 1 - m(t) = 1, \text{ for all } t \in \Omega$$

and $|g_0|(K) = \{1\}$, so $g_0 \in S_{\mathcal{C}_0(\Omega)}$

Now we will check that

$$\|g_0 - f_0\| \leq \eta < \varepsilon. \tag{28}$$

If $t \in U$, then $|f_0(t)| > 1 - \eta$, so

$$|g_0(t) - f_0(t)| = \left|\frac{f_0(t)}{|f_0(t)|} m(t) - m(t) f_0(t)\right| = m(t)|1 - |f_0(t)|| < \eta.$$

In the case that $t \in \Omega \setminus U$ we have that $m(t) = 0 = h_0(t)$, so $g_0(t) - f_0(t) = 0$. Now we will provide the new set of continuous functionals that satisfy the desired condition. For each $j \in F$ let us define the functional

$$z_j^*(f) := \frac{1}{|\mu_j|(K_j)} \int_{K_j} f \frac{\bar{f_0}}{|f_0|} d|\mu_j| \ (f \in \mathcal{C}_0(\Omega)).$$

Obviously $\{z_j^* : j \in F\} \subset S_{\mathcal{C}_0(\Omega)^*}$, and for every $f \in \mathcal{C}_0(\Omega)$ we have

$$\begin{aligned}
&|y_j^*(f) - z_j^*(f)| \\
&\leq \left|\int_{\Omega \setminus K_j} f h_j d|\mu_j|\right| + \left|\int_{K_j} \left(f - f\frac{\bar{f_0}}{|f_0|}\bar{h}_j\right) h_j d|\mu_j|\right| \\
&\qquad + \left(\frac{1}{|\mu_j|(K_j)} - 1\right)\left|\int_{K_j} f \frac{\bar{f_0}}{|f_0|} d|\mu_j|\right| \\
&\leq |\mu_j|(\Omega \setminus K_j)\|f\| + \int_{K_j} \left|\left(1 - \frac{f_0}{|f_0|}\bar{h}_j\right) f\right| |d||\mu_j| + \left(\frac{1}{|\mu_j|(K_j)} - 1\right)|\mu_j|(K_j)\|f\| \\
&\leq \eta\|f\| + \int_{K_j} \sqrt{2\eta}\|f\| d|\mu_j| + \left(1 - |\mu_j|(K_j)\right)\|f\| \text{ (by (27) and Lemma (3.3.14))} \\
&\leq (\eta + \sqrt{2\eta} + \eta)\|f\| = (2\eta + \sqrt{2\eta})\|f\| \text{ (by (26)).}
\end{aligned}$$

Hence

$$\|y_j^* - z_j^*\| \leq 2\eta + \sqrt{2\eta} < \varepsilon, \text{ for all } j \in F. \tag{29}$$

Finally, for each $j \in F$ we have that

$$z_j^*(g_0) = \frac{1}{|\mu_j|(K_j)} \int_{K_j} g_0 \frac{\bar{f_0}}{|f_0|} d|\mu_j| = \frac{1}{|\mu_j|(K_j)} \int_{K_j} \frac{f_0}{|f_0|}\frac{\bar{f_0}}{|f_0|} d|\mu_j| = 1$$

In view of (28),(29) and the last equality, the proof is completed.

Corollary (3.3.19)[157]: For every Hausdorff and compact topological space K, the space $Y = \mathcal{C}(K)$, of real or complex-valued and continuous functions on K, satisfies that (Y, Y^*) has the Approximate Hyperplane Series property.

Corollary (3.3.20)[157]: If $X = c_0$ (real or complex case), then (X, X^*) has the Approximate Hyperplane Series property.

Proposition (3.3.21)[157]: If H is a Hilbert space and $X = K(H)$, the space of compact operators on H, then (X, X^*) has the Approximate Hyperplane Series property.

Proof. In the finite-dimensional case it suffices to use Proposition (3.3.17). So we can assume that H is infinite-dimensional.

We will use the standard identification of $K(H)^*$ and the space of nuclear operators on H, endowed with the nuclear norm (see for instance [175], Theorem 1 , p. 46, [174], Theorem 5.6 or [168], Theorem 16.50). Indeed we will prove that the assumptions of Corollary (3.3.13) are satisfied.

Given $\varepsilon > 0$, we choose $0 < \eta < \frac{1}{2}$ such that

$$2\sqrt{2\eta} + 5\eta < \varepsilon$$

Let us fix any element $S_0 \in S_{K(H)}$. By using the polar decomposition of S_0 (see for instance [174]), there are orthonormal systems (y_n) and (x_k) in H such that

$$S_0 := \sum_{n=1}^{\infty} a_k y_k^* \otimes x_k$$

where (a_k) is a decreasing sequence of real numbers convergent to zero, $1 = \|S_0\| = \max\{a_n : n \in \mathbb{N}\}$ and $y_k^*(x) = (x \mid y_k)$ for each $k \in \mathbb{N}$ and $x \in H$, where (.|.) denotes the inner product of H. Let us take $s := 1 - \eta$ and

$$B := \{n \in \mathbb{N} : a_n > s\}.$$

It is clear that B is finite and non-empty. So the operator T_0 on H, defined as

$$T_0 := \sum_{n \in B} y_k^* \otimes x_k + \sum_{n \in \mathbb{N} \setminus B} a_k y_k^* \otimes x_k,$$

belongs to $S_{K(H)}$ and satisfies

$$\|T_0 - S_0\| = \max\{1 - a_n : n \in B\} < 1 - s = \eta < \varepsilon. \tag{30}$$

Now take $t := 1 - \eta^4$ and choose any element $z^* \in S_{K(H)^*}$ such that $\operatorname{Re} z^*(S_0) > t$. By the description of $K(H)^*$ there are a sequence $(b_n) \in \ell_1$ of nonnegative real numbers, and orthonormal systems (f_n) and (e_n) in H such that

$$z^*(T) = \sum_{n=1}^{\infty} b_n (T(e_n) \mid f_n), \text{ for all } T \in K(H).$$

Also, it is satisfied that $1 = \|z^*\| = \sum_{n=1}^{\infty} b_n$ (see for instance [175] or [174]). In such a case we will write $z^* = \sum_{n=1}^{\infty} b_n f_n^* \otimes e_n$. We have

$$1 - \eta^4 = t < \operatorname{Re} z^*(S_0) = \operatorname{Re} \sum_{n=1}^{\infty} b_n (S_0(e_n) \mid f_n)$$

If we consider the set C given by $C := \{n \in \mathbb{N} : \operatorname{Re}(S_0(e_n) \mid f_n) \geq 1 - \eta^3\}$, then Lemma (3.3.11) with $r := 1 - \eta^3$ gives

$$\sum_{n \in C} b_n > 1 - \eta > 0, \text{ and so } \sum_{n \in \mathbb{N} \setminus C} b_n < \eta. \tag{31}$$

Now we define the element $y^* \in K(H)^*$ given by

$$y^* = \frac{1}{\sum_{n \in C} b_n} \sum_{n \in C} b_n f_n^* \otimes e_n$$

Then $\|y^*\| \leq 1$, and in view of (31) we have that

$$
\begin{aligned}
\|y^* - z^*\| &= \left\| \frac{1}{\sum_{n\in C} b_n} \sum_{n\in C} b_n f_n^* \otimes e_n - \sum_{n=1}^{\infty} b_n f_n^* \otimes e_n \right\| \\
&\le \left(\frac{1}{\sum_{n\in C} b_n} - 1 \right) \left\| \sum_{n\in C} b_n f_n^* \otimes e_n \right\| + \left\| \sum_{n\in \mathbb{N}\setminus C} b_n f_n^* \otimes e_n \right\| \\
&\le 2\left(1 - \sum_{n\in C} b_n \right) < 2\eta. \qquad (32)
\end{aligned}
$$

Now for each $n \in C$ we will use the parallelogram law for the elements $S_0(e_n)$ and f_n and obtain that

$$
\begin{aligned}
&r^2 + 1 + 2r + \|S_0(e_n) - f_n\|^2 \\
&\quad \le \left(\mathrm{Re}\big((S_0(e_n) + f_n) \mid f_n \big) \right)^2 + \|S_0(e_n) - f_n\|^2 \\
&\quad \le \|S_0(e_n) + f_n\|^2 + \|S_0(e_n) - f_n\|^2 \\
&\quad = 2\left(\|S_0(e_n)\|^2 + \|f_n\|^2 \right) \le 4.
\end{aligned}
$$

We deduce that

$$
\|S_0(e_n) - f_n\|^2 \le 4 - (r^2 + 2r + 1) < 4\eta^3, \text{ for all } n \in C.
$$

Now by using (30) we deduce

$$
\begin{aligned}
\|T_0(e_n) - f_n\| &\le \|T_0(e_n) - S_0(e_n)\| + \|S_0(e_n) - f_n\| \\
&\le \|T_0 - S_0\| + 2\eta \le 3\eta, \ \forall n \in C. \qquad (33)
\end{aligned}
$$

We also know that $\|S_0(e_n)\| \ge r$ for every $n \in C$. Let us denote by P the orthogonal projection on H onto the subspace Y generated by $\{y_k : k \in B\}$. For each $n \in C$ we have that

$$
\begin{aligned}
r^2 &\le \|S_0(e_n)\|^2 \\
&= \left\| \sum_{k\in B} a_k (e_n \mid y_k) x_k + \sum_{k\in \mathbb{N}\setminus B} a_k (e_n \mid y_k) x_k \right\|^2 \\
&\le \sum_{k\in B} |a_k|^2 |(P(e_n) \mid y_k)|^2 + \sum_{k\in \mathbb{N}\setminus B} |a_k|^2 |(e_n - P(e_n) \mid y_k)|^2 \\
&\le \|P(e_n)\|^2 + s^2 \|e_n - P(e_n)\|^2 \\
&= \|P(e_n)\|^2 + s^2\left(1 - \|P(e_n)\|^2\right) = \|P(e_n)\|^2 (1 - s^2) + s^2
\end{aligned}
$$

Hence

$$
\|P(e_n)\|^2 \ge \frac{r^2 - s^2}{1 - s^2} = 1 - \frac{\eta^2(2 - \eta^3)}{2 - \eta} > 1 - \eta > 0
$$

that is,

$$
\|e_n - P(e_n)\|^2 < \eta
$$

As a consequence

$$
\begin{aligned}
\left\| e_n - \frac{P(e_n)}{\|P(e_n)\|} \right\|^2 &= \left\| P(e_n) - \frac{P(e_n)}{\|P(e_n)\|} \right\|^2 + \|e_n - P(e_n)\|^2 \\
&= |1 - \|P(e_n)\||^2 + \|e_n - P(e_n)\|^2 \le 2\eta.
\end{aligned}
$$

We just checked that

$$\left\| e_n - \frac{P(e_n)}{\|P(e_n)\|} \right\| \leq \sqrt{2\eta}.$$

Let us denote $\tilde{e}_n := \frac{P(e_n)}{\|P(e_n)\|}$ for each $n \in C$. Then we know that

$$\tilde{e}_n \in S_H \cap Y \text{ and } \|\tilde{e}_n - e_n\| \leq \sqrt{2\eta}, \text{ for all } n \in C. \tag{34}$$

Finally we take the functional $x^* \in K(H)^*$ given by

$$x^* = \frac{1}{\sum_{n\in C} b_n} \sum_{n\in C} b_n T_0(\tilde{e}_n)^* \otimes \tilde{e}_n.$$

It is clear that $x^* \in B_{K(H)*}$. It is also clear that $\|T_0(\tilde{e}_n)\| = 1$ for every $n \in C$, and so

$$x^*(T_0) = \frac{1}{\sum_{n\in C} b_n} \sum_{n\in C} b_n \big(T_0(\tilde{e}_n) \mid T_0(\tilde{e}_n)\big) = 1, \text{ hence } x^* \in S_{K(H)^*}. \tag{35}$$

Now we estimate

$$\begin{aligned}
\sum_{n\in C} b_n \|x^* - y^*\| &= \left\| \sum_{n\in C} b_n T_0(\tilde{e}_n)^* \otimes \tilde{e}_n - \sum_{n\in C} b_n f_n^* \otimes e_n \right\| \\
&\leq \left\| \sum_{n\in C} b_n T_0(\tilde{e}_n)^* \otimes \tilde{e}_n - \sum_{n\in C} b_n T_0(\tilde{e}_n)^* \otimes e_n \right\| \\
&\quad + \left\| \sum_{n\in C} b_n T_0(\tilde{e}_n)^* \otimes e_n - \sum_{n\in C} b_n T_0(e_n)^* \otimes e_n \right\| \\
&\quad + \left\| \sum_{n\in C} b_n T_0(e_n)^* \otimes e_n - \sum_{n\in C} b_n f_n^* \otimes e_n \right\| \\
&\leq 2 \sum_{n\in C} b_n \|\tilde{e}_n - e_n\| + \sum_{n\in C} b_n \|T_0(e_n) - f_n\| \\
&\leq \sum_{n\in C} b_n (2\sqrt{2\eta} + 3\eta) \text{ (by (34) and (33))}
\end{aligned}$$

We have obtained that

$$\|x^* - y^*\| \leq 2\sqrt{2\eta} + 3\eta.$$

Finally, in view of (32) we have that

$$\begin{aligned}
\|x^* - z^*\| &\leq \|x^* - y^*\| + \|y^* - z^*\| \\
&\leq 2\sqrt{2\eta} + 5\eta < \varepsilon.
\end{aligned}$$

By (30), (35) and the previous inequality, we can apply Corollary (3.3.13), and this completes the proof.

Choi and Song [166] proved that there is no version of the Bishop-Phelps-Bollobás Theorem for bilinear forms. Indeed it happens for ℓ_1. Hence, by Theorem (3.3.15) the pair (ℓ_1, ℓ_1^*) does not have the AHSP. Actually we will show that for every infinite-dimensional $L_1(\mu)$ the pair $(L_1(\mu), L_1(\mu)^*)$ fails the AHSP.

Proposition (3.3.22)[157]: For every infinite-dimensional space $L_1(\mu)$, the pair $(L_1(\mu), L_1(\mu)^*)$ fails the Approximate Hyperplane Series property.

Proof. Since $L_1(\mu)$ is infinite-dimensional, there exists a sequence $(A_n)_{n=1}^{\infty}$ of pairwise disjoint measurable sets with $0 < \mu(A_n) < \infty$ for every $n \in \mathbb{N}$. Assume that the pair $(L_1(\mu), L_1(\mu)^*)$

has the AHSP. Given $0 < \varepsilon < \frac{1}{2}$, let $0 < \delta, \eta < \varepsilon$ be the positive real numbers satisfying the conditions in Definition (3.3.10). Let us choose $n \in \mathbb{N}, n \geq 2$ such that $\frac{1}{n} < \min\{\delta, \eta\}$ and take

$$f_0 = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\mu(A_j)} \chi_{A_j} \text{ and } g_i = \sum_{j=1, j\neq i}^{n} \chi_{A_j}$$

for $1 \leq i \leq n$. It is clear that $f_0 \in S_{L_1(\mu)}$ and $g_i \in L_\infty(\mu)$ with $\|g_i\|_\infty = 1$ for every i, where $\|\cdot\|_\infty$ denotes the usual norm in $L_\infty(\mu)$. Since $0 < \mu(A_j) < \infty$ for every j, g_i is associated to an element in the unit sphere of $L_1(\mu)^*$. We will denote by x_i^* the element in $L_1(\mu)^*$ corresponding to g_i for $1 \leq i \leq n$. It is satisfied that

$$x_i^*(f_0) = \int_\Omega g_i f_0 d\mu = \frac{1}{n} \sum_{j=1, j\neq i}^{n} \frac{1}{\mu(A_j)} \int_\Omega \chi_{A_j} d\mu = \frac{n-1}{n},$$

for $i = 1, \ldots, n$. Thus, the convex series $\sum_{i=1}^{n} \frac{1}{n} x_i^*$ satisfies

$$\left(\sum_{i=1}^{n} \frac{1}{n} x_i^*\right)(f_0) = \frac{n-1}{n} > 1 - \delta$$

Since we are assuming that the AHSP is satisfied, there exist $C \subset \{1, \ldots, n\}$ and $\{y_k^* : k \in C\} \subset S_{L_1(\mu)^*}$ and $f \in S_{L_1(\mu)}$ such that $\frac{\text{card}(C)}{n} > 1 - \delta$, $\|y_k^* - x_k^*\| < \varepsilon$, $y_k^*(f) = 1$, for all $k \in C$, and $\|f - f_0\|_1 < \varepsilon$, where $\|\cdot\|_1$ denotes the usual norm in $L_1(\mu)$. Now let us notice that for every function $h \in S_{L_\infty(\mu)}$ satisfying $\left|\int_\Omega f h d\mu\right| = 1$, if for some measurable set A we have $\|h\chi_A\|_\infty < 1$, since

$$1 = \left|\int_\Omega f h d\mu\right| \leq \int_A \|h\chi A\|_\infty |f| d\mu + \int_{\Omega \setminus A} |f| d\mu = 1 + \left(\|h\chi_A\|_\infty - 1\right) \int_\Omega |f| \chi A d\mu,$$

then it follows that $|f|\chi_A = 0$ almost everywhere. In our case, since the support of f is a countable union of measurable sets of finite measure and every A_i has finite measure, if we take $B := \operatorname{supp} f \cup \bigcup_{j=1}^{n} A_j$, the restriction of y_k^* to $L_1(\mu_{|B})$ is represented by a function $h_k \in L_\infty(\mu)$ for every $k \in C$. Hence we have that

$$y_k^*(f) = \int_\Omega f h_k d\mu = 1, \ \|h_k - g_k\|_\infty \leq \|y_k^* - x_k^*\| < \varepsilon, \text{ for all } k \in C,$$

and so $\|h_k \chi_{A_k}\|_\infty < \varepsilon$ for $k \in C$. Thus $f\chi_{A_k} = 0$ a.e. for every $k \in C$. As a consequence

$$\|f - f_0\|_1 \geq \int_{\bigcup_{k\in C} A_k} |f_0| d\mu = \frac{\text{card}(C)}{n} > 1 - \delta > \frac{1}{2} > \varepsilon,$$

which is a contradiction

Chapter 4
Bishop-Phelps-Bollobás Type Theorems and Property

We use the weak *-to-norm fragmentability of weak *-compact subsets of the dual of Asplund spaces and we need to observe a Urysohn type result producing peak complex-valued functions in uniform algebras that are small outside a given open set and whose image is inside a Stolz region. New examples of Banach spaces Y with AHSP are provided. We also obtain that certain ideals of Asplund operators satisfy the Bishop-Phelps-Bollobás property. We show a general BPB-type theorem for Γ-flat operators acting to a space with ACK_ρ structure and show that uniform algebras and spaces with the property β have ACK_ρ structure. We also study the stability of the ACK_ρ structure under some natural Banach space theory operations. We discover many new examples of spaces Y such that the Bishop-Phelps-Bollobás property for Asplund operators is valid for all pairs of the form (X, Y).

Section (4.1): Uniform Algebras:

Mathematical optimization is associated to maximizing or minimizing real functions. James's compactness theorem [193] and Bishop-Phelps's theorem [182] are two landmark results along this line in functional analysis. The former characterizes reflexive Banach spaces X as those for which continuous linear functionals $x^* \in X^*$ attain their norm in the unit sphere S_X. The latter establishes that for any Banach space X every continuous linear functional $x^* \in X^*$ can be approximated (in norm) by linear functionals that attain the norm in S_X. We concerned with the study of a strengthening of Bishop-Phelps's theorem that mixes ideas of Bollobás [183] see Theorem (4.1.9) here - and Lindenstrauss [196] - who initiated the study of the Bishop-Phelps property for bounded operators between Banach spaces. Our starting point is the following definition brought in by Acosta, Aron, García and Maestre in 2008:

Definition (4.1.1)[177]: ([178]). A pair of Banach spaces (X, Y) is said to have the Bishop-Phelps-Bollobás property (BPBp for short) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $\tilde{T} \in S_{L(X,Y)}$ satisfying

$$\|\tilde{T}(u_0)\| = 1,\ \|x_0 - u_0\| < \varepsilon \text{ and } \ \| T - \tilde{T} \| < \varepsilon.$$

A good number regarding BPBp have been written during the last years, see [180],[184],[185]. Very recently, a general result has been proved in [179], that in particular says that pairs of the form $(X, C(K))$ do have the BPBp whenever X is an Asplund space and $C(K)$ is the space of continuous functions defined on a compact Hausdorff space: this result provided the first examples of pairs of the kind (c_0, Y) with BPBp for Y infinite dimensional Banach space. We extend and sharpen the results of [179] and prove Theorem (4.1.14).

For $\mathfrak{A} = C(K)$ the above result was proved in [179] with worse estimates. The key points for the known proof when $\mathfrak{A} = C(K)$ were, on one hand, the asplundness of T hidden in Lemma 2.3 of [179] that led to a suitable open set $U \subset K$ and, on the other hand, the Urysohn's lemma that applied to an arbitrary $t_0 \in U$ produces a function $f \in C(K)$ satisfying

$$f(t_0) = \| f \|_\infty = 1,\ f(K) \subset [0,1] \text{ and } \operatorname{supp}(f) \subset U.$$

With all this setting, $\tilde{T}$ was explicitly defined by

$$\tilde{T}(x)(t) = f(t) \cdot y^*(x) + \big(1 - f(t)\big) \cdot T(x)(t),\ x \in X, t \in K, \qquad (1)$$

where $y^* \in S_{X^*}$ was chosen satisfying, amongst other things, satisfying $1 = |y^*(u_0)| = \|u_0\|$ and $\|x_0 - u_0\| \lesssim \varepsilon$. The provisos about y^* and f were used then to prove that T and $\tilde{T}$ were close and that that $1 = \| \tilde{T} \| = \|\tilde{T}u_0\|$, but he or she will have to make use of the fact that $f(K) \subset [0,1]$. Once this is said, it becomes clear that the arguments above cannot work for a proof of Theorem (4.1.14) for a general uniform algebra $\mathfrak{A} \subset C(K)$. Certainly, $\mathfrak{A}$ could be too rigid (for instance the disk algebra) to allow the construction of $f \in \mathfrak{A}$ peaking at t_0 and with $f(K) \subset [0,1]$. To overcome

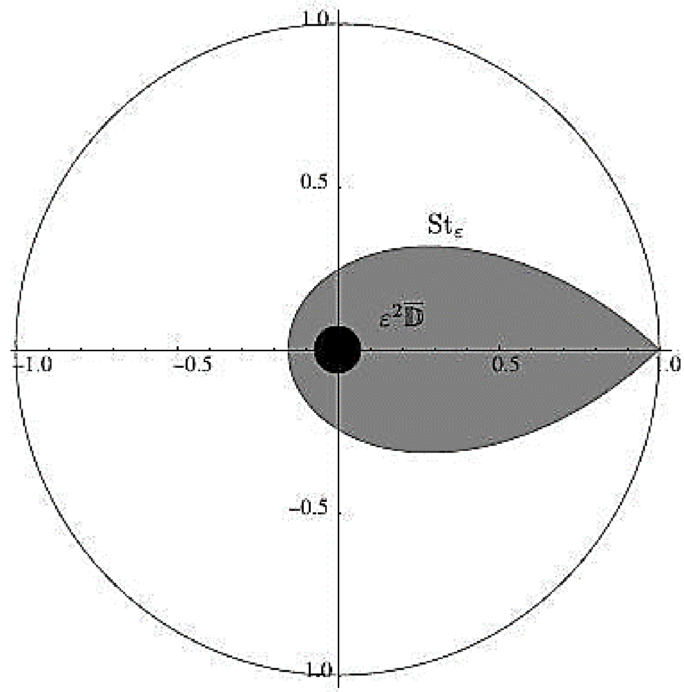


Fig. 1[177]. The Stolz region.

these difficulties we observe in Lemma (4.1.5) below an easy but useful statement about the existence of peak functions $f \in \mathfrak{A}$ that are small outside an open set and with $f(K)$ contained in the Stolz region

$$\mathrm{St}_\varepsilon = \{z \in \overline{\mathbb{D}}: |z| + (1-\varepsilon)|1-z| \leq 1\},$$

see Fig. 1.

Let $\mathfrak{A} \subset C(K)$ be a unital uniform algebra and Γ_0 its Choquet boundary. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \| f \|_\infty = 1, |f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset \mathrm{St}_\varepsilon$, i.e.

$$|f(t)| + (1-\varepsilon)|1-f(t)| \leq 1, \quad \text{for all } t \in K. \tag{2}$$

With Lemma (4.1.5) in mind we can appeal at the full power of Lemma 2.3 of [179], that is also suited for a boundary instead of K, to produce U and then modify the definition of $\tilde{T}$ in (1) with an auxiliary ε' as

$$\tilde{T}(x)(t) = f(t) \cdot y^*(x) + (1-\varepsilon')\big(1-f(t)\big) \cdot T(x)(t), \; x \in X, t \in K. \tag{3}$$

Here $\mathcal{L}$ is linked to ε' and U via Lemma (4.1.5). Inequality (2) allows us to prove again $1 = \| \tilde{T} \| = \|\tilde{T}u_0\|$ and the other thesis in Lemma (4.1.5) imply $\| T - \tilde{T} \| < 2\varepsilon$.

The explanations above cover the relevant results and isolate the difficulties we have had to overcome to prove them. We should stress that our results are proved for unital and non unital uniform algebras, and that to the best of our knowledge these results are not known even for the Bishop-Phelps property.

Then, We devoted to prove the existence of peak functions for uniform algebras with values in St_ε: this is what we observe as our Urysohn type lemmas, see Lemma (4.1.5) and Lemma (4.1.7), that are needed to establish our main result, Theorem (4.1.14). The difficulty to prove the existence of peak functions in uniform algebras with values in our needed St_ε is the same that when St_ε is replaced by the closure of any bounded simply connected region with simple boundary points: for this reason we have observed these general facts too in Proposition (4.1.8). We devoted to prove Theorem (4.1.14), its preparatives and its consequences.

By letters X and Y we always denote Banach spaces. Unless otherwise stated our Banach spaces can be real or complex. B_X and S_X are the closed unit ball and the unit sphere of X. By X^* - respectively X^{**} - we denote the topological dual - respectively bidual of X. Given a complex Banach space X we will write $X_{\mathbb{R}}$ to denote X but with its subjacent real Banach structure. The weak topology in X is denoted by w, and w^* is the weak* topology in X^*. $L(X,Y)$ stands for the space of norm bounded linear operators from X into Y endowed with its usual norm of uniform convergence on bounded sets of X. A subset B of the dual unit ball B_{X^*} is said to be 1-norming if for every $x \in X$ we have $\| x \| = \sup\{|x^*(x)|: x^* \in B\}$. Given a convex subset $C \subset X$ we denote by ext (C) the set of extreme points of C, i.e., those points in C that are not midpoints of non-degenerate segments in C. Given $C \subset X, x^* \in X^*$ and $\alpha > 0$ we write

$$S(x^*, C, \alpha) := \left\{ y \in C : \operatorname{Re} x^*(y) > \sup_{z \in C} \operatorname{Re} x^*(z) - \alpha \right\}.$$

$S(x^*, C, \alpha)$ is called a slice of C. In particular, if $C \subset X^*$ and $x^* = x$ is taken in the predual X we say that the slice $S(x, C, \alpha)$ is a w^*-slice of C. A classical Choquet's lemma says that for a convex and w^*-compact set $C \subset X^*$, given a point $x^* \in \operatorname{ext}(C)$, the family of w^*-slices

$$\{S(x, C, \alpha): \alpha > 0, x \in X, x^* \in S(x, C, \alpha)\}$$

forms a neighborhood base of x^* in the relative w^*-topology of C-see [186].

The letters K and L are reserved to denote compact and locally compact Hausdorff spaces respectively. $C(K)$ stands for the space of complex-valued continuous functions defined on K and $\|\cdot\|_\infty$ denotes the supremum norm on $C(K)$. A uniform algebra is a $\|\cdot\|_\infty$-closed subalgebra $A \subset C(K)$ equipped with the supremum norm, that separates the points of K (that is, for every $x \neq y$ in K there exists $f \in A$ such that $(x) \neq f(y)$). Given $x \in K$, we denote by $\delta_x : A \to \mathbb{C}$ the evaluation functional at x given by $\delta_x(f) = f(x)$, for $f \in A$. The natural injection $t: K \to A^*$ defined by $l(t) = \delta_t$ for $t \in K$ is a homeomorphism from K onto $(t(K), w^*)$. A set $S \subset K$ is said to be a boundary for the uniform algebra A if for every $f \in A$ there exists $x \in S$ such that $|f(x)| = \| f \|_\infty$. We say that the uniform algebra $A \subset C(K)$ is unital if the constant function 1 belongs to A. Given $x \in K$ we denote by $\mathcal{N}_x$ the family of the open sets in K containing x.

In what follows $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk of the complex plane, $\overline{\mathbb{D}} = \{z \in \mathbb{C}: |z| \leq 1\}$ is the closed unit disk and $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$ is the unit circle. By $A(\mathbb{D})$ we denote the disk algebra, i.e., the uniform subalgebra of $C(\overline{\mathbb{D}})$ made of functions whose restrictions to $\mathbb{D}$ are analytic. Given $z \in \mathbb{C}$ and $r > 0$, we write $D(z; r)$ - respectively $D[z; r]$ - to denote the open disk $z + r\mathbb{D}$-respectively the closed disk $z + r\overline{\mathbb{D}}$.

See [192] for Banach space theory, [187] for Banach algebras, [200] for complex analysis and [194] for harmonic analysis.

As we mentioned we extend [179] to any uniform algebra. As noted, this result in [179] depends on Urysohn's lemma, that for a compact K allows us to find for a given $x \in K$ and $U \in \mathcal{N}_x$, a continuous real valued function of norm one, taking value 1 at x and vanishing on $K \setminus U$.

We cannot use this lemma in the setting of a general uniform algebra A, because the resulting function does not necessarily belong to A. Therefore, our first task here is to prove a Urysohn type lemma for uniform algebras on which we can rely on.

Throughout this A is a unital uniform algebra on K. If

$$S:=\{x^*\in A^*:\|x^*\|=1,x^*(\mathbf{1})=1\},\qquad(4)$$

then $\Gamma_0=\{t\in K:\delta_t\in\operatorname{ext}(S)\}$ is a boundary for A that is called the Choquet boundary of A, see ([187], Lemma 4.3.2 and Proposition 4.3.4).

A stronger version of Lemma (4.1.2) below can be proved taking into account that in unital uniform algebras the Choquet boundary consists exactly of the strong boundary points of K for the algebra, see [187] (see also Proposition (4.1.8) where this result is applied). We prefer to state Lemma (4.1.2) as follows because this is exactly what is needed to prove our main result. On the other hand the proof that we provide makes this part self-contained and our arguments will be later adapted when proving the corresponding result for non-unital algebras, see Lemma (4.1.6).

Lemma (4.1.2)[177]: Let $A\subset C(K)$ be as above. Then, for every open set $U\subset K$ with $U\cap\Gamma_0\neq\emptyset$ and $\delta>0$, there exists $f=f_\delta\in A$ and $t_0\in U\cap\Gamma_0$ such that $\|f\|_\infty=f(t_0)=1$ and $|f(t)|<\delta$ for every $t\in K\setminus U$.

Proof. Observe first that $l(U)$ is a w^*-open set in $l(K)$. Therefore, there exists a w^*-open set $V\subset S$ such that $l(U)=V\cap l(K)$. Fix $x\in U\cap\Gamma_0$. Since δ_x is an extreme point of the w^* compact set S and δ_x belongs to $V\subset S$, Choquet's lemma ensures the existence of $f_0\in A$ and $r\in\mathbb{R}$ such that the w^*-slice of $S,\{x^*\in S:\operatorname{Re}x^*(f_0)>r\}$, is included into $V\cap S$ and contains δ_x. In particular, $\operatorname{Re}f_0(x)>r$ and $\operatorname{Re}f_0(t)\le r$ for all $t\in K\setminus U$.

Note that $\max_{t\in K}\operatorname{Re}f_0(t)=:m>r$ and consider $g(t):=e^{f_0(t)}$ for $t\in K$. It is clear that $g\in A$ - see Lemma (4.1.3)-,$g(K)\subset e^m\overline{\mathbb{D}}$ and that g maps $K\setminus U$ into $e^r\overline{\mathbb{D}}$, i.e., strictly inside of $e^m\overline{\mathbb{D}}$. Since Γ_0 is a boundary for A, there exists $t_0\in U\cap\Gamma_0$ such that $|g(t_0)|=e^m$. Now, take $n\in\mathbb{N}$ such that $e^{n(r-m)}<\delta$. Then, the function defined by

$$f(t)=\left(\frac{g(t)}{g(t_0)}\right)^n,\quad\text{for } t\in K,$$

is the one that we need.

We also need the following two lemmas that gather some basic and known results about uniform algebras. Lemma (4.1.4) that we write down without a proof can be proved in several different easy ways; it also appears as a very particular and straightforward consequence of some other much stronger result, see for instance Mergelyan's theorem [200].

Lemma (4.1.3)[177]: Let $A\subset C(K)$ be a uniform algebra, $M\subset\mathbb{C}$ and $g:M\to\mathbb{C}$ a function that is the uniform limit of a sequence of complex polynomials restricted to M. For every $f\in A$ with $f(K)\subset M$ the following statements hold true:

(i) If A is unital, then $g\circ f\in A$.

(ii) If A is non-unital, $0\in M$ and $g(0)=0$, then $g\circ f\in A$.

Proof. Let us fix a sequence $p_n:\mathbb{C}\to\mathbb{C}$ of polynomials that converges uniformly to g on M. In case (i), $p_n\circ f\in A$ for $n\in\mathbb{N}$ and $g\circ f$ is the uniform limit on K of $(p_n\circ f)_n$, and therefore $g\circ f\in A$. In case (ii), we define $q_n:=p_n-p_n(0)$ for every $n\in\mathbb{N}$. Now, $q_n\circ f\in A$ for $n\in\mathbb{N}$ and $g\circ f$ is the uniform limit on K of $(q_n\circ f)_n$, and therefore $g\circ f\in A$.

Lemma (4.1.4)[177]: Every $\phi \in A(\mathbb{D})$ is the uniform limit of a sequence of complex polynomials on $\overline{\mathbb{D}}$.

As already recalled for $0 < \varepsilon < 1$ the Stolz region is defined by
$$\mathrm{St}_\varepsilon := \{z \in \mathbb{C}: |z| + (1-\varepsilon)|1-z| \leq 1\}.$$
Let us note that St_ε is convex, $\mathrm{St}_\varepsilon \subset \overline{\mathbb{D}}$ and 1 is the only point of the unit circle $\mathbb{T}$ that belongs to (the boundary of) St_ε. Note also that $\varepsilon^2\overline{\mathbb{D}} \subset \mathrm{St}_\varepsilon$ and therefore 0 is an interior point of St_ε. Indeed, for every $z \in \varepsilon^2\overline{\mathbb{D}}$ we have that
$$|z| + (1-\varepsilon)|1-z| \leq \varepsilon^2 + (1-\varepsilon)(1+\varepsilon^2) < \varepsilon^2 + (1-\varepsilon)(1+\varepsilon) = 1.$$
Theorem 14.19 of [200] implies that the Stolz region has the following property.

Finally we can prove the auxiliary lemma:

Lemma (4.1.5)[177]: Let $A \subset C(K)$ be a unital uniform algebra. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in A$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \| f \|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and $f(K) \subset \mathrm{St}_\varepsilon$, i.e.
$$|f(t)| + (1-\varepsilon)|1-f(t)| \leq 1, \quad \text{for all } t \in K. \tag{5}$$
Proof. Let $\phi \in A(\mathbb{D})$ be the function. The set $\phi^{-1}(\varepsilon^2\mathbb{D}) \subset \mathbb{D}$ is an open neighborhood of 0. Let $\delta > 0$ be such that $\delta\mathbb{D} \subset \phi^{-1}(\varepsilon^2\mathbb{D})$ and let f_δ be the function of norm one and t_0 the corresponding point in $U \cap \Gamma_0$ provided by Lemma (4.1.2). Then the function $f = \phi \circ f_\delta$ is the one that we need. Indeed, on one hand Lemma (4.1.3) and Lemma (4.1.4) assure us that $f \in A$. On the other hand, we have that $f(K) \subset \mathrm{St}_\varepsilon$ that gives us inequality (5), and also $f(t_0) = \phi\big(f_\delta(t_0)\big) = 1 = \| f \|_\infty$. Finally we have that,
$$f(K \setminus U) = \phi(f_\delta(K \setminus U)) \subset \phi(\delta\mathbb{D}) \subset \varepsilon^2\mathbb{D} \subset \varepsilon\mathbb{D}.$$
Thus, $|f(z)| < \varepsilon$ for every $t \in K \setminus U$ and the proof is finished.

Throughout this B is a non-unital uniform algebra, that is, a closed subalgebra of $C(K)$, separating points and with $\mathbf{1} \notin B$. Denote by $A := \{c\mathbf{1} + f : c \in \mathbb{C}, f \in B\}$ the $\|\cdot\|_\infty$- closed subalgebra generated by $B \cup \{\mathbf{1}\}$. Since the natural embedding of A into the space of continuous functions on the set of characters of A is an isometry, we can assume without loss of generality that K is the Gelfand compactum - i.e. set of characters - of A. Consider the Choquet boundary of $A, \Gamma_0(A) \subset K$. Since B is a maximal ideal of A (note that it is 1-codimensional), Gelfand-Mazur theorem assures us that there exists $v \in K$ such that $B = \{f \in A: \delta_v(f) = 0\}$. Denote $\Gamma_0 = \Gamma_0(A) \setminus \{v\}$. Observe that Γ_0 is a boundary for B. For general background on Gelfand representation theory see [190].

With a bit of extra work in the proof of Lemma (4.1.2), its non-unital version is proved below.

Lemma (4.1.6)[177]: Let $B \subset C(K)$ be as above. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $\delta > 0$, there is $f \in B$ and $t_0 \in U \cap \Gamma_0$ such that $\| f \|_\infty = f(t_0) = 1$ and $|f(t)| < \delta$ for every $t \in K \setminus U$.

Proof. Without loss of generality we can assume that $v \notin U$. We use the natural identification of K with $l(K)$ as we did in the proof of Lemma (4.1.2). Let us fix $x \in U \cap \Gamma_0$. Since x is an extreme point of S as defined in (4), by Choquet's lemma, there exists a w^*-slice of S that contains x and lies inside U. This slice that can be assumed generated by an element $f_0 \in B$- note that $\mathbf{1}$ is constant on S - is of the form $\{y^* \in S : \mathrm{Re}\, y^*(f_0) > r\}$ for some $r \in \mathbb{R}$. So, $\mathrm{Re}\, f_0(x) > r$, and for every $t \in K \setminus U$ we have $\mathrm{Re}\, f_0(t) \leq r$ and in particular $0 = \mathrm{Re}\, f_0(v) \leq r$.

Note that $\max_{t\in K}\operatorname{Re} f_0(t) =: m > r$. Since Γ_0 is a boundary for B, there exists a $t_0 \in \Gamma_0 \cap U$ such that $\operatorname{Re} f_0(t_0) = m$. Define $g(t) = e^{f_0(t)} - 1, t \in K$. Then we have that $g \in B$ after Lemma (4.1.3), $g(K) \subset e^m\overline{\mathbb{D}} - 1$, and $g(K \setminus U) \subset e^r\overline{\mathbb{D}} - 1$, i.e., strictly inside of $e^m\overline{\mathbb{D}} - 1$. Observe that $0 \in e^m\overline{\mathbb{D}} - 1$ because $m > r \geq \operatorname{Re} f_0(v) = 0$. Now, consider a Möbius transformation $h(z) = \frac{az+b}{cz+d}$ that conformally maps $e^m\mathbb{D} - 1$ onto $\mathbb{D}$, the boundary of $e^m\mathbb{D} - 1$ onto the boundary of $\mathbb{D}$ and such that $h(0) = 0$. Since $g(t_0) = e^{f_0(t_0)} - 1$ belongs to the boundary of $e^m\overline{\mathbb{D}} - 1$, its image $h(g(t_0))$ belongs to the boundary of $\overline{\mathbb{D}}$. Then

$$f(t) := \left(\frac{(h \circ g)(t)}{(h \circ g)(t_0)}\right)^n, \ t \in K,$$

for suitable $n \in \mathbb{N}$, is the function that we need.

The main result reads as follows:

Lemma (4.1.7)[177]: Let $B \subset C(K)$ be as in the previous lemmas. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and $0 < \varepsilon < 1$, there exist $f \in B$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \| f \|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in K \setminus U$ and

$$|f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K.$$

For our applications to the Bishop-Phelps-Bollobás property we just need the Lemmas (4.1.5) and (4.1.7) as presented already. We might have realized that our previous arguments work for arbitrary bounded simply connected region with simple boundary points. Although we do not need it we complete with a few comments about this general case.

Recall that a boundary point β of a simply connected region Ω of $\mathbb{C}$ is said to be a simple boundary point of Ω if β has the following property: to every sequence $(z_n)_n$ in Ω such that $z_n \to \beta$ there corresponds a curve $\gamma: [0,1] \to \mathbb{C}$ and a sequence $(t_n)_n$,

$$0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots \text{ with } t_n \to 1,$$

such that $\gamma(t_n) = z_n$ for every $n \in \mathbb{N}$ and $\gamma([0,1]) \subset \Omega$, see [200]. All points in the boundary of $\overline{\mathbb{D}}$ and St_ε are simple boundary points.

Every bounded simply connected region Ω such that all points in its boundary $\partial\Omega$ are simple has the property that every conformal mapping of Ω onto $\mathbb{D}$ extends to a homeomorphism of $\bar{\Omega}$ onto $\overline{\mathbb{D}}$, see [200].

Proposition (4.1.8)[177]: Let $A \subset C(K)$ be a unital uniform algebra, $\Omega \subset \mathbb{C}$ a bounded simply connected region such that all points in its boundary $\partial\Omega$ are simple. Let us fix two different points a and b with $b \in \partial\Omega, a \in \bar{\Omega}$ and a neighborhood $V_a \subset \bar{\Omega}$ of a. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \emptyset$ and for every $t_0 \in U \cap \Gamma_0$, there exists $f \in A$ such that

(i) $f(K) \subset \bar{\Omega}$;

(ii) $f(t_0) = b$;

(iii) $f(K \setminus U) \subset V_a$.

Proof. According to [187] any point $t_0 \in \Gamma_0$ is a strong boundary point for A and therefore for every $\delta > 0$ there exists a function $g_\delta \in A$ such that $g_\delta(t_0) = 1 = \|g_\delta\|_\infty$ and $g_\delta(K \setminus U) \subset \delta\mathbb{D}$.

We distinguish two cases for the proof:

Case 1: $a \in \Omega$. According to [200] we can produce a homeomorphism $\phi: \overline{\mathbb{D}} \to \bar{\Omega}$ such that ϕ is a conformal mapping from $\mathbb{D}$ onto Ω with $\phi(1) = b$ and $\phi(0) = a$. Using and adequate

g_δ as described above and ϕ the proof goes along the path that we followed in the proof of Lemma (4.1.5).

Case 2: $a \in \partial\Omega$. Since $\operatorname{int}(V_a) \cap \Omega \neq \emptyset$ we can take $a' \in \Omega$ and $\delta' > 0$ such that $D(a', \delta') \subset V_a \cap \Omega$. Now, we apply case 1 to a', its neighborhood $D(a', \delta')$ and b. The thesis follows.

Needless to say that in the non-unital case other results in the vein of the above proposition with the right hypothesis could be proved too.

The result below that appears as Theorem 1 in [183] is known nowadays in the literature as the Bishop-Phelps-Bollobás theorem:

Theorem (4.1.9)[177]: Let X be a Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $|1 - x_0^*(x_0)| \leq \varepsilon^2/2$ $(0 < \varepsilon < 1/2)$. Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| < \varepsilon + \varepsilon^2.$$

It is easily seen that in the real case, if we assume that $x_0^*(x_0) \geq 1 - \varepsilon^2/4$ then the points x^* and x above can be taken satisfying $\|x_0^* - x^*\| \leq \varepsilon$ and $\|x_0 - x\| \leq \varepsilon$.

Note that a direct application of Brøndsted-Rockafellar variational principle, [199], gives a better result:

Corollary (4.1.10)[177]: Let X be a real Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $x_0^*(x_0) \geq 1 - \varepsilon^2/2$ $(0 < \varepsilon < \sqrt{2})$. Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| \leq \varepsilon.$$

We remark that in the previous corollary the hypothesis $x_0^*(x_0) \geq 1 - \varepsilon^2/2$ cannot be weakened if we still wish to obtain the estimates (6), see [183].

Corollary (4.1.10) is easily extended to the complex case. Recall that given a complex Banach space X, the canonical map $\Re: X^* \to (X_{\mathbb{R}})^*$ defined by $\operatorname{Re}(x^*)(x) := \operatorname{Re} x^*(x)$, for $x^* \in X^*$ and $x \in X$, is an isometry and also an homeomorphism from (X^*, w^*) onto $((X_{\mathbb{R}})^*, w^*)$.

Corollary (4.1.11)[177]: Let X be a Banach space, $x_0^* \in S_{X^*}$ and $x_0 \in S_X$ such that $|x_0^*(x_0)| \geq 1 - \varepsilon^2/2$ $(0 < \varepsilon < \sqrt{2})$. Then there exists $x^* \in S_{X^*}$ that attains the norm at some $x \in S_X$ such that

$$\|x_0^* - x^*\| \leq \varepsilon \text{ and } \|x_0 - x\| \leq \varepsilon. \tag{6}$$

Proof. Let us take $\lambda \in \mathbb{C}$ such that $|x_0^*(x_0)| = \lambda x_0^*(x_0)$. Then, we can apply Corollary (4.1.10) to the norm one real functional $\Re(x_0^*)$ and the norm one vector λx_0, to obtain $u^* \in S_{(X_{\mathbb{R}})^*}$ and $u \in S_X$ with $u^*(u) = 1$ and such that

$$\|u^* - \Re(x_0^*)\| \leq \varepsilon \text{ and } \|u_0 - \lambda x_0\| \leq \varepsilon.$$

If we set $x^* = \Re^{-1}(u^*)$ and $x = \lambda^{-1}u$, then x^* is a norm one complex continuous functional on X that satisfies $|x^*(x)| = |\lambda^{-1}| = 1$. On the other hand $\|x_0 - x\| = \|\lambda x_0 - u\| \leq \varepsilon$. Since $\Re$ is an isometry, we deduce that $\|x_0^* - x^*\| = \|\operatorname{Re}(x_0^*) - u^*\| \leq \varepsilon$, and the proof is over.

A complex Banach space X is said to be an Asplund space if its underlying real space $X_{\mathbb{R}}$ is Asplund, that is, whenever ψ is a convex continuous real valued function defined on an open convex subset U of X, the set of all points of U where ψ is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [181] under the name strong differentiability space. Combined efforts of Namioka, Phelps and Stegall led to Theorem (4.1.12) below that is valid both for real and complex Banach spaces. This result already hints at the power of the concepts involved.

Theorem (4.1.12)[177]: ([198],[201],[202]). Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space;

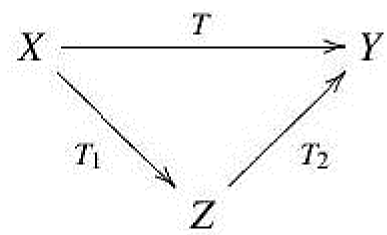
(ii) every w^*-compact subset of (X^*, w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

For the notion of the Radon-Nikodým property see [189] and for the concept of fragment ability see [197].

An operator $T \in L(X,Y)$ is said to be an Asplund operator if it factors through an Asplund space,



i.e., there are an Asplund space Z and operators $T_1 \in L(X,Z), T_2 \in L(Z,Y)$ such that $= T_2 \circ T_1$, see [191],[203]. Note that every weakly compact operator $T \in \mathcal{W}(X,Y)$ factors through a reflexive Banach space, see [188], and hence T is an Asplund operator.

A careful reading of [179] together with the fact that (i) $\Leftrightarrow$ (ii) in Theorem (4.1.12), for real and complex spaces, should give the tools to establish the validity of the following lemma. As usual T^* denotes the adjoint of T.

Lemma (4.1.13)[177]: Let $T: X \to Y$ be an Asplund operator with $\|T\| = 1$ and $x_0 \in S_X$ such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2} (0 < \varepsilon < \sqrt{2})$. For any given 1-norming set $\Gamma \subset B_{Y^*}$ if we write $M = T^*(\Gamma)$ then, for every $r > 0$ there exist:

(i) a w^*-open set $U_r \subset X^*$ with $U_r \cap M \neq \emptyset$, and

(ii) points $y_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|y_r^*(u_r)| = 1$ such that

$$\|x_0 - u_r\| \le \varepsilon \text{ and } \|z^* - y_r^*\| \le r + \frac{\varepsilon^2}{2} + \varepsilon \text{ for every } z^* \in U_r \cap M. \tag{7}$$

We can prove our main result as application of all the above.

Theorem (4.1.14)[177]: Let $\mathfrak{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathfrak{A}$ be an Asplund operator with $\| T \| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{L(X,2)}$ satisfying that

$$\|\tilde{T}u_0\| = 1, \ \|x_0 - u_0\| \le \varepsilon \text{ and } \| T - \tilde{T} \| < 2\varepsilon.$$

Proof. Fix arbitrary $r > 0$ and $0 < \varepsilon' < 1$. If $\mathfrak{A} = A$ is unital then take $\Gamma_0 = \Gamma(A)$ the Choquet boundary of A. If $\mathfrak{A} = B$ is not unital then change K and take Γ_0 as we did at the beginning. In any case, we can assume that we are dealing with an Asplund operator $T: X \to \mathfrak{A} \subset (C(K), \| \_\|_\infty)$ for which we can apply Lemma (4.1.13) for $Y := \mathfrak{A}, \Gamma = \{\delta_s \in \mathfrak{A}^*: s \in \Gamma_0\}, r$ and $\varepsilon > 0$. We produce the w^*-open set U_r, the point u_r and the functional $y_r^* \in S_{X^*}$ satisfying the properties in the aforementioned lemma. Since $U_r \cap M \neq \emptyset$ we can pick $s_0 \in \Gamma_0$ such that $T^*\delta_{s_0} \in U_r$. The w^*-continuity of T^* ensures that $U = \{s \in K: T^*\delta_s \in U_r\}$ is an open neighborhood of s_0. Using Lemma (4.1.5) - or Lemma (4.1.7) in the not unital case - for the open set U - that clearly satisfies $U \cap \Gamma_0 \neq \emptyset$ - and ε' we obtain a function $f \in \mathfrak{A}$ and $t_0 \in U \cap \Gamma_0$ satisfying

$$f(t_0) = \| f \|_\infty = 1, \tag{8}$$

$$|f(t)| < \varepsilon' \quad \text{for every } t \in K \setminus U \tag{9}$$

and

$$|f(t)| + (1-\varepsilon')|1 - f(t)| \leq 1 \quad \text{for every } t \in K. \tag{10}$$

Define now the linear operator $\tilde{T}: X \to \mathfrak{A}$ by the formula

$$\tilde{T}(x)(t) = f(t)y_r^*(x) + (1-\varepsilon')\big(1 - f(t)\big)T(x)(t). \tag{11}$$

It is easily checked that $\tilde{T}$ is well-defined. Bearing in mind (10) we prove that $\|\tilde{T}\| \leq 1$. On the other hand,

$$1 = |y_r^*(u_r)| \overset{(8)}{=} \left|\tilde{T}(u_r)(t_0)\right| \leq \|\tilde{T}(u_r)\| \leq 1$$

and therefore $\tilde{T}$ attains the norm at the point $u_0 = u_r \in S_X$ for which we already had that $\|u_0 - x_0\| \leq \varepsilon$.

Now, for every $x \in B_X$, since Γ_0 is a boundary for $\mathfrak{A}$, we have that

$$\begin{aligned} \|Tx - \tilde{T}^x\|_\infty &= \sup_{t\in\Gamma_0} |f(t)(y_r^*(x) - T(x)(t)) - \varepsilon'(1 - f(t))T(x)(t)| \\ &\leq \sup_{t\in\Gamma_0} \{|f(t)||y_r^*(x) - T^*\delta_t(x)| + \varepsilon'|1 - f(t)||T(x)(t)|\} \\ &\overset{(8)}{\leq} \sup_{t\in\Gamma_0} \{|f(t)|\|y_r^* - T^*\delta_t\|\} + 2\varepsilon'. \end{aligned}$$

On one hand, since $T^*\delta_t \in U_r \cap M$ for every $t \in U \cap \Gamma_0$, we deduce that

$$\sup_{t\in U\cap\Gamma_0} |f(t)|\|y_r^* - T^*\delta_t\| \overset{(7)}{\leq} r + \frac{\varepsilon^2}{2} + \varepsilon.$$

On the other hand, since $t \in \Gamma_0 \setminus U$ implies $t \in K \setminus U$, we obtain that

$$\sup_{t\in\Gamma_0\setminus U} |f(t)|\|y_r^* - T^*\delta_t\| \overset{(9)}{\leq} 2\varepsilon'.$$

Gathering the information of the last three inequalities we conclude that

$$\|T - \tilde{T}\| \leq \max\{4\varepsilon', 2\varepsilon' + r + \varepsilon^2/2 + \varepsilon\}.$$

Since $r > 0$ and $0 < \varepsilon' < 1$ are arbitrary, for suitable values

$$\max\{4\varepsilon', 2\varepsilon' + r + \varepsilon^2/2 + \varepsilon\} < 2\varepsilon.$$

To finish the proof we show that $\tilde{T}$ is also an Asplund operator. To this end it suffices to observe that Asplund operators between Banach spaces form an operator ideal, and that $\tilde{T}$ in (11) appears as a linear combination of a rank one operator, the operator T and the operator $x \mapsto f \cdot T\underset{T}{T}(x)$. The latter is the composition of a bounded operator from $\mathfrak{A}$ into itself with T. Therefore $\tilde{T}$ is an Asplund operator and the proof is over.

We conclude with a list of remarks concerning the peculiarities and scope of the results that we have proved here:

R1: If we denote by $\mathcal{A}$ the ideal of Asplund operators between Banach spaces and $\mathcal{I} \subset \mathcal{A}$ is a sub-ideal, Theorem (4.1.14) naturally applies for any operator $T \in \mathcal{I}(X, \mathfrak{A})$ and the provided $\tilde{T}$ belongs again to $\mathcal{I}(X, \mathfrak{A})$.

R2: Theorem (4.1.14) applies in particular to the ideals of finite rank operators $\mathcal{F}$, compact operators $\mathcal{K}, p$-summing operators Π_p and of course to the weakly compact operators $\mathcal{W}$ themselves. To the best of our knowledge even in the case $\mathcal{W}(X, \mathfrak{A})$ the Bishop-Phelps property that follows from Theorem (4.1.14) is a brand new result.

R3: Let L be a scattered and locally compact space. The space of continuous functions vanishing at infinity $C_0(L)$ on L endowed with its sup norm $\|\cdot\|_\infty$ is an Asplund space, see comments after Corollary 2.6 in [179]. Therefore $(C_0(L), \mathfrak{A})$ has the BPBp for any uniform algebra. More in particular, for any set Γ the pair, $(c_0(\Gamma), \mathfrak{A})$ has the BPBp. Note that [179] provided the first example of an infinite dimensional Banach space Y such that (c_0, Y) has the Bishop-Phelps-Bollobás property, namely for any $Y = C_0(L)$ as before. In a different order of ideas, it has been established in [195] that (c_0, Y) has the BPBp for every uniformly convex Banach space Y.

Section (4.2): Certain Spaces of Operators:

E. Bishop and R. Phelps in [210] proved that every continuous linear functional x^* on a Banach space X can be uniformly approximated on the closed unit ball of X by a continuous linear functional y^* that attains its norm. This result is called the Bishop-Phelps Theorem. B. Bollobás [211] showed that this approximation can be obtained with the additional property that the point at which x^* almost attains its norm is close in norm to a point at which y^* attains its norm. This is a "quantitative version" of the Bishop-Phelps Theorem, known as the Bishop-Phelps-Bollobás Theorem.

X and Y will be Banach spaces over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. As usual, S_X, B_X and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual of X, respectively.

Theorem (4.2.1)[204]: (Bishop-Phelps-Bollobás Theorem). (See [212].) Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X*}$ such that $y^*(y) = 1, \| y - x \| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$

K will be a compact Hausdorff space and μ will be a σ-finite measure. Different versions of the Bishop-Phelps-Bollobás Theorem for operators were proved in [205]. Amongst them it is shown a characterization of the Banach spaces Y satisfying an analogous result to the Bishop-Phelps-Bollobás Theorem for operators from ℓ_1 into Y. There are also positive results for operators from $L_1(\mu)$ into $L_\infty[0,1]$ [208],[214] and for operators from an Asplund space into $C(K)$ [207]. See also [213],[223],[224]

We provide classes of spaces satisfying a version of the Bishop-Phelps-Bollobás Theorem for operators. By $\mathcal{L}(X, Y)$ we denote the Banach space of bounded linear operators from X into Y. We need the following definitions.

The next property was introduced in [205].

Definition (4.2.2)[204]: Let X and Y be both real or complex Banach spaces. The pair (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon) \ \text{ and } \ \| S - T \| < \varepsilon.$$

In this case, we also say that the space $\mathcal{L}(X, Y)$ has the Bishop-Phelps-Bollobás property.

When the operator T (in the definition above) belongs to a certain class, we expect that S also belongs to the same class. Therefore we introduce the following notion.

Definition (4.2.3)[204]: Let X and Y be both real or complex Banach spaces and M a subspace of $\mathcal{L}(X,Y)$. We say that M satisfies the Bishop-Phelps-Bollobás property if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for any $T \in S_M$, if $x_0 \in S_X$ satisfies that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_M$ satisfying the following conditions:

$$\|Su_0\| = 1,\ \|u_0 - x_0\| < \varepsilon \text{ and } \| S - T \| < \varepsilon$$

To study the Bishop-Phelps-Bollobás property for operators on ℓ_1, the following geometric property was introduced in [205].

Definition (4.2.4)[204]: A Banach space X has the approximate hyperplane series property (AHSP) if for every $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{t\to 0+}\gamma(t) = 0$ such that for every sequence $(x_k) \subset S_X$(or $(x_k) \subset B_X$) and every convex series $\sum_{k\geqslant 1}\alpha_k$ satisfying

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1 - \eta(\varepsilon)$$

there exist a subset $D \subset \mathbb{N}, \{z_k : k \in D\} \subset S_X$ and $x^* \in S_{X^*}$ such that

(i) $\sum_{k\in D}\alpha_k > 1 - \gamma(\varepsilon)$,

(ii) $\|z_k - x_k\| < \varepsilon$ for all $k \in D$,

(iii) $x^*(z_k) = 1$ for all $k \in D$.

Note that X has AHSP if whenever we have a convex series of vectors in B_X whose norm is very close to 1 , then a preponderance of these vectors are uniformly close to unit vectors that lie in the same affine hyperplane. For instance, finite-dimensional spaces, uniformly convex spaces, $C(K)$ and $L_1(\mu)$ have AHSP [205].

We characterize the Banach spaces Y such that certain subspaces of operators from $L_1(\mu)$ into Y satisfy the Bishop-Phelps-Bollobás property. As a consequence, we show that the following conditions are equivalent:

(a) Y satisfies AHSP.

(b) $\mathcal{F}(L_1(\mu), Y)$ (finite-rank operators) has the Bishop-Phelps-Bollobás property.

(c) $\mathcal{K}(L_1(\mu), Y)$ (compact operators) has the Bishop-Phelps-Bollobás property.

(d) $\mathcal{W}(L_1(\mu), Y)$ (weakly compact operators) has the Bishop-Phelps-Bollobás property.

(e) $\mathcal{RN}(L_1(\mu), Y)$ (Radon-Nikodým operators) has the Bishop-Phelps-Bollobás property.

We also deal with the Bishop-Phelps-Bollobás property for Asplund operators. We extend Theorem 2.4 and Corollary 2.5 of [207] to some spaces of vector valued continuous functions. As a consequence, we obtain new spaces of operators satisfying the Bishop-Phelps-Bollobás property. We prove that the pairs $\left(X, \mathcal{K}\left(Y, C(K)\right)\right)$, $\left(X, \mathcal{W}\left(Y, C(K)\right)\right)$, and $\left(X, \mathcal{L}\left(Y, C(K)\right)\right)$ satisfy the Bishop-Phelps-Bollobás property if X is an Asplund space and Y has property α of Schachermayer [227] (for instance $Y = \ell_1$). Finally, new examples of spaces having AHSP are provided, for instance $\mathcal{K}(X, C(K))$ and $\mathcal{L}(X, C(K))$ whenever X is uniformly smooth.

It will be convenient to begin by recalling a few definitions and results related to Radon-Nikodým operators. Let (Ω, Σ, μ) be a finite measure space. A bounded linear operator $T: L_1(\mu) \to Y$ is said to be representable if there exists $g \in S_{L_\infty(\mu,Y)}$ such that

$$T(f) = \int_\Omega gf d\mu \text{ for all } f \in L_1(\mu)$$

(see [218], p. 61 or [220], Definition 5.5.15)

We recall that a Radon-Nikodým operator is an operator $T: X \to Y$ such that TS is representable for every operator $S: L_1(\mu) \to X$ (see [220], Definition 5.5.12 and Theorem 5.5.19). A bounded operator $T: L_1(\mu) \to Y$ is representable if and only if T is a Radon-Nikodým operator (see [220], Proposition 5.5.18). Also, a Banach space Y has the Radon-Nikodým property if and only if every operator $T: L_1(\mu) \to Y$ is a Radon-Nikodým operator (see [220], Proposition 5.5.16).

Following [217], an operator ideal $\mathcal{J}$ is a subclass of the class $\mathcal{L}$ such that for any pair of Banach spaces $(X, Y), \mathcal{J}(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$ which contains the finite rank operators and satisfies the so-called "ideal property". That is, given arbitrary Banach spaces X_0, Y_0, we have $R \circ S \circ T \in \mathcal{J}(X, Y)$ for any S in $\mathcal{J}(X_0, Y_0), T$ in $\mathcal{L}(X, X_0)$, and R in $\mathcal{L}(Y_0, Y)$, and for every Banach spaces X and Y. The operator ideal $\mathcal{J}$ is said to be closed if the subspace $\mathcal{J}(X, Y)$ is closed in $\mathcal{L}(X, Y)$ for all Banach spaces X and Y

As mentioned above, we denote by $\mathcal{RN}$ the closed operator ideal of all Radon-Nikodým operators. Also we have $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{RN}$ (see [220], Proposition 5.5.20)

The elementary result below will be useful.

Lemma (4.2.5)[204]: (See [205], Lemma 3.3.) Let (c_n) be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for some convex series $\sum_n \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $D := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate

$$\sum_{i \in D} \alpha_i \geqslant 1 - \frac{\eta}{1 - r}$$

The following result is a refinement of [214].

Proposition (4.2.6)[204]: Let (Ω, Σ, μ) be a measure space such that $L_1(\mu)$ is infinite-dimensional, Y a Banach space, and M a subspace of $\mathcal{L}(L_1(\mu), Y)$ containing all finite-rank operators. If M has the Bishop-Phelps-Bollobás property, then Y has $AHSP$

Proof. For every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ satisfying Definition (4.2.3).

Now, given $0 < \varepsilon < \frac{1}{9}$, we will prove that Y satisfies AHSP for the functions $\bar{\eta}(\varepsilon) = \min\{\eta(\varepsilon^3), \varepsilon\}$ and γ given by

$$\gamma(\varepsilon) := 8\varepsilon(1 - \varepsilon) + \varepsilon + \varepsilon^3(1 - \varepsilon). \tag{12}$$

It is clear that $\gamma(\varepsilon) > 0$ and $\lim_{\varepsilon \to 0} \gamma(\varepsilon) = 0$ as it is required in Definition (4.2.4).

Let (y_n) be a sequence in S_Y and a convex series $\sum_n \alpha_n$ satisfying

$$\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \bar{\eta}(\varepsilon).$$

Fix N such that

$$\left\| \sum_{n=1}^{N} \alpha_n y_n \right\| > 1 - \bar{\eta}(\varepsilon) \geqslant 1 - \varepsilon > 0. \tag{13}$$

If we write $\tilde{\alpha}_n = \frac{\alpha_n}{\sum_{k=1}^{N} \alpha_k}$ then

$$\left\| \sum_{k=1}^{N} \tilde{\alpha}_k y_k \right\| \geqslant \left\| \sum_{k=1}^{N} \alpha_k y_k \right\| > 1 - \eta(\varepsilon^3) \text{ and } \sum_{k=1}^{N} \tilde{\alpha}_k = 1. \tag{14}$$

By assumption, there is a sequence (E_n) of pairwise disjoint subsets in Σ satisfying $0 < \mu(E_n) < \infty$ for each n. For every positive integer n, let x_n^* be the functional on $L_1(\mu)$ associated to χ_{E_n}, that is,

$$x_n^*(f) := \int_{E_n} f d\mu \ (f \in L_1(\mu))$$

Now we define the finite-rank operator (therefore in M)$T: L_1(\mu) \to Y$ by

$$T(f) = \sum_{k=1}^{N} x_k^*(f) y_k \ (f \in L_1(\mu))$$

Note that $\| T \| \leqslant 1$ and $\|T(\chi_{E_k})\| = \|\chi_{E_k}\|_1$ for all $k \leqslant N$, then $T \in S_M$.

Define $f_0 := \sum_{k=1}^{N} \tilde{\alpha}_k \frac{\chi_{E_k}}{\mu(E_k)}$. By (14), $\|f_0\|_1 = 1$ and $\|T(f_0)\| = \|\sum_{k=1}^{N} \tilde{\alpha}_k y_k\| > 1 - \eta(\varepsilon^3)$. Since M has the Bishop-Phelps-Bollobás property, there exist $g_0 \in S_{L_1(\mu)}$ and $S \in S_M$ satisfying

$$\|S g_0\| = 1, \ \|g_0 - f_0\|_1 < \varepsilon^3 \ \text{ and } \ \| S - T \| < \varepsilon^3. \tag{15}$$

Proceeding as in [214] we obtain

$$\sum_{k=1}^{N} \operatorname{Re} x_k^*(g_0) > 1 - \varepsilon^3. \tag{16}$$

Let $s = 1 - \frac{\varepsilon^2}{8}$ and $D := \{k \in \mathbb{N} : k \leqslant N, \operatorname{Re} x_k^*(g_0) > s x_k^*(|g_0|)\}$. By (16) and following the proof of [214] we obtain

$$\sum_{k \in D} \operatorname{Re} x_k^*(g_0) > 1 - \frac{\varepsilon^3}{1 - s} = 1 - 8\varepsilon > 0. \tag{17}$$

Thus $D \neq \emptyset$.

Combining (17) and (15) and using $\varepsilon < \frac{1}{9}$ we deduce that

$$\sum_{k \in D} \tilde{\alpha}_k \geqslant \sum_{k \in D} \operatorname{Re} x_k^*(g_0) - \|g_0 - f_0\|_1 > 1 - 8\varepsilon - \|g_0 - f_0\|_1 > 1 - 8\varepsilon - \varepsilon^3 > 0.$$

By (13) and the previous inequality

$$\sum_{k \in D} \alpha_k = \left(\sum_{k \in D} \tilde{\alpha}_k\right)\left(\sum_{k=1}^{N} \alpha_k\right) > (1 - 8\varepsilon - \varepsilon^3)(1 - \bar{\eta}(\varepsilon))$$
$$\geqslant (1 - 8\varepsilon - \varepsilon^3)(1 - \varepsilon) = 1 - \gamma(\varepsilon)$$

Therefore, condition (i) of Definition (4.2.4) is satisfied. Now, note that for a complex number w with $|w| \leqslant 1$ and $\operatorname{Re} w > r > 0$ it is satisfied $|1 - w|^2 = 1 + |w|^2 - 2\operatorname{Re} w < 2(1 - r)$. So for every $k \in D$ we have

$$\left|1 - \frac{x_k^*(g_0)}{x_k^*(|g_0|)}\right|^2 < 2(1 - s) = \frac{\varepsilon^2}{4}. \tag{18}$$

For $k \in \mathbb{N}$ we define $z_k = S\left(\frac{g_0 \chi_{E_k}}{x_k^*(|g_0|)}\right)$ if $x_k^*(|g_0|) \neq 0$ and 0 otherwise. In particular, $\|z_k\| \leqslant 1$ for every k. We write $\Omega_1 = \Omega \setminus \bigcup_{k=1}^{\infty} E_k$. Let us notice that $g_0 = \sum_{k=1}^{\infty} g_0 \chi_{E_k} + g_0 \chi_{\Omega_1}$ and the series is norm convergent. Then

$$S(g_0)=\sum_{k=1}^{\infty} S\big(g_0\chi_{E_k}\big)+S\big(g_0\chi_{\Omega_1}\big)=\sum_{k=1}^{\infty} x_k^*(|g_0|)z_k+S\big(g_0\chi_{\Omega_1}\big)$$

By the Hahn-Banach Theorem, there is a functional $y^* \in S_{Y^*}$ attaining its norm at $S(g_0)$. Then

$$1=y^*\big(S(g_0)\big)=\sum_{k=1}^{\infty} x_k^*(|g_0|)y^*(z_k)+y^*\left(S\big(g_0\chi_{\Omega_1}\big)\right)$$

$$\leqslant \sum_{k=1}^{\infty}\left(\int_{E_k}|g_0|d\mu\right)+\|g_0\chi_{\Omega_1}\|=\|g_0\|_1=1$$

Therefore

$$y^*(z_k)=1 \text{ for all } k\in\mathbb{N} \text{ with } x_k^*(|g_0|)\neq 0$$

In particular, $z_k \in S_Y$ for $k \in D$ and condition (iii) of Definition (4.2.4) is also satisfied.

Now for every $k \in D$ we have that $x_k^*(g_0) \neq 0$ and $T\left(\frac{g_0\chi_{E_k}}{x_k^*(g_0)}\right)=y_k$. Hence by (15) for every $k \in D$ we deduce that

$$\left\|z_k-\frac{x_k^*(g_0)}{x_k^*(|g_0|)}y_k\right\|=\left\|S\left(\frac{g_0\chi_{E_k}}{x_k^*(|g_0|)}\right)-T\left(\frac{g_0\chi_{E_k}}{x_k^*(|g_0|)}\right)\right\|\leqslant\| S-T \|<\varepsilon^3$$

Finally, by (18), for every $k \in D$ we obtain

$$\|z_k-y_k\|\leqslant\left\|z_k-\frac{x_k^*(g_0)}{x_k^*(|g_0|)}y_k\right\|+\left\|\left(\frac{x_k^*(g_0)}{x_k^*(|g_0|)}-1\right)y_k\right\|\leqslant\varepsilon^3+\frac{\varepsilon}{2}<\varepsilon$$

and Y has AHSP.

Improving [214], we give a partial converse of Proposition (4.2.6).

Theorem (4.2.7)[204]: Let (Ω,Σ,μ) be a finite measure space, Y a Banach space with AHSP and M a subspace of $\mathcal{L}(L_1(\mu),Y)$ such that contains all finite-rank operators and it is contained in the subspace of all representable operators. Also, assume that the operator $S_A(f)=S(f\chi_A)$ belongs to M whenever $S\in M$ and A is any measurable subset of Ω. Then M has the Bishop-Phelps-Bollobás property for operators.

Proof. By assumption Y has AHSP; let γ and η be the functions satisfying Definition (4.2.4). Given $0<\varepsilon<1$, we choose $0<\delta<\frac{\varepsilon}{6}$ such that $0<\gamma(\delta)<\frac{\varepsilon}{6}$ and $0<\delta'<\min\left\{\frac{\varepsilon}{6},\frac{\eta(\delta)}{4}\right\}$. Define $\rho(\varepsilon):=\frac{\eta(\delta)}{2}$ and assume that $T\in S_M$ and $f_0\in S_{L_1(\mu)}$ satisfy that $\|Tf_0\|>1-\rho(\varepsilon)$. There is a function $h\in L_\infty(\mu)$ such that $|h(t)|=1$ for every $t\in\Omega$ and satisfying also that $h(t)f_0(t)=|f_0(t)|$ for every $t\in\Omega$. Now we define a surjective linear isometry $\psi: L_1(\mu)\to L_1(\mu)$ given by

$$\psi(f)=hf \ (f\in L_1(\mu)),$$

that satisfies $\psi(f_0)(t)\in\mathbb{R}_0^+$ for every $t\in\Omega$

We write $R=T\psi^{-1}$ and $u_0=\psi(f_0)$. Clearly, we have $\|R(u_0)\|=\|T(f_0)\|>1-\rho(\varepsilon)$, with $u_0\in S_{L_1(\mu)}$ nonnegative and $R\in S_{\mathcal{L}(L_1(\mu),Y)}$

Since T is a representable operator, R is also representable. So there is $g\in L_\infty(\mu,Y)$ such that

$$R(f)=\int_\Omega gf\,d\mu \ \text{ for all } f\in L_1(\mu)$$

By [218], g also satisfies that $\|g\|_\infty = \|R\| = 1$. By [218], there exist a measurable function $h: \Omega \to Y$, whose range is countable, and a μ-null subset E of Ω such that $\|(g-h)\chi_{\Omega\setminus E}\|_\infty < \frac{\varepsilon}{4}$. Write $h = \sum_{n=1}^{\infty} \chi_{B_n} w_n$ (pointwise convergence) with $(w_n) \subset Y$ and (B_n) a sequence of pairwise disjoint measurable sets of Ω with $\cup_n B_n = \Omega$. Hence, fixed $n \in \mathbb{N}$ and $s, t \in B_n \setminus E$ we have

$$\|g(s)-g(t)\| \leqslant \|g(s)-h(s)\| + \|h(s)-h(t)\| + \|h(t)-g(t)\| < \frac{\varepsilon}{2}$$

Both functions g and $g\chi_{\Omega\setminus E}$ represent R, then we may assume that

$$\|g(s)-g(t)\| < \frac{\varepsilon}{2} \text{ for all } s, t \in B_n \text{ and } n \in \mathbb{N}. \tag{19}$$

By the Monotone Convergence Theorem the sequence $(u_0\chi_{\cup_{k=1}^n B_k})$ converges to u_0 in $L_1(\mu)$. Since $1-\rho(\varepsilon) < \|R(u_0)\|$, for some m large enough we have

$$1-\rho(\varepsilon) < \left\|R\left(u_0\chi_{\cup_{k=1}^m B_k}\right)\right\| \text{ and } \left\|u_0 - u_0\chi_{\cup_{k=1}^m B_k}\right\| < \delta'. \tag{20}$$

We write $B = \cup_{i=1}^m B_k$. Since u_0 is a non-negative function in $S_{L_1(\mu)}$, there is a non-negative simple function v_0 in $B_{L_1(\mu)}$ with support contained in B satisfying $\|v_0 - u_0\chi_B\| < \delta'$ and $\|v_0\| = \|u_0\chi_B\|$ and so $0 < 1-\delta' \leqslant \|v_0\| \leqslant 1$. The element $s_0 = \frac{v_0}{\|v_0\|}$ belongs to $S_{L_1(\mu)}$. Its support is contained in B and also satisfies that

$$\begin{aligned}\|s_0 - u_0\chi_B\| &\leqslant \|s_0 - v_0\| + \|v_0 - u_0\chi_B\| = 1 - \|v_0\| + \|v_0 - u_0\chi_B\| < 2\delta' \\ &< \min\left\{\frac{\varepsilon}{3}, \frac{\eta(\delta)}{2}\right\}.\end{aligned} \tag{21}$$

Hence, there is a finite number of pairwise disjoint measurable sets in $B, \{A_1, \dots, A_N\}$, such that s_0 belongs to the space generated by $\{\chi_{A_i}: 1 \leqslant i \leqslant N\}$.

Let $\{C_i: 1 \leqslant i \leqslant p\}$ be the family of pairwise disjoint measurable subsets obtained by indexing the set $\{A_i \cap B_j: 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant m, \mu(A_i \cap B_j) > 0\}$. Write $s_0 = \sum_{k=1}^p \beta_k \chi_{C_k}$ with $\beta_k \geqslant 0$ and $\sum_{k=1}^p \beta_k \mu(C_k) = \|s_0\| = 1$.

From (20) and (21) we obtain that

$$\begin{aligned}1-\eta(\delta) = 1-\rho(\varepsilon) - \frac{\eta(\delta)}{2} &< \|R(u_0\chi_B)\| - \frac{\eta(\delta)}{2} < \|R(s_0)\| \\ &= \left\|\sum_{k=1}^p \beta_k \mu(C_k) R\left(\frac{\chi_{C_k}}{\mu(C_k)}\right)\right\|\end{aligned}$$

Since $R \in S_{\mathcal{L}(L_1(\mu),Y)}, y_k = R\left(\frac{\chi_{C_k}}{\mu(C_k)}\right) \in B_Y$ for $1 \leqslant k \leqslant p$ and

$$1-\eta(\delta) < \left\|\sum_{k=1}^p \beta_k \mu(C_k) y_k\right\|. \tag{22}$$

Observe that by (19), for every $k \leqslant p$ and $t \in C_k$ we have that

$$\begin{aligned}\|g(t) - y_k\chi_{C_k}(t)\| &= \left\|\int_{C_k} \frac{g(t)}{\mu(C_k)} d\mu(u) - \int_{C_k} \frac{g(u)}{\mu(C_k)} d\mu(u)\right\| \\ &\leqslant \int_{C_k} \frac{\| g(t) - g(u) \|}{\mu(C_k)} d\mu(u) \leqslant \frac{\varepsilon}{2}.\end{aligned} \tag{23}$$

Since Y has AHSP and $\sum_{k=1}^{p}\beta_k\mu(C_k)=1$, by (22), there are sets $D\subset\{1,\dots,p\},\{z_k: k\in D\}\subset S_Y$ and $y^*\in S_{Y^*}$ satisfying

$$y^*(z_k)=1,\ \|z_k-y_k\|<\delta \ \text{ for all } k\in D \ \text{ and } \sum_{k\in D}\beta_k\mu(C_k)>1-\gamma(\delta)>0. \quad (24)$$

Now define the function $g_1:\Omega\to Y$ given by $g_1=g\chi_{\Omega\setminus C}+\sum_{k\in D}z_k\chi_{C_k}$, where $C=\bigcup_{k\in D}C_k$. It is clear that $g_1\in B_{L_\infty(\mu,Y)}$. By (23) and (24), we have

$$\|g_1-g\|_\infty=\|(g_1-g)\chi_C\|_\infty<\delta+\frac{\varepsilon}{2}<\varepsilon.$$

Let R_1 be the element in $\mathcal{L}(L_1(\mu),Y)$ associated to g_1. Then $\|R_1\|\leqslant 1$ and

$$\|R_1-R\|=\|g_1-g\|_\infty<\varepsilon. \quad (25)$$

Let $s_1=\sum_{k\in D}\beta_k\chi_{C_k}$, which by (24) is nonzero and satisfies

$$\|s_1\|=\sum_{k\in D}\beta_k\mu(C_k)=y^*\left(\sum_{k\in D}\beta_k\mu(C_k)z_k\right)=y^*\big(R_1(s_1)\big)\leqslant\|y^*\|\|R_1\|\|s_1\|=\|s_1\|$$

Then, $\|R_1\|\leqslant 1$ and R_1 attains its norm at $s_2=\frac{s_1}{\|s_1\|}\cdot$ By (20), (21) and (24) we have

$$\begin{aligned}\|s_2-u_0\| &\leqslant \left\|\frac{s_1}{\|s_1\|}-s_1\right\|+\|s_1-s_0\|+\|s_0-u_0\|=|1-\|s_1\||\\ &\quad+\sum_{k\leqslant p,k\notin D}\beta_k\mu(C_k)+\|s_0-u_0\|\\ &\leqslant 2\sum_{k\leqslant p,k\notin D}\beta_k\mu(C_k)+\|s_0-u_0\chi_B\|+\|u_0\chi_B-u_0\|\\ &\leqslant 2\gamma(\delta)+\frac{\varepsilon}{3}+\delta'<\varepsilon.\end{aligned}$$

Now, define $T_1=R_1\psi$ and $f_2=\psi^{-1}s_2$. Since ψ is an isometry, $T_1\in S_{\mathcal{L}(L_1(\mu),Y)}, f_2\in S_{L_1(\mu)}$ and T_1 attains its norm at f_2. By (25), $\|T_1-T\|=\|R_1-R\|<\varepsilon$, also $\|f_2-f_0\|<\varepsilon$.

Let us notice that R_1-R is the operator associated to the function g_1-g. Hence, for every $f\in L_1(\mu)$ we have

$$(R_1-R)(f)=(R_1-R)(f\chi_C)=\sum_{k\in D}\left(\int_{C_k}f\,d\mu\right)z_k-R(f\chi_C)=S(f)-R_C(f)$$

where $R_C(f)=R(f\chi_C)$ and S is the finite-rank operator given by $S(f)=\sum_{k\in D}\left(\int_{C_k}f\,d\mu\right)z_k$. Hence

$$T_1-T=(R_1-R)\psi=(S-R_C)\psi. \quad (26)$$

To show that $T_1\in M$ note that

$$R_C(\psi(f))=R_C(hf)=R(hf\chi_C)=T\big(\psi^{-1}(hf\chi_C)\big)=T\big(\bar{h}hf\chi_C\big)=T_C(f),$$

where $\bar{h}$ stands for the conjugate of h.

Now, the hypothesis on M implies that $R_C\circ\psi$ also belongs to M. On the other hand, M contains all finite-rank operators, thus (26) gives that T_1 is in M. Therefore M has the Bishop-Phelps-Bollobás property.

As a consequence of Theorem (4.2.7), if $\mathcal{J}$ is an operator ideal such that $\mathcal{J}(L_1(\mu), Y) \subset \mathcal{RN}(L_1(\mu), Y)$, Y has AHSP and μ is any finite measure, then the space $\mathcal{J}(L_1(\mu), Y)$ satisfies the Bishop-Phelps-Bollobás property. By Proposition (4.2.6), we deduce the following:

Corollary (4.2.8)[204]: Let Y be a Banach space and (Ω, Σ, μ) a finite measure space such that $L_1(\mu)$ is infinitedimensional. The following conditions are equivalent:

(a) Y satisfies AHSP
(b) $\mathcal{F}(L_1(\mu), Y)$ has the Bishop-Phelps-Bollobás property.
(c) $\mathcal{K}(L_1(\mu), Y)$ has the Bishop-Phelps-Bollobás property.
(d) $\mathcal{W}(L_1(\mu), Y)$ has the Bishop-Phelps-Bollobás property.
(e) $\mathcal{RN}(L_1(\mu), Y)$ has the Bishop-Phelps-Bollobás property.

There are very different Banach spaces having AHSP. For instance, finite-dimensional spaces, uniformly convex spaces, $C(K), L_1(\mu)$ ($\mu\sigma$-finite) and $\mathcal{K}(H)^*(\mathcal{K}(H) =$ compact operators on a Hilbert space) satisfy this property (see [205] and [206]). Also every lush space has AHSP [215] (see also [213]). We will provide later some examples of spaces of operators satisfying AHSP.

We recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be an Asplund operator if T^* is a Radon-Nikodým operator (see [220], Definition 5.5.22). We denote by $\mathcal{A}$ the closed operator ideal of all Asplund operators.

A Banach space Y is said to have property β (of Lindenstrauss [225]) if there are two sets $\{y_\alpha : \alpha \in \Lambda\} \subset S_Y$, $\{y_\alpha^* : \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leqslant \rho < 1$ such that the following conditions hold

(a) $y_\alpha^*(y_\alpha) = 1$,
(b) $|y_\alpha^*(y_\gamma)| \leqslant \rho < 1$ if $\alpha \neq \gamma$
(c) $\| y \| = \sup\{|y_\alpha^*(y)| : \alpha \in \Lambda\}$, for all $y \in Y$.

Aron, Cascales and Kozhushkina in ([207], Theorem 2.4 and Corollary 2.5) proved that $\mathcal{A}(X, C(K))$ has the Bishop-Phelps-Bollobás property. We extend this result to some spaces of vector-valued continuous functions $C(K, Y)$ (Theorem (4.2.9)).

In general, it is known that not every operator into a $C(K)$ space can be approximated by norm attaining operators (see [226], Theorem A or [222], Corollary 2). Moreover, in view of [209], we have to introduce some restrictions on Y in order to get a positive result of Bishop-Phelps-Bollobás property for operators into $C(K, Y)$

We recall that a subspace Z of Y^* is said to be norming for Y, if for every $y \in Y$, we have $\| y \| = \sup\{|\phi(y)| : \phi \in B_Z\}$ for any $y \in Y$. We also say that a subset C of Y^* is $1-$ norming, if $\| y \| = \sup\{|\phi(y)| : \phi \in C\}$ for every $y \in Y$. We denote by $\sigma(Y, Z)$ the topology on Y of pointwise convergence on Z. If Z is any norming subspace for Y and τ is any linear topology on Y with $\sigma(Y, Z) \subset \tau \subset n$ where n is the norm topology then $C(K, (Y, \tau))$ is a Banach space with the norm induced by $\ell_\infty(K, Y)$. Also $C(K, (Y, \tau))$ is stable under products by elements of $C(K)$

Theorem (4.2.9)[204]: Let Y be a Banach space satisfying property β for the subset of functionals $\Delta = \{y_\alpha^* : \alpha \in \Lambda\}$ and Z the closed subspace of Y^* generated by Δ. Let τ be a linear topology on Y with $\sigma(Y, Z) \subseteq \tau \subseteq n$. Then for every closed operator ideal $\mathcal{J}$ such that $\mathcal{J} \subseteq \mathcal{A}$, we have that $\mathcal{J}(X, C(K, (Y, \tau)))$ has the BishopPhelps-Bollobás property for every Banach space X and every compact Hausdorff topological space K.

Proof. Let us fix T in the unit sphere of $\mathcal{J}(X, C(K, (Y, \tau)))$, $0 < \varepsilon < 1$ and $x_0 \in S_X$ such that

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$$

We will prove that there exist $u_0 \in S_X$ and R in the unit sphere of $\mathcal{J}(X, \mathcal{C}(K, (Y, \tau)))$ such that

$$\|R(u_0)\| = 1, \ \|u_0 - x_0\| < \varepsilon \ \text{ and } \ \| R - T \| \leqslant \varepsilon\left(3 + \frac{8\rho}{1-\rho}\right)$$

where ρ is the constant appearing in the definition of property β.

Since Y has property β, the set

$$B := \{\delta_t \otimes y_\alpha^* : t \in K, \alpha \in \Lambda\}$$

is a 1-norming subset of $B_{\mathcal{C}(K,(Y,\tau))^*}$. By [207] one can find a w^*-open subset U of X^* so that $U \cap T^*(B) \neq \emptyset$ and two elements $u_0 \in S_X, u_0^* \in S_{X^*}$ such that

$$u_0^*(u_0) = 1, \ \|u_0 - x_0\| < \varepsilon \ \text{ and } \ \|x^* - u_0^*\| < 3\varepsilon \ \text{ for all } x^* \in U \cap T^*(B). \quad (27)$$

Since $U \cap T^*(B)$ is nonempty, we can find some $t_0 \in K$ and $\alpha_0 \in \Lambda$ such that $T^*\left(\delta_{t_0} \otimes y_{\alpha_0}^*\right) \in U$. Consider the set

$$W := \left\{t \in K : T^*\left(\delta_t \otimes y_{\alpha_0}^*\right) \in U\right\}$$

which is open and contains t_0.

By Urysohn's Lemma, there is a continuous function $f: K \to [0,1]$ whose support is contained in W such that $f(t_0) = 1$. Define the operator $S: X \to \mathcal{C}(K, (Y, \tau))$ by

$$S(x)(t) = T(x)(t) + \left((1+\eta)u_0^*(x) - T^*\left(\delta_t \otimes y_{\alpha_0}^*\right)(x)\right)f(t)y_{\alpha_0} \ (x \in X, t \in K)$$

where $\eta = \frac{4\varepsilon\rho}{1-\rho}$. The operator S is clearly bounded and linear.

Our aim now is to show that S belongs to $\mathcal{J}(X, \mathcal{C}(K, (X, \tau)))$. In order to do that, we consider the bounded linear operators $R: X \to \mathcal{C}(K, (Y, \tau))$ and $F, M_f: \mathcal{C}(K, (Y, \tau)) \to \mathcal{C}(K, (Y, \tau))$ given by

$$R(x)(t) = (1+\eta)u_0^*(x)f(t)y_{\alpha_0} \ (x \in X, t \in K)$$

$$M_f(g)(t) = f(t)g(t) \ \text{ and } \ F(g)(t) = y_{\alpha_0}^*(g(t))y_{\alpha_0} \ (g \in \mathcal{C}(K, (Y, \tau)), t \in K)$$

It is clearly satisfied that $S = T + R - F \circ M_f \circ T$. Since $\mathcal{J}$ is an operator ideal we have that the rank-one operators $R, F \circ M_f \circ T$ and so S belong to $\mathcal{J}(X, \mathcal{C}(K, (Y, \tau)))$.

We will check that $\|S\| = \|S(u_0)\| = 1 + \eta$. Indeed, we have that

$$y_{\alpha_0}^*\left(S(u_0)(t_0)\right) = (1+\eta)u_0^*(u_0) = 1 + \eta. \quad (28)$$

On the one hand, for $t \in K \setminus W$ we know that $f(t) = 0$, so $S(x)(t) = T(x)(t)$, hence

$$\|S(x)(t)\| \leqslant 1 \ \text{ for all } x \in B_X. \quad (29)$$

On the other hand, if $t \in W$ we distinguish two cases to estimate $|y_\alpha^*(S(x)(t))|$.

For $\alpha = \alpha_0$ we obtain that

$$\begin{aligned} \left|y_{\alpha_0}^*(S(x)(t))\right| &= \left|y_{\alpha_0}^*(T(x)(t)) + \left((1+\eta)u_0^*(x) - T^*\left(\delta_t \otimes y_{\alpha_0}^*\right)(x)\right)f(t)\right| \\ &= \left|(1-f(t))y_{\alpha_0}^*(T(x)(t)) + (1+\eta)u_0^*(x)f(t)\right| \\ &\leqslant \left|(1-f(t))y_{\alpha_0}^*(T(x)(t)) + f(t)u_0^*(x)\right| + \eta|u_0^*(x)| \leqslant 1 + \eta, \end{aligned} \quad (30)$$

since $(1-f(t))y_{\alpha_0}^*(T(x)(t)) + f(t)u_0^*(x)$ is a convex combination of $y_{\alpha_0}^*(T(x)(t))$ and $u_0^*(x)$.

For $\alpha \in \Lambda \setminus \{\alpha_0\}$, since t is in W, by (27) we know that $\left\|u_0^* - T^*\left(\delta_t \otimes y_{\alpha_0}^*\right)\right\| < 3\varepsilon$. Thus,

$$|y_\alpha^*(S(x)(t))| \leqslant |y_\alpha^*(T(x)(t))| + \left|\left(u_0^* - T^*\left(\delta_t \otimes y_{\alpha_0}^*\right)\right)(x) + \eta u_0^*(x)\right| \left|y_\alpha^*\left(y_{\alpha_0}\right)\right| f(t)$$
$$\leqslant 1 + (3\varepsilon + \eta)\rho < 1 + \eta. \quad (31)$$

By (29), (30) and (31), we have that $\|S\| \leqslant 1 + \eta$ and by (28) we obtain $\|S\| = 1 + \eta$ and $\|S(u_0)\| = 1 + \eta$. We will check that $\|S - T\| \leqslant \varepsilon\left(3 + \frac{4\rho}{1-\rho}\right)$. If $t \in K \setminus W$ then $S(x)(t) = T(x)(t)$. If $t \in W$ then by (27)

$$\|S(x)(t) - T(x)(t)\| = \left\|\left((1+\eta)u_0^*(x) - T^*\left(\delta_t \otimes y_{\alpha_0}^*(x)\right)\right) f(t) y_{\alpha_0}\right\|$$
$$\leqslant 3\varepsilon + \eta = \varepsilon\left(3 + \frac{4\rho}{1-\rho}\right).$$

Finally, taking $R = \frac{S}{\|S\|}$ we get

$$\|R - T\| \leqslant \left\|\frac{S}{\|S\|} - S\right\| + \|S - T\| = (1 - \|S\|) + \|S - T\|$$
$$\leqslant \eta + \varepsilon\left(3 + \frac{4\rho}{1-\rho}\right) = \varepsilon\left(3 + \frac{8\rho}{1-\rho}\right)$$

which completes the proof.

We provide examples of pairs of Banach spaces with the Bishop-Phelps-Bollobás property for operators. Recall that the spaces $\mathcal{C}(K, Y^*), \mathcal{C}\left(K, (Y^*, w)\right)$ and $\mathcal{C}\left(K, (Y^*, w^*)\right)$ can be isometrically identified with $\mathcal{K}(Y, \mathcal{C}(K)), \mathcal{W}(Y, \mathcal{C}(K))$ and $\mathcal{L}(Y, \mathcal{C}(K))$, respectively (see [219], Theorem VI.7.1, p. 490). It is also known that $\mathcal{L}(X, Y) = \mathcal{A}(X, Y)$ whenever X is an Asplund space. The following property will be required.

A Banach space Y is said to have property α (of Schachermayer) if there are two sets $\{y_\alpha : \alpha \in \Lambda\} \subset S_Y, \{y_\alpha^* : \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leqslant \rho < 1$ such that the following conditions hold

(a) $y_\alpha^*(y_\alpha) = 1$ for all $\alpha \in \Lambda$

(b) $\left|y_\alpha^*\left(y_\gamma\right)\right| \leqslant \rho < 1$ for $\alpha, \gamma \in \Lambda, \alpha \neq \gamma$

(c) the unit ball of Y is the closed, circled convex hull of $\{y_\alpha : \alpha \in \Lambda\}$

For every set Λ the space $\ell_1(\Lambda)$ has property α. Property α is quite general if we admit equivalent norms (see [227] and [221]). It is clear that Y^* has property β whenever Y has property α. Hence, we obtain the following corollary:

Corollary (4.2.10)[204]: Let X be an Asplund space and Y a Banach space satisfying property α. Then $(X, \mathcal{K}(Y, \mathcal{C}(K)))$, $(X, \mathcal{W}(Y, \mathcal{C}(K)))$, and $(X, \mathcal{L}(Y, \mathcal{C}(K)))$ have the Bishop-Phelps-Bollobás property for operators for every compact Hausdorff topological space K.

It is known that uniformly convex spaces have AHSP (see [205], Proposition 3.8). Hence X^* has AHSP whenever X is uniformly smooth. We will generalize this fact by providing some spaces of operators satisfying the same property.

We recall that a Banach space X is uniformly convex if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_X, \ \frac{\|u + v\|}{2} > 1 - \delta \Rightarrow \|u - v\| < \varepsilon$$

In such a case, the modulus of convexity of X is given by

$$\delta(\varepsilon) := \inf\left\{1 - \frac{\|u + v\|}{2} : u, v \in B_X, \| u - v \| \geqslant \varepsilon\right\}.$$

Given a (non-empty) bounded subset A of X, an element $x^* \in X^*$ and $\alpha > 0$, the slice $S(A, x^*, \alpha)$ is the subset of A given by

$$S(A, x^*, \alpha) := \left\{z \in A : \operatorname{Re} x^*(z) > \sup_{x \in A} \operatorname{Re} x^*(x) - \alpha\right\}.$$

The following elementary fact will be useful below.

Lemma (4.2.11)[204]: (See [206], Lemma 2.1.) If X is uniformly convex, then for every $\varepsilon > 0$,

$$\operatorname{diam} S(B_X, x^*, \delta(\varepsilon)) \leqslant \varepsilon \text{ for all } x^* \in S_{X^*}$$

Theorem (4.2.12)[204]: Let X be a uniformly convex Banach space and τ be a linear topology on X satisfying $w \subseteq \tau \subseteq n$. Then the space $\mathcal{C}(K, (X, \tau))$ has AHSP for any compact Hausdorff topological space K.

Proof. We write $Y = \mathcal{C}(K, (X, \tau))$ and denote by δ the modulus of convexity of X. Take $(f_i)_{i=1}^n \subset B_Y$ and a finite convex series $\sum_{i=1}^n \alpha_i$ satisfying

$$\left\| \sum_{i=1}^n \alpha_i f_i \right\| > 1 - \varepsilon\delta(\varepsilon).$$

Choose $x_0^* \in S_{X^*}$ and $t_0 \in K$ so that

$$x_0^* \left(\sum_{i=1}^n \alpha_i f_i(t_0) \right) > 1 - \varepsilon\delta(\varepsilon)$$

By Lemma (4.2.5), the set $D := \{1 \leqslant i \leqslant n \ : \ \operatorname{Re} x_0^*\big(f_i(t_0)\big) > 1 - \delta(\varepsilon)\}$ satisfies that $\sum_{k \in D} \alpha_k > 1 - \varepsilon$. Consider the subset U of K given by

$$U = \bigcap_{i \in D} f_i^{-1}\big(S(B_X, x_0^*, \delta(\varepsilon))\big)$$

Since $w \subseteq \tau, U$ is open and it clearly contains t_0. By Urysohn's Lemma, there exists a continuous function $\phi: K \to [0,1]$ with $\operatorname{supp}(\phi) \subset U$ and $\phi(t_0) = 1$

By assumption X is reflexive, so there is $x_0 \in S_X$ so that $x_0^*(x_0) = 1$. For each $i \in D$, define $g_i \in B_Y$ by

$$g_i = \phi x_0 + (\mathbf{1} - \phi) f_i$$

For $i \in D$, we have that $g_i(t_0) = x_0$ and by Lemma (4.2.11) we obtain

$$\begin{aligned} \|g_i - f_i\| &= \|\phi(x_0 \cdot \mathbf{1} - f_i)\| \leqslant \sup_{t \in U} \|x_0 - f_i(t)\| \\ &\leqslant \operatorname{diam} S(B_X, x_0^*, \delta(\varepsilon)) \leqslant \varepsilon \end{aligned}$$

On the other hand, the element $x_0^* \circ \delta_{t_0}$ belongs to S_{Y^*} and $\big(x_0^* \circ \delta_{t_0}\big)(g_i) = x_0^*\big(g_i(t_0)\big) = 1$ for every $i \in D$.

$\mathcal{C}(K, X)$ has AHSP whenever X also satisfies AHSP [216]. As we already noticed, sometimes vector-valued spaces of continuous functions can be identified with spaces of operators. Hence, we deduce the following result.

Corollary (4.2.13)[204]: Let X be a Banach space whose dual has $AHSP$. Then the space $\mathcal{K}(X, \mathcal{C}(K))$ has $AHSP$ for every compact Hausdorff topological space K.

The above corollary implies that $\mathcal{L}(X, \mathcal{C}(K))$ has AHSP for any finite-dimensional space X. It is a natural question whether or not there are infinite-dimensional spaces with the previous property. The answer is positive since it is not difficult to show that for every set I, the space

$(\oplus_{i\in I} Y)_{\ell_\infty}$ has AHSP whenever Y satisfies AHSP. Hence the space $\mathcal{L}(\ell_1, Y) = (\oplus_{n\in\mathbb{N}} Y)_{\ell_\infty}$ has also AHSP. We will provide another example that follows from the main result.

Corollary (4.2.14)[204]: The spaces $\mathcal{L}(X, \mathcal{C}(K))$ and $\mathcal{K}(X, \mathcal{C}(K))$ have AHSP for every uniformly smooth Banach space X and every compact Hausdorff topological space K.

Section (4.3): Γ-Flatness and Operators:

X, Y are Banach spaces (real or complex), $\mathbb{K}$ stands for the field of scalars $\mathbb{R}$ or $\mathbb{C}$, $L(X, Y)$ is the space of all bounded linear operators $T: X \to Y, L(X) = L(X, X), B_X$ and S_X denote the closed unit ball and the unit sphere of X, respectively and aco A stands for the absolute convex hull of the set A.

According to [229], a pair (X, Y) has the Bishop-Phelps-Bollobás property (BPB property) for operators if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every operator $T \in L(X, Y)$ of norm 1 , if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying $\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon$, and $\| T - S \| < \varepsilon$.

If an analogous definition is valid for operators T, S from a subspace $\mathcal{I} \subset L(X, Y)$, then we say that (X, Y) has the Bishop-Phelps-Bollobás property for operators from $\mathcal{I}$.

The original Bishop-Phelps-Bollobás theorem [236] says that for every X, the pair $(X, \mathbb{K})$ has the BPB property for operators. Also, see Acosta, Aron, García, and Maestre [229], if Y has the Lindenstrauss' property β, then for every Banach space X the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators.

In 2011 Aron, Cascales, and Kozhushkina [232] showed that for every X and every compact Hausdorff space K the pair $(X, C(K))$ has the BPB property for Asplund operators. In 2013 Cascales, Guirao and Kadets [237] extended this result to uniform algebras $\mathcal{A} \subset C(K)$. The exact statement of the last result is given below.

Theorem (4.3.1)[228]: ([237], Theorem 3.6). Let $\mathcal{A} \subset C(K)$ be a uniform algebra and $T: X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X,\mathcal{A})}$ satisfying that:

$$\|Su_0\| = 1, \ \|x_0 - u_0\| \leq \varepsilon \ \text{ and } \ \| T - S \| < 2\varepsilon.$$

In the same vein, Acosta, Becerra Guerrero, García, Kim, and Maestre [230] generalized [232] to some spaces of continuous vector-valued functions.

We extend all these results to a wider class of Banach spaces and to a wider class of operators. The main difference of our approach is that instead of proving a Bishop-Phelps-Bollobás kind theorem for each space separately (and thus repeating essential parts of the proof many times), we introduce a new Banach space property (called ACK_ρ structure) which extracts all the useful technicalities for the BPB type of approximation. We prove a general Bishop-Phelps-Bollobás type theorem for Γ-flat operators (see Definition (4.3.8)) acting to a space with ACK_ρ structure and show that uniform algebras and spaces with the property β have ACK_ρ structure. After that, we study the stability of the ACK_ρ structure under some natural Banach space theory operations which as a consequence gives us a wide collection of examples of pairs (X, Y) possessing the BPB property for Asplund operators.

We collect the necessary definitions (in particular that of Asplund operators and of Γ-flat operators) and prove an important Basic Lemma. We introduce the central concept of ACK_ρ structure and prove a general BPB type theorem for this class of Banach spaces. Finally, we perform the announced study of spaces with ACK_ρ structure which, on the one hand, gives a unified proof of several results from [229],[230],[232] and [237], and on the other hand, leads to new BPB type theorems in concrete spaces.

For the non-defined notions used through, see [240].

Let (B,τ) be a topological space, ρ be a metric on B (possibly, not related with τ). B is said to be fragmented by ρ, if for every non-empty subset $A \subset B$ and for every $\varepsilon > 0$ there exists a τ-open U such that $U \cap A \neq \emptyset$ and $\operatorname{diam}(U \cap A) < \varepsilon$. Some important examples of fragmented topological spaces come from Banach space theory. For instance, every weakly compact subset of a Banach space is fragmented by the norm (i.e., by the metric $\rho(x,y) = \| x - y \|$), see [244].

A Banach space X is called an Asplund space if, whenever f is a convex continuous function defined on an open subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [231] under the name strong differentiability space. This concept has multiple characterizations via topology or measure theory, as in the following:

Theorem (4.3.2)[228]: ([245],[249],]22],[251]). Let X be a Banach space. Then the following conditions are equivalent:

(i) X is an Asplund space;

(ii) every w^*-compact subset of (X,w^*) is fragmented by the norm;

(iii) each separable subspace of X has separable dual;

(iv) X^* has the Radon-Nikodým property.

According to the above, every reflexive space and every separable space whose dual is separable is an Asplund space. Classical example of Asplund spaces are L_p and ℓ_p with $1 < p < \infty$, and also c_0; examples of spaces that are not Asplund are $C[0,1], \ell_1, \ell_\infty, L_1[0,1]$ and $L_\infty[0,1]$, see [239].

Definition (4.3.3)[228]: ([251]). An operator $T \in L(X,Y)$ is said to be an Asplund operator if it factors through an Asplund space, i.e., there exist an Asplund Banach space Z and operators $T_1 \in L(X,Z), T_2 \in L(Z,Y)$ such that $= T_2 \circ T_1$.

Compact and weakly compact operators are Asplund operators (every weakly compact operator factorizes through a reflexive space).

Theorem (4.3.2) yields the following result:

Definition (4.3.4)[228]: Let Y be a Banach space. Y is said to have the BishopPhelps-Bollobás property for Asplund operators (A-BPBp for short) if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for every Banach space X and every Asplund operator $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \| T - S \| < \varepsilon.$$

Definition (4.3.5)[228]: ([241]). Let A and B be topological spaces. A function $f\colon A \to B$ is said to be quasi-continuous, if for every non-empty open subset $U \subset A$, every $z \in U$ and every neighborhood V of $f(z)$ there exists a non-empty open subset $W \subset U$ such that $f(W) \subset V$.

Note that a similar concept of fragmentability of maps was introduced in [242].

Definition (4.3.6)[228]: Let A be a topological space and (M,d) be a metric space. A function $f\colon A \to M$ is said to be openly fragmented, if for every nonempty open subset $U \subset A$ and every $\varepsilon > 0$ there exists a non-empty open subset $V \subset U$ with $d - \operatorname{diam}(f(V)) < \varepsilon$.

Every continuous or quasi-continuous function $f: A \to M$ is openly fragmented. In particular, if A is a discrete topological space then every $f: A \to M$ is openly fragmented. For every metric space M, every left continuous $f:[0,1] \to M$ and every right-continuous function $f:[0,1] \to M$ are openly fragmented. Every $f: A \to M$ with a dense set of continuity points is openly fragmented. Every separately continuous function of two variables $f:[0,1] \times [0,1] \to M$ is quasi-continuous [234] and, consequently, openly fragmented. Some other easy but useful examples are given in the following theorem:

Theorem (4.3.7)[228]: Let A, B be topological spaces, ρ be a metric on B (possibly, not related with the original topology), and $f: A \to B$ be a function.

(i) If B is fragmented by ρ, and f is continuous in the original topologies, then $f: A \to (B, \rho)$ is openly fragmented.

(ii) If A is fragmented by some metric ρ_1 and $f:(A, \rho_1) \to (B, \rho)$ is uniformly continuous, then $f: A \to (B, \rho)$ is openly fragmented.

Let, moreover, $(B, \|\cdot\|)$ be a Banach space. Then

(iii) If $f, g: A \to (B, \|\cdot\|)$ are openly fragmented then $f + g: A \to (B, \|\cdot\|)$ is openly fragmented.

(iv) If $f: A \to (B, \|\cdot\|)$ and $g: A \to \mathbb{K}$ are openly fragmented then $gf: A \to (B, \|\cdot\|)$ is openly fragmented.

The statements (ii), (iii) and (iv) are routine.

Definition (4.3.8)[228]: Let X, Y be Banach spaces and $\Gamma \subset Y^*$. An operator $T \in L(X, Y)$ is said to be Γ-flat, if $T^*|_\Gamma: (\Gamma, w^*) \to (X^*, \|\cdot\|_{X^*})$ is openly fragmented. In other words, for every w^*-open subset $U \subset Y^*$ with $U \cap \Gamma \neq \emptyset$ and every $\varepsilon > 0$ there exists a w^*-open subset $V \subset U$ with $V \cap \Gamma \neq \emptyset$ such that $\operatorname{diam}(T^*(V \cap \Gamma)) < \varepsilon$. The set of all Γ-flat operators in $L(X, Y)$ will be denoted by $\mathrm{Fl}_\Gamma(X, Y)$.

Statements (iii) and (iv) of the previous theorem imply that $\mathrm{Fl}_\Gamma(X, Y)$ is a linear subspace of $L(X, Y)$. Let us list some examples of Γ-flat operators.

Example (4.3.9)[228]: Every Asplund operator $T \in L(X, Y)$ is Γ-flat for every $\Gamma \subset B_{Y^*}$. This follows from Theorem (4.3.7), (i).

Example (4.3.10)[228]: If $(\Gamma, w^*) \subset Y^*$ is norm fragmented, then every bounded operator in $L(X, Y)$ is Γ-flat (Theorem (4.3.7), (ii)). In particular, we have the next concrete example.

Example (4.3.11)[228]: If $(\Gamma, w^*) \subset Y^*$ is discrete, then every operator $T \in L(X, Y)$ is Γ-flat.

The notion of Γ-flat generalizes the property of Asplund operators that allowed to prove [232]. The immediate generalization of that lemma is the following result:

The proof of this fact is a modification of that of [232]. First, we use the following fact:

Proposition (4.3.12)[228]: ([247], Corollary 2.2). Let X be a real Banach space, $z^* \in S_{X^*}, z \in S_X, \eta > 0$ and $z^*(z) \geq 1 - \eta$. Then for every $k \in (0,1)$ there exist $y^* \in S_{X^*}$ and $u \in S_X$ such that

$$y^*(u) = 1, \ \| z - u \| \leq \frac{\eta}{k}, \ \|z^* - y^*\| \leq 2k.$$

In the next proposition, we relax the condition $z^* \in S_X$ allowing $\|z^*\|$ to be smaller than 1. Note that x^* plays the role of z^*.

Proposition (4.3.13)[228]: Let X be a Banach space, $\varepsilon \in (0, 2/3), x \in S_X, x^* \in B_{X^*}$ and $|x^*(x)| \geq 1 - \varepsilon$. Then, for every $k \in \left[\frac{\varepsilon}{2(1-\varepsilon)}, 1\right)$ there exist $y^* \in S_{X^*}$ and $u \in S_X$ such that

$$|y^*(u)| = 1, \ \| x - u \| \leq \frac{\varepsilon}{k}, \ \|x^* - y^*\| \leq 2k.$$

Proof. Without loss of generality we can assume that $x^*(x) \geq 1 - \varepsilon$. Then $\|x^*\| \geq 1 - \varepsilon$. Set $z^* := x^*/\|x^*\|, z := x$. Then $z^*(z) \geq 1 - \eta$ for $\eta = 1 - (1-\varepsilon)\|x^*\|^{-1} \in [0, \varepsilon]$. If $\eta = 0$, then $z^*(z) = 1$, so we can take $y^* = z^*$ and $u = x$, which satisfy the inequalities we want. So we may assume that $0 < \eta \leq \varepsilon$. Set $k_0 := \frac{k\eta}{\varepsilon} \in (0,1)$. So, according to Proposition (4.3.12), there exist $y^* \in S_{X^*}$ and $u \in S_X$ such that

$$y^*(u) = 1, \ \| z - u \| \leq \frac{\eta}{k_0}, \ \|z^* - y^*\| \leq 2k_0.$$

Therefore, $\| x - u \| \leq \eta/k_0 = \varepsilon/k$. Also, we have

$$\begin{aligned}\|x^* - y^*\| &\leq \|x^* - z^*\| + \|z^* - y^*\| \leq \left\|x^* - \frac{x^*}{\|x^*\|}\right\| + 2k_0 \\ &= 1 - \|x^*\| + 2k_0 = 1 - \|x^*\| + \frac{2k}{\varepsilon}\left(1 - \frac{1-\varepsilon}{\|x^*\|}\right).\end{aligned}$$

Observe that the function $\psi(t) = 1 - t + \frac{2k}{\varepsilon}\left(1 - \frac{1-\varepsilon}{t}\right)$ is increasing when $t \in \left(0, \sqrt{\frac{2k(1-\varepsilon)}{\varepsilon}}\right)$.

So, if $k \geq \frac{\varepsilon}{2(1-\varepsilon)}$, we have $\psi(\|x^*\|) \leq \psi(1) = 2k$. In this case, we get our conclusion.

Lemma (4.3.14)[228]: (Basic Lemma). Let X, Y be Banach spaces, $\Gamma \subset B_{Y^*}$ be a 1-norming set, $T \in \mathrm{Fl}_\Gamma(X, Y)$ be a Γ-flat operator with $\| T \| = 1$, $0 < \varepsilon < 2/3$, and $x_0 \in S_X$ be such that $\|Tx_0\| > 1 - \varepsilon$. Then for every $r > 0$ and for every $k \in \left[\frac{\varepsilon}{2(1-\varepsilon)}, 1\right)$ there exist:

(i) a w^*-open set $U_r \subset Y^*$ with $U_r \cap \Gamma \neq \emptyset$, and

(ii) points $x_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|x_r^*(u_r)| = 1$ such that

$$\|x_0 - u_r\| \leq \frac{\varepsilon}{k} \text{ and } \|T^*z^* - x_r^*\| \leq r + 2k \quad \text{for every } z^* \in U_r \cap \Gamma. \tag{32}$$

Proof. Use that $\Gamma \subset B_{Y^*}$ is 1 -norming and pick $y_0^* \in \Gamma$ such that

$$|T^*(y_0^*)(x_0)| = |y_0^*(Tx_0)| > 1 - \varepsilon.$$

Set $U := \{y^* \in Y^* : |T^*y^*(x_0)| > 1 - \varepsilon\}$. We have that $y_0^* \in U \cap \Gamma \subset B_{Y^*}$. Since U is w^*-open in Y^* and $U \cap \Gamma \neq \emptyset$, according to Definition (4.3.8), for every $r > 0$ there exists a w^*-open subset $U_r \subset U$ with $U_r \cap \Gamma \neq \emptyset$ such that $\mathrm{diam}\big(T^*(U_r \cap \Gamma)\big) < r$.

Fix some $y_1^* \in U_r \cap \Gamma$ and set $x_1^* = T^*y_1^*$. Then, $1 \geq \|x_1^*\| \geq |x_1^*(x_0)| > 1 - \varepsilon$ which, by applying Proposition (4.3.13) to any $\frac{\varepsilon}{2(1-\varepsilon)} \leq k < 1$, gives $x_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|x_r^*(u_r)| = 1$ and such that

$$\|x_0 - u_r\| \leq \frac{\varepsilon}{k} \ \text{ and } \ \|x_1^* - x_r^*\| \leq 2k.$$

Finally, let $z^* \in U_r \cap \Gamma$ be arbitrary. Then,

$$\|T^*z^* - x_r^*\| \leq \|T^*z^* - x_1^*\| + \|x_1^* - x_r^*\| \leq r + 2k,$$

which finishes the proof.

In the definition below we extract the structural properties of $C(K)$ and its uniform subalgebras that were essential in the proof of [237]. The name "ACK structure" comes from the words "Asplund" and "$C(K)$".

Definition (4.3.15)[228]: Let X be a Banach space and $\mathcal{O}$ be a non-emtpy subset of $L(X)$. We will say that X has $\mathcal{O}$-ACK structure with parameter ρ, for some $\rho \in [0,1)\big(X \in \mathcal{O} - \text{ACK}_\rho$, for short) whenever there exists a 1-norming set $\Gamma \subset B_{X^*}$ such that for every $\varepsilon > 0$ and every non-empty relatively w^*-open subset $U \subset \Gamma$ there exist a non-empty subset $V \subset U$, vectors $x_1^* \in V$, $e \in S_X$ and an operator $F \in \mathcal{O}$ with the following properties:

(I) $\| Fe \| = \| F \| = 1$;

(II) $x_1^*(Fe) = 1$;

(III) $F^* x_1^* = x_1^*$;

(IV) denoting $V_1 = \{x^* \in \Gamma : \|F^* x^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x^*)\| \leq 1\}$, then $|v^*(Fe)| \leq \rho$ for every $x^* \in \Gamma \setminus V_1$;

(V) $\text{dist}(F^* x^*, \text{aco}\,\{0, V\}) < \varepsilon$ for every $x^* \in \Gamma$; and

(VI) $|v^*(e) - 1| \leq \varepsilon$ for every $v^* \in V$.

The Banach space X is said to have simple $\mathcal{O} - ACK$ structure ($X \in \mathcal{O}$ ACK) if $V_1 = \Gamma$. In other words, for $X \in \mathcal{O} - \text{ACK}$ the above definition holds true with the following modification: the property (IV) becomes

(IV)' $\|F^* x^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x^*)\| \leq 1$ for every $x^* \in \Gamma$.

In case of $\mathcal{O} = L(X)$, we will simply say ACK_ρ (and simple ACK) structure.

Definition (4.3.16)[228]: A linear subspace $\mathcal{J} \subset L(X,Y)$ is said to be a Γ-flat ideal, if all elements of $\mathcal{J}$ are Γ-flat operators, $\mathcal{J}$ contains all operators of finite rank, and for every $T \in \mathcal{J}$ and every $F \in L(Y)$ their composition $F \circ T$ belongs to $\mathcal{J}$.

Observe that the subspace of Asplund operators in $L(X,Y)$ is an example of Γ-flat ideal. The theorem below motivates the above definition.

Lemma (4.3.17)[228]: Under the conditions of Definition (4.3.16) above, for every $k \in (\varepsilon/(2(1-\varepsilon)),1)$ and for every

$$\nu > 2k\left(1 + \frac{2}{1-\rho+2k}\right),$$

there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying $\|Su_0\| = 1, \|x_0 - u_0\| \leq \frac{\varepsilon}{k}$ and $\| T - S \| < \nu$. In the case of $Y \in \text{ACK}$ the same is true for every $\nu > 2k$.

If, moreover, T belongs to a Γ-flat ideal $\mathcal{J}$, then S can be chosen from $\mathcal{J}$ as well.

Proof. First, consider the more involved case of $Y \in \text{ACK}_\rho$. Fix $r > 0$ and $0 < \varepsilon' < 2/3$. Now, we can apply Lemma (4.3.14) with Y, Γ, r and $\varepsilon > 0$. We produce a w^*-open set $U_r \subset Y^*$ with $U_r \cap \Gamma \neq \emptyset$, and points $x_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|x_r^*(u_r)| = 1$ such that (32) holds true.

Since $U_r \cap \Gamma \neq \emptyset$, we can apply Definition (4.3.15) to $U = U_r \cap \Gamma$ and ε' and obtain a non-empty $V \subset U, y_1^* \in V, e \in S_Y, F \in L(Y)$ and $V_1 \subset \Gamma$ which satisfy properties (I) - (VI). In particular, for every $z^* \in V \subset U_r \cap \Gamma$ according to (32) we have

$$\|T^* z^* - x_r^*\| \leq r + 2k. \tag{33}$$

Define now the linear operator $S: X \to Y$ by the formula

$$S(x) := x_r^*(x) Fe + (1-\tilde{\varepsilon})(I_Y - F)Tx, \tag{34}$$

where the value of $\tilde{\varepsilon} \in [\varepsilon', 1)$ will be specified below in such a way that $\| S \| \leq 1$. In order to do this, bearing in mind the fact that Γ is 1-norming, we can write

$$\| S \| = \|S^*\| = \sup\ \{\|S^* y^*\| : y^* \in \Gamma\}.$$

So our first goal is to estimate

$$\|S^* y^*\| = \|y^*(Fe) x_r^* + (1-\tilde{\varepsilon}) T^*(I_{Y^*} - F^*)(y^*)\| \tag{35}$$

from above for all $y^* \in \Gamma$. For $y^* \in V_1$, the sought estimate $\|S^*y^*\| \leq 1$ follows immediately from the definition of V_1 (see property (IV)). So, it remains to consider the case $y^* \in \Gamma \setminus V_1$. Thanks to (V), for every $y^* \in \Gamma$, there exists an element $v^* = \sum_{k=1}^n \lambda_k v_k^*$ with

$$\|F^*y^* - v^*\| < \varepsilon' \tag{36}$$

such that $\{v_k^*\}_{k=1}^n \subset V$, and $\sum_{k=1}^n |\lambda_k| \leq 1$. According to (33) we have $\|T^*v_k^* - x_r^*\| \leq r + 2k$, consequently

$$\begin{aligned} \|v^*(e)x_r^* - T^*v^*\| &\leq \sum_{k=1}^n |\lambda_k| \|v_k^*(e)x_r^* - T^*v_k^*\| \\ &\overset{\text{(VI)}}{\leq} \varepsilon' + \sum_{k=1}^n |\lambda_k| \|x_r^* - T^*v_k^*\| \leq \varepsilon' + r + 2k. \end{aligned} \tag{37}$$

Now, for every $y^* \in \Gamma \setminus V_1$

$$\begin{aligned} \|S^*y^*\| &\leq \tilde{\varepsilon}|y^*(Fe)| + (1-\tilde{\varepsilon})\|y^*(Fe)x_r^* + T^*y^* - T^*F^*y^*\| \\ &\overset{\text{(IV)}}{\leq} \tilde{\varepsilon}\rho + (1-\tilde{\varepsilon})\|T^*y^*\| + (1-\tilde{\varepsilon})\|(F^*y^*)(e)x_r^* - T^*F^*y^*\| \\ &\overset{(36)}{\leq} \tilde{\varepsilon}\rho + (1-\tilde{\varepsilon}) + 2\varepsilon'(1-\tilde{\varepsilon}) + (1-\tilde{\varepsilon})\|v^*(e)x_r^* - T^*v^*\| \\ &\overset{(37)}{\leq} \tilde{\varepsilon}\rho + (1-\tilde{\varepsilon}) + 2\varepsilon'(1-\tilde{\varepsilon}) + (1-\tilde{\varepsilon})(\varepsilon' + r + 2k) \\ &\leq \tilde{\varepsilon}\rho + (1-\tilde{\varepsilon})(1 + 3\varepsilon' + r + 2k). \end{aligned}$$

This means, that if we choose $\tilde{\varepsilon} = (3\varepsilon' + r + 2k)/(1 - \rho + 3\varepsilon' + r + 2k)$, then we have $\| S \| \leq 1$. In this case,

$$1 = |x_r^*(u_r)| \overset{\text{(II)}}{=} |y_1^*(x_r^*(u_r)Fe)| \overset{\text{(III)}}{=} \left|y_1^*\big(S(u_r)\big)\right| \leq \|S(u_r)\| \leq 1.$$

Therefore, $\| S \| = 1$ and S attains the norm at the point $u_0 := u_r \in S_X$ for which by (32) we already had that $\|u_0 - x_0\| \leq \frac{\varepsilon}{k}$.

Now, let us estimate

$$\begin{aligned} \| S - T \| &= \|S^* - T^*\| = \sup_{y^* \in \Gamma} \|S^*y^* - T^*y^*\| \\ &\leq \sup_{y^* \in \Gamma} \|y^*(Fe)x_r^* - T^*F^*y^*\| + 2\tilde{\varepsilon}. \end{aligned} \tag{38}$$

For every $y^* \in \Gamma$ we can proceed the same way as before. Namely,

$$\begin{aligned} \|(F^*y^*)(e)x_r^* - T^*F^*y^*\| &\overset{(36)}{\leq} 2\varepsilon' + \|v^*(e)x_r^* - T^*v^*\| \\ &\overset{(37)}{\leq} 3\varepsilon' + r + 2k. \end{aligned}$$

Combining this with the inequalities (38) and the value of $\tilde{\varepsilon}$ we conclude that

$$\| T - S \| \leq 3\varepsilon' + r + 2k + 2\frac{3\varepsilon' + r + 2k}{1 - \rho + 3\varepsilon' + r + 2k}. \tag{39}$$

Since $r > 0$ and $0 < \varepsilon' < 2/3$ are arbitrary, for suitable values we will have the desired estimate $\| T - S \| < \nu$.

To finish the proof in the case of $Y \in \mathrm{ACK}_\rho$ we observe that if T belongs to a Γ-flat ideal $\mathcal{J}$ then $S \in \mathcal{J}$.

Now the simpler case of $Y \in \mathrm{ACK}$. In this case $\|S^*y^*\| \leq 1$ for all $y^* \in \Gamma$ thanks to (IV)'. So, $\|S\| \leq 1$ for all values of $\tilde{\varepsilon} \in [\varepsilon', 1)$ and we can simply take $\tilde{\varepsilon} = \varepsilon'$. With such a choice of

$\tilde{\varepsilon}$ the estimate (39) changes to $\|T-S\| \le 5\varepsilon' + r + 2k$, which again for small values of r and ε' gives us $\|T-S\| < \nu$ for the ν which corresponds to this case.

Theorem (4.3.18)[228]: Let X be a Banach space, $Y \in \mathrm{ACK}_\rho, \Gamma \subset Y^*$ be the corresponding 1-norming set from Definition (4.3.15) and $T \in L(X,Y)$ be a Γ flat operator with $\| T \| = 1$. Let $0 < \varepsilon \le 1/2$ and let $x_0 \in S_X$ be such that $\|Tx_0\| > 1-\varepsilon$. Then there exist $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ with $\|Su_0\| = 1$ such that

$$\max \ \{\|x_0 - u_0\|, \| T - S \|\} < \sqrt{2\varepsilon}\left(1 + \frac{2}{1-\rho+\sqrt{2\varepsilon}}\right).$$

Moreover, if $Y \in \mathrm{ACK}$ then the estimate can be improved to

$$\max \ \{\|x_0 - u_0\|, \| T - S \|\} < \sqrt{2\varepsilon}.$$

Additionally, S can be chosen from $\mathcal{J}$ whenever T belongs to a Γ-flat ideal $\mathcal{J}$. In particular, every $Y \in \mathrm{ACK}_\rho$ (ACK) has the A-BPBp.

Proof. First, select $\varepsilon_0 \in (0,\varepsilon)$ in such a way that the inequality $\|Tx_0\| > 1-\varepsilon_0$ is still valid. Now we apply Lemma (4.3.17) with ε_0 instead of ε and substitute $k = \sqrt{\varepsilon_0/2}$. In the case of $Y \in \mathrm{ACK}_\rho$ we take $\nu \in \left(\sqrt{2\varepsilon_0}\left(1+\frac{2}{1-\rho+\sqrt{2\varepsilon_0}}\right), \sqrt{2\varepsilon}\left(1+\frac{2}{1-\rho+\sqrt{2\varepsilon}}\right)\right)$, and in the case of $Y \in$ ACK we take $\nu \in \left(\sqrt{2\varepsilon_0}, \sqrt{2\varepsilon}\right)$.

Also, a look at the proof of Lemma (4.3.14) shows that the condition of T being Γ-flat can be weaken in the following way: for every $y \in B_Y$ and every $\delta > 0$ if the w^*-slice $S(\Gamma, x, \delta) := \{y^* \in \Gamma : \mathrm{Re}\, y^*(y) > 1-\delta\}$ is not empty, then for every $\varepsilon > 0$ there exists a non-empty relatively w^*-open subset $V \subset S(\Gamma, x, \delta)$ such that $\mathrm{diam}(T^*(V)) < \varepsilon$.

There are two reasons why we have selected the more restrictive variants. Firstly, with the restrictive definition of (IV) we are able to prove a nice stability result, and secondly, all the examples with "relaxed" versions of (IV) and of Γ-flatness that we have in hand, satisfy the restrictive variant of (IV) and of Γ-flatness.

We presenting those natural examples of Banach spaces having ACK structure as well as showing the stability of the ACK structure under some operations, such us ℓ_∞-sums or injective tensor products.

First of all, let us introduce the first natural class of Banach spaces with ACK structure. As commented above, Definition (4.3.15), comes from an analysis of the proofs in [237]. We shall show next that, indeed, every uniform algebra $\mathcal{A}$ has simple ACK structure. The key tool is Lemma (4.3.20), that was proved in [237], and is about the existence of peak functions $f \in S_{\mathcal{A}}$ whose range is contained in the Stolz's region

$$\mathrm{St}_\varepsilon = \{z \in \mathbb{C} : |z| + (1-\varepsilon)|1-z| \le 1\}.$$

For a topological space (T,τ), we denote by $C_b(T)$ the space of bounded continuous functions $f : T \to \mathbb{K}$ equipped with the sup-norm.

Definition (4.3.19)[228]: Let (T,τ) be a topological space. A subalgebra $\mathcal{A} \subset C_b(T)$ is said to be an ACK-subalgebra, if for every non-empty open set $W \subset T$ and $0 < \varepsilon < 1$, there exist $f \in \mathcal{A}$ and $t_0 \in W$ such that $f(t_0) = \| f \|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in T \setminus W$ and $f(T) \subset \mathrm{St}_\varepsilon$.

Lemma (4.3.20)[228]: Let $\mathcal{A} \subset C(K)$ be a uniform algebra. Then there exists a topological space $\Gamma_{\mathcal{A}}$ such that $\mathcal{A}$ is isometric to an ACK-subalgebra of $C_b(\Gamma_{\mathcal{A}})$. In the case of K being the space of multiplicative functionals on $\mathcal{A}$ the corresponding $\Gamma_{\mathcal{A}}$ can be selected as a topological subspace of K.

We will use the following elementary property of $\mathrm{St}_{\varepsilon}$.

Lemma (4.3.21)[228]: If z belongs to the Stolz region $\mathrm{St}_{\varepsilon}$, then $z^n \in \mathrm{St}_{\varepsilon}$.

Proof. For every $z \in \mathrm{St}_{\varepsilon}$ it holds

$$\begin{aligned} |z^n| + (1-\varepsilon)|1-z^n| &= |z^n| + (1-\varepsilon)|1-z||1+z+\cdots+z^{n-1}| \\ &\le |z|^n + (1-|z|)|1+z+\cdots+z^{n-1}| \\ &\le |z|^n + (1-|z|)(1+|z|+\cdots+|z|^{n-1}) \\ &= |z|^n + (1-|z|^n) = 1, \end{aligned}$$

which finishes the proof.

The following simple lemma gives an essential property that turns uniform algebras into Banach spaces with simple ACK structure.

Lemma (4.3.22)[228]: Let $\mathcal{A} \subset C_b(\Gamma_{\mathcal{A}})$ be an ACK-subalgebra. Then, for every non-empty open set $W \subset \Gamma_{\mathcal{A}}$ and $0 < \varepsilon < 1$, there exist a non-emtpy subset $W_0 \subset W$, functions $f, e \in \mathcal{A}$, and $t_0 \in W_0$ such that $f(t_0) = \| f \| = 1$, $e(t_0) = \| e \| = 1, |f(t)| < \varepsilon$ for every $t \in \Gamma_{\mathcal{A}} \setminus W_0, |1-e(t)| < \varepsilon$ for every $t \in W_0$ and $f(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_{\varepsilon}$.

Proof. By using Definition (4.3.19) for the open set $W \subset \Gamma_{\mathcal{A}}$ and ε, we get a function $e \in \mathcal{A}$ and $t_0 \in W$ such that $e(t_0) = \| e \| = 1, |e(t)| < \varepsilon$ for every $t \in \Gamma_{\mathcal{A}} \setminus W$ and $e(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_{\varepsilon}$. Let $W_0 := \{t \in W : |1-e(t)| < \varepsilon\}$. Define the function $f_n : \Gamma_{\mathcal{A}} \to \mathbb{K}$ by $f_n(t) := (e(t))^n$ whose range, by Lemma (4.3.21), is contained in $\mathrm{St}_{\varepsilon}$. From the very definition of W_0 and the fact that $e(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_{\varepsilon}$, we deduce that $|e(t)| \le 1 - \varepsilon(1-\varepsilon) < 1$ for every $t \in \Gamma_{\mathcal{A}} \setminus W_0$. Thus, taking a suitable $n_0 \in \mathbb{N}$, we can assume that $\left|f_{n_0}(t)\right| = |e(t)|^{n_0} < \varepsilon$ on $\Gamma_{\mathcal{A}} \setminus W_0$. Therefore, $f := f_{n_0} \in \mathcal{A}$ gives the conclusions of the lemma.

Theorem (4.3.23)[228]: Let $\mathcal{A} \subset C_b(\Gamma_{\mathcal{A}})$ be an ACK-subalgebra, and let X be a subspace $\mathcal{A} \subset X \subset C_b(\Gamma_{\mathcal{A}})$ that has the following property: $fx \in X$ for every $x \in X$ and $f \in \mathcal{A}$. Then $X \in$ ACK with the corresponding 1-norming subset of B_{X^*} being $\Gamma = \{\delta_t : t \in \Gamma_{\mathcal{A}}\}$.

Proof. Fix $\varepsilon > 0$ and a non-emtpy relatively w^*-open subset $= \{\delta_t : t \in W \subset \Gamma_{\mathcal{A}}\} \subset \Gamma$. Observe that $W \subset \Gamma_{\mathcal{A}}$ is open. Now, by applying Lemma (4.3.22) to W with ε we obtain the corresponding $W_0 \subset \Gamma_{\mathcal{A}}, t_0 \in W_0$, $f, e_{\mathcal{A}} \in \mathcal{A}$. Let us define $V \subset U, x_1^* \in V, e \in S_X$ and $F \in L(X)$ as follows:

$$V := \{\delta_t : t \in W_0\},\ x_1^* := \delta_{t_0},\ e := e_{\mathcal{A}},\ Fx := fx, \text{ for } x \in X.$$

Then, $F^*x^* = f(t)x^*$ for every $x^* = \delta_t \in \Gamma$. We shall show that properties (I) $-$ (VI) are satisfied. First, $\| F \| \le 1$ and $\| Fe \| = e(t_0)f(t_0) = 1$, which proves (I). Property (II) is straightforward from $x_1^*(Fe) = x_1^*(fe) = e(t_0)f(t_0) = 1$. From $(F^*x_1^*)(x) = x(t_0)f(t_0) = x(t_0) = x_1^*(x)$ we deduce that $F^*x_1^* = x_1^*$, which is (III). To show (IV)', take $x^* = \delta_t \in \Gamma$ and estimate

$$\begin{aligned} &\|F^*x^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x^*)\| \\ &\qquad \le |f(t)| + (1-\varepsilon)|1-f(t)| \le 1. \end{aligned}$$

Let us show now (V). Take $x^* = \delta_t \in \Gamma$. In case t belongs to $\Gamma_{\mathcal{A}} \setminus W_0$, then $\|F^*x^*\| = |f(t)| < \varepsilon$. Otherwise, $t \in W_0$ (that is, $x^* \in V$), using that $F^*x^* = f(t)x^*$ and that $f \in S_X$, we deduce that $f(t)x^* \in \mathrm{aco}\{0, V\}$. Hence, in both cases

$$\mathrm{dist}(F^*x^*, \mathrm{aco}\{0, V\}) < \varepsilon.$$

Finally, for every $v^* \in V$ we have that $v^*(e) = e(t)$ for some $t \in W_0$. So,

$$|v^*(e)-1| = |e(t)-1| \le \varepsilon,$$
which shows (VI) and finishes the proof.

From Lemma (4.3.20) and Theorem (4.3.23) taking $X = \mathcal{A}$ we obtain the promised example.

Corollary (4.3.24)[228]: Every uniform algebra $\mathcal{A}$ has simple ACK structure.

Theorem (4.3.23) gives more examples of spaces with simple ACK structure. For instance, let $\mathbb{T}$ be the unit disk in $\mathbb{C}, A(\mathbb{T}) \subset C(\mathbb{T})$ be the disc-algebra, i.e., $A(\mathbb{T})$ is the closure in $C(\mathbb{T})$ of the set $\{\sum_{k=0}^{m} a_k z^k : a_k \in \mathbb{C}, m \in \mathbb{N}\}$ of all polynomials. For a given $n \in \mathbb{N}$ denote $A_n(\mathbb{T})$ the closure in $C(\mathbb{T})$ of the set $\{\sum_{k=-n}^{m} a_k z^k : a_k \in \mathbb{C}, m \in \mathbb{N}\}$. Then $A(\mathbb{T})$ and $X = A_n(\mathbb{T})$ satisfy all the conditions of Theorem (4.3.23), so $A_n(\mathbb{T}) \in$ ACK, but $A_n(\mathbb{T})$ is not an algebra. Another example: let $c_0 \subset X \subset \ell_\infty$. Then $X \in$ ACK.

The first example is of illustrative character, because the space $A_n(\mathbb{T})$ is isometric to the algebra $A(\mathbb{T})$. In contrast, the second example gives a big variety of mutually non-isomorphic spaces with ACK structure. Observe that the simple ACK structure of those X such that $c_0 \subset X \subset \ell_\infty$ can be also deduced from Theorem (4.3.26) below.

Now we show that Banach spaces with Lindenstrauss' property β (see for instance [246]) have ACK structure.

Definition (4.3.25)[228]: A Banach space X is said to have the property β if there exist two sets $\{x_\alpha : \alpha \in \Lambda\} \subset S_X, \{x_\alpha^* : \alpha \in \Lambda\} \subset S_{X^*}$ and $\rho \in [0,1)$ such that the following conditions hold:

(i) $x_\alpha^*(x_\alpha) = 1$;

(ii) $|x_\alpha^*(x_\gamma)| \le \rho < 1$ if $\alpha \neq \gamma$; and

(iii) $\|x\| = \sup\{|x_\alpha^*(x)| : \alpha \in \Lambda\}$, for all $x \in X$.

Theorem (4.3.26)[228]: Let X have the property β. Then $X \in \mathrm{ACK}_\rho$ with the same value of ρ as in Definition (4.3.25) and with $\Gamma = \{x_\alpha^* : \alpha \in \Lambda\}$ from that definition. Moreover, if X has property β with $\rho = 0$, then $X \in$ ACK.

Proof. Since X has property β, the set $\Gamma = \{x_\alpha^* : \alpha \in \Lambda\}$ is a 1-norming subset of B_{X^*}. Observe that property β implies that (Γ, w^*) is a discrete topological space. Fix $\varepsilon > 0$ and a non-empty relatively w^*-open subset $U \subset \Gamma$. Take $x_{\alpha_0}^* \in U$. Let us define the corresponding $V, x_1^* \in V, e \in S_X$, and $F \in L(X)$ as follows:
$$V := \{x_{\alpha_0}^*\} \subset U,\ x_1^* := x_{\alpha_0}^*,\ e := x_{\alpha_0},\ F(x) := x_{\alpha_0}^*(x) x_{\alpha_0}.$$
It is clear that $F^* x^* = x^*(x_{\alpha_0}) x_{\alpha_0}^*$ for every $x^* \in X^*$. We shall show that properties (I) - (VI) of Definition (4.3.15) hold true. Properties (I) - (III) are routine. To show (IV) observe first that
$$\|F^* x_{\alpha_0}^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x_{\alpha_0}^*)\| = \|x_{\alpha_0}^*(x_{\alpha_0}) x_{\alpha_0}^*\| = 1,$$
that is, $x_{\alpha_0}^* \in V_1$. Consequently, whenever $v^* = x_\alpha^* \in \Gamma \setminus V_1$, then $\alpha \neq \alpha_0$ and thus $|v^*(Fe)| = |x_\alpha^*(x_{\alpha_0})| \le \rho$.

In case that $\rho = 0$, we have that $F^* x_\alpha^* = 0$ for every $\alpha \neq \alpha_0$, so
$$\|F^* x_\alpha^*\| + (1-\varepsilon)\|(I_{X^*} - F^*) x_\alpha^*\| = (1-\varepsilon)\|x_\alpha^*\| < 1,$$
i.e., $V_1 = \Gamma$.

Property (V) is a consequence of the fact that $F^* x^* \in \mathrm{aco}\{0, V\}$ for every $x^* = x_\alpha^* \in \Gamma$, because $F^* x^* = x_\alpha^*(x_{\alpha_0}) x_{\alpha_0}^*$. Finally, property (VI) and in turn our conclusions are consequence of the fact that the unique $v^* \in V$ is $v^* = x_{\alpha_0}^*$, so $|v^*(e) - 1| = 0 \le \varepsilon$.

Corollary (4.3.27)[228]: ([229], Theorem 2.2). Let Y have property β. Then, for every Banach space X, the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators.

Proof. In the proof of Theorem (4.3.26), (Γ, w^*) is a discrete topological space. Therefore every operator $T \in L(X, Y)$ is Γ-flat (Example (4.3.11) after Definition (4.3.8)). Now the application of Theorem (4.3.18) completes the proof.

Now we show the stability of the ACK structure with respect to the operations of ℓ_∞-sum and injective tensor product of two spaces (Theorem (4.3.28) and Theorem (4.3.29))

Theorem (4.3.28)[228]: Let X, Y be Banach spaces having ACK structure with parameters ρ_X and ρ_Y respectively. Then $Z := X \oplus_\infty Y \in \text{ACK}_\rho$ with $\rho = \max\{\rho_X, \rho_Y\}$. Moreover, $Z \in \text{ACK}$ whenever $X, Y \in \text{ACK}$.

Proof. Observe that both X and Y have ACK structure with parameter ρ. Let $\Gamma_X \subset B_{X^*}$ and $\Gamma_Y \subset B_{Y^*}$ be the corresponding 1 -norming subsets in Definition (4.3.15). Then, the set

$$\Gamma := \{(x^*, 0): x^* \in \Gamma_X\} \cup \{(0, y^*): y^* \in \Gamma_Y\}$$

is a 1-norming subset of B_{Z^*}. Take a non-empty relatively w^*-open subset $U \subset \Gamma$. Then, there exist relatively w^*-open subsets $U_X \subset \Gamma_X$ and $U_Y \subset \Gamma_Y$ that are not both empty and such that $(U_X \times \{0\}) \cup (\{0\} \times U_Y) \subset U$. Without loss of generality we may assume that $U_X \neq \emptyset$.

Fix $\varepsilon > 0$. By using Definition (4.3.15) for X, ε, and U_X we obtain a nonempty subset $V_X \subset U_X, x_1^* \in V_X, e_X \in S_X, F_X \in L(X)$ with the properties (I) - (VI). Thus, we can define the corresponding $V \subset U, z_1^* \in V$, $e \in S_Z$ and $F \in L(Z)$ as follows:

$$V := \{(x^*, 0): x^* \in V_X\} \subset U,\ z_1^* := (x_1^*, 0),\ e := (e_X, 0),$$

and for $(x, y) \in Z$,

$$F(x, y) := (F_X(x), 0).$$

Let us check the required properties. It is clear that $\| F \| = 1$ and that $\| Fe \| = \|F_X(e_X)\| = 1$, which shows (I). (II) follows easily; $z^*(Fe) = x_1^*(F_X e_X) = 1$. Due to the fact that $(F_X x_1^*, 0) = (x_1^*, 0)$, we deduce that $F^* z_1^* = z_1^*$, showing that (III) holds. Now, for every $z^* = (x^*, 0) \in V$ with $x^* \in V_{X,1}$ we have

$$\begin{aligned} \|F^* z^*\| &+ (1 - \varepsilon)\|(I_{Z^*} - F^*)(z^*)\| \\ &= \|F_X^* x^*\| + (1 - \varepsilon)\|(I_{X^*} - F_X^*)(x^*)\| \\ &\leq 1, \end{aligned}$$

which can be easily deduced from $F^* z^* = (F_X^* x^*, 0)$. Consequently, for every $x^* \in V_{X,1}$ we have $z^* = (x^*, 0) \in V_1$. (Observe that in the case of simple ACK structure we have already proved (IV)'). Let $v^* \in \Gamma \setminus V_1$. Then, either $v^* = (0, y^*)$, or $v^* = (x^*, 0)$ with $x^* \in \Gamma_X \setminus V_{X,1}$. On the one hand, when $v^* = (0, y^*)$, we have $|v^*(Fe)| = 0 \leq \rho$. On the other hand, whenever $v^* = (x^*, 0)$ with $x^* \in \Gamma_X \setminus V_{X,1}$, then $|v^*(Fe)| = |x^*(F_X e_X)| \leq \rho$, which proves (IV). Now, let $z^* \in \Gamma$. Whenever $z^* = (0, y^*)$ we have $F^* z^* = 0$. Otherwise, $z^* = (x^*, 0)$ and we have $\operatorname{dist}(F_X^* x^*, \operatorname{aco}\{0, V_X\}) < \varepsilon$. Thus, in both cases

$$\operatorname{dist}(F^* z^*, \operatorname{aco}\{0, V\}) < \varepsilon.$$

Finally, for every $v^* = (x^*, 0) \in V$ we have $|v^*(e) - 1| = |x^*(e_X) - 1| \leq \varepsilon$, which proves (VI) and concludes our proof.

Recall, that given two normed spaces X and Y, one can define their injective tensor product $X \hat{\otimes}_\varepsilon Y$, as the completion of $(X \otimes Y, \|\cdot\|_\varepsilon)$, where

$$\| z \|_\varepsilon := \sup\ \{|\langle x^* \otimes y^*, z\rangle| : x^* \in B_{X^*}, y^* \in B_{Y^*}\},$$

for every $z \in X \otimes Y$ and $\langle x^* \otimes y^*, x \otimes y\rangle := x^*(x)y^*(y)$, for every $x \otimes y \in X \otimes Y$ and for every $x^* \in X^*$ and $y^* \in Y^*$.

An important example of such a product is the Banach space $C(K) \hat{\otimes}_\varepsilon Y$, which can be naturally identified with $C(K, Y)$, that is, the Banach space of continuous $(Y, \|\cdot\|)$-valued functions defined on K, endowed with the supremum norm $\|f\| = \sup\{\|f(t)\|: t \in K\}$.

Note that it follows from the definition of the injective norm that if $X_0 \subset B_{X^*}$ and $Y_0 \subset B_{Y^*}$ are 1-norming, then for every $z \in X \hat{\otimes}_\varepsilon Y$ the following equality holds:

$$\| z \|_\varepsilon = \sup \ \{|\langle x^* \otimes y^*, z\rangle|: x^* \in X_0, y^* \in Y_0\}.$$

Recall also that $\|x^* \otimes y^*\|_{(X \hat{\otimes}_\varepsilon Y)^*} = \|x^*\| \cdot \|y^*\|$ for every $x^* \in X^*$ and $y^* \in Y^*$.

This is all the information about tensor products that will be used in Theorem (4.3.29) below. See Ryan [248] for tensor products theory in general and the above definitions and statements in particular.

Theorem (4.3.29)[228]: Let X and Y be Banach spaces both of which have ACK (resp. ACK_ρ) structure. Then, $X \hat{\otimes}_\varepsilon Y$ has ACK (resp. ACK_ρ) structure.

Proof. Since X and Y have ACK (resp. ACK_ρ) structure, there exist $1-$ norming sets $\Gamma_X \subset S_{X^*}$ and $\Gamma_Y \subset S_{Y^*}$ satisfying Definition (4.3.15). Define the map $\phi: (B_{X^*}, w^*) \times (B_{Y^*}, w^*) \to \left(B_{(X \hat{\otimes}_e Y)^*}, w^*\right)$ by $\phi(x^*, y^*) = x^* \otimes y^*$, for every $x^* \in B_{X^*}$ and for every $y^* \in B_{Y^*}$.

First, we shall show that the map ϕ is continuous. Let $\{(x^*_\alpha, y^*_\alpha)\}_{\alpha \in \Lambda}$ be a convergent net to $(x^*, y^*) \in B_{X^*} \times B_{Y^*}$. Then, for every $x \otimes y \in X \otimes Y$, we can estimate

$$\begin{aligned} |\langle \phi(x^*_\alpha, y^*_\alpha) - \phi(x^*, y^*), x \otimes y\rangle| &= |x^*_\alpha(x) y^*_\alpha(y) - x^*(x) y^*(y)| \\ &\le |(x^*_\alpha(x) - x^*(x)) y^*_\alpha(y)| + |x^*(x)(y^*_\alpha(y) - y^*(y))| \\ &\le |x^*_\alpha(x) - x^*(x)| \|y^*_\alpha\| \| y \| + \|x^*(x)\| |y^*_\alpha(y) - y^*(y)| \\ &\le |x^*_\alpha(x) - x^*(x)| \| y \| + \| x \| |y^*_\alpha(y) - y^*(y)|, \end{aligned}$$

which tends to zero. This argument extends easily to every element in $X \otimes Y$ and, in turn, to every $z \in X \hat{\otimes}_\varepsilon Y$ (due to the boundedness of the range of the map ϕ).

The 1-norming set Γ that we need for our theorem can be introduced as follows:

$$\Gamma := \{x^* \otimes y^*: x^* \in \Gamma_X, y^* \in \Gamma_Y\} = \phi(\Gamma_X \times \Gamma_Y).$$

Let $\varepsilon > 0$ and U be a non-empty relatively w^*-open subset of Γ. Let $x^*_0 \in \Gamma_X$ and $y^*_0 \in \Gamma_Y$ be such that $\phi(x^*_0, y^*_0) \in U$. The continuity of ϕ ensures that there exist non-empty relatively w^*-open subsets $W_X \subset \Gamma_X$, $W_Y \subset \Gamma_Y$ such that $x^*_0 \in W_X, y^*_0 \in W_Y$ and $\phi(W_X \times W_Y) \subset U$.

We can apply Definition (4.3.15) to X and Y, to the former with $\varepsilon/2$ and W_X and to the latter with $\varepsilon/2$ and W_Y, to find two non-empty sets $V_X \subset W_X$ and $V_Y \subset W_Y$, two functionals $x^*_1 \in V_X$ and $y^*_1 \in V_Y$, two points $e_X \in S_X$ and $e_Y \in S_Y$ and finally, two operators $F_X \in L(X)$ and $F_Y \in L(Y)$, satisfying respectively the properties (I) - (VI), or with their corresponding modifications for the the simple ACK structure. Denote also by $V_{X,1}$ and $V_{Y,1}$ the corresponding variants for X and Y of the set V_1 from property (IV) of Definition (4.3.15).

Now, define the non-emtpy set $V \subset U$ and corresponding $z^*_1 \in V, e \in S_{X \otimes_\varepsilon Y}, F \in L\left(X \hat{\otimes}_\varepsilon Y\right)$ as follows: $:= \phi(V_X \times V_Y) \subset U, z^*_1 := \phi(x^*_1, y^*_1) = x^*_1 \otimes y^*_1, e := e_X \otimes e_Y$, and $F(x \otimes y) := F_X(x) \otimes F_Y(y)$ for every $x \otimes y \in X \otimes Y$. It remains to check the properties (I) - (VI). First, observe that $F^*(x^* \otimes y^*) = F^*_X x^* \otimes F^*_Y y^*$ for every $x^* \in X^*$ and $y^* \in Y^*$.

(I) Let z belong to $B_{X \hat{\otimes}_e Y}$, then

$$\begin{aligned}\| Fz \|_\varepsilon &= \sup_{x^*\in\Gamma_X}\sup_{y^*\in\Gamma_Y} |\langle x^*\otimes y^*, Fz\rangle| = \sup_{x^*\in\Gamma_X}\sup_{y^*\in\Gamma_Y} |\langle F^*(x^*\otimes y^*), z\rangle| \\ &= \sup_{x^*\in\Gamma_X}\sup_{y^*\in\Gamma_Y} |\langle F_X^*x^*\otimes F_Y^*y^*, z\rangle| \le \sup_{x^*\in\Gamma_X}\sup_{y^*\in\Gamma_Y} \|F_X^*x^*\|\|F_Y^*y^*\| \\ &\le \|F_X^*\|\|F_Y^*\| \le 1,\end{aligned}$$

which implies that $\| F \| = 1$, since

$$\| Fe \| = \|F_Xe_X \otimes F_Ye_Y\| = \|F_Xe_X\|\|F_Ye_Y\| = 1.$$

(II) $z_1^*(Fe) = (x_1^*\otimes y_1^*)(F_Xe_X\otimes F_Ye_Y) = x_1^*(F_Xe_X)y_1^*(F_Ye_Y) = 1.$

(III) $F^*z_1^* = z_1^*$, since for every $x\otimes y \in X\otimes Y$ we have

$$(F^*z_1^*)(x\otimes y) = (x_1^*\otimes y_1^*)(F_Xx\otimes F_Yy) = (F_X^*x_1^*)(x)(F_Y^*y_1^*)(y),$$

which, in turn, implies that $(F^*z_1^*)(x\otimes y) = x_1^*(x)y_1^*(y) = z_1^*(x\otimes y)$.

(IV) For $(x^*, y^*) \in \Gamma_X \times \Gamma_Y$, denote $z^* = x^*\otimes y^*$. Firstly, let us show that for every $x^* \in V_{X,1}$ and $y^* \in V_{Y,1}$ the functional z^* belongs to V_1, i.e., that

$$\|F^*z^*\| + (1-\varepsilon)\left\|\left(I_{(X\hat{\otimes}_\varepsilon Y)^*} - F^*\right)(z^*)\right\| \le 1.$$

First of all, observe that

$$\begin{aligned}&\|x^* \otimes y^* - F_X^*x^*\otimes F_Y^*y^*\| \\ &\quad= \|x^*\otimes(y^* - F_Y^*y^*) - (x^* - F_X^*x^*)\otimes F_Y^*y^*\| \\ &\quad\le \|y^* - F_Y^*y^*\| + \|F_Y^*y^*\|\|(x^* - F_X^*x^*)\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|F_X^*x^*\|\|F_Y^*y^*\| + (1-\varepsilon)\|x^*\otimes y^* - F_X^*x^*\otimes F_Y^*y^*\| \\ &\quad= \|F_Y^*y^*\|(\|F_X^*x^*\| + (1-\varepsilon)\|(x^* - F_X^*x^*)\|) + (1-\varepsilon)\|y^* - F_Y^*y^*\| \\ &\quad\le \|F_Y^*y^*\| + (1-\varepsilon)\|y^* - F_Y^*y^*\| \le 1.\end{aligned}$$

This implies that for every $z^* = x^*\otimes y^* \in \Gamma\setminus V_1$ we have two possibilities: either $x^* \notin V_{X,1}$ or $y^* \notin V_{Y,1}$. By symmetry, it is sufficient to consider $x^* \notin V_{X,1}$. In this case $|x^*(F_Xe_X)| \le \rho$, so

$$|z^*(Fe)| = |x^*(F_Xe_X)||y^*(F_Ye_Y)| \le |x^*(F_Xe_X)| \le \rho.$$

(V) We shall show that $\operatorname{dist}(F^*z^*, \operatorname{aco}\{0, V\}) < \varepsilon$ for every $z^* = x^*\otimes y^* \in \Gamma$. Due to the facts that $\operatorname{dist}(F_X^*x^*, \operatorname{aco}\{0, V_X\}) < \varepsilon/2$ and that $\operatorname{dist}(F_Y^*y^*, \operatorname{aco}\{0, V_Y\}) < \varepsilon/2$, there exist $v_X^* \in \operatorname{aco}\{0, V_X\}$ and $v_Y^* \in \operatorname{aco}\{0, V_Y\}$ such that $\|F_X^*x^* - v_X^*\| < \varepsilon/2$ and $\|F_Y^*y^* - v_Y^*\| < \varepsilon/2$. Then $v^* := v_X^*\otimes v_Y^*$ belongs to $\operatorname{aco}\{0, V\}$ and

$$\begin{aligned}\|F^*z^* - v^*\| &\le \|(F_X^*x^* - v_X^*)\otimes F_Y^*y^*\| + \|v_X^*\otimes(F_Y^*y^* - v_Y^*)\| \\ &\le \|F_X^*x^* - v_X^*\|\|F_Y^*y^*\| + \|v_X^*\|\|F_Y^*y^* - v_Y^*\| \le \varepsilon.\end{aligned}$$

(VI) For every $v^* = x^*\otimes y^* \in V$ we get

$$\begin{aligned}|v^*(e) - 1| &= |x^*(e_X)y^*(e_Y) - 1| \le |x^*(e_X)y^*(e_Y) - y^*(e_Y)| \\ &\quad+ |y^*(e_Y) - 1| \le \frac{\varepsilon}{2}|y^*(e_Y)| + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

This finishes the proof.

As we mentioned, Acosta, Becerra Guerrero, García, Kim, and Maestre considered A-BPBp in spaces of continuous vector-valued functions. Let us recall their result explicitly. Here, as usual, $\sigma(Z, \Delta)$ denotes the weakest topology on Z in which all elements of $\Delta \subset Z^*$ are continuous.

Theorem (4.3.29)[228]: ([230], Theorem 3.1). Let X, Z be Banach spaces, K be a compact Hausdorff topological space. Let Z satisfy property β for the subset of functionals $\Delta = \{z_\alpha^* : \alpha \in \Delta\}$. Let $\tau \supseteq \sigma(Z, \Delta)$ be a linear topology on Z dominated by the norm topology. Then for every closed operator ideal $\mathcal{J}$ contained in the ideal of Asplund operators, we have that $(X, C(K,(Z,\tau)))$ has the Bishop-Phelps-Bollobás property for operators from $\mathcal{J}$.

The next proposition together with Theorem (4.3.18) generalize Theorem (4.3.29) for the case of Z endowed with its strong topology.

Proposition (4.3.30)[228]: Let K be a compact Hausdorff topological space. Then,

$$\left(Y \in \mathrm{ACK}_\rho\right) \Rightarrow \left(C(K,Y) \in \mathrm{ACK}_\rho\right);$$
$$(Y \in \mathrm{ACK}) \Rightarrow (C(K,Y) \in \mathrm{ACK}).$$

Proof. Bearing in mind Corollary (4.3.24) and Theorem (4.3.29), the fact that the space $C(K) \hat{\otimes}_\varepsilon Y$ is isometric to $C(K,Y)$ concludes the proof.

Our aim now is showing a generalization of Theorem (4.3.29) in the spirit of the ACK structure, that covers all topologies τ from that theorem.

For a topological space T and a Banach space Z denote by $C_{\mathrm{bof}}(T,Z)$ the space of all bounded openly fragmented (see Definition (4.3.6)) functions $f: T \to Z$ equipped with the sup-norm. For a topology τ on Z denote by $C_b(T,(Z,\tau))$ the space of bounded τ-continuous functions $f: T \to Z$ equipped with the sup-norm.

Definition (4.3.31)[228]: Let $Z \in \mathrm{ACK}_\rho$ and let $\Gamma \subset B_{Z^*}$ be the corresponding 1-norming set. A linear topology τ on Z is said to be Γ-acceptable, if it is dominated by the norm topology and dominates $\sigma(Z,\Gamma)$.

The following result simultaneously generalizes our Theorem (4.3.23) and Theorem (4.3.29). We state the result in the most general settings, which makes the statement bulky. Some "elegant" partial cases will be given as corollaries.

Theorem (4.3.32)[228]: Let $\mathcal{A} \subset C_b(\Gamma_\mathcal{A})$ be an ACK-subalgebra. Let Z be a Banach space and $\mathcal{O} \subset L(Z)$ such that $Z \in \mathcal{O} - \mathrm{ACK}_\rho (Z \in \mathcal{O}\text{-ACK})$ with $\Gamma_Z \subset B_{Z^*}$ being the corresponding 1-norming set. Finally, let τ be a Γ_Z- acceptable topology on Z. Let $X \subset C_b(\Gamma_\mathcal{A},(Z,\tau))$ be a Banach space satisfying the following properties:

(i) For every $x \in X$ and $f \in \mathcal{A}$ the function fx belongs to X.

(ii) X contains all functions of the form $f \otimes z, f \in \mathcal{A}, z \in Z$.

(iii) $F \circ x \in X$ for every $x \in X$ and $F \in \mathcal{O}$.

(iv) For every finite collection $\{x_k\}_{k=1}^n \subset X$ the corresponding function of two variables $\varphi: \Gamma_\mathcal{A} \times (\Gamma_Z, w^*) \to \mathbb{K}^n$, defined by $\varphi(t,z^*) = \left(z^*(x_k(t))\right)_{k=1}^n$, is quasi-continuous.

Then $X \in \mathrm{ACK}_\rho (X \in \mathrm{ACK}$, respectively) with the corresponding 1norming subset of B_{X^*} being $\Gamma = \{\delta_t \otimes z^*: t \in \Gamma_\mathcal{A}, z^* \in \Gamma_Z\}$, where the functional $\delta_t \otimes z^* \in X^*$ acts as follows: $(\delta_t \otimes z^*)(x) = z^*(x(t))$.

Proof. Fix $\varepsilon > 0$ and a non-empty relatively w^*-open subset $U \subset \Gamma$. Let $t_0 \in \Gamma_\mathcal{A}$ and $z_0^* \in \Gamma_Z$ be such that $\delta_{t_0} \otimes z_0^* \in U$. Since U is relatively w^*-open, there exist $\{x_k\}_{k=1}^n \subset X$ such that $\delta_t \otimes z^* \in \Gamma$ belongs to U whenever

$$\max_{1 \le k \le n} \left|\left\langle \left(\delta_{t_0} \otimes z_0^*\right) - (\delta_t \otimes z^*), x_k \right\rangle\right| < 1.$$

Consider the non-emtpty open set

$$B := \left\{t \in \Gamma_\mathcal{A}: \left|z_0^*(x_k(t)) - z_0^*\left(x_k(t_0)\right)\right| < 1 \text{ for } 1 \le k \le n\right\},$$

and define the following non-empty relatively w^*-open subset of Γ_Z :

$$D := \left\{z^* \in \Gamma_Z: \left|z^*\left(x_k(t_0)\right) - z_0^*\left(x_k(t_0)\right)\right| < 1 \text{ for } 1 \le k \le n\right\}.$$

Using property (iv) for $\{x_k\}_{k=1}^n \subset X$ we can find a non-empty open subset $B_1 \subset B$ and a non-empty relatively w^*-open subset $D_1 \subset D$ such that for every $t \in B_1$ and every $z^* \in D_1$ it holds

$$\max_{1\le k\le n} \left|z^*(x_k(t)) - z_0^*\big(x_k(t_0)\big)\right| < 1.$$

Define the non-empty subset $W := \{\delta_t \otimes z^* : t \in B_1, z^* \in D_1\} \subset \Gamma$. It is clear that $W \subset U$.

By applying Definition (4.3.15) to Z, Γ_Z, D_1 and $(\varepsilon/2)$, we get $V_Z \subset D_1$, $z_1^* \in V_Z, e_Z \in S_Z$ and $F_Z \in \mathcal{O}$ satisfying (I)−(VI). Denote also $V_{Z,1} \subset \Gamma_Z$, the subset that appears in property (IV) (in the case of $Z \in$ ACK we have $V_{Z,1} = \Gamma_Z$). By applying Lemma (4.3.22) to $\mathcal{A}, \Gamma_{\mathcal{A}}$, the non-empty open set B_1 and $(\varepsilon/2)$, we find a non-empty subset $B_2 \subset B_1$, functions $f_0, e_{\mathcal{A}}$ (both belonging to $\mathcal{A}$) and $s_0 \in B_2$, satisfying its conclusions.

Finally, let us define the requested non-empty subset $V \subset U$ and corresponding $x_1^* \in V, e \in S_X, F \in L(X)$ as follows:

$$V := \{\delta_t \otimes z^* : t \in B_2, z^* \in V_Z\} \subset W \subset U$$

$$x_1^* := \delta_{s_0} \otimes z_1^*,\ e(t) := e_{\mathcal{A}}(t)e_Z, \text{ for every } t \in \Gamma_{\mathcal{A}}$$

(condition (ii) implies $e \in X$), and

$$(Fx)(t) := f_0(t)F_Z(x(t)),$$

for every $x \in X$ and for every $t \in \Gamma_{\mathcal{A}}$. Conditions (i) and (iii) ensure that $F(x) \in X$. Observe that for every $x^* = \delta_t \otimes z^* \in \Gamma$

$$F^*x^* = f_0(t)(\delta_t \otimes F_Z^* z^*).$$

It remains to check the properties (I) −(VI).

(I) It is clear that $\| F \| = \|F_Z\| = 1$ and $\| Fe \| = \|f_0 e_{\mathcal{A}}\| \|F_Z(e_Z)\| = 1$.

(II) $x_1^*(Fe) = z_1^*\big(f_0(s_0)e_{\mathcal{A}}(s_0)F_Z(e_Z)\big) = 1$.

(III) $F^*x_1^* = x_1^*$, since for every $x \in X$ we have

$$(F^*x_1^*)(x) = z_1^*\big(f_0(s_0)F_Z x(s_0)\big) = (F_Z^* z_1^*)\big(x(s_0)\big) = z_1^*\big(x(s_0)\big) = x_1^*(x).$$

(IV) For every $x^* \in \Gamma$, we have $x^* = \delta_t \otimes z^*, t \in \Gamma_{\mathcal{A}}$ and $z^* \in \Gamma_Z$. First, consider the case $z^* \in V_{Z,1}$ and observe that

$$\begin{aligned}\|(I_{X^*} - F^*)(x^*)\| &= \|z^* - f_0(t)F_{Z^*}^* z^*\| \\ &\le |1 - f_0(t)|\|z^*\| + |f_0(t)| \cdot \|(I_{Z^*} - F_Z^*)(z^*)\| \\ &= |f_0(t)| \cdot \|(I_{Z^*} - F_Z^*)(z^*)\| + |1 - f_0(t)|.\end{aligned}$$

Therefore, in this case

$$\begin{aligned}&\|F^*x^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x^*)\| \\ &= |f_0(t)| \cdot \|F_Z^* z^*\| + (1-\varepsilon)\|z^* - f_0(t)F_{Z^*}z^*\| \\ &\le |f_0(t)|(\|F_Z^* z^*\| + (1-\varepsilon)\|(I_{Z^*} - F_Z^*)(z^*)\|) + (1-\varepsilon)|1 - f_0(t)| \\ &\le |f_0(t)| + (1-\varepsilon)|1 - f_0(t)| \le 1.\end{aligned}$$

Whenever $Z \in$ ACK, then $V_{Z,1} = \Gamma_Z$, so the above inequality holds for every $z^* \in \Gamma_Z$. Thus, we have proved (IV)'. If $Z \in \text{ACK}_\rho$ we still must consider those x^* belonging to $\Gamma \setminus V_1$. The above inequality implies that $z^* \notin V_{Z,1}$ and, consequently, $|z^*(F_Z e_Z)| \le \rho$ which, in turn, implies that

$$|x^*(Fe)| = |f_0(t)e_{\mathcal{A}}(t)z^*(F_Z e_Z)| \le \rho.$$

(V) Let $x^* = \delta_t \otimes z^* \in \Gamma$. Recall that $F^*x^* = f_0(t)\delta_t \otimes F_Z^* z^*$. Set $V_{\mathcal{A}} := \{\delta_t : t \in B_2\}$. In the proof of Theorem (4.3.23) it was proved that for every $t \in \Gamma_{\mathcal{A}}$ it holds

$$\operatorname{dist}(f(t)\delta_t, \operatorname{aco}\{0, V_{\mathcal{A}}\}) < \frac{\varepsilon}{2}.$$

On the other hand, by our construction, we deduce that

$$\operatorname{dist}(F_Z^* z^*, \operatorname{aco}\{0, V_Z\}) < \frac{\varepsilon}{2}.$$

Thus, there exist $a^* \in \operatorname{aco}\{0, V_{\mathcal{A}}\}$ and $b^* \in \operatorname{aco}\{0, V_Z\}$ such that

$$\|f(t)\delta_t - a^*\| < \frac{\varepsilon}{2} \text{ and } \|F_Z^* z^* - b^*\| < \frac{\varepsilon}{2}.$$

In particular, since $a^* \otimes b^*$ belongs to aco $\{0, V\}$, we can deduce that

$$\begin{aligned} \operatorname{dist}(F^* x^*, \operatorname{aco}\{0, V\}) &\leq \|f_0(t)\delta_t \otimes F_Z^* z^* - a^* \otimes b^*\| \\ &\leq \|f_0(t)\delta_t \otimes F_Z^* z^* - f_0(t)\delta_t \otimes b^*\| + \\ &+ \|f_0(t)\delta_t \otimes b^* - a^* \otimes b^*\| \\ &\leq \|F_Z^* z^* - b^*\| + \|f_0(t)\delta_t - a^*\| < \varepsilon. \end{aligned}$$

(VI) For every $x^* = \delta_t \otimes z^* \in V$ we have $t \in B_2$ and $z^* \in V_Z$. Consequently, $|e_{\mathcal{A}}(t) - 1| \leq \frac{\varepsilon}{2}$ and $|z^*(e_Z) - 1| \leq \frac{\varepsilon}{2}$. From this we get

$$|x^*(e) - 1| = |e_{\mathcal{A}}(t) z^*(e_Z) - 1| = |e_{\mathcal{A}}(t)(z^*(e_Z) - 1) + (e_{\mathcal{A}}(t) - 1)| \leq \varepsilon,$$

which completes the proof.

Conditions (i) - (iii) in Theorem (4.3.32) are easily verified in concrete examples. In contrast, condition (iv) looks technical. So, in order to make Theorem (4.3.32) more applicable, we shall present easy-to-verify sufficient conditions for (iv).

Before passing to these sufficient conditions, observe that the function of two variables $\varphi: \Gamma_A \times (\Gamma_Z, w^*) \to \mathbb{K}^n$ from condition (iv) is separately continuous. Therefore, the role of sufficient condition for (iv) can be played by any theorem about quasi-continuity of a separately continuous function $f: U \times V \to W$. There is a number of such theorems (see Encyclopedia of Mathematics article "Separate and joint continuity" or the introduction to [235]). For example, according to Namioka's theorem [243] this (and a much stronger result) occurs for U being a regular, strongly countably complete topological space, V being a locally compact σ-compact space and W being a pseudo-metric space. The results of the kind "separate continuity implies quasi-continuity" that we list and apply below do not pretend to be new.

Proposition (4.3.33)[228]: Let U, V, W be topological spaces, V be discrete and $f: U \times V \to W$ be separately continuous. Then, f is continuous (and consequently quasi-continuous).

If Z has property β, the corresponding (Γ_Z, w^*) is a discrete topological space. Thus, the above proposition guaranties the validity of (iv) of Theorem (4.3.32) in this case.

Corollary (4.3.34)[228]: Under the conditions of Theorem (4.3.29), $(K, (Z, \tau)) \in \mathrm{ACK}_\rho$, where ρ is the parameter from the property β of Z. If $\beta = 0$, then $C(K, (Z, \tau)) \in \mathrm{ACK}$. In particular, this implies the conclusion of Theorem (4.3.29).

Proposition (4.3.33) also guaranties (iv) of Theorem (4.3.32) in the case of $\Gamma_{\mathcal{A}} = \mathbb{N}$ (just change the roles of U and V in Proposition (4.3.33)). If we apply Theorem (4.3.32) with $\mathcal{A} = c_0 \subset C_b(\mathbb{N}) = \ell_\infty$, this leads to the following result:

Corollary (4.3.35)[228]: Let $Z \in \mathrm{ACK}_\rho (Z \in \mathrm{ACK}), c_0(Z) \subset X \subset \ell_\infty(Z)$, and X has the following property: $(Fz_1, Fz_2, \dots) \in X$ for every $x = (z_1, z_2, \dots) \in X$ and $F \in L(Z)$. Then $X \in \mathrm{ACK}_\rho (X \in \mathrm{ACK}$ respectively).

This corollary is applicable to $c_0(Z)$ and $\ell_\infty(Z)$ themselves and also for some intermediate spaces like $c_0(Z, w)$ of weakly null sequences in Z.

Proposition (4.3.36)[228]: Let Z be a Banach space, $(\Gamma_{\mathcal{A}}, \tau)$ be a topological space, $\Gamma_Z \subset (B_{Z^*}, w^*)$, and $x_k: \Gamma_{\mathcal{A}} \to Z$ for $k \in \{1,2, \dots, n\}$ be $\tau - \sigma(Z, \Gamma_Z)$ continuous and $\tau - \|\cdot\|$-openly fragmented functions. Then, the function $\varphi: (\Gamma_{\mathcal{A}}, \tau) \times (\Gamma_Z, w^*) \to \mathbb{K}^n$ given by $\varphi(t, z^*) = \left(z^*(x_k(t))\right)_{k=1}^{n}$ is quasicontinuous.

Proof. Fix $(t_0, z_0^*) \in \Gamma_{\mathcal{A}} \times \Gamma_Z$. Let $U_{\mathcal{A}} \subset \Gamma_{\mathcal{A}}, U_Z \subset \Gamma_Z$ be open and w^*-open neighborhoods of t_0 and z_0^* respectively. Set $U := U_{\mathcal{A}} \times U_Z$. We have to show that, for a given $\varepsilon > 0$, there exist a non-empty open subset $W_{\mathcal{A}} \subset U_{\mathcal{A}}$ and a non-empty relatively w^*-open subset $W_Z \subset U_Z$ such that for every $t \in W_{\mathcal{A}}$ and every $z^* \in W_Z$

$$\max_{1 \le k \le n} \left| z^*(x_k(t)) - z_0^*\big(x_k(t_0)\big) \right| < \varepsilon. \tag{40}$$

Fix $\delta < \varepsilon/4$ and define

$$V_{\mathcal{A}} := \left\{ t \in U_{\mathcal{A}} : \max_{1 \le k \le n} \left| z_0^*(x_k(t)) - z_0^*\big(x_k(t_0)\big) \right| < \delta \right\}.$$

The set $V_{\mathcal{A}} \subset U_{\mathcal{A}}$ is a non-emtpy open neighborhood of t_0 because of the $\tau - \sigma(Z, \Gamma_Z)$ continuity of x_k (the map $z_0^* \circ x_k$ is a $\mathbb{K}$-valued τ-continuous function). Applying inductively the definition of openly fragmented function, we define a non-empty open set $W_{\mathcal{A}} \subset (V_{\mathcal{A}}, \tau)$ in such a way that for all $k = 1, \dots, n$ it holds

$$\operatorname{diam}\big(x_k(W_{\mathcal{A}})\big) < \delta.$$

Fix a $t_1 \in W_{\mathcal{A}}$ and define the non-empty relatively w^*-open subset $W_Z \subset U_Z$ as follows:

$$W_Z := \left\{ z^* \in U_Z : \max_{1 \le k \le n} \left| z^*\big(x_k(t_1)\big) - z_0^*\big(x_k(t_1)\big) \right| < \delta \right\}.$$

Let us show, for every $t \in W_{\mathcal{A}}$ and every $z^* \in W_Z$, the validity of inequality (40):

$$\begin{aligned} \left| z_0^*\big(x_k(t_0)\big) - z^*(x_k(t)) \right| \quad &\le \left| z_0^*\big(x_k(t_0)\big) - z_0^*(x_k(t)) \right| \\ &+ \left| z_0^*(x_k(t)) - z_0^*\big(x_k(t_1)\big) \right| \\ &+ \left| z_0^*\big(x_k(t_1)\big) - z^*\big(x_k(t_1)\big) \right| \\ &+ \left| z^*\big(x_k(t_1)\big) - z^*(x_k(t)) \right|. \end{aligned}$$

The first summand in the right-hand side of the previous inequality does not exceed δ since $t \in V_{\mathcal{A}}$. Accordingly, the second and fourth summands are both bounded by δ since $z_0^*, z^* \in B_{Z^*}$ and $\|x_k(t) - x_k(t_1)\| < \delta$ since $t, t_1 \in W_{\mathcal{A}}$ and $\operatorname{diam}\big(x_k(W_{\mathcal{A}})\big) < \delta$. Finally, the corresponding third summand is bounded by δ since $z^* \in W_Z$. Therefore,

$$\left| z_0^*\big(x_k(t_0)\big) - z^*(x_k(t)) \right| \le 4\delta < \varepsilon,$$

which completes the proof of (40) and that of the proposition.

As an application of the previous proposition we get the following corollaries which contain as a particular case the space $C_w(K, Z)$ of Z-valued weakly continuous functions for $Z \in \mathrm{ACK}_\rho$ (or $Z \in \mathrm{ACK}$).

Corollary (4.3.37)[228]: Let $Z \in \mathcal{O} - \mathrm{ACK}_\rho$ (or $Z \in \mathcal{O}$-ACK) and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$, where Γ_Z is the 1-norming set given by the ACK structure of Z. Denote by $\mathcal{A}_{\sigma(Z,H)}(K, Z)$ the following subspace of $(K, (Z, \sigma(Z, H)))$:

$$\mathcal{A}_{\sigma(Z,H)}(K, Z) = \left\{ f \in Z^K : z^* \circ f \in \mathcal{A} \text{ for all } z^* \in H \right\}.$$

Let us assume that

(i) $F^* H \subset H$ for every $F \in \mathcal{O}$.

(ii) $(f(K), \sigma(Z, H))$ is fragmented by the norm for every f belonging to $\mathcal{A}_{\sigma(Z,H)}(K, Z)$.

Then, $\mathcal{A}_{\sigma(Z,H)}(K, Z) \in \mathrm{ACK}_\rho$ (resp. $\mathcal{A}_{\sigma(Z,H)}(K, Z) \in \mathrm{ACK}$).

Sketch of the proof: It relays on the use of Theorem (4.3.32). Let $\Gamma_{\mathcal{A}} \subset K$ be the corresponding subset from Lemma (4.3.20). Then, restrictions of elements of $\mathcal{A}$ to $\Gamma_{\mathcal{A}}$ form an ACK-subalgebra $C_b(\Gamma_A)$ isometric to $\mathcal{A}$ (that we identify with $\mathcal{A}$) and restrictions of elements of $\mathcal{A}_{\sigma(Z,H)}(K,Z)$ to $\Gamma_{\mathcal{A}}$ form a subspace $X \subset C_b(\Gamma_{\mathcal{A}}, (Z, \sigma(Z,H)))$ isometric to $\mathcal{A}_{\sigma(Z,H)}(K,Z)$. The conditions (i) and (ii) of Theorem (4.3.32) follow from the definition of $\mathcal{A}_{\sigma(Z,H)}(K,Z)$. The condition (iii) of Theorem (4.3.32) is reduced to the present condition (i). And, finally, the condition (iv) of Theorem (4.3.32) is reduced to the present (ii) by using Proposition (4.3.36).

The condition (i) above could be quite demanding, for instance, when $\mathcal{O} = L(Z)$ in which case H is forced to be Z^*. However, in all concrete examples that we know of ACK structure, the family $\mathcal{O}$ can be taken really small. Thus, for concrete examples of Z, the condition (i) could be easily satisfied for every election of H.

By using the results from [233] it can be shown that condition (ii) above is satisfied for every H whenever (Z, w) is Lindelöf. Indeed, given f belonging to $\mathcal{A}_{\sigma(Z,H)}(K,Z), f(K) \subset Z$ is $\sigma(Z,H)$-compact, thus, it is also Lindelöf. A straightforward application of [233] ensures that $(f(K), \sigma(Z,H))$ is norm-fragmented. Hence, in this case, Corollary (4.3.37) can be simplified as follows:

Corollary (4.3.38)[228]: Let $Z \in \mathcal{O} - ACK_\rho$ (or $Z \in \mathcal{O}\text{-}ACK$) such that (Z, w) is Lindelöf and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$ such that $F^*H \subset H$ for every $F \in \mathcal{O}$, where Γ_Z is the 1-norming set given by the ACK structure of Z. Then, $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}_\rho$ (resp. $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}$).

Observe that when Z has property β, the set $\mathcal{O}$ coincides with the set $\{x_\alpha^*(\cdot)x_\alpha : \alpha \in \Lambda\}$. Therefore, in this case, $F^*H \subset H$ for every H and for every $F \in \mathcal{O}$. Thus, we have proved the following corollary.

Corollary (4.3.39)[228]: Let Z be a Banach space with property β such that (Z, w) is Lindelöf and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$, where $\Gamma_Z = \{x_\alpha^* : \alpha \in \Lambda\}$. Then, $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}_\rho$.

However, this technique can not fully generalize Theorem (4.3.29) by Acosta et al. to the case of vector-valued uniform algebras, since here the Lindelöf property is essential and property β does not imply in general weak Lindelöf. Observe that nevertheless the original statement of Theorem (4.3.29) is covered completely by our Corollary (4.3.34).

Chapter 5
Bishop-Phelps-Bollobás Property and Moduli

We show that every infinite-dimensional separable Banach space can be renormed to fail the BPBp-nu. In particular, this shows that the Radon-Nikodým property (even reflexivity) is not enough to get BPBp-nu. We calculate the two moduli for Hilbert spaces and also present many examples for which the moduli have the maximum possible value (among them, there are $C(K)$ spaces and $L_1(\mu)$ spaces). We show that if a Banach space has the maximum possible value of any of the moduli, then it contains almost isometric copies of the real space $\ell_\infty^{(2)}$ and present an example showing that this condition is not sufficient. We deduce that the subspaces of finite-rank operators, compact operators and weakly compact operators on $L_1(\mu)$ have the BPBp-ν.

Section (5.1): Numerical Radius:

For X be a (real or complex) Banach space and X^* its dual space. The unit sphere of X will be denoted by S_X. We write $\mathcal{L}(X)$ for the space of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, its numerical radius is defined by

$$v(T) = \sup\{|x^*Tx| : (x, x^*) \in \Pi(X)\}, \tag{1}$$

where $\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. It is clear that v is a seminorm on $\mathcal{L}(X)$. See [253],[254] for background. An operator $T \in \mathcal{L}(X)$ attains its numerical radius if there exists $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^*Tx_0|$.

We will discuss the density of numerical radius attaining operators, actually on a stronger property called Bishop-Phelps-Bollobás property for numerical radius. Let us present first a short account on the known results about numerical radius attaining operators. Motivated by the study of norm attaining operators initiated by J. Lindenstrauss in the 1960s, Sims [255] asked in 1972 whether the numerical radius attaining operators are dense in the space of all bounded linear operators on a Banach space. Berg and Sims [256] gave a positive answer for uniformly convex spaces and Cardassi showed that the answer is positive for $\ell_1, c_0, C(K)$ (where K is a metrizable compact), $L_1(\mu)$, and uniformly smooth spaces [257]-[259]. Acosta showed that the numerical radius attaining operators are dense in $C(K)$ for every compact Hausdorff space K [260]. Acosta and Payá showed that numerical radius attaining operators are dense in $\mathcal{L}(X)$ if X has the Radon-Nikodým property [261]. On the other hand, Payá [262] showed in 1992 that there is a Banach space X such that the numerical radius attaining operators are not dense in $\mathcal{L}(X)$, which gave a negative answer to Sims' question. Some also paid attention to the study of denseness of numerical radius attaining nonlinear mappings [263]-[266].

Motivated by the work [267] of Acosta et al. on the Bishop-Phelps-Bollobás property for operators, Guirao and Kozhushkina [268] introduced very recently the notion of Bishop-Phelps-Bollobás property for numerical radius.

Definition (5.1.1)[252]: (see [268]). A Banach space X is said to have the Bishop-Phelps-Bollobás property for numerical radius (in short, $BPBp - nu$) if, for every $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exit $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy| = 1, \ \|T - S\| < \varepsilon,$$
$$\|x - y\| < \varepsilon, \ \|x^* - y^*\| < \varepsilon. \tag{2}$$

Notice that if a Banach space X has the BPBp-nu, then the numerical radius attaining operators are dense in $\mathcal{L}(X)$. We show that the converse result is no longer true. It is shown in [268] that the real or complex spaces c_0 and ℓ_1 have the BPBp-nu. This result has been extended to the real space $L_1(\mathbb{R})$ by Falcó [269]. Aviles et al. [270] give sufficient conditions on a compact space K for the real space $C(K)$ to have the BPBp-nu which, in particular, include all metrizable compact spaces.

We introduce a modulus of the BPBp-nu analogous to the one introduced in [271] for the Bishop-Phelps-Bollobás property for the operator norm. As easy applications, we prove that finite-dimensional spaces always have the BPBp-nu and that a reflexive space has the BPBp-nu if and only if its dual does. We devoted to prove that Banach spaces which are both uniformly convex and uniformly smooth satisfy a weaker version of the BPBp-nu and to discuss such weaker version. In particular, it is shown that $L_p(\mu)$ spaces have the BPBp-nu for every measure μ when $1 < p < \infty, p \neq 2$. We show that, given any measure μ, the real or complex space $L_1(\mu)$ has the BPBp-nu. Finally, we prove that every separable infinite-dimensional Banach space can be equivalently renormed to fail the BPBp-nu (actually, to fail the weaker version). In particular, this shows that reflexivity (or even superreflexivity) is not enough for the BPBp-nu, while the Radon-Nikodým property was known to be sufficient for the density of numerical radius attaining operators.

The n dimensional space with the ℓ_1 norm is denoted by $\ell_1^{(n)}$. Given a family $\{X_k\}_{k=1}^{\infty}$ of Banach spaces, $[\oplus_{k=1}^{\infty} X_k]_{c_0}$ (resp., $[\oplus_{k=1}^{\infty} X_k]_{\ell_1}$) is the Banach space consisting of all sequences $(x_k)_{k=1}^{\infty}$ such that each x_k is in X_k and $\lim_{k\to\infty}\|x_k\| = 0$ (resp., $\sum_{k=1}^{\infty}\|x_k\| < \infty$) equipped with the norm $\|(x_k)_{k=1}^{\infty}\| = \sup_k \|x_k\|$ (resp., $\|(x_k)_{k=1}^{\infty}\| = \sum_{k=1}^{\infty}\|x_k\|$)

Analogously to what is done in [271] for the BPBp for the operator norm, we introduce here a modulus to quantify the Bishop-Phelps-Bollobás property for numerical radius.

Notation (5.1.2)[252]: Let X be a Banach space. Consider the set

$$\Pi_{\mathrm{nu}}(X) = \{(x, x^*, T): (x, x^*) \in \Pi(X), \quad T \in \mathcal{L}(X), v(T) = 1 = |x^*Tx|\}, \tag{3}$$

which is closed in $S_X \times S_{X^*} \times \mathcal{L}(X)$ with respect to the following metric:

$$\mathrm{dist}\big((x, x^*, T), (y, y^*, S)\big) = \max\{\| x - y \|, \|x^* - y^*\|, \| T - S \|\}. \tag{4}$$

The modulus of the Bishop-Phelps-Bollobás property for numerical radius is the function defined by

$$\eta_{\mathrm{nu}}(X)(\varepsilon) = \inf\{1 - |x^*Tx|: (x, x^*) \in \Pi(X), T \in \mathcal{L}(X), v(T) = 1, \mathrm{dist}\big((x, x^*, T), \Pi_{\mathrm{nu}}(X)\big) \geqslant \varepsilon\} \tag{5}$$

for every $\varepsilon \in (0,1)$. Equivalently, $\eta_{\mathrm{nu}}(X)(\varepsilon)$ is the supremum of those scalars $\eta > 0$ such that, whenever $T \in \mathcal{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta$, there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy| = 1, \ \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \ \|x^* - y^*\| < \varepsilon. \tag{6}$$

It is immediate that a Banach space X has the BPBp-nu if and only if $\eta_{\text{nu}}(\varepsilon) > 0$ for every $0 < \varepsilon < 1$. By construction, if a function $\varepsilon \mapsto \eta(\varepsilon)$ is valid in the definition of the BPBp-nu, then $\eta_{\text{nu}}(\varepsilon) \geqslant \eta(\varepsilon)$.

An immediate consequence of the compactness of the unit ball of a finite-dimensional space is the following result. It was previously known to A. Guirao (private communication).

Proposition (5.1.3)[252]: Let X be a finite-dimensional Banach space. Then X has the Bishop-Phelps-Bollobás property for numerical radius.

Proof. Let $K = \{S \in \mathcal{L}(X): v(S) = 0\}$. Then K is a norm-closed subspace of $\mathcal{L}(X)$. Hence $\mathcal{L}(X)/K$ is a finitedimensional space with two norms:

$$\begin{aligned} v([T]) &:= \inf\{v(T-S): S \in K\} = v(T), \\ \|[T]\| &:= \inf\{\| T-S \|: S \in K\}, \end{aligned} \tag{7}$$

where $[T]$ is the class of T in the quotient space $\mathcal{L}(X)/K$. Hence there is a constant $0 < c \leqslant 1$ such that

$$c\|[T]\| \leqslant v(T) \leqslant \|[T]\|. \tag{8}$$

Suppose that X does not have the BPBp-nu. Then, there is $0 < \varepsilon < 1$ such that $\eta_{\text{nu}}(X)(\varepsilon) = 0$. That is, there are sequences $(x_n, x_n^*) \in \Pi(X)$ and $(T_n) \in \mathcal{L}(X)$ with $v(T_n) = 1$ such that

$$\begin{gathered} \operatorname{dist}\big((x_n, x_n^*, T_n), \Pi_{\text{nu}}(X)\big) \geqslant \varepsilon \ (n \in \mathbb{N}), \\ \lim_n |x_n^* T_n x_n| = 1. \end{gathered} \tag{9}$$

By compactness, we may assume that $\lim_n \|[T_n] - [T_0]\| = 0$ for some $T_0 \in \mathcal{L}(X)$ and $v(T_0) = 1$. Hence there exists a sequence $\{S_n\}_n$ in K such that $\lim_n \|T_n - (T_0 + S_n)\| = 0$. Observe that $v(T_0 + S_n) = v(T_0) = 1$ for every $n \in \mathbb{N}$.

By compactness again, we may assume that (x_n, x_n^*) converges to $(x_0, x_0^*) \in X \times X^*$. This implies that $(x_0, x_0^*) \in \Pi(X)$, and $|x_0^*(T_0 + S_n)x_0| = v(T_0 + S_n) = 1$, that is, $(x_0, x_0^*, T_0 + S_n) \in \Pi_{\text{nu}}(X)$ for all n. This is a contradiction with the fact that

$$\begin{aligned} 0 &= \lim_n \operatorname{dist}\big((x_n, x_n^*, T_n), (x_0, x_0^*, T_0 + S_n)\big) \\ &\geqslant \lim_n \operatorname{dist}\big((x_n, x_n^*, T_n), \Pi_{\text{nu}}(X)\big) \geqslant \varepsilon. \end{aligned} \tag{10}$$

We may also give the following easy result concerning duality.

Proposition (5.1.4)[252]: Let X be a reflexive space. Then

$$\eta_{nu}(X)(\varepsilon) = \eta_{nu}(X^*)(\varepsilon) \tag{11}$$

for every $\varepsilon \in (0,1)$. In particular, X has the BPBp-nu if and only if X^* has the BPBp-nu.

We will use that $v(T^*) = v(T)$ for all $T \in \mathcal{L}(X)$, where T^* denotes the adjoint operator of T. This result can be found in [253], but it is obvious if X is reflexive.

Proof. By reflexivity, it is enough to show that $\eta_{\text{nu}}(X)(\varepsilon) \leqslant \eta_{\text{nu}}(X^*)(\varepsilon)$. Let $\varepsilon \in (0,1)$ be fixed. If $\eta_{\text{nu}}(X)(\varepsilon) = 0$, there is nothing to prove. Otherwise, consider $0 < \eta < \eta_{\text{nu}}(X)(\varepsilon)$. Suppose that $T_1 \in \mathcal{L}(X^*)$ and $(x_1^*, x_1) \in \Pi(X^*)$ satisfy

$$v(T_1) = 1,\ |x_1 T_1 x_1^*| > v(T_1) - \eta. \tag{12}$$

By considering $T_1^* \in \mathcal{L}(X)$, we may find $S_1 \in \mathcal{L}(X)$ and $(y_1, y_1^*) \in \Pi(X)$ such that

$$\begin{gathered} |y_1^* S_1 y_1| = v(S_1) = 1,\ \|y_1 - x_1\| < \varepsilon, \\ \|y_1^* - x_1^*\| < \varepsilon,\ \|T_1^* - S_1\| < \varepsilon. \end{gathered} \tag{13}$$

Then $S_1^* \in \mathcal{L}(X^*)$ and $(y_1^*, y_1) \in \Pi(X^*)$ satisfy

$$\begin{gathered} |\langle y_1, S_1^* y_1^* \rangle| = v(S_1) = 1,\ \|y_1^* - x_1^*\| < \varepsilon, \\ \|y_1 - x_1\| < \varepsilon,\ \|T_1 - S_1^*\| < \varepsilon. \end{gathered} \tag{14}$$

This implies that $\eta_{\mathrm{nu}}(X^*)(\varepsilon) \geqslant \eta$. We finish by just taking supremum on η.

We do not know whether the result above is valid in the nonreflexive case.

For a Banach space which is both uniformly convex and uniformly smooth, we get a property which is weaker than BPBp-nu. This result was known to A. Guirao (private communication).

Proposition (5.1.5). Let X be a uniformly convex and uniformly smooth Banach space. Then, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T_0 \in \mathcal{L}(X)$ with $v(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\begin{gathered} v(S) = |y^* S y|, \quad \| x - y \| < \varepsilon, \\ \|x^* - y^*\| < \varepsilon, \quad \|S - T_0\| < \varepsilon \end{gathered} \tag{15}$$

Proof. Notice that the uniform smoothness of X is equivalent to the uniform convexity of X^*. Let $\delta_X(\varepsilon)$ and $\delta_{X^*}(\varepsilon)$ be the moduli of convexity X and X^*, respectively. Given $0 < \varepsilon < 1$, consider

$$\eta(\varepsilon) = \frac{\varepsilon}{4} \min\left\{\delta_X\left(\frac{\varepsilon}{4}\right), \delta_{X^*}\left(\frac{\varepsilon}{4}\right)\right\} > 0. \tag{16}$$

Consider $T_0 \in \mathcal{L}(X)$ with $v(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfying $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$. Define $T_1 \in \mathcal{L}(X)$ by

$$T_1 x = T_0 x + \lambda_1 \left(\frac{\varepsilon}{4}\right) x_0^*(x) x_0 \tag{17}$$

for all $x \in X$, where λ_1 is the scalar satisfying $|\lambda_1| = 1$ and $|x_0^* T_0 x_0 + \lambda_1(\varepsilon/4)| = |x_0^* T_0 x_0| + \varepsilon/4$. Now, choose $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that $|x_1^*(x_1)| = 1, x_1^*(x_0) = |x_1^*(x_0)|$, and

$$|x_1^* T_1 x_1| \geqslant v(T_1) - \eta\left(\frac{\varepsilon^2}{4^2}\right). \tag{18}$$

Now we define a sequence (x_n, x_n^*, T_n) in $S_X \times S_{X^*} \times \mathcal{L}(X)$ inductively. Indeed, suppose that we have a defined sequence $\left(x_j, x_j^*, T_j\right)$ for $0 \leqslant j \leqslant n$ and let

$$T_{n+1} x = T_n x + \lambda_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} x_n^*(x) x_n. \tag{19}$$

Then choose $x_{n+1} \in S_X$ and $x_{n+1}^* \in S_{X^*}$ such that $|x_{n+1}^*(x_{n+1})| = 1$ and $|x_{n+1}^*(x_n)| = x_{n+1}^*(x_n)$:

$$|x_{n+1}^* T_{n+1} x_{n+1}| \geqslant v(T_{n+1}) - \eta\left(\frac{\varepsilon^{n+2}}{4^{n+2}}\right). \tag{20}$$

Notice that, for all $n \geqslant 0$, we have

$$\begin{gathered} \|T_{n+1} - T_n\| \leqslant \frac{\varepsilon^{n+1}}{4^{n+1}}, \\ v(T_{n+1}) - v(T_n) \,|\leqslant \frac{\varepsilon^{n+1}}{4^{n+1}}. \end{gathered} \tag{21}$$

This implies that (T_n) is a Cauchy sequence and assume that it converges to $S \in \mathcal{L}(X)$. Then we have

$$\begin{gathered} \lim_n T_n = S, \ \|T_0 - S\| < \varepsilon \\ \lim_n |x_n^* T_n x_n| = \lim_n v(T_n) = v(S). \end{gathered} \tag{22}$$

We will show that both sequences (x_n) and (x_n^*) are Cauchy. From the definition, we have

$$
\begin{aligned}
&v(T_{n+1}) - \eta\left(\frac{\varepsilon^{n+2}}{4^{n+2}}\right)\\
&\quad \leqslant |x_{n+1}^* T_{n+1} x_{n+1}|\\
&\quad \leqslant \left| x_{n+1}^* T_n x_{n+1} + \lambda_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} x_n^*(x_{n+1}) x_{n+1}^*(x_n)\right|\\
&\quad \leqslant v(T_n) + \frac{\varepsilon^{n+1}}{4^{n+1}} x_{n+1}^*(x_n),
\end{aligned}
$$

$$
\begin{aligned}
v(T_{n+1}) &\\
&\geqslant |x_n^* T_{n+1} x_n| = \left| x_n^* T_n x_n + \lambda_{n+1}\frac{\varepsilon^{n+1}}{4^{n+1}}\right|\\
&= |x_n^* T_n x_n| + \frac{\varepsilon^{n+1}}{4^{n+1}} \geqslant v(T_n) - \eta\left(\frac{\varepsilon^{n+1}}{4^{n+1}}\right) + \frac{\varepsilon^{n+1}}{4^{n+1}}.
\end{aligned} \tag{23}
$$

In summary, we have

$$
\begin{aligned}
&v(T_n) + \frac{\varepsilon^{n+1}}{4^{n+1}} x_{n+1}^*(x_n)\\
&\quad \geqslant v(T_n) - \eta\left(\frac{\varepsilon^{n+1}}{4^{n+1}}\right) + \frac{\varepsilon^{n+1}}{4^{n+1}} - \eta\left(\frac{\varepsilon^{n+2}}{4^{n+2}}\right).
\end{aligned} \tag{24}
$$

Hence

$$
\begin{aligned}
x_{n+1}^*(x_n) &\geqslant 1 - 2\frac{4^{n+1}}{\varepsilon^{n+1}} \eta\left(\frac{\varepsilon^{n+1}}{4^{n+1}}\right)\\
&= 1 - \frac{1}{2}\min\left\{\delta_X\left(\frac{\varepsilon^{n+1}}{4^{n+2}}\right), \delta_{X^*}\left(\frac{\varepsilon^{n+1}}{4^{n+2}}\right)\right\},\\
\left\|\frac{x_n + x_{n+1}}{2}\right\| &\geqslant x_{n+1}^*\left(\frac{x_n + x_{n+1}}{2}\right) \geqslant 1 - \delta_X\left(\frac{\varepsilon^{n+1}}{4^{n+2}}\right)\\
\left\|\frac{x_n^* + x_{n+1}^*}{2}\right\| &\geqslant \frac{x_n^* + x_{n+1}^*}{2}(x_n) \geqslant 1 - \delta_{X^*}\left(\frac{\varepsilon^{n+1}}{4^{n+2}}\right).
\end{aligned} \tag{25}
$$

This means that

$$
\begin{aligned}
\|x_n - x_{n+1}\| &\leqslant \frac{\varepsilon^{n+1}}{4^{n+2}},\\
\|x_n^* - x_{n+1}^*\| &\leqslant \frac{\varepsilon^{n+1}}{4^{n+2}},
\end{aligned} \tag{26}
$$

for all n. So (x_n) and (x_n^*) are Cauchy. Let $x_\infty = \lim_n x_n$ and $x_\infty^* = \lim_n x_n^*$. Then $\|x_0 - x_\infty\| < \varepsilon/4$ and $\|x_0^* - x_\infty^*\| < \varepsilon/4$. Hence, $|x_\infty^*(x_\infty)| = \lim_n |x_n^*(x_n)| = 1$ and

$$
v(S) = \lim_n v(T_n) = \lim_n |x_n^* T_n x_n| = |x_\infty^* S x_\infty|. \tag{27}
$$

Let $\alpha = x_\infty^*(x_\infty), y^* = \bar{\alpha} x_\infty^*$, and $y = x_\infty$. Then we have $y^*(y) = 1, v(S) = |y^* S y|$, and $\|y - x_0\| < \varepsilon$. Notice that

$$
\begin{aligned}
|\alpha - 1| &= |x_\infty^*(x_\infty) - x_0^*(x_0)|\\
&\leqslant |(x_\infty^* - x_0^*)(x_\infty)| + |x_0^*(x_\infty) - x_0^*(x_0)| < \frac{\varepsilon}{2}.
\end{aligned} \tag{28}
$$

Therefore

$$\|y^* - x^*\| \leqslant \|\bar{\alpha}y^* - y^*\| + \|y^* - x^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon. \quad (29)$$

This completes the proof.

We discuss a little bit about the equivalence between the property in the result above and the BPBp-nu. For convenience, let us introduce the following definition.

Definition (5.1.6)[252]: A Banach space X has the weak Bishop-PhelpsBollobás property for the numerical radius (in short weak $BPBp - nu$); if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T_0 \in \mathcal{L}(X)$ with $v(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\begin{gathered} v(S) = |y^* S y|, \quad \| x - y \| < \varepsilon, \\ \|x^* - y^*\| < \varepsilon, \quad \| S - T \| < \varepsilon. \end{gathered} \quad (30)$$

Notice that the only difference between this concept and the BPBp-nu is the normalization of the operator S by the numerical radius. Of course, if the numerical radius and the operator norm are equivalent, these two properties are the same. This equivalence is measured by the so-called numerical index of the Banach space, as follows. For a Banach space X, the numerical index of X is defined by

$$n(X) = \inf\{v(T): T \in \mathcal{L}(X), \|T\| = 1\}. \quad (31)$$

It is clear that $0 \leqslant n(X) \leqslant 1$ and $n(X)\|T\| \leqslant v(T) \leqslant \|T\|$ for all $T \in \mathcal{L}(X)$. The value $n(X) = 1$ means that v equals the usual operator norm. This is the case of $X = L_1(\mu)$ and $X = C(K)$, among many others. On the other hand, $n(X) > 0$ if and only if the numerical radius is equivalent to the norm of $\mathcal{L}(X)$. See [272] for more information and background.

The following result is immediate. We include a proof for the sake of completeness.

Proposition (5.1.7)[252]: Let X be a Banach space with $n(X) > 0$. Then, X has the BPBp-nu if and only if X has the weak-BPBp-nu.

Proof. The necessity is clear. For the converse, assume that we have $\eta(\varepsilon) > 0$ satisfying the conditions of the weak-BPBp-nu for all $0 < \varepsilon < 1$. If $T \in \mathcal{L}(X)$ with $v(T) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* T x_0| > 1 - \eta(\varepsilon)$ for $0 < \varepsilon < 1$, then there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\begin{gathered} v(S) = |y^* S y|, \qquad \|S - T\| < \varepsilon, \\ \|x - y\| < \varepsilon, \qquad \|x^* - y^*\| < \varepsilon. \end{gathered} \quad (32)$$

As $v(S) > 0$ by the above, let $S_1 = (1/v(S))S$. Then we have

$$\begin{gathered} 1 = v(S_1) = |y^* S_1 y|, \ \|x - y\| < \varepsilon \\ \|x^* - y^*\| < \varepsilon. \end{gathered} \quad (33)$$

Finally, we have

$$\begin{aligned} \|S_1 - T\| &\leqslant \left\| \frac{1}{v(S)} S - S \right\| + \|S - T\| \\ &= \frac{\| S \|}{v(S)} |v(S) - 1| + \|S - T\| \\ &\leqslant \frac{1}{n(X)} |v(S) - v(T)| + \|S - T\| \\ &\leqslant \left(\frac{1}{n(X)} + 1 \right) \|S - T\| < \frac{n(X) + 1}{n(X)} \varepsilon. \end{aligned} \quad (34)$$

An obvious change of parameters finishes the proof.

We do not know whether the hypothesis of $n(X) > 0$ can be omitted in the above result.

Putting together Propositions (5.1.5) and (5.1.7), we get the following.

Corollary (5.1.8)[252]: Let X be a uniformly convex and uniformly smooth Banach space with $n(X) > 0$. Then X has the BPBpnu.

We comment that every complex Banach space X satisfies $n(X) \geqslant 1/e$, so the above corollary automatically applies in the complex case. In the real case, this is no longer true, as the numerical index of a Hilbert space of dimension greater than or equal to two is 0. On the other hand, it is proved in [273] that real $L_p(\mu)$ spaces have nonzero numerical index for every measure μ when $p \neq 2$. Therefore, we have the following examples.

Example (5.1.9)[252]: (a) Complex Banach spaces which are uniformly smooth and uniformly convex satisfy the BPBp-nu.

(b) In particular, for every measure μ, the complex spaces $L_p(\mu)$ have the BPBp-nu for $1 < p < \infty$.

(c) For every measure μ, the real spaces $L_p(\mu)$ have the BPBp-nu for $1 < p < \infty, p \neq 2$.

H. J. Lee, M. Martín, and J. Merí have proved that Proposition (5.1.7) can be extended to some Banach spaces with numerical index zero as, for instance, real Hilbert spaces. Hence, they have shown that Hilbert spaces have the BPBp-nu. These results will appear elsewhere.

We will show that $L_1(\mu)$ has the BPBp-nu for every measure μ. In the proof, we are dealing with complex integrable functions since the real case is followed easily by applying the same proof.

As a first step, we have to start dealing with finite regular positive Borel measures, for which a representation theorem for operators exists.

To prove this proposition, we need some background on representation of operators on Lebesgue spaces on finite regular positive Borel measures and several preliminary lemmas.

Let m be a finite regular positive Borel measure on a compact Hausdorff space Ω. If μ is a complex-valued Borel measure on the product space $\Omega \times \Omega$, then define their marginal measures μ^i on $\Omega (i = 1,2)$ as follows: $\mu^1(A) = \mu(A \times \Omega)$ and $\mu^2(B) = \mu(\Omega \times B)$, where A and B are Borel measurable subsets of Ω.

Let $M(m)$ be the complex Banach lattice of measures consisting of all complex-valued Borel measures μ on the product space $\Omega \times \Omega$ such that $|\mu|^i$ are absolutely continuous with respect to m for $i = 1, 2$, endowed with the norm

$$\left\| \frac{d|\mu|^1}{dm} \right\|_\infty . \tag{35}$$

Each $\mu \in M(m)$ defines a bounded linear operator T_μ from $L_1(m)$ to itself by

$$\langle T_\mu(f), g \rangle = \int_{\Omega \times \Omega} f(x) g(y) d\mu(x, y), \tag{36}$$

where $f \in L_1(m)$ and $g \in L_\infty(m)$. Iwanik [274] showed that the mapping $\mu \mapsto T_\mu$ is a lattice isometric isomorphism from $M(m)$ onto $\mathcal{L}(L_1(m))$. Even though he showed this for the real case, it can be easily generalized to the complex case. For details, see ([274], Theorem 1 and [275], IV Theorem 1.5(ii), Corollary 2).

We will also use that, given an arbitrary measure μ, every $T \in \mathcal{L}(L_1(\mu))$ satisfies $v(T) = \|T\|$ [276] (that is, the space $L_1(\mu)$ has numerical index 1).

Lemma (5.1.10)[252]: (see [267], Lemma 3.3). Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ such that, for a convex series $\sum \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N}^2 : \operatorname{Re} c_i > r\}$ satisfies the estimate

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1 - r}. \tag{37}$$

From now on, m will be a finite regular positive Borel measure on the compact Hausdorff space Ω.

Lemma (5.1.11)[252]: Suppose that there exist a nonnegative simple function $f \in S_{L_1(m)}$ and a function $g \in S_{L_\infty(m)}$ such that

$$\operatorname{Re}\langle f, g\rangle > 1 - \frac{\varepsilon^3}{16}. \tag{38}$$

Then there exist a nonnegative simple function $f_1 \in S_{L_1(m)}$ and a function $g_1 \in S_{L_\infty(m)}$ such that

$$\begin{gathered} g_1(x) = \chi_{\operatorname{supp}(f_1)}(x) + g(x)\chi_{\Omega \setminus \operatorname{supp}(f_1)}(x), \\ \langle f_1, g_1\rangle = 1,\ \|f - f_1\|_1 < \varepsilon \\ \|g - g_1\|_\infty < \sqrt{\varepsilon},\ \operatorname{supp}(f_1) \subset \operatorname{supp}(f). \end{gathered} \tag{39}$$

Proof. Let $f = \sum_{j=1}^{m} \left(\beta_j / m(B_j)\right) \chi_{B_j}$ for some (β_j) such that $\beta_j \geqslant 0$ for all j and $\sum_{j=1}^{m} \beta_j = 1$, and B_j 's are mutually disjoint. By the assumption, we have

$$\operatorname{Re}\langle f, g\rangle = \sum_{j=1}^{n} \beta_j \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x)\, dm(x) > 1 - \frac{\varepsilon^3}{16}, \tag{40}$$

and letting

$$J = \left\{ j : 1 \leqslant j \leqslant n, \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x) dm(x) > 1 - \frac{\varepsilon^2}{4} \right\}, \tag{41}$$

we have by Lemma (5.1.10)

$$\sum_{j \in J} \beta_j > 1 - \frac{\varepsilon}{4}. \tag{42}$$

For each $j \in J$, we have

$$\begin{aligned} 1 - \frac{\varepsilon^2}{4} \quad &< \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x) dm(x) \\ &= \frac{1}{m(B_j)} \int_{B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\}} \operatorname{Re} g(x) dm(x) \\ &\quad + \int_{B_j \cap \{\operatorname{Re} g > 1 - \varepsilon\}} \operatorname{Re} g(x) dm(x) \\ &\leqslant \frac{1}{m(B_j)} \big((1 - \varepsilon) m(B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\}) \\ &= 1 - \varepsilon \frac{m(B_j \cap \{\operatorname{Re} g > 1 - \varepsilon\})\big)}{m(B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\})}. \end{aligned} \tag{43}$$

This implies that

$$\frac{m\left(B_j \cap \{\operatorname{Re} g \leqslant 1-\varepsilon\}\right)}{m\left(B_j\right)} < \frac{\varepsilon}{4}. \quad (44)$$

Define $\tilde{B}_j = B_j \cap \{\operatorname{Re} g > 1-\varepsilon\}$ for all $j \in J$,

$$f_1 = \left(1/\sum_{j\in J} \beta_j\right)\sum_{j\in J} \beta_j \left(\chi_{\tilde{B}_j}/m\left(\tilde{B}_j\right)\right), \quad (45)$$

and $g_1(x) = 1$ on $\operatorname{supp}(f_1)$ and $g_1(x) = g(x)$ elsewhere. Then it is clear that $\operatorname{supp}(f_1) \subset \operatorname{supp}(f)$, $\|g - g_1\|_\infty < \sqrt{\varepsilon}$, and $\langle f_1, g_1\rangle = 1$. Finally we will show that $\|f - f_1\| < \varepsilon$. Notice first that

$$\begin{aligned}
&\left\|\sum_{j\in J} \beta_j \frac{\chi_{\tilde{B}_j}}{m\left(\tilde{B}_j\right)} - \sum_{j\in J} \beta_j \frac{\chi_{B_j}}{m\left(B_j\right)}\right\| \\
&\leqslant \left\|\sum_{j\in J} \beta_j \frac{\chi_{\bar{B}_j}}{m\left(\tilde{B}_j\right)} - \sum_{j\in J} \beta_j \frac{\chi_{\bar{B}_j}}{m\left(B_j\right)}\right\| \\
&\quad + \left\|\sum_{j\in J} \beta_j \frac{\chi_{\bar{B}_j}}{m\left(B_j\right)} - \sum_{j\in J} \beta_j \frac{\chi_{B_j}}{m\left(B_j\right)}\right\| \quad (46)\\
&= 2\sum_{j\in J} \beta_j \frac{m\left(B_j \setminus \tilde{B}_j\right)}{m\left(B_j\right)} < \frac{\varepsilon}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|f - f_1\| &\leqslant \left\|\frac{1}{\sum_{j\in J}\beta_j}\sum_{j\in J}\beta_j \frac{\chi_{\tilde{B}_j}}{m\left(\tilde{B}_j\right)} - \sum_{j\in J}\beta_j \frac{\chi_{\tilde{B}_j}}{m\left(\tilde{B}_j\right)}\right\| + \left\|\sum_{j\in J}\beta_j \frac{\chi_{\bar{B}_j}}{m\left(\tilde{B}_j\right)} - f\right\| \\
&\leqslant \frac{1 - \sum_{j\in J}\beta_j}{\sum_{j\in J}\beta_j}\left\|\sum_{j\in J}\beta_j \frac{\chi_{\bar{B}_j}}{m\left(\tilde{B}_j\right)}\right\| + \left\|\sum_{j\in J} \beta_j \frac{\chi_{\bar{B}_j}}{m\left(\tilde{B}_j\right)} - \sum_{j\in J} \beta_j \frac{\chi_{B_j}}{m\left(B_j\right)}\right\| \\
&= \left(1 - \sum_{j\in J} \beta_j\right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \quad (47)
\end{aligned}$$

Lemma (5.1.12)[252]: (see [277], Lemma 3.3). Suppose that T_μ is a normone element in $\mathcal{L}(L_1(m))$ for some $\mu \in M(m)$ and there is a nonnegative simple function f_0 such that f_0 is a norm-one element of $L_1(m)$ and $\|T_\mu f_0\| \geqslant 1 - \varepsilon^3/2^6$ for some $0 < \varepsilon < 1$. Then there exist a norm-one bounded linear operator T_ν for some $\nu \in M(m,m)$ and a nonnegative simple function f_1 in $S_{L_1(m)}$ such that $\|T_\mu - T_\nu\| \leqslant \varepsilon, \|f_1 - f_0\| \leqslant 3\varepsilon$, and $(d|\nu|^1/dm)(x) = 1$ for all $x \in \operatorname{supp}(f_1)$.

Lemma (5.1.13)[252]: Suppose that $T_\nu \in \mathcal{L}(L_1(m))$ is a norm-one operator, $f = \sum_{i=1}^n \beta_i\left(\chi_{B_i}/m(B_i)\right)$, where $m\left(B_j\right) > 0$ for all $1 \leqslant j \leqslant n$ and $\left\{B_j\right\}_{j=1}^n$ are mutually disjoint Borel subsets of Ω, is a norm-one nonnegative simple function, and g is an element of $S_{L_\infty(m)}$ such that

$$\operatorname{Re}\langle g, T_v f\rangle \geqslant 1-\frac{\varepsilon^6}{2^7} \tag{48}$$

for some $0<\varepsilon<1$ and

$$\frac{d|v|^1}{dm}(x)=1,\ g(x)=1 \tag{49}$$

for all x in the support of f.

Then there exist a nonnegative simple function $\tilde{f} \in S_{L_1(m)}$, a function $\tilde{g} \in S_{L_\infty(m)}$, and an operator $T_{\tilde{v}}$ in $\mathcal{L}(L_1(m), L_1(m))$ such that

$$\begin{gathered}\left\langle\tilde{g}, T_{\tilde{v}} \tilde{f}\right\rangle=\|T_{\tilde{v}}\|=1,\ \|T_v-T_{\tilde{v}}\| \leqslant 2\varepsilon \\ \| f-\tilde{f} \| \leqslant 3\varepsilon,\ \| g-\tilde{g} \| \leqslant \sqrt{\varepsilon},\ \langle\tilde{f}, \tilde{g}\rangle=1.\end{gathered} \tag{50}$$

Proof. Since

$$\operatorname{Re}\langle g, T_v f\rangle \geqslant 1-\frac{\varepsilon^6}{2^7}, \tag{51}$$

we have

$$\begin{aligned} 1-\frac{\varepsilon^6}{2^7} &< \operatorname{Re}\langle g, T_v f\rangle \\ &= \int_{\Omega \times \Omega} f(x) \operatorname{Re} g(y) d\mu(x, y) \\ &= \sum_{j=1}^n \beta_j \int_{\Omega \times \Omega} \frac{\chi_{B_j}(x)}{m(B_j)} \operatorname{Re} g(y) dv(x, y). \end{aligned} \tag{52}$$

Let

$$J=\left\{j: \int_{\Omega \times \Omega}\left(\frac{\chi_{B_j}(x)}{m(B_j)}\right) \operatorname{Re} g(y) dv(x, y)>\frac{1-\varepsilon^3}{2^6}\right\} \tag{53}$$

Then from Lemma (5.1.10) we have $\sum_{j \in J} \beta_j>1-\varepsilon^3/2$. Let $f_1=\sum_{j \in J} \tilde{\beta}_j\left(\chi_{B_j}/m(B_j)\right)$, where $\tilde{\beta}_j=\beta_j/\left(\sum_{j \in J} \beta_j\right)$ for all $j \in J$. Then

$$\|f_1-f\| \leqslant\left\|\sum_{j \in J}(\tilde{\beta}_j-\beta_j) \frac{\chi_{B_j}}{m(B_j)}\right\|+\sum_{j \in J} \beta_j \leqslant \varepsilon^3 \leqslant \varepsilon. \tag{54}$$

Note that there is a Borel measurable function h on $\Omega \times \Omega$ such that $dv(x, y)= h(x, y) d|v|(x, y)$ and $|h(x, y)|=1$ for all $(x, y) \in \Omega \times \Omega$. Let

$$C=\left\{(x, y):|g(y) h(x, y)-1|<\frac{\sqrt{\varepsilon}}{2^{3/2}}\right\}. \tag{55}$$

Define two measures v_f and v_c as follows:

$$v_f(A)=v(A \setminus C),\ v_c(A)=v(A \cap C) \tag{56}$$

for every Borel subset A of $\Omega \times \Omega$. It is clear that

$$\begin{gathered} dv=dv_f+dv_c,\ d\left|v_f\right|=\bar{h} dv_f, \\ d|v_c|=\bar{h} dv_c,\ d|v|=d\left|v_f\right|+d|v_c|. \end{gathered} \tag{57}$$

Since $(d|v|^1/dm_1)(x)=1$ for all $x \in \bigcup_{j=1}^n B_j$, we have

$$1=\frac{d|v|^1}{dm_1}(x)=\frac{d|v_f|^1}{dm_1}(x)+\frac{d|v_c|^1}{dm_1}(x) \tag{58}$$

for all $x \in B = \bigcup_{j=1}^{n} B_j$, and we deduce that $|v|^1(B_j) = m_1(B_j)$ for all $1 \leqslant j \leqslant n$.

We claim that $|v_f|^1(B_j)/m_1(B_j) \leqslant \varepsilon^2/2^2$ for all $j \in J$. Indeed, if $|g(y)h(x,y)-1| \geqslant \sqrt{\varepsilon}/2^{3/2}$, then

$$\operatorname{Re}(g(y)h(x,y)) \leqslant 1-\frac{\varepsilon}{2^4}. \tag{59}$$

So we have

$$\begin{aligned}1-\frac{\varepsilon^3}{2^6} &\leqslant \frac{1}{m_1(B_j)}\operatorname{Re}\int_{\Omega\times\Omega}\chi_{B_j(x)}g(y)dv(x,y)\\ &=\frac{1}{m_1(B_j)}\int_{\Omega\times\Omega}\chi_{B_j(x)}\operatorname{Re}(g(y)h(x,y))d|v|(x,y)\\ &=\frac{1}{m_1(B_j)}\int_{\Omega\times\Omega}\chi_{B_j(x)}\operatorname{Re}(g(y)h(x,y))d|v_f|(x,y)\\ &\quad+\frac{1}{m_1(B_j)}\int_{\Omega\times\Omega}\chi_{B_j(x)}\operatorname{Re}(g(y)h(x,y))d|v_c|(x,y)\\ &\leqslant\frac{1}{m_1(B_j)}\left(\left(1-\frac{\varepsilon}{2^4}\right)|v_f|^1(B_j)+|v_c|^1(B_j)\right)\\ &=1-\frac{\varepsilon}{2^4}\frac{|v_f|^1(B_j)}{m_1(B_j)}.\end{aligned} \tag{60}$$

This proves our claim.

We also claim that, for each $j \in J$, there exists a Borel subset $\tilde{B}_j$ of B_j such that

$$\begin{gathered}\left(1-\frac{\varepsilon}{2}\right)m_1(B_j)\leqslant m_1(\tilde{B}_j)\leqslant m_1(B_j)\\ \frac{d|v_f|^1}{dm_1}(x)\leqslant\frac{\varepsilon}{2}\end{gathered} \tag{61}$$

for all $x \in \tilde{B}_j$. Indeed, set $\tilde{B}_j = B_j \cap \left\{x \in \Omega : \left(d|v_f|^1/dm_1\right)(x) \leqslant \varepsilon/2\right\}$. Then

$$\begin{aligned}\int_{B_j\setminus\bar{B}_j}\frac{\varepsilon}{2}dm_1(x) &\leqslant \int_{B_j}\frac{d|v_f|^1}{dm_1}(x)dm_1(x)\\ &=|v_f^1|(B_j)\leqslant\frac{\varepsilon^2}{2^2}m_1(B_j)\end{aligned} \tag{62}$$

This shows that $m_1(B_j \setminus \tilde{B}_j) \leqslant (\varepsilon/2)m_1(B_j)$. This proves our second claim.

Now, we define $\tilde{g}$ by $\tilde{g}(y) = g(y)/|g(y)|$ if $|g(y)| \geqslant 1-\sqrt{\varepsilon}/2^{3/2}$ and $\tilde{g}(y) = g(y)$ if $|g(y)| < 1-\sqrt{\varepsilon}/2^{3/2}$, and we write $\tilde{f} = \sum_{j\in J}\tilde{\beta}_j\left(\chi_{\bar{B}_j}/m_1(\tilde{B}_j)\right)$. It is clear that $\tilde{g} \in S_{L_\infty(m)}$, $\| g-\tilde{g} \| < \sqrt{\varepsilon}$, and $\tilde{g}(y) = 1$ for all $x \in \operatorname{supp}\tilde{f}$.

Finally, we define the measure

$$d\tilde{v}(x,y) = \sum_{j\in J} \chi_{\bar{B}_j}(x)\overline{\tilde{g}(y)h(x,y)}dv_c(x,y)\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} + \chi_{J_1\setminus\bar{B}}(x)dv(x,y) \tag{63}$$

where $\tilde{B} = \bigcup_{j\in J}\tilde{B}_j$. It is easy to see that $(d|\tilde{v}|^1/dm_1)(x) = 1$ on $\tilde{B}$ and $(d|\tilde{v}|^1/dm_1)(x) \leqslant 1$ elsewhere. Note that

$$d(\tilde{v}-v)(x,y) = \sum_{j\in J} \chi_{\bar{B}_j}(x)\left[\frac{\tilde{g}(y)h(x,y)}{\tilde{g}}\left[\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} - 1\right]dv'\right] - \sum_{j\in J} \chi_{\tilde{B}_j}(x)dv_f(x,y). \tag{64}$$

If $(x,y)\in C$, then $|g(y)| \geqslant 1-\sqrt{\varepsilon}/2^{3/2} \geqslant 1-1/2^{3/2}$ and

$$|\overline{\tilde{g}(y)h(x,y)}-1| = \left|\frac{g(y)}{|g(y)|}h(x,y)-1\right| \leqslant \frac{|g(y)h(x,y)-1|}{|g(y)|} + \frac{|1-|g(y)||}{|g(y)|} \leqslant 2\frac{|g(y)h(x,y)-1|}{|g(y)|} \leqslant 2\frac{\sqrt{\varepsilon}}{2^{\frac{3}{2}}}\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}}-1} \leqslant 2\sqrt{\varepsilon}. \tag{65}$$

Hence, for all $(x,y)\in C$, we have

$$\left|\overline{\tilde{g}(y)h(x,y)}\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} - 1\right| \leqslant |\overline{\tilde{g}(y)h(x,y)}-1|\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} - 1\right| \leqslant 2\sqrt{\varepsilon}\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} - 1\right|. \tag{66}$$

So, we have, for all $x\in J_1$,

$$\frac{d|\tilde{v}-v|^1}{dm_1}(x) \leqslant \sum_{j\in J} \chi_{\bar{B}_j}(x)\left[2\sqrt{\varepsilon}\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} - 1\right|\right]\frac{d|v_c|^1}{dm_1}(x)$$

$$+\sum_{j\in J}\chi_{\bar{B}_j}(x)\frac{d\left|v_f\right|^1}{dm_1}(x)$$

$$\leqslant\sum_{j\in J}\chi_{\bar{B}_j}(x)\left(2\sqrt{\varepsilon}+\left(1-\frac{d|v_c|^1}{dm_1}(x)\right)\right)$$

$$+\sum_{j\in J}\chi_{\bar{B}_j}(x)\left(\frac{d\left|v_f\right|^1}{dm_1}(x)\right)$$

$$\leqslant 2\sqrt{\varepsilon}+\varepsilon<3\sqrt{\varepsilon}. \tag{67}$$

This gives that $\|T_\nu-T_{\bar{\nu}}\|<3\sqrt{\varepsilon}$. Note also that, for all $j\in J$,

$$\left\langle T_{\tilde{\nu}}\frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)},\tilde{g}\right\rangle$$

$$=\int_{\Omega\times\Omega}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\tilde{g}(y)d\tilde{v}(x,y)$$

$$=\int_{\Omega\times\Omega}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\frac{h(x,y)}{h}\frac{d|v_c|^1}{dm_1}(x)\Bigg)^{-1}dv_c(x,y)$$

$$=\int_{\Omega}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1}d|v_c|^1(x)$$

$$=\int_{\Omega}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}dm_1(x)=1. \tag{68}$$

Hence we get $\langle T_{\bar{\nu}}\tilde{f},\tilde{g}\rangle=1$, which implies that $\|T_{\bar{\nu}}\tilde{f}\|=1$. Finally,

$$\left\|\tilde{f}-f\right\|\leqslant\|\tilde{f}-f_1\|+\|f_1-f\|$$

$$=\left\|\sum_{j\in J}\tilde{\beta}_j\frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)}-\sum_{j\in J}\tilde{\beta}_j\frac{\chi_{B_j}}{m_1(B_j)}\right\|+\varepsilon$$

$$\leqslant\sum_{j\in J}\tilde{\beta}_j\left(\left\|\frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)}-\frac{\chi_{B_j}}{m_1(\tilde{B}_j)}\right\|\right.$$

$$\left.+\left\|\frac{\chi_{B_j}}{m_1(\tilde{B}_j)}-\frac{\chi_{B_j}}{m_1(B_j)}\right\|\right)+\varepsilon$$

$$=2\sum_{j\in J}\tilde{\beta}_j\frac{m_1(B_j\setminus\tilde{B}_j)}{m_1(\tilde{B}_j)}+\varepsilon$$

$$\leqslant 2\sum_{j\in J}\tilde{\beta}_j\frac{\left(\frac{\varepsilon}{2}\right)m_1(B_j)}{m_1(\tilde{B}_j)}+\varepsilon\leqslant\frac{\varepsilon}{1-\frac{\varepsilon}{2}}+\varepsilon<3\varepsilon. \tag{69}$$

We are now ready to present the proof of the main result in the case of finite regular positive Borel measure

Proposition (5.1.14)[252]: Let m be a finite regular positive Borel measure on a compact Hausdorff space Ω. Then $L_1(m)$ has the Bishop-Phelps-Bollobás property for numerical radius. More precisely, given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ (which is independent of the measurem) such that if a norm-one element T in $\mathcal{L}(L_1(m))$ and $(f_0, g_0) \in \Pi(L_1(m))$ satisfy $|\langle Tf_0, g_0\rangle| > 1 - \eta(\varepsilon)$, then there exist an operator $S \in \mathcal{L}(L_1(m)), (f_1, g_1) \in \Pi(L_1(m))$ such that

$$|\langle Sf_1, g_1\rangle| = \|S\| = 1,\ \|f_0 - f_1\| \leqslant \varepsilon,$$
$$\|g_0 - g_1\| \leqslant \varepsilon,\ \|T - S\| \leqslant \varepsilon. \tag{70}$$

Proof. Let $\delta_1 = \delta_2^3/(5 \cdot 2^4), \delta_2 = \delta_3^{12}/(3^2 \cdot 2^{14})$, and $\delta_3 = (\varepsilon/10)^2$ for some $0 < \varepsilon < 1$. Suppose that $T \in \mathcal{L}(L_1(m))$ with $\| T \| = 1$ and that there is an $f_0 \in S_{L_1(m)}$ and $g_0 \in S_{L_\infty(m)}$ such that $\langle f_0, g_0\rangle = 1$ and $|\langle Tf_0, g_0\rangle| > 1 - \delta_1^3/2^6$. Then there is an isometric isomorphism Ψ from $L_1(m)$ onto itself such that $\Psi(f_0) = |f_0|$ and there is a scalar number α in $S_{\mathbb{R}}$ such that $|\langle Tf_0, g_0\rangle| = \langle \alpha Tf_0, g_0\rangle$. Then letting $f_1 = \Psi f_0$, $g_1 = (\Psi^{-1})^* g_0$, and $T_1 = \alpha\Psi T\Psi^{-1}$, we have

$$\langle Sf_1, g_1\rangle = \langle \alpha\Psi T_0\Psi^{-1}\Psi f_0, (\Psi^{-1})^* g_0\rangle$$
$$= \langle \alpha Tf_0, g_0\rangle > 1 - \frac{\delta_1^3}{2^6},$$
$$\langle f_1, g_1\rangle = \langle \Psi f_0, (\Psi^{-1})^* g_0\rangle = 1. \tag{71}$$

Since $\|T_1 f_1\| > 1 - \delta(\delta_1^3/2^6)$, by Lemma (5.1.12), there exists a norm-one bounded operator T_v and a nonnegative simple function $f_2 \in S_{L_1(m)}$ such that $\|T_1 - T_v\| \leqslant \delta_1, \|f_2 - f_1\| \leqslant 3\delta_1$, and $(d|v|^1/dm_1)(x) = 1$ for all $x \in \operatorname{supp}(f_2)$. Then

$$\langle T_v f_2, g_1\rangle = \langle T_1 f_1, g_1\rangle - \langle T_1 f_1 - T_1 f_2, g_1\rangle - \langle T_1 f_2 - T_v f_2, g_1\rangle$$
$$\geqslant \langle T_1 f_1, g_1\rangle - \|f_1 - f_2\| - \|T_1 - T_v\|$$
$$\geqslant 1 - \frac{\delta_2^3}{16} - 2\sqrt{\delta_2} \geqslant 1 - 3\sqrt{\delta}_2 = 1 - \frac{\delta_3^6}{2^7}. \tag{72}$$

Notice also that

$$\langle f_2, g_1\rangle = \langle f_1, g_1\rangle - \langle f_1 - f_2, g_1\rangle \geqslant 1 - \|f_1 - f_2\|$$
$$\geqslant 1 - 3\delta_1 \geqslant 1 - 5\delta_1 = 1 - \frac{\delta_2^3}{16}. \tag{73}$$

By Lemma (5.1.11) there are a nonnegative simple function $f_3 \in S_{L_1(m)}$ and a function $g_3 \in S_{L_\infty(m)}$ such that

$$g_3(x) = \chi_{\operatorname{supp} f_3}(x) + g_2(x)\chi_{\Omega\setminus\operatorname{supp} f_3}(x)$$
$$\|f_2 - f_3\| \leqslant \delta_2,\ \|g_3 - g_1\| \leqslant \sqrt{\delta_2},\ \langle f_3, g_3\rangle = 1. \tag{74}$$

So we have

$$\langle T_v f_3, g_3\rangle = \langle T_v f_2, g_1\rangle - \langle T_v f_2 - T_v f_3, g_1\rangle - \langle T_v f_3, g_1 - g_3\rangle$$
$$\geqslant 1 - \frac{\delta_2^3}{16} - 2\sqrt{\delta_2} \geqslant 1 - 3\sqrt{\delta}_2 = 1 - \frac{\delta_3^6}{2^7}. \tag{75}$$

By Lemma (5.1.13), there exist $f_4 \in S_{L_1(m)}$ and $g_4 \in S_{L_\infty(m)}$ and an operator T_4 such that

$$\langle g_4, T_4 f_4\rangle = 1 = \|T_4\|,\ \|T_4 - T_v\| \leqslant 2\delta_3,$$
$$\|f_4 - f_3\| \leqslant 3\delta_3,\ \|g_4 - g_3\| \leqslant \sqrt{\delta_3},\ \langle f_4, g_4\rangle = 1. \tag{76}$$

So we have

$$\|T_4 - T_1\| \leqslant \|T_4 - T_v\| + \|T_v - T_1\|$$
$$\leqslant \delta_1 + 2\delta_3 \leqslant 3\delta_3,$$
$$\|f_1 - f_4\| \leqslant \|f_1 - f_2\| + \|f_2 - f_3\| + \|f_3 - f_4\|$$

$$\leqslant 3\delta_1 + \delta_2 + 3\delta_3 \leqslant 10\delta_3, \quad (77)$$
$$\|g_1 - g_4\| \leqslant \|g_1 - g_3\| + \|g_3 - g_4\|$$
$$\leqslant \delta_2 + \sqrt{\delta_3} \leqslant 2\sqrt{\delta_3}.$$
Let $S = \alpha\Psi^{-1}T_4\Psi, \tilde{f} = \Psi^{-1}f_4$, and $\tilde{g} = \Psi^* g_4$; then we have
$$\|T - S\| = \|T - \alpha\Psi^{-1}T_4\Psi\| = \|\alpha\Psi T\Psi^{-1} - T_4\|$$
$$= \|T_1 - T_4\| \leqslant 3\delta_3,$$
$$\|f_0 - \tilde{f}\| = \|f_0 - \Psi^{-1}f_4\| = \|f_1 - f_4\| \leqslant 10\delta_3,$$
$$\|g_0 - \tilde{f}\| = \|g_0 - \Psi^* g_4\| = \|(\Psi^{-1})^* g_0 - g_4\|$$
$$= \|g_1 - g_4\| \leqslant 2\sqrt{\delta_3} \quad (78)$$
$$\langle \tilde{f}, \tilde{g} \rangle = \langle \Psi^{-1}f_4, \Psi^* g_4 \rangle = \langle f_4, g_4 \rangle = 1,$$
$$|\langle S\tilde{f}, \tilde{g} \rangle| = |\langle \alpha\Psi^{-1}T_4\Psi\Psi^{-1}f_4, \Psi^* g_4 \rangle| = |\alpha| = 1.$$
This completes the proof.

Finally, we may give the proof of the main result in full generality.

Theorem (5.1.15)[252]: Let μ be a measure. Then $L_1(\mu)$ has the BishopPhelps-Bollobás property for numerical radius. More precisely, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ (which does not depend on μ) such that whenever $T_0 \in \mathcal{L}(L_1(\mu))$ with $v(T_0) = 1$ and $(f_0, g_0) \in \Pi(L_1(\mu))$ satisfy $|\langle T_0 f_0, g_0 \rangle| > 1 - \eta(\varepsilon)$, then there exist $T \in \mathcal{L}(L_1(\mu)), (f_1, g_1) \in \Pi(L_1(\mu))$ such that
$$|\langle Tf_1, g_1 \rangle| = v(T) = 1, \ \|f_0 - f_1\| < \varepsilon$$
$$\|g_0 - g_1\| < \varepsilon, \ \|T - T_0\| < \varepsilon. \quad (79)$$
Proof. Notice that the Kakutani representation theorem (see [278] for a reference) says that, for every σ-finite measure v, the space $L_1(v)$ is isometrically isomorphic to $L_1(m)$ for some positive Borel regular measure on a compact Hausdorff space. Then, by Proposition (5.1.14), there is a universal function $\varepsilon \mapsto \eta(\varepsilon) > 0$ which gives the BPBp-nu for $L_1(v)$ for every σ-finite measure v.

Fix $\varepsilon > 0$. Suppose that $T_0 \in \mathcal{L}(L_1(\mu))$ with $v(T_0) = 1$ and $(f_0, f_0^*) \in \Pi(L_1(\mu))$ satisfy
$$|\langle f_0^*, T_0 f_0 \rangle| > 1 - \eta(\varepsilon). \quad (80)$$
Choose a sequence $\{f_n\}$ in $L_1(\mu)$ such that $\sup_n \|T_0 f_n\| = 1$ and let G be the closed linear span of
$$\{T_0^n f_m : n, m \in \mathbb{N} \cup \{0\}\}. \quad (81)$$
As G is separable, there is a dense subset $\{g_n : n \in \mathbb{N}\}$ of G and let $E = \bigcup_{n=1}^{\infty} \operatorname{supp} g_n$, where $\operatorname{supp} g_n$ is the support of g_n. Then the measure $\mu|_E$ is σ-finite. Let
$$Y = \{f \in L_1(\mu) : \operatorname{supp}(f) \subset E\} \quad (82)$$
be a closed subspace of $L_1(\mu)$. It is clear that $L_1(\mu) = Y \oplus_1 Z$ and Y is isometrically isomorphic to $L_1(\mu|_E)$. So Y has the BPBp-nu with $\eta(\varepsilon)$.

Now, write $S_0 = T_0|_Y : Y \to Y$, consider $y_0 = f_0 \in S_Y, y_0^* = f_0^*|_Y \in S_{Y^*}$, and observe that $y_0^*(y_0) = 1$ and $|y_0^*(S_0 y_0)| = |f_0^*(T_0 f_0)| > 1 - \eta(\varepsilon)$. Hence, there exist $S \in \mathcal{L}(Y)$ and $(\tilde{y}_0, \tilde{y}_0^*) \in \Pi(Y)$ such that
$$|\tilde{y}_0^*(S\tilde{y}_0)| = 1 = v(S), \ \|S - S_0\| < \varepsilon,$$
$$\|y_0 - \tilde{y}_0\| < \varepsilon, \ \|y_0^* - \tilde{y}_0^*\| < \varepsilon. \quad (83)$$
Finally consider the operator $T \in \mathcal{L}(L_1(\mu))$ given by
$$T(y, z)$$
$$= (Sy, 0) + T_0(0, z)((y, z) \in L_1(\mu) \equiv Y \oplus_1 Z). \quad (84)$$

We have $\|T\| = 1$ (and so $v(T) = 1$). Indeed,

$$\|T(y,z)\| = \|(Sy,0)\| + \|T_0(0,z)\| \leqslant \|y\| + \|z\| = \|(y,z)\| \tag{85}$$

for all $(y,z) \in L_1(\mu)$ and $\|T(\tilde{y}_0,0)\| = \|(S\tilde{y}_0,0)\| = \|S\tilde{y}_0\| = 1$. Let $x = (\tilde{y}_0,0)$ and $x^* = (\tilde{y}_0^*, f_0|_Z)$. Then $(x,x^*) \in \Pi(L_1(\mu))$. Moreover, we have

$$\begin{aligned} |x^*Tx| &= |\tilde{y}_0^* S y_0| = 1 = v(T), \\ \|x - f_0\| &= \|y - y_0\| < \varepsilon, \\ \|x_0^* - f_0^*\| &= \max\{\|y - f_0^*|_Y\|, \|f_0^*|_Z - f_0^*|_Z\|\} \\ &= \|y^* - y_0^*\| < \varepsilon, \\ \|T - T_0\| &= \sup_{\|y\|+\|z\|\leqslant 1} \|T(y,z) - T_0(y,z)\| \\ &= \sup_{\|y\|\leqslant 1} \|Sy - S_0 y\| = \|S - S_0\| < \varepsilon. \end{aligned} \tag{86}$$

This completes the proof

We prove that the density of numerical radius attaining operators does not imply the BPBp-nu. Actually, we will show that, among separable spaces, there is no isomorphic property implying the BPBp $-$ nu other than finitedimensionality.

We need to relate the BPBp-nu to the Bishop-PhelpsBollobás property for operators which, as mentioned in the introduction, was introduced in [267]. A pair (X,Y) of Banach spaces has the Bishop-Phelps-Bollobás property for operators (in short, $BPBp$); if given $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that, given $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x \in S_X$ such that $\|Tx\| > 1 - \eta(\varepsilon)$, then there exist $z \in S_X$ and $S \in \mathcal{L}(X,Y)$ satisfying

$$\begin{aligned} \|S\| = \|Sz\| = 1,\ \|x - z\| < \varepsilon, \\ \|T - S\| < \varepsilon. \end{aligned} \tag{87}$$

See [267],[271],[277] for more information and background. Among the interesting results on the BPBp, we emphasize that a pair (X,Y) when X is finite-dimensional does not necessarily have the BPBp. For instance, if Y is a strictly convex space which is not uniformly convex, then the pair $\left(\ell_1^{(2)}, Y\right)$ fails to have the BPBp (this is contained in [267]; see [271]). The next result relates the BPBp-nu to the BPBp for operators in a particular case. We will deduce our example from it.

Before proving this proposition, we will use it to get the main examples. The first example shows that the density of numerical radius attaining operators does not imply the BPBp-nu.

Example (5.1.16)[252]: There is a reflexive space (and so numerical radius attaining operators on it are dense) which fails to have the $BPBp - nu$. Indeed, let Y be a reflexive separable space which is not superreflexive and we may suppose that Y is strictly convex. Observe that Y cannot be uniformly convex since it is not superreflexive. Now, $X = \ell_1^{(2)} \oplus_1 Y$ is reflexive, but the pair $\left(\ell_1^{(2)}, Y\right)$ fails the BPBp since Y is strictly convex but not uniformly convex [271]. Therefore, Theorem (5.1.20) gives us that X does not have the BPBp-nu.

The example above can be extended to get the result that every infinite-dimensional separable Banach space can be renormed to fail the BPBp-nu. This follows from the fact that every infinite-dimensional separable Banach space can be renormed to be strictly convex but not uniformly convex (this result can be proved "by hand"; an alternative categorical argument for it can be found in [279]). With a little more of effort, we may get the main result.

We need the following result which is surely well known. We include a nice and easy proof kindly given to us by Vladimir Kadets. We recall that, given a Banach space Y, the set of all equivalent norms on Y can be viewed as a metric space using the Banach-Mazur distance.

Lemma (5.1.17)[252]: Let Y be an infinite-dimensional separable Banach space. Then the set of equivalent norms on Y which are strictly convex and are not (locally) uniformly convex is dense in the set of all equivalent norms on Y (with respect to the Banach-Mazur distance).

Proof. Fix $e \in S_Y$ and $e_1^* \in S_{Y^*}$ such that $e_1^*(e) = 1$. For a fixed $\varepsilon \in (0,1/2)$, denote

$$q(y) = \max\{(1-\varepsilon)\|y\|, |e_1^*(y)|\} \ (y \in Y). \tag{88}$$

Evidently, $(1-\varepsilon)\|y\| \leqslant q(y) \leqslant \|y\|$ for every $y \in Y$. Fix a sequence $\{e_k^*: k \geqslant 2\}$ of norm-one functionals separating the points of Y, and denote

$$p(y) = \sqrt{\sum_{k=1}^{\infty} \frac{1}{2^k} \left|e_k^*(y)\right|^2} \ (y \in Y). \tag{89}$$

Then, p is a strictly convex norm on $Y, p(e) \geqslant 1/\sqrt{2}$, and $p(y) \leqslant \|y\|$ for all $y \in X$. Finally, write

$$\|y\|_1 = (1-\varepsilon)q(y) + \varepsilon \frac{p(y)}{p(e)} \ (y \in Y). \tag{90}$$

Then, $\|\cdot\|_1$ is a strictly convex norm on Y and

$$(1-\varepsilon)^2\|y\| \leqslant \|y\|_1 \leqslant (1+\varepsilon)\|y\| \ (y \in Y). \tag{91}$$

We will finish the proof by showing that $\|\cdot\|_1$ is not uniformly convex (actually, it is not locally uniformly convex). Indeed, for each $n \in \mathbb{N}$ we select $y_n \in \bigcap_{k=1}^{n} \ker e_k^*$ with $\|y_n\| = 1$ and consider $e_n = e + (\varepsilon/4)y_n$. Then, $q(e) = 1, q(e_n) = 1$, and $q(e + e_n) = 2$. At the same time, $p(y_n) \to 0$, so $p(e_n) \to p(e)$ and $p(e + e_n) \to 2p(e)$. Consequently,

$$\|e\|_1 = 1, \ \|e_n\|_1 \longrightarrow 1, \ \|e + e_n\|_1 \longrightarrow 2, \tag{92}$$

but $\|e - e_n\|_1 = (\varepsilon/4)\|y_n\|_1 \geqslant (1-\varepsilon)^2(\varepsilon/4)$, which means the absence of local uniform convexity at e.

Theorem (5.1.18)[252]: Every infinite-dimensional separable Banach space can be renormed to fail the weak-BPBp-nu (and so, in particular, to fail the BPBp-nu).

Proof. Let X be an infinite-dimensional separable Banach space. Take a closed subspace Y of X of codimension two. By [280], the map carrying every equivalent norm on Y to its numerical index is continuous and so, the set of values of the numerical index of Y up to reforming is a nontrivial interval [280]. Then Lemma (5.1.17) allows us to find an equivalent norm $|\cdot|$ on Y in such a way that $(Y, |\cdot|)$ is strictly convex and is not uniformly convex, and $n(Y, |\cdot|) > 0$. Now, the space $\tilde{X} = \ell_1^{(2)} \oplus_1 (Y, |\cdot|)$ is an equivalent renorming of X which does not have the BPBp-nu (indeed, otherwise, the pair $\left(\ell_1^{(2)}, (Y, |\cdot|)\right)$ would have the BPBp for the operator norm and so, $(Y, |\cdot|)$ would be uniformly convex by [271], a contradiction.) Moreover, as

$$n(\tilde{X}) = \min\left\{n\left(\ell_1^{(2)}\right), n(Y, |\cdot|)\right\} > 0 \tag{93}$$

(see [272], for instance), $\tilde{X}$ also fails the weakBPBp-nu by Proposition (5.1.7) .

To finish with the promised proof of Theorem (5.1.20), we first see the following stability result.

Lemma (5.1.19)[252]: Let $X=[\oplus_{k=1}^{\infty}X_k]_{c_0}$ or $[\oplus_{k=1}^{\infty}X_k]_{\ell_1}$. If X has the Bishop-Phelps-Bollobás property for numerical radius with a function η, then each Banach space X_i has the Bishop-PhelpsBollobás property for numerical radius with $\eta_{nu}(X_i)\geqslant\eta$. That is, $\inf_i\eta_{nu}(X_i)(\varepsilon)\geqslant\eta_{nu}(X)(\varepsilon)$ for all $0<\varepsilon<1$.

Proof. Let $P_i: X\to X_i$ and $P_i': X^*\to X_i^*$ be the natural projections, and let $Q_i: X_i\to X$ and $Q_i': X_i^*\to X^*$ be the natural embeddings.

Assume that an operator $T_i: X_i\to X_i$ and a pair $(x_i,x_i^*)\in\Pi(X_i)$ satisfy that

$$v(T_i)=1,\ |x_i^*T_ix_i|>1-\eta(\varepsilon). \tag{94}$$

We define an operator $T: X\to X$ and $(x,x^*)\in\Pi(X)$ by

$$T=Q_i\circ T_i\circ P_i,\ (x,x^*)=(Q_ix_i,Q_i'x_i^*); \tag{95}$$

then clearly we see that

$$|x^*Tx|=|x_i^*T_ix_i|>1-\eta(\varepsilon). \tag{96}$$

From the assumption, there exist $S: X\to X$ and a pair $(y,y^*)\in\Pi(X)$ such that

$$\begin{gathered}|y^*Sy|=1=v(S),\ \|S-T\|<\varepsilon,\\ \|y^*-x^*\|<\varepsilon,\ \|y-x\|<\varepsilon.\end{gathered} \tag{97}$$

Since this clearly shows that

$$\begin{gathered}\|P_i\circ S\circ Q_i-T_i\|<\varepsilon,\ \|P_i'y^*-x_i^*\|<\varepsilon,\\ \|P_iy-x_i\|<\varepsilon,\end{gathered} \tag{98}$$

we only need to show that $|P_i'y^*(P_i\circ S\circ Q_i)P_iy|=1$.

We first show the case of c_0 sum. Since

$$\|P_jy\|=\|P_jy-P_jx\|\leqslant\|y-x\|<\varepsilon \tag{99}$$

for every $j\neq i$, we have

$$\begin{aligned}1&=y^*(y)=\sum_{j\in\mathbb{N}}P_j'y^*(P_jy)\leqslant\sum_{j\in\mathbb{N}}\|P_j'y^*\|\|P_jy\|\\ &\leqslant\|P_i'y^*\|+\varepsilon\sum_{j\in\mathbb{N},j\neq i}\|P_j'y^*\|\leqslant\|y^*\|=1.\end{aligned} \tag{100}$$

This shows that $\|P_i'y^*\|=1$ and $P_j'y^*=0$ for every $j\neq i$. So $y^*=Q_i'P_i'y^*$ and $P_i'y^*(P_iy)=1$. This and the fact that $\|y-Q_iP_iy\|<\varepsilon$ imply that

$$\left(Q_iP_iy+\left(\frac{1}{\varepsilon}\right)(y-Q_iP_iy),Q_i'P_i'y^*\right)\in\Pi(X). \tag{101}$$

So we get that $(Q_i'P_i'y^*)S\big(Q_iP_iy+(1/\varepsilon)(y-Q_iP_iy)\big)v(S)=1$. Hence, we have

$$\begin{aligned}1&=|y^*Sy|=|(Q_i'P_i'y^*)Sy|\\ &=\left|(1-\varepsilon)(Q_i'P_i'y^*)S(Q_iP_iy)\right.\\ &\left.\quad+\varepsilon(Q_i'P_i'y^*)S\left(Q_iP_iy+\frac{1}{\varepsilon}(y-Q_iP_iy)\right)\right|\leqslant1,\end{aligned} \tag{102}$$

and so we get $|P_i'y^*(P_i\circ S\circ Q_i)P_iy|=|(Q_i'P_i'y^*)S(Q_iP_iy)|=1$.

We next show the case of ℓ_1 sum. The proof is almost the same as that of the c_0 case. However, for the sake of completeness, we provide it here.

Since $\|P_j'y^*\|=\|P_j'y^*-P_j'x^*\|\leqslant\|y^*-x^*\|<\varepsilon$ for every $j\neq i$, we have

$$\begin{aligned} 1 &= y^*(y) = \sum_{j\in\mathbb{N}} P_j'y^*(P_jy) \leqslant \sum_{j\in\mathbb{N}} \|P_j'y^*\|\|P_jy\| \\ &\leqslant \|P_iy\| + \varepsilon \sum_{j\in\mathbb{N}, j\neq i} \|P_jy\| \leqslant \|y\| = 1, \end{aligned} \tag{103}$$

which shows $\|P_iy\| = 1$ and $P_jy = 0$ for every $j \neq i$. Since this implies $\left(Q_iP_iy, Q_i'P_i'y^* + (1/\varepsilon)(y^* - Q_i'P_i'y^*)\right) \in \Pi(X)$, we get that $\left|\left(Q_i'P_i'y^* + (1/\varepsilon)(y^* - Q_i'P_i'y^*)\right)S(Q_iP_iy)\right| \leqslant v(S) = 1$. Hence, we have

$$\begin{aligned} 1 &= |y^*Sy| = |y^*S(Q_iP_iy)| \\ &= \left|(1-\varepsilon)(Q_i'P_i'y^*)S(Q_iP_iy) \right. \\ &\quad \left. + \varepsilon\left(Q_i'P_i'y^* + \frac{1}{\varepsilon}(y^* - Q_i'P_i'y^*)\right)S(Q_iP_iy)\right| \leqslant 1, \end{aligned} \tag{104}$$

and so $|P_i'y^*(P_i \circ S \circ Q_i)P_iy| = |(Q_i'P_i'y^*)S(Q_iP_iy)| = 1$.

Theorem (5.1.20)[252]: If $L_1(\mu) \oplus_1 X$ has the $BPBp - nu$, then the pair $(L_1(\mu), X)$ has the BPBp for operators.

Proof. Note that $\eta_{\text{nu}}(L_1(\mu) \oplus_1 X)(\varepsilon) \to 0$ as $\varepsilon \to 0$. Fix $0 < \varepsilon_0 < 1$ and choose $0 < \varepsilon < 1$ such that $6\varepsilon + \eta_{\text{nu}}(L_1(\mu) \oplus_1 X)(\varepsilon) < \varepsilon_0$. Let $\eta(\varepsilon_0) = \eta_{\text{nu}}(L_1(\mu) \oplus_1 X)(\varepsilon)$.

Suppose that $T_0 \in \mathcal{L}(L_1(\mu), X)$ with $\|T_0\| = 1$ and $f_0 \in S_{L_1(\mu)}$ satisfy

$$\|T_0f_0\| > 1 - \eta(\varepsilon_0). \tag{105}$$

For any measurable subset B, let

$$L_1(\mu|_B) = \{f|_B : f \in L_1(\mu)\} \tag{106}$$

with the norm $\|f|_B\| = \|f\chi_B\|_1$. Then it is easy to see that $L_1(\mu|_B)$ is isometrically isomorphic to a complemented subspace of $L_1(\mu)$. Let $P_B: L_1(\mu) \to L_1(\mu|_B)$ be the restriction defined by $P_B(f) = f|_B$ for all $f \in L_1(\mu)$ and let $J_B: L_1(\mu|_B) \to L_1(\mu)$ be the extension defined by $J_B(f)(\omega) = f(\omega)$ if $\omega \in B$ and $J_B(f)(\omega) = 0$ otherwise. It is clear that $P_BJ_B = \text{Id}_{L_1(\mu|_B)}$ and $J_BP_B(f) = f\chi_B$ for all $f \in L_1(\mu)$. Notice also that $L_1(\mu)$ is isometrically isomorphic to $L_1(\mu|_B) \oplus_1 L_1(\mu|_{B^c})$.

Let $A = \operatorname{supp} f_0$ and $g_0 = P_Af_0$. Then

$$\|T_0J_Ag_0\| = \|T_0f_0\| > 1 - \eta(\varepsilon_0) > 0 \tag{107}$$

and define the operator $T_A: L_1(\mu|_A) \to X$ by $T_Af = T_0J_Af/\|T_0J_A\|$ for every $f \in L_1(\mu|_A)$. Then,

$$\|T_Ag_0\| \geqslant \|T_0f_0\| > 1 - \eta(\varepsilon_0). \tag{108}$$

Since $\mu|_A$ is σ-finite, $L_1(\mu|_A)^* = L_\infty(\mu|_A)$. Let $g_0^* \in S_{L_\infty(\mu|_A)}$ be a function such that $\langle g_0^*, g_0\rangle = 1$, and choose $x_0^* \in S_{X^*}$ such that $x_0^*(T_Ag_0) = \|T_Ag_0\|$. Define the operator $S_0 \in \mathcal{L}(L_1(\mu|_A) \oplus_1 X)$ by

$$S_0(f, x) = (0, T_Af) \ ((f, x) \in L_1(\mu|_A) \oplus_1 X) \tag{109}$$

and observe that $\|S_0\| = v(S_0) = 1$. Indeed,

$$\begin{aligned} \|S_0\| &\leqslant 1 = \|T_A\| \\ &= \sup\left\{|x^*T_Af| : x^* \in S_{X^*},\ f \in S_{L_1(\mu|_A)}\right\} \\ &= \sup\left\{|(f^*, x^*)S_0(f, x)| : \left((f^*, x^*), (f, x)\right) \in \Pi(L_1(\mu|_A) \oplus_1 X)\right\} \\ &= v(S_0) \leqslant \|S_0\|. \end{aligned} \tag{110}$$

It is immediate that

$$(g_0^*, x_0^*)S_0(g_0, 0) = x_0^*(T_A g_0) = \|T_A g_0\| > 1 - \eta(\varepsilon_0). \tag{111}$$

By Lemma (5.1.19), $L_1(\mu|_A) \oplus_1 X$ has the BPBp-nu with the function η. Therefore, there exist $S_1 \in \mathcal{L}(L_1(\mu|_A) \oplus_1 X)$, $(g_1, x_1) \in S_{L_1(\mu|_A)\oplus_1 X}$, and $(g_1^*, x_1^*) \in S_{L_\infty(\mu|_A)\oplus_\infty X^*}$ such that

$$\begin{gathered}\|(g_1, x_1) - (g_0, 0)\| < \varepsilon,\ \|(g_1^*, x_1^*) - (f_0^*, x_0^*)\| < \varepsilon \\ \|S_1 - S_0\| < \varepsilon,\ \langle (g_1^*, x_1^*), (g_1, x_1)\rangle = 1, \\ |(g_1^*, x_1^*)S_1(g_1, x_1)| = v(S_1) = 1.\end{gathered} \tag{112}$$

Claim (5.1.21)[252]: We claim that $x_1 = 0$.

Otherwise,

$$\begin{aligned}1 &= \mathrm{Re}\langle (g_1^*, x_1^*), (g_1, x_1)\rangle \\ &= \|g_1\|\mathrm{Re}\left\langle (g_1^*, x_1^*), \frac{(g_1, 0)}{\|g_1\|}\right\rangle \\ &\quad + \|x_1\|\mathrm{Re}\left\langle (g_1^*, x_1^*), \frac{(0, x_1)}{\|x_1\|}\right\rangle \leqslant 1.\end{aligned} \tag{113}$$

We deduce that

$$\left(\frac{(g_1, 0)}{\|g_1\|}, (g_1^*, x_1^*)\right), \left(\frac{(0, x_1)}{\|x_1\|}, (g_1^*, x_1^*)\right) \in \Pi(L_1(\mu|_A) \oplus_1 X). \tag{114}$$

Since $\|S_1((0, x_1)/\|x_1\|)\| = \|(S_1 - S_0)((0, x_1)/\|x_1\|)\| < \varepsilon$, we get that

$$\begin{aligned}1 &= |\langle (g_1^*, x_1^*), S_1(g_1, x_1)\rangle| \\ &= \left| \|g_1\| \left\langle (g_1^*, x_1^*), S_1\left(\frac{(g_1, 0)}{\|g_1\|}\right)\right\rangle \right. \\ &\quad \left. + \|x_1\| \left\langle (g_1^*, x_1^*), S_1\left(\frac{(0, x_1)}{\|x_1\|}\right)\right\rangle \right| \\ &\leqslant \|g_1\| v(S_1) + \varepsilon\|x_1\| < \|g_1\| + \|x_1\| = 1,\end{aligned} \tag{115}$$

a contradiction. This proves the claim.

We define the operator $S_2: L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A) \oplus_1 X$ by $S_2(f, x) = S_1(f, 0)$ for every $f \in L_1(\mu|_A)$ and for every $x \in X$. Then we have

$$v(S_2) = |(g_1^*, x_1^*)S_2(g_1, 0)| = 1,\ \|S_1 - S_2\| \leqslant \varepsilon. \tag{116}$$

Indeed, from Claim (5.1.21), we have

$$\begin{aligned}v(S_1) &= |(g_1^*, x_1^*)S_1(g_1, x_1)| = |(g_1^*, x_1^*)S_1(g_1, 0)| \\ &= |(g_1^*, x_1^*)S_2(g_1, 0)| \leqslant v(S_2).\end{aligned} \tag{117}$$

On the other hand, we see that

$$|(f^*, x^*)S_2(f, x)| = |(f^*, x^*)S_1(f, 0)| \leqslant \| f \|\ v(S_1) \leqslant v(S_1) \tag{118}$$

for every $((f^*, x^*), (f, x)) \in \Pi(L_1(\mu|_A) \oplus_1 X)$. Therefore, $v(S_2) \leqslant v(S_1)$. Also,

$$\begin{aligned}\|S_1 - S_2\| &\leqslant \sup_{x \in S_X} \|S_1(0, x)\| \\ &= \sup_{x \in S_X} \|S_1(0, x) - S_0(0, x)\| \leqslant \varepsilon.\end{aligned} \tag{119}$$

Claim (5.1.22)[252]: There exists $S_3: L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A) \oplus_1 X$ such that $\|S_3(g_1, 0)\| = \|S_3\| = 1, S_3(0, x) = 0, S_3(f, x) \in \{0\} \oplus_1 X$ for every $(f, x) \in L_1(\mu|_A) \oplus_1 X$, and $\|S_3 - S_2\| < 4\varepsilon$.

Indeed, write $S_1 = (D_1, D_2)$, where $D_1: L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A)$ and $D_2: L_1(\mu|_A) \oplus_1 X \to X$. We have that

$$\begin{aligned}
&\sup\{|g^* D_1(g_1, 0) + x^* D_2(g_1, 0)|: \\
&\qquad x^* \in S_{X^*}, \langle g^*, g_1 \rangle = 1, g^* \in S_{L_\infty(\mu|_A)}\} \\
&= \sup\{|g^* D_1(g_1, 0)| + \|D_2(g_1, 0)\|: \\
&\qquad \langle g^*, g_1 \rangle = 1, g^* \in S_{L_\infty(\mu|_A)}\} \\
&= \sup\{|g^* D_1(g_1, 0)|: \\
&\qquad \langle g^*, g_1 \rangle = 1, g^* \in S_{L_\infty(\mu|_A)}\} + \|D_2(g_1, 0)\| \\
&\leqslant v(S_2) = |(g_1^*, x_1^*) S_2(g_1, 0)| \\
&= |g_1^* D_1(g_1, 0) + x_1^* D_2(g_1, 0)|.
\end{aligned} \tag{120}$$

This implies that

$$|x_1^* D_2(g_1, 0)| = \|D_2(g_1, 0)\|$$
$$|g_1^* D_1(g_1, 0)| = \sup\{|g^* D_1(g_1, 0)|: \langle g^*, g_1 \rangle = 1, g^* \in L_\infty(\mu|_A)\} \tag{121}$$

Therefore, $|g_1^*|$ equals 1 on the support of $D_1(g_1, 0)$. As $|\langle g_1^*, g_1 \rangle| = 1$, we also have that $|g_1^*|$ equals 1 on the support of g_1. Changing the values of g_1^* by the ones of f_0^* on $\setminus \left(\operatorname{supp}(D_1(g_1, 0)) \cup \operatorname{supp}(g_1)\right)$, we may and do suppose that $|g_1^*| = 1$ on the whole A.
We also have $\|D_2(g_1, 0)\| > 0$. Indeed,

$$\begin{aligned}
&\|S_2(g_1, 0) - S_0(g_0, 0)\| \\
&\leqslant \|S_2(g_1, 0) - S_0(g_1, 0)\| + \|S_0(g_1, 0) - S_0(g_0, 0)\| \\
&< 2\varepsilon + \varepsilon = 3\varepsilon
\end{aligned} \tag{122}$$

So we have

$$\begin{aligned}
&\|D_2(g_1, 0) - T_A g_0\| \\
&\leqslant \|D_1(g_1, 0)\| + \|D_2(g_1, 0) - T_A g_0\| \\
&= \|\left(D_1(g_1, 0), D_2(g_1, 0)\right) - (0, T_A g_0)\| \\
&= \|S_2(g_1, 0) - S_0(g_0, 0)\| < 3\varepsilon
\end{aligned} \tag{123}$$

and $\|D_2(g_1, 0)\| > \|T_A g_0\| - 3\varepsilon \geqslant 1 - \eta(\varepsilon_0) - 3\varepsilon > 0$.

Finally define the operator S_3 by

$$S_3(f, x) = \left(0, D_2(f, 0) + g_1^*\left(D_1(f, 0)\right) \frac{D_2(g_1, 0)}{x_1^* D_2(g_1, 0)}\right) \tag{124}$$

for $(f, x) \in L_1(\mu|_A) \oplus_1 X$. It is clear that

$$\|S_3\| \leqslant \sup_{f \in S_{L_1}(\mu_A)} (\|D_2(f, 0)\| + |g_1^* D_1(f, 0)|). \tag{125}$$

Notice also that

$$\begin{aligned}
&\|D_1(f, 0)\| \\
&\leqslant \|D_1(f, 0)\| + \|D_2(f, 0) - T_A f\| \\
&= \|(D_1(f, 0), D_2(f, 0)) - (0, T_A f)\| \\
&= \|S_2(f, x) - S_0(f, x)\|
\end{aligned} \tag{126}$$

for all $(f, x) \in L_1(\mu|_A) \oplus_1 X$. Hence we have

$$\begin{aligned}&\|S_3 - S_2\| \\ &= 2 \sup_{f \in S_{L_1(\mu|A)}} \|D_1(f,0)\| \leqslant 2\|S_2 - S_0\| < 4\varepsilon. \end{aligned} \tag{127}$$

On the other hand, let $G: L_1(\mu|_A) \to L_1(\mu|_A)$ be defined by $G(f) = \overline{g_1^*} f$ for every $f \in L_1(\mu|_A)$. Then, we have

$$\begin{aligned} v(S_2) &= \sup\{|z^* S_2 z| : (z, z^*) \in \Pi(L_1(\mu|_A) \oplus_1 X)\} \\ &\geqslant \sup\left\{\left|x^* D_2\left(G\left(\frac{1}{\mu(C)}\chi_C\right), 0\right)\right.\right. \\ &\qquad \left. + g_1^* D_1\left(G\left(\frac{1}{\mu(C)}\chi_C\right), 0\right)\right| : \\ &\qquad \left. x^* \in S_{X^*}, C \in \Sigma_A, \mu(C) > 0\right\} \\ &= \sup\left\{\left\|D_2\left(G\left(\frac{1}{\mu(C)}\chi_C\right), 0\right)\right\|\right. \\ &\qquad + \left|g_1^* D_1\left(G\left(\frac{1}{\mu(C)}\chi_C\right), 0\right)\right| : \\ &\qquad \left. C \in \Sigma_A, \mu(C) > 0\right\}, \end{aligned} \tag{128}$$

where Σ_A is the family of measurable subsets of A. Hence, for any simple function $= \sum_{i=1}^{n} \left(\left(\alpha_i / \mu(A_i)\right)\chi_{A_i}\right) \in S_{L_1(\mu|_A)}$, where $\{A_i\}_i$ is a family of disjoint measurable subsets with strictly positive measure, we have

$$\begin{aligned} v(S_2) &\geqslant \sum_{i=1}^{n} |\alpha_i| \left(\left\|D_2\left(G\left(\frac{1}{\mu(A_i)}\chi_{A_i}\right), 0\right)\right\|\right. \\ &\qquad \left. + \left|g_1^* D_1\left(G\left(\frac{1}{\mu(A_i)}\chi_{A_i}\right), 0\right)\right|\right) \\ &\geqslant \|D_2(G(s), 0)\| + |g_1^* D_1(G(s), 0)|. \end{aligned} \tag{129}$$

Since $|g_1^*| = 1, G$ is an isometric isomorphism, so for each $f \in S_{L_1(\mu|_A)}$ there exists a sequence of norm-one simple functions (s_k) such that $G(s_k)$ converges to f. Therefore,

$$v(S_2) \geqslant \sup_{f \in S_{L_1(\mu|_A)}} (\|D_2(f,0)\| + |g_1^* D_1(f,0)|) \geqslant \|S_3\|. \tag{130}$$

On the other hand, we see that

$$\begin{aligned} \|S_3\| &\geqslant |(g_1^*, x_1^*) S_3(g_1, 0)| \\ &= |x_1^* D_2(g_1, 0) + g_1^* D_1(g_1, 0)| = v(S_2) = 1. \end{aligned} \tag{131}$$

Therefore, $1 = \|S_3\| = \|S_3(g_1, 0)\|$ which proves Claim (5.1.22).

Finally, set $S_3 = (0, \tilde{T})$ for a suitable $\tilde{T}: L_1(\mu|_A) \oplus_1 X \to X$ and define the operator $T_1: L_1(\mu) \to X$ by

$$T_1(f) = T_0(f\chi_{A^c}) + \tilde{T}(P_A f, 0) \tag{132}$$

for every $f \in L_1(\mu)$. Then, we have

$$\|T_1(f)\| \leqslant \|T_0\| \|f\chi_{A^c}\| + \left\|\tilde{T}\right\| \|f\chi_A\| = \|f\| \tag{133}$$

for every $f \in L_1(\mu)$, so $\|T_1\| \leqslant 1$. Also,

$$\|T_1(J_A g_1)\| = \|S_3(g_1, 0)\| = \|S_3\| = 1, \tag{134}$$

so T_1 attains its norm at $J_A g_1 \in L_1(\mu)$, and

$$\|J_A g_1 - f_0\| = \|g_1 - g_0\| < \varepsilon. \quad (135)$$

We also have that, for any $f \in S_{L_1(\mu)}$,

$$\begin{aligned} \|T_0(f) - T_1(f)\| &= \|T_0(f\chi_A) - \tilde{T}(P_A f, 0)\| \\ &\leqslant \|T_0(J_A P_A f) - T_A(P_A f)\| \\ &\quad + \|T_A(P_A f) - \tilde{T}(P_A f, 0)\| \\ &\leqslant \|T_0 J_A - T_A\| + \|S_0 - S_3\| \\ &< \eta(\varepsilon_0) + 6\varepsilon. \end{aligned} \quad (136)$$

Hence $\|T_0 - T_1\| \leqslant \eta(\varepsilon_0) + 6\varepsilon < \varepsilon_0$.

Section (5.2): Banach Space:

The classical Bishop-Phelps theorem of 1961 [285] states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. A few years later, B. Bollobás [286] gave a sharper version of this theorem allowing to approximate at the same time a functional and a vector in which it almost attains the norm. We study the best possible approximation of this kind that one may have in each Banach space, measuring it by using two moduli which we define.

We first present the original result by Bollobás which nowadays is known as the Bishop-Phelps-Bollobás theorem. Given a (real or complex) Banach space X, we write B_X and S_X to denote the closed unit ball and the unit sphere of the space, and X^* denotes the (topological) dual of X. We will also use the notation

$$\Pi(X) := \{(x, x^*) \in X \times X^* : \| x \| = \|x^*\| = x^*(x) = 1\}$$

Theorem (5.2.1)[281]: (Bishop-Phelps-Bollobás theorem). (See [286].) Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| \leqslant \varepsilon^2/2$ for some $0 < \varepsilon < 1/2$. Then there exists $(y, y^*) \in \Pi(X)$ such that $\| x - y \| < \varepsilon + \varepsilon^2$ and $\|x^* - y^*\| \leqslant \varepsilon$.

The idea is that given $(x, x^*) \in S_X \times S_{X^*}$ such that $x^*(x) \sim 1$, there exist $y \in S_X$ close to x and $y^* \in S_{X^*}$ close to x^* for which $y^*(y) = 1$. This result has many applications, especially for the theory of numerical ranges, see [286],[287].

Our objective is to introduce two moduli which measure, for a given Banach space, what is the best possible Bollobás theorem in this space, that is, how close can be y to x and y^* to x^* in the result above depending on how close is $x^*(x)$ to 1. In the first modulus, we allow the vector and the functional to have norm less than or equal to one, whereas in the second modulus we only consider norm-one vectors and functionals.

Definitions (5.2.2)[281]: (Bishop-Phelps-Bollobás moduli). Let X be a Banach space. The Bishop-Phelps-Bollobás modulus of X is the function $\Phi_X : (0,2) \to \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\| x - y \| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

The spherical Bishop-Phelps-Bollobás modulus of X is the function $\Phi_X^S : (0,2) \to \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X^S(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\| x - y \| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

Evidently, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta)$, so any estimation from above for $\Phi_X(\delta)$ is also valid for $\Phi_X^S(\delta)$ and, viceversa, any estimation from below for $\Phi_X^S(\delta)$ is also valid for $\Phi_X(\delta)$.

Recall that the dual of a complex Banach space X is isometric (taking real parts) to the dual of the real subjacent space $X_{\mathbb{R}}$. Also, $\Pi(X)$ does not change if we consider X as a real

Banach space (indeed, if $(x,x^*) \in \Pi(X)$ then $x^* \in S_{X^*}$ and $x \in S_X$ satisfies $x^*(x)=1$ so, obviously, $\operatorname{Re} x^*(x)=1$ and $(x, \operatorname{Re} x^*) \in \Pi(X_{\mathbb{R}})$). Therefore, only the real structure of the space is playing a role in the above definitions. We prefer to develop the theory for real and complex spaces which, actually, does not suppose much more effort. This is mainly because for classical sequence or function spaces, the real space underlying the complex version of the space is not equal, in general, to the real version of the space. Unless otherwise is stated, the (arbitrary or concrete) spaces we are dealing with will be real or complex and the results work in both cases.

The following notations will help to the understanding and further use of

Let X be a Banach space and fix $0<\delta<2$. Write

$$A_X(\delta) := \{(x,x^*) \in B_X \times B_{X^*} : \operatorname{Re} x^*(x) > 1-\delta\},\ A_X^S(\delta): = \{(x,x^*) \in S_X \times S_{X^*} : \operatorname{Re} x^*(x) > 1-\delta\}.$$

It is clear that

$$\Phi_X(\delta) = \sup_{(x,x^*)\in A_X(\delta)} \inf_{(y,y^*)\in\Pi(X)} \max\{\| x-y \|, \|x^*-y^*\|\},$$

$$\Phi_X^S(\delta) = \sup_{(x,x^*)\in A_X^S(\delta)} \inf_{(y,y^*)\in\Pi(X)} \max\{\| x-y \|, \|x^*-y^*\|\}.$$

We denote $d_H(A,B)$ the Hausdorff distance between $A,B \subset X \times X^*$ associated to the ℓ_∞-distance dist in $_\infty \times X^*$, that is,

$$\operatorname{dist}_\infty\big((x,x^*),(y,y^*)\big) = \max\{\| x-y \|, \|x^*-y^*\|\}$$

for $(x,x^*),(y,y^*) \in X \times X^*$, and

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} \operatorname{dist}_\infty(a,b), \sup_{b\in B}\inf_{a\in A} \operatorname{dist}_\infty(a,b)\right\}$$

for $A,B \subset X \times X^*$. We clearly have that

$$\Phi_X(\delta) = d_H(A_X(\delta),\Pi(X)) \text{ and } \Phi_X^S(\delta) = d_H(A_X^S(\delta),\Pi(X))$$

for every $0<\delta<2$ (observe that $\Pi(X) \subset A_X(\delta)$ and $\Pi(X) \subset A_X^S(\delta)$ for every δ).

The following result is immediate.

Routine computations and the fact that the Hausdorff distance does not change if we take closure in one of the sets, provide the following observations.

Observe that the smaller are the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$, the better is the approximation on the space. It can be deduced from the Bishop-Phelps-Bollobás theorem that there is a common upper bound for $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ for all Banach spaces X. In the next section we present the best possible upper bound. We will show that

$$\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}\ \big(0<\delta<2, X \text{ Banach space}\big). \tag{137}$$

This follows from a result by R. Phelps [294]. A version for $\Phi_X^S(\delta)$ for small's can be also deduced from the BrøndstedRockafellar variational principle [295], as claimed in [288]. The sharpness of (137) can be verified by considering the real space $X=\ell_\infty^{(2)}$.

We prove that for every Banach space X, the moduli $\Phi_X(\delta)$ and $\Phi_X^S(\delta)$ are continuous in δ. We prove that $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$ and $\Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$. Finally, we show that $\Phi_X(\delta)=\sqrt{2\delta}$ if and only if $\Phi_X^S(\delta)=\sqrt{2\delta}$.

Examples of spaces for which the two moduli are computed are presented. Among other results, the moduli of $\mathbb{R}$ and of every real or complex Hilbert space of (real)-dimension greater than one are calculated, and there are presented a number of spaces for which the value of both moduli are $\sqrt{2\delta}$ (i.e. the maximal possible value) for small δ 's: namely c_0, ℓ_1 and, more in general, $L_1(\mu)$, $C_0(L)$, unital C^*-algebras with non-trivial centralizer...

The main result states that if a Banach space X satisfies $\Phi_X(\delta_0) = \sqrt{2\delta_0}$ (equivalently, $\Phi_X^S(\delta_0) = \sqrt{2\delta_0}$) for some $\delta_0 \in (0,1/2)$, then X contains almost isometric copies of the real space $\ell_\infty^{(2)}$. We provide, for every $\delta \in (0,1/2)$, an example of a three-dimensional real space Z containing an isometric copy of $\ell_\infty^{(2)}$ for which $\Phi_Z(\delta) < \sqrt{2\delta}$.

Our first result is the promised best upper bound of the Bishop-Phelps-Bollobás moduli.

We deduce the above result from [294], which was stated for general bounded convex sets on real Banach spaces. Particularizing the result to the case of the unit ball of a Banach space, using a routine argument to change non-strict inequalities to strict inequalities, and taking into account that the dual of a complex Banach space is isometric (taking real parts) to the dual of the real subjacent space, we get the following result.

Proposition (5.2.3)[281]: (Particular case of [294]). Let X be Banach space. Suppose that $z^* \in S_{X^*}, z \in B_X$ and $\eta > 0$ are given such that $\operatorname{Re} z^*(z) > 1 - \eta$. Then, for any $k \in (0,1)$ there exist $\tilde{y}^* \in X^*$ and $\tilde{y} \in S_X$ such that

$$\|\tilde{y}^*\| = \tilde{y}^*(\tilde{y}), \ \| z - \tilde{y} \| < \frac{\eta}{k}, \ \|z^* - \tilde{y}^*\| < k$$

Theorem (5.2.4)[281]: For every Banach space X and every $\delta \in (0,2), \Phi_X(\delta) \leqslant \sqrt{2\delta}$ and so, $\Phi_X^S(\delta) \leqslant \sqrt{2\delta}$

Proof. We have to show that given $(x, x^*) \in B_X \times B_{X^*}$ with Re $x^*(x) > 1 - \delta$, there exists $(y, y^*) \in \Pi(X)$ such that $\| x - y \| < \sqrt{2\delta}$ and $\|x^* - y^*\| < \sqrt{2\delta}$. We first prove the case of $\delta \in (0,1)$. In this case,

$$0 < 1 - \delta < \|x^*\| \leqslant 1,$$

so, if we write $\eta = \frac{\|x^*\| - 1 + \delta}{\|x^*\|} > 0, z^* = x^*/\|x^*\|$ and $z = x$, one has

$$\operatorname{Re} z^*(z) > 1 - \eta.$$

Next, we consider $k = \eta/\sqrt{2\delta}$ and claim that $0 < k < 1$. Indeed, as the function

$$\varphi(t) = \frac{t - 1 + \delta}{\sqrt{2\delta} t} \ (t \in \mathbb{R}^+) \tag{138}$$

is strictly increasing, $k = \varphi(\|x^*\|)$ and $1 - \delta < \|x^*\| \leqslant 1$, we have that

$$0 = \varphi(1 - \delta) < k \leqslant \varphi(137) = \frac{\sqrt{\delta}}{\sqrt{2}} < 1,$$

as desired. Therefore, we apply Proposition (5.2.3) with $z^* \in S_{X^*}, z \in B_X, \eta > 0$ and $0 < k < 1$ to obtain $\tilde{y}^* \in X^*$ and $\tilde{y}_X \in S_X$ satisfying

$$\|\tilde{y}^*\| = \tilde{y}^*(\tilde{y}), \ \| z - \tilde{y} \| < \frac{\eta}{k} = \sqrt{2\delta}, \ \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| < k = \frac{\|x^*\| - 1 + \delta}{\|x^*\|\sqrt{2\delta}}.$$

As $k < 1$, we get $\tilde{y}^* \neq 0$ and we write $y^* = \frac{\tilde{y}^*}{\|\tilde{y}^*\|}, y = \tilde{y}$, to get that $(y, y^*) \in \Pi(X)$. We already have that $\| x - y \| < \sqrt{2\delta}$. On the other hand, we have

$$\|x^* - y^*\| = \left\| x^* - \frac{\tilde{y}^*}{\|\tilde{y}^*\|} \right\| \leqslant \|x^* - \|x^*\| \tilde{y}^* \| + \| \| x^* \left\| \tilde{y}^* - \frac{\tilde{y}^*}{\|\tilde{y}^*\|} \right\|$$

$$\leqslant \|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + |\|x^*\|\|\tilde{y}^*\| - 1|$$
$$\leqslant \|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + |\|x^*\|\|\tilde{y}^*\| - \|x^*\||+| 1 - \|x^*\|$$
$$\leqslant \|x^*\| \left[\left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + |\|\tilde{y}^*\| - 1|\right] + 1 - \|x^*\|$$
$$\leqslant 2\|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + 1 - \|x^*\|$$
$$< \frac{2}{\sqrt{2\delta}}(\|x^*\| - 1 + \delta) + 1 - \|x^*\|.$$

Now, as the function

$$\gamma(t) = \frac{2}{\sqrt{2\delta}}(t - 1 + \delta) + 1 - t \ (t \in [0,1])$$

is strictly increasing (for this we only need $0 < \delta < 2$), we get $\gamma(\|x^*\|) \leqslant \gamma(137) = \frac{2\delta}{\sqrt{2\delta}} = \sqrt{2\delta}$. It follows that $\|x^* - y^*\| < \sqrt{2\delta}$, as desired.

Let us now prove the case when $\delta \in [1,2)$. Here, it can be routinely verified that

$$\frac{\delta - 1}{\sqrt{2\delta} - 1} < \sqrt{2\delta} - 1$$

so, writing

$$\psi(\delta) = \frac{1}{2}\left(\frac{\delta - 1}{\sqrt{2\delta} - 1} + \sqrt{2\delta} - 1\right)$$

we get

$$\frac{\delta - 1}{\sqrt{2\delta} - 1} < \psi(\delta) < \sqrt{2\delta} - 1 \ (\delta \in [1,2)). \tag{139}$$

Now, we distinguish two situations. First suppose that $\|x^*\| \leqslant \psi(\delta)$. Then, we take any $y \in S_X$ such that $\| x - y \| \leqslant 1$ and take $y^* \in S_{X^*}$ such that $y^*(y) = 1$. Then, $(y, y^*) \in \Pi(X)$, $\| x - y \| \leqslant 1 < \sqrt{2\delta}$ and

$$\|x^* - y^*\| \leqslant 1 + \|x^*\| \leqslant 1 + \psi(\delta) < \sqrt{2\delta}$$

by (139). Otherwise, suppose $\|x^*\| > \psi(\delta)$. We then write $\eta = \frac{\|x^*\| - 1 + \delta}{\|x^*\|} > 0$ and $k = \eta/\sqrt{2\delta}$ as in the previous case, and we show that $k < 1$. This is trivial for the case $\delta = 1$ and for $\delta > 1$, we use that the function φ defined in (138) is now strictly decreasing to get that

$$k = \varphi(\|x^*\|) < \varphi(\psi(\delta)) < \varphi\left(\frac{\delta - 1}{\sqrt{2\delta} - 1}\right) = 1.$$

Then, the rest of the proof follows the same lines of the case when $\delta \in (0,1)$ since this hypothesis is no longer used.

Notice that the above proof is much simpler if we restrict to $x^* \in S_{X^*}$ (in particular, to the spherical modulus $\Phi_X^S(\delta)$), but the result for non-unital functionals is stronger. Actually, the following stronger version can be deduced by modifying the selection of k in the proof of Theorem (5.2.4).

We observe that, given $0 < \theta < 1$, the hypothesis above is not empty only when $1 - \theta < \delta$. On the other hand, in the proof it is sufficient to consider only the case of $\delta < 1 + \theta$. Otherwise, the evident inequality $\operatorname{Re} x^*(x) > -\theta = 1 - (1 + \theta)$ implies that there is a pair $(y, y^*) \in \Pi(X)$ satisfying $\| x - y \| < \sqrt{2(1+\theta)}$ and $\|x^* - y^*\| < \sqrt{2(1+\theta)}$. Hence the statement of our remark holds true with $\rho := \sqrt{2\delta} - \sqrt{2(1+\theta)}$.

Next, we rewrite Theorem (5.2.4) in two equivalent ways.

Corollary (5.2.5)[281]: Let X be a Banach space.

(a) Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_X *$ satisfy

$$\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2$$

Then, there exists $(y, y^*) \in \Pi(X)$ such that

$$\| x - y \| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

(b) Let $0 < \delta < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy

$$\operatorname{Re} x^*(x) > 1 - \delta.$$

Then, there exists $(y, y^*) \in \Pi(X)$ such that

$$\| x - y \| < \sqrt{2\delta} \text{ and } \|x^* - y^*\| < \sqrt{2\delta}.$$

As the last result, we present an example of a Banach space for which the estimate in Theorem (5.2.4) is sharp.

Example (5.2.6)[281]: Let X be the real space $\ell_\infty^{(2)}$. Then, $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for all $\delta \in (0,2)$.

Proof. Fix $0 < \delta < 2$. We consider

$$z = (1 - \sqrt{2\delta}, 1) \in S_X \text{ and } z^* = \left(\frac{\sqrt{2\delta}}{2}, 1 - \frac{\sqrt{2\delta}}{2}\right) \in S_{X^*},$$

and observe that $z^*(z) = 1 - \delta$. Now, suppose we may find $(y, y^*) \in \Pi(X)$ such that $\| z - y \| < \sqrt{2\delta}$ and $\|z^* - y^*\| < \sqrt{2\delta}$. By the shape of B_X, we only have two possibilities: either y is an extreme point of B_X or y^* is an extreme point of B_{X^*} (this is actually true for all two-dimensional real spaces). Suppose first that y is an extreme point of B_X, which has the form $y = (a, b)$ with $a, b \in \{-1,1\}$. As

$$\| z - y \| = \mathrm{m} \ \{|1 - \sqrt{2\delta} - a|, |1 - b|\} < \sqrt{2\delta},$$

we are forced to have $b = 1$ and $a = -1$. Now, we have $y^* = (-t, 1 - t)$ for some $0 \leqslant t \leqslant 1$ and

$$\|z^* - y^*\| = \frac{\sqrt{2\delta}}{2} + t + \left|t - \frac{\sqrt{2\delta}}{2}\right| = \mathrm{m} \ \{\sqrt{2\delta}, 2t\} \geqslant \sqrt{2\delta},$$

a contradiction. On the other hand, if y^* is an extreme point of B_{X^*}, then either $y^* = (a, 0)$ or $y^* = (0, b)$ for suitable $a, b \in \{-1,1\}$. In the first case, as

$$\|z^* - y^*\| = \left|\frac{\sqrt{2\delta}}{2} - a\right| + 1 - \frac{\sqrt{2\delta}}{2} < \sqrt{2\delta},$$

we are forced to have $a = 1$ and so, $y = (1, s)$ for suitable $s \in [-1,1]$. But then $\| z - y \| \geqslant \sqrt{2\delta}$, which is impossible. In case $y^* = (0, b)$ with $b = \pm 1$, we have

$$\|z^* - y^*\| = \frac{\sqrt{2\delta}}{2} + \left|1 - \frac{\sqrt{2\delta}}{2} - b\right| < \sqrt{2\delta},$$

so $b=-1$ and therefore, $y=(s,-1)$ for suitable $s\in[-1,1]$, giving $\| z-y \|\geqslant 2$, a contradiction.

Our first result is the continuity of the Bishop-Phelps-Bollobás moduli.

We need the following three lemmas which could be of independent interest.

Lemma (5.2.7)[281]: For every pair $(x_0,x_0^*)\in B_X\times B_{X^*}$ there is a pair $(y,y^*)\in\Pi(X)$ with
$$\operatorname{Re}[y^*(x_0)+x_0^*(y)]\geqslant 0.$$
Moreover, if actually $\operatorname{Re} x_0^*(x_0)>0$ then $(y,y^*)\in\Pi(X)$ can be selected to satisfy
$$\operatorname{Re}[y^*(x_0)+x_0^*(y)]\geqslant 2\sqrt{\operatorname{Re}\big(x_0^*(x_0)\big)}$$
Proof. (a) Take $y_0\in S_X\cap\ker x_0^*$ and let y_0^* be a supporting functional at y_0. Then
$$\operatorname{Re}[y_0^*(x_0)+x_0^*(y_0)]=\operatorname{Re} y_0^*(x_0)$$
If the right-hand side is positive we can take $y=y_0, y^*=y_0^*$, in the opposite case take $y=-y_0, y^*=-y_0^*$.

(b) Take $y=\frac{x_0}{\|x_0\|}$ and let y^* be a supporting functional at y. Then, since for a fixed $a>0$ the minimum of $f(t):=t+\frac{a}{t}$ for $t>0$ equals $2\sqrt{a}$, we get
$$\operatorname{Re}[y^*(x_0)+x_0^*(y)]=\|x_0\|+\frac{1}{\|x_0\|}\operatorname{Re} x_0^*(x_0)\geqslant 2\sqrt{\operatorname{Re} x_0^*(x_0)}$$
as desired.

The above lemma allows us to prove the following result which we will use to show the continuity of the Bishop-Phelps-Bollobás modulus.

Lemma (5.2.8)[281]: Let X be a Banach space. Suppose $(x_0,x_0^*)\in A_X(\delta_0)$ with $0<\delta<\delta_0<2$. Then:

Case 1: If $\delta,\delta_0\in]0,1]$ then
$$\operatorname{dist}_\infty\big((x_0,x_0^*),A_X(\delta)\big)\leqslant 2\frac{\sqrt{1-\delta}-\sqrt{1-\delta_0}}{1-\sqrt{1-\delta_0}}.$$
Case 2: If $\delta,\delta_0\in[1,2)$ then
$$\operatorname{dist}_\infty\big((x_0,x_0^*),A_X(\delta)\big)\leqslant 2\frac{2-\delta_0}{\delta_0}\cdot\frac{\delta_0-\delta}{\delta_0-1+\sqrt{1-2\delta+\delta\delta_0}}$$
Proof. Denote $t=\operatorname{Re} x_0^*(x_0)$. Let $(y,y^*)\in\Pi(X)$ be from the previous lemma (in Case 1 we use part 2 of the lemma, in Case 2 we use part 1). For every $\lambda\in[0,1]$ we define $x_\lambda=(1-\lambda)x_0+\lambda y$ and $x_\lambda^*=(1-\lambda)x_0^*+\lambda y^*$. Both x_λ and x_λ^* belong to corresponding balls, and $\operatorname{dist}_\infty\big((x_0,x_0^*),(x_\lambda,x_\lambda^*)\big)\leqslant 2\lambda$. We have
$$\operatorname{Re} x_\lambda^*(x_\lambda)=(1-\lambda)^2t+\lambda(1-\lambda)\operatorname{Re}[y^*(x_0)+x_0^*(y)]+\lambda^2, \tag{140}$$
so in Case 1
$$\operatorname{Re} x_\lambda^*(x_\lambda)\geqslant(1-\lambda)^2t+2\lambda(1-\lambda)\sqrt{t}+\lambda^2=((1-\lambda)\sqrt{t}+\lambda)^2.$$
Now we are looking for a possibly small value of λ, for which $(x_\lambda,x_\lambda^*)\in A_X(\delta)$. If $\delta\geqslant 1-t$, the value $\lambda=0$ is already OK and $\operatorname{dist}_\infty\big((x_0,x_0^*),A_X(\delta)\big)=0$. If $0<\delta<1-t$ then the positive solution in λ of the equation $((1-\lambda)\sqrt{t}+\lambda)^2=1-\delta$ is
$$\lambda_t=\frac{\sqrt{1-\delta}-\sqrt{t}}{1-\sqrt{t}}.$$

Evidently, $\lambda_t \in [0,1]$, so $(x_{\lambda_t}, x_{\lambda_t}^*) \in A_X(\delta)$. Since λ_t decreases in t,

$$\operatorname{dist}_\infty\big((x_0, x_0^*), A_X(\delta)\big) \leqslant 2\lambda_t \leqslant 2\lambda_{1-\delta_0} = 2\frac{\sqrt{1-\delta} - \sqrt{1-\delta_0}}{1-\sqrt{1-\delta_0}}.$$

This completes the proof of Case 1 .

In Case 2 we may assume $t \leqslant 1-\delta$ (otherwise the corresponding distance is 0 and the job is done), so $t \leqslant 0$. By part 1 of the previous lemma and (140)

$$\operatorname{Re} x_\lambda^*(x_\lambda) \geqslant (1-\lambda)^2 t + \lambda^2,$$

so we are solving in λ the equation

$$(1-\lambda)^2 t + \lambda^2 - 1 + \delta = 0, \text{ i.e. } (1+t)\lambda^2 - 2t\lambda + (t-1+\delta) = 0$$

The discriminant of this equation is $D = -t\delta - \delta + 1$. Note that $D \geqslant -(1-\delta)\delta - \delta + 1 = (1-\delta)^2 \geqslant 0$ and $t - 1 + \delta \leqslant 0$, so there is a positive solution of our equation given by

$$\lambda_t = \frac{1}{1+t}(t + \sqrt{D}) = \frac{1}{1+t}(t + \sqrt{1 - t\delta - \delta}).$$

This λ_t decreases in t, so

$$\lambda_t \leqslant \lambda_{1-\delta_0} = \frac{1}{\delta_0}\left(1 - \delta_0 + \sqrt{1 - 2\delta + \delta\delta_0}\right) = \frac{2+\delta_0}{\delta_0} \cdot \frac{\delta_0 - \delta}{\delta_0 - 1 + \sqrt{1 - 2\delta + \delta\delta_0}}$$

which finishes the proof.

For the continuity of the spherical modulus, we need the following result.

Lemma (5.2.9)[281]: Let X be a Banach space. Suppose $(x_0, x_0^*) \in A_X^S(\delta_0)$ with $0 < \delta < \delta_0 < 2$.

Case 1: If $\delta < 1$, then

$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X^S(\delta)\right) \leqslant \frac{4(\delta_0 - \delta)}{\delta_0}.$$

Case 2: If $\delta \in [1,2)$ and $2 - \sqrt{2-\delta_0} < \delta < \delta_0$, then

$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X^S(\delta)\right) \leqslant \frac{2(\delta_0 - \delta)}{2-\delta}.$$

Proof. Let us start with Case 1. Fix $\xi \in (0,\delta)$. As $\|x_0^*\| = 1$, we may find $y_\xi \in S_X$ satisfying $x_0^*(y_\xi) > 1 - \xi$. For every $\lambda \in [0,1]$ we define

$$x(\lambda, \xi) = \lambda x_0 + (1-\lambda) y_\xi.$$

Consider $\lambda_\xi = \frac{\delta - \xi}{\delta_0 - \xi} \in [0,1]$ and write $x_\xi = x(\lambda_\xi, \xi)$. A straightforward verification shows that

$$\operatorname{Re} x_0^*(x_\xi) > 1 - \delta$$

and so, as $1 - \delta \geqslant 0$, we have that $x_\xi \neq 0$ and also that

$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right) > 1 - \delta.$$

Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right) \in A_X^S(\delta)$. We have

$$\begin{aligned}\left\|x_0 - \frac{x_\xi}{\|x_\xi\|}\right\| &\leqslant \|x_0 - x_\xi\| + \left\|x_\xi - \frac{x_\xi}{\|x_\xi\|}\right\| \leqslant 2\left(\frac{\delta_0 - \delta}{\delta_0 - \xi}\right) + \big|\|x_\xi\| - 1\big| \\ &\leqslant 2\left(\frac{\delta_0 - \delta}{\delta_0 - \xi}\right) + \big|\|x_\xi\| - \|x_0\|\big| \leqslant 2\left(\frac{\delta_0 - \delta}{\delta_0 - \xi}\right) + \|x_\xi - x_0\| \leqslant 4\left(\frac{\delta_0 - \delta}{\delta_0 - \xi}\right).\end{aligned}$$

We get the result by just letting $\xi \to 0$.

Let us prove Case 2. If $\operatorname{Re} x_0^*(x_0) > 1-\delta$, then the proof is done. Suppose that
$$1-\delta \geqslant \operatorname{Re} x_0^*(x_0) > 1-\delta_0.$$
Fix $\in \left(0, \min\left\{2-\delta_0, \frac{4\delta-2-\delta_0-\delta^2}{\delta-1}\right\}\right)$ (observe that $\frac{4\delta-2-\delta_0-\delta^2}{\delta-1} > 0$ by the conditions on δ). As $\|x_0^*\| = 1$, we may find $y_\xi \in S_X$ satisfying $x_0^*(y_\xi) > 1-\xi$. Now, we consider
$$\lambda_\xi = \frac{\delta_0-\delta}{2-\delta-\xi} \text{ and } x_\xi = x_0 + \lambda_\xi y_\xi.$$
Notice that $\lambda_\xi \in (0,1)$ (since $\delta < \delta_0$ and $\xi < 2-\delta_0$) and
$$\|x_\xi\| \geqslant \|x_0\| - \lambda\|y_\xi\| = 1 - \lambda\xi > 0.$$
Also, observe that
$$\operatorname{Re} x_0^*(x_\xi) \leqslant 1-\delta+\lambda_\xi = \frac{(1-\delta)(2-\delta-\xi)+\delta_0-\delta}{2-\delta-\xi}$$
so, $\operatorname{Re} x_0^*(x_\xi) \leqslant 0$ since $\xi \leqslant \frac{4\delta-2-\delta_0-\delta^2}{\delta-1}$. Now,
$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right) \geqslant \operatorname{Re} x_0^*\left(\frac{x_\xi}{1-\lambda_\xi}\right) > \frac{1-\delta_0+\lambda_\xi(1-\xi)}{1-\lambda_\xi} = 1-\delta.$$
Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right) \in A_X^S(\delta)$. We have
$$\begin{aligned}\left\|x_0 - \frac{x_\xi}{\|x_\xi\|}\right\| &\leqslant \|x_0 - x_\xi\| + \left\|x_\xi - \frac{x_\xi}{\|x_\xi\|}\right\| \leqslant \frac{\delta_0-\delta}{2-\delta-\xi} + \big|\|x_\xi\| - 1\big| \\ &\leqslant \frac{\delta_0-\delta}{2-\delta-\xi} + \big|\|x_\xi\| - \|x_0\|\big| \leqslant \frac{\delta_0-\delta}{2-\delta-\xi} + \|x_\xi - x_0\| \leqslant 2\left(\frac{\delta_0-\delta}{2-\delta-\xi}\right).\end{aligned}$$
Consequently, letting $\xi \to 0$, we get
$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X^S(\delta)\right) \leqslant \frac{2(\delta_0-\delta)}{2-\delta}$$
as we desired.

Proposition (5.2.10)[281]: Let X be a Banach space. Then, the functions
$$\delta \mapsto \Phi_X(\delta) \text{ and } \delta \mapsto \Phi_X^S(\delta)$$
are continuous in $(0,2)$.

Proof. Let us give the proof for $\Phi_X(\delta)$. Observe that for $\delta_1, \delta_2 \in (0,2)$ with $\delta_1 < \delta_2$, one has
$$\begin{aligned}0 < \Phi_X(\delta_2) - \Phi_X(\delta_1) &= d_H(A_X(\delta_2), \Pi(X)) - d_H(A_X(\delta_1), \Pi(X)) \\ &\leqslant d_H\big(A_X(\delta_2), A_X(\delta_1)\big).\end{aligned}$$
Now, the continuity follows routinely from Lemma (5.2.8).

An analogous argument allows to prove the continuity of $\Phi_X^S(\delta)$ from Lemma (5.2.9).

The following lemma will be used to show that the approximation in the space is not worse than the approximation in the dual. It is actually an easy application of the Principle of Local Reflexivity.

Lemma (5.2.11)[281]: For $\varepsilon > 0$, let $(x, x^*) \in B_X \times B_{X^*}$ and let $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(X^*)$ such that
$$\|x^* - \tilde{y}^*\| < \varepsilon \text{ and } \|x - \tilde{y}^{**}\| < \varepsilon.$$
Then there is a pair $(y, y^*) \in \Pi(X)$ such that
$$\| x - y \| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$

Proof. First chose $\varepsilon' < \varepsilon$ such that still
$$\|x^* - \tilde{y}^*\| < \varepsilon' \text{ and } \|x - \tilde{y}^{**}\| < \varepsilon'$$
Now, we consider $\xi > 0$ such that
$$(1+\xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon$$
and use the Principle of Local Reflexivity (see [282], for instance) to get an operator T: Lin $\{x, \tilde{y}^{**}\} \to X$ satisfying
$$\| T \|, \|T^{-1}\| \leqslant 1+\xi,\ T(x) = x,\ \tilde{y}^*\left(T(\tilde{y}^{**})\right) = y^{**}(\tilde{y}^*) = 1.$$
Next, we consider $\tilde{x} = \frac{T(\tilde{y}^{**})}{\|T(\tilde{y}^{**})\|} \in S_X$ and $\tilde{x}^* = \tilde{y}^* \in S_{X^*}$, observe that
$$\operatorname{Re} \tilde{x}^*(\tilde{x}) > \frac{1}{1+\xi} = 1 - \frac{\xi}{1+\xi},$$
and we use Corollary (5.2.5) to get $(y, y^*) \in \Pi(X)$ satisfying that
$$\| \tilde{x} - y \| < \sqrt{\frac{2\xi}{1+\xi}} \text{ and } \|\tilde{x}^* - y^*\| < \sqrt{\frac{2\xi}{1+\xi}}.$$
Let us show that $(y, y^*) \in \Pi(X)$ fulfill our requirements.
$$\| x - y \| \leqslant \|T(x) - T(\tilde{y}^{**})\| + \|T(\tilde{y}^{**}) - \tilde{x}\| + \| \tilde{x} - y \|$$
$$< (1+\xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon$$
and, analogously,
$$\|x^* - y^*\| \leqslant \|x^* - \tilde{y}^*\| + \|\tilde{y}^* - y^*\| < \varepsilon' + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon,$$
getting the desired result.

Proposition (5.2.12)[281]: Let X be a Banach space. Then
$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta) \text{ and } \Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$$
for every $\delta \in (0,2)$.

Proof. The proof is the same for both moduli, so we are only giving the case of $\Phi_X(\delta)$. Fix $\delta \in (0,2)$. We consider any $\varepsilon > 0$ such that $\Phi_{X^*}(\delta) < \varepsilon$ and for a given $(x, x^*) \in A_X(\delta)$ consider $(x^*, x) \in A_{X^*}(\delta)$ (we identify X as a subspace of X^{**}) and so we may find $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(Y^*)$ such that
$$\|x^* - \tilde{y}^*\| < \varepsilon \text{ and } \|x - \tilde{y}^{**}\| < \varepsilon.$$
From Lemma (5.2.11), we find $(y, y^*) \in \Pi(X)$ such that
$$\| x - y \| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon.$$
This means that $\Phi_X(\delta) \leqslant \varepsilon$ and, therefore, $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$, as desired.

We do not know whether the inequalities in Proposition (5.2.12) can be strict. Of course, this cannot be the case when the space is reflexive.

Corollary (5.2.13)[281]: For every reflexive Banach space X, one has $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ and $\Phi_X^S(\delta) = \Phi_{X^*}^S(\delta)$ for every $0 < \delta < 2$.

Our last result states that when the Bishop-Phelps-Bollobás modulus is the worst possible, then the spherical Bishop-Phelps-Bollobás modulus is also the worst possible.

Proposition (5.2.14)[281]: Let X be a Banach space. For every $\delta \in (0,2)$, the condition $\Phi_X(\delta) = \sqrt{2\delta}$ is equivalent to the condition $\Phi_X^S(\delta) = \sqrt{2\delta}$ Proof. Since $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, the implication $\left[\Phi_X^S(\delta) = \sqrt{2\delta}\right] \Rightarrow \left[\Phi_X(\delta) = \sqrt{2\delta}\right]$ is evident. Let us prove the inverse implication. Let $\Phi_X(\delta) = \sqrt{2\delta}$. Then there is a sequence of pairs $(x_n, x_n^*) \in B_X \times B_{X^*}$ such that $\operatorname{Re} x_n^*(x_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$ we have

$$\|x_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} \text{ or } \|x_n^* - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n}.$$

An application gives us that $\|x_n^*\| \to 1$ as $n \to \infty$. As the duality argument given in Lemma (5.2.11) implies the dual version, we also have $\|x_n\| \to 1$ as $n \to \infty$. Denote $\tilde{x}_n = \frac{x_n}{\|x_n\|}$, $\tilde{x}_n^* = \frac{x_n^*}{\|x_n^*\|}$. In the case when $\delta \in (0,1]$, we have $\operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$

$$\|\tilde{x}_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|x_n - \tilde{x}_n\| \text{ or } \|\tilde{x}_n^* - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|\tilde{x}_n^* - x_n^*\|.$$

Since the right-hand sides of the above inequalities go to $\sqrt{2\delta}$, we get the condition $\Phi_X^S(\delta) = \sqrt{2\delta}$.

In the case of $\delta \in (1,2)$, we no longer know that $\operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) > 1 - \delta$, but what we do know is that $\liminf \operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) \geqslant 1 - \delta$, and that gives us the desired condition $\Phi_X^S(\delta) = \sqrt{2\delta}$ thanks to the continuity of the spherical modulus (Proposition (5.2.10)).

We start with the simplest example of $X = \mathbb{R}$.

Example (5.2.15)[281]: $\Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leqslant 1, \\ \sqrt{\delta - 1} + 1 & \text{if } 1 < \delta < 2, \end{cases}$ $\Phi_{\mathbb{R}}^S(\delta) = 0$ for every $\delta \in (0,2)$.

Proof. We first fix $\delta \in (0,1]$. First observe that taking $x = 1 - \delta, x^* = 1$, it is evident that $\Phi_{\mathbb{R}}(\delta) \geqslant \delta$. For the other inequality, we fix $x, x^* \in [-1,1]$ with $x^* x > 1 - \delta$. Then, x and x^* have the same sign and we have that $|x| > 1 - \delta$ and $|x^*| > 1 - \delta$. Indeed, if $|x| < 1 - \delta$, as $|x^*| \leqslant 1$, one has $x^* x = |x^* x| < 1 - \delta$, a contradiction; the other inequality follows in the same manner. Finally, one deduces that $|x - \operatorname{sign}(x)| < \delta$ and $|x^* - \operatorname{sign}(x^*)| < \delta$, as desired.

Second, fix $\delta \in (1,2)$. On the one hand, taking $x = \sqrt{\delta - 1}, x^* = -\sqrt{\delta - 1}$, one has $x^* x = 1 - \delta$. As $|x + 1| = \sqrt{\delta - 1} + 1$ and $|x^* - 1| = \sqrt{\delta - 1} + 1$, it follows that $\Phi_{\mathbb{R}}(\delta) \geqslant \sqrt{\delta - 1} + 1$. For the other inequality, we fix $x, x^* \in [-1,1]$ with $x^* x > 1 - \delta$. If x and x^* have the same sign, which we may and do suppose positive, then $|x - 1| \leqslant 1 < \delta$ and $|x^* - 1| \leqslant 1 < \delta$ and the same is true if one of them is null. Therefore, to prove the last case we may and do suppose that $x > 0$ and $x^* < 0$. Now, if we suppose, for the sake of contradiction, that

$$|x - (-1)| \geqslant \sqrt{\delta - 1} + 1 \text{ and } |x^* - 1| \geqslant \sqrt{\delta - 1} + 1,$$

we get $x \geqslant \sqrt{\delta - 1}$ and $-x^* \geqslant \sqrt{\delta - 1}$, so $-x^* x \geqslant \delta - 1$ or, equivalently, $x^* x \leqslant 1 - \delta$, a contradiction. Therefore, either $|x - (-1)| < \sqrt{\delta - 1} + 1$ and $|x^* - (-1)| < 1 < \sqrt{\delta - 1} + 1$ or $|x^* - 1| < \sqrt{\delta - 1} + 1$ and $|x - 1| < 1 < \sqrt{\delta - 1} + 1$.

The result for $\Phi_{\mathbb{R}}^S$ is an obvious consequence of the fact that $S_{\mathbb{R}} = \{-1,1\}$.

Let us observe that the above proof gives actually a lower bound for $\Phi_X(\delta)$ for every Banach space X when $\delta \in (0,1]$.

We do not know a result giving a lower bound for $\Phi_X(\delta)$ when $\delta > 1$, outside of the trivial one $\Phi_X(\delta) \geqslant 1$. Also, we do not know if the lower bound for the behavior of $\Phi_X(\delta)$ in a neighborhood of 0 given in the remark above can be improved for Banach spaces of dimension greater than or equal to two.

We next calculate the moduli of a Hilbert space of (real) dimension greater than one.

Example (5.2.16)[281]: Let H be a Hilbert space of dimension over $\mathbb{R}$ greater than or equal to two. Then:

(a) $\Phi_H^S(\delta) = \sqrt{2-\sqrt{4-2\delta}}$ for every $\delta \in (0,2)$.

(b) For $\delta \in (0,1], \Phi_H(\delta) = \max\{\delta, \sqrt{2-\sqrt{4-2\delta}}\}$. For $\delta \in (1,2), \Phi_H(\delta) = \sqrt{\delta}$.

Proof. As we commented in the introduction, both Φ_H and Φ_H^S only depend on the real structure of the space, so we may and do suppose that H is a real Hilbert space of dimension greater than or equal to 2. Let us also recall that H^* identifies with H and that the action of a vector $y \in H$ on a vector $x \in H$ is nothing but their inner product denoted by $\langle x, y\rangle$. In particular,

$$\Pi(H) = \{(z,z) \in S_H \times S_H\}.$$

Therefore, for every $\delta \in (0,2), \Phi_H(\delta)$ (resp. $\Phi_H^S(\delta)$) is the infimum of those $\varepsilon > 0$ such that whenever $x, y \in B_H$ (resp. $x, y \in S_H$) satisfies $\langle x, y\rangle \geqslant 1-\delta$, there is $z \in S_H$ such that $\| x - z \| \leqslant \varepsilon$ and $\| y - z \| \leqslant \varepsilon$.

We will use the following (easy) claim in both the proofs of (a) and (b).

Claim (5.2.17)[281]:. Given $x, y \in S_H$ with $x + y \neq 0$, write $z = \frac{x+y}{\|x+y\|}$ to denote the normalized midpoint. Then

$$\| x - z \| = \| y - z \| = \sqrt{2-\sqrt{2+2\langle x, y\rangle}}.$$

Indeed, we have $\| x - z \|^2 = 2 - 2\langle x, z\rangle$ and

$$2\langle x, z\rangle = \frac{2\langle x, x+y\rangle}{\| x+y \|} = \frac{2+2\langle x, y\rangle}{\sqrt{2+2\langle x, y\rangle}},$$

giving $\| x - z \| = \sqrt{2-\sqrt{2+2\langle x, y\rangle}}$, being the other equality true by symmetry.

(a) We first prove that $\Phi_H^S(\delta) \leqslant \sqrt{2-\sqrt{4-2\delta}}$. Take $x, y \in S_H$ with $\langle x, y\rangle \geqslant 1-\delta$(so $x + y \neq 0$), consider $z = \frac{x+y}{\|x+y\|} \in S_H$ and use the claim to get that

$$\| x - z \| = \| y - z \| = \sqrt{2-\sqrt{2+2\langle x, y\rangle}} \leqslant \sqrt{2-\sqrt{4-2\delta}}.$$

To get the other inequality, we fix an ortonormal basis $\{e_1, e_2, \ldots\}$ of H, consider

$$x = \sqrt{1-\delta/2}e_1 + \sqrt{\delta/2}e_2 \in S_H \text{ and } y = \sqrt{1-\delta/2}e_1 - \sqrt{\delta/2}e_2 \in S_H$$

and observe that $\langle x, y\rangle = 1-\delta$. Now, given $z \in S_H$, we write $z_1 = \langle z, e_1\rangle, z_2 = \langle z, e_2\rangle$, and observe that

$\max\{\| z - x \|^2, \| z - y \|^2\}$

$$= \max_{\pm}\left\{\left|z_1-\sqrt{1-\delta/2}\right|^2+\left|z_2\pm\sqrt{\delta/2}\right|^2+1-z_1^2-z_2^2\right\}$$
$$= z_1^2+1-\delta/2-2z_1\sqrt{1-\delta/2}+\max_{\pm}\left|z_2\pm\sqrt{\delta/2}\right|^2+1-z_1^2-z_2^2$$
$$= 2-2z_1\sqrt{1-\delta/2}+2|z_2|\sqrt{\delta/2}\geqslant 2-2\sqrt{1-\delta/2}.$$

It follows that $\Phi_H^S(\delta)\geqslant\sqrt{2-\sqrt{4-2\delta}}$, as desired.

(b) We first fix $\delta\in(0,1)$ and write $\varepsilon_0=\max\{\delta,\sqrt{2-\sqrt{4-2\delta}}\}$. The inequality $\Phi_H(\delta)\geqslant\varepsilon_0$ follows, the fact that $\Phi_H(\delta)\geqslant\Phi_H^S(\delta)$ and the result in item (a). To get the other inequality, we first observe that

$$\Phi_H(\delta)\leqslant\Phi_{\mathrm{Lin}\{x,y\}}(\delta)\ \forall x,y\in B_H \text{ with } \langle x,y\rangle=1-\delta. \tag{141}$$

This follows from the obvious fact that $\Phi.(\delta)$ increases when we restrict to subspaces. This implies that it is enough to show that for $P=(p_1,0),Q=(q_1,q_2)\in B_{\ell_2^{(2)}}$ such that $p_1,q_2>0$, $\|P\|>\|Q\|$, and $\langle P,Q\rangle\geqslant 1-\delta$ where $\ell_2^{(2)}$ is the 2 -dimensional Hilbert space, there exists $z\in S_{\ell_2^{(2)}}$ so that $\|P-z\|\leqslant\varepsilon_0$ and $\|Q-z\|\leqslant\varepsilon_0$. Now, it is straightforward to check that we have $\|P\|\in[\sqrt{1-\delta},1]$, and $q_1=\frac{1-\delta}{\|P\|}\in[1-\delta,\sqrt{1-\delta}]$. Fig. 1 helps to the better understanding of the rest of the proof.

Consider $M=\left(\sqrt{\frac{1-\delta+\|P\|}{2\|P\|}},\sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which is the normalized midpoint between $A=(1,0)$ and $B=\left(\frac{1-\delta}{\|P\|},\sqrt{1-\left(\frac{1-\delta}{\|P\|}\right)^2}\right)$ and write Δ to denote the arc of the unit sphere of H between A and M. We claim that $Q\in\bigcup_{z\in\Delta}B(z,\varepsilon_0)$ and $P\in\bigcap_{z\in\Delta}B(z,\varepsilon_0)$. Observe that this gives that there is $z\in\Delta\subset S_H$ whose distance to P and Q is less than or equal to ε_0, finishing the proof. Let us prove the claim. First, we show that $Q=(q_1,q_2)\in\bigcup_{z\in\Delta}B(z,\varepsilon_0)$. If $q_2\leqslant\sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}$, the ball of radius ε_0 centered in the point of Δ with second coordinate equal to q_2 contains the point Q since $\varepsilon_0\geqslant\mathrm{dist}_\infty\big((q_1,0),A\big)\geqslant\mathrm{dist}(Q,\Delta)$. For greater values of q_2, write first $C=\left(q_1,\sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which belongs to $B(M,\varepsilon_0)$ by the previous argument. Also, as M is the normalized midpoint between A and B, we have by the claim at the beginning of this proof that

$$\|M-B\|=\sqrt{2-\sqrt{2+2\langle A,B\rangle}}=\sqrt{2-\sqrt{2+2\frac{1-\delta}{\|P\|}}}\leqslant\sqrt{2-\sqrt{4-2\delta}}\leqslant\varepsilon_0$$

so, also, $\|M-D\|\leqslant\varepsilon_0$. Therefore, both the points C and D belong to $B(M,\varepsilon_0)$, so also the whole segment $[C,D]$ is contained there, and this proves the first part of the claim. To show the second part of the claim, that $P\in\bigcap_{z\in\Delta}B(z,\varepsilon_0)$, we consider the function

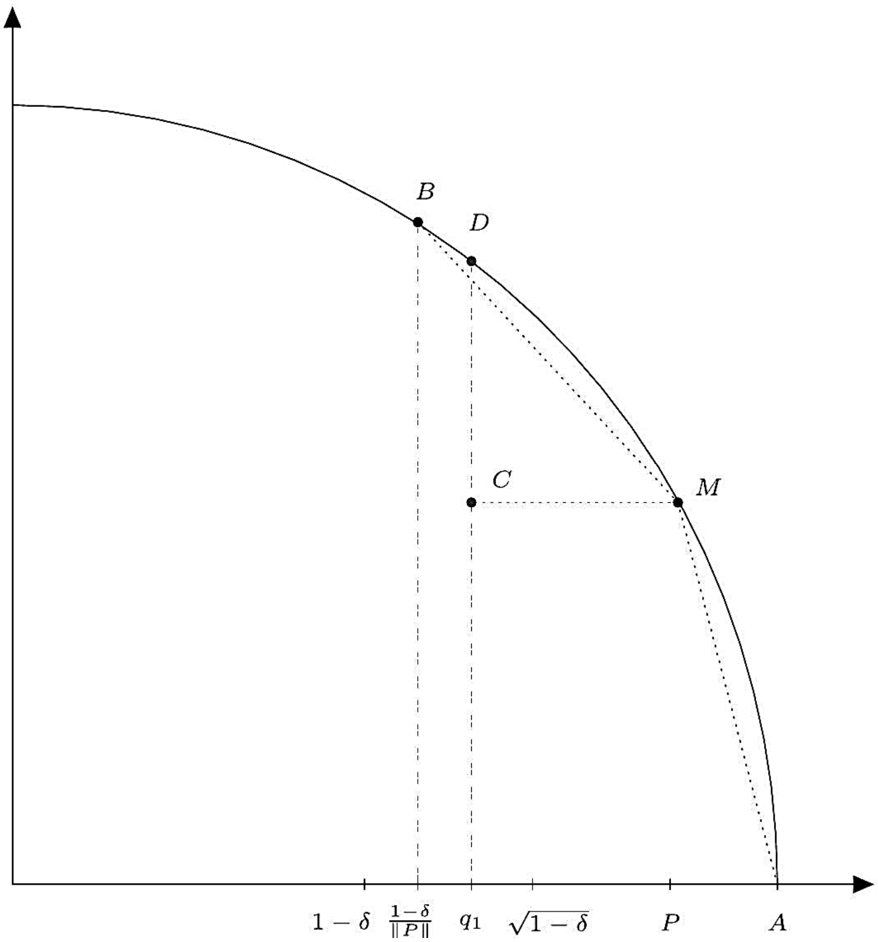


Fig. 1[281]: Calculating $\Phi_H(\boldsymbol{\delta})$ for $\boldsymbol{\delta} \in (\mathbf{0}, \mathbf{1})$.

$$f(p) := 1 + p^2 - \sqrt{2p(p+1-\delta)} \; (p \in [\sqrt{1-\delta}, 1])$$

and observe that it is a convex function, so

$$f(p) \leqslant \mathrm{m} \;\; \{f(137), f(\sqrt{1-\delta})\} \leqslant \varepsilon_0^2.$$

It follows that

$$\| P - M \| = \sqrt{1 + \| P \|^2 - \sqrt{2 \| P \| (\| P \| + 1 - \delta)}} \leqslant \varepsilon_0,$$

hence $M \in B(P, \varepsilon_0)$. As also $A \in B(P, \varepsilon_0)$, it follows that the whole circular arc Δ is contained in $B(P, \varepsilon_0)$ or, equivalently, that $P \in \bigcap_{z \in \Delta} B(z, \varepsilon_0)$.

Fix $\delta \in (1,2)$. Analogously to what we did before in Eq. (141), to show that $\Phi_H(\delta) \leqslant \sqrt{\delta}$, it is enough to consider the two-dimensional case and that, given $p = (\| p \|, 0) \in B_H, q = (q_1, q_2) \in B_H$ with $q_2 \geqslant 0$, to find $z \in S_H$ such that $\| z - P \|, \| z - Q \| \leqslant \sqrt{\delta}$. Routine computations show that

$$z = \left(\frac{\| p \| + q_1}{2}, \sqrt{1 - \left(\frac{\| p \| + q_1}{2} \right)^2} \right) \in S_H$$

does the job. For the other inequality, we fix an orthonormal basis $\{e_1, e_2, \ldots\}$ of H, consider

$$P = \sqrt{\delta - 1} e_1 \in B_H, \; Q = -\sqrt{\delta - 1} e_1 \in B_H$$

and observe that $\langle P, Q \rangle = 1 - \delta$. For any $z \in S_H$, we write $z_1 = \langle z, e_1 \rangle$ and we compute

$\max\{\| z - P \|^2, \| z - Q \|^2\}$

$$= \left\{ \left| z_1 - \sqrt{\delta - 1} \right|^2 + 1 - |z_1|^2, \left| z_1 + \sqrt{\delta - 1} \right|^2 + 1 - |z_1|^2 \right\}$$
$$= \max_{\pm} \left| z_1 \pm \sqrt{\delta - 1} \right|^2 + 1 - |z_1|^2 = \left(|z_1| + \sqrt{\delta - 1} \right)^2 + 1 - |z_1|^2$$
$$= \delta + 2\sqrt{\delta - 1}|z_1| \geqslant \delta.$$
It follows that $\Phi_H(\delta) \geqslant \sqrt{\delta}$, as desired. We present a number of examples for which the values of the Bishop-Phelps-Bollobás moduli are the maximum possible, namely $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for small δ's. As we always have $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, it is enough if we prove the formally stronger result that $\Phi_X^S(\delta) = \sqrt{2\delta}$ for small δ 's (actually, the two facts are equivalent, see Proposition (5.2.14)), and this is what we will show. It happens that all of the examples have in common that they contains an isometric copy of the real space $\ell_\infty^{(2)}$ or $\ell_1^{(2)}$. We show that the latter is a necessary condition that it is not actually sufficient.

The first result is about Banach spaces admitting an L-decomposition. As a consequence we will calculate the moduli of $L_1(\mu)$ spaces.

Proposition (5.2.18)[281]: Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus\ _1$. Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$

Proof. Fix $\delta \in (0,1/2]$ and consider $(y_0, y_0^*) \in \Pi(Y)$ and $(z_0, z_0^*) \in \Pi(Z)$ and write
$$x_0 = \left(\frac{\sqrt{2\delta}}{2} y_0, \left(1 - \frac{\sqrt{2\delta}}{2} \right) z_0 \right) \in S_X,\ x_0^* = \left((1 - \sqrt{2\delta}) y_0^*, z_0^* \right) \in S_{X^*}$$
It is clear that $\operatorname{Re} x_0^*(x_0) = 1 - \delta$. Now, suppose that we may choose $(x, x^*) \in \Pi(X)$ such that
$$\|x_0 - x\| < \sqrt{2\delta} \text{ and } \|x_0^* - x^*\| < \sqrt{2\delta}.$$
Write $x = (y, z) \in Y \oplus_1 Z, x^* = (y^*, z^*) \in Y^* \oplus_\infty Z^*$ and observe that
$$1 = \operatorname{Re} x^*(x) = \operatorname{Re} y^*(y) + \operatorname{Re} z^*(z) \leqslant \|y^*\| \parallel y \parallel + \|z^*\| \parallel z \parallel \leqslant \parallel y \parallel + \parallel z \parallel = 1,$$
therefore, we have
$$\operatorname{Re} y^*(y) = \|y^*\| \parallel y \parallel. \tag{142}$$
Now, we have
$$\left| (1 - \sqrt{2\delta}) - \|y^*\| \right| \leqslant \left\| (1 - \sqrt{2\delta}) y_0^* - y^* \right\| < \sqrt{2\delta}$$
from which follows that $\|y^*\| < 1$ and so, $y = 0$ by (142), giving $\parallel z \parallel = \parallel x \parallel = 1$. But then,
$$\|x_0 - x\| = \left\| \frac{\sqrt{2\delta}}{2} y_0 \right\| + \left\| \left(1 - \frac{\sqrt{2\delta}}{2} \right) z_0 - z \right\| \geqslant \frac{\sqrt{2\delta}}{2} + \left| \left(1 - \frac{\sqrt{2\delta}}{2} \right) - \parallel z \parallel \right| = \sqrt{2\delta},$$
a contradiction. We have proved that $\Phi_X(\delta) \geqslant \sqrt{2\delta}$, being the other inequality always true.

The result above produces the following example.

Example (5.2.19)[281]: Let (Ω, Σ, μ) be a measure space such that $L_1(\mu)$ has dimension greater than one and let E be any non-zero Banach space. Then, $\Phi_{L_1(\mu,E)}(\delta) = \Phi_{L_1(\mu,E)}^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

Indeed, we may find two measurable sets $A, B \subset \Omega$ with empty intersection such that $\Omega = A \cup B$. Then $Y = L_1(\mu|_A, E)$ and $Z = L_1(\mu|_B, E)$ are non-null, $L_1(\mu, E) = Y \oplus_1 Z$ and so the results follows from Proposition (5.2.18).

Particular cases of the above example are ℓ_1 and $L_1[0,1]$.

It is immediate that, using a dual argument than the one given in Proposition (5.2.18), it is possible to deduce the same result for a Banach space which decomposes as an ℓ_∞-sum. Actually, in this case we will get a better result using ideals instead of subspaces.

Proposition (5.2.20)[281]: Let X be a Banach space. Suppose that $X^* = Y \oplus_1 Z$ where Y and Z are (non-trivial) subspaces of X^* such that $\bar{Y}^{w^*} \neq X^*$ and $\bar{Z}^{w^*} \neq X^*$(w^* is the weak -topology $\sigma(X^*, X)$). Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

Proof. We claim that there are $y_0, z_0 \in S_X$ and $y_0^* \in S_Y$ and $z_0^* \in S_Z$ such that

$$\operatorname{Re} y_0^*(y_0) = 1,\ \operatorname{Re} z_0^*(z_0) = 1,\ y^*(z_0) = 0\ \forall y^* \in Y,\ z^*(y_0) = 0\ \forall z^* \in Z.$$

Indeed, we define y_0 and y_0^*, being z_0 and z_0^* analogous. By assumption there is $y_0 \in S_X$ such that $z^*(y_0) = 0$ for every $z^* \in Z$ and we may choose $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(y_0) = 1$ and we only have to prove that $x^* \in Y$ and then write $y_0^* = x^*$. But we have $x^* = y^* + z^*$ with $y^* \in Y, z^* \in Z$ and

$$1 = \operatorname{Re} x^*(y_0) = \operatorname{Re} y^*(y_0) \leqslant \|y^*\| \leqslant \|y^*\| + \|z^*\| = 1,$$

so $z^* = 0$ and $x^* \in Y$.

We now define

$$x_0^* = \left(\frac{\sqrt{2\delta}}{2} y_0^*, \left(1 - \frac{\sqrt{2\delta}}{2}\right) z_0^*\right) \in S_{X^*}\ x_0 = (1 - \sqrt{2\delta}) y_0 + z_0 \in X$$

and first observe that $\|x_0\| \leqslant 1$. Indeed, for every $x^* = y^* + z^* \in S_{X^*}$ one has

$$|x^*(x_0)| = \left|(1 - \sqrt{2\delta}) y^*(y_0) + z^*(z_0)\right| \leqslant (1 - \sqrt{2\delta})\|y^*\| + \|z^*\| \leqslant \|y^*\| + \|z^*\| = 1.$$

It is clear that $\operatorname{Re} x_0^*(x_0) = 1 - \delta$. Now, suppose that we may choose $(x, x^*) \in \Pi(X)$ such that

$$\|x_0 - x\| < \sqrt{2\delta} \text{ and } \|x_0^* - x^*\| < \sqrt{2\delta}.$$

We consider the semi-norm $\|\cdot\|_Y$ defined on X by $\| x \|_Y := \sup\{|y^*(x)| : y^* \in S_Y\}$ which is smaller than or equal to the original norm, write $x^* = y^* + z^*$ with $y^* \in Y$ and $z^* \in Z$, and observe that

$$1 = \operatorname{Re} x^*(x) = \operatorname{Re} y^*(x) + \operatorname{Re} z^*(x) \leqslant \|y^*\| \| x \|_Y + \|z^*\| \| x \| \leqslant \|y^*\| + \|z^*\| = 1.$$

Therefore, we have, in particular, that

$$\operatorname{Re} y^*(x) = \|y^*\| \| x \|_Y . \tag{143}$$

Now, we have

$$\left|(1 - \sqrt{2\delta}) - \| x \|_Y\right| = \left|(1 - \sqrt{2\delta})\|y_0\|_Y - \| x \|_Y\right| \leqslant \left\|(1 - \sqrt{2\delta}) y_0 - x\right\|_Y < \sqrt{2\delta}$$

from which follows that $\| x \|_Y < 1$ and so, $y^* = 0$ by (143) and $\|z^*\| = \|x^*\| = 1$. But then,

$$\|x_0^* - x^*\| = \left\|\frac{\sqrt{2\delta}}{2} y_0^*\right\| + \left\|\left(1 - \frac{\sqrt{2\delta}}{2}\right) z_0^* - z^*\right\| \geqslant \frac{\sqrt{2\delta}}{2} + \left|\left(1 - \frac{\sqrt{2\delta}}{2}\right) - \|z^*\|\right| = \sqrt{2\delta},$$

a contradiction. Again, we have proved that $\Phi_X(\delta) \geqslant \sqrt{2\delta}$, the other inequality always being true.

The first consequence of the above result is to Banach spaces which decompose as ℓ_∞-sum of two subspaces. Indeed, if $X = Y \oplus_\infty Z$ for two (non-trivial) subspaces Y and Z, then $X^* = Y^\perp \oplus_1 Z^\perp$ and $Y^\perp$ and $Z^\perp$ are w^*-closed, so far away of being dense. Therefore, Proposition (5.2.20) applies. We have proved the following result.

Corollary (5.2.21)[281]: Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus \infty$. Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$

As a consequence, we obtain the following examples, analogous to the ones presented in Example (5.2.19).

Examples (5.2.22)[281]:

(a) Let (Ω, Σ, μ) a measure space such that $L_\infty(\Omega)$ has dimension greater than one and let E be any non-zero Banach space. Then,

$$\Phi_{L_\infty(\mu,E)} = \Phi^S_{L_\infty(\mu,E)}(\delta) = \sqrt{2\delta}\ (\delta \in (0,1/2]).$$

(b) Let Γ be a set with more than one point and let E be any non-zero Banach space. Then,

$$\Phi_{c_0(\Gamma,E)} = \Phi^S_{c_0(\Gamma,E)}(\delta) = \sqrt{2\delta} \text{ and } \Phi_{c(\Gamma,E)} = \Phi^S_{c(\Gamma,E)}(\delta) = \sqrt{2\delta}\ (\delta \in (0,1/2]).$$

We deduce from Proposition (5.2.20) that also arbitrary $C(K)$ spaces have the maximum moduli and for this we have to deal with the concept of M-ideal. Given a subspace J of a Banach space X, J is called M-ideal if $J^\perp$ is an L-summand on X^* (use [291] for background). In this case, $X^* = J^\perp \oplus_1 J^\#$ where $J^\# = \{x^* \in X^*: \|x^*\| = \|x^*|_J\|\} \equiv J^*$. Now, if X contain a non-trivial M-ideal J, one has $X^* = J^\perp \oplus_1 J^\#$ and to apply Proposition (5.2.20) we need that $J^\#$ to be not $\sigma(X^*, X)$-dense. Actually, $J^\#$ is not dense in X^* if and only if there is $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \| y \|\}$ for every $y \in J$ (this is easy to verify and a proof can be found in [284]). Let us enunciate what we have shown.

Corollary (5.2.23)[281]: Let X be a Banach space. Suppose that there is a non-trivial M-ideal J of X and a point $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \| y \|\}$ for every $y \in J$. Then, $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$. With the above corollary we are able to prove that the moduli of any non-trivial $C_0(L)$ space are maximum.

Example (5.2.24)[281]: Let L be a locally compact Hausdorff topological space with at least two points and let E be any non-zero Banach space. Then $\Phi_{C_0(L,E)}(\delta) = \Phi^S_{C_0(L,E)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

Indeed, we may find a non-empty non-dense open subset U of L and consider the subspace

$$J = \{f \in C_0(L, E): f|_U = 0\},$$

which is an M-ideal of $C_0(L, E)$ by [291] (use the simpler [291] for the scalar-valued case) and it is non-zero since $L \setminus U$ has non-empty interior. As U is open and non-empty, we may find a non-null function $x_0 \in C_0(L, E)$ whose support is contained in U. It follows that $\|x_0 + y\| = \max\{\|x_0\|, \| y \|\}$ for every $y \in J$ by disjointness of the supports.

A sufficient condition to be in the hypotheses of Corollary (5.2.23) is that a Banach space X contains two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. In this case, J_1 and J_2 are complementary M-summands in $J_1 + J_2$ [291]. Let us comment that this is actually what happens in $C(K)$ when K has more than one point.

Corollary (5.2.25)[281]: Let X be a Banach space. Suppose there are two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. Then $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$

A sufficient condition for a Banach space to have two non-intersecting M-ideals is that its centralizer is non-trivial (i.e. has dimension at least two). We are not going into details, but roughly speaking, the centralizer $Z(X)$ of a Banach space X is a closed subalgebra of $L(X)$ isometrically isomorphic to $C(K_X)$ where K_X is a Hausdorff topological space, and it is possible to see X as a $C(K_X)$-submodule of $\prod_{k\in K_X} X_k$ for suitable X_k's. We refer to [283] and [291] for details. It happens that every M-ideal of $C(K_X)$ produces an M-ideal of X in a suitable way (see

[283]) and if $Z(X)$ contains more than one point, then two non-intersecting M-ideals appear in X, so our corollary above applies.

Corollary (5.2.26)[281]: Let X a Banach space. If $Z(X)$ has dimension greater than one, then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,2]$.

To give some new examples coming from this corollary, we recall that the centralizer of a unital (complex) C^*-algebra identifies with its center (see [291] or [283]).

Example (5.2.27)[281]: Let A be a unital C^*-algebra with non-trivial center. Then, $\Phi_A(\delta) = \Phi_A^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

It would be interesting to see whether the algebra $L(H)$ for a finite- or infinite-dimensional Hilbert space H has the maximum Bishop-Phelps-Bollobás moduli. None of the results of this section applies to it since its center is trivial and, despite it containing $K(H)$ as an M-ideal, there is no element $x_0 \in L(H)$ satisfying the requirements of Corollary (5.2.23) (see [3, p. 538]). Let us also comment that the bidual of $L(H)$ is a C^*-algebra with non-trivial centralizer, so $\Phi_{L(H)*}(\delta) = \Phi_{L(H)^{**}}^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$. If there is $\delta \in (0,1/2]$ such that $\Phi_{L(H)}(\delta) < \sqrt{2\delta}$, then this would be an example when the inequality in Proposition (5.2.12) is strict.

We finish with two pictures: one with the Bishop-Phelps-Bollobás moduli of $\mathbb{R}$, $\mathbb{C}$ and $\ell_\infty^{(2)}$, and another one with the corresponding values of the spherical Bishop-Phelps-Bollobás moduli. (See Figs. 2 and 3.)

We show that Banach spaces with the greatest possible moduli contain almost isometric copies of the real ℓ_∞^2. Let us first recall the following definition.

Definition (5.2.28)[281]: Let X, E be Banach spaces. X is said to contain almost isometric copies of E if, for every $\varepsilon > 0$ there is a subspace $E_\varepsilon \subset X$ and there is a bijective linear operator $T: E \to E_\varepsilon$ with $\| T \| < 1 + \varepsilon$ and $\|T^{-1}\| < 1 + \varepsilon$.

The next result is well-known and has a straightforward proof.

Lemma (5.2.29)[281]: A real Banach space E contains an isometric copy of $\ell_\infty^{(2)}$ if and only if there are elements $u, v \in S_E$ such that $\| u - v \| = \| u + v \| = 2$. E contains almost isometric copies of $\ell_\infty^{(2)}$ if and only if there are elements $u_n, v_n \in S_E, n \in \mathbb{N}$ such that $\|u_n - v_n\| \to 2$ and $\|u_n + v_n\| \to 2$ as $n \to \infty$

The class of spaces X that do not contain almost isometric copies of $\ell_\infty^{(2)}$ was deeply studied by James [292] (see also the exposition in Van Dulst [290]), who gave to such spaces the name "uniformly non-square". He proved in particular, that

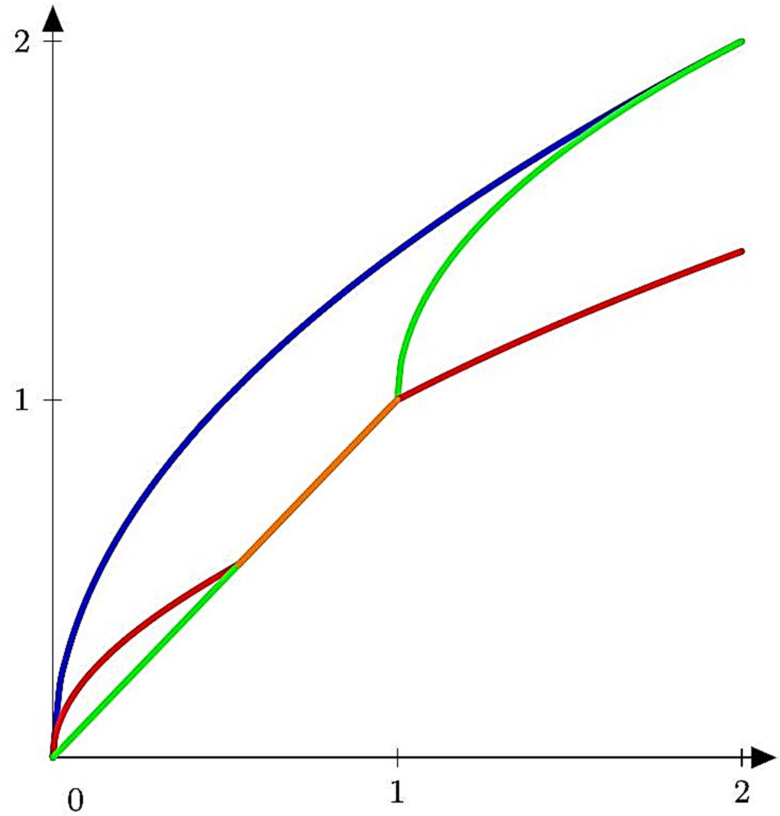


Fig. 2[281]: The value of $\Phi_X(\delta)$ for $\mathbb{R}$ (green), $\mathbb{C}$ (red), $\ell_\infty^{(2)}$ (blue). (For interpretation of the references to color in this figure legend, see [281]).

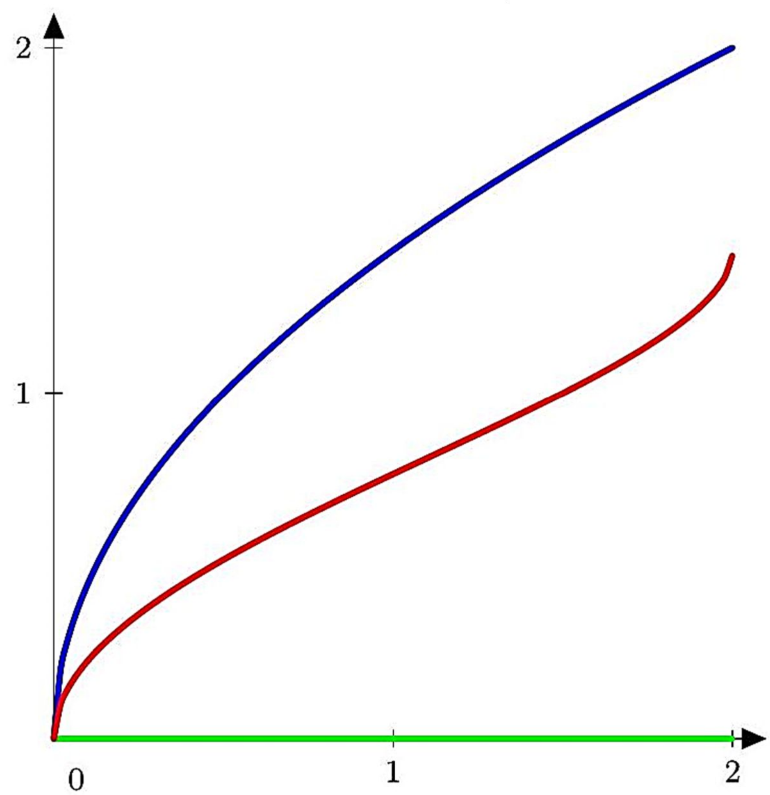


Fig. 3[281]: The value of $\Phi_X^S(\delta)$ for $\mathbb{R}$ (green), $\mathbb{C}$ (red), $\ell_\infty^{(2)}$ (blue). (For interpretation of the references to color in this figure legend, see [281].)

every uniformly non-square space must be reflexive, that this property is stable under passing to subspaces, quotient spaces and duals. In fact, a general result is true [293]: for every 2-

dimensional space E if a real Banach space X does not contain almost isometric copies of E then X is reflexive.

We prove that if a real Banach space X satisfies that its Bishop-Phelps-Bollobás modulus is $\sqrt{2\delta}$ in at least one point $\delta \in (0,1/2)$, then X (and, equivalently, the dual space) contains almost isometric copies of $\ell_\infty^{(2)}$. Actually, as shown , $\Phi_X(\delta) = \sqrt{2\delta}$ if and only if $\Phi_X^S(\delta) = \sqrt{2\delta}$. Therefore, we may use the formally stronger hypothesis of $\Phi_X^S(\delta) = \sqrt{2\delta}$.

We will use some lemmas and ideas of Bishop and Phelps [285], see corresponding lemmas in Diestel [289].

From now on, X will denote a real Banach space. For $t > 1$ and $x^* \in S_{X^*}$, we denote

$$K(t, x^*) := \{x \in X : \| x \| \leqslant t x^*(x)\}.$$

Observe that $K(t, x^*)$ is a convex cone with non-empty interior.

Lemma (5.2.30)[281]: (Particular case of [289], Chapter 1, Lemma 1). For every $z \in B_X$, every $x^* \in S_{X^*}$ and every $t > 1$, there is $x_0 \in S_X$ such that $x_0 - z \in K(t, x^*)$ and $[K(t, x^*) + x_0] \cap B_X = \{x_0\}$

Lemma (5.2.31)[281]: (See [289], Chapter 1, Lemma 2, with a little modification that follows from the proof there.) Let $x^*, y^* \in S_{X^*}$ and suppose that $x^*(\ker y^* \cap S_X) \subset (-\infty, \varepsilon/2]$. Then

$$\operatorname{dist}(x^*, \operatorname{Lin} y^*) \leqslant \varepsilon/2 \text{ and } \mathrm{m}\ \ \{\|x^* - y^*\|, \|x^* + y^*\|\} \leqslant \varepsilon$$

Lemma (5.2.32)[281]: Let $z \in B_X, x^* \in S_{X^*}, t > 1$, and let $x_0 \in S_X$ be from Lemma (5.2.30). Denote $y^* \in S_{X^*}$ a functional that separates $x_0 + K(t, x^*)$ from B_X, so $y^*(x_0) = 1$ and $y^*\big(K(t, x^*)\big) \subset [0, \infty)$. Then $x^*(\ker y^* \cap S_X) \subset (-\infty, 1/t]$ and so, dist $(x^*, \operatorname{Lin} y^*) \leqslant 1/t$ and $\min\{\|x^* - y^*\|, \|x^* + y^*\|\} \leqslant 2/t$

Proof. This also can be extracted from [289], but it is better to give a proof. For every $w \in \ker y^* \cap S_X$ we have that w does not belong to the interior of $K(t, x^*)$, so $1 = \| w \| \geqslant t x^*(w)$, i.e. $x^*(\ker y^* \cap S_X) \subset (-\infty, 1/t]$. An application of Lemma (5.2.31) completes the proof.

Now we are passing to our results. At first, for the sake of simplicity, we consider the easier finite-dimensional case.

Lemma (5.2.33)[281]: Let X be a finite-dimensional real space. Fix $\varepsilon \in (0,1)$. Suppose that $(x, x^*) \in S_X \times S_{X^*}$ satisfies that $x^*(x) = 1 - \frac{\varepsilon^2}{2}$ and that

$$\mathrm{m}\ \ \{\| y - x \|, \|y^* - x^*\|\} \geqslant \varepsilon$$

for every pair $(y, y^*) \in \Pi(X)$. Then for $t = \frac{2}{\varepsilon}$, there exists $y_0 \in [x + K(t, x^*)] \cap S_X$ such that $x^*(y_0) = 1$.

Proof. Consider a sequence $t_n > t, n \in \mathbb{N}$, with $\lim_n t_n = t$. Using Lemma (5.2.30), we get $y_n \in S_X$ such that

$$y_n - x \in K(t_n, x^*) \text{ and } (K(t_n, x^*) + y_n) \cap B_X = \{y_n\}. \tag{144}$$

Let $y_n^* \in X^*$ be a functional that separates $K(t_n, x^*) + y_n$ from B_X, i.e. $y_n^*(y_n) = 1$ and $y_n^*\big(K(t_n, x^*)\big) \subset [0, \infty)$. Then, according to Lemma (5.2.32),

$$\min\{\|x^* - y_n^*\|, \|x^* + y_n^*\|\} \leqslant \frac{2}{t_n} < \varepsilon. \tag{145}$$

But

$$\|x^* + y_n^*\| \geqslant (x^* + y_n^*)(y_n) = 1 + x^*(y_n) = 1 + x^*(x) + x^*(y_n - x)$$

$$= 2 - \frac{\varepsilon^2}{2} + x^*(y_n - x).$$
Since $(y_n - x) \in K(t_n, x^*)$, we have $x^*(y_n - x) \geqslant \|(y_n - x)\|/t_n \geqslant 0$ so
$$\|x^* + y_n^*\| \geqslant 2 - \frac{\varepsilon^2}{2} > \varepsilon$$
(we have used here that $0 < \varepsilon < 1$). Comparing with (145), we get $\|x^* - y_n^*\| < \varepsilon$, so the condition of our lemma says that $\|x - y_n\| \geqslant \varepsilon$. Without loss of generality (passing to a subsequence if necessary) we can assume that y_n tend to some y_0. Then
$$\varepsilon \leqslant \lim_n \|y_n - x\| \leqslant \lim_n t_n x^*(y_n - x) = t(x^*(y_0) - x^*(x)) \leqslant \frac{2}{\varepsilon}\left(x^*(y_0) - 1 + \frac{\varepsilon^2}{2}\right)$$
$$\leqslant \frac{2}{\varepsilon}\left(1 - 1 + \frac{\varepsilon^2}{2}\right) = \varepsilon.$$
This means that all the inequalities in the above chain are in fact equalities. In particular, $x^*(y_0) = 1$ and
$$\|y_0 - x\| = \lim_n \|y_n - x\| = t(x^*(y_0) - x^*(x)),$$
i.e. $y_0 \in [x + K(t, x^*)] \cap S_X$.

Lemma (5.2.34)[281]: Under the conditions of Lemma (5.2.33), there are $y^* \in S_{X^*}$ and $\alpha \geqslant 1 - \frac{\varepsilon}{2}$ with
$$\|x^* - \alpha y^*\| \leqslant \frac{\varepsilon}{2} \text{ and } \|x^* - y^*\| \geqslant \varepsilon, \tag{146}$$
and there is $v \in S_X$ such that
$$x^*(v) = y^*(v) = 1. \tag{147}$$
Proof. Let y_0 be from Lemma (5.2.33). Fix a strictly increasing sequence of $t_n > 1$ with $\lim_n t_n = t$ and let us consider two cases. Case 1: Suppose there exists $m_0 \in \mathbb{N}$ with int $\left[K\left(t_{m_0}, x^*\right) + x\right] \cap B_X \neq \emptyset$. Then, using the fact that for every closed convex set with non-empty interior, the closure of the interior is the whole set, we get
$$y_0 \in [x + K(t, x^*)] \cap B_X = \overline{\text{int}[x + K(t, x^*)] \cap B_X} = \overline{\bigcup \text{int}[x + K(t_n, x^*)] \cap B_X}.$$
So, we can pick
$$z_n \in [x + K(t_n, x^*)] \cap B_X \tag{148}$$
such that $z_n \to y_0$. In particular, $x^*(z_n) \to 1$. Let us apply Lemma (5.2.30): there are $v_n \in S_X$ such that
$$v_n - z_n \in K(t_n, x^*) \text{ and } [K(t_n, x^*) + v_n] \cap B_X = \{v_n\}. \tag{149}$$
Then $x^*(v_n - z_n) \geqslant 0$, i.e. $1 \geqslant x^*(v_n) \geqslant x^*(z_n) \to 1$, so $x^*(v_n) \to 1$. Condition (148) implies that $z_n - x \in K(t_n, x^*)$ which, together with (149), mean that $v_n - x \in K(t_n, x^*)$. Consequently,
$$\|v_n - x\| \leqslant t_n x^*(v_n - x) \leqslant t_n \frac{\varepsilon^2}{2} < \varepsilon.$$
If we denote $y_n^* \in S_{X^*}$ to the functional that separates $v_n + K(t_n, x^*)$ from B_X, then $(v_n, y_n^*) \in \Pi(X)$. Since we are working under the conditions of Lemma (5.2.33), it follows that
$$\|y_n^* - x^*\| \geqslant \varepsilon.$$
Also, by Lemma (5.2.32), $\text{dist}(x^*, \text{Lin } y_n^*) \leqslant 1/t_n$, so there are $\alpha_n \in \mathbb{R}$ such that

$$\|x^* - \alpha_n y_n^*\| \leqslant 1/t_n.$$

Again, without loss of generality, we may assume that the sequences $(\alpha_n), (v_n)$ and (y_n^*) have limits. Let us denote $\alpha := \lim_n \alpha_n, y^* := \lim_n y_n^*$, and $v := \lim_n v_n$. Then $\| v \| = 1, \|y^*\| = 1, x^*(v) = \lim_n x^*(v_n) = 1$, and $y^*(v) = \lim_n y_n^*(v_n) = 1$. This proves (147). Also,

$$\|x^* - \alpha y^*\| = \lim_n \|x^* - \alpha_n y_n^*\| \leqslant \frac{1}{t} = \frac{\varepsilon}{2}.$$

Consequently,

$$\frac{\varepsilon}{2} \geqslant \|x^* - \alpha y^*\| \geqslant (x^* - \alpha y^*)(v) = 1 - \alpha, \tag{150}$$

so, $\alpha \geqslant 1 - \frac{\varepsilon}{2}$.

Case 2: Assume that for every $n \in \mathbb{N}$ we have $\mathrm{int}[K(t_n, x^*) + x] \cap B_X = \emptyset$. Let us separate $x + \mathrm{int}(K(t_n, x^*))$ from B_X by a norm-one functional y_n^*, that is,

$$y_n^*(x + \mathrm{int}[K(t_n, x^*)]) > 1,$$

so, in particular, $y_n^*(x) \geqslant 1$.

Again, passing to a subsequence, we can assume that there exists $y^* = \lim_n y_n^*$ which satisfies $\|y^*\| = 1, 1 \geqslant y^*(x) \geqslant \lim_n y_n^*(x) \geqslant 1$. So, $y^*(x) = 1$, i.e. $(x, y^*) \in \Pi(X)$. By the conditions of our lemma, this implies that

$$\|y^* - x^*\| = \max\{\| x - x \|, \|y^* - x^*\|\} \geqslant \varepsilon.$$

Since

$$y_0 \in x + K(t, x^*) = \overline{\bigcup_{n\in\mathbb{N}} \mathrm{int}[x + K(t_n, x^*)]},$$

we can select $z_n \in \mathrm{int}[x + K(t_n, x^*)]$ in such a way that $z_n \to y_0$. Then

$$y^*(y_0) = \lim_n y_n^*(z_n) \geqslant 1,$$

hence, $y^*(y_0) = 1$. This means that condition (147) works for $v := y_0$. The remaining conditions can be deduced from Lemma (5.2.32) the same way as in Case 1. We state and prove the main result in the finite-dimensional case.

Theorem (5.2.35)[281]: Let X be a finite-dimensional real Banach space. Suppose that there is a $\delta \in (0,1/2)$ such that $\Phi_X(\delta) = \sqrt{2\delta}$(or, equivalently, $\Phi_X^S(\delta) = \sqrt{2\delta}$). Then X^* contains an isometric copy of $\ell_\infty^{(2)}$ (hence, X also contains an isometric copy of $\ell_\infty^{(2)}$).

Proof. Denote $\varepsilon := \sqrt{2\delta} \in (0,1)$. There is a sequence of pairs $(x_n, x_n^*) \in S_X \times S_{X^*}$ such that $x_n^*(x_n) > 1 - \delta = 1 - \frac{\varepsilon^2}{2}$ and

$$\max\{\|y - x_n\|, \|y^* - x_n^*\|\} \geqslant \varepsilon - \frac{1}{n}$$

for every pair $(y, y^*) \in \Pi(X)$. Since the space is finite-dimensional, we can find a subsequence of (x_n, x_n^*) that converges to a pair $(x, x^*) \in S_X \times S_{X^*}$. This pair satisfies that $x^*(x) \geqslant 1 - \delta$ and for every $(y, y^*) \in \Pi(X)$,

$$\begin{aligned}\max\{\| y - x \|, \|y^* - x^*\|\} &\geqslant \max\{\|y - x_n\|, \|y^* - x_n^*\|\} - \max\{\|x - x_n\|, \|x^* - x_n^*\|\} \\ &\geqslant \varepsilon - \frac{1}{n} - \max\{\|x - x_n\|, \|x^* - x_n^*\|\} \to \varepsilon\end{aligned}$$

Since by Theorem (5.2.4), $x^*(x)$ cannot be strictly smaller than $1 - \delta$, we have $x^*(x) = 1 - \delta$. Therefore, we may apply Lemma (5.2.34) to find $y^* \in S_{X^*}$ and $\alpha \geqslant 1 - \frac{\varepsilon}{2}$ for which conditions

(146) and (147) are fulfilled. Now we claim that in fact there is only one number $\gamma \in \mathbb{R}$ for which

$$\|x^* - \gamma y^*\| \leqslant \frac{\varepsilon}{2} \tag{151}$$

and this γ equals $1 - \frac{\varepsilon}{2}$. So $\alpha = 1 - \frac{\varepsilon}{2}$ and, we also claim that

$$\|x^* - \alpha y^*\| = \frac{\varepsilon}{2} \text{ and } \|x^* - y^*\| = \varepsilon. \tag{152}$$

Indeed, when we were proving Eq. (150), we proved that every $\gamma \in \mathbb{R}$ that fulfill (151) satisfies $\gamma \geqslant 1 - \frac{\varepsilon}{2}$. On the other hand, the function $\gamma \mapsto \|x^* - \gamma y^*\|$ is convex, so the set G of those $\gamma \in \mathbb{R}$ satisfying (151) is also convex; but $1 \notin G$, so $\gamma < 1$. Finally, according to (146),

$$\frac{\varepsilon}{2} \geqslant 1 - \gamma = \|y^* - \gamma y^*\| \geqslant \|x^* - y^*\| - \|x^* - \gamma y^*\| \geqslant \frac{\varepsilon}{2}.$$

This means that all the inequalities above are equalities, so $\gamma \leqslant 1 - \frac{\varepsilon}{2}$, and also (152) is true. The claim is proved. Now, let us define

$$u^* := \frac{x^* - \alpha y^*}{\|x^* - \alpha y^*\|} = \frac{2}{\varepsilon}\left(x^* - \left(1 - \frac{\varepsilon}{2}\right) y^*\right)$$

and let us show that functionals u^* and y^* span a subspace of X^* isometric to $\ell_\infty^{(2)}$. According to Lemma (5.2.29), it is sufficient to show that $\|u^* - y^*\| = \|u^* + y^*\| = 2$. At first,

$$\|u^* - y^*\| = \left\|\frac{2}{\varepsilon}\left(x^* - \left(1 - \frac{\varepsilon}{2}\right) y^*\right) - y^*\right\| = \frac{2}{\varepsilon}\|x^* - y^*\| = 2.$$

At second,

$$2 \geqslant \|u^* + y^*\| = \left\|\frac{2}{\varepsilon}\left(x^* - \left(1 - \frac{\varepsilon}{2}\right) y^*\right) + y^*\right\| = \frac{2}{\varepsilon}\|x^* - y^* + \varepsilon y^*\|$$
$$\geqslant \frac{2}{\varepsilon}(x^* - y^* + \varepsilon y^*)(v) = 2.$$

Let us comment that for complex Banach spaces, we cannot expect that Theorem (5.2.35) provides a complex copy of $\ell_\infty^{(2)}$ in the dual of the space. Namely, the two-dimensional complex space $X = \ell_1^{(2)}$ satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for $\delta \in (0,1/2)$ but it does not contain the complex space $\ell_\infty^{(2)}$ (of course, it contains the real space $\ell_\infty^{(2)}$ as a subspace since $\ell_1^{(2)}$ and $\ell_\infty^{(2)}$ are isometric in the real case). We do not know whether it is true a result saying that if a complex space X satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for some $\delta \in (0,1/2)$, then X contains a copy of the complex space $\ell_1^{(2)}$ or a copy of the complex space $\ell_\infty^{(2)}$.

We extend the result of Theorem (5.2.35) to the infinite-dimensional case. We proceed as in the proof of such theorem, but instead of selecting convergent subsequences, we select subsequences such that their numerical characteristics (like norms of elements, pairwise distances, or values of some important functionals) have limits.

Theorem (5.2.36)[281]: Let X be an infinite-dimensional Banach space. Suppose that there is $\delta \in (0,1/2)$ such that $\Phi_X(\delta) = \sqrt{2\delta}$(or, equivalently, $\Phi_X^S(\delta) = \sqrt{2\delta}$). Then X^* (and hence also X) contains almost isometric copies of $\ell_\infty^{(2)}$.

Proof. Denote $\varepsilon := \sqrt{2\delta}$. There is a sequence of pairs $(x_n, x_n^*) \in S_X \times S_{X^*}$ such that $x_n^*(x_n) > 1 - \delta = 1 - \frac{\varepsilon^2}{2}$ and

$$\max\{\|y-x_n\|,\|y^*-x_n^*\|\} \geqslant \varepsilon - \frac{1}{n} \tag{153}$$

for every pair $(y,y^*) \in \Pi(X)$. Since we have $x_n^*(x_n) \leqslant 1-\left(\varepsilon-\frac{1}{n}\right)^2/2$ by Theorem (5.2.4), we deduce that $\lim_n x_n^*(x_n) = 1-\delta$. Denote $t=\frac{2}{\varepsilon}$. As in the proof of Lemma (5.2.33), we find a sequence (y_n) of elements in S_X such that

$$\lim_n \|y_n - x_n\| \leqslant t \lim_n x_n^*(y_n - x_n) \text{ and } \lim_n x_n^*(y_n) = 1. \tag{154}$$

Pick a sequence (t_n) with $t_n > t, n \in \mathbb{N}$ and $\lim_n t_n = t$. Using Lemma (5.2.30), for every $n \in \mathbb{N}$ we get $y_n \in S_X$ such that

$$y_n - x_n \in K(t_n, x_n^*) \text{ and } (K(t_n,x_n^*)+y_n) \cap B_X = \{y_n\}. \tag{155}$$

For given $n \in \mathbb{N}$, let $u_n^* \in S_{X^*}$ be a functional that separates $K(t_n,x_n^*)+y_n$ from B_X, that is, satisfying $u_n^*(y_n)=1$ and $u_n^*\big(K(t_n,x_n^*)\big) \subset [0,\infty)$. Then, according to Lemma (5.2.32), we have

$$\min\{\|x_n^*-u_n^*\|,\|x_n^*+u_n^*\|\} \leqslant 2/t_n < \varepsilon.$$

As we have

$$\begin{aligned}\|x_n^*+u_n^*\| &\geqslant (x_n^*+u_n^*)(y_n) = 1 + x_n^*(y_n) = 1 + x_n^*(x_n) + x_n^*(y_n - x_n)\\ &\geqslant 2 - \frac{\varepsilon^2}{2} > \varepsilon,\end{aligned}$$

we get $\|x_n^*-u_n^*\| < \varepsilon$, so (153) says that $\|x_n - y_n\| \geqslant \varepsilon - \frac{1}{n}$. Without loss of generality, passing to a subsequence if necessary, we can assume that the following limits exist: $\lim_n \|x_n - y_n\|, \lim_n x_n^*(y_n - x_n)$ and $\lim_n x_n^*(y_n)$. Then

$$\begin{aligned}\varepsilon &\leqslant \lim_n \|y_n-x_n\| \leqslant \lim_n t_n x_n^*(y_n-x_n) = t \lim_n x_n^*(y_n-x_n) \leqslant \frac{2}{\varepsilon}\left(\lim_n x_n^*(y_n) - 1 + \frac{\varepsilon^2}{2}\right)\\ &\leqslant \frac{2}{\varepsilon}\left(1-1+\frac{\varepsilon^2}{2}\right) = \varepsilon.\end{aligned}$$

This means that all the inequalities in the above chain are in fact equalities. In particular, $\lim_n x_n^*(y_n) = 1$, and

$$\varepsilon = \lim_n \|y_n - x_n\| = t \lim_n x_n^*(y_n - x_n), \tag{156}$$

so the analogue of Lemma (5.2.33) is proved.

Now, we proceed with analogue of Lemma (5.2.34): we need to show that there are $y_n^* \in S_{X^*}$ and $\alpha_n \geqslant 0, \alpha_n \to 1-\frac{\varepsilon}{2}$ with

$$\|x_n^* - \alpha_n y_n^*\| \leqslant \frac{\varepsilon}{2} \text{ and } \|x_n^* - y_n^*\| \geqslant \varepsilon, \tag{157}$$

and there is a sequence of $v_n \in S_X$ such that

$$\lim_n x_n^*(v_n) = \lim_n y_n^*(v_n) = 1. \tag{158}$$

Case 1: Assume that there exist $r>0$ and $n \in \mathbb{N}$ such that, for all $m>n$,

$$([K(t-r,x_m^*)+x_m] \cap B_X) \setminus (x_m + rB_X) \neq \emptyset.$$

This means that for all $m>n$ there is z_m such that

$$\|z_m - x_m\| > r,\ \|z_m\| \leqslant 1 \text{ and } \|z_m - x_m\| \leqslant (t-r)x_m^*(z_m - x_m).$$

For $\lambda \in (0,1)$ denote $y_{m,\lambda} := \lambda z_m + (1-\lambda) y_m$. Clearly, $y_{m,\lambda} \in B_X$. Denote also

$$\lambda_m = \mathrm{i}\ \{\lambda : y_{m,\lambda} \in x_m + K(t, x_m^*)\},$$

and let us show that

$$\lim_m \lambda_m = 0. \tag{159}$$

Observe first that λ_m is smaller than every value of λ for which

$$\|y_{m,\lambda} - x_m\| \leqslant tx^*(y_{m,\lambda} - x_m).$$

On the one hand, if $\|y_m - x_m\| - tx_m^*(y_m - x_m) \leqslant 0$, then $\lambda = 0$ belongs to the set in question, and the job is done. On the other hand, if $\|y_m - x_m\| - tx_m^*(y_m - x_m) \geqslant 0$, then there is λ for which

$$\lambda\|z_m - x_m\| + (1-\lambda)\|y_m - x_m\| = t\lambda x_m^*(z_m - x_m) + t(1-\lambda)x_m^*(y_m - x_m)$$

is positive and belongs to the set in question. This means that

$$\lambda_m \leqslant \frac{\|y_m - x_m\| - tx_m^*(y_m - x_m)}{\|y_m - x_m\| - tx_m^*(y_m - x_m) + tx_m^*(z_m - x_m) - \|z_m - x_m\|},$$

but the limit of the right-hand side equals 0 thanks to (156). So condition (159) is proved. This means that $y_m, \lambda_m \in x_m + K(t, x_m^*)$ and $\|y_{m,\lambda_m} - y_m\| \leqslant 2\lambda_m \to 0$. Let us pick a little bit bigger $\tilde{\lambda}_m > \lambda_m$ in such a way that we still have $\| y_{m,\tilde{\lambda}_m} - y_m \| \to 0$, but for some $\tilde{t}_n < t$ with $\tilde{t}_n \to t$, we have

$$\|y_{m,\tilde{\lambda}_m} - x_m\| \leqslant \tilde{t}_n x_m^*(y_{m,\tilde{\lambda}_m} - x_m). \tag{160}$$

Then, in particular, $\lim_n x_n^*(y_{n,\tilde{\lambda}_n}) = \lim_n x_n^*(y_n) = 1$. Let us apply Lemma (5.2.30). There are $v_n \in S_X$ such that

$$v_n - y_{n,\tilde{\lambda}_n} \in K(\tilde{t}_n, x_n^*) \text{ and } [K(\tilde{t}_n, x_n^*) + v_n] \cap B_X = \{v_n\}. \tag{161}$$

Then $x_n^*(v_n - y_{n,\tilde{\lambda}_n}) \geqslant 0$, i.e. $1 \geqslant x_n^*(v_n) \geqslant x_n^*(y_{n,\tilde{\lambda}_n}) \to 1$, so $x_n^*(v_n) \to 1$. This proves the first part of (158). Condition (160) implies that $y_{n,\tilde{\lambda}_n} - x_n \in K(\tilde{t}_n, x_n^*)$ which, together with (161), mean that $v_n - x_n \in K(\tilde{t}_n, x_n^*)$. Consequently,

$$\|v_n - x_n\| \leqslant \tilde{t}_n x_n^*(v_n - x_n) \leqslant \tilde{t}_n \frac{\varepsilon^2}{2} < \varepsilon.$$

If we denote by $y_n^* \in S_{X^*}$ the functional that separates $v_n + K(\tilde{t}_n, x^*)$ from B_X, then $(v_n, y_n^*) \in \Pi(X)$(this proves the second part of (158) even in a stronger form) so, thanks to (153),

$$\|y_n^* - x_n^*\| \geqslant \varepsilon - \frac{1}{n}.$$

Also, by Lemma (5.2.32), $\mathrm{dist}(x_n^*, \mathrm{Lin}\ y_n^*) \leqslant 1/\tilde{t}_n$, so there are $\alpha_n \in \mathbb{R}$ such that

$$\|x^* - \alpha_n y_n^*\| \leqslant 1/\tilde{t}_n.$$

Again, without loss of generality, we may assume that the sequences (α_n) and $\|x_n^* - \alpha_n y_n^*\|$ converge. Then,

$$\lim_n \|x_n^* - \alpha_n y_n^*\| \leqslant \frac{1}{t} = \frac{\varepsilon}{2}.$$

Consequently,

$$\frac{\varepsilon}{2} \geqslant \lim_n \|x_n^* - \alpha_n y_n^*\| \geqslant \lim_n (x_n^* - \alpha_n y_n^*)(v) = 1 - \lim_n \alpha_n,$$

so $\lim_n \alpha_n \geqslant 1 - \frac{\varepsilon}{2}$. Starting at this point, (157) can be deduced in the same way as it was done for (152).

Case 2: Assume that there is a sequence of $r_n > 0, r_n \to 0$ and that there is a subsequence of (x_m, x_m^*) (that we will again denote (x_m, x_m^*)) such that

$$([K(t - r_m, x_m^*) + x_m] \cap B_X) \setminus (x_m + r_m B_X) = \emptyset \text{ (for all } m \in \mathbb{N}).$$

Then also

$$[K(t - r_m, x_m^*) + x_m] \cap (1 - r_m)B_X = \emptyset \text{ (for all } m \in \mathbb{N}).$$

Let us separate

$$\frac{1}{1 - r_m}[K(t - r_m, x_m^*) + x_m]$$

from B_X by a norm-one functional y_n^*, that is,

$$y_n^*(K(t - r_m, x_m^*) + x_m) > 1 - r_m \quad (162)$$

so, in particular, $y_m^*(x_m) \geqslant 1 - r_m$ and $\lim_m y_m^*(x_m) = 1$. By the Bishop-Phelps-Bollobás theorem, there is a sequence $(\tilde{x}_n, \tilde{y}_n^*) \in \Pi(X)$, such that

$$\max\{\|\tilde{x}_n - x_n\|, \|\tilde{y}_n^* - y_n^*\|\} \to 0 \text{ as } n \to \infty.$$

Again, passing to a subsequence, we can assume that all the numerical characteristics that appear here have the corresponding limits. According to (153), for n big enough, we have

$$\|\tilde{y}_n^* - x_n^*\| = \max\{\|\tilde{x}_n - x_n\|, \|\tilde{y}_n^* - x_n^*\|\} \geqslant \varepsilon - \frac{1}{n},$$

so $\lim_n \|y_n^* - x_n^*\| \geqslant \varepsilon$. We can select $z_n \in x_n + K(t - r_n, x_n^*)$ in such a way that $\|z_n - y_n\| \to 0$. Then

$$1 \geqslant \lim_n y_n^*(y_n) = \lim_n y_n^*(z_n) \geqslant \lim_n (1 - r_n) = 1.$$

This means that condition (158) works for $v_n := y_n$.

Now consider an arbitrary $w \in \ker y_n^* \cap S_X$. Taking a convex combination with an element h of the unit sphere where $y_n^*(h)$ almost equals -1, we can construct an element $\tilde{w} \in B_X$ such that $\| \tilde{w} - w \| \leqslant 2r_n$ and $y_n^*(\tilde{w}) = -r_n$. Then, by (162), $\tilde{w} \notin \operatorname{int}(K(t - r_n, x_n^*))$, so $\| \tilde{w} \| \geqslant (t - r_n)x_n^*(\tilde{w})$. Consequently,

$$x_n^*(w) \leqslant x_n^*(\tilde{w}) + 2r_n \leqslant \frac{1}{t - r_n} + 2r_n.$$

Observe that we have shown that the values of the functional x_n^* on $\ker y_n^* \cap S_X$ do not exceed $\frac{1}{t-r_n} + 2r_n$. Therefore, by Lemma (5.2.31),

$$\operatorname{dist}(x_n^*, \operatorname{Lin} y_n^*) \leqslant \frac{1}{t - r_n} + 2r_n \to \frac{1}{t}$$

and so there are $\alpha_n \in \mathbb{R}$ such that

$$\lim_n \|x_n^* - \alpha_n y_n^*\| \leqslant \frac{1}{t}.$$

The remaining conditions in (157) and (158) can be deduced from the same way as in Case 1.

Finally, (157) and (158) imply that $\lim_n \|x_n^* - y_n^*\| = \lim_n \|x_n^* - y_n^*\| = 2$: the proof does not differ much from the corresponding part of Theorem (5.2.35) demonstration.

Corollary (5.2.37)[281]: Let X be a uniformly non-square Banach space. Then, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) < \sqrt{2\delta}$ for every $\delta \in (0,1/2)$. Consequently, every superreflexive Banach space can be equivalently renormed in such a way that, in the new norm, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) < \sqrt{2\delta}$ for all $\delta \in (0,1/2)$

It would be interesting to obtain a quantitative version of the above corollary.

For every $\delta \in (0,1/2)$ we denote $\varepsilon = \sqrt{2\delta}$, so $0 < \varepsilon < 1$. We denote $B_\varepsilon^3 \subset \mathbb{R}^3$ the absolute convex hull of the following 11 points $A_k, k = 1, \dots, 11$ (or, what is the same, the convex hull of 22 points $\pm A_k, k = 1, \dots, 11$)

$$A_1 = \left(0,0,\frac{3}{4}\right),$$

$$A_2 = \left(1-\varepsilon, 1, \frac{\varepsilon}{2}\right),\ A_3 = \left(1-\varepsilon, -1, \frac{\varepsilon}{2}\right),\ A_4 = \left(\varepsilon - 1, 1, \frac{\varepsilon}{2}\right),\ A_5 = \left(\varepsilon - 1, -1, \frac{\varepsilon}{2}\right),$$

$$A_6 = \left(1, 1-\varepsilon, \frac{\varepsilon}{2}\right),\ A_7 = \left(-1, 1-\varepsilon, \frac{\varepsilon}{2}\right),\ A_8 = \left(1, \varepsilon - 1, \frac{\varepsilon}{2}\right),\ A_9 = \left(-1, \varepsilon - 1, \frac{\varepsilon}{2}\right),$$

$$A_{10} = (1,1,0),\ A_{11} = (1,-1,0).$$

Denote D_ε ("D" from "Diamond") the normed space $(\mathbb{R}^3, \|\cdot\|)$, for which B_ε^3 is its unit ball. Then D_ε^* can be viewed as $\mathbb{R}^3$ with the polar of B_ε^3 as the unit ball, and the action of $x^* \in D_\varepsilon^*$ on $x \in D_\varepsilon$ is just the standard inner product in $\mathbb{R}^3$. Let us list, without proof, some properties of D_ε whose verification is straightforward:

- The subspace of D_ε formed by vectors of the form $(x_1, x_2, 0)$ is canonically isometric to $\ell_\infty^{(2)}$.
- There are no other isometric copies of $\ell_\infty^{(2)}$ in D_ε.
- The subspace of D_ε^* formed by vectors of the form $(x_1, x_2, 0)$ is canonically isometric to $\ell_1^{(2)}$ (and so, is isometric to $\ell_\infty^{(2)}$.
- There are no other isometric copies of $\ell_\infty^{(2)}$ in D_ε^*.
- The following operators act as isometries both on D_ε and D_ε^*: $(x_1, x_2, x_3) \mapsto (x_2, x_1, x_3), (x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$. In other words, changing the sign of one coordinate or rearranging the first two coordinates do not change the norm of an element.

The following theorem shows that the existence of an $\ell_\infty^{(2)}$-subspace does not imply that $\Phi_X(\delta) = \sqrt{2\delta}$, even in dimension 3.

Theorem (5.2.38)[281]: Let $\delta \in (0,1/2), \varepsilon = \sqrt{2\delta}$, and $X = D_\varepsilon$. Then $\Phi_X(\delta) < \sqrt{2\delta}$.

Proof. Assume contrary that $\Phi_X(\delta) = \sqrt{2\delta}$. Like in the proof of Theorem (5.2.35), this implies the existence of a pair $(x, x^*) \in S_X \times S_{X^*}$ with the following properties: $x^*(x) = 1 - \delta$ and

$$\max\{\| z - x \|, \|z^* - x^*\|\} \geqslant \varepsilon \text{ for every pair } (z, z^*) \in \Pi(X). \qquad (163)$$

Also, repeating the proof of Theorem (5.2.35) for this $x^* \in S_{X^*}$, we can find $u^*, y^* \in S_{X^*}$ such that the pair (u^*, y^*) is 1-equivalent to the canonical basis of $\ell_1^{(2)}$ and

$$u^* = \frac{2}{\varepsilon}\left(x^* - \left(1 - \frac{\varepsilon}{2}\right) y^*\right).$$

This means that $x^* = \frac{\varepsilon}{2} u^* + \left(1 - \frac{\varepsilon}{2}\right) y^*$. What can be this (u^*, y^*) if we take into account that there is only one isometric copy of $\ell_1^{(2)}$ in X^* ? It can be either $u^* = (1,0,0), y^* = (0,1,0)$, or a pair of vectors that can be obtained from this one by application of isometries, i.e. just 8 possibilities. Consequently, x^* either equals to the vector $(\varepsilon/2, 1 - \varepsilon/2, 0)$, or to a vector that can be obtained from this one by application of isometries, again just 8 possibilities.

By duality argument, there are $u, y \in S_X$ such that the pair (u, y) is 1-equivalent to the canonical basis of $\ell_1^{(2)}$ and

$$x = \frac{\varepsilon}{2}u + \left(1 - \frac{\varepsilon}{2}\right)y.$$

Since the only (up to isometries) pair $u, y \in S_X$ of this kind is $u = (1,1,0), y = (1,-1,0)$, we get $x = (1,1-\varepsilon,0)$, or can be obtained from this one by application of isometries. So there are $8 \times 8 = 64$ possibilities for the pair (x, x^*). Taking into account that $x^*(x) = 1 - \delta$ we reduce this number to 8 possibilities: $x = (1-\varepsilon,1,0), x^* = (\varepsilon/2, 1-\varepsilon/2, 0)$ and images of this pair under remaining 7 reflections and rotations of the underlying $\mathbb{R}^2$. If we show that this choice of (x, x^*) do not satisfy condition (163) then, by symmetry, the remaining choices would not satisfy (163) neither, and this would give us the desired contradiction.

Indeed, the pair $(z, z^*) \in \Pi(X)$ that do not satisfy (163) for $x = (1-\varepsilon,1,0), x^* = (\varepsilon/2, 1-\varepsilon/2, 0)$ is the following one: $z = (1-\varepsilon, 1, \varepsilon/2), z^* = (\varepsilon/2, 1-\varepsilon/2, \varepsilon)$. Let us check the required properties. At first, $z = A_2 \in S_X$. Then, $z^*(z) = 1$. The last property means, that $\|z^*\| \geqslant 1$, so in order to check that $\|z^*\| = 1$ it remains to show that $|z^*(A_k)| \leqslant 1$ for all k. This is true for $\varepsilon < 1$. Finally, $\| z - x \| = \| (0,0,\varepsilon/2) \| = \frac{\varepsilon}{2}\left\|\frac{4}{3}A_1\right\| = \frac{2}{3}\varepsilon < \varepsilon$, and $\|z^* - x^*\| = \| (0,0,\varepsilon) \| = \langle (0,0,\varepsilon), A_1 \rangle = \frac{3}{4}\varepsilon < \varepsilon$.

Section (5.3): Numerical Radius of Operators on $L_1(\mu)$:

We provide a version of Bishop-Phelps-Bollobás theorem for numerical radius for operators. For a Banach space X, B_X and S_X will be the closed unit ball and the unit sphere of X, respectively. We will denote by X^* the topological dual of X and by $\mathcal{L}(X)$ the space of bounded linear operators on X endowed with the operator norm. The symbols $\mathcal{F}(X)$, $\mathcal{K}(X)$ and $\mathcal{WC}(X)$ denote the spaces of finite-rank operators, compact operators and weakly compact operators on X, respectively. It is well known that $\mathcal{F}(X) \subset \mathcal{K}(X) \subset \mathcal{WC}(X)$. The normed spaces will be either real or complex.

Bishop-Phelps-Bollobás theorem states that for any Banach space X, given $0 < \varepsilon < 1$, and $(x, x^*) \in B_X \times S_{X^*}$ such that $|x^*(x) - 1| < \frac{\varepsilon^2}{2}$, there is a pair $(y, y^*) \in S_X \times S_{X^*}$ satisfying

$$\| y - x \| < \varepsilon, \|y^* - x^*\| < \varepsilon \text{ and } y^*(y) = 1$$

(see [300],[301] or [302]).

After some interesting about denseness of the set of norm attaining operators, in 2008 it was initiated the study of versions of Bishop-Phelps-Bollobás Theorem for operators [297]. It was considered the problem of obtaining versions of such results for numerical radius of operators (see [307]). We just mention that the numerical radius of an operator is a continuous semi-norm in the space $\mathcal{L}(X)$ for every Banach space X.

Guirao and Kozhushkina proved that the spaces c_0 and ℓ_1 satisfy the Bishop-PhelpsBollobás property for numerical radius (BPBp-v) in the real case as well as in the complex case [307]. Falcó showed the same result for $L_1(\mathbb{R})$ in the real case [306]. Choi, Kim, Lee and Martín extended the previous result to $L_1(\mu)$ for any positive measure μ [304]. Avilés, Guirao and Rodríguez provided sufficient conditions on a compact Hausdorff space K in order that $C(K)$ has the BPBp-v in the real case [299]. For instance, a metrizable space K satisfies the previous condition [299]. It is an open problem whether or not such result is satisfied for any compact Hausdorff space K in the real case. In the complex case there are no results until now for $C(K)$ spaces. Motivated by Definition 1.2 of [307], we introduce the notion of the BPBp-ν for subspaces of the space of bounded linear operators. A Banach space X satisfies the BPBp-ν, introduced in [307], if and only if the space $\mathcal{M} = \mathcal{L}(X)$ satisfies the BPBp-ν (Definition (5.3.1)). Then, we give some sufficient conditions on a subspace $\mathcal{M}$ of $\mathcal{L}(L_1(\mu))$ to satisfy the BPBp- ν, for any finite measure μ. We show that $\mathcal{M}$ has the BPBp- ν if $\mathcal{M}$ contains the space of finite-rank operators on $L_1(\mu)$, is contained in the class of representable operators on $L_1(\mu)$ (see Definition (5.3.5)) and $T_{|A} \in \mathcal{M}$ for every $T \in \mathcal{M}$ and any measurable set A, where $T_{|A}$ is the operator on $L_1(\mu)$ given by $T_{|A}(f) = T(f\chi_A)$ for all $f \in L_1(\mu)$. As a consequence of the main result we obtain that for any σ-finite measure μ, the spaces of finite-rank operators, compact operators and weakly compact operators on $L_1(\mu)$ have the BPBp- ν. The results are valid in the real as well as in the complex case.

If X is a Banach space and $T \in \mathcal{L}(X)$, we recall that the numerical radius of $T, \nu(T)$, is defined by

$$\nu(T) = \sup\ \{|x^*(T(x))| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

In general the numerical radius is a semi-norm on $\mathcal{L}(X)$ satisfying $\nu(T) \leq \| T \|$ for each $T \in \mathcal{L}(X)$. The numerical index of $X, n(X)$ is defined by

$$n(X) = \inf\ \{\nu(T) : T \in S_{\mathcal{L}(X)}\}.$$

Hence, $n(X)$ is the greatest constant t such that $t \| T \| \leq \nu(T)$ for each $T \in \mathcal{L}(X)$. It is always satisfied that $0 \leq n(X) \leq 1$ and, in case that $n(X) = 1$, it is said that X has numerical index equal to 1. In such case it is satisfied that $\nu(T) = \| T \|$ for each $T \in \mathcal{L}(X)$. It is well known that the spaces $L_1(\mu)$ and $C(K)$ have numerical index equal to 1 for any measure μ and any compact Hausdorff space K [303].

Guirao and Kozhushkina [307] introduced the definition of the BPBp- ν. We will use a little different concept by admitting subclasses of the space of bounded linear operators on a Banach space X.

Definition (5.3.1)[296]: Let X be a Banach space and $\mathcal{M}$ a subspace of $\mathcal{L}(X)$. We will say that $\mathcal{M}$ has the Bishop-Phelps-Bollobás property for numerical radius (BPBp- ν) if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $S \in \mathcal{M}, \nu(S) = 1, x_0 \in S_X$ and $x_0^* \in S_{X^*}$ are such that $x_0^*(x_0) = 1$ and $|x_0^*(S(x_0))| > 1 - \eta(\varepsilon)$, there are $T \in \mathcal{M}, x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that

i) $x_1^*(x_1) = 1$,

ii) $|x_1^*(T(x_1))| = \nu(T) = 1$,

iii) $\nu(T - S) < \varepsilon, \|x_1 - x_0\| < \varepsilon$ and $\|x_1^* - x_0^*\| < \varepsilon$.

We notice that for spaces with numerical index equal to one, Definition (5.3.1) can be reformulated by using the usual norm of the space $\mathcal{L}(X)$ instead of the numerical radius. The following simple technical lemmas will be useful. Next lemma is a straightforward consequence of [297].

Lemma (5.3.2)[296]: Assume that $\{z_k : k \in \mathbb{N}\} \subset \{z \in \mathbb{C} : |z| \leq 1\}$ and $\{\beta_k : k \in \mathbb{N}\} \subset \mathbb{C}$ satisfies that $\sum_{k=1}^{\infty} |\beta_k| = 1$. If $0 < \varepsilon < 1$ and $\mathrm{Re}(\sum_{k=1}^{\infty} \beta_k z_k) > 1 - \varepsilon^2$, then

$$\sum_{k \in B} |\beta_k| > 1 - \varepsilon,$$

where $B = \{k \in \mathbb{N} : \mathrm{Re}(\beta_k z_k) > (1 - \varepsilon)|\beta_k|\}$.

Next result is a generalization of Lemma (5.3.2) to $L_1(\mu)$. Also it extends [307] where the state the analogous result for the sequence space ℓ_1.

Lemma (5.3.3)[296]: Let (Ω, Σ, μ) be a measure space. Assume that $0 < \varepsilon < 1, f \in B_{L_1(\mu)}$ and $g \in B_{L_\infty(\mu)}$ are such that

$$1 - \varepsilon^2 < \operatorname{Re} \int_\Omega fg d\mu.$$

Then the set C given by

$$C = \{t \in \Omega : \operatorname{Re} f(t)g(t) > (1 - \varepsilon)|f(t)|\},$$

satisfies that

$$\operatorname{Re} \int_C fg d\mu > 1 - \varepsilon.$$

Proof. It is clear that the set C is measurable. By assumption we have

$$1 - \varepsilon^2 < \operatorname{Re} \int_\Omega fg d\mu \leq \operatorname{Re} \int_C fg d\mu + (1 - \varepsilon) \int_{\Omega \setminus C} |f| d\mu$$

$$\leq \varepsilon \operatorname{Re} \int_C fg d\mu + (1 - \varepsilon) \left(\int_C |f| d\mu + \int_{\Omega \setminus C} |f| d\mu \right) \leq \varepsilon \operatorname{Re} \int_C fg d\mu + 1 - \varepsilon.$$

Hence,

$$\operatorname{Re} \int_C fg d\mu > 1 - \varepsilon.$$

Lemma (5.3.4)[296]: Let z be a complex number, $0 < \varepsilon < 1$ and assume that

$$\operatorname{Re} z > (1 - \varepsilon)|z|.$$

Then

$$|z - |z|| < \sqrt{2\varepsilon}|z|.$$

Proof. We write $z = x + iy$, where $x, y \in \mathbb{R}$. Since $x^2 + y^2 = |z|^2$ and $= \operatorname{Re} z > (1 - \varepsilon)|z|$, we have $y^2 \leq |z|^2 - (1 - \varepsilon)^2|z|^2 = (2\varepsilon - \varepsilon^2)|z|^2$. It follows that

$$|z - |z|^2 = (|z| - x)^2 + y^2 < (\varepsilon|z|)^2 + (2\varepsilon - \varepsilon^2)|z|^2 = 2\varepsilon|z|^2.$$

We recall the following notion (see for instance [305], Definition III.3).

Definition (5.3.5)[296]: Let (Ω, Σ, μ) be a finite measure space and Y a Banach space. An operator $T \in \mathcal{L}(L_1(\mu), Y)$ is called Riesz representable (or simply representable) if there is $h \in L_\infty(\mu, Y)$ such that $T(f) = \int_\Omega hf d\mu$ for all $f \in L_1(\mu)$. We say that the function h is a representation of T.

We will use the following identification.

Proposition (5.3.6)[296]: ([305], Lemma III.4, p. 62) Let (Ω, Σ, μ) be a finite measure space and Y be a Banach space. There is a linear isometry Φ from the space $\mathcal{R}$ of representable operators in $\mathcal{L}(L_1(\mu), Y)$ into $L_\infty(\mu, Y)$ such that if $T \in \mathcal{R}$ and $\Phi(T) = h$, then it is satisfied that

$$T(f) = \int_\Omega hf d\mu, \text{ for all } f \in L_1(\mu).$$

It is known that $\mathcal{WC}(L_1(\mu))$ is a subset of the representable operators into $L_1(\mu)$ whenever μ is any finite measure (see for instance [305], Theorem III.12, p. 75). We will write $\mathcal{R}(L_1(\mu))$ for the space of representable operators into $L_1(\mu)$. Given $T \in \mathcal{L}(L_1(\mu))$ and a measurable subset A of Ω, we will denote by $T_{|A}$ the operator on $L_1(\mu)$ given by $T_{|A}(f) = T(f\chi_A)$ for all $f \in L_1(\mu)$.

In [298] it was proved that a subspace of $\mathcal{L}(L_1(\mu), Y)$ that contains the subspace of finite-rank operators and is contained in the space of representable operators and that satisfies also an additional assumption has the Bishop-Phelps-Bollobás property for operators whenever Y has the so called AHSp, a property satisfied by $L_1(\mu)$. Now we will prove a parallel result for numerical radius for subspaces of $\mathcal{L}(L_1(\mu))$. Such proof is more involved since we have to approximate one pair of elements (x, x^*) in the product of $S_{L_1(\mu)} \times S_{(L_1(\mu))^*}$ instead of one element in the unit sphere of $L_1(\mu)$.

In the proof of the next result we will write $g(f)$ instead of $\int_\Omega g(t)f(t)d\mu$ for each element $f \in L_1(\mu)$ and $g \in L_\infty(\mu)$.

Theorem (5.3.7)[296]: Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{M}$ be a subspace of $\mathcal{L}(L_1(\mu))$ such that $\mathcal{F}(L_1(\mu)) \subseteq \mathcal{M} \subseteq \mathcal{R}(L_1(\mu))$. Assume also that for each measurable subset A of Ω and each $T \in \mathcal{M}$ it is satisfied $T_{|A} \in \mathcal{M}$. Then $\mathcal{M}$ has the $BPBp$ - ν, and the function η satisfying Definition (5.3.1) is independent from the measure space and also from $\mathcal{M}$.

Proof. Let us fix $0 < \varepsilon < 1$. We take $\eta(= \eta(\varepsilon)) = \frac{\varepsilon^8}{2^{33}}$. Assume that $T_0 \in S_{\mathcal{M}}$, $f_0 \in S_{L_1(\mu)}$ and $g_0 \in S_{L_\infty(\mu)}$ satisfy $g_0(f_0) = 1$ and $|g_0(T_0(f_0))| > 1 - \eta$. Let λ_0 be a scalar with $|\lambda_0| = 1$ and such that $|g_0(T_0(f_0))| = \operatorname{Re} \lambda_0 g_0(T_0(f_0))$. By changing T_0 by $\lambda_0 T_0$ we may assume that $\operatorname{Re} g_0(T_0(f_0)) = |g_0(T_0(f_0))|$. In view of Proposition (5.3.6) there is a function $h_0 \in S_{L_\infty(\mu, L_1(\mu))}$ associated to the operator T_0. Since the proof is long we divided it into five steps.

Step 1. In this step we will approximate the pair of functions (f_0, g_0) by a new pair (f_1, g_1) such that f_1 and g_1 take a countable set of values and also there are subsets where f_1, g_1 are constant and h_0 has small oscillation on these subsets.

More concretely, we will show that there are functions $f_1 \in S_{L_1(\mu)}$ and $g_1 \in S_{L_\infty(\mu)}$ and a countable family $\{D_k : k \in J\} \subset \Omega$ of pairwise disjoint measurable sets such that $\mu(D_k) > 0$ for all $k \in J$, $\mu(\Omega \setminus \cup_{k \in J} D_k) = 0$ and such that the following conditions are satisfied

$$\|f_1 - f_0\|_1 < \frac{\varepsilon}{4},\ \|g_1 - g_0\|_\infty < \frac{\varepsilon}{4}, \tag{164}$$

$$\operatorname{Re} g_1(f_1) > 1 - \eta,\ \operatorname{Re} g_1(T_0(f_1)) > 1 - \eta, \tag{165}$$

$$\text{for each } k \in J, f_1 \text{ and } g_1 \text{ are constant on } D_k \tag{166}$$

$$\sup\ \{\|h_0(s) - h_0(t)\|_1 : s, t \in D_k\} \le \eta,\ \forall k \in J, \tag{167}$$

and

$$1 = \|h_0\|_\infty = \sup\ \{\|h_0(t)\|_1 : t \in \cup_{k \in J} D_k\}. \tag{168}$$

Since the set of simple functions is dense in both $L_1(\mu)$ and $L_\infty(\mu)$, there are simple functions $f_1 \in S_{L_1(\mu)}$ and $g_1 \in S_{L_\infty(\mu)}$ satisfying (164) and (165).

On the other hand, by [305] there is a measurable subset E_1 of Ω such that $\mu(E_1) = 0$ and $h_0(\Omega \setminus E_1)$ is a separable subset of $L_1(\mu)$. Suppose that the set $\{y_i : i \in \mathbb{N}\}$ is dense in $h_0(\Omega \setminus E_1)$. Since f_1 and g_1 are simple functions, we can assume that $\operatorname{Im}(f_1) = \{a_r : r = 1, \dots, n\}$ and $\operatorname{Im}(g_1) = \{b_l : l = 1, \dots, m\}$. Now, for $i \in \mathbb{N}$, $r \in \{1, \dots, n\} = N$ and $l \in \{1, \dots, m\} = M$ we consider the following subsets of Ω

$$A_{(1,r,l)} = h_0^{-1}\left(B_{\frac{\eta}{2}}(y_1)\right) \cap (\Omega \setminus E_1) \cap f_1^{-1}(a_r) \cap g_1^{-1}(b_l)$$

and

$$A_{(i,r,l)} = \left(h_0^{-1}\left(B_{\frac{\eta}{2}}(y_i)\right) \setminus \cup_{e=1}^{i-1} h_0^{-1}\left(B_{\frac{\eta}{2}}(y_e)\right)\right) \cap (\Omega \setminus E_1) \cap f_1^{-1}(a_r) \cap g_1^{-1}(b_l),\ \forall i \ge 2.$$

It is clear that the elements of the family $\{A_{(i,r,l)}: (i,r,l) \in \mathbb{N} \times N \times M\}$ are measurable subsets of Ω and pairwise disjoint. Now, let $W = \{(i,r,l) \in \mathbb{N} \times N \times M: \mu(A_{(i,r,l)}) = 0\}$ and $E_2 = \bigcup_{(i,r,l)\in W} A_{(i,r,l)}$. By the definition of W it is trivially satisfied that E_2 is measurable and $\mu(E_2) = 0$. On the other hand there exists a measurable subset E_3 of $\Omega \setminus (E_1 \cup E_2)$ such that $\mu(E_3) = 0$ and $\| h \|_\infty = \sup\{\| h(t) \|_1: t \in \Omega \setminus E_3\}$. Assume that $\{D_k: k \in J\}$ is the family of pairwise disjoint measurable subsets obtained by indexing the set $\{A_{(i,r,l)} \setminus E_3: (i,r,l) \in (\mathbb{N} \times N \times M) \setminus W\}$. Then, we have that $\mu(D_k) > 0$ for all $k \in J, \mu(\Omega \setminus \bigcup_{k\in J} D_k) = 0$ and also the family $\{D_k: k \in J\}$ satisfies the conditions (166), (167) and (168). Therefore, by (166) there are sets of scalars $\{\alpha_k: k \in J\}$ and $\{\gamma_k: k \in J\}$ such that

$$f_1 = \sum_{k\in J} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}, \ \sum_{k\in J} |\alpha_k| = 1, \ g_1 = \sum_{k\in J} \gamma_k \chi_{D_k}, \ |\gamma_k| \leq 1, \ \forall k \in J. \tag{169}$$

Step 2. In this step we will define another simple function $f_2 \in S_{L_1(\mu)}$ which is an approximation of f_1, and can be expressed as a finite sum instead of the countable sum appearing in the expression of f_1 given in (169).

By (169) and (165) there is a finite subset F of J such that

$$\sum_{k\in F} |\alpha_k| > 1 - \eta > 0, \ \text{Re}\, g_1\left(\sum_{k\in F} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}\right) > 1 - \eta. \tag{170}$$

and also

$$\text{Re}\, g_1\left(T_0\left(\sum_{k\in F} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}\right)\right) > 1 - \eta. \tag{171}$$

For each $k \in F$ we put $\beta_k = \frac{\alpha_k}{\sum_{k\in F}|\alpha_k|}$ and define $f_2 = \sum_{k\in F'} \beta_k \frac{\chi_{D_k}}{\mu(D_k)}$. In view of (170) and (171) we have that

$$\text{Re}\, g_1(f_2) = \text{Re}\, g_1\left(\sum_{k\in F} \beta_k \frac{\chi_{D_k}}{\mu(D_k)}\right) > 1 - \eta \tag{172}$$

and

$$\text{Re}\, g_1(T_0(f_2)) = \text{Re}\, g_1\left(T_0\left(\sum_{k\in F} \beta_k \frac{\chi_{D_k}}{\mu(D_k)}\right)\right) > 1 - \eta. \tag{173}$$

Clearly $f_2 \in S_{L_1(\mu)}$ and by (169), (170) we have that

$$\begin{aligned}\|f_2 - f_1\|_1 &= \left\|\sum_{k\in F} \beta_k \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in J} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\ &= \left\|\sum_{k\in F} \beta_k \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in J\setminus F} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\ &\leq \sum_{k\in F} |\beta_k - \alpha_k| + \sum_{k\in J\setminus F} |\alpha_k| = 1 - \sum_{k\in F} |\alpha_k| + \sum_{k\in J\setminus F} |\alpha_k|\end{aligned}$$

$$= 2\left(1 - \sum_{k\in F} |\alpha_k|\right) < 2\eta < \frac{\varepsilon}{4}. \tag{174}$$

Step 3. Now, we approximate the function h_0 by a new one h_2 such that for each $k \in F$ the new function is constant on each D_k. So we also approximate the operator T_0 by a new one.

For this aim we choose an element t_k in D_k, for any $k \in F$, put $\psi_k = h_0(t_k) \in L_1(\mu)$ and define $h_1 \in L_\infty(\mu, L_1(\mu))$ by

$$h_1 = h_0\chi_{\Omega\setminus(\cup_{k\in F} D_k)} + \sum_{k\in F} \psi_k \chi_{D_k}.$$

By (168) we have that $\|h_1\|_\infty \leq 1$. If $T_1 \in \mathcal{L}(L_1(\mu))$ is the operator associated to h_1, then T_1 is the sum of $T_{0|\Omega\setminus\setminus\cup_{k\in F}D_k)}$ and a finite-rank operator, so $T_1 \in B_{\mathcal{M}}$. By using (167), we clearly have

$$\begin{aligned} \|T_1 - T_0\| &= \|h_1 - h_0\|_\infty \leq \sup\ \left\{\|\psi_k - h_0(t)\|_1 : t \in D_k, k \in F\right\} \\ &= \sup\ \left\{\|h_0(t_k) - h_0(t)\|_1 : t \in D_k, k \in F\right\} \leq \eta. \end{aligned} \tag{175}$$

Since $\|T_0\| = 1$ we get that $0 < 1 - \eta \leq \|T_1\| \leq 1$. Now we define $T_2 = \frac{T_1}{\|T_1\|}$ and so we have that

$$\|T_2 - T_1\| = 1 - \|T_1\| \leq \eta.$$

In view of the previous inequality and (175) we obtain that

$$\|T_2 - T_0\| \leq \|T_2 - T_1\| + \|T_1 - T_0\| \leq 2\eta < \frac{\varepsilon}{4}. \tag{176}$$

From (173) and (176) we get that

$$\operatorname{Re} g_1\big(T_2(f_2)\big) \geq \operatorname{Re} g_1\big(T_0(f_2)\big) - \|T_2 - T_0\| > 1 - 3\eta. \tag{177}$$

On the other hand, it is clear that

$$T_1(f_2) = \int_\Omega h_1 f_2 d\mu = \int_{\Omega\setminus\cup_{k\in F}D_k} h_1 f_2 d\mu + \sum_{k\in F} \int_{D_k} h_1 f_2 d\mu = \sum_{k\in F} \beta_k \psi_k.$$

For simplicity, for each $k \in F$, put $\phi_k = \frac{\psi_k}{\|T_1\|}$. So we have that

$$T_2(f_2) = \sum_{k\in F} \beta_k \phi_k.$$

It is clear that $\phi_k \in B_{L_1(\mu)}$ for every $k \in F$. From (172) and (177) we obtain that

$$\operatorname{Re} g_1\left(\sum_{k\in F} \frac{\beta_k}{2}\left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k\right)\right) = \operatorname{Re} g_1\left(\frac{f_2 + T_2(f_2)}{2}\right) > 1 - 2\eta.$$

Step 4. In this step we will obtain approximations f_3, T_3 of f_2 and T_2, respectively. We will check in the final step that T_3 attains its norm at f_3, a necessary condition for our purpose. In fact f_3 and T_3 are the final approximations to f_0 and T_0.

Define the set G as follows

$$G = \left\{k \in F : \operatorname{Re} g_1\left(\frac{\beta_k}{2}\left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k\right)\right) > (1 - \sqrt{2\eta})|\beta_k|\right\}.$$

In view of Lemma (5.3.2) we have that

$$\sum_{k\in G} |\beta_k| > 1 - \sqrt{2\eta} = 1 - \frac{\varepsilon^4}{2^{16}}. \tag{178}$$

It is immediate that

$$\operatorname{Re} \beta_k g_1\left(\frac{\chi_{D_k}}{\mu(D_k)}\right) > (1 - 2\sqrt{2\eta})|\beta_k| = \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k|, \ \forall k \in G.$$

So, for each $k \in G$ we have

$$\operatorname{Re} \beta_k \gamma_k = \operatorname{Re} \beta_k g_1\left(\frac{\chi_{D_k}}{\mu(D_k)}\right) > \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k| \geq \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k \gamma_k|.$$

Hence, we obtain that $\beta_k \neq 0$ for $k \in G$ and also that

$$|\gamma_k| > 1 - \frac{\varepsilon^4}{2^{15}} > 0, \ \forall k \in G. \tag{179}$$

By using also Lemma (5.3.4) we get

$$|\beta_k \gamma_k - |\beta_k \gamma_k|| < \frac{\varepsilon^2}{2^7}|\beta_k \gamma_k|.$$

Hence,

$$\left|\beta_k - \frac{|\beta_k \gamma_k|}{\gamma_k}\right| < \frac{\varepsilon^2}{2^7}|\beta_k| \text{ and } \left|\gamma_k - \frac{|\beta_k \gamma_k|}{\beta_k}\right| < \frac{\varepsilon^2}{2^7}|\gamma_k|, \ \forall k \in G, \tag{180}$$

so

$$\left|\frac{\gamma_k}{|\gamma_k|} - \frac{|\beta_k|}{\beta_k}\right| < \frac{\varepsilon^2}{2^7}, \ \forall k \in G. \tag{181}$$

The element f_3 given by

$$f_3 = \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\chi_{D_k}}{\mu(D_k)}$$

belongs to the unit sphere of $L_1(\mu)$. Now, by using (178) and (180) we get that

$$\begin{aligned}
\|f_3 - f_2\|_1 &= \left\|\frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F} \beta_k \frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\
&= \left\|\frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in G} \beta_k \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F\setminus G} \beta_k \frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\
&\leq \sum_{k\in G} \left|\frac{1}{\sum_{k\in G} |\beta_k|} \frac{|\beta_k \gamma_k|}{\gamma_k} - \beta_k\right| + \sum_{k\in F\setminus G} |\beta_k| \qquad (182) \\
&\leq \sum_{k\in G} \left|\frac{1}{\sum_{k\in G} |\beta_k|} \frac{|\beta_k \gamma_k|}{\gamma_k} - \frac{|\beta_k \gamma_k|}{\gamma_k}\right| + \sum_{k\in G} \left|\frac{|\beta_k \gamma_k|}{\gamma_k} - \beta_k\right| + \sum_{k\in F\setminus G} |\beta_k| \\
&\leq 1 - \sum_{k\in G} |\beta_k| + \sum_{k\in G} \frac{\varepsilon^2}{2^7}|\beta_k| + \sum_{k\in F\setminus G} |\beta_k| \\
&\leq 2\left(1 - \sum_{k\in G} |\beta_k|\right) + \frac{\varepsilon^2}{2^7} \leq \frac{\varepsilon}{8}.
\end{aligned}$$

In view of (164),(174) and (182), we obtain that

$$\|f_3 - f_0\|_1 \le \|f_3 - f_2\|_1 + \|f_2 - f_1\|_1 + \|f_1 - f_0\|_1 < \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \quad (183)$$

Now notice obviously that

$$\operatorname{Re} \beta_k g_1(\phi_k) > (1 - 2\sqrt{2\eta})|\beta_k| > \left(1 - \frac{\varepsilon^4}{2^{14}}\right)|\beta_k|, \ \forall k \in G.$$

For each $k \in G$, define P_k as follows

$$P_k = \left\{t \in \Omega : \operatorname{Re} \beta_k g_1(t)\phi_k(t) > \left(1 - \frac{\varepsilon^2}{2^7}\right)|\beta_k \phi_k(t)|\right\}.$$

Clearly P_k is a measurable set. According to Lemma (5.3.3), for each $k \in G$ we have

$$\operatorname{Re} \int_{P_k} \beta_k g_1 \phi_k d\mu > \left(1 - \frac{\varepsilon^2}{2^7}\right)|\beta_k|,$$

so

$$\int_{P_k} |\phi_k| d\mu > 1 - \frac{\varepsilon^2}{2^7} > 0. \quad (184)$$

Let us fix $k \in G$ and $t \in P_k$. Notice that $\beta_k g_1(t) \neq 0$. By Lemma (5.3.4) it follows

$$|\beta_k g_1(t)\phi_k(t) - |\beta_k g_1(t)\phi_k(t)|| < \frac{\varepsilon}{2^3}|\beta_k g_1(t)\phi_k(t)|,$$

So

$$\left|\phi_k(t) - \frac{|\beta_k g_1(t)\phi_k(t)|}{\beta_k g_1(t)}\right| < \frac{\varepsilon}{2^3}|\phi_k(t)|, \ \forall k \in G, t \in P_k. \quad (185)$$

For each $k \in G$ we can define the element φ_k in $L_1(\mu)$ by

$$\varphi_k = \frac{\gamma_k}{|\gamma_k|} \frac{|\phi_k|}{\int_{P_k} |\phi_k| d\mu} \frac{|g_1|}{g_1} \chi_{P_k}.$$

It is immediate that $\varphi_k \in S_{L_1(\mu)}$. From (184) and (185), for each $k \in G$ we have

$$\begin{aligned}
&\|\varphi_k - \phi_k\|_1 \\
&\quad \le \|\varphi_k - \phi_k \chi_{P_k}\|_1 + \|\phi_k \chi_{\Omega \setminus P_k}\|_1 \\
&\quad < \|\varphi_k - \phi_k \chi_{P_k}\|_1 + \frac{\varepsilon^2}{2^7} \\
&\quad \le \left\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\right\|_1 + \left\|\frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k} - \frac{|\beta_k|}{\beta_k}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\right\|_1 \\
&\qquad + \left\|\frac{|\beta_k|}{\beta_k}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k} - \phi_k \chi_{P_k}\right\|_1 + \frac{\varepsilon^2}{2^7} \\
&\quad \le \left\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\right\|_1 + \left|\frac{\gamma_k}{|\gamma_k|} - \frac{|\beta_k|}{\beta_k}\right| + \frac{\varepsilon}{2^3} + \frac{\varepsilon^2}{2^7} \\
&\quad \le \left\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\right\|_1 + \frac{\varepsilon}{4} \quad \text{(by(181))} \\
&\quad = \left\|\frac{\gamma_k}{|\gamma_k|}\frac{|\phi_k|}{\int_{P_k}|\phi_k| d\mu}\frac{|g_1|}{g_1}\chi_{P_k} - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\right\|_1 + \frac{\varepsilon}{4}
\end{aligned}$$

$$= 1 - \int_{P_k} |\phi_k| d\mu + \frac{\varepsilon}{4} < \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}. \tag{186}$$

Let the function h_3 be defined as follows

$$h_3 = \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega \setminus \cup_{k \in G} D_k} + \sum_{k \in G} \varphi_k \chi_{D_k}.$$

It is easy to see that h_3 belongs to the unit sphere of $L_\infty(\mu, L_1(\mu))$. Let $T_3 \in S_{\mathcal{L}(L_1(\mu))}$ be the operator associated to the function h_3 in view of Proposition (5.3.6). Since G is a finite set, $\mathcal{F}(L_1(\mu)) \subset \mathcal{M}$ and $T_1 \in \mathcal{M}$, by using the assumptions on $\mathcal{M}$ we know that $T_3 \in S_{\mathcal{M}}$.

We also have that

$$\begin{aligned}
\|T_3 - T_2\| &= \left\| h_3 - \frac{h_1}{\|h_1\|_\infty} \right\|_\infty \\
&= \left\| h_3 \chi_{\Omega \setminus (\cup_{k \in G} D_k)} + \sum_{k \in G} h_3 \chi_{D_k} - \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega \setminus (\cup_{k \in G} D_k)} - \sum_{k \in G} \frac{h_1}{\|h_1\|_\infty} \chi_{D_k} \right\|_\infty \\
&= \left\| \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega \setminus (\cup_{k \in G} D_k)} + \sum_{k \in G} \varphi_k \chi_{D_k} - \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega \setminus (\cup_{k \in G} D_k)} - \sum_{k \in G} \phi_k \chi_{D_k} \right\|_\infty \\
&= \left\| \sum_{k \in G} (\varphi_k - \phi_k) \chi_{D_k} \right\|_\infty = \sup_{k \in G} \|\varphi_k - \phi_k\|_1 \le \frac{\varepsilon}{2}
\end{aligned}$$

By the previous inequality and (176) we obtain

$$\|T_3 - T_0\| \le \|T_3 - T_2\| + \|T_2 - T_0\| < \varepsilon. \tag{187}$$

Step 5. Finally, we are going to find an approximation of g_1 and complete our proof.

We put $A = \left\{ t \in \Omega : |g_1(t)| \ge 1 - \frac{\varepsilon^2}{2^7} \right\}$ and let the function g_2 be defined by $g_2 = \frac{g_1}{|g_1|} \chi_A + g_1 \chi_{\Omega \setminus A}$. Since $g_1 \in S_{L_\infty(\mu)}$, we have that $g_2 \in S_{L_\infty(\mu)}$. It is also clear that

$$\|g_2 - g_1\|_\infty \le \frac{\varepsilon^2}{2^7}. \tag{188}$$

By using (164) and (188) we also have that

$$\|g_2 - g_0\|_\infty \le \|g_2 - g_1\|_\infty + \|g_1 - g_0\|_\infty \le \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \varepsilon. \tag{189}$$

By (179) we know that $|\gamma_k| > 1 - \frac{\varepsilon^4}{2^{15}}$ for each $k \in G$. Since $G \subset J$, in view of (169), the restriction of g_1 to D_k coincides with γ_k and so $D_k \subset A$ for all $k \in G$. Hence,

$$g_{2|D_k} = \frac{\gamma_k}{|\gamma_k|}, \ \forall k \in G.$$

Therefore, we deduce that

$$\begin{aligned}
g_2(f_3) &= g_2 \left(\frac{1}{\sum_{k \in G} |\beta_k|} \sum_{k \in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\chi_{D_k}}{\mu(D_k)} \right) \\
&= \frac{1}{\sum_{k \in G} |\beta_k|} \sum_{k \in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{1}{\mu(D_k)} g_2(\chi_{D_k})
\end{aligned}$$

$$= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\gamma_k}{|\gamma_k|} = 1. \quad (190)$$

For each $k \in G$, from the definition of P_k and A, we deduce that $P_k \subset A$, so

$$g_2(\varphi_k) = \int_{P_k} \frac{\gamma_k}{|\gamma_k|} \frac{|\phi_k|}{\int_{P_k} |\phi_k| d\mu} d\mu = \frac{\gamma_k}{|\gamma_k|}. \quad (191)$$

Since

$$T_3(f_3) = \int_\Omega h_3 f_3 d\mu = \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \varphi_k,$$

by using (191) we have that

$$\begin{aligned} g_2(T_3(f_3)) &= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} g_2(\varphi_k) \\ &= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\gamma_k}{|\gamma_k|} = 1. \quad (192) \end{aligned}$$

We have shown that there are elements $T_3 \in S_{\mathcal{M}}, f_3 \in S_{L_1(\mu)}$ and $g_2 \in S_{L_\infty(\mu)}$ that in view of (183),(187),(189),(190) and (192) satisfy

$$\|T_3 - T_0\| < \varepsilon,\ \|f_3 - f_0\|_1 < \varepsilon,\ \|g_2 - g_0\|_\infty < \varepsilon$$

and also

$$g_2(f_3) = g_2(T_3(f_3)) = 1.$$

So we showed that $\mathcal{M}$ has the BPBp- ν with the function η given by $\eta(\varepsilon) = \frac{\varepsilon^8}{2^{33}}$.

In case that μ is a σ-finite measure, there is a finite measure ζ and a linear isometry Φ from $L_1(\mu)$ onto $L_1(\zeta)$. From this fact we deduce the following result which generalizes Theorem (5.3.7) for some well-known classes of operators.

Corollary (5.3.8)[296]: Let (Ω, Σ, μ) be a σ-finite measure space. The following subspaces of $\mathcal{L}(L_1(\mu))$ have the $BPBp$-ν and the function η satisfying Definition (5.3.1) is independent from the measure space.

(a) The subspace of all finite-rank operators on $L_1(\mu)$.

(b) The subspace of all compact operators on $L_1(\mu)$.

(c) The subspace of all weakly compact operators on $L_1(\mu)$.

In case that μ is finite, then the subspace of all representable operators on $L_1(\mu)$ also has the $BPBp - \nu$.

Proof. Assume first that μ is a finite measure. It is known that $\mathcal{F}(L_1(\mu)) \subset \mathcal{K}(L_1(\mu)) \subset \mathcal{WC}(L_1(\mu)) \subset \mathcal{R}(L_1(\mu))$ and $T_{|A}(B_{L_1(\mu)}) \subset T(B_{L_1(\mu)})$ for each $T \in \mathcal{L}(L_1(\mu))$ and every measurable subset A of Ω. Also, it is clear that $T_{|A} \in \mathcal{R}(L_1(\mu))$ for any $T \in \mathcal{R}(L_1(\mu))$ and every measurable subset A of Ω. Therefore, the spaces $\mathcal{F}(L_1(\mu)), \mathcal{K}(L_1(\mu)), \mathcal{WC}(L_1(\mu))$ and $\mathcal{R}(L_1(\mu))$ satisfy the assumptions of Theorem (5.3.7), and so the above statements hold in case that μ is finite.

Now, let μ be a σ-finite measure. We will show that the space $\mathcal{F}(L_1(\mu))$ satisfies the BPBp- ν. There is a finite measure ζ and a surjective linear isometry Φ from $L_1(\mu)$ into $L_1(\zeta)$. The mapping Φ induces a surjective linear isometry from $\mathcal{F}(L_1(\mu)))$ into $\mathcal{F}(L_1(\zeta)))$ given by

$T \mapsto \Phi \circ T \circ \Phi^{-1}$. Since Φ is an isometry, it follows that $\nu(T) = \nu(\Phi \circ T \circ \Phi^{-1})$ for every $T \in \mathcal{F}(L_1(\mu))$. On the other hand, it is satisfied that $(f, g) \in \Pi(L_1(\mu))$ if and only if $(\Phi(f), (\Phi^{-1})^t(g)) \in \Pi(L_1(\zeta))$. Also $(\Phi^{-1})^t(g)(\Phi \circ T \circ \Phi^{-1}(\Phi(f))) = g(T(f))$ for every $T \in \mathcal{F}(L_1(\mu))\big)$. Since $\mathcal{F}(L_1(\zeta))$ has the BPBp- ν we deduce the same property for $\mathcal{F}(L_1(\mu))$.

The proofs of the statements b) and c) are analogous.

Corollary (5.3.9)[365]: Let (Ω, Σ, μ) be a measure space. Assume that $0 < \varepsilon < 1, f^j \in B_{L_1(\mu)}$ and $g^j \in B_{L_\infty(\mu)}$ are such that

$$1 - \varepsilon^2 < \operatorname{Re} \int_\Omega \sum f^j g^j d\mu.$$

Then the set C given by

$$C = \{t \in \Omega: \operatorname{Re} \sum f^j(t) g^j(t) > (1 - \varepsilon) \sum |f^j(t)|\},$$

satisfies that

$$\operatorname{Re} \int_C \sum f^j g^j d\mu > 1 - \varepsilon.$$

Proof. It is clear that the set C is measurable. By assumption we have

$$1 - \varepsilon^2 < \operatorname{Re} \int_\Omega \sum f^j g^j d\mu \leq \operatorname{Re} \int_C \sum f^j g^j d\mu + (1 - \varepsilon) \int_{\Omega \backslash C} \sum |f^j| d\mu$$

$$\leq \varepsilon \operatorname{Re} \int_C \sum f^j g^j d\mu + (1 - \varepsilon) \sum \left(\int_C |f^j| d\mu + \int_{\Omega \backslash C} |f^j| d\mu \right)$$

$$\leq \varepsilon \operatorname{Re} \int_C \sum f^j g^j d\mu + 1 - \varepsilon.$$

Hence,

$$\operatorname{Re} \int_C \sum f^j g^j d\mu > 1 - \varepsilon.$$

Corollary (5.3.10)[365]: Let z_j be a complex number, $0 < \varepsilon < 1$ and assume that

$$\operatorname{Re} \sum z_j > (1 - \varepsilon) \sum |z_j|.$$

Then

$$\sum |z_j - |z_j|| < \sqrt{2\varepsilon} \sum |z_j|.$$

Proof. We write $z_j = x_j + iy_j$, where $x_j, y_j \in \mathbb{R}$. Since $x_j^2 + y_j^2 = |z_j|^2$ and $\sum x_j = \operatorname{Re} \sum z_j > (1 - \varepsilon) \sum |z_j|$, we have $\sum y_j^2 \leq |\sum z_j|^2 - (1 - \varepsilon)^2 \sum |z_j|^2 = (2\varepsilon - \varepsilon^2) \sum |z_j|^2$. It follows that $\sum \left|z_j - |z_j|\right|^2 = \sum (|z_j| - x_j)^2 + \sum y_j^2 < \sum (\varepsilon |z_j|)^2 + (2\varepsilon - \varepsilon^2) \sum |z_j|^2 = 2\varepsilon \sum |z_j|^2$.

Corollary (5.3.13)[365]: Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{M}$ be a subspace of $\mathcal{L}(L_1(\mu))$ such that $\mathcal{F}(L_1(\mu)) \subseteq \mathcal{M} \subseteq \mathcal{R}(L_1(\mu))$. Assume also that for each measurable subset A of Ω and each $T^j \in \mathcal{M}$ it is satisfied $T^j_{|A} \in \mathcal{M}$. Then $\mathcal{M}$ has the $BPBp$ -ν, and the function η satisfying Definition (5.3.1) is independent from the measure space and also from $\mathcal{M}$.

Proof. Let us fix $0 < \varepsilon < 1$. We take $\eta (= \eta(\varepsilon)) = \frac{\varepsilon^8}{2^{33}}$. Assume that $\left(T^j\right)_0 \in S^j_{\mathcal{M}}$, $\left(f^j\right)_0 \in S_{L_1(\mu)}$ and $\left(g^j\right)_0 \in S_{L_\infty(\mu)}$ satisfy $\sum \left(g^j\right)_0 \left(\left(f^j\right)_0\right) = 1$ and $\sum \left|\left(g^j\right)_0 \left(\left(T^j\right)_0 \left(\left(f^j\right)_0\right)\right)\right| > 1 -$

η. Let $(\lambda^j)_0$ be a scalar with $\sum\left|(\lambda^j)_0\right| = 1$ and such that $\sum\left|(g^j)_0\left((T^j)_0\left((f^j)_0\right)\right)\right| = \operatorname{Re}\sum(\lambda^j)_0(g^j)_0\left((T^j)_0\left((f^j)_0\right)\right)$. By changing $(T^j)_0$ by $(\lambda^j)_0(T^j)_0$ we may assume that $\operatorname{Re}\sum(g^j)_0\left((T^j)_0\left((f^j)_0\right)\right) = \sum\left|(g^j)_0\left((T^j)_0\left((f^j)_0\right)\right)\right|$. In view of Proposition (5.3.6) there is a function $(h^j)_0 \in S_{L_\infty(\mu,L_1(\mu))}$ associated to the operator $(T^j)_0$. Since the proof is long we divided it into five steps.

Step 1. In this step we will approximate the pair of functions $\left((f^j)_0,(g^j)_0\right)$ by a new pair $\left((f^j)_1,(g^j)_1\right)$ such that $(f^j)_1$ and $(g^j)_1$ take a countable set of values and also there are subsets where $(f^j)_1,(g^j)_1$ are constant and $(h^j)_0$ has small oscillation on these subsets.

More concretely, we will show that there are functions $(f^j)_1 \in S_{L_1(\mu)}$ and $(g^j)_1 \in S_{L_\infty(\mu)}$ and a countable family $\{D_k : k \in J\} \subset \Omega$ of pairwise disjoint measurable sets such that $\mu(D_k) > 0$ for all $k \in J$, $\mu\left(\Omega \setminus \cup_{k\in J} D_k\right) = 0$ and such that the following conditions are satisfied

$$\sum\left\|(f^j)_1 - (f^j)_0\right\|_1 < \frac{\varepsilon}{4},\ \sum\left\|(g^j)_1 - (g^j)_0\right\|_\infty < \frac{\varepsilon}{4},$$

$$\operatorname{Re}\sum(g^j)_1\left((f^j)_1\right) > 1-\eta,\ \operatorname{Re}\sum(g^j)_1\left((T^j)_0\left((f^j)_1\right)\right) > 1-\eta,$$

$$\text{for each } k \in J, (f^j)_1 \text{ and } (g^j)_1 \text{ are constant on } D_k$$

$$\sup \sum\left\{\left\|(h^j)_0(s) - (h^j)_0(t)\right\|_1 : s,t \in D_k\right\} \le \eta,\ \forall k \in J,$$

and

$$1 = \sum\left\|(h^j)_0\right\|_\infty = \sup \sum\left\{\left\|(h^j)_0(t)\right\|_1 : t \in \cup_{k\in J} D_k\right\}.$$

Since the set of simple functions is dense in both $L_1(\mu)$ and $L_\infty(\mu)$, there are simple functions $(f^j)_1 \in S_{L_1(\mu)}$ and $(g^j)_1 \in S_{L_\infty(\mu)}$ satisfying (164) and (165).

On the other hand, by [305, Theorem II.2, p. 42] there is a measurable subset E_1 of Ω such that $\mu(E_1) = 0$ and $(h^j)_0(\Omega \setminus E_1)$ is a separable subset of $L_1(\mu)$. Suppose that the set $\{y_i : i \in \mathbb{N}\}$ is dense in $(h^j)_0(\Omega \setminus E_1)$. Since $(f^j)_1$ and $(g^j)_1$ are simple functions, we can assume that $\operatorname{Im}\left((f^j)_1\right) = \{a_r^j : r = 1,\dots,n\}$ and $\operatorname{Im}\left((g^j)_1\right) = \{b_l^j : l = 1,\dots,m\}$. Now, for $i \in \mathbb{N}$, $r \in \{1,\dots,n\} = N$ and $l \in \{1,\dots,m\} = M$ we consider the following subsets of Ω

$$A_{(1,r,l)} = h_0^{-j}\left(B_{\frac{\eta}{2}}\left((y_j)_1\right)\right) \cap (\Omega \setminus E_1) \cap f_1^{-j}(a_r^j) \cap g_1^{-j}(b_l^j)$$

and

$$A_{(i,r,l)} = \left(h_0^{-j}\left(B_{\frac{\eta}{2}}\left((y_j)_i\right)\right) \setminus \cup_{e=1}^{i-1} h_0^{-j}\left(B_{\frac{\eta}{2}}\left((y_j)_e\right)\right)\right) \cap (\Omega \setminus E_1) \cap f_1^{-j}(a_r^j) \cap g_1^{-j}(b_l^j),\quad \forall i \ge 2.$$

It is clear that the elements of the family $\left\{A_{(i,r,l)} : (i,r,l) \in \mathbb{N} \times N \times M\right\}$ are measurable subsets of Ω and pairwise disjoint. Now, let $W = \left\{(i,r,l) \in \mathbb{N} \times N \times M : \mu\left(A_{(i,r,l)}\right) = 0\right\}$ and $E_2 = \cup_{(i,r,l)\in W} A_{(i,r,l)}$. By the definition of W it is trivially satisfied that E_2 is measurable and $\mu(E_2) =$

0. On the other hand there exists a measurable subset E_3 of $\Omega \setminus (E_1 \cup E_2)$ such that $\mu(E_3) = 0$ and $\|h^j\|_\infty = \sup \sum\{\| h^j(t) \|_1 : t \in \Omega \setminus E_3\}$. Assume that $\{D_k : k \in J\}$ is the family of pairwise disjoint measurable subsets obtained by indexing the set $\{A_{(i,r,l)} \setminus E_3 : (i,r,l) \in (\mathbb{N} \times N \times M) \setminus W\}$. Then, we have that $\mu(D_k) > 0$ for all $k \in J, \mu(\Omega \setminus \cup_{k \in J} D_k) = 0$ and also the family $\{D_k : k \in J\}$ satisfies the conditions (166), (167) and (168). Therefore, by (166) there are sets of scalars $\{\alpha_k^j : k \in J\}$ and $\{\gamma_k^j : k \in J\}$ such that

$$(f^j)_1 = \sum_{k \in J} \sum \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)}, \ \sum_{k \in J} \sum |\alpha_k^j| = 1,$$
$$(g^j)_1 = \sum_{k \in J} \sum \gamma_k^j \chi_{D_k}, \ \sum |\gamma_k^j| \le 1, \qquad \forall k \in J.$$

Step 2. In this step we will define another simple function $(f^j)_2 \in S_{L_1(\mu)}$ which is an approximation of $(f^j)_1$, and can be expressed as a finite sum instead of the countable sum appearing in the expression of $(f^j)_1$ given in (169).

By (169) and (165) there is a finite subset F of J such that

$$\sum_{k \in F} \sum |\alpha_k^j| > 1 - \eta > 0, \ \operatorname{Re} \sum (g^j)_1 \left(\sum_{k \in F} \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right) > 1 - \eta.$$

and also

$$\operatorname{Re} \sum (g^j)_1 \left((T^j)_0 \left(\sum_{k \in F} \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right) \right) > 1 - \eta.$$

For each $k \in F$ we put $\beta_k^j = \sum \frac{\alpha_k^j}{\sum_{k \in F} |\alpha_k^j|}$ and define $(f^j)_2 = \sum_{k \in F'} \sum \beta_k^j \frac{\chi_{D_k}}{\mu(D_k)}$. In view of (170) and (171) we have that

$$\operatorname{Re} \sum (g^j)_1 \left((f^j)_2 \right) = \operatorname{Re} \sum (g^j)_1 \left(\sum_{k \in F} \beta_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right) > 1 - \eta$$

and

$$\operatorname{Re} \sum (g^j)_1 \left((T^j)_0 \left((f^j)_2 \right) \right) = \operatorname{Re} \sum (g^j)_1 \left((T^j)_0 \left(\sum_{k \in F} \beta_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right) \right) > 1 - \eta.$$

Clearly $(f^j)_2 \in S_{L_1(\mu)}$ and by (169), (170) we have that

$$\sum \|(f^j)_2 - (f^j)_1\|_1 = \left\| \sum_{k \in F} \sum \beta_k^j \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k \in J} \sum \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right\|_1$$
$$= \left\| \sum_{k \in F} \sum \beta_k^j \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k \in F} \sum \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k \in J \setminus F} \sum \alpha_k^j \frac{\chi_{D_k}}{\mu(D_k)} \right\|_1$$

$$\leq \sum_{k\in F}\sum\left|\beta_k^j-\alpha_k^j\right| + \sum_{k\in J\setminus F}\sum\left|\alpha_k^j\right| = 1-\sum_{k\in F}\sum\left|\alpha_k^j\right| + \sum_{k\in J\setminus F}\sum\left|\alpha_k^j\right|$$
$$= 2\left(1-\sum_{k\in F}\sum\left|\alpha_k^j\right|\right) < 2\eta < \frac{\varepsilon}{4}.$$

Step 3. Now, we approximate the function $\left(h^j\right)_0$ by a new one $\left(h^j\right)_2$ such that for each $k\in F$ the new function is constant on each D_k. So we also approximate the operator $\left(T^j\right)_0$ by a new one.

For this aim we choose an element t_k in D_k, for any $k\in F$, put $\psi_k^j = \left(h^j\right)_0(t_k)\in L_1(\mu)$ and define $\left(h^j\right)_1 \in L_\infty(\mu, L_1(\mu))$ by
$$\left(h^j\right)_1 = \sum\left(h^j\right)_0 \chi_{\Omega\setminus(\cup_{k\in F}D_k)} + \sum_{k\in F}\sum \psi_k^j\chi_{D_k}.$$
By (168) we have that $\left\|\sum\left(h^j\right)_1\right\|_\infty \leq 1$. If $\left(T^j\right)_1\in\mathcal{L}(L_1(\mu))$ is the operator associated to $\left(h^j\right)_1$, then $\left(T^j\right)_1$ is the sum of $\left(T^j\right)_{0|\Omega\setminus\setminus\cup_{k\in F}D_k)}$ and a finite-rank operator, so $\left(T^j\right)_1\in B_{\mathcal{M}}^j$. By using (167), we clearly have
$$\sum\left\|\left(T^j\right)_1-\left(T^j\right)_0\right\|$$
$$= \sum\left\|\left(h^j\right)_1-\left(h^j\right)_0\right\|_\infty \leq \sup\ \sum\left\{\left\|\psi_k^j-\left(h^j\right)_0(t)\right\|_1 : t\in D_k, k\in F\right\}$$
$$= \sup\ \sum\left\{\left\|\left(h^j\right)_0(t_k)-\left(h^j\right)_0(t)\right\|_1 : t\in D_k, k\in F\right\} \leq \eta.$$
Since $\sum\left\|\left(T^j\right)_0\right\| = 1$ we get that $0<1-\eta\leq\left\|\sum\left(T^j\right)_1\right\|\leq 1$. Now we define $\left(T^j\right)_2 = \sum\frac{\left(T^j\right)_1}{\left\|\left(T^j\right)_1\right\|}$ and so we have that
$$\sum\left\|\left(T^j\right)_2-\left(T^j\right)_1\right\| = 1-\sum\left\|\left(T^j\right)_1\right\| \leq \eta.$$
In view of the previous inequality and (175) we obtain that
$$\sum\left\|\left(T^j\right)_2-\left(T^j\right)_0\right\| \leq \sum\left\|\left(T^j\right)_2-\left(T^j\right)_1\right\| + \sum\left\|\left(T^j\right)_1-\left(T^j\right)_0\right\| \leq 2\eta < \frac{\varepsilon}{4}.$$
From (173) and (176) we get that
$$\mathrm{Re}\sum\left(g^j\right)_1\left(\left(T^j\right)_2\left(\left(f^j\right)_2\right)\right)$$
$$\geq \mathrm{Re}\sum\left(g^j\right)_1\left(\left(T^j\right)_0\left(\left(f^j\right)_2\right)\right) - \sum\left\|\left(T^j\right)_2-\left(T^j\right)_0\right\| > 1-3\eta.$$
On the other hand, it is clear that
$$\left(T^j\right)_1\left(\left(f^j\right)_2\right) = \int_\Omega\sum\left(h^j\right)_1\left(f^j\right)_2 d\mu$$
$$= \int_{\Omega\setminus\cup_{k\in F}D_k}\sum\left(h^j\right)_1\left(f^j\right)_2 d\mu + \sum_{k\in F}\int_{D_k}\sum\left(h^j\right)_1\left(f^j\right)_2 d\mu = \sum_{k\in F}\sum\beta_k^j\psi_k^j.$$

For simplicity, for each $k \in F$, put $\phi_k^j = \sum \frac{\psi_k^j}{\|T_1^j\|}$. So we have that

$$\left(T^j\right)_2\left(\left(f^j\right)_2\right) = \sum_{k\in F} \sum \beta_k^j \phi_k^j.$$

It is clear that $\phi_k^j \in B_{L_1(\mu)}$ for every $k \in F$. From (172) and (177) we obtain that

$$\mathrm{Re} \sum \left(g^j\right)_1 \left(\sum_{k\in F} \frac{\beta_k^j}{2}\left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k^j\right)\right) = \mathrm{Re} \sum \left(g^j\right)_1 \left(\frac{\left(f^j\right)_2 + \left(T^j\right)_2\left(\left(f^j\right)_2\right)}{2}\right) > 1 - 2\eta.$$

Step 4. In this step we will obtain approximations $\left(f^j\right)_3$, $\left(T^j\right)_3$ of $\left(f^j\right)_2$ and $\left(T^j\right)_2$, respectively. We will check in the final step that $\left(T^j\right)_3$ attains its norm at $\left(f^j\right)_3$, a necessary condition for our purpose. In fact $\left(f^j\right)_3$ and $\left(T^j\right)_3$ are the final approximations to $\left(f^j\right)_0$ and $\left(T^j\right)_0$.

Define the set G as follows

$$G = \left\{k \in F : \mathrm{Re} \sum \left(g^j\right)_1 \left(\frac{\beta_k^j}{2}\left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k^j\right)\right) > \sum \left(1 - \sqrt{2\eta}\right)\left|\beta_k^j\right|\right\}.$$

In view of Lemma (5.3.2) we have that

$$\sum_{k\in G} \sum \left|\beta_k^j\right| > 1 - \sqrt{2\eta} = 1 - \frac{\varepsilon^4}{2^{16}}.$$

It is immediate that

$$\mathrm{Re} \sum \beta_k^j \left(g^j\right)_1 \left(\frac{\chi_{D_k}}{\mu(D_k)}\right) > \left(1 - 2\sqrt{2\eta}\right) \sum \left|\beta_k^j\right| = \left(1 - \frac{\varepsilon^4}{2^{15}}\right) \sum \left|\beta_k^j\right|, \qquad \forall k \in G.$$

So, for each $k \in G$ we have

$$\mathrm{Re} \sum \beta_k^j \gamma_k^j = \mathrm{Re} \sum \beta_k^j \left(g^j\right)_1 \left(\frac{\chi_{D_k}}{\mu(D_k)}\right) > \left(1 - \frac{\varepsilon^4}{2^{15}}\right) \sum \left|\beta_k^j\right|$$
$$\geq \left(1 - \frac{\varepsilon^4}{2^{15}}\right) \sum \left|\beta_k^j \gamma_k^j\right|.$$

Hence, we obtain that $\beta_k^j \neq 0$ for $k \in G$ and also that

$$\sum \left|\gamma_k^j\right| > 1 - \frac{\varepsilon^4}{2^{15}} > 0, \ \forall k \in G.$$

By using also Lemma (5.3.4) we get

$$\sum \left|\beta_k^j \gamma_k^j - \left|\beta_k^j \gamma_k^j\right|\right| < \frac{\varepsilon^2}{2^7} \sum \left|\beta_k^j \gamma_k^j\right|.$$

Hence,

$$\sum \left|\beta_k^j - \frac{\left|\beta_k^j \gamma_k^j\right|}{\gamma_k^j}\right| < \frac{\varepsilon^2}{2^7} \sum \left|\beta_k^j\right| \text{ and} \sum \left|\gamma_k^j - \frac{\left|\beta_k^j \gamma_k^j\right|}{\beta_k^j}\right|$$
$$< \frac{\varepsilon^2}{2^7} \sum \left|\gamma_k^j\right|, \ \forall k \in G,$$

so

$$\sum\left|\frac{\gamma_k^j}{|\gamma_k^j|}-\frac{|\beta_k^j|}{\beta_k^j}\right|<\frac{\varepsilon^2}{2^7},\quad \forall k\in G.$$

The element $(f^j)_3$ given by

$$(f^j)_3=\sum\frac{1}{\sum_{k\in G}|\beta_k^j|}\sum_{k\in G}\sum\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}\frac{\chi_{D_k}}{\mu(D_k)}$$

belongs to the unit sphere of $L_1(\mu)$. Now, by using (178) and (180) we get that

$$\begin{aligned}
&\sum\left\|(f^j)_3-(f^j)_2\right\|_1\\
&=\sum\left\|\frac{1}{\sum_{k\in G}|\beta_k^j|}\sum_{k\in G}\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}\frac{\chi_{D_k}}{\mu(D_k)}-\sum_{k\in F}\beta_k^j\frac{\chi_{D_k}}{\mu(D_k)}\right\|_1\\
&=\sum\left\|\frac{1}{\sum_{k\in G}|\beta_k^j|}\sum_{k\in G}\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}\frac{\chi_{D_k}}{\mu(D_k)}-\sum_{k\in G}\beta_k^j\frac{\chi_{D_k}}{\mu(D_k)}-\sum_{k\in F\setminus G}\beta_k^j\frac{\chi_{D_k}}{\mu(D_k)}\right\|_1\\
&\le\sum_{k\in G}\sum\left|\frac{1}{\sum_{k\in G}|\beta_k^j|}\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}-\beta_k^j\right|+\sum_{k\in F\setminus G}\sum|\beta_k^j|\\
&\le\sum_{k\in G}\sum\left|\frac{1}{\sum_{k\in G}|\beta_k^j|}\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}-\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}\right|+\sum_{k\in G}\sum\left|\frac{|\beta_k^j\gamma_k^j|}{\gamma_k^j}-\beta_k^j\right|+\sum_{k\in F\setminus G}\sum|\beta_k^j|\\
&\le 1-\sum_{k\in G}\sum|\beta_k^j|+\sum_{k\in G}\sum\frac{\varepsilon^2}{2^7}|\beta_k^j|+\sum_{k\in F\setminus G}\sum|\beta_k^j|\\
&\le 2\left(1-\sum_{k\in G}\sum|\beta_k^j|\right)+\frac{\varepsilon^2}{2^7}\le\frac{\varepsilon}{8}.
\end{aligned}$$

In view of (164),(174) and (182), we obtain that

$$\begin{aligned}
&\sum\left\|(f^j)_3-(f^j)_0\right\|_1\\
&\le\sum\left\|(f^j)_3-(f^j)_2\right\|_1+\sum\left\|(f^j)_2-(f^j)_1\right\|_1+\sum\left\|(f^j)_1-(f^j)_0\right\|_1\\
&<\frac{\varepsilon}{8}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\varepsilon.
\end{aligned}$$

Now notice obviously that

$$\operatorname{Re}\sum\beta_k^j(g^j)_1(\phi_k^j)>(1-2\sqrt{2\eta})\sum|\beta_k^j|>\left(1-\frac{\varepsilon^4}{2^{14}}\right)\sum|\beta_k^j|,\ \forall k\in G.$$

For each $k\in G$, define P_k as follows

$$P_k=\sum\left\{t\in\Omega:\operatorname{Re}\beta_k^j(g^j)_1(t)\phi_k^j(t)>\left(1-\frac{\varepsilon^2}{2^7}\right)|\beta_k^j\phi_k^j(t)|\right\}.$$

Clearly P_k is a measurable set. According to Lemma (5.3.3), for each $k\in G$ we have

$$\operatorname{Re}\int_{P_k}\sum\beta_k^j(g^j)_1\phi_k^j d\mu>\left(1-\frac{\varepsilon^2}{2^7}\right)\sum|\beta_k^j|,$$

so

$$\int_{P_k} \sum \left|\phi_k^j\right| d\mu > 1 - \frac{\varepsilon^2}{2^7} > 0.$$

Let us fix $k \in G$ and $t \in P_k$. Notice that $\sum \beta_k^j \left(g^j\right)_1 (t) \neq 0$. By Lemma (5.3.4) it follows

$$\sum \left|\beta_k^j\left(g^j\right)_1(t)\phi_k^j(t) - \left|\beta_k^j\left(g^j\right)_1(t)\phi_k^j(t)\right|\right| < \frac{\varepsilon}{2^3}\sum \left|\beta_k^j\left(g^j\right)_1(t)\phi_k^j(t)\right|,$$

So

$$\sum \left|\phi_k^j(t) - \frac{\left|\beta_k^j\left(g^j\right)_1(t)\phi_k^j(t)\right|}{\beta_k^j(g^j)_1(t)}\right| < \frac{\varepsilon}{2^3}\sum\left|\phi_k^j(t)\right|,\ \forall k \in G, t \in P_k.$$

For each $k \in G$ we can define the element φ_k^j in $L_1(\mu)$ by

$$\varphi_k^j = \sum \frac{\gamma_k^j}{\left|\gamma_k^j\right|} \frac{\left|\phi_k^j\right|}{\int_{P_k}\left|\phi_k^j\right| d\mu} \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1} \chi_{P_k}.$$

It is immediate that $\varphi_k^j \in S_{L_1(\mu)}$. From (184) and (185), for each $k \in G$ we have

$$\begin{aligned}
\sum \left\|\varphi_k^j - \phi_k^j\right\|_1 &\le \sum \left\|\varphi_k^j - \phi_k^j \chi_{P_k}\right\|_1 + \sum \left\|\phi_k^j \chi_{\Omega\setminus P_k}\right\|_1 \\
&< \sum \left\|\varphi_k^j - \phi_k^j \chi_{P_k}\right\|_1 + \frac{\varepsilon^2}{2^7} \\
&\le \sum \left\|\varphi_k^j - \frac{\gamma_k^j}{\left|\gamma_k^j\right|}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k}\right\|_1 \\
&\quad + \sum \left\|\frac{\gamma_k^j}{\left|\gamma_k^j\right|}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k} - \frac{\left|\beta_k^j\right|}{\beta_k^j}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k}\right\|_1 \\
&\quad + \sum \left\|\frac{\left|\beta_k^j\right|}{\beta_k^j}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k} - \phi_k^j\chi_{P_k}\right\|_1 + \frac{\varepsilon^2}{2^7} \\
&\le \sum \left\|\varphi_k^j - \frac{\gamma_k^j}{\left|\gamma_k^j\right|}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k}\right\|_1 + \sum\left|\frac{\gamma_k^j}{\left|\gamma_k^j\right|} - \frac{\left|\beta_k^j\right|}{\beta_k^j}\right| + \frac{\varepsilon}{2^3} + \frac{\varepsilon^2}{2^7} \\
&\le \sum \left\|\varphi_k^j - \frac{\gamma_k^j}{\left|\gamma_k^j\right|}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k}\right\|_1 + \frac{\varepsilon}{4} \text{(by (181))} \\
&= \sum \left\|\frac{\gamma_k^j}{\left|\gamma_k^j\right|} \frac{\left|\phi_k^j\right|}{\int_{P_k}\left|\phi_k^j\right| d\mu} \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k} - \frac{\gamma_k^j}{\left|\gamma_k^j\right|}\left|\phi_k^j\right| \frac{\left|\left(g^j\right)_1\right|}{(g^j)_1}\chi_{P_k}\right\|_1 + \frac{\varepsilon}{4} \\
&= 1 - \int_{P_k}\sum\left|\phi_k^j\right| d\mu + \frac{\varepsilon}{4} < \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.
\end{aligned}$$

Let the function $\left(h^j\right)_3$ be defined as follows

$$\left(h^j\right)_3 = \sum \frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty} \chi_{\Omega \setminus \cup_{k\in G} D_k} + \sum_{k\in G} \sum \varphi_k^j \chi_{D_k}.$$

It is easy to see that $\left(h^j\right)_3$ belongs to the unit sphere of $L_\infty(\mu, L_1(\mu))$. Let $\left(T^j\right)_3 \in S_{\mathcal{L}(L_1(\mu))}$ be the operator associated to the function h_3^j in view of Proposition (5.3.6). Since G is a finite set, $\mathcal{F}(L_1(\mu)) \subset \mathcal{M}$ and $\left(T^j\right)_1 \in \mathcal{M}$, by using the assumptions on $\mathcal{M}$ we know that $\left(T^j\right)_3 \in S_{\mathcal{M}}^j$.

We also have that

$$\sum \left\|\left(T^j\right)_3 - \left(T^j\right)_2\right\| = \sum \left\|\left(h^j\right)_3 - \frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty}\right\|_\infty$$

$$= \sum \left\|\left(h^j\right)_3 \chi_{\Omega\setminus(\cup_{k\in G} D_k)} + \sum_{k\in G} \left(h^j\right)_3 \chi_{D_k} - \frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} - \sum_{k\in G} \frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty} \chi_{D_k}\right\|_\infty$$

$$= \sum \left\|\frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} + \sum_{k\in G} \varphi_k^j \chi_{D_k} - \frac{\left(h^j\right)_1}{\|(h^j)_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} - \sum_{k\in G} \phi_k^j \chi_{D_k}\right\|_\infty$$

$$= \left\|\sum_{k\in G} \sum \left(\varphi_k^j - \phi_k^j\right) \chi_{D_k}\right\|_\infty = \sup_{k\in G} \sum \left\|\varphi_k^j - \phi_k^j\right\|_1 \le \frac{\varepsilon}{2}$$

By the previous inequality and (176) we obtain

$$\sum \left\|\left(T^j\right)_3 - \left(T^j\right)_0\right\| \le \sum \left\|\left(T^j\right)_3 - \left(T^j\right)_2\right\| + \sum \left\|\left(T^j\right)_2 - \left(T^j\right)_0\right\| < \varepsilon.$$

Step 5. Finally, we are going to find an approximation of $\left(g^j\right)_1$ and complete our proof.

We put $A = \left\{t \in \Omega: \sum \left|\left(g^j\right)_1(t)\right| \ge 1 - \frac{\varepsilon^2}{2^7}\right\}$ and let the function $\left(g^j\right)_2$ be defined by $\left(g^j\right)_2 = \sum \frac{\left(g^j\right)_1}{\left|\left(g^j\right)_1\right|} \chi_A + \sum \left(g^j\right)_1 \chi_{\Omega\setminus A}$. Since $\left(g^j\right)_1 \in S_{L_\infty(\mu)}$, we have that $\left(g^j\right)_2 \in S_{L_\infty(\mu)}$. It is also clear that

$$\sum \left\|\left(g^j\right)_2 - \left(g^j\right)_1\right\|_\infty \le \frac{\varepsilon^2}{2^7}.$$

By using (164) and (188) we also have that

$$\sum \left\|\left(g^j\right)_2 - \left(g^j\right)_0\right\|_\infty \le \sum \left\|\left(g^j\right)_2 - \left(g^j\right)_1\right\|_\infty + \sum \left\|\left(g^j\right)_1 - \left(g^j\right)_0\right\|_\infty \le \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \varepsilon.$$

By (179) we know that $\sum \left|\gamma_k^j\right| > 1 - \frac{\varepsilon^4}{2^{15}}$ for each $k \in G$. Since $G \subset J$, in view of (169), the restriction of $\left(g^j\right)_1$ to D_k coincides with γ_k^j and so $D_k \subset A$ for all $k \in G$. Hence,

$$\left(g^j\right)_{2|D_k} = \sum \frac{\gamma_k^j}{\left|\gamma_k^j\right|}, \ \forall k \in G.$$

Therefore, we deduce that

$$\left(g^j\right)_2 \left(\left(f^j\right)_3\right) = \sum \left(g^j\right)_2 \left(\frac{1}{\sum_{k\in G} \left|\beta_k^j\right|} \sum_{k\in G} \frac{\left|\beta_k^j \gamma_k^j\right|}{\gamma_k^j} \frac{\chi_{D_k}}{\mu(D_k)}\right)$$

$$= \sum \frac{1}{\sum_{k\in G} |\beta_k^j|} \sum_{k\in G} \frac{|\beta_k^j \gamma_k^j|}{\gamma_k^j} \frac{1}{\mu(D_k)} (g^j)_2(\chi_{D_k})$$
$$= \sum \frac{1}{\sum_{k\in G} |\beta_k^j|} \sum_{k\in G} \frac{|\beta_k^j \gamma_k^j|}{\gamma_k^j} \frac{\gamma_k^j}{|\gamma_k^j|} = 1.$$
For each $k \in G$, from the definition of P_k and A, we deduce that $P_k \subset A$, so
$$(g^j)_2(\varphi_k^j) = \int_{P_k} \sum \frac{\gamma_k^j}{|\gamma_k^j|} \frac{|\phi_k^j|}{\int_{P_k} |\phi_k^j| d\mu} d\mu = \sum \frac{\gamma_k^j}{|\gamma_k^j|}.$$
Since
$$(T^j)_3\left((f^j)_3\right) = \int_\Omega \sum (h^j)_3 (f^j)_3 d\mu = \sum \frac{1}{\sum_{k\in G} |\beta_k^j|} \sum_{k\in G} \frac{|\beta_k^j \gamma_k^j|}{\gamma_k^j} \varphi_k^j,$$
by using (191) we have that
$$(g^j)_2\left((T^j)_3\left((f^j)_3\right)\right) = \sum \frac{1}{\sum_{k\in G} |\beta_k^j|} \sum_{k\in G} \frac{|\beta_k^j \gamma_k^j|}{\gamma_k^j} (g^j)_2(\varphi_k^j)$$
$$= \sum \frac{1}{\sum_{k\in G} |\beta_k^j|} \sum_{k\in G} \frac{|\beta_k^j \gamma_k^j|}{\gamma_k^j} \frac{\gamma_k^j}{|\gamma_k^j|} = 1.$$
We have shown that there are elements $(T^j)_3 \in S_{\mathcal{M}}^j, (f^j)_3 \in S_{L_1(\mu)}$ and $(g^j)_2 \in S_{L_\infty(\mu)}$ that in view of (183),(187),(189),(190) and (192) satisfy
$$\sum \left\|(T^j)_3 - (T^j)_0\right\| < \varepsilon, \ \sum \left\|(f^j)_3 - (f^j)_0\right\|_1 < \varepsilon,$$
$$\sum \left\|(g^j)_2 - (g^j)_0\right\|_\infty < \varepsilon$$
and also
$$\sum (g^j)_2\left((f^j)_3\right) = \sum (g^j)_2\left((T^j)_3\left((f^j)_3\right)\right) = 1.$$
So we showed that $\mathcal{M}$ has the BPBp- ν with the function η given by $\eta(\varepsilon) = \frac{\varepsilon^8}{2^{33}}$.

Corollary (5.3.12)[365]: Let (Ω, Σ, μ) be a σ-finite measure space. The following subspaces of $\mathcal{L}(L_1(\mu))$ have the $BPBp$-ν and the function η satisfying Definition (5.3.5) is independent from the measure space.

(a) The subspace of all sequence of finite-rank operators on $L_1(\mu)$.
(b) The subspace of all sequence of compact operators on $L_1(\mu)$.
(c) The subspace of all sequence of weakly compact operators on $L_1(\mu)$.

In case that μ is finite, then the subspace of all representable operators on $L_1(\mu)$ also has the $BPBp - \nu$.

Proof. Assume first that μ is a finite measure. It is known that $\mathcal{F}(L_1(\mu)) \subset \mathcal{K}(L_1(\mu)) \subset \mathcal{WC}(L_1(\mu)) \subset \mathcal{R}(L_1(\mu))$ and $T_{|A}^j(B_{L_1(\mu)}) \subset T^j(B_{L_1(\mu)})$ for each $T^j \in \mathcal{L}(L_1(\mu))$ and every measurable subset A of Ω. Also, it is clear that $T_{|A}^j \in \mathcal{R}(L_1(\mu))$ for any $T^j \in \mathcal{R}(L_1(\mu))$ and every measurable subset A of Ω. Therefore, the spaces $\mathcal{F}(L_1(\mu)), \mathcal{K}(L_1(\mu)), \mathcal{WC}(L_1(\mu))$ and

$\mathcal{R}(L_1(\mu))$ satisfy the assumptions of Theorem (5.3.7), and so the above statements hold in case that μ is finite.

Now, let μ be a σ-finite measure. We will show that the space $\mathcal{F}(L_1(\mu))$ satisfies the BPBp- ν. There is a finite measure ζ and a surjective linear isometry Φ^j from $L_1(\mu)$ into $L_1(\zeta)$. The mapping Φ^j induces a surjective linear isometry from $\mathcal{F}(L_1(\mu)))$ into $\mathcal{F}(L_1(\zeta)))$ given by $T^j \mapsto \Phi^j \circ T^j \circ \Phi^{-j}$. Since Φ^j is an isometry, it follows that $\nu(T^j) = \sum \nu\left(\Phi^j \circ T^j \circ \Phi^{-j}\right)$ for every $T^j \in \mathcal{F}(L_1(\mu))$. On the other hand, it is satisfied that $(f^j, g^j) \in \Pi(L_1(\mu))$ if and only if $\sum(\Phi^j(f^j), (\Phi^{-j})^t(g^j)) \in \Pi(L_1(\zeta))$. Also $\sum\left(\Phi^{-j}\right)^t(g^j)\left(\Phi^j \circ T^j \circ \Phi^{-j}\left(\Phi^j(f^j)\right)\right) = g^j(T^j(f^j))$ for every $T^j \in \mathcal{F}(L_1(\mu))\Big)$. Since $\mathcal{F}(L_1(\zeta))$ has the BPBp- ν we deduce the same property for $\mathcal{F}(L_1(\mu))$.

Chapter 6
Bishop-Phelps-Bollobás Version and Theorem

We study Banach spaces Y such that (X, Y) has the Bishop-Phelps-Bollobás property for every Banach space X. In this case, we show that there is a universal function $\eta Y(\varepsilon)$ such that for every X, the pair (X, Y) has the BPBP with this function. This implies that this property of Y is strictly stronger than Lindenstrauss property B. We get these results is the study of the Bishop-Phelps-Bollobás property for c_0-, ℓ_1- and ℓ_∞- sums of Banach spaces. The Bishop-Phelps-Bollobás theorem holds when the range space is finite-dimensional, an $L_1(\mu)$-space for a σ-finite measure μ, a $C(K)$-space for a compact Hausdorff space K, or a uniformly convex Banach space. This result implies that Bishop-Phelps-Bollobás theorem holds for operators from ℓ_1 into such direct sums of Banach spaces. We also show that the direct sum of two spaces with the approximate hyperplane series property has such property whenever the norm of the direct sum is absolute.

Section (6.1): Lindenstrauss Properties A and B:

In 1963, J. Lindenstrauss [327] examined the extension of the Bishop-Phelps theorem, on denseness of the family of norm-attaining scalar-valued functionals on a Banach space, to vector-valued linear operators. He introduced two universal properties, A and B, that a Banach space might have. Seven years later, B. Bollobás observed that there is a numerical version of the Bishop-Phelps theorem, and his contribution is known as the Bishop-Phelps-Bollobás theorem. Vector-valued versions of this result have been studied (see, e.g., [310]). We introduce and study analogues of properties A and B of vector-valued versions of the Bishop-Phelps-Bollobás theorem.

This time giving the necessary background material to help make entirely accessible. The Bishop-Phelps-Bollobás property was introduced in 2008 [310] as an extension of the Bishop-Phelps-Bollobás theorem to the vector-valued case. It can be regarded as a "quantitative version" of the study of norm-attaining operators initiated by J. Lindenstrauss in 1963. Let X and Y be Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will use the common notation S_X, B_X, X^* for the unit sphere, the closed unit ball and the dual space of X respectively, $L(X, Y)$ for the Banach space of all bounded linear operators from X into Y, and $NA(X, Y)$ for the subset of all norm-attaining operators. (We say that an operator $T \in L(X, Y)$ attains its norm if $\| T \| = \| Tx \|$ for some $x \in S_X$.) We will abbreviate $L(X, X)$, resp. $NA(X, X)$, by $L(X)$, resp. $NA(X)$.

Definition (6.1.1)[308]: ([310], Definition 1.1). A pair of Banach spaces (X, Y) is said to have the Bishop-Phelps-Bollobás property ($BPBp$ for short) if for every $\varepsilon \in (0,1)$ there is $\eta(\varepsilon) > 0$ such that for every $T_0 \in L(X, Y)$ with $\|T_0\| = 1$ and every $x_0 \in S_X$ satisfying

$$\|T_0(x_0)\| > 1 - \eta(\varepsilon),$$

there exist $S \in L(X, Y)$ and $x \in S_X$ such that

$$1 = \| S \| = \| Sx \|, \ \|x_0 - x\| < \varepsilon \ \text{ and } \ \|T_0 - T\| < \varepsilon.$$

In this case, we will say that (X, Y) has the BPBp with function $\varepsilon \longmapsto \eta(\varepsilon)$.

The study of the "denseness of norm-attaining things" goes back to the celebrated Bishop-Phelps theorem [315] which appeared in 1961. This theorem simply states that $NA(X, \mathbb{K})$ is dense in X^* for every Banach space X. The problem of the denseness of $NA(X, Y)$ in $L(X, Y)$ was given, and in 1963J. Lindenstrauss [327] provided a simple example to show that it is not true in general. On the other hand, motivated by some problems in numerical range theory, in

1970 B. Bollobás [316] gave a "quantitative version" of the Bishop-Phelps theorem which, in the above language, states that the pair $(X, \mathbb{K})$ has the BPBp for every Banach space X. Nowadays, this result is known as the Bishop-Phelps-Bollobás theorem. See the expository [309] for a detailed account on norm-attaining operators; any result not explicitly referred to below can be found there.

As the problem of the denseness of norm attaining operators is so general, J. Lindenstrauss [327] introduced and studied the following two properties. A Banach space X is said to have Lindenstrauss property A if $\overline{NA(X,Z)} = L(X,Z)$ for every Banach space Z. A Banach space Y is said to have Lindenstrauss property B if $\overline{NA(Z,Y)} = L(Z,Y)$ for every Banach space Z. We remark that both properties A and B are isometric in nature (that is, they depend upon the particular norm).

Lindenstrauss property A is trivially satisfied by finite-dimensional spaces by compactness of the closed unit ball, and it is also satisfied by reflexive spaces [327]. J. Bourgain [317] proved in 1977 that the Radon-Nikodým property characterizes Lindenstrauss property A isomorphically: a Banach space has the Radon-Nikodým property if and only if it has Lindenstrauss property A in every equivalent norm. From the isometric viewpoint, W. Schachermayer [330] introduced property α, which implies Lindenstrauss property A. It is satisfied, for instance, by ℓ_1, and it is also satisfied in many Banach spaces, including all separable spaces, after an equivalent renorming. There is a weakening of property α, called property quasi-α introduced by Y. S. Choi and H. G. Song [321] which still implies Lindenstrauss property A. On the other hand, examples of Banach spaces failing Lindenstrauss property A are c_0, non-atomic $L_1(\mu)$ spaces, $C(K)$ spaces for infinite and metrizable K, and the canonical predual of any Lorentz sequence space $d(w,1)$ with $w \in \ell_2 \setminus \ell_1$. Finally, we mention that Lindenstrauss property A is stable under arbitrary ℓ_1-sums [321].

Less is known about Lindenstrauss property B. The base field $\mathbb{K}$ clearly has it, since this is just the Bishop-Phelps theorem. However, it is unknown whether every finite-dimensional space has Lindenstrauss property B, even for two-dimensional Euclidean space. J. Lindenstrauss gave an isometric sufficient condition for property B, called property β, which is satisfied by polyhedral finite-dimensional spaces, and by any space between c_0 and ℓ_∞ inclusive. It was proved later by J. Partington [329] that every Banach space can be equivalently renormed to have property β and, therefore, to have Lindenstrauss property B. There is a weakening of property β, called property quasi- β and introduced by M. Acosta, F. Aguirre and R. Payá [313], which provides new examples of spaces with Lindenstrauss property B, such as some non-polyhedral finite-dimensional spaces and the canonical predual of any Lorentz sequence space $d(w,1)$ with $w \in c_0 \setminus \ell_1$. Among spaces without Lindenstrauss property B we find infinite-dimensional $L_1(\mu)$ spaces, $C[0,1], d(w,1)$ with $w \in \ell_2 \setminus \ell_1$ and every infinite-dimensional strictly convex space (or $\mathbb{C}$-rotund space in the complex case); in particular ℓ_p for $1 < p < \infty$ all fail Lindenstrauss property B. We finish by mentioning that Lindenstrauss property B is stable under arbitrary c_0-Sums [313].

Since its introduction in 2008, quite a few papers regarding the Bishop-Phelps-Bollobás property have been published (see, e.g., [312],[314],[318], [322], [326]). Among others, the following pairs have been shown to have the BPBp: $(L_1(\mu), L_\infty[0,1])$ for every σ-finite measure μ, $(X, C_0(L))$ for every Asplund space X and every Hausdorff locally compact space L, (X, Y) when X is uniformly convex or Y has property β, or if both X and Y are finite-dimensional.

We will deal with the following definitions, which are exactly the BPB versions of Lindenstrauss properties A and B.

Definition (6.1.2)[308]: Let X and Y be Banach spaces. We say that X is a universal BPB domain space if for every Banach space Z, the pair (X,Z) has the BPBp. We say that Y is a universal BPB range space if for every Banach space Z, the pair (Z,Y) has the BPBp.

The following assertions are clearly true:

(a) a universal BPB domain space has Lindenstrauss property A,

(b) a universal BPB range space has Lindenstrauss property B.

The converse of (a) is known to be false: the space ℓ_1 has Lindenstrauss property A but fails to be a universal BPB domain space [310]. Even more, every finite-dimensional Banach space clearly has Lindenstrauss property A, but ℓ_1^2 fails to be a universal BPB domain space. (We will prove this later in Corollary (6.1.15), but it can be found by "surfing" into the details of the proofs in [310].) Also, a twodimensional real space is a universal BPB domain space if and only if it is uniformly convex [326] (see Corollary (6.1.17) below).

The validity of the converse of (b) has been pending from the beginning of the study of the BPBp, since the basic examples of spaces with Lindenstrauss property B, i.e. those having property β, are actually universal BPB range spaces [310]. We will provide an example of a Banach space having Lindenstrauss property B but failing to be a universal BPB range space.

We compare the function $\eta(\varepsilon)$ appearing in the definition of the BPBp for different pairs of spaces.

Notation (6.1.3)[308]: Fix a pair (X,Y) of Banach spaces and write

$$\Pi(X,Y)=\{(x,T)\in X\times L(X,Y):\| T \|=\| x \|=\| Tx \|=1\}.$$

For every $\rho\in(0,1)$, let

$$\mathcal{S}(X,Y)(\rho):=\{(S,x)\in L(X,Y)\times X:\| S \|=\| x \|=1,\| Sx \|>1-\rho\}.$$

Also, for every $\varepsilon\in(0,1)$ we define $\eta(X,Y)(\varepsilon)$ to be the supremum of the set consisting of 0 and those $\rho>0$ such that for all pairs $(T_0,x_0)\in\mathcal{S}(X,Y)(\rho)$, there exists a pair $(T,x)\in\Pi(X,Y)$ such that $\|x_0-x\|<\varepsilon$ and $\|T_0-T\|<\varepsilon$. Equivalently,

$$\eta(X,Y)(\varepsilon)=\inf\{1-\| Tx \|:x\in S_X,T\in L(X)$$
$$\| T \|=1,\operatorname{dist}((x,T),\Pi(X,Y))\geqslant\varepsilon\}$$

where $\operatorname{dist}((x,T),\Pi(X,Y))=\inf\{\max\{\| x-y \|,\| T-S \|\}:(y,S)\in\Pi(X,Y)\}$.

It is clear that the pair (X,Y) has the BPBp if and only if $\eta(X,Y)(\varepsilon)>0$ for every $\varepsilon\in(0,1)$. By construction, if a function $\varepsilon\longmapsto\eta(\varepsilon)$ is valid in the definition of the BPBp for the pair (X,Y), then $\eta(\varepsilon)\leqslant\eta(X,Y)(\varepsilon)$. That is, $\eta(X,Y)(\varepsilon)$ is the best function (i.e. the largest) we can find to ensure that (X,Y) has the BPBp. It is also immediate that $\eta(X,Y)(\varepsilon)$ is increasing with respect to ε.

We first study the behavior of the BPBp with respect to direct sums of Banach spaces. Specifically, we prove that given two families $\{X_i:i\in I\}$ and $\{Y_j:j\in J\}$ of Banach spaces, if X is the c_0-, ℓ_1^-or ℓ_∞-sum of $\{X_i\}$ and Y is the c_0-, ℓ_1-or ℓ_∞-sum of $\{Y_j\}$, then

$$\eta(X,Y)(\varepsilon)\leqslant\eta(X_i,Y_j)(\varepsilon)\ (i\in I,j\in J)$$

Therefore, if the pair (X,Y) has the BPBp, then every pair (X_i,Y_j) does with a non-worse function η. The main consequence of this result is that every universal BPB space has a "universal" function η. That is, if X is a universal BPBp domain space, then

$$\inf\{\eta(X,Z)(\varepsilon):Z\text{ Banach space }\}>0\ (\varepsilon\in(0,1))$$

if Y is a universal BPBp range space, then

$$\inf\{\eta(Z,Y)(\varepsilon): Z \text{ Banach space }\} > 0 \ (\varepsilon \in (0,1)).$$

With this fact in mind, we see from [326] (where the existence of a universal function was hypothesized) some necessary conditions for a Banach space to be a universal BPB domain space. In particular, a real, two-dimensional universal BPB domain space must be uniformly convex. Another related result is that for Banach spaces X and Y and a compact Hausdorff space K, if the pair $(X, C(K,Y))$ has the BPBp, then so does (X,Y). We also provide a new result on the stability of density of norm-attaining operators from which we can deduce that $NA(X)$ is not dense in $L(X)$ for the space $X = C[0,1] \oplus_1 L_1[0,1]$. With respect to domain spaces, we obtain the following result: If a Banach space contains a non-trivial L-summand, then it is not a universal BPB domain space. Hence every Banach space of dimension greater than one can be equivalently renormed so as not to be a universal BPB domain space.

With respect to range spaces, we provide an example of a Banach space $\mathcal{Y}$ which is the c_0-sum of a family of polyhedral spaces of dimension two such that $(\ell_1^2, \mathcal{Y})$ does not have the BPBp. This is the announced result of a Banach space having Lindenstrauss property B (in fact, even having property quasi- β of [313]) and yet failing to be a universal BPB range space. It also follows that being a universal BPB range space is not stable under infinite c_0 - or ℓ_∞-sums.

Contains the results on direct sums which will be the main tools for the rest. We give the results related to domain spaces, while our results related to range spaces are discussed.

We study the relationship between the BPBp and certain direct sums of Banach spaces. We will use these results later to get the main theorems.

Our principal deals with c_0-, ℓ_1-, and ℓ_∞-sums of Banach spaces. Given a family $\{X_\lambda: \lambda \in \Lambda\}$ of Banach spaces, we denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_0}$) the c_0-sum (resp. ℓ_1-sum, ℓ_∞-sum) of the family. In case Λ has just two elements, we use the simpler notation $X \oplus \infty Y$ and $X \oplus 1$.

Theorem (6.1.4)[308]: Let $\{X_i: i \in I\}$ and $\{Y_j: j \in J\}$ be families of Banach spaces, let X be the c_0-, ℓ_1-, or $\ell_\infty -$ sum of $\{X_i\}$ and let Y be the c_0-, ℓ_1 -, or ℓ_∞ sum of $\{Y_j\}$. If the pair (X,Y) has the BPBp with $\eta(\varepsilon)$, then the pair (X_i, Y_j) also has the $BPBp$ with $\eta(\varepsilon)$ for every $i \in I, j \in J$. In other words,

$$\eta(X,Y) \leqslant \eta(X_i, Y_j) \ (i \in I, j \in J)$$

This theorem will be obtained by simply combining Proposition (6.1.5), Proposition (6.1.8), and Proposition (6.1.9) below. In some cases, we are able to provide partial converses.

Before providing these propositions, and therefore the proof of Theorem (6.1.4), we present its main consequence: universal BPB spaces have "universal" functions η. We will frequently appeal to the following result.

Corollary (6.1.4)[308]: Let X and Y be Banach spaces.

(a) If X is a universal BPB domain space, then there is a function $\eta_X : (0,1) \longrightarrow \mathbb{R}^+$ such that for every Banach space Y, (X,Y) has the $BPBp$ with η_X. In other words, for every $Y, \eta(X,Y) \geqslant \eta_X$.

(b) If Y is a universal BPB range space, then there is a function $\eta_Y: (0,1) \longrightarrow \mathbb{R}^+$ such that for every Banach space X, (X,Y) has the BPBp with η_Y. In other words, for every $X, \eta(X,Y) \geqslant \eta_Y$.

Proof. (a) Assume that (X,Y) has the BPBp for every Banach space Y, but no such universal function η_X exists. Then, for some $\varepsilon > 0$, there exists a sequence of Banach spaces $\{Y_n\}$ such that (X,Y_n) has the BPBp and $\eta(X,Y_n)(\varepsilon) \longrightarrow 0$ when $n \to \infty$. But if we consider the space $Y = \left[\oplus_{n\in\mathbb{N}} Y_j\right]_{c_0}$, then (X,Y) has the BPBp by the BPB universality of X, and Theorem (6.1.4) (actually Proposition (6.1.5) below) gives $\eta(X,Y_n)(\varepsilon) \geqslant \eta(X,Y)(\varepsilon) > 0$, a contradiction.

(b) The same idea as in (a) works, with ℓ_1-sums of domain spaces substituting $c_0 -$ sums of range spaces.

We are now ready to provide the following three propositions which will give Theorem (6.1.4). The first is the most natural case: ℓ_1-sums of domain spaces and c_0- or ℓ_∞-sums of range spaces.

We observe that this result appeared in [320], but not interested in controlling the function $\eta(\varepsilon)$, which will be of relevance to us.

Proposition (6.1.5)[308]: Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be families of Banach spaces, $X = [\oplus_{i\in I} X_i]_{\ell_1}$ and $Y = \left[\oplus_{j\in J} Y_j\right]_{\ell_\infty}$ or $Y = \left[\oplus_{j\in J} Y_j\right]_{c_0}$. If the pair (X,Y) has the $BPBp$ with $\eta(\varepsilon)$, then (X_i, Y_j) also has the $BPBp$ with $\eta(\varepsilon)$ for every $i \in I$, $j \in J$. In other words,

$$\eta(X,Y)(\varepsilon) \leqslant \eta(X_i, Y_j)(\varepsilon) \quad (i \in I, j \in J).$$

Proof. Let E_i and F_j denote the natural isometric embeddings of X_i and Y_j into X and Y, respectively, and let P_i and Q_j denote the natural normone projections from X and Y onto X_i and Y_j, respectively. For $T \in L(X,Y)$, we can easily see that

$$\| T \| = \sup\{\|Q_j T\| : j \in J\} = \sup\{\|T E_i\| : i \in I\}$$

(A proof of this fact can be found in [328], for instance.) Hence

$$\| T \| = \sup\{\|Q_j T E_i\| : i \in I, j \in J\}.$$

Fix $h \in I$ and $k \in J$. To show that the pair (X_h, Y_k) satisfies the BPBp with function $\eta(\varepsilon)$, suppose that $\|T(x_h)\| > 1 - \eta(\varepsilon)$ for $T \in S_{L(X_h,Y_k)}$ and $x_h \in S_{X_h}$. Consider the linear operator $\tilde{T} = F_k T P_h \in L(X,Y)$. Note that $Q_j\tilde{T} = 0$ for $j \neq k$ and $Q_k \tilde{T} E_i = 0$ for $i \neq h$, while $Q_k \tilde{T} E_h = T$, $\| \tilde{T} \| = \| T \| = 1$ and $\|\tilde{T}(E_h x_h)\| > 1 - \eta(\varepsilon)$.

Since the pair (X,Y) has the BPBp, there exists $(x_0, \tilde{S}) \in \Pi(X,Y)$ such that

$$\| \tilde{T} - \tilde{S} \| < \varepsilon \text{ and } \|x_0 - E_h(x_h)\| < \varepsilon.$$

Let $S = Q_k \tilde{S} E_h \in L(X_h, Y_k)$. Clearly $\| S \| \leqslant 1$ and $\| S - T \| \leqslant \| \tilde{S} - \tilde{T} \| < \varepsilon$. Now we want to show that S attains its norm at $P_h(x_0)$ and that $\|P_h(x_0)\| = 1$. Indeed, for $j \in J, j \neq k$ one has

$$\|Q_j \tilde{S} x_0\| = \|Q_j \tilde{S} x_0 - Q_j \tilde{T} x_0\| \leqslant \| \tilde{S} - \tilde{T} \| < \varepsilon < 1$$

Hence $\|\tilde{S} x_0\| = 1 = \|Q_k \tilde{S} x_0\|$, which shows that $\|Q_k \tilde{S}\| = 1$ and $Q_k \tilde{S}$ attains its norm at x_0. Similarly, for $i \in I, i \neq h$ we have

$$\|Q_k \tilde{S} E_i\| = \|Q_k \tilde{S} E_i - Q_k \tilde{T} E_i\| \leqslant \| \tilde{S} - \tilde{T} \| < \varepsilon < 1$$

and so $\|Q_k \tilde{S}\| = 1 = \|Q_k \tilde{S} E_h\| = \| S \|$. Since

$$\begin{aligned} 1 &= \|Q_k \tilde{S} x_0\| \leqslant \sum_{i\in I} \|Q_k \tilde{S} E_i P_i x_0\| = \|S P_h x_0\| + \sum_{i\in I, i\neq h} \|Q_k \tilde{S} E_i P_i x_0\| \\ &\leqslant \|P_h x_0\| + \varepsilon \sum_{i\in I, i\neq h} \|P_i x_0\| \leqslant \|P_h x_0\| + \sum_{i\in I, i\neq h} \|P_i x_0\| = 1 \end{aligned}$$

we have $\|P_i x_0\| = 0$ for $i \neq h$ and $\|S(P_h x_0)\| = \|P_h x_0\| = 1$. Further,
$$\|P_h x_0 - x_h\| = \|P_h\big(x_0 - E_h(x_h)\big)\| \leqslant \|x_0 - E_h(x_h)\| < \varepsilon.$$
The next proposition gives when the converse result is possible, but only for range spaces.

Proposition (6.1.6)[308]: Let X be a Banach space and let $\{Y_j : j \in J\}$ be a family of Banach spaces. Then, for both $Y = \left[\oplus_{j\in J} Y_j\right]_{c_0}$ and $Y = \left[\oplus_{j\in J} Y_j\right]_{\ell_\infty}$, one has
$$\eta(X,Y) = \inf_{j\in J} \eta\big(X, Y_j\big).$$
Consequently, the following four conditions are equivalent:

(i) $\inf_{j\in J} \eta\big(X, Y_j\big)(\varepsilon) > 0$ for all $\varepsilon \in (0,1)$,

(ii) every pair $\big(X, Y_j\big)$ has the $BPBp$ with a common function $\eta(\varepsilon) > 0$,

(iii) the pair $\left(X, \left[\oplus_{j\in J} Y_j\right]_{\ell_\infty}\right)$ has the $BPBp$,

(iv) the pair $\left(X, \left[\oplus_{j\in J} Y_j\right]_{c_0}\right)$ has the $BPBp$.

Proof. The inequality $\leqslant$ follows from Proposition (6.1.5). So let us prove the converse inequality. To do this, we fix $\varepsilon \in (0,1)$. Write $\eta(\varepsilon) = \inf_{j\in J} \eta\big(X, Y_j\big)$ and suppose $\eta(\varepsilon) > 0$ (otherwise there is nothing to prove), so $\eta\big(X, Y_j\big) \geqslant \eta(\varepsilon) > 0$. Let F_j and Q_j be as in the proof of Proposition (6.1.5). Suppose that $T \in S_{L(X,Y)}$ and $x_0 \in S_X$ satisfy
$$\|T x_0\| > 1 - \eta(\varepsilon).$$
Then there is $j_0 \in I$ such that $\|Q_{j_0} T x_0\| > 1 - \eta(\varepsilon)$. By the assumption on ε, there are an operator $S_{j_0} : X \longrightarrow Y_{j_0}$ and a vector $u \in S_X$ such that
$$\|S_{j_0}\| = \|S_{j_0} u\| = 1,\ \|S_{j_0} - Q_{j_0} T\| < \varepsilon,\ \text{ and } \|x_0 - u\| < \varepsilon.$$
Define $S : X \longrightarrow Y$ by
$$S = \sum_{j\neq j_0} F_j Q_j T + F_{j_0} S_{j_0}$$
Clearly $\| S \| \leqslant 1$ and $\| Su \| \geqslant \|S_{j_0} u\| = 1$, so $(u,S) \in \Pi(X,Y)$. On the other hand,
$$\| T - S \| = \sup_{j\in J} \|Q_j(T - S)\| = \|Q_{j_0} T - S_{j_0}\| < \varepsilon.$$
We have proved that (X,Y) has the BPBp with the function $\eta(\varepsilon)$. In other words, $\eta(X,Y)(\varepsilon) \geqslant \eta(\varepsilon)$

One particular case of Proposition (6.1.6) is that the BPBp is stable for finite ℓ_∞- sums of range spaces.

Corollary (6.1.7)[308]: Let $X, Y_1, \dots, Y_m$ be Banach spaces and write $Y = Y_1 \oplus_\infty \cdots \oplus_\infty Y_m$. Then, (X,Y) has the $BPBp$ if and only if $\big(X, Y_j\big)$ has the $BPBp$ for every $j = 1, \dots, m$. As a consequence, $Y_1, \dots, Y_m$ are universal BPB range spaces if and only if Y is.

Note that since ℓ_1 is not a universal BPB domain space, we cannot expect an analogue of Proposition (6.1.6) for ℓ_1-sums of domain spaces. Indeed, even Corollary (6.1.7) has no counterpart for finite ℓ_1-sums of domain spaces. This follows from the fact that ℓ_1^2 fails to be a universal BPB domain space (Corollary (6.1.15) below or [326], Corollary 9).

The second result deals with c_0-or $\ell_\infty -$ Sums of domain spaces.

Proposition (6.1.8)[308]: Let $\{X_i : i \in I\}$ be a family of Banach spaces, $X = [\oplus_{i\in I} X_i]_{c_0}$ or $X = [\oplus_{i\in I} X_i]_{\ell_\infty}$, and let Y be a Banach space. If the pair (X,Y) has the BPBp with $\eta(\varepsilon)$, then the pair (X_i, Y) also has the $BPBp$ with $\eta(\varepsilon)$ for every $i \in I$. In other words, $\eta(X,Y) \leqslant \eta(X_i,Y)$ for every $i \in I$.

Proof. For every $i \in I$ we may write $X = X_i \oplus_\infty Z$ for a suitable Banach space Z. Thus, without loss of generality, we may assume that $X = X_1 \oplus_\infty X_2$.

For fixed $\varepsilon \in (0,1)$, suppose that $\|T(x_0)\| > 1-\eta(\varepsilon)$ for some $T \in L(X_1,Y)$ with $\| T \| = 1$ and some $x_0 \in S_{X_1}$. Define a linear operator $\tilde{T} \in L(X,Y)$ by

$$\tilde{T}(x_1,x_2) = Tx_1 \ \big((x_1,x_2) \in X\big).$$

Observe that $\| \tilde{T} \| = 1$ and $\|\tilde{T}(x_0,0)\| > 1-\eta(\varepsilon)$. Since the pair (X,Y) has the BPBp, there exist $\tilde{S} \in L(X,Y)$ with $\| \tilde{S} \| = 1$ and $(x_1', x_2') \in S_X$ such that

$$\| \tilde{S} \| = \|\tilde{S}(x_1',x_2')\| = 1, \ \ \| \tilde{T} - \tilde{S} \| < \varepsilon \ \text{ and } \ \|(x_1',x_2') - (x_0,0)\| < \varepsilon.$$

From the last inequality, we see that

$$\|x_1' - x_0\| < \varepsilon, \ \|x_2'\| < \varepsilon < 1 \ \text{ and } \ \|x_1'\| = 1. \tag{1}$$

If we define $S \in L(X_1,Y)$ by $S(x_1) = \tilde{S}(x_1,0)$, then $\| S \| \leqslant \| \tilde{S} \| = 1$ and $\| S - T \| \leqslant \| \tilde{T} - \tilde{S} \| < \varepsilon$. So, by (1), it suffices to show that $\|S(x_1')\| = 1$. Indeed, using (1) again, we have that $\|\varepsilon^{-1}x_2'\| \leqslant 1$, so $(x_1', \varepsilon^{-1}x_2') \in B_X$. We can write

$$\tilde{S}(x_1',x_2') = (1-\varepsilon)\tilde{S}(x_1',0) + \varepsilon\tilde{S}(x_1',\varepsilon^{-1}x_2')$$

and, since $\|\tilde{S}(x_1',x_2')\| = 1$, we get $\|\tilde{S}(x_1',0)\| = 1$. This is exactly $\|S(x_1')\| = 1$, as desired.

Considered as a real Banach space, ℓ_∞^2 is not a universal BPB domain space (it is isometric to ℓ_1^2. As a consequence, we cannot expect the converse of Proposition (6.1.8) to be true, even for finite sums.

The third proposition deals with the remaining case of ℓ_1-sums of range spaces.

Proposition (6.1.9)[308]: Let $\{Y_j : j \in J\}$ be a family of Banach spaces, $Y = \left[\oplus_{j\in J} Y_j\right]_{\ell_1}$, and let X be a Banach space. If the pair (X,Y) has the BPBp with $\eta(\varepsilon)$, then the pair (X, Y_j) also has the $BPBp$ with $\eta(\varepsilon)$ for every $j \in J$. In other words, $\eta(X,Y) \leqslant \eta(X,Y_j)$ for every $j \in J$.

Proof. Arguing as in the previous proof, we may assume that $Y = Y_1 \oplus_1 Y_2$.

For $\varepsilon \in (0,1)$ fixed, suppose that $\|T(x_1)\| > 1-\eta(\varepsilon)$ for some $T \in L(X,Y_1)$ with $\| T \| = 1$ and some $x_1 \in S_X$. Define a linear operator $\tilde{T} \in L(X,Y)$ by

$$\tilde{T}(x) = (Tx, 0) \ (x \in X).$$

Observe that $\| \tilde{T} \| = 1$ and $\|\tilde{T}(x_1)\| > 1-\eta(\varepsilon)$. Since the pair (X,Y) has the BPBp, there exist $\tilde{S} \in L(X,Y)$ with $\| \tilde{S} \| = 1$ and $x_0 \in S_X$ such that

$$\| \tilde{S} \| = \|\tilde{S}(x_0)\| = 1, \ \ \| \tilde{T} - \tilde{S} \| < \varepsilon \ \text{ and } \ \|x_1 - x_0\| < \varepsilon.$$

Write $\tilde{S}(x) = (S_1(x), S_2(x))$ for every $x \in X$, where $S_j \in L(X, Y_j)$ for $j = 1,2$, and observe that

$$\| \tilde{S}x - \tilde{T}x \| = \|S_1x - Tx\| + \|S_2x\| < \varepsilon \qquad (x \in B_X). \tag{2}$$

In particular,

$$\|S_1 - T\| < \varepsilon \ \text{ and } \ \|S_2\| < \varepsilon.$$

Now, consider $y^* = (y_1^*, y_2^*) \in Y^* \equiv Y_1^* \oplus_\infty Y_2^*$ with $\|y^*\| = 1$ such that $\operatorname{Re} y^*\left(\tilde{S}(x_0)\right) = 1$. We have that

$$1 = \operatorname{Re} y^*\left(\tilde{S}(x_0)\right) = \operatorname{Re} y_1^*(S_1x_0) + \operatorname{Re} y_2^*(S_2x_0) \leqslant \|S_1x_0\| + \|S_2x_0\| = \|\tilde{S}(x_0)\| = 1$$

Therefore, we get that

$$\operatorname{Re} y_1^*(S_1x_0) = \|S_1x_0\| \ \text{ and } \ \operatorname{Re} y_2^*(S_2x_0) = \|S_2x_0\|$$

Finally, we define $S \in L(X, Y_1)$ by

$$Sx = S_1x + y_2^*(S_2x)\frac{S_1x_0}{\|S_1x_0\|}\ (x \in X)$$

(Observe that $\|S_1x_0\| = 1 - \|S_2x_0\| \geqslant 1 - \|S_2\| > 1 - \varepsilon > 0$.) Then, for $x \in B_X$ we have

$$\| Sx \| \leqslant \|S_1x\| + |y_2^*(S_2x)| \leqslant \|S_1x\| + \|S_2x\| = \| \tilde{S}(x) \| \leqslant 1,$$

so $\| S \| \leqslant 1$. Furthermore,

$$\begin{aligned}\|Sx_0\| &= \left\|S_1x_0 + y_2^*(S_2x_0)\frac{S_1x_0}{\|S_1x_0\|}\right\| \geqslant \operatorname{Re} y_1^*\left(S_1x_0 + y_2^*(S_2x_0)\frac{S_1x_0}{\|S_1x_0\|}\right) \\ &= \operatorname{Re} y_1^*(S_1x_0) + \operatorname{Re} y_2^*(S_2x_0) = 1\end{aligned}$$

Hence S attains its norm at x_0. As $\|x_0 - x_1\| < \varepsilon$, it remains to prove that $\| S - T \| < \varepsilon$. Indeed, for $x \in B_X$, we have

$$\| Sx - Tx \| \leqslant \|S_1x - Tx\| + |y_2^*(S_2x)| \leqslant \|S_1x - Tx\| + \|S_2x\|$$

so $\| S - T \| < \varepsilon$ by (2)

As ℓ_1 does not have Lindenstrauss property B, we cannot expect that a converse of Proposition (6.1.9) can be true in general. In fact, we don't even know whether such a converse is true for finite sums, that is, whether $(X, Y_1 \oplus_1 Y_2)$ has the BPBp whenever (X, Y_j) does for $j = 1,2$.

Another result in the same direction is the following generalization of [319] where it is proved for $X = \ell_1$.

Proposition (6.1.10)[308]: Let X and Y be Banach spaces and let K be a compact Hausdorff space. If $(X, C(K, Y))$ has the $BPBp$ with a function $\eta(\varepsilon)$, then (X, Y) has the $BPBp$ with the same function $\eta(\varepsilon)$. In other words, $\eta(X, Y) \geqslant \eta(X, C(K, Y))$.

Proof. Given $\varepsilon > 0$, consider $T \in S_{L(X,Y)}$ and $x_0 \in S_X$ satisfying

$$\|Tx_0\| > 1 - \eta(\varepsilon).$$

The bounded linear operator $\tilde{T}: X \longrightarrow C(K, Y)$, defined by $[\tilde{T}(x)](t) = T(x)$ for all $x \in X$ and $t \in K$, satisfies $\|\tilde{T}x_0\| = \|Tx_0\| > 1 - \eta(\varepsilon)$. By the assumption, there exist $x_1 \in S_X$ and $\tilde{S} \in L(X, C(K, Y))$ such that

$$\| \tilde{S} \| = \|\tilde{S}(x_1)\| = 1,\ \ \| \tilde{T} - \tilde{S} \| < \varepsilon,\ \|x_0 - x_1\| < \varepsilon$$

Moreover, there is $t_1 \in K$ such that $1 = \|\tilde{S}(x_1)\| = \|[\tilde{S}(x_1)](t_1)\|$. We can now see that the bounded linear operator $S: X \longrightarrow Y$ defined by $S(x) = [\tilde{S}(x)](t_1)$ for all $x \in X$ satisfies that

$$\| S \| = \sup_{x \in B_X} \|[\tilde{S}(x)](t_1)\| = \|[\tilde{S}(x_1)](t_1)\| = \|S(x_1)\| = 1$$

and

$$\begin{aligned}\| S - T \| &= \sup_{x \in B_X} \| S(x) - T(x) \| = \sup_{x \in B_X} \|[\tilde{S}(x)](t_1) - [\tilde{T}(x)](t_1)\| \\ &\leqslant \sup_{x \in B_X} \| \tilde{S}(x) - \tilde{T}(x) \| = \| \tilde{S} - \tilde{T} \| < \varepsilon\end{aligned}$$

As we have already known that $\|x_1 - x_0\| < \varepsilon$, we obtain that (X, Y) has the BPBp with the function $\eta(\varepsilon)$.

We observe that the converse implication in the above proposition is false, as $Y = \mathbb{K}$ is a universal BPBp range space by the Bishop-Phelps-Bollobás theorem but $C[0,1]$ does not have even Lindenstrauss property B [331].

A more general way of stating Proposition (6.1.10) is that if $(X, C(K) \hat{\otimes}_\epsilon Y)$ has the BPBp with $\eta(\epsilon)$, then so does (X, Y). We do not know what other spaces, besides $C(K)$, have this property. In an analogous way, noting that $L_1(\mu, X) = L_1(\mu) \hat{\otimes}_\pi X$, we remark that we do not know if a result similar to Proposition (6.1.10) can be obtained for vector-valued L_1-spaces in the domain; that is, we do not know if the fact that $(L_1(\mu, X), Y)$ has the BPBp implies that (X, Y) does as well (where X and Y are Banach spaces and μ is a positive measure).

We finish with a discussion on some analogues to Propositions (6.1.5), (6.1.8), and (6.1.9) for norm-attaining operators. Let X, Y, X_1, X_2, Y_1, and Y_2 be Banach spaces. It has been proved in [328] that Proposition (6.1.5) has a counterpart for norm-attaining operators; that is, if $NA(X_1 \oplus 1X_2, Y_1 \oplus_\infty Y_2)$ is dense in $L(X_1 \oplus 1X_2, Y_1 \oplus_\infty Y_2)$, then $NA(X_i, Y_j)$ is dense in $NA(X_i, Y_j)$ for $i, j \in \{1,2\}$. If we consider ℓ_1-sums in the range space, it is possible to adapt the proof of Proposition (6.1.9) to this case, obtaining that if $NA(X, Y_1 \oplus_1 Y_2)$ is dense in $L(X, Y_1 \oplus_1 Y_2)$, then $NA(X, Y_j)$ is dense in $L(X, Y_j)$ for $j = 1,2$. As far as we know, this result is new. On the other hand, if we consider ℓ_∞-sums of domain spaces, we do not know how to adapt the proof of Proposition (6.1.8) to the case of norm-attaining operators. We do not know if the result is true or not; that is, we do not know whether the fact that $NA(X_1 \oplus_\infty X_2, Y)$ is dense in $L(X_1 \oplus_\infty X_2, Y)$ forces $NA(X_i, Y)$ to be dense in $L(X_i, Y), i = 1,2$.

Let us summarize this new result here (actually, we state a formally more general result) and deduce from it what we believe is an interesting consequence.

Proposition (6.1.11)[308]: Let $\{Y_j : j \in J\}$ be a family of Banach spaces, $Y = \left[\oplus_{j \in J} Y_j\right]_{\ell_1}$, and let X be a Banach space. If $NA(X, Y)$ is dense in $L(X, Y)$, then $NA(X, Y_j)$ is dense in $L(X, Y_j)$ for every $j \in J$.

Example (6.1.12)[308]: Consider the Banach space $X = C[0,1] \oplus_1 L_1[0,1]$. Then, $NA(X, X)$ is not dense in $L(X, X)$. Indeed, if $NA(X, X)$ were dense in $L(X, X)$, then $NA(L_1[0,1], X)$ would be dense in $L(L_1[0,1], X)$ by [328]. But the above proposition would imply that $NA(L_1[0,1], C[0,1])$ is dense in $L(L_1[0,1], C[0,1])$, a result which was proved to be false by W. Schachermayer [331].

We have two objectives. First, we will show that every Banach space of dimension greater than one can be renormed to not be a universal BPBp domain space. One should compare this result with the result by J. Bourgain [317] asserting that a Banach space has Lindenstrauss property A in every equivalent norm if and only if the space has the Radon-Nikodým property. We will study conditions which ensure that a Banach space is a universal BPB domain space.

Lemma (6.1.13)[308]: Let X be a Banach space containing a non-trivial L-summand (i.e. $X = X_1 \oplus_1 X_2$ for some non-trivial subspaces X_1 and X_2) and let Y be a strictly convex Banach space. If the pair (X, Y) has the BPBp, then Y is uniformly convex.

Proof. For $j = 1,2$, we pick $e_j \in S_{X_j}$ and $e_j^* \in S_{X^*}$ such that $e_j^*(e_j) = 1$ and $e_1^*(X_2) = 0$ and $e_2^*(X_1) = 0$ (just identify X_j^* with a subspace of X^* and extend the functional to be zero on the other subspace).

Fix $\varepsilon \in (0,1/2)$ and recall that $\eta(X, Y)(\varepsilon) > 0$ by the BPBp. Consider $y_1, y_2 \in S_Y$ such that

$$\|y_1 + y_2\| > 2 - 2\eta(X, Y)(\varepsilon)$$

We define an operator $T \in L(X, Y)$ by

$$T(x_1,x_2)=e_1^*(x_1)y_1+e_2^*(x_2)y_2\ \big((x_1,x_2)\in X\big),$$
which satisfies $\| T \|=1$. As
$$\left\|T\left(\frac{1}{2}e_1,\frac{1}{2}e_2\right)\right\|=\frac{1}{2}\|y_1+y_2\|>1-\eta(X,Y)(\varepsilon),$$
there are $(x_1,x_2)\in S_X$ and $S\in L(X,Y)$ such that
$$\| S \|=\|S(x_1,x_2)\|=1,\ \left\|\frac{1}{2}e_1-x_1\right\|+\left\|\frac{1}{2}e_2-x_2\right\|<\varepsilon \text{ and } \| T-S \|<\varepsilon.$$
We deduce that $\left|\frac{1}{2}-\|x_j\|\right|<\varepsilon<1/2$, so $0<\|x_j\|<1$ and
$$\left\|e_j-\frac{x_j}{\|x_j\|}\right\|\leqslant 2\left\|\frac{1}{2}e_j-x_j\right\|+\left\|2x_j-\frac{x_j}{\|x_j\|}\right\|\leqslant 2\varepsilon+\big|1-2\|x_j\|\big|<4\varepsilon$$
for $j=1,2$. If we write
$$z_1=S\left(\frac{x_1}{\|x_1\|},0\right)\in B_Y \text{ and } z_2=S\left(0,\frac{x_2}{\|x_2\|}\right)\in B_Y$$
we have that
$$1=\|S(x_1,x_2)\|=\|\| x_1\|z_1+\|x_2\|z_2\| \text{ and } \|x_1\|+\|x_2\|=1.$$
As Y is strictly convex, it follows that $z_1=z_2$. Now,
$$\begin{aligned}\|y_1-y_2\| &=\|T(e_1,0)-T(0,e_2)\|\\ &\leqslant \|T(e_1,0)-S(e_1,0)\|+\|T(0,e_2)-S(0,e_2)\|\\ &+\|S(e_1,0)-z_1\|+\|S(0,e_2)-z_2\|\\ &<2\| T-S \|+\left\|S(e_1,0)-S\left(\frac{x_1}{\|x_1\|},0\right)\right\|+\left\|S(0,e_2)-S\left(0,\frac{x_2}{\|x_2\|}\right)\right\|\\ &\leqslant 2\varepsilon+\| S \|\left(\left\|e_1-\frac{x_1}{\|x_1\|}\right\|+\left\|e_2-\frac{x_2}{\|x_2\|}\right\|\right)<10\varepsilon.\end{aligned}$$
This implies that Y is uniformly convex, as desired.

Theorem (6.1.14)[308]: The base field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ is the unique Banach space which is a universal $BPBp$ domain space in any equivalent renorming.

Proof. If $\dim(X)>1$, we consider any one-codimensional subspace Z of X and observe that $X\simeq\tilde{X}:=\mathbb{K}\oplus_1 Z$. Now, consider any strictly convex Banach space Y which is not uniformly convex, and the above lemma gives that $(\tilde{X},Y)$ does not have the BPBp.

We next give the following particular case of Lemma (6.1.13), which can be deduced from arguments given in [310].

Corollary (6.1.15)[308]: Let Y be a strictly convex Banach space. If (ℓ_1^2,Y) has the $BPBp$, then Y is uniformly convex.

A nice consequence of the above corollary is the following example.

Example (6.1.16)[308]: There exists a reflexive Banach space X such that the pair (X,X) fails the $BPBp$. Indeed, let Y be a reflexive strictly convex space which is not uniformly convex and consider the reflexive space $X=\ell_1^2\oplus_1 Y$. If the pair (X,X) had the BPBp, then so would (ℓ_1^2,Y) by Theorem (6.1.4), a contradiction with Corollary (6.1.15).

We give some necessary conditions for a Banach space to be a universal BPB domain space. Let us recall that if a Banach space X has Lindenstrauss property A, then the following hold [327]: (a) if X is isomorphic to a strictly convex space, then S_X is the closed convex hull of its extreme points, and (b) if X is isomorphic to a locally uniformly convex space, then S_X is the closed convex hull of its strongly exposed points. These results have been strengthened to the case of universal BPBp domain spaces in [326] but with the additional hypothesis that there is a common function η giving the BPBp for all range spaces. Thanks to Corollary (6.1.4), this hypothesis is unnecessary.

Corollary (6.1.17)[308]: Let X be a universal BPB domain space. Then,

(a) in the real case, there is no face of S_X which contains a non-empty relatively open subset of S_X

(b) if X is isomorphic to a strictly convex Banach space, then the set of all extreme points of B_X is dense in S_X;

(c) if X is superreflexive, then the set of all strongly exposed points of B_X is dense in S_X.

In particular, if X is a real 2-dimensional Banach space which is a universal BPB domain space, then X is uniformly convex.

We don't know if a universal BPB domain space has to be uniformly convex. We can improve Corollary (6.1.17)(c) to get a slightly stronger result. To do this, we follow Lindenstrauss [327] to say that a family $\{x_\alpha\}_\alpha \subset S_X$ is uniformly strongly exposed (with respect to a family $\{f_\alpha\}_\alpha \subset S_{X^*}$) if there is a function $\varepsilon \in (0,1) \longmapsto \delta(\varepsilon) > 0$ having the following properties:

(i) $f_\alpha(x_\alpha) = 1$ for every α, and

(ii) for any $x \in B_X$, $\operatorname{Re} f_\alpha(x) > 1 - \delta(\varepsilon)$ implies $\|x - x_\alpha\| < \varepsilon$.

In a uniformly convex Banach space, the unit sphere is a uniformly strongly exposed family (actually, this property characterizes uniform convexity).

Corollary (6.1.18)[308]: Let X be a superreflexive universal BPB domain space. Then for every $\varepsilon_0 \in (0,1)$ there exists an ε_0-dense uniformly strongly exposed family. In particular, the set of all strongly exposed points of B_X is dense in S_X.

Proof. We write $\|\cdot\|$ for the given norm of X and consider an equivalent norm $|\|\cdot\||$ on X for which the Banach space $(X, \| | \cdot | \|)$ is uniformly convex. We may assume that $\|x \| \leqslant \| x \|$ for all $x \in X$ (see [323], for instance). For each $m \in \mathbb{N}$, we define an equivalent norm on X by

$$\| x \|_m := \left(\| x \|^2 + \frac{1}{m} |\|\| x \||\|^2\right)^{1/2} \quad (x \in X).$$

We observe that $X_m = (X, \|\cdot\|_m)$ is uniformly convex, so we may consider the function $\varepsilon \longmapsto \delta_m(\varepsilon)$ (from the uniform convexity of X_m), which gives that S_{X_m} is a uniformly strongly exposed family (with respect to $S_{X_m^*}$). That is, for every $y \in S_{X_m}$ there is $y^* \in S_{X_m^*}$ such that $y^*(y) = 1$ and

$$\text{if } z \in B_{X_m} \text{ is such that } \operatorname{Re} y^*(z) > 1 - \delta_m(\varepsilon), \text{ then } \| z - y \|_m < \varepsilon. \tag{3}$$

Let $I_m : (X, \|\cdot\|) \longrightarrow (X_m, \|\cdot\|_m)$ be the formal identity operator and write

$$T_m = \frac{I_m}{\|I_m\|} \quad \text{and} \quad a_m = \frac{\sqrt{m+1}}{\sqrt{m}}$$

for every $m \in \mathbb{N}$. Since

$$1 \leqslant \|I_m\| \leqslant a_m \quad \text{and} \quad a_m^{-1} \leqslant \|I_m^{-1}\| \leqslant 1,$$

it follows that

$$a_m^{-1} \leqslant \|T_m^{-1}\| \leqslant a_m.$$

Therefore, if $S \in L(X, X_m)$ satisfies $\|T_m - S\| < a_m^{-1} \leqslant \|T_m^{-1}\|^{-1}$, then S is invertible (see [324], Corollary 18.12) and

$$\|T_m^{-1} - S^{-1}\| \leqslant \frac{\|T_m^{-1}\|^2 \|T_m - S\|}{1 - \|T_m^{-1}\| \|T_m - S\|} \leqslant \frac{a_m^2 \|T_m - S\|}{1 - a_m \|T_m - S\|}$$

(see [309] where it is done for the case $X_m = X$, but the general case easily follows). Then

$$\|S^{-1}\| \leqslant \|T_m^{-1}\| + \|T_m^{-1} - S^{-1}\| \leqslant a_m + \frac{a_m^2 \|T_m - S\|}{1 - a_m \|T_m - S\|} = \frac{a_m}{1 - a_m \|T_m - S\|}. \quad (4)$$

Finally, let $\varepsilon \longmapsto \eta_X(\varepsilon) > 0$ be the universal BPBp function for X given by Corollary (6.1.4).

We need to show that for every fixed $\varepsilon_0 \in (0,1)$, there exists a function $\varepsilon \longmapsto \delta_{\varepsilon_0}(\varepsilon) > 0$ satisfying the following conditions: For each $x_0 \in S_X$ there exist $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that

(i) $\|x_1 - x_0\| < \varepsilon_0$

(ii) $x_1^*(x_1) = 1$, and

(iii) for every $\varepsilon > 0$, if $x \in B_X$ satisfies $\operatorname{Re} x_1^*(x) > 1 - \delta_{\varepsilon_0}(\varepsilon)$, then $\|x_1 - x\| < \varepsilon$. Indeed, fix $\varepsilon_0 \in (0,1)$ and choose $m \in \mathbb{N}$ satisfying

$$a_m^{-1} = \frac{\sqrt{m}}{\sqrt{1+m}} > 1 - \eta_X(\varepsilon_0) \text{ and } a_m^{-1} > \varepsilon_0.$$

Consider $\delta_{\varepsilon_0}(\varepsilon) = \delta_m\left(\frac{1 - a_m \varepsilon_0}{a_m} \varepsilon\right)$ $(\varepsilon > 0)$ for this m. Observe that the operator $T_m \in L(X, X_m)$ satisfies $\|T_m\| = 1$ and

$$\|T_m x_0\|_m = \frac{\|x_0\|_m}{\|I_m\|} \geqslant \|x_0\| a_m^{-1} > 1 - \eta_X(\varepsilon_0)$$

Hence, by the BPB property of (X, X_m), there exist both an operator $S \in L(X, X_m)$ with $\| S \| = 1$ and $x_1 \in S_X$ such that

$$\|x_0 - x_1\| < \varepsilon_0, \ \|S(x_1)\|_m = 1, \text{ and } \|T_m - S\| < \varepsilon_0.$$

Now, for $y = Sx_1 \in S_{X_m}$, let $y^* \in S_{X_m^*}$ be the functional satisfying (3) (i.e. y^* strongly exposed Sx_1 with the function $\delta_m(\cdot)$). So $y^*(Sx_1) = 1$, and for $x \in B_X$ and $\varepsilon' > 0$

$$\operatorname{Re} y^*(Sx) > 1 - \delta_m(\varepsilon') \text{ implies } \|Sx - Sx_1\| < \varepsilon'.$$

Consider $x_1 \in S_X$ (which satisfies $\|x_1 - x_0\| < \varepsilon_0$) and $x_1^* = S^*(y^*) \in X^*$ (which satisfies $x_1^*(x_1) = 1$ and $\|x_1^*\| = 1$). Suppose that for some $\varepsilon > 0, x \in B_X$ satisfies $\operatorname{Re} x_1^*(x) > 1 - \delta_{\varepsilon_0}(\varepsilon)$. Then

$$\operatorname{Re} y^*(Sx) = \operatorname{Re} x_1^*(x) > 1 - \delta_{\varepsilon_0}(\varepsilon) = 1 - \delta_m\left(\frac{1 - a_m \varepsilon_0}{a_m} \varepsilon\right)$$

so $\|Sx - Sx_1\| < \frac{1 - a_m \varepsilon_0}{a_m} \varepsilon$. On the other hand, as $\|T_m - S\| < \varepsilon_0 < a_m^{-1} \leqslant \|T_m^{-1}\|^{-1}$, it follows that S is invertible and we get from (4) that $\|S^{-1}\| \leqslant \frac{a_m}{1 - a_m \|T_m - S\|}$. Therefore

$$\begin{aligned} \|x - x_1\| & \leqslant \|S^{-1}\| \|Sx - Sx_1\| < \|S^{-1}\| \frac{1 - a_m \varepsilon_0}{a_m} \varepsilon \\ & \leqslant \frac{a_m}{1 - a_m \|T_m - S\|} \frac{1 - a_m \varepsilon_0}{a_m} \varepsilon < \varepsilon \end{aligned}$$

One can ask whether it is actually possible to deduce uniform convexity from the above corollary. This is not the case, even in the finite-dimensional case. To see this, just consider a three-dimensional space X in which the set M of strongly exposed points is dense but not all of S_X. (For instance, we may modify the Euclidean sphere in such a way that there are two diametrically opposite small line segments and the rest of the points are still strongly exposed.) Now, for every $\varepsilon_0 > 0$, consider the set of those points in S_X whose distance to $S_X \setminus M$ is greater than or equal to $\varepsilon_0/2$ (which is contained in M). Then this set is a closed subset of S_X consisting of strongly exposed points, and so it is uniformly strongly exposed by compactness. On the other hand, it is clearly ε_0-dense.

We give an example of a Banach space having Lindenstrauss property B which is not a universal BPB range space. We recall that, as a particular case of [310], finite-dimensional real polyhedral spaces are universal BPB range spaces since they have property β (we will not introduce the definition of property β here since we are not going to work with it).

We are now able to present the main example.

Example (6.1.19)[308]: For $k \in \mathbb{N}$, consider $Y_k = \mathbb{R}^2$ endowed with the norm

$$\| (x,y) \| = \text{m } \left\{|x|, |y| + \frac{1}{k}|x|\right\} \ (x,y \in \mathbb{R}).$$

Observe that B_{Y_k} is the absolutely convex hull of the set $\left\{(0,1), \left(1,1-\frac{1}{k}\right), \left(-1,1-\frac{1}{k}\right)\right\}$, so Y_k is polyhedral and, therefore, it is a universal BPB range space by [2, Theorem 2.2]. Then, we have that

$$\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\varepsilon) = 0$$

for every $\varepsilon \in (0,1/2)$. Therefore, if we consider

$$\mathcal{Y} = \left[\bigoplus_{i=1}^{\infty} Y_k\right]_{c_0}, \ \mathcal{Z} = \left[\bigoplus_{i=1}^{\infty} Y_k\right]_{\ell_1} \text{ and } \mathcal{W} = \left[\bigoplus_{i=1}^{\infty} Y_k\right]_{\ell_\infty}$$

then none of the pairs $(\ell_1^2, \mathcal{Y}), (\ell_1^2, \mathcal{Z})$ and $(\ell_1^2, \mathcal{W})$ has the BPBp.

Proof. Define $T_k \in L(\ell_1^2, Y_k)$ by

$$T_k(e_1) = \left(-1,1-\frac{1}{k}\right) \text{ and } T_k(e_2) = \left(1,1-\frac{1}{k}\right).$$

Clearly $\|T_k\| = 1$ and $T_k\left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) = \left(0,1-\frac{1}{k}\right)$. Hence, $\left\|T_k\left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right\| = 1 - \frac{1}{k}$. Assume that for some $1/2 > \varepsilon > 0$ we have

$$\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\varepsilon) > 0$$

and take $\eta(\varepsilon)$ such that $\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\varepsilon) > \eta(\varepsilon) > 0$. Then, for every $k \in \mathbb{N}$ such that $1 - \frac{1}{k} > 1 - \eta(\varepsilon)$, we can find $S_k \in L(\ell_1^2, Y_k)$ with $\|S_k\| = 1$ and $u_k \in S_{\ell_1^2}$ such that

$$\|S_k u_k\| = 1, \ \|T_k - S_k\| < \varepsilon \text{ and } \left\|u_k - \left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right\| < \varepsilon.$$

Now, as $\left\|u_k - \left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right\| < 1/2$, we have that u_k lies in the interior of the interval $[e_1, e_2] \subset S_{\ell_1^2}$ and that $\|S_k(u_k)\| = 1$. It follows that the entire interval $[S_k(e_1), S_k(e_2)]$ lies in the unit sphere of Y_k, and so $\|S_k(e_1) - S_k(e_2)\| \leqslant 1$. Since $\|T_k - S_k\| < \varepsilon$, we get that

$$\|T_k(e_1) - S_k(e_2)\| \leqslant \|T_k(e_1) - S_k(e_1)\| + \|S_k(e_1) - S_k(e_2)\| < \varepsilon + 1 < 3/2.$$

On the other hand, since $\|T_k(e_1) - T_k(e_2)\| = 2$,

$$\|T_k(e_1) - S_k(e_2)\| \geqslant \|T_k(e_1) - T_k(e_2)\| - \|T_k(e_2) - S_k(e_2)\| > 2 - \varepsilon > 3/2$$

a contradiction.

The last assertion is a direct consequence of Proposition (6.1.5) (for the c_0- and ℓ_∞-sums) and Proposition (6.1.9) (for the ℓ_1-sum).

Here is the main consequence of the example above.

Theorem (6.1.20)[308]: Lindenstrauss property B does not imply being a universal BPB range space.

Proof. Just consider $\mathcal{Y}$ in the above example. Then $\mathcal{Y}$ has Lindenstrauss property B as a sum of Y_k 's, each of which has it, and this property is stable under c_0-sums (see [313]). However, $\mathcal{Y}$ is not a universal BPB range space since $(\ell_1^2, \mathcal{Y})$ does not have BPBp.

In [313], a property called quasi- β was introduced as a weakening of property β which still implies property B. The above argument shows that property quasi- β does not imply being a universal BPB range space:

Corollary (6.1.21)[308]: Property quasi- β does not imply being a universal BPB range space.

Proof. As in the theorem above, consider the space $\mathcal{Y}$ of Example (6.1.19). Then, $\mathcal{Y}$ has property quasi- β since it is a c_0-sum of spaces with property β, and we may use [313].

We now list more consequences of Example (6.1.19), showing first that there is no infinite counterpart to Corollary (6.1.7).

Corollary (6.1.22)[308]: The $BPBp$ is not stable under infinite $c_0 -$ or l_∞-sums of the range space. Even more, being a universal BPB range space is not stable under infinite $c_0 -$ or $\ell_\infty -$ sums.

It is shown in [325] that if Y is a real strictly convex Banach space which is not uniformly convex, then (c_0, Y) fails to have the BPBp. We are now able to show the same for some non-strictly convex spaces.

Corollary (6.1.23)[308]: Let $\mathcal{Y}, \mathcal{Z}$ and $\mathcal{W}$ be the spaces of Example (6.1.19). Then none of $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ is strictly convex and, in the real case, none of the pairs $(c_0, \mathcal{Y})$, $(c_0, \mathcal{Z})$, $(c_0, \mathcal{W})$ has the $BPBp$

Proof. Notice that c_0 and $\ell_1^2 \oplus_\infty c_0$ are isometric, so we consider $\ell_1^2 \oplus_\infty c_0$. If $(\ell_1^2 \oplus_\infty c_0, \mathcal{Y})$ has the BPBp, then from Proposition (6.1.8), $(\ell_1^2, \mathcal{Y})$ has the BPB, contradicting Example (6.1.19). The same argument works for $(c_0, \mathcal{Z})$.

Finally, we have the following negative results on the stability of the so-called AHSP. Recall that a Banach space Y has the Approximate Hyperplane Series Property $(AHSP)$ [310] if it satisfies a geometrical condition which is equivalent to the fact that (ℓ_1, Y) has the BPBp.

Corollary (6.1.24)[308]: The AHSP is not stable under infinite $c_0-, \ell_1 -$ or $\ell_\infty - sums$.

Proof. Let $\mathcal{Y} = [\oplus_{i=1}^\infty Y_k]_{c_0}$ be the space given in Example (6.1.19). As $\ell_1 \equiv \ell_1^2 \oplus_1 \ell_1$, it follows from Proposition (6.1.6) that $(\ell_1, \mathcal{Y})$ does not have the BPBp, from which we deduce that $\mathcal{Y}$ does not have the AHSP [310]. On the other hand, all the Y_k have the AHSP since they are finite-dimensional [310]. For the ℓ_1-sum and the ℓ_∞-sum, the argument is the same considering $\mathcal{Z} = [\oplus_{i=1}^\infty Y_k]_{\ell_1}$ and $\mathcal{W} = [\oplus_{i=1}^\infty Y_k]_{\ell_\infty}$.

Section (6.2): Theorem for Operators:

The celebrated Bishop-Phelps theorem states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. The study of when a theorem of this type holds in the vector valued case has produced a theory with deep and elegant results. Lindenstrauss in [337] proved that for certain Banach spaces X and Y, the subset of norm attaining operators from X into Y is not norm dense in the space of all continuous and linear operators $L(X, Y)$. There are also remarkable situations in which a Bishop-Phelps theorem for operators does hold, such as when the domain space is reflexive [337] or, more generally, when it has the Radon-Nikodým property [335].

Given a Banach space X, we denote the unit sphere of X by S_X and the closed unit ball by B_X. X^* will be the topological dual of X. Bollobás in [334], [333] proved a "quantitative version" of the Bishop-Phelps theorem [332] (known as the Bishop-Phelps-Bollobás theorem) that can be stated as follows.

Let $\varepsilon > 0$ be arbitrary. If $x \in B_X$ and $x^* \in S_{X^*}$ are such that $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, then there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1, \| y - x \| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

Bollobás proved this result in order to be able to apply it to the study of the numerical range of an operator.

For a Banach space X, we let $\Pi(X)$ denote the subset of $X \times X^*$ given by $\Pi(X) := \{(x, x^*) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$. Given a bounded function $\Phi : S_X \to X$, its numerical range is $V(\Phi) := \{x^*(\Phi(x)) : (x, x^*) \in \Pi(X)\}$. The properties of the set $\Pi(X)$ play a crucial role in the study of numerical range. What Bollobás proved was that the ordered pairs of $X \times X^*$ that "almost belong" to $\Pi(X)$ can be approximated, in the product norm, by elements of $\Pi(X)$. The numerical range of an operator allows the recovery of some properties of the operator. Thanks, among other things, to the Bishop-Phelps-Bollobás theorem, the theory of numerical range is far richer that one might expect at first glance (see [334]). Since this theory studies operators from a Banach space into itself it may be of interest to consider possible extensions of the Bishop-Phelps-Bollobás theorem to operators between two Banach spaces.

Since it is false in general that for every pair of Banach spaces X and Y, the subset of norm attaining operators from X into Y is norm dense in the space $L(X, Y)$, we cannot expect a version of the Bishop-Phelps-Bollobás theorem for operators to hold in full generality. That is why we introduce the following property.

Definition (6.2.1)[331]: Let X and Y be real or complex Banach spaces. We say that the couple (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators (or that the Bishop-Phelps-Bollobás theorem holds for all bounded operators from X into Y) if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0} \beta(t) = 0$ such that for all $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon), \ \text{ and } \ \| S - T \| < \varepsilon$$

Note that an independent concept, the Bishop-Phelps-Bollobás property for a pair (X, Y), has been studied for closed subspaces $X \subset Y$ by M. Martín, J. Merí, and R. Payá in [338] in their work on intrinsic and spatial numerical range.

In the study of the Bishop-Phelps theorem for operators between Banach spaces two kind of questions are usually considered:

(a) For which X is it true that for every Banach space Y, the norm attaining operators are dense in $L(X, Y)$? (b) For which Y is it true that for every Banach space X, the norm attaining operators are dense in $L(X, Y)$?

Schachermayer in [340] introduced property α as a sufficient condition on a Banach space X to fulfill (a). A sufficient condition for (b) was given by Lindenstrauss [337] introducing property β. These two properties generalize in some sense the geometric situations of the classical Banach spaces ℓ_1 and c_0, respectively. We study whether these properties still work not just for the Bishop-Phelps theorem for operators but for the Bishop-Phelps-Bollobás theorem for operators.

We prove that the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators for every Banach space X whenever Y has property β. This implies a general positive result about the Bishop-Phelps-Bollobás theorem for operators between Banach spaces whenever the range space is fixed. Looking at the dual case, we concentrate on ℓ_1, since it is the typical example of space having property α. We characterize when the Bishop-Phelps-Bollobás theorem holds for operators from ℓ_1 into Y. In order to do this, we introduce property AHSP and show that there are many spaces having this property, including finite-dimensional normed spaces, $L_1(\mu)$ for every σ-finite measure μ, $C(K)$ for any compact Hausdorff space K, and every uniformly convex Banach space. A consequence of our study is that property α of Schachermayer is no longer a sufficient condition for a Banach space X to satisfy that the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators for every Banach space Y. We show that a version of the Bishop-Phelps-Bollobás theorem holds when $X=\ell_{\infty}^{n}$ and Y is uniformly convex. Finally, following Lindenstrauss' fundamental paper [337], it seems reasonable to ask if there is a version of the Bishop-Phelps-Bollobás theorem that involves the second duals of X and Y. We provide an example to show that no such result holds in general.

We will provide a partial positive result concerning the Bishop-PhelpsBollobás theorem for operators under an additional assumption. The result will use an isometric condition on the range space Y, called property β, that was introduced by Lindenstrauss [337].

Definition (6.2.2)[331]: A Banach space Y is said to have property β (of Lindenstrauss) if there are two sets $\{y_\alpha: \alpha \in \Lambda\} \subset S_Y, \{y_\alpha^*: \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leqslant \rho < 1$ such that the following conditions hold:

(a) $y_\alpha^*(y_\alpha)=1$.

(b) $\left|y_\alpha^*(y_\beta)\right| \leqslant \rho < 1$ if $\alpha \neq \beta$

(c) $\| y \|=\sup_\alpha\{|y_\alpha^*(y)|\}$, for all $y \in Y$.

Clearly, $c_0(\Lambda)$ and $\ell_\infty(\Lambda)$ satisfy the above property for $\{y_\alpha: \alpha \in \Lambda\}=\{e_\alpha: \alpha \in \Lambda\}$ and $\{y_\alpha^*: \alpha \in \Lambda\}$ the biorthogonal functionals, and $\rho=0$ in this case.

Theorem (6.2.3)[331]: Let X and Y be Banach spaces such that Y has property β. Then the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators. Indeed, if $T \in S_{L(X,Y)}, \varepsilon>0$ and $x_0 \in S_X$ satisfy $\|T(x_0)\|>1-\frac{\varepsilon^2}{4}$, then for each real number η such that $\eta>\frac{\rho}{1-\rho}\left(\varepsilon+\frac{\varepsilon^2}{4}\right)$, there are $S \in L(X, Y), z_0 \in S_X$ such that:

$$\|S z_0\|=\| S \|, \|z_0-x_0\|<\varepsilon, \quad \| S-T \|<\eta+\varepsilon+\frac{\varepsilon^2}{4}$$

Proof. Since Y has property β, there is $\alpha_0 \in \Lambda$ such that $\left|y_{\alpha_0}^*\left(T(x_0)\right)\right|>1-\frac{\varepsilon^2}{4}$. By the Bishop-Phelps-Bollobás theorem, there exist $z_0^* \in S_{X^*}$ and $z_0 \in S_X$ such that $|z_0^*(z_0)|=1, \|z_0-x_0\|<\varepsilon$ and $\left\|z_0^*-\frac{T^{\mathrm{t}}(y_{\alpha_0}^*)}{\|T^{\mathrm{t}}(y_{\alpha_0}^*)\|}\right\|<\varepsilon$(see [334]). Hence we obtain that

$$\|z_0^*-T^{\mathrm{t}}(y_{\alpha_0}^*)\| \leqslant\left\|z_0^*-\frac{T^{\mathrm{t}}(y_{\alpha_0}^*)}{\|T^{\mathrm{t}}(y_{\alpha_0}^*)\|}\right\|+\left\|\frac{T^{\mathrm{t}}(y_{\alpha_0}^*)}{\|T^{\mathrm{t}}(y_{\alpha_0}^*)\|}-T^{\mathrm{t}}(y_{\alpha_0}^*)\right\|$$

$$< \varepsilon + \left| \|T^{\mathrm{t}}(y_{\alpha_0}^*)\| - 1 \right| \leqslant \varepsilon + \frac{\varepsilon^2}{4}$$

For a real number η satisfying $\eta > \frac{\rho}{1-\rho}\left(\varepsilon + \frac{\varepsilon^2}{4}\right)$, we define the operator $S \in L(X,Y)$ by

$$S(x) = T(x) + \left[(1+\eta)z_0^*(x) - T^{\mathrm{t}}(y_{\alpha_0}^*)(x)\right]y_{\alpha_0} \ (x \in X)$$

Note that S is a rank one perturbation of T, and so $S - T$ is compact. Thus for all $y^* \in Y^*$,

$$S^{\mathrm{t}}(y^*) = T^{\mathrm{t}}(y^*) + y^*(y_{\alpha_0})\left[(1+\eta)z_0^* - T^{\mathrm{t}}(y_{\alpha_0}^*)\right]$$

Since the set $\{y_\alpha^* : \alpha \in \Lambda\}$ is norming for Y it follows that $\| S \| = \sup_\alpha \|S^{\mathrm{t}}(y_\alpha^*)\|$. Let us estimate the norm of S. Clearly,

$$S^{\mathrm{t}}(y_{\alpha_0}^*) = (1+\eta)z_0^*$$

and thus

$$\| S \| \geqslant \|S^{\mathrm{t}}(y_{\alpha_0}^*)\| = (1+\eta)\|z_0^*\| = 1+\eta.$$

On the other hand, for $\alpha \neq \alpha_0$, by the choice of η, we obtain

$$\begin{aligned} \|S^{\mathrm{t}}(y_\alpha^*)\| &\leqslant 1 + \rho\left(\|z_0^* - T^{\mathrm{t}}(y_{\alpha_0}^*)\| + \eta\|z_0^*\|\right) \\ &< 1 + \rho\left(\varepsilon + \frac{\varepsilon^2}{4} + \eta\right) < 1 + \eta \end{aligned}$$

Therefore,

$$\begin{aligned} \| S \| &= \|S^{\mathrm{t}}(y_{\alpha_0}^*)\| = (1+\eta)\|z_0^*\| = (1+\eta)|z_0^*(z_0)| \\ &= \left|y_{\alpha_0}^*(Sz_0)\right| \leqslant \|Sz_0\| \leqslant \| S \| \end{aligned}$$

so S attains its norm at z_0, and moreover we have that

$$\|z_0 - x_0\| < \varepsilon \ \text{ and } \ \| S - T \| < \eta + \varepsilon + \frac{\varepsilon^2}{4}$$

Since property β is not restrictive at all from an isomorphic point of view (see [339], we deduce the following consequence.

Corollary (6.2.4)[331]: For every Banach space Y, there is a space Z isomorphic to Y such that the Bishop-Phelps-Bollobás theorem holds for the operators from any other Banach space X to Z. In fact, the function that controls the distance between the original operator T and its norm attaining approximation S depends just on Y.

Now, we are going to prove that for finite-dimensional spaces, the Bishop-Phelps-Bollobás theorem holds for operators. The following result is true.

Proposition (6.2.5)[331]: Let X and Y be finite-dimensional Banach spaces. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $T \in S_{L(X,Y)}$, there is a linear operator $R \in S_{L(X,Y)}$ such that the following conditions hold:

(i) $\| R - T \| < \varepsilon$, and

(ii) for all $x \in S_X$ satisfying $\| T(x) \| > 1 - \delta$, there is $\tilde{x} \in S_X$ such that $\| R(\tilde{x}) \| = 1$ and such that $\| x - \tilde{x} \| < \varepsilon$

In other words, we have a Bishop-Phelps-Bollobás theorem for finite-dimensional spaces X and Y that is uniform in the following sense. Given X, Y and ε, there is a δ such that for any $T: X \to Y$ there is $R: X \to Y$, as above, with $\| T - R \| < \varepsilon$ and such that for any unit vector x at which T is within δ of attaining the norm, there is a unit vector $\tilde{x}$ within ε of x at which R attains its norm. That is, the same R "works" for all such x. On the other hand, unlike the classical

Bishop-Phelps-Bollobás theorem, the constant δ depends not only on ε but also on X and Y. This is true, even in the case when $Y=\mathbb{R}$ or $\mathbb{C}$.

Proof. The proof is by contradiction. If the result is false for some ε_0 then for every n, we can find $T_n \in S_{L(X,Y)}$ such that for all $R \in S_{L(X,Y)}$ with $\|T_n - R\| \leqslant \varepsilon_0$, there is $x_{n,R} \in S_X$ satisfying $\|T_n(x_{n,R})\| > 1-\frac{1}{n}$ and such that $\operatorname{dist}(x_{n,R}, NA(R)) \geqslant \varepsilon_0$ (where $NA(R)=\{z \in S_X : \| R(z) \| = 1\}$). By taking subsequences, we may assume that $(T_n) \to T_0 \in S_{L(X,Y)}$. Putting $x_n := x_{n,T_0}$, we can also assume that $(x_n) \to x_0 \in S_X$. Now, $\|T_0(x_0)\| = 1$ although for all large n, $\varepsilon_0 \leqslant \operatorname{dist}(x_n, NA(T_0)) \leqslant \|x_n - x_0\| \to 0$, which is the desired contradiction.

In order to give versions of Bishop-Phelps theorem for operators from a fixed Banach space X, Schachermayer in [340] introduced the isometric property α, which has a certain duality relationship with property β. The most typical example of a space having property α is ℓ_1. Our aim will be to characterize when the Bishop-Phelps-Bollobás theorem holds for operators from ℓ_1 into an arbitrary Banach space Y. We introduce the awkwardly named property AHSP that we use to get such a characterization, and we show the richness of this property by proving that several classes of spaces enjoy it.

Definition (6.2.6)[331]: A Banach space X is said to have property AHSP if for every $\varepsilon > 0$ there exists $0 < \eta < \varepsilon$ such that for every sequence $(x_k) \subset S_X$ and every convex series $\sum_{k=1}^{\infty}\alpha_k$ with

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1-\eta$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\}$ satisfying

(i) $\sum_{k\in A}\alpha_k > 1-\eta$, and

(ii)(a) $\|z_k - x_k\| < \varepsilon$ for all $k \in A$,

(b) $x^*(z_k) = 1$ for a certain $x^* \in S_{X^*}$ and all $k \in A$.

It is immediate that the above property holds if it is satisfied just for finite convex combinations (instead of infinite convex series). In Definition (6.2.6) we can consider sequences $(x_k)_{k=1}^{\infty}$ of vectors in the unit ball of X. A characterization of property AHSP is the following.

Geometrically, X has AHSP if whenever we have a convex series of vectors in B_X whose norm is very close to 1 , then a preponderance of these vectors are uniformly close to unit vectors that lie in the same hyperplane $(x^*)^{-1}(1)$ where x^* has norm 1.

The following elementary lemma will be very useful to check that some Banach spaces have property AHSP.

Lemma (6.2.7)[331]: Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum\alpha_n$, $\operatorname{Re}\sum_{n=1}^{\infty}\alpha_n c_n > 1-\eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate

$$\sum_{i\in A} \alpha_i \geqslant 1-\frac{\eta}{1-r}$$

Proof. By the assumption we have that

$$1-\eta \leqslant \operatorname{Re}\sum_{i=1}^{\infty} \alpha_i c_i = \sum_{i=1}^{\infty} \alpha_i \operatorname{Re} c_i \leqslant \sum_{i\in A} \alpha_i + r\sum_{i\notin A} \alpha_i = (1-r)\sum_{i\in A} \alpha_i + r$$

Then we obtain that

$$\sum_{i\in A}\alpha_i \geqslant \frac{1-\eta-r}{1-r} = 1-\frac{\eta}{1-r}$$

The next result will be used to check that finite-dimensional spaces have AHSP.

Lemma (6.2.8)[331]: Let X be a finite-dimensional normed space. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $x^* \in S_{X^*}$, there exists $y^* \in S_{Y^*}$ such that $\operatorname{dist}(x, D(y^*)) < \varepsilon$ for all $x \in \{x \in S_X : \operatorname{Re} x^*(x) > 1-\delta\}$, where $D(y^*) := \{y \in B_X : y^*(y) = 1\}$.

Proof. We argue by contradiction. Assume that there is some positive real number ε_0 not satisfying the above condition. Thus, for every positive $\delta > 0$ there exists $x_\delta^* \in S_{X^*}$ such that for all $y^* \in S_{X^*}, \operatorname{dist}(x, D(y^*)) \geqslant \varepsilon_0$ for some $x \in \{y \in S_X : \operatorname{Re} x_\delta^*(y) > 1-\delta\}$. Hence, we can find sequences $(r_n) \to 1, (x_n^*) \subset S_{X^*}$ such that for all $y^* \in S_{X^*}, \{x \in S_X : x_n^*(x) > \ r_n\} \cap \{x \in S_X : \operatorname{dist}(x, D(y^*)) \geqslant \varepsilon_0\} \neq \emptyset$. By compactness of the unit sphere, we may assume $(x_n^*) \to x^*$ for some $x^* \in S_{X^*}$. By the previous condition there is a sequence $(x_n) \subset S_X$ so that $r_n < \operatorname{Re} x_n^*(x_n) \leqslant 1$ for every n and such that for all $n \in \mathbb{N}$,

$$\operatorname{dist}(x_n, D(x^*)) \geqslant \varepsilon_0. \tag{5}$$

We may also assume that (x_n) converges to some $x \in S_X$. Since $(x_n^*(x_n)) \to 1$ and both sequences are convergent, it follows that $x^*(x) = 1$; that is $x \in D(x^*)$. We obtain that $\operatorname{dist}(x_n, D(x^*)) \leqslant \|x_n - x\|$ for every n. Since (x_n) converges to x, this inequality contradicts (5).

Lemma (6.2.8) should be compared with the Bishop-Phelps-Bollobás theorem that is valid in finite-dimensional spaces. Here, the functional y^* depends only on x^*, whereas in the general case y and y^* depend on the choice of x and x^*. Note that this strengthened version comes at the cost of having δ depend not only on ε but also on the particular space X. The condition appearing in Lemma (6.2.8) is a strengthening of property AHSP, as we will check below.

Proposition (6.2.9)[331]: Every finite-dimensional normed space has AHSP.

Proof. If X is a finite-dimensional normed space, then we have just seen that for each $\varepsilon > 0$, there is $\delta > 0$ satisfying the condition in Lemma (6.2.8). We may assume that $\delta < \varepsilon < 1$. Now assume that

$$\left\|\sum_{k=1}^{\infty} \alpha_k y_k\right\| > 1-\delta^2$$

for some convex series $\sum \alpha_k y_k$ of elements $\{y_k\}$ in B_X. If $\operatorname{Re} x^*(\sum_{k=1}^{\infty}\alpha_k y_k) > 1-\delta^2$ for some $x^* \in S_{X^*}$, then the subset

$$G := \{n \in \mathbb{N} : \operatorname{Re} x^*(y_n) > 1-\delta\}$$

is such that $\sum_{k\in G}\alpha_k > 1-\delta$ in view of Lemma (6.2.7). Hence the above lemma provides an element $y^* \in S_{X^*}$ and a subset $\{z_k : k \in G\} \subset S_X$ such that $y^*(z_k) = 1$ for all $k \in G$ with $\|y_k - z_k\| < \varepsilon$, as we wanted to show.

Now we will show that some classical Banach spaces have AHSP.

Proposition (6.2.10)[331]: For every σ-finite measure μ, the real or complex space $L_1(\mu)$ has AHSP.

Proof. The following proof for the complex case works for real $L_1(\mu)$ as well. Assume that $0 < \varepsilon < 1$ and take

$$s(\varepsilon) := \sqrt{\frac{4}{4+\varepsilon^2}},\ r(\varepsilon) := \frac{4+\varepsilon(s(\varepsilon)-1)}{4} \text{ and } \eta(\varepsilon) := \varepsilon\big(1-r(\varepsilon)\big). \tag{6}$$

Note that $0 < s(\varepsilon) < r(\varepsilon) < 1$ and so $\eta(\varepsilon) > 0$

Assume that (f_n) is a sequence in $B_{L_1(\mu)}$ such that a certain convex series $\sum_n \alpha_n f_n$ satisfies $\|\sum_n \alpha_n f_n\|_1 > 1 - \eta(\varepsilon)$. We choose a functional x^* in the unit sphere of the dual of $L_1(\mu)$ such that $\operatorname{Re} x^*(\sum_n \alpha_n f_n) > 1 - \eta(\varepsilon)$. We may assume that x^* is an extreme point of the unit ball of $(L_1(\mu))^*$, and we denote the corresponding function by $h \in L_\infty(\mu)$. Since x^* is an extreme point, we may assume that $|h| = 1$. By using a convenient isometry we may also assume that the function $h \in L_\infty(\mu)$ that represents the functional x^* is the constant function, $h \equiv 1$.

Now we define

$$A := \{n \in \mathbb{N} : \operatorname{Re} x^*(f_n) > r(\varepsilon)\} = \left\{n \in \mathbb{N} : \int_\Omega \operatorname{Re} f_n d\mu > r(\varepsilon)\right\}$$

By Lemma (6.2.7) we know that

$$\sum_{i\in A} \alpha_i > 1 - \frac{\eta(\varepsilon)}{1-r(\varepsilon)}$$

and so we take $\gamma(\varepsilon) := \frac{\eta(\varepsilon)}{1-r(\varepsilon)} = \varepsilon$. Letting $E_n := \{t \in \Omega : \operatorname{Re} f_n(t) > s(\varepsilon)|f_n(t)|\}$ for each $n \in A$, we clearly have

$$\begin{aligned} r(\varepsilon) \quad &< \int_\Omega \operatorname{Re} f_n d\mu = \int_{E_n} \operatorname{Re} f_n d\mu + \int_{\Omega\setminus E_n} \operatorname{Re} f_n d\mu \\ &\leqslant \int_{E_n} \operatorname{Re} f_n d\mu + \int_{\Omega\setminus E_n} s(\varepsilon)|f_n| d\mu \\ &\leqslant \int_{E_n} \operatorname{Re} f_n d\mu + s(\varepsilon)\left(1 - \int_n |f_n| d\mu\right) \\ &\leqslant \int_{E_n} \operatorname{Re} f_n d\mu + s(\varepsilon)\left(1 - \int_{E_n} \operatorname{Re} f_n d\mu\right) \\ &= (1-s(\varepsilon)) \int_{E_n} \operatorname{Re} f_n d\mu + s(\varepsilon) \end{aligned}$$

Then we obtain

$$\int_{E_n} \operatorname{Re} f_n d\mu > \frac{r(\varepsilon)-s(\varepsilon)}{1-s(\varepsilon)}. \tag{7}$$

Hence

$$\int_{\Omega\setminus E_n} |f_n| d\mu \leqslant 1 - \int_{E_n} |f_n| d\mu \leqslant 1 - \int_{E_n} \operatorname{Re} f_n d\mu < 1 - \frac{r(\varepsilon)-s(\varepsilon)}{1-s(\varepsilon)}. \tag{8}$$

If $t \in E_n$ we have $(\operatorname{Re} f_n(t))^2 > s(\varepsilon)^2((\operatorname{Re} f_n(t))^2 + (\operatorname{Im} f_n(t))^2)$ and so $(\varepsilon)|\operatorname{Im} f_n(t)| \leqslant \sqrt{1-s(\varepsilon)^2}|\operatorname{Re} f_n(t)|$. Hence we obtain the following upper-estimate:

$$\int_{E_n} |\operatorname{Im} f_n| d\mu \leqslant \sqrt{\frac{1-s(\varepsilon)^2}{s(\varepsilon)^2}}. \tag{9}$$

For each $n \in A$, we define $g_n \in L_1(\mu)$ by

$$g_n := \frac{(\operatorname{Re} f_n)\chi_{E_n}}{\left\|(\operatorname{Re} f_n)\chi_{E_n}\right\|_1} \quad (n \in A)$$

It is clear that $\|g_n\|_1 = 1$ and also $x^*(g_n) = \int_\Omega g_n d\mu = 1$ for every $n \in A$.

To complete the proof, by finding an upper-estimate of $\|g_n - f_n\|_1$. We have

$$\begin{aligned}\|g_n - f_n\|_1 &\leqslant \left\|g_n - f_n\chi_{E_n}\right\|_1 + \|f_n\chi\Omega \setminus E_n\|_1 \\ &\leqslant \|g_n - (\operatorname{Re} f_n)\chi E_n\|_1 + \left\|(\operatorname{Re} f_n - f_n)\chi_{E_n}\right\|_1 + \left\|f_n\chi_{\Omega\setminus E_n}\right\|_1 \\ &\leqslant 1 - \left\|\operatorname{Re} f_n\chi_{E_n}\right\|_1 + \left\|(\operatorname{Im} f_n)\chi_{E_n}\right\|_1 + 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)} \quad \text{(by (8))}\end{aligned}$$

Hence we have proved that $L_1(\mu)$ has AHSP.

Proposition (6.2.11)[331]: The real or complex spaces $C(K)$ have AHSP for any compact Hausdorff space K

Proof. Once again, the proof will only deal with $\mathbb{C}$-valued functions on K and it is valid in both cases. Fix $0 < \varepsilon < 1$, and let $(f_k)_{k=1}^{\infty} \subset B_{C(K)}$ and a convex series $(\alpha_k)_{k=1}^{\infty}$ satisfy

$$\left\|\sum_{k=1}^{\infty} \alpha_k f_k\right\| > 1 - \left(\frac{\varepsilon}{4}\right)^4$$

Consider a point $t_0 \in K$ and a scalar $\lambda, |\lambda| = 1$, satisfying

$$1 \geqslant \operatorname{Re}\left(\lambda \sum_{k=1}^{\infty} \alpha_k f_k(t_0)\right) > 1 - \left(\frac{\varepsilon}{4}\right)^4$$

We take $A := \left\{k \in \mathbb{N} : \operatorname{Re}\big(\lambda f_k(t_0)\big) > 1 - \left(\frac{\varepsilon}{4}\right)^2\right\}$ and $\delta = \varepsilon^2/4^2$. Then

$$\begin{aligned}1 - \left(\frac{\varepsilon}{4}\right)^4 &< \operatorname{Re}\left(\lambda \sum_{k=1}^{\infty} \alpha_k f_k(t_0)\right) = \sum_{k=1}^{\infty} \alpha_k \operatorname{Re}\big(\lambda f_k(t_0)\big) \\ &\leqslant \sum_{k\in A} \alpha_k + \left(1 - \left(\frac{\varepsilon}{4}\right)^2\right) \sum_{k\notin A} \alpha_k\end{aligned}$$

As $\sum_{k=1}^{\infty} \alpha_k = 1$, we obtain $1 - \left(\frac{\varepsilon}{4}\right)^4 < 1 - \left(\frac{\varepsilon}{4}\right)^2 \sum_{k\notin A} \alpha_k$. Hence $\sum_{k\notin A} \alpha_k < \left(\frac{\varepsilon}{4}\right)^2$ and so $\sum_{k\in A} \alpha_k > 1 - \left(\frac{\varepsilon}{4}\right)^2$.

For each $k \in A$, we choose a function $u_k \in C(K)$ such that

$$\operatorname{supp} u_k \subset |f_k|^{-1}((1 - \delta, 1]),\ 0 \leqslant u_k \leqslant 1,\ u_k(t_0) = 1.$$

If we define g_k on K by $g_k := \lambda\left(f_k + u_k\left(-f_k + \frac{f_k}{|f_k|}\right)\right)$ on $\operatorname{supp} u_k$ and $g_k = \lambda f_k$ on $K \setminus \operatorname{supp} u_k$, then g_k is continuous on K. Also, g_k is in the unit sphere of $C(K)$ since f_k is in the unit ball,

$$\left|f_k + u_k\left(-f_k + \frac{f_k}{|f_k|}\right)\right| \leqslant |f_k| + |1 - |f_k|| = 1$$

$$\leqslant 1 - \|\mathrm{Re}\, f_n \chi_{E_n}\|_1 + \sqrt{\frac{1 - s(\varepsilon)^2}{s(\varepsilon)^2}} + 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)} \text{ (by (9))}$$

$$\leqslant \sqrt{\frac{1 - s(\varepsilon)^2}{s(\varepsilon)^2}} + 2\left(1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)}\right) \text{ (by (7) and (6))}$$

$$= \varepsilon.$$

and $|g_k(t_0)| = 1$. In addition, $\|g_k - \lambda f_k\| < \delta$ since this function is zero outside the set supp u_k and for $t \in \mathrm{supp}\, u_k$ we know that

$$|g_k(t) - \lambda f_k(t)| \leqslant \big|1 - |f_k(t)|\big| < \delta. \quad (10)$$

Writing $a := 2\left(\frac{\varepsilon}{4}\right)^2$, we see that

$$\mathrm{Re}\, g_k(t_0) > \mathrm{Re}\, \lambda f_k(t_0) - \delta > 1 - \left(\frac{\varepsilon}{4}\right)^2 - \delta = 1 - a$$

and so

$$|\mathrm{Im}\, g_k(t_0)| < \sqrt{2a}$$

Hence

$$|g_k(t_0) - 1| < \sqrt{a^2 + 2a}. \quad (11)$$

Now for every $k \in A$ we set $h_k := \mu_k \bar{\lambda} g_k$, where $\mu_k = \overline{g_k(t_0)}$, so that $h_k \in S_{C(K)}$. The element $x^* = \lambda \delta_{t_0}$ is an element of $S_{C(K)^*}$ and satisfies $x^*(h_k) = 1$ for all $k \in A$. Indeed, in view of (11), (10) and the choice of δ, for every $k \in A$,

$$\begin{aligned} \|h_k - f_k\| &= \|\mu_k \bar{\lambda} g_k - f_k\| = \|\mu_k g_k - \lambda f_k\| \\ &\leqslant \|\mu_k g_k - g_k\| + \|g_k - \lambda f_k\| \\ &\leqslant |\mu_k - 1| + \|g_k - \lambda f_k\| \\ &\leqslant \sqrt{a^2 + 2a} + \delta = \sqrt{\frac{\varepsilon^4}{4^3} + \frac{\varepsilon^2}{4}} + \left(\frac{\varepsilon}{4}\right)^2 < \varepsilon. \end{aligned}$$

In the above proof we need only that every point t_0 in K has a basis of compact neighborhoods. Hence the same argument shows that $C_0(\Omega)$, the Banach space of continuous functions on Ω that vanish at ∞, also has AHSP for any locally compact space Ω.

We now show that many spaces that are completely different from $C(K)$ and $L_1(\mu)$ also have AHSP. To do so, we recall that a Banach space X is uniformly convex if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$\text{for all } u, v \in B_X \text{ such that } \frac{\| u + v \|}{2} > 1 - \delta, \text{ we have } \| u - v \| < \varepsilon$$

In such a case, the modulus of convexity is given by

$$\delta(\varepsilon) := \inf\left\{1 - \frac{\| u + v \|}{2} : u, v \in B_X, \| u - v \| \geqslant \varepsilon\right\}$$

Proposition (6.2.12)[331]: A uniformly convex Banach space has AHSP.

Proof. Let X be a uniformly convex Banach space, let $\varepsilon > 0$ be arbitrary, and let $\delta = \delta(\varepsilon)$ be as in the definition of uniformly convex.

Let us fix $0 < \varepsilon < 1$ and take $r(\varepsilon) = 1 - \delta(\varepsilon), \eta(\varepsilon) = \frac{\varepsilon(1-r(\varepsilon))}{2}$ and $\gamma(\varepsilon) = \frac{\varepsilon}{2}$. Assume that $\{x_n : n \in \mathbb{N}\} \subset B_X$ is a subset such that for some convex series $\sum_{n=1}^{\infty} \alpha_n x_n$, $\|\sum_{n=1}^{\infty} \alpha_n x_n\| > 1 - \eta(\varepsilon)$. We choose a functional $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(\sum_{n=1}^{\infty} \alpha_n x_n) > 1 - \eta(\varepsilon)$ and let $A = \{n \in \mathbb{N}: \operatorname{Re} x^*(x_n) > r(\varepsilon)\}$. By Lemma (6.2.7) we know that $\sum_{n\in A} \alpha_n > 1 - \frac{\eta(\varepsilon)}{1-r(\varepsilon)} = 1 - \frac{\varepsilon}{2}$. For $n, m \in A$ we have that $\|x_n + x_m\| \geqslant |x^*(x_n + x_m)| > 2r(\varepsilon) = 2 - 2\delta(\varepsilon)$ and, by using the uniform convexity of X, we obtain $\|x_n - x_m\| < \varepsilon$. Since $A \neq \emptyset$, we can choose $n_0 \in A$ and define $z_n = x_{n_0}$ for every $n \in A$. Hence we have that

$$\|z_n - x_n\| < \varepsilon, \text{ for all } n \in A, \text{ and } \sum_{n\in A} \alpha_n > 1 - \frac{\varepsilon}{2} = 1 - \gamma(\varepsilon)$$

Finally, if we choose a functional $x^* \in S_{X^*}$ such that $x^*(x_{n_0}) = 1$, we see that the three requirements for property AHSP have been met.

The following proposition shows that every strictly convex Banach space which is not uniformly convex fails AHSP. In particular, the reflexive space $X = \oplus_2 \ell_\infty^n$ does not satisfy AHSP (see [336], Theorems 9.18,9.14 and 8.17).

Proposition (6.2.13)[311]: A strictly convex Banach space having AHSP is uniformly convex.

Proof. Recall that a Banach space Z is said to be strictly convex if every point of its unit sphere is an extreme point of the unit ball. Assume that the Banach space X has AHSP. By assumption, for $\varepsilon > 0$ small enough such that $\gamma(\varepsilon) < 1/2$, we have the following. Whenever $, y \in B_X, \| x + y \| > 2 - 2\eta(\varepsilon)$, then there exist $u, v \in S_X$ such that $\| u - x \| < \varepsilon, \| v - y \| < \varepsilon$, and $\| u + v \| = 2$. If we use the strict convexity of X, it follows that $u = v$ so $\| x - y \| < 2\varepsilon$. It follows that X is uniformly convex.

We are going to characterize those Banach spaces Y having the property that the Bishop-Phelps-Bollobás theorem holds for operators from ℓ_1 into Y. To do so, we will use the property AHSP that was introduced.

Theorem (6.2.14)[331]: A Banach space Y is such that the couple (ℓ_1, Y) has the Bishop-Phelps-Bollobás property for operators if, and only if, Y satisfies $AHSP$.

Proof. Our proof will be given for the case of complex Banach spaces. (In fact, the case of real Banach spaces is simpler and gives a better order of approximation.)

Let Y be a Banach space with AHSP. Given $\varepsilon > 0$, we will use the functions $\gamma(\varepsilon)$ and $\eta(\varepsilon)$ satisfying the conditions. We can assume that ε is small enough such that $0 < \gamma(\varepsilon) < 1$. Given $T \in S_{L(\ell_1,Y)}$, we take $x_0 = (x_0(n))_{n=1}^{\infty} \in S_{\ell_1}$, such that $\|Tx_0\| > 1 - \eta(\varepsilon)$. By composing with an isometry, we may assume that $x_0(n) = \operatorname{Re} x_0(n) \geqslant 0$ for every positive integer n

By the assumptions on T and x_0, we can apply AHSP to the convex series $\sum x_0(n)$ and for the elements $x_n = T(e_n), n \in \mathbb{N}$, where (e_n) is the canonical basis of ℓ_1. Hence, there is a subset $A \subset \mathbb{N}$ and $\{y_n : n \in A\} \subset S_Y$ such that

$$\sum_{n\in A} x_0(n) > 1 - \gamma(\varepsilon), \ \|y_n - x_n\| < \varepsilon, \text{ for all } n \in A. \tag{12}$$

and

$$\left\|\sum_{n\in A} x_0(n) y_n\right\| = \sum_{n\in A} x_0(n). \tag{13}$$

There is a linear bounded operator S of norm 1 from ℓ_1 to Y such that

$$S(e_n) = \begin{cases} y_n & \text{if } n \in A \\ T(e_n) & \text{if } n \notin A \end{cases}$$

In view of (12) we obtain that

$$\begin{aligned} \| T - S \| &= \sup_n \|(S - T)(e_n)\| \\ &= \sup_{n \in A} \|y_n - x_n\| \leqslant \varepsilon. \end{aligned}$$

Since $\gamma(\varepsilon) < 1$, in view of (12), then $A \neq \emptyset$. If $P_A(x_0) = \sum_{n \in A} x_0(n) e_n$, then the element $z_0 = \frac{P_A(x_0)}{\|P_A(x_0)\|} \in S_{\ell_1}$ is such that

$$\begin{aligned} \|x_0 - z_0\| &\leqslant \|x_0 - P_A x_0\| + \left\| P_A x_0 - \frac{P_A x_0}{\|P_A x_0\|} \right\| \\ &= \sum_{n \notin A} x_0(n) + |1 - \|P_A x_0\|| \ \text{(by (12))} \\ &= 2 \sum_{n \notin A} x_0(n) < 2\gamma(\varepsilon) \end{aligned}$$

Also, by using (13), we know that

$$\|S z_0\| = \frac{\|\sum_{n \in A} x_0(n) y_n\|}{\|P_A x_0\|} = \frac{\|\sum_{n \in A} x_0(n) y_n\|}{\sum_{n \in A} x_0(n)} = 1$$

Hence, by taking $\beta(\varepsilon) = 2\gamma(\varepsilon)$, we obtain that (ℓ_1, Y) satisfies the Bishop-Phelps-Bollobás property for operators.

Conversely, assume that Y is a complex Banach space such that (ℓ_1, Y) satisfies the BishopPhelps-Bollobás property for operators. Given $0 < \rho < 1$, we choose s such that $0 < s < 1$ and $0 < \sqrt{2(1 - s)} < \frac{\rho}{2}$

Let $\eta(\varepsilon)$ and $\beta(\varepsilon)$ be the positive numbers that appear in the definition of the Bishop-Phelps-Bollobás property for operators. Choose $\varepsilon = \varepsilon(\rho)$ such that $0 < \varepsilon < \frac{\rho}{2} < 1$ and $\frac{\beta(\varepsilon)}{1-s} < \frac{\rho}{2}$. Let $(y_n) \subset S_Y$ be a sequence and let $\sum \alpha_n$ be a convex series such that

$$\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \eta(\varepsilon)$$

There is a bounded linear operator $T: \ell_1 \to Y$ such that $T(e_n) = y_n$ for all n. We have $\| T \| = 1$ and the element $x_0 = \sum_{n=1}^{\infty} \alpha_n e_n \in S_{\ell_1}$ satisfies that

$$\|T(x_0)\| = \left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \eta(\varepsilon). \tag{14}$$

We apply the assumption that (ℓ_1, Y) satisfies the Bishop-Phelps-Bollobás property to obtain a norm one operator $S \in L(\ell_1, Y)$ and an element $u_0 \in S_{\ell_1}$ such that

$$\|S u_0\| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon), \ \| S - T \| < \varepsilon.$$

It then follows that

$$\sum_{n=1}^{\infty} (\alpha_n - \operatorname{Re} u_0(n)) \leqslant \sum_{n=1}^{\infty} |u_0(n) - \alpha_n| = \|u_0 - x_0\| < \beta(\varepsilon), \tag{15}$$

and so

$$\sum_{n=1}^{\infty} \operatorname{Re} u_0(n) > 1 - \beta(\varepsilon). \tag{16}$$

Let us consider the set

$$A := \{n \in \mathbb{N} : \operatorname{Re} u_0(n) > s|u_0(n)|\}$$

By using (16) we obtain that

$$\begin{aligned} 1 - \beta(\varepsilon) &< \sum_{n=1}^{\infty} \operatorname{Re} u_0(n) \\ &= \sum_{n\in A} \operatorname{Re} u_0(n) + \sum_{n\notin A} \operatorname{Re} u_0(n) \\ &\leqslant \sum_{n\in A} \operatorname{Re} u_0(n) + s\sum_{n\notin A} |u_0(n)| \\ &= \sum_{n\in A} \operatorname{Re} u_0(n) + s\left(1 - \sum_{n\in A} |u_0(n)|\right) \\ &\leqslant \sum_{n\in A} \operatorname{Re} u_0(n) + s\left(1 - \sum_{n\in A} \operatorname{Re} u_0(n)\right) \end{aligned}$$

So

$$\sum_{n\in A} \operatorname{Re} u_0(n) > 1 - \frac{\beta(\varepsilon)}{1-s}. \tag{17}$$

Hence

$$\begin{aligned} \sum_{n\in A} \alpha_n &\geqslant \sum_{n\in A} \operatorname{Re} u_0(n) - \|u_0 - x_0\| \\ &> 1 - \frac{\beta(\varepsilon)}{1-s} - \|u_0 - x_0\| \\ &> 1 - \frac{\beta(\varepsilon)}{1-s} - \beta(\varepsilon) \end{aligned}$$

We take $\gamma(\rho) := \beta(\varepsilon) + \frac{\beta(\varepsilon)}{1-s} < \rho$ and so $\lim_{t\to 0}\gamma(t) = 0$. Now, if $z \in \mathbb{C}$ satisfies $|z| = 1$ and $\operatorname{Re} z > t > 0$, then we know that

$$|1 - z|^2 = 1 + |z|^2 - 2\operatorname{Re} z < 2(1 - t).$$

Thus, for $n \in A$, by the choice of s, it follows that

$$\left|1 - \frac{u_0(n)}{|u_0(n)|}\right|^2 < 2(1 - s) < \frac{\rho^2}{4}. \tag{19}$$

If we write $z_n := S(e_n)$, then

$$1 = \|S(u_0)\| = \left\|\sum_{n=1}^{\infty} u_0(n)z_n\right\|.$$

Hence, there is an element $y^* \in S_{Y^*}$ such that

$$u_0(n)y^*(z_n) = |u_0(n)| \tag{20}$$

for all $n \in \mathbb{N}$. Thus, for all $n \in A$, z_n belongs to S_Y. Also we know that for $n \in A$ we have

$$\|z_n - y_n\| = \|S(e_n) - T(e_n)\| < \varepsilon < \frac{\rho}{2},$$

and so

$$\begin{aligned}\left\|\frac{u_0(n)}{|u_0(n)|}z_n - y_n\right\| &\leqslant \left\|\frac{u_0(n)}{|u_0(n)|}z_n - z_n\right\| + \|z_n - y_n\| \\ &\leqslant \left|\frac{u_0(n)}{|u_0(n)|} - 1\right| + \|z_n - y_n\| \\ &< \frac{\rho}{2} + \frac{\rho}{2} = \rho \ \text{ (by (19))}.\end{aligned}$$

In view of (18), the previous inequality and (20), we have checked that

$$\sum_{n\in A} \alpha_n > 1 - \gamma(\rho),\ \left\|\frac{u_0(n)}{|u_0(n)|}z_n - y_n\right\| < \rho,\ \text{ and }\ y^*\left(\frac{u_0(n)}{|u_0(n)|}z_n\right) = 1,\ \text{ for all } n \in A,$$

and so Y satisfies AHSP.

We show that for every $n \in N$ and for every uniformly convex space Y, the pair (ℓ_∞^n, Y) satisfies the Bishop-Phelps-Bollobás property for operators.

We begin with the following result for operators from c_0 into a uniformly convex Banach space. In order to state it, let us recall that for $A \subset \mathbb{N}, P_A: c_0 \to c_0$ is defined by $P_A(x) = \sum_{n\in A} x(n)e_n$.

Lemma (6.2.15)[331]: Let Y be a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$. Let $\varepsilon > 0$. If $T \in S_{L(c_0,Y)}$, and $A \subset \mathbb{N}$ has the property that $\|TP_A\| > 1 - \delta(\varepsilon)$, then we have that $\|T(I - P_A)\| \leqslant \varepsilon$

Proof. Since Y is uniformly convex, for each $\varepsilon > 0$, there is $0 < \delta(\varepsilon) < 1$ such that whenever u and v are in B_X, satisfying $\| u + v \| \geqslant 2 - 2\delta(\varepsilon)$, it follows that $\| u - v \| < \varepsilon$. Assume that $T \in L(c_0, Y)$ satisfies $\| T \| = 1$ and let $A \subset \mathbb{N}$ with $\|TP_A\| > 1 - \delta(\varepsilon)$. Choose $x_0 \in P_A(c_0) \cap S_{c_0}$ such that $\|TP_A(x_0)\| > 1 - \delta(\varepsilon)$.

Since $1 = \| T \| \geqslant \|T(x_0 \pm z)\|$ for every element $z \in B_{c_0}$ whose support lies outside A, we obtain that $\|T(x_0) \pm T(I - P_A)(y)\| \leqslant 1$ for any $y \in B_{c_0}$. Also, we have that

$$\begin{aligned}&\|T(x_0 + (I - P_A)(y)) + T(x_0 - (I - P_A)(y))\| \\ &= \|2T(x_0)\| = \|2TP_A(x_0)\| > 2 - 2\delta(\varepsilon).\end{aligned}$$

Thus, by using the uniform convexity of Y we obtain that

$$\|2T(I - P_A)(y)\| = \|(Tx_0 + T(I - P_A)(y)) - (Tx_0 - T(I - P_A)(y))\| < 2\varepsilon.$$

Since y is an arbitrary element of the unit ball of c_0, we finally get that $\|T(I - P_A)\| \leqslant \varepsilon$.

We prove the promised result that (ℓ_∞^n, Y) satisfies the Bishop-Phelps-Bollobás property for operators for every n whenever Y is a uniformly convex Banach space. Unfortunately, our method involves constants that depend on n, and we do not know whether the result can be extended to, say, (c_0, Y) or (ℓ_∞, Y) if Y is uniformly convex.

Theorem (6.2.16)[331]: Let Y be a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$. Let $n \in \mathbb{N}, 0 < \varepsilon < 1, 0 < \varepsilon' < \varepsilon$ with $\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}} < \min\left\{\delta(\varepsilon), \frac{3}{2}\left(\varepsilon + \varepsilon^{2/3}\right)\right\}$. For any $x_0 \in B_{\ell_\infty^n}$ and $T \in S_{L(\ell_\infty^n, Y)}$ such that $\|Tx_0\| > 1 - \varepsilon'$, there exist $z_0 \in B_{\ell_\infty^n}$ and $V \in S_{L(\ell_\infty^n, Y)}$ such that

$$\|Vz_0\| = 1,\ \|z_0 - x_0\| < \varepsilon^{1/4} + \varepsilon^{1/3},\ \| V - T \| \leqslant \varepsilon + 6n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) + \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right)$$

Proof. Let $T \in L(\ell_\infty^n, Y)$ be a norm one operator and $x_0 \in B_{\ell_\infty^n}$ satisfying $\|Tx_0\| > 1 - \varepsilon'$. By composing with an isometry on ℓ_∞^n if necessary, we may assume that $x_0(i) \geqslant 0$ for each $i \leqslant n$. Let $y^* \in S_{Y^*}$ be such that $y^*T(x_0) = \operatorname{Re}(T^t y^*)(x_0) > 1 - \varepsilon'$.

Define

$$\begin{aligned} E \quad &:= \left\{i \leqslant n : \operatorname{Re}(T^t y^*)(e_i)x_0(i) > \left(1 - \varepsilon^{1/3}\right)|(T^t y^*)(e_i)|\right\} \\ &\subset \left\{i \leqslant n : \operatorname{Re}(T^t y^*)(e_i) > 0, x_0(i) > 1 - \varepsilon^{1/3}\right\} \end{aligned}$$

Since $T^t y^* \in (\ell_\infty^n)^* \equiv \ell_1^n$ and $\|T^t y^*\| \leqslant \|y^*\| = 1$,

$$\sum_{k=1}^{n} |(T^t y^*)(e_k)| \leqslant 1$$

If $A := \sum_{i \notin E} |(T^t y^*)(e_i)|$, we will check that $A < \frac{\varepsilon'}{\varepsilon^{1/3}}$. Indeed,

$$\begin{aligned} 1 - \varepsilon' < \operatorname{Re}(T^t y^*)(x_0) \quad &= \sum_{i=1}^{n} \operatorname{Re}(T^t y^*)(e_i)x_0(i) \\ &\leqslant \sum_{i \in E} |(T^t y^*)(e_i)x_0(i)| + \sum_{i \notin E} \operatorname{Re}(T^t y^*)(e_i)x_0(i) \\ &\leqslant \sum_{i \in E} |(T^t y^*)(e_i)| + \left(1 - \varepsilon^{1/3}\right)\sum_{i \notin E} |(T^t y^*)(e_i)| \\ &\leqslant 1 - A + \left(1 - \varepsilon^{1/3}\right)A = 1 - \varepsilon^{1/3}A \end{aligned}$$

so

$$A < \frac{\varepsilon'}{\varepsilon^{1/3}}. \tag{21}$$

Thus,

$$\begin{aligned} 1 - \varepsilon' \quad &< \operatorname{Re}(T^t y^*)(x_0) \leqslant \sum_{i \in E} \operatorname{Re}(T^t y^*)(e_i)x_0(i) + \sum_{i \notin E} |(T^t y^*)(e_i)x_0(i)| \\ &\leqslant \sum_{i \in E} \operatorname{Re}(T^t y^*)(e_i) + \sum_{i \notin E} |(T^t y^*)(e_i)| < \sum_{i \in E} \operatorname{Re}(T^t y^*)(e_i) + \frac{\varepsilon'}{\varepsilon^{1/3}}, \end{aligned}$$

and sO

$$\sum_{i \in E} \operatorname{Re}(T^t y^*)(e_i) > 1 - \varepsilon' - \frac{\varepsilon'}{\varepsilon^{1/3}}$$

By the choice of ε' we have

$$\|TP_E\| \geqslant \left\|TP_E\left(\sum_{i \in E} e_i\right)\right\| \geqslant \left|(T^t y^*)\left(\sum_{i \in E} e_i\right)\right|$$

$$> 1 - \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{\frac{1}{3}}}\right) > 1 - \delta(\varepsilon). \tag{22}$$

By Lemma (6.2.15) we obtain that

$$\|T(I - P_E)\| \leqslant \varepsilon. \tag{23}$$

Setting $e_0 = \sum_{i \in E} e_i$ in $B_{P_E(\ell_\infty^n)}$ and

$$x_0^* = \sum_{i \in E} \frac{1}{|E|} e_i^*$$

in $(\ell_\infty^n)^*$, by the definition of E, we have that

$$\|P_E(x_0) - e_0\| < \varepsilon^{1/3} \text{ and } x_0^*(e_0) = 1. \tag{24}$$

Define the operator $S: \ell_\infty^n \to Y$ by

$$S(x) := TP_E(x) + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) x_0^*(P_E(x)) \frac{T(e_0)}{\|T(e_0)\|} \ (x \in \ell_\infty^n)$$

Let $\tau = \frac{\sqrt{\varepsilon}}{2|E|}$. We claim that $\|e - e_0\| < \varepsilon^{1/4}$ for all $e \in \text{Ext}\left(B_{P_E(\ell_\infty^n)}\right)$ satisfying $|x_0^*(e) - 1| < \tau$. Indeed, if $|x_0^*(e) - 1| < \tau$ then $\left|\sum_{i \in E} e(i) - |E|\right| < \tau|E|$, and so $\text{Re}(1 - e(i)) < \tau|E|$ for all $i \in E$. Hence $|e(i) - 1| = \sqrt{2 - 2\text{Re}(e(i))} < \sqrt{2\tau|E|} = \varepsilon^{1/4}$ for all $i \in E$, and the claim follows.

By (22) we obtain

$$\|S(e_0)\| = \|TP_E(e_0)\| + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) \geqslant 1 - \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{\frac{1}{3}}}\right) + 3n\left(\sqrt{\varepsilon} + \varepsilon^{\frac{1}{6}}\right) \tag{25}$$

and we also know that

$$\| S(e) \| \leqslant 1 + 3n\left(\sqrt{\varepsilon} + \varepsilon^{\frac{1}{6}}\right)(1 - \tau), \tag{26}$$

for all $e \in \text{Ext}\left(B_{P_E(\ell_\infty^n)}\right)$ such that $|x_0^*(e)| \leqslant 1 - \tau$. By the choice of ε', the upper bound in (26) is less than the lower bound in (25), so the operator $S = S \circ P_E$ attains its norm at some point e in $\text{Ext}\left(B_{P_E(\ell_\infty^n)}\right)$ with $1 - |x_0^*(e)| < \tau$. So, by the claim above, S attains its norm at λe for some number λ of modulus one such that $\|\lambda e - e_0\| < \varepsilon^{1/4}$. Hence S also attains its norm at $z_0 = \lambda e + (I - P_E)(x_0)$ and by (24) we have

$$\|z_0 - x_0\| = \|\lambda e - P_E(x_0)\| \leqslant \|\lambda e - e_0\| + \|e_0 - P_E(x_0)\| < \varepsilon^{1/4} + \varepsilon^{1/3}.$$

From the definition of S and by (25) and (26) we have

$$1 - \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right) + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) \leqslant \| S \| \leqslant 1 + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right)$$

and putting $V := \frac{S}{\|S\|}$ it follows that

$$\begin{aligned} \| T - V \| &\leqslant \| T - S \| + \| S - V \| \\ &\leqslant \|T(I - P_E)\| + \|TP_E - S\| + \left\| S - \frac{S}{\| S \|} \right\| \\ &\leqslant \|T(I - P_E)\| + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) + |\, \| S \| - 1| \\ &\leqslant \varepsilon + 3n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) + |\, \| S \| - 1| \end{aligned}$$

$$\leqslant \varepsilon + 3n\left(\sqrt{\varepsilon} + \varepsilon^{\frac{1}{6}}\right) + \max\left\{\varepsilon' + \frac{\varepsilon'}{\varepsilon^{\frac{1}{3}}}, 3n\left(\sqrt{\varepsilon} + \varepsilon^{\frac{1}{6}}\right)\right\}$$
$$\leqslant \varepsilon + 6n\left(\sqrt{\varepsilon} + \varepsilon^{1/6}\right) + \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right)$$

and the proof is complete.

On a general vector valued result of Bishop-Phelps type, Lindenstrauss [337] proved the denseness of the subset of operators between Banach spaces whose second adjoints attain their norms. Thus, instead of asking whether or not every pair of Banach spaces (X, Y) has the Bishop-Phelps-Bollobás property for operators, one could begin by asking the following question:

Is there a function $\gamma: \mathbb{R}^+ \to (0,1), \lim_{t\to 0}\gamma(t) = 0$, such that the following holds: for all $T \in S_{L(X,Y)}$ and $x_0 \in S_X$ with $\|Tx_0\| > 1 - \gamma(\varepsilon)$, there exist $S \in S_{L(X,Y)}$ and $x_0^{**} \in S_{X^{**}}$ satisfying,

$$\|S^{\mathrm{tt}}x_0^{**}\| = 1, \ \| S - T \| < \varepsilon, \ \|x_0^{**} - x_0\| < \varepsilon?$$

Unfortunately, even this question has a negative answer in general. We will use the original idea of Lindenstrauss to show that.

Lemma (6.2.17)[331]: Let Y be a strictly convex Banach space.

(a) Let $T: \ell_\infty \to Y$ be an operator such that $T(e_n) \neq 0$ for all n. If T attains its norm at a point $z \in B_{\ell_\infty}$, then $|z(n)| = 1$, for all $n \in \mathbb{N}$.

(b) If $T: c_0 \to Y$ is an operator attaining its norm, then T is a finite rank operator.

Proof. (a) Suppose that there exists a point $z \in S_{\ell_\infty}$ at which T attains its norm. If we assume that there exists n so that $|z(n)| < 1$, then $\|z \pm (1 - |z(n)|)e_n\| \leqslant 1$ and so, by convexity,

$$\| T \| = \| T(z) \| = \|T(z \pm (1 - |z(n)|)e_n)\|$$

Since Y is strictly convex we get that $T(e_n) = 0$. This is a contradiction.

(b) Let $z \in S_{c_0}$ be such that T attains its norm at z. Since there exists an n_0 with $|z(n)| < 1$ for all $n \geqslant n_0$, the above argument implies that $T(e_n) = 0$ for all $n \geqslant n_0$.

The argument of the proof of part (b) of the above lemma actually shows that if for some operator $T: c_0 \to Y$ the Bishop-Phelps-Bollobás theorem holds, then T can be approximated by finite-rank operators and so it is compact.

By taking second adjoints we obtain the following proposition.

Proposition (6.2.18)[331]: Let $T_0: c_0 \to Y$ be an isomorphism. Assume that Y^{**} is strictly convex and $T \in L(c_0, Y)$ is such that

$$\|T - T_0\| < \inf_n\{\|T_0(e_n)\|\}$$

Then

$$\left\{y \in B_{\ell_\infty}: \|T^{\mathrm{tt}}(y)\| = \| T \|\right\} \subset \left\{y \in B_{\ell_\infty}: |y(n)| = 1, \text{for all } n \in \mathbb{N}\right\}.$$

Example (6.2.19)[331]: Applying the above proposition to $X = c_0, Y$ any Banach space isomorphic to c_0 such that Y^{**} is strictly convex and $T_0 = I$, the identity mapping, gives a negative answer to the above question. Indeed, given $T \in L(c_0, Y)$ such that $\| T - I \| < \inf_n\{\|e_n\|Y\}$, then $|z(n)| = 1$, for all $n \in \mathbb{N}$ and all $z \in B_{\ell_\infty}$ with $\|T^{\mathrm{tt}}(z)\| = \| T \|$. So $\mathrm{dist}\left(z, B_{c_0}\right) = 1$.

Section (6.3): Approximate Hyperplane Series Properties:

The motivation comes from recent intensive study of the famous BishopPhelps Theorem [351], which states that every Banach space is subreflexive, i.e., the set of norm attaining (continuous and linear) functionals on a Banach space is dense in its topological dual.

The first who initiated the study of the denseness of norm-attaining operators between two Banach spaces was Lindenstrauss [363]. Later a lot of attention was devoted to extend Bishop-Phelps result in the setting of operators on Banach spaces (see, e.g., [343],[354]).

In 1970, Bollobás showed the following "quantitative version" which is now called Bishop-Phelps-Bollobás Theorem [352]. To state this result we mention that for a normed space X, we denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. As usual, X^* denotes the dual Banach space of X.

The mentioned above version of the Bishop-Phelps-Bollobás Theorem from [353] states that if X is a Banach space and $0 < \varepsilon < 1$, then given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2/4$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1, \| y - x \| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

For a refinement of the above result see [356]. In 2008 Acosta, Aron, García and Maestre initiated the study of parallel versions of this result for operators [344]. For two normed spaces X and Y over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C}), \mathcal{L}(X, Y)$ denotes the space of (bounded and linear) operators from X into Y, endowed with the usual operator norm.

We recall the following definition from [344].

Definition (6.3.1)[341]: Let X and Y be both either real or complex Banach spaces. It is said that the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators $(BPBp)$, if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ is such that $\| Tx \| > 1 - \eta(\varepsilon)$, then there exist an element u in S_X and an operator S in $S_{\mathcal{L}(X,Y)}$ satisfying the following conditions

$$\| Su \| = 1, \quad \| u - x \| < \varepsilon \quad \text{and} \quad \| S - T \| < \varepsilon.$$

During the last years there are a number of interesting results where it is shown versions of Bishop-Phelps-Bollobás Theorem for operators (see [348],[355] and [361]). It is known that the pair (X, Y) has the BPBp whenever X and Y are finite dimensional spaces (see [344]). If a Banach space Y has the property β of Lindenstrauss, then (X, Y) has the BPBp for every Banach space X (see [344]). In the case when $X = \ell_1$ a characterization of the Banach spaces Y such that the pair (ℓ_1, Y) has the BPBp was given in [344].

It should be pointed out that very little is known about the stability under direct sums of the property that a pair of Banach spaces (X, Y) has the Bishop-Phelps-Bollobás property for operators. In order to state some results of this kind we recall the following notion used in [345]. Given two Banach spaces X and Y (both real or complex), we say that Y has property $\mathcal{P}_X$ if the pair (X, Y) has the BPBp for operators.

It was shown in [349] that the pairs $\left(X, (\oplus \sum_{n=1}^{\infty} Y_n)_{c_0}\right)$ and $\left(X, (\oplus \sum_{n=1}^{\infty} Y_n)_{\ell_\infty}\right)$ satisfy the Bishop-Phelps-Bollobás property for operators whenever all pairs (X, Y_n) have the Bishop-Phelps-Bollobás property for operators "uniformly". In general the analogous stability result does not hold for every Banach sequence lattice E instead of c_0. For instance, the subset of norm attaining operators from any Banach space X into $\ell_p (1 \le p < \infty)$ is not dense in the space of operators from X into ℓ_p ([359],[342]) for every Banach space X. Indeed it is a longstanding open question if for every (real) Banach space X, the subset of norm attaining operators from X into the euclidean space $\mathbb{R}^2$ is dense in the corresponding space of operators. However, it is also known that $\mathcal{P}_{\ell_1}$ is stable under finite ℓ_p-sums for $1 \le p \le \infty$ (see [345]).

We provide two nontrivial extensions of the above stability results. On one hand we prove that the property $\mathcal{P}_{\ell_1}$ is stable under absolute summands (Theorem (6.3.9)). This extends the above mentioned result for finite ℓ_p-sums. We also prove under mild additional assumptions, that the property $\mathcal{P}_{\ell_1}$ is stable under E-sums, being E a uniformly monotone Banach sequence lattice (Theorem (6.3.13)). As a consequence we deduce, for instance, that if $\{X_k: k \in \mathbb{N}\}$ is a sequence of spaces such that X_k is either some $C(K)$ or $L_1(\mu)$ or a Hilbert space, then the pair $\left(\ell_1, (\sum_{k=1}^{\infty} X_k)_{\ell_p}\right)$ has the BPBp for operators (Corollary (6.3.14)).

On the other hand, in case that the range is a Hilbert space, we also prove some optimal stability result of BPBp under ℓ_1-sums on the domain (Proposition (6.3.6)). This result extends [362], where the above result for the ℓ_1-sum of copies of the same space.

As we already mentioned there is a characterization of the Banach spaces Y such that the pair (ℓ_1, Y) has the Bishop-Phelps-Bollobás property for operators [344]. The property on Y equivalent to the previous fact was called the AHSp.

We will need the following definition, where in what follows by a convex series we mean a series $\sum \alpha_n$, where $0 \le \alpha_n \le 1$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = 1$.

Definition (6.3.2)[341]: A Banach space X has the approximate hyperplane series property (AHSp) if and only if for every $0 < \varepsilon < 1$ there exists $0 < \eta < \varepsilon$ such that for every sequence $\{x_n\}$ in S_X and every convex series $\sum \alpha_n$ with

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| > 1 - \eta.$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k: k \in A\} \subset S_X$ satisfying

(a) $\sum_{k \in A} \alpha_k > 1 - \varepsilon$,

(b) $\|z_k - x_k\| < \varepsilon$ for all $k \in A$ and

(c) there is $x^* \in S_{X^*}$ such that $x^*(z_k) = 1$ for every $k \in A$.

We will use the following characterization of the AHSp (see [345], Proposition 1.2.)

Proposition (6.3.3)[341]: Let X be a Banach space. The following conditions are equivalent:

(a) X has the AHSp.

(b) For every $0 < \varepsilon < 1$ there exist $\gamma_X(\varepsilon) > 0$ and $\eta_X(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \gamma_X(\varepsilon) = 0$ such that for every sequence $\{x_n\}$ in B_X and every convex series $\sum_n \alpha_n$ with $\|\sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta_X(\varepsilon)$, there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k \in A} \alpha_k > 1 - \gamma_X(\varepsilon)$, an element $x^* \in S_{X^*}$, and $\{z_k: k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.

(c) For every $0 < \varepsilon < 1$ there exists $0 < \eta < \varepsilon$ such that for any sequence $\{x_n\}$ in B_X and every convex series $\sum_n \alpha_n$ with $\|\sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta$, there are a subset $A \subset \mathbb{N}$ with $\sum_{k \in A} \alpha_k > 1 - \varepsilon$, an element $x^* \in S_{X^*}$, and $\{z_k: k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.

(d) The same statement holds as in (c) but for every sequence $\{x_n\}$ in S_X.

We study the Bishop-Phelps-Bollobás property for operators between special types of Banach spaces. In particular we are interested in stability of this property when the domain is an ℓ_1 sum of Banach spaces. We consider either real or complex Banach spaces.

We will need the following lemma (see [344], Lemma 3.3).

Lemma (6.3.4)[341]: Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for each n and let $\eta > 0$ be such that there is some sequence $\{\alpha_n\}$ of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} \alpha_n \leq 1$ and $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta}{1-r}.$$

We also need the following technical lemma. For the sake of completeness we include a proof.

Lemma (6.3.5)[341]: Let H be a real or complex Hilbert space and assume that $u, v \in S_H$. Then there is a surjective linear isometry Φ on H such that $\Phi(u) = v$ and $\| \Phi - I \| = \| u - v \|$.

Proof. The result is obvious in the case $\dim H = 1$. Assume that $\dim H \geq 2$. Thus there is an element $v^{\perp} \in S_H$ orthogonal to v and such that $[u, v] \subset [v, v^{\perp}]$, where $[x, y]$ is the linear span of the vectors x and y in H. Let $u_1, u_2 \in \mathbb{K}$ such that $u = u_1 v + u_2 v^{\perp}$ and write $u^{\perp} = -\overline{u_2} v + \overline{u_1} v^{\perp}$. It is clearly satisfied that

$$1 = \| u \|^2 = |u_1|^2 + |u_2|^2 \text{ and } \langle u, u^{\perp} \rangle = 0.$$

Let M be a subspace of H orthogonal to $[v, v^{\perp}] = [u, u^{\perp}]$ and such that $H = [u, u^{\perp}] \oplus M$. Define the mapping $\Phi : H \longrightarrow H$ given by

$$\Phi(zu + wu^{\perp} + m) = zv + wv^{\perp} + m, \ \forall (z, w) \in \mathbb{K}^2, m \in M,$$

which is a surjective linear isometry on H. It clearly satisfies $\Phi(u) = v$ and $\Phi(u^{\perp}) = v^{\perp}$.

Clearly $(\Phi - I)(u) = v - u$, $(\Phi - I)(u^{\perp}) = v^{\perp} - u^{\perp}$ and $\| u - v \| = \|u^{\perp} - v^{\perp}\|$. Also we have that

$$\langle v - u, v^{\perp} - u^{\perp} \rangle = -(\langle v, u^{\perp} \rangle + \langle u, v^{\perp} \rangle) = 0.$$

Hence $\Phi - I$ restricted to $[u, u^{\perp}]$ is a multiple of a linear isometry from this subspace into itself. As a consequence $\| \Phi - I \| = \| u - v \|$. The next result uses the argument outlined in [362] in the case that the domain is the ℓ_1-sum of one space.

Proposition (6.3.6)[341]: Assume that $\{X_i : i \in I\}$ is a family of Banach spaces, H is a Hilbert space such that the pair (X_i, H) has the $BPBp$ for operators for every $i \in I$ and with the same function η. Then the pair $\left((\oplus \sum_{i \in I} X_i)_{\ell_1}, H\right)$ has the BPBp.

Proof. We write $Z = (\oplus \sum_{i \in I} X_i)_{\ell_1}$. Given $0 < \varepsilon < 1$, we choose positive real numbers r, s and t such that

$$r < \frac{\varepsilon}{4},\ s < \min\left\{\frac{\varepsilon}{4}, \frac{\delta_H(r)}{3}\right\} \text{ and } t < \min\left\{\frac{\varepsilon}{4}, \eta(s), \frac{\delta_H(r)}{3}\right\}, \tag{27}$$

where δ_H is the modulus of convexity of H.

Assume that $z_0 = \{z_0(i)\} \in S_Z$ and $T \in S_{\mathcal{L}(Z,H)}$ satisfies $\|T(z_0)\| > 1 - t^2$. For every $i \in I$, we denote by T_i the restriction of T to X_i, that is embedded in Z in a natural way.

Assume that $y^* \in S_{H^*}$ satisfies that $\operatorname{Re} y^*\big(T(z_0)\big) = \|T(z_0)\| > 1 - t^2$.

Denote by $B = \big\{i \in I : \operatorname{Re} y^*\big(T_i(z_0(i))\big) > (1 - t)\|z_0(i)\|\big\}$. We clearly have that

$$\begin{aligned} 1 - t^2 < \operatorname{Re} y^*\big(T(z_0)\big) &= \sum_{i \in I} \operatorname{Re} y^*\left(T_i\big(z_0(i)\big)\right) \\ &= \sum_{i \in B} \operatorname{Re} y^*\big(T_i(z_0(i))\big) + \sum_{i \in I \setminus B} \operatorname{Re} y^*\big(T_i(z_0(i))\big) \end{aligned}$$

$$\leq \sum_{i\in B} \|z_0(i)\| + \sum_{i\in I\setminus B} (1-t)\|z_0(i)\|$$
$$= 1 - t\sum_{i\in I\setminus B} \|z_0(i)\|.$$

Hence

$$\sum_{i\in I\setminus B} \|z_0(i)\| \leq t. \tag{28}$$

By assumption, for every $i \in B$ there is an operator $S_i \in S_{\mathcal{L}(X_i,H)}$ and an element $x_i \in S_{X_i}$ such that

$$\left\|S_i - \frac{T_i}{\|T_i\|}\right\| < s, \ \left\|x_i - \frac{z_0(i)}{\|z_0(i)\|}\right\| < s \ \text{ and } \ \|S_i(x_i)\| = 1, \ \forall i \in B. \tag{29}$$

It follows by (29) that for every $i, j \in B$ we have that

$$\begin{aligned}
\|S_i(x_i) + S_j(x_j)\| &\geq \left\|\frac{S_i(z_0(i))}{\|z_0(i)\|} + \frac{S_j(z_0(j))}{\|z_0(j)\|}\right\| - 2s \\
&\geq \left\|\frac{T_i(z_0(i))}{\|T_i\|\|z_0(i)\|} + \frac{T_j(z_0(j))}{\|T_j\|\|z_0(j)\|}\right\| - 4s \\
&\geq 2(1-t) - 4s \\
&> 2(1-\delta_H(r)).
\end{aligned}$$

As a consequence $\|S_i(x_i) - S_j(x_j)\| \leq r$ for each $i, j \in B$.

Since $B \neq \emptyset$, we choose some element $i_0 \in B$ and define $y_0 = S_{i_0}(x_{i_0})$. By Lemma (6.3.5), for every $i \in B$, there is a linear surjective isometry $\Phi_i: H \to H$ such that $\Phi_i(S_i(x_i)) = y_0$ and $\|\Phi_i - I\| = \|S_i(x_i) - y_0\| \leq r$.

We define an operator $R = \{R_i\}_{i\in I} \in \mathcal{L}(Z, H)$ by

$$R_i = \Phi_i \circ S_i, \ \forall i \in B \ \text{ and } \ R_i = T_i, \ \forall i \in I \setminus B.$$

Clearly that R is in the unit ball of $\mathcal{L}(Z, H)$ and it satisfies

$$\begin{aligned}
\| R - T \| &= \sup\ \{\|R_i - T_i\| : i \in B\} \\
&\leq \sup\ \{\|\Phi_i - I\| : i \in B\} + \sup\ \{\|S_i - T_i\| : i \in B\} \\
&\leq r + \sup\ \left\{\left\|S_i - \frac{T_i}{\|T_i\|}\right\| : i \in B\right\} + \sup\ \left\{\left\|\frac{T_i}{\|T_i\|} - T_i\right\| : i \in B\right\} \\
&\leq r + s + \sup\ \{|1 - \|T_i\|| : i \in B\} \\
&\leq r + s + t < \varepsilon.
\end{aligned}$$

Let P_B be the natural projection on the subspace of elements in Z whose support is contained in B.

Now observe that x_0 given by

$$x_0(i) = \begin{cases} \dfrac{\|z_0(i)\|x_i}{\|P_B(z_0)\|}, & \text{if } i \in B \\ 0 & \text{if } i \in I \setminus B \end{cases}$$

belongs to S_Z and also satisfies

$$\begin{aligned}\|x_0 - z_0\| &\le \|x_0 - \|P_B(z_0)\|x_0\| + \|\| P_B(z_0)\|x_0 - z_0\chi_B\| + \|z_0\chi_{I\setminus B}\| \\ &\le |1 - \|P_B(z_0)\|| + \sum_{i\in B} \|\| z_0(i)\|x_i - z_0(i)\| + \|z_0\chi_{I\setminus B}\| \\ &\le 2\|z_0\chi_{I\setminus B}\| + s\sum_{i\in B} \|z_0(i)\| \text{ (by (29))} \\ &\le 2t + s \text{ (by (28))} \\ &< \varepsilon.\end{aligned}$$

It remains to check that R attains its norm at x_0. Indeed,

$$\begin{aligned}\|R(x_0)\| &= \frac{1}{\|P_B(z_0)\|}\left\|\sum_{i\in B}\right\| z_0(i)\|R_i(x_i)\| \\ &= \frac{1}{\|P_B(z_0)\|}\left\|\sum_{i\in B}\right\| z_0(i)\|\Phi_i(S_i(x_i))\| \\ &= \frac{1}{\|P_B(z_0)\|}\left\|\sum_{i\in B}\right\| z_0(i)\|y_0\| = 1.\end{aligned}$$

Hence $R \in S_{\mathcal{L}(Z,H)}$ and $\|R(x_0)\| = 1$. This completes the proof that the pair (Z, H) has the BPBp.

It follows from [349] that (X_i, H) has the BPBp for every $i \in I$ with the same function η provided that $((\bigoplus \sum_{i\in I} X_i)_{\ell_1}, H)$ has the BPBp. This shows that the assumption in Proposition (6.3.6) is a necessary condition. Now we prove stability results of the Bishop-Phelps-Bollobás property for operators when the domain is ℓ_1.

As we already mentioned it was proved that the pair (ℓ_1, Y) has the BPBp for operators if, and only if, Y has the approximate hyperplane series property (see [344]). Since the AHSp is an isometric property, if a space is the (topological) direct sum of two subspaces with the AHSp, in general it does not have the AHSp. However, we will prove that this property is stable under sums involving an absolute (or monotone) norm. First we recall this notion.

Definition (6.3.7)[341]: Let X and Y be Banach spaces, and $Z = X \bigoplus Y$, a norm $\|\cdot\|_f$ in Z is said to be absolute if there is a function $f: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ such that

$$\| x + y \|_f = f(\| x \|, \| y \|),\ \forall x \in X, y \in Y. \tag{30}$$

The absolute norm is normalized if $f(1,0) = 1 = f(0,1)$.

It is immediate to check that in case that the equality (30) gives a norm in Z, the function f can be extended to a norm $|\cdot|$ on $\mathbb{R}^2$ satisfying $|(r, s)| = f(|r|, |s|)$ for every pair of real numbers (r, s).

We also recall that the norm $|\cdot|_f$ is absolute on $\mathbb{R}^2$ if, and only if, it satisfies

$$|r| \le |s|, |t| \le |u| \Rightarrow f(r, t) \le f(s, u)$$

(see [353]).

Clearly the usual ℓ_p-norm of the sum of two Banach spaces is an absolute norm for every $1 \le p \le \infty$.

Next result is a far reaching extension of Proposition 2.1, Theorems 2.3 and 2.6 in [345], where the ℓ_p-norm on $\mathbb{R}^2$ for $1 \le p < \infty$ is considered. Part of the essential idea of the argument we will use is contained there, however our proof is simpler.

The following technical lemma will be useful in the proof of the main result.

Lemma (6.3.8)[341]: Let $|\cdot|$ be an absolute and normalized norm on $\mathbb{R}^2$. For every $\varepsilon > 0$ there is $\delta > 0$ satisfying the following conditions:

$$(r,s) \in \mathbb{R}^2,\ |(r,s)| = 1, s > 1-\delta \Rightarrow \exists t \in \mathbb{R}: |(t,1)| = 1 \text{ and } |t-r| < \varepsilon$$

and

$$(r,s) \in \mathbb{R}^2, |r,s| = 1, r > 1-\delta \Rightarrow \exists t \in \mathbb{R}: |1,t| = 1 \text{ and } |t-s| < \varepsilon.$$

Proof. Of course it suffices to check only the first assertion. Assume that it is not true. Hence there is some $\varepsilon_0 > 0$ such that

$$\forall \delta > 0 \exists (r_\delta, s_\delta) \in S_{(\mathbb{R}^2,|\cdot|)}, s_\delta > 1-\delta \text{ and } t \in \mathbb{R} \text{ with } |(t,1)| = 1 \Rightarrow |t - r_\delta| \geq \varepsilon_0.$$

We choose any sequence $\{\delta_n\}$ of positive real numbers converging to 0 . By assumption there is a sequence $\{(r_n, s_n)\}$ in $S_{(\mathbb{R}^2,|\cdot|)}$ satisfying for each $n \in \mathbb{N}$ that

$$s_n > 1-\delta_n \text{ and } |t-r_n| \geq \varepsilon_0\ \forall t \in \mathbb{R} \text{ with } |(t,1)| = 1. \tag{31}$$

By passing to a subsequence, we may assume that $(r_n, s_n) \to (r,s)$. Since $|(0,1)| = 1$ and the norm is absolute on $\mathbb{R}^2$ it is satisfied

$$s = |(0,s)| \leq |(r,s)| = 1.$$

Since $s_n > 1-\delta_n$ for each n we also have $s \geq 1$. So $s = 1$. So $|(r,1)| = 1$. We also know that $r_n \to r$, hence $(r_n, s_n) \to (r,1)$ and this contradicts condition (31).

Theorem (6.3.9)[341]: Assume that $|\cdot|$ is an absolute and normalized norm on $\mathbb{R}^2$. Let X be a (real or complex) Banach space that can be decomposed as $X = M \oplus N$ for certain subspaces M and N and such that

$$\|(m,n)\| = |\|m\|, \|n\||,\ \forall m \in M, n \in N.$$

Then X has the AHSp if, and only if, both M and N has the AHSp. In such case, both subspaces satisfy Definition (6.3.2) with the same function η.

Proof. We can clearly assume that both M and N are non-trivial. Let P and Q be the natural projections from X onto M and N, respectively.

First we check the necessary condition. So assume that X has the AHSp and we show that M also has the AHSp. Let us fix $0 < \varepsilon < 1$ and let η_0 be the positive number satisfying Definition (6.3.2) for the space X and $\varepsilon/2$.

Assume that $\sum_{k=1}^{\infty} \alpha_k m_k$ is a convex series with $\{m_k : k \in A\} \subset S_M$ satisfying

$$\left\| \sum_{k=1}^{\infty} \alpha_k m_k \right\| > 1 - \eta_0.$$

By the assumption there are $A \subset \mathbb{N}$ and $\{x_k : k \in \mathbb{N}\} \subset S_X$ such that

$$\sum_{k \in A} \alpha_k > 1 - \frac{\varepsilon}{2} > 0,\ \|x_k - m_k\| < \frac{\varepsilon}{2},\ \forall k \in A \text{ and } \operatorname{co}\{x_k : k \in A\} \subset S_X.$$

So $A \neq \emptyset$.

Since the norm $|\cdot|$ on $\mathbb{R}^2$ is an absolute norm it is satisfied

$$\|P(x_k) - m_k\| = \|P(x_k - m_k)\| \leq \|x_k - m_k\| < \frac{\varepsilon}{2}, \tag{32}$$

and

$$\|Q(x_k)\| \leq \|x_k - m_k\| < \frac{\varepsilon}{2}.$$

Hence we have that

$$\|P(x_k)\| > 1 - \frac{\varepsilon}{2} \text{ and } \|Q(x_k)\| < \frac{\varepsilon}{2}, \ \forall k \in A. \tag{33}$$

On the other hand, since $\mathrm{co}\{x_k : k \in A\} \subset S_X$ there is $x^* \in S_{X^*}$ that can be decomposed as $x^* = m^* + n^*$, for some $m^* \in M^*$ and $n^* \in N^*$ and such that for each $k \in A$ it is satisfied

$$\begin{aligned} 1 &= \operatorname{Re} x^*(x_k) \\ &= \operatorname{Re} m^*\big(P(x_k)\big) + \operatorname{Re} n^*\big(Q(x_k)\big) \\ &\leq \|m^*\|\|P(x_k)\| + \|n^*\|\|Q(x_k)\| \\ &= (\|m^*\|, \|n^*\|)(\|P(x_k)\|, \|Q(x_k)\|) \\ &\leq \|x^*\|\|x_k\| = 1. \end{aligned}$$

As a consequence, we obtain that

$$m^*\big(P(x_k)\big) = \|m^*\|\|P(x_k)\|, \ \forall k \in A. \tag{35}$$

Let us fix $k \in A$. If $m^* = 0$, in view of (34) we obtain that $\|Q(x_k)\| = 1$, which contradicts (33). By using again (33) we also know that $P(x_k) \neq 0$, so we can write $u_k = \frac{P(x_k)}{\|P(x_k)\|}$. By (35) we obtain that

$$\frac{m^*}{\|m^*\|}(u_k) = 1 \ \forall k \in A$$

and clearly $\frac{m^*}{\|m^*\|} \in S_{M^*} \subset S_{X^*}$.

For $k \in A$ we also have

$$\begin{aligned} \|u_k - m_k\| &\leq \left\| \frac{P(x_k)}{\|P(x_k)\|} - P(x_k) \right\| + \|P(x_k) - m_k\| \\ &\leq |1 - \|P(x_k)\|| + \|P(x_k) - m_k\| \\ &< \varepsilon \text{ (by (33) and (32))}. \end{aligned}$$

We checked that M has the AHSp.

Conversely, assume that M and N have the AHSp. We will prove that X also has the AHSp. Let ε be a real number with $0 < \varepsilon < 1$. In view of Lemma (6.3.8) there is $0 < \delta < 1$ satisfying the following conditions

$$(a, b) \in S_{(\mathbb{R}^2, |\cdot|)}, \ b > 1 - \delta \Rightarrow \exists c \in \mathbb{R}: |(c, 1)| = 1 \text{ and } |a - c| < \frac{\varepsilon}{5} \tag{36}$$

and

$$(a, b) \in S_{(\mathbb{R}^2, |\cdot|)}, \ a > 1 - \delta \Rightarrow \exists c \in \mathbb{R}: |(1, c)| = 1 \text{ and } |b - c| < \frac{\varepsilon}{5}. \tag{37}$$

Let us choose $0 < \varepsilon_1 < \frac{\varepsilon}{8}$. Assume that the pair (ε_1, η_1) satisfy condition (c) in Proposition (6.3.3) for both M and N. We also fix real numbers r, s and ε_0 such that

$$0 < s < \min\left\{\frac{\delta}{2}, \frac{\eta_1}{2}\right\}, \ 0 < r < \min\left\{\frac{\delta}{2}, s^2\eta_1\right\} \text{ and } 0 < \varepsilon_0 < \frac{r\varepsilon}{8}. \tag{38}$$

By [344] finite-dimensional spaces have the AHSp. So for every $\varepsilon_0 > 0$ there is $0 < \eta_0 < \varepsilon_0$ satisfying condition (d) in Proposition (6.3.3) for $\mathbb{R}^2$ endowed with the norm $|\cdot|$.

Let $\{x_k\}$ be a sequence in S_X and $\sum \alpha_k$ be a convex series such that $\|\sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta_0$. Hence we have

$$1 - \eta_0 < \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| = \left\| \sum_{k=1}^{\infty} \alpha_k \big(P(x_k) + Q(x_k)\big) \right\|$$

$$= \left|\left(\left\|\sum_{k=1}^{\infty} \alpha_k P(x_k)\right\|, \left\|\sum_{k=1}^{\infty} \alpha_k Q(x_k)\right\|\right)\right|$$
$$\leq \left|\left(\sum_{k=1}^{\infty} \alpha_k \|P(x_k)\|, \sum_{k=1}^{\infty} \alpha_k \|Q(x_k)\|\right)\right|$$
$$= \left|\sum_{k=1}^{\infty} \alpha_k (\|P(x_k)\|, \|Q(x_k)\|)\right|.$$

Since $(\mathbb{R}^2, |\cdot|)$ has the AHSp, it follows that for the convex series $\sum_{k=1}^{\infty} \alpha_k (\|P(x_k)\|, \|Q(x_k)\|)$, there are a subset $A \subset \mathbb{N}, \{(r_k, s_k): k \in A\} \subset S_{\mathbb{R}^2}$ and $(\alpha, \beta) \in S_{(\mathbb{R}^2)^*}$ satisfying

$$\sum_{k\in A} \alpha_k > 1 - \varepsilon_0,\ r_k, s_k \geq 0,\ \alpha r_k + \beta s_k = 1,\ \forall k \in A, \tag{39}$$

and

$$|\|P(x_k)\| - r_k| < \varepsilon_0,\ |\|Q(x_k)\| - s_k| < \varepsilon_0,\ \forall k \in A. \tag{40}$$

It is clearly satisfied that

$$\begin{aligned} \left\|\sum_{k\in A} \alpha_k x_k\right\| &\geq \left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| - \left\|\sum_{k\in \mathbb{N}\setminus A} \alpha_k x_k\right\| \\ &\geq \left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| - \sum_{k\in \mathbb{N}\setminus A} \alpha_k \\ &> 1 - \eta_0 - \varepsilon_0 \ \big(\text{by } (2.13)\big) \\ &> 1 - 2\varepsilon_0. \end{aligned} \tag{41}$$

Now fix arbitrary elements $m_0 \in S_M$ and $n_0 \in S_N$ and define the following elements:

$$m_k := \begin{cases} \dfrac{r_k P(x_k)}{\|P(x_k)\|} & \text{if } k \in A \text{ and } P(x_k) \neq 0 \\ r_k m_0 & \text{if } k \in A \text{ and } P(x_k) = 0 \end{cases}$$

and

$$n_k := \begin{cases} \dfrac{s_k Q(x_k)}{\|Q(x_k)\|} & \text{if } k \in A \text{ and } Q(x_k) \neq 0 \\ s_k n_0 & \text{if } k \in A \text{ and } Q(x_k) = 0. \end{cases}$$

Next we write $y_k := m_k + n_k$ for all $k \in A$. Since $|(r_k, s_k)| = 1$ for every $k \in A$, it is clear that $\{y_k: k \in A\} \subset S_X$ and in view of (40) we obtain

$$\|y_k - x_k\| \leq |r_k - \|P(x_k)\|| + |s_k - \|Q(x_k)\|| < 2\varepsilon_0,\ \forall k \in A. \tag{42}$$

By the previous inequality and bearing in mind (41) we have

$$\left\|\sum_{k\in A} \alpha_k y_k\right\| > \left\|\sum_{k\in A} \alpha_k x_k\right\| - 2\varepsilon_0 > 1 - 4\varepsilon_0.$$

In view of Hahn-Banach theorem there is a functional $x^* \in S_{X^*}$ such that

$$\operatorname{Re} x^*\left(\sum_{k\in A} \alpha_k y_k\right) = \left\|\sum_{k\in A} \alpha_k y_k\right\| > 1 - 4\varepsilon_0.$$

Now we define $B = \{k \in A : \operatorname{Re} x^*(y_k) > 1 - r\}$. In view of Lemma (6.3.4) we have that

$$\sum_{k\in B} \alpha_k > 1 - \frac{4\varepsilon_0}{r} > 0. \tag{43}$$

If we decompose $x^* = m^* + n^*$, for each $k \in B$ we have that

$$\begin{aligned} 1 - r \quad &< \operatorname{Re} x^*(y_k) = \operatorname{Re}\big(m^*(m_k) + n^*(n_k)\big) \\ &\leq \|m^*\|\|m_k\| + \operatorname{Re} n^*(n_k) \\ &\leq \|m^*\|\|m_k\| + \|n^*\|\|n_k\| \leq 1. \end{aligned} \tag{44}$$

As a consequence of (44), for each $k \in B$, we also have that

$$\|m^*\| r_k = \|m^*\|\|m_k\| \leq \operatorname{Re} m^*(m_k) + r \tag{45}$$

and

$$\|n^*\| s_k = \|n^*\|\|n_k\| \leq \operatorname{Re} n^*(n_k) + r. \tag{46}$$

In order to show the result we will consider three cases:

Case 1) Assume that $\|m^*\| \leq s$.

Since $\|n^*\| \leq \|x^*\| = 1$, in view of (44) we know that

$$s_k \geq \|n^*\| s_k \geq 1 - r - s > 1 - \delta, \ \forall k \in B. \tag{47}$$

By using also (46) we obtain that

$$\operatorname{Re} n^*(n_k) \geq 1 - 2r - s > 1 - \eta_1, \ \forall k \in B.$$

Since N has the AHSp there are $C \subset B, \{v_k : k \in C\} \subset S_N$ and $n_1^* \in S_{N^*}$ such that

$$\sum_{k\in C} \alpha_k > (1 - \varepsilon_1) \sum_{k\in B} \alpha_k, n_1^*(v_k) = 1 \text{ and } \|v_k - n_k\| < \varepsilon_1, \ \forall k \in C. \tag{48}$$

By (47) we can use (36), and so for every $k \in C$ there is $a_k \in \mathbb{R}$ such that

$$|(a_k, 1)| = 1, \ |a_k - r_k| < \frac{\varepsilon}{5}. \tag{49}$$

So we define the subset $\{z_k : k \in C\} \subset X$ by

$$z_k = a_k \frac{m_k}{\|m_k\|} + v_k \text{ if } m_k \neq 0, \ z_k = a_k m_0 + v_k \text{ if } m_k = 0, \ \forall k \in C.$$

Clearly we have that

$$\|z_k\| = |(a_k, 1)| = 1, \ \forall k \in C.$$

By (42), (49) and (48) we obtain that

$$\begin{aligned} \|z_k - x_k\| \quad &\leq \|z_k - y_k\| + \|y_k - x_k\| \\ &\leq |a_k - r_k| + \|v_k - n_k\| + 2\varepsilon_0 \\ &\leq \frac{\varepsilon}{5} + \varepsilon_1 + 2\varepsilon_0 \\ &< \varepsilon. \end{aligned}$$

We also have that

$$n_1^*(z_k) = n_1^*(v_k) = 1, \ \forall k \in C.$$

Finally from (48) and (43) we also know that

$$\sum_{k\in C} \alpha_k > (1 - \varepsilon_1) \sum_{k\in B} \alpha_k > (1 - \varepsilon_1)\left(1 - \frac{4\varepsilon_0}{r}\right) > 1 - \varepsilon_1 - \frac{4\varepsilon_0}{r} > 1 - \varepsilon.$$

So the proof is finished in this case.

Case 2) Assume that $\|n^*\| \leq s$.

We can proceed in the same way that in Case 1, but by using that M has the AHSp.

Case 3) Assume that $\|m^*\|, \|n^*\| > s$.

We define the set B_1 given by

$$B_1 = \{k \in B : r_k \geq s\}.$$

For each element $k \in B_1$, in view of (45) we have that

$$\frac{\operatorname{Re} m^*(m_k)}{\|m^*\| r_k} \geq 1 - \frac{r}{\|m^*\| r_k} \geq 1 - \frac{r}{s^2} > 1 - \eta_1.$$

Since M has the AHSp there is a set $D_1 \subset B_1, \{u_k : k \in D_1\} \subset S_M$ and $m_1^* \in S_{M^*}$ such that

$$\sum_{k \in D_1} \alpha_k \geq (1 - \varepsilon_1) \sum_{k \in B_1} \alpha_k \geq \sum_{k \in B_1} \alpha_k - \varepsilon_1 \tag{50}$$

and

$$\left\| u_k - \frac{m_k}{r_k} \right\| < \varepsilon_1,\ m_1^*(u_k) = 1,\ \forall k \in D_1. \tag{51}$$

In an analogous way, we can proceed by defining the set $C_1 = \{k \in B : s_k \geq s\}$ and by using that N has the AHSp we obtain that there is a set $F_1 \subset C_1, \{v_k : k \in F_1\} \subset S_N$ and $n_1^* \in S_{N^*}$ such that

$$\sum_{k \in F_1} \alpha_k \geq (1 - \varepsilon_1) \sum_{k \in C_1} \alpha_k \geq \sum_{k \in C_1} \alpha_k - \varepsilon_1 \tag{52}$$

and

$$\left\| v_k - \frac{n_k}{s_k} \right\| < \varepsilon_1,\ n_1^*(v_k) = 1,\ \forall k \in F_1. \tag{53}$$

Let us notice that for $k \in B \setminus B_1$ we have that $r_k \leq s$ and since $1 = |(r_k, s_k)| \leq s + s_k < \frac{1}{2} + s_k$ then $s_k > \frac{1}{2} > s$. Hence $k \in C_1$. Hence we checked that

$$B \setminus B_1 \subset C_1 \text{ and so } B \setminus C_1 \subset B_1. \tag{54}$$

Clearly we have that

$$\begin{aligned} \sum_{k \in B_1 \cap C_1} \alpha_k &\leq \sum_{k \in D_1 \cap F_1} \alpha_k + \sum_{k \in B_1 \setminus D_1} \alpha_k + \sum_{k \in C_1 \setminus F_1} \alpha_k \\ &\leq \sum_{k \in D_1 \cap F_1} \alpha_k + 2\varepsilon_1 \text{ (by (50) and (52))} \end{aligned} \tag{55}$$

We also obtain

$$\begin{aligned} \sum_{k \in B \setminus B_1} \alpha_k &= \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \sum_{k \in B \setminus (B_1 \cup F_1)} \alpha_k \\ &\leq \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \sum_{k \in C_1 \setminus F_1} \alpha_k \text{ (by (2.28))} \\ &\leq \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \varepsilon_1 \text{ (by (52))}. \end{aligned} \tag{56}$$

By arguing as above we get

$$
\begin{aligned}
\sum_{k\in B\setminus C_1}\alpha_k &\le \sum_{k\in(B\setminus C_1)\cap D_1}\alpha_k + \sum_{k\in B\setminus(C_1\cup D_1)}\alpha_k\\
&\le \sum_{k\in(B\setminus C_1)\cap D_1}\alpha_k + \sum_{k\in B_1\setminus D_1}\alpha_k \text{ (by (54))}\\
&\le \sum_{k\in(B\setminus C_1)\cap D_1}\alpha_k + \varepsilon_1 \text{ (by (50))}
\end{aligned} \tag{57}
$$

Now we take the set C given by $C=(D_1\cap F_1)\cup\big((B\setminus B_1)\cap F_1\big)\cup\big((B\setminus C_1)\cap D_1\big)$. Let us notice that in view of (54) the three subsets whose union is C are pairwise disjoint.

We deduce that

$$
\begin{aligned}
\sum_{k\in C}\alpha_k &= \sum_{k\in D_1\cap F_1}\alpha_k + \sum_{k\in(B\setminus B_1)\cap F_1}\alpha_k + \sum_{k\in(B\setminus C_1)\cap D_1}\alpha_k\\
&\ge \sum_{k\in B_1\cap C_1}\alpha_k + \sum_{k\in B\setminus B_1}\alpha_k + \sum_{k\in B\setminus C_1}\alpha_k - 4\varepsilon_1 \text{ by (55), (56) and (57))}\\
&= \sum_{\mathrm{k}\in\mathrm{B}}\alpha_k - 4\varepsilon_1\\
&> 1-\frac{4\varepsilon_0}{r} - 4\varepsilon_1 \text{ (by (43))}\\
&> 1-\varepsilon.
\end{aligned}
$$

If $D_1=\emptyset$, then $C=(B\setminus B_1)\cap F_1$. In this case we choose any elements $u_0\in S_M$ and $m_1^*\in S_{M^*}$ with $m_1^*(u_0)=1$. Analogously, in case that $F_1=\emptyset$, we have $C=(B\setminus C_1)\cap D_1$ and we choose $v_0\in S_N$ and $n_1^*\in S_{N^*}$ such that $n_1^*(v_0)=1$. Otherwise $D_1\neq\emptyset$ and $F_1\neq\emptyset$ and so the elements m_1^* and n_1^* satisfying (51) and (53) attain their norms; so in this case we can choose $u_0\in S_M$ and $v_0\in S_N$ with $m_1^*(u_0)=1$ and $n_1^*(v_0)=1$.

For each $k\in C$ we define

$$
z_k=\begin{cases} r_k u_k + s_k v_k & \text{if } k\in D_1\cap F_1\\ r_k u_0 + s_k v_k & \text{if } k\in(B\setminus B_1)\cap F_1\\ r_k u_k + s_k v_0 & \text{if } k\in(B\setminus C_1)\cap D_1.\end{cases}
$$

We claim that $\|z_k-x_k\|<\varepsilon$ for each $k\in C$. To see this observe that for $k\in D_1\cap F_1$ we have

$$
\begin{aligned}
\|z_k-x_k\| &\le \|z_k-y_k\|+\|y_k-x_k\|\\
&\le \left|\left(r_k\left\|u_k-\frac{m_k}{r_k}\right\|, s_k\left\|v_k-\frac{n_k}{s_k}\right\|\right)\right| + 2\varepsilon_0 \text{ (by (2.16) by (51) and (53))}\\
&\le |(r_k\varepsilon_1, s_k\varepsilon_1)| + 2\varepsilon_0\\
&\le \varepsilon_1 + 2\varepsilon_0 < \varepsilon
\end{aligned}
$$

For $k\in(B\setminus B_1)\cap F_1$ we have that

$$
\begin{aligned}
\|z_k-x_k\| \le \|z_k-y_k\|+\|y_k-x_k\| &\le 2r_k + s_k\left\|v_k-\frac{n_k}{s_k}\right\| + 2\varepsilon_0\\
&\le 2s+\varepsilon_1+2\varepsilon_0 \ \big(\text{by (53)}\big)\\
&< \varepsilon.
\end{aligned}
$$

In case when $k \in (B \setminus C_1) \cap D_1$,

$$\begin{aligned} \|z_k - x_k\| &\le \|z_k - y_k\| + \|y_k - x_k\| \\ &\le r_k \left\| u_k - \frac{m_k}{r_k} \right\| + 2s_k + 2\varepsilon_0 \\ &\le \varepsilon_1 + 2s + 2\varepsilon_0 \ \text{ (by (51))} \\ &< \varepsilon \end{aligned}$$

and this proves the claim.

Now we observe that $\alpha m_1^* + \beta n_1^* \in X^*$ and $\|\alpha m_1^* + \beta n_1^*\| = |(\alpha, \beta)|^* = 1$. In view of (51), (53) and the choice of u_0 and v_0, for each $k \in C$ one clearly has

$$\begin{aligned} (\alpha m_1^* + \beta n_1^*)(z_k) &= \alpha m_1^*\big(P(z_k)\big) + \beta n_1^*\big(Q(z_k)\big) \\ &= \alpha r_k + \beta s_k = 1. \end{aligned}$$

Before we state and prove a stability result of AHSp for some infinite sums of Banach spaces that includes infinite ℓ_p-sums, we recall the following notion that was introduced in [357], [358].

Definition (6.3.10)[341]: A Banach space X has the approximate hyperplane property (AHp) if there exists a function $\delta: (0,1) \longrightarrow (0,1)$ and a 1-norming subset C of S_{X^*} satisfying the following property.

Given $\varepsilon > 0$ there is a function $\Upsilon_{X,\varepsilon}: C \longrightarrow S_{X^*}$ with the following condition

$$x^* \in C, x \in S_X, \operatorname{Re} x^*(x) > 1 - \delta(\varepsilon) \Rightarrow \operatorname{dist}\left(x, F\left(\Upsilon_{X,\varepsilon}(x^*)\right)\right) < \varepsilon,$$

where $F(y^*) = \{y \in S_X: \operatorname{Re} y^*(y) = 1\}$ for any $y^* \in S_{X^*}$.

A family of Banach spaces $\{X_i: i \in I\}$ has AHp uniformly if every space X_i has property AHp with the same function δ.

Clearly we can assume that the 1-norming subset C in the previous definition satisfies $\mathrm{T}C \subset C$, where T is the unit sphere of the scalar field.

Let us notice that a similar property to AHp was implicitly used to prove that several classes of spaces have AHSp (see [344]).

It is known that property AHp implies AHSp (see [357]). Examples of spaces having AHp are finite-dimensional spaces, uniformly convex spaces, $L_1(\mu)$ for every measure μ and also $C(K)$ for every compact Hausdorff topological space K (see [344], Propositions 3.5,3.8,3.6 and 3.7 and also [357], Corollary 2.12).

In what follows we will use the standard notation from the theory of Banach lattices as presented for example in [364]. We denote by ω the space of all real sequences. As usual, the order $|x| := (|x_n|) \le |y|$ for $x = (x_n), y = (y_n) \in \omega$ means that $|x_n| \le |y_n|$ for each $n \in \mathbb{N}$.

A (real) Banach space $E \subset \omega$ is solid whenever $x \in w, y \in E$ and $|x| \le |y|$ then $x \in E$ and $\| x \|_E \le \| y \|_E$. E is said to be a Banach sequence lattice (or Banach sequence space) if $E \subset \omega, E$ is solid and there exists $u \in E$ with $u > 0$. A Banach sequence lattice E is said to be order continuous if for every $0 \le f_n \downarrow 0$, it follows that $\|f_n\|_E \to 0$. If E is an order-continuous Banach sequence lattice, then E^* can be identified in a natural way with the Köthe dual space $(E', \|\cdot\|_{E'})$ of all $x = (x_k) \in \omega$ equipped with the norm

$$\| x \|_{E'} := \sup_{(y_k) \in B_E} \sum_{k=1}^{\infty} |x_k y_k|.$$

Let E be a Banach sequence lattice. For a given sequence $\left(X_k, \|\cdot\|_{X_k}\right)_{k=1}^{\infty}$ of Banach spaces the vector space of sequences $x = (x_k)_{k=1}^{\infty}$, with $x_k \in X_k$ for each $k \in \mathbb{N}$ and with $(\|x_k\|) \in E$, becomes a Banach space when equipped with the norm

$$\|(x_k)\| = \left\|\left(\|x_k\|_{X_k}\right)\right\|_E ;$$

this space will be denoted by $(\bigoplus \sum_{k=1}^{\infty} X_k)_E$.

Finally we recall that a Banach lattice E is uniformly monotone (UM) if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x \in S_E, y \in E$ and $x, y \geq 0$ the condition $\| x + y \| \leq 1 + \delta$ implies that $\| y \| \leq \varepsilon$. It is known that every UM Banach lattice is order continuous (see [350]).

We will use the following duality result which is well known in the case $E = \ell_p$ with $1 \leq p < \infty$ or $E = c_0$ (see, e.g., [347]). Since the proof of the general case is similar we omit it.

Theorem (6.3.11)[341]: Let E be an order continuous Banach sequence lattice and let (X_n) be a sequence of Banach spaces. Then the mapping $(\bigoplus \sum_{n=1}^{\infty} X_n^*)_{E'} \ni x^* = (x_n^*) \mapsto \phi_{x^*}$ defined by

$$\phi_{x^*}(x_n) = \sum_{n=1}^{\infty} x_n^*(x_n),\ (x_n) \in \left(\bigoplus \sum_{n=1}^{\infty} X_n\right)_E .$$

is an isometrical isomorphism from $(\bigoplus \sum_{n=1}^{\infty} X_n^*)_{E'}$ onto $((\bigoplus \sum_{n=1}^{\infty} X_n)_E)^*$.

The following technical result will be useful.

Lemma (6.3.12)[341]: Let E be a Banach sequence lattice which is order continuous and $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces. For each natural number k assume that $C_k \subset S_{X_k^*}$ is a 1-norming set for X_k. Then the set C given by

$$C = \{(e_k^* \lambda_k x_k^*): e^* \in S_{E'}, e^* \geq 0, \lambda_k \in \mathbb{K}, |\lambda_k| = 1, x_k^* \in C_k, \forall k \in \mathbb{N}\}$$

is a subset of S_{Z^*}, a 1-norming set for Z, where $\mathbb{K}$ is the scalar field and $Z = (\bigoplus \sum_{k=1}^{\infty} X_k)_E$.

Proof. By Theorem (6.3.11) the set C is contained in S_{Z^*}. Let $z = (z_k) \in Z$ and $\varepsilon > 0$. By assumption we know that $(\|z_k\|) \in E$. In view of Theorem (6.3.11), E^* coincides with E', so there is a nonnegative element $e^* \in S_{E'}$ such that $e^*\big((\|z_k\|)\big) = \|(\|z_k\|)\|_E = \| z \|$. For each $k \in \mathbb{N}, C_k$ is a 1-norming set for X_k and so there exists $z_k^* \in C_k$ and a scalar λ_k with $|\lambda_k| = 1$ such that $\operatorname{Re} \lambda_k z_k^*(z_k) > \|z_k\| - \frac{\varepsilon}{(e_k^*+1)2^k}$. The element $z^* = (e_k^* \lambda_k z_k^*) \in C$ and

$$\operatorname{Re} z^*(z) = \sum_{\mathrm{k}=1}^{\infty} \operatorname{Re} e_k^* \lambda_k z_k^*(z_k) > \sum_{\mathrm{k}=1}^{\infty} e_k^* \left(\|z_k\| - \frac{\varepsilon}{(e_k^*+1)2^k}\right) \geq \| z \| - \varepsilon.$$

We proved that C is a 1-norming set for Z.

Now we are ready to prove the stability of the AHSp.

Theorem (6.3.13)[341]: Let E be a Banach sequence lattice with the AHSp and such that it is uniformly monotone. Assume that $\{X_k : k \in \mathbb{N}\}$ has property AHp uniformly. Then the space $(\bigoplus \sum_{k=1}^{\infty} X_k)_E$ has the AHSp.

Proof. We take $M = \{k \in \mathbb{N} : X_k \neq 0\}$. If M is infinite, there is no loss of generality in assuming that $M = \mathbb{N}$. Otherwise the proof of the statement is essentially the same but easier.

So we assume that $X_k \neq \{0\}$ for each k. We put $Z := (\bigoplus \sum_{k=1}^{\infty} X_k)_E$.

Let us fix $0 < \varepsilon < 1$. By assumption, $\{X_k : k \in \mathbb{N}\}$ has AHp uniformly, so there is $\ :$ $(0,1) \longrightarrow (0,1)$ satisfying Definition (6.3.10) for each $k \in \mathbb{N}$. We choose $0 < \eta <$

$\min\left\{\frac{\varepsilon}{4},\delta\left(\frac{\varepsilon}{4}\right)\right\}$. Since E is uniformly monotone, we can use condition ii) in [360], so there is $0<\alpha<\varepsilon/4<1$ satisfying that

$$e\in S_E,\ e\geq 0,\ A\subset\mathbb{N},\ \|e\chi_A\|_E>\frac{\varepsilon}{4}\Rightarrow\left\|e\chi_{\mathbb{N}\setminus A}\right\|_E<1-\alpha. \tag{58}$$

For $r=(1+2\eta-\alpha\eta)/(1+2\eta)$, we choose $0<\varepsilon'<(1-r)\varepsilon/3$. Then by our assumption, it follows that there is $0<\eta'<\varepsilon'$ such that E satisfies the statement (d) in Proposition (6.3.3) for (ε',η').

In order to prove that Z satisfies the AHSp we will show that condition (d) in Proposition (6.3.3) is satisfied for (ε,η').

Assume that (z_n) is a sequence in S_Z and $\sum\alpha_n$ is a convex series such that $\|\sum_{n=1}^{\infty}\alpha_n z_n\|>1-\eta'$.

Then

$$\begin{aligned}1-\eta' &< \left\|\sum_{n=1}^{\infty}\alpha_n z_n\right\| \\ &= \left\|\left(\left\|\sum_{n=1}^{\infty}\alpha_n z_n(k)\right\|\right)_k\right\|_E \\ &\leq \left\|\left(\sum_{n=1}^{\infty}\alpha_n\|z_n(k)\|\right)_k\right\|_E \\ &= \left\|\sum_{n=1}^{\infty}\alpha_n(\|z_n(k)\|)_k\right\|_E .\end{aligned} \tag{59}$$

Combining our hypothesis that E has the AHSp with $(\|z_n(k)\|)_k\in S_E$ for each positive integer n, we conclude that there is a finite subset $A\subset\mathbb{N}$ and $\{r_n: n\in A\}\subset S_E$ such that

$$\sum_{n\in A}\alpha_n>1-\varepsilon' \tag{60}$$

and also

$$r_n\geq 0, \|r_n-(\|z_n(k)\|)_k\|_E<\varepsilon' \text{ and there is } r^*\in S_{E'} \text{ with } r^*(r_n)=1, \text{for all } n\in A. \tag{61}$$

Hence from (59) and (60) we obtain that

$$1-\eta'-\varepsilon'<\left\|\sum_{n\in A}\alpha_n z_n\right\|. \tag{62}$$

For each $k\in\mathbb{N}$ we choose an element $x_k\in S_{X_k}$ and define for every $n\in A$ the element u_n in Z given by

$$u_n(k)=\begin{cases} r_n(k)\dfrac{z_n(k)}{\|z_n(k)\|} & \text{if } z_n(k)\neq 0\\ r_n(k)x_k & \text{otherwise}.\end{cases}$$

By (61) it is clearly satisfied that

$$\|u_n-z_n\|=\|r_n-(\|z_n(k)\|)_k\|_E<\varepsilon',\ \forall n\in A. \tag{63}$$

So in view of (62) we obtain that

$$1-\eta'-2\varepsilon' < \left\| \sum_{n\in A} \alpha_n u_n \right\|. \tag{64}$$

By assumption, $\{X_k: k \in \mathbb{N}\}$ has AHp uniformly. For each $k \in \mathbb{N}$ let $G_k \subset S_{X_k^*}$ be the 1-norming set for X_k satisfying Definition (6.3.10). We can also assume that $G_k = \{\lambda x^*: \lambda \in \mathbb{K}, |\lambda| = 1, x^* \in G_k\}$ for each $k \in \mathbb{N}$. By Lemma (6.3.12) there is $z^* \in S_{Z^*}$ that can be written as $z^* \equiv (z_k^*) = (e_k^* x_k^*)$ where $e^* \in S_{E'}, e^* \geq 0$ and $x_k^* \in G_k$ for each $k \in \mathbb{N}$ satisfying that

$$1-\eta'-2\varepsilon' < \operatorname{Re} z^*\left(\sum_{n\in A} \alpha_n u_n\right).$$

Now we define the set C by $C = \{n \in A: \operatorname{Re} z^*(u_n) > r\}$. By Lemma (6.3.4) we obtain that

$$\sum_{n\in C} \alpha_n > 1 - \frac{\eta' + 2\varepsilon'}{1-r} > 1-\varepsilon > 0.$$

For each element $n \in C$ we have that

$$\begin{aligned} r &< \operatorname{Re} z^*(u_n) = \sum_{k=1}^{\infty} \operatorname{Re} z_k^*\big(u_n(k)\big) \\ &\leq \sum_{k=1}^{\infty} \left|z_k^*\big(u_n(k)\big)\right| \\ &\leq \sum_{k=1}^{\infty} \|z_k^*\|\|u_n(k)\| \\ &\leq \|(\|z_k^*\|)\|_{E'}\|(\|u_n(k)\|)_k\|_E \\ &= 1. \end{aligned} \tag{66}$$

For each $n \in C$ and $k \in \mathbb{N}$ we put

$$d_n(k) = \|z_k^*\|\|u_n(k)\| - \operatorname{Re} z_k^*\big(u_n(k)\big).$$

The chain of inequalities (66) implies that

$$\sum_{k=1}^{\infty} d_n(k) \leq 1-r,\ \forall n \in C. \tag{67}$$

We now fix a positive integer k. If $z_k^* = 0$, then $d_n(k) = 0$ for every $n \in C$. If $n \in C$ and $u_n(k) = 0$ for some $k \in \mathbb{N}$ then $d_n(k) = 0$. Otherwise it is satisfied that

$$\operatorname{Re} \frac{z_k^*}{\|z_k^*\|}\left(\frac{u_n(k)}{\|u_n(k)\|}\right) = 1 - \frac{d_n(k)}{\|z_k^*\|\|u_n(k)\|}. \tag{68}$$

In what follows, for each $n \in C$, we consider the following subset

$$B_n = \{k \in \mathbb{N}: d_n(k) < \eta\|z_k^*\|\|u_n(k)\|\}.$$

By (66) we know that

$$\begin{aligned} r &< \sum_{k=1}^{\infty} \|z_k^*\|\|u_n(k)\| \\ &= \sum_{k\in B_n} \|z_k^*\|\|u_n(k)\| + \sum_{k\in \mathbb{N}\setminus B_n} \|z_k^*\|\|u_n(k)\| \end{aligned}$$

$$\leq \sum_{k\in B_n} \|z_k^*\|\|u_n(k)\| + \frac{1}{\eta}\sum_{k\in\mathbb{N}\setminus B_n} d_n(k)$$
$$\leq \sum_{k\in B_n} \|z_k^*\|\|u_n(k)\| + \frac{1}{\eta}(1-r) \text{ (by (67))}.$$

As a consequence,

$$\sum_{k\in B_n} \|z_k^*\|\|u_n(k)\| > r - \frac{1-r}{\eta} > 0 \tag{69}$$

and in view of (66) we deduce that

$$\sum_{k\in\mathbb{N}\setminus B_n} \|z_k^*\|\|u_n(k)\| < 1 - r + \frac{1-r}{\eta},\ \forall n \in C. \tag{70}$$

In view of (68), for every $n \in C$ and $k \in B_n$ it is satisfied that

$$\operatorname{Re} x_k^*\left(\frac{u_n(k)}{\|u_n(k)\|}\right) = \operatorname{Re}\frac{z_k^*}{\|z_k^*\|}\left(\frac{u_n(k)}{\|u_n(k)\|}\right) = 1 - \frac{d_n(k)}{\|z_k^*\|\|u_n(k)\|} > 1 - \eta.$$

Now we will use that for each k the space X_k has the property AHp for the function $\delta, \eta < \delta\left(\frac{\varepsilon}{4}\right)$ and $x_k^* \in G_k$. Hence for each $k \in \cup_{l\in C} B_l$, there is $y_k^* \in S_{X_k^*}$ such that if $n \in C$ and $k \in B_n$ there is $m_n(k) \in S_{X_k}$ with

$$\left\|m_n(k) - \frac{u_n(k)}{\|u_n(k)\|}\right\| < \frac{\varepsilon}{4},\ \text{ and } \operatorname{Re} y_k^*\big(m_n(k)\big) = 1,\ \forall n \in C,\ \forall k \in B_n. \tag{71}$$

Let $D = \mathbb{N} \setminus \bigcup_{l\in C} B_l$. For each $k \in D$, we choose any element $y_k^* \in S_{X_k^*}$ such that $y_k^*(x_k) = 1$.

For each $n \in C$, we write $C_n = \bigcup_{l\in C} B_l \setminus B_n$ and define $v_n \in Z$ by

$$v_n(k) = \begin{cases} r_n(k)m_n(k) & \text{if } k \in B_n \\ r_n(k)m_{p(k)}(k) & \text{if } k \in C_n \\ r_n(k)x_k & \text{if } k \in D, \end{cases}$$

where $p(k) = \min\{s \in C : k \in B_s\}$ if $k \in \bigcup_{l\in C} B_l$. It is clear that $\|v_n\| = \|r_n\|_E = 1$ for each $n \in C$.

We clearly have that

$$\begin{aligned} \|r_n\chi_{B_n}\|_E &= \|u_n\chi_{B_n}\| \geq \operatorname{Re} z^*\big(u_n\chi_{B_n}\big) \\ &= \operatorname{Re}\sum_{k\in B_n} z_k^*\big(u_n(k)\big) \\ &= \sum_{k=1}^{\infty} \operatorname{Re} z_k^*\big(u_n(k)\big) - \sum_{k\in\mathbb{N}\setminus B_n} \operatorname{Re} z_k^*\big(u_n(k)\big) \\ &> r - \sum_{k\in\mathbb{N}\setminus B_n} \operatorname{Re} z_k^*\big(u_n(k)\big) \ \big(\text{by } (2.40)\big) \\ &\geq r - \sum_{k\in\mathbb{N}\setminus B_n} \|z_k^*\|\|u_n(k)\| \ \text{ (by (70))} \\ &> r - \left(1 - r + \frac{1-r}{\eta}\right) \end{aligned}$$

$$= 2r - 1 - \frac{1-r}{\eta} = 1 - \alpha. \quad (72)$$

Since $0 \le r_n$ for each $n \in C$ and $\{r_n : n \in C\} \subset S_E$, from (72) and (58) it follows that

$$\left\| r_n \chi_{\mathbb{N} \setminus B_n} \right\|_E \le \frac{\varepsilon}{4}. \quad (73)$$

For every $n \in C$ and $k \in B_n$, in view of (71) we have that

$$\begin{aligned} \|v_n(k) - u_n(k)\| &= \|r_n(k) m_n(k) - u_n(k)\| \\ &\le \frac{\varepsilon}{4} r_n(k). \end{aligned} \quad (74)$$

Hence from (74), for every $n \in C$ we have that

$$\begin{aligned} \|v_n - u_n\| &\le \left\| (v_n - u_n) \chi_{B_n} \right\| + \left\| v_n \chi_{\mathbb{N} \setminus B_n} \right\| + \left\| u_n \chi_{\mathbb{N} \setminus B_n} \right\| \\ &\le \frac{\varepsilon}{4} \|r_n\|_E + 2 \left\| r_n \chi_{\mathbb{N} \setminus B_n} \right\|_E \ \text{(by (74))} \\ &\le \frac{3\varepsilon}{4} \ \text{(by (73))}. \end{aligned}$$

Combining with (63), we conclude that for each $n \in C$,

$$\begin{aligned} \|v_n - z_n\| &\le \|v_n - u_n\| + \|u_n - z_n\| \\ &\le \frac{3\varepsilon}{4} + \varepsilon' < \varepsilon. \end{aligned}$$

Let v^* be the element in Z^* given by $v^* = \{r_k^* y_k^*\}$. By Theorem (6.3.11) it is satisfied that $\|v^*\| = \|r^*\|_{E'} = 1$. For each $n \in C$ we clearly have that

$$\begin{aligned} v^*(v_n) &= \sum_{k=1}^{\infty} r_k^* y_k^*(v_n(k)) \\ &= \sum_{k \in B_n} r_k^* r_n(k) y_k^*\left(m_n(k)\right) + \sum_{k \in C_n} r_k^* r_n(k) y_k^*\left(m_{p(k)}(k)\right) + \sum_{k \in D} r_k^* r_n(k) y_k^*(x_k) \\ &= \sum_{k=1}^{\infty} r_k^* r_n(k) \ \text{(by (71))} \\ &= r^*(r_n) = 1 \ \text{(by (61))}. \end{aligned}$$

From (65) we also know that $\sum_{n \in C} \alpha_n > 1 - \varepsilon$, so the proof is finished.

As we mentioned above uniformly convex spaces have AHp. Indeed in this case the modulus of convexity plays the role of the function δ satisfying Definition (6.3.10) and the identity function on the unit sphere of the dual plays the role of the function Υ_δ [346]. So a family $\{X_i : i \in I\}$ of uniformly convex Banach spaces has the AHp uniformly in case that $\inf\{\delta_i(\varepsilon) : i \in I\} > 0$, for any $\varepsilon > 0$, being δ_i the modulus of convexity of X_i. Also $C(K)$ spaces and $L_1(\mu)$ have AHp uniformly for any compact Hausdorff space K and any measure μ [357]. As a consequence of Theorem (6.3.13) and [344] we deduce, for instance, the following result.

Corollary (6.3.14)[341]: Let $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces such that any of them is either a uniformly convex space or $C(K)$ (some compact K) or $L_1(\mu)$ (some measure μ). Let $A = \{k \in \mathbb{N} : X_k$ is a uniformly convex space $\}$ and assume that $\inf\{\delta_k(\varepsilon) : k \in A\} > 0$ for every $\varepsilon > 0$, being δ_k the modulus of convexity of X_k. Then the pair $\left(\ell_1, (\oplus \sum_{k=1}^{\infty} X_k)_{\ell_p}\right)$ satisfies the $BPBp$ for every $1 \le p < \infty$.

We remark that in general AHSp is not stable under infinite ℓ_1-sums (see [349]). So in order to have the stability result in Theorem (6.3.13) some additional restriction is needed. Now we show the following partial converse of Theorem (6.3.13) that extends to some infinite sums the necessary condition obtained in Theorem (6.3.9).

Proposition (6.3.15)[341]: Let $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces and E be an order continuous Banach sequence lattice. Assume that the space $Z = (\bigoplus \sum_{k=1}^{\infty} X_k)_E$ has the approximate hyperplane series property. Then there is a function $\tilde{\eta}: (0,1) \to (0,1)$ such that X_k satisfies the approximate hyperplane series property with the function $\tilde{\eta}$ for every $k \in \mathbb{N}$. One can take the function given by $\tilde{\eta}(\varepsilon) = \eta\left(\frac{\varepsilon}{2}\right)$, where η is the function satisfying Definition (6.3.2) for Z.

Proof. It suffices to prove that X_1 has the property AHSp for $\tilde{\eta}$. Consider the subspace Z_1 of Z given by

$$Z_1 = \{z \in Z : z(k) = 0, \forall k \geq 2\}.$$

Notice that the mapping from Z_1 into X_1 given by $z \mapsto z(1)\|e_1\|_E$ is a linear isometry, where e_1 is the sequence given by $e_1(k) = \delta_1^k$ for each natural number k. Since AHSp is clearly preserved by linear isometries (and the function η satisfying AHSp also) then it suffices to prove that Z_1 satisfies AHSp with the function $\tilde{\eta}$.

So let us fix $0 < \varepsilon < 1$. Assume that $\alpha_n \geq 0, u_n \in S_{Z_1}$ for every $n, \sum_{n=1}^{\infty} \alpha_n = 1$ and it is also satisfied that

$$\left\|\sum_{n=1}^{\infty} \alpha_n u_n\right\| > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

By assumption Z has the AHSp, so there is a subset $A \subset \mathbb{N}$ such that $\sum_{n \in A} \alpha_n > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon, z^* \in S_{Z^*}$ and $\{z_n : n \in A\} \subset S_Z$ such that

$$\|z_n - u_n\| < \frac{\varepsilon}{2} \text{ and } z^*(z_n) = 1, \ \forall n \in A. \tag{75}$$

For every $n \in A$ we define the element $y_n \in Z_1$ given by

$$y_n(1) = z_n(1), \ y_n(k) = 0, \ \forall k \geq 2.$$

Let us fix $n \in A$. We clearly have that

$$\|y_n - u_n\| = \|(\|y_n(k) - u_n(k)\|)\|_E \leq \|(\|z_n(k) - u_n(k)\|)\|_E = \|z_n - u_n\| < \frac{\varepsilon}{2}. \tag{76}$$

Since we know that

$$\|y_n\| \leq \|z_n\| = 1, \ \forall n \in A,$$

in view of (76) we deduce that

$$1 - \frac{\varepsilon}{2} \leq \|y_n\| \leq 1, \ \forall n \in A. \tag{77}$$

As a consequence of Theorem (6.3.11) we know that $z^* \in (\bigoplus \sum_{k=1}^{\infty} X_k^*)_{E'}$ and we also have

$$z^*(1)\big(y_n(1)\big) = z^*(1)\big(z_n(1)\big) = \|z^*(1)\|\|z_n(1)\| = \|z^*(1)\|\|y_n(1)\|, \ \forall n \in A. \tag{78}$$

On the other hand, it is satisfied that

$$
\begin{aligned}
\left|z^*(1)\big(y_n(1)\big)\right| &= |z^*(y_n)| \\
&\geq |z^*(z_n)| - |z^*(y_n - z_n)| \\
&\geq 1 - \|z_n - y_n\| \\
&\geq 1 - \|z_n - u_n\| - \|u_n - y_n\| \\
&\geq 1 - 2\|z_n - u_n\| \ \ \text{(by (76))} \\
&> 1 - \varepsilon > 0 \ \ \text{(by (75))}\,.
\end{aligned} \tag{79}
$$

We denote by w^* the element in Z^* given by

$$w^*(1) = z^*(1),\ w^*(k) = 0,\ \text{ if } k \geq 2.$$

Notice that $\|e_1\|_{E'}\|e_1\|_E = 1$. So it is clearly satisfied

$$
\begin{aligned}
\operatorname{Re} w^*(y_n) &= \operatorname{Re} z^*(y_n) \\
&= \|z^*(1)\|\|y_n(1)\| \text{ (by (78))} \\
&= \frac{\|w^*\|}{\|e_1\|_{E'}}\frac{\|y_n\|}{\|e_1\|_E} \\
&= \|w^*\|\|y_n\|,
\end{aligned}
$$

and bearing in mind (79) we deduce that $w^*(y_n) \neq 0$.

Since for each $n \in A$ we have also that

$$
\begin{aligned}
\left\|u_n - \frac{y_n}{\|y_n\|}\right\| &\leq \|u_n - y_n\| + \left\|y_n - \frac{y_n}{\|y_n\|}\right\| \\
&< \frac{\varepsilon}{2} + 1 - \|y_n\| \leq \varepsilon \text{ (by (76) and (77))},
\end{aligned}
$$

we checked that Z_1 has the AHSp for the function $\tilde{\eta}$ as we wanted to show.

Corollary (6.3.16)[365]: Let H be a real or complex Hilbert space and assume that $u_j, v_j \in S_H^j$. Then there is a surjective linear isometry Φ^j on H such that $\Phi^j(u_j) = v_j$ and $\sum\left\|\Phi^j - I\right\| = \sum\left\|u_j - v_j\right\|$.

Proof. The result is obvious in the case $\dim H = 1$. Assume that $\dim H \geq 2$. Thus there is an element $v_j^\perp \in S_H^j$ orthogonal to v_j and such that $[u_j, v_j] \subset \left[v_j, v_j^\perp\right]$, where $[x_j, y_j]$ is the linear span of the vectors x_j and y_j in H. Let $(u_j)_1, (u_j)_2 \in \mathbb{K}$ such that $u_j = (u_j)_1 v_j + (u_j)_2 v_j^\perp$ and write $u_j^\perp = -\overline{(u_j)_2} v_j + \overline{(u_j)_1} v_j^\perp$. It is clearly satisfied that

$$1 = \| u_j \|^2 = \left|(u_j)_1\right|^2 + \left|(u_j)_2\right|^2 \ \text{ and } \left\langle u_j, u_j^\perp\right\rangle = 0.$$

Let M be a subspace of H orthogonal to $\left[v_j, v_j^\perp\right] = \left[u_j, u_j^\perp\right]$ and such that $H = \left[u_j, u_j^\perp\right] \oplus M$. Define the mapping $\Phi^j : H \longrightarrow H$ given by

$$\Phi^j\left(z_j u_j + w_j u_j^\perp + m\right) = z_j v_j + w_j v_j^\perp + m, \quad \forall (z_j, w_j) \in \mathbb{K}^2, m \in M,$$

which is a surjective linear isometry on H. It clearly satisfies $\Phi^j(u_j) = v_j$ and $\Phi^j\left(u_j^\perp\right) = v_j^\perp$.

Clearly $\sum(\Phi^j - I)(u_j) = \sum\left(v_j - u_j\right), \sum(\Phi^j - I)\left(u_j^\perp\right) = \sum\left(v_j^\perp - u_j^\perp\right)$ and $\sum \| u_j - v_j \| = \sum\left\|u_j^\perp - v_j^\perp\right\|$. Also we have that

$$\sum\left\langle v_j - u_j, v_j^\perp - u_j^\perp\right\rangle = -\sum\left(\left\langle v_j, u_j^\perp\right\rangle + \left\langle u_j, v_j^\perp\right\rangle\right) = 0.$$

Hence $\Phi^j - I$ restricted to $\left[u_j, u_j^\perp\right]$ is a multiple of a linear isometry from this subspace into itself. As a consequence $\sum \| \Phi^j - I \| = \sum \| u_j - v_j \|$.

Corollary (6.3.17)[365]: Assume that $\{X_i : i \in I\}$ is a family of Banach spaces, H is a Hilbert space such that the pair (X_i, H) has the $BPBp$ for operators for every $i \in I$ and with the same function η. Then the pair $\left((\bigoplus \sum_{i\in I} X_i)_{\ell_1}, H\right)$ has the BPBp.

Proof. We write $Z = (\bigoplus \sum_{i\in I} X_i)_{\ell_1}$. Given $0 < \varepsilon < 1$, we choose positive real numbers r, s and t such that

$$r < \frac{\varepsilon}{4},\ s < \min\left\{\frac{\varepsilon}{4}, \frac{\delta_H(r)}{3}\right\} \text{ and } t < \min\left\{\frac{\varepsilon}{4}, \eta(s), \frac{\delta_H(r)}{3}\right\},$$

where δ_H is the modulus of convexity of H.

Assume that $(z_j)_0 = \left\{(z_j)_0(i)\right\} \in S_{z_j}^j$ and $T^j \in S_{\mathcal{L}(z_j,H)}^j$ satisfies $\sum \left\|T^j\left((z_j)_0\right)\right\| > 1 - t^2$. For every $i \in I$, we denote by T_i^j the restriction of T^j to X_i, that is embedded in Z in a natural way.

Assume that $y_j^* \in S_{H^*}^j$ satisfies that $\sum \operatorname{Re} y_j^*\left(T^j\left((z_j)_0\right)\right) = \sum \left\|T^j\left((z_j)_0\right)\right\| > 1 - t^2$.

Denote by $B = \left\{i \in I : \sum \operatorname{Re} y_j^*\left(T_i^j\left((z_j)_0(i)\right)\right) > (1-t)\sum \left\|(z_j)_0(i)\right\|\right\}$. We clearly have that

$$\begin{aligned} 1 - t^2 \quad &< \sum \operatorname{Re} y_j^*\left(T^j\left((z_j)_0\right)\right) = \sum_{i\in I} \sum \operatorname{Re} y_j^*\left(T_i^j\left((z_j)_0(i)\right)\right) \\ &= \sum_{i\in B} \sum \operatorname{Re} y_j^*\left(T_i^j\left((z_j)_0(i)\right)\right) + \sum_{i\in I\setminus B} \sum \operatorname{Re} y_j^*\left(T_i^j\left((z_j)(i)\right)\right) \\ &\le \sum_{i\in B} \sum \left\|(z_j)_0(i)\right\| + \sum_{i\in I\setminus B} \sum (1-t)\left\|(z_j)_0(i)\right\| \\ &= 1 - t \sum_{i\in I\setminus B} \sum \left\|(z_j)_0(i)\right\|. \end{aligned}$$

Hence

$$\sum_{i\in I\setminus B} \sum \left\|(z_j)_0(i)\right\| \le t.$$

By assumption, for every $i \in B$ there is an operator $S_i^j \in S_{\mathcal{L}(X_i,H)}^j$ and an element $(x_j)_i \in S_{X_i}^j$ such that

$$\sum \left\|S_i^{\mathrm{j}} - \frac{T_i^j}{\|T_i^j\|}\right\| < s,\ \sum \left\|(x_j)_i - \frac{(z_j)_0(i)}{\left\|(z_j)_0(i)\right\|}\right\| < s$$

$$\text{and } \sum \left\|S_i^j\left((x_j)_i\right)\right\| = 1,\ \forall i \in B.$$

It follows by (29) that for every $i, j_0 \in B$ we have that

$$\sum \left\|S_i^j\left((x_j)_i\right) + S_j^j\left((x_j)_{j_0}\right)\right\| \ge \sum \left\|\frac{S_i^j\left((z_j)_0(i)\right)}{\left\|(z_j)_0(i)\right\|} + \frac{S_j^j\left((z_j)_0(j_0)\right)}{\left\|(z_j)_0(j_0)\right\|}\right\| - 2s$$

$$\geq \sum \left\| \frac{T_i^j\left(\left(z_j\right)_0(i)\right)}{\left\|T_i^j\right\| \left\|\left(z_j\right)_0(i)\right\|} + \frac{T_{j_0}^j\left(\left(z_j\right)_0(j_0)\right)}{\left\|T_{j_0}^j\right\| \left\|\left(z_j\right)_0(j_0)\right\|} \right\| - 4s$$
$$\geq 2(1-t) - 4s$$
$$> 2(1-\delta_H(r)).$$

As a consequence $\sum \left\|S_i^j\left(\left(x_j\right)_i\right) - S_{j_0}^j\left(\left(x_j\right)_{j_0}\right)\right\| \leq r$ for each $i, j_0 \in B$.

Corollary (6.3.18)[365]: Let $|\cdot|$ be an absolute and normalized norm on $\mathbb{R}^2$. For every $\varepsilon > 0$ there is $\delta > 0$ satisfying the following conditions:

$$(r,s) \in \mathbb{R}^2,\ |(r,s)| = 1, s > 1-\delta \Rightarrow \exists t \in \mathbb{R}: |(t,1)| = 1 \text{ and } |t-r| < \varepsilon$$

and

$$(r,s) \in \mathbb{R}^2, |r,s| = 1, r > 1-\delta \Rightarrow \exists t \in \mathbb{R}: |1,t| = 1 \text{ and } |t-s| < \varepsilon.$$

Proof. Of course it suffices to check only the first assertion. Assume that it is not true. Hence there is some $\varepsilon_0 > 0$ such that

$$\forall \delta > 0 \exists (r_\delta, s_\delta) \in S^j_{(\mathbb{R}^2,|\cdot|)}, s_\delta > 1-\delta$$
$$\text{and } t \in \mathbb{R} \text{ with } |(t,1)| = 1 \Rightarrow |t - r_\delta| \geq \varepsilon_0.$$

We choose any sequence $\{\delta_n\}$ of positive real numbers converging to 0. By assumption there is a sequence $\{(r_n, s_n)\}$ in $S^j_{(\mathbb{R}^2,|\cdot|)}$ satisfying for each $n \in \mathbb{N}$ that

$$s_n > 1-\delta_n \text{ and } |t - r_n| \geq \varepsilon_0\ \forall t \in \mathbb{R} \text{ with } |(t,1)| = 1.$$

By passing to a subsequence, we may assume that $(r_n, s_n) \to (r,s)$. Since $|(0,1)| = 1$ and the norm is absolute on $\mathbb{R}^2$ it is satisfied

$$s = |(0,s)| \leq |(r,s)| = 1.$$

Corollary (6.3.19)[365]: Let E be a Banach sequence lattice which is order continuous and $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces. For each natural number k assume that $C_k \subset S^j_{X_k^*}$ is a 1-norming set for X_k. Then the set C given by

$$C = \left\{\left(e_k^* \lambda_k^j \left(x_j^*\right)_k\right) : e^* \in S^j_{E'}, e^* \geq 0, \lambda_k^j \in \mathbb{K}, \left|\lambda_k^j\right| = 1, \left(x_j^*\right)_k \in C_k, \forall k \in \mathbb{N}\right\}$$

is a subset of $S^j_{Z^*}$, a 1-norming set for Z, where $\mathbb{K}$ is the scalar field and $Z = (\oplus \sum_{k=1}^\infty X_k)_E$.

Proof. By Theorem (6.3.11) the set C is contained in S_{Z^*}. Let $z_j = \left(\left(z_j\right)_k\right) \in Z$ and $\varepsilon > 0$. By assumption we know that $\left(\left\|\left(z_j\right)_k\right\|\right) \in E$. In view of Theorem (6.3.11), E^* coincides with E', so there is a nonnegative element $e^* \in S^j_{E'}$ such that $\sum e^*\left(\left(\left\|\left(z_j\right)_k\right\|\right)\right) = \sum \left\|\left(\left\|\left(z_j\right)_k\right\|\right)\right\|_E = \sum \| z_j \|$. For each $k \in \mathbb{N}$, C_k is a 1-norming set for X_k and so there exists $\left(z_j^*\right)_k \in C_k$ and a scalar λ_k^j with $\left|\lambda_k^j\right| = 1$ such that $\operatorname{Re} \lambda_k^j \left(z_j^*\right)_k \left(\left(z_j\right)_k\right) > \left\|\left(z_j\right)_k\right\| - \frac{\varepsilon}{(e_k^*+1)2^k}$. The element $z_j^* = \left(e_k^* \lambda_k^j \left(z_j^*\right)_k\right) \in C$ and

$$\sum \operatorname{Re}\left(z_j^*\right)_k (z_j) = \sum_{k=1}^\infty \sum \operatorname{Re} e_k^* \lambda_k^j \left(z_j^*\right)_k \left(\left(z_j\right)_k\right)$$

$$> \sum_{k=1}^{\infty} \sum e_k^* \left(\left\| (z_j)_k \right\| - \frac{\varepsilon}{(e_k^* + 1)2^k} \right) \geq \sum \|z_j\| - \varepsilon.$$

We proved that C is a 1-norming set for Z.

Corollary (6.3.20)[365]: Let $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces and E be an order continuous Banach sequence lattice. Assume that the space $Z = (\bigoplus \sum_{k=1}^{\infty} X_k)_E$ has the approximate hyperplane series property. Then there is a function $\tilde{\eta}: (0,1) \to (0,1)$ such that X_k satisfies the approximate hyperplane series property with the function $\tilde{\eta}$ for every $k \in \mathbb{N}$. More precisely, one can take the function given by $\tilde{\eta}(\varepsilon) = \eta\left(\frac{\varepsilon}{2}\right)$, where η is the function satisfying Definition (6.3.2) for Z.

Proof. It suffices to prove that X_1 has the property AHSp for $\tilde{\eta}$. Consider the subspace Z_1 of Z given by

$$Z_1 = \{z_j \in Z : z_j(k) = 0, \forall k \geq 2\}.$$

Notice that the mapping from Z_1 into X_1 given by $z_j \mapsto z_j(1)\|e_1\|_E$ is a linear isometry, where e_1 is the sequence given by $e_1(k) = \delta_1^k$ for each natural number k. Since AHSp is clearly preserved by linear isometries (and the function η satisfying AHSp also) then it suffices to prove that Z_1 satisfies AHSp with the function $\tilde{\eta}$.

So let us fix $0 < \varepsilon < 1$. Assume that $\sum \alpha_n^j \geq 0, (u_j)_n \in S_{Z_1}^j$ for every n, $\sum_{n=1}^{\infty} \sum \alpha_n^j = 1$ and it is also satisfied that

$$\left\| \sum_{n=1}^{\infty} \sum \alpha_n^j (u_j)_n \right\| > 1 - \eta\left(\frac{\varepsilon}{2}\right).$$

By assumption Z has the AHSp, so there is a subset $A \subset \mathbb{N}$ such that $\sum_{n \in A} \sum \alpha_n^j > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon, z_j^* \in S_{Z^*}^j$ and $\left\{(z_j)_n : n \in A\right\} \subset S_Z$ such that

$$\sum \left\| (z_j)_n - (u_j)_n \right\| < \frac{\varepsilon}{2} \text{ and } \sum z_j^* \left((z_j)_n \right) = 1, \ \forall n \in A.$$

For every $n \in A$ we define the element $(y_j)_n \in Z_1$ given by

$$(y_j)_n(1) = (z_j)_n(1), \ (y_j)_n(k) = 0, \ \forall k \geq 2.$$

Let us fix $n \in A$. We clearly have that

$$\sum \left\| (y_j)_n - (u_j)_n \right\| = \sum \left\| \left(\left\| (y_j)_n(k) - (u_j)_n(k) \right\| \right) \right\|_E$$
$$\leq \sum \left\| \left(\left\| (z_j)_n(k) - (u_j)_n(k) \right\| \right) \right\|_E = \sum \left\| (z_j)_n - (u_j)_n \right\| < \frac{\varepsilon}{2}.$$

Since we know that

$$\sum \left\| (y_j)_n \right\| \leq \sum \left\| (z_j)_n \right\| = 1, \ \forall n \in A,$$

in view of (76) we deduce that

$$1 - \frac{\varepsilon}{2} \leq \sum \left\| (y_j)_n \right\| \leq 1, \qquad \forall n \in A.$$

As a consequence of Theorem (6.3.11) we know that $z_j^* \in (\bigoplus \sum_{k=1}^{\infty} X_k^*)_{E'}$ and we also have

$$\sum z_j^*(1)\left((y_j)_n(1) \right) = \sum z_j^*(1)\left((z_j)_n(1) \right) = \sum \|z_j^*(1)\| \left\| (z_j)_n(1) \right\|$$

$$= \sum \|z_j^*(1)\| \left\|\left(y_j\right)_n(1)\right\|, \ \forall n \in A.$$

On the other hand, it is satisfied that

$$\begin{aligned}
\sum \left|z_j^*(1)\left(\left(y_j\right)_n(1)\right)\right| = \sum \left|z_j^*\left(\left(y_j\right)_n\right)\right| \\
&\geq \sum \left|z_j^*\left(\left(z_j\right)_n\right)\right| - \sum \left|z_j^*\left(\left(y_j\right)_n - \left(z_j\right)_n\right)\right| \\
&\geq 1 - \sum \left\|\left(z_j\right)_n - \left(y_j\right)_n\right\| \\
&\geq 1 - \sum \left\|\left(z_j\right)_n - \left(u_j\right)_n\right\| - \sum \left\|\left(u_j\right)_n - \left(y_j\right)_n\right\| \\
&\geq 1 - 2\sum \left\|\left(z_j\right)_n - \left(u_j\right)_n\right\| \ \text{(by (76))} \\
&> 1 - \varepsilon > 0 \ \text{(by (75))}.
\end{aligned}$$

We denote by w_j^* the element in Z^* given by

$$w_j^*(1) = z_j^*(1), \ w_j^*(k) = 0, \ \text{if } k \geq 2.$$

Notice that $\|e_1\|_{E'}\|e_1\|_E = 1$. So it is clearly satisfied

$$\begin{aligned}
\sum \operatorname{Re} w_j^*\left(\left(y_j\right)_n\right) &= \sum \operatorname{Re} z_j^*\left(\left(y_j\right)_n\right) \\
&= \sum \|z_j^*(1)\| \left\|\left(y_j\right)_n(1)\right\| \ \text{(by (78))} \\
&= \sum \frac{\|w_j^*\|}{\|e_1\|_{E'}} \frac{\left\|\left(y_j\right)_n\right\|}{\|e_1\|_E} \\
&= \sum \|w_j^*\| \left\|\left(y_j\right)_n\right\|,
\end{aligned}$$

and bearing in mind (79) we deduce that $\sum w_j^*\left(\left(y_j\right)_n\right) \neq 0$.

List of Symbols

Symbol		Page
co	: convex	1
WLD	: Weakly Lindelöf	1
diam	: diameter	2
sup	: Supremum	2
RN	: Randon-Nikodym	6
RNP	: Randon-Nikodym Property	6
min	: minimum	7
PRI	: Projectional resolution of identity	11
dens	: dense	11
LUR	: Locally uniformly rotund	11
ℓ^∞	: Banach space of sequences	12
inf	: Infimum	12
dist	: distance	15
WLD	: Weakly Lindelöf determined	17
supp	: support space	20
L^p	: Lebesgue space	21
L^q	: Dual of Lebesgue Space	21
BPBP	: Bishop-Phelps-Bollobás Property	22
L_1	: Lebesgue integral on the Real line	22
$\bigoplus$	: Orthogonal Summand	28
NRA	: Numerical radius attaining	29
arg	: argument	30
Re	: Real	30
Im	: Imaginary	38
AHSP	: Approximate Hyperplane series property	41
ess	: essential	48
ℓ_∞^2	: essential Hilbert space	48
ℓ_p	: Banach space	55
$\otimes$	: Tensor product	55

max	: Maximum	63
card	: cardinality	89
ACK	: Asplund and $C(k)$	90
ext	: extreme	92
aco	: absolute conves hull	111
int	: interior	174
UM	: Uniformly monotone	175

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