Sudan University of Sciences and Technology

College of Graduate Studies

Group Classification of Invariant Solutions of Nonlinear Wave Equations

تصنيف الزمرة للحلول اللامتغيرة لمعادلات الموجة غير الخطية

A Thesis Submitted in Fulfillment of the Requirements for Degree of Ph.D. in Mathematics

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June 2022

Dedication

To: My father and my mother happiness springs.

To: My wife and sons.

To: My brothers and sisters.

To: My teachers in Mathematics.

To: my colleagues companions long road.

To: People who illuminated the path in front of me.... And spread joy around me, and took part in the Farhi and sorrow To: All fans of knowledge.

Acknowledgements

Thanks be firstly to Allah Who created me in the best calendar, and made me Muslim. All thanks to my parents who have made their utmost so that the male stage. I express my sincere thanks and exalted appreciation to P. Mohammed Ali Basher, which had a holy grail in the heuristic guideline in the research writing until he came out in the picture that between your hands. And it is so prisoners to the grace of God facts new science. As well as Thank you very much and bilateral those who helped me as much of knowledge and also to thank Co.Supervisor D/Blegiss Abdulaziz Abdalrahman Ebyed, who made every effort to light the long road listened guiding value and its likes. And also thanks to the teachers of the Mathematics Department. And also thanks to the fellow who Divided long road with me the hardships and all the professors the Faculty of Graduate Studies and College of Science have all of us very much thanks and appreciation and gratitude.

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Abstract

We perform complete group classification of the general class of quasilinear wave equations in two variables. This class may be seen as a generalization of the nonlinear d'Alembert, Liouville, sin/sinh-Gordon and Tzitzeica equations. We derive a number of new genuinely nonlinear invariant models with high symmetry properties. In particular, we obtain four classes of nonlinear wave equations that admit five-dimensional invariance groups.

الخلاصة

قمنا بتنفيذ تصنيف الزمرة التامة للعائلة العامة لمعادلات الموجة شبه الخطية في متغيرين. هذه العائلة قد ينظر إليها كتعميم لمعادلات دليمبيرت وليوفيل وجوردون-sin/sinh وتزيتزيكا. قمنا بإشتقاق عدد لنماذج غير متغيرة غير خطية بصدق جديدة مع خصائص متماثلة عالية. بصفة خاصة قمنا بالحصول على أربعة عائلات لمعادلات الموجة غير الخطية التي لها قبول الزمر اللامتغيرة ذات البعد الخامس.

CHAPTER ONE

VECTOR FIELDS AND INTEGRAL CURVES IN R^n

Chapter One

Vector Fields and Integral Curves in R^n

Introduction:

In this research we reviews a treatment of differential equations using methods from Lie group theory. Symmetry group methods are amongst the most powerful universal tools for the study of differential equation .There has been rapid progress on these methods over the last few decades. Methods and algorithms for classifying subalgebras of Lie algebras, new results on the structure and classification of abstract finite and infinite dimensional Lie algebra, and methods for solving group classification problems for differential equations greatly facilitated to systematically obtain exact analytic solutions by quadratures to ordinary differential equations and group-invariant solutions to partial differential equations and to identify equivalent equations. The application of Lie groups to differential equations has a long history. In the second half of the nineteenth century, Norwegian mathematician Sophus Lie (1842-1899) introduced continuous groups of transformations , to give a unified and systematic theory for the study of properties of solutions of differential equations just like Evariste Galois's (1811-1831) dream to solve algebraic equations by radicals, which led to the theory of Galois.The theory of Lie groups and algebras originated precisely in the context of differential equations. Over the years, these transformations evolved into the modern theory of abstract Lie groups and algebras.

As far as differential equations are concerned, the main observation was that much of the known solutions methods were actually specific cases of a general solution (general or particular solution) method based on the invariance of a system of differential equations under a continuous group of transformations (called symmetry group of the system).

Symmetry group of a partial differential equation (PDE) can be used to reduce the number of independent variables, to transform known simple solutions to new solutions. For an ordinary differential equation (ODE), even reduction in order can be made. Under some special structure of the symmetry group, reduction can even go all the way down to an algebraic equation, from which general solution can be obtained.

The computation of the symmetry group of a system of differential equations can be computationally complicated, but nevertheless completely algorithmic. Computer algebra systems can automate most of the steps of the Lie symmetry algorithm.

It should be emphasized that applications of Lie group methods using classical group of point transformations are not only restricted to differential equations. They, and when applicable, their generalizations to higher order symmetries can be also be carried over to conservation laws, Hamiltonnian systems, difference and differential-difference equations, integro-differential equations, delay differential equations, fractional differential equations.

Basic ideas, definitions, theorems and results needed to be able to apply Lie group methods for solving differential equations are presented. Many examples with mathematical and physical applications are considered to illustrate applications to integration of (ODEs) by the method of reduction of order, construction of group- invariant solutions to PDEs, identification of equivalent equations based on the existence of isomorphic symmetry groups, generating new solutions from known ones. Group -classification problem and construction of invariant differential equations.

Hyperbolic type second-order nonlinear PDEs in two equivalent variables play a fundamental role in modern mathematical physics. Equations of the type are utilized to describe various types of wave propagation. They are used in differential geometry, in various fields of hydrodynamics and gas dynamics, chemical technology, superconductivity, crystal dislocation to mention only a few applications areas. Surprising the list of equations utilized is rather narrow. In fact, it is comprised by the Liouville, sine/sinh-Gordan, Goursat, d'Alembert, and Tzitzeica equations and a couple of others. Popularity of these very models has a natural group-theoretical interpretation, namely, all of them have nontrivial Lie or Lie-$Bäcklund$ symmetry. By this very reason some of them are integrable by the inverse problem methods.

Knowing symmetry group of the equation under study provides us with the powerful equation exploration tool. So it is natural to attempt classifying a reasonable extensive class of nonlinear hyperbolic type PDEs into subclasses of equations enjoying the best symmetry properties. Saying reasonably extensive we mean this class should contain the above enumerated equations as for applications. The list of the so obtained invariant equations will contain candidates for realistic nonlinear mathematical models of the physical and chemical processes mentioned above.

The modern formulation of the problem of group classification of PDEs was suggested by Ovsyannikov, he developed a regular method (we will refer to it as the Lie-Ovsyannikov method) for classifying differential equations with nontrivial symmetry and performed complete group classification of the nonlinear heat conductivity equation. In a number of subsequent publication more general types of nonlinear heat equations were classified.

However, even a very quick analysis of the research on group classification of PDEs reveals that an overwhelming majority of them deals with equations whose arbitrary elements (functions) depend on one variable only. The reason for this is that Lie-Ovsyannikov method becomes inefficient for PDEs containing arbitrary functions of several variables. To achieve a complete classification one either needs to specify the transformation group realization or restrict somehow an arbitrariness of the functions contained in the equation under study.

We have recently, developed an efficient approach enabling to overcome this difficulty for the low dimensional PDEs. Utilizing it we have derived the complete group classification of the general quasilinear heat conductivity equation in two independent variables. In this research we apply the approach in question to perform group classification of the most general quasilinear hyperbolic PDE in two independent variables.

(1-1) Vector Fields and Integral Curves:

We begin with a brief review of some essential objects that will be employed throughout. Let M be a differentiable manifold of dimension n. a curve γ at a point x of M is a differentiable map $\gamma: I \to M,$ where I subinterval of R. Such that $\gamma(0) = x, 0 \in I$.

A vector field v of the manifold M is a C^∞ −section of TM. In other words a C^∞ mapping from M to TM that assigns to each point x of M a vector in T_xM. In the local coordinate system $x = (x_1, \dots, x_n) \in M, v$ can be expressed as

$$v|_x = \sum_{i=1}^{n} \xi_i(x)\partial_{x_i}, \qquad (1)$$

where $\xi_i(x) \epsilon\, C^\infty(M),\ i = 1, \dots, n.$

An integral curve of the vector field at the point x is the curve $\gamma\ at\ x$ whose tangent vector $\gamma^{'}(t)$ coincides with v at the point $x = \gamma(t)$ such that $\gamma^{'}(t) = v|_{(t)}$ for such $t \in I$. In the local representation of the curve γ it amounts to saying that the curve satisfies an autonomous system of first order ordinary differential equations

$$\frac{d\gamma_i}{dt} = \xi_i(\gamma(t)), \qquad i = 1, \dots, n \qquad (2)$$

The existence and uniqueness theorem for systems of ODES ensures that there is a unique solution to the system with the initial data $\gamma(0) = x_0$ (the Cauchy Problem). This gives rise to the existence of a unique maximal curve $\gamma(t)$ passing through the point $x_0 = \gamma(0)\, \epsilon M$. We call such a maximal integral curve the flow of $v\ at\ x = \gamma(t)$ an denote $\phi(t, x)$ with the basic properties

$$\phi(0,x) = x, \quad \phi(s,\phi(t,x)) = \phi(t+s,x), \quad \frac{d}{d}\phi(t,x) = v|_{\phi(t,x)},$$

$$x\epsilon M, (3)$$

For all sufficiently small $t, s\ \epsilon R$. A more suggestive notation for the flow is $\phi(t,x) = exp(tv)x$. The reason is simply that it satisfies the ordinary exponential rules. The second property implies that $\phi(-t,x) = \phi^{-1}(t,x)$ or

$exp(tv)^{-1}x = exp(-tv)x$. On can in finitesimally express the flow

$$exp(tv)x = x + tv_{|_x} + \vartheta(t^2). \quad (4)$$

The flow $exp(tv)x$ generated by the vector field v is sometimes called a one ----parameter group of transformations as it arises as the action of the Lie group R on the manifold M.

Conversely, given a flow with the first two properties of (3). We can reconstruct its generating vector field v by differentiating the flow:

$$v_{|_x} = \frac{d}{dt} exp(tv)x|_{t=0}, \quad x\epsilon M \quad .$$

The inverse process of constructing the flow is usually called exponentiation (or integration) of v.

Rectification of the vector field v in a neighborhood of a regular point (a point x at which $v|_x$ dose not vanish) is always possible.

Theorem (1.1.1):

If x_0 is a regular point of v, then there exist local rectifying (or straightening out) coordinates $y = (y_1, \dots, y_n)$ near x_0 such that $y = \partial_{y_1}$ generates the translational flow $exp(tv)y = (y_1 + t, y_2, \dots, y_n)$.

(1-2) Differential Equations and their Symmetry Group:

We consider a system of $n-th$ order differential equations ε

$$\varepsilon : \mathrm{E}_v(x, u^{(n)}) = \mathrm{E_v}(\mathrm{x}, \mathrm{u}, \mathrm{u}^1, \dots, \mathrm{u}^{(\mathrm{n})}) = 0, \quad \mathrm{v} = 1,2, \dots, \mathrm{N}, \qquad (5)$$

where $x = (x_1, \dots, x_p) \epsilon R^p,\ u = (u_1, \dots, u_q) \epsilon R^q\ (p, q \epsilon Z^+)$ are the independent and dependent variables, which form local coordinates on the space of independent and dependent variables $\mathrm{E} = X \times U \approx R^p \times R^q$. The derivatives of u are denoted by $u_{\alpha,J} = {\partial^J_{u\alpha}}/{\partial_{xJ}}$, where $J = (j_1, \dots, j_k)$,
$1 \le j_v \le p, k = j_1 + \dots + j_k$, is a symmetric multi- index of order $k = \#J$.

$u^{(k)}$ Denotes all partial derivatives of order $\le k$ of the components u_α $of\ u$, which provide coordinates on the jet space $J^n(x, u^{(n)}) = J^n\mathrm{E}$. If there is a single independent and dependent variable, namely $p = 1\ and\ q = 1$, then the system becomes a scalar ordinary differential equation. In that case, we simply write

$$\mathrm{E}(x, u, u_1, u_2, \dots, u_n) = 0,$$

where $u_1 = u_x, u_2 = u_{xx}, \dots, u_n = u^{(n)}$.

The system $\mathrm{E}_v = 0$ defined by a collection smooth functions $\mathrm{E} = (\mathrm{E}_1, \dots, \mathrm{E}_N)$ can be identified with a variety $S_\mathrm{E} = \{(x, u^{(n)}) : \mathrm{E} = 0\}$ contained in the $n - th$ order jet space J^n with local coordinates (x, u^n) has maximal rank

$$rank\left(\frac{d\mathrm{E}_v}{dx_i}, \frac{d\mathrm{E}_v}{du_{\alpha j}}\right) = N,$$

at each $(x, u^{(n)})$ satisfying the system.

A classical symmetry group of (5) is a local group G of point transformations ϕ: E → E, a locally defined invertible map on the space of independent and dependent variables, mapping solutions of the system to solutions

$$\phi: (x^{\sim}, u^{\sim}) = g.(x, u) = (\phi_1(x, u), \phi_2(x, u).$$

Such transformations act on solutions $u = f(x)$ bye mapping point wise their graphs. More precisely, if $\Gamma_f = \{(x, f(x))\}$ is the graph of $f(x)$, then the mapped graph will have the graph

$$\Gamma_f = \{(x^{\sim}, f^{\sim}(x^{\sim})\} = g.\Gamma_f \equiv \{g.(x, f(x)\}.$$

Contact or generalized transformations where $\emptyset$ depends on higher order derivatives will not be treaded here.

Definition (1.2.1):

A local Lie group of point transformations G is called a symmetry group of the system of partial differential equations (5) if $\hat{f} = g.f$ is a solution whenever f is.

To find the symmetry group, prolonged transformation $pr^{(n)}\emptyset: J^n \to J^n$ is required to preserve the differential structure of the equation manifold S_{E}.

In order to find the symmetry group Lie,s infinitesimal approach will be used. We need to use the prolongation tool for the group transformation and the vector field generating it. Let $\phi_\varepsilon = exp(\varepsilon v)$ be a one-parameter subgroup of the connected group G and let

$$v = \sum_{i=1}^{p} \xi_i\,(x, u)\partial_{x_i}$$

$$+ \sum_{\alpha=1}^{q} \varphi_\alpha\,(x, u)\partial_{u\alpha}, \qquad (6)$$

be the infinitesimal generator of $\emptyset_\varepsilon$. The infinitesimal generator of the prolonged one-parameter subgroup $pr^{(n)}\phi_\varepsilon$ is defined to be the prolongation of the vector field v.

Definition (1.2.2):

The $n-th$prolongation $pr^{(n)}v$ of v is a vector field on the $n-th$ jet space J^n defined by

$$pr^{(n)}v|_{(x,u^{(n)})}$$

$$=\frac{d}{d\varepsilon}|_{\varepsilon=0}pr^{(n)}\phi_{\varepsilon}\big(x,u^{(n)}\big), \qquad (7)$$

For every $(x,u^{(n)}\in J^n$.

If we integrate $pr^{(n)}v$ we find the prolongation of the group action $pr^{(n)}\emptyset_{\varepsilon}$ on the space J^n. The prolonged vector field $pr^{(n)}v$ has the form

$$pr^{(n)}v=\sum_{i=1}^{p}\xi_i\,\partial_{x_i}+\sum_{\alpha=i}^{q}\sum_{\#J\leq n}\varphi_{\alpha}^{j}\,\partial_{u_{\alpha,J}}, \qquad (8)$$

where the coefficients φ_{α}^{J} are given by the formula

$$\varphi_{\alpha}^{J}=D_J(\varphi_{\alpha}-\sum_{i=1}^{p}\xi_i\,u_{\alpha,j})$$

$$+\sum_{i=1}^{p}\xi_i\,u_{\alpha,J,i}, \qquad (9)$$

where $u_{\alpha,i}={\partial u_{\alpha}}/{\partial x_i}, u_{\alpha,J,i}={\partial u_{\alpha,J}}\Big/{\partial x_i}$ and $D_J=D_{j_1}\ldots D_{j_k}, 1\leq J_v\leq$

p is the J-th total derivative operator. Here D_i is the total differentiation operator defined by

$$D_i=\frac{\partial}{\partial x_i}+\sum_{\alpha=1}^{q}\sum_{J}u_{\alpha,J,i}\frac{\partial}{\partial u_{\alpha,J}}.$$

D_i involves infinite summation, but its application to a particular differential function will only require finitely many terms of order $0\leq\#J\leq n,$ where n is the highest order derivative in the differential function on which D_i acts. There is a useful recursive formula for the coefficients of the prolonged vector field in (8)

$$\varphi_\alpha^{J,i} = D_i\varphi_\alpha^J - \sum_{k=1}^{p}(D_i\,\xi_k)u_{\alpha,J,k}. \qquad (10)$$

If n-th prolongation is known, the (n+1)-th prolongation can be calculated by the formula (10). In particular, the coefficient of the first order derivatives u_x, in (8) are then given by

$$\varphi_\alpha^i = D_j\varphi_\alpha - \textstyle\sum_{k=1}^{p}(D_i\,\xi_k)u_{\alpha,k}.$$

In the special case p=q=1, the recursion formula (10) simplifies to

$$\varphi^J = D_x\varphi^{j-1} - (D_x\xi)u^{(j)}, \quad j = 1,2,\ldots \qquad (11)$$

the coefficients of the second prolongation of the vector field $v = \xi(x,u)\partial_x + \varphi(x,u)\partial_u$ corresponds to $j = 1,2$ with the convention $\varphi^0 = \varphi$

$$\varphi^x = D_{x\varphi} - (D_x\xi)\mathrm{u_x}, \;\; \varphi^{xx} = \mathrm{D_x}\varphi^{x} - (\mathrm{D_x}\xi)\mathrm{u_{xx}},$$

$$\mathrm{Q} = \varphi - \xi\mathrm{u_x},$$

the prolongations of vector fields sat isfy the linearity

$$pr^{(n)}(av + bw) = apr^{(n)}v + bpr^{(n)}w. \qquad (12)$$

For constants a,b and the Lie algebra property

$$pr^{(n)}[v,w] = [pr^{(n)}v, pr^{(n)}w] \qquad (13)$$

hence, the prolongation process defines a Lie algebra homomorphism from the space of vector felids J^0 to the space of vector fields on J^n . if the vector fields v form a Lie algebra, then their prolongations realize an isomorphic Lie algebra of the vector fields on J^n.

Example (1.2.1):

Let us consider the smooth projective vector field

$$v = x^2\partial_x + xu\partial_u$$

generating the one-parameter local projective group

$$\emptyset_\varepsilon(x,u): x^{\sim}(x,u;\varepsilon) = \frac{x}{1-\varepsilon x}, u^{\sim}(x,u;\varepsilon) = \frac{u}{1-\varepsilon x},$$

whenever $1-\varepsilon x \neq 0$ the vector field v can be recovered by differentiating $\emptyset_\varepsilon(x,u)$ at $\varepsilon = 0$.

The first and second prolonged group transformations are derived by the usual chain rule of the ordinary derivatives:

$$pr^{(1)}\emptyset_\varepsilon(x,u,u_1) = (x^{\sim}, u^{\sim}, u_1^{\sim}), \quad u_1^{\sim} = u_1 + \varepsilon(u - xu_1)),$$

and

$$pr^{(2)}\emptyset_\varepsilon(x,u,u_1,u_2) = (x^{\sim}, u^{\sim}, u_1 + \varepsilon(u - xu_1), (1 - \varepsilon x)^3 u_2).$$

Applying definition 1-2 we find the prolonged vector fields

$$pr^{(1)}v = v + (u - xu_1)\partial_{u_1}, \quad pr^{(2)}v = v + (u - xu_1)\partial_{u_1} - 3xu_2\partial_{u_2}. \quad (14)$$

The following the theorem determines the Lie algebra of the symmetry group G and known as the infinitesimal criterion of invariance of (5).

Theorem (1.2.1):

A connected local group of transformations G is a symmetry group of the system ε of (5) if and only if the $n-th$ prolongation $pr^n v$ annihilates the system on solutions, namely

$$pr^{(n)}v(\mathrm{E}_v) = 0, \quad v = 1,2,\ldots,N, \quad (15)$$

whenever $u = f(x)$ is a solution to the system (5) for every infinitesimal generator $v\ of\ G$.

Eqs. (15) are known as the determining equations of the symmetry group for the system. They form a large over-determining linear system of partial differential equations for the coefficients ξ_i and φ_α of v. This criterion has been applied to many differential equations arising in different branches of mathematics, physics, and engineering to compute symmetry groups. The computation of symmetry group using the infinitesimal approach have been implemented in several computer algebra systems, such as mathematic, maple, reduced [6]. There are packages dedicated to the symmetry group calculations which make considerably easy the routine steps of finding the determining system and partial integration of them.

Some packages are capable of triangularize the over determined system using differential Gröbner basis method. Packages equipped with automatic integrators can usually fail to provide the general solution of the determining system depending on the complexity of the system.

There is an alternative formulation of the prolongation formula, which is useful in prolongation computations. This is requires the formalism of the evolutionary vector fields. Given the vector field v as in (1), we define the q-tuple $Q(x, u^{(1)}) = (Q_1, \dots, Q_q)$ defined by

$$Q_\alpha(x, u^{(1)}) = \varphi_\alpha(x, u) - \sum_{i=1}^{p} \xi_i\,(x, u) u_{\alpha,i} \quad \alpha = 1,2, \dots, q.$$

The functions Q_α are called the characteristics of the vector field v, then, we have

$$\varphi_\alpha^J = D_J Q_\alpha + \sum_{i=1}^{p} \xi_i u_{\alpha,J,i,}\, D_J = D_{j_1} \dots D_{j_k}, \ \ 1 \le j_v \le p \qquad (16)$$

and the $n - th$ prolongation of v can be expressed as where

$$pr^{(\alpha)} v = pr^{(\alpha)} v_Q + \sum_{i=1}^{p} \xi_i D_i,$$

$$v_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(1)})\partial_{u_\alpha}, \quad pr^{(\alpha)}v_Q = \sum_{\alpha=1}^{q}\sum_{J} D_J\, Q_\alpha \partial_{u_{\alpha,,J}},$$

obviously, v_Q and their prolongations do not act on the independent variables x_i. in terms of characteristics Q_α, the infinitesimal transformations can be written as

$$x_i^{\sim} = x_i \quad u_\alpha^{\sim} = u_\alpha + \varepsilon Q_\alpha + \vartheta(\varepsilon^2).$$

Since $D_i \mathrm{E}_v = 0$ on solutions, we can replace the infinitesimal symmetry condition (15) by the simpler formula

$$pr^{(n)}v_Q|_{\mathrm{E}_v=0} = 0 \qquad (17)$$

Example (1.2.2):

We show that the Laplace equation $\Delta u(x,y) = u_{xx} + u_{yy} = 0$ in the plane is invariant under the symmetry group generated by the vector field

$$v = \xi(x,y)\partial_x + \eta(x,y)\partial_y, \qquad (18)$$

where ξ and η satisfy the Cauchy-Riemann equations $\xi_x = \eta_y$, $\xi_y = -\eta_{x,}$ in other words ξ, η are harmonic functions and therefore v generates an infinite dimensional symmetry group of the two-dimensional Laplace equations.

The second prolongation of v is

$$Pr^{(2)}v = v + \varphi^x\partial_{u_x} + \varphi^y\partial_{u_y} + \varphi^{xx}\partial_{u_{xx}} + \varphi^{yy}\partial_{u_{yy}},$$

the coefficients $\varphi^{xx}, \varphi^{yy}$ are calculated from the general prolongation formula

$$\varphi^{xx} = -(2\eta_x u_{xy} + 2\xi_x u_{xx} + \eta_{xx}u_y + \xi_{xx}u_x),$$
$$\varphi^{yy} = -(2\xi_y u_{xy} + 2\eta_y u_{yy} + \eta_{yy}u_y + \xi_{yy}u_x).$$

So from the Cauchy- Riemann equations we find that the infinitesimal criterion of invariance is satisfied

$$pr^{(2)}v(\Delta u) = \varphi^{xx} + \varphi^{yy} = -2\xi_x \Delta u = 0,$$

on the solution surface. The linearity of the equation implies that at also admits the additional trivial symmetries $u\partial_u$ and $\rho(x,y)\partial_u$ with $\Delta\rho = 0$ (one can multiply solutions by constants and add them). The symmetry condition then becomes

$pr^{(2)}(u\partial_u)(\Delta u) = \Delta u = 0 \; and \; pr^{(2)}(\rho\partial_u)(\Delta u) = \Delta\rho = 0$ on solutions. Obviously, the second one is satisfied if ρ is an arbitrary harmonic function.

The special choice $(\xi,\eta) = (x^2 - y^2, 2xy)$ leads to the conformal invariance of the Laplace equation. The one-parameter conformal symmetry group corresponding to the vector field

$c_x = (x^2 - y^2)\partial_x + 2xy\partial_y$ is easily obtained solving the following complex initial value problem for $z^{\sim}(x,y,\varepsilon)$ (more precisely, integrating the analytic vector field $z^2\partial_z$)

$$\frac{d\hat{z}}{d\varepsilon} = z^{\sim^2} = (x^{\sim^2} - y^{\sim^2}) + 2ix^{\sim}y^{\sim},$$

with the condition $z^{\sim}(x,y;0) = z(x,y) = x + iy$. The flow is given by $\hat{z} = \frac{z}{1-\varepsilon z} = \frac{z-\varepsilon|z|^2}{(1-\varepsilon z)(1-\varepsilon z^{\sim})}$.

We separate the real and complex parts of $z^{\sim}$ to obtain the following (well-defined) symmetry group $exp(\varepsilon c_x)(x,y)$

$$\hat{x} = \frac{x-\varepsilon(x^2+y^2)}{1-2\varepsilon x+\varepsilon^2(x^2+y^2)}, \quad \hat{y} = \frac{y}{1-2\varepsilon x+\varepsilon^2(x^2+y^2)} \qquad (19)$$

possessing the invariant function $\zeta(x,y) = y(x^2+y^2)^{-1}$, y-component of $I(x,y)$ satisfying $\zeta(x^{\sim},y^{\sim}) = \zeta(x,y)$ on $R^2/(0,0)$. $\zeta(x,y)$ is readily obtained by eliminating the group parameter ε in (19).

It is a well-know fact that the inversion map $\mathrm{I(x,y)} = (\mathrm{x}^2 + \mathrm{y}^2)^{-2}(\mathrm{x,y})$,

$(x,y) \neq 0$ $(an\ involution: I^{-1} = 1)$ is a discrete (not connected) symmetry, I.e. if $f(x,y)$ satisfies the Laplace equation, so dose $f((x^2 + y^2)^{-2}x, (x^2+y^2)^{-2}y)$. We observe that the map $(\hat{x},\hat{y}) = I(x,y)$ also provides the coordinates rectifying $c_x\ to\ -\partial_x$.

Conjugating any symmetry of the equation by I will produce a new symmetry (a conformal mapping here). Indeed, the push forward I_* of the vector field $-\partial_x$ through I is $I_*(-\partial_x) = \hat{c}_{\hat{x}}$, where tilde means that the vector field is written in the new coordinates. So $exp(\varepsilon c_x)(x,y)$ can be recovered by conjugating the translational group along the $x - axis: x \to x - \varepsilon\ by\ I$

$$exp\{\varepsilon c_x\}(x,y) = I(x,y)^\circ\ exp\{-\varepsilon\partial_x\}^\circ\ I(x,y).$$

Similarly, since $I_*(-\partial_y) = \hat{c}_{\hat{y}} = 2\hat{x}\hat{y}\partial_{\hat{x}} + (\hat{y}^2 - \hat{x}^2)\partial_{\hat{y}}, conjugating\ -\partial_y$ by I, $exp\{\varepsilon c_y\}(x,y) = I(x,y)^\circ\ exp\{-\varepsilon\partial_y\}\ ^\circ I(x,y)$.

Generate another conformal transformation

$$\hat{x} = \frac{x}{1 - 2\varepsilon y + \varepsilon^2(x^2 + y^2},$$

$$\hat{y} = \frac{y - \varepsilon(x^2+y^2)}{1 - 2\varepsilon y + \varepsilon^2(x^2+y^2)}. \quad (20)$$

The one-parameter group transformations generated by the elements of the abelian sub algebra $\{c_x, c_y\}$are conformal because they leave form invariant the planar metric:

$$d\hat{x}^2 + d\hat{y}^2 = \lambda(x,y;\varepsilon)(dx^2 + dy^2),$$

for some function λ (conformal factor).

$$For\ \ c_x, \lambda = (1 - 2\varepsilon x + \varepsilon^2(x^2+y^2))^{-2}$$

Note: that the inversion itself is also conformal mapping with

$$\lambda = (x^2+y^2)^{-2}.$$

We conclude that action of this group on solutions states that $u = f(\hat{x},\hat{y})$ is also a solution, whenever $f(x,y)$ is solution to the Laplace equation. For example, with the help of invariant ζ, the radial solution $f(x,y) = log(x^2+y^2)$ or the angular solution $f(x,y) = arctan(y/x)$, among many others (homogeneous harmonics) can be happed to produce the new solutions

$$u = log\frac{x^2+y^2}{1-2\varepsilon x+\varepsilon^2(x^2+y^2)}, \quad u = arctan\frac{y}{x-\varepsilon(x^2+y^2)}.$$

Adding to $c_x\ and\ c_y$ sub algebras obtained by other choices $(\xi,\eta) = (1,0),(0,1),(\xi,\eta) = (-y,x)\ and,(\xi,\eta) = (x,y)$ leading to the translational, rotational and dilatational invariance, in terms of vector fields, $P_x = \partial_x, P_y = \partial_y,\ j = -y\partial_x + x\partial_y,\ \ d = x\partial_x + y\partial_y$, respectively, we obtain the 6-dimensional Lie algebra of the conformal group $Conf(R^2)$ of the Euclidean plane R^2, isomorphic to $SO(3,1)$, the Lorentz group of four-dimensional Minkowski space [7]. Obviously, the sub algebra spanned by $\{P_x, d, c_x\}$ is conformal group is the two-dimensional analogue of the full conformal group in dimensions $n \geq 3$. Note that the full conformal group in the plane $R^2 \cong C$ is infinite-dimensional, with the Lie group $SO(3,1)$ at its maximal finite-dimensional subgroup, because any analytic function $f: C \to C$ leads to a conformal transformation.

$(In\ our\ case f(z) = z, iz, z^2)$. We have excluded the trivial symmetry algebra stemming from the linearity of the PDF. Their non-zero commutators satisfy

$$[P_{x,y}, d] = P_{x,y},\ [j, P_x] = -P_y,\ [j, P_y] = P_x,\ \ [P_1, c_x] = [P_2, c_y] = 2d.$$

$$[P_x, c_y] = -[P_y, c_x] = -2j,\ \ [d, c_{x,y}] = c_{x,y}, [j, c_x] = -c_y, [j, c_y] = c_x.$$

A nonlinear variant of the Laplace equation, known as the conformal scalar curvature equation, or the elliptic Liouville,s equation, occurs in the study of isothermal coordinates in differential geometry and has the form

$$u_{xx} + u_{yy} = Ke^{u}, \quad (21)$$

where K is constant (Gaussian curvature) .

The conformal symmetry structure of this equation on the $(x, y) - plane$ is preserved. The vector field generating the symmetry group G of the equation is given by

$$v = \xi\partial_x + \eta\partial_y - 2\xi_x\partial_u,$$

where $\xi(x, y)$, $\eta(x, y)$ satisfy the Cauchy-Riemann equations. For $(\xi, \eta) = (x^2 - y^2, 2xy)$, $v = (x^2 - y^2)\partial_x + 2xy\partial_y - 4x\partial_u$.

We solve the initial value problem $d\hat{u}/d\varepsilon = -4\hat{x}, \hat{u}(x, y; 0) = u(x, y)$

and find the transformation of u under the group action:

$$\hat{u}(x, y; \varepsilon) = 2log\sigma(x, y; \varepsilon) + u(x, y), \sigma(x, y) = 1 - 2\varepsilon x + \varepsilon^2(x^2 + y^2). \quad (22)$$

Application of the one-parameter transformation group defined by (19) and (22) to a solution $f(x, y)$ where the coordinates (x, y) are written in terms of $(\hat{x}, \hat{y})$ leads to the transformed new solution $u_\varepsilon(x, y)$ (after the tildes are removed)

$$u_\varepsilon(x, y) = -2log\sigma(x, y; -\varepsilon) + f(\sigma(x, y; -\varepsilon)^{-1}(x + \varepsilon(x^2 + y^2)), \sigma(x, y; -\varepsilon)^{-1}y).$$

Note that $\sigma(x, y; \varepsilon) = \sigma(\hat{x}, \hat{y}; \varepsilon)^{-1}$.

Remark (1.2.1):

The Laplace equation in R^n with $n \geq 3$ is invariant only under a finite dimensional conformal Lie symmetry group of R^n with dimension

$\binom{n+2}{2} = (n+1)(n+2)/2$,consisting of the groups of translations, notations, dilation and conformal transformations (obtained by conjugating the n-components of the translational group via inversion $I(x,y) = |x|^{-2}x)$ on ${R^n}/_{\{0\}}$.

The linear wave equation $u_{tt} = \Delta u, u = u(t,x), (t,x) \in R^{n+1}$ is invariant under a Lie point symmetry algebra isomorphic to the Lorentz group $SO(n+1,2),\ of dimension\ \binom{n+3}{2} = {(n+2)(n+3)}/_{2}, n \geq 2$ in a Minkowski space with an inderlying metric $ds^2 = dt^2 - dx_1^2 - \cdots - dx_n^2$.

The nonlinear wave (or Klein-Gordon) equation $u_{tt} - u_{xx} = f(u)$ is invariant under the Poincare P(1.1) of 1+1-dimensional MinKowski plane,

for any $f(u)$ with $f^n \neq 0$. Its Lie symmetry algebra is generated by the translational and Lorenz vector fields

$$v_1 = \partial_t, \quad v_2 = \partial_x, \quad v_3 = t\partial_x + x\partial_t.$$

For two specific forms of $f(u)$, the symmetry algebra is larger.

The additional vector field for

$f(u) = f_0 u^k$ is $v_4 = t\partial_t + x\partial_x + \frac{2}{1-k}u\partial_u,\ and\ v_4 = t\partial_t + x\partial_x - \frac{2}{\lambda}\partial_u$ for $f(u) = f_0 e^{\lambda u}$. The linear case $f^n = 0$ is quite different, the symmetry group is the infinite-dimensional conformal group.

(1-3) Differential Invariants:

Given a Lie algebra g, characterization of all invariant equations, equations that remain invariant under the symmetry group $G\ of\ g$ requires the notion of differential invariants, which are functions unaffected by the action of G on some manifold M. An ordinary invariant is a $C^\infty(J(x,u))$ function $I(x,u)$ on $J(x,u) \subset M$, which satisfies $I(g.(x,u) = I(x,u)$ for all group elements $g \in G$ and coordinates (x,u).

Definition (1.3.1):

A differential invariant of order n of a connected transformation group G is a differential function $I(x, u^{(n)})$ on the jet space J^n $if\ I(g^{(n)}.(x, u^{(n)})\ for\ g \in G\ and\ (x, u^{(n)}) \in J^n.$

An ordinary invariant is a differential invariant of order 0. The following infinitesimal invariance criterion for differential invariants serves to determine differential invariants of a given connected group transformations in a simple manner by just solving a system of linear first order $PDEs$.

Proposition (1.3.1):

A differential invariant of order n of a connected group G if and only if it is annihilated by all the prolonged vector fields (infinitesimal generators)

$$v^{(n)}(I) \equiv pr^{(n)}v(I) = 0 \qquad (23)$$

For all $v \in g$.

An function I is an ordinary invariant if and only if $v(I) = 0$. For a general vector field

$$v = \sum_{i=1}^{n} \xi_i(x)\partial_x,$$

the coordinates $y = \eta(x)$ rectifying $v = \partial_{y1}$ are found by solving the first order partial differential equations $v(\eta_1) = 1,\ v(\eta_i) = 0,\ i > 1$. So the new coordinates $y_i(x)$ are the functionally independent invariants of the one-parameter group generated by v.

Remark (1.3.1):

The dimension of the space J^n is $dimJ^n = p + q\binom{p+n}{n}$. The number of derivatives of order exactly n is given by $q_n = dimJ_n - dimJ_{n-1} = q\binom{p+n-1}{n}$.

The number of functionally independent differential invariants of order n is equal to

$$k = dimJ^n - (dimG - dimG_0), \qquad (24)$$

where G_0 is the stabilizer subgroup (also called the isotropy group) of a generic point on J^n. $dimG - dimG_0$ is the dimension of the orbit of G at a generic point. An equivalent formula for k_n is

$$k_n = dimJ^n - rankZ \geq 0,$$

where Z is the matrix of size $r \times dimJ^n$ formed by the coefficients of the $n-th$ prolongations $pr^{(n)}(V_v), v = 1,2,\ldots,r$ of the basis vector fields $v_1,\ldots,v_r$ of Lie algebra g of the group G as rows

$$V_v = \sum_{i=1}^{p} \xi_{i,v}(x,u)\partial_{x_i} + \sum_{\alpha=1}^{q} \varphi_{\alpha,v}(x,u)\partial_{u_\alpha}, \quad v = 1,2,\ldots,r.$$

The rank $m\ of\ Z$ is calculated at a generic point of J^n. For the special case of one independent and one dependent variable $p = q = 1$ we have $dimJ^n = n + 2$ and the number of functionally independent invariants is $k_n = n + 2 - m.$

The set of $n-th$ order differential invariants $I_1, I_2, \ldots, I_k$ of g will be denoted by $\tau_n(g)$. This set is an R-algebra. This means $\tau_n(g)$ is a vector space over the field R and satisfies the property that any arbitrary smooth function $H(I_1,\ldots,I_k)$ of the set of differential invariants $I_1,\ldots,I_k$ is also a differential invariant, i,e.if $I_1, I_2, \ldots, I_k \in \tau_n(g),\ then\ H(I_1,\ldots,I_k) \in \tau_n(g)$. They also satisfy the inclusions

$$\tau_0(g) \subset \tau_i(g) \subset \cdots \tau_n(g) \subset \cdots$$

the algebra $U_{n=1}^{\infty}\tau_n(g)$ is called the algebra of differential invariants.

(1-4) Invariant Differentiation:

Lie [23, 22, 24] and Tresses [12] introduced the notation of "invariant"

Differential operators to obtain$(n+1)-st$ order differential invariants for $n-th$ order ones. This enables one to produce all the higher order functionally independent invariants by successive application of the invariant operators to lower order invariants. The situation is easier when there is only one dependent variable. Let the group G with the Lie algebra g act on the basic space $\mathrm{E} = J^0(R, R^n) \approx X \times U$.

Proposition (1.4.1):

Suppose that $I(x, u^{(n)}$ are functionally independent invariants, at least one of which has order exactly n. Then the ratio ${D_x J}/{D_x I}$ (the Tresse derivatives) is an $(n+1)-st$ order differential invariant.

If $I(x, u^{(n)})$ is any given differential invariant, then $D = (D_x I)^{-1} D_x$ is an invariant differential operator so that iterating on D one can generate an hierarchy $D^k J, k = 0,1,2, \ldots$, of higher order differential invariants.

If $z = I(x, u)\ and\ w = J(x, u, u_1)$ are a complete set of functionally independent invariants of the first prolongation $pr^{(1)} g, i.e.$ they form the basis of $\tau_1(g), then\ I, J$ together with the derivatives $D^k J = {d^k w}/{dz^k}, k = 1,2, \ldots, n-1$ generate a complete set of functionally independent invariants for the prolonged algebra $pr^{(n)}(g)\ for\ n \geq 1$. They all satisfy the infinitesimal invariant condition $pr^{(n)} v(D^k J) = 0$ for a vector field $v \in g$.

Theorem (1.4.1):

If the differential functions

$$I_1(x, u^{(n)}), I_2(x, u^{(n)}), \ldots, I_k(x, u^{(n)}) \in \tau_n(g) \subset J^n$$

form a set of functionally independent $n-th$ order differential invariants of G, then a system of $n-th$ order differential equations are invariant under G if and only if it can be written in terms of the differential invariants:

$$\mathrm{E}_v(x,u^{(n)}) = H_v(I_1,I_2,\dots,I_k) = 0, v = 1,2,\dots,N. \qquad (25)$$

Invariant equations obtained in this way are called strongly invariant $pr^{(n)}H_v(I_1,I_2,\dots,I_k) = 0$ is satisfied everywhere).

Example (1.4.1):

We find all second order equations invariant under the abelian subgroup of the projective group $SL(3,R)$ generated by

$$v_1 = x^2\partial_x + xy\partial_y, v_2 = xy\partial_x + y^2\partial_y .$$

From example (1.2) we know the following set of second order differential invariants

$$I_1 = \frac{y}{x}, I_2 = y - xy_1, \ \ I_3 = x^3y_2.$$

The second prolongation of v_2 is

$$pr^{(2)}v_2 = v_2 + y_1(y - xy_1)\partial_{y_1} - 3xy_1y_2\partial_{y_2}.$$

Imposing the condition $pr^{(2)}v_2(H) = 0$ and changing to the invariants as new coordinates we find that $H(I_1,I_2,I_3)$ satisfies

$$I_2^2\frac{\partial H}{\partial I_2} + 3I_2I_3\frac{\partial H}{\partial I_3} = 0.$$

Solving this PDE by the method of characteristics, it follows that there are two independent invariants $I = I_1 = {}^{y}/_{x}$ and $J = I_3I_2^{-3} = x^3(y - xy_1)^{-3}y_2$.

The most general equation can now be written as

$$y_2 = x^{-3}(y - xy_1)^3G(\frac{y}{x}). \qquad (26)$$

Where G is an arbitrary function.

The rectifying coordinate for v_1 to $\hat{v}_1 = \partial_x$ are $r = {}^{y}/_{x}$ (an invariant) an $s = {}^{-1}/_{x}$. In terms of r, s, v_2 gets transformed to $\hat{v}_2 = r\partial_s$. The invariant equation corresponding to the abelian algebra $\{\partial_s, r\partial_s\}$

(compare with the canonical realization $A_{2,2}$ is a linear one ${d^2s}/{dr^2} = G(r)$. We conclude that the same transformation linearizes Eq.(26).

If we replace $v_2\ by\, v_2 = x\partial_x + ky\partial_y$, this time $H(I_1, I_2, I_3)$ has to satisfy the zero-degree quasi-homogeneous function PDE

$$(k-1)I_1\frac{\partial H}{\partial I_1} + kI_2\frac{\partial H}{\partial I_2} + (k+1)I_3\frac{\partial H}{\partial I_3} = 0.$$

Integrating the characteristic equations of this PDE we obtain the invariants $I = I_1^{k/(1-k)}I_2, J = I_1^{(k+1)/(1-k)}I_3, k \neq 1,\ while\, for k = 1, I = I_1$ and$J = I_3 I_2^{-2}$.

In the former case, the invariant equation will have the form $J = G(I), or$

$$y_2 = x^{-(n+3)}y^n G(I), \qquad I = (\frac{y}{x})^{-(n+1)/2}(y - xy_1), \qquad k = \frac{n+1}{n-1}.$$

In this case, the algebra belongs to the nonabelian realization $A_{2,3}$ for the specific choice of $G = k = const$. It reduces to the celebrated Emden-Forwler equation.

In the latter case, we have the nonabelian algebra of linearly connected (or rank-one) vector fields with $v_1 = xv_2$, which is the canonical form $\{\partial_x, s\partial_s\}$, up to change of coordinates $r = {y}/{x},\ s = {-1}/{x}$. The corresponding invariant equation is

$$y_2 = (y - xy_1)^2 F(\frac{y}{x}).$$

Changing to new coordinates(r,s) linearizes this equation to

$$\frac{d^2s}{dr^2} + F(r)\frac{ds}{dr} = 0.$$

Example (1.4.2):

We know the vector fields $v = x^2\partial_x + xu\partial_u$ on $J^0(R,R)$ has the first order differential invariants $I = u/x\,, J = xu_1 - u,$ which satisfy $v(I) = 0, pr^{(1)}v(J) = 0$. The tresse derivative $DJ/DI = J^{-1}(x^3u_2)$ give the second order differential invariant. We can take it as $J_2 = JDJ/DJ = x^3u_2$ as it is needed on only up to functional independence. Iterating the tresse derivatives and multiplying by J we obtain the sequence of all other differential invariants as $J_k = x^2D_xJ_{k-1}, k = 3,4, \ldots.$

Determination of a complete set of functionally independent differential invariants ($the\ basis\ \tau_n(g)$) allows us to construct classes of differential equations with a prescribed symmetry algebra g.

Example (1.4.3):

We construct all third order ODEs invariant under the solvable group $E(2)$ of rigid motions in the plane (isometries of the Euclidean space R^2) of $e(2)$ algebra of symmetries composed of translations along x and y axes and planar rotations

$$v_1 = \partial_x, v_2 = \partial_y, \;\; v_3 = -y\partial_x + x\partial_y,$$

with non-zero commutators

$$[v_1, v_3] = v_2, \qquad [v_2, v_3] = -v_1.$$

Prolongations of $v_1\ and\ v_1$ do not alter their local form, but the third order prolongation of v_3 is given by

$$pr^{(3)}v_3 = v_3 + (1 + y_1^2)\partial_{y_1} + 3y_1y_2\partial_{y_2} + (4y_1y_3 + 3y_2^2)\partial_{y_3}.$$

We note that the Euclidean group $E(2)$ has no ordinary invariants on the space $(x,y)(k = n + 2 - m = 2 - 2 = 0)$, nor differential invariant of the first order $(k = 1 + 2 - 3 = 0)$ because the group acts transitively on (x,y) and (x,y,y_1), but there are differential invariants of order≥ 2. We can not use invariant differentiation process to find higher order ones. If $I(x,y,y_1,y_2,y_3)$ is differential invariant of order three of $e(2)$, then

$pr^{(3)}v_1(I) = 0, pr^{(3)}v_2(I) = 0, pr^{(3)}v_3(I) = 0$. From the first two equations, I must be independent of x and y coordinates, namely $I(y_1, y_2, y_3)$. We solve the characteristic system to find the two independent differential invariants, satisfying $pr^{(3)}v_3(I) = 0$. The characteristic system is given by

$$\frac{dy_1}{1+y_1^2} = \frac{dy_2}{3y_1y_2} = \frac{dy_3}{4y_1y_3 + 3y_2^2}.$$

From the first characteristic equation, a second order invariant is $k = (1+y_1^2)^{-3/2}y_2$ (the curvature). The other one is obtained by replacing $y_2\ by\ k(1+y_1^2)^{3/2}$, with k treated constant, in the last term and then integrating the first linear ODE

$$\frac{dy_3}{dy_1} - \frac{4y_1}{1+y_1^2}y_3 = 3k^2.$$

This provides as the differential invariant $\xi(y_1, y_2, y_3) = (1 + y_1^2)y_2^{-2}y_3 - 3y_1$ so that $k\ and\ \xi$ form a basis of third order invariants of $e(2)$ (a set of functionally independent invariants). The invariant equation now can be written as

$$(1+y_1^2)y_3 - 3y_1y_2^2 = y_2^2H(k),$$

where H is any smooth function of the curvature.

In view of proposition higher order differential invariants and invariant ODEs can be constructed using the tresse derivatives of $k\ and\ \xi$. For instance, the ratio $D_x\xi/D_x\,k$ gives a fourth order invariant.

CHAPTER TWO

REDUCTION OF ORDER FOR ORDINARY DIFFERENTIAL EQUATIONS

Chapter two

(2.1) Reduction of Order for Ordinary Differential Equations:

Theorem (2.1.1):

Let the scalar ordinary differential equation

$$\mathrm{E}(x, y^{(n)}) = \mathrm{E}(x, y, y_1, \dots, y_n) = 0, \quad \frac{\partial \mathrm{E}}{\partial y_n} \neq 0,$$

admit a one-parameter symmetry group G generated by $v \in g$. All non tangential solutions ($v \in g$ *is nowher tangent to the graph of the solution*) can be found by quadruture from the solutions to the reduced $ODE(\mathrm{E}/\mathrm{G})(\mathrm{x}, \mathrm{y}^{(\mathrm{n}-1)})$.

Proof: if we introduce rectifying (canonical or normal) coordinates $r = r(x, y)$ and $s = s(x, y)$ in which v generates a group of translation $r \to r, s \to s + \varepsilon$ with the corresponding normal form $\hat{v} = \partial_s$. Its prolongation $pr^{(n)}\hat{v} = \partial_s$ and therefore the derivatives in the new coordinates remain unchanged so that from the invariance condition (22) it follows that the equation in normal form should be independent of the variable s, but can depend on the derivatives. Therefore we have reduced our equation to one of order $n - 1, (\mathrm{E}/G)(r, s', \dots, s^{(n-1)})$ for the derivative $z = \vartheta(r) = ds/dr = s'\,(r)$. Once we know the solution of the reduced equation, the solution to the original one is obtained by a quadrature $s = f\vartheta(r)dr$.

The rectifying coordinates r, s are constructed as solutions of the partial differential equations $(s) = 0, v(s) = 1$. Note that the coordinate r is an invariant of $v(r)$ and r, s can be replaced by any arbitrary functions of r and s.

For a first order equation $y' = F(x, y)$ admitting the symmetry generated by $v = \xi(x, y)\partial_x + \eta(x, y)\partial_y$, the determining equation is

$$\xi F_x + \eta F_y = (\eta_y - \xi_x)F - \xi_y F^2 + \eta_x. \quad (27)$$

This equation admits the solution $\eta = F\xi$. But the corresponding vector fields $v = \xi(x, y)(\partial_x + F(x, y)\partial_y)$ are everywhere tangential to solutions and do not serve our purpose for reduction because finding canonical coordinates equally require integrating the equation itself. Other than these trivial symmetries, the above determining equation can allow particular solutions for given F leading to one-parameter symmetry groups. Transforming to canonical coordinates reduces the equation to quadrature. It is quite straightforward to see that, if $\eta - F\xi \neq 0$, the infinitesimal symmetry condition (27) can be re-expressed as

$$\mu_x + (\mu F)_y = 0, \quad \mu(x, y) = (\eta - F\xi)^{-1},$$

implies the existence of an integrating factor $\mu(x, y)$ of the equation.

In practice it is more feasible to solve the inverse problem of constructing the most general first order ODEs admitting a given group as a symmetry group. The same problem for higher order equations are equally useful.

Example (2.1.1):

We consider the following one-parameter local group of transformation (a special form of the so-called fiber-preserving transformation in which the changes in x are not affected by the dependent variable y)

$$\phi_\varepsilon(x, y): \quad \hat{x} = X(x; \varepsilon), \quad \hat{y} = Y(x; y) \quad (28)$$

with the infinitesimal generator

$$v = \xi(x)\partial_x + \eta(x)y\partial_y, \quad (29)$$

and its prolongation

$$pr^{(1)}v = v + [\eta' y + (\eta - \xi')y']\partial_{y'}.$$

Solving the characteristic equation of the first order PDE $pr^{(1)}v(I) = 0$

$$\frac{dx}{\xi(x)} = \frac{dy}{\eta(x)y} = \frac{dy'}{\eta' y + (\eta - \xi')y'},$$

we find the first order fundamental invariants

$$r(x,y) = v(x)y, \;\; w(x,y,y') = v(x)(\xi y' - \eta y), \;\; v(x) = exp\left\{-\int^x \frac{\eta(t)}{\xi(t)} dt\right\}.$$

The corresponding ODE can be expressed in terms of invariants in the form

$F(r,w) = 0 \quad or\; w = H(r),$ more precisely

$$y' - \frac{\eta(x)}{\xi(x)} y = \frac{H(v(x)y)}{v(x)\xi(x)}, \qquad (30)$$

where F and H are arbitrary functions of a single argument. In terms of the canonical coordinates it has the form, which is independent of s,

$$\frac{dr}{ds} = H(r), \quad w = \frac{dr}{ds}, \quad s = \int^x \frac{dt}{\xi(t)}. \qquad (31)$$

This means that Eq. (31) can be integrated by quadrature. This equation involves interesting types of ODEs like Abel's equation of second type or Riccati equation, if we choose ξ arbitrary, $\eta = -1$ and $H(\tau) = A\tau^{-1} + \tau + B$ we have

$$yy' = \frac{A}{\xi v} y + \frac{B}{\xi v^2}, \qquad v(x) = e^{s(x)},$$

with the one-parameter symmetry group generated by $v = \xi(x)\partial_x - y\partial_y$. In terms of r, s, it is reduced to the quadrature

$$\int \frac{rdr}{r^2 + Ar + B} = s + c,$$

where c is an integration constant.

The choice $H(\tau) = Ar^2 + B\tau + C$ leads to the Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x), \qquad (32)$$

where

$$a(x) = \frac{Av}{\xi}, \;\; b(x) = \frac{B+\eta}{\xi}, \qquad c(x) = \frac{C}{v\xi},$$

invariant under the symmetry $v = \xi\partial_x + \eta y\partial_y$. Its separable form in coordinates r,s is the following

$$\int \frac{dr}{Ar^2 + Br + C} = s + c.$$

The particular choice $\xi = 1,\ \eta = k$ (constant) leads to the Reccati equation

$$y' = Ae^{-kx}y^2 + By + Ce^{kx},$$

Admitting the symmetry group $(x, y) \to (x + \varepsilon, e^{ke}y)$ generated by $v = \partial_x + ky\partial_y$.

Remark (2.1.1):

Lie showed that the Lie symmetry algebra the second order ODE can be of dimension $dim = 0,1,2,3,8$. The maximum dimension is attained if and only if can be mapped by a point transformation to the canonical linear equation $y'' = 0$, which admits a symmetry group isomorphic to the $SL(3, R)$ group, acting as the group of projective transformations of the Euclidean plane with the coordinates (x, y). Moreover, he classified equations with point symmetries into equivalence classes under the action of the infinite dimensional group Diff (2,C) of all local diffeomorphisms of a complex plane C.

Remark (2.1.2):

There are two isomorphism classes of two-dimensional Lie algebras exist (over R and over C). Each of them can be realized in two different, ways:

$A_{2,1}$: $v_1 = \partial_x$, $v_2 = \partial_y$, $A_{2,2}$: $v_1 = \partial_y$, $v_2 = x\partial_y$, (33)

$A_{2,3}$: $v_1 = \partial_y$, $v_2 = x\partial_x + y\partial_y$, $A_{2,4}$: $v_1 = \partial_y$, $v_2 = y\partial_y$. (34)

In the terminology of Lie, for the realizations $A_{2,1}, A_{2,4}$, the vector fields v_1 and v_2 are called linearly connected, for $A_{2,1}, A_{2,3}$,`they are vector fields $v_1 and\, v_2$ are called linearly connected, for $A_{2,1}, A_{2,3}$, they are called linearly unconnected. The invariant equations are $y'' = F(y')\, and\, y'' = F(x)\, for\, A_{2,1}$, and $A_{2,2}; xy'' = F(y')$ and $y'' = F(x)y'$ for $A_{2,3}, and\, A_{2,4}$, respectively.

Any second order ODE with a symmetry group G of dimension≥ 2 (a nonlinear second order ODE can be invariant at most under a three-dimensional Lie group) can be integrated by two quadratures except for the rotation group $SO(3, R)$ when $dim\, G = 3$, which has no two-dimensional subgroup. In the latter case, there is method based on first integrals to find the general solution without integration.

The following theorem is useful when performing reductions.

Theorem (2.1.2):

Suppose E$(x, y^{(n)} = 0$ is an $n - th$ order ODE with a symmetry group G. Let H be a one-parameter subgroup of G. then the ODE reduced by H, E$/H$, admits the quotient group $Nor_G\, {}^{(H)}/_{H}$, where $Nor_G(H) = \{g\epsilon G: g.H.g^{-1} \subset H\}$ is the normalize subgroup of H in G, as a symmetry group (often called inherited symmetry group of E$/H$.

The normalize algebra in g of the sub algebra $\mathfrak{h} \subset g$ is the maximal sub algebra satisfying

$$nor_g \mathfrak{h} = \{v \in g: [v, \mathfrak{h}] \subset \mathfrak{h}\}. \quad (35)$$

Remark (2.1.3):

The normalize of a sub algebra $\mathfrak{h} \subset g$ in the Lie algebra g of $dimg = r$ a basis$\{v_1, \ldots, v_2\}$ is easily obtained by solving a linear algebra problem. Let the subalgebra $\mathfrak{h}\, of\, dim\, \mathfrak{h} = k$ be spanned by $\{v_1, \ldots, v_2\}$. Now for

$v = \sum_{j=1}^{r} a_j\, v_j, a_j \in R\ and\ v_\alpha \in \mathfrak{h}, 1 \leq \alpha \leq k,$ we impose that requirement (35)

$$[v, v_\alpha] = \left[\sum_{j=1}^{r} \alpha_j v_j, v_\alpha\right] = \sum_{j=1}^{r} [v_j, v_\alpha] a_j$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} C_{j\alpha}^{i}\, a_j v_i = \sum_{i=1}^{k} \lambda_{\alpha i}, v_i$$

for some constants $\lambda_{\alpha,i}$. Here$C_{j\alpha}^{i}$ are the structure constants of g. This implies the following set of linear algebraic equations to be solved for the coefficients $a_j, 1 \leq j \leq r$

$$\sum_{j=1}^{r} C_{j\alpha}^{i} a_j = \lambda_{\alpha i}, \quad i = 1, \dots, k, \quad \alpha = 1, \dots, k,$$

$$\sum_{j=1}^{r} C_{j\alpha}^{i}\, a_j = 0, \quad i = k+1, \dots, r.$$

Remark (2.1.4):

If $\mathfrak{h}$ is already and ideal of g, then $nor_g \mathfrak{h} = g. If\ nor_g \mathfrak{h} = \mathfrak{h}$, then $\mathfrak{h}$ is called self-normalizing.

Theorem (2.1.3):

Infinitesimally states that if the Lie symmetry algebra $g\ of\ G$ has a basis $v_1, \dots, v_2$. Then the ODE reduced by v_1 can be reduced one more if $\hat{v}_2 = span\{v_2, \dots, v_r\}$ is chosen to satisfy $[\hat{v}_2, v_1] = kv_1$ for some real constant k, meaning that v_1 is an ideal (normal subalgebra) of $\hat{v}_2$. One can reiterate this process to achieve a full reduction.

Theorem (2.1.4):

Applied to a two-parameter symmetry group with Lie algebra g being one of the isomorphy classes of Remark 2.2 and satisfying the commutation relation $[v_1, v_2] = kv_1$ ensures that given a second order ODE invariant under g. Reducing its order by one by the ideal (normal subgroup) $v_1\, in\, g$ will lead to a first order ODE inheriting the one-parameter subgroup generated by v_2 as a symmetry group. Note that the normalize of $\{v_1\}\, is\, \{v_1, v_2\}$ an v_2 belong to $nor_g\, \mathfrak{h}/\mathfrak{h}$. This reduction procedure makes possible the integration of the ODE by two successive quadratures.

If the reduction is preformed in the reverse order, the reduced ODE will in general not inherit of the original equation, so we may not be able to complete the full integration.

Example (2.1.2):

The second order invariant equation with the same symmetry $v_1 = \xi\,\partial_x + \eta y\,\partial_y$ the previous example can be expressed in terms of the second order invariants; r, w as defined in (30) and

$$\zeta(x, y, y', y'') = v(x)[\xi^2 y'' + \xi(\xi' - 2\eta)y' + (\eta^2 - \xi\eta')y] \qquad (36)$$

as

$$y'' + p(x)y' + q(x)y = \frac{H(r, w)}{v\xi^2} \qquad (37)$$

where

$$p(x) = \xi^{-1}(\xi' - 2\eta), \qquad q(x) = \xi^{-2}(\eta^2 - \xi\eta'), \quad \xi = 0.$$

Elimination of η gives the relation

$$\tfrac{1}{4}(2\xi\xi'' - \xi'^2) + \xi^2 I(x) = 0, \quad I(x) = q(x) - \tfrac{1}{2}p'(x) - \tfrac{1}{4}p(x)^2. \qquad (38)$$

If $\xi = x^2\ and\, \eta = x$ are chosen, v_1 generates an inversional group and this equation simplifies to

$$y'' = x^{-3}H(r, w), \quad r = \frac{y}{x}, \quad w = xy' - y.$$

If we additionally ask the equation to be invariant under the scaling $(x, y) \to (\lambda x, \lambda^{\alpha} y), \lambda > 0$ generated by $v_2 = x\partial_x + \alpha y \partial_y$, the following condition on H should be imposed

$$(\alpha - 1) r H_r + a w H_w = (\alpha + 1) H.$$

Thus, if $\alpha \neq 1, H$ is restricted to

$$H = r^{(\alpha+1)/(\alpha-1)} \widehat{H}(\sigma), \quad \sigma = r^{\alpha/(1-\alpha)} w,$$

and to $H = w^2 \widehat{H}(r)\ if\ \alpha = 1.$

Choosing $\widehat{H} = K = const, \alpha = (n+1)/(n-1), n \neq 1$ we obtain the special form of the famous Emden- Fowler equation

$$y'' = K x^{-(n+3)} y^n, \quad n \neq 0,1, \quad K \neq 0 \qquad (39)$$

with a two- parameter symmetry group generated by the two-dimensional Lie algebra g of type $A_{2,3}$

$$v_1 = x^2 \partial_x + x y \partial_y, \quad v_2 = x\partial_x + \frac{n+1}{n-1} y \partial_y, \quad [v_1, v_2] = -v_1.$$

Theorem (2.1.5):

Guarantees that integration is completed using two quadratures by reductions in the order of v_1 (an ideal of g) first and then v_2. We remark that apart from the special case $y'' = K x^{-5} y^2$, which is obtained for $n = 2$ from (39), there are only two other values of the exponent m in the equation $y'' = K x^m y^2$ for which the symmetry algebra is a two-dimensional (rank-two) none- abelian one. They are $m = -15/7\ and\ m = -20/7$. For all values $m\ with\ m \notin \{-5, --15/7, -20/7\}$, the symmetry algebra is one-dimensional and generated by the scaling symmetry $v = x\partial_x - (m + 2) y \partial_y$.

The case $n = -3$ gives the Ermakov-Pinney equation $y'' = K y^{-3}$ with solution

$y = \pm\sqrt{A + 2Bx + Cx^2}, AC - B^2 = K,$ which arises in many applications. This equation admits the $sI(2, R)$ algebra as the symmetry algebra with the basis

$$w_1 = \partial_x, \quad w_2 = x\partial_x + \frac{1}{2}y\partial_y, \quad w_3 = x^2\partial_x + xy\partial_y.$$

Just as in the previous case, the choice $\xi = x^{2-k}, \eta = (1-k)x^{1-k}$ in v_1 of (37), combined with the scaling group generated by

$$v_2 = x\partial_x + \frac{\delta + 2}{1 - m}y\partial_y, \qquad m \neq 1$$

can be shown to produce the following integrable variant of the generalized Lane- Emden-Fowler equation

$$y'' + \frac{k}{x}y' = Kx^{\delta}y^m, \quad m \neq 0,1, \quad K \neq 0, \qquad (40)$$

if the condition $3 + \delta + m - k(m + 1) = 0$ is satisfied. The case when $\delta = 0$ $(m = (k - 3)/(1 - k))$ appeared in [8] as one of the reductions of a radially symmetric nonlinear porous- medium equation. A suitable basis of algebra in the case $k \neq 1,2$ is

$$\hat{v}_1 = (1 - k)v_1, \qquad \hat{v}_2 = (1 - k)^{-2}v_2 = \frac{1}{k - 1}x\partial_x + \frac{1}{k - 2}y\partial_y,$$

with commutation relation $[\hat{v}_1, \hat{v}_2] = \hat{v}_1(type\ A_{2,3})$. In terms of canonical (or normal) coordinates

$$r = \frac{x^{k-2}y^{(k-2)/(k-1)}}{k - 1}, \qquad s = \frac{x^{k-1}}{k - 1},$$

we find the standard form of the corresponding equation

$$rs''(r) = -K(k - 2)s'(r)^3 + (k - 1)^3 s'(r),$$

being invariant under the algebra $\{\partial_s, r\partial_r + s\partial_s\}$. On solving by two quadratures, implicit solution of the original equation in obtained.

Another particular case where $m = -3\ and\ \delta = -2k$ leads to an $sI(2, R)$ invariant equation. Symmetry vector fields for $k \neq 1$ are

$$v_1 = x^{2-k}\partial_x + (1-k)x^{1-k}y\partial_y,\ \ v_2 = x\partial_x + \frac{1-k}{2}y\partial_y,\ \ \ v_3 = x^k\partial_x,$$

and otherwise

$$v_1 = 2xlnx\partial_x + y\partial_y,\quad v_2 = x(lnx)^2\partial_x + ylnx\partial_y,\qquad v_3 = x\partial_x.$$

This equation reduces to the standard Ermakov-pinney equation $y''(t) = Ky^{-3}(t)$ by change of the independent variable, $t = x^{1-k}/(1-k), k \neq 1,$ and $t = lnx\ for\ k = 1.$

We can extract a similar integrable class from (37) by imposing invariance under a two-parameter symmetry group extended by $v_2 = \partial_x$. To do this we require the commutation relation $[v_2, v_1] = \mu v_1$ to hold, which implies that $\xi = e^{\mu x}, \eta = \alpha e^{\mu x}$ for some constant α and that Eq. (37) be autonomous. For this choice of ξ, η the right hand side of (37) is independent of x. The left hand side is also true if $\mu = \alpha(1-m)/2$ and $H = Kr^m, m \neq 1\ (\mu \neq 0),\ \ K = constant.$ The condition (38) is automatically satisfied. This gives us the integrable equation (a type of Emden-fowler equation known as force-free generalized Duffing oscillator)

$$y'' + py' + qy = Ky^m,\ p = -\frac{\alpha(m+3)}{2},$$

$$q = \frac{\alpha^2(m+1)}{2} \qquad (41)$$

with symmetry algebra generated by

$$v_1 = e^{\mu x}(\partial_x + \alpha y\partial_y),\ \ v_2 = \partial_x,\ \ \alpha = \frac{2\mu}{1-m} \qquad (42)$$

if α is eliminated between the coefficients p and q we find the integrability condition

$$q = \frac{2(m+1)}{(m+3)^2}p^2,\ \ m \neq -3 \qquad (43)$$

Under this condition, (41) passes the painlev'e test.

In terms of the canonical coordinates of v_1

$$\hat{x} = \mu^{-1}e^{-\mu x}, \quad \hat{y} = ye^{-\alpha x}, \quad \mu = \frac{m-1}{m+3}p, \quad \alpha = -\frac{2p}{m+3},$$

Eq. (41) is reduced to $d^2\hat{y}/d\hat{x}^2 = K\hat{y}^m$, which is invariant under the symmetry group generated by

$$\hat{v}_1 = \partial_{\hat{x}}, \quad \hat{v}_2 = \hat{x}\partial_{\hat{x}} + \frac{2}{1-m}\hat{y}\partial_{\hat{y}},$$

and can be integrated by two quadratures.

The travelling wave solutions of Fisher's (also called Kolmogorov-Petrovsky- Piscunov) equation satisfy the ODE[1]

$$y'' + cy' + y(1-y) = 0. \tag{44}$$

If we identify (44) with (41) we find $p = c, \ q = 1 \ and \ m = 2$ and the integrablity condition imposes the constraint on the wave speed: $c = \pm 5/\sqrt{6}$.

The same type of solutions for the Newell-Whitehead-Segel equation satisfy

$$y'' + cy' + y(1-y^2) = 0. \tag{45}$$

The integrability condition (43) for $p = c, \ q = 1 \ and \ m = 3$ then requires $c = \pm 3/\sqrt{2}$. Solutions are found in terms of Jacobi elliptic functions.

Surprisingly, the special case $m = 3$ of (41) turns up in seeking localized stationary solutions of the form $u(x,t) = e^{-i\lambda t}u(x)$ of the one-dimensional nonlinear $Schrödinger$ equation with inhomogeneous nonlinearity

$$iu_t + u_{xx} = V(x)u + g(x)|u|^2u, \quad x \in R,$$

where $V(x)$ is an external potential and $g(x)$ describes the spatial modulation of the nonlinearity (see for example [2]). For some special choice of pairs (V,g), dictated by the presence of a Lie point symmetry, $u(x)$ satisfies

$$u'' + 2Cu^1 + Eu = g_0 u^3.$$

This equation is integrble if $E = 8/9C^2$.

In Eqs. (44) and (45), there is no loss of generality in assuming $c > 0$, because using discrete transformation $x \to -x$, we can put $c \to -c$.

The case $m = 3$ $(\mu = -\alpha \neq 0\ arbitrary)$ of (41) is also known as the usual Duffing oscillator and under the condition $q = (2/9)\,p^2$ its exact solutions can be found in terms of Jacobi elliptic functions from integrating the first integral (energy)

$$I = \frac{1}{2}\hat{y}'^2 - \frac{K}{4}\hat{y}^4 = \frac{1}{4}e^{-4\alpha x}[2y'^2 - 4\alpha y y' + 2\alpha^2 y^2 - K y^4].$$

On the other hand, the case $m = -3$ $(p = 0, q = -\alpha^2,\ \mu = 2\alpha)$ is recognized to be the celebrated Ermakov-Pinny equation

$$y'' - \alpha^2 y = K y^{-3}. \qquad (46)$$

Its symmetry algebra (42) is extended by one additional element

$$v_3 = e^{-2\alpha x}(\partial_x - \alpha y \partial_y),$$

or in coordinates $(\hat{x}, \hat{y})(up\ to\ mutiple\ of\ -4\alpha^2\ )\ by$ the projective element

$$\hat{v}_3 = \hat{x}^2 \partial_{\hat{x}} + \hat{x}\hat{y}\partial_{\hat{y}}.$$

It is isomorphic to the $sI(2, R)$ algebra. The general solution depending on two independent arbitrary constants is gives by

$$y^2 = Ae^{2\alpha x} + B + Ce^{-2\alpha x}, \quad (4AC - B^2)\alpha^2 = K.$$

In order obtain another interesting subclass integrable by quadratures, we now let Eq. (37) be invariant under the scaling transformation generated by $v_2 = y\partial_y$, equivalently choosing $H(r, w) = Aw + Br$, where A,B are arbitrary constants, then we obtain the variable coefficient linear invariant equation

$$y'' + \hat{p}(x)y' + \hat{p}(x)y = 0, \qquad (47)$$

$$\hat{p}(x) = \xi^{-1}(\xi' - 2\eta - A), \quad \hat{q}(x) = \xi^{-2}(\eta^2 - \xi\eta' + A\eta - B). \qquad (48)$$

Eliminating η results in the relation

$$(2\xi\xi'' - \xi'^2) + 4\xi^2 I(x) = -(A^2 + AB) = -D^2, \qquad (49)$$

where $I(x)$ is the (semi)-invariant of (47), namely

$$I(x) = \hat{q}(x) - \frac{1}{2}\hat{p}'(x) - \frac{1}{4}\hat{p}(x)^2.$$

Eq. (49) is related to the Ermakov-Pinney equation

$$\chi'' + I(x)\chi = -\frac{D^2}{4}\chi^{-3}$$

by the transformation $\xi = \chi^2$ and to the linear equation

$$\xi''' + 4I\xi' + 2I'\xi = 0$$

by differentiation.

Eq. (47) admits a symmetry group isomorphic to the $SL(3, R)$ group (the projective group of the plane (x, y), preserving the straight lines in the (x, y) plane as the symmetry group and can be integrated by quadratures using the two-parameter abelian subgroup generated by $\{v_1, v_2\}$. Passing to the canonical coordinates r and s to the constant coefficient linear equation

$$r''(s) - Ar'(s) - Br(s) = 0, \qquad (50)$$

preserving the homogeneity property in r (invariance under the scaling $r\partial_r$).

In the special case of the inversional group $((\xi = x^2, \eta = x)$, the corresponding invariant ODE becomes

$$y'' - \frac{A}{x^2}y' + \frac{Ax - B}{x^4}y = 0. \qquad (51)$$

Its normal form easy to obtain from the relation (49) as

$$v'' + I(x)v = 0,$$

$$I(x) = -\frac{D^2}{4x^4}, \;\; y = exp\left\{-\frac{A}{2x}\right\} v. \qquad (52)$$

Then, we have $r = y/x$, $s = -1/x$, and we can easily solve (50) to find the general solution of (51)

$$y(x) = x\left[c_1 exp\left(-\frac{\lambda_+}{x}\right) + c_2 exp\left(-\frac{\lambda -}{x}\right)\right], \qquad (53)$$

where $\lambda_\pm = (A \pm D)/2$, $D^2 = A^2 + 4B > 0$ are the real roots of the characteristic equation $\lambda^2 - A\lambda - B = 0$. If the discriminant $D^2$is zero or negative, the solution should be modified appropriately.

Alternatively, we can use the differential invariant approach. By means of the differential invariant $z = {}^{y'}\!/_{y} = {}^{v'}\!/_{v} \;\; of \; v_2^{(1)} = y\partial_y + y'\partial_{y'}$ we can express (52) as a Riccati equation

$$\frac{dz}{dx} + z^2 = \frac{D^2}{4x^4},$$

which inherits the symmetry $\hat{v}_1 = x^2\partial_x + (1 - 2xz)\partial_z$. Using the canonical coordinates $\rho = x(xz - 1), s = {}^{-1}\!/_{x}$ satisfying $\hat{v}_1(\rho) = 0, \hat{v}_1(s) = 1$, it can be written as a separable equation, invariant under the group of transformations $(\rho, s) \to (\rho, s + \varepsilon)$,

$$\frac{d\rho}{ds} = \frac{D^2}{4} - \rho^2$$

with solution $\rho(s) = {}^{D}\!/_{2} \, tanh\left[\frac{D}{2}(s + c_0)\right]$. Finally, from the relation, by integration

$$z = \frac{y'}{y} = \frac{1}{x} + \frac{\rho}{x^2}$$

we recover the general solution (53), after some manipulation with the arbitrary constants.

The following equation arises in the integrability analysis of the variable coefficient Basener-Ross model [9]

$$y''(x) + I(x)y = 0,\ \ I(x) = -\frac{e^{2x} + 4\tau e^{x} + \tau^2}{4(\tau + 2e^{x})^2}, \qquad (54)$$

where τ is a certain constant. The relation (49) for the choice $A = 1, B = -3/16$ ($D^2 = A^2 + 4B = 1/4$) and

$$\xi = \frac{1}{2}(2 + \tau e^{-x})$$

gives precisely $I(x)$ as above. One can check that

$$v_1 + \xi(x)\left(\partial_x - \frac{1}{2}y\partial_y\right)$$

generates one-parameter symmetry group of (54). So we have $(x) = -\xi(x)/2$,
$v(x) = e^{x/2},\ s(x) = ln(\tau + 2e^{x}),\ and\ r(x) = v(x)y.$ In canonical coordinates $(s, r), r$ satisfies the constant coefficient equation (ODE) (50)

$$r''(s) - r'(s) + \frac{3}{16}r(s) = 0.$$

Solving this equation and changing to (x, y) coordinates we obtain the general solution

$$y(x) = e^{-x/2}\left[c_1(\tau + 2e^{x})^{1/4} + c_2(\tau + 2e^{x})^{3/4}\right].$$

In general, gives $\xi(x), A, B,$ one can solve (49) for $I(x)$ and thus construct an invariant equation of the form (47) with symmetry $v = \xi\partial_x + \eta y\partial_y$.
$\eta(x)$ is found from solving $\hat{p}(x) = 0\ as\, \eta(x) = (\xi' - A)/2$. If $\eta(x)$ is substituted into $\hat{q}(x)$ of (48) it follows that $\hat{q} = I(x)$ as expected. Solution is readily obtained by transforming into the constant coefficient

linear equation (50) by the transformation $y(x) = \sqrt{\xi} exp[-(A/2)s]r(s)$.

The knowledge of invariance of an $n-th$ order ODE under an r-parameter symmetry group can be useful in reducing in order more than once. But the full reduction to an equation of order $n-r$ can only be guaranteed if the symmetry group is solvable.

A Lie algebra g is solvable if the derived series defined recursively by the chain of sub algebras

$$g = g^{(1)} \supseteq g^{(1)} \ldots \supseteq g^{(k)} \supseteq \cdots g^{(k)}$$
$$= \left[g^{(k-1)}, g^{(k-1)}\right] \qquad (55)$$

terminates, namely there exists $k \in N$ such that $g^{(k)} = 0$. The algebra of commutators $D_g = g^{(1)} = [g, g]$ is called the derived algebra. For the solvable algebra $D^k g = \{0\}$ holds.

Theorem (2.1.6):

Let $E(x, y^{(n)}) = 0$ be an n-th order ODE. If $E(x, y^{(n)}) = 0$ admits a solvable r-parameter group of symmetries G such that, then the general solution of the equation can be found by quadratures from the general solution of an (n-r) –th order reduced ODE E/G. In particular, if the ODE admits a solvable n-parameter group of symmetries, then the general solutions can be found by quadratures alone.

A solvable three-dimensional Lie algebra g always contains a two-dimensional abelian ideal, which is unique up to conjugacy under inner automorphisms unless g is abelian or nilpotent. An integration strategy for an ODE with a three-parameter solvable symmetry group is to first reduce the equation by this two-dimensional ideal and then to use the remaining symmetry to complete the integration by quadratures.

Example (2.1.3):

We turn to Eq. (26). As this equation admits a three-dimensional solvable algebra $g = e(2)$ as the symmetry algebra we can integrate it by three consecutive quadratures. The derived series of g is

$$g \supset D_g = \{v_1, v_2\}, \qquad D^2 g = g^{(2)} = \{0\}.$$

The third order differential invariants of the ideal $\{v_1, v_2\}$ $are\ z = y_1$, $\rho = y_2, \rho'(z) = d\rho/dz = z_2/z_1$, in the terms of which Eq.(26) reduces to the first order ODE

$$(1 + z^2)\rho'(z) - 3z\rho = \rho H(k), \qquad k = (1 - z^2)^{-3/2}\rho. \qquad (56)$$

This equation should retain the final (inherited) symmetry v_3, which, in terms of z, ρ, has the reduced from (from restriction of $pr^{(2)} v_3\ to\ z, \rho$ coordinates)

$$\hat{v}_3 = (1 + z^2)\partial_x + 3z\rho\partial_\rho$$

the coordinates $k = (1 + z^2)^{-3/2}\rho, \chi = aretan\, z$ retifies the vector field $\hat{v}_3 = \partial_z$. In terms of k, χ,(56) becomes a separable equation

$$\frac{dk}{d\chi} = kH(k) \qquad (57)$$

with implicit solution $\hat{H}(k) = \chi + c_1 or\ solving\ for\ \rho = y_2 = G(y_1, c_1)$, which is invariant under the translational group $\{v_1, v_2\}$ and can be integrated by two further quadratures.

The special case $H = \lambda = const. leads\ to$ $(1 + y_1^2)y_3 = (3y_1 + \lambda)y_2^2$ with the additional symmetry $v_4 = x\partial_x + y\partial_y$. Form (57), the solution of reduced equation is $k = c_1 e^{\lambda x}$, and with the original variables, it is the second order ODE $(1 + y_1^2)^{-3/2} y_2 = c_1 exp\{\lambda arctan y_1\}$.

For $\lambda = 0$, the solutions are curves with the constant curvature $k = c_1, i.e.$

The family of circles with radius c_1^{-1}: $(x-c_2)^2+(y-c_3)^2=c_1^{-2}$. In this case the equation admits two further additional symmetries $v_5=(x^2-y^2)\partial_x+2xy\partial_y$ and $v_6=2xy\partial_x+(y^2-x^2)\partial_y$. The maximal symmetry algebra of the equation is the six-dimensional Lorentz algebra $so(3,1)$.

When $\lambda\neq 0$, the corresponding second order ODE can be integrated using two-parameter translational group. More conveniently, a parametric solution can be produced by the introduction of the parametrization $\tau=arctan y_1$ in the form

$$x(\tau)=c_1e^{-\lambda\tau}(\lambda cos\tau-sin\tau)+c_2, y(\tau)=c_1e^{-\lambda\tau}(\lambda sin\tau+cos\tau)+c_3.$$

To see what happens when the symmetry group is not solvable we consider the following example of a third-order ODE, known as the Schwarzian equation,

$$\frac{y_3}{y_1}-\frac{3}{2}\left(\frac{y_2}{y_1}\right)^2=F(x), \tag{58}$$

where F is any function of its argument, admitting the nonsolvable symmetry group $SL(2,R)$, with Lie algebra having the basis

$$v_1=\partial_y,\quad v_2=y\partial_y,\quad v_3=y^2\partial_y \tag{59}$$

The corresponding Lie group $SL(2,R)$ is the group of linear fractional transformations

$$(x,y)\to\left(x,\frac{ay+b}{cy+d}\right),\quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,R).$$

The expression on the left-hand side of (58) is called the Schwarzian derivative of y with respect to x and is denoted by the symbol $\{y,x\}$. It is invariant under the $Möbius$ transformation in y: $\{(ay+b)/(cy+d);x\}=\{y,x\}, ad-bc=1$(a unique differential invariant of order≤ 3 of the algebra (59)). A two- dimensional solvable subalgebra g_0 generated by $\{v_1, v_2\}$ can be used to reduce the equation to one of first order. The second-order differential invariants of g_0 $are\ x\ and\ w = y_2/y_1$, in terms of which the reduced equation becomes

$$\frac{dw}{dx} = \frac{1}{2}w^2 + F(x),$$

which is recognized as a Riccati equation. This equation does not inherit the symmetry v_3 of the original equation. Indeed, the reduced vector field $\hat{v}_3$ in terms of the invariants $x\ and\ w$ nonlocal (the so-called exponential vector field)

$$\hat{v}_3 = 2e^{\int w dx}\partial_w.$$

On the other hand, the well-known Hopf-cole transformation $w = -2\frac{\theta'(x)}{\theta(x)}$

($y_1 = y' = \theta^{-2}$) linearizes to

$$\theta'' + \frac{F(x)}{2}\theta = 0. \quad (60)$$

Let $\theta\ and\ \psi$ be two independent solutions of (60). Then, we have $\psi/\theta = W\int\theta^{-2}\,dx + c$, where W is the (constant) Wronskian of θ and ψ, (we can put $W = 1$ by scaling ψ) and c is an arbitrary constant, that can be absorbed to ψ so that we can put c=0 without loss of generality. The solution y can now be expressed as a ratio $y = \psi/\theta$ of two linearly independent solutions to (60).

When $F(x) = 0$ (known as the Kummer-Schwarz equation; this equation is also encountered in the study of geodesic curves in spaces of constant curvature), the symmetry algebra g becomes $six - dimensional$ and has a direct-sum structure

$g = sI(2, R) \oplus sI(2, R)$ spanned by the vector fields (59) and

$$v_4 = \partial_x, \;\; v_5 = x\partial_x, \;\;\; v_6 = x^2\partial_x. \qquad (61)$$

The symmetry group is then a linear functional group of both $x \; and \; y$ coordinates. The corresponding solution is the linear fractional (or $M\ddot{o}bius$) transformation in x: $y = (ax + b)/(c\,x + d), \;\; ad - bc \neq 0$.

(2-2) Integration of Ordinary Differential Equations:

Lie made the remarkable observation that virtually all the classical methods for solving specific types of ordinary differential equations (separable, homogeneous, exact, etc.) are particular examples of a general method for integrating ordinary differential equations that admit a group of symmetries. In particular, knowledge of a one-parameter group of symmetries of an ordinary differential equation allows us to reduce its order by one. Before beginning, though, we must remark that the method cannot be used to find every solution to the equation.

Definition (2.2.1):

Let v be a vector field on the space of independent and dependent variables. A function $u = f(x)$ is called nontangential provided v is nowhere tangent to the graph of f.

Theorem (2.2.1):

Let $\Delta\big(x, u^{(n)}\big) = 0$ be an n^{th} order scalar ordinary differential equation admitting a regular one-parameter symmetry group G. Then all nontangential solutions can be found by quadrature from the solutions to an ordinary differential equation $(\Delta/G)\big(x, u^{(n-1)}\big) = 0 \; of \; order \; n - 1$, called the symmetry reduced equation.

Remark (2.2.1):

Note that a solution is everywhere tangential if and only if it is invariant under G, so the method will not, in particular, produce invariant solutions. However, in the scalar case, the graph of an invariant function must locally coincide with a one-dimensional orbit of the group, and hence the invariant solutions can be determined by inspection, for applications to envelopes and separatrices.

Proof: Let us introduce rectifying coordinates $y = \chi(x, u), v = \psi(x, u)$, in terms of which the infinitesimal generator of G is the vertical translation field ∂_u. If v is not tangent to the graph Γ of a solution $u = f(x)$, then, in terms of the new y, v coordinates, Γ remains transverse, and therefore will locally coincide with the graph of a smooth function $v = h(y)$. In the new coordinates, the group transformations, and their prolongations, are simply given by translation $v \mapsto v + \varepsilon$ in the v coordinate alone, the derivative coordinate remaining fixed. (the infinitesimal form is $v^{(n)} = \partial_v, n \geq 0$.) therefore the equation variety $s_\Delta = \{\Delta(y, v^{(n)}) = 0\} \subset J^n$ is invariant if and only if it dose not depend on the variable v, and hence the equation is equivalent to one that dose not depend explicitly on v itself (although it dose depend on the derivatives of v with respect to y). Therefore, replacing $v\, by\, w = v_y$ reduces the equation to one of order $n - 1\, for\, w = w(y)$, moreover, we recover the solution to our original equation by quadrature: $v = f w(y) dy$.

In particular, a first order equation $u_x = P(x, u)$ admitting a one-parameter symmetry group can be solved by quadrature. However, the symmetry must be nontangential ; the trivial symmetries $v = \xi(x, u)(\partial_x + P(x, u)\partial_u)$ are everywhere tangential to solutions; moreover, the characteristic method for finding the rectifying coordinates of such a vector field is the essentially same problem of determining the must general symmetry group of the first order equation is some complicated than solving the equation itself, so we can only successfully apply Lie's method if, by inspection (perhaps motivated by geometry or physics), we can detect a relatively simple symmetry group.

Example (2.2.1):

A classical example is provided by the homogeneous equation;

$$\frac{du}{dx} = x^{m-1}F\left(\frac{u}{x^m}\right),$$

which admits the scaling group $(x,u) \to (\lambda x, \lambda^m u)$ with infinitesimal generator $x\partial_x + mu\partial_x. For\ x \neq 0,$ rectifying coordinates are given by $y = u/x^m\ , v = logx,$ in terms of which the equation reduces to

$$\frac{dv}{dy} = \frac{1}{F(y) - my}.$$

This can clearly be integrated, $v = h(y) = \int dy/\{F(y) - my\}$, thereby defining u implicitly: $Logx = h(u/x^m)$. (The reader might enjoy comparing this method with the one taught. In elementary ordinary differential equation texts for solving homogeneous equations).

In this case, the nontangentiality condition requires that $du/dx \neq mu/x$, meaning that $F(y) \neq my$, and this method (and the standard one) break down at such singularities. In particular, the scale-invariant function $u = cx^m$, which will be a solution provided $F(c) = mc$, cannot be recovered by this approach, and constitutes a singular solution to the equation.

Example (2.2.2):

Consider the second order ordinary differential equation

$$uu_{xx} = \alpha u_x^2,$$

where α is a nonzero constant. The equation admits three obvious symmetries: a translation $x \to x + \varepsilon$ in the independent variable, reflecting the fact that the equation is autonomous, and two independent scaling transformations $(x,u) \to (\lambda x, \mu u)$. To reduce with respect to the translation group, we set $y = u, v = x, so\ that\ u_x = 1/v_y, u_{xx} =$

$-v_{yy}/v_y^3$, and the equation reduces to a linear equation $yv_{yy} + \alpha v_y = 0$, with solution $v = cy^{1-\alpha} + d \; for \; \alpha \neq 1$, or $v = clogy + d \; for \; \alpha = 1$. The corresponding solution of above equation is then $u = (ax + b)^{1/(1-\alpha)}, or \; u = exp(ax + b)$. (Note that the translationally invariant solutions $u = constant$ are recovered as limiting cases of these solutions.)

Alternatively, if we use the scaling symmetry in u, then the appropriate coordinates are $x \; and \; v = logu$, in terms of which the equation becomes $v_{yy} = (\alpha - 1)v_y^2$, which reduces to a homogeneous (separable) equation for $\omega = v_y$. The reader may enjoy seeing what happens if we reduce with respect to the other scaling symmetry $x \to \lambda x$ instead.

Example (2.2.3):

Finally, consider a general homogeneous second order linear equation

$$u_{xx} + f(x)u_x + g(x)u = 0.$$

This clearly admits the scaling symmetry generated by $u\partial_u$. According to the general reduction procedure, as long as $u \neq 0$, we can introduce the new variable $v = logu$, in terms of which the equation becomes $v_{xx} + v_x^2 + f(x)v_x + g(x) = 0$, which is a first order Riccati equation for $w = v_x = u_x/u$. We have thus recovered the well-known correspondence between second order linear equations and first order Riccati equations.

If a higher order equation admits several symmetries, then it may be reducible in order more than once. However, unless the symmetry group has additional structure, we may not be able to make a full reduction since the reduced equation. (On the other hand, they may admit additional symmetries not shared by the original system!) that is full details of the reduction techniques available.

Theorem (2.2.2):

Suppose $\Delta = 0$ is an n^{th} order ordinary differential equation admitting a symmetry group G. Let $H \subset G$ be a one-parameter subgroup. Then the H-reduced equation $\Delta/H = 0$ admits the quotient group $G_H/H, where\ G_H = \{g/g\,.H.\,g^{-1} \subset H\}$ is the normalize subgroup, as a symmetry group.

Proof: Let $y = \chi(x,u), v = \psi(x,u)$ be the rectifying coordinates for the infinitesimal generator $v = \partial_x$ of the subgroup H. The original differential equation reduces to an $(n-1)^{st}$ order equation for $w = v_y = w(x,u,u_x)$. Consider an infinitesimal symmetry $w \in g$ of the original equation, which we re-express in terms of the rectifying coordinates: $w = \eta(y,v)\partial_y + \zeta(y,v)\partial_v$. Clearly, the vector field w will induce a point symmetry of the reduced equation if and only if its first prolongation $w^{(1)} = \eta\partial_y + \zeta\partial_v + \zeta^y\partial_{v_y}$ can be reduced to a local vector field $\widetilde{w} = \eta(y,w)\partial_y + \theta(y,w)\partial_w$ depending on just y and w. This will happen if and only if $\eta\ and\ \zeta^y$ are independent of v. This occurs if and only if $\zeta = \alpha(y) + cv$ for c constant. Since $v = \partial_v$, this condition is equivalent to the requirement that $[v,w] = cv$. That $w \in g_H$, the subalgebra of g corresponding to the normalize subgroup G_H, thereby completing the proof.

Example (2.2.4):

Consider a second order equation of the form $x^2u_{xx} = F(xu_x - u)$, which admits the two-parameter symmetry group $(x,u) \mapsto (\lambda x, \mathrm{u} + \mu\mathrm{x})$ with infinitesimal generators $v = x\partial_x, w = x\partial_u$. Since $[v,w] = w$, if we reduce with respect to w, then the resulting first order equation will retain a symmetry corresponding to v, and hence theorem 2-8 guarantees that it can be integrated. In this case, we set $v = u/x\,, w = v_x = x^{-2}(xu_x - u)$, so that reduces to $x^3w_x = F(x^2w) + 2x^2w$. This equation admits a scaling symmetry generated by reduced vector field $\tilde{v} = x\partial_x - 2w\partial_w$, which means that it is of homogeneous can be integrated. On the other hand, if we try to reduce the original equation with respect to v initially, using variables $\tilde{y} = u, \tilde{v} = logx$, reduces to a first order equation $w_y = -w[1 + F(w^{-1} - y)]$ having no obvious symmetry. It is worth remarking that the latter equation can be solved-just reverse the procedure so as to replace it by the original second order equation, and use the first reduction method.

An r-dimensional Lie group G is called solvable if there exists a sequence of subgroups $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{r-1} \subset G_r = G$ such that each G_i is a normal subgroups of G_{i+1}. This is equivalent to the requirement that the corresponding subalgebras of g satisfy $[g_i, g_{i+1}] \subset g_i$. The ordinary differential equation admits a (sufficiently regular) r-dimensional solvable symmetry group, then its solutions can be determined, by quadrature, from those to a reduced equation of order $n - r$.

Finally, it is worth reiterating the fact that not every integration method for ordinary differential equations is based on symmetry. Indeed, the equation appearing in provides a simple example of an equation with no symmetries, but which can, nevertheless, be explicitly solved.

(2-3) Characterization of Invariant Differential Equations:

One of the most important uses of differential invariants is the construction of general systems of differential equations (and variational problems) which admits a prescribed symmetry group. This is especially important in modern physical theories, where one begins by postulating the basic “symmetry group” of the theory, and then determines which field equations are admissible. The basic result that allows us to immediately write down the most general system of differential equations which is invariant under a prescribed transformation group is direct corollary of characterizing invariant systems of algebraic equations. Note that this result is valid for both ordinary and partial differential equations.

Theorem (2.3.1):

Let G be a Lie group acting on E. Assume that the n^{th} prolongation of G acts regularly and has a complete set of functionally independent n^{th} order differential invariants $I_1, \dots, I_k$ on an open subset $W^{(n)} \subset J^n$. A system of n^{th} order differential equations admits G as a symmetry group if and only if, on $W^{(n)}$, it can be rewritten in terms of the differential invariants:

$$\Delta_v\left(x, u^{(n)}\right) = F_v\left(I_1\left(x, u^{(n)}\right)\right), \dots, I_k\left(x, u^{(n)}\right)) = 0, \qquad v = 1, \dots, l.$$

Example (2.3.1):

Suppose we have just one independent variable and one dependent variable, and consider the usual rotation group $SO(2)$ acting on $E = R \times R$. There are two first order differential invariants-the radius $r = \sqrt{x^2 + u^2}$ and the angular invariant $w = (xu_x - u)/(x + uu_x)$. Therefore, the most general first order ordinary differential equation admitting $SO(2)$ as a symmetry group can be written in the form $F(r, w) = 0$. Solving for w, we deduce that the equation has the explicit form

$$\frac{xu_x - u}{x + uu_x} = H\left(\sqrt{x^2 + u^2}\right) \quad or \quad u_x = \frac{u + xH\left(\sqrt{x^2 + u^2}\right)}{x - uH\left(\sqrt{x^2 + u^2}\right)}.$$

In terms of the polar coordinates $r, \theta,$ in above equation takes the separable form $\theta_r = r^{-1}H(r)$ and can thus be integrated.

The most general second order differential equation admitting a rotational symmetry group can be written in the form $F(r, w, k) = 0, where\ k = (1 + u_x^2)^{-3/2}u_{xx}$ is the curvature. Solving for u_{xx}, we find

$$u_{xx} = (1 + u_x^2)^{3/2} H\left(\sqrt{x^2 + u^2}, \frac{xu_x - u}{x + uu_x}\right).$$

This second order equation can also be rewritten in terms of polar coordinates:

$$r\theta_{rr} = (1 + r^2\theta_r^2)H(r, r\theta_r) - r^2\theta_r^3 - 2\theta_r.$$

Since the latter equation dose not explicitly depend on θ, it can be reduced to a first order equation by introducing the variable $v = d\theta/dr$.

Example (2.3.2):

Let $E = R^2 \times R$, and consider the action of the rotation group $SO(2)$ acting on the independent variables x, y only. Every first order $SO(2) - invariant$ partial differential equation has the form $H\big(xu_x + yu_y, yu_x - xu_y, u, r\big) = 0$. Similarly, every second order $SO(2) - invariant$ partial differential equation can be written in terms of the second (and lower) order differential invariants

$$U = x^2u_{xx} + 2xyu_{xy} + y^2u_{yy},$$

$$V = -xyu_{xx} + (x^2 - y^2)u_{xy} + xyu_{yy},$$

$$W = y^2u_{xx} - 2xyu_{xy} + x^2u_{yy}.$$

In particular, the rotational invariance of the Helmholtz equation follows from the identity

$$\Delta u + \lambda u = u_{xx} + u_{yy} + \lambda u = r^{-2}(U + W) + \lambda u.$$

The construction theorem (2.9) suggests an alternative reduction method for ordinary differential equations invariant under a symmetry group. Under the assumptions of theorem 2.9, we can rewrite any n^{th} order ordinary differential equation admitting and r parameter symmetry group in the form

$$F\left(y, w, \frac{dw}{dy}, \dots, \frac{d^{n-r}w}{dy^{n-r}}\right) = 0,$$

involving only the two fundamental differential invariants: $y = I\big(x, u^{(s)}\big), w = J\big(x, u^{(r)}\big)$, which have orders $s < r = dim\, G, with\, s = r - 1\, unless\, G$ is either intransitive, in which case $s = 0$, or pseudostabilizes, in which case $s = r - 2$. Therefore, we have reduced the original n^{th} order differential equation to an $(n-r)^{th}$ order differential equation in the differential invariants. However, once we have solved above equation for $w = h(y)$, we then must solve an auxiliary r^{th} order differential equation

$$J\left(x, u^{(r)}\right) = h\left[I\left(x, u^{(s)}\right)\right],$$

in order to recover u as a function of x, only involves the first differential invariant y, the Lie reduction method discussed above can be applied, although, unless the Lie group is solvable, we will not in general be able to integrate it by quadrature alone.

A particularly interesting class of applications is provided by the three inequivalent planar actions of the special linear group $SL(2)$. All three actions are related via a process of prolongation and projection; this provides a ready means of simultaneously classifying and reducing the corresponding invariant ordinary differential equation admitting the symmetry group generated by $\partial_x, x\partial_x, x^2\partial_x$ can be written in the form

$$\frac{d^{n-3}s}{du^{n-3}} = H\left(u, s, \frac{ds}{du}, \dots, \frac{d^{n-4}s}{du^{n-4}}\right),$$

where

$$u \ \ and \ \ s = \frac{u_x u_{xxx} - \frac{3}{2}u_{xx}^2}{u_x^4},$$

are the two fundamental differential invariants. Once we know the solution $s = F(u)$ by a pair of quadratures. The function $\psi(u) = \sqrt{u_x}$ is a solution to the second order, homogeneous, linear $Schrödinger$ equation

$$\frac{d^2\psi}{du^2} - \frac{1}{2}F(u)\psi = 0.$$

We can recover $u(x)$ by a single quadrature:

$$\int^{u} \frac{d\tilde{u}}{\psi(\tilde{u})^2} = x + k.$$

Note that, according to the method of variation of parameters, if $\psi(u)$ is one solution to the linear ordinary differential equation, then a second, linearly independent solution, is given by

$$\psi(u) = \psi(u) \int^{u} \frac{d\tilde{u}}{\psi(\tilde{u})^2}.$$

Comparing with

$$\int^{u} \frac{d\tilde{u}}{\psi(\tilde{u})^2} = x + k.$$

And absorbing the integration constant, we conclude that the general solution to the invariant equation is given, parametrically, as a ratio $x = \varphi(u)/\psi(u)$ of two arbitrary linearly independent solutions the linear $Schrödinger$ equation. The differential invariant s can be identified with the negative of the Schwarzian derivative of $x = x(u)$; therefore, our symmetry reduction, provides a direct proof of a classical theorem due to Schwarz.

Theorem (2.3.2):

The general solution to the Schwarzian equation

$$\frac{x_u x_{uuu} - \frac{3}{2}x_{uu}^2}{x_u^2} = \tilde{F}(u),$$

has the form $x = \varphi(u)/\psi(u)$, where $\varphi(u)$ and $\psi(u)$ form two linearly independent, but otherwise arbitrary, solutions to the linear $Schrödinger$ equation $\psi_{uu} + \frac{1}{2}\tilde{F}(u)\psi = 0$. Alternatively, $w = x_{uu}/x_u$ is an arbitrary solution to the Riccati equation $w_u = \frac{1}{2}w^2 + \tilde{F}(u)$.

(2-4) Linearization of Partial Differential Equations:

Because of the preceding results, trivial linearizable ordinary differential equations are (as far as we know) uniquely characterized by the property that they admit a symmetry group of the maximal, finite dimension. Symmetry groups can also be used effectively to characterize linearizable systems of partial differential equations. The key remark is that a linear system of partial differential equations $D[u] = 0,$ where D is an n^{th} order linear differential operator, has (assuming the system is not over determined) an infinite-dimensional symmetry group since we can add any other solution to a given solution. The infinitesimal generators of the relevant infinite-dimensional symmetry group take the form

$$v_{\psi} = \sum_{\alpha=1}^{q} \psi^{\alpha}(x) \frac{\partial}{\partial u^{\alpha}},$$

where $\psi = (\psi^1(x), \dots, \psi^q(x))$ is any solution to the linear system $D[\psi] = 0.$ Note that the vector fields above commute, and hence generate an infinite-dimensional abelian symmetry group. Assuming the system $D[\psi] = 0$ is locally solvable, the operators span a q-dimensional subspace, namely the space of vertical tangent directions in TE. Any equivalent system of partial differential equations must also admit such an infinite-dimensional symmetry group, and hence any system of partial differential equations which has only a finite-dimensional symmetry group is certainly not linearizable (at least by a local transformation).

Theorem (2.4.1):

Let $\Delta(x, u^{(n)}) = 0$ be an n^{th} order system of q independent partial differential equations in $p \geq 2$ independent variables and q unknowns. If the system admits an infinite-dimensional abelian symmetry group, haveng q-dimensional orbits, and infinitesimal generators depending linearly on the general solution to an n^{th} order system of q independent linear partial differential equations $D[\psi] = 0$, then it can, by a change of variables, be mapped to an inhomogeneous form of the linear system $D[u] = f.$

Proof: Any abelian transformation group with q-dimensional orbits can, through a change of variables, be mapped to a group generated by vector fields of the last equation. The additional hypothesis implies that the coefficient functions $\psi(x)$ form the general solution to linear n^{th} order system of partial differential equations $D[\psi]=0$. Note that, in terms of the original coordinates, the generators take the form

$$v_{\psi}=\sum_{\alpha=1}^{q}\psi^{\alpha}\big(\eta^{1}(x,u),\dots,\eta^{p}(x,u)\big)w_{\alpha},$$

where the vector fields $w_1,\dots,w_q$ are linearly independent, commute, and satisfy $w_{\alpha}\left(\eta^{i}\right)=0\ for\ all\ \alpha,i$. Now, in the new coordinates, the system must be equivalent to the inhomogeneous form of the linear system. To see, it suffices to note that the only n^{th} order differential invariants of the infinite-dimensional group generated by the vector fields, are the components of $D[u]$ and the variables x. Therefore, any invariant system of differential equations must be isomorphic to one of the form $H(D[u],x)=0$. But, since the system consists of q independent equations, we can solve the system for the components of $D[u]$, placing the system into the desired inhomogeneous form.

Example (2.4.1):

The nonlinear diffusion equation $u_t=u_x^{-2}u_{xx}$ admits the following six symmetry generators:

$$\partial_t,\quad \partial_u,\quad x\partial_x,\quad 2t\partial_t+u\partial_u,$$
$$xu\partial_x-2t\partial_u,\quad (2xt+xu^2)\partial_x-4t^2\partial_t-4tu\partial_u,$$

as well as the infinite dimensional abelian subalgebra

$$\alpha(t,u)\partial_x,\qquad where\qquad \alpha_t=\alpha_{uu}.$$

The hodograph transformation $v=x, y=u$, to the (homogeneous) heat equation $v_t=v_{yy}$.

CHAPTER THREE

GROUP CLASSIFICATION OF INVARIANT SOLUTIONS OF DIFFERENTIAL EQUATIONS

Chapter three

Group Classification of Invariant Solutions of Differential Equations

(3-1) Group Classification Problem:

When a system involves an arbitrary parameter or arbitrary function, the symmetry group can be have richer symmetry for certain specific forms of these arbitrary terms. The problem of identifying such arbitrary parameters or functions is known as the group classification problem. When there are only parameters or arbitrary functions of a single argument of independent or dependent variable then the problem is easily tackled solving the determining system where splitting is almost always possible. The arbitrary functions are found by solving an ODE(the so-called classifying ODE). The situation becomes rather complicated when the system depends on arbitrary functions of more than one argument. In this case, the group classification problem is often solved by reducing it to the classification of realizations of low-dimensional abstract Lie algebras combined with the notion of equivalence group and the knowledge of abstract Lie theory.

Example (3.1.1):

A classification problem: we wish to determine all possible forms of $F(y)$ for which the following ODE allows a two-dimensional symmetry algebra:

$$y'' = \mu y' + F(y), \qquad \mu \neq 0, \;\; F'' \neq 0. \qquad (62)$$

Under the reflection $x \to -x$, we have $\mu \to -\mu > 0$ so we can assume $\mu > 0$.

The special nonlinearity $F(y) = ay^3 + by$ is recognized as Duffing's equation.

The trivial case $\mu = 0$ is clearly integrable. It has the energy first integral

$$\tfrac{1}{2}{y'}^2 - G(y) = constant, \quad F(y) = G'(y).$$

The symmetry classification for $\mu \neq 0$ naturally characterizes all possible integrable cases.Eq. (62) arises in obtaining travelling- wave solutions of the nonlinear heat (diffusion) equation

$$u_t = u_{xx} + F(u), \qquad F'' \neq 0. \tag{63}$$

These are solutions of the form $u(x,t) = y(z) = y(x - ct)$, being invariant under a combination of time and space translational symmetries. In what follows we take $z \equiv x$.

We can put $\mu = 1\ in$ scaling the independent variable $x \to \mu^{-1}x$ and redefining $\mu^{-2}F\ as\ F$. The equation is invariant under translation of x that $v_1 = \partial_x$ generates a symmetry. For certain functions F, the symmetry group will be two-dimensional. The general symmetry algebra is generated by vector fields of the form

$$v = \xi(x,y)\partial_x + \eta(x,y)\partial_y.$$

The second prolongation formula for vector field v is

$$pr^{(2)}v = \xi(\mathrm{x,y})\,\partial_\mathrm{x} + \eta(\mathrm{x,y})\,\partial_\mathrm{y} + \eta'(\mathrm{x,y,y'})\,\partial_{\mathrm{y'}} + \eta^2(\mathrm{x,y,y',y''})\,\partial_{\mathrm{y''}},$$

where

$$\eta' = D_x\eta - y'D_x\xi, \qquad \eta^2 = D_x\eta^1 - y''D_x\xi.$$

Higher order prolongation coefficients can be calculated from the recursion formula

$$\eta^k = D_x\eta^{k-1} - y_kD_x\xi, \qquad y_k = y^{(k)},$$

or in terms of the characteristic $Q(x,y,y') = \eta - y'\xi$ as

$$\eta^k = D_x^kQ + \xi y_{k+1}.$$

From the infinitesimal symmetry we find $\eta^2 - \eta^1 - \eta F' = 0$ on the solutions.

Replacing y'' by $y' + F(y)$ and setting the coefficients of the derivative y' equal to zero we obtain the determining system for the coefficients ξ, η

$$3F\xi_y + \xi_x - 2\eta_{xy} + \xi_{xx} = 0, \quad (64)$$

$$2\xi_y - \eta_{yy} + 2\xi_{xy} = 0, \quad (65)$$

$$\xi_{yy} = 0, \quad (66)$$

$$F'\eta - (\eta_y - 2\xi_x)F - \eta_{xx} + \eta_x = 0, \quad (67)$$

differentiating twice the first two equations with respect to y and using the third one and the condition $F'' \neq 0$ $(F\ is\ not\ linear)$ we find

$$\xi_y = 0, \quad \eta_{yy} = 0.$$

So we take $\xi = b(x)\ and\ \eta = c(x)y + d(x)$. The determining equations are then reduced to solving the classifying ODE

$$(cy + d)F' - (c - 2b')F = (c'' - c')y + d'' - d' \quad (68)$$

with the relation

$$2c' - (b' + b'') = 0, \quad (69)$$

from which we have

$$c = \frac{1}{2}(L + b' + b), \quad (70)$$

where L is a constant.

Now we shall require the equation under study be invariant under a two-dimensional symmetry algebra with commutation relations

$$[v_1, v_2] = kv_2$$

and $v_1 = \partial_x$. The case $k = 0$ is abelian.

We shall find all possible forms of X_2 that obey commuation relation and the corresponding F (not linear in y). The commutation relation implies that we must have

$$b' = kb, \qquad c' = kc, \quad d' = kd,$$

and then

$$b = b_0 e^{kx}, \quad c = c_0 e^{kx}, \quad d = d_0 e^{kx},$$

$$c'' - c' = c_0 k(k-1)e^{kx}, \qquad d'' - d' = d_0 k(k-1)e^{kx}.$$

The classifying ODE then becomes

$$(c_0 y + d_0)F' - (c_0 - 2kb_0)F = k(k-1)(c_0 y + d_0), \qquad (71)$$

where b_0, c_0, d_0 are constants. The relation (34) gives

$$k[2c_0 - (k+1)b_0] = 0.$$

If $k = 0$ then b, c and d are arbitrary constants and F should be linear in y. this means the symmetry algebra should be nonabelian.

If $k = -1 \; then \; c_0 = 0 \; and \; b_0 \; is \; arbitrary$. The ODE becomes now

$$F' - \beta F = 2, \quad d_0 = \frac{2}{\beta} b_0, \qquad \beta \neq 0$$

which integrates to $F(y) = \alpha e^{\beta y} - \frac{2}{\beta}$. The symmetry algebra spanned by

$$v_1 = \partial_x, \qquad v_2 = e^{-x}[\partial_x + \frac{2}{\beta}\partial_y] \qquad (72)$$

Leaves invariant

$$y'' = y' + \alpha e^{\beta y} - \frac{2}{\beta}. \qquad (73)$$

By scaling transformation $y \to {}^{\beta}/_{2y}$, followed by transformation $y \to y + ln\sqrt{\alpha}$ for $\alpha > 0$, we can put $\beta = 2, and\ \alpha = 0$, re spectively. Our respresentative equation simplifies to

$$y'' = y' + e^{2y} - 1 \qquad (74)$$

If $k = 1$ then we have $b_0 = c_0 \neq 0$ and

$$(y+\delta)F' + F = 0, \quad \delta = \frac{d_0}{b_0}.$$

The corresponding F and the symmetry algebra are given by

$$F = \alpha(y+\delta)^{-1}, \; v_1 = \partial_x, \;\; v_2 = e^x[\partial_x + (y+\delta)\partial_y].$$

Now we let $k \neq 0, -1$: If we assume $c_0 \neq 0$, from (34) we have

$$b_0 = \frac{2c_0}{k+1} \neq 0.$$

If we put $d_0 = b_0 \frac{\delta(k+1)}{2} = c_0\delta$, the ODE (36) can be written as

$$(y+\delta)F' - \gamma F = \frac{2(\gamma^2-1)}{(\gamma+3)^2}(y+\delta), \;\; \gamma = \frac{1-3k}{k+1} \qquad (75)$$

The solution of the ODE (40) is given by

$$F = M(y+\delta)^{\gamma} - \frac{2(\gamma+1)}{(\gamma+3)^2}(y+\delta), \qquad \gamma \neq 1, -3. \qquad (76)$$

The symmetry algebra is given by

$$v_1 = \partial_x, \;\; v_2 = e^{kx}[\partial_x + \frac{k+1}{2}(y+\delta)\partial_y], \; k = \frac{1-\gamma}{\gamma+3}. \qquad (77)$$

Note that the above result also contains the sub case $k = 1$ (but not $k = -1$).

By change of basis $v_1 \to -v_1,\ v_2 \to v_2\ and\ v_1 \to kv_1\ , v_2 \to v_2$ the commutation relation takes the standard form for a two-dimensional nonabelian algebra: $[v_1, v_2] = v_2$.

Is not changed by the linear transformation $\hat{y} = py + q$. The arbitrary coefficient F(y) gets changed into $\hat{F}(\hat{y}) = pf(\frac{\hat{y}-q}{p})$. By a suitable choice of p and q we can set $M = 1\ and\ \delta = 0$.

The two-dimensional symmetry algebra makes of the original equation possible to quadratures by methods of differential invariants or canonical coordinates. We consider the case $k \neq 0$ with two-parameter symmetry group generated by

$$v_1 = \partial_x,\ \ v_2 = e^{kx}\left[\partial_x + \frac{k+1}{2} y\partial_y\right].$$

The function F in the invariant is given by

$$F(y) = y^\gamma + \frac{(k^2 - 1}{4} y, \gamma$$
$$= \frac{1 - 3k}{k + 1} \qquad (78)$$

we start to reduce by the normal sub algebra v_2 which, in terms of the coordinates

$$s = k^{-1}e^{-kx},\ \ r$$
$$= ye^{-(k+1)x/2}, \qquad (79)$$

has the canonical form $\tilde{v}_2 = -\partial_s$. The transformed equation becomes

$$\frac{d^2r}{ds^2} = e^{\frac{(3k-1)}{2}x}\left[-\frac{k^2-1}{4} y + F(y)\right] =$$
$$r^\gamma, \qquad (80)$$

which retains the symmetry v_1. In terms of the invariants $r\ and\ \rho = {dr}/{ds}$ of v_1, it reduces to the first order ODE $\rho d\rho/dr = r^\gamma$ with solution

$\rho^2 = 2lnr + c_1\ if\ \gamma = -1;\ \rho^2 = 2(\gamma+1)^{-1}r^{\gamma+1} + c_1,$ otherwise. The general solution is obtained implicitly after a second quadrature from $(dr/ds)^2 = \rho^2(r)$. The alterative ay is to integrate using the integrating factor $I = r'(s)$. When $\gamma = 2,3$, the solutions are obtained in terms of Jacobi elliptic functions or weiestrass $\wp$ function, when $\gamma > 3$ in terms of hyperelliptic integrals.

The case $k = -1$ is treated similarly. The canonical coordinates $r = y - x,\ \ s = e^x$ maps Eq. (76) to $r''(s) = e^{2r}$ with symmetry $\{\partial_s, s\partial_s - \partial_r\}$. The corresponding solution is given by

$$y = x + ln\left\{\frac{2\sqrt{c_1}exp\{\sqrt{c_1}e^x\}}{1 - c_2^2 exp\{2\sqrt{C_1}e^x\}}\right\}, \qquad c_1 > 0$$

or

$$y = x + \tfrac{1}{2}ln\{-c_1 sec^2(\sqrt{-c_1}e^x + c_2)\}, \qquad c_1 < 0.$$

Remark (3.1):

The special case $\gamma = 2,\ k = -1/5\ and\ F(y) = y(y - 6/25)$ of (78) deserves a special attention, because this relates, up to a scaling of y, to the travelling – wave solutions of Fisher's equation (or nonlinear reaction-diffusion equation)

$$u_t = u_{xx} + au(1-u). \tag{81}$$

This equation originated in 1936 to model the propagation of a gene in population. Notice that under the change of variable $u \to 1 - u$, the parameter a changes sign. The corresponding travelling- wave solutions are then reduced to integrating the second order $ODE\ r''(s) = r^2$. The general solution of this equation is $r = \wp(s/\sqrt{6} + a; 0, g_3)$, where a, g_3 are arbitrary constants. Here

$\wp(s; g_2, g_3)$ is weierstrass $\wp$ function with invariants g_2 and g_3.

Changing to the original variables using $s = -5e^{\frac{x}{5}},\ r = ye^{-2x/5}$ from (44), for $g_3 \neq 0$, the solutions can be written as

$$y(x) = e^{2x/5}\wp\left(\frac{5}{\sqrt{6}}e^{x/5} + a; 0, g_3\right). \quad (82)$$

We note that $\wp$ is an even function since $g_2 = 0$ and g_3 is arbitrary. They are doubly periodic with an infinite number of poles on the real axis. The solutions of biological interest are are obtained for $g_3 = 0$. Using the fact that $\wp(s; 0,0) = s^{-2}$ we find

$$y(x) = \left[\frac{5}{\sqrt{6}} + ae^{-x/5}\right]^{-2}$$

satisfying the boundary conditions $\lim_{x\to\infty} y(x) = 6/25$ and $\lim_{x\to\infty} y(x) = 0$. In [1], it was shown that for the special wave speed $c = 5/\sqrt{6}$, for which the Fisher's equation passed the necessary condition to be of painleve type, can be reduced to the canonical form $r'' = 6r^2$ by a transformation of the both independent and dependent variables of the form $y(x) = f(x)r(s), s = h(x)$ with $f and$ h appropriately chosen.

Another context in which $\gamma = 2$ appears is the study of travelling solutions of the two-dimensional Kortweg-de Vries-Burgers ($\mu \neq 0$) and Kadomtsev-petviashvili ($\mu = 0$) equations for the special quadratic nonlinearity F(u) [6].

We recall that the case $\gamma = 3, k = -1/3$ cotrresponds to the travelling wave solutions for the Newell-Whitehead- segel equation. The rectifying transformation is $r = -3e^{x/3}, \qquad s = e^{-x/3}y$.

(3-2) Group-Invariant Solutions:

One of the main applications of the notation of symmetry group to PDEs is to construct group-invariant solutions. Suppose that G is a symmetry group of the system (11). A solution $u = f(x)$ is called group-invariant if $g.f = f \ for \ all$
$g \in G$. This means that a group-invariant solution does not change under the symmetry group transformations. For example, if G is the group of rotations in the space of independent variables x, then a solution invariant under G well be a function of the radius alone in the form $u = F(|x|)$. Travelling wave solutions are solutions invariant under the group of translations. Self-similar (or similarity) solutions that frequently arise in applications correspond to scaling symmetries.

Theorem (3.2.1):

Suppose that the symmetry group G acts on the space of independent and dependent variables $E = X \times U$ and sweeps out generic orbits of dimension d and of codimension $p + q - d$ (the number of functionally independent invariants of the group G). Then all the group-invariant solutions to $E = 0$ can be found by solving a reduced system of differential equations $E/G = 0$ in d fewer independent variables.

For example, if we have a system of partial differential equations in two independent variables, then the solutions invariant under a one-parameter symmetry group can all be found by integrating a system of ordinary differential equations.

Reduction in the number of independent variables will be possible if the orbit dimension d satisfies the inequality $p \leq d$. When $d = p,$ the reduced system $E/G = 0$ is a system of algebraic equations, while if $d > p$ there are no group-invariant solutions. In particular, if $d = p - 1$ we have a system of ODEs.

Let G be a local Lie group of transformations with infinitesimal generators $v_1, \ldots, v_r$, and the associated characteristics $Q_1, \ldots, Q_r$. Then a function is invariant under G if and only if it is a solution to the system of quasilinear first order partial differential equations characterizing the functions invariant under G

$$Q_{\alpha,l}(x, u^{(1)}) = 0, \qquad \alpha = 1,2, \ldots, q, \ \ l = 1,2, \ldots, r. \tag{83}$$

The group-invariant solution thus will satisfy both the original system together with the invariance constraints (83), which form an over determined system of PDEs. The method of system reduction consists of solving (83) in terms of invariant coordinates and substituting these into the original system. In the final stem all non-invariant coordinates will drop out of the resulting reduced system. What remains to derive group-invariant solutions is to solve the reduced system depending on fewer independent variables.

Given a solution invariant under a subgroup H of the full symmetry group G of a system, it can be transformed to other group-invariant solutions by elements $g \in G$ not in the subgroup H. Two group-invariant solutions are called inequivalent if one can not be transformed to the other by some group transformation $g \in G$. The corresponding reduced systems also have to be inquivalent. Two subgroups which are conjugate under the symmetry group G will produce equivalent reduced systems (systems connected with a transformation in the symmetry group).

Let $H \subset G$be a s-parameter subgroup. If $u = f(x)$ is solution invariant under H and $g \in G$ is any other group element, then the transformed function $u = \hat{f}(x) = g.f(x)$ is solution invariant under the conjugate subgroup $K_g(H) = g.H.g^{-1}$.

For example, the stationary solutions $u = f(x)$ of an evolution equation such as head equation and KdV (Korteweg-de Vries) equation invariant under Galilean group v_b can be conjugated by the Galilean boosts $e^{-cv_b}\partial_t e^{cv_b}$ to map to travelling wave solutions $u = f(x - ct)$ and vice versa.

So the collection of all group-invariant solutions are partitioned into equivalence classes. If g is the Lie algebra of G, we are interested in obtaining a representative list of subalgebras of g (also called an optimal system of subalgebras) such that every subalgebra of g is conjugated to precisely one algebra in the list and no two subalgebras in the list are conjugate. Two algebras $g\ and\ g'$ are conjugate under G if $g = Gg'G^{-1}$. Consequently, the problem of classifying group-invariant solutions is completely tantamount to finding the optimal system of subalgebras. When this task has been finished, it is sufficient to construct only group-invariant solutions corresponding to these representative subalgebra in the list. All other solutions can be obtained by applying the symmetry transformations to the representative classes of solutions.

For a finite dimensional Lie algebra of dimension≥ 2, this is in general a complicated problem. If g is a direct sum of two or more algebras, there is an algorithemic classification method, which is adaptation to Lie algebras of the Goursat’s method for direct products of discrete groups. When g is Levi de-composition of a semi-simple algebra and its radical (maximal solvable ideal).

In physical applications, one usually encounters with low-dimensional algebras as a symmetry algebras. In such cases, subalgebra classification problem is solved by using the adjoint transformations using the Baker-Compbell-Housedorff formula (or the Lie series)

$Ad(exp(\varepsilon v))w = exp(\varepsilon v)w\, exp(-\varepsilon v) = w + \varepsilon[v,w] + \frac{\varepsilon^2}{2!}\big[v,[v,w]\big] + \cdots$, (84)

where $v, w \in g$ and ε is the group parameter. This series terminates or can be easily summed up when the vector field v is nilpotent element or generates a translation, rotation or scaling group. The basic idea is to take a general vector field $v = span\,\{v_1, \dots, v_r\}\ in\, g$ and to simplify it as much as possible using adjoint transformations.

Example (3.2.1):

We give an example of one-dimensional subalgebras for the simple algebra $so(3, R)$ with commutation relations

$$[v_1, v_2] = v_3, \quad [v_3, v_1] = v_2, \quad [v_2, v_3] = v_1.$$

Let $v = a_1v_1 + a_2v_2 + a_3v_3 \in so(3, R). For\ a_3 \neq 0$, using the formula (84) and the above commulation relations and summing infinite series in ε we find

$$Ad(exp(\varepsilon v_2))v_1 = cos\varepsilon v_1 + sin\varepsilon v_3, \quad Ad(exp(\varepsilon v_2))v_3 = cos\varepsilon v_3 - sin\varepsilon v_1,$$

and from the linearity of the adjoint transformation

$$v' = Ad(exp(\varepsilon v_2))v = a_1'v_1 + a_2'v_2 + a_3'v_3,$$

where

$$a_1' = a_1cos\varepsilon - a_3sin\varepsilon, \quad a_2' = a_2, \quad a_3' = a_1sin\varepsilon + a_3cos\varepsilon.$$

Note that $a_1'^2 + a_2'^2 + a_3'^2 = a_1^2 + a_2^2 + a_3^2\ \ for\ any\ \varepsilon \in R$. This means that the positive-definite function $I(v) = a_1^2 + a_2^2 + a_3^2$ is invariant under the corresponding adjoint transformation. It can be seen that I is an invariant of the full adjoint action: $I(Ad_g(v)) = I(v), g \in SO(3, R)$, the Lie group of $so(3, R)$. If $I \neq 0$, we can choose ε such that the coefficient a_3' is zero. So we may assume that $a_3 = 0$ and $v = a_1v_1 + a_2v_2$. We further conjugate $v\ by\ v_3$ (or apply an inner automorphism of the algebra) and find

$v' = Ad(exp(\alpha v_2))v = (a_1 cos\alpha + a_2 sin\alpha)v_1 + (a_2 cos\alpha - a_1 sin\alpha)v_2.$

Now we can arrange for α to be such that $a_2 cos\alpha - a_1 sin\alpha = 0$ and hence v is conjugate to $a_1 v_1$ for $a_1 \neq 0$ or to v_1 by scaling. The optimal system contains only one vector field v_1.

Example (3.2.2):

We consider the diffusion equation with constant drift b

$$u_t = xu_{xx} + bu_x, \quad b > 0. \tag{85}$$

As this equation contains a parameter the structure of its symmetry group will crucially depend on the parameter. The problem of determination of all possible symmetries can be regarded as a group classification problem. Let the symmetry group of (85) be generated by the vector fields

$$v = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \varphi(t,x,u)\partial_u, \tag{86}$$

where the coefficients τ, ξ and φ will be determined from the invariance criterion (22). We need to know the second prolongation of v

$$pr^{(2)}v = v + \varphi^t \partial_{u_t} + \varphi^x \partial_{u_x} + \varphi^{xx} \partial_{u_{xx}} + \cdots.$$

Applying the criterion to $E(t,x,u,u_t,u_x,u_{xx}) = u_t - xu_{xx} - bu_x$ gives on solutions

$$\varphi^t - x\varphi^{xx} - \xi u_{xx} - b\varphi^x = 0,$$

where, from (23), in terms of the characteristic $Q = \varphi - \xi u_x - \tau u_t$,

$$\varphi^t = D_t Q + \xi u_{xt} + \tau u_{tt}, \qquad \varphi^x = D_x Q + \xi u_{xx} + \tau u_{xt},$$

and

$$\varphi^{xx} = D_x^2 Q + \xi u_{xxx} + \tau u_{xxt}.$$

Eliminating u_{xx} using $E = 0$ and splitting with respect to the derivatives u_t, u_x, u_{xt} we find that $\tau_x = \tau_u = 0,\ \xi_u = 0\ and\ \varphi_{uu} = 0$, which mean $\tau = \tau(t)$,

$\xi = \xi(t,x), \varphi = \phi(t,x)u + \psi(t,x)$. Using this information and setting equal to zero the coefficients of $u, u_t, u_x,$ the determining system is simplified to

$$2x\xi_x - \dot{\tau}x - \xi = 0, \quad x^2(\xi_{xx} - 2\phi_x) - x\xi_1 - bx\xi_x + b\xi = 0,$$
$$\phi_x = x\phi_{xx} + b\phi_x, \qquad \psi_t = x\psi_{xx} + b\psi_x. \qquad (87)$$

Solving from the first equation for ξ and using in the remaining first two equations we find $\xi(t,x) = f(t)\sqrt{x} + \dot{\tau}x, and\ 4x\dot{f} + (1-2b)f + 4x^{3/2}(\ddot{\tau} + 2\phi_x) = 0$,

from which by integrating

$$\phi(t,x) = -\frac{1}{4}\left[2\ddot{\tau}x + 4\sqrt{x}\dot{f} + \frac{(2b-1)f}{\sqrt{x}}\right] + g(t).$$

Substituting ϕ in find final equation we find

$$(2b-1)(2b-3)f = 0. \quad \ddot{f} = 0. \quad \dddot{\tau} = 0, \quad b\ddot{\tau} + 2\dot{g} = 0. \qquad (88)$$

The last two equations give $\tau(t) = \tau_2 t^2 + \tau_1 t + \tau_0\ and\ g(t) = -b/2\tau + g_0$

where $\tau_2, \tau_1, \tau_0, g_0$ are the integration constants. The last equation in (87) decouples from the others, therefore we find that for any solution $\psi(t,x)$ of the (85), the vector field $v_\psi = \psi\partial_u$ gives rise to a symmetry. Which simply reflects

	v_1	v_2	v_3	v_4	v_5
v_1	0	v_1	$2v_2$	0	v_4
v_2	$-v_1$	0	v_3	$-v_4/2$	$v_5/2$
v_3	$-2v_2$	$-v_3$	0	$-v_5$	0
v_4	0	$v_4/2$	v_5	0	$-v_0/2$
v_5	$-v_4$	$-v_5/2$	0	$v_0/2$	0

Table 1: Commutator Table

The superposition rule of the linear equations. Hereafter, we shall factor out this infinite-dimensional algebra. The constant g_0 corresponds to the scaling symmetry $v_0 = u\partial_u$. This means we can multiply solutions by constants. There are two cases to consider

$b \neq 1/2 . 3/2$ $(f = 0)$: v depends on four integration constants. The symmetry algebra g_4 is four-dimensional. A suitable basis is given by

$$v_1 = \partial_t, \ \ v_2 = t\partial_t + x\partial_x,$$
$$v_3 = t^2\partial_t + 2tx\partial_x - (x + bt)u\partial_u, \ \ v_0 = u\partial_u. \tag{89}$$

$b = 1/2\ or\ 3/2$ $(f \neq 0)$: We have $f(t) = f_1 t + f_0$. The symmetry algebra g_0 is six-dimensional. In this case, the basis (89) for $b = 1/2$ is extended by two additional elements

$$v_4 = \sqrt{x}\partial_x, \quad v_5 = tv_4 - \sqrt{x}u\partial_u, \tag{90}$$

and for $b = 3/2$ by the following two

$$v_4 = \sqrt{x}\partial_x - \frac{1}{2\sqrt{x}}u\partial_u, \ \ v_5 = tv_4 - \sqrt{x}u\partial_u. \tag{91}$$

v_0 is the center element of the symmetry algebra.

From the commutator table (1) we are that the algebra g_4 has the structure of a direct sum $g_4 \simeq sI(2,R) \oplus \{v_0\}, where\ sI(2,R) \sim \{v_1, v_2, v_3\}$, where as the algebra $g_0 \simeq sI(2,R) \uplus h(1), where\ h(1) \sim \{v_1, v_2, v_3\}$ is three-dimensional Heisenberg algebra with center v_0. The symmetry algebra g_0 is isomorphic to the well-known symmetry algebra of the constant coefficient heat equation $u_t = u_{xx}$.

The existence of such an isomorphism is a necessary (but not sufficient) condition for a point transformation $\emptyset: (t, x, u) \to (\tilde{t}, \tilde{x}, \tilde{u})$ to exist, mapping equations into each other. Indeed, in the case of $b = 1/2$, where is a point transformation (not necessarily unique)

$$\emptyset: \tilde{t} = -\frac{1}{t}, \quad \tilde{x} = \frac{2\sqrt{x}}{t}, \quad \tilde{u} = \sqrt{t} exp\left(\frac{x}{t}\right) u, \tag{92}$$

And

$$\emptyset: \tilde{t} = -\frac{1}{t}, \quad \tilde{x} = \frac{2\sqrt{x}}{t}, \quad \tilde{u} = \sqrt{tx} exp\left(\frac{x}{t}\right) u, \tag{93}$$

if $b = 3/2$, mapping (85) to the first canonical form $\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}$. If $b = 1/2$, a simple change of space variable $\tilde{x} = 2\sqrt{x}$ alone also does the job.

Let $b = 3/2$. The pushforwards $\tilde{v}_i = \emptyset_*(v_i)$ of the vector fields $v_i, i = 0, \ldots, 5$ of the algebra g_6 via the map $\emptyset$ are easily calculated:

$$\tilde{v}_1 = t^2 \partial_t + tx \partial_x - \frac{1}{4}(x^2 + 2t) u \partial_u, \tilde{v}_2 = -\left(t \partial_t + \frac{x}{2} \partial_x\right) + \frac{1}{2} \tilde{v}_0, \tilde{v}_3 = \partial_t,$$

$$\tilde{v}_4 = -\left(t \partial_x - \frac{x}{2} u \partial_u\right),$$

$$\tilde{v}_5 = \partial_x, \quad \tilde{v}_0 = v_0 = u \partial_u, \tag{94}$$

where all coordinates should be replaced by tildes (written in the new coordinates). The Lie algebra $\tilde{g}$ spanned by $\{\tilde{v}_i\}$ is recognized to be the symmetry algebra of $u_t = u_{xx}$, having the Levi decomposition $\tilde{g} = \{\tilde{v}_3, \tilde{v}_2, \tilde{v}_1\} \uplus \{\tilde{v}_5, \tilde{v}_4, \tilde{v}_0\} \simeq sI(2, R) \uplus h(1)$. This can be verified by using the same steps of symmetry calculation done for Eq. (85).

For $b = 1/2$, the pushforwards $\tilde{v}_i = \emptyset_*(v_i)$ of the vector fields $v_i, i = 0,1,2, \dots, 5$ of the algebra g_6 spanned by (89)-(90) via the map $\emptyset: (t, x, u) \to (t, 2\sqrt{x}, u)$ are exactly the basis vectors of the heat equation algebra:

$$\tilde{v}_1 = \partial_t,\ \ \tilde{v}_2 = t\partial_t + \frac{x}{2}\partial_x, \tilde{v}_3 = t^2\partial_t + xt\partial_x - \frac{1}{4}(x^2 + 2t)v_0,$$

$$\tilde{v}_4 = \partial_x, \qquad \tilde{v}_5 = t\partial_t - \frac{x}{2}v_0, \quad v_0 = u\partial_u. \tag{95}$$

The corresponding one-parameter symmetry groups are time translations, scaling t, x, projective (or inversional) transformation, Galilei boosts, space translations and scaling in u, respectively.

In the case of $b \notin \{1/2, 3/2\}$, the equation transforms to the second canonical form

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} + \frac{\mu}{\tilde{x}^2}\tilde{u}, \quad \mu = \frac{(2b-1)(2b-3)}{4} \tag{96}$$

via the transformation

$$\tilde{t} = t, \ \ \tilde{x} = 2\sqrt{x}, \quad u = x^{(1-2b)/4}\tilde{u}.$$

Admitting the symmetry algebra spanned by

$$v_1 = \partial_{\tilde{t}}, \qquad v_2 = \tilde{t}\partial_{\tilde{t}} + \frac{\tilde{x}}{2}\partial_{\tilde{x}},$$

$$v_3 = \tilde{t}^2\partial_{\tilde{t}} + \tilde{t}\tilde{x}\partial_{\tilde{x}} - \frac{1}{4}(\tilde{x}^2 + 2\tilde{t})\tilde{u}\partial_{\tilde{u}}, \ \ v_0 = \tilde{u}\partial_{\tilde{u}}. \tag{97}$$

We thus have given the identification of the symmetry algebras of two locally equivalent differential equations, (85) and $u_t = u_{xx}$. In general, two ferential equation with a symmetry g is mapped to another one by a point transformation $\emptyset$, then $\tilde{g} = \emptyset . g . \emptyset^{-1}$ is a symmetry of the second equation.

The well-known solutions of $u_t = u_{xx}$ can be mapped to solutions of (85) when $b = 1/2\,, 3/2$. We note that the equation is also invariant under the discrete group of time and space reflections: $(t, x, u) \to (-t, x, u), (t, x, u) \to (t, -x, u)$.

Symmetry group of the equation will now be used to derive explicit group invariant solutions. We illustrate how to construct group-invariant fundamental solution of (85) for any value of b using the most general element

$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$ of the Lie algebra g_4. A fundamental solution $K(t, x, y)$ as distributions satisfies (85) with the initial condition $lim_{t \to 0} K(t, x, y) = \delta(x - y), where\ \delta(x - y)$ is Dirac distribution with singularity at y. The initial condition puts the following restriction on the coefficients

$$\tau(0) = 0, \quad \xi(0, y) = 0, \quad \Phi(0, y) + \xi_x(0, y) = 0. \qquad (98)$$

These condition are satisfied by the vector field (a projective type symmetry)

$$v = t^2 \partial_t + 2tx\partial_x - (x - y + bt)u\partial_u. \qquad (99)$$

The fundamental solution will be looked for as a solution invariant the symmetry group generated by (99). Its invariants are found by solving the invariance condition $Q(x, t, u, u_t, u_x) = \varphi - \tau u_t - \xi u_x = 0\ as\ \eta = x/t^2\,, \zeta = ut^2 exp[-(x + y)/t]$.

The group-invariant solution has the form

$$u = t^{-b} exp\left\{-\frac{x+y}{t}\right\} F(\eta).$$

Substituting the above solution u into (85) we find that F satisfies

$$\eta F'' + bF' - yF = 0.$$

We note that this ODE is free the non-invariant coordinates. The solution F, which is bounded near zero is given by

$$F = \left(\frac{\eta}{y}\right)^{(1-b)/2} I_{b-1}\left(2\sqrt{y\eta}\right),$$

where I_v denotes the usually modified Bessel function of the first kind and order v. We recall the following asymptotic relations for small values of x

$$I_v \sim \frac{1}{2^v v!} x^v, \quad I_{-v} \sim \frac{2^v}{(-v)!} x^{-v} \qquad v \notin Z.$$

We thus have constructed the following fundamental solution, up to a nonzero constant C.

$$K(t,x,y) = \frac{C}{t}\left(\frac{x}{y}\right)^{\frac{1-b}{2}} exp\left\{-\frac{x+y}{t}\right\} I_{b-1}\left(\frac{2\sqrt{xy}}{t}\right).$$

The value of C is determined from the normalization condition.

The special case $b = 1/2 \ \ with\, C = \pi^{-1/2}$ produces the elementary solution

$$K(t,x,y) = \frac{1}{\sqrt{\pi t y}} exp\left\{-\frac{x+y}{t}\right\} cosh\left(\frac{2\sqrt{xy}}{t}\right). \qquad (100)$$

Translation along t gives the full heat kernel $K(t,x,t_0,x_0) = K(t - t_0, x; x_0)$.

Similarly, in the case $b = 3/2 \ with\ C = \pi^{-1/2}, K$ is again elementary

$$K(t,x,y) = \frac{1}{\sqrt{\pi t x}} exp\left\{-\frac{x+y}{t}\right\} sinh\left(\frac{2\sqrt{xy}}{t}\right). \quad (101)$$

Both fundamental solutions satisfy the normalization condition

$$\int_0^\infty K(0,x,y)dx = 1$$

we remark that solutions (100)-(101) can be recovered using the transformations

(92)-(93) and the separable solutions (solutions invariant under $\tilde{v}_3 + yv_0, y > 0$ a parameter)

$$\tilde{u}(\tilde{t},\tilde{x}) = Ce^{y\tilde{t}} \begin{cases} cosh(\sqrt{y}\tilde{x}), \\ sinsh(\sqrt{y}\tilde{x}), \end{cases}$$

of the heat equation $\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}$.

In the case when $b \neq 1/2\,,3/2$, group-invariant solutions can be derived using the one dimensional subalgebras (an optimal system of subalgebras) of the algebra (89)

$$v_1 + av_0,\ \ v_2 + av_0,\ \ v_1 + v_3 + av_0,\ \ v_0,\ \ a \in R. \quad (102)$$

Now let as look at the action of subgroup, say v_3 of (89), on the solution $u(t,x). exp\{\varepsilon v_3\}u(t,x)$. We find by exponentiating v_3 the group action generated by v_3

$$\tilde{u}_\varepsilon(t,x) = (1+\varepsilon t)^{-b} exp\left\{\frac{-\varepsilon t}{1+\varepsilon t}\right\} u\left(\frac{x}{(1+\varepsilon t)^2}, \frac{t}{1+\varepsilon t}\right). \quad (103)$$

Formula (103) states that if $u(t,x)$ is a solution of (85), then $\tilde{u}_\varepsilon(t,x)$ is also a solution. Simple solutions like stationary (time-independent) solutions can be mapped to new solutions. A stationary solution $u(x)$ satisfies the linear ODE $xu_{xx} = c_0 + c_1 log\ x$ if $b = 1$. They get mapped to the following new solutions, respectively:

$$u_\varepsilon(t,x) = (1+\varepsilon t)^{-1} exp\left\{\frac{-\varepsilon}{1+\varepsilon t}\right\}\left[c_0 + c_1(1+\varepsilon t)^{2(b-1)}x^{1-b}\right],$$

$$b \neq 1.$$

$$u_\varepsilon(t,x) = (1+\varepsilon t)^{-1} exp\left\{\frac{-\varepsilon x}{1+\varepsilon t}\right\}\left[c_0 + c_1 log\frac{\sqrt{x}}{1+\varepsilon t}\right], \; b$$

$$= 1. \qquad (104)$$

The remaining symmetry group of translation in t and scaling in t and $x: (t,x) \to (\lambda(t+t_0), \lambda x)$ can be applied to these solutions to obtain new ones depending on two more group parameters t_0, λ (solutions invariant under the group $SL(2,R)$).

(3-3) Linearization by Symmetry Structure:

We known that a linear differential equation or one that is linearizable by a point transformation admits an infinite-dimensional symmetry algebra. By checking the existence of such a symmetry structure we can construct linearizing transformations. A well-known example is the potential Burger's equation

$$u_t = u_{xx} + u_x^2$$

$$= 0, \qquad (105)$$

admitting the infinite-dimensional symmetry algebra generated by the vector fields

$$v_1 = \partial_x, \;\; v_2 = \partial_t, \quad v_3 = \partial_x, \;\; v_4 = 2t\partial_x - x\partial_u,$$

$$v_5 = 2t\partial_t + x\partial_x, \quad v_6 = 4t^2\partial_t + xt\partial_x - (2t + x^2)\partial_u,$$

$$v_\rho = \rho(x,t)e^{-u}\partial_u,$$

where ρ is a solution to the linear heat equation $\rho_t = \rho_{xx}$. It is easy to see that the point transformation $\tilde{u} = e^u$ maps the basis vector fields to (95) of the linear equation together with $\tilde{v}_\rho = \rho(x,t)\partial_{\tilde{u}}$ and Eq. (105) to the linear heat equation $\tilde{u}_t = \tilde{u}_{xx}$. The transformation $v = u_x$ relates (105) to the usual Burger's equation

$$v_t = v_{xx} + 2vv_x. \qquad (106)$$

So we have established the celebrated Cole-Hopf transformation $v = (log\tilde{u})_x = \tilde{u}_x/\tilde{u}$ taking solutions of (106) to positive solutions of the heat equation.

As well, for higher order PDEs, we can use the same method of looking at the maximal symmetry algebra (if possible) and constructing a linearizing transformation from its infinite-dimensional symmetry involving an arbitrary function as being solution to a linear PDE.

Example (3.3.1):

We consider a third order KdV-type evolution equation

$$u_t = u_{xxx} + 6u^{-1}u_x u_{xx} + u^{-2}u_x^3.$$

The symmetry algebra is infinite-dimensional with basis elements

$$v_1 = \partial_t,\ \ v_2 = \partial_x,\ \ v_3 = 3t\partial_t + x\partial_x,\ \ v_4 = u\partial_u,\ \ v_\rho = \rho(t,x)u^{-2}\partial_u,$$

where ρ is any solution to the linear KdV equation $\rho_t = \rho_{xxx}$. The transformation $\tilde{u} = u^3$ maps the symmetry algebra to that of the linear KdV equation $\tilde{u}_t = \tilde{u}_{xxx}$, and hence the original equation to the linear one.

It may not always be practical to calculate the maximal infinite-dimensional symmetry algebra by application of the Lie symmetry algorithm. When this is the case, it is usually useful to give some tests involving only certain finite-dimensional subalgebra of the maximal symmetry algebra. But, in this case, for PDEs the construction of linearizing transformation is a tricky task.

For ODEs, we already encountered such a test. It is sufficient to identify two-dimensional subalgebra equivalent to the canonical (abelian) forms $A_{2,2}$ or $A_{2,4}$ (nonabelian) as the eight-dimensional symmetry algebra of a linearizable second order ODE. Finding linearizing coordinates is immediate. A further example is the following second member of the

Riccati chain, known to linearizable to $z_3 = z^m = 0$ by the Hopf-Cole transformation $y = z'(x)/z(x)$.

Example (3.3.2):

$E(x, y^{(2)}) = y_2 + 3yy_1 + y^3 =$
$0 \qquad (107)$

this equation admits a linearly connected (rank-one) nonabelian 2-dimensional algebra spanned by

$$v_1 = y\partial_x - y^3\partial_y, \quad v_2 = \lambda(x,y)v_1, \qquad \lambda(x,y) = \frac{x}{y}\left(1 - \frac{1}{2}xy\right)$$

with commutation relation $[v_1, v_2] = v_1$. This is verified by noting

$$pr^{(2)}v_1(E) = -3(y^2 + y_1)E,$$

and

$$pr^{(2)}v_2(E) = \frac{1}{2}[3x^2(y^2 + y_1) - 4]E,$$

then checking the infinitesimal invariance criterion (22). This knowledge ensures as that Eq. (107) has a eight-dimensional symmetry algebra and thus can be linearized.

A linearizing coordinate transformation $r = r(x,y),\ s = s(x,y)$ is found by solving the set of linear first order PDEs $v_1(r) = 0, v_1(s) = 1, v_2(r) = 0,\ v_2(s) = s$ as

$$r = x - \frac{1}{y}, \quad s = \frac{x}{y} - \frac{x^2}{2}.$$

Note that we simply have chosen r as joint invariant and $s = \lambda(x,y)$. The transformed algebra is the canonical form $A_{2,4}$: $\{\partial_s, s\partial_s\}$. This transformation linearized Eq. (107) to $s''(r) = 0$. This readily seen by calculating the following derivative and using the given equation:

$$\frac{d^2s}{dr^2} = -\frac{y^3(y_2 + 3yy_1 + y^3)}{(y^2 + y_1)^3} = 0.$$

We note that the full eight-dimensional symmetry algebra of the equation is isomorphic to that of the equation $s'' = 0$. This is the $sI(3, R)$ algebra generated by the vector fields

$$\partial_r, \quad r\partial_r, \quad s\partial_r, \quad rs\partial_r + s^2\partial_s, \ \partial_s, \ s\partial_s, \qquad r^2\partial_r + rs\partial_s.$$

The corresponding group is just the projective group SL(3,R) in the (r, s) plane

$$(\tilde{r}, \tilde{s}): \quad (r, s) \to \left(\frac{a_1 r + a_2 s + a_3}{a_7 r + a_8 s + a_9}, \frac{a_4 r + a_5 s + a_6}{a_7 r + a_8 s + a_9}\right)$$

with the condition

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \neq 0.$$

The projective group $SL(3, R)$ maps the family of straight lines (and also quadrics) onto themselves. Another interesting fact is that this group has the lowest order differential invariant starting at order 7.but obviously there are lower order relative differential invariants such as the second order one $I = s''(r)$. This means that, for some differential function $\mu, \tilde{s}''(\tilde{r}) = \mu s''(r)$ under the projective group. A differential function $I: J^n \to R$ is a relative differential invariant of order n of a Lie algebra g if for some differential function $\lambda(x, u^{(n)})$

$$pr^{(n)}v(I) = \lambda(x, u^{(n)})I$$

for every prolonged vector field $pr^{(n)}v \in g^{(n)}$. If $\lambda = 0{,}1$ is a differential invariant of order n. The seventh order differential invariant can be expressed in terms of relative differential invariant.

Example (3.3.3):

$$y_2 + \rho(x)y = \sigma \frac{y_1^2}{y}, \qquad (108)$$

where σ is an arbitrary constant. This equation is invariant under the symmetry algebra represented by the vector fields

$$v_1 = y\partial_y, \quad v_2(\rho) = \rho(x)y^\sigma \partial_y,$$

where $\rho(x)$ satisfies the linear equation

$$\rho''(x) + (1-\sigma)\rho(x) = 0. \qquad (109)$$

We note that under the condition (109)

$$pr^{(2)}v_2(E) = \lambda(x,y)E, \quad \lambda(x,y) = \sigma\rho(x)y^\sigma,$$

where $E = y_2 + py - \sigma y^{-1}y_1^2$. The vector fields $v_1, v_2(\rho)$ reflect the linearity of the equation modulo a transformation. Indeed, the point transformation $y = z^{1/(1-\sigma)}, \quad \sigma \neq 1$ maps $v_2(f)\ to\ \tilde{v}_2(\rho) = \rho(x)\partial_z$ (preserving homogeneity in z) and the equation to the linear homogeneous one

$$z_2 + (1-\sigma)p(x)z = 0.$$

If we extend Eq. (108) to

$$y_2 + p(x)y = \sigma\frac{y_1^2}{y} + h(x,y) \qquad (110)$$

and impose invariance under $v_2(f)$ we find $h(x,y) = g(x)y^\sigma, \sigma \neq 1$. If $\sigma = 1$ we go back to the case $h = 0$. The resulting equation is nonhomogeneous linear one

$$z_2 + (1-\sigma)p(x)z = (1-\sigma)g(x).$$

(3-4) Lie's Linearization Theorem:

This theorem states that for second-ordinary differential equations Eq. $y_2 = f(x,y,p)$ with $p = y_1$ is point equivalent to the trivial free-particle equation $y_2 = 0$ if and only if the following fourth-order tresses (absolute) invariants.

$$I_1 = f_{pppp} = 0, I_2$$
$$= \widetilde{D}_x^2 f_{pp} - 4\widetilde{D}_x f_{yp} - f_p \widetilde{D}_x f_{pp} + 6f_{yy} - 3f_y f_{pp} + 4f_p f_{yp}$$
$$= 0, \tag{111}$$

are identically zero. Here $\widetilde{D}_x = \partial_x + p\partial_y + f\partial_p$ is the truncation of the total derivative operator D_x. The first condition implies the f should be at most cubic in $p = y_1$. For example, any equation admitting a $A_{2,3} - type$ symmetry can be linearized if and only if it has the form

$$xy_2 = ay_1^3 + \left(1 + \frac{b^2}{3a}\right)y_1 + \frac{b(9a + b^2)}{27a^2} \tag{112}$$

where $a \neq 0$ and b are arbitrary constants, it has been shown that an equivalence group transformation shifting only the parameter b as $b \to b + \varepsilon$ can be used to put $b = 0$ by choosing the group parameter $\varepsilon = -b$. Such a transformation is given by

$$\tilde{x} = \sqrt{e^{-b}(1 + x^2) - 1}, \quad \tilde{y} = e^{-b/2}\left(y + \frac{b}{3a}x\right). \tag{113}$$

So (112) has effectively been reduced to

$$\tilde{x}\tilde{y}_2 = a\tilde{y}_1^3 + \tilde{y}_1. \tag{114}$$

Moreover, a can be transformed to $a = \epsilon = \pm$ by scaling y $or (x)$. Two equivalent equations have isomorphic symmetry groups so that there should be a point transformation mapping the two Lie algebras an Eqs. (114) and $y_2 = 0$ into each other. For the construction of such a map. An casier way of achieving linearization is by using a two-dimensional subalgebra of type $A_{2,2}$ of 4.4 of its entire $sI(3, R)$ symmetry algebra. It is given by

$$v_1 = \frac{1}{x}\partial_{\tilde{x}}, \quad v_2 = \frac{\tilde{y}}{\tilde{x}}\partial_{\tilde{x}}.$$

Suitable canonical coordinates are $r = \tilde{y}$ (an invariant), $s = \tilde{x}^2/2$, which maps the subalgebra to$\{\partial_s, r\partial_s\}$ and Eq. (114) to the linear equation $s''(r) = \tilde{y}_1^{-3}(\tilde{y}_1 - \tilde{x}\tilde{y}_2) = -a$. Consequently, the solution in tilde coordinates is

$$a\tilde{y}^2 + \frac{2b}{3}xy + \left(1 + \frac{b^2}{9a}\right)x^2 + C_1\left(y + \frac{b}{3a}x\right) + C_2 = 0,$$

which geometrically describes a family of ellipses or hyperbolas as its diseriminant $\Delta = -4a$ varies for $a < 0 \; or \; a > 0$, respectively.

Eq.of Example 3.3.2 with two-dimensional symmetry group generated by $\{\partial_x, x\partial_x - y\partial_y\}$ of the type $A_{2,3} \; of$ 4.4 (modulo a point transformation $(x, y) \rightarrow (y^{-1}, x + y^{-1})$) belongs to the class (112) with $a = -1, b = 6$ or to (114) with $a = -1$ and hence once again its linearizability has been established.

In general, if the second order equation $y_2 = f(x, y, p)$ is known to satisfy linearizability conditions (111), it is sufficient to pick a two-dimensional intransitive subalgebra of type $A_{2,2} \; or \; A_{2,4}$ (up to change of basis) of the $sI(3, R)$ symmetry algebra to transform to linear equation.

Linearization criteria (111) for the ODE (110) is fulfilled if $h(x, y)$ satisfies the PDE

$$y^2 h_{yy} - \sigma y h_y + \sigma h = 0,$$

allowing the general solution $h = g(x)y^\sigma + f(x)y \; for \; \sigma \neq 0$. The arbitrary function $f(x)$ can be put to zero by taking a linear combination with $p(x)$. For $\sigma = 1$, we find $h = g(x)ylny,\; y > 0$. Then, the vector fields $v(\rho) = \rho(x)y\partial_y$, where ρ solves the linear ODE $\rho'' - 9\rho = 0$ is a symmetry because

$$pr^{(2)}V(E) = -\rho(x)E = 0 \; on \; E = y_2 + py - y^{-1}y_1^2 - g(x)ylny = 0.$$

It permits us to find the linearizing transformation $y = e^z$. The linearized ODE is

$$z'' - g(x)z + p(x) = 0.$$

We comment that there are also similar tests for second order systems of ODEs, a general approach to linearizability can be found in the research.

Chapter Four

Group Classification of Invariant Solutions of Algebra

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(4-1) Group Classification Algorithm:

While classifying a given class of differential equations into subclasses, one can use different classifying features, like linearity, order, the number of independent or dependent variables, atc. In group analysis of differential equations the principal classifying features are symmetry properties of equations under study. This means that classification objects are equations considered together with their symmetry groups. This point of view is based on the well-known fact that any PDE admits a (possibly trivial) Lie transformation group. And what is more, any transformation group corresponds to a class of PDEs, which are invariant under this group. So the problem of group classification of a class of PDEs reduces to describing all possible (inequivalent) pairs (PDE, maximal invariance group), where PDE should belong to the class of equations under consideration.

We perform group classification of the following class of quailinear wave equations:

$$u_{tt} = u_{xx} + F(t, x, u, u_x) \qquad (115)$$

where F is an arbitrary smooth function, $u = u(t, x)$. Hereafter we adopt notations $u_t = \partial u / \partial t\,, u_x = \partial u / \partial x\,, u_{tt} = \partial^2 u / \partial t^2\,, etc.$

Our aim is describing all equations of the form (115) that admit nontrivial symmetry groups.

The challenge of this task is in the word all. If, for example, we somehow constrain the form of the invariance group to be found, then the classification problem simplifies enormously. A slightly more cumbersome (but still tractable with the standard Lie-Ovsyannikov approach) is the problem of group classification of equation with arbitrary functions of, at most, one variable.

As equation invariant under similar Lie groups are identical within the group-theoretic framework, it makes sense to consider nonsimilar transformation groups only. The important example of similar Lie groups is provided by Lie transformation groups obtained one from another by a suitable change of variables. Consequently, equations obtained one from another by a change of variables have similar symmetry groups and cannot be distinguished within the group theoretical viewpoint. That is why, we perform group classification of (115) within a change of variables preserving the class of PDEs (115).

The problem of group classification of linear hyperbolic type equation

$$u_{tx} + A(t,x)u_x + B(t,x)u_x + C(t,x)u = 0 \qquad (116)$$

with $u = u(t,x)$, was solved by Lie, in view of this fact, we consider only those equations of the form (115) which are not (locally) equivalent to the linear equation (116).

As we have already mentioned in the Introduction, the Lie-Ovsyannikov method of group classification of differential equations. Utilizing this method enabled solving the group classification problem for a number of important one-dimensional nonlinear wave equations:

$$u_{tt} = u_{xx} + F(u),$$

$$u_{tt} = [f(u)u_x]_x,$$

$$u_{tt} = f(u_x)u_{xx},$$

$$u_{tt} = F(u_x)u_{xx} + H(u_x),$$

$$u_{tt} = F(u_{xx})$$

$$u_{tt} = u_x'''u_{xx} + F(u),$$

$$u_{tt} + f(u)u_t = (g(u)u_x)_x + h(u)u_x,$$

$$u_{tt} = (f(x,u)u_x)_x,$$

analysis of the above list shows the most of the all arbitrary elements (=arbitrary functions) depend on one variable. This not coincidental. As the already mentioned, the Lie- Ovsyannikov approach works smoothly for the case when the arbitrary elements are functions of one variable. The reason for this is that obtained system of determining equations is still over-determined. So it can be effectively solved within the same techniques used to compute maximal symmetry group of PDEs containing no arbitrary elements.

The situation becomes much more complicated for the case arbitrary elements are functions of two (or more) arguments. By this very reason the group classification of nonlinear wave equations,

$$u_{tt} + \lambda u_{xx} = g(u, u_x),$$
$$u_{tt} = [f(u)u_x + g(x, u)]_x,$$
$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x),$$

is not complete.

We suggest an efficient approach to the problem of group classification of low dimensional PDEs. This approach is based on the Lie-Ovsyannikov infinitesimal method and classification results for abstract finite-dimensional Lie algebras. It enables as to obtain the complete solution of the group classification problem for the general heat equation with a nonlinear source

$$u_{tt} = u_{xx} + F(t, x, u, u_x).$$

Later on, we perform complete group classification of the most general quasilinear evolution equation,

$$u_t = f(t, x, u, u_x)u_{xx} + g(t, x, u, u_x).$$

We utilize the above approach to obtain complete solution of the group classification problem of Eqs. (115).

Our algorithm of group classification of the class of PDEs (115) is implemented in the following three steps:

(I) Using the infinitesimal Lie method we derive the system of the determining equations for coefficients of the first-order operator that generates symmetry group of equation (115).

(Note that the determining equations which explicitly depend on the function F and is derivatives are called classifying equations.) Integrating equations that not depend on F we obtain the form of the most general infinitesimal operator admitting by Eqs. (115) under arbitrary F. Another task of this step is calculating the equivalence group ε of the class of PDEs (115).

(II) We construct all realizations of Lie algebras $A_{11}$of the dimension $n \geq 3$ in the class operators obtained at the first step within the equivalence relation defined by transformations form the equivalence group ε. Inserting the so obtained operators into classifying equations we select those realizations that can be symmetry algebras of a differential equation of the form (115).

(III) we compute all possible extensions of realizations constructed at the previous step to realizations of higher dimensional $(n > 3)$ Lie algebras. Since extending symmetry algebras results in reducing arbitrariness of the function F, at some point this function will contain either arbitrary functions of at most one variable or arbitrary constants. At this point, we apply the standard classification method (which is due to Lie and Ovsyannikov) the derive the maximal symmetry group of the equation study. This completes group classification of (115).

Performing the above enumerated steps yields the complete list of inequivalent equations of the form (115) together with their maximal (in Lie sense) symmetry algebras.

We say that the group classification problem is completely solved when it is proved that

(a) The constructed symmetry algebras are maximal invariance algebras of the equations under consideration;

(b) The list of invariant equations contains only inequivalent ones, namely, no equation can be transformed into another one from the list by atransformed into another one from the list by a transformation from the equivalence group ε.

(4-2) Preliminary Group Classification of Eq. (115):

We look for the infinitesimal operator of symmetry group of equation (115) in the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \quad (117)$$

where τ, ξ, η are smooth functions defined on an open domain Ω of the space $V = R^2 \times R^1$ of independent (117) generates one-parameter invariance group of (115) iff its coefficients $\tau, \xi, \eta, \epsilon$ satisfy the equation (Lie's invariance criterion)

$$\varphi^{tt} - \varphi^{xx} - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{u_x}|_{(1,1)}$$
$$= 0, \quad (118)$$

where

$$\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\varphi^{tt} = D_t(\varphi') - u_{tt} D_t(\tau) - u_{tt} D_t(\xi),$$

$$\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),$$

and D_t, D_x are operators of total differentiation with respect the variables t, x. As customary, by writing (115) we mean that one needs to replace u_{tt} and its differential consequences in (118).

After a simple transformations algebra we reduce (118) to the form

(1) $\xi_u = \tau_u = \eta_{uu} = 0,$

(2) $\tau_t - \xi_x = 0,\ \xi_t - \tau_x = 0,$

(119)

(3) $2\eta_{tu} + \tau_x F_{u_x} = 0,$

(4) $\eta_{tt} - \eta_{xx} - 2u_x\eta_{xu} + [\eta_u - 2\tau_t] \times\ F - \tau F_t - \xi F_x - \eta F_{tt} - [\eta_x + u_x(\eta_u - \xi_x)]F_{u_x} = 0.$

The first two groups of PDEs from (119) are to be used to derive the form of the most general infinitesimal operator admitted by (115). The remaining PDEs are classifying equations.

Theorem (4.2.1):

Provided $F_{u_t u_x} \neq 0$, the maximal invariance group of equation (115) is generated by the following infinitesimal operator:

$$Q = (\lambda \mathrm{t} + \lambda_1)\, \partial_{\mathrm{t}} + (\lambda \mathrm{x} + \lambda_2)\, \partial_{\mathrm{x}} + [\mathrm{h(x)u} + \mathrm{r(t,x)}]\, \partial_{\mathrm{u}}, \qquad (120)$$

where $\lambda, \lambda_1, \lambda_2$ are real constants and $h = h(x), r = r(t,x), F = F(t,x,u,u_x)$ are functions obeying the constraint

$$r_{tt} - r_{xx} - \frac{d^2h}{dx^2}u - 2\frac{dh}{dx}u_x + (h - 2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x - (hu + r)F_u - \left(r_x + \frac{dh}{dx}u + (h - \lambda)u_x\right)F_{u_x} = 0. \qquad (121)$$

If $F = g(t,x,u)u_x + f(t,x,u), g_{tt} \neq 0$, then the maximal invariance group of equation (115) is generated by infinitesimal operator (120), where $\lambda, \lambda_1, \lambda_2$ are real constants h, r, g, f are functions satisfying system of two equations

$$-2h' - \lambda g = (\lambda t + \lambda_1)g_t + (\lambda x + \lambda_2)g_u, \qquad (122)$$

$$-h''u + r_{tt} - r_{xx} + (h - 2\lambda)f = (\lambda t + \lambda_1)f_t + (\lambda x + \lambda_2)f_x + (hu + r)f_u + g(h'u + r_x).$$

Next, if $F = g(t,x)u_x + f(t,x,u), q \neq 0, F_{uu} \neq 0$, then the infinitesimal operator of the invariance group of equation (115) reads as

$$Q = \tau(t,x)\partial_t + \xi(t,x)\partial_x + (h(t,x)u + r(t,x))\partial_u,$$

where τ, ξ, h, r, g, f are functions satisfying system of PDEs

$$\tau_t - \xi_x = 0, \quad \xi_t - \tau_x = 0,$$

$$2h_t = -\tau_x g, \ \ 2h_x = -\tau g - \tau g_t - \xi g_x,$$

$$(h_{tt} - h_{xx})u + r_{tt} - r_{xx} + f(h - 2\tau_t) - \tau f_t - \xi f_x - (hu + r)f_u - (h_x u + r_x) = 0.$$

Finally, if $F = f(t,x,u), f_{uu} \neq 0$, then the maximal invariance group of equation (115) reduces to the one of classifying equations of more specific forms,

$$u_{tt} = u_{xx} + F(t,x,u,u_x), F_{u_x u_x} \neq 0, \qquad (123)$$

$$u_{tt} = u_{xx} + g(t,x,u)u_x + f(t,x,u), g_u \neq 0, \qquad (124)$$

$$u_{tt} = g(t,x)u_x + f(t,x,u), \qquad g_x \neq 0, f_{uu} \neq 0, \qquad (125)$$

$$u_{tx} = f(t,x,u), \qquad f_{uu} \neq 0. \qquad (126)$$

Note that condition $g_x \neq 0$ is essential, since otherwise (125) is locally equation (126).

Summing up, we conclude that the problem of group classification of (115) reduces to classifying more specific classes of PDEs (123)-(126).

First, we consider equations (124)-(126).

(4-3) Group Classification of Eq. (124):

According to theorem I invariance group of equation (124) is generated by infinitesimal operator (120). And what is more, the real constants $\lambda, \lambda_1, \lambda_2$ and functions h, r, g, f satisfy equation (122). System (122) is to be used to specify both the form of functions f, g from (124) and functions h, r and constants $\lambda, \lambda_1, \lambda_2$ in (120). It is called the determining (sometimes classifying) equations.

Efficiency of the Lie method for calculation of maximal invariance group of PDE is essentially based on the fact that routinely system of determining equations is over-determined. This is clearly not the case, since we have only one equationfor four (!) arbitrary functions and the three of the latter depend on two variables. By this very reason direct application of Lie approach in the ovsyannikov's pint is not longer efficient when we attempt classifying PDEs with arbitrary functions of several variables.

Compute the equivalence group E of equation (124). This group is generated by invertible transformations the space V preserving the differential structure of equation (124). Saying it another way, group transformation from ε

$$\bar{t} = \alpha(t,x,u), \qquad \bar{x} = \beta(t,x,u), \qquad v = U(t,x,u), \frac{D(\overline{t,x,v})}{D(t,x,u)} \neq 0,$$

should reduce (124) to equation of the same form

$$v_{\bar{t}\bar{t}} = v_{\overline{xx}} + \tilde{g}(\overline{t,x,v})v_{\bar{x}} + \tilde{f}(\overline{t,x,v}), \tilde{g}_v \neq 0$$

with possibly different $\tilde{f}, \tilde{g}$.

As proved by Ovsyannikov, it is possible to modify the Lie's infinitesimal approach to calculate equivalence group in essentially same way as invariance group. We omit the simple intermediate calculations and present the final result.

Assertion 1: The maximal equivalence group ε of Eq. (124) reads as

$$\bar{t} = kt + k_1, \qquad \bar{x} = \epsilon kx + k_2, \quad v = \mathrm{X}(x)u + Y(t,x) \qquad (127)$$

where $k \neq 0, \mathrm{X} \neq 0, \epsilon = \pm 1, k, k_1, k_2 \in R, and$ are arbitrary smooth functions.

This completes the first step of the algorithm.

(4-3-1) Preliminary Group Classification of Eq. (124):

First, we derive inequivalent classes of equations of the form (124) admitting one-parameter invariance groups.

Lemma (4.3.1): There exist transformations (127) that reduce operator (120) to one of the six possible forms,

$$Q = m(t\partial_t + x\partial_x), \qquad m \neq 0, Q = \partial_t + \beta\partial_x, \qquad \beta \geq 0,$$
$$Q = \partial_t + \sigma(x)u\partial_u, \quad \sigma \neq 0, \quad Q = \partial_x, \qquad (128)$$
$$Q = \sigma(x)u\partial_u, \qquad \sigma \neq 0, Q = \theta(t,x)\partial_u, \qquad \theta \neq 0.$$

Proof: Change of variables (127) reduces operator (120) to become

$$\tilde{Q} = k(\lambda t + \lambda_1)\partial_t + \epsilon k(\lambda x + \lambda_2)\partial_x + [Y_1(\lambda t + \lambda_1) + (\lambda x + \lambda_2)(\chi' u + Y_x) + \chi(hu + r)]\partial_v. \qquad (129)$$

If $\lambda \neq 0$ in (120), then setting $k_1 = \lambda^{-1}\lambda_1 k, k_2 = \epsilon\lambda^{-1}\lambda_2 k$, and taking as $\chi, Y(\chi \neq 0)$ integrals of system of PDEs,

$$\chi'(\lambda x + \lambda_2) + \chi h = 0,$$
$$Y_t(\lambda t + \lambda_1) + Y_x(\lambda x + \lambda_2) + \chi r = 0,$$

we reduce (129) to the form

$$\tilde{Q} = \lambda(\overline{t\partial_{\bar{t}}} + \overline{x\partial_{\bar{x}}}).$$

Provided $\lambda = 0 \; and \; \lambda_1 \neq 0$, we similarly obtain

$$\tilde{Q} = \partial_{\bar{t}} + \beta\partial_{\bar{x}}, \;\; \beta \geq 0, \qquad Q = \partial_{\bar{t}} + \sigma(\bar{x})v\partial_v, \quad \sigma \neq 0.$$

Next, if $\lambda = \lambda_1 = 0, \lambda_2 \neq 0$ in (120), then setting $k = \epsilon\lambda_2^{-1}, and \; taking \; as \; X, Y(X \neq 0)$ integrals of equation

$$\lambda_2 X' + hX = 0, \quad Y_x + rX = 0,$$

we reduce operator (129) to become $\tilde{Q} = \partial_{\bar{x}}$.

Finally, the case $\lambda = \lambda_1 = \lambda_2 = 0$, gives rise to operators $\tilde{Q} = \sigma(\bar{x})v\partial_v, \tilde{Q} = \theta(\overline{t,x})\partial_v$.

Rewriting the above operators in the initial variables t, x completes the proof.

Theorem (4-3-1):

There are exactly five inequivalent equations of the form (124) that admit one-parameter transformation groups. Below we list these equations together with one-dimensional Lie algebras generating their invariance groups (note that we do not present the full form of invariant PDEs we just give the functions f and g),

$$A_1^1 = \langle t\partial_t + x\partial_x \rangle, g = x^{-1}\tilde{g}(\psi, u),$$
$$f = x^{-2}\tilde{f}(\psi, u), \psi = tx^{-1}, \tilde{g}_u \neq 0,$$
$$A_1^2 = \langle \partial_t + \beta\partial_x \rangle, \quad g = \tilde{g}(\eta, u), \; f = \tilde{f}(\eta, u),$$
$$\eta = x - \beta t, \quad \beta \geq 0, \tilde{g}_u \neq 0,$$
$$A_1^3 = \langle \partial_t + \sigma(x)u\partial_u \rangle, g = -2\sigma'\sigma^{-1}ln|u| + \tilde{g}(\rho, x),$$
$$f = (\sigma'\sigma^{-1})^2 u ln^2|u| - \sigma'\sigma^{-1}\tilde{g}(\rho, x)u ln|u| - \sigma^{-1}\sigma'' u ln|u|$$
$$+ u\tilde{f}(\rho, x),$$

$$\rho = u\, exp(-t\sigma), \quad \sigma \neq 0,$$
$$A_1^4 = \langle \partial_x \rangle, \; g = \tilde{g}(t, u), \quad \tilde{g}_u \neq 0,$$
$$A_1^5 = \langle \sigma(x)u\partial_u \rangle, g = -2\sigma'\sigma^{-1}ln|u| + \tilde{g}(t, x), \; f = (\sigma'\sigma^{-1})^2 u ln^2|u|$$
$$-(\sigma^{-1}\sigma'' + \sigma^{-1}\sigma'\tilde{g}(t, x))u ln|u| + u\tilde{f}(t, x), \sigma' \neq 0,$$

Proof: If Eq. (124) admits a one-parameter invariance group, then it is generated by operator of the form(120). According to lemma 1. The later is equivalent to one of the six operators (128). That is why, all we need to do is integrate six systems of determining equations corresponding to operators (122). For the first five operators solutions of determining equations are easily shown to have the form given in the statement of the theorem.

We consider in more detail the operator $Q = \theta(t,u)\partial_u$. Determining Eqs. (122) for this operator reduce to the form

$$\theta_{tt} - \theta_{xx} = \theta f_u + \theta_x g, \ \ \theta g_{tt} = 0,$$

whence we get $g_{tt} = 0$. Consequently, the system of determining equations is incompatible and the corresponding invariant equation fails to exist.

Nonequivalence of the invariant equations follows from nonequivalence of the corresponding symmetry operators.

The theorem is proved.

Note that in the sequel we give the formulations of theorems omitting routine proofs. The detailed proofs of the most of the statements presented in the research can be found.

It is a common knowledge that there exist two inequivalent two-dimensional solvable Lie algebras

$$A_{21} = \langle e_1, e_2\rangle, \ [e_1, e_2] = 0,$$

$$A_{22} = \langle e_1, e_2\rangle, \quad [e_1, e_2] = e_2.$$

To construct all possible realizations of the above algebras we take as the first basis element one of the realizations of one-dimensional invariance algebras listed in lemma 1. The second operator is looked for in the generic form (120).

Algebra $A_{2,1}$: Let operator e_1 be of the form $\partial_t + x\partial_x$ and operator e_2 read as (120). Then it follows form the relation $[e_1, e_2] = 0\, that\, \lambda_1 = \lambda_2 = xh' = 0$, $tr_t + xr_x = 0$. Consequently, we can choose basis elements of the algebra in question in the form $\langle t\partial_t + x\partial_x, (mu + r(\psi))\partial_{tt}\rangle, when\, m \in R, \psi = tx^{-1}$. Provided m=0, the operator e_2 becomes $r(\psi)\partial_u$. It is straightforward to verify that this realization dose not satisfy the determining equations. Hence, $m \neq 0$.

Making the change of variables

$$\bar{t} = t, \qquad \bar{x} = x, \qquad v = u + m^{-1}r(\psi)$$

reduces the basis operators in question to the form $\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}, mv\partial_v$. That is why we can restrict ore considerations to the realization $\langle t\partial_t + x\partial_x, u\partial_u\rangle$.

The second determining equation from (122) takes the form $u\partial_u = 0$. Hence it follows that the realization under consideration dose not satisfy the determining equations. Consequently, the realization A_1^1 cannot be extended to a realization of the two-dimensional algebra $A_{2,1}$.

Algebra $A_{2.2}$: If operator e_1 is of the form $t\partial_t + x\partial_x$, then it follows from $[e_1, e_2] = e_2$ that $\lambda = \lambda_1 = \lambda_2 = 0, xh' = h, tr_1 + xr_x = r$.
Next, if e_2 reads as $t\partial_t + x\partial_x$, then we get from $[e_1, e_2] = e_2$ the erroneous equality $1 = 0$.

That is why, the only possible case is when $e_2 = (mxu + xr(\psi))\partial_u, m \neq 0, \psi = tx^{-1}$, which gives rise to the following realization of the algebra $A_{2.2}$: $\langle t\partial_t + x\partial_x, xu\partial_u\rangle$.

This is indeed invariance algebra of an equation from the class (124) and the functions f and g read as $g = -2x^{-1}ln|u| + x^{-1}\tilde{g}(\psi), f = x^{-2}uln^2|u| - x^{-2}\tilde{g}(\psi)u\, ln|u| + x^{-2}u\tilde{f}(\psi),\ \psi = tx^{-1}$.

Analysis of the remaining realizations of one-dimensional Lie algebras yields 10 inequivalent $A_{2.1}$-and $A_{2.2}$-invariant equations (see the assertions below). What is more, the obtained (two-dimensional) algebras are maximal symmetry agebras of the corresponding equations.

Theorem (4.3.2):

There are, at most, four inequivalent $A_{2.1}$-invariant nonlinear equations (124) .
Below we list the realizations of $A_{2.1}$ and the corresponding expressions for f and g.

(1) $\langle \partial_t, \sigma(x)u\partial_u \rangle$, $g = -2\sigma'\sigma^{-1}ln|u|$,

$$f = (\sigma'\sigma^{-1})^2 u ln^2|u| - \sigma^{-1}\sigma' u\, ln|u| + u\tilde{f}(x), \qquad \sigma' \neq 0,$$

(2) $\langle \partial_t, \partial_x \rangle$, $g = \tilde{g}(u)$, $f = \tilde{f}(u)$, $\tilde{g}_u \neq 0$,

(3) $\langle \partial_x, \partial_t + u\partial_u \rangle, g = \tilde{g}(\omega)$, $f = exp(t)\tilde{f}(\omega)$,

$\omega = exp(-t)$, $\tilde{g}_u \neq 0$,

(4) $\langle \sigma(x)u\partial_u, \partial_t - \frac{1}{2}k\sigma(x)\Psi(x)u\partial_u \rangle, g = -2\sigma'\sigma^{-1}ln\,|u| + kt + \tilde{g}(x)$,

$$k \neq 0, \quad \sigma' \neq 0, \quad \psi = \int \sigma^{-1}\, dx.$$

Theorem (4.3.3):

There exist, at most, six inequivalent $A_{2.2}$-invariant nonlinear equation (124). Below we list the realizations of $A_{2.1}$ and the corresponding expressions f and g.

(1) $\langle t\partial_t + x\partial_x, k^{-1}|x|^k u\partial_u \rangle, g = x^{-1}(-2kln|x| + \tilde{g}(\psi))$,

$$f = x^{-2}u(-k^2 ln^2|u| + k\tilde{g}(\psi)ln|u| + k(k-1)ln|u| + \tilde{f}(\psi),$$

(2) $\langle \partial_t + \beta\partial_x, exp(\beta^{-1}x)u\partial_u \rangle$, $g = -2\beta^{-1}ln|u| + \tilde{g}(\eta)$,

$$f = \beta^2 u ln^2|u| - (\beta^{-2} + \beta^{-1}\tilde{g}(\eta))uln|u| + u\tilde{f}(\eta),$$

$$\beta > 0, \qquad \eta = x - \beta t,$$

(3) $\langle -t\partial_t - x\partial_x, \partial_t + \beta\partial_x \rangle$, $g = \eta^{-1}\tilde{g}(u)$, $f = \eta^{-2}\tilde{f}(u), \beta \geq 0$,

$$\eta = x - \beta t, \qquad \tilde{g}_u \neq 0,$$

(4) $\langle -t\partial_t - x\partial_x, \partial_t + mx^{-1}u\partial_u \rangle$, $g = x^{-1}(2m\psi + \tilde{g}(\omega))$,

$$f = x^{-1}[-2m\psi u - 2m\psi - 2 - \tilde{g}(\omega) + exp(m\psi)\tilde{g}(\omega)],$$

$$m > 0, \quad \omega = uexp(-m\psi), \qquad \psi = tx^{-1}, \qquad \tilde{g}_u \neq 0,$$

(5) $\langle \partial_x, e^x u\partial_u \rangle, g = -2ln|u| + \tilde{g}(t), f = uln^2|u| - uln|u|(1 + \tilde{g}(t) + u\tilde{f}(t)$,

(6) $\langle -t\partial_t - x\partial_x, \partial_x \rangle, g = t^{-1}\tilde{g}(u), f = t^{-2}\tilde{f}(u)$, $\tilde{g}_u \neq 0$.

(4-3-2) Completing Group Classification of Eq. (124):

As the invariant equations obtained in the previous subsection contain functions of, at most one variable, we can now apply the standard Lie-ovsyannikov routine to complete the group classification of (124). We give the computation details for the case of the first $A_{2.1}$-invariant equation, the remaining cases are handled in similar way.

Setting $g = -2\sigma'\sigma^{-1}ln|u|, f = (\sigma'\sigma^{-1})uln^2|u| - \sigma'\sigma''uln|u| + u\tilde{f}(x), \sigma = \sigma(x), \sigma' \neq 0$ we rewrite the first determining equation to become

$$-2h' + 2\lambda\sigma'\sigma^{-1}ln|u| = -2(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'_x ln|u| - 2h\sigma'\sigma^{-1} - 2r\sigma'\sigma^{-1}u^{-1}.$$

As $h = f(x), \sigma = \sigma(x), r = r(t,x), \lambda, \lambda_2 \in R$, the above relation is equivalent to the following ones:

$$h' = \sigma'\sigma^{-1}h, \;\; r = 0, \;\; \lambda\sigma'\sigma^{-1} = -(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'.$$

If σ is an arbitrary function, then $\lambda = \lambda_2 = r = 0, h = C\sigma, C \in R$ and we get $\langle \partial_t, \sigma(x)u\partial_u \rangle$ as the maximal symmetry algebra. Hence, extension of the symmetry algebra is only possible when the function $\psi = \sigma'\sigma^{-1}$ is a (nonvanishing identically) solution of the equation

$$(\alpha x + \beta)\psi' + \alpha\psi = 0, \;\; \alpha, \beta \in R, \quad |\alpha| + |\beta| \neq 0.$$

If $\alpha \neq 0$, then at the expense of displacements by x we can get $\beta = 0$, so that $\psi = mx^{-1}, m \neq 0$. Integrating the remaining determining equations yields

$$g = -2mx^{-1}ln|u|, f = mx^{-2}[muln^2|u| - (m-1)uln|u| + mu], m \neq 0, m, n \in R.$$

The maximal invariance algebra of the obtained equation is the three-dimensional Lie algebra $\langle \partial_t, |x|^m u\partial_u, t\partial_t + x\partial_x \rangle$ isomorphic to $A_{3.7}$.

Next, if $\alpha = 0$, then $\beta \neq 0 \; and \; \psi = m, m \neq 0$. If this is the case, we have

$$g = ln|u|, f = \frac{1}{4} u ln^2 |u| - \frac{1}{4} u ln|u| + nu, \quad n \in R.$$

The maximal invariance algebra of the above equation reads as

$$\langle \partial_t, \partial_x, exp\left(-\frac{1}{2}x\right) u\partial_u \rangle.$$

It is isomorphic to $A_{3.2}$.

Similarly, we prove that the list of inequivalent equations of the form (124) admitting three-dimensional symmetry algebras is exhausted by the equations given below. Note that the presented algebras are maximal. This means, in particular, that maximal symmetry algebra of Eq. (124) is, at most, three dimensional.

$A_{3.2}$-invariant equations,

(1) $u_{tt} = u_{xx} + u_x ln|u| + \frac{1}{4} u ln^2 |u| - \frac{1}{4} u ln|u| + nu (n \in R), \langle \partial_t, \partial_x, exp\left(-\frac{1}{2}x\right) u\partial_u \rangle$,

(2) $u_{tt} = u_{xx} + m[ln|u| - t]u_x + (m^2/4)u[(ln|u| - t - 1)] + nu (m > 0, n \in R), \langle \partial_x, \partial_t + u\partial_u, exp\left(-\frac{1}{2}mx\right) u\partial_u \rangle$.

$A_{3.4}$-invariant equations,

(1) $u_{tt} = u_{xx} + x^{-1}[2ln|u| + mx^{-1}t + n]u_x + x^{-2} u ln|u| + (mx^{-1}t + n - 2)x^{-2} u ln\,|u| + \frac{1}{4} m^2 x^{-4} t^2 u + \frac{1}{2} m(n - 3)x^{-3} tu + px^{-2}u (m \neq 0, n, p \in R), \; \langle t\partial_t + x\partial_x, x^{-1} u\partial_u, \partial_t - (m/2)x^{-1} ln|x| u\partial_u \rangle$.

$A_{3.5}$-invariant equations,

(1) $u_{tt} = u_{xx} + |u|^m u_x + n|u|^{1+2m}, (m \neq 0, n \in R), \langle \partial_t, \partial_x, t\partial_t + x\partial_x - m^{-1} u\partial_u \rangle$,

(2) $u_{tt} = u_{xx} + e^u u_x + ne^{2u} \; (n \in R), \langle \partial_t, \partial_x, t\partial_t + x\partial_x - \partial_u \rangle$,

(3) $u_{tt} = u_{xx} - x^{-1}[2ln\,|u| - mx^{-1}t - n]u_x + x^{-2}uln^2|u| - x^{-2}(mx^{-1}t + n)uln|u| + ux^{-2}[(m/4)x^{-2}t^2] + [(m/2)(n - 1)x^{-1}t + p](m, n, p \in R)$, $\langle t\partial_t + x\partial_x, xu\partial_u, \partial_t + (m/4)x^{-1}u\partial_u\rangle$.

$A_{3.7}$-invariant equations,

(1) $u_{tt} = u_{xx} - 2mx^{-1}u_x ln|u| + mx^{-2}[muln^2|u| - (m-1)uln|u| + nu] \times (m \neq 0,0,1; n \in R)$,
$\langle \partial_t, |x|^m u\partial_u, t\partial_t + x\partial_x\rangle$,

(2) $u_{tt} = u_{xx} - x^{-1}[2k + ln|u| - mx^{-1}t - n]u_x + k^2x^{-2}uln^2|u| - kx^{-2}[mtx^{-1} + k + n - 1]uln|u| + \frac{1}{2}m(k - 2 + n)tx^{-3}u + \frac{1}{4}m^2t^2x^{-4}u + px^{-2}u$ $(|k| \neq 0,1; m \neq 0, n, p \in R), \langle t\partial_t + x\partial_x, |x|^k u\partial_u, \partial_t + [m/2\,(1+k)]x^{-1}u\partial_u\rangle$.

This completes the group classification of nonlinear equations (124).

(4-3-3) Group Classification of Eq. (125):

Omitting calculation details we present below the determining equations for symmetry operators admitted by Eq. (125).

Assertion 2: The maximal invariance group of PDE (125) is generated by the infinitesimal operator:

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + [h(t)u + r(t,x)]\partial_u, \quad (130)$$

where τ, ξ, h, r, f, g are smooth functions satisfying the conditions

$$r_{tx} + f[h - \tau_t - \xi_x] = gr_x + \tau f_t + \xi f_x + [hu + r]f_u,$$

$$h_t = \tau_t g + \tau g_1 + \xi g_x. \quad (131)$$

Assertion 3: The equivalence group ε of (125) is formed by the following transformations of the space V:

(1) $\bar{t} = T(t), \ \bar{x} = X(x), \ v = V(t)u + Y(t,x), \ t'X'V \neq 0$,

(2) $\bar{t} = T(x), \ \bar{x} = X(t), \ v = \psi(x)\phi(t,x)u + Y(t,x)$, (132)

$$\phi(t,x) = exp\left(-\int g(t,x)dt\right), g_x \neq 0.$$

As the direct verification shows, given arbitrary functions $g\ and\ f$, it follows from (131) that $\tau = h = \xi = r = 0$. So that in the generic case the maximal invariance group of (125) is the trivial group of identical transformations.

We begin classification of (125) by constructing equations that admit one-dimensional symmetry algebras. The following assertions hold.

Lemma (4.3.3): There exist transformations (132) reducing operator (130) to one of the seven canonical forms given below

$$Q = t\partial_t + x\partial_x,\ \ Q = \partial_t,\ \ Q = \partial_x + tu\partial_u,$$

$$Q = \partial_x + \epsilon u\partial_u, \epsilon = 0,1,\ Q = tu\partial_u, \qquad (133)$$

$$Q = u\partial_u,\ \ Q = r(t,x)\partial_u, \qquad r \neq 0.$$

Theorem (4.3.4):

There exist, at most, three inequivalent nonlinear equations (125) that admit one-dimensional invariance algebras the form of functions f, g and the corresponding symmetry algebras are given below:

$$A_1^1 = \langle t\partial_t + x\partial_x\rangle, g = t^{-1}\tilde{g}(\omega), f = t^{-2}f(u,\omega), \omega = tx^{-1}, \tilde{g}_\omega \neq 0, f_{u\omega} \neq 0,$$

$$A_1^2 = \langle\partial_t\rangle, g = \tilde{g}(x), f = \tilde{f}(x,u), \tilde{g}' \neq 0, \tilde{f}_{uu} \neq 0,$$

$$A_1^3 = \langle\partial_x + tu\partial_u\rangle, g = x + \tilde{g}(t), f = e^{tx}\tilde{f}(t,\omega), \omega = e^{-tx}u, \tilde{f}_{\omega\omega} \neq 0,$$

we proceed now to analyzing Eqs. (125) admitting two-dimensional symmetry algebras.

Theorem (4.3.5):

There exist, at most, three inequivalent nonlinear equations (125) that admit two-dimensional symmetry algebras, all of them being $A_{2.2}$-

invariant equations. The forms of functions $f\ and\ g$ and the corresponding realizations of the Lie algebra $A_{2.2}$ read as

$$A_{2.2}^1 = \langle t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x + mut\partial_u\rangle \quad (m \in R),$$

$$g = [mt + (k-m)x]t^{-1}(t-x)^{-1}, \qquad k \neq 0,$$

$$f = |t-x|^{m-2}|x|^{-m}\tilde{f}(\omega),$$

$$\omega = u|t-x|^{-m}|x|^m, \tilde{f}_{\omega\omega} \neq 0,$$

$$A_{2.2}^2 = \langle t\partial_t + x\partial_x, t^2\partial_t + mtu\partial_u\rangle \quad (m \in R),$$

$$g = t^{-2}[kx + mt], k \neq 0, f = |t|^{m-2}|x|^{-m}\tilde{f}(\omega),$$

$$\omega = |t|^{-m}|x|^m u, \qquad \tilde{f}_{\omega\omega} \neq 0,$$

$$A_{2.2}^3 = \langle t\partial_t + x\partial_x, x^2\partial_x + tu\partial_u\rangle,$$

$$g = (tx)^{-1}(mx - t)(m \in R), f = x^{-2}exp(-tx^{-1})\tilde{f}(\omega),$$

$$\omega = u\ exp(tx^{-1}), \tilde{f}_{\omega\omega} \neq 0.$$

Note that if the function $\tilde{f}$ is arbitrary, then the invariance algebras given in the statement 0f the theorem (4-3-5) are maximal.

It turns out that the above theorems provide complete group classification of the class PDEs (125). Namely, the following assertion holds true.

Theorem (4.3.5):

A nonlinear equation (125) having nontrivial symmetry properties is equivalent to one of the equations listed in theorem (4-3-4) and (4-3-5).

(4-3-4) Group Classification of Eq. (126):

As earlier, we present the results of the first step of group classification algorithm skipping derivation details.

Assertion 4: Invariance group of equation (126) is generated by infinitesimal operator

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + (ku + r(t,x))\partial_u, \quad (134)$$

where k is a constant τ, ξ, r, f are functions satisfying the relation

$$r_{tx} + [k - \tau' - \xi']f = \tau f_t + [ku + r]f_u. \quad (135)$$

Assertion 5: Equivalence group ε of the class of equations (126) is formed by the following transformations:

$$(1)\ \bar{t} = T(t),\ \bar{x} = X(x),\quad v = mu + Y(t,x),$$

$$(2)\ \bar{t} = T(x),\quad \bar{x} = X(t),\ v = mu + Y(t,x), T'X'm \neq 0. \quad (136)$$

Note that given an arbitrary function f, it follows from (135) that $\tau = \xi = k = r = 0, i.e.$, the group admitted is trivial. To obtain equations with nontrivial symmetry we need to specify properly the function f. To this end we perform classification of equations ander study admitting one-dimensional invariance algebras. The following assertions give exhaustive classification of those.

Lemma (4.3.4): there exist transformations from the group ε (136) that reduce (134) to one of the four canonical forms,

$$Q = \partial_t + \partial_x + \epsilon u \partial_u (\epsilon = 0,1),$$

Number	function f	symmetry operator	type
1	$e^{t}\tilde{f}(\omega)$, $\omega = ue^{-t}, \tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u, \partial_x$	$A_{2.1}$
2	$e^{t+x}\tilde{f}(\omega)$, $\omega = ue^{-t-x}, \tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u$, $\partial_t + u\partial_u$,	$A_{2.1}$
3	$(t-x)^{-3}\tilde{f}(\omega)$, $\omega = (t-x)u, \tilde{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x + u\partial_u$, $\partial_t + \partial_x$	$A_{2.2}$
4	$x^{-1}\tilde{f}(\omega)$, $\omega = x^{-1}u, \ddot{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x - u\partial_u$, ∂_t	$A_{2.2}$
5	$(t-x)^{-2}\tilde{f}(u)$, $\tilde{f}_{uu} \neq 0$	$\partial_t + \partial_x$, $t\partial_t + x\partial_x$, $t^2\partial_t + x^2\partial_x$	$sI(2,R)$
6	$exp(x^{-1}u)$	$-t\partial_t + x\partial_u$, $\partial_t, t\partial_t - 1/mu\partial_u$,	$A_{2.2}\oplus A_1$
7	$\lambda\lvert x\rvert^{-m-2}\lvert u\rvert^{m+1}$ $\lambda \neq 0, m \neq 0,-1,-2$	$\partial_t, t\partial_t - 1/mu\partial_u$, $x\partial_x + m + 1/mu\partial_u$	$A_{2.2}\oplus A_1$
8	$\tilde{f}(u), \tilde{f}_{uu} \neq 0$	$\partial_t, \partial_x, -t\partial_t - x\partial_x$	$A_{1.6}$
9	$\lambda\lvert u\rvert^{n+1}, \lambda \neq 0, n \neq 0,-1$	$t\partial_t - 1/nu\partial_u$ ∂_t, ∂_x	$A_{2.2}\oplus A_{2.2}$

Table 2. Invariant Equations (126) Invariance Algebra:

$$Q = \partial_t + \epsilon u\partial_u \quad (\epsilon = 0,1),$$

$$Q = u\partial_u, \qquad Q = g(t,x)\partial_u \quad (g \neq 0),$$

Theorem (4.3.6):

There exist exactly two nonlinear equations of the form (126) admitting one-dimensional invariance algebras. The corresponding expressions for function f and invariance algebras are given below.

$$A_1^1 = \langle \partial_t + \partial_x + \epsilon u\partial_u \rangle \quad (\epsilon = 0,1), \qquad f = e^{\epsilon t}\tilde{f}(\theta,\omega),$$

$$\theta = t - x, \ \ \omega = e^{-\epsilon t}u, \ \ \tilde{f}_{\omega\omega} \neq 0,$$

$$A_1^2 = \langle \partial_x + \epsilon u \partial_u \rangle \quad (\epsilon = 0,1), f = e^{\epsilon t}\tilde{f}(x,\omega), \omega = e^{-\epsilon t}u, \tilde{f}_{\omega\omega} \neq 0.$$

Our analysis of Eqs. (126) admitting higher dimensional invariance algebras yields the following assertion.

Theorem (4.3.7):

The Liouville equation $u_{tx} = \lambda e^u, \lambda \neq 0$, has the highest symmetry among equations (126). Its maximal invariance algebra is infinite-dimensional and is spanned by the following infinite set of basis operators:

$$Q = h(t)\partial_t + g(x)\partial_x - (h'(t)) + g'(x))\partial_u,$$

where h and g are arbitrary smooth functions. Next, there exist exactly nine inequivalent equations of the form (126), whose maximal invariance algebras have dimension higher that one. We give these equations and their invariance algebras in table 2.

CHAPTER FIVE

GROUP CLASSIFICATION OF NONLINEAR WAVE EQUATIONS

Chapter Five

Group Classification of Nonlinear Wave Equations

(5-1) Group Classification of Eq. (123):

The first step of the algorithm of group classification of (123),

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \quad F_{u_x, u_x} \neq 0,$$

has been partially performed in see. It follows from theorem (4-1) that the invariance group of Eq. (123) is generated by infinitesimal operator (124). What is more, the real constants $\lambda, \lambda_1, \lambda_2$ and group of the class of equations (123) is formed by transformations (127).

With these facts in hand we can utilize results of group classification of Eq. (124) in order to classify Eq. (123). In particular, using lemma 1 and lemma 2, it is straightforward to verify that the following assertions hold true.

Theorem (5.1.1):

There are, at most, seven inequivalent classes of nonlinear equations (123) invariant under the one-dimensional Lie algebras.

Below we give the full list of the invariant equations and the corresponding invariance algebras,

$$A_1^1 = \langle t\partial_t + x\partial_x \rangle, Ft^{-2}G(\xi, u, \omega), \xi = tx^{-1}, \omega = xu_x,$$

$$A_1^2 = \langle \partial_t + k\partial_x \rangle \;\; (k > 0), F = G(\eta, u, u_x), \eta = x - kt,$$

$$A_1^3 = \langle \partial_x \rangle, \qquad F = G(t, u, u_x),$$

$$A_1^4 = \langle \partial_t \rangle, \quad F = G(x, u, u_x),$$

$$A_1^5 = \langle \partial_t + f(x)u\partial_u \rangle, (f \neq 0),$$

$$F = -tf''u + t^2(f')^2u - 2tf'u_x + e^{tf}G(x, v, \omega),$$

$$v = e^{tf}u, \ \ \omega = u^{-1}u_x - f'f^{-1}ln|u|,$$

$$A_1^6 = \langle f(x)u\partial_u\rangle \ (f \neq 0), F = -f^{-1}f''uln|u| - 2f^{-1}f'u_xln|u| + f^{-2}(f')^2uln^2|u| + uG(t, x, \omega), \omega = u^{-1}u_x - f'f^{-1}ln|u|,$$

$$A_1^7 = \langle f(t, x)\partial_u\rangle \ \ (f \neq 0), F = f^{-1}(f_u - f_{xx})u + G(t, x, \omega),$$

$$\omega = u_x - f^{-1}f_xu.$$

Note that if the functions F and G are arbitrary, then the presented algebras are maximal (in Lie's sense) symmetry algebras of the respective equations.

Theorem (5.1.2):

An equation of the form (123) cannot admit Lie algebra which has a subalgebra having nontrivial Livi factor. With account of the above facts we conclude that nonlinear equations (123) whose invariance algebras are two-dimensional solvable Lie algebras.

Below we present the list of invariant equations and the corresponding realizations of the two-dimensional invariance algebras.

(1) $A_{2.1}$-invariant equations,

$$A_{2.1}^1 = \langle t\partial_t + x\partial_x, u\partial_u\rangle, \qquad F = x^{-2}uG(\xi, \omega),$$

$$\xi = tx^{-1}, \omega = u^{-1}xu_x,$$

$$A_{2.1}^{2} = \langle t\partial_t + x\partial_x, \sigma(\xi)\partial_u \rangle \quad (\sigma \neq 0, \xi = tx^{-1}),$$

$$F = x^{-1}[\sigma^{-1}((1-\xi^2)\sigma'' - 2\xi\sigma')u + G(\xi,\omega)],$$

$$\omega = \xi\sigma' u + \sigma x u_x,$$

$$A_{2.1}^{3} = \langle \partial_t + k\partial_x, u\partial_u \rangle \quad (k > 0), F = uG(\eta,\omega),$$

$$\eta = x - kt, \qquad \omega = u^{-1}u_x,$$

$A_{2.1}^{4} = \langle \partial_t + k\partial_x, \varphi(\eta)\partial_u \rangle$ $(k > 0, \eta = x - kt, \varphi \neq 0),$

$F = (k^2 - 1)\varphi''\varphi^{-1}u + G(\eta,\omega), \omega = \varphi u_x - \varphi' u,$

$$A_{2.1}^{5} = \langle \partial_t + k\partial_x, u\partial_u \rangle \ (k > 0),$$

$F = e^{\eta}G(\omega, v), \eta = x - kt, \omega = ue^{-\eta}, v = u^{-1}u_x,$

$$A_{2.1}^{6} = \langle \partial_t, \partial_x \rangle, \ F = G(u, u_x),$$

$A_{2.1}^{7} = \langle \partial_x, u\partial_u \rangle, \quad F = uG(t,\omega), \ \omega = u^{-1}u_x,$

$$A_{2.1}^{8} = \langle \partial_x, \varphi(t)\partial_u \rangle \quad (\varphi \neq 0),$$

$$F = \varphi^{-1}\varphi'' u + G(x, u_x),$$

$$A_{2.1}^{9} = \langle \partial_t, \partial_u \rangle,\ F = G(x, u_x),$$

$$A_{2.1}^{10} = \langle \partial_t, f(x)u\partial_u \rangle \quad (f \neq 0),$$

$$F = -u^{-1}u_x^2 + uG(t, \omega),$$

$$\omega = u^{-1}u_x - f'f^{-1}ln\,|u|,$$

$A_{2.1}^{11} = \langle \partial_t + f(x)u\partial_u, g(x)u\partial_u \rangle (\delta = f^{-1}f' - g^{-1}g' \neq 0),$

$F = -g^{-}g''uln|u| - 2g^{-1}g'u_x ln|u|$

$+g^{-2}(g')^2 uln^2|u| - 2f\delta tu_x + 2f\delta g'g^{-1}tuln|u| + f^2\delta^2 t^2 u + f(g^{-1}g'' - f^{-1}f'')tu + uG(x, \omega),$

$$\omega = u^{-1}u_x - g'g^{-1}ln|u| - tf\delta,$$

$$A_{2.1}^{12} = \langle \partial_t + f(x)u\partial_u, e^{tf}\partial_u \rangle \ (f \neq 0),$$

$F = [f^2 - tf'' + t^2(f')^2]u - 2tf'u_x + e^{tf}G(x, \omega),$

$$\omega = e^{-tf}(u_x - tf'u),$$

$A_{2.1}^{13} = \langle f(x)u\partial_u, g(x)u\partial_u \rangle \quad (\delta = f'g - g'f \neq 0),$

$$F = -u^{-1}u_x^2 - \delta^{-1}\delta' u_x + \delta^{-1}[f''g' - g''f']u\, ln|u| + uG(t,x),$$

$$A_{2.1}^{14} = \langle \varphi(t)\partial_u, \psi(t)\partial_u \rangle \quad (\varphi'\psi - \varphi\psi' \neq 0),$$

$$F = \varphi^{-1}\varphi'' u + G(t,x,u_x), \; \varphi''\psi - \varphi\psi'' = 0.$$

(II) $A_{2.2}$-invariant equations,

$$A_{2.2}^{1} = \langle t\partial_t + x\partial_x, xu\partial_u \rangle,$$

$$F = x^{-2}u ln^2|u| - 2x^{-1}u_x ln|x| + t^{-2}uG(\xi,\omega),$$

$$\xi = tx^{-1},$$

$$\omega = xu^{-1}u_x - ln|u|,$$

$$A_{2.2}^{2} = \langle t\partial_t + x\partial_x, t\varphi(\xi)\partial_u \rangle \quad (\varphi \neq 0, \xi = tx^{-1}),$$

$F = t^{-2}(1-\xi^2)\varphi^{-1}\xi(2\varphi' + \xi\varphi'')u + t^{-2}G(\xi,\omega),$

$$\omega = x\varphi u_x + \xi\varphi' u,$$

$$A_{2.2}^{3} = \langle \partial_t + k\partial_x, exp(k^{-1}x)u\partial_u \rangle (k > 0),$$

$$F = k^{-2}uln^2|u| - 2k^{-1}u_x ln|u| - k^{-2}uln|u| + uG(\eta, \omega),$$

$$\eta = x - kt, \omega = u^{-1}u_x - k^{-1}ln|u|,$$

$$A_{2.2}^4 = \langle \partial_t + k\partial_x, e^t\varphi(\eta)\partial_u \rangle \; (\eta = x - kt, k > 0, \varphi \neq 0),$$

$$F = ((k^2 - 1)\varphi''\varphi^{-1} - 2k\varphi'\varphi^{-1} + 1)u + G(\eta, \omega),$$

$$\omega = \varphi u_x - \varphi' u, \;\; \varphi' = \frac{d\varphi}{d\eta},$$

$$A_{2.2}^5 = \langle -t\partial_t - x\partial_x, \partial_t + k\partial_x \rangle (k > 0),$$

$$F = \eta^{-2}G(u, \omega), \;\; \eta = x - kt, \omega = u_x\eta,$$

$$A_{2.2}^6 = \langle -t\partial_t - x\partial_x + mu\partial_u, \partial_t + k\partial_x \rangle \; (k > 0, m \neq 0),$$

$$F = |\eta|^{-2-m}G(v, \omega), \eta = x - kt,$$

$$\omega = u|\eta|^m, \quad v = u_x|\eta|^{m+1},$$

$$A_{2.2}^7 = \langle \partial_x, e^x u\partial_u \rangle, F = uln^2|u| - uln|u| - 2u_x ln|u| + uG(t, \omega)$$

$$\omega = u^{-1}u_x - ln|u|,$$

$$A_{2.2}^{8} = \langle \partial_x, e^x \varphi(t) \partial_u \rangle \quad (\varphi \neq 0),$$

$$F = (\varphi^{-1}\varphi'' - 1)u + G(t, \omega), \ \omega = u_x - u,$$

$$A_{2.2}^{9} = \langle -t\partial_t - x\partial_x, \partial_t \rangle, \ F = t^{-2}G(u, tu_x),$$

$$A_{2.2}^{10} = \langle -t\partial_t - x\partial_x + ku\partial_u, \partial_x \rangle \ (k \neq 0),$$

$$F = |t|^{-2-k}G(v, \omega), \ v = |t|^k u, \ \omega = |t|^{k+1}u_x,$$

$$A_{2.2}^{11} = \langle \partial_t, e^t \partial_u \rangle, \ F = u + G(x, u_x),$$

$$A_{2.2}^{12} = \langle -t\partial_t - x\partial_x, \partial_t \rangle, \qquad F = x^{-2}G(u, \omega), \omega = xu_x,$$

$$A_{2.2}^{13} = \langle \partial_t + f(x)u\partial_u, e^{(1+f)t}\partial_u \rangle (f \neq 0),$$

$$F = -(tf'' - t^2(f')^2 - (1 + f^2))u - 2tf'u_x + e^{tf}G(x, \omega),$$

$$\omega = e^{-tf}(u_x - f'(t + f^{-1})u),$$

$$A_{2.2}^{14} = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u \rangle \quad (k > 0),$$

$$F = -2ktx^{-3}u + k^2t^2x^{-4}u + x^{-2}exp(ktx^{-1})G(v, \omega),$$

$$v = exp(-kx^{-1}t)u, \omega = xu^{-1}u_x + ln|u|,$$

$$A_{2.2}^{15} = \langle k(t\partial_t + x\partial_x), |x|^{k-1}u\partial_u\rangle \quad (k \neq 0,1),$$

$$F = -k^{-2}(1-k)x^{-2}uln|u| - 2k^{-1}u_x ln|u| + k^{-2}x^{-2}uln^2|u| + x^{-2}uG(v,\omega),$$

$$v = tx^{-1}, \;\; \omega = xu^{-1}u_x - k^{-1}ln|u|,$$

$$A_{2.2}^{16} = \langle k(t\partial_t + x\partial_x), |t|^{k^{-1}}\varphi(\xi)\partial_u\rangle (k \neq 0,1, \varphi \neq 0, \xi = tx^{-1}),$$

$$F = [k^{-1}(k^{-1}-1) + 2\xi(k^{-1}-\xi^2)\varphi^{-1}\varphi' + \xi^2(1-\xi)^2\varphi^{-1}\varphi''] \times (t^{-2}u + t^{-2}G(\xi,\omega), \omega = x\varphi u_x + \xi\varphi' u.$$

In the above formulas G stands for an arbitrary smooth function. As customary, the prime denotes the derivatives of a function of one variable.

(5-2) Group Classification of the Equation $u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)uln|u| + uD(t,x)$:

Before analyzing Eqs. (123) admitting algebras of the dimension higher than two we perform group classification of the equation

$$u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u\, ln|u| + uD(t,x). \qquad (137)$$

Here A(x), B(x), D(t,x) are arbitrary smooth functions. Note that the above class of PDEs contains $A_{21}^{13} - invariant\ equation$. Importantly, class (137) contains a major part of equations of the form (123), whose maximal symmetry algebras have dimension three or four. This fact is used to simplify group classification of Eqs. (123).

The complete account of the symmetry properties of PDE (137) is given the following assertions.

Lemma (5.2.1):

If A, B, and D are arbitrary, then the maximal invariance algebra of PDE (137) is the two-dimensional Lie algebra equivalent to $A_{2.1}^{13}$ and (137) reduces to $A_{2.1}^{13}$-invariant equation.

Next, if the maximal symmetry algebra of an equation of the form (137) is three-dimensional (we denote it as A_3), then this equation is equivalent to one of the following ones:

$$(I)A_3 \sim A_{3.1}, \quad A_3 = \langle \partial_t, f(x)u\partial_u, \varphi(x)u\partial_u \rangle,$$

$$A = -\sigma^{-1}\sigma', \ \ B = \sigma^{-1}\rho, \ \ D = 0 \quad \sigma = f'\varphi - f\varphi' \neq 0,$$

$$\rho = \varphi' f'' - \varphi'' f',$$

$$(II)A_3 \sim A_{3.1}, \quad A_3 = \langle f(x)u\partial_u, \varphi(x)u\partial_u, \partial_t + \psi(x)u\partial_u \rangle,$$

$$A = -\sigma^{-1}\sigma', \qquad B = \sigma^{-1}\rho,$$

$$D = t\sigma^{-1}[\sigma'\psi' - \psi\rho - \sigma\psi''],$$

$$\sigma = f'\varphi - \varphi' f \neq 0, \ \ \rho = f''\varphi' - \varphi'' f',$$

$$f'\psi - f\psi' \neq 0, \quad \varphi'\psi - \varphi\psi' \neq 0,$$

$$(III)D = x^{-2}G(\xi), \ \ \xi = tx^{-1}, \qquad G \neq 0,$$

$$(1)A_3 \sim A_{3.2}, \qquad A_3 = \langle t\partial_t + x\partial_x, u\partial_u, |x|^{1-n}u\partial_u \rangle,$$

$$A = nx^{-1} \ \ (n \neq 0), \qquad B = 0,$$

$$(2)A_3 \sim A_{3.3}, \qquad A_3 = \langle t\partial_t + x\partial_x, u\partial_u, u \ ln|x|\partial_u \rangle,$$

$$A = x^{-1}, \ \ B = 0,$$

$$(3)A_3 \sim A_{3.4}, \qquad A = \langle t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, \sqrt{|x|} \ ln|x|u\partial_u \rangle,$$

$$A = 0, \qquad B = \frac{1}{4}x^{-2},$$

$(4) A_3 \sim A_{3.9}, A_3$

$$= \langle t\partial_t + x\partial_x, \sqrt{|x|} cox\left(\frac{1}{2}\beta ln|x|\right) u\partial_u, \sqrt{|x|} sin\left(\frac{1}{2}\beta ln|x|\right) u\partial_u \rangle,$$

$$(A = 0, \quad B = mx^{-2},)$$

$$m > \frac{1}{4}, \quad \beta = \sqrt{4m-1},$$

$$(5) A_3 \sim A_{3.7}, A_3 = \langle t\partial_t + x\partial_x, \left(\sqrt{|x|}\right)^{1+\beta} u\partial_u, \left(\sqrt{|x|}\right)^{1-\beta} u\partial_u \rangle,$$

$$A = 0, \quad B = mx^{-2}, m < \frac{1}{4}, \quad m \neq 0, \quad \beta = \sqrt{1-4m,}$$

$$(6) A_3 \sim A_{3.8}, A_3 = \langle t\partial_t + x\partial_x, cos\left(\sqrt{m} ln|x|\right) u\partial_u, sin\left(\sqrt{m} ln|x|\right) u\partial_u \rangle,$$

$$A = x^{-1}, \; B = mx^{-2} \; m > 0,$$

$$(7) A_3 \sim A_{3.6}, A_3 = \langle t\partial_t + x\partial_x, |x|^{\sqrt{m}} u\partial_u, |x|^{-\sqrt{m}} u\partial_u \rangle,$$

$$A = x^{-1}, \quad B = mx^{-2}, m < 0,$$

$$(8) A_3 \sim A_{3.4}, A_3 = \langle t\partial_t + x\partial_x, \left(\sqrt{|x|}\right)^{1-n} u\partial_u, \left(\sqrt{|x|}\right)^{1-n} \times ln|x| u\partial_u \rangle,$$

$$A = nx^{-1} \; (n \neq 0,1), \quad B = \frac{1}{4}(n-1)^2 x^{-1},$$

$$(9) \; A_1 \sim A_{3.9}, A_3 =$$

$$\langle t\partial_t + x\partial_x, \left(\sqrt{|x|}\right)^{1-p} \cos\left(\frac{1}{2}\beta ln|x|\right) u\partial_u, \left(\sqrt{|x|}\right)^{1-p} sin\left(\frac{1}{2}\beta ln|x|\right) u\partial_u \rangle,$$

$$A = nx^{-1} \;\; (n \neq 0,1),$$

$$B = mx^{-2} \;\; \left(m > \frac{1}{4}(n-1)^2\right), \quad \beta = \sqrt{4m-(n-1)^2},$$

$$(10) A_3 \sim A_{3.7}, A_3 = \langle t\partial_t + x\partial_x, \left(\sqrt{|x|}\right)^{1-\beta-n} u\partial_u, \left(\sqrt{|x|}\right)^{1-n+\beta} \times u\partial_u \rangle,$$

$$A = nx^{-1} \; (n \neq 0,1), \; B = mx^{-2}$$

$$\left(m < \frac{1}{4}(n-1)^2, m \neq 0\right), \quad \beta = \sqrt{(n-1)^2 - 4m.}$$

$$(IV) \; D = G(t),$$

$$(1) A_3 \sim A_{3.3}, \quad A_3 = \langle \partial_x, u\partial_u, xu\partial_u \rangle,$$

$$A = B = 0,$$

$$(2) \; A_3 = A_{3.2}, \quad A_3 = \langle \partial_x, u\partial_u, e^x u\partial_u \rangle,$$

$$A=-1, \qquad B=1$$

$$(3)\ A_3 \sim A_{3.8}, A_3=\langle \partial_x, cos(x)u\partial_u, sin(x)u\partial_u \rangle,$$

$$A=0, \quad B=1,$$

$$(4)\ A_3 \sim A_{3.6},\ \ A_3=\langle \partial_x, e^x u\partial_u, e^{-x}u\partial_u \rangle,$$

$$A=0, \ \ B=-1,$$

$$(5)\ A_3 \sim A_{3.4}, A_3=\langle \partial_x, exp\left(\tfrac{1}{2}x\right)u\partial_u, exp\left(\tfrac{1}{2}x\right)xu\partial_u \rangle,$$

$$A=-1, \ \ B=\frac{1}{4},$$

$$(6)\ A_3 \sim A_{3.7},\ \ A_3=\langle \partial_x, exp\left(\tfrac{1}{2}(1+\beta)x\right)u\partial_u, exp\left(\tfrac{1}{2}(1-\beta)x\right)u\partial_u \rangle,$$

$$A=-1, B=m \ \left(m<\frac{1}{4}\right), \qquad m\neq 0, \qquad \beta=\sqrt{1-4m},$$

$$(7)\ A_3 \sim A_{3.9}, A_3$$

$$=\langle \partial_x, exp\left(\tfrac{1}{2}x\right)cos\left(\tfrac{1}{2}\beta x\right)u\partial_u, exp\left(\tfrac{1}{2}x\right)sin\left(\tfrac{1}{2}\beta x\right)u\partial_u \rangle,$$

$$A=-1, B=m \ \left(m>\tfrac{1}{4}\right),\ \ \beta=\sqrt{4m-1},$$

$$(V)\ D=G(\eta),\ \ \eta=x-kt,\ \ k>0,$$

$$(1)\ A_3 \sim A_{3.3},\ \ A_3=\langle \partial_t+k\partial_x, u\partial_u, xu\partial_u \rangle,$$

$$A=B=0,$$

$$(2)\ A_3=A_{3.3}, \qquad A_3=\langle \partial_t+k\partial_x, u\partial_u, e^x u\partial_u \rangle,$$

$$A=-1, \quad B=0,$$

$$(3) A_3 \sim A_{3.8}, \qquad A_3=\langle \partial_t+k\partial_x, cos(x)u\partial_u, sin(x)u\partial_u \rangle,$$

$$A=0, \qquad B=1,$$

$$(4) A_3 \sim A_{3.6}, \qquad A_3=\langle \partial_t+k\partial_x, e^x u\partial_u, e^{-x}u\partial_u \rangle,$$

$$A=n, \quad B=-1,$$

$$(5)\ A_3 \sim A_{3.4}, \qquad A_3=\langle \partial_t+k\partial_x, exp\left(\tfrac{1}{2}x\right)u\partial_u, exp\left(\tfrac{1}{2}x\right)xu\partial_u \rangle,$$

$$A=-1, \ \ B=\frac{1}{4},$$

$(6) A_3 \sim A_{3.7}, A_3$

$$= \langle \partial_t + k\partial_u, exp\left(\tfrac{1}{2}(1+\beta)x\right)u\partial_u, exp\left(\tfrac{1}{2}(1-\beta)x\right)u\partial_u\rangle,$$

$$A = -1, B = m \ \left(m < \frac{1}{4}\right), m \neq 0, \qquad \beta = \sqrt{1-4m},$$

$(7)\ A_3 \sim A_{3.9}, A_3$

$$= \langle \partial_t + k\partial_x, exp\left(\tfrac{1}{2}x\right)cos\left(\tfrac{1}{2}\beta x\right)u\partial_u, exp\left(\tfrac{1}{2}x\right)sin\left(\tfrac{1}{2}\beta x\right)u\partial_u\rangle,$$

$$A = -1,\ \ B = m\ \left(m > \tfrac{1}{4}\right),\ \ \beta = \sqrt{4m-1}.$$

Theorem (5.2.3):

Equation $u_{tt} = u_{xx} - u^{-1}u_x^2$ has the wides symmetry group amongst equations of the form (137). Its maxmal invariance algebra is the five-dimensional Lie algebra,

$$A_5^1 = \langle \partial_t, \partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u\rangle,$$

there are no equations of the form (137) which are inequivalent to the above equation and admit invariance algebra of the dimension higher than four. Inequivalent equations (137) admitting four-dimensional algebras are listed below together with their symmetry algebras.

$$(1) D = 0,$$

$$(1) A_4 \sim A_{3.6} \oplus A_1, A_4 = \langle \partial_t, \partial_x, u\ ch(\beta x)\partial_u, u\ sinh(\beta x)\partial_u\rangle,$$

$$A = 0,\ \ B = -\beta^2,\ \ \beta \neq 0,$$

$$(2)\ A_4 \sim A_{3.8} \oplus A_1, \qquad A_4 = \langle \partial_t, \partial_x, u\ cos(\beta x)\partial_u, u\ sin(\beta x)\partial_u\rangle,$$

$$A = 0, \qquad B = \beta^2,\ \ \beta \neq 0,$$

$$(3) A_4 \sim A_{2.1} \oplus A_{2.2}, \quad A_4 = \langle \partial_t, \partial_x, u\partial_u, e^{-x}u\partial_u\rangle, A = 1, B = 0,$$

$$(4) A_4 \sim A_{3.4}, \qquad A_4 = \langle \partial_t, \partial_x, e^{-x}u\partial_u, xe^{-x}u\partial_u\rangle, \qquad A = 2, \qquad B = 1,$$

$$(5) A_4 \sim A_{3.9} \oplus A_1, \qquad A_4 = \langle \partial_t, \partial_x, ue^{-x}cos(\beta x)\partial_u, ue^{-x}sin(\beta x)\partial_u\rangle,$$

$$A = 2, \qquad B = m,\ \ m > 1, \qquad \beta = \sqrt{m-1},$$

$$(6) A_4 \sim A_{3.7} \oplus A_1, \qquad A_4 = \langle \partial_t, \partial_x, ue^{-x}ch(\beta x)\partial_u, ue^{-x}sinh(\beta x)\partial_u\rangle,$$

$$A = 2,\ \ B = m, \qquad m > 1, \qquad m \neq 0, \qquad \beta = \sqrt{1 - m},$$

$$(7) A_4 \sim A_{4.2},\ \ A_4 = \langle \partial_t, t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, u\sqrt{|x|}ln|x|\partial_u \rangle,$$

$$A = 0, \qquad B = \frac{1}{4}x^{-2},$$

$$(8) A_4 \sim A_{4.5}, \qquad A_4 = \langle \partial_t, t\partial_t + x\partial_x |x|^{\frac{1}{2}+\beta}u\partial_u, |x|^{\frac{1}{2}-\beta}u\partial_u \rangle, A = 0,$$

$$B = mx^{-2}, \qquad m < \tfrac{1}{4},\ \ m \neq 0,\ \ \beta = \sqrt{\tfrac{1}{4} - m},$$

$$(9) A_4 \sim A_{4.6}, A_4 = \langle \partial_t, t\partial_t + x\partial_x, \sqrt{|x|}cos(\beta ln|x|)u\partial_u, \sqrt{|x|}sin(\beta ln|x|)u\partial_u \rangle,$$

$$A = 0, \qquad B = mx^{-2}, \qquad m > \tfrac{1}{4},\ \ \beta = \sqrt{m - \tfrac{1}{4}},$$

$$(10) A_4 \sim A_{4.3}, \qquad A_4 = \langle \partial_t, t\partial_t + x\partial_x, u\ ln|x|\partial_u, u\partial_u \rangle, A = x^{-1}, B = 0,$$

$$(11) A_4 \sim A_{3.7} \oplus A_1, \qquad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{1-n}u\partial_u, u\partial_u \rangle,$$

$$A = nx^{-1}, B = 0, \qquad n \neq 0,1,$$

$$(12) A_4 \sim A_{4.5}, \qquad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)}u\partial_u, |x|^{\frac{1}{2}(1-n)}u\ ln|x|\partial_u \rangle,$$

$$A = nx^{-1}, \qquad B = \frac{1}{4}(n-1)^2 x^{-2}, \qquad n \neq 0,1,$$

$$(13) A_4 \sim A_{4.5},\ \ A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n+\beta)}u\partial_u, |x|^{\frac{1}{2}(1-n-\beta)}u\partial_u \rangle,$$

$$A = nx^{-1}, \qquad B = mx^{-2},\ \ m < \frac{1}{4}(n-1)^2, \qquad m \neq 0, n \neq 0,$$

$$\beta = \sqrt{(n-1)^2 - 4m},$$

$$(14) A_4 \sim A_{4.6}, A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)}cos(\beta ln|x|)u\partial_u, |x|^{\frac{1}{2}(n-1)}sin(\beta ln|x|)u\partial_u \rangle,$$

$$A = nx^{-1}, \qquad B = mx^{-2},$$

$$m \neq 0, n \neq 0, \qquad m > \frac{1}{4}(n-1)^2,\ \ \beta = \sqrt{m - \frac{1}{4}(n-1)^2},$$

$$(II) D = ktx^{-3}, \qquad k > 0,$$

$$(1) A_4 \sim A_{4.1},\ A_4 = \langle \partial_t - \frac{1}{2} k x^{-1} u \partial_u, t\partial_t + x\partial_x, x u \partial_u, u\partial_u \rangle, A = B = 0,$$

$$(2) A_4 \sim A_{4.2}, A_4 = \langle \partial_t - \frac{4}{9} k x^{-1} u \partial_u, t\partial_t + x\partial_x, \sqrt{|x|} u \partial_u, \sqrt{|x|} ln|x| u \partial_u \rangle,$$

$$A = 0,\ \ B = \frac{1}{4} x^{-2},$$

$$(3) A_4 \sim A_{4.5}, A_4 = \langle \partial_t - [k/(m+2)] x^{-1} u \partial_u, t\partial_t + x\partial_x, |x|^{\frac{1}{2}+\beta} u \partial_u, |x|^{\frac{1}{2}-\beta} u \partial_u \rangle,$$

$$A = 0, \quad B = m x^{-2}, \quad m \neq 0, -2, \quad m < \frac{1}{4},\ \beta = \sqrt{\frac{1}{4} - m},$$

$$(4) A_4 \sim A_{4.2}, A_4 = \langle \partial_t + \frac{1}{9} k x^{-1} (1 + 3\, ln|x| u) \partial_u, t\partial_t + x\partial_x, x^2 u \partial_u, x^{-1} u \partial_u \rangle,$$

$$A = 0, \quad B = -2x^{-2},$$

$$(5) A_4 \sim A_{4.6}, A_4 = \langle \partial_t - [k/(m+2)] x^{-1} u \partial_u, t\partial_t + x\partial_x, \sqrt{|x|} u cos(\beta\, ln|x|) \partial_u, \sqrt{|x|} u\, sin(\beta\, ln|x|) \partial_u \rangle,$$

$$A = 0, \quad B = m x^{-2}, \quad m > \frac{1}{4},\ \ \beta = \sqrt{m - \frac{1}{4}},$$

$$(6) A_4 \sim A_{4.3},\ \ A_4 = \langle \partial_t - k x^{-1} u \partial_u, t\partial_t + x\partial_x, u\partial_u, u\, ln|x| \partial_u \rangle,$$

$$A = x^{-1}, \quad B = 0,$$

$$(7) A_4 \sim A_{3.4} \oplus A_1, A_4 = \langle \partial_t + k x^{-1} (1 + ln|x|) u \partial_u, t\partial_t + x\partial_x, u\partial_u, x^{-1} u \partial_u \rangle,$$

$$A = 2x^{-1},\ \ B = 0,$$

$$(8) A_4 \sim A_{3.7} \oplus A_1, A_4 = \langle \partial_t + [k/(n-2)] x^{-1} u \partial_u, t\partial_t + x\partial_x, u\partial_u, |x|^{1-n} u \partial_u \rangle,$$

$$A = n x^{-1},\ \ B = 0,\ \ n \neq 0,1,2,$$

$$(9) A_4 = A_{4.4},\ \ A_4 = \langle \partial_t - \frac{1}{2} k x^{-1} ln^2|x| u \partial_u, t\partial_t + x\partial_x, x^{-1} u \partial_u, x^{-1} ln|x| u \partial_u \rangle,$$

$$A = 3x^{-1}, \quad B = x^{-2},$$

(10)$A_2 = A_{4.4}, A_4 = \langle \partial_t - [4k/(n-3)^2]x^{-1}u\partial_u, t\partial_t + x\partial_x, x^{-1}u\partial_u, x^{-1}ln|x|u\partial_u \rangle,$

$$A = nx^{-1}, \ \ B = \frac{1}{4}(n-1)^2x^{-2}, \qquad n \neq 0,3,$$

(11)$A_4 \sim A_{4.5}, A_4$

$$= \langle t\partial_t + x\partial_x, \partial_t - [k(2-n+m)]x^{-1}u\partial_u, |x|^{\frac{1}{2}(1-n+\beta)}u\partial_u |x|^{\frac{1}{2}(1-n-\beta)}u\partial_u \rangle,$$

$$A = nx^{-1}, \quad B = mx^{-2},$$

$$n \neq 0,2, \ \ m \neq n-2, \ \ m < \frac{1}{2}(n-1)^2, \ \ \beta = \sqrt{(n-1)^2 - 4m},$$

(12)$A_4 \sim A_{4.2}, A_4 = \langle t\partial_t + x\partial_x, \partial_t + [k/(3-n)]x^{-1}ln|x|u\partial_u, x^{-1}u\partial_u, |x|^{2-n}u\partial_u \rangle,$

$$A = nx^{-1}, B = (n-2)x^{-2}, \quad n \neq 0,2,3,$$

(13)$A_4 \sim A_{4.6}, \quad A_4$

$$= \langle t\partial_t + x\partial_x, \partial_t - [k/(2-n+m)]x^{-1}u\partial_u, |x|^{\frac{1}{2}(1-n)}ucos(\beta \ ln|x|)\partial_u, |x|^{\frac{1}{2}(1-n)}usin(\beta \ ln|x|)\partial_u \rangle,$$

$$A = nx^{-1}, \qquad B = mx^{-2}, n \neq 0, \qquad m \neq 0, \qquad m > \frac{1}{4}(n-1)^2,$$

$$\beta = \sqrt{m - \frac{1}{4}(n-1)^2},$$

$$(III) D = kt, \qquad k > 0,$$

(1)$A_4 \sim A_{4.1}, \qquad A_4 = \langle \partial_x, \partial_t - \frac{1}{2}kx^2u\partial_u, xu\partial_u, u\partial_u \rangle, A = B = 0,$

(2)$A_4 \sim A_{4.3}, A_4 = \langle \partial_x, \partial_t - kxu\partial_u, e^{-x}u\partial_u, u\partial_u \rangle, \qquad A = 1, B = 0,$

(3)$A_4 \sim A_{3.8} \oplus A_1,$

$$A_4 = \langle \partial_x, \partial_t - k\beta^{-2}u\partial_u, ucos(\beta x)\partial_u, usin(\beta x)\partial_u \rangle,$$

$$A = 0, \ \ B = \beta^2, \ \ \beta \neq 0,$$

(4)$A_4 \sim A_{3.6} \oplus A_1, \quad A_4$

$$= \langle \partial_x, \partial_t + k\beta^{-2}u\partial_u, u \ ch(\beta x)\partial_u, u \ sinh(\beta x)\partial_u \rangle,$$

$$A = 0, \ \ B = -\beta^2, \ \ \beta \neq 0,$$

(5)$A_4 \sim A_{3.4} \oplus A_1,$

$$A_4 = \langle \partial_x, \partial_t - 4ku\partial_u, exp\left(-\frac{1}{2}x\right)u\partial_u, x \ exp\left(-\frac{1}{2}x\right)u\partial_u \rangle,$$

$$A = 1, \ \ B = \frac{1}{4},$$

$(6) A_4 \sim A_{3.7} \oplus A_1, A_4$
$$= \langle \partial_x, \partial_t - km^{-1}u\partial_u, exp\left(-\tfrac{1}{2}(1-\beta)x\right)u\partial_u, exp\left(-\tfrac{1}{2}(1+\beta)x\right)u\partial_u \rangle,$$

$$A = 1, \quad B = m, \ m < \frac{1}{4}, \ \ m \neq 0, \ \beta = \sqrt{1-4m},$$

$(7) A_4 \sim A_{3.9} \oplus A_1, A_4 =$
$\langle \partial_x, \partial_t - km^{-1}u\partial_u, exp\left(-\tfrac{1}{2}x\right)cos(\beta x)u\partial_u, exp\left(-\tfrac{1}{2}x\right)sin(\beta x)u\partial_u \rangle,$

$$A = 1, \ B = m, \quad m > \frac{1}{4}, \ \ \beta = \sqrt{m - \frac{1}{4}},$$

$$(IV) D = kx^{-2}, \ \ k \neq 0,$$

$$A_4 \sim A_{4.8} \ \ (q = -1), A_4 = \langle \partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle, \ \ A = B = 0,$$

$$(V) D = m(x - kt)^{-2}, \ \ k > 0, \ \ m \neq 0,$$

$$A_4 \sim A_{4.8} \ \ (q = -1), \ \ A_4 = \langle \partial_t + k\partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle, A = B = 0.$$

(5-3) Nonlinear Equations (123) Invariant Under Three-Dimensional Lie Algebras:

Equations of the form (123) cannot be invariant under the algebra which is isomorphic to a Lie algebra with a nontrivial Levi ideal. That is why, to complete the second step of our classification algorithm it suffices to consider three-dimensional solvable real Lie algebras.

We begin by considering two decomposable three-dimensional solvable Lie algebras.

Note that while classifying invariant equations (123) we skip those belonging to the class (137), since the latter has already been analyzed.

(5-4) Invariance Under Decomposable Lie Algebras:

As $A_{3.1} = 3A_1 = A_{2.1} \oplus A_1, A_{3.2} = A_{2.2} \oplus A_1$, to construct all realizations of $A_{3.1}$ it suffices to compute all possible extensions of the (already known) realizations of the algebras $A_{2.1} = \langle e_1, e_2 \rangle$ and $A_{2.2} = \langle e_1, e_2 \rangle$. To this end we need to supplement the latter by a basis operator e_3 of the form (120) in order to satisfy the commutation relations

$$[e_1, e_3] = [e_2, e_3] = 0. \qquad (138)$$

What is more, to simplify the form of e_3 we may use those transformations from ε that do not alter the remaining basis operators of the corresponding two-dimensional Lie algebras.

Calculation steps needed to extend $A_{2.1}$ to a realization of $A_{3.1}$.

Consider the realization $A^1_{2.1}$. Upon checking commutation relation (138), where e_3 is of form (120), we get

$$\lambda_1 = \lambda_2 = r(t, x) = 0, \;\; h = k = const.$$

Consequently, e_3 is the linear combination of e_1, e_2, namely, $e_3 = \lambda e_1 + ke_2$, which is impossible by the assumption that the algebra under study is three dimensional. Hence we conclude that the above realization of $A^1_{2.1}$ cannot be extended to a realization of the algebra $A_{3.1}$.

Turn now to the realization $A^2_{2.1}$. Checking commutation relations (138), where e_3

is of form (120) yields the following realization of $A_{3.1}$:

$$\langle t\partial_t + x\partial_x, \sigma(\xi)\partial_u, \gamma(\xi)\partial_u \rangle, \quad \xi = tx^{-1},$$

where $\gamma'\sigma - \gamma\sigma' \neq 0$ however, the corresponding invariant equation (123) is linear.

Finally, consider the realization $A^3_{2.1}$. inserting its basis operators and the operator e_3 of the form (120) into (138) and solving the obtained equations gives the following realization of $A_{3.1}$:

$$\langle \partial_t, \partial_x, u\partial_u \rangle.$$

Inserting the obtained coefficients for e_3 into the classifying equation (121) we get invariant equation

$$u_{tt} = u_{xx} + uG(w), \qquad w = u^{-1}u_x,$$

Where (to ensure nonlinearity) we need to have $G_{ww} \neq 0$.

Similar analysis of the realizations $A_{2.1}^{i}(i=4,5,\dots,12,14)$ yields three new invariant equations. For two of thus obtained $A_{3.1}-$ $invariant\ equations$ the corresponding three-dimensional algebras are maximal. The other two may admit four-dimensional invariance algebras provided arbitrary elements are properly specified.

Handing in a similar way the extensions of $A_{2.2}$ up to realizations of $A_{3.2}$ gives 10 inequivalent nonlinear equations whose maximal invariance algebras are realizations of the three-dimensional algebra $A_{3.2}$ and four inequivalent equations (123) whose maximal symmetry algebras are realizations of three-dimensional Lie algebras $A_{3.1}$ and $A_{3.2}$.

$A_{3.1}-invariant\ equations,$

$$A_{3.1}^{1}=\langle\partial_t,\partial_x,u\partial_u\rangle,$$

$$F=uG(w),\qquad w=u^{-1}u_x,$$

$$A_{3.1}^{2}=\langle\partial_x,\varphi(t)\partial_u,\psi(t)\partial_u\rangle,$$

$$\sigma=\psi'\varphi-\psi\varphi'\neq0,\qquad\sigma'=0,$$

$$F=\varphi^{-1}\varphi''u+G(t,u_x),$$

$A_{3.2}-invariant\ equations,$

$$A_{3.2}^{1}=\langle\partial_t,\partial_x,e^{x}u\partial_u\rangle,$$

$$F=-u^{-1}u_x^2-u\,ln|u|+G(w),$$

$$w=u^{-1}u_x-ln|u|,$$

$$A_{3.2}^{2}=\langle-t\partial_t-x\partial_x,\partial_t+k\partial_x,u\partial_u\rangle\ \ (k\geq0),$$

$$F=u\eta^{-2}G(w),\ \ \eta=x-kt,$$

$$w=\eta u^{-1}u_x,$$

$$A_{3.2}^{2}=\langle-t\partial_t-x\partial_x,\partial_t+k\partial_x,u\partial_u\rangle\qquad(k\geq0),$$

$$F=u\eta^{-2}G(w),\qquad\eta=x-kt,$$

$$w=\eta u^{-1}u_x,$$

$$A_{3.2}^{3}=\langle-t\partial_t-x\partial_x+mu\partial_u,\partial_t+k\partial_x,|\eta|^{-m}\partial_u\rangle,$$

$$(\eta=x-kt,k=m=0\ or\ k>0,m\in R),$$

$$F = m(k^2-1)(m+1)\eta^{-2}u + |\eta|^{-2-m}G(w),$$

$$w = |\eta|^m(mu + \eta u_x),$$

$$A_{3.2}^4 = \langle \partial_x, e^x u\partial_u, \partial_t + mu\partial_u \rangle \ (m > 0),$$

$$F = -u^{-1}u_x^2 - u_x + uG(w),$$

$$w = u^{-1}u_x - ln|u| + mt,$$

$$A_{3.2}^5 = \langle -t\partial_t - x\partial_x, \partial_x, u\partial_u \rangle,$$

$$F = ut^{-2}G(w), w = tu^{-1}u_x,$$

$$A_{3.2}^6 = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, u\partial_u \rangle \ \ (k > 0),$$

$$F = 2ktx^{-2}u_x - 2ktx^{-3}u + k^2t^2x^{-4}u + x^{-2}uG(w),$$

$$w = xu^{-1}u_x + ktx^{-1},$$

$$A_{3.2}^7 = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, exp(ktx^{-1})\partial_u \rangle \quad (k > 0),$$

$$F = 2ktx^{-2}u_x + (k^2t^2x^{-4} - 2ktx^{-3} + k^2x^{-2})u$$

$$+ x^{-2}exp(ktx^{-1})G(w),$$

$$w = exp(-ktx^{-1})(xu_x + ktx^{-1}u),$$

$$A_{3.2}^8 = \langle \frac{1}{2K}(\partial_t + k\partial_t), e^{x+kt}\partial_u, e^{\eta}\partial_u \rangle \quad (k > 0, \eta = x - kt),$$

$$F = (x^2-1)u + G(\eta, w), \ \ w = u_x - u,$$

$$A_{3.2}^9 = \langle \partial_t + f(x)u\partial_u, e^{(1+f(x))}\partial_u, f(x)e^{f(x)t}\partial_u \rangle,$$

$$F = -\left(tf'' - t^2(f')^2 - (1+f)^2u - 2tf'u_x + e^{tf}G(x,w)\right),$$

$$w = e^{-tf}(u_x - f'(t + f^1)u), \qquad f'' + 2f'^2 + f = 0, \qquad f \neq 0,$$

$$A_{3.2}^{10} = \langle k(t\partial_t + x\partial_x), |t|^{k^{-1}}|\xi|^{(k-1)/2k}\partial_u, |\xi|^{(k-1)/2k}\partial_u \rangle \ (k \neq 0; 1),$$

$$F = \left[\frac{1-k}{k}\xi^2 + \frac{1-k}{4k^2}(1-\xi^2)\right]t^{-2}u + t^{-2}G(\xi, w),$$

$$w = |\xi|^{(k-1)/2k}\left[xu_x + \frac{k-1}{2k}u\right], \xi = tx^{-1}.$$

(5-5) Invariance Under Nondecomposable Three-Dimensional Solvable Lie Algebras:

There exist seven nondecomposable three-dimensional solvable Lie algebras over the field of real numbers. All those algebras contain a subalgebra which is the two- dimensional Abelian ideal.

Consequently, we can use the results of classification of $A_{2.1} - invariant$ equations in order to describe equations admitting nondecomposable three-dimensional solvable real Lie algebras. We remind that equations of the form (137) has already been analyzed and therefore are not considered in the sequel. Note that there are nonlinear PDEs of the considered form that admits four-dimensional invariance algebras. As four-dimensional algebras will be considered separately in the next section, we give below only those nonlinear invariant equations whose maximal symmetry algebras are three-dimensional nondecomposable solvable real Lie algebras.

$A_{3.3} - invariant\ equations,$

$$A_{3.4}^1 = \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + (mu + t|\eta|^{m-1})\partial_u\rangle$$

$$(\eta = x - kt, k > 0, m \neq 0),$$

$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u - 2k(m-1)\eta^{m-2}ln|\eta| + |\eta|^{m-1}G(w),$$

$$w = [\eta u_x - (m-1)u]|\eta|^{-m},$$

$$A_{3.4}^2 = \langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u\rangle \quad (k \geq 0),$$

$$F = e'G(\eta, w),\ \ \eta = x - kt,\ \ w = e^{-t}u_x,$$

$$A_{3.4}^3 = \langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{3}{2}u\partial_u\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + u_x^{-1}G(\xi, w),\quad \xi = tx^{-1},\quad w = x^{-1}u_x^2,$$

$$A_{3.3}^4 = \langle \partial_u, -t\partial_u, \partial_t + k\partial_x\rangle \qquad (k \geq 0),$$

$$F = G(\eta, u_x),\ \ \eta = x - kt.$$

$A_{3.4} - invariant\ equations,$

$$A^1_{3.4} = \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + (mu + t|\eta|^{m-1})\partial_u\rangle,$$

$$(\eta = x - kt, k > 0, m \neq 1),$$

$$F = (k^2 - 1)(m-1)(m-2)\eta^{-2}u - 2k(m-1)\eta^{m-2}ln|\eta| + |\eta|^{m-2}G(w),$$

$$w = [\eta u_x - (m-1)u]|\eta|^{-m},$$

$$A^2_{3.4} = \langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u\rangle \quad (k \geq 0),$$

$$F = e'G(\eta, w), \qquad \eta = x - kt, \qquad w = e^{-t}u_x,$$

$$A^3_{3.4} = \langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{3}{2}u\partial_u\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + u_x^{-1}G(\xi, w), \qquad \xi = tx^{-1}, \qquad w = x^{-1}u_x^2,$$

$$A^4_{3.4} = \langle kx^{-1}u\partial_u, \partial_t - kx^{-1}ln|x|u\partial_u, t\partial_t + x\partial_x\rangle \quad (k > 0),$$

$$F = -3ktx^{-3}u - 2x^{-2}uln|u| - u^{-1}\,u_x^2 + x^{-2}uG(w),$$

$$w = xu^{-1}u_x + ln|u| + ktx^{-1},$$

$$A^5_{3.4} = \langle exp(ktx^{-1})\partial_u, \partial_t + kx^{-1}u\partial_u, t\partial_t + x\partial_x + \big(u + texp(ktx^{-1})\big)\partial_u\rangle,$$

$$(k > 0),$$

$$F = k^2x^{-4}u(t^2 + x^2) + 2x^{-1}(ktx^{-1} + 1)u_x + 2kexp(ktx^{-1})x^{-1}ln|x| + x^{-1}exp(ktx^{-1})G(w), \quad w = exp(-ktx^{-1})(u_x + ktx^{-2}u).$$

$A_{3.6} - invariant\ equations,$

$$A^1_{3.5} = \langle \partial_t + k\partial_x, |\eta|^{m+1}\partial_u, \partial_t + x\partial_x + mu\partial_u\rangle \qquad (k > 0, m \neq 1),$$

$$F = (k^2 - 1)(m-1)(m-2)\eta^{-2}u + |\eta|^{m-2}G(w),$$

$$w = |\eta|^{-m}[\eta u_x - (m-1)u], \qquad \eta = x - kt,$$

$$A^2_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x\rangle,$$

$$F = u_x^2G(u),$$

$$A^3_{3.5} = \langle \partial_t, \partial_x, t\partial_t + mu\partial_u\rangle \quad (m \neq 0),$$

$$F = |u|^{1-(2/m)}G(w), \;\; w = |u_x|^m|u|^{1-m},$$

$$A^4_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + \partial_u\rangle,$$

$$F = e^{-2u}G(w), \quad w = e^{u}u_x,$$

$$A_{3.5}^{5} = \langle \partial_t, x^{-1}u\partial_u, t\partial_t + x\partial_x \rangle,$$

$$F = -u^{-1}u_x^2 - 2x^{-2}u\, ln|u| + x^{-2}uG(w),$$

$$w = xu^{-1}u_x + ln|u|,$$

$$A_{3.5}^{6} = \langle \partial_t + kx^{-1}u\partial_u, exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + u\partial_u \rangle \quad (k > 0),$$

$$F = kx^{-4}u[kt^2 - 2tx + kx^2] + 2ktx^{-2}u_x + x^{-1}exp(ktx^{-1})G(w),$$

$$w = exp(-ktx^{-1})(u_x + ktx^{-2}u),$$

$$A_{3.5}^{7} = \langle \varphi(t)\partial_u, \psi(t)\partial_u, \partial_x + u\partial_u \rangle \quad (\varphi'\psi - \varphi\psi' \neq 0),$$

$$F = \varphi^{-1}\varphi''u + u_xG(t, w),$$

$$w = e^{-x}u_x, \quad \varphi''\psi - \varphi\psi'' = 0.$$

$A_{3.6} - invariant\ equations,$

$$A_{3.6}^{1} = \langle \partial_t + k\partial_x, |\eta|^{m+1}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle \quad (k > 0, m \neq -1),$$

$$F = m(k^2 - 1)(m + 1)\eta^{-2}u + |\eta|^{m-2}G(w),$$

$$w = |\eta|^{1-m}[u_x - \eta^{-1}(m + 1)u], \qquad \eta = x - kt,$$

$$A_{3.6}^{2} = \langle \partial_t + mx^{-1}u\partial_u, xu\partial_u, t\partial_t + x\partial_x \rangle \quad (m \geq 0),$$

$$F = -u^{-1}u_x^2 - 2mtx^{-3}u + x^{-2}uG(w),$$

$$w = xu^{-1}u_x - ln|u| + 2mtx^{-1},$$

$$A_{3.6}^{3} = \langle \partial_t + kx^{-1}u\partial_u, exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x - u\partial_u \rangle \quad (k > 0),$$

$$F = x^{-4}[k^2x^2 - 2ktx + k^2t^2]u + 2ktx^{-2}u_x + x^{-3}exp(ktx^{-1})G(w),$$

$$w = exp(-ktx^{-1})(x^2u_x + ktu),$$

$$A_{3.6}^{4} = \langle e^{-t}\partial_u, e^{t}\partial_u, \partial_t + k\partial_x \rangle \qquad (k \geq 0),$$

$$F = u + G(\eta, u_x), \quad \eta = x - kt,$$

$$A_{3.6}^{5} = \langle |t|^{\frac{1}{2}}\partial_u, |t|^{\frac{1}{2}}\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u \rangle,$$

$$F = \tfrac{1}{2}t^{-2}u + |t|^{-3/2}G(\xi, w), \qquad \xi = tx^{-1}, \qquad w = x^{-1}u_x^2.$$

$A_{3.7} - invariant\ equations,$

$$A_{3.7}^{1} = \langle \partial_t + k\partial_x, |\eta|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle$$

$$(k > 0, m \neq q, 0 < |q| < 0),$$

$$F = (k^2 - 1)(m - q - 1)\eta^2 u + |\eta|^{m-2} G(w),$$

$$w = |\eta|^{1-m}[u_x - (m - q)\eta^{-1} u], \quad \eta = x - kt,$$

$$A_{3.7}^2 = \langle \partial_t + kx^{-1} u \partial_u, exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle,$$

$$(k > 0, 0 < |q| < 1),$$

$$F = [k^2 x^{-2} + k^2 x^4 x^2 - 2ktx^{-3}]u + 2ktx^{-2} u_x$$

$$+ |x|^{q-2} exp(ktx^{-1}) G(w),$$

$$w = |x|^{1-q} exp(-ktx^{-1})(u_x + ktx^{-2} u),$$

$$A_{3.7}^3 = \langle |t|^{\frac{1}{2}q} \partial_u, |t|^{1-\frac{1}{2}q} \partial_u, t\partial_t + x\partial_x + \left(1 + \frac{1}{2}q\right) u\partial_u \rangle \quad (q \neq 0, \pm 1),$$

$$F = \frac{1}{4} q(q-2) t^{-2} u + |t|^{\frac{1}{2}(q-2)} G(\xi, w),$$

$$\xi = tx^{-1}, w = |t|^{\frac{1}{2}q} u_x,$$

$$A_{3.7}^4 = \langle exp\left(\frac{1}{2}(q-1)t\right)\partial_u, exp\left(\frac{1}{2}(1-q)t\right)\partial_u, \partial_t + k\partial_x$$

$$+ \frac{1}{2}(1+q) u\partial_u \rangle$$

$$(q \neq 0, \pm 1; k \geq 0),$$

$$F = \frac{1}{4}(q-1)^2 u + exp\left(\frac{1}{2}(1+q)t\right) G u_x,$$

$$A_{3.7}^5 = \langle \partial_t + kx^{-1} u\partial_u, |x|^{-q} u\partial_u, t\partial_t + x\partial_x \rangle \quad (k \geq 0, q \neq 0, \pm 1),$$

$$F = -u^{-1} u_x^2 - q(q+1) x^{-2} u \, ln|u| + k(q-1)(q+2) t x^{-2} u$$

$$+ u x^{-2} G(w),$$

$$w = x u^{-1} u_x + q \, ln|u| + k(1-q) t x^{-1}.$$

$A_{3.8} - invariant\ equations$

$$A_{3.8}^1 = \langle \cos t\partial_u, -\sin t\partial_u, \partial_t + k\partial_x \rangle \quad (k \geq 0),$$

$$F = -u + G(\eta, u_x), \quad \eta = x - kt,$$

$$A_{3.8}^2 = \langle |t|^{\frac{1}{2}} \cos(ln|t|)\partial_u, -|t|^{\frac{1}{2}} \sin(ln|t|)\partial_u, t\partial_t + x\partial_x + \frac{1}{2} u\partial_u \rangle,$$

$$F = -\frac{5}{4} t^{-2} u + |t|^{-3/2} G(\xi, w),$$

$$\xi = tx^{-1}, \qquad w = |t|^{1/2} u_x.$$

$A_{3.9} - invariant\ equations$

$$A_{3.9}^1 = \langle \sin t\partial_u, \cos t\partial_u, \partial_t + k\partial_x + qu\partial_u \rangle \quad (k \geq 0, q > 0),$$

$$F=-u+e^{qt}G(\eta,w),\qquad \eta=x-kt, w=e^{-qt}u_x,$$

$$A_{3.9}^2=\langle |t|^{\frac{1}{2}}\sin(ln|t|)\partial_u, |t|^{\frac{1}{2}}\cos(ln|t|)\partial_u, t\partial_t+x\partial_x+\left(\frac{1}{2}+q\right)u\partial_u\rangle$$

$$(q\neq 0), F=--\frac{3}{4}t^{-2}u+|t|^{q-\frac{3}{2}}G(\xi,w),$$

$$\xi=tx^{-1}, w=|t|^{\frac{1}{2}-q}u_x.$$

(5-6) Complete Group Classification of Eq. (123):

The aim of this section is finalizing group classification of (123). The majority of invariant equations obtained in the preceding section contain arbitrary functions of one variable. So that we can utilize the standard Lie –Ovsyannikov approach in order to complete their group classification.

(I) Equations Depending on an Arbitrary Function of One Variable:

Note that equations belonging to the already investigated class of (137) are not considered. As our computations show, new results could be obtained for the equations.

$$u_{tt}=u_{xx}+uG(w),\ w=u^{-1}u_x, \tag{139}$$

$$u_{tt}=u_{xx}+G(u_x). \tag{140}$$

Only, below we give ($without\ proof$) the assertions describing their group properties.

Assertion 6:

Equation (139) admits wider symmetry group iff it is equivalent to the following equation:

$$u_{tt}=u_{xx}+mu^{-1}u_x^2\quad (m\neq 0,-1). \tag{141}$$

The maximal invariance algebra of (141) is four-dimensional Lie algebra,

$$A_4\sim A_{3.5}\oplus A_1,\ A_4=\langle\partial_t,\partial_x,t\partial_t+x\partial_x,u\partial_u\rangle.$$

Assertion 7: Equation (140) admits wider symmetry group iff it is equivalent to one of the following PDEs:

$$u_{tt} = u_{xx} + e^{u_x}, \qquad (142)$$

$$u_{tt} = u_{xx} + m ln|u_x|, \quad m > 0, \qquad (143)$$

$$u_{tt} = u_{xx} + |u_x|^k, \quad k \neq 0,1. \qquad (144)$$

The maximal invariance algebras of the above equations are five-dimensional solvable Lie algebras listed below,

$$A_5^2 = \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + (u - x)\partial_u \rangle,$$

$$A_3^3 = \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + \left(2u + \tfrac{1}{2}mt^2\right)\partial_u \rangle,$$

$$A_5^4 = \langle \partial_t, \partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + \frac{k-2}{k-1}u\partial_u \rangle.$$

Analyzing the remaining equations containing arbitrary functions of one variable we come to conclusion that one of them can admit wider invariance groups iff either

(1) It is equivalent to PDE of the form (137)

(2) It is equivalent to PDE of the form (141)

To finalize the procedure of group classification of Eqs. (123) we need to consider invariant equations obtained in the preceding section that contain arbitrary functions of two variables.

(II) Classification of Equations with Arbitrary Functions of Two Variables:

In the case under study the standard Lie-Ovsyannikov method is inefficient and we apply our classification algorithm. In order to do this we perform extension of three-dimensional solvable Lie algebras to all possible realizations of four-dimensional solvable Lie algebras. The next step will be to check which of the obtained realizations are symmetry algebras of nonlinear equations of the form (123). In what follows we use the results, where all inequivalent (within the action of inner automorphism group) four- dimensional solvable abstract Lie algebras are given.

Here we summarize the obtained results as follows:

(1) If the functions contained in the equations under study are arbitrary, then the corresponding realizations are their maximal invariance algebras, and

(2) Except for Eq. (140), all the equations in question do not allow for extension of their symmetry.

Below we give the complete list of PDEs (123) invariant under four-dimensional solvable Lie algebras that are obtained through group analysis of equations with arbitrary functions of two variables.

$$A_{2.2} \oplus 2A_1 - invariant$$

$(1)\langle \partial_x, \partial_t + u\partial_u, e^t\partial_u, e^{-t}\partial_u \rangle,\ F = u + e^t G(w),\ \ w = u^{-t}u_x,$

$(2)\langle \frac{1}{2k}(\partial_t + k\partial_x), e^{x+kt}\partial_t, e^{\eta}\partial_u, \partial_x + u\partial_u \rangle \quad (k > 0, \eta = x - kt),$

$$F = (k^2 - I)u + e^{\eta}G(w), \quad w = e^{-\eta}(u_x - u).$$

$2A_{2.2}$

$- invariant\ equations,$

$$(1)\langle \partial_t + \epsilon u\partial_u, \partial_x, e^{x+kt}\partial_u \rangle \quad (\epsilon = 0{,}1; k > 0),$$

$$F = (k^2 - 1)u + e^{\epsilon t}G(w), \quad w = e^{-\epsilon t}(u_x - u),$$

$$(2)\langle \alpha\partial_x - u\partial_u, \partial_t + k\partial_x, e^{-t}\partial_u, e^t\partial_u \rangle \qquad (k \geq 0, \alpha > 0),$$

$$F = u + exp(-\alpha^{-1}\eta)G(w),\ \ \eta = x - kt,\ \ w = exp(\alpha^{-1}\eta)u_x$$

$A_{3.3} \oplus A_1$

$- invariant\ equations,$

$$(1)\ \langle \partial_t, \partial_x, \partial_u, t\partial_u \rangle,\ \ G = (U_X).$$

$A_{3.4} \oplus A_1$

$- invariant\ equations$

$$(1)\langle \partial_u, \partial_x, t\partial_t + x\partial_x + (u + x)\partial_u, t\partial_u \rangle,$$

$$F = t^{-1}G(w), \quad w = u_x - ln|t|,$$

$$(2)\langle \partial_t + u\partial_u, \partial_x, t\partial_u, \partial_u\rangle, F = e^t G(w), \quad w = e^{-t}u_x,$$

$$(3)\langle x^{-t}\partial_u, \partial_x - x^{-1}(u + ln|x|)\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u\rangle,$$

$$F = 2x^{-1}u_x + x^{-2} + t^{-1}x^{-1}G(w), \qquad w = xu_x + u - ln|tx^{-1}|.$$

$A_{3.5}\oplus A_1 - invariant\ equations,$

$$(1)\langle \partial_x, \partial_u, t\partial_t + x\partial_x + u\partial_u, t\partial_u\rangle, \qquad F = t^{-1}\, G(u_x),$$

$$(2)\ \langle x^{-1}\partial_u, \partial_x - x^{-1}u\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u\rangle,$$

$$F = -2x^{-2}u + 2t^{-1}(u_x + x^{-1}u)ln|t(u_x + x^{-1}u)|$$

$$+ t^{-1}(u_x + x^{-1}u)G(w),$$

$$w = xu_x + u.$$

$A_{3.6}\oplus A_1 - invariant\ equations,$

$$(1)\ \langle exp\left(-\tfrac{1}{2}(1-q)t\right)\partial_u, exp\left(\tfrac{1}{2}(1-q)t\right)\partial_u, \partial_t + \tfrac{1}{2}(1+q)u\partial_u, \partial_x\rangle$$

$$(q \neq 0, \pm 1), \qquad F = \tfrac{1}{4}(1-q)^2 u + exp\left(\tfrac{1}{2}(1+q)t\right)G(w),$$

$$w = exp\left(-\tfrac{1}{2}(1+q)t\right)u_x,$$

$$(2)\ \langle \partial_x, |t|^{\frac{1}{2}(1-q)}\partial_u, |t|^{\frac{1}{2}(1+q)}\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}(1+q)u\partial_u\rangle,$$

$$(q \neq 0, \pm 1), \qquad F = \tfrac{1}{4}(q^2 - 1)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(w),$$

$$w = |t|^{\frac{1}{2}(1-q)}u_x,$$

$$(3)\ \langle |t|^{-1/q}|\xi|^{(q+1)/2}\partial_u, \partial_x - \frac{1+q}{2q}x^{-1}u\partial_u$$

$$- q(t\partial_t + x\partial_x), |\xi|^{(1+q)/2q}\partial_u\rangle,$$

$$(q \neq 0, \pm 1), \qquad F\left[\frac{1-q^2}{4q^2}(t^{-2} + x^{-2})\right]u + \frac{1+q}{q}x^{-1}u_x$$

$$+ t^{-2}|\xi|^{(1+q)/2q}G(w),$$

$$\xi = tx^{-1}, \qquad w = |\xi|^{(q-1)/2q}\left[xu_x + \frac{q+1}{2q}u\right].$$

$A_{3.8}\oplus A_1 - invariant\ equations,$

$$(1)\langle sint\partial_u, \cos t\partial_u, \partial_t, \partial_u\rangle, F = -u + G(u_x).$$

$A_{3.9}\oplus A_1 - invariant\ equations,$

$$(1)\langle sint\partial_u, \cos t\partial_u, \partial_t + qu\partial_u, \partial_x\rangle \quad (q > 0),$$

$$F = -u + e^{qt}G(w), \qquad w = e^{-qt}u_x .$$

$A_{4.1} - invariant\ equations,$

$$(1)\langle \partial_u, -t\partial_u, \partial_x, \partial_t - tx\partial_u\rangle, F = G(w), \ \ w = u_x + \tfrac{1}{2}t^2 ,$$

$$(2)\langle \partial_u, -t\partial_u, \alpha\partial_x + kx\partial_x\rangle \ \ (k \geq 0), (\alpha > 0),$$

$$F = \alpha^{-1}(x - kt) + G(u_x).$$

$A_{4.2} - invariant\ equations,$

$$(1)\,\langle |t|^{1-\frac{1}{2}q}\partial_u, |t|^{\frac{1}{2}q}\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\left(1 + \tfrac{1}{2}q\right)u + x|t|^{\frac{1}{2}q}\right]\partial_u\rangle$$

$$(q \neq 0{,}1), F = \tfrac{1}{4}q(q-2)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(w),$$

$$w = |t|^{\frac{1}{2}(1-q)}u_x - 2|t|^{\frac{1}{2}},$$

$$(2)\,\langle \partial_x, \sqrt{|t|}\partial_u, \sqrt{|t|}ln|t|\partial_u, t\partial_t + x\partial_x + \left(q + \tfrac{1}{2}\right)u\partial_u\rangle$$

$$(q \neq 0), \qquad F = -\tfrac{1}{4}t^{-2}u + |t|^{q-\frac{3}{2}}G(w), \ \ w = |t|^{\frac{1}{2}-q}u_x.$$

$A_{4.3} - invariant\ equations,$

$$(1)\,\langle \partial_x, |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + |t|^{\frac{-3}{2}}G(w), \ \ w = |t|^{\frac{1}{2}}u_x,$$

$$(2)\ \ \langle \partial_x, t\partial_u, \partial_u, t\partial_t + x\partial_x\rangle, \qquad F = t^{-2}G(w), w = tu_x,$$

$$(3)\langle e^{kt}\partial_u, \partial_t + ku\partial_u, \beta\partial_x + te^{kt}\partial_u, e^{-kt}\partial_u\rangle \quad (k \neq 0, \beta > 0),$$

$$F = k^2u + 2k\beta^{-1}xe^{kt} + e^{kt}G(w), \qquad w = e^{-kt}u_x,$$

$$(4)\langle e^{x+kt}\partial_u, e^{\eta}\partial_u, \alpha(\partial_x + u\partial_u) + 2kte^{\eta}\partial_u, -(1/2k)(\partial_t + k\partial_x)\rangle$$

$$(\alpha \neq 0, k > 0), \qquad F(k^2 - 1)u - 4k^2\alpha^{-1}\eta e^{\eta} + e^{\eta}G(w),$$

$$w = e^{-\eta}(u_x - u), \quad \eta = x - kt.$$

$A_{4.4} - invariant\ equations,$

$$(1)\,\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}ln|t|\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\tfrac{1}{2}u - x|t|^{\frac{1}{2}}ln|t|\right]\partial_u\rangle,$$

$$F = \tfrac{1}{4}t^{-2}u + |t|^{-\frac{1}{2}}G(w), \quad w = |t|^{-\frac{1}{2}}u_x + \tfrac{1}{2}ln^2|t|.$$

$A_{4.5} - invariant\ equations,$

$$(1)\langle \partial_x, |t|^{m-\alpha}\partial_u, |t|^{1-m+\alpha}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle$$

$$\left(m \neq \frac{1}{2}(1+\alpha), \frac{1}{2} + \alpha; \alpha \neq 0\right),$$

$$F = (m-\alpha)(m-\alpha-1)t^{-2}u + |t|^{m-2}G(w), \qquad w = |t|^{1-m}u_x.$$

$A_{4.6} - invariant\ equations,$

$$(1)\ \langle \partial_x, |t|^{\frac{1}{2}}\sin(q^{-1}ln|t|)\partial_u\,, |t|^{\frac{1}{2}}\cos(q^{-1}ln|t|)\partial_u\,, qt\partial_t$$

$$+ qx\partial_x \left(\frac{1}{2}q + p\right)u\partial_u \rangle$$

$$(q \neq 0, p \geq 0),$$

$$F = (m-\alpha)(m-\alpha)t^{-2}u + |t|^{m-2}G(w), \ \ w = |t|^{1-m}u_x.$$

$A_{4.7} - invariant\ equations,$

$$(1)\ \langle \partial_u, -t\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + \left(2u - \frac{1}{2}t^2\right)\partial_u \rangle \quad (k \geq 0),$$

$$F = -ln|\eta| + G(w), \quad w = \eta^{-1}u_x, \quad \eta = x - kt.$$

$A_{4.8} - invariant\ equations,$

$$(1)\langle \partial_t + \epsilon u\partial_u, \partial_x, e^x\partial_u, te^x\partial_u \rangle \qquad (\epsilon = 0; 1),$$

$$F = -u + e^{\epsilon t}G(w), \ \ w = e^{-\epsilon t}(u_x - u),$$

$$(2)\langle |x|^{m-q}\partial_u, \partial_x, t|x|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle \quad (q \neq 0, m \in R),$$

$$F = -(m-q)(m-q-1)x^{-2}u + |x|^{m-2}G(w),$$

$$w = |t|^{1-m}[u_x - (m-q)x^{-1}u],$$

$$(3)\langle \partial_t + k\partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle \quad (k > 0, q \in R),$$

$$F = |\eta|^{q-2}G(w), \ \ w = |\eta|^{1-q}u_x, \qquad \eta = x - kt,$$

$$(4)\langle x^{-1}\partial_u, \partial_t + \partial_x - x^{-1}u\partial_u, tx^{-1}\partial_u, t\partial_t + x\partial_x \rangle,$$

$$F = 2x^{-1}u_x + x^{-1}(t-x)^{-1}G(w), \qquad w = xu_x + u.$$

$$(5)\langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u, \alpha\partial_x + u\partial_u \rangle \ \ (\alpha \neq 0, k \geq 0),$$

$$F = exp(\alpha^{-1}\eta + t)G(w), \ \ w = exp(-\alpha^{-1}\eta - t)u_x, \qquad \eta = x - kt.$$

$A_{4.10} - invariant\ equations,$

$$(1)\langle \sin t\partial_u, \cos t\partial_u, \partial_x + u\partial_u, \partial_t + k\partial_u \rangle \qquad (k \geq 0),$$

$$F = -u + e^{\eta}G(w), \ \ w = e^{-\eta}u_x, \qquad \eta = x - kt.$$

In the above formulas $G = G(w)$ is an arbitrary function satisfying the condition $F_{u_x u_x} \neq 0$.

Concluding Remarks:

Let us briefly summarize the results obtained in this research.

We prove that the problem of group classification of the general quasilinear hyperbolic type equation (115) reduces to classifying equations of more specific forms,

$$(i) u_{tt} = u_{xx} + F(t, x, u, u_x), \; F_{u_x u_x} \neq 0,$$

$$(ii) u_{tt} = u_{xx} + g(t, x, u) u_x + f(t, x, u), g_u \neq 0,$$

$$(iii) u_{xx} = g(t, x) u_x + f(t, x, u), \qquad g_x \neq 0, \; f_{uu} \neq 0,$$

$$(iv) u_{tx} = f(t, x, u), \quad f_{uu} \neq 0.$$

If we denote as $D\varepsilon$ the set of PDEs (ii)-(iii), then the results of application of our algorithm for group classification of equations (i)-(iv) can be summarized as follows.

(1) We perform complete group classification of the class $D\varepsilon$. We prove that the Liouville equation has the highest symmetry properties among equations from $D\varepsilon$. Next, we prove that the only equation belonging to this class and admitting the four-dimensional invariance algebra is the nonlinear d'Alembert equations. It is established that there are 12 inequivalent equations from $D\varepsilon$ invariant under three-dimensional Lie algebras. We give the lists of all inequivalent equations from $D\varepsilon$ that admit one-and two-dimensional symmetry algebras.

(2) We have studied the structure of invariance algebras admitted by nonlinear equations from the class (1). It is proved, in particular, that their invariance algebras are necessary solvable.

(3) We perform complete group classification of nonlinear equations from the class of PDEs (1).

We prove that the highest symmetry algebras admitted by those equations are five dimensional and construct all inequivalent classes of equations invariant with respect to five-dimensional Lie algebras. We also construct all inequivalent equations of the form (1) admitting one, two, three, and four-dimensional Lie algebras.

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