Sudan University of Science and Technology

College of Graduate Studies

Fully Measurable and Approximation Problems with Modular Inequalities in Variable Lebesgue Spaces

قابلية القياس تماما ومسائل التقريب مع المتباينات المعيارية في فضاءات لبيق

A thesis Submitted in Fulfillment of the Requirement for the Degree of Ph.D in Mathematics

by

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Dedication

To my Family

Acknowledgements

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Abstract

We show the new properties and weighted fully measure and fully measurable of small, grand and iterated grand Lebesgue spaces with their applications and the maximal theorem .Direct and inverse theorems of approximation theory in variable Lebesgue and Smirnov spaces are discussed . The trigonometric and polynomial approximation of functions and problems in generalized Lebesgue spaces with variable exponent and Smirnov spaces with nonstandard growth are studied . The maximal function and atomic decomposition of Hardy spaces with variable exponents and its application to bounded linear operators are considered .We characterize the modular inequalitis for the Calderon and maximal operators in variable Lebesgue spaces.

الخلاصة

قمنا بتوضيح خصائص جديدة وقياس تام مرجح وقابلية قياس تامة لفضاءات لبيق الصغيرة والكبيرة والكبيرة المتكررة مع تطبيقاتها و مبرهنة الحد الأقصى . تمت مناقشة المبرهنات المباشرة والإنعكاسية لنظرية التقريب في فضاءات لبيق المتغيرة وسميرنوف . قمنا بدراسة تقريب حساب المثلثات وكثيرة الحدود للدوال والمسائل في فضاءات لبيق المعممة مع أسية المتغير وفضاءات سميرنوف مع النمو غير القياسي. تم إعتبار دالة الحد الأقصى والتفكيك الذري لفضاءات هاردي مع أسيات المتغير وتطبيقاتها إلى المؤثرات الخطية المحدودة . تم تشخيص المتباينات المعيارية لأجل مؤثرات كالدرون والحد الأقصى في فضاءات لبيق المتغيرة.

Introduction

We show some new properties of the small Lebesgue spaces introduced by Fiorenza in [2]. Combining these properties with the Poincar$\acute{e}$-Sobolev inequalities for the relative rearrangement. The norm of the grand Lebesgue spaces is defined through the supremum of Lebesgue norms, balanced by an infinitesimal factor. We consider the spaces defined by a norm with an analogous expression, where Lebesgue norms are replaced by grand Lebesgue norms. Without the use of interpolation theory, we prove an iteration-type theorem, and we establish that the new norm is a gain equivalent to the norm of grand Lebesgue spaces. Let $1 < p < \infty$.Given $\Omega \subset \mathbb{R}^n$ a measurable set of finite Lebesgue measure, the norm of the grand Lebesgue spaces $L^{p)}(\Omega)$ isgivenby

$$|f|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|}\int_\Omega |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}.$$

we consider the norm $|f|_{L^{p),\delta}(\Omega)}$ obtained replacing $\varepsilon^{\frac{1}{p-\varepsilon}}$ by a generic nonnegative measurable function $\delta(\varepsilon)$.

The approximation properties of N$\ddot{o}$rlund (N_n) and Riesz (R_n) means of trigonometric Fourier series are investigated in generalized Lebesgue spaces $L^{p(x)}$. We investigate the approximation properties of the trigonometric system in $L_{2\pi}^{p(\cdot)}$. We deals with basic approximation problems such as direct, inverse and simultaneous theorems of trigonometric approximation of functions of weighted Lebesgue spaces with a variable exponent on weights satisfying a variable Muckenhoupt $A_{p(.)}$ type condition.

If $P, Q : [0, \infty) \rightarrow$ are increasing functions and T is the Calderón operator defined on positive or decreasing functions, then optimal modular inequalities $\int P(Tf) < C\int Q(f)$ are proved. We give continuity conditions on the exponent function $p(x)$ which are sufficient for the Hardy–Littlewood maximal operator to be bounded on the variable Lebesgue space $L^{p(x)}(\Omega)$, where Ω is any open subset of R^n .

Using variable exponents, we build a new class of rearrangement-invariant Banach function spaces, independent of the variable Lebesgue spaces, whose function norm is $\rho(f) = \text{esssup}_{x\in(0,1)}\, \rho_{p(x)}(\delta(x)f(\cdot))$, where $\rho_{p(x)}$ denotes the norm of the Lebesgue space of exponent $p(x)$ (assumed measurable and possibly infinite) and δ is measurable, too. Anatriello and Fiorenza introduced the fully measurable grand Lebesgue spaces on the interval $(0,1) \subset \mathbb{R}$, which contain some known Banach spaces of functions, among which there are the classical and the grand Lebesgue spaces, and the $E\,X\,P_\alpha$ spaces $(\alpha > 0)$. We introduce the weighted fully measurable grand Lebesgue spaces and we prove the boundedness of the Hardy–Littlewood maximal function. We build a new class of Banach function spaces, whose function norm is

$$\rho_{(p[\cdot],\delta[\cdot]}(f) = \inf_{\sum_{k=1}^{f=\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in(0,1)} \rho_{p(x)}\big(\delta(x)^{-1} f_k(\cdot)\big) \quad ,$$

where $\rho_{p(x)}$ denotes the norm of the Lebesgue space of exponent $p(x)$ (assumed measurable and possibly infinite), constant with respect to the variable of f, and δ is measurable, too. Such class contains some known Banach spaces of functions, among which are the classical and the small Lebesgue spaces, and the Orlicz space $L(\log L)^{\alpha}, \alpha > 0$.

We consider the Lebesgue space $L_{2\pi}^{p(x)}$ with variable exponent $p(x)$. It consists of measurable functions $f(x)$ for which the integral$\int_0^{2\pi} |f(x)|^{p(x)} dx$ exists. We establish an analogue of Jackson's second theorem in the case when the $2\,\pi$-periodic variable exponent $p(x) \geq 1$ satisfiesthe condition

$$|p(x) - p(y)| \cdot \ln \frac{2\pi}{|x-y|} \leq d, \qquad x, y \in [0,2\pi].$$

We investigate the inverse approximation problems in the Lebesgue and Smirnov spaces with weights satisfying the so-called Muckenhoupt's A_p condition in terms of the α-th mean modulus of smoothness, $\alpha > 0$. In the variable exponent Lebesgue space, the r-th modulus of smoothness $(r = 1, 2, \ldots)$ is defined and in this term, the direct and inverse theorems of approximation theory are proved.

We generalize the classical $L \log L$ inequalities of Wiener and Stein for the Hardy-Littlewood maximal operator to variable L^p spaces where the exponent function $p(\cdot)$ approaches 1 in value. As applications of atomic decomposition results of Hardy spaces with variable exponents, we shall prove the boundedness of commutators and the fractional integral operators as well as the Hardy operators. A now classical result in the theory of variable Lebesgue spaces due to Lerner (2005) is that a modular inequality for the Hardy–Littlewood maximal function in $L^{p(\cdot)}(\mathbb{R}^n)$ holds if and only if the exponent is constant. We generalize this result and give a new and simpler proof. We then find necessary and sufficient conditions for the validity of the weaker modular inequality

$$\int_{\Omega} Mf(x)^{p(x)}\, dx \leqslant c_1 \int_{\Omega} |f(x)|^{q(x)}\, dx + c_2,$$

where c_1, c_2 are non-negative constants and Ω is any subset of $\mathbb{R}^n$.

The Contents

Chapter 1
Iterated and New Properties

We derive some new and precises estimates either for small LebesgueSobolev spaces or for quasilinear equations with data in the small Lebesgue spaces. We show that the expression involved satisfy the axioms of Banach Function Spaces, and we find explicit values of the constants of the equivalence. Analogous results are proved for small Lebesgue spaces. We find necessary and sufficient conditions on δ in order to get a functional equivalent to a Banach function norm, and we determine the ‘‘interesting’’ class B_p of functions δ, with the property that every generalized function norm is equivalent to a function norm built with $\delta \in B_p$. We then define the $L^{p),\delta}(\Omega)$ spaces, prove some embedding results and conclude with the proof of the generalized Hardy inequality.

Section (1.1): Small Lebesgue Spaces and Their Applications

In [2], we shall give some new properties of the small Lebesgue spaces introduced by Fiorenza (3]), denoted by $L^{(p}(\Omega)$ for a bounded set $\Omega, 1 < p < +\infty$. This set is smaller than the Lebesgue space $L^p(\Omega)$and contains all $L^{p+\varepsilon}(\Omega)$ for all $\varepsilon > 0$.
If we denote by

$$g_{**}(s) = \frac{1}{s}\int_0^s |g|_*(t)dt\ , \qquad s \in (0, meas(\Omega)) = \Omega_*,$$

g_* being the monotone rearrangement of $|g|$, then $g_{**} \in L^{(p}(\Omega_*)$ if $g \in L^{(p}(\Omega)$ and

$$\|g\|_{L^{(p}(\Omega)} \le |g_{**}|_{L^{(p}(\Omega_*)} \le p'\|g\|_{L^{(p}(\Omega)}\,. \qquad (1)$$

These spaces satisfy the Levi monotone convergence property. A first consequence of such a property is that if $(E_m)_{m\ge 0}$ is a monotone sequence of measurable sets with $meas\ (E_m)\ \overrightarrow{m \to \infty}\ 0$, then for any $f \in L^{(p}(\Omega)$, one has $\left|f\chi_{E_m}\right|_{L^{(p}(\Omega)} \longrightarrow 0$.
Here χ_{E_m} is the characteristic function of E_m. We shall give a direct application of some of the properties of the small Lebesgue spaces, by proceeding on a direct proof of some precise pointwise estimate.
Namely, $W^{1,(N}(\Omega) \subset L^\infty(\Omega) \cap C(\Omega)$ (if Ω is a bounded connected Lipschitz set) and

$$\underset{B(x,t)}{osc} u \le \frac{\omega_N^{1/N'}}{\omega_{N-1}} |\Omega|^{1/N'} (N')^{1/N'} \left||\nabla u|\chi_{B(x,t)}\right|_{(N}\,. \qquad (2)$$

(We denote by $|\cdot|_{(p}$ the Banach Function norm in $L^{(p)}$. The inclusion in $C(\Omega)$ is a consequence of the work of [3] (see also [4,5]) since

$$W^{1,(N}(\Omega) \subset \{v \in W^{1,1}(\Omega) : |\nabla v| \in L^{N,1}(\Omega)\}.$$

The inclusion in L^∞ is given in [6]. The main part is the estimate (2). Furthermore, if $u \in W_0^{1,p}(\Omega)$ solution of

$$(P)\quad Au = -div(\hat{a}(x,u,\nabla u)) = f \in L^{(\frac{N}{p}}(\Omega)\quad 1 < p < N$$

then u is bounded if $p \leq 2$. Furthermore if $p < \frac{2N}{N+1}$ and $|\Omega| = 1$, then we havethe following precise estimate:

$$|u|_*(s) \leq c_{\alpha Np}\left(\int_0^1 s^{-\frac{p'}{N'}}\varphi(s)^{\frac{1}{p-1}}\,ds\right)|g_{*|u|}|_{L^{(\frac{N}{p}}(\Omega_*)}^{\frac{1}{p-1}} < +\infty, \qquad (3)$$

with $\varphi(s) = \sup_{0<\varepsilon<q-1}(\varepsilon s)^{\frac{1}{q-\varepsilon}}$, $q = \frac{N}{N-p}$, $c_{\alpha Np} = \frac{1}{\alpha^{\frac{p'}{p}}\left(N\omega_N^{\frac{1}{N}}\right)^{p'}}$ $\quad g = |f|$.

If $p = 2$ then one has:

$$|u|_*(s) \leq c_{\alpha N2}\left(\frac{N}{N-2}\right)^2 \quad |g|_{(\frac{N}{2}} \qquad (4)$$

The above estimate (4) shows, in particular, that if we consider the operator $(-\Delta)^{-1} : L^{(\frac{N}{2}}(\Omega) \longrightarrow L^\infty(\Omega)$ then we have the following estimate of the norm

$$(-\Delta)^{-1} = \sup_{g\neq 0}\frac{|(-\Delta)^{-1}g|_\infty}{|g|_{(\frac{N}{2}}} \leq \left[\frac{1}{(N-2)\omega_N^{\frac{1}{N}}}\right]^2.$$

For $p > 2$, we shall introduce the following vector space

$$V_p = \left\{g \in L^{(\frac{N}{p}}(\Omega) :\ g_{**}^{\frac{1}{p-1}} \in L^{(\frac{N}{p'}}(\Omega_*)\right\}$$

If $f \in V_p$ then u is bounded.We shall show that V_p is different from $L^{(\frac{N}{p}}(\Omega)$ by producing an element g of $L^{(\frac{N}{p}}(\Omega)$ whose decreasing monotone rearrangement is $g_*(s) = s^{-\frac{p}{N}}|\ln s|^{1-p}$, with $s \in (0,1/2)$, $g_*(s) = g_*(\frac{1}{2})$ for $\frac{1}{2} \leq s \leq 1$, and g is not in V_p.
Moreover ,we shall prove that if Ω is a ball of measure 1 centered at the origin and $u \in W_0^{1,1}(\Omega)$ the unique radial solution of :

$$-\Delta_p u = -div(|\nabla u|^{p-1}\nabla u) = g_*(\omega_N|x|^N),$$

then there exists a constant $c > 0$ (depending only on Ω, N, p) such that $u(x) \geq c\ln|\ln|x||$ near the origin. This last result implies that the above L^∞ result is optimal in the frame of small Lebesgue spaces in the sense that there are functions $f \in L^{(\frac{N}{p}}(\Omega), f \notin V_p$ for which the solution u of (P) is not bounded.
We recall that if $f \in L^s(\Omega), s > \frac{N}{p}$, then the boundedness of solutions is known (see for instance [5]) and is not true if $s = \frac{N}{p}$. The L^∞-estimate is also known in the frame of Lorentz spaces that is if f is in $L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)$ the solution $u \in W_0^{1,p}(\Omega)$ of $Au = f$ is

bounded (see for instance [6]). However, the techniques we employ here are slightly differents from those references and the estimates we have , seem to be sharp in the class of small Sobolev spaces.

When $p \leq 2$,we have the strict inclusion that $L^{(\frac{N}{p}}(\Omega) \subset L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)$.For $p > 2, L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)$ does not contain $L^{(\frac{N}{p}}(\Omega)$.

We remark here that for $p > 2$ it is easy to show that $V_p \subset L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)$. In fact, applying the Hölder inequality for small and grand Lebesgue spaces, we have

$$|g|^{\frac{1}{p-1}}_{L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)} \leq \int_{\Omega_*} \left[t^{\frac{p}{N}} g_{**}(t)\ \frac{dt}{t}\right]^{\frac{1}{p-1}} = \int_{\Omega_*} t^{\frac{p'}{N}-1}\ g_{**}^{\frac{1}{p-1}}(t)dt$$
$$\leq |\ t^{\frac{p'}{N}-1}|_{(N/p')')}|g_{**}^{\frac{1}{p-1}}|_{(N/p'}$$

and the right hand side is finite because $g \in V_p$.

The proofs of relations (2) and (3) rely on the techniques developed in [7], based on the Poincaré-Sobolev pointwise relation for the relative rearrangement. These techniques are different from those introduced by Talenti [8] since we don't make appeal to the derivative of the distribution function $m_u(t) = \text{meas}\{x : u(x) > t\}$, for instance. We shall denote by Ω an open set of $\mathbb{R}^N$ (bounded or unbounded).

If $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N)$, then $(x, y) = x \cdot y = \sum_{i=1}^{N} x_i y_i$ is the euclidian product and $|x| = (x, x)^{\frac{1}{2}}$, the associated norm, $B(x, t)$ is the ball of $\mathbb{R}^N$centered at a point x of radius $t > 0$. For $1 \leq p \leq +\infty, L^p(\Omega)$ is the usual Lebesgue space endowed with the usual norm, denoted by $|\cdot|_p$. $L^0(\Omega)$ denotes the set of all measurable functions on Ω. The usual Sobolev space $W^{1,p}_{(\Omega)}$ is endowed with the following norm$|u|_{1,p} = |u|_p + |\nabla u|_p$.
For a measurable set E of $\mathbb{R}^N$, we shall denote by$|E|$its Lebesgue measureand if $u : \Omega \to \mathbb{R}$ is a measurable function then,

$$\{u > t\} = \{x \in \Omega : u(x) > t\} and |u > t| = |\{u > t\}|, \{u = t\} = \{x \in \Omega : u(x) = t\}.$$

A plateau of value t is the set $\{u = t\}$satisfying $|u = t| > 0$. If $u \in L^p(\Omega)$, we set $P(u) = \bigcup_{t \in D} \{u = t\}$ the plateau u;(D is at most countable if Ω is bounded or $1 \leq p < +\infty$for unbounded domain Ω). For $g \geq 0$ measurable on Ω, we shall associate quantity introduced in [7]:

$$|g|_{(p'} = \operatorname*{Inf}_{\substack{g=\sum_{k=1}^{+\infty} g_k \\ g_k \geq 0}} \left\{ \sum_{k=1}^{+\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} g_k^{(p-\varepsilon)'}\ dx \right)^{\frac{1}{(p-\varepsilon)'}} \right\} ,$$

with $p' = \frac{p}{p-1}$, $\fint_{\Omega}$ is the average on Ω, $1 < p < +\infty$.

Definition $(\mathbf{1.1.1})]\mathbf{1}]$: Let u be in $L^p(\Omega), 1 \leq p \leq +\infty (u \geq 0$ if Ω is unbounded). The distribution function associated to u is the real function: $m_u : t \rightarrow |\{u > t\}| = |u > t|$. The following is proved in $([8], p. 135)$

Theorem $(\mathbf{1.1.2})]\mathbf{1}]$: The space defined by

$$L^{(p'}(\Omega) = \{g \in L^0(\Omega): \|g\|_{(p'} < +\infty\}$$

is a Banach space under the norm given by: $g \in L^{(p'}(\Omega) \longmapsto \|g\|_{(p'}$. This space is called small Lebesgue space.

Furthermore, $\forall \varepsilon > 0$:

$$L^{(p'+\varepsilon}(\Omega) \underset{\neq}{\subset} L^{(p'}(\Omega) \underset{\neq}{\subset} L^{p'}(\Omega).$$

The above spaces are associated to the so-called Grand Lebesgue spaces introduced in [9] (see [10]). For Ω a bounded open set:

$$L^{p}(\Omega) = \left\{ v \in L^1(\Omega) : [v] = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |v|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}} < +\infty \right\}.$$

The norm in this space is [11] denoted also by $|v|_{L^{N)}(\Omega)}$.

Definition $(\mathbf{1.1.3})]\mathbf{1}]$: We define on $[0, |\Omega|]$ the function u_* by setting:

$$u_*(s) = Inf \{t \in \mathbb{R}, |u > t| \leq s\}, s \in \Omega_*$$

and $u_*(0) = \text{ess sup}_\Omega \, u, u_*(|\Omega|) = \underset{\Omega}{\text{ess inf}} \, u$.

The function u_* is called the decreasing rearrangement of u.

Let $v \in L^p(\Omega), u \in L^r(\Omega), 1 \leq p < +\infty, 1 \leq r < +\infty$.

If Ω is unbounded we assume that $u \geq 0$ and the restriction of v to $\{u = 0\}$ is nonnegative. Furthermore, if v $(resp. u)$ satisfies the conditions:

$$|v > t| < +\infty \quad (resp. |u > t| < +\infty) \, \forall t > 0,$$

then p $(resp. r)$ can be infinite. For Ω bounded, p or r can be infinite. Consider the function $w : \overline{\Omega}_* \rightarrow \mathbb{R}$, defined by

$$w(s) = \int_{\{u>u_*(s)\}} v(x) \, dx + \int_0^{s-|u>u_*(s)|} \left(v|_{\{u=u_*(s)\}} \right)_* (\sigma) d\sigma$$

where $v|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$. The following result summarizes all those obtained in [12].

(a) If Ω is bounded then one has:

$\frac{(u+\lambda v)_* - u_*}{\lambda} \xrightarrow[\lambda \to 0]{} \frac{dw}{ds}$ in $L^p(\Omega_*)$-weak if $1 \leq p < +\infty$ and in $L^\infty(\Omega_*)$weak-star if $p = +\infty$.

(b) If Ω is unbounded then one has:

$i.)$ $w \in W^{1,p}_{loc}([0, +\infty)$,

$ii.)$ $\dfrac{dw}{ds} \in L^p(0, +\infty)$,

$iii.)\ \frac{(u+\lambda v)_{+*}-u_*}{\lambda}\overrightarrow{\lambda\to 0}\ \frac{dw}{ds}$ in $L^p(0,+\infty)$-weak if $1<p<+\infty$ (weak-star for $p=+\infty$) and in $L^1(0,M)$-weak,$\forall M$ finite.
In any case, $\left|\frac{dw}{ds}\right|_{L^p(\Omega_*)}\le |v|_{L^p(\Omega)}$.

Definition $(\mathbf{1.1.4})$]$\mathbf{1}$]: The function $\frac{dw}{ds}$ is called the relative rearrangement of v with respect to u and is denoted by $v_{*u}=\frac{dw}{ds}$.
In the case of a bounded domain, a similar notion can be found in [13] (see also [14]).
Poincar´e-Sobolev inequality for the relative rearrangement

Definition $(\mathbf{1.1.5})[\mathbf{1}]$: LetΩ be abounded open set. We will say that a subset V of $W^{1,1}(\Omega)$ satisfies the Poincar´e-Sobolev inequality for a relative rearrangement if there exists a function $K(.,\Omega,V)$ from Ω_* into $\mathbb{R}_+$, such that for all $u\in V$:
$(a)\ u_*\in W^{1,1}_{loc}(\Omega_*)$
$(b)-u'_*(s)\le K(s,\Omega,V)|\nabla u|_{*u}(s)$ for almost $s\in\Omega_*$.
For simplicity, we shall call PSR the above property. We shall use the following result proved in [15]:

Lemma $(\mathbf{1.1.6})[\mathbf{1}]$:Let Ωbe a ball of radius $R>0$.Then $V=W^{1,1}(\Omega)$ satisfies the PSR property. Furthermore,

$$K(s,\Omega,V)=\frac{1}{\omega_{N-1}}\left(\frac{\omega_N}{2}\right)^{1-\frac{1}{N}}Max\left(s^{\frac{1}{N}-1},(\omega_N R^N-s)^{\frac{1}{N}-1}\right).$$

Here, ωm denotes the volume of the unit ball of $\mathbb{R}^m$.
The Levi's theorem of monotone convergence for small Lebesgue spaces
Before stating and proving Levi's theorem, we give a few lemmas. The first one tells that in the expression of the norm of $L^{(p'}(\Omega)$ we can take also $\varepsilon\in]0,\frac{p-1}{2}]$ instead of $\varepsilon\in]0,p-1]$:

Lemma $(\mathbf{1.1.7})[\mathbf{1}]$: The following norms are equivalent:

$$\|g\|_{(p'}=\inf_{g=\sum g_k}\left\{\sum_k\inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\int_\Omega|g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\right\}$$

$$|||g|||_{(p'}=\inf_{g=\sum g_k}\left\{\sum_k\inf_{0<\varepsilon<\frac{p-1}{2}}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\int_\Omega|g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}\right\},$$

Proof: Of course $\|g\|_{(p'}\le|||g|||_{(p'}$.
On the other hand, fix $k\in\mathbb{N}$ such that

$$\inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\int_\Omega|g_k|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}=$$

$$= \inf_{\frac{p-1}{2}<\sigma<p-1} \sigma^{-\frac{1}{p-\sigma}} \left(\int_\Omega |g_k|^{(p-\sigma)'}\, dx\right)^{\frac{1}{(p-\sigma)'}}.$$

We have

$$\inf_{0<\varepsilon<\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'}\, dx\right)^{\frac{1}{(p-\varepsilon)'}}$$

$$\leq \left(\frac{p-1}{2}\right)^{-\frac{1}{p-\left(\frac{p-1}{2}\right)}} \left(\int_\Omega |g_k|^{\left(p-\frac{p-1}{2}\right)'}\, dx\right)^{\frac{1}{\left(p-\frac{p-1}{2}\right)'}}$$

$$= c_p \left(\int_\Omega |g_k|^{\left(p-\frac{p-1}{2}\right)'}\, dx\right)^{\frac{1}{\left(p-\frac{p-1}{2}\right)'}}$$

$$= \frac{c_p}{\sigma^{-\frac{1}{p-\sigma}}}\, \sigma^{-\frac{1}{p-\sigma}} \left(\int_\Omega |g_k|^{\left(p-\frac{p-1}{2}\right)'}\, dx\right)^{\frac{1}{\left(p-\frac{p-1}{2}\right)'}}$$

$$\leq \frac{c_p}{(p-1)^{-\frac{1}{p-\sigma}}}\, \sigma^{-\frac{1}{p-\sigma}} \left(\int_\Omega |g_k|^{(p-\sigma)'}\, dx\right)^{\frac{1}{(p-\sigma)'}}$$

$$\forall\ \frac{p-1}{2} < \sigma < p-1.$$

Therefore

$$\inf_{0<\varepsilon<\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'}\, dx\right)^{\frac{1}{(p-\varepsilon)'}}$$

$$\leq c'_p \inf_{\frac{p-1}{2}<\sigma<p-1} \sigma^{-\frac{1}{p-\sigma}} \left(\int_\Omega |g_k|^{(p-\sigma)'}\, dx\right)^{\frac{1}{(p-\sigma)'}}$$

$$= c'_p \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega |g_k|^{(p-\varepsilon)'}\, dx\right)^{\frac{1}{(p-\varepsilon)'}}$$

From the above computation we get easily $c'_p = (p-1)\left(\frac{p-1}{2}\right)^{-\frac{2}{p+1}}$. The conclusion is

$$\frac{1}{p-1}\left(\frac{p-1}{2}\right)^{\frac{2}{p+1}} |||g|||_{(p'} \leq \|g\|_{(p'} \leq |||g|||_{(p'}$$

and the lemma is proved.

Throughout the following we will use $|||.|||_{(p'}$ instead of $\|.\|_{(p'}$.Next lemma is an elementary inequality.

Lemma (1.1.8)[1]:

$$a \geq b \geq 0, \qquad p \geq 1 \Rightarrow a^p - b^p \geq (a-b)^p.$$

Lemma $(\mathbf{1.1.9})[\mathbf{1}]$: If $0 \leq b < a, \tau > 0, a \geq (1+\tau)b, 0 < \alpha_0 \leq \alpha < 1$, thenthere exists $c = c(\tau, \alpha_0)$ such that

$$(a-b)^{\alpha} \leq c(a^{\alpha} - b^{\alpha}).$$

Proof: By elementary computations we have

$$\sup_{\substack{0 \leq b < a \\ a \geq (1+\tau)b}} \frac{(a-b)^{\alpha}}{a^{\alpha} - b^{\alpha}} = \sup_{t \geq 1+\tau} \frac{(t-1)^{\alpha}}{t^{\alpha} - 1} = \frac{\tau^{\alpha}}{(1+\tau)^{\alpha} - 1} \quad \forall 0 < \alpha_0 \leq \alpha < 1.$$

On the other hand

$$\sup_{\alpha_0 \leq \alpha < 1} \frac{\tau^{\alpha}}{(1+\tau)^{\alpha} - 1} = \frac{\tau^{\alpha_0}}{(1+\tau)^{\alpha_0} - 1},$$

therefore the lemma is proved with

$$c = c(\tau, \alpha_0) = \frac{\tau^{\alpha_0}}{(1+\tau)^{\alpha_0} - 1}.$$

We have prove the following

Theorem $(\mathbf{1.1.10})[\mathbf{1}]$: Let (f_m) be a monotone nondecreasing sequence (i.e. $f_m \leq f_{m+1}$) such that $M = \sup_m \|f_m\|_{(p'} < +\infty$. Then the function $f = \sup\limits_m f_m$ is such that

$(i).\ f \in L^{(p'}$

$(ii).\ f_m \nearrow f\ a.e.$

$(iii).\ f_m \longrightarrow f \text{ in } L^{(p'}$

Proof: Without loss of the generality we may assume that the sequence $\left(|||f_m|||_{(p'}\right)_{m\in\mathbb{N}}$ is convergent, where $|||.|||_{(p'}$ is the expression equivalent to $\|.\|_{(p'}$, given in Lemma (1.1.7): if it is not the case, we can extract a subsequence of (f_m) and we prove first the theorem for such subsequence. The assertion in general then easily follows from the order-preserving property of $\|.\|_{(p'}$.

Now let $r, s \in \mathbb{N}, r > s$, and let $\sigma > 0$. Let $\left(f_r^{(k)}\right)_{k\in\mathbb{N}}$ be a decomposition of f_r:

$$f_r = \sum_k f_r^{(k)} \tag{5}$$

such that

$$\sum_k \operatorname{onf}_{0<\varepsilon\leq\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_{\Omega} \left|f_r^{(k)}\right|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}} \leq |||f_r|||_{(p'} + \sigma. \tag{6}$$

Let $\sigma_k \in]0, \frac{p-1}{2}]$ be such that

$$\sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_{\Omega} \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

$$\leq \inf_{0<\varepsilon\leq\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_{\Omega} \left|f_r^{(k)}\right|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}} + \frac{\sigma}{2k} \quad \forall k \in \mathbb{N}. \tag{7}$$

On the other hand, since $r > s$ and (f_m) is an increasing sequence, we have $f_s \leq f_r$, and by Fiorenza [12], we know that there exists $\left(f_s^{(k)}\right)$ such that

$$f_s = \sum_k f_s^{(k)} \tag{8}$$

$$f_s^{(k)} \leq f_r^{(k)} \ \forall k \in \mathbb{N} \tag{9}$$

From (5), (8), (9) we get

$f_r - f_s = \sum_k \left(f_r^{(k)} - f_s^{(k)}\right)$
and therefore

$$|||f_r - f_s|||_{(p'} \leq \inf_{0<\varepsilon\leq\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega \left|f_r^{(k)} - f_s^{(k)}\right|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}}$$

$$\leq \sum_k \sigma_k{}^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)} - f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

By Lemma (1.1.8)

$$|||f_r - f_s|||_{(p'} \leq \sum_k \sigma_k{}^{-\frac{1}{p-\sigma_k}} - \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx - \int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

Now fix $0 < \tau < 1$, and let

$$A_\tau = \left\{k \in \mathbb{N} : \int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx < (1+\tau)\int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right\}$$

$B_\tau = \mathbb{N} - A_\tau$.
We have

$$\sum_{k\in A_\tau} \sigma_k{}^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx - \int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

$$\leq \sum_{k\in A_\tau} \tau^{\frac{1}{(p-\sigma_k)'}} \cdot \sigma_k{}^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

$$\leq \tau^{\frac{1}{\left(p-\frac{p-1}{2}\right)'}} \sum_{k\in\mathbb{N}} \sigma_k{}^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

and by (7)

$$\ldots \le \tau^{\frac{1}{\left(\frac{p+1}{2}\right)'}} \left[\sigma + \sum_{k\in\mathbb{N}} \inf_{0<\varepsilon\le\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}} \right]$$

and by (6)

$$\ldots \le \tau^{\frac{p-1}{p+1}} \left[\||f_r|\|_{(p'} + 2\sigma\right] \le \tau^{\frac{p-1}{p+1}} [M + 2\sigma]$$

On the other hand, by lemma (1.1.9)

$$\sum_{k\in B_\tau} \sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx - \int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}$$

$$\le c\left(\tau, \frac{p-1}{p+1}\right) \sum_{k\in B_\tau} \left[\sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}} - \sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}\right]$$

$$\le c\left(\tau, \frac{p-1}{p+1}\right) \sum_{k\in \mathbb{N}} \left[\sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_r^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}} - \sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}\right]$$

and as above, by (6) and (7)

$$\ldots \le c\left(\tau, \frac{p-1}{p+1}\right) \left[\||f_r|\|_{(p'} + 2\sigma - \sum_{k\in\mathbb{N}} \sigma_k^{-\frac{1}{p-\sigma_k}} \left(\int_\Omega \left|f_s^{(k)}\right|^{(p-\sigma_k)'} dx\right)^{\frac{1}{(p-\sigma_k)'}}\right]$$

$$\le c\left(\tau, \frac{p-1}{p+1}\right) \left[\||f_r|\|_{(p'} - \||f_s|\|_{(p'} + 2\sigma\right]$$

Until now we proved that $\forall\sigma > 0, \forall 0 < \tau < 1$

$$\||f_r - f_s|\|_{(p'} \le \tau^{\frac{p-1}{p+1}} [M + 2\sigma] + c\left(\tau, \frac{p-1}{p+1}\right) \left[\||f_r|\|_{(p'} - \||f_s|\|_{(p'} + 2\sigma\right]$$

Letting $\sigma \to 0$ we get, for any $r, s, \in \mathbb{N}, r > s$,

$$\||f_r - f_s|\|_{(p'} \le \tau^{\frac{p-1}{p+1}} M + c\left(\tau, \frac{p-1}{p+1}\right) \left[\||f_r|\|_{(p'} - \||f_s|\|_{(p'}\right]. \qquad (10)$$

Now let $\varepsilon > 0$, and fix $\tau = \bar{\tau}_\varepsilon$ such that

$$\bar{\tau}_\varepsilon^{\frac{p-1}{p+1}} M < \frac{\varepsilon}{2} \qquad (11)$$

On the other hand, since the sequence $\left(|||f_r|||_{(p'}\right)_{r\in\mathbb{N}}$ is convergent, there exists $m_\varepsilon \in \mathbb{N}$ such that

$$c\left(\tau_\varepsilon, \frac{p-1}{p+1}\right)\left[|||f_r|||_{(p'} - |||f_s|||_{(p'}\right] < \frac{\varepsilon}{2} \quad \forall r > s > m_\varepsilon. \tag{12}$$

By (10), (11), (12) we have that

$$\forall\varepsilon > 0 \exists m_\varepsilon \in \mathbb{N} : |||f_r - f_s|||_{(p'} < \varepsilon \qquad \forall r > s > m_\varepsilon,$$

therefore, by Lemma (1.1.7), (f_m) is a Cauchy sequence in $L^{(p'}$ and converges to some function $f \in L^{(p'}$.

From the imbedding of $L^{(p'}$in L^1 it follows that the limit f coincides a.e. with $\sup_m f_m$, which is also the a.e. limit of (f_m).

Corollary $(\mathbf{1.1.11})[\mathbf{1}]$: Let $f \in L^{(p'}(\Omega)$ and let (E_m) be a sequence of measurable sets in Ω such that

$i)$ $\Omega \supseteq E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots \supseteq E_m \supseteq \cdots$

$ii)$ $|E_m| \longrightarrow 0$

Then

$$\left\|f\chi_{E_m}\right\|_{L^{(p'}(\Omega)} \longrightarrow 0.$$

Proof: Without loss of generality we may assume that $f \geq 0$. Let us set $f_m = f - f\chi_{E_m}$ $\forall m \in \mathbb{N}$. By $i)$ the sequence (f_m) is increasing. Moreover, since $f_m \leq f \in L^{(p'}(\Omega)$ $\forall m$, we have

$$\sup_m \|f_m\|_{L^{(p'}(\Omega)} \leq \|f\|_{L^{(p'}(\Omega)} = M$$

Finally, by $ii)$, we have that $f = \sup_m f_m$.

By the theorem proved, $f_m \to f$ in $L^{(p'}(\Omega)$, therefore

$$\left\|f\chi_{E_m}\right\|_{L^{(p'}(\Omega)} = \|f - f_m\|_{L^{(p'}(\Omega)} \longrightarrow 0$$

Proposition $(\mathbf{1.1.12})[\mathbf{1}]$: Let g be in $L^{(p'}(\Omega)$ and for $\sigma \in \Omega_*$, set

$$g_{**}(\sigma) = \frac{1}{\sigma}\int_0^\sigma |g|_*(t)\,dt.$$

Then

$$\|g\|_{L^{(p'}(\Omega)} \leq |g_{**}|_{L^{(p'}(\Omega_*)} \leq p\|g\|_{L^{(p'}(\Omega)}.$$

Proof: Let $g_k \geq 0$ be an admissiblede composition of $|g|$ $i.e$ $|g| = \sum_{k=1}^{+\infty} g_k$.Setting

$$g_{k**}(\sigma) = \frac{1}{\sigma}\int_0^\sigma g_{k*}(t)\,dt,$$

then from Hardy’s inequality, we have:

$$|g_{k**}|_{L^{(p-\varepsilon)'}(\Omega_*)} \leq p|g_k|_{L^{(p-\varepsilon)'}(\Omega)}. \tag{13}$$

Moreover, we have from the Hardy-Littlewood property:

$$g_{**}(\sigma) \le \sum_{k=1}^{+\infty} g_{k**}(\sigma) \ . \tag{14}$$

From relations (13) and (14), we have, by using Lemma 2.1 of [14]:

$$|g_{**}|_{L^{(p'}(\Omega_*)} \le \sum_{k=1}^{+\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_{\Omega_*} g_{k**}(t)^{(p-\varepsilon)'}\, dt \right)^{\frac{1}{(p-\varepsilon)'}}$$
$$\le p \sum_{k=1}^{+\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\int_{\Omega_*} g_{k}(t)^{(p-\varepsilon)'}\, dt \right)^{\frac{1}{(p-\varepsilon)'}},$$

from which we get the upper bound.

For the lower bound we have

$$|g|_* \le g_{**}. \tag{15}$$

Since the norm in $L^{(p'}$ is a rearrangement invariant norm, we then have:

$$\|g\|_{L^{(p'}(\Omega)} = ||g|_*|_{L^{(p'}(\Omega_*)} \le |g_{**}|_{L^{(p'}(\Omega_*)}$$

The following result has been proved in [15]:

Lemma $(\mathbf{1.1.13})[\mathbf{1}]$: For $g \in L^{(p'}(\Omega), v \in L^1(\Omega)$, we have

$$\|g_{*v}\|_{L^{(p'}(\Omega_*)} \le \|g\|_{L^{(p'}(\Omega)} .$$

Theorem $(\mathbf{1.1.14})[\mathbf{1}]$: Let Ω be a bounded Lipschitz connected set and

$$W^{1,(N}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{(N}(\Omega) \right\}.$$

Then

$$W^{1,(N}(\Omega) \subset L^\infty(\Omega) \cap C(\Omega).$$

Moreover, we have the following rate of convergence:

$$\underset{B(x,t)}{osc}\, u \le \frac{\omega_N^{\frac{1}{N'}}}{\omega_{N-1}} |\Omega|^{\frac{1}{N}} (N')^{\frac{1}{N'}} \left| |\nabla u| \chi_{B(x,t)} \right|_{L^{(N}(\Omega)} .$$

whenever $B(x,t) \subset \Omega, t > 0$. Here, ω_m is the volume of the unit ball of $\mathbb{R}^m$.

Proof: The fact that $W^{1,(N}(\Omega) \subset L^\infty(\Omega)$ has been proved directly in [15]. Let (x,t) be such that $B(x,t) \subset \Omega, t > 0$. Let $u \in W^{1,(N}(\Omega)$ and u the restriction of u to $B(x,t)$. From the Lemma (1.1.6) (Poincar´e-Sobolev pointwise relations), we have (see [16])

$$\underset{B(x,t)}{osc}\, u \le \frac{\omega_N^{1-\frac{1}{N}}}{\omega_{N-1}} \int_0^{\omega_N t^N} s^{\frac{1}{N}-1} (|\nabla \bar{u}|_{*\bar{u}})_*(s) ds \tag{16}$$

By the H¨older inequality (see [17]), we then have:

$$\underset{B(x,t)}{osc} u \le \frac{\omega_N^{2-\frac{1}{N}}}{\omega_{N-1}} t^N \left| s^{\frac{1}{N}-1} \right|_{L^{N')}(0,\omega_N t^N)} \cdot ||\nabla \bar{u}|_{*\bar{u}}|_{L^{N)}(0,\omega_N t^N)} \tag{17}$$

Since the norm of $L^{N)}$ is a rearrangement invariant norm, we then deduce from the result of [18]:

$$||\nabla\bar{u}|_{*\bar{u}}|_{L^{N)}(0,\omega_N t^N)} \leq \|\nabla\bar{u}\|_{L^{N)}(B(x,t))} . \tag{18}$$

and coming back to the definition of the norm

$$\|\nabla\bar{u}\|_{L^{N)}(B(x,t))} \leq \frac{|\Omega|^{\frac{1}{N}}}{(\omega_N t^N)^{\frac{1}{N}}} \left||\nabla u|\chi_{B(x,t)}\right|_{L^{N)}(\Omega)} \tag{19}$$

Moreover, we have:

$$\left|s^{\frac{1}{N}-1}\right|_{L^{N')}(0,\omega_N t^N)} = \sup_{0<\varepsilon<N'-1}\left(\frac{\varepsilon}{\omega_N t^N}\int_0^{\omega_n t^N} s^{-\frac{1}{N'}(N'-\varepsilon)}\, ds\right)^{\frac{1}{N'-\varepsilon}}$$

$$\left|s^{\frac{1}{N}-1}\right|_{L^{N')}(0,\omega_N t^N)} = (N')^{\frac{1}{N'}}\omega_N^{-\frac{1}{N'}}t^{-\frac{N}{N'}} . \tag{20}$$

Combining relations (17), (19) and (20), we have

$$\underset{B(x,t)}{osc}\, u \leq \frac{\omega_N^{\frac{1}{N'}}}{\omega_{N-1}} |\Omega|^{\frac{1}{N}}(N')^{\frac{1}{N'}}\left||\nabla u|\chi_{B(x,t)}\right|_{L^{(N}(\Omega)} .$$

Notice that from corollary (1.1.11) of theorem (1.1.10), we deduce that

$$\underset{B(x,t)}{osc}\, u \to 0 \; as \;\; t \to 0$$

therefore u is continuous.

The second application concerns the regularity of quasilinear equations. For simplicity, in, we shall assume that $|\Omega| = 1$. We shall need the following assumption:

Assumption $(\mathbf{1.1.15})[\mathbf{1}]$: Let $\hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a nonlinear Caratheodory map satisfying the following conditions:

i) For $a.e.\, x, \forall(u,\xi) \in \mathbb{R} \times \mathbb{R}^N$:

$$\hat{a}\,(x,u,\xi)\cdot\xi \geq \alpha|\xi|^p$$

for some $\alpha > 0, 1 < p < N$.

ii)

$$\forall(\phi,\psi) \in W^{1,p}(\Omega)^2, \quad \hat{a}\,(x,\phi,\nabla\phi)\cdot\nabla\psi \in L^1(\Omega)$$

Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P) :

$$\int_\Omega \hat{a}\,(x,u,\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega f\varphi\, dx \;\forall\varphi \in W_0^{1,p}(\Omega).$$

We recall that if $p > 2$ we shall consider

$$V_p = \left\{g \in L^{\left(\frac{N}{p}\right.}(\Omega) : g_{**}^{\frac{1}{p-1}} \in L^{\left(\frac{N}{p'}\right.}(\Omega_*)\right\}.$$

First, we shall show that V_p is different from $L^{\left(\frac{N}{p}\right.}(\Omega)$.

Proposition $(\mathbf{1.1.16})[\mathbf{1}]$: Let Ω be a set of measure 1.We define the function

$$\mu(s) = s^{-\frac{p}{N}}|Lns|^{1-p},$$

for $s \in \left(0,\frac{1}{2}\right)$, $\mu(s) = \mu\left(\frac{1}{2}\right)$ for $\frac{1}{2} \leq s \leq 1$.Then, for $p > 2$,

1. $\mu \in L^{\left(\frac{N}{p}\right.}(0,1)$ and $\mu^{\frac{1}{p-1}} \notin L^{\left(\frac{N}{p'}\right.}(0,1)$;

2. There exists a function $g \in L^{\left(\frac{N}{p}\right.}(\Omega)$ such that $g \notin V_p$.

Proof: Since there are two constants $c_1 > 0, c_2 > 0$ such that, for m large:

$$c_1\, Lns \leq -Ln\mu(s) \leq c_2 Lns, for\ all\ s \in \left(0,\frac{1}{2^m}\right)$$

it follows easily that μ belongs to the Zygmund spaces $L^{\frac{N}{p}}(Ln\,L)^{\beta\frac{N}{p}-1}$ for all $\beta \in]1, p-1[$ and μ is not in $L^{\frac{N}{p}}(Ln\,L)^{\frac{N}{p'}-1}$. Combining the results of Greco ([18]) and Fiorenza ([19]), we know that

$$L^{\frac{N}{p}}(Ln\,L)^{\beta\frac{N}{p}-1} \subset L^{\left(\frac{N}{p}\right.}\ for\ all\ \beta > 1,$$

and

$$L^{\left(\frac{N}{p'}\right.} \subset L^{\left(\frac{N}{p'}\right.}(Ln\,L)^{\frac{N}{p'}-1}.$$

We conclude that $\mu \in L^{\left(\frac{N}{p}\right.}(0,1)$ and $\mu^{\frac{1}{p-1}}$ cannot be in $L^{\left(\frac{N}{p'}\right.}(0,1)$.

From the Lyapunov theorem (see [18]), there exists a function $g : \Omega \to \mathbb{R}$ such that $g_* = \mu$ (since μ is decreasing). Since $L^{\left(\frac{N}{p}\right.}(\Omega)$ is a rearrangement invariant space, therefore, $g \in L^{\left(\frac{N}{p}\right.}(\Omega)$.

We have $g_{**}^{\frac{1}{p-1}} \notin L^{\left(\frac{N}{p'}\right.}(0,1)$.If not ,since $\mu = g_* \leq g_{**}$, then $\mu^{\frac{1}{p-1}}$ will belong to $L^{\left(\frac{N}{p'}\right.}(0,1)$ which is not true. Thus $g \notin V_p$.

Theorem (1.1.17)[1]: If $f \in L^{\left(\frac{N}{p}\right.}(\Omega)$, then u is bounded if $p \leq 2$.

Moreover if $p < \frac{2N}{N+1}$ and $|\Omega| = 1$ then

$$|u|_\infty \leq c_{\alpha N}\left(\int_0^1 s^{-\frac{p'}{N'}}\varphi(s)^{\frac{1}{p-1}}\,ds\right)|g_{*v}|^{\frac{1}{p-1}}_{L^{\left(\frac{N}{p}\right.}(\Omega_*)} < +\infty.$$

(and $|g_{*v}|_{L^{\left(\frac{N}{p}\right.}(\Omega)} \leq |f|_{L^{\left(\frac{N}{p}\right.}(\Omega)}$). Here, $\varphi(s) = \sup_{0<\varepsilon<q-1}(\varepsilon s)^{\frac{1}{q-\varepsilon}}$,

with $q = \frac{N}{N-p}$, $c_{\alpha Np} = \frac{1}{\alpha^{\frac{p'}{p}}\left(N\omega_N^{\frac{1}{N}}\right)^{p'}}$, ω_N is the measure of the unit ball, $g = |f|, v = |u|$.

If $p = 2$ then $|u|_\infty \leq c_{\alpha N2}\left(\frac{N}{N-2}\right)^2 |g|_{\left(\frac{N}{2}\right.}$.

If $p > 2, f \in V_p$, then u is bounded.

Proof: Let $g = |f|, v = |u|$. It has been proved in [19] that for almost all s:

$$-\frac{dv_*}{ds}(s) \leq c_{\alpha Np}\, s^{-\frac{p'}{N'}} \left[\int_0^s g_{*v}(\sigma)\, d\sigma\right]^{\frac{p'}{p}} . \qquad (21)$$

1st case: $p < \frac{2N}{N+1}$

From Hölder inequality, we have:

$$\int_0^s g_{*v}(t)\, dt \leq \left|\chi_{[0,s]}\right|_{L^{\left(\frac{N}{p}\right)'})} \cdot |g_{*v}|_{L^{(\frac{N}{p}}} . \qquad (22)$$

Setting $\varphi(s) = \left|\chi_{[0,s]}\right|_{L^{\left(\frac{N}{p}\right)'})} = \sup_{0<\varepsilon<q-1}(\varepsilon s)^{\frac{1}{q-\varepsilon}}$, $q = \frac{N}{N-p}$, we derive from relations (21) and (22) that, for all $\sigma \in (0,1)$

$$v_*(\sigma) \leq c_{\alpha Np} \left(\int_0^1 s^{-\frac{p\prime}{N\prime}} \varphi(s)^{\frac{1}{p-1}} ds\right) |g_{*v}|^{\frac{1}{p-1}}_{L^{(\frac{N}{p}}(\Omega_*)} < +\infty. \qquad (23)$$

From the lemma (1.1.13), we know that:

$$|g_{*v}|_{L^{(\frac{N}{p}}(\Omega_*)} \leq |f|_{L^{(\frac{N}{p}}(\Omega)} < +\infty. \qquad (24)$$

It remains to show that the quantity

$$\int_0^1 s^{-\frac{p\prime}{N\prime}} \varphi(s)^{\frac{1}{p-1}} ds < +\infty \; if \; 1 < p < \frac{2N}{N+1} .$$

Indeed, from the result of Fiorenza ([20]), we have

$$\varphi(s) \underset{s \to 0}{\simeq} s^{\frac{1}{q}} |Lns|^{-\frac{1}{q}}, \; q = \frac{N}{N-p} .$$

Thus, there exist constants $c_1 > 0, a > 1$

$$\varphi(s) \leq c_1 s^{\frac{1}{q}} |Lns|^{-\frac{1}{q}}, \quad for\; 0 < s \leq \frac{1}{a} . \qquad (25)$$

From (25), we the deduce:

$$\int_0^{\frac{1}{a}} s^{-\frac{p\prime}{N\prime}} \varphi(s)^{\frac{1}{p-1}} ds \leq c_2 \int_0^{\frac{1}{a}} \frac{ds}{s|Lns|^{\gamma}} = c_2 I, with \; \gamma = \frac{N-p}{N(p-1)} .$$

With the change of variables, $\sigma = -\ln s$, we have:

$$I = \int_{\ln a}^{+\infty} \frac{d\sigma}{\sigma\gamma} = \frac{(\ln a)^{1-\gamma}}{\gamma - 1} < +\infty \; forp < \frac{2N}{N+1} . \qquad (26)$$

From (23), (24) and (26), we get:

$$|u|_\infty \leq c_{\alpha Np} \left(\int_0^1 s^{-\frac{p'}{N'}} \phi(s)^{\frac{1}{p-1}} ds\right) |f|^{\frac{1}{p-1}}_{L^{(\frac{N}{p}}(\Omega)} < +\infty.$$

2nd case: $\frac{2N}{N+1} \leq p \leq 2$ From the work of Greco ([21]), we know that $L^{\left(\frac{N}{p}\right)'},\left(\frac{1}{p-1}\right)'(\Omega) \subset L^{\left(\frac{N}{p}\right)')}(\Omega)\; if\; p \leq 2$. By using the result of Fiorenza ([22]) on associate spaces, one has the inclusion that $L^{\left(\frac{N}{p}\right.}(\Omega) \subset L^{\frac{N}{p},\frac{1}{p-1}}(\Omega)$. Applying the boundedness result of [23], we get the fact that u is bounded.

We can get a precise estimate when $p = 2$ or $f \in V_p, p > 2$. First, we derive from relation (21) and Hardy-Littlewood inequality:

$$v_*(s) \leq c_{\alpha Np} \int_0^1 t^{\frac{p'}{p}-\frac{p'}{N'}} \cdot [(g_{*v})_{**}]^{\frac{p'}{p}}\, dt \qquad (27)$$

Moreover, from the result of [22], [23] (see also [24]) we deduce $(g_{*v})_{**} \leq g_{**}$. Thus relation (27) with Hölder inequality yields for $p \geq 2$:

$$v_*(s) \leq c_{\alpha Np} \left| t^{\frac{p'}{p}-\frac{p'}{N'}} \right|_{L^{\left(\frac{N}{p'}\right)')}} \cdot \left| g_{**}^{\frac{1}{p-1}} \right|_{L^{\left(\frac{N}{p'}\right.}} \qquad (28)$$

and using the definition of the norm of the Grand Sobolev space, we have:

$$\left| t^{\frac{p'}{p}-\frac{p'}{N'}} \right|_{L^{\left(\frac{N}{p}\right)')}} = \frac{N}{N-p'}. \qquad (29)$$

If $p = 2$, knowing from proposition (1.1.12) that

$$|g_{**}|_{\left(\frac{N}{2}\right.} \leq \frac{N}{N-2} \cdot |g|_{\left(\frac{N}{2}\right.} \qquad (30)$$

we obtain from (28) to (30) the result. If $p > 2, f \in V_p$, we derive from (28) and (29) that

$$|u|_\infty \leq c_{\alpha Np} \frac{N}{N-p'} \left| g_{**}^{\frac{1}{p-1}} \right|_{L^{\left(\frac{N}{p'}\right.}}.$$

To show that the last statement of Theorem (1.1.17) cannot be improved in the frame of small Lebesgue spaces, we shall prove its optimality. We shall use the same notations and functions as for proposition (1.1.16).

Proposition $(\mathbf{1.1.18})[\mathbf{1}]$: Let Ω be a ball centered at the origin of measure 1 and let u be the unique radial solution u in $W_0^{1,1}(\Omega)$ of

$$-\Delta_p u = \mu(\omega_N |x|^N).$$

Then there exists a constant $c > 0$ and a neighborhood Ω' of the origin such that for all

$$x \in \Omega', u(x) \geq c \ln|\ln|x||.$$

In particular, if we consider $g(x) = (\omega_N |x|^N), p^2 < (p-1)N, p > 2$ then $g \in L^{\left(\frac{N}{p}\right.}(\Omega) \cap L^{p'}(\Omega), u \in W_0^{1,p}(\Omega)$ and $u \notin L^\infty(\Omega)$.

Proof: Following the work of [25], one can show directly that the solution u can be written as:

$$u(x) = \frac{1}{N^{\frac{1}{p-1}}\left(\omega_N^{\frac{1}{N}}\right)^{p'}} \int_{\omega_N|x|^N}^{1} \left[t^{\left(\frac{1}{N}-1\right)p} \int_0^t \mu(\sigma) d\sigma dt \right]^{\frac{1}{p-1}}.$$

From the expression of μ and the fact that it is decreasing:

$$u(x) \geq c_1 \int_{\omega_N|x|^N}^{\frac{1}{2}} \left[t^{\frac{p}{N}-p} \cdot t \cdot t^{-\frac{p}{N}} |\ln t|^{1-p} \right]^{\frac{1}{p-1}} dt.$$

$$\geq c_1 \int_{\omega_N|x|^N}^{\frac{1}{2}} t^{-1}|\ln t|^{-1}\, dt \geq c \ln|\ln|x|| \; near \; x = 0.$$

The function $g(x) = \mu(\omega_N|x|^N)$ satisfies $g_* = \mu$. Thus $g \in L^{\left(\frac{N}{p}\right.}(\Omega)$. If $p^2 < (p-1)N$, then $g \in L^{p'}(\Omega)$. By a classical result $u \in W_0^{1,p}(\Omega)$.

Section (1.2): Grand and Small Lebesgue Spaces

The norm of the grand Lebesgue spaces

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \|f\|_{p-\epsilon} (1 < p < \infty)$$

was introduced by Iwaniec and Sbordone in [27], in the framework of the study of the integrability properties of the Jacobian determinant. Since then, such norm attracted the interest of several researchers , either in Harmonic Analysis (see e.g. [28]) in Interpolation Extrapolation Theory (see e.g. [29]), either in $P.D.Es$ (see e.g. [30]). Much attention has been devoted to the problem to identify the associate space of the grand Lebesgue spaces. The first characterization of the norm, which originated the small Lebesgue spaces, was given in [31]; another characterization appeared in [32] (see also [33]).

Both grand and small Lebesgue spaces are Banach Function Spaces in the sense of Benentt and Sharpley [34]; however, while in [35] there is an explicit proof that the expression of the norm of the small Lebesgue spaces satisfy all axioms of Banach Function Spaces (see [36] for the Fatou property), the (much simpler) proof that the corresponding axioms for the grand Lebesgue spaces seems missing in literature, even if actually this fact is commonly well known. We begin by establishing that the grand Lebesgue spaces are Banach Function Spaces. The proof will be given for a (already well known) generalized expression of the norm$\|\cdot\|_{p)}$, namely, for

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} (1 < p < \infty)$$

where θ is a positive parameter. Of course, when $\theta = 1$ the expression$\|f\|_{p),\theta}$ gives back the original norm$\|f\|_{p)}$.

We study norm obtained replacing, in the expression of $\|.\|_{p),\theta}$ above, the Lebesgue norm $\|f\|_{p-\epsilon}$ by a grand Lebesgue norm. The resulting formula is (the single parameter θ will be replaced by the couple of parameters α, β)

$$\|f\|_{p)),\alpha,\beta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\alpha}{p-\epsilon}} \|f\|_{p-\epsilon),\beta} \, 1 < p < \infty \, \alpha > 0, \qquad \beta > 0. \quad (31)$$

This expression can play a key role in iteration-type results about grand Lebesgue spaces, typical of interpolation theory: be sides the pioneering [29] very recently a development in this direction seems announced in [30]. We first prove that (31) is a Banach Function Norm and then, in Corollary (1.2.5), we prove that the norm in (31) gives back a norm in a grand Lebesgue space. The proofs are direct, do not require any background of the literature quoted above, and constants of the equivalences are given explicitly. Analogous results are stated and proved for the small Lebesgue spaces, whose norm has a less simple expression to deal with. Very recently the small Lebesgue spaces have been characterized as the optimal rearrangement-invariant Banach Function Spaces for the freedimensional Sobolev estimate, see [31].

For $\Omega \subset R^N, N \geq 1$, be a set of Lebesgue measure $|\Omega| = 1$, and let $1 < p < \infty, \theta > 0$. Let M_0 be the set of all the measurable, real-valued functions on $\Omega$and let $L^{p),\theta}$ be the set of the functions f in M_0 such that $\|f\|_{p),\theta} < \infty$.

The main result is that $L^{p),\theta}$ is a special case of Banach Function Space, namely, its norm satisfies the following properties, where f, g, f_n are in $M_0, \lambda \geq 0$, and E is any measurable subset of Ω:

(i). $\|f\|_{p),\theta} \geq 0$

(ii). $\|f\|_{p),\theta} = 0 \; iff f = 0$ a.e. in Ω

(iii). $\|\lambda f\|_{p),\theta} = \lambda \|f\|_{p),\theta}$

(iv) . $\|f + g\|_{p),\theta} \leq \|f\|_{p),\theta} + \|g\|_{p),\theta}$

(v). $if \; |g| \leq |f| a.e.$ in Ω, then $\|g\|_{p),\theta} \leq \|f\|_{p),\theta}$

(vi). if $0 \leq f_n \uparrow f$ a.e. in Ω, then$\|f_n\|_{p),\theta} \uparrow \|f\|_{p),\theta}$

(vii). $\|\chi_E\|_{p),\theta} < +\infty$

$(viii)$. $\int_E |f| dx \leq C(p, \theta, E) \|f\|_{p),\theta}$

Proposition $(\mathbf{1.2.1})[\mathbf{26}]$: Let $1 < p < \infty, \theta > 0$. The space $L^{p),\theta}$ is a Banach Function Space.

Proof: We have to prove the properties (1)– (8).

The first three properties follow directly from the corresponding properties true for Lebesgue spaces.

. It is

$$\|f + g\|_{p-\epsilon} \leq \|f\|_{p-\epsilon} + \|g\|_{p-\epsilon} \;\; \forall \epsilon \in]0, p-1[\, .$$

Hence

$$\|f+g\|_{p),\theta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f+g\|_{p-\epsilon}$$

$$\leq \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} + \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|g\|_{p-\epsilon}$$

$$= \|f\|_{p),\theta} + \|g\|_{p),\theta}.$$

Since $g \leq f$ a.e. in Ω, then

$$\epsilon^{\frac{\theta}{p-\epsilon}}\|g\|_{p-\epsilon} \leq \epsilon^{\frac{\theta}{p-\epsilon}}\|f\|_{p-\epsilon} \,\forall \epsilon \in]0, p-1[$$

therefore, passing to the supremum over $\epsilon \in]0, p-1[$,

$$\|g\|_{p),\theta} \leq \|f\|_{p),\theta}.$$

If $0 \leq f_n \uparrow f$ a.e. in Ω,

$$\|f_n\|_{p),\theta} \uparrow \sup_n \|f_n\|_{p),\theta} = \sup_n \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f_n\|_{p-\epsilon} = \sup_{0<\epsilon<p-1} \sup_n \epsilon^{\frac{\theta}{p-\epsilon}}\|f_n\|_{p-\epsilon}$$

$$= \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} = \|f\|_{p),\theta}.$$

Since $\epsilon^{\frac{1}{p-\epsilon}}$ is increasing in $\epsilon \in]0, p-1[$,

$$\|\chi_E\|_{p),\theta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|\chi_E\|_{p-\epsilon} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} |E|^{\frac{1}{p-\epsilon}} = (p-1)^{\theta} |E|^{\frac{1}{p}}.$$

Fix $0 < \epsilon < p-1$.

By Hölder's inequality we have

$$\int_E |f|\, dx = \int_\Omega |f|\chi_E\, dx \leq \|f\|_{p-\epsilon} \left(\int_\Omega \chi_E^{(p-\epsilon)'}\, dx\right)^{1-\frac{1}{p-\epsilon}},$$

where $(p-\epsilon)$denotes, as usual, the conjugate exponent of $p-\epsilon$. Therefore

$$\int_E |f|\, dx \leq |E|^{1-\frac{1}{p-\epsilon}} \epsilon^{-\frac{\theta}{p-\epsilon}} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} \leq |E|^{1-\frac{1}{p-\epsilon}} \epsilon^{-\frac{\theta}{p-\epsilon}} \|f\|_{p),\theta}$$

from which

$$\int_E |f|\, dx \leq \frac{|E|^{1-\frac{1}{p}}}{(p-1)^{\theta}} \|f\|_{p),\theta}$$

and hence

$$\int_E |f|\, dx \leq C(p,\theta,E)\|f\|_{p),\theta},$$

where $C(p,\theta,E) = \frac{|E|^{1-\frac{1}{p}}}{(p-1)^{\theta}}$.

The name "grand Lebesgue space" comes from the continuous embedding $L^p \subset L^{p),\theta}$: by Hölder's inequality, and by the monotonicity in $\epsilon$of the function $\epsilon^{\frac{1}{p-\epsilon}}$,

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} \leq (p-1)^{\theta}\|f\|_{p} .$$

This shows that the following continuous embeddings hold:

$$L^{p} \subset L^{p),\theta} \subset L^{p-\epsilon} \ 0 < \epsilon < p-1$$

but these embeddings can be refined in the framework of Orlicz spaces (see [35]; see [36] and [37] for a sharp result):

$$\frac{L^{p}}{(log\ L)^{\theta}} \subset L^{p),\theta} \subset \frac{L^{p}}{(log\ L)^{\theta+\epsilon}} \ \epsilon > 0.$$

As to the embeddings between two grand Lebesgue spaces $L^{p),\theta}$ we note that, in terms of inclusions, these spaces increase with θ: in fact, if $0 < \phi < \theta$,

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta-\phi}{p-\epsilon}} \epsilon^{\frac{\phi}{p-\epsilon}} \|f\|_{p-\epsilon}$$

and, being

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta-\phi}{p-\epsilon}} = (p-1)^{\theta-\phi} ,$$

we have

$$\|f\|_{p),\theta} \leq (p-1)^{\theta-\phi} \|f\|_{p),\phi} .$$

The space $L^{p),\theta}$ is rearrangement-invariant but not separable, neither reflexive. The set of the bounded functions is not dense, and the closure of L^{∞} in the norm of $L^{p),\theta}$ can be characterized (see [38]) by the functions f such that

$$\lim_{\epsilon\to 0} \sup \epsilon^{\frac{\theta}{p-\epsilon}} \|f\|_{p-\epsilon} = 0 .$$

The fundamental function is equivalent to $\varphi_{p),\theta}(t) = t^{1/p} \left[log\left(\frac{1}{t}\right)\right]^{-\theta/p}$. Note that the supremum in the definition of the norm, carried over the interval $]0, p-1[$, can be equivalently considered over the interval $]0, \epsilon_0[$, for any positive ϵ_0 smaller than $p-1$ (see [39]). The associate space of $L^{p),\theta}$, defined by

$$\left(L^{p),\theta}\right)' = \left\{f \in M_0 : \sup \int_{\Omega} fg\, dx < \infty\right\},$$

where the sup is computed over all $g \in M_0$ such that $\|g\|_{p),\theta} \leq 1$, can be characterized as the Banach Function Space whose norm is given by

$$\|f\|_{p'),\theta} = \inf_{f=\sum_{k=1}^{\infty} f_k} \left\{\sum_{k=1}^{\infty} \inf_{0<\epsilon<p-1} \epsilon^{-\frac{\theta}{p-\epsilon}} \|f_k\|_{(p-\epsilon)'}\right\}$$

and such norm is in turn equivalent to the quasinorm

$$[f]_{(p',\theta} = \int_0^1 (1-\log t)^{\frac{\theta}{p-1}} \left(\int_0^t f^*(s)^{p'} ds\right)^{\frac{1}{p'}} \frac{dt}{t} .$$

For a systematic study of these spaces see [29].

we study the space defined through the norm$\|f\|_{p)),\alpha,\beta}$ in (31). We begin with the following

Proposition $(1.2.2)$[26]: If $1 < p < \infty, \alpha, \beta > 0$, then $\|.\|_{p)),\alpha,\beta}$ is a Banach Function Norm. Proof The proof of properties $(1.2.1) - (1.2.6)$ is analogous to the ones of the corresponding properties in Proposition (1.2.1).

$$\|\chi_E\|_{p)),\alpha,\beta} = \sup_{0<\epsilon<p-1} \epsilon^{\frac{\alpha}{p-\epsilon}} \|\chi_E\|_{p-\epsilon),\beta}$$

$$\leq \sup_{0<\epsilon<p-1} \epsilon^{\frac{\alpha}{p-\epsilon}} (p-\epsilon-1)^{\beta} |E|^{\frac{1}{p-\epsilon}} \leq (p-1)^{\alpha+\beta} |E|^{\frac{1}{p}}$$

Fix $0 < \epsilon < p-1$. By Hölder's inequality we have

$$\int_E |f| dx \leq \frac{|E|^{1-\frac{1}{p-\epsilon}}}{(p-\epsilon-1)^{\beta}} \|f\|_{p-\epsilon),\beta} \;\forall \epsilon \in]0, p-1[\,.$$

Further

$$\int_E |f| dx \leq \frac{|E|^{1-\frac{1}{p-\epsilon}}}{(p-\epsilon-1)^{\beta}} \epsilon^{-\frac{\alpha}{p-\epsilon}} \|f\|_{p)),\alpha,\beta} \;\forall \epsilon \in]0, p-1[$$

and therefore

$$\int_E |f| dx \leq |E|^{\frac{p-1}{p+1}} \left(\frac{2}{p-1}\right)^{\frac{\beta p+\beta+2\alpha}{p+1}} \|f\|_{p)),\alpha,\beta}\,,$$

where the constant in the right hand side comes setting, for instance, $\epsilon = \frac{p-1}{2}$.

Next results show the comparison between the iterated grand Lebesgue spaces and the grand Lebesgue spaces. The final conclusion that such spaces fall again within the scale of the grand Lebesgue spaces, will be stated after, as immediate corollary.

Theorem $(1.2.3)$[26]: If $1 < p < \infty, \alpha, \beta > 0$, then

$$\|f\|_{p)),\alpha,\beta} \leq max\left\{(p-1)^{\beta\left(1-\frac{1}{p}\right)} ,1\right\} \|f\|_{p),\alpha+\beta}.$$

Proof: Since $0 < \frac{1}{p-\epsilon-\eta} - \frac{1}{p-\eta} < 1 - \frac{1}{p} \;\forall \epsilon \in]0, p-1[, \eta \in]0, p-\epsilon-1[$,

$$\|f\|_{p)),\alpha,\beta} = \sup_{0<\epsilon<p-1} \sup_{0<\eta<p-\epsilon-1} \epsilon^{\frac{\alpha}{p-\epsilon}} \eta^{\frac{\beta}{p-\epsilon-\eta}} \|f\|_{p-\epsilon-\eta}$$

$$= \sup_{0<\epsilon<p-1} \sup_{0<\eta<p-\epsilon-1} \epsilon^{\frac{\alpha}{p-\epsilon}} \eta^{\beta\left(\frac{1}{p-\epsilon-\eta} - \frac{1}{p-\eta}\right)} \eta^{\frac{\beta}{p-\eta}} \|f\|_{p-\epsilon-\eta}$$

$$\leq max\left\{(p-1)^{\beta\left(1-\frac{1}{p}\right)} ,1\right\} \sup_{0<\zeta<p-1} \zeta^{\frac{\alpha+\beta}{p-\zeta}} \|f\|_{p-\zeta}$$

$$= max\left\{(p-1)^{\beta\left(1-\frac{1}{p}\right)} ,1\right\} \|f\|_{p),\alpha+\beta}$$

Theorem $(1.2.4)$[26]: If $1 < p < \infty, \alpha, \beta > 0$, then

$$\|f\|_{p),\alpha+\beta} \leq 2^{\alpha+\beta}\, max\left\{(p-1)^{\alpha\left(1-\frac{1}{p}\right)}\ \ ,1\right\}\|f\|_{p)),\alpha,\beta}.$$

Proof ;

$$\|f\|_{p),\alpha+\beta}$$

$$= \sup_{0<\sigma<p-1} \sigma^{\frac{\alpha+\beta}{p-\sigma}}\|f\|_{p-\sigma} = \sup_{0<\epsilon<\frac{p-1}{2}} (2\epsilon)^{\frac{\alpha+\beta}{p-2\epsilon}}\|f\|_{p-2\epsilon}$$

$$\leq 2^{\alpha+\beta} \sup_{0<\epsilon<\frac{p-1}{2}} \epsilon^{\frac{\alpha}{p-2\epsilon}}\, \epsilon^{\frac{\beta}{p-2\epsilon}}\, \|f\|_{p-2\epsilon} = 2^{\alpha+\beta} \sup_{0<\epsilon<\frac{p-1}{2}} \epsilon^{\frac{\alpha}{p-\epsilon}}\epsilon^{\frac{\alpha}{p-2\epsilon}-\frac{\alpha}{p-\epsilon}}\epsilon^{\frac{\beta}{p-2\epsilon}}\|f\|_{p-2\epsilon}$$

$$\leq 2^{\alpha+\beta}\ max\left\{(p-1)^{\alpha\left(1-\frac{1}{p}\right)}\ \ ,1\right\} \sup_{0<\epsilon<\frac{p-1}{2}} \epsilon^{\frac{\alpha}{p-\epsilon}}\, \epsilon^{\frac{\beta}{p-2\epsilon}}\, \|f\|_{p-2\epsilon}$$

$$\leq 2^{\alpha+\beta}\ max\left\{(p-1)^{\alpha\left(1-\frac{1}{p}\right)}\ \ ,1\right\} \sup_{0<\epsilon<p-1} \epsilon^{\frac{\alpha}{p-\epsilon}} \sup_{0<\eta<p-\epsilon-1} \eta^{\frac{\beta}{p-\epsilon-\eta}}\ \|f\|_{p-\epsilon-\eta}\ .$$

Hence

$$\|f\|_{p),\alpha+\beta} \leq 2^{\alpha+\beta}\ max\left\{(p-1)^{\alpha\left(1-\frac{1}{p}\right)}\ \ ,1\right\}\|f\|_{p)),\alpha,\beta}.$$

By Theorem (1.2.3) and Theorem (1.2.4) we get immediately the following

Corollary $(\mathbf{1.2.5})[\mathbf{26}]$: If $1 < p < \infty, \alpha, \beta > 0,$ then$\|\cdot\|_{p)),\alpha,\beta}$ is equivalent to$\|\cdot\|_{p),\alpha+\beta}$.

We study the "dual" functional of (31), namely, the functional obtained inserting the norm of the small spaces inside the norm of the small spaces. From the formal point of view, the resulting functional has a quite voluminous expression, and, after proving its equivalence to a norm of a small Lebesgue space, the fact that it is exactly a Banach Function Norm loses of interest. Therefore, here we limit ourselves to prove the analogous result of Corollary (1.2.5) for the functional [where, as usual, $1 < p < \infty, \alpha > 0, \beta > 0$ and $p' = p/(p-1)$]

$$\|f\|_{((p,\alpha,\beta} = \inf_{f=\sum_{j=1}^{\infty} f_j} \left\{\sum_{j=1}^{\infty} \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}}\left\|f_j\right\|_{((p'-\epsilon)',\beta}\right\}.$$

Theorem $(\mathbf{1.2.6})[\mathbf{26}]$: If $1 < p < \infty, \alpha, \beta > 0$, then $\|f\|_{(p,\alpha+\beta} \leq \|f\|_{((p,\alpha,\beta}$.

Proof : Step 1 First observe that, fixed $\epsilon \in]0, p'-1[$and $\sigma \in]0, p'-\epsilon-1[$, we have

$$(\epsilon+\sigma)^{-\frac{\alpha}{p'-\epsilon-\sigma}} \leq \epsilon^{-\frac{\alpha}{p'-\epsilon}}$$

and therefore

$$(\epsilon+\sigma)^{-\frac{\alpha+\beta}{p'-\epsilon-\sigma}} = (\epsilon+\sigma)^{-\frac{\alpha}{p'-\epsilon-\sigma}}(\epsilon+\sigma)^{-\frac{\beta}{p'-\epsilon-\sigma}} \leq \epsilon^{-\frac{\alpha}{p'-\epsilon}}\sigma^{-\frac{\beta}{p'-\epsilon-\sigma}}\ .$$

Hence, if $h \in M_0$, setting $\tau = \epsilon+\sigma$, we have

$$\tau^{-\frac{\alpha+\beta}{p\prime-\tau}}\|h\|_{(p'-\tau)'} \leq \epsilon^{-\frac{\alpha}{p\prime-\epsilon}}\sigma^{-\frac{\beta}{p\prime-\epsilon-\sigma}}\|h\|_{(p'-\epsilon-\sigma)'},$$

therefore

$$\inf_{0<\tau<p'-1} \tau^{-\frac{\alpha+\beta}{p\prime-\tau}} \|h\|_{(p'-\tau)'} \leq \epsilon^{-\frac{\alpha}{p\prime-\epsilon}}\sigma^{-\frac{\beta}{p\prime-\epsilon-\sigma}}\|h\|_{(p'-\epsilon-\sigma)'}$$

from which

$$\inf_{0<\tau<p'-1} \tau^{-\frac{\alpha+\beta}{p\prime-\tau}} \|h\|_{(p'-\tau)'} \leq \epsilon^{-\frac{\alpha}{p\prime-\epsilon}} \inf_{0<\sigma<p\prime-\epsilon-1} \sigma^{-\frac{\beta}{p\prime-\epsilon-\sigma}}\|h\|_{(p'-\epsilon-\sigma)'}. \quad (32)$$

Step 2

Let $g \in M_0, g = \sum_k h_k$ and $0 < \epsilon < p' - 1$,

By (32)

$$\sum_k \inf_{0<\tau<p\prime-1} \tau^{-\frac{\alpha+\beta}{p\prime-\tau}} \|h_k\|_{(p'-\tau)'} \leq \epsilon^{-\frac{\alpha}{p\prime-\epsilon}} \sum_k \inf_{0<\sigma<p'-\epsilon-1} \sigma^{-\frac{\beta}{p\prime-\epsilon-\sigma}} \|h_k\|_{(p'-\epsilon-\sigma)'}$$

then

$$\|g\|_{(p,\alpha+\beta} \leq \inf_{g=\sum_k h_k} \epsilon^{-\frac{\alpha}{p\prime-\epsilon}} \sum_k \inf_{0<\sigma<p'-\epsilon-1} \sigma^{-\frac{\beta}{p\prime-\epsilon-\sigma}} \|h_k\|_{(p'-\epsilon-\sigma)'}$$
$$= \epsilon^{-\frac{\alpha}{p\prime-\epsilon}}\|g\|_{((p'-\epsilon)',\beta}$$

and passing to the infimum over $\epsilon \in]0, p' - 1[$, we get

$$\|g\|_{(p,\alpha+\beta} \leq \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p\prime-\epsilon}}\|g\|_{((p'-\epsilon)',\beta} \;. \quad (33)$$

Step 3

Let $f \in M_0, f = \sum_j f_j$.

By (33)

$$\sum_j \|f_j\|_{(p,\alpha+\beta} \leq \sum_j \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p\prime-\epsilon}} \|f_j\|_{((p'-\epsilon)',\beta}$$

and, being

$$\|f\|_{(p,\alpha+\beta} \leq \sum_j \|f_j\|_{(p,\alpha+\beta} \;,$$

we have

$$\|f\|_{(p,\alpha+\beta} \leq \sum_j \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p\prime-\epsilon}} \|f_j\|_{((p'-\epsilon)',\beta}.$$

Passing to the infimum over all the decompositions $f = \sum_j f_j$ on the right hand side, we get the assertion

$$\|f\|_{(p,\alpha+\beta} \leq \|f\|_{((p,\alpha,\beta} \,.$$

Theorem (1.2.7)[26]: If $1 < p < \infty, \alpha, \beta > 0$, then

$$\|f\|_{((p,\alpha,\beta} \leq 2^{\alpha+\beta}\, max\left\{(p'-1)^{\alpha\left(2-\frac{1}{p'}\right)+\beta}, 1\right\} \|f\|_{(p,\alpha+\beta}.$$

Proof : Fix f and fix a decomposition $f = \sum f_j$. We have, for any j,

$$\|f_j\|_{((p'-\epsilon)',\beta} \leq \inf_{0<\eta<p'-\epsilon-1} \eta^{-\frac{\beta}{p'-\epsilon-\eta}} \|f_j\|_{(p'-\epsilon-\eta)'}.$$

Let $0 < \epsilon < p'-1$. Setting

$$\sigma_\epsilon = \epsilon\left(2-\frac{\epsilon}{p'-1}\right)$$

we note that σ_ϵ is an increasing function of ϵ in$]0, p'-1[$and

$$\epsilon \in]0, p'-1[\Leftrightarrow \sigma_\epsilon \in]0, p'-1[\,. \tag{34}$$

Let $\eta = \sigma_\epsilon - \epsilon$. We have

$$\|f_j\|_{((p'-\epsilon)',\beta} \leq (\sigma_\epsilon-\epsilon)^{-\frac{\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}$$

from which

$$\epsilon^{-\frac{\alpha}{p'-\epsilon}}\|f_j\|_{((p'-\epsilon)',\beta}$$

$$\leq \epsilon^{-\frac{\alpha}{p'-\epsilon}}\epsilon^{-\frac{\beta}{p'-\sigma_\epsilon}}\left(1-\frac{\epsilon}{p'-1}\right)^{-\frac{\beta}{p'-\sigma_\epsilon}} \sigma^{\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}$$

$$= \epsilon^{-\frac{\alpha}{p'-\sigma_\epsilon}}\epsilon^{-\frac{\alpha}{p'-\epsilon}}\left(1-\frac{\epsilon}{p'-1}\right)^{-\frac{\beta}{p'-\sigma_\epsilon}}\left(2-\frac{\epsilon}{p'-1}\right)^{\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}$$

and therefore

$$\inf_{0<\epsilon<p'-1}\epsilon^{-\frac{\alpha}{p'-\sigma_\epsilon}}\|f_j\|_{((p'-\epsilon)',\beta}$$

$$\leq \left(2-\frac{\epsilon}{p'-1}\right)^{\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\epsilon^{-\frac{\alpha}{p'-\sigma_\epsilon}}\epsilon^{-\frac{\alpha}{p'-\epsilon}}\left(1-\frac{\epsilon}{p'-1}\right)^{-\frac{\beta}{p'-\sigma_\epsilon}}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}$$

$$\leq 2^{\alpha+\beta}\, max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)}, 1\right\}\left(1-\frac{\epsilon}{p'-1}\right)^{-\frac{\beta}{p'-\sigma_\epsilon}}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}$$

and, fixing $\theta \in]0, p'-1[$, we obtain

$$\inf_{0<\epsilon<p'-1}\epsilon^{-\frac{\alpha}{p'-\sigma_\epsilon}}\|f_j\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta}\, max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)}, 1\right\}\cdot \tag{35}$$

$$\inf_{0<\epsilon<p'-1-\theta}\left(1-\frac{\epsilon}{p'-1}\right)^{-\frac{\beta}{p'-\sigma_\epsilon}}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}$$

$$\leq 2^{\alpha+\beta}\, max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)}, 1\right\}\left(\frac{\theta}{p'-1}\right)^{-\beta}\inf_{0<\epsilon<p'-1-\theta}\sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}}\|f_j\|_{(p'-\sigma_\epsilon)'}.$$

On the other hand, fix $\gamma \in [p'-1-\theta, p'-1[$,

$$\inf_{0<\epsilon<p'-1-\theta} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'} \leq \sigma_{p'-1-\theta}^{-\frac{\alpha+\beta}{p'-\sigma_{p'-1-\theta}}} \|f_j\|_{\left(p'-\sigma_{p'-1-\theta}\right)'}$$

$$\leq \sigma_{p'-1-\theta}^{-\frac{\alpha+\beta}{p'-\sigma_{p'-1-\theta}}} \| f_j\|_{(p'-\sigma_\gamma)'} = \sigma_{p'-1-\theta}^{-\frac{\alpha+\beta}{p'-\sigma_{p'-1-\theta}}} \sigma_\gamma^{\frac{\alpha+\beta}{p'-\sigma_\gamma}} \sigma_\gamma^{-\frac{\alpha+\beta}{p'-\sigma_\gamma}} \| f_j\|_{(p'-\sigma_\gamma)'}.$$

Being

$$\sigma_\gamma^{\frac{\alpha+\beta}{p'-\sigma_\gamma}} \leq (p'-1)^{\alpha+\beta}$$

and

$$\sigma_{p'-1-\theta}^{-\frac{\alpha+\beta}{p'-\sigma_{p'-1-\theta}}} = \left(\frac{(p'-1)^2-\theta^2}{p'-1}\right)^{-\frac{(\alpha+\beta)(p'-1)}{p'-1+\theta^2}} = \left(\frac{p'-1}{(p'-1)^2-\theta^2}\right)^{\frac{(\alpha+\beta)(p'-1)}{p'-1+\theta^2}},$$

we get

$$\inf_{0<\epsilon<p'-1-\theta} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}$$

$$\leq (p'-1)^{\alpha+\beta} \left(\frac{p'-1}{(p'-1)^2-\theta^2}\right)^{\frac{(\alpha+\beta)(p'-1)}{p'-1+\theta^2}} \inf_{p'-1-\theta\leq\gamma<p'-1} \sigma_\gamma^{-\frac{\alpha+\beta}{p'-\sigma_\gamma}} \| f_j\|_{(p'-\sigma_\gamma)'} . \quad (36)$$

Then, by (35) and (36),

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \|f_j\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)}, 1\right\} \cdot$$

$$\frac{(p'-1)^{\alpha+2\beta}}{\theta^\beta} max\left\{\left(\frac{p'-1}{(p'-1)^2-\theta^2}\right)^{\alpha+\beta}, 1\right\} \inf_{p'-1-\theta\leq\gamma<p'-1} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'} . \, (37)$$

Now we need to estimate the left hand side of (35) and (37) by

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \|f_j\|_{(p'-\epsilon)'} .$$

To this goal, we note that by (34)

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \|f_j\|_{(p'-\epsilon)'} = \inf_{0<\epsilon<p'-1} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}$$

$$= min\left\{\inf_{0<\epsilon<p'-1-\theta} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}, \inf_{p'-1-\theta\leq\gamma<p'-1} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}\right\}$$

and we distinguish three cases. If

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \|f_j\|_{(p'-\epsilon)'} = \inf_{0<\epsilon<p'-1-\theta} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \|f_j\|_{(p'-\sigma_\epsilon)'}$$

by (35) we have

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \|f_j\|_{((p'-\epsilon)',\beta}$$

$$\leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\} \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{(p'-\epsilon)'}. \tag{38}$$

If

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{(p'-\epsilon)'} = \inf_{0<\epsilon<p'-1-\theta} \sigma_\epsilon^{-\frac{\alpha+\beta}{p'-\sigma_\epsilon}} \left\|f_j\right\|_{(p'-\sigma_\epsilon)'}$$

we observe that

$$\frac{1}{p'-1} < \frac{p'-1}{(p'-1)^2-\theta^2} < \infty\ 0 < \theta < p'-1$$

and we consider the cases $1 < p' \leq 2$ and $p' > 2$.
Let $1 < p' \leq 2$. By (37), we have

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\}\cdot$$

$$\cdot \frac{(p'-1)^{\alpha+2\beta}}{\theta^\beta} \left(\frac{p'-1}{(p'-1)^2-\theta^2}\right)^{\alpha+\beta} \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha+\beta}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)'}$$

and therefore

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)',\beta} \tag{39}$$

$$\leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\} \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha+\beta}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)'}.$$

Let $p' > 2$. We have

$$\min_{0<\theta<p'-1} max\left\{\left(\frac{p'-1}{(p'-1)^2-\theta^2}\right)^{\alpha+\beta},1\right\} = 1$$

and therefore, by (37) also, we have

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\}\cdot \tag{40}$$

$$\cdot (p'-1)^{\alpha+\beta} \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha+\beta}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)'}.$$

By (38), (39), (40), we have

$$\inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\}\cdot$$

$$\cdot max\{(p'-1)^{\alpha+\beta},1\}\cdot \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha+\beta}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)'}$$

and, summing over j,

$$\sum_j \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha}{p'-\epsilon}} \left\|f_j\right\|_{((p'-\epsilon)',\beta} \leq 2^{\alpha+\beta} max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\}\cdot$$

$$\cdot max\{(p'-1)^{\alpha+\beta},1\}\cdot\sum_j \inf_{0<\epsilon<p'-1} \epsilon^{-\frac{\alpha+\beta}{p'-\epsilon}} \|f_j\|_{((p'-\epsilon)'}.$$

Passing to the infimum over all the decompositions $f=\sum f_j$, we get the assertion, taking into account that

$$max\left\{(p'-1)^{\alpha\left(1-\frac{1}{p'}\right)},1\right\}\cdot max\{(p'-1)^{\alpha+\beta},1\} = max\left\{(p'-1)^{\alpha\left(2-\frac{1}{p'}\right)+\beta},1\right\}.$$

By Theorem (1.2.6) and Theorem (1.2.7) we get immediately the following

Corollary $(\mathbf{1.2.8})[26]$:If $1<p<\infty,\alpha,\beta>0$, then$\|\cdot\|_{((p,\alpha,\beta}$ is equivalent to $\|\cdot\|_{(p,\alpha+\beta}$.

Section (1.3): Grand Lebesgue Spaces with Measurable Functions

For $\Omega\subset\mathbb{R}^n, n\geq 2$,be a measurable set of Lebesgue measure $|\Omega|<+\infty$. In 1992 Iwaniec and Sbordone [48] studied the integrability properties of the Jacobi an determinant, and introduced the grand Lebesgue space $L^{n)}(\Omega)$ as a space such that

$$|Df|\in L^{n)}(\Omega)\Rightarrow |Jf|\in L^1_{loc}(\Omega)$$

for all Sobolev mappings $f:\Omega\to\mathbb{R}^n, f=(f_1,\ldots,f_n)$.

Since then the grand Lebesgue spaces play an important role in $PDEs$ theory (see e.g. [48]) and in Function Spaces Theory (see e.g. [49]). It turns out that such spaces are Banach Function Spaces in the sense of [50]: namely (here and in the following we will use the letter p instead of n,assuming $1<p<\infty$)

$$L^{p)}(\Omega)=\left\{f\in M_o:\|f\|_{p)}=\rho(|f|)=\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |f|_{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}<+\infty\right\},$$

where M_o is the set of all real valued measurable functions on Ω, and, denoting by M_o^+ the subset of M_o of then on negative function $s,\rho: M_o^+\to[0,+\infty]$ is such that for all $f,g,f_n(n=1,2,3,\ldots)$ in M_o^+, for all constants $\lambda\geq 0$, and for all measurable subsets $E\subset\Omega$,the following properties hold:

(i) $\rho(f)=0\Leftrightarrow f=0 a.e.$in Ω

(ii) $\rho(\lambda f)=\lambda\rho(f)$

(iii) $\rho(f+g)\leq\rho(f)+\rho(g)$

(iv) $0\leq g\leq f\ a.e.\ in\ \Omega\Rightarrow\rho(g)\leq\rho(f)$

(v) $0\leq f_n\uparrow f\ a.e.\ in\ \Omega\Rightarrow\rho(f_n)\uparrow\rho(f)$

(vi) $E\subset\Omega\Rightarrow\rho(\chi_E)<+\infty$

(vii) $E\subset\Omega\Rightarrow\int_E f dx\leq C_E\rho(f)$

for some constant $C_E, 0<C_E<\infty$, depending on E and ρ, but independent of f.

Grand Lebesgue spaces belong to a special category of Banach Function Spaces: they are rearrangement-invariant, namely,setting

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}| \ \forall \lambda \geq 0 \tag{41}$$

it is $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$.
A generalization of the grand Lebesgue spaces are the spaces $L^{p),\theta}, \theta \geq 0$,defined by (see e.g. [49])

$$\|f\|_{L^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \frac{1}{|\Omega|} \int_\Omega |f|_{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} .$$

When $\theta = 0$ the spaces $L^{p),0}(\Omega)$ reduce to Lebesgue spaces $L^p(\Omega)$ and when $\theta = 1$the spaces $L^{p),1}(\Omega)$ reduce to grand Lebesgue spaces $L^{p)}(\Omega)$.
A useful property of the norm, used in [51] is the fact that the supremum over $(0, p-1)$ in the norm of $L^{p)}(\Omega)$ can be computed also in any smaller interval $(0, \varepsilon_0)$: the result is an equivalent expression of the norm (i.e. each expression can be majorized by the other, multiplied by a constant not depending on f). Obviously, the constants involved in the equivalence will depend on p and ε_0. This phenomenon has been clarified also in a more general context in [52]. Were call also the continuous embeddings, easy consequence from the definition,

$$L^p(\Omega) \subset L^{p),\theta}(\Omega) \subset L^{p-\epsilon}(\Omega), 0 < \epsilon \leq p-1 \theta > 0.$$

Let $\delta : (0, p-1) \to [0, +\infty[$ be a measurable function, and for all $f \in M_o^+$ set

$$\rho_{p),\delta}(f) = \operatorname*{ess\,sup}_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_\Omega f^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}, \tag{42}$$

where $\int_\Omega$ stands for $\frac{1}{|\Omega|}\int_\Omega$ .For $1 \leq r < \infty$, we will also write$\|f\|_r$ to denote the normalized norm of f in $L^r(\Omega)$:

$$\|f\|_r = \left(\int_\Omega f^r dx \right)^{\frac{1}{r}} .$$

By convention, we establish that the right handside of (42) is ∞ if for some $0 < \varepsilon < p-1$the function $f \notin L^{p-\varepsilon}(\Omega)$: this position gives always a meaning to the eess sup, also when the indeterminate form $0 \cdot \infty$ appears. The case $\delta(\varepsilon) = \varepsilon^\theta, \theta > 0$, gives back the norm of the $L^{p),\theta}(\Omega)$ spaces.
We find a necessary and sufficient condition on δ such that $\rho_{p),\delta}$ is equivalent to a Banach function norm, i.e. equivalent to a functional satisfying all the properties (1)–(7) listed in the previous.
It is clear that the first way to prove that $\rho_{p),\delta}$ is equivalent to a Banach function norm is to try to reproduce the analogous proof, valid for grand L^p spaces. This latter proof is an easy consequence of the classical properties of the norm of Lebesgue spaces, and it seems, for this reason, almost absent in literature. The problem when considering the functional $\rho_{p),\delta}$

is that δ is defined almost everywhere, and the expression $\delta(\varepsilon)$ does not have the meaning of value attained in ε. Moreover, the estimate of $\rho_{p),\delta}$looks much less evident when, for instance, δ attains the value zero in finite times in a neighborhood of zero. Besides solving completely the problems above, we will show in particular that for any measurable bounded $\delta, \rho_{p),\delta}$is equivalent to a Banach function norm, and that the same resulting space can be obtained by using a new function$\bar{\delta}$, defined everywhere, whose expression is explicitly shown. After this step, also in the case of bounded measurable functions, the proof of being equivalent to a Banach function norm can be considered equally trivial as in the case of grand Lebesgue spaces.

Going back, an immediate necessary condition is suggested by property (6),when $E = \Omega$: $since_{\rho_{p),\delta}}(\chi_\Omega)$ must be finite, it must be $\delta \in L^\infty(0,p-1)$. The Theorem we will prove is that this condition is actually also sufficient.

Theorem $(\mathbf{1.3.1})[\mathbf{47}]$: Let $1 < p < \infty$ and let $\delta : (0,p-1) \to [0,+\infty[$ be a measurable function, not identically zero. The mapping $\rho_{p),\delta}$ is equivalent to a Banach function norm if and only if $\delta \in L^\infty(0,p-1)$.

The proof of Theorem (1.3.1) requires some intermediate results of independent interest. As a byproduct, we will determine the ''interesting'' class $B_p$of functions $\delta \in L^\infty(0,p-1)$,with the property that every $\rho_{p),\delta}$, obtained from a generic bounded measurable δ, is equivalent to a function norm built with $\delta \in B_p$.

Lemma $(\mathbf{1.3.2})[\mathbf{47}]$: If $\delta_1, \delta_2 : (0,p-1) \to [0,+\infty[$ are measurable functions such that

$$\operatorname*{esssup}_{0<\varepsilon<\sigma} \delta_1(\varepsilon)^{\frac{1}{p-\varepsilon}} = \operatorname*{esssup}_{0<\varepsilon<\sigma} \delta_2(\varepsilon)^{\frac{1}{p-\varepsilon}}, \qquad \sigma \in]0,p-1], \tag{43}$$

then $\rho_{p),\delta_1} = \rho_{p),\delta_2}$.

Proof: Since we may exchange the roles of $\rho_{p),\delta_1}$ and $\rho_{p),\delta_2}$, it is sufficient to prove that for all $f \in M_o^+$

$$\rho_{p),\delta_1}(f) \le \rho_{p),\delta_2}(f). \tag{44}$$

If f is identically zero, (44) is trivially true, therefore we may work with functions f having positive Lebesgue norm. If for some ε it is $f \notin L^{p-\varepsilon}(\Omega)$, then both sides of (44) are ∞, therefore we may consider the functions f such that

$$0 < \|f\|_{p-\varepsilon} < \infty \; for\ all\ 0 < \varepsilon < p-1. \tag{45}$$

If $\rho_{p),\delta_1}(f)$ or $\rho_{p),\delta_2}(f)$ is zero, then the other one is also zero: in fact, if for instance $\rho_{p),\delta_1}(f) = 0$, from (45) we get that $\delta_1 = 0$ a.e.in $(0,p-1)$. By (43) used with $\sigma = p-1$,we deduce that $\delta_2 = 0$ also, and our claim is proved.

Consider the case $0 < \rho_{p),\delta_1}(f) < \infty$and fix $\eta > 0$. By the definition of $\rho_{p),\delta_1}(f)$ there exists a set of positive measure$T_\eta \subset (0,p-1)$ such that

$$\delta_1(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} > \rho_{p),\delta_1}(f) - \eta, \qquad \varepsilon \in T_\eta$$

from which

$$\delta_1(\varepsilon)^{\frac{1}{p-\varepsilon}} > \frac{\rho_{p),\delta_1}(f)-\eta}{\|f\|_{p-\varepsilon}}\ , \qquad \varepsilon \in T_\eta\,. \tag{46}$$

Now set

$$e'_\eta = \operatorname*{ess\,inf}_{T_\eta} x \in [0, p-1[$$

$$e''_\eta = \operatorname*{ess\,sup}_{T_\eta} x \in]0, p-1]$$

and fix $\sigma, e'_\eta < \sigma < e''_\eta$. From (46) and the monotonicity of the (normalized) norm $\|f\|_r$ with respect or (due to the Hölder's inequality),

$$\operatorname*{ess\,sup}_{0<\zeta<\sigma} \delta_1(\zeta)^{\frac{1}{p-\zeta}} > \frac{\rho_{p),\delta_1}(f)-\eta}{\|f\|_{p-\varepsilon}}, \qquad \varepsilon \in]0,\sigma[$$

and by (43)

$$\operatorname*{ess\,sup}_{0<\zeta<\sigma} \delta_2(\zeta)^{\frac{1}{p-\zeta}} > \frac{\rho_{p),\delta_1}(f)-\eta}{\|f\|_{p-\varepsilon}}, \qquad \varepsilon \in]0,\sigma[.$$

We deduce the existence of a set $T'_\eta \subset (0,\sigma)$ of positive measure such that

$$\delta_2(\zeta)^{\frac{1}{p-\zeta}} > \frac{\rho_{p),\delta_1}(f)-\eta}{\|f\|_{p-\varepsilon}}, \qquad \zeta \in T'_\eta, \qquad \varepsilon \in]0,\sigma[$$

from which

$$\delta_2(\varepsilon)^{\frac{1}{p-\varepsilon}} > \frac{\rho_{p),\delta_1}(f)-\eta}{\|f\|_{p-\varepsilon}}, \qquad \varepsilon \in T'_\eta$$

$$\operatorname*{ess\,sup}_{0<\varepsilon<\sigma} \delta_2(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} > \rho_{p),\delta_1}(f)-\eta.$$

Since η can be arbitrarily small, we get the assertion.

Finally, if $\rho_{p),\delta_1}(f) = \infty$,we may follow the same argument as before, replacing $\rho_{p),\delta_1}(f)-\eta$ by any $M > 0$.The lemma is therefore proved.

The next proposition shows that monotone functions play an important role in our study. The term increasing for a function δ means that if $\varepsilon_1 < \varepsilon_2$,then $\delta(\varepsilon_1) \leq \delta(\varepsilon_2)$.

Proposition $(\mathbf{1.3.3})[\mathbf{47}]$: If $\delta \in L^\infty(0,p-1), 0 \leq \delta \leq 1$,then there exists $\bar{\delta} \in L^\infty(0,p-1)$ such that:

(i) $0 \leq \bar{\delta} \leq 1$

(ii) $\bar{\delta}(\varepsilon)^{\frac{1}{p-\varepsilon}}$ increasing in ε and left continuous

(iii) $\rho_{p),\delta} = \rho_{p),\bar{\delta}}$.

Proof: For any given $\delta \in L^\infty(0,p-1), 0 \leq \delta \leq 1$,the function ¯

$$\bar{\delta}(\varepsilon) = \left[\operatorname*{ess\,sup}_{0<\zeta<\varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}}\right]^{p-\varepsilon}, \varepsilon \in (0,p-1)$$

has trivially the property (i). Property (ii) follows by a well known characterization of increasing, left continuous functions, see e.g. [48,Theorem 8.19]. We have to prove only that $\rho_{p),\delta} = \rho_{p),\bar{\delta}}$.

On one hand, the definition of $\bar{\delta}$gives immediately that

$$\bar{\delta}(\varepsilon)^{\frac{1}{p-\varepsilon}} = \operatorname*{ess\,sup}_{0<\zeta<\varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}}, \varepsilon \in (0, p-1) \quad (47)$$

and, on the other hand, by (ii),

$$\bar{\delta}(\varepsilon)^{\frac{1}{p-\varepsilon}} = \operatorname*{ess\,sup}_{0<\zeta<\varepsilon} \bar{\delta}(\zeta)^{\frac{1}{p-\zeta}}, \varepsilon \in (0, p-1). \quad (48)$$

Combining (47) and (48),we get

$$\operatorname*{ess\,sup}_{0<\zeta<\varepsilon} \delta(\zeta)^{\frac{1}{p-\zeta}} = \operatorname*{ess\,sup}_{0<\zeta<\varepsilon} \bar{\delta}(\zeta)^{\frac{1}{p-\zeta}}, \varepsilon \in (0, p-1).$$

By Lemma (1.3.2),we get (iii).

The following definition plays a crucial role in the study of the generalization of the grand Lebesgue spaces with respect to measurable functions.

Definition $(1.3.4)$[47]: Let $1 < p < \infty$.A function δ, left continuous on $(0, p-1)$, is said to be in the class B_p if

(j) $\delta(0+) = 0$

(jj) $0 < \delta \leq 1$

(jjj) $\delta(\varepsilon)^{\frac{1}{p-\varepsilon}}$ isincreasingin ε.

It is easy to check that functions in B_p are increasing. Moreover, the left continuity of its functions permit us to write more simply sup instead of $esssup$ in the expressions related to $\rho_{p),\delta}$.

We have now the prerequisites for the

Let $\delta \in L^\infty(0, p-1)$ be nonnegative. We may think to divide δ by its (positive) $L^\infty(\Omega)$ norm, therefore without loss of generality we may assume that $0 \leq \delta \leq 1$. Moreover, by Proposition (1.3.3), without loss of generality we may assume that (ii) holds true, therefore, in particular, δ increasing and left continuous. Therefore it makes sense to compute $\delta(0+)$.

If $\delta(0+) > 0$,for any measurable function f on Ω, possibly not in $L^p(\Omega)$,it is

$$\delta(0+)\|f\|_p = \delta(0+) \sup_{0<\varepsilon<p-1} \|f\|_{p-\varepsilon} \leq \rho_{p),\delta}(f) \leq \|f\|_p.$$

If $\delta(0+) = 0$,since δ is increasing, there are two possibilities :there exists, or not, $0 < \varepsilon < p-1$such that $\delta(\varepsilon) = 0$. In the first case let $\varepsilon_0 = max\{\varepsilon : 0 < \varepsilon < p-1, \delta(\varepsilon) = 0\}$.It is $0 < \varepsilon_0 < p-1$and

$$\rho_{p),\delta}(f) = \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} = \sup_{\varepsilon_0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon}.$$

After a change of variable in the sup,

$$\rho_{p),\delta}(f) = \sup_{0<\varepsilon<p-\varepsilon_0-1} \delta(\varepsilon+\varepsilon_0)^{\frac{1}{p-\varepsilon_0-\varepsilon}} \|f\|_{p-\varepsilon_0-\varepsilon}.$$

Setting $\tilde{\delta}(\varepsilon) = \delta(\varepsilon+\varepsilon_0)$ and $r = p-\varepsilon_0$,we get

$$\rho_{p),\delta}(f) = \rho_{r),\tilde{\delta}}(f).$$

Since, by the maximality of ε_0, it is $\tilde{\delta}(\varepsilon) > 0$ when $\varepsilon > 0$, the first case we are studying will be concluded after the examinationofthesecondcase. The assumptions we have now on δ imply that $\delta \in B_p$. At this point, all the axioms of the Banach function norms are straight forward to prove.

As a byproduct of the proof of Theorem (1.3.1),we have the following

Theorem $(\mathbf{1.3.5})[\mathbf{47}]$: Let $1 < p < \infty$and let $\delta \in L^{\infty}(0, p-1), \delta \geq 0, \delta \not\equiv 0$.The mapping $\rho_{p),\delta}$ is equivalent to $\rho_{p),\bar{\delta}}$, where $\bar{\delta}$ is the increasing, left continuous function defined by

$$\bar{\delta}(\varepsilon) = \left[\operatorname*{ess\,sup}_{0<\zeta<\varepsilon}\left(\frac{\delta(\zeta)}{\|\delta\|_{\infty}}\right)^{\frac{1}{p-\zeta}}\right]^{p-\varepsilon} , \varepsilon \in (0, p-1).$$

Again by the proof of Theorem (1.3.1), it is clear that the interesting functions δ to consider in the Banach function norm are those ones in the class B_p. This motivates the following

Definition $(\mathbf{1.3.6})[\mathbf{47}]$: Let $\Omega \subset \mathbb{R}^n, n \geq 1$,be a measurable set of Lebesgue measure $|\Omega| < +\infty$, let $1 < p < \infty$and let $\delta \in B_p$. The grand L^p space over Ω with respect to δ is the Banach Function Space defined by

$$L^{p),\delta}(\Omega) = \left\{f \in M_o : \|f\|_{p),\delta} = \rho_{p),\delta}(|f|) = \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} < +\infty\right\}.$$

It is immediate from the definition that the spaces $L^{p),\delta}(\Omega)$ are rearrangement-invariant, include $L^p(\Omega)$ and are included in each $L^{p-\varepsilon}(\Omega)$,$0 < \varepsilon < p-1$.

We conclude by showing a sufficient condition, and a necessary condition , for the embedding between grand L^p spaces built from two functions $\delta_1, \delta_2 \in B_p$. We will need the following simple lemma, which extends the useful property of the grand Lebesgue spaces mentioned.

Lemma $(\mathbf{1.3.7})[\mathbf{47}]$**:** Let $1 < p < \infty$and $0 < \sigma < p-1$.If $\delta \in B_p$, there exists a constant $c = c(p, \delta, \sigma)$ such that

$$\sup_{0<\varepsilon<\sigma} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_{\Omega} f^{p-\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}} \leq \rho_{p),\delta}(f) \leq c \sup_{0<\varepsilon<\sigma} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \left(\int_{\Omega} f^{p-\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}}.$$

Proof: The left wing in equality is trivial, therefore we need to prove only the right wing one. Fix $\varepsilon \geq \sigma$ and $0 < \mu < \sigma$.By Hölder's inequality,$\|\cdot\|_r$ is increasing in r, therefore we have

$$\sup_{\sigma\leq\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} \leq \|f\|_{p-\mu} = \delta(\mu)^{-\frac{1}{p-\mu}} \delta(\mu)^{\frac{1}{p-\mu}} \|f\|_{p-\mu}$$

from which

$$\sup_{\sigma\leq\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} \leq \delta(\mu)^{-\frac{1}{p-\mu}} \sup_{\sigma\leq\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon}, \mu \in]0, \sigma[.$$

Passing to the infimum over μ on the right hand side, and recalling that $0 < \delta \leq 1$, we get the desired inequality with $c = \delta(\sigma)^{-\frac{1}{p-\sigma}}$.

Proposition $(\mathbf{1.3.8})[\mathbf{47}]$: Let $\Omega \subset \mathbb{R}^n, n \geq 1$,be a measurable set of Lebesgue measure $|\Omega| < +\infty$, let $1 < p < \infty$, and let $\delta_1, \delta_2 \in B_p$. Then

$$\lim_{\varepsilon\to 0} \sup_{0<\sigma<\varepsilon} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} < \infty \Rightarrow L^{p),\delta_2}(\Omega) \subset L^{p),\delta_1}(\Omega) \Rightarrow \lim_{\varepsilon\to 0} \inf_{0<\sigma<\varepsilon} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} < \infty.$$

Proof: If

$$\lim_{\varepsilon\to 0} \sup_{0<\sigma<\varepsilon} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = M < \infty,$$

then for small $\varepsilon_0 > 0$ it is

$$\delta_1(\varepsilon) < (M+1)\delta_2(\varepsilon), \varepsilon \in (0, \varepsilon_0)$$

and this immediately implies that $L^{p),\delta_2}(\Omega) \subset L^{p),\delta_1}(\Omega)$. On the other hand, if this inclusion holds, assume, on the contrary, that

$$\lim_{\varepsilon\to 0} \inf_{0<\sigma<\varepsilon} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = \infty.$$

For any fixed $M > 1$, there exists a small $\varepsilon_0 > 0$such that

$$\delta_1(\varepsilon) > M\delta_2(\varepsilon), \varepsilon \in (0, \varepsilon_0).$$

Raising both sides to the power $\frac{1}{p-\varepsilon}$,multiplying by $\|f\|_{p-\varepsilon}$ and taking the supremum over $(0, \varepsilon_0)$, by Lemma (1.3.7) we get

$$\|f\|_{p),\delta_1} \geq cM^{\frac{1}{p}} \|f\|_{p),\delta_2}$$

Which is in contradiction with the assumed embedding.

Corollary $(\mathbf{1.3.9})[\mathbf{47}]$: Let $\Omega \subset \mathbb{R}^n, n \geq 1$,be a measurable set of Lebesgue measure $|\Omega| < +\infty$, let $1 < p < \infty$,and let $\delta_1, \delta_2 \in B_p$ be equivalent n a neighborhood of the origin. Then $L^{p),\delta_1}(\Omega) = L^{p),\delta_2}(\Omega)$.

The classical Hardy inequality states that

Theorem $(\mathbf{1.3.10})[\mathbf{47}]$: Let $p > 1$and f be a measurable, non negative function in (0,1).Then

$$\left(\int_0^1 \left(\int_0^x f dt\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^1 f^p dx\right)^{1/p}. \qquad (49)$$

we extend the Hardy inequality in the context of $L^{p),\delta}(0,1)$ spaces. We will follow closely the proofs given in [51].

Theorem $(\mathbf{1.3.11})[\mathbf{47}]$: Let $1 < p < \infty$ and $\delta \in B_p$. There exists a constant $c(p,\delta) > 1$such that

$$\left\|\int_0^x f dt\right\|_{p),\delta} \leq c(p,\delta)\|f\|_{p),\delta} \qquad (50)$$

for all non negative measurable functions f in (0,1).

Proof: Let $0 < \sigma < p-1$.We have

$$\left\| \int_0^x f dt \right\|_{p),\delta}$$

$$= max\left\{ \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} , \sup_{\sigma\le\epsilon<p-1} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \right\}$$

$$= max\left\{ \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} , \sup_{\sigma\le\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \left(\int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \right\}$$

$$\le max\left\{ \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} , \right.$$

$$\left. \times \sup_{\sigma\le\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \delta(\sigma)^{-\frac{1}{p-\sigma}} \delta(\sigma)^{\frac{1}{p-\sigma}} \left(\int_0^1 \left(\int_0^x f dt \right)^{p-\sigma} dx \right)^{\frac{1}{p-\sigma}} \right\},$$

$$\le max\left\{ \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} , \right.$$

$$\left. \times \sup_{\sigma\le\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \delta(\sigma)^{-\frac{1}{p-\sigma}} \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \right\}.$$

Therefore,

$$\left\| \int_0^x f dt \right\|_{p),\delta} \le \max_{\sigma\le\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \delta(\sigma)^{-\frac{1}{p-\sigma}} \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}}$$

Since $\max_{\sigma\le\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \delta(\sigma)^{-\frac{1}{p-\sigma}} \ge 1$.

Now take $0 < \epsilon \le \sigma$, so that $p-\epsilon > 1$. Applying the Hardy inequality (49) with the exponent p replaced by $p-\epsilon$, and multiplying both sides by $\delta(\epsilon)^{\frac{1}{p-\epsilon}}$, we get

$$\left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \le \frac{p-\epsilon}{p-\epsilon-1} \left(\delta(\epsilon) \int_0^1 f^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} .$$

If we pass to the sup over $0 < \epsilon < \sigma$ on both sides, the previous in equality implies

$$\sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 \left(\int_0^x f dt \right)^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \le \frac{p-\sigma}{p-\sigma-1} \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon) \int_0^1 f^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}}$$

and therefore

$$\left\|\int_0^x f\,dt\right\|_{p),\delta} \leq \max_{\sigma\leq\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \;\; \delta(\sigma)^{-\frac{1}{p-\sigma}} \frac{p-\sigma}{p-\sigma\;-1} \sup_{0<\epsilon<\sigma} \left(\delta(\epsilon)\int_0^1 f^{p-\epsilon}\,dx\right)^{\frac{1}{p-\epsilon}}$$

$$\leq \max_{\sigma\leq\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \;\; \delta(\sigma)^{-\frac{1}{p-\sigma}} \frac{p-\sigma}{p-\sigma\;-1} \sup_{0<\epsilon<p-1} \left(\delta(\epsilon)\int_0^1 f^{p-\epsilon}\,dx\right)^{\frac{1}{p-\epsilon}}.$$

Setting

$$c(p,\delta) := \inf_{0<\sigma<p-1} \max_{\sigma\leq\epsilon<p-1} \delta(\epsilon)^{\frac{1}{p-\epsilon}} \;\; \delta(\sigma)^{-\frac{1}{p-\sigma}} \frac{p-\sigma}{p-\sigma\;-1} \geq 1,$$

we get the inequality (50).

Chapter2
Trigonometric and Polynomial Approximation of Functions

We show that the deviations $\|f - N_n(f)\|_{p(x)}$ and $\|f - R_n(f)\|_{p(x)}$ are estimated by $n^{-\alpha}$ for $f \in Lip(\alpha, p(x))(0 < \alpha \leq 1)$.We consider the fractional order moduli of smoothness and obtain direct, converse approximation theorems together with a constructive characterization of a Lipschitz-type class. Several applications of these results help us transfer the approximation results for weighted variable Smirnov spaces of functions defined on sufficiently smooth finite domains of complex plane $\mathbb{C}$.

Section (2.1): Generalized Lebesgue Spaces $L^{p}(x)$

For $p : R \to [1, \infty)$ be a measurable 2π –periodic function, that is $p(x + 2\pi) = p(x)$. Denote by $L^{p(x)} = L^{p(x)}([0,2\pi])$ the set of all measurable 2π –periodic functions f such that $m_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$,where

$$m_p(f) := \int_0^{2\pi} |f(x)|^{p(x)}\, dx.$$

$L^{p(x)}$ becomesa Banach space with respect to the norm

$$\|f\|_{p(x)} = \inf\left\{\lambda > 0 : m_p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

If $p(x) \equiv p$ is constant $(1 \leq p < \infty)$, then the space $L^{p(x)}$ is isometrically isomorphic to the Lebesgue space L^p.

If the function p satisfies

$$1 < p_- := \operatorname*{ess\,inf}_{x\in[0,2\pi]} p(x) \;, \qquad p_+ := \operatorname*{ess\,sup}_{x\in[0,2\pi]} p(x) < \infty \qquad (1)$$

then the function

$$p'(x) := \frac{p(x)}{p(x) - 1}$$

is well defined and satisfies (1) itself.

The space $L^{p(x)}$ consists of all measurable 2π –periodicfunctions f such that

$$\int_0^{2\pi} |f(x)g(x)|\, dx < \infty$$

for all measurable g with $m_{p'}(g) \leq 1$ and

$$\|f\|^*_{p(x)} = \sup\left\{\int_0^{2\pi} |f(x)g(x)|\, dx : m_{p'}(g) \leq 1\right\}$$

is also a norm on $L^{p(x)}$. It is known that the inequalities

$$\|f\|_{p(x)} \leq \|f\|^*_{p(x)} \leq r_p \|f\|_{p(x)}$$

satisfied for all functions $f \in L^{p(x)}$, where

$$r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+},$$

and hence the norms $\|f\|_{p(x)}$and $\|f\|^{*}_{p(x)}$ are equivalent. See [67], [68], [69] and [70] for properties above and for more general informationabout $L^{p(x)}$spaces.
Denote by M the Hardy-Littlewoodmaximal operator,defined for $f \in L^1$ by

$$M(f)(x) = \sup_I |I| \int_I |f(t)|dt \quad , x \in [0,2\pi],$$

where the supremumis taken over all intervals I with $x \in I$. The boundedness problem of the operator M on the space $L^{p(x)}$ was studied by many ([71]). In [72] it was provedthat if the function p satisfies (1) and the condition

$$|p(x) - p(y)| \leq \frac{C}{-ln|x-y|}, 0 < |x-y| \leq \frac{1}{2}, \qquad (2)$$

then the maximal operator M is boundedon $L^{p(x)}$, that is,

$$\|M(f)\|_{p(x)} \leq c\|f\|_{p(x)} \qquad (3)$$

for all $f \in L^{p(x)}$, where c is a constant dependsonly on p. The set of all measurable 2π – periodic functions $p: R \to [1,\infty)$ satisfies the con-ditions (1) and (2) will be denoted by M. Let $p \in M$ and $f \in L^{p(x)}$. The modulus of continuity of the function f is defined by

$$\Omega_{p(x)}(f,\delta) = \sup_{|h|\leq \delta} \|T_h(f)\|_{p(x)} \ , \delta > 0, \qquad (4)$$

where

$$T_h(f)(x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)|dt \ . \qquad (5)$$

The existence of $\Omega_{p(x)}(f,\delta)$ follows from (3), and also the inequality

$$\Omega_{p(x)}(f,\delta) \leq c\|f\|_{p(x)}$$

satisfied for all $\delta > 0$.
The modulus $\Omega_{p(x)}(f,\cdot)$ is nonnegative, continuous function such that

$$\lim_{\delta \to 0} \Omega_{p(x)}(f,\delta) = 0 \ , \ \Omega_{p(x)}(f_1 + f_2,\cdot) \leq \Omega_{p(x)}(f_1,\cdot) + \Omega_{p(x)}(f_2,\cdot).$$

In the Lebesgue spaces L^p $(1 < p < \infty)$, the classical modulus of continuity $\omega_p(f,\cdot)$ is defined by

$$\omega_p(f,\delta) = \sup_{|h|\leq \delta} \|T'_h(f)\|_p , \qquad \delta > 0, \qquad (6)$$

where

$$T'_h(f)(x) := f(x+h) - f(x).$$

It is known that in the Lebesgue spaces Lp the moduli of continuity (4) and (6) are equivalent(see [74]).
We define in the spaces $L^{p(x)}$ the modulus of continuity by using the shift (5), because the space $L^{p(x)}$ is not translation invariant, in general (see, for example [73, Example2.9]).
Let $p \in M$ and $0 < \alpha \leq 1$. We define the Lipschitz class $Lip(\alpha, p(x))$ as

$$Lip(\alpha, p(x)) = \{f \in L^{p(x)} : \Omega_{p(x)}(f,\delta) = O(\delta^{\alpha}), \delta > 0\}.$$

Let $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \ . \qquad (7)$$

Denote by $S_n(f)(x), n = 0,1, \ldots$ the nth partial sums of the series (7) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} A_k(f)(x) \ ,$$

where

$$A_0(f)(x) = \frac{a_0}{2}, A_k(f)(x) = a_k \cos kx + b_k \sin kx\,, \qquad k = 1,2, \ldots .$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. We consider two means of the series (7) defined by

$$N_n(f)(x) = \frac{1}{p_n} \sum_{m=0}^{n} p_{n-m} S_m(f)(x)$$

and

$$R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_m S_m(f)(x) \ ,$$

where $P_n := \sum_{m=0}^{n} p_m \ , p_{-1} = P_{-1} := 0$. The means $N_n(f)$ and $R_n(f)$ are called the N¨orlund and Riesz means of the series (7),respectively. In thecase $p_n = 1, n \geq 0$, both of $N_n(f)$ and $R_n(f)$ are equal to the Cesàro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f)(x) \ .$$

If we take $p_n = A_n^{\beta-1} (\beta > 0)$, where

$$A_0^\beta := 1, \qquad A_k^\beta := \frac{\beta(\beta+1)\ldots\beta+k}{k!}, \qquad k \geq 1,$$

the mean $N_n(f)$ be the generalized Cesàro mean $\sigma_n^\beta(f)(x)$, that is

$$N_n(f)(x) = \frac{1}{A_n^\beta} \sum_{m=0}^{n} A_{n-m}^{\beta-1} S_m(f)(x) \ .$$

The approximate on properties of the Cesàro means σ_n in Lipschitz classes $Lip(\alpha, p), 1 \leq p < \infty, 0 < \alpha \leq 1$ were investigated by Quade in [79]. The generalizations of Quade's results were studied by Mohapatra and Russell [76], Chandra ([78]) and Leindler [79]. In [80], Chandra obtained estimates for $\|f - N_n(f)\|_p$, where $1 < p < \infty$. Chandra also gave estimates for the difference $\|f - R_n(f)\|_p$, where $f \in Lip(\alpha, p), 1 < p < \infty, 0 < \alpha \leq 1$ [82]. [84], Chandra gave some conditions on the sequence $\{p_n\}_0^\infty$ and obtained very satisfactory results about approximation by the means $N_n(f)$ and $R_n(f)$ in $Lip(\alpha, p), 1 \leq p < \infty, 0 < \alpha \leq 1$. Later, Leindler in [85] weakened the conditions given by Chandra on the sequence $\{p_n\}_0^\infty$ and generalized his

results. In [81], the analogues of Chandra's results was obtained for weighted Lebesgue spaces.
we give $L^{p(x)}$ analogues of the results obtained by Leindler in [85] and Chandra in [84].
We shall use the notations

$$\Delta g_n := g_n - g_{n+1}, \qquad \Delta_m g(n,m) := g(n,m) - g(n,m+1).$$

A sequenceof positive real numbers $\{p_n\}_0^\infty$ is called almost monotone decreasing (increasing) if there exists a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the inequality

$$p_n \leq cp_m \ (cp_n \geq p_m)$$

holds. Such sequences will be denoted by $\{p_n\}_0^\infty \in AMDS$ ($\{p_n\}_0^\infty \in AMIS$).Our main results are the following.

Theorem $(\mathbf{2.1.1})[\mathbf{66}]$: Let $p \in M$, $0 < \alpha < 1, f \in Lip(\alpha, p(x))$ and $\{p_n\}_0^\infty$ be a sequence of positive numbers. If

$$\{p_n\}_0^\infty \in AMDS,$$

or

$$\{p_n\}_0^\infty \in AMIS \ and \ (n+1)p_n = O(P_n),$$

then

$$\|f - N_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

Theorem $(\mathbf{2.1.2})[\mathbf{66}]$: Let $p \in M$, $f \in Lip(1, p(x))$ and $\{p_n\}_0^\infty$ be a sequence of positive numbers. If

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n),$$

or

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right),$$

then the estimate

$$\|f - N_n(f)\|_{p(x)} = O(n^{-1})$$

holds for n =1,2,....

Theorem $(\mathbf{2.1.3})[\mathbf{66}]$: Let $p \in M$, $0 < \alpha \leq 1, f \in Lip(\alpha, p(x))$ and$\{p_n\}_0^\infty$be a sequence of positive numbers. If

$$\sum_{m=0}^{n-1} \left|\Delta\left(\frac{P_m}{m+1}\right)\right| = O\left(\frac{P_n}{n+1}\right), \qquad (8)$$

then for $n = 1,2,\ldots$ the estimate

$$\|f - R_n(f)\|_{p(x)} = O(n^{-\alpha})$$

holds. In the classical Lebesgue spaces L^p, the analogues of Theorem (2.1.1) and Theorem (2.1.2) were proved by Leindler in [75], and Theorem (2.1.3) in L^pspaces was obtained by Chandra [74].

we will denote the constants (in general, different in different relations) depend only on quantities that are not important for the questions of interest.

Let $p \in M$. Denote by $E_n(f)_{p(x)}(n = 0,1,\dots)$ the best approximation of $f \in L^{p(x)}$ in Π_n (the set of trigonometric polynomials of degree at most n), that is

$$E_n(f)_{p(x)} = \inf\{\|f - t_n\|_{p(x)} : t_n \in \Pi_n\}.$$

It follows that, for example from Theorem 1.1 in [76], there exists a trigonometric polynomial $t_n^* \in \Pi_n$ such that

$$E_n(f)_{p(x)} = \|f - t_n^*\|_{p(x)}$$

for $n = 0,1,\dots$.

By $W^{p(x)} = W^{p(x)}([0,2\pi])$ we denote the set of absolutely continuous functions f such that $f' \in L^{p(x)}$.

Lemma $(2.1.4)$[66]: Let $p \in M$ and $f \in W^{p(x)}$. Then the estimate

$$E_n(f)_{p(x)} = O\left(\frac{1}{n}\|f'\|_{p(x)}\right) \qquad (9)$$

holds for $n = 1,2,\dots$.

Proof: It follows from Theorem 6.1 of [78] that

$$\|f - S_n(f)\|_{p(x)} \to 0, n \to \infty.$$

It is easy to see that

$$A_k(\widetilde{f'})(x) = kA_k(f)(x), \qquad k = 1,2,\dots,$$

where $\widetilde{f'}$ is the conjugate function of f'. By considering the uniform boundedness of $\{S_n\}_0^\infty$ and the boundedness of the conjugation operator in the space $L^{p(x)}$ (see [71]), we get

$$\|f - S_n(f)\|_{p(x)} = \left\|\sum_{k=n+1}^{\infty} A_k(f)\right\|_{p(x)} = \left\|\sum_{k=n+1}^{\infty} \frac{1}{k} A_k(\widetilde{f'})\right\|_{p(x)}$$

$$= \left\|\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\left(S_k(\widetilde{f'}) - \widetilde{f'}\right) + \frac{1}{n+1}\left(S_n(\widetilde{f'}) - \widetilde{f'}\right)\right\|_{p(x)}$$

$$\le \sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\left\|S_k(\widetilde{f'}) - \widetilde{f'}\right\|_{p(x)} + \frac{1}{n+1}\left\|S_n(\widetilde{f'}) - \widetilde{f'}\right\|_{p(x)}$$

$$\le c\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right\}\|f'\|_{p(x)} + \frac{1}{n+1}\|f'\|_{p(x)}$$

$$\le \frac{c}{n}\|f'\|_{p(x)},$$

and hence (9) follows.

Lemma $(2.1.5)$[66]: If $p \in M$, the Jackson type inequality

$$E_n(f)_{p(x)} = O\left(\Omega_{p(x)}\left(f,\frac{1}{n}\right)\right), \qquad n = 1,2,\ldots$$

holds for $f \in L^{p(x)}$.

Proof: Let $f \in L^{p(x)}$. Consider the transform

$$U_\delta(f)(x) := \frac{2}{\delta}\int_{\delta/2}^{\delta}\left(\frac{1}{h}\int_0^h f(x+t)dt\right) dh, \qquad \delta > 0.$$

It is clear that $U_\delta(f) \in W^{p(x)}$ for each $\delta > 0$ and

$$\left(U_\delta(f)\right)'(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta}\int_h^1 (f(x+h) - f(x))\, dh$$

for almost all x. Since

$$\left|\left(U_\delta(f)\right)'(x)\right| \le \frac{4}{\delta}\left(\frac{1}{\delta}\int_0^\delta |f(x+h) - f(x)|\, dh\right),$$

it follows from definition of $\Omega_{p(x)}(f,\delta)$ that

$$\left\|\left(U_\delta(f)\right)'\right\|_{p(x)} \le \frac{4}{\delta}\left\|\frac{1}{\delta}\int_0^\delta |f(\cdot+h) - f|dh\right\|_{p(x)}$$

$$\le \frac{4}{\delta}\,\Omega_{p(x)}(f,\delta).$$

On the other hand,since

$$U_\delta(f)(x) - f(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta}\left(\frac{1}{h}\int_0^h (f(x+t) - f(x))dt\right)dh,$$

we get

$$\|U_\delta(f) - f'\|_{p(x)} \le \frac{2}{\delta}\int_{\delta/2}^{\delta}\left\|\frac{1}{h}\int_0^h |f(\cdot+t) - f|dt\right\|_{p(x)} dh$$

$$\le \sup_{\delta/2\le h\le\delta}\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\|\frac{1}{h}\int_0^h |f(\cdot+t) - f|dt\right\|_{p(x)} dh$$

$$\le \frac{2}{\delta}\int_{\delta/2}^{\delta}\left(\sup_{\delta/2\le h\le\delta}\left\|\frac{1}{h}\int_0^h |f(\cdot+t) - f|dt\right\|_{p(x)} dh\right)$$

$$= \sup_{\delta/2\le h\le\delta}\left\|\frac{1}{h}\int_0^h |f(\cdot+t) - f|dt\right\|_{p(x)} \le \Omega_{p(x)}(f,\delta).$$

Hence, by subadditivityof the best approximation and (9) we obtain

$$E_n(f)_{p(x)} \le E_n\left(f - U_{1/n}(f)\right)_{p(x)} + E_n\left(U_{1/n}(f)\right)_{p(x)}$$

$$\le \left\|f - U_{1/n}(f)\right\|_{p(x)} + \frac{c}{n}\left\|\left(U_{1/n}(f)\right)'\right\|_{p(x)}$$

$$\le \Omega_{p(x)}\left(f, \frac{1}{n}\right) + \frac{c}{n} 4n\Omega_{p(x)}\left(f, \frac{1}{n}\right),$$

which finishes the proof.

Lemma (2.1.6)[66]: Let $p \in M$ and $0 < \alpha \le 1$. Then for every $f \in Lip(\alpha, p(x))$ the estimate

$$\|f - S_n(f)\|_{p(x)} = O(n^{-\alpha}), \qquad n = 1,2,\ldots$$

holds.

Proof: Let t_n^* $(n = 0,1,\ldots)$ be the trigonometric polynomial of best approximation to $f \in Lip(\alpha, p(x))$. By Lemma(2.1.5)

$$\|f - t_n^*\|_{p(x)} = O\left(\Omega_{p(x)}(f, 1/n)\right),$$

and hence

$$\|f - t_n^*\|_{p(x)} = O(n^{-\alpha}).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the space $L^{p(x)}$ (see [71]), we get

$$\|f - S_n(f)\|_{p(x)} \le \|f - t_n^*\|_{p(x)} + \|t_n^* - S_n(f)\|_{p(x)}$$
$$= \|f - t_n^*\|_{p(x)} + \|S_n(t_n^* - f)\|_{p(x)}$$
$$= O\left(\|f - t_n^*\|_{p(x)}\right)$$
$$= O(n^{-\alpha}).$$

Lemma (2.1.7)[66]: Let $p \in M$. If $f \in Lip(1, p(x))$, then f is absolutely continuous and $f' \in L^{p(x)}$, that is $f \in W^{p(x)}$.

Proof: Let $f \in Lip(1, p(x))$ and $\delta > 0$. Since $p_- \le p(x)$ almost everywhere, by Theorem 2.8 of [73] the space $L^{p(x)}$is continuously embedded in L^{p_-}. Hence we have

$$\|T_h(f)\|_{p_-} \le c\|T_h(f)\|_{p(x)}$$

for every h with $|h| \le \delta$. This inequality and equivalence of $\omega_{p_-}(f,\cdot)$ and $\Omega_{p_-}(f,\cdot)$yield

$$\omega_{p_-}(f, \delta) \le c\, \Omega_{p(x)}(f, \delta).$$

Hence, $f \in Lip(1, p(x))$ implies $\omega_{p_-}(f, \delta) = O(\delta)$, and this implies that f is absolutely continuous and $f' \in L^{p_-}$([76, pp. 51–54]). Since the relation

$$\frac{f(x+t) - f(x)}{t} \to f'(x), t \to 0$$

holds almost everywhere, for almost all x we get

$$\frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{|f(x+t) - f(x)|}{t} dt \to |f'(x)|, \delta \to 0+.$$

By Fatou Lemma, for every measurablefunction g with $m_{p'}(g) \le 1$,

$$\int_0^{2\pi} |f'(x)||g(x)|\ dx = \int_0^{2\pi}\left(\lim_{\delta\to 0+}\frac{2}{\delta}\int_{\delta/2}^{\delta}\frac{|f\ (x+t)-\ f(x)|}{t}dt\right)|g(x)|dx$$
$$\le \liminf_{\delta\to 0+}\int_0^{2\pi}\left(\frac{2}{\delta}\int_{\delta/2}^{\delta}\frac{|f\ (x+t)-\ f(x)|}{t}dt\right)|g(x)|dx$$
$$\le \liminf_{\delta\to 0+}\frac{4}{\delta}\int_0^{2\pi}\left(\frac{1}{\delta}\int_{\delta/2}^{\delta}|f\ (x+t)-\ f(x)|\,dt\right)|g(x)|dx$$
$$= \liminf_{\delta\to 0+}\frac{4}{\delta}\int_0^{2\pi} T_\delta(f)(x)|g(x)|\ dx$$
$$\le \liminf_{\delta\to 0+}\frac{4}{\delta}\ \|T_\delta(f)\|_{p(x)} \le \liminf_{\delta\to 0+}\frac{4}{\delta}\ \Omega_{p(x)}(f,\delta\ )$$
$$= \liminf_{\delta\to 0+}\frac{4}{\delta}\ O(\ \delta\ ) = O(1),$$
and this means that $f' \in L^{p(x)}$.

Lemma $(2.1.8)[66]$: Let $p \in M$ and $f\ \in Lip(1, p(x))$. Then for $n = 1,2, \ldots$ the estimate
$$\|S_n(f) -\ \sigma_n(f)\|_{p(x)}\ = O(n^{-1})$$
holds.

Proof: By Lemma (2.1.7), $f\ \in W^{p(x)}$. If f has the Fourier series
$$f(x) \sim \sum_{k=0}^{\infty} A_k(f)(x)\ ,$$
then the Fourier series of the conjugate function $\widetilde{f}'$ be
$$\widetilde{f}'(x) \sim \sum_{k=1}^{\infty} kA_k(f)(x)\ \ .$$
On the other hand,
$$S_n(f)(x) -\ \sigma_n(f)(x) = \sum_{k=1}^{n}\frac{k}{n+1}A_k(f)(x)$$
$$= \frac{1}{n+1}\ S_n\big(\widetilde{f}'\big)(x).$$
Hence, by considering the uniform boundedness of the partial sums and the conjugation operator in the space $L^{p(x)}$ (see [81]), we obtain
$$\|S_n(f) -\ \sigma_n(f)\|_{p(x)}\ = O(n^{-1})$$
for $n = 1,2, \ldots$.

The following Lemma was proved in [85].

Lemma $(2.1.9)[66]$: Let $\{p_n\}_0^\infty$ be a sequence of positive numbers. If $\{p_n\}_0^\infty \in AMDS$, or $\{p_n\}_0^\infty \in AMIS$ and $(n+1)p_n\ = O(P_n)$, then

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O(n^{-\alpha} P_n)$$

for $0 < \alpha < 1$.

Since

$$f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \ f(x),$$

we have

$$f(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \ \{f(x) - S_m(f)(x)\}.$$

By Lemma (2.1.6) and Lemma (2.1.9) we obtain

$$\|f - N_n(f)\|_{p(x)} \leq \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \ \|f - S_m(f)\|_{p(x)}$$

$$= \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \ O(m^{-\alpha}) + \frac{p_n}{P_n} \ \|f - S_0(f)\|_{p(x)}$$

$$= \frac{1}{P_n} O(n^{-\alpha} P_n) + O\left(\frac{1}{n+1}\right)$$

$$= O(n^{-\alpha}).$$

It is clear that

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} P_{n-m} A_m(f)(x).$$

By Abel transform,

$$S_n(f)(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=1}^{n} (P_n - P_{n-m}) A_m(f)(x)$$

$$= \frac{1}{P_n} \sum_{m=1}^{n} \Delta_m \left(\frac{P_n - P_{n-m}}{m}\right) \left(\sum_{k=1}^{m} k A_k(f)(x)\right) + \frac{1}{n+1} \sum_{k=1}^{n} k A_k(f)(x),$$

and hence

$$\|S_n(f) - N_n(f)\|_{p(x)} \leq \frac{1}{P_n} \sum_{m=1}^{n} \left|\Delta_m \left(\frac{P_n - P_{n-m}}{m}\right)\right| \left\|\sum_{k=1}^{m} k A_k(f)\right\|_{p(x)}$$

$$+ \frac{1}{n+1} \left\|\sum_{k=1}^{n} k A_k(f)\right\|_{p(x)}.$$

Since

$$S_n(f)(x) - \sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=1}^{n} kA_k(f)\,(x),$$

by Lemma (2.1.8) we get

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n} kA_k(f)\right\|_{p(x)}$$

$$= \|S_n(f) - \sigma_n(f)\|_{p(x)} = O(n^{-1}).$$

Hence,

$$\|S_n(f) - N_n(f)\|_{p(x)} = O\left(\frac{1}{P_n}\right) \sum_{m=1}^{n} \left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right| + O(n^{-1}). \tag{10}$$

Suppose the condition

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$$

holds. This implies that (see [87])

$$\sum_{m=1}^{n} \left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right| = O\left(\frac{P_n}{n}\right),$$

and hence by (10) we have

$$\|S_n(f) - N_n(f)\|_{p(x)} = O(n^{-1}).$$

This relation and Lemma (2.1.6) yield

$$\|f - N_n(f)\|_{p(x)} = O(n^{-1}).$$

Now let

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right). \tag{11}$$

A simple calculation yields

$$\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right) = \frac{1}{m(m+1)}\left(\sum_{k=n-m}^{n} p_k - (m+1)p_{n-m}\right),$$

and by induction one can easily get

$$\left|\sum_{k=n-m}^{n} p_k - (m+1)p_{n-m}\right| \le \sum_{k=1}^{m} k|p_{n-k+1} - p_{n-k}| \ .$$

Thus,

$$\sum_{m=1}^{n} \left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right| \le \sum_{m=1}^{n} \frac{1}{m(m+1)} \left(\sum_{k=1}^{m} k|p_{n-k+1} - p_{n-k}|\right)$$

$$= \sum_{k=1}^{n} k|p_{n-k+1} - p_{n-k}| \left(\sum_{m=k}^{n} \frac{1}{m(m+1)} \right)$$
$$\leq \sum_{k=1}^{n} k|p_{n-k+1} - p_{n-k}| \left(\sum_{m=k}^{\infty} \frac{1}{m(m+1)} \right)$$
$$= \sum_{k=1}^{n} k|p_{n-k+1} - p_{n-k}| = \sum_{k=0}^{n-1} |\Delta p_k| \ .$$

Combiningthis, the assumption (11) and (10) we get

$$\|S_n(f) - N_n(f)\|_{p(x)} = O(n^{-1}),$$

and considering Lemma (2.1.6) again we obtain the desired result.

Let $0 < \alpha < 1$. By definition of $R_n(f)(x)$,

$$f(x) - R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_m \{f(x) - S_m(f)(x)\} \quad .$$

From Lemma (2.1.6), we get

$$\|f - R_n(f)\|_{p(x)} \leq \frac{1}{P_n} \sum_{m=0}^{n} p_m \|f - S_m(f)\|_{p(x)} \qquad (12)$$
$$= O\left(\frac{1}{P_n}\right) \sum_{m=1}^{n} p_m m^{-\alpha} + \frac{p_0}{P_n} \|f - S_0(f)\|_{p(x)}$$
$$= O\left(\frac{1}{P_n}\right) \sum_{m=1}^{n} p_m m^{-\alpha} \ .$$

By Abel transform,

$$\sum_{m=1}^{n} p_m m^{-\alpha} = \sum_{m=1}^{n-1} P_m \{m^{-\alpha} - (m+1)^{-\alpha}\} + n^{-\alpha} P_n$$
$$\leq \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} + n^{-\alpha} P_n,$$

and

$$\sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} = \sum_{m=1}^{n-1} \Delta \left(\frac{P_m}{m+1}\right) \left(\sum_{k=1}^{m} k^{-\alpha} \right) + \frac{P_n}{n+1} \sum_{m=1}^{n-1} m^{-\alpha}$$
$$= O(n^{-\alpha} P_n)$$

by condition (8). This yields

$$\sum_{m=1}^{n} p_m m^{-\alpha} = O(n^{-\alpha} P_n)$$

and from this and (12) we get

$$\|f - R_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

Let's consider the case $\alpha = 1$.

By Abel transform,

$$R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n-1}\{P_m\big(S_m(f)(x) - S_{m+1}(f)(x)\big) + P_nS_n(f)(x)\}$$

$$= \frac{1}{P_n}\sum_{m=0}^{n-1} P_m\big(-A_{m+1}(f)(x)\big) + S_n(f)(x),$$

and hence

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n}\sum_{m=0}^{n-1} P_mA_{m+1}(f)(x) \ .$$

Using Abel transform again yields

$$\sum_{m=0}^{n-1} P_mA_{m+1}(f)(x) = \sum_{m=0}^{n-1}\frac{P_m}{m+1}\,(m+1)A_{m+1}(f)\,(x)$$

$$= \sum_{m=0}^{n-1}\Delta\left(\frac{P_m}{m+1}\right)\left(\sum_{k=0}^{m}(k+1)A_{k+1}(f)(x)\right)$$

$$+\frac{P_n}{n+1}\sum_{k=0}^{n-1}(k+1)A_{k+1}(f)(x)\,.$$

Thus, by considering Lemma (2.1.8) and (8) we obtain

$$\left\|\sum_{m=0}^{n-1} P_mA_{m+1}(f)\right\|_{p(x)} \le \sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right| \left\|\sum_{k=0}^{m}(k+1)A_{k+1}(f)\right\|_{p(x)}$$

$$+\frac{P_n}{n+1}\left\|\sum_{k=0}^{n-1}(k+1)A_{k+1}(f)\right\|_{p(x)}$$

$$= \sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right|(m+2)\|S_{m+1}(f) - \sigma_{m+1}(f)\|_{p(x)}$$

$$+P_n\|S_n(f) - \sigma_n(f)\|_{p(x)}$$

$$= O(1)\sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right| + O\left(\frac{P_n}{n}\right).$$

This gives

$$\|R_n(f) - S_n(f)\|_{p(x)} = \frac{1}{P_n}\left\|\sum_{m=0}^{n-1} P_m A_{m+1}(f)\right\|_{p(x)}$$
$$= \frac{1}{P_n}\, O\left(\frac{P_n}{n}\right) = O\left(\frac{1}{n}\right).$$

Combiningthis estimate with Lemma (2.1.6) yields

$$\|f - R_n(f)\|_{p(x)} = O(n^{-1}).$$

Section (2.2): Generalized Lebesgue Spaces with Variable Exponent

Generalized Lebesgue spaces $L^{p(x)}$ with variable exponent and corresponding Sobolev-type spaces have waste applications in elasticity theory, fluid mechanics, differential operators [90], nonlinear Dirichlet boundary-value problems [92], nonstandard growth and variational calculus [91].

These spaces appeared first in [92] as an example of modular spaces [93] and Sharapudinov [96] has been obtained topological properties of $L^{p(x)}$. Furthermore if $p^* := \text{ess}\sup_{x\in T} p(x) < \infty$, then $L^{p(x)}$ is a particular case of Musielak–Orlicz spaces [96]. Later various mathematicians investigated the main properties of these spaces [97]. In $L^{p(x)}$there is a rich theory of boundedness of integral transforms of various type [98].

For $p(x) := p, 1 < p < \infty, L^{p(x)}$ is coincide with Lebesgue space L^p and basic problems of trigonometric approximation in L^p are investigated by several mathematicians (among others [99], ...). Approximation by algebraic polynomials and rational functions in Lebesgue spaces, Orlicz spaces, symmetric spaces and their weighted versions on sufficiently smooth complex domains and curves was investigated in [100–103, 115, 118, 116]. For a complete treatise of polynomial approximation see [105, 108].

In harmonic and Fourier analysis some of operators (for example partial sum operator of Fourier series, conjugate operator, differentiation operator, shift operator $f \to\to f(\cdot + h), h \in \mathbb{R}$) have been extensively used to prove direct and converse type approximation inequalities. Unfortunately the space$L^{p(x)}$ is not $p(\cdot)-$continuous and not translation invariant [94]. Under various assumptions (including translation invariance) on modular space Musielak [97] obtained some approximation theorems in modular spaces with respect to the usual moduli of smoothness. Since $L^{p(x)}$ is not translation invariant using Butzer–Wehrens type moduli of smoothness (see [99]) Israfilov et all. [107] obtained direct and converse trigonometric approximation theorems in $L^{p(x)}$. we investigate the approximation properties of the trigonometric system in $L^{p(\cdot)}_{2\pi}$. We consider the fractional order moduli of smoothness and obtain direct, converse approximation theorems together with a constructive characterization of a Lipschitz-type class.

Let $T := [-\pi, \pi]$ and P be the class of 2π-periodic, Lebesgue measurable functions $p = p(x): T \to (1, \infty)$ such that $p^* < \infty$. We define class $L_{2\pi}^{p(\cdot)} := L_{2\pi}^{p(\cdot)}(T)$ of 2π-periodic measurable functions f defined on T satisfying

$$\int_T |f(x)|^{p(x)}\, dx < \infty.$$

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space [94] with norms

$$\|f(x)\|_{p,\pi} := \|f(x)\|_{p,\pi,T} := \inf\left\{\alpha > 0 : \int_T \left|\frac{f(x)}{\alpha}\right|^{p(x)} |dx| \leq 1\right\}$$

and

$$\|f(x)\|_{p,\pi}^* := \sup\left\{\int_T |f(x)g(x)|\, dx : g \in L_{2\pi}^{p'(\cdot)}, \int_T |g(x)|^{p'(x)}\, dx \leq 1\right\} b$$

having the property [1]

$$\|f\|_{p,\pi} \asymp \|f\|_{p,\pi}^*, \tag{13}$$

where $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$.

The variable exponent $p(x)$ which is defined on T is said to be satisfy Dini– Lipschitz property DL_γ of order γ on T if

$$\sup_{x_1, x_2 \in T} \{|p(x_1) - p(x_2)| : |x_1 - x_2| \leq \delta\} \left(ln\frac{1}{\delta}\right) \gamma \leq c, \qquad 0 < \delta < 1.$$

Let $f \in L_{2\pi}^{p(\cdot)}$, $p \in P$ satisfy DL_1, $0 < h \leq 1$ and let

$$\sigma_h f(x) := \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)dt, \qquad x \in T,$$

be Steklov's mean operator. In this case the operator σ_h is bounded [97] in $L_{2\pi}^{p(\cdot)}$. Using these facts and setting $x, t \in T, 0 \leq \alpha < 1$ we define

$$\sigma_h^\alpha f(x) := (I - \sigma_h)^\alpha f(x) =$$

$$= \sum_{k=0}^{\infty} X(-1)^k \binom{\alpha}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \dots \int_{-h/2}^{h/2} f(x + u_1 + \dots + u_k)\, du_1 \dots du_k, \tag{14}$$

where $f \in L_{2\pi}^{p(\cdot)}$, $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$ for $k > 1$, $\binom{\alpha}{1} := \alpha$, $\binom{\alpha}{0} := 1$ and I is the identity operator.

Since the Binomial coefficients $\binom{\alpha}{k}$ satisfy $[104, p. 14]$

$$\binom{\alpha}{k} \leq \frac{c(\alpha)}{k^{\alpha+1}}, k \in \mathbb{Z}^+,$$

we get

$$C(\alpha) := \sum_{k=0}^{\infty} \binom{\alpha}{k} < \infty$$

and therefore

$$\|\sigma_h^\alpha f\|_{p,\pi} \le c\|f\|_{p,\pi} < \infty \qquad (15)$$

provided $f \in L_{2\pi}^{p(\cdot)}, p \in P$ satisfy DL_1 and $0 < h \le 1$.

For $0 \le \alpha < 1$ and $r = 1,2,3,\ldots$ we define the fractional modulus of smoothness of index $r + \alpha$ for $f \in L_{2\pi}^{p(\cdot)}, p \in P$, satisfy DL_1 and $0 < h \le 1$ as

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} := \sup_{0\le h_i, h\le\delta} \left\| \prod_{i=1}^{r} \left(I - \sigma_{h_i}\right) \sigma_h^\alpha f \right\|_{p,\pi}$$

and

$$\Omega_\alpha(f,\delta)_{p(\cdot)} := \sup_{0\le h\le\delta} \|\sigma_h^\alpha f\|_{p,\pi} \ .$$

We have by (15) that

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} \le c\|f\|_{p,\pi}$$

where $f \in L_{2\pi}^{p(\cdot)}, p \in P$ satisfy $DL_1, 0 < h \le 1$ and the constant $c > 0$ dependent onlyon α, r and p.

Remark $(\mathbf{2.2.1})[\mathbf{89}]$: The modulus of smoothness $\Omega_\alpha(f,\delta)_{p(\cdot)}, \alpha \in \mathbb{R}^+$, has the following properties for $p \in P$ satisfying DL_1:

(i) $\Omega_\alpha(f,\delta)_{p(\cdot)}$ is non-negative and nondecreasing function of $\delta \ge 0$, (ii) $\Omega_\alpha(f_1+f_2,\cdot)_{p(\cdot)} \le \Omega_\alpha(f_1,\cdot)_{p(\cdot)} + \Omega_\alpha(f_2,\cdot)_{p(\cdot)}$, (iii) $\lim_{\delta\to 0} \Omega_\alpha(f,\delta)_{p(\cdot)} = 0$.

Let

$$E_n(f)_{p(\cdot)} := \inf_{T\in T_n} \|f - T\|_{p,\pi} \ , n = 0,1,2,\ldots,$$

be the approximation error of function $f \in L_{2\pi}^{p(\cdot)}$ where T_n is the class of trigonometric polynomials of degree not greater than n. For a given $f \in L^1$, assuming

$$\int_T f(x)dx = 0, \qquad (16)$$

we define α-th fractional ($\alpha \in \mathbb{R}^+$) integral of f as $[93, v.2, p.134]$

$$I_\alpha(x,f) := \sum_{k\in\mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx} \ ,$$

where $c_k := \int_T f(x)e^{-ikx}dx$ for $k \in \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$ and

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha \, sign k}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define fractional derivative of a function $f \in L^1$, satisfying (16), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+[\alpha]-\alpha}(x,f)$$

provided the right-hand side exists, where $[x]$ denotes the integer part of a real number x. Let $W_{p(\cdot)}^\alpha, p \in P, \alpha > 0$, be the class of functions $f \in L_{2\pi}^{p(\cdot)}$ such that $f^{(\alpha)} \in L_{2\pi}^{p(\cdot)}$. $W_{p(\cdot)}^\alpha$ becomes a Banach space with the norm

$$\|f\|_{W^{\alpha}_{p(\cdot)}} := \|f\|_{p,\pi} + \left\|f^{(\alpha)}\right\|_{p,\pi}.$$

Main results are following.

Theorem (2.2.2)[89]: Let $f \in W^{\alpha}_{p(\cdot)}, \alpha \in \mathbb{R}^+$, and $p \in P$ satisfy DL_γ with $\gamma \geq 1$, then for every natural n there exists a constant $c > 0$ independent of n such that

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}$$

holds.

Corollary (2.2.3)[89]: Under the conditions of Theorem (2.2.2)

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} \left\|f^{(\alpha)}\right\|_{p,\pi}$$

with a constant $c > 0$ independent of $n = 0,1,2,3,\ldots$.

Theorem (2.2.4)[89]: If $\alpha \in \mathbb{R}^+, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$, then there exists a constant $c > 0$ dependent only on α and p such that for $n = 0,1,2,3,\ldots$

$$E_n(f)_{p(\cdot)} \leq c\Omega_\alpha\left(f, \frac{2\pi}{n+1}\right)_{p(\cdot)}$$

holds.

The following converse theorem of trigonometric approximation holds.

Theorem (2.2.5)[89]: If $\alpha \in \mathbb{R}^+, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$, then for $n = 0,1,2,3,\ldots$

$$\Omega_\alpha\left(f, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot)}$$

hold where the constant $c > 0$ dependent only on α and p.

Corollary (2.2.6)[89]: Let $\alpha \in \mathbb{R}^+, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$. If

$$E_n(f)_{p(\cdot)} = O(n^{-\sigma}), \quad \sigma > 0, \quad n = 1,2,\ldots,$$

then

$$\Omega_\alpha(f,\delta)_{p(\cdot)} = \begin{cases} O(\delta^\sigma), & \alpha > \sigma, \\ O(\delta^\sigma |log(1/\delta)|), & \alpha = \sigma, \\ O(\delta^a), & \alpha < \sigma, \end{cases}$$

hold.

Definition (2.2.7)[89]: For $0 < \sigma < \alpha$ we set

$$Lip\,\sigma\,(\alpha, p(\cdot)) := \left\{f \in L^{p(\cdot)}_{2\pi} : \Omega_\alpha(f,\delta)_{p(\cdot)} = O(\delta^\sigma), \delta > 0\right\}.$$

Corollary (2.2.8)[89]: Let $0 < \sigma < \alpha, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$ be fulfilled. Then the following conditions are equivalent:

(a) $f \in Lip\sigma\,(\alpha, p(\cdot))$,

(b) $E_n(f)_{p(\cdot)} = O(n^{-\sigma}), n = 1,2,\ldots$.

Theorem(2.2.9)[89]: Let $p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L_{2\pi}^{p(\cdot)}$. If $\beta \in (0,\infty)$ and

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1} E_\nu(f)_{p,\pi} < \infty$$

then $f \in W_{p(\cdot)}^{\beta}$ and

$$E_n\left(f^{(\beta)}\right)_{p(\cdot)} \leq c\left((n+1)^\beta E_n(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} E_\nu(f)_{p(\cdot)}\right)$$

hold where the constant $c > 0$ dependent only on β and p.

Corollary (2.2.10)[89]: Let $p \in P$ satisfy DL_γ with $\gamma \geq 1, f \in L_{2\pi}^{p(\cdot)}, \beta \in (0,\infty)$ and

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot)} < \infty$$

for some $\alpha > 0$. In this case for $n = 0,1,2,\ldots$ there exists a constant $c > 0$ dependent only on α, β and p such that

$$\Omega_\beta\left(f^{(\alpha)}, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq c\,(n+1)^\beta \sum_{\nu=0}^{n} (\nu+1)^{\alpha+\beta-1} E_\nu(f)_{p(\cdot)} + c \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot)}$$

hold.

The following simultaneous approximation theorem holds.

Theorem (2.2.11)[89]: Let $\beta \in [0,\infty), p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L_{2\pi}^{p(\cdot)}$. Then there exist a $T \in T_n$ and a constant $c > 0$ depending only on α and p such that

$$\left\|f^{(\beta)} - T^{(\beta)}\right\|_{p,\pi} \leq cE_n\left(f^{(\beta)}\right)_{p(\cdot)}$$

holds.

Definition (2.2.12)[89]: (Hardy space of variable exponent $H^{p(\cdot)}$ on the unit disc $\mathbb{D}$with the boundary $\mathbb{T} := \partial\mathbb{D}$) [90]. Let $p(z): \mathbb{T} \to (1,\infty)$, be measurable function. We say that a complex valued analytic function Φ in $\mathbb{D}$ belongs to the Hardy space $H^{p(\cdot)}$ if

$$\sup_{0<r<1} \int_0^{2\pi} \left|\Phi\left(re^{i\vartheta}\right)\right|^{p(\vartheta)} d\vartheta < +\infty$$

where $p(\vartheta) := p\left(e^{i\vartheta}\right), \vartheta \in [0,2\pi]$ (and therefore $p(\vartheta)$ is 2π-periodic function). Let $\underline{p} := \inf_{z\in\mathbb{T}} p(z)$ and $\bar{p} := \sup_{z\in\mathbb{T}} p(z)$. If $p > 0$, then it is obvious that $H^{\bar{p}} \subset H^{p(\cdot)} \subset \subset H^{\underline{p}}$. Therefore if $f \in H^{p(\cdot)}$ and $\underline{p} > 0$, then there exist nontangential boundary-values

$f(e^{i\theta})a.e.$ on $\mathbb{T}$ and $f(e^{i\theta}) \in L_{2\pi}^{p(\cdot)}(\mathbb{T})$. Under the conditions $1 < p$ and $p < \infty, H^{p(\cdot)}$ becomes a Banach space with the norm

$$\|f\|_{H^{p(\cdot)}} := f(e^{i\theta})_{p,\pi}, \mathbb{T} = \inf\left\{\lambda > 0: \int_{\mathbb{T}} \left|\frac{f(e^{i\theta})}{\lambda}\right|^{p(\theta)} d\theta \leq 1\right\}.$$

Theorem $(\mathbf{2.2.13})$[89]: If $p \in P$ satisfy DL_γ with $\gamma \geq 1, f$ belongs to Hardy space $H^{p(\cdot)}$ on $\mathbb{D}$ and $r \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} \leq c\Omega_r\left((e^{i\theta}), \frac{1}{n+1}\right)_{p(\cdot)}, n = 0,1,2,\ldots,$$

where $a_k(f), k = 0,1,2,3,\ldots$, are the Taylor coefficients of f at the origin.

We begin with the following lemma.

Lemma $(\mathbf{2.2.14})$[89]: [100]. For $r \in \mathbb{R}^+$ we suppose that

(i) $a_1 + a_2 + \ldots + a_n + \ldots,$

(ii) $a_1 + 2^r a_2 + \ldots + n^r a_n + \ldots$

be two series in a Banach space $(B, \|\cdot\|)$. Let

$$R_n^{\langle r \rangle} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) a_k$$

and

$$R_n^{\langle r \rangle *} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) k^r a_k$$

for $n = 1,2,\ldots$. Then

$$\left\| R_n^{\langle r \rangle *} \right\| \leq c, \quad n = 1,2,\ldots,$$

for some $c > 0$ if and only if there exists a $R \in B$ such that

$$\left\| R_n^{\langle r \rangle *} - R \right\| \leq \frac{C}{n^r},$$

where c and C are constants depending only on one another.

Lemma $(\mathbf{2.2.15})$[89]: [98]. If $p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L_{2\pi}^{p(\cdot)}$ then there are constants $c, C > 0$ such that

$$\|\tilde{f}\|_{p,\pi} \leq c\|f\|_{p,\pi} \tag{17}$$

and

$$\|S_n(\cdot, f)\|_{p,\pi} \leq C\|f\|_{p,\pi} \tag{18}$$

hold for $n = 1,2,\dots.$

Remark $(\mathbf{2.2.16})$[**89**]: Under the conditions of Lemma (2.2.15) (i) It can be easily seen from (17) and (18) that there exists constant $c > 0$ such that

$\|f - S_n(\cdot, f)\|_{p,\pi} \leq cE_n(f)_{p(\cdot)} \asymp E_n(\tilde{f})_{p(\cdot)}.$

(ii) From generalized Hölder inequality [104] (Theorem 2.1) we have

$$L_{2\pi}^{p(\cdot)} \subset L^1.$$

For a given $f \in L^1$ let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad (19)$$

and

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty}(a_k \sin kx + b_k \cos kx)$$

be the Fourier and the conjugate Fourier series of f, respectively. Putting $A_k(x) :=:= c_k e^{ikx}$ in (19) we define

$$S_n(f) := S_n(x, f) := \sum_{k=0}^{n} \left(A_k(x) + A_{-k}(x)\right) =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), n = 0,1,2,\dots,$$

$$R_n^{\langle\alpha\rangle}(f, x) := \sum_{k=0}^{n}\left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right)\left(A_k(x) + A_{-k}(x)\right)$$

and

$$\Theta_m^{\langle r\rangle} := \frac{1}{1 - \left(\frac{m+1}{2m+1}\right)^r} R_{2m}^{\langle r\rangle} - \frac{1}{\left(\frac{2m+1}{m+1}\right)^r - 1} R_m^{\langle r\rangle}, for\ m = 1,2,3,\dots. \qquad (20)$$

Under the conditions of Lemma (2.2.15) using (18) and Abel's transformation we get

$$\left\|R_n^{\langle\alpha\rangle}(f, x)\right\|_{p,\pi} \leq c\|f\|_{p,\pi}, \ n = 1,2,3,\dots,\ x \in T,\ f \in L_{2\pi}^{p(\cdot)}, \quad (21)$$

and therefore from (20) and (21)

$$\left\|\Theta_m^{\langle r\rangle}(f, x)\right\|_{p,\pi} \leq c\|f\|_{p,\pi}, m = 1,2,3,\dots, \qquad x \in T, \qquad f \in L_{2\pi}^{p(\cdot)}.$$

From the property [105] ((16))

$$\Theta_m^{\langle r\rangle}(f)(x) = $$

$$= \frac{1}{\sum_{k=m+1}^{2m}[(k + 1)^r - k^r]} \sum_{k=m+1}^{2m} [(k + 1)^r - k^r] S_k(x,f), x \in T, \qquad f \in L^1,$$

it is known [105] ((18)) that

$$\Theta_m^{\langle r\rangle}(T_m) = T_m \tag{22}$$

for $T_m \in T_m, m = 1,2,3,\ldots$.

Lemma $(\mathbf{2.2.17})$[89]: Let $T_n \in T_n, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $r \in \mathbb{R}^+$. Then there exists a constant $c > 0$ independent of n such that

$$\left\|T_n^{(r)}\right\|_{p,\pi} \leq cn^r \|T_n\|_{p,\pi}$$

holds.

Proof: Without loss of generality one can assume that $\|T_n\|_{p,\pi} = 1$. Since

$$T_n = \sum_{k=0}^{n} \left(A_k(x) + A_{-k}(x)\right)$$

we get

$$\frac{\tilde{T}_n}{n^r} = \sum_{k=1}^{n}\left[\left(A_k(x) - A_{-k}(x)\right)/n^r\right]$$

and

$$\frac{T_n^{(r)}}{n^r} = i^r \sum_{k=1}^{n} k^r\left[\left(A_k(x) - A_{-k}(x)\right)/n^r\right].$$

In this case we have by (21) and (17) that

$$\left\|R_n^{\langle r\rangle}\left(\frac{\tilde{T}_n}{n^r}\right)\right\|_{p,\pi} \leq \frac{c}{n^r}\left\|\tilde{T}_n\right\|_{p,\pi} \leq \frac{c}{n^r} \|T_n\|_{p,\pi} = \frac{c}{n^r}$$

and hence applying Lemma (2.2.14) (with $R = 0$) to the series

$$\sum_{k=1}^{n}\left[\left(A_k(x) - A_{-k}(x)\right)/n^r\right] + 0 + 0 + \ldots + 0 + \ldots,$$

$$\sum_{k=1}^{n} k^r\left[\left(A_k(x) - A_{-k}(x)\right)/n^r\right] + 0 + 0 + \ldots + 0 + \ldots,$$

we find

$$\left\| \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^{r} \right) k^{r} \left[\left(A_k(x) - A_{-k}(x) \right) / n^{r} \right] \right\|_{p,\pi} \leq c,$$

namely

$$\left\| R_n^{\langle r \rangle} \left(\frac{T_n^{(r)}}{n^r} \right) \right\|_{p,\pi} = \left\| i^r \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^{r} \right) k^{r} \left[\left(A_k(x) - A_{-k}(x) \right) / n^{r} \right] \right\|_{p,\pi} =$$

$$= \left\| \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^{r} \right) k^{r} \left[\left(A_k(x) - A_{-k}(x) \right) / n^{r} \right] \right\|_{p,\pi} \leq c_*.$$

Since $R_n^{\langle r \rangle}(cf) = cR_n^{\langle r \rangle}(f)$ for every real c we obtain from (22) and the last inequality that

$$\left\| T_n^{(r)} \right\|_{p,\pi} = \left\| \Theta_n^{\langle r \rangle} \left(T_n^{(r)} \right) \right\|_{p,\pi} = n^r \left\| \frac{1}{n^r} \Theta_n^{\langle r \rangle} \left(T_n^{(r)} \right) \right\|_{p,\pi} =$$

$$= n^r \left\| \Theta_n^{\langle r \rangle} \left(\frac{T_n^{(r)}}{n^r} \right) \right\|_{p,\pi} \leq c_* n^r = c_* n^r \| T_n \|_{p,\pi} .$$

General case follows immediately from this.

Lemma $(\mathbf{2.2.18})$[**89**]: If $p \in P$ satisfy DL_γ with $\gamma \geq 1, f \in W^2_{p(\cdot)}$ and $r = 1,2,3,\ldots$, then

$$\Omega_r(f,\delta)_{p(\cdot} \leq c\delta^2 \Omega_{r-1}(f'',\delta)_{p(\cdot)}, \delta \geq 0,$$

with some constant $c > 0$.

Proof: Putting

$$g(x) := \prod_{i=2}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

we have

$$\left(I - \sigma_{h_1} \right) g(x) = \prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

and

$$\prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f(x) = \frac{1}{h_1} \int_{-h_1/2}^{h_1/2} \left(g(x) - g(x+t) \right) dt =$$

$$= -\frac{1}{h_1}\int_0^{h_1/2}\int_0^{2t}\int_{-u/2}^{u/2} g"(x + s)\, dsdudt.$$

Therefore from (13)

$$\left\|\prod_{i=1}^{r}\left(I - \sigma_{h_i}\right) f\,(x)\right\|_{p,\pi} \leq$$

$$\leq \frac{c}{2h_1}\sup\left\{\int_T \left|\int_0^{h_1/2}\int_0^{2t}\int_{-u/2}^{u/2} g"(x + s)\, dsdudt\right| |g_0(x)|dx: g_0 \in L_{2\pi}^{p'(\cdot)}\ and\ \int_T |g_0(x)|^{p'(x)}\, dx \leq 1\right\} \leq$$

$$\leq \frac{c}{2h_1}\int_0^{h_1/2} u \int_0^{2t}\left\|\frac{1}{u}\int_{-u/2}^{u/2} g"(x + s)\, ds\right\|_{p,\pi} dudt \leq$$

$$\leq \frac{c}{2h_1}\int_0^{h_1/2}\int_0^{2t} u\|g"\|_{p,\pi}\, dudt = ch_1^2\|g"\|_{p,\pi}\ .$$

Since

$$g"(x) = \prod_{i=2}^{r}\ (I - \sigma_{h_i})\, f"(x),$$

we obtain that

$$\Omega_r\,(f,\delta)_{p(\cdot)} \leq \sup_{\substack{0<h_i\leq\delta \\ i=1,2,\dots,r}} ch_1^2\|g"\|_{p,\pi} = c\delta^2 \sup_{\substack{0<h_i\leq\delta \\ i=1,2,\dots,r}}\left\|\prod_{i=2}^{r}\ (I - \sigma_{h_i})\, f"(x)\right\|_{p,\pi} =$$

$$= c\delta^2 \sup_{\substack{0<h_j\leq\delta \\ j=1,2,\dots,r-1}}\left\|\prod_{j=1}^{r-1}\ (I - \sigma_{h_j})\, f"(x)\right\|_{p,\pi} = c\delta^2\Omega_{r-1}\,(f",\delta)_{p(\cdot)}\ .$$

Lemma (2.2.18) is proved.

Corollary $(\mathbf{2.2.19})[\mathbf{89}]$: If $r = 1,2,3,\dots, p \in P$ satisfy DL_γ with $\gamma \geq 1$ and $f \in W_{p(\cdot)}^{2r}$,
then

$$\Omega_r\,(f,\delta)_{p(\cdot)} \leq c\delta^{2r}\left\|f^{(2r)}\right\|_{p,\pi}, \delta \geq 0,$$

with some constant $c > 0$.

Lemma $(\mathbf{2.2.20})[\mathbf{89}]$: Let $\alpha \in \mathbb{R}^+, p \in P$ satisfy DL_γ with $\gamma \geq 1$, $n = 0,1,2,\dots$ and $T_n \in T_n$. Then

$$\Omega_\alpha\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha}\left\|T_n^{(\alpha)}\right\|_{p,\pi}$$

hold where the constant $c > 0$ dependent only on α and p.

Proof: Firstly we prove that if $0 < \alpha < \beta, \alpha, \beta \in > 0$.

Lemma $(\mathbf{2.2.21})[\mathbf{89}]$: Let $\alpha \in \mathbb{R}^+$ then

$$\Omega_\beta(f,\cdot)_{p(\cdot)} \leq c\Omega_\alpha(f,\cdot)_{p(\cdot)} . \tag{23}$$

It is easily seen that if $\alpha \leq \beta, \alpha, \beta \in \mathbb{Z}^+$, then

$$\Omega_\beta(f,\cdot)_{p(\cdot)} \leq c(\alpha,\beta,p)\Omega_\alpha(f,\cdot)_{p(\cdot)} . \tag{24}$$

Now, we assume that $0 < \alpha < \beta < 1$. In this case putting $\Phi(x) := \sigma_h^\alpha f(x)$ we have

$$\sigma_h^{\beta-\alpha}\ \Phi(x) = \sum_{j=0}^{\infty}(-1)^j\binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}\Phi(x+u_1+\dots u_j)du_1\dots du_j =$$

$$= \sum_{j=0}^{\infty}(-1)^j\binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}\left[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\frac{1}{h^k}\int_{-h/2}^{h/2}\dots\dots\int_{-h/2}^{h/2}f\,(x\right.$$

$$\left. + u_1 + \dots u_j + u_{j+1} + \dots u_{j+k})du_1\dots du_j du_{j+1}\dots du_{j+k}\right] =$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta-\alpha}{j}\binom{\alpha}{k} \times$$

$$\times\left[\frac{1}{h^{j+k}}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}f(x+u_1+\dots u_{j+k})du_1\dots du_{j+k}\right] =$$

$$= \sum_{\upsilon=0}^{\infty}(-1)^\upsilon\binom{\beta}{\upsilon}\frac{1}{h^\upsilon}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}f\,(x+u_1+\dots u_\upsilon)d_{u_1}\dots d_{u_\upsilon} = \sigma_h^\beta f\,(x) \qquad a.e.$$

Then

$$\left\|\sigma_h^\beta f(x)\right\|_{p,\pi} = \left\|\sigma_h^{\beta-\alpha}\ \Phi(x)\right\|_{p,\pi} \leq c\left\|\sigma_h^\beta f(x)\right\|_{p,\pi}$$

and

$$\Omega_\beta(f,\cdot)_{p(\cdot)} \leq c\Omega_\alpha(f,\cdot)_{p(\cdot)} . \tag{25}$$

We note that if $r_1, r_2 \in \mathbb{Z}^+, \alpha_1, \beta_1 \in (0,1)$ taking $\alpha := r_1 + \alpha_1, \beta := r_2 + \beta_1$ for the remaining cases $r_1 = r_2, \alpha_1 < \beta_1$ or $r_1 < r_2, \alpha_1 = \beta_1$ or $r_1 < r_2, \alpha_1 < \beta_1$ it can easily be obtained from (24) and (25) that the required inequality (23) holds. Using (23), Corollary (2.2.19) and Lemma (2.2.17) we get

$$\Omega_\alpha\left(T_n,\frac{\pi}{n+1}\right)_{p(\cdot)} \le c\Omega_{[\alpha]}\left(T_n,\frac{\pi}{n+1}\right)_{p(\cdot)} \le c\left(T_n,\frac{\pi}{n+1}\right)^{2[\alpha]}\left\|T_n^{(2[\alpha])}\right\|_{p,\pi} \le$$

$$\le \frac{c}{(n+1)^{2[\alpha]}}\ (n+1)^{[\alpha]-(\alpha-[\alpha])}\left\|T_n^{(\alpha)}\right\|_{p,\pi} = \frac{c}{(n+1)^{\alpha}}\left\|T_n^{(\alpha)}\right\|_{p,\pi}$$

the required result.

Definition $(\mathbf{2.2.22})[\mathbf{89}]$: For $p \in P, f \in L_{2\pi}^{p(\cdot)}$, $\delta > 0$ and $r = 1,2,3,...$ the Peetre Kfunctional is defined as

$$K\left(\delta,f;L_{2\pi}^{p(\cdot)},W_{p(\cdot)}^{r}\right) := \inf_{g\in W_{p(\cdot)}^{r}}\left\{\|f-g\|_{p,\pi} + \delta\|g(r)\|_{p,\pi}\right\}. \quad (26)$$

Theorem $(\mathbf{2.2.23})[\mathbf{89}]$: If $p \in P$ satisfy DL_γ with $\gamma \ge 1$ and $f \in L_{2\pi}^{p(\cdot)}$, then the Kfunctional $K\left(\delta^{2r},f;L_{2\pi}^{p(\cdot)},W_{p(\cdot)}^{2r}\right)$ in (26) and the modulus $\Omega_r\ (f,\delta)_{p(\cdot)}$, $r = 1,2,3,..$.are equivalent.

Proof: If $h \in W_{p(\cdot)}^{2r}$, then we have by Corollary (2.2.19) and (26) that

$\Omega_r\ (f,\delta)_{p(\cdot)} \le c\|f-h\|_{p,\pi} + c\delta^{2r}\left\|h^{(2r)}\right\|_{p,\pi} \le cK\left(\delta^{2r},f;L_{2\pi}^{p(\cdot)},W_{p(\cdot)}^{2r}\right)$.

We estimate the reverse of the last inequality. The operator Lδ defined by

$$(L_\delta f)(x) := 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\int_{-u/2}^{u/2} f\,(x+s)\,dsdudt,\ \ x \in T,$$

is bounded in $L_{2\pi}^{p(\cdot)}$ because

$$\|L_\delta f\|_{p,\pi} \le \int_0^{\delta/2}\int_0^{2t} u\|\sigma_u f\|_{p,\pi}\,dudt \le c\|f\|_{p,\pi}\ .$$

We prove

$$\frac{d^2}{dx^2}\ \|L_\delta f\|f = \frac{c}{\delta^2}(I-\sigma_\delta)f$$

with a real constant c. Since

$$(L_\delta f)(x) = 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\int_{-u/2}^{u/2} f\,(x+s)\,dsdudt =$$

$$= 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\left[\int_0^{x+u/2} f\,(s)\,ds - \int_0^{x-u/2} f\,(s)\,ds\right]dudt$$

using Lebesgue Differentiation Theorem

$$\frac{d}{dx}\,(L_\delta f)(x) = \;3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\left[\frac{d}{dx}\int_0^{x+u/2} f\,(s)\,ds - \frac{d}{dx}\int_0^{x-u/2} f\,(s)\,ds\right]dudt\; =$$

$$= \;3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}[f\,(x\;+\;u/2) - f\,(x-u/2)]\;dudt\; =$$

$$= \;6\delta^{-3}\int_0^{\delta/2}\left[\int_x^{x+t} f\,(u)\,du + \int_x^{x-t} f\,(u)\,du\right]dt \qquad a.e.$$

Using Lebesgue Differentiation Theorem once more

$$\frac{d^2}{dx^2}(L_\delta f)(x)\; = \;6\delta^{-3}\int_0^{\delta/2}\left[\frac{d}{dx}\int_x^{x+t} f\,(u)\,du + \frac{d}{dx}\int_x^{x-t} f\,(u)\,du\right]dt\; =$$

$$= \;6\delta^{-3}\int_0^{\delta/2}\;[f\,(x\;+\;t) - f\,(x)\;+\;f\,(x-t) - f(x)]\;dt\; =$$

$$= \frac{6}{\delta^3}\left[\int_0^{\delta/2} f\,(x\;+\;t)\,dt + \int_0^{\delta/2} f\,(x-\;t)\,dt - \delta f(x)\right] =$$

$$= \frac{6}{\delta^2}\left[\frac{1}{\delta}\int_0^{\delta/2} f\,(x\;+\;t)\,dt + \frac{1}{\delta}\int_{-\delta/2}^{0} f\,(x+\;t)\,dt - f(x)\right] =$$

$$= \frac{6}{\delta^2}\left[\frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f\,(x\;+\;t)\,dt + -f(x)\right] == \frac{-6}{\delta^2}\left[f(x) - \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f\,(x\;+\;t)\,dt\right]$$

$$= \frac{-6}{\delta^2}(I\;-\sigma_\delta)f(x) \qquad a.e.$$

The last equality implies by induction on r that

$$\frac{d^{2r}}{dx^{2r}}L_\delta^r\,f\; = \frac{c}{\delta^{2r}}\;(I\;-\sigma_\delta)^r\,f, \qquad r\; = \;1,2,3,\ldots a.e.$$

Indeed, for $r\; = \;2$

$$\frac{d^4}{dx^4}L_\delta^2 f\; = \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta^2\,f\right) = \;\frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta\;(L_\delta f\; = : u)\right) =$$

$$= \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta u\right) = \;\frac{d^2}{dx^2}\left(\frac{-6}{\delta^2}(I - \sigma_\delta)u\right) =$$

$$= \;\frac{-6}{\delta^2}\;\left(\frac{d^2}{dx^2}(I\;-\sigma_\delta)u\right) = \;\frac{-6}{\delta^2}\;\left(\frac{d^2}{dx^2}(I\;-\sigma_\delta)L_\delta f\right)\;a.e.$$

Since $\frac{d^2}{dx^2}\sigma_\delta\ (L_\delta f) = \ \sigma_\delta\left(\frac{d^2}{dx^2}L_\delta f\right)$ we get

$$\frac{d^2}{dx^2}(I\ -\sigma_\delta)L_\delta f = \frac{d^2}{dx^2}L_\delta f\ - \frac{d^2}{dx^2}\sigma_\delta\ (L_\delta f)\ == \frac{d^2}{dx^2}L_\delta f\ -\sigma_\delta\left(\frac{d^2}{dx^2}L_\delta f\right)$$

$$= \ (I\ -\sigma_\delta)\left(\frac{d^2}{dx^2}L_\delta f\right)\quad a.e.$$

and therefore

$$\frac{d^4}{dx^4}L_\delta^2\ f\ = \ \frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}(I\ -\sigma_\delta)L_\delta f\right) = \ \frac{-6}{\delta^2}(I\ -\sigma_\delta)\left[\frac{d^2}{dx^2}L_\delta f\right] =$$

$$= \ \frac{-6}{\delta^2}(I\ -\sigma_\delta)\left[\frac{-6}{\delta^2}(I\ -\sigma_\delta)f\right] = \ \frac{c}{\delta^4}\ (I\ -\sigma_\delta)^2\ f\ a.e.$$

Now let be $\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}\ f\ = \frac{c}{\delta^{2(r-1)}}\ (I\ -\sigma_\delta)^{(r-1)}\ f\ a.e.$ Then

$$\frac{d^{2r}}{dx^{2r}}L_\delta^2\ f\ = \frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}(L_\delta\ :=\ u)\right] = \ \frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}u\right] =$$

$$= \frac{d^2}{dx^2}\ \left[\frac{c}{\delta^{2(r-1)}}\ (I\ -\sigma_\delta)^{(r-1)}u\right] = \ \frac{d^2}{dx^2}\left[\frac{c}{\delta^{2(r-1)}}(I\ -\sigma_\delta)^{(r-1)}\ L_\delta\right] =$$

$$= \frac{c}{\delta^{2(r-1)}}\ (I\ -\sigma_\delta)^{(r-1)}\left[\frac{d^2}{dx^2}L_\delta\right] = \ \frac{c}{\delta^{2r}}\ (I\ -\sigma_\delta)^r\ f\ a.e.$$

Letting $A_\delta^r\ :=\ I\ -\ (I\ -L_\delta^r)^r$ we prove that $\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^r f\right\|_{p,\pi}\ \leq\ c\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^r f\right\|_{p,\pi}$ and $A_\delta^r f\ \in\ W_{p(\cdot)}^{2r}$. For $r\ =\ 1$ we have $A_\delta^1 f :=\ I-\left(I\ -L_\delta^1 f\right)^1\ =\ L_\delta^1 f$ and $\left\|\frac{d^2}{dx^2}A_\delta^1 f\right\|_{p,\pi} =$ $= \left\|\frac{d^2}{dx^2}L_\delta^1 f\right\|_{p,\pi}$. Since $\frac{d^2}{dx^2}L_\delta f\ =\ \frac{c}{\delta^2}\ (I\ -\sigma_\delta)f$ we get $A_\delta^1 f\ \in\ W_{p(\cdot)}^2$. For $r\ ==$ $2,3,\ldots$ using

$$A_\delta^r :=\ I\ -(I\ -L_\delta^r)^r = \sum_{j=0}^{r-1}(-1)^{r-j+1}\binom{r}{j}L_\delta^{r(r-j)}$$

we obtain

$$\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^r f\right\|_{p,\pi}\ \leq \sum_{j=0}^{r-1}\binom{r}{j}\ \left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r(r-j)}\ f\right\|_{p,\pi}.$$

We estimate $\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r(r-j)}\ f\right\|_{p,\pi}$ as the following

$$\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^{r(r-j)}\ f\right\|_{p,\pi} = \left\|\frac{d^{2r}}{dx^{2r}}L_\delta^r\ \left(L_\delta^{r(r-j)}f\ =:u\right)\right\|_{p,\pi} =$$

$$= \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r \, u \right\|_{p,\pi} = \left\| \frac{c}{\delta^{2r}} \ (I - \sigma_\delta)^r \, u \right\|_{p,\pi} =$$

$$= \left\| \frac{c}{\delta^{2r}} \ (I - \sigma_\delta)^r \ \left[L_\delta^{r(r-j)} f \right] \right\|_{p,\pi} = \frac{c}{\delta^{2r}} \left\| (I - \sigma_\delta)^r \ \left[L_\delta^{r(r-j)} f \right] \right\|_{p,\pi} \leq$$

$$\leq \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i \left[L_\delta^{r(r-j)} f \right] \right\|_{p,\pi} .$$

Since $\sigma_\delta(L_\delta f) = L_\delta \ (\sigma_\delta f)$ we have $\sigma_\delta^i \left[L_\delta^{r(r-j)} f \right] = L_\delta^{r(r-j)} \left(\sigma_\delta^i f \right)$and hence

$$\left\| \frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} \ f \right\|_{p,\pi} \leq \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i \left[L_\delta^{r(r-j)} f \right] \right\|_{p,\pi} \leq$$

$$\leq \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} \ (-1)^i \binom{r}{i} L_\delta^{r(r-j)} \left(\sigma_\delta^i f \right) \right\|_{p,\pi} =$$

$$= \frac{c}{\delta^{2r}} \left\| L_\delta^{r(r-j)} \left[\sum_{i=0}^{r} \ (-1)^i \binom{r}{i} \left(\sigma_\delta^i f \right) \right] \right\|_{p,\pi} \leq \frac{C}{\delta^{2r}} \left\| \sum_{i=0}^{r} \ (-1)^i \binom{r}{i} \left(\sigma_\delta^i f \right) \right\|_{p,\pi} =$$

$$= \frac{C}{\delta^{2r}} \| (I - \sigma_\delta)^r f \|_{p,\pi} = \left\| \frac{C}{\delta^{2r}} (I - \sigma_\delta)^r f \right\|_{p,\pi} = c_1 \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi} .$$

From the last inequality

$$\left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{p,\pi} \leq c \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi} \ and \ A_\delta^r f \in W_{p(\cdot)}^{2r}.$$

Therefore we find

$$\left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{p,\pi} \leq c \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi} = \frac{c}{\delta^{2r}} \| (I - \sigma_\delta)^r f \|_{p,\pi} \leq \frac{c}{\delta^{2r}} \Omega_r(f, \delta)_{p(\cdot)}.$$

Since

$$I - L_\delta^r = (I - L_\delta) \sum_{j=0}^{r-1} L_\delta^j$$

we get

$$\| (I - L_\delta^r)^g \|_{p,\pi} \leq c \| (I - L_\delta)^g \|_{p,\pi} \leq$$

$$\leq 3c\delta^{-3} \int_0^{\delta/2} \int_0^{2t} u \| (I - \sigma_u)^g \|_{p,\pi} \, dudt \leq c \sup_{0<u\leq\delta} \| (I - \sigma_u)^g \|_{p,\pi} .$$

Taking into account

$$\| f - A_\delta^r f \|_{p,\pi} = \| (I - L_\delta^r)^r f \|_{p,\pi}$$

by a recursive procedure we obtain

$$\|f - A_\delta^r f\|_{p,\pi} \le c \sup_{0<t_1\le\delta} \left\|\left(I - \sigma_{t_1}\right)(I - L_\delta^r)^{r-1} f\right\|_{p,\pi} \le$$

$$\le c \sup_{0<t_1\le\delta} \sup_{0<t_2\le\delta} \left\|\left(I - \sigma_{t_1}\right)\left(I - \sigma_{t_2}\right)(I - L_\delta^r)^{r-2} f\right\|_{p,\pi} \le \dots$$

$$\dots \le c \sup_{\substack{0<t_i\le\delta \\ i=1,2,\dots,r}} \left\|\prod_{i=1}^{r} \left(I - \sigma_{t_i}\right) f(x)\right\|_{p,\pi} = c\Omega_r(f,\delta)_{p(\cdot)}.$$

Theorem (2.2.23) is proved.

We set $A_k(x,f) := a_k \cos kx + + b_k \sin kx$. Since the set of trigonometric polynomials is dense [92] in $L_{2\pi}^{p(\cdot)}$ for given $f \in L_{2\pi}^{p(\cdot)}$ we have $E_n(f)_{p(\cdot)} \to 0$ as $n \to \infty$. From the first inequality in Remark (2.2.16), we have $f(x) = \sum_{k=0}^{\infty} A_k(x,f)$ in $\|\cdot\|_{p,\pi}$ norm. For $k = 1,2,3,\dots$ we can find

$$A_k(x,f) = a_k \cos k\left(x + \frac{\alpha\pi}{2k} - \frac{\alpha\pi}{2k}\right) + b_k \sin k\left(x + \frac{\alpha\pi}{2k} - \frac{\alpha\pi}{2k}\right) =$$

$$= A_k\left(x + \frac{\alpha\pi}{2k}, f\right)\cos\frac{\alpha\pi}{2} + A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right)\sin\frac{\alpha\pi}{2}$$

and

$$A_k\left(x, f^{(\alpha)}\right) = k^\alpha A_k\left(x + \frac{\alpha\pi}{2k}, f\right).$$

Therefore

$$\sum_{k=0}^{\infty} A_k(x,f) =$$

$$= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, f\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right) =$$

$$= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right)$$

and hence

$$f(x) - S_n(x,f) = \cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right).$$

Since

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) =$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) - \left(S_{k-1}\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)\right] =$$

$$= \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)$$
$$- -(n+1)^{-\alpha}\left(S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)$$

and

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right) = \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left(S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right) -$$
$$-(n+1)^{-\alpha}\left(S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right)$$

we obtain

$$\|f(\cdot) - S_n(\cdot, f)\|_{p,\pi} \leq \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\|S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi} +$$
$$+(n+1)^{-\alpha}\left\|S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi} +$$
$$+ \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\|S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p,\pi}$$
$$+ +(n+1)^{-\alpha}\left\|S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p,\pi} \leq$$
$$\leq c\left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}\right] +$$
$$+c\left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) E_k\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)}\right].$$

Consequently from equivalence in Remark (2.2.16) (i) we have

$$\|f(x) - S_n(x, f)\|_{p,\pi} \leq$$
$$\leq c\left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\right]\left\{E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)}\right\} \leq$$
$$\leq cE_n\left(f^{(\alpha)}\right)_{p(\cdot)} \left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\right] \leq \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}.$$

Theorem (2.2.2) is proved.

We put $r - 1 < \alpha < r, r \in \mathbb{Z}^+$. For $g \in W_{p(\cdot)}^{2r}$ we have by Corollary (2.2.3), (26) and Theorem (2.2.23) that

$$E_n(f)_{p(\cdot)} \leq E_n\,(f - g)_{p(\cdot)} + E_n(g)_{p(\cdot)} \leq c\left[\|f - g\|_{p,\pi} + (n+1)^{-2r}\left\|g^{(2r)}\right\|_{p,\pi}\right]$$
$$\leq$$
$$\leq cK\left((n+1)^{-2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right) \leq c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot)}$$

as required for $r \in \mathbb{Z}^+$. Therefore by the last inequality

$$E_n(f)_{p(\cdot)} \leq c\Omega_r(f, 1/(n+1))_{p(\cdot)} \leq c\Omega_r(f, 2\pi/(n+1))_{p(\cdot)}, n = 0,1,2,3,\ldots,$$

and by (24) we get

$$E_n(f)_{p(\cdot)} \leq c\Omega_r(f, 2\pi/(n+1))_{p(\cdot)} \leq c\Omega_a(f, 2\pi/(n+1))_{p(\cdot)}$$

and the assertion follows.

Let $T_n \in T_n$ be the best approximating polynomial of $f \in\in L_{2\pi}^{p(\cdot)}$ and let $m \in \mathbb{Z}^+$. Then by Remark (2.2.1) (ii)

$$\Omega_a(f, \pi/n+1)_{p(\cdot)} \leq \Omega_a(f - T_{2^m}, \pi/(n+1))_{p(\cdot)} + \Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)} \leq$$
$$\leq cE_{2^m}(f)_{p(\cdot)} + \Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)}.$$

Since

$$T_{2^m}^{(\alpha)}(x) = T_1^{(\alpha)}(x) + \sum_{\nu=0}^{m-1}\left\{T_{2^{\nu+1}}^{(\alpha)}(x) - T_{2^\nu}^{(\alpha)}(x)\right\}$$

we get by Lemma (2.2.20) that

$$\Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha}\left\{\left\|T_1^{(\alpha)}\right\|_{p,\pi} + \sum_{\nu=0}^{m-1}\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi}\right\}.$$

Lemma (2.2.17) gives

$$\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi} \leq c2^{\nu\alpha}\left\|T_{2^{\nu+1}} - T_{2^\nu}\right\|_{p,\pi} \leq c2^{\nu\alpha+1}E_{2^\nu}(f)_{p(\cdot)}$$

and

$$\left\|T_1^{(\alpha)}\right\|_{p,\pi} = \left\|T_1^{(\alpha)} - T_0^{(\alpha)}\right\|_{p,\pi} \leq cE_0(f)_{p(\cdot)}.$$

Hence

$$\Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + \sum_{\nu=0}^{m-1} 2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)}\right\}.$$

Using

$$2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)} \leq c^* \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1}E_\mu(f)_{p(\cdot)}, \qquad \nu = 1,2,3,\ldots,$$

we obtain

$$\Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)} \leq$$
$$\leq \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + 2^\alpha E_1(f)_{p(\cdot)} + c\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1}E_\mu(f)_{p(\cdot)}\right\} \leq$$

$$\leq \frac{c}{(n+1)^{\alpha}}\left\{E_0(f)_{p(\cdot)} + \sum_{\mu=1}^{2^m} \mu^{\alpha-1} E_{\mu}(f)_{p(\cdot)}\right\}$$

$$\leq \frac{c}{(n+1)^{\alpha}} \sum_{\nu=0}^{2^m-1} (\nu+1)^{\alpha-1} E_{\nu}(f)_{p(\cdot)}.$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_a(T_{2^m}, \pi/(n+1))_{p(\cdot)} \leq \frac{c}{(n+1)^{\alpha}} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_{\nu}(f)_{p(\cdot)},$$

$$E_{2^m}(f)_{p(\cdot)} \leq E_{2^{m-1}}(f)_{p(\cdot)} \leq \frac{c}{(n+1)^{\alpha}} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_{\nu}(f)_{p(\cdot)}.$$

Last two inequalities complete the proof.

For the polynomial T_n of the best approximation to f we have by Lemma (2.2.17) that

$$\left\|T^{(\beta)}_{2^{i+1}} - T^{(\beta)}_{2^i}\right\|_{p,\pi} \leq C(\beta)2^{(i+1)\beta} \left\|T_{2^{i+1}} - T_{2^i}\right\|_{p,\pi} \leq 2C(\beta)2^{(i+1)\beta} E_{2^i}(f)_{p(\cdot)}.$$

Hence

$$\sum_{i=1}^{\infty} \left\|T_{2^{i+1}} - T_{2^i}\right\|_{W^{\beta}_{p(\cdot)}} = \sum_{i=1}^{\infty} \left\|T^{(\beta)}_{2^{i+1}} - T^{(\beta)}_{2^i}\right\|_{p,\pi} + \sum_{i=1}^{\infty} \left\|T_{2^{i+1}} - T_{2^i}\right\|_{p,\pi} \leq$$

$$\leq c \sum_{m=2}^{\infty} m^{\beta-1} E_m(f)_{p(\cdot)} < \infty.$$

Therefore

$$\left\|T_{2^{i+1}} - T_{2^i}\right\|_{W^{\beta}_{p(\cdot)}} \to 0 \text{ as } i \to \infty.$$

This means that $\{T_{2^i}\}$ is a Cauchy sequence in $L^{p(\cdot)}_{2\pi}$. Since $T_{2^i} \to f$ in $L^{p(\cdot)}$ 2π and $W^{\beta}_{p(\cdot)}$ is a Banach space we obtain $f \in W^{\beta}_{p(\cdot)}$. On the other hand since

$$\left\|f^{(\beta)} - S_n\left(f^{(\beta)}\right)\right\|_{p,\pi} \leq$$

$$\leq \left\|S_{2^{m+2}}\left(f^{(\beta)}\right) - S_n\left(f^{(\beta)}\right)\right\|_{p,\pi} + \sum_{k=m+2}^{\infty} \left\|S_{2^{k+1}}\left(f^{(\beta)}\right) - S_{2^k}\left(f^{(\beta)}\right)\right\|_{p,\pi}$$

we have for $2^m < n < 2^{m+2}$

$$\left\|S_{2^{m+2}}\left(f^{(\beta)}\right) - S_n\left(f^{(\beta)}\right)\right\|_{p,\pi} \leq c2^{(m+2)\beta} E_n(f)_{p(\cdot)} \leq c(n+1)^{\beta} E_n(f)_{p(\cdot)}.$$

On the other hand we find

$$\sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}\left(f^{(\beta)}\right) - S_{2^k}\left(f^{(\beta)}\right) \right\|_{p,\pi} \leq c \sum_{k=m+2}^{\infty} 2^{(k+1)\beta} E_{2^k}(f)_{p(\cdot)} \leq$$

$$\leq c \sum_{k=m+2}^{\infty} \sum_{\mu=2^{k-1}+1}^{2^k} \mu^{\beta-1} E_{\mu}(f)_{p(\cdot)} =$$

$$= c \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\beta-1} E_{\nu}(f)_{p(\cdot)} \leq c \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} E_{\nu}(f)_{p(\cdot)} .$$

Theorem (2.2.9) is proved.

We set $W_n(f) := W_n(x,f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x,f)\,, n == 0,1,2,\ldots$. Since

$$W_n\left(\cdot, f^{(\alpha)}\right) = W_n^{(\alpha)}(\cdot, f)$$

we have

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{p,\pi} \leq \left\| f^{(\alpha)}(\cdot) - W_n\left(\cdot, f^{(\alpha)}\right) \right\|_{p,\pi} +$$

$$+ \left\| T_n^{(\alpha)}\left(\cdot, W_n(f)\right) - T_n^{(\alpha)}(\cdot, f) \right\|_{p,\pi} + \left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}\left(\cdot, W_n(f)\right) \right\|_{p,\pi} :=$$

$$:= I_1 + I_2 + I_3.$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_{2\pi}^{p(\cdot)}$. In this case, from the boundedness of the operator S_n in $L_{2\pi}^{p(\cdot)}$ we obtain the boundedness of operator W_n in $L_{2\pi}^{p(\cdot)}$ and there holds

$$I_1 \leq \left\| f^{(\alpha)}(\cdot) - T_n^*\left(\cdot, f^{(\alpha)}\right) \right\|_{p,\pi} + \left\| T_n^*\left(\cdot, f^{(\alpha)}\right) - W_n\left(\cdot, f^{(\alpha)}\right) \right\|_{p,\pi} \leq$$

$$\leq c E_n\left(f^{(\alpha)}\right)_{p(\cdot)} + \left\| W_n\left(\cdot, T_n^*\left(f^{(\alpha)}\right) - f^{(\alpha)}\right) \right\|_{p,\pi} \leq c E_n\left(f^{(\alpha)}\right)_{p(\cdot)} .$$

From Lemma (2.2.17) we get

$$I_2 \leq c n^\alpha \left\| T_n\left(\cdot, W_n(f)\right) - T_n(\cdot, f) \right\|_{p,\pi}$$

and

$$I_3 \leq c(2n)^\alpha \| W_n(\cdot, f) - T_n(\cdot, W_n(f)) \|_{p,\pi} \leq c(2n)^\alpha E_n\left(W_n(f)\right)_{p(\cdot)} .$$

Now we have

$$\left\| T_n\left(\cdot, W_n(f)\right) - T_n(\cdot, f) \right\|_{p,\pi} \leq$$

$$\leq \left\| T_n\left(\cdot, W_n(f)\right) - W_n(\cdot, f) \right\|_{p,\pi} + \| W_n(\cdot, f) - f(\cdot) \|_{p,\pi} + \| f(\cdot) - T_n(\cdot, f) \|_{p,\pi} \leq$$

$$\leq c E_n\left(W_n(f)\right)_{p(\cdot)} + c E_n(f)_{p(\cdot)} + c E_n(f)_{p(\cdot)} .$$

Since $E_n\left(W_n(f)\right)_{p(\cdot)} \leq c E_n(f)_{p(\cdot)}$ we get

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi} \le cE_n\left(f^{(\alpha)}\right)_{p(\cdot)} + cn^{\alpha}E_n\left(W_n(f)\right)_{p(\cdot)} +$$
$$+ cn^{\alpha}E_n(f)_{p(\cdot)} + c(2n)^{\alpha}E_n\left(W_n(f)\right)_{p(\cdot)} \le cE_n\left(f^{(\alpha)}\right)_{p(\cdot)} + cn^{\alpha}E_n(f)_{p(\cdot)}.$$

Since by Theorem (2.2.2)

$$E_n(f)_{p(\cdot)} \le \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}$$

we obtain

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\pi} \le cE_n\left(f^{(\alpha)}\right)_{p(\cdot)}.$$

Theorem (2.2.11) is proved.

Let $f \in H^{p(\cdot)}(\mathbb{D})$. First of all if $p(x)$, defined on T, satisfy Dini–Lipschitz property DL_{γ} for $\gamma \ge 1$ on T, then $p\left(e^{ix}\right), x \in T$, defined on T, satisfy Dini–Lipschitz property DL_{γ} for $\gamma \ge 1$ on T. Since $H^{p(\cdot)} \subset H^1(\mathbb{D})$ for $1 < p$, let $\sum_{k=-\infty}^{\infty} \beta_k e^{ik\theta}$ be the Fourier series of the function $f\left(e^{i\theta}\right)$, and $S_n(f,\theta) :=:= \sum_{k=-n}^{n} \beta_k e^{ik\theta}$ be its nth partial sum. From $f\left(e^{i\theta}\right) \in H^1(\mathbb{D})$, we have [11, p. 38]

$$\beta_k = \begin{cases} 0, & for \;\; k < 0; \\ a_k(f), & for \; k \ge 0. \end{cases}$$

Therefore

$$\left\|f(z)-\sum_{k=0}^{n} a_k(f)z^k\right\|_{H^{p(\cdot)}} = \|f - S_n(f,\cdot)\|_{p,\pi}. \qquad (27)$$

If t_n^* is the best approximating trigonometric polynomial for $f\left(e^{i\theta}\right)$ in $L_{2\pi}^{p(\cdot)}$, then from (18), (27) and Theorem (2.2.4) we get

$$\left\|f(z)-\sum_{k=0}^{n} a_k(f)z^k\right\|_{H^{p(\cdot)}} \le \left\|f\left(e^{i\theta}\right)-t_n^*(\theta)\right\|_{p,\pi} + \|S_n(f - t_n^*,\theta)\|_{p,\pi} \le$$
$$\le cE_n\left(f\left(e^{i\theta}\right)\right)_{p(\cdot)} \le c\Omega_r\left(f\left(e^{i\theta}\right),\frac{1}{n+1}\right)_{p(\cdot)}.$$

Theorem (2.2.13) is proved.

Section (2.3): Weighted Lebesgue and Smirnov Spaces with Nonstandard Growth

For functions of weighted Lebesgue spaces $L_{\omega}^{p(.)}$ with nonstandard growth, it was proved in [135] that

$$E_n(f)_{p(.),\omega} \le c\Omega_r\left(f,\frac{1}{n+1}\right)_{p(.),\omega}, n+1, r = 1,2,3,\ldots, \qquad (28)$$

and its weak inverse

$$\Omega_r\left(f,\frac{1}{n}\right)_{p(.),\omega} \le \frac{C}{n^{2r}}\sum_{v=0}^{n}(v+1)^{2r-1}E_v(f)_{p(.)}\ , \qquad n,r=1,2,3,\dots\ , \qquad (29)$$

holds provided the weight $\omega$and the exponent $p(.)$ are such that the Hardy–Littlewood maximal operator M is bounded on the space $L_\omega^{p(x)}$, where

$$E_n(f)_{p(.),\omega} := \inf_{T\in T_n}\|f-T\|_{p(.),\omega}\ , \qquad n=0,1,2,\dots, f\ \in L_\omega^{p(.)},$$

T_n is the class of trigonometric polynomials of degree not greater than n,

$$\Omega_r(f,\delta)_{p(.),\omega} := \sup_{0\le h_i\le\delta}\left\|\prod_{i=1}^{r}(I-\sigma_{h_i})f\right\|_{p(.),\omega}\ ,$$

$$f\ \in L_\omega^{p(.)}, \qquad \delta\le 0, \qquad r=1,2,3,\dots,$$

is the modulus of smoothness of degree r ([136]), I is the identity operator and

$$\sigma_h f(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt\ \ for\ h\in\mathbb{R}\ and\ x\in T:=[-\pi,\pi]\ .$$

In equalities (28) , (29) and their several consequences were given in [135]. In the recent [135] and [136] we considered the weighted fractional moduli of smoothness, i.e. $\Omega_r(f,.)_{p,\omega}$ with $r\in(0,\infty)$, to obtain inequalities of types (28) and (29) in weighted Orlicz spaces. Fractional smoothness is not a new concept for nonweightedLebesguespaces;Butzer[143],Taberski[145],Tikhonov–Simonov[144] and Akgün–Israfilov [140] applied the fractional moduli of smoothness successfully to solve approximation problems in Lebesgue and Smirnov spaces. As a consequence of these facts, defining the weighted fractional moduli of smoothness ([138]), we consider basic approximation problems such as direct, inverse and simultaneous theorems of trigonometric approximation of functions of weighted Lebesgue spaces with variable exponent for weights satisfying a variable Muckenhoupt condition $A_{p(.)}$. Several applications of these results help us to transfer approximation results for weighted Smirnov spaces of functions defined on a finite domain with sufficiently smooth boundary.

Generalized Lebesgue spaces $L^{p(.)}$ with variable exponent (with nonstandard growth) appeared first in [136] as an example of modular spaces ([137,135]), and the corresponding Sobolev type spaces have extensive applications in fluid mechanics, differential operators ([142,138]), elasticity theory, nonlinear Dirichlet boundary value problems ([134]), nonstandard growth and variational calculus ([140]). If $p^*(T) :=$ ess $\sup_{x\in T} p(x) < \infty$, then $L^{p(.)}$ is a particular case of Musielak– Orliczspaces[135]. For a constant $p(x) := p, 1 < p < \infty$, the corresponding generalized Lebesgue spaces $L^{p(.)}$ with nonstandard growth become classical Lebesgue spaces L^p having deep approximation results. The main properties of $L^{p(.)}$ are investigated in [142]. The boundedness of classical integral transforms on $L^{p(x)}$ and weighted $L^{p(x)}$ is obtained . Let $P(T)$ be the class of Lebesgue measurable functions $p = p(x)$: $T \to (1, \infty)$ such that

$$1 < p^*(T) \coloneqq \operatorname*{ess\,inf}_{x\in T} p(x) \le p^* < \infty.$$

We define a class $L_{2\pi}^{p(.)}$ of 2π -periodic measurable functions $f : T \to \mathbb{C}$ satisfy-ing

$$\int_{-\pi+c}^{\pi+c} |f(x)|^{p(x)} dx < \infty$$

For any real number c and $p \in \mathcal{P}(T)$.

The class $L_{2\pi}^{p(.)}$ is a Banach space ([34]) with any of the following equivalent norms :

$$\|f\|_{T,p(.)} \coloneqq \inf_{\alpha>0}\left\{\int_T \left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \le 1\right\}$$

And

$$\|f\|_{T,p(.)}^{*} \coloneqq \sup_{g\in L_{2\pi}^{p'(.)}}\left\{\int_T |f(x)g(x)|dx : \int_T |g(x)|^{p'(x)} dx \le 1\right\} \qquad (30)$$

Where $p'(x) \coloneqq p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$.

Let $\omega : T \to [0, \infty]$ be a 2π periodic weight, $i.e.$, a Lebesgue measurable and $a.e.$ positive function. Denote by $L_{\omega}^{p(.)}$ the class of Lebesgue measurable functions $f : T \to \mathbb{C}$ satisfying $\omega f \to L_{2\pi}^{p(.)}$. Weighted Lebesgue spaces with nonstandard growth $L_{\omega}^{p(.)}$ are Banach spaces with the norm $\|f\|_{p(.),\omega} \coloneqq \|\omega f\|_{T,p(.)}$.

For given $p \in \mathcal{P}(T)$the class of weights ω satisfying the condition ([141])

$$\left\|\omega^{p(x)}\right\|_{A_{p(.)}} \coloneqq \sup_{B\in\mathfrak{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\|\frac{1}{\omega^{p(x)}}\right\|_{B,(p'(.)/p(.))} < \infty$$

is denoted by $A_{p(.)}(T)$. Here $p_B \coloneqq \left(\frac{1}{|B|}\int_B \frac{1}{p(x)} dx\right)^{-1}$ and $\mathfrak{B}$ is the class of all balls in T .

The variable exponent $p(x)$ is said to satisfy the local log- H$\ddot{o}$lder continuity condition if

$$|p(x_1) - p(x_2)| \le \frac{c}{\log(e + 1/|x_1 - x_2|)} \text{ forall } x_1, x_2 \in T . \qquad (31)$$

We denote by $P_{\pm}^{log}(T)$ the class of $p \in \mathcal{P}(T)$ satisfying (31).

Let $f \in L_{\omega}^{p(.)}$ and

$$A_h f(x) \coloneqq \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t)dt \;\; x_2 \in T .$$

be Steklov's mean operator. For $P_{\pm}^{log}(T)$ and $f \in L_{\omega}^{p(.)}$, it was proved in [141] that

The Hardy–Littlewood maximal function M is bounded in $L_{\omega}^{p(.)}$ if and only if

$$\omega \in A_{p(.)}(T). \qquad (32)$$

Therefore if $p \in P_{\pm}^{log}(T)$ and $\omega \in A_{p(.)}(T)$, then A_h is bounded in $L_{\omega}^{p(.)}$. Using these facts and setting $x, h \in T$, $0 \leq r$ we define via binomial expansion , for $f \in L_{\omega}^{p(.)}$,

$$\sigma_h^r f(x) := (I - A_h)^r f(x)$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2} f(x + u_1 + \cdots + u_k) du_1 \cdots du_k .$$

Since the binomial coefficients $\binom{r}{k}$ satisfy ([141, p. 14])

$$\binom{r}{k} \leq \frac{c(r)}{k^{r+1}}, \qquad k \in \mathbb{Z}^+;$$

we get

$$\sum_{k=0}^{\infty} \binom{r}{k} < \infty$$

and therefore

$$\|\sigma_h^r\|_{p(.),\omega} \leq c\|f\|_{p(.),\omega} < \infty \qquad (33)$$

Provided $p \in P_{\pm}^{log}(T), \omega \in A_{p(.)}(T)$ and $f \in L_{\omega}^{p(.)}$.

For $0 \leq r$ we can now define ([48]) the fractional moduli of smoothness of the index r for $\in P_{\pm}^{log}(T), \omega \in A_{p(.)}(T)$ and $f \in L_{\omega}^{p(.)}$ as

$$\Omega_r(f, \delta)_{p(.),\omega} := \sup_{0<h_i,t\leq\delta} \left\| \prod_{i=1}^{[r]} (I - A_{h_i}) \sigma_t^{r-[r]} f \right\|_{p(.),\omega}, \quad \delta \geq 0 \bullet$$

where $\Omega_0(f, \delta)_{p(.),\omega} := \|f\|_{p(.),\omega}, \prod_{i=1}^{0} (I - A_{h_i}) \sigma_t^r f := \sigma_t^r f$ for $0 < r < 1$,and .$[r]$ denotes the integer part of the nonnegative real number r.

We have by (33) that

$$\Omega_r(f, \delta)_{p(.),\omega} \leq c \|f\|_{p(.),\omega}$$

where $p \in P_{\pm}^{log}(T), \omega \in A_{p(.)}(T), f \in L_{\omega}^{p(.)}$ and the constant $c > 0$ depends only on r and p.

Remark (2.3.1)[133]: The modulus of smoothness $\Omega_r(f, \delta)_{p(.),\omega}, r \in \mathbb{R}^+$, has the following properties for $p \in P_{\pm}^{log}(T), \omega \in A_{p(.)}(T)$ and $f \in L_{\omega}^{p(.)}$:

(i) $\Omega_r(f, \delta)_{p(.),\omega}$ is a nonnegative and nondecreasing function of $\delta \geq 0$,

(ii) $\Omega_r(f_1 + f_2, .)_{p(.),\omega} \leq \Omega_r(f_1, .)_{p(.),\omega} + \Omega_r(f_2, .)_{p(.),\omega}$,

(iii) $\lim_{\delta \to 0^+} \Omega_r(f, \delta)_{p(.),\omega} = 0$.

If $p \in P_{\pm}^{log}(T)$ and $\omega \in A_{p(.)}(T)$, then $\omega^{p(x)} \in L^1(T)$. This implies that the set of trigonometric polynomials is dense in $L_{\omega}^{p(.)}$([142]). Therefore approximation problems make sense in $L_{\omega}^{p(.)}$. On the other hand, if $p \in P_{\pm}^{log}(T)$ and $\omega \in A_{p(.)}(T)$,then $L_{\omega}^{p(.)} \subset L^1(T)$.

For given $f \in L^1(T)$, let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k(f)e^{ikx} \qquad (34)$$

and

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty}(a_k \sin kx + b_k \cos kx)$$

be respectively the Fourier and the conjugate Fourier series of f. We set

$$L_0^1(T) := \{f \in L^1(T) c_0(f) = 0 \text{for the series in (34)}\}.$$

Let $\alpha \in \mathbb{R}^+$ be given. We define the fractional derivative of a function $f \in L_0^1(T)$ as

$$f^{(\alpha)}(x) := \sum_{k=-\infty}^{\infty} c_k(f)(ik)^{\alpha}e^{ikx}$$

provided the right-hand side, where $(ik)^{\alpha} = |k|^{\alpha}\, e^{(1/2)\pi i\alpha \operatorname{sign} k}$, exists as principalvalue. We say that a function $f \in L_{\omega}^{p(.)}$ has the fractional derivative of degree$\alpha \in \mathbb{R}^+$ if there exists a function $g \in L_{\omega}^{p(.)}$ such that its Fourier coefficients satisfy $c_k(g) = c_k(f)(ik)^{\alpha}$. In that case, we write $f^{(\alpha)} = g$.

For $p \in P(T)$ and $\delta > 0$, let $W_{p(.),\omega}^{\alpha}$ be the class of functions $f \in L_{\omega}^{p(.)}$ such that $f^{(\alpha)} \in L_{\omega}^{p(.)}$. Then $W_{p(.),\omega}^{\alpha}$ becomes a Banach space with the norm

$$\|f\|_{W_{p(.),\omega}^{\alpha}} := \|f\|_{p(.),\omega} + \left\|f^{(\alpha)}\right\|_{p(.),\omega}.$$

The main results are as follows.

Theorem (2.3.2)[133]: If $p \in P_{\pm}^{log}(T)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \; for\ some\ p_0 \in \left(1, p_*(T)\right)$$

$\alpha \in \mathbb{R}^+$ and $f \in W_{p(.),\omega}^{\alpha}$, then for every $n = 0,1,2,3,\cdots$ there exists a constant $c > 0$ independent of n such that

$$E_n(f)_{p(.),\omega} \le \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(.),\omega}$$

holds.

Corollary (2.3.3)[133]: Under the conditions of Theorem (2.3.2),

$$E_n(f)_{p(.),\omega} \le \frac{c}{(n+1)^{\alpha}} \left\|f^{(\alpha)}\right\|_{p(.),\omega}$$

with a constant $c > 0$ independent of n.

Theorem (2.3.4)[133]: If $p \in P_{\pm}^{log}(T)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \; for\ some\ p_0 \in \left(1, p_*(T)\right)$$

and $f \in L_{\omega}^{p(.)}$, then there exists a constant $c > 0$ dependent only on r and p such that

$$E_n(f)_{p(.),\omega} \leq c\,\Omega_r\left(f, \frac{1}{n+1}\right)_{p(.),\omega}$$
holds for $r \in \mathbb{R}^+$ and $n = 0,1,2,3,\cdots$.
The following inverse theorem of trigonometric approximation holds.
Theorem $(\mathbf{2.3.5})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), the inequality
$$\Omega_r\left(f, \frac{1}{n+1}\right)_{p(.),\omega} \leq \frac{c}{(n+1)^r}\sum_{v=0}^{n}(v+1)^{r-1}E_v(f)_{p(.),\omega}$$
holds for $r \in \mathbb{R}^+$ and $n = 0,1,2,3,\cdots$, where the constant $c > 0$ depends only on r and p.
Corollary $(\mathbf{2.3.6})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), if the condition
$$E_n(f)_{p(.),\omega} = \mathcal{O}(n^{-\sigma}), \qquad n = 1,2,\cdots,$$
is satisfied for some $\sigma > 0$, then
$$\Omega_r(f,\delta)_{p(.),\omega} = \begin{cases} \mathcal{O}(\delta^{\sigma}), & r > \sigma, \\ \mathcal{O}(\delta^{-\sigma}|\log(1/\delta)|), & r = \sigma, \\ \mathcal{O}(\delta^{r}), & r < \sigma, \end{cases}$$
holds for $r \in \mathbb{R}^+$.
Definition $(\mathbf{2.3.7})[\mathbf{133}]$: For $0 < \sigma < r$ we set
$$Lip\sigma(r, p(.), \omega) := \left\{f \in L_\omega^{p(.)} : \Omega_r(f,\delta)_{p(.),\omega} = \mathcal{O}(\delta^{\sigma}), \delta > 0\right\}.$$
Corollary $(\mathbf{2.3.8})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), if $0 < \sigma < r$ and
$$E_n(f)_{p(.),\omega} = \mathcal{O}(n^{-\sigma}), \;\; for \;\; n = 1,2,\cdots,$$

then $f \in Lip\sigma(r, p(.), \omega)$.
Corollary $(\mathbf{2.3.9})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), if $0 < \sigma < r$, then the following conditions are equivalent:
(a) $f \in Lip\sigma(r, p(.), \omega)$.
(b) $E_n(f)_{p(.),\omega} = \mathcal{O}(n^{-\sigma}), \qquad n = 1,2,\cdots.$
Theorem $(\mathbf{2.3.10})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), if
$$\sum_{v=1}^{\infty} v^{\alpha-1}E_v(f)_{p(.),\omega} < \infty$$
for some $\alpha \in (0,\infty)$, then $f \in W^{\alpha}_{p(.),\omega}$ And
$$E_n\left(f^{(\alpha)}\right)_{p(.),\omega} \leq c\left((n+1)^{\alpha}E_n(f)_{p(.),\omega} + \sum_{v=n+1}^{\infty} v^{\alpha-1}E_v(f)_{p(.),\omega}\right)$$
hold, where the constant $c > 0$ depends only on α and p.
The latter theorem gives rise to
Corollary $(\mathbf{2.3.11})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), if $r \in (0,\infty)$ and

$$\sum_{v=1}^{\infty} v^{\alpha-1} E_v(f)_{p(.),\omega} < \infty$$

for some $\alpha > 0$, then there exists a constant $c > 0$ depending only on α, r and p such that

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n+1}\right)_{p(.),\omega} \leq \frac{c}{(n+1)^r} \sum_{v=0}^{n} (v+1)^{\alpha+r-1} E_v(f)_{p(.),\omega} + c \sum_{v=n+1}^{\infty} v^{\alpha-1} E_v(f)_{p(.),\omega}$$

holds.

The following simultaneous approximation theorem is valid.

Theorem $(\mathbf{2.3.12})[\mathbf{133}]$: If $p \in P_{\pm}^{\log}(T)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \ for\ some\ p_0 \in \left(1, p_*(T)\right),$$

$\alpha \in [0, \infty)$, and $f \in W_{p(.),\omega}^{\alpha}$, then there exist $T \in T_n, n = 1,2,3, \cdots$, and a constant $c > 0$ depending only on α and p such that

$$\left\|f^{(\alpha)} - T^{(\alpha)}\right\|_{p(.),\omega} \leq cE_n\left(f^{(\alpha)}\right)_{p(.),\omega}$$

holds.

Theorem $(\mathbf{2.3.13})[\mathbf{133}]$: If $p \in P_{\pm}^{\log}(T)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \ for\ some\ p_0 \in (1, p_*),$$

f belongs to the Hardy space $H^{p(.)}$ with a variable exponent on the unit circumference $\mathbb{D}$ and $r \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that

$$\left\|f(z) - \sum_{k=0}^{n} \eta_k(f) z^k\right\|_{p(.),\omega} \leq c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(.),\omega}, \quad n = 0,1,2, \ldots,$$

where $\eta_k(f), k = 0,1,2,3, \ldots$, are the Taylor coefficients of f at the origin.

We begin with

Lemma $(\mathbf{2.3.14})[\mathbf{133}]$: ([137]). For $\alpha \in \mathbb{R}^+$ we suppose that

(i) $a_1 + a_2 + \cdots + a_n + \cdots$

and

$(ii)\ a_1 + 2^\alpha a_2 + \cdots + n^\alpha a_n + \cdots$

are two series in the Banach space .$(B, \|.\|)$. Let

$$R_n^{\langle a\rangle} := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{a}\right) a_k$$

and

$$R_n^{\langle a\rangle *} := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{a}\right) k^\alpha\, a_k$$

for $n = 1,2, \dots$. Then

$$\left\| R_n^{\langle a\rangle *} \right\| \le c, \qquad n = 1,2, \dots,$$

for some $c > 0$ if and only if there exists $R \in B$ such that

$$\left\| R_n^{\langle a\rangle} - R \right\| \le \frac{C}{n^\alpha},$$

where c and C are constants depending only on each other.
Putting $A_k(x) := c_k(f) e^{ikx}$ in (34), we define

$$S_n(f) := S_n(x,f) := \sum_{k=0}^{n} \big(A_k(x) + A_{-k}(x)\big) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 0,1,2,\cdots,$$

$$R_n^{\langle \alpha\rangle}(f,x) := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right) \big(A_k(x) + A_{-k}(x)\big)$$

and

$$\Theta_m^{\langle \alpha\rangle} := \frac{1}{1 - \left(\frac{m+1}{2m+1}\right)^{\alpha}} R_{2m}^{\langle \alpha\rangle} - \frac{1}{\left(\frac{2m+1}{m+1}\right)^{\alpha} - 1} R_m^{\langle \alpha\rangle}\ for\ m = 1,2,3,\cdots. \qquad (35)$$

Lemma $(\mathbf{2.3.15})[\mathbf{133}]$: Under the conditions of Theorem (2.3.4), there are constants $c, C > 0$ such that

$$\|\tilde{f}\|_{p(.),\omega} \le c\|f\|_{p(.),\omega} \qquad (36)$$

And

$$\|S_n(.,f)\|_{p(.),\omega} \le C\|f\|_{p(.),\omega} \quad for\ n = 1,2,\cdots \qquad (37)$$

hold.

Proof: Let $S_*(f) := S_*(f,x) := \sup_{k\geq 0}|S_k(f,x)|.f;x/j$. Then using Theorem 4.16 of [143] we obtain

$$\|S_n(.,f)\|_{p(.),\omega} \leq \|S_*(f)\|_{p(.),\omega} \leq C\|f\|_{p(.),\omega}.$$

For (36) we use extrapolation Theorem 3.2 of [33]. For any $p > 1$ we have ([18])

$$\|\tilde{f}\|_{p,\omega} \leq c\|f\|_{p,\omega}$$

and[143,Theorem3.2, (3.3)] is satisfied for $p = p_0 = q_0$ and $q(x) = p(x)$.Therefore (36) holds,

$$\|\tilde{f}\|_{p(.),\omega} \leq c\|f\|_{p(.),\omega} .$$

Remark$(\mathbf{2.3.16})$[133]:Under the conditions of Theorem (2.3.4), it can be easily seen from(36) and (37) that there exists a constant $c > 0$ such that

$$\|f - S_n(.,f)\|_{p(.),\omega} \leq cE_n(f)_{p(.),\omega} \asymp E_n(\tilde{f})_{p(.),\omega}.$$

Under the conditions of Theorem (2.3.4), using (37)and the Abel transform , we get

$$\left\|R_n^{\langle\alpha\rangle}(f,x)\right\|_{p(.),\omega} \leq c\|f\|_{p(.),\omega}\ for\ n = 1,2,3,\cdots, x \in T, f \in L_\omega^{p(.)} \quad (38)$$

and therefore (35) and (38) imply

$$\left\|\Theta_m^{\langle\alpha\rangle}(f,x)\right\|_{p(.),\omega} \leq c\|f\|_{p(.),\omega}\ for\ m = 1,2,3,\cdots, x \in T, f \in L_\omega^{p(.)} .$$

From the property

$$\Theta_m^{\langle\alpha\rangle}(f)(x) = \frac{1}{\sum_{k=m+1}^{2m}[(k+1)^\alpha - k^\alpha]}\sum_{k=m+1}^{2m}[(k+1)^\alpha - k^\alpha]S_k(x,f)\ for\ x \in T, f \in L^1$$

it follows that

$$\Theta_m^{\langle\alpha\rangle}(T_m) = T_m, \quad (39)$$

where $T_m \in \mathcal{T}_m$ for $m = 1,2,3,\cdots$.

Lemma $(\mathbf{2.3.17})$[133]: Under the conditions of Theorem (2.3.4), if $T_n \in \mathcal{T}_n$ and $\alpha \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that

$$\left\|T_n^{\langle\alpha\rangle}\right\|_{p(.),\omega} \leq cn^\alpha\|T_n\|_{p(.),\omega}$$

holds.

Proof: Without loss of generality one can assume that $\|T_n\|_{p(.),\omega} = 1$. Since

$$T_n = \sum_{k=0}^{n}\big(A_k(x) + A_{-k}(x)\big), \frac{\tilde{T}_n}{n^\alpha} = \sum_{k=1}^{n}\big[\big(A_k(x) - A_{-k}(x)\big)/n^\alpha\big]$$

and

$$\frac{T_n^{\langle\alpha\rangle}}{(in)^\alpha} = \sum_{k=0}^{n}k^\alpha\big[\big(A_k(x) - A_{-k}(x)\big)/n^\alpha\big] ,$$

we have by (38) and (36) that

$$\left\|R_m^{\langle\alpha\rangle}\left(\frac{\tilde{T}_n}{n^\alpha}\right)\right\|_{p(.),\omega} \leq \frac{c}{n^\alpha}\left\|\tilde{T}_n\right\|_{p(.),\omega} \leq \frac{c}{n^\alpha}\|T_n\|_{p(.),\omega} = \frac{c}{n^\alpha}$$

and by Lemma (2.3.14)

$$\left\|R_m^{\langle\alpha\rangle}\left(\frac{T_n^{\langle\alpha\rangle}}{(in)^\alpha}\right)\right\|_{p(.),\omega} \leq c.$$

Hence by (39)

$$\left\|T_n^{\langle\alpha\rangle}\right\|_{p(.),\omega} = n^\alpha\left\|\Theta_m^{\langle\alpha\rangle}\left(\frac{T_n^{\langle\alpha\rangle}}{(in)^\alpha}\right)\right\|_{p(.),\omega} \leq cn^\alpha\|T_n\|_{p(.),\omega}.$$

A general case follows immediately from this.

Lemma (2.3.18)[133]: If $p \in P_{\pm}^{log}(T)$, $\omega \in A_{p(.)}(T)$ and $f \in W_{p(.)}^2$, then there exists aconstant $c > 0$ such that for $r = 1,2,3,\cdots$ and $\delta \geq 0$

$$\Omega_r(f,\delta)_{p(.),\omega} \leq c\delta^2\Omega_{r-1}(f'',\delta)_{p(.),\omega}$$

holds.

Proof: Putting

$$g(x) := \prod_{i=2}^{r}\left(I - A_{h_i}\right)f(x),$$

we have

$$\left(I - A_{h_1}\right)g(x) = \prod_{i=1}^{r}\left(I - A_{h_i}\right)f(x)$$

And

$$\prod_{i=1}^{r}\left(I - A_{h_i}\right)f(x) = \frac{1}{h_1}\int_{-h_1/2}^{h_1/2}\left(g(x) - g(x+t)\right)dt$$

$$= -\frac{1}{2h_1}\int_0^{h_1/2}\int_0^{2t}\int_{-u/2}^{u/2}g''(x+s)\,ds\,du\,dt.$$

Therefore from (30)

$$\left\|\prod_{i=1}^{r}\left(I-A_{h_i}\right)f(x)\right\|_{p(.),\omega}$$

$$\leq \frac{c}{2h_1}\sup_{g_0\in L_{2\pi}^{p'(.)}}\left\{\int_T \left|\int_0^{h_1/2}\int_0^{2t}\int_{-u/2}^{u/2} g''(x+s)ds\,du\,dt\right|\omega(x)|g_0(x)|dx\right.$$

$$\left.:\int_T |g_0(x)|^{p'(x)}dx\leq 1\right\}$$

$$\leq \frac{c}{2h_1}\int_0^{h_1/2}\int_0^{2t} u\left\|\frac{1}{u}\int_{-u/2}^{u/2} g''(x+s)ds\right\|_{p(.),\omega} du\,dt$$

$$\leq \frac{c}{2h_1}\int_0^{h_1/2}\int_0^{2t} u\|g''\|_{p(.),\omega}du\,dt = Ch_1^2\|g''\|_{p(.),\omega}\,.$$

Since

$$g''^{(x)} = \prod_{i=2}^{r}\left(I-A_{h_i}\right)f''(x)\,,$$

we obtain

$$\Omega_r(f,\delta)_{p(.),\omega}\leq C\sup_{\substack{0<h_i\leq\delta\\ i=1,2,\dots,r}} h_1^2\|g''\|_{p(.),\omega}$$

$$= c\delta^2\sup_{\substack{0<h_i\leq\delta\\ i=2,3,\dots,r}}\left\|\prod_{i=2}^{r}\left(I-A_{h_i}\right)f''(x)\right\|_{p(.),\omega}$$

$$= c\delta^2\sup_{\substack{0<h_i\leq\delta\\ j=1,2,\dots,r-1}}\left\|\prod_{i=2}^{r-1}\left(I-A_{h_i}\right)f''(x)\right\|_{p(.),\omega}$$

$$= C\delta^2\Omega_{r-1}(f'',\delta)_{p(.),\omega}$$

and Lemma (2.3.18) is proved.

Corollary $(\mathbf{2.3.19})[\mathbf{133}]$: If $p\in P_{\pm}^{log}(T)$, $\omega\in A_{p(.)}(T)$, $r=1,2,3,\dots$, and $f\in W_{p(.),\omega}^{2r}$,then there exists a constant $c > 0$ depending only on r and p such that

$$\Omega_r(f\,,\delta)_{p(.),\omega}\leq c\delta^{2r}\left\|f^{(2r)}\right\|_{p(.),\omega}$$

holds for $\delta\leq 0$.

Lemma $(\mathbf{2.3.20})[\mathbf{133}]$: If $p \in P_{\pm}^{log}(T), \omega \in A_{p(.)}(T), n = 0,1,2,\dots, T_n \in T_n$ and $r \in \mathbb{R}^+$,then there exists a constant $c > 0$ depending only on r and p such that

$$\Omega_r\left(T_n, \frac{1}{n+1}\right)_{p(.),\omega} \leq \frac{c}{(n+1)^r}\left\|T_n^{(r)}\right\|_{p(.),\omega}$$

holds.

Proof: First we prove that if $0 < \alpha < \beta, \alpha, \beta \in \mathbb{R}^+$, then

⍰$\Omega_\beta(f\,,.)_{p(.),\omega} \leq c$⍰$\Omega_\alpha(f\,,.)_{p(.),\omega}$. (40)

It is easily seen that if $\alpha \leq \beta, \alpha, \beta \in \mathbb{Z}^+$, then

⍰$\Omega_\beta(f\,,.)_{p(.),\omega} \leq c(\alpha,\beta,p)\Omega_\alpha(f\,,.)_{p(.),\omega}$. (41)

Now, we assume that $0 < \alpha < \beta < 1$. In that case, putting

$$\Phi(x) \coloneqq \sigma_h^\alpha f(x)$$

we have

$$\sigma_h^{\beta-\alpha}\Phi(x)$$

$$= \sum_{j=0}^{\infty}(-1)^j\binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}\Phi\big(x+u_1+\dots+u_j\big)\,du_1\dots du_j$$

$$= \sum_{j=0}^{\infty}(-1)^j\binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}$$

$$\times\left[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\frac{1}{h^k}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}f\big(x+u_1+\dots+u_j+u_{j+1}+\dots\right.$$

$$\left.+u_{j+k}\big)du_1\dots du_j du_{j+1}\dots du_{j+k}\right]$$

$$= \sum_{j=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+k}\binom{\beta-\alpha}{j}\binom{\alpha}{k}$$

$$\times\left[\frac{1}{h^{j+k}}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}f\big(x+u_1+\dots+u_{j+k}\big)du_1\dots du_{j+k}\right]$$

$$= \sum_{v=0}^{\infty}(-1)^v\binom{\beta}{v}\frac{1}{h^v}\int_{-h/2}^{h/2}\dots\int_{-h/2}^{h/2}f(x+u_1+\dots+u_v)du_1\dots du_v$$

$$= \sigma_h^\beta f(x) \quad a.e.$$

Then by (33)

$$\left\|\sigma_h^\beta f(.)\right\|_{p(.),\omega}$$

and

$$\Omega_\beta(f\ ,.)_{p(.),\omega} \le\ c\Omega_\alpha(f\ ,.)_{p(.),\omega}\ . \tag{42}$$

We note that if $r_1, r_2\ \in \mathbb{Z}^+, \alpha_1, \beta_1 \in\ (0,1)$, taking $\alpha \coloneqq r_1 + \alpha_1, \beta \coloneqq r_2 + \beta_1$ for the remaining cases $r_1 = r_2, \alpha_1 < \beta_1$ or $r_1 <\ r_2, \alpha_1 = \beta_1$ or $r_1 <\ r_2, \alpha_1 < \beta_1$, it can be easily obtained from (41) and (42) that the required inequality (40) holds.

Using (40), Corollary (2.3.19) and Lemma (2.3.17), we get

$$\begin{aligned}\Omega_r\left(T_n, \frac{1}{n+1}\right)_{p(.),\omega} &\le c\Omega_{[r]}\left(T_n, \frac{1}{n+1}\right)_{p(.),\omega}\\ &\le c\left(\frac{1}{n+1}\right)^{2[r]}\left\|T_n^{(2[r])}\right\|_{p(.),\omega}\\ &\le \frac{c}{(n+1)^{2[r]}}(n+1)^{[r]-(r-[r])}\left\|T_n^{(r)}\right\|_{p(.),\omega}\\ &= \frac{c}{(n+1)^r}\left\|T_n^{(r)}\right\|_{p(.),\omega}\end{aligned}$$

which is the required result.

Definition (2.3.21)[133]: For $p\ \in\ P(T)\ , f\ \in L_\omega^{p(.)}\ , \delta >\ 0$ and $r = 1,2,3, \dots$ the PeetreK-functional is defined as

$$K\left(\delta, f; L_\omega^{p(.)}, W_{p(.),\omega}^r\right) \coloneqq \inf_{g\in W_{p(.),\omega}^r}\left\{\|f-g\|_{p(.),\omega} + \delta\left\|g^{(r)}\right\|_{p(.),\omega}\right\}. \tag{43}$$

Theorem (2.3.22)[133]: If $p \in P_\pm^{log}(T)\ , \omega \in A_{p(.)}(T), r = 1,2,3, \dots$,and $f \in L_\omega^{p(.)}$, then $K\left(\delta^{2r}, f; L_{2\pi}^{p(.)}, W_{p(.),\omega}^{2r}\right)$ in (43)and the modulus $\Omega_r(f,\delta)_{p(.),\omega}$ are equivalent.

Proof: If $h \in W_{p(.),\omega}^{2r}$, then we have by Corollary (2.3.19) and (43) that

$$\begin{aligned}\Omega_r(f,\delta)_{p(.),\omega} &\le c\|f-h\|_{p(.),\omega} + \delta^{2r}\left\|h^{(2r)}\right\|_{p(.),\omega}\\ &\le K\left(\delta^{2r}, f; L_{2\pi}^{p(.)}, W_{p(.),\omega}^{2r}\right).\end{aligned}$$

Putting

$$(L_\delta f)\ (x) \coloneqq\ 3\delta^{-3}\int_0^{\delta/2}\int_0^{2t}\int_{-u/2}^{u/2} g''(x+s)\,ds\,dudt\quad for\ x \in\ T\ ,$$

we have

$$\frac{d^r}{dx^r}L_\delta f\ =\ \frac{c}{\delta^2}(I -\ A_\delta)f$$

and hence

$$\frac{d^{2r}}{dx^{2r}}\ L_\delta^r f = \frac{c}{\delta^{2r}}\ (I -\ A_\delta)^r\ for\ r = 1,2,3, \dots.$$

On the other hand, we find

$$\|L_\delta f\|_{p(.),\omega} \leq 3\delta^{-3}\int_0^\delta \int_0^{2t} u\|A_u f\|_{p(.),\omega}\ dudt\ \leq\ c\|f\|_{p(.),\omega}.$$

Now, let $F_\delta^r := I-(I-L_\delta^r)^r$. Then $F_\delta^r f \in W_{p(.),\omega}^{2r}$ and

$$\left\|\frac{d^{2r}}{dx^{2r}} F_\delta^r f\right\|_{p(.),\omega} \leq c\left\|\frac{d^{2r}}{dx^{2r}} L_\delta^r f\right\|_{p(.),\omega} = \frac{c}{\delta^{2r}}\|(I-A_\delta)^r f\|_{p(.),\omega}$$

$$\leq \frac{c}{\delta^{2r}}\Omega_r(f,\delta)_{p(.),\omega}.$$

Since

$$I-L_\delta^r = (I-L_\delta)\sum_{j=0}^{r-1} L_\delta^j\ ,$$

we get

$$\begin{aligned}\|(I-L_\delta^r)g\|_{p(.),\omega} &\leq c\|(I-L_\delta)g\|_{p(.),\omega}\\ &\leq 3c\delta^{-3}\int_0^\delta\int_0^{2t} u\|(I-A_u)g\|_{p(.),\omega}\,dudt\\ &\leq c\sup_{0<u\leq\delta}\|(I-A_u)g\|_{p(.),\omega}\end{aligned}$$

Taking into account
$\|f-F_\delta^r f\|_{p(.),\omega} = \|(I-L_\delta^r)^r f\|_{p(.),\omega}$,

by a recursive procedure we obtain

$$\|f-F_\delta^r f\|_{p(.),\omega} \leq c\sup\nolimits_{0<t_1\leq\delta}\left\|\left(I-A_{t_1}\right)(I-L_\delta^r)^{r-1}f\right\|_{p(.),\omega}$$

$$\leq c\sup_{0<t_1\leq\delta}\ \sup_{0<t_2\leq\delta}\left\|\left(I-A_{t_1}\right)\left(I-A_{t_2}\right)(I-L_\delta^r)^{r-1}f\right\|_{p(.),\omega}$$

$$\vdots$$

$$\leq C\sup_{\substack{0<t_i\leq\delta\\ i=1,2,\dots,r}}\left\|\prod_{i=1}^r\left(I-A_{t_i}\right)f(.)\right\|_{p(.),\omega} = C\Omega_r(f,\delta)_{p(.),\omega}$$

and the proof is completed.

First of all we note that by (32) and Theorem 3.2 of [143], the condition

$$''\omega^{-p_0} \in\in A_{\left(\frac{p(.)}{p_0}\right)'}(T)\ for\ some\ p_0 \in \left(1, p_*(T)\right)''$$

implies that $\omega \in A_{p(.)}(T)$. We set $A_k(x,f) := a_k \cos kx + b_k \sin kx$. Since the set of trigonometric polynomials is dense in $L_\omega^{p(.)}$, for given $f \in L_\omega^{p(.)}$ we have

$$E_n(f)_{p(.),\omega} \to 0 \; as \; n \to \infty.$$

By the first inequality in Remark (2.3.16) we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x,f)$$

in $\|.\|_{p(.),\omega}$ norm. For $k = 1,2,3,\ldots$ we know that

$$\begin{aligned} A_k(x,f) &= a_k \cos k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) + b_k \sin k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) \\ &= A_k\left(x + \frac{\alpha\pi}{2k}, f\right)\cos\frac{\alpha\pi}{2} + A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right)\sin\frac{\alpha\pi}{2} \end{aligned}$$

and

$$A_k\left(x, f^{(\alpha)}\right) = k^{\alpha} A_k\left(x + \frac{\alpha\pi}{2k}, f\right).$$

Therefore

$$\begin{aligned} \sum_{k=0}^{\infty} A_k(x,f) &= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, f\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} A_k\left(x + \frac{\alpha\pi}{2k}, \tilde{f}\right) \\ &= A_0(x,f) + \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right) \end{aligned}$$

and hence

$$f(x) - S_n(x,f) = \cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha}} A_k\left(x, f^{(\alpha)}\right) + \sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha}} A_k\left(x, \tilde{f}^{(\alpha)}\right).$$

Since

$$\begin{aligned} \sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, f^{(\alpha)}\right) &= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[\left(S_k\left(., f^{(\alpha)}\right) - f^{(\alpha)}(.)\right) - \left(S_{k-1}\left(., f^{(\alpha)}\right) - f^{(\alpha)}(.)\right)\right] \\ &= \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(., f^{(\alpha)}\right) - f^{(\alpha)}(.)\right) \\ &\quad - (n+1)^{-\alpha}\left(S_n\left(., f^{(\alpha)}\right) - f^{(\alpha)}(.)\right) \end{aligned}$$

and

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right)$$
$$= \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left(S_k\left(.\,, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(.)\right) - (n+1)^{-\alpha} \left(S_n\left(.\,, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(.)\right)$$

we obtain

$\|f(.) - S_n(.\,, f)\|_{p(.),\omega} \le \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\|S_k\left(.\,, f^{(\alpha)}\right) - f^{(\alpha)}(.)\right\|_{p(.),\omega} + (n+1)^{-\alpha} \left\|S_n\left(.\,, f^{(\alpha)}\right) - f^{(\alpha)}(.)\right\|_{p(.),\omega} + \sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) \left\|S_k\left(.\,, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(.)\right\|_{p(.),\omega} + (n+1)^{-\alpha} \left\|S_n\left(.\,, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(.)\right\|_{p(.),\omega} \le c \left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) E_k\left(f^{(\alpha)}\right)_{p(.),\omega} + (n+1)^{-\alpha} E_n\left(f^{(\alpha)}\right)_{p(.),\omega}\right]$ Consequently, from the equivalence in Remark (2.3.16) we have

$$\|f(.) - S_n(.\,, f)\|_{p(.),\omega} \le c \left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\right] \times \left\{E_k\left(f^{(\alpha)}\right)_{p(.),\omega} + E_n\left(\tilde{f}^{(\alpha)}\right)_{p(.),\omega}\right\}$$
$$\le c E_n\left(f^{(\alpha)}\right)_{p(.),\omega} \left[\sum_{k=n+1}^{\infty} (k^{-\alpha} - (k+1)^{-\alpha}) + (n+1)^{-\alpha}\right]$$
$$\le \frac{c}{(n+1)^{-\alpha}} E_n\left(f^{(\alpha)}\right)_{p(.),\omega}$$

and Theorem (2.3.2) is proved.

First we give the proof for $r \in \mathbb{Z}^+$. In case $g \in W_{p(.),\omega}^{2r}$ we have by Corollary (2.3.19), (43) and Theorem (2.3.22) that

$$E_n(f)_{p(.),\omega} \le E_n(f-g)_{p(.),\omega} + E_n(g)_{p(.),\omega}$$
$$\le c\left[\|f-g\|_{p(.),\omega} + (n+1)^{-2r} \left\|g^{(2r)}\right\|_{p(.),\omega}\right]$$
$$\le cK\left((n+1)^{-2r}, f; L_\omega^{p(.)}, W_{p(.),\omega}^{2r}\right) \le c\Omega_r \left(f, \frac{1}{n+1}\right)_{p(.),\omega}$$

as required for any $r \in \mathbb{Z}^+$. Therefore by the last inequality and (40) we get

$$E_n(f)_{p(.),\omega} \le c\Omega_{[r]+1} \left(f, \frac{1}{n+1}\right)_{p(.),\omega} \le c\Omega_r \left(f, \frac{1}{n+1}\right)_{p(.),\omega}, \qquad n = 0,1,2,3,\cdots,$$

and the assertion follows for general $r > 0$.

Let $T_n \in T_n$ be the best approximating polynomial of the function $f \in L_\omega^{p(.)}$ and let $m \in \mathbb{Z}^+$. Then by Remark (2.3.1)(ii)

$$\Omega_r\left(f,\frac{1}{n+1}\right)_{p(.),\omega} \le \Omega_r\left(f-T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega} + \Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega}$$
$$\le cE_{2^m}(f)_{p(.),\omega} + \Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega}.$$

By Lemma (2.3.20) we have

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega} \le c\left(\frac{1}{n+1}\right)^r \left\|T_{2^m}^{(r)}\right\|_{p(.),\omega}.$$

Since

$$T_{2^m}^{(r)}(.) = T_1^{(r)}(.) + \sum_{v=0}^{m-1}\left\{T_{2^{v+1}}^{(r)}(.) - T_{2^v}^{(r)}(.)\right\}$$

we get

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega} \le \frac{c}{(n+1)^r}\left\{\left\|T_1^{(r)}\right\|_{p(.),\omega} + \sum_{v=0}^{m-1}\left\|T_{2^{v+1}}^{(r)} - T_{2^v}^{(r)}\right\|_{p(.),\omega}\right\}.$$

Lemma (2.3.17) gives

$$\left\|T_{2^{v+1}}^{(r)} - T_{2^v}^{(r)}\right\|_{p(.),\omega} \le c2^{vr}\left\|T_{2^{v+1}} - T_{2^v}\right\|_{p(.),\omega}$$
$$\le c2^{vr+1}E_{2^v}(f)_{p(.),\omega}$$

and

$$\left\|T_1^{(r)}\right\|_{p(.),\omega} = \left\|T_1^{(r)} - T_0^{(r)}\right\|_{p(.),\omega} \le cE_0(f)_{p(.),\omega}.$$

Hence

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega} \le \frac{c}{(n+1)^r}\left\{E_0(f)_{p(.),\omega} + \sum_{v=0}^{m-1} 2^{(v+1)r}\, E_{2^v}(f)_{p(.),\omega}\right\}.$$

It is easy to see that

$$2^{(v+1)r}E_{2^v}(f)_{p(.),\omega} \le c^* \sum_{\mu=2^{v-1}+1}^{2^v} \mu^{r-1}\, E_\mu(f)_{p(.),\omega}, \quad v = 1,2,3,\dots,$$

where

$$c^* = \begin{cases} 2^{r+1}, & 0 < r < 1, \\ 2^{2r}, & r \ge 1. \end{cases}$$

Therefore

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega}$$

$$\leq \frac{c}{(n+1)^r}\left\{E_0(f)_{p(.),\omega}+2^rE_1(f)_{p(.),\omega}+c\sum_{v=1}^{m}\sum_{\mu=2^{v-1}+1}^{2^v}\mu^{r-1}E_\mu(f)_{p(.),\omega}\right\}$$

$$\leq \frac{c}{(n+1)^r}\left\{E_0(f)_{p(.),\omega}+\sum_{\mu=1}^{2^m}\mu^{r-1}E_\mu(f)_{p(.),\omega}\right\}$$

$$\leq \frac{c}{(n+1)^r}\sum_{v=0}^{2^m-1}(v+1)^{r-1}E_v(f)_{p(.),\omega}.$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_r\left(T_{2^m},\frac{1}{n+1}\right)_{p(.),\omega} \leq \frac{c}{(n+1)^r}\sum_{v=0}^{n}(v+1)^{r-1}E_v(f)_{p(.),\omega},$$

$$E_{2^m}(f)_{p(.),\omega} \leq E_{2^{m-1}}(f)_{p(.),\omega}$$

$$\leq \frac{c}{(n+1)^r}\sum_{v=0}^{n}(v+1)^{r-1}E_v(f)_{p(.),\omega}.$$

the last two inequalities complete the proof.

For the polynomial T_n of the best trigonometric approximation for $f \in L_\omega^{p(.)}$ we have

$$\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),\omega} \leq 2E_{2^i}(f)_{p(.),\omega}$$

and from Lemma (2.3.17) it follows that

$$\left\|T_{2^{i+1}}^{(a)}-T_{2^i}^{(a)}\right\|_{p(.),\omega} \leq c2^{(i+1)\alpha}E_{2^i}(f)_{p(.),\omega}.$$

Hence

$$\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{W_{p(.),\omega}^{\alpha}} = \sum_{i=1}^{\infty}\left\|T_{2^{i+1}}^{(a)}-T_{2^i}^{(a)}\right\|_{p(.),\omega}+\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p(.),\omega}$$

$$\leq c\sum_{m=2}^{\infty}m^{\alpha-1}E_m(f)_{p(.),\omega} < \infty.$$

Therefore

$$\left\|T_{2^{i+1}}-T_{2^i}\right\|_{W_{p(.),\omega}^{\alpha}} \to 0 \text{ } as \text{ } i \to \infty.$$

This means that $\{T_{2^i}\}$is a Cauchy sequence in $L_\omega^{p(.)}$. Since $T_{2^i} \to f$ in $L_\omega^{p(.)}$ and $W_{p(.),\omega}^{\alpha}$is a Banach space, we obtain $f \in W_{p(.),\omega}^{\alpha}$.On the other hand, since

$$\|f^{(a)} - S_n(f^{(a)})\|_{p(.),\omega} \leq \|S_{2^{m+2}}(f^{(a)}) - S_n(f^{(a)})\|_{p(.),\omega}$$

$$+ \sum_{k=m+2}^{\infty} \|S_{2^{k+2}}(f^{(a)}) - S_{2^k}(f^{(a)})\|_{p(.),\omega},$$

we have for $2^m < n < 2^{m+1}$

$$\|S_{2^{m+2}}(f^{(a)}) - S_n(f^{(a)})\|_{p(.),\omega} \leq c2^{(m+2)\alpha} E_n(f)_{p(.),\omega}$$

$$\leq c2^{(m+2)\alpha} E_n(f)_{p(.),\omega}.$$

Thus, we find

$$\sum_{k=m+2}^{\infty} \|S_{2^{k+2}}(f^{(a)}) - S_{2^k}(f^{(a)})\|_{p(.),\omega}$$

$$\leq c \sum_{k=m+2}^{\infty} 2^{(k+1)\alpha} E_{2^k}(f)_{p(.),\omega}$$

$$\leq c \sum_{k=m+2}^{\infty} \sum_{\mu=2^{k-1}+1}^{2^k} \mu^{a-1} E_\mu(f)_{p(.),\omega}$$

$$= c \sum_{v=2^{m+1}+1}^{\infty} v^{a-1} E_v(f)_{p(.),\omega}$$

$$\leq c \sum_{v=n+1}^{\infty} v^{a-1} E_v(f)_{p(.),\omega}$$

and Theorem (2.3.10) is proved.

In the case of $\alpha = 0$ the result follows from Remark (2.3.16) and the property $S_n(f) \in T_n$:

$$\|f - S_n(f)\|_{p(.),\omega} \leq cE_n(f)_{p(.),\omega}.$$

For $\alpha > 0$ we set

$$W_n(f) \coloneqq W_n(x,f) \coloneqq \frac{1}{n+1} \sum_{v=n}^{2n} S_v(x,f) \quad for\ n = 0,1,2, \ldots.$$

Since

$$W_n(.,f^{(a)}) := W_n^{(a)}(.,f),$$

we have

$$\left\|f^{(a)}(.) - S_n^{(a)}(.,f)\right\|_{p(.),\omega} \le \|f^{(a)}(.) - W_n(.,f^{(a)})\|_{p(.),\omega}$$
$$+ \left\|S_n^{(a)}(.,W_n(f)) - S_n^{(a)}(.,f)\right\|_{p(.),\omega}$$
$$+ \left\|W_n^{(a)}(.,f) - S_n^{(a)}(.,W_n(f))\right\|_{p(.),\omega}$$
$$:= I_1 + I_2 + I_3.$$

In this case, from the boundedness of the operator S_n in $L_\omega^{p(.)}$ we obtain the boundedness of the operator W_n in $L_\omega^{p(.)}$ and there holds

$$I_1 \le \|f^{(a)}(.) - S_n(.,f^{(a)})\|_{p(.),\omega} + \|S_n(.,f^{(a)}) - W_n(.,f^{(a)})\|_{p(.),\omega}$$

$$\le cE_n(f^{(a)})_{p(.),\omega} + \|W_n(.,S_n(.,f^{(a)}) - f^{(a)})\|_{p(.),\omega}$$

$$\le cE_n(f^{(a)})_{p(.),\omega}.$$

From Lemma (2.3.17) we get

$$I_2 \le cn^\alpha \|S_n(.,W_n(f)) - S_n(.,f)\|_{p(.),\omega}$$

and

$$I_3 \le c(2n)^\alpha \|W_n(.,f) - S_n(.,W_n(f))\|_{p(.),\omega} \le c(2n)^\alpha E_n(W_n(f))_{p(.),\omega}.$$

Now we have

$$\|S_n(.,W_n(f)) - S_n(.,f)\|_{p(.),\omega} \le \|S_n(.,W_n(f)) - W_n(.,f)\|_{p(.),\omega}$$
$$+\|W_n(.,f) - f(.)\|_{p(.),\omega}$$
$$+\|f(.) - S_n(.,f)\|_{p(.),\omega}$$
$$\le cE_n(W_n(f))_{p(.),\omega} + cE_n(f)_{p(.),\omega}$$
$$+cE_n(f)_{p(.),\omega}.$$

Since

$$E_n(W_n(f))_{p(.),\omega} + cE_n(f)_{p(.),\omega},$$

we get

$$\left\| f^{(a)}(.) - S_n^{(a)}(.,f) \right\|_{p(.),\omega} \leq cE_n(f^{(a)})_{p(.),\omega} + cn^{\alpha}E_n(W_n(f))_{p(.),\omega}$$

$$+cn^{\alpha}E_n(f)_{p(.),\omega}$$
$$+c(2n)^{\alpha}E_n(W_n(f))_{p(.),\omega}$$
$$\leq cE_n(f^{(a)})_{p(.),\omega} + Cn^{\alpha}E_n(f)_{p(.),\omega}.$$

Since by Theorem (2.3.2)

$$E_n(f)_{p(.),\omega} \leq \frac{c}{(n+1)^{\alpha}} E_n(f^{(a)})_{p(.),\omega},$$

we obtain

$$\left\| f^{(a)}(.) - S_n^{(a)}(.,f) \right\|_{p(.),\omega} \leq cE_n(f^{(a)})_{p(.),\omega}$$

and the proof is completed.

Let $\sum_{k=-\infty}^{\infty} c_k(g)e^{ik\theta}$ be the Fourier series of the bound-ary function g of $f \in H^{p(.)}(\mathbb{D})$, and $S_n(g,\theta) \coloneqq \sum_{k=-n}^{n} c_k(g)e^{ik\theta}$ be its n th partial sum. Since $g \in H^1(\mathbb{D})$, we have $([13, p. 38])$

$$c_k(g) = \begin{cases} 0 & k < 0, \\ \eta_k(f) & for\ k \geq 0. \end{cases}$$

Therefore

$$\left\| f(z) - \sum_{k=0}^{n} \eta_k(f)z^k \right\|_{p(.),\omega} = \|g - S_n(g,.)\|_{p(.),\omega} \qquad (44)$$

If t_n^* is the best approximating trigonometric polynomial for g in $L_{\omega}^{p(.)}$, thenfrom $(37), (44)$ and Theorem (2.3.4) we get

$$\left\| f(z) - \sum_{k=0}^{n} \eta_k(f)z^k \right\|_{p(.),\omega} \leq \|g - t_n^*\|_{p(.),\omega} + \|S_n(g - t_n^*,.)\|_{p(.),\omega}$$

$$\leq cE_n(g)_{p(.),\omega} + E_n(f)_{p(.),\omega}$$

$$\leq c\Omega_r\left(f, \frac{1}{n+1}\right)_{p(.),\omega}$$

and the proof of Theorem (2.3.13) is completed.

Some of the above results can be extended to the complex case.

Let G_0 and G_1be, respectively, the bounded and the unbounded components of a closed rectifiable curve Γ of the complex plane $\mathbb{C}$. Without loss of generality we may assume that $0 \in G_0$. Let $w = \varphi(z)$ and $w = \varphi_1(z)$ be the conformal mappingsof $G_1$and G_0 onto the complement $\mathbb{D}_\infty$ of $\mathbb{D}$, normalized by the conditions

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \varphi(z)/z > 0$$

and

$$\varphi_1(0) = \infty, \qquad \lim_{z\to\infty} z\varphi_1(z) > 0,$$

respectively. We denote by and ψ, the inverse mappings of $\varphi$and φ_1, respectively.
Denote by $P(\Gamma)$ the class of Lebesgue measurable functions $p = p(z): \Gamma \to (1, \infty)$ with $1 < p_* (\Gamma) := \operatorname{ess\,inf}_{z\in\Gamma} p(z) \leq p^*(\Gamma) := \operatorname{ess\,sup}_{z\in\Gamma} p(z) < \infty$.
Let $p \in P(\Gamma)$ be a bounded measurable function and let $\omega: \Gamma \to [0, \infty]$ be a weight with

$$|\{t \in \Gamma: \omega(t) = 0\}| = 0.$$

For these p and ω we denote by $L_\omega^{p(.)}(\Gamma)$ the class of functions $f: \Gamma \to \mathbb{C}$ for which

$$\int_\Gamma |f(z)\omega(z)|^{p(z)} |dz| < \infty.$$

The space $L_\omega^{p(.)}(\Gamma)$ is a Banach space with the norm

$$\|f\|_{\Gamma,p(.),\omega} := \inf_{\alpha>0}\left\{\int_\Gamma \frac{|f(z)\omega(z)|^{p(z)}}{\alpha}|dz| < 1\right\}.$$

If p andω are as above, the set of bounded rational functions defined on Γ is dense in $L_\omega^{p(.)}(\Gamma)(cf. [141])$. If $1 < p_* (\Gamma) \leq p(z) \leq p^*(\Gamma) < \infty$for $z \in \Gamma$ and $\omega \equiv 1$, then the space $L_\omega^{p(.)}(\Gamma)$ coincides with

$$\left\{f: \left|\int_\Gamma f(z)g(z)dz\right| < \infty \; for\ all\ g \in L_\omega^{p'(.)}(\Gamma)\right\},$$

where $p'(z) := p(z)/(p(z) - 1)$ is the conjugate exponent of $p(z)$.
We define for $p \in P(\Gamma)$ and a weightω

$$E_\omega^{p(.)}(G_0) := \left\{f \in E^1(G_0): f \in L_\omega^{p(.)}(\Gamma)\right\},$$
$$E_\omega^{p(.)}(G_\infty) := \left\{f \in E^1(G_\infty): f \in L_\omega^{p(.)}(\Gamma)\right\}$$

and

$$\tilde{E}_\omega^{p(.)}(G_\infty) := \left\{f \in E_\omega^{p(.)}(G_\infty): f(\infty) = 0\right\},$$

where $E^p(X), 1 \leq p < \infty$, is a Smirnov space of analytic functions defined on a simply connected domain $X \subset \mathbb{C}$. If $p(z) = p$ is constant, then $E_\omega^{p(.)}(X)$ coincides with a usual weighted Smirnov space on X.
Basic approximation problems in the spaces $E^p(G_0)$ were proposed by several mathematicians. Walsh and Russel [146] gave the results in $E^p(G_0), 1 < p < \infty$, for polynomial approximation orders in the case of an analytic boundary. Al'per [146] proved

direct and inverse approximation theorems by algebraic poly nomials in the spaces $E^p(G_0), 1 < p < \infty$, for a Dini smooth boundary. Kokilashvili [148] improved Al'per's direct and inverse results for algebraic polynomial approximation and, assuming that the Cauchy singular integral operator is bounded (corners permitted), he obtained the improved direct and inverse approximation theorems in the Smirnov spaces $E^p(G_0), 1 < p < \infty$([140]). Andersson [147] proved that Kokilashvili's results also hold in $E^1(G_0)$. When the boundary is a Carlesoncurve ,the approximation of functions of $E^p(G_0), 1 < p < \infty$, by the partial sum of Faber series was investigated by Israfilov in [149] and [149]. These results are generalized to the Muckenhoupt weighted case in [150] and [151]. The approximation properties of Faber series inso-called weighted and non weighted Smirnov–Orlicz spaces are investigated in [154] and [153]. Most of the above results use the partial sum of Faber series as approximation tool.
we prove the main theorems of approximation, respectively, by algebraic polynomials and rational functions in the weighted variable Smirnov spaces $E_\omega^{p(.)}(G_0)$ and $\tilde{E}_\omega^{p(.)}(G_0)$.
A smooth Jordan curve Γ will be called Dini-smooth ([157]) if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arc length s, has the modulus of continuity $\Omega(\theta, s)$ satisfying the Dini condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s} ds < \infty, \qquad \delta > 0.$$

If Γ is Dini-smooth, then ([147])

$$0 < c < |\psi'(w)| < C < \infty, \qquad |w| \geq 1, \tag{45}$$

with some constants c and C. Similar inequalities hold also for ψ_1' and φ_1' in the case of $|w| = 1$ and $z \in \Gamma$, respectively.
Let $P_\pm^{log}(\Gamma) := \{p \in P(\Gamma)$: p $satisfies$ (31)$with\ the\ replacements x_1 \to z_1, x_2 \to z_2$ and $T \to \Gamma\}$.

For given $p \in P(\Gamma)$ the class of weights ωsatisfying the condition

$$\|\omega^{p(z)}\|_{A_{p(.)}(\Gamma)} := \sup_{B \in B(\Gamma)} \frac{1}{|B|^{p_B}} \|\omega^{p(z)}\|_{L^1(B)} \left\|\frac{1}{\omega^{p(z)}}\right\| \in_{B,(p'(.)/p(.))} < \infty$$

is denoted by $A_{p(.)}(\Gamma)$. Here $p_B := \left(\frac{1}{|B|}\int_B \frac{1}{p(z)}|dz|\right)^{-1}$ and

$$B(\Gamma) := \{B(z,r) \cap \Gamma : B(z,r) \text{ is a ball in } \mathbb{C} \text{ of radius r with } z \in \Gamma\}.$$

For given $f \in L_\omega^{p(.)}(\Gamma)$ we define

$$f_0(e^\theta) := f\left(\psi(e^\theta)\right), f_1(e^\theta) := f\left(\psi_1(e^\theta)\right) \text{ for } \theta \in T$$

And

$$\omega_0(e^\theta) := \omega\left(\psi(e^\theta)\right), \omega_1(e^\theta) := \omega\left(\psi_1(e^\theta)\right) \text{ for } \theta \in T.$$

Theorem $(2.3.23)$[133]: Let Γ be a Dini-smooth curve, $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \ for\ some\ p_0 \in \left(1, p_*(\Gamma)\right),$$

$r > 0$ and $f \in L_{\omega}^{p(.)}(\Gamma)$. Then there is a constant $c > 0$ such that for any natural number n

$$\|f - R_n(.,f)\|_{\Gamma,p(.),w} \leq c\left\{\Omega_r\left(f,\frac{1}{n+1}\right)_{\Gamma,p(.),w} + \widetilde{\Omega}_r\left(f,\frac{1}{n+1}\right)_{\Gamma,p(.),w}\right\},$$

where $R_n(.,f)$is the nth partial sum of the Faber–Laurent series of f .

Corollary $(2.3.24)$[133]: Let Γ be a Dini-smooth curve, $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \ for\ some\ p_0 \in \left(1, p_*(\Gamma)\right),$$

$r > 0$ and $f \in E_{\omega}^{p(.)}(G_0)$. Then there is a constant $c > 0$ such that for any natural number n

$$\|f - P_n(.,f)\|_{\Gamma,p(.),w} \leq c\ \Omega_r\left(f,\frac{1}{n+1}\right)_{\Gamma,p(.),w},$$

where $P_n(.,f)$ is the nth partial sum of the Faber series of f .

Corollary $(2.3.25)$[133]: Let Γ be a Dini-smooth curve, $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(T) \ for\ some\ p_0 \in \left(1, p_*(\Gamma)\right),$$

$r > 0$ and $f \in \tilde{E}_{\omega}^{p(.)}(G_{\infty})$. Then there is a constant $c > 0$ such that for any natural number n

$$\|f - R_n(.,f)\|_{\Gamma,p(.),w} \leq c\ \widetilde{\Omega}_r\left(f,\frac{1}{n+1}\right)_{\Gamma,p(.),w},$$

where $R_n(.,f)$ is as in Theorem (2.3.23).

Theorem $(2.3.26)$[133]: Under the conditions of Corollary (2.3.24), the inequality

□

$$\Omega_r\left(f,\frac{1}{n}\right)_{\Gamma,p(.),w} \leq \frac{c}{n^r}\left\{E_0(f)_{\Gamma,p(.),w} + \sum_{k=1}^{n} k^{r-1}\, E_k(f)_{\Gamma,p(.),w}\right\}$$

holds with a constant $c > 0$.

Corollary $(2.3.27)$[133]: Under the conditions of Corollary (2.3.24), if

$$E_n(f)_{\Gamma,p(.),\omega} = O(n^{-\alpha}), \qquad \alpha > 0, \qquad n = 1,2,3,\dots,$$

then

$$\Omega_r(f,\delta)_{\Gamma,p(.),\omega} = \begin{cases} \mathcal{O}(\delta^{\alpha}), & r > \alpha, \\ \mathcal{O}\left(\delta^{\alpha}\left|\log\frac{1}{\delta}\right|\right), & r = \alpha, \\ \mathcal{O}(\delta^{r}), & r < \alpha, \end{cases}$$

Definition$(\mathbf{2.3.28})[\mathbf{133}]$:Let $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(\Gamma) \text{ for some } p_0 \in \left(1, p_*(\Gamma)\right)$$

and $r \in \mathbb{R}^+$. If $f \in E_{\omega}^{p(.)}(G_0)$, then for $0 < \sigma < r$ we set

$$Lip\sigma(r, \Gamma, p(.), \omega) := \left\{f \in E_{\omega}^{p(.)}(G_0) : \Omega_r(f, \delta)_{\Gamma, p(.), \omega} = \mathcal{O}(\delta^{\alpha}), \delta > 0\right\}$$

and

$$\widetilde{Lip}\sigma(r, \Gamma, p(.), \omega) := \left\{f \in \tilde{E}_{\omega}^{p(.)}(G_{\infty}) : \widetilde{\Omega}_r(f, \delta)_{\Gamma, p(.), \omega} = \mathcal{O}(\delta^{\alpha})\right\}.$$

Corollary$(\mathbf{2.3.29})[\mathbf{133}]$: Let $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(\Gamma) \text{ for some } p_0 \in \left(1, p_*(\Gamma)\right)$$

and $r \in \mathbb{R}^+$. If $f \in E_{\omega}^{p(.)}(G_0), 0 < \sigma < r$ and $E_n(f)_{\Gamma, p(.), \omega} = O(n^{-a})$for $n = 1,2, \cdots$, then $f \in Lip\sigma(r, \Gamma, p(.), \omega)$.

By Corollary (2.3.24) and Corollary (2.3.27) we have the constructive characterization of the classes $Lip\sigma(\alpha, \Gamma, p(.), \omega)$.

Corollary $(\mathbf{2.3.20})[\mathbf{133}]$: Let $p \in P_{\pm}^{log}(\Gamma)$,

$$\omega^{-p_0} \in A_{\left(\frac{p(.)}{p_0}\right)'}(\Gamma) \text{ for some } p_0 \in \left(1, p_*(\Gamma)\right),$$

$0 < \alpha < r$ and $f \in E_{\omega}^{p(.)}(G_0)$. Then the following conditions are equivalent:

(a) $f \in Lip\sigma(r, \Gamma, p(.), \omega)$.

(b) $E_n(f)_{\Gamma, p(.), \omega} = O(n^{-a}), n = 1,2, \cdots$.

The inverse theorem for unbounded domains is formulated as follows.

Theorem $(\mathbf{2.3.31})[\mathbf{133}]$: Under the conditions of Corollary (2.3.25), there is a constant $c > 0$ such that for every natural number n

$$\widetilde{\Omega}_r\left(f, \frac{1}{n}\right)_{\Gamma, p(.), \omega} \leq \frac{c}{n^r}\left\{\tilde{E}_0(f)_{\Gamma, p(.), \omega} + \sum_{k=1}^{n} k^{r-1}\tilde{E}_k(f)_{\Gamma, p(.), \omega}\right\}$$

holds.

In a similar way as for $E_{\omega}^{p(.)}(G_0)$ we obtain the following corollaries.

Corollary$(\mathbf{2.3.32})[\mathbf{133}]$: Under the conditions of Corollary (2.3.25), if

$$\tilde{E}_n(f)_{M, \Gamma, \omega} = O(n^{-a}), \qquad \alpha > 0, \qquad n = 1,2,3, \cdots,$$

then

$$\widetilde{\Omega}_r(f, \delta)_{\Gamma, p(.), \omega} = \begin{cases} \mathcal{O}(\delta^{\alpha}), & r > \alpha, \\ \mathcal{O}\left(\delta^{\alpha}\left|\log\frac{1}{\delta}\right|\right), & r = \alpha, \\ \mathcal{O}(\delta^{r}), & r < \alpha, \end{cases}$$

Using Corollary (2.3.32) and Definition (2.3.21) we get

Corollary$(\mathbf{2.3.33})[\mathbf{133}]$: Under the conditions of Corollary (2.3.25), if

$$\tilde{E}_n(f)_{\Gamma,p(.),\omega} = O(n^{-a}), \qquad \sigma > 0, \qquad n = 1,2,3,\cdots,$$

then $f \in \widetilde{Lip}\sigma(r,\Gamma,p(.),\omega)$.

By Corollary (2.3.32) and (2.3.33) we have

Corollary $(\mathbf{2.3.34})[\mathbf{133}]$: Let $0 < \sigma < r$ and the conditions of Corollary (2.3.25) be fulfilled. Then the following conditions are equivalent.

$(a)\ \in \widetilde{Lip}\sigma(r,\Gamma,p(.),\omega)$,

$(b)\ \tilde{E}_n(f)_{\Gamma,p(.),\omega} = O(n^{-a}), \quad n = 1,2,3,\cdots.$

Chapter 3
Modular Inequalities and Variable L^p Spaces

We show that if $P = Q$, the condition on P is both necessary and sufficient for the modular inequality. In addition, we establish general interpolation theorems for modular spaces. Further, our conditions are necessary on R. Our result extends the recent work of

Pick and Ruzicka [184], Diening [183] and Nekvinda [185]. We also show that under much weaker assumptions on $p(x)$, the maximal operator satisfies a weak-type modular inequality.

Section (3.1): The Calderon Operator

For (M,μ) and (N,v) be two σ −finite measure spaces, and let $L_0(\mu)$ and $L_0(v)$ be the sets of measurable functions defined on M and N respectively. An operator $T : L_0(v) \to L_0(\mu)$ is called quasilinear if $|T(\lambda f)(x)| = |\lambda||Tf(x)|$ and if there exists a constant $K > 0$ independent of f and g such that $|T(f + g)(x)| \leq K(|Tf(x)| + |\Gamma g(x)|)$. If $K = 1, T$ is said to be sublinear.

A function $Q : [0,\infty) \to [0,\infty)$ is called a modular function if Q is an increasing (non-decreasing) function and $Q(0+) = 0$. If, in addition, Q satisfies the Δ_2-condition

$$Q(2t) < CQ(t)$$

for any $t > 0$, then Q is called a Δ_2-modular function and we write $Q \in \Delta_2$.

Let Q be a modular function and set

$$L_Q(\mu) = L_Q = \left\{f \in L_0(\mu); \|f\|_Q = \int_M Q(|f(x)|)d\mu(x) < \infty\right\}.$$

Then, we want to study mapping properties for which $T : L_Q(\mu) \text{—>} L_P(v)$ is bounded, for certain operators T.

Modular inequalities have been studied previously by several authors $(cf.$ [184]) in connection with weight characterizations. However, unlike the case treated here, the functions P and Q are typically Young's or N-functions, and the optimality of P and Q is not in general considered.

Recall that if T is an operator of weak type (a,a) and (b,b), $0 < a < b < \infty$; that is, $v(\{x \in N, |Tf(x)| > y\}) < (C\|f\|_{\alpha,\mu}/y)^\alpha$ where $a = a$ and $\alpha = b$, then

$$\int_N P(|Tf(x)|)\ dv(x) < C \int_M Q(|Tf(x)|)\ d\mu(x) \quad (1)$$

is satisfied for $P(x) = Q(x) — |x|^p$ and $a < p < b$. Moreover, such operators satisfy the rearrangement inequality

$$(Tf)_v^*(t) \leq C\left(\frac{1}{t^{1/a}}\int_0^t f_\mu^*(s)s^{1/a-1}ds + \frac{1}{t^{1/b}}\int_0^\infty f_\mu^*(s)s^{1/b-1}ds\right) \quad (2)$$

where $f_\mu^*(s)\inf\{s > 0;\ \lambda_f^\mu(s) \le t\}$ is the rearrangement decreasing function of f and $\lambda_f^\mu(y) = \mu(\{x;\ |f(x)| > y\})$ is the distribution function of f. Similarly it is understood for $(Tf)_v^*$. The term in parenthesis on the right of (2) is called the Calderón operator.

In order to prove (1) for general modular functions, observe that for Q modular, an elementary argument shows that

$$\int_M Q(|f(x)|)\, d\mu(x) = \int_0^\infty Q\left(f_\mu^*(t)\right) dt = \int_0^\infty \lambda_f^\mu(y) dQ(y),$$

such that a general (P,Q) modular inequality will follows if

$$\int_0^\infty P\left[C\left(\left(S_a f_\mu^*\right)(t) + \left(\tilde{S}_b f_\mu^*\right)(t)\right)\right] dt \le C_1 \int_0^\infty Q\left(f_\mu^*(t)\right) dt$$

holds, where

$$S_a f(t) = \frac{1}{t^{1/a}} \int_0^t f(s) s^{1/a-1} ds$$

and

$$\tilde{S}_b f(t) = \frac{1}{t^{1/b}} \int_0^t f(s) s^{1/b-1} ds.$$

Note that $S_1 = S$ is the Hardy averaging operator and $\tilde{S}_\infty = \tilde{S}$ is the conjugate Hardy operator. we provide optimal conditions characterizing modular pairs P and Q, for which (P,Q) (and in case $Q = P, (P)$) modular inequalities

$$\int_0^\infty P\left(S_a f(t)\right) dt \le C \int_0^\infty Q(f(t))\, dt$$

and

$$\int_0^\infty P\left(\tilde{S}_b f(t)\right) dt \le C \int_0^\infty Q(f(t))\, dt$$

are satisfied for $0 < \alpha, b < \infty$. The case where $b = \infty$ and T is bounded on L^∞ is also considered. These results yield sharper estimates and interpolation theorems for several classical operators.

We characterize (P,Q) modular inequalities for $S_a, 0 < a < 1$ (Theorem (3.1.2)) and give a corresponding characterization in the case when $a = 1$ and $f$is decreasing for a reverse Hardy modular inequality (Theorem (3.1.1)). In order to prove corresponding (P,Q) modular inequalities for $S_a, a > 1$ and $\tilde{S}_b, 0 < b < \infty$, some general modular results are required. These are proved and yield general modular interpolation theorems (Corollary (3.1.8)). Finally contains the (P,Q) and (P) modular inequalities for $S_a, a > 1$ and $\tilde{S}_b, 0 < b \le \infty$. A characterization of P,Q modular functions for which a (P,Q) modular inequality for the Hubert transform holds and a short proof of an interpolation theorem of Miyamoto ([185]) for modular functions are also given.

The notation usedis standard: If f/g is bounded above and below by positive constants, we write $f \approx g$ and say that $f$and g are equivalent functions. Constants denoted by C, sometimes with subscripts, are assumed to be positive and independent of the functions involved, and may differ at different places. If $0 \leq g$ is decreasing, we write $g^{**}(x) = (1/x)\int_0^x g$, where the measure under which the rearrangement occurs is deleted when there is no ambiguity. χ_E is the characteristic function of the set E and its Lebesgue measure is denoted by $|E|$.

Finally, inequalities, such as (1), are interpreted in the sense that if the right side is finite, so is the left side and the inequality holds. Unless indicated to the contrary, we assume that P and Q are modular functions or are equivalent to modular functions.

We begin by proving (P, Q) modular inequalities for the Hardy averaging operator.

Theorem $(\mathbf{3.1.1})[\mathbf{182}]$: (i) There exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^\infty P\left(\frac{\int_0^t f}{t}\right) dt \leq C_1 \int_0^\infty Q\big(C_2 f(t)\big)\ dt \tag{3}$$

is satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $t > 0$,

$$P(t) + t\int_0^t \frac{P(y)}{y^2} dy \leq C_3 Q(C_4 t). \tag{4}$$

(ii) The inequality (3) is reversed for every decreasing nonnegative function f if and only if the inequality (4) is reversed.

(iii) There exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\sup_{t>0} t\, P\left(\frac{\int_0^t f}{t}\right) dt \leq C_1 \int_0^\infty Q(C_2 f(t))\ dt$$

is satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $r > 0$,

$$\sup_{u \leq r} \frac{P(u)}{u} \leq C_3 \frac{Q(C_4 r)}{r}.$$

We thank J. Soria for pointing out that the argument in proving (i) applies also to the proof of (iii).

Proof, (i) To show the necessary condition, let us take $f(s) = t\chi_{[o,r)}(s)$ Then, we have that

$$\int_0^\infty P\left(\frac{t}{x}\min(r, x)\right) dx \leq C_1 r Q(C_2 t);$$

that is,

$$\int_0^r P(t)\, dx + \int_r^\infty P\left(\frac{tr}{x}\right) dx = rP(t) + rt\int_0^t \frac{P(y)}{y^2} dy \leq C_1 r Q(C_2 t).$$

from which the result follows with $C_3 = C_1$ and $C_4 = C_2$.

Conversely, if (4) holds, then we may assume that for small $t > 0, \int_0^t P(y)/y^2\, dy < \infty$, and from this it follows that $P(y)/y^2 \to 0$ as $y \to 0$.

Now, writing $f^{**}(t) = (1/t)\int_0^t f(s)\,ds$, we have

$$\int_0^\infty P(f^{**}(t))\,dt = \int_0^\infty \lambda_{f^{**}}(z)\,dP(z).$$

where the distribution function of f^{**}satisfies (see [183])

$$\frac{1}{2s}\lambda_f^f(s) \le \lambda_{f^{**}}(s) \le \frac{2}{s}\lambda_f^f(s/2), \tag{5}$$

and hence

$$\int_0^\infty \lambda_{f^{**}}(z)\,dP(z) \le \int_0^\infty \frac{2}{z}\left(\int_{\{x:f(x)>z/2\}} f(x)dx\right)dP(z)$$
$$= 2\int_0^\infty f(x)\left(\int_0^{2f(x)} \frac{dP(z)}{z}\right)dx$$
$$= 2\int_0^\infty f(x)\left(\frac{P\big(2f(x)\big)}{f(x)} + \int_0^{2f(x)} \frac{P(z)}{z}dz\right)dx$$
$$\le 2C_3\int_0^\infty Q(2C_4 f(x))\,dx.$$

That is, (3) holds with $C_1 = 2C_3$ and $C_2 = 2C_4$.

(ii) The proof follows as in (i), but now the first inequality of (5) is applied.

(iii) The weak type characterization follows analogously.

Theorem $(\mathbf{3.1.2})[\mathbf{182}]$: Let $0 < a < 1$. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^\infty P\left(\frac{1}{t^{1/a}}\int_0^t f(s)\,s^{a-1}ds\right)dt \le C_1\int_0^\infty Q\big(C_2 f(t)\big)\,dt \tag{6}$$

satisfied for every decreasing nonnegative function f if and only if there exist constants $C_3 > 0$ and $C_4 > 0$ such that, for every $t > 0$,

$$P(t) + t^a\int_0^t \frac{P(y)}{y^{a+1}}dy \le C_3 Q(C_4 t). \tag{7}$$

Proof: Let $g(s) = af(s^a)$. Then obvious change of variables shows that (6) is equivalent to

$$\int_0^\infty P\left(\frac{1}{t}\int_0^t g(s)\,ds\right)t^{a-1}\,dt \le C_1\int_0^\infty Q\left(\frac{C_2}{a}g(t)\right)t^{a-1}\,dt \tag{8}$$

For the necessary condition, it is enough to apply the hypothesis to the functions $f(s) = tX_{(\theta,r)}(s)$ Then (7) follows with $C_3 = C_1$ and $C_4 = C_2/a$.

For the converse, first observe that we can assume $\int_0^t P(y)/y^{a+1}\,dy < \infty$ for small t, since otherwise the result is trivial. Also, in this case, $\lim_{y\to 0} P(y)/y^a = 0$.

To show that (7) implies (8), note that, interchanging the order of integration and applying (5), we obtain

$$\int_0^\infty P\left(\frac{1}{x}\int_0^x g\right)x^{a-1}dx = \frac{1}{a}\int_0^\infty \left[\lambda_{g^{**}}(z)\right]^a dP(z)$$
$$\leq \frac{2^a}{a}\int_0^\infty \left(\frac{1}{z}\lambda_g^g(z/2)\right)^a dP(z)$$

$$= \frac{2^a}{a}\int_0^\infty \left(\int_{\{g(x)>z/2\}} g(x)dx\right)^a \frac{dP(z)}{z^a}$$
$$= \frac{2^a}{a}\int_0^\infty \left(\int_0^{\lambda_g(z/2)} g(x)dx\right)^a \frac{dP(z)}{z^a}$$
$$= \frac{2^a}{a}\int_0^\infty \left(\int_0^{\lambda_g(z/2)} \left(\int_0^x g\right)^{a-1} g(x)dx\right) \frac{dP(z)}{z^a}.$$

But, since g is decreasing and $0 < a < 1$, it follows that $\left(\int_0^x g\right)^{a-1} \leq (xg(x))^{a-1}$ and hence

$$\int_0^\infty P\big(g^{**}(x)\big)\, x^{a-1}dx \leq 2^a \int_0^\infty \left(\int_0^{\lambda_g(z/2)} x^{a-1} g^a(x)dx\right)\frac{dP(z)}{z^a}$$
$$= 2^a \int_0^\infty x^{a-1}\, g^a(x)\left(\int_0^{2g(x)} \frac{dP(z)}{z^a}\right)dx$$
$$= 2^a \int_0^\infty x^{a-1}\, g^a(x)\left(\frac{P(2g(x))}{g(x)^a} + a\int_0^{2g(x)} \frac{dP(z)}{z^{a+1}}\right)dx$$
$$\leq C_3 \int_0^\infty x^{a-1}\, Q(2C_4 g(x))dx,$$

where the last inequality follows from (7) with $t = 2g(x)$. Hence, (8) holds with $C_1 = 2C_4 a$.

Clearly the arguments in proving Theorem (3.1.3) do not apply to obtain (P, Q) modular inequalities for S_a with $a > 1$. In order to obtain such estimates for S_a and $\tilde{S}_b, 0 < b \leq \infty$, we need some general results for quasilinear operators and the notion of admissible functions. As a consequence, we obtain a number of weak type estimates and general interpolation theorems.

Our first result shows that, under a simple condition on T, a (P) modular inequality implies $P \in \Delta_2$.

Let $L \subset L_0(\mu)$ be a set such that $\boldsymbol{R}^+ L \subset L$. For us, L will be either $L_0(\mu)$ or the set of measurable decreasing functions on $\boldsymbol{R}^+$.

Proposition $(\mathbf{3.1.3})[\mathbf{182}]$: Suppose that T satisfies $a\ (P)$ modular inequality for every function in L. If there exist a measurable set E such that $\chi_E \subset L$ and $\mu(E) < \infty$ and a constant $d > 1$ such that

$$v(\{x; |T\chi_E(x)| > d\}) \neq 0,$$

then $P \in \Delta_2$.

Proof: Take $\lambda > 0$ and $f(x) = \lambda\chi_E(x)$. Then, since

$$P(y)\lambda_{Tf}^{v}(y) \leq C\int_M P(|f(x)|)\,d\mu(x).$$

we get

$$P(y)v(\{x;\ |\lambda||T\chi_E(x)| > y\}) < CP(\lambda)\mu(E).$$

Choose now $y = d\lambda$. Then we get

$$P(d\lambda) \leq \frac{C\mu(E)}{v(\{x; |T\chi_E(x)| > d\})}P(\lambda).$$

from which the $\Delta_2$condition for P follows.

Now, for our next purpose, we need to give the following definition:

Definition $(\mathbf{3.1.4})[\mathbf{182}]$: We say that a function $A : [0,\infty) \to [0,\infty)$ with $A(0) = 0$ is admissible for T and L if, for every function $f \in L$,

$$\lambda_{Tf}^{v}(I) \leq \int_M A(|f(x)|)\,d\mu(x).$$

Remark $(\mathbf{3.1.5})[\mathbf{182}]$: (i) In terms of the decreasing rearrangement the above inequality is

$$(Tf)_v^*\left[\int_M A(|f(x)|)\,d\mu(x)\right] \leq 1.$$

Since we are assuming $R^+L \subset L$, for every admissible function A for T and L and every $y > 0$, it holds that for any $f \in L$

$$\lambda_{Tf}^{v}(y) \leq \int_M A\left(\frac{|f(x)|}{y}\right)d\mu(x)\,. \tag{9}$$

(ii) If B is a modular function such that $B(x) > 1$ for every $x > 1$ and, for every $f \in L$,

$$\int_N B(|Tf(x)|)\ dv(x) \leq \int_M A(|f(x)|)\,d\mu(x),$$

then

$$\begin{aligned}\sup_{y>0} y\,B[(Tf)_v^*(y)] &\leq \sup_{y>0}\int_0^y B[(Tf)_v^*(t)]\ dt\\ &= \int_N B(|Tf(x)|)\ dv(x) \leq \int_M A(|f(x)|)\,d\mu(x).\end{aligned}$$

In particular, if $y = \int_M A(|f(x)|)\,d\mu(x)$, then $B\left[(Tf)_v^*\left(\int_M A(|f(x)|)\,d\mu(x)\right)\right] \leq 1$. Then, by the hypothesis of B, this implies $(Tf)_v^*\left(\int_M A(|f(x)|)\,d\mu(x)\right) \leq 1$, and hence A is admissible for T and L.

(iii) If T is of weak type (p,p) with $p > 0$, then $A(t) = \|T\|_{(p,p)}t^p$ is an admissible function for T and $L_0(\mu)$.

Observe that, for $0 < \alpha < \infty, \|S_a f\|_\infty \le a\|f\|_\infty$ and that if f is decreasing, then for $0 \le b \le \infty$, $\operatorname{supp}(\tilde{S}_b f) \subset \operatorname{supp} f$. For operators which satisfy conditions of this type we have the following result:

Lemma $(\mathbf{3.1.6})[\mathbf{182}]$: Let L be a set as above and T a quasilinear operator defined on L.

(i) Let $\tilde{L} = \{g = f\chi_{\{|f|>y\}};\ f \in L, y > 0\}$ and A an admissible function for T and L. Suppose that $T : L^\infty(\mu) \to L^\infty(v)$ is bounded with (operator) norm less than or equal to M. Then, for every $f \in L$ and every $y > 0$,

$$\lambda^v_{Tf}(y) \le \int\limits_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right) du(x), \qquad (10)$$

where K is the constant arising from the quasilinearity of T.

(ii) Let $\tilde{\tilde{L}} = \{g = f\chi_{\{|f|\le y\}}; f \in L, y > 0\}$ and A an admissible function for T and $\tilde{\tilde{L}}$.

If there exists a constant $C > 0$ such that $v(\operatorname{supp} Tf) < C\mu(\operatorname{supp} f)$, then, for every $\varepsilon > 0$,

every $y > 0$ and every $f \in L$,

$$\lambda^v_{Tf}(y) \le \int\limits_{\{|f(x)|\le y\}} A\left(\frac{(1+\varepsilon)K|f(x)|}{y}\right) du(x) + C\lambda^\mu_f(y) \qquad (11)$$

Proof: (i) Fix $y > 0$ and write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > y/(2MK)$ and zero otherwise. Then,

$$v(\{x;\ |Tf(x)| > y\}) \le v\left(\{x;\ |Tf_1(x)| > \frac{y}{2K}\}\right) + v\left(\{x;\ |Tf_2(x)| > \frac{y}{2K}\}\right) .$$

But, since $\|Tf_2\|_{\infty,v} \le M\|f_2\|_{\infty,\mu} \le y/2K$, the second term is zero, and hence, since $f_1 \in \tilde{L}$, we obtain by (9) that

$$\lambda^v_{Tf}(y) \le \int\limits_M A\left(\frac{2K|f_1(x)|}{y}\right) du(x) = \int\limits_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right) du(x)$$

To show (ii), fix $y > 0$ and write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|/|f(x)| > y$ and zero otherwise. Then, for every $\varepsilon > 0$,

$$\begin{aligned}
\lambda^v_{Tf}(y) &\le v\left(\left\{x;\ |Tf_1(x)| \ge \frac{\varepsilon y}{K(1+\varepsilon)}\right\}\right) + v\left(\left\{x;\ |Tf_2(x)| \ge \frac{y}{K(1+\varepsilon)}\right\}\right) \\
&\le v(\{x;\ |Tf_1(x)| > 0\}) + v(\{x; K(1+\varepsilon)\ |Tf_2(x)| > y\}) \\
&\le C\mu(\{x;\ |f(x)| > y\}) + \int\limits_M A\left(\frac{(1+\varepsilon)K|f_2(x)|}{y}\right) du(x) \\
&= C\lambda^\mu_f(y) + \int\limits_{\{|f(x)|\le y\}} A\left(\frac{(1+\varepsilon)K|f(x)|}{y}\right) du(x).
\end{aligned}$$

The lemma implies now the following (P, Q) interpolation theorem:

Theorem $(\mathbf{3.1.7})[\mathbf{182}]$: (i) Let $\tilde{L}$ be as in Lemma (3.1.6) (i). Let T be a quasilinear operator such that $T : L^{\infty}(\mu) \to L^{\infty}(y)$ is bounded with norm M. If there exist a constant C and an admissible function A for T and $\tilde{L}$ such that, for every $t > 0$,

$$\int_{0}^{2MKt} A\left(\frac{2Kt}{y}\right) dP(y) \le CQ(t), \tag{12}$$

then T satisfies a (P, Q)modular inequality for every function in L.

(ii) Let $\tilde{\tilde{L}}$ be as in Lemma (3.1.6) (ii). Suppose that for every $f \in L$, $v(\text{supp } Tf) \le C\mu(\text{supp } f)$for some constant C independent of f. If there exist a constant C and an admissible functin A for T and $\tilde{\tilde{L}}$ such that, for every $t > 0$, $\lim_{z\to 0} P(t/z)A(z) = 0$ and, for some ε

$$P(t) + \int_{t}^{\infty} A\left(\frac{(1+\varepsilon)t}{z}\right) dP(z) \le CQ(t), \tag{13}$$

then T satisfies a (P, Q) modular inequality for every function in L.

Proof, (i) By (10) and (12),

$$\int_{N} P(|Tf(x)|)dv(x) = \int_{0}^{\infty} \lambda_{Tf}^{v}(y)dP(y)$$

$$\le \int_{0}^{\infty} \left[\int_{\{|f(x)|>y/(2MK)\}} A\left(\frac{2K|f(x)|}{y}\right) du(x) \right] dP(y)$$

$$= \int_{M} \left[\int_{0}^{2MK|f(x)|} A\left(\frac{2K|f(x)|}{y}\right) dP(y) \right] du(x) \le C \int_{M} Q(|f(x)|)du(x),$$

which proves (i).

The proof of (ii) follows in the same way, using now (11) and (13).

Note that if $S(x) = X_{(1,\infty)}(x)$, then $L_S(v) = \{f;\ \lambda_f^v(1) < \infty\}$. Similarly, if one defines $L^0(\mu)$ by $L^0(\mu) = \{f;\ \mu(\text{supp } f) < \infty\}$, then Theorem (3.1.7) has the following formulation:

Corollary $(\mathbf{3.1.8})[\mathbf{182}]$: Suppose that $T : L_A(\mu) \to L_S(v)$ is bounded.

(i) If $\tilde{L}$ is as in Lemma (3.1.6) (i), $T : L^{\infty}(\mu) \to L^{\infty}(y)$ is bounded with norm M and A is an admissible function for T and $\tilde{L}$, then $T : L_Q(\mu) \to L_P(v)$ is bounded for (P, Q) satisfying (12).

(ii) If $\tilde{\tilde{L}}$ is as in Lemma (3.1.6) (ii), $T : L^0(\mu) \to L^0(v)$ is bounded and A is an admissible function for T and $\tilde{\tilde{L}}$, then $T : L_Q(\mu) \to L_P(v)$ bounded for (P, Q) satisfying (13).

We now derive (P) and (P,Q) modular inequalities for S_a with $a > 1$ and $\tilde{S}_b, 0 < b < \infty$, as well as for $\tilde{S}$. In addition, we give a short proof and an extension of an interpolation theorem of Miyamoto [187].

Proposition $(\mathbf{3.1.9})[\mathbf{182}]$: Assume that $a > 1$ and $\chi_{(0,1)} \in L. Then, if\ S_a$ satisfies a (P) modular inequality for $L, P \in \Delta_2$.

Proof: Let $E = (0,1)$ in Proposition (3.1.3). Then, it suffices to show that, for some $d > 1, \left|\{x;\ |S_a\chi_{(0,1)}(x))| > d\}\right| \neq 0$. But, since $a > 1$, we can choose $a > d > 1$, and hence, since

$$S_a\chi_{(0,1)}(x) = \begin{cases} a & if\ x < 1 \\ a/x^{1/a} & if\ x \geq 1 \end{cases},$$

we get

$$\left|\{x;\ |S_a\chi_{(0,1)}(x))| > d\}\right| = \left(\frac{a}{d}\right)^a \neq 0.$$

The main result for S_a is the following:

Theorem $(\mathbf{3.1.10})[\mathbf{182}]$: Let $a > 1$ and assume that, for every $r > 0, \chi_{(0,r)} \in L$. Then the following hold.

(i) If Sa satisfies $a(P,Q)$ modular inequality for L, there exists a constant C such that, for every $t > 0$,

$$t^a \int_0^t \frac{P(y)}{y^{a+1}}\, dy \leq CQ(t). \tag{14}$$

(ii) If there exists an $\varepsilon > 0$ such that

$$P(2t) + t^{a+\varepsilon} \int_0^{2t} \frac{P(y)}{y^{a+\varepsilon+1}}\, dy \leq CQ(t). \tag{15}$$

then S_a satisfies $a\ (P,Q)$ modular inequality for L.

(iii) S_a satisfies $a\ (P)$ modular inequality for L if and only if $P \in \Delta_2$ and there exists a constant C such that, for every $t > 0$,

$$t^a \int_0^t \frac{P(y)}{y^{a+1}}\, dy \leq CP(t). \tag{16}$$

Proof: (i) It is enough to check the hypothesis on the functions $f = t\chi_{(0,r)}$.

(ii) Clearly, $S_a : L^\infty \to L^\infty$ is bounded with norm a and $S_a : L^{a,\infty} \to L^{a,\infty}$ is bounded. Therefore, by inteφolation, $S_a : L^{a+\varepsilon} \to L^{a+\varepsilon}$ is bounded for every $\varepsilon > 0$, and hence, for some constant C, $A(t) = Ct^{a+\varepsilon}$ is an admissible function for S_a and every subset of $L_0(\boldsymbol{R}^+)$.

Now, by Theorem (3.1.7) with $M = a$ and $A(t) = Ct^{a+\varepsilon}$, the linear operator S_a satisfies the (P,Q) modular inequality provided that

$$(2t)^{a+\varepsilon}\int_0^{2at}\frac{dP(y)}{y^{a+\varepsilon}}\leq CQ(t). \qquad (17)$$

But since (15) is satisfied, it follows that $\lim_{y\leftrightarrow 0+}P(y)/y^{a+\varepsilon}=0$, and hence an integration by parts shows that (17) is equivalent to (15).

(iii) If S_a satisfies a (P) modular inequality, then by Proposition (3.1.9), $P\in\Delta_2$and (16) now follows from (14) with $Q=P$. Conversely, if $P\in\Delta_2$, then there exists $q>a$ such that $P(y)/y^q$ is decreasing (see [185]), and hence, by (16)

$$CP(t)\geq t^a\int_0^t\frac{P(y)}{y^{a+1}}\,dy\leq C_1P(t).$$

For $m=0,1,2,\ldots$, define

$$A_m=t^a\int_0^t\frac{P(y)}{y^{a+1}}\frac{(\log(t/y))^m}{m!}\,dy.$$

Then, by (16),

$$A_m=t^a\int_0^t\frac{P(y)}{y^{a+1}}\left(\int_y^t\frac{(\log(t/s))^{m-1}}{(m-1)!}\frac{ds}{s}\right)dy$$
$$=t^a\int_0^t\frac{(\log(t/s))^{m-1}}{m!\,s^{a+1}}\left(S^a\int_y^s\frac{P(y)}{y^{a+1}}dy\right)ds \qquad \leq CA_{m-1}.$$

Therefore, $A_m\leq C^mA_0=C^{m+1}P(t)$. Choose $1<C<M$ and $\varepsilon<1/M$. Then

$$\sum_{m=0}^{\infty}\varepsilon^mA_m\leq\sum_{m=0}^{\infty}\frac{A_m}{M^m}\leq CP(t)\sum_{m=0}^{\infty}\left(\frac{C}{M}\right)^m=C_2P(t).$$

Also

$$\sum_{m=0}^{\infty}\varepsilon^mA_m=\sum_{m=0}^{\infty}t^a\int_0^t\frac{P(y)}{y^{a+1}}\frac{(\varepsilon\log(t/y))^m}{m!}dy=t^a\int_0^t\frac{P(y)}{y^{a+1}}\left(\frac{t}{y}\right)^{\varepsilon}dy,$$

and hence

$$t^{a+\varepsilon}\int_0^t\frac{P(y)}{y^{a+\varepsilon+1}}\,dy\leq C_2P(t).$$

But, since $P\in\Delta_2$, this implies (15) with $Q=P$, and so S_a satisfies the (P) modular inequality.

We now consider the operator $\tilde{S}_b$ with $b>0$.

Proposition $(3.1.11)$[182]: Assume that $b>0$ and $\chi_{(0,1)}\in L$. Then, if $\tilde{S}_b$ satisfies a (P) modular inequality for $L,P\in\Delta_2-$ Moreover, in this case, the Δ_2 constant for P is less than or equal to $C((2+b)/b)^b$, where C is the constant arising from the (P) modular inequality.

Proof: It is enough to see that the set $E = (0,1)$ and $d = 2$ satisfies the condition of Proposition (3.1.3). But $\tilde{S}_b\chi_{(0,1)}(x) = b(x^{-1/b} - 1)\chi_{(0,1)}(x)$. It then follows that $\left|\{x; \left|\tilde{S}_b\chi_{(0,1)}(x)\right| > 2\}\right| = (b/(b+2))^b \neq 0$, and hence

$$P(y) \leq C\left(\frac{b+2}{b}\right)^b P\left(\frac{y}{2}\right), \ \ y > 0.$$

We shall also need the following lemma.

Lemma $(\mathbf{3.1.12})[\mathbf{182}]$: Let $M \geq 0$. If f is a decreasing function on $[M, \infty)$ and $0 < p \leq q \leq \infty$, then, for every $x \geq 2M$,

$$\left(\int_x^\infty \left(t^{1/p} f(t)\right)^q \frac{dt}{t}\right)^{1/q} \leq C\left(\int_M^\infty f^p(t)\, dt\right)^{1/p}.$$

where the constant depends only on p and q.

Proof: The result follows from a straight forward modification of the case $M = 0$ given in [189].

Theorem $(\mathbf{3.1.13})[\mathbf{182}]$: Let $0 < b < \infty$ and assume that $\tilde{S}_b$ is defined on decreasing functions.

(i) If $\tilde{S}_b$ satisfies a (P, Q) modular inequality, then there exists a constant C such that, for every $t > 0$,

$$t^b \int_t^\infty \frac{P(y)}{y^{b+1}}\, dy \leq CQ(t). \tag{18}$$

(ii) If there exists an $\varepsilon > 0$ such that

$$t^{b-\varepsilon} \int_t^\infty \frac{P(y)}{y^{b-\varepsilon+1}}\, dy \leq CQ(t), \tag{19}$$

then $\tilde{S}_b$ satisfies a (P, Q) modular inequality.

(iii) $\tilde{S}_b$ satisfies a (P) modular inequality if and only if there exists a constant C such that, for every $t > 0$, (18) holds for $Q = P$.

Proof: (i) It is enough to check the hypothesis on the functions $f = t\chi_{(0,r)}$

(ii) Let us consider first the case $b > 1$. Choose $\varepsilon > 0$ such that $b - \varepsilon > 1$. Then, it follows from the weighted (conjugate) Hardy inequalities ([185]) that $\tilde{S}_b : L^{b-\varepsilon} \to L^{b-\varepsilon}$ is bounded and therefore, for some constant C, the function $A(t) = Ct^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and every subset of $L_0(R^+)$. Consequently, the function $A(t) = C_1 t^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and $\tilde{\tilde{L}}$ and, since $\left|\operatorname{supp} \tilde{S}_b f\right| \leq |\operatorname{supp} f|$, we can apply Theorem (3.1.7)(ii). Hence, if for some ε',

$$P(t) + C_1(1+\varepsilon')^{b-\varepsilon} t^{b-\varepsilon} \int_t^\infty \frac{dP(z)}{z^{b-\varepsilon}} \leq CQ(t),$$

then we see that $\tilde{S}_b$ satisfies a (P, Q) modular inequality. Since we may assume that the integral on the left side of (19) is bounded, it follows that $P(y)/y^{b-\varepsilon} \to 0$ as $y \to$

∞. Integration by parts argument then shows that (19) implies the above inequality. Let now $0 < b \leq 1$. Then, we do not know if the $A(t) = C_1 t^{b-\varepsilon}$ is an admissible function for $\tilde{S}_b$ and L, but the inequality (11) still holds. To see this, we have to apply Lemma (3.1.12) as follows. Let f be a decreasing function and set $g = f\chi_{\{|f| \leq y\}}$ with $y > 0$. Choose $\varepsilon > 0$ such that $\alpha = b - \varepsilon > 0$. Applying Lemma (3.1.12) with $p = \alpha$ and $q = 1$, it then follows that, if $x \geq 2\lambda_f(y)$,

$$\tilde{S}_b g(x) = x^{-1/b} \int_0^\infty g(s) s^{1/b-1}\, ds = x^{-1/b} \int_0^\infty g(s) s^{1/b-1} s^{1/\alpha} s^{-1/\alpha}\, ds$$
$$\leq x^{-1/\alpha} \int_0^\infty g(s) s^{1/\alpha - 1}\, ds \leq C x^{-1/\alpha} \|g\|_\alpha.$$

Therefore, for every $z > 0$,

$$\left|\{x > 0; \left|\tilde{S}_b g(x)\right| > z\}\right| \leq 2\lambda_f(y) + \left|\{x \geq \lambda_f(y); \left|\tilde{S}_b g(x)\right| > z\}\right|$$
$$\leq 2\lambda_f(y) + \left|\{x \geq \lambda_f(y); C x^{-1/\alpha} \|g\|_\alpha > z\}\right|$$
$$\leq 2\lambda_f(y) + \left(\frac{C\|g\|_\alpha}{z}\right)^\alpha,$$

and hence, for every $\varepsilon > 0$,

$$\lambda_{\tilde{S}_b f}(y) \leq \left|\left\{x > 0; \left|\tilde{S}_b(f-g)(x)\right| \geq \frac{\varepsilon y}{(1+\varepsilon)}\right\}\right| + \left|\left\{x ; \left|\tilde{S}_b g(x)\right| \geq \frac{y}{(1+\varepsilon)}\right\}\right|$$
$$\leq \left|\{x ; \left|\tilde{S}_b(f-g)(x)\right| > 0\}\right| + \left|\{x; (1+\varepsilon)\left|\tilde{S}_b g(x)\right| > y\}\right|$$
$$\leq C\big(\left|\{x ; |f(x)| > y\}\right| + \left|\{x \leq 2\lambda_f(y); (1+\varepsilon)\left|\tilde{S}_b g(x)\right| > y\}\right|$$
$$+ \left|\{x > 2\lambda_f(y); (1+\varepsilon)\left|\tilde{S}_b g(x)\right| > y\}\right|\big)$$
$$\leq C\lambda_f(y) + \int_{\{|f(x)| \leq y\}} \left(\frac{(1+\varepsilon)|f(x)|}{y}\right)^\alpha dx,$$

which is the inequality (11). The proof now proceeds as for the case $b > 1$.

(iii) If $\tilde{S}_b$ satisfies $a\,(P)$ modular inequality, then by (i), (18) holds with $Q = P$. Conversely, if (18) holds with $Q = P$, then it follows that $P(y)/y^b$ tends to zero when y tends to infinity, and an integration by parts shows that (18) is equivalent to

$$\int_t^\infty \frac{dP(y)}{y^b} \leq C t^{-b} \int_0^t dP(y).$$

This implies that dP satisfies a B_b condition (see [186]), and hence it is known (see for example Lemma 3 of [188]) that there exists an $\varepsilon > 0$ such that $dP \in B_{p-\varepsilon}$. Again an integration by parts shows that

$$t^{b-\varepsilon} \int_t^\infty \frac{P(y)}{y^{b-\varepsilon+1}} dy \leq CP(t),$$

and the result follows from (ii).
If $b = \infty$, we have the following result for the conjugate Hardy operator.
Theorem $(3.1.14)$[182]: Assume that, for every $r > 0, \chi_{(0,r)} \in L$. Then
(i)
$$\int_0^\infty P\left(\int_t^\infty \frac{f(s)}{s} ds\right) dt \leq C \int_0^\infty P(f(t))\, dt, \quad f \in L \tag{20}$$
if and only if $P \in \Delta_2$.
(ii) If either P or $Q \in \Delta_2$, then $\tilde{S}$ satisfies a (P,Q) modular inequalities if and only if $P \leq CQ$.
Proof: (i) If the inequality (20) holds, we have that $P \in \Delta_2$ by Proposition (3.1.3), since obviously
$$\left|\left\{x; \left|\tilde{S}\chi_{(0,1)}(x)\right| > 2\right\}\right| = e^{-2} \neq 0.$$
Conversely, if $P \in \Delta_2$, then (see [189]) there exists $p > 0$ such that $P(t)/t^p$ is equivalent to a decreasing function and hence
$$\int_t^\infty \frac{P(y)}{y^{p+1}} dy \leq C \frac{P(t)}{t^{p+1}}.$$
An integration by parts shows that $t^{p+1} \int_t^\infty (1/y^{p+1})\, dP(y) \leq CP(t)$ and, since we already know that $A(t) = t^{b+1}$ is admissible for 5, we get (i) from Theorem (3.1.7) (ii).

(ii) Suppose P or Q satisfies Δ_2. Then, by (i) a (P) or (Q) modular inequality is satisfied. Since $P \leq CQ$, we get the (P,Q) modular inequality in either case.
Conversely, if we apply the (P,Q) modular inequality to the functions $f(x) = t\chi_{(0,1)}(x)$ we get
$$\int_0^1 P\left(t\ \log\frac{1}{x}\right) dx \leq CQ(t),$$
and with $z = t\, log(1/x)$, we obtain
$$\frac{1}{e} P(t) \leq \frac{1}{t} \int_t^\infty P(z)\, e^{-z/t} dz \leq \int_0^1 P\left(t\ \log\frac{1}{x}\right) dx \leq CQ(t).$$
Theorem (3.1.1)(i) and Theorem (3.1.14) now yields a characterization of a (P,Q) modular inequality for the Hubert transform.
Corollary $(3.1.15)$[182]: Suppose either P or Q satisfies the $\Delta_2$condition. Then, the (P,Q) modular inequality for the Hubert transform
$$\int_R P(|Hf(x)|)\, dx < C \int_R Q(|f(x)|) dx \tag{21}$$
is satisfied for $f \in L_0(dx)$ if and only if $P \leq CQ$ and
$$t \int_0^t \frac{P(s)}{s^2} ds \leq CQ(t). \tag{22}$$
Proof: Clearly (21) is equivalent to

$$\int_0^\infty P\big((Hf)^*(x)\big)\,dx \ \le C\ \int_0^\infty Q\big(f^*(x)\big)dx.$$

But, since (see [190])

$$(Hf)^*(x) \le C_1\left[\frac{1}{x}\int_0^x f^*(t)\,dt + \int_x^\infty \frac{f^*(t)}{t}\,dt\right] \le C_2(Hf^*)^*(x),$$

it follows that (21) is satisfied if and only if

$$\int_0^\infty P\left[\frac{1}{x}\int_0^x f^*(t)\,dt\right]dx \le C\int_0^\infty Q\big(f^*(x)\big)dx,$$

and

$$\int_0^\infty P\left[\int_x^\infty \frac{f^*(t)}{t}\,dt\right]dx \le C\int_0^\infty Q\big(f^*(x)\big)dx$$

is satisfied. Then, by Theorem (3.1.1)(i) and Theorem (3.1.13) (ii), this holds if and only if $P \ \le CQ$ and (22) holds.

Finally, we give a short proof of an interpolation theorem proved by Miyamoto in [189] in the case where P is continuous, $P \in \Delta_2$ and $P(x) \ = \ 0$ if and only if $x \ = \ 0$. As we shall see, these conditions can be removed.

Theorem $(\mathbf{3.1.16})[\mathbf{182}]$: Let T be a quasilínear operator such that T is of weak type (a,a) and (b,b), where $0 \ < \ a \ < \ b \ < \infty$. Then, T satisfies a (P,Q) modular inequality for every measurable function f with

$$Q(t) = \max\left(t^a\int_0^t \frac{P(s)}{s^{a+1}}\,ds, t^b\int_t^\infty \frac{P(s)}{s^{b+1}}\,ds\right).$$

Proof: It follows from the definition of Q that

$$\lim_{t\to 0}\frac{P(t)}{t^a} = \lim_{t\to\infty}\frac{P(t)}{t^b} = 0.$$

Now, fix $y > \ 0$ and write $f \ = \ f_1 + f_2$, where $f_1(x) \ = \ f(x)$ if $|f(x)| \ > \ y$ and zero otherwise. Then, by assumption

$$\lambda_{Tf}^v(y) \le \lambda_{Tf_1}^v(y/2K) + \lambda_{Tf_2}^v(y/2K)$$
$$\le C\left[\int_{\{|f(x)|>y\}}\left(\frac{|f(x)|}{y}\right)^a d\mu(x) + \int_{\{|f(x)|<y\}}\left(\frac{|f(x)|}{y}\right)^b d\mu(x)\right],$$

and therefore

$$\int_N P(|Tf(x)|)\,dv(x) = \int_0^\infty \lambda_{Tf}^v(y)\,dP(y)$$
$$\le C\int_0^\infty\left[\int_{\{|f(x)|>y\}}\left(\frac{|f(x)|}{y}\right)^a d\mu(x) + \int_{\{|f(x)|<y\}}\left(\frac{|f(x)|}{y}\right)^b d\mu(x)\right]dP(y)$$
$$= C\left[\int_M |f(x)|^a\int_0^{|f(x)|}\frac{dP(y)}{y^a}\,d\mu(x) + \int_M |f(x)|^b\int_{|f(x)|}^\infty \frac{dP(y)}{y^b}\,d\mu(x)\right]$$

$$= C[I_1 + I_2].$$

Since $P(t) \leq bQ(t)$, using an integration by parts, we obtain that

$$I_1 = \int_M |f(x)|^\alpha \left[\frac{P(|f(x)|)}{|f(x)|^\alpha} + \int_0^{|f(x)|} \frac{P(y)}{y^{\alpha+1}} dy\right] d\mu(x) \leq C \int_M Q(|f(x)|)\, d\mu(x).$$

The estimate for h follows similarly.

Section (3.2): The Maximal Function

Given an open set $\Omega \subset R^n$, and a measurable function $p: \Omega \to [1, \infty)$, let $L^{p(x)}(\Omega)$ denote the Banach function space of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega |f(x)/\lambda|^{p(x)}\ dx < \infty,$$

with norm

$$\|f\|_{p(x),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)}\ dx \leq 1\right\}.$$

These spaces are a special case of the Musielak–Orlicz spaces (cf. Musielak [198]). When $p(x) = p_0$ is constant, $L^{p(x)}(\Omega)$ becomes the standard Lebesgue space $L^{p_0}(\Omega)$.

Functions in these spaces and the associated Sobolev spaces $W^{k,p(x)}(\Omega)$ have been considered by a number of authors: see, for example, [199]and [200]. They appear in the study of variational integrals and partial differential equations with non-standard growth conditions. Some of the properties of the Lebesgue spaces readily generalize to the spaces $L^{p(x)}(\Omega)$: see, for example, Kováčik and Rákosník [195]. On the other hand, elementary properties, such as the continuity of translation, often fail to hold (see [195] or [200]), and for applications it is an important and open problem to determine which results from harmonic analysis remain true in the variable exponent setting. We consider the Hardy–Littlewood maximal operator,

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\, dy, \tag{23}$$

where the supremum is taken over all balls B which contain x and for which $|B \cap \Omega| > 0$. It is well known (cf. Duoandikoetxea [195]) that the maximal operator satisfies the following weak and strong-type inequalities:

$$|\{x \in \Omega : Mf(x) > t\}| \leq \frac{C}{t^p} \int_\Omega |f(y)|^p\ dy, \qquad 1 \leq p < \infty,$$

$$\int_\Omega Mf(y)^p\ dy \leq C \int_\Omega |f(y)|^p\ dy,\ 1 < p < \infty.$$

We prove analogous inequalities for functions in $L^{p(x)}(\Omega)$. Strong-type inequalities have been studied. Pick and Růžička [197] constructed examples which showed that the following uniform continuity condition on $p(x)$ is necessary (in some sense) for the maximal operator to be bounded on $L^{p(x)}(\Omega)$:

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \qquad x, y \in \Omega, |x-y| < \frac{1}{2}\,. \tag{24}$$

This condition appears to be natural in the study of variable L^p spaces; see [201], [202] and the references contained therein.
Diening [203] has shown that this condition is sufficient on bounded domains. To state his result, let $p_* = \inf\{p(y) : y \in \Omega\}, p^* = \sup\{p(y) : y \in \Omega\}$.
Theorem $(\mathbf{3.2.1})[\mathbf{197}]$: (Diening). Let $\Omega \subset R^n$ be an open, bounded domain, and let $p : \Omega \to [1, \infty)$ satisfy (24) and be such that $1 < p_* \le p^* < \infty$. Then the maximal operator is bounded on $L^{p(x)}(\Omega)$: $\|Mf\|_{p(x),\Omega} \le C(p(x), \Omega)\|f\|_{p(x),\Omega}$.
Theorem $(\mathbf{3.2.2})[\mathbf{197}]$: (Nekvinda). Let $p: R^n \to [1, \infty)$ satisfy (24) and be such that $1 < p_* \le p^* < \infty$. Suppose further that there is a constant $p_\infty > 1$ suchthat $p(x) = p_\infty + \phi(x)$, where there exists $R > 0$ such that $\phi(x) \ge 0$ if $|x| > R$,and $\beta > 0$ such that

$$\int_{\{x \in R^n : \phi(x) > 0\}} \phi(x) \beta^{1/\phi(x)} \, dx < \infty. \tag{25}$$

Then the maximal operator is bounded on $L^{p(x)}(R^n)$.
Theorem $(\mathbf{3.2.3})[\mathbf{197}]$: Given an open set $\Omega \subset R^n$, let $p: \Omega \to [1, \infty)$ be such that$1 < p_* \le p^* < \infty$. Suppose that $p(x)$ satisfies (24) and

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x|)}, \qquad x, y \in \Omega, |y| \ge |x|. \tag{26}$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$.
Condition (26) is the natural analogue of (24) at infinity. It implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions. It is also necessary (in some sense) on R, as the next example shows.
Theorem $(\mathbf{3.2.4})[\mathbf{197}]$: Fix $p_\infty, 1 < p_\infty < \infty$, and let $\phi: [0, \infty) \to [0, p_\infty - 1)$ be such that $\phi(0) = 0, \phi$ is decreasing on $[1, \infty), \phi(x) \to 0$ as $x \to \infty$, and

$$\lim_{x \to \infty} \phi(x) log(x) = \infty. \tag{27}$$

Define the function $p: R \to [1, \infty)$ by

$$p(x) = \begin{cases} p_\infty, & x \le 0, \\ p_\infty - \phi(x), & x > 0; \end{cases}$$

then the maximal operator is not bounded on $L^{p(x)}(R)$.
The assumption in Theorem (3.2.3) that $p^* < \infty$ again holds automatically: it follows from (26). However, the assumption that $p_* > 1$ is necessary, as the following example shows.
Theorem $(\mathbf{3.2.5})[\mathbf{197}]$: Let $\Omega \subset R^n$ be open, and let $p: \Omega \to [1, \infty)$ be upper semi-con tinuous. If $p_* = 1$ then the maximal operator is not bounded on $L^{p(x)}(\Omega)$.
In passing, we note that an immediate application of Theorem (3.2.3) has been given by Diening [204]: he has shown that if $\partial\Omega$ is Lipschitz, and the maximal operator is bounded on $L^{p(x)}(\Omega)$, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$.
Unlike the case of the strong-type inequalities, we appear to be the first to prove an analogue of the weak (p, p) inequality for the maximal operator. Our weak-type result is somewhat surprising, since it requires no continuity assumptions on $p(x)$, and it is satisfied

by unbounded functions. To state it, we need a definition. Given a non-negative, locally integrable function u on R^n, we say that $u \in RH_\infty$ if there exists a constant C such that for every ball B,

$$u(x) \leq \frac{C}{|B|} \int_B u(y)\, dy \; a.e. \; x \in B.$$

Denote the smallest constant C such that this inequality holds by $RH_\infty(u)$. The RH∞ condition is satisfied by a variety of functions u: for instance, if there exist positive constants A and B such that $A \leq u(x) \leq B$ for all x. More generally, $u \in RH_\infty$ if $u(x) = |x|a, a > 0$, or if there exists $r > 0$ such that $u - r$ is in the Muckenhoupt class A_1. For further information about RH_∞, see Cruz-Uribe and Neugebauer [201].

Theorem $(\mathbf{3.2.6})[\mathbf{197}]$: Given an open set Ω, suppose the function $p: \Omega \to [1, \infty)$can be extended to R^n in such a way that $1/p \in RH_\infty$. Then for all $f \in L^{p(x)}(\Omega)$ and $t > 0$,

$$|\{x \in \Omega : Mf(x) > t\}| \leq C \int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} dy. \qquad (28)$$

The proof of Theorem (3.2.3) requires a series of lemmas. Throughout, let $\alpha(x) = (e + |x|)^{-n}$.

The first lemma is due to Diening [203, Lemma 3.1]. For completeness we include its short proof.

Lemma $(\mathbf{3.2.7})[\mathbf{197}]$: Given an open set Ω and a function $p: \Omega \to [1, \infty)$ whichsatisfies (24), then for any ball B such that$|B \cap \Omega| > 0$,

$$|B|^{p_*(B\cap\Omega) - p^*(B\cap\Omega)} \leq C.$$

Proof: Since $p_*(B \cap \Omega) - p^*(B \cap \Omega) \leq 0$, we may assume that if r is theradius of B, then $r < \frac{1}{4}$. But in that case, (24) implies that

$$p^*(B \cap \Omega) - p_*(B \cap \Omega) \leq \frac{C}{log(1/2r)} .$$

Therefore,

$$|B|^{p_*(B \cap \Omega) - p^*(B \cap \Omega)} \leq cr^{-n(p^*(B \cap \Omega) - p_*(B \cap \Omega))} \leq cr^{-nC/log(1/2r)} \leq C.$$

Though our proof of the following lemma is not directly dependent on Nekvinda [209].

Lemma $(\mathbf{3.2.8})[\mathbf{197}]$: Given a set G and two non-negative functions $r(x)$ and $s(x)$, suppose that for each $x \in G$,

$$0 \leq s(x) - r(x) \leq \frac{C}{log(e + |x|)} .$$

Then for every function f ,

$$\int_G |f(x)|^{r(x)}\, dx \leq C \int_G |f(x)|^{s(x)}\, dx + \int_G \alpha(x)^{r_*(G)}\, dx.$$

Proof: Let $G^\alpha = \{x \in G : |f(x)| \geq \alpha(x)\}$. Then

$$\int_G |f(x)|^{r(x)}\, dx = \int_{G^\alpha} |f(x)|^{r(x)}\, dx + \int_{G \setminus G^\alpha} |f(x)|^{r(x)}\, dx,$$

and we estimate each integral separately. First, since $\alpha(x) \leq 1$,

$$\int_{G\setminus G^{\alpha}} |f(x)|^{r(x)}\,dx \le \int_{G\setminus G^{\alpha}} \alpha(x)^{r(x)}\,dx \le \int_{G} \alpha(x)^{r_*(G)}\,dx.$$

On the other hand, if $x \in G^{\alpha}$, then

$$|f(x)|^{r(x)} = |f(x)|^{s(x)}|f(x)|^{r(x)-s(x)} \le |f(x)|^{s(x)}\alpha(x)^{-C/log(e+|x|)} \le C|f(x)|^{s(x)}.$$

The desired inequality now follows immediately.

The next two lemmas generalize the key step in Diening's proof of Theorem (3.2.1) (see [203, Lemma 3.2]).

Lemma $(3.2.9)$[197]: Given Ω and p as in the statement of Theorem (3.2.3), suppose that $|f|_{p(x),\Omega} \le 1$, and $|f(x)| \ge 1$ or $f(x) = 0, x \in \Omega$. Then for all $x \in \Omega$,

$$Mf(x)^{p(x)} \le CM\left(|f(\cdot)|^{p(\cdot)/p_*}\right)(x)^{p_*} + C\alpha(x)^{p_*}, \tag{29}$$

where $\alpha(x) = (e + |x|)^{-n}$.

Proof: Without loss of generality, we may assume that f is non-negative. Fix $x \in \Omega$, and fix a ball B of radius $r > 0$ containing x such that $|B \cap \Omega| > 0$. Let $B_\Omega = B \cap \Omega$. It will suffice to show that (29) holds with the left-hand side replaced by

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)},$$

and with a constant independent of B. We will consider three cases.

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Case 1: $r < |x|/4$. Define $\bar{p}(x) = p(x)/p_*$. Then $\bar{p}(x) \ge 1$, and (26) holdswith p replaced by $\bar{p}$. In particular, by our assumption on r, if $y \in B_\Omega$,

$$0 \le \bar{p}(y) - \bar{p}_*(B_\Omega) \le \frac{C}{\log(e + |y|)}. \tag{30}$$

Therefore, by Hölder's inequality and by Lemma (3.2.8) with $r(x)$ replaced by the constant $\bar{p}_*(B_\Omega)$ and $s(x)$ by $\bar{p}(y)$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)dy\right)^{p(x)} \le \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}_*(B_\Omega)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)}$$

$$\le \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy + \frac{1}{|B|}\int_{B_\Omega} \alpha(y)^{\bar{p}_*(B_\Omega)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)};$$

since $r < |x|/4$ and $p(x)/\bar{p}_*(B_\Omega) \le p_* < \infty$,

$$\le \left(\frac{C}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy + C\alpha(x)^{\bar{p}_*(B_\Omega)}\right)^{p(x)/\bar{p}_*(B_\Omega)}$$

$$\le 2^{p^*}C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)} + 2^{p^*}C\alpha(x)^{p(x)}.$$

If $|B| \ge 1$, then by H¨older's inequality and since $|f|_{p(x),\Omega} \le 1$,

$$\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy \le \left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \le \left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \le 1.$$

Hence, since $p(x)/\bar{p}_*(B_\Omega) \geq p_*$ and $\alpha(x) \leq 1$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*}$$
$$\leq CM\big(f(\cdot)^{\bar{p}(\cdot)}\big)(x)^{p_*} + C\alpha(x)^{p_*}.$$

If, on the other hand, $|B| \leq 1$, then, again since $|f|_{p(x),\Omega} \leq 1$,

$$\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy \leq |B_\Omega|^{1/p'_*}\left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{1/p_*} \leq 1.$$

Therefore,

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq C|B|^{-p(x)/\bar{p}_*(B_\Omega)}\left(\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p(x)/\bar{p}_*(B_\Omega)} + C\alpha(x)^{p_*}$$
$$\leq C|B|^{-p(x)/\bar{p}_*(B_\Omega)+p_*}\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*}.$$

Since $|B| \leq 1$, and since

$$-p(x)/\bar{p}_*(B_\Omega) + p_* = (p_*/p_*(B_\Omega))(p_*(B_\Omega) - p(x))$$
$$\geq (p_*/p_*(B_\Omega))\,(p_*(B_\Omega) - p^*(B_\Omega)),$$

by Lemma (3.2.7),

$$\leq C\left(\frac{1}{|B|}\int_{B_\Omega} f(y)^{\bar{p}(y)}\,dy\right)^{p_*} + C\alpha(x)^{p_*} \leq CM\big(f(\cdot)^{\bar{p}(\cdot)}\big)\,(x)^{p_*} + C\alpha(x)^{p_*}.$$

This is precisely what we wanted to prove.

Case 2: $|x| \leq 1$ and $r \geq |x|/4$. The proof is essentially the same as in the previous case: since $|x| \leq 1, \alpha(x) \approx 1$, so inequality (30) and the subsequent argument still hold.

Case 3: $|x| \geq 1$ and $r \geq \frac{|x|}{4}$. Since $f(x) \geq 1, p_* \geq 1$ and $|f|_{p(x),\Omega} \leq 1$,

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq |B|^{-p(x)}\left(\int_{B_\Omega} f(y)^{p(y)}\,dy\right)^{p(x)}$$
$$\leq Cr^{-np(x)}|f|_{p(x),\Omega}^{p(x)} \leq C|x|^{-np_*} \leq C\alpha(x)^{p_*}$$
$$\leq CM\big(f(\cdot)^{\bar{p}(\cdot)}\big)\,(x)^{p_*} + C\alpha(x)^{p_*}.$$

This completes the proof.

Definition $(3.2.10)$[197]: Given a function f on Ω, we define the Hardy operator H by

$$Hf(x) = \big|B_{|x|}(0)\big|^{-1}\int_{B_{|x|}(0)\cap\Omega} |f(y)|\,dy.$$

Lemma $(3.2.11)$[197]: Given Ω and p as in the statement of Theorem (3.2.3), suppose that $|f|_{p(x),\Omega} \leq 1$, and $|f(x)| \leq 1, x \in \Omega$. Then for all $x \in \Omega$,

$$Mf(x)^{p(x)} \leq CM\big(|f(\cdot)|^{p(\cdot)/p_*}\big)(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}, \quad (31)$$

where $\alpha(x) = (e + |x|)^{-n}$.

Proof: We may assume without loss of generality that f is non-negative. We argue almost exactly as we did in the proof of Lemma (3.2.9). In that proof we only used the fact that $f(x) \geq 1$ in Case 3, so it will suffice to fix $x \in \Omega, |x| \geq 1$, and a ball B containing x with radius $r > |x|/4$, and prove that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq CM\left(|f(\cdot)|^{p(\cdot)/p_*}\right)(x)^{p_*} + C\alpha(x)^{p_*} + CHf(x)^{p(x)}.$$

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Since $p^* < \infty$, we have that

$$\left(\frac{1}{|B|}\int_{B_\Omega} f(y)\,dy\right)^{p(x)} \leq 2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega\cap B_{|x|}(0)} f(y)\,dy\right)^{p(x)}$$

$$+2^{p^*}\left(\frac{1}{|B|}\int_{B_\Omega/B_{|x|}(0)} f(y)\,dy\right)^{p(x)};$$

since $r > |x|/4$,

$$\leq C\left(|B_{|x|}(0)|^{-1}\int_{B_{|x|}(0)\cap\Omega} |f(y)|\,dy\right)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega/B_{|x|}(0)} f(y)\,dy\right)^{p(x)}$$

$$= CHf(x)^{p(x)} + C\left(\frac{1}{|B|}\int_{B_\Omega/B_{|x|}(0)} f(y)\,dy\right)^{p(x)}.$$

To estimate the last term, note that if $y \in B_\Omega/B_{|x|}(0)$ then (30) holds and$\alpha(y) \leq \alpha(x)$, so the argument in Case 1 of the proof of Lemma (3.2.9) goes through. This shows that

$$\left(\frac{1}{|B|}\int_{B_\Omega/B_{|x|}(0)} f(y)\,dy\right)^{p(x)} \leq CM\left(|f(\cdot)|^{p(\cdot)/p_*}\right)(x)^{p_*} + C\alpha(x)^{p_*},$$

and this completes the proof.

Lemma $(\mathbf{3.2.12})[\mathbf{197}]$: If $i(x)$ is a radial, increasing function, $i_* > 1$, and if $|f(x)| \leq 1$, $t$hen

$$\int_\Omega Hf(y)^{i(y)}\;dy \leq C(n, i(x))\int_\Omega |f(y)|^{i(y)}\;dy.$$

Proof: Without loss of generality we may assume that f is non-negative. Also, for clarity of notation, we extend f to all of R^n by setting it equal to zero on $R^n\backslash\Omega$.

We first assume only that $i_* \geq 1$. Recall that $|B_{|x|}(0)| = |B_1(0)||x|^n$. Let Sdenote the unit sphere in R^n. Then by switching to polar coordinates and making a change of variables, we get that

$$Hf(x)^{i(x)} = \left(|B_1(0)|^{-1}|x|^{-n}\int_{B_{|x|}(0)} f(y)\;dy\right)^{i(x)}$$

$$= \left(|B_1(0)|^{-1} |x|^{-n} \int_S \int_0^{|x|} f(r\theta) r^{n-1} \, dr d\theta \right)^{i(x)}$$
$$= \left(|B_1(0)|^{-1} \int_S \int_0^1 f(|x| r\theta) r^{n-1} \, dr d\theta \right)^{i(x)}$$

$$= \left(|B_1(0)|^{-1} \int_{B_1(0)} f(|x|y) \, dy \right)^{i(x)}$$
$$\leq |B_1(0)|^{-1} \int_{B_1(0)} f(|x|y)^{i(x)} \, dy,$$

by Hölder's inequality.

Now let $r > 1$; the exact value of r will be chosen below. By Minkowski's integral inequality, and again by switching to polar coordinates, \

$$\left\| Hf(\cdot)^{i(\cdot)} \right\|_{r,R^n} \leq C \left(\int_{R^n} \left(\int_{B_1(0)} f(|x|y)^{i(x)} \, dy \right)^r dx \right)^{1/r}$$
$$\leq C \int_{B_1(0)} \left(\int_{R^n} f(|x|y)^{ri(x)} \, dx \right)^{1/r} dy$$
$$= C \int_S \int_0^1 \left(\int_{R^n} f(|x|s\theta)^{ri(x)} dx \right)^{1/r} s^{n-1} \, ds d\theta$$
$$= C \int_S \int_0^1 s^{-n/r} \left(\int_{R^n} f(|x|\theta)^{ri(x/s)} dx \right)^{1/r} s^{n-1} \, ds d\theta,$$

by a change of variables in the inner integral. Since i is a radial increasing function, $i(x/s) \geq i(x)$; since $f(|x|\theta) \leq 1$,

$$\leq \int_S \int_0^1 s^{-n/r} \left(\int_{R^n} f(|x|\theta)^{ri(x)} dx \right)^{1/r} s^{n-1} \, ds d\theta$$
$$\leq C \int_S \left(\int_{R^n} f(|x|\theta)^{ri(x)} dx \right)^{1/r} d\theta.$$

Since S has constant, finite measure, by Hölder's inequality,

$$\leq C \left(\int_S \int_{R^n} f(|x|\theta)^{ri(x)} dx \, d\theta \right)^{1/r}.$$

Since i is a radial function, if we rewrite the inner integral in polar coordinates, we get that

$$= C \left(\int_S \int_S \int_0^\infty f(u\theta)^{ri(u)} u^{n-1} \, du d\phi d\theta \right)^{1/r}$$
$$= C \left(\int_S \int_0^\infty f(u\theta)^{ri(u)} u^{n-1} \, du d\theta \right)^{1/r} = C \left(\int_{R^n} f(y)^{ri(y)} \, dy \right)^{1/r}.$$

To complete the proof, we repeat the above argument with $i(x)$ replaced by $\bar{\imath}(x) = i(x)/i_*$ and with $r = i_*$, since $i_* > 1$.

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Without loss of generality we may assume that f is non-negative. We first show there exists a constant C such that if $|f|_{p(x),\Omega} \le 1$, then $|Mf|_{p(x)},\Omega \le C$. Fix f , $|f|_{p(x),\Omega} \le 1$. Let $f = f_1 + f_2$, where

$$f_1(x) = f(x)\chi_{\{x:f(x)\ge 1\}}(x).$$

Then for $i = 1,2$, $|f_i|_{p(x),\Omega} \le 1$. Since $p^* < \infty$,

$$\int_\Omega Mf(y)^{p(y)}\ dy \le 2^{p^*}\int_\Omega Mf_1(y)^{p(y)} + 2^{p^*}\int_\Omega Mf_2(y)^{p(y)}\ dy.$$

We will show that each integral on the right-hand side is bounded by a constant. Since $|f_2(x)| \le 1$, by Lemma (3.2.11), f_2 satisfies inequality (31). Therefore, if we integrate over Ω we get that

$$\int_\Omega Mf_2(y)^{p(y)}\ dy \le C\int_\Omega \left(Mf_2(\cdot)^{p(\cdot)/p_*}\right)(y)^{p_*}dy + C\int_\Omega \alpha(y)^{p_*}\ dy$$
$$+ C\int_\Omega Hf_2(y)^{p(y)}\ dy.$$

Since $p_* > 1$, M is bounded on $L^{p_*}(\Omega)$ and $\alpha(x) \in L^{p_*}(\Omega)$, so

$$\le C\int_\Omega f_2(y)^{p(y)}\ dy + C + C\int_\Omega Hf_2(y)^{p(y)}\ dy \le C + C\int_\Omega Hf_2(y)^{p(y)}\ dy.$$

Given a function p, define its increasing, radial minorant i_p to be the function

$$i_p(x) = \inf_{|y|\ge|x|} p(y) \ .$$

Clearly, i_p is a radial, increasing function. Further, (26) implies that for all $x \in \Omega$,

$$0 \le p(x) - i_p(x) \le \frac{C}{log(e + |x|)} \ .$$

Therefore, since $f_2(x) \le 1$ and $(i_p)_* = p_*$, by Lemmas (3.2.12) and (3.2.8),

$$\int_\Omega Hf_2(y)^{p(y)}\ dy \le C\int_\Omega Hf_2(y)^{i_p(y)}\ dy \le C\int_\Omega f_2(y)^{i_p(y)}\ dy$$
$$\le C\int_\Omega f_2(y)^{p(y)}\ dy + C\int_\Omega \alpha(y)^{p_*}\ dy \le C.$$

Hence, $|Mf_2|_{p(x),\Omega} \le C$.

Avery similar argument using Lemma (3.2.9) shows that $|Mf_1|_{p(x),\Omega} \le C$. Therefore, we have shown that if $|f|_{p(x),\Omega} \le 1$, then $|Mf|_{p(x),\Omega} \le C$. Since $C > 1$, it follows that

$$\int_\Omega (C^{-1}Mf(x))^{p(x)}\ dx \le 1,$$

which in turn implies that

$$\|Mf\|_{p(x),\Omega} \le C.$$

To complete the proof we fix a function $g \in L^{p(x)}(\Omega)$, and let $f(x) = g(x)/\|g\|_{p(x),\Omega}$. Then $\|f\|_{p(x),\Omega} \leq 1$, so $|f|_{p(x),\Omega} \leq 1$. Hence,

$$\|Mg\|_{p(x),\Omega} = \|g\|_{p(x),\Omega}\|Mf\|_{p(x),\Omega} \leq C\|g\|_{p(x),\Omega}.$$

Our proof is closely modeled on the construction given by Pick and R$\dot{u}\check{z}i\check{c}$ka in [210]. By inequality (27), we have that

$$\lim_{x\to\infty} \left(1 - \frac{p_\infty}{p(2x)}\right) log(x) = -\infty,$$

which in turn implies that

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)} = 0.$$

Therefore, we can form a sequence $\{c_n\}_{n=1}^{\infty}$, $c_{n+1} < 2c_n \leq -1$, such that

$$|c_n|^{1-p_\infty/p(2|c_n|)} \leq 2^{-n}.$$

Let $d_n = 2c_n < c_n$, and define the function f on R by

$$f(x) = \sum_{n=1}^{\infty} |c_n|^{-1/p(|d_n|)} \chi_{(d_n, c_n)}(x).$$

We claim that $|f|_{p(x),R} \leq 1$ and $|Mf|_{p(x),R} = \infty$; it follows immediately from this that $\|f\|_{p(x),R} \leq 1$ and $\|Mf\|_{p(x),R} = \infty$, so the maximal operator is not bounded on $L^{p(x)}(R)$. First, we have that

$$|f|_{p(x),R} = \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p(x)/p(|d_n|)}\, dx = \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p_\infty/p(|d_n|)}\, dx$$

$$= \sum_{n=1}^{\infty} |c_n|^{1-p_\infty/p(|d_n|)} \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

On the other hand, if $x \in (|c_n|, |d_n|)$, then

$$Mf(x) \geq \frac{1}{2|d_n|} \int_{d_n}^{|d_n|} f(y)\, dy \geq \frac{1}{2|d_n|} \int_{d_n}^{c_n} f(y)\, dy$$

$$= \frac{|c_n|^{1-1/p(|d_n|)}}{2|d_n|} = \frac{1}{4} |c_n|^{-1/p(|d_n|)}.$$

Therefore, since $p(x)$ is an increasing function and $|c_n| \geq 1$,

$$|Mf|_{p(x),R} \geq \frac{1}{4} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(x)/p(|d_n|)} \geq \frac{1}{4} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(|d_n|)/p(|d_n|)})$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} 1 = \infty.$$

Fix $k \geq 1$. Since $p_* = 1$, Ω is open and p is uppersemi-continuous, there exists $x_k \in \Omega$ and $\varepsilon_k > 0$ such that $B_k = B_{\varepsilon_k}(x_k) \subset \Omega$, and such that if $x \in B_k, p(x) < 1 + 1/k$. We define the function $f_k(x) = |x_k - x|^{-nk/(k+1)}\chi_{B_k}(x)$. Then $f_k \in L^{p(x)}(\Omega)$. On the other hand, for $x \in B_k$, let $r = |x - x_k|$; then

$$Mf_k(x) \geq \frac{c}{|B_r(x_k)|}\int_{B_r(x_k)} f_k(y)\,dy = c(k+1)f_k(x).$$

Hence, $\|Mf_k\|_{p(x),\Omega} \geq c(k+1)\|f_k\|_{p(x),\Omega}$; since we may take k arbitrarily large, the maximal operator is not bounded on $L^{p(x)}(\Omega)$.

We begin with a lemma which, intuitively, plays the role that Hölder's inequality does in the standard proof that the maximal operator is weak (p,p).

Lemma $(\mathbf{3.2.13})[\mathbf{197}]$: Given an open set Ω, a function $p: R^n \to [1,\infty)$ such that $1/p$ is locally integrable, f in $L^{p(x)}(\Omega)$ and $t > 0$, suppose that B is a ball such that

$$\frac{1}{|B|}\int_{B\cap\Omega} |f(y)|\,dy > t.$$

Then

$$\int_B \frac{dx}{p(x)} \leq \frac{1}{p_*(B)}\int_{B\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

Proof: Fix a sequence of simple functions $\{s_n(x)\}$ on B, such that $s_n(x) \geq p_*(B)$ and such that the sequence increases monotonically to $p(x)$ on B. For each n we have that

$$s_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j}\chi_{A_{n,j}}(x),$$

where the $A_{n,j}$'s are disjoint sets whose union is B. Let $t_n(x)$ be the conjugate function associated to $s_n(x)$; then $t_n(x)$ decreases to $q(x)$, the conjugate function of $p(x)$. By Hölder's and Young's inequalities,

$$\int_{B\cap\Omega}\frac{|f(y)|}{t}\,dy = \sum_{j=1}^{k_n}\int_{A_{n,j}\cap\Omega}\frac{|f(y)|}{t}\,dy$$

$$\leq \sum_{j=1}^{k_n}\left(\int_{A_{n,j}\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy\right)^{1/\alpha_{n,j}} |A_{n,j}|^{1/\alpha'_{n,j}}$$

$$\leq \sum_{j=1}^{k_n}\left(\frac{1}{\alpha_{n,j}}\int_{A_{n,j}\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{\alpha_{n,j}} dy + \frac{|A_{n,j}|}{\alpha'_{n,j}}\right)$$

$$\leq \sum_{j=1}^{k_n}\left(\frac{1}{p_*(B)}\int_{A_{n,j}\cap\Omega}\left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_{A_{n,j}}\frac{dy}{t_n(y)}\right)$$

$$\leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{s_n(y)} dy + \int_B \frac{dy}{t_n(y)} .$$

Since this is true for all n, by the monotone convergence theorem,

$$\int_B \frac{|f(y)|}{t} \, dy \leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy + \int_B \frac{dy}{q(y)} .$$

Therefore,

$$\int_B \frac{dy}{p(y)} = |B| - \int_B \frac{dy}{q(y)} < \int_{B\cap\Omega} \frac{|f(y)|}{t} dy - \int_B \frac{dy}{q(y)}$$

$$\leq \frac{1}{p_*(B)} \int_{B\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

For each $N > 0$, define the operator M_N by

$$M_N f(x) = \sup \frac{1}{|B|} \int_{B\cap\Omega} |f(y)| \, dy,$$

where the supremum is taken over all balls containing x such that $|B| \leq N$. The sequence $\{M_N f(x)\}$ is increasing and converges to $Mf(x)$ for each $x \in \Omega$. Thus, by the monotone convergence theorem, for each $t > 0$,

$$|\{x \in \Omega : Mf(x) > t\}| = \lim_{N\to\infty} |\{x \in \Omega : M_N f(x) > t\}| .$$

Therefore, it will suffice to prove (28) with M replaced by M_N , and with a constant independent of N .

Fix $t > 0$ and let $E_N = \{x \in \Omega : M_N f(x) > t\}$. Then for each $x \in E_N$,there exists a ball B_x containing x, $|B_x| \leq N$, such that

$$\frac{1}{|B_x|} \int_{B_x\cap\Omega} |f(y)| \, dy > t.$$

By a weak variant of the Vitali covering lemma (cf. Stein [211, $p.$ 9]), there exists a collection of disjoint balls, $\{B_k\}$, contained in $\{B_k : x \in E_N\}$, and a constant C depending only on the dimension n, such that

$$|E_n| \leq C \sum_k |B_k| .$$

Therefore, by Lemma (3.2.13),

$$|E_n| \leq C \sum_k |B_k| \leq \sum_k |B_k| \left(\int_{B_k} \frac{dy}{p(y)}\right)^{-1} \int_{B_k} \frac{dy}{p(y)}$$

$$\leq \sum_k \left(\frac{1}{|B_k|} \int_{B_k} \frac{dy}{p(y)}\right)^{-1} \frac{1}{p_*(B_k)} \int_{B_k\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy;$$

since $p_*(B_k) - 1 = (1/p)^*(B_k)$, by the definition of RH_∞,

$$\leq RH_\infty(1/p) \sum_k \int_{B_k\cap\Omega} \left(\frac{|f(y)|}{t}\right)^{p(y)} dy \leq C \int_\Omega \left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

Chapter 4
Fully Measurable and Maximal Theorem

We show the class contains some known Banach spaces of functions, among which the classical and the grand Lebesgue spaces, and the EXP_α spaces ($\alpha > 0$). We analyze the function norm and we prove a boundedness result for the Hardy-Littlewood maximal operator, via a Hardy type inequality. We show that

$$\|f\|_{p[\cdot],\delta(\cdot),w} = \operatorname*{ess\,sup}_{x\in(0,1)}\left(\int_0^1 \left(\delta(x)f\,(t)\right)^{p(x)} w(t)dt\right)^{\frac{1}{p(x)}},$$

where w is a weight, $0 < \delta(\cdot) \le 1 \le p(\cdot) < \infty$,we show that if $p^+ = \|p\|_\infty < +\infty$,the inequality

$$\|Mf\|_{p[\cdot],\delta(\cdot),w} \le c\|f\|_{p[\cdot],\delta(\cdot),w}$$

holds with some constant c independent of f if and only if the weight w belongs to the Muckenhoupt class A_{p^+}. We show the following Hölder-type inequality

$$\int_0^1 fgdt \le \rho_{p[\cdot]),\delta[\cdot]}(f)\,\rho_{(p'[\cdot],\delta[\cdot]}(g),$$

where $\rho_{p[\cdot]),\delta[\cdot]}(f)$ is the norm of fully measurable grand Lebesgue spaces introduced by Anatriello and Fiorenza in [295]. For suitable choices of $p(x)$ and $\delta(x)$ it reduces to the classical Hölder's inequality for the spaces $EXP_{1/\alpha}$ and $L(\log L)^\alpha, \alpha > 0$.

Section (4.1): Fully Measurable Grand Lebesgue Spaces

Let $I = (0,1), p = p(\cdot)$ a variable exponent defined a. e. in I ([230, $Sect.$ 2.1]), which for simplicity we assume finite, and f a nonnegative Lebesgue measurable function defined a. e. in I. It is clear that the norm of the variable exponent Lebesgue spaces cannot be neither the expression

$$\left(\int_I f(x)^{p(x)}\,dx\right)^{\frac{1}{p(x)}} \quad \text{nor} \left(\int_I f(t)^{p(x)}\,dt\right)^{\frac{1}{p(x)}} \qquad (1)$$

the main reason being the fact that both depend on $x \in I$ and therefore (except trivial cases) are not nonnegative real numbers. If $p = p(\cdot)$ is constant, both expressions coincide with the usual norm of the Lebesgue space

$$\|f\|_p = \|f\|_{L^p(I)} = \left(\int_I f(t)^p \ dt\right)^{\frac{1}{p}}$$

The first expression in (1) must be modified in order to define the correct norm in the variable Lebesgue spaces $\|f\|_{p(\cdot)}$ see [224]). The second expression can be synthetically written with the symbol $\|f\|_{p(x)}$, which has been used in [225] to denote $\|f\|_{p(\cdot)}$ (a little bit improperly, but a reason has been explained); such expression gives an operator which has been considered, in an independent context from the variable Lebesgue spaces. Namely, the function $p(\cdot)$ used is $p(x) = p - x$, when the norm of the so called grand Lebesgue spaces (originated in [225]) are considered (here the variable x appears changed in ϵ):

$$\|f\|_{p)} = \operatorname*{esssup}_{0<\epsilon<p-1} \epsilon^{\frac{1}{p^{-\epsilon}}} \left(\int_I f(t)^{p^{-\epsilon}} \ dt\right)^{\frac{1}{p^{-\epsilon}}} \qquad (1 < p < \infty)$$

In [227] the norm $\|f\|_{p)}$ has been generalized, and the space of the functions f such that

$$\|f\|_{p),\delta} = \operatorname*{esssup}_{0<\epsilon<p-1} \epsilon^{\frac{1}{p^{-\epsilon}}} \left(\int_I (\delta(\epsilon) f(t))^{p^{-\epsilon}} \ dt\right)^{\frac{1}{p^{-\epsilon}}} < \infty, \qquad (2)$$

where δ is a measurable function in I, has been considered. It has been shown that the interesting case is that one where δ is left continuous, increasing $(i.e.\, 0 < \epsilon_1 < \epsilon_2 < p - 1 \Rightarrow \delta(\epsilon_1) \le \delta(\epsilon_2))$ and such that $\delta(0+) = 0, 0 < \delta \le 1$. Note that in (2), differently from [227], we put the function δ inside the integral: in view of the theory developed through several, [225] where the power $\frac{1}{p^{-\epsilon}}$ appears quite frequently, we think that it is worth to make this choice to simplify the volume of the formulas. We consider the further generalization of (2) where $p^{-\epsilon}$ is changed into a general measurable function, thus dealing with the operator $\|f\|_{p(x)}$. We note that the variable ϵ varies in $(0, p-1)$ while x varies in (0,1); for simplicity we can deal with functions defined in $I = (0,1)$ because the important values of this variable are those ones close to 0, and the supremum of the interval is not influent on the norm of the space ([228]). The plan the following: we will build a new class of rearrangement-invariant Banach function spaces, which contains also some Orlicz Exp-type spaces. Then, , we will prove some reduction theorems and, finally, we will get the boundedness result for the Hardy-Littlewood maximal operator, via a Hardy type inequality.

A recent investigation involving the link between the variable Lebesgue spaces and the grand Lebesgue spaces is in [231].

For the sequel it is important to note that it makes sense to consider the function

$x \in I \to \|f\|_{p(x)}$

which is measurable, because it is a composition of the borelian function $\|f\|_{(\cdot)}$ (it is well known that $\|f\|_{(\cdot)}$ is continuous where it is finite, see e.g. [235, $Ch.\,3\ ex.\,4b$)], see also

$[229, \text{Theorem } 3.5.7\ p.94]$) and of the measurable function $p(\cdot)$(we recall that the composition of two measurable functions is not necessarily a measurable function, but the composition of a borelian function with a measurable function it is, see e.g. $[229, Sect. 1.5\ p.41], [233, Ch. 8\ p.231]$). We observe also that for any measurable function δ in I it makes sense to consider the function (by convention we set the esssup equal to 0 if computed over the null set)

$$x \in I \ \rightarrow \ \underset{y \in I : p(y) \geq p(x)}{\text{esssup}} \ \delta(y)$$

which is measurable, because it is a composition of the borelian function (it is borelian since it is monotone, see e.g. $[228, p.298]$)

$$t \in (0, \infty) \ \rightarrow \ \underset{y \in I : p(y) \geq t}{\text{esssup}} \ \delta(y)$$

and of the measurable function $p(\cdot)$.

Let M be the set of all Lebesgue measurable functions in I with values in $[-\infty, +\infty], M^+$ the subset of the nonnegative functions, M_0 the subset of the real valued functions, and M_0^+ the subset of the real valued, nonnegative functions. Let $p(\cdot) \in M, p(\cdot) \geq 1\ a.e.$ and$\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \leq 1$. For $f \in M^+$ we set

$$\rho_{p[\cdot],\delta(\cdot)}(f) = \operatorname*{ess\,sup}_{x \in I} \rho_{p(x)} \ (\delta(x) f(\cdot))$$

where

$$\rho_{p(x)}(\delta(x) f(\cdot)) = \begin{cases} \left(\int_I \big(\delta(x) f(t)\big)^{p(x)} \, dt \right)^{\frac{1}{p(x)}} & if\ 1 \leq p(x) < \infty\,; \\ \operatorname*{ess\,sup}_{t \in I} \big(\delta(x) f(t)\big) & if\ p(x) = \infty \end{cases} \qquad (3)$$

The assumption $\delta > 0\ a.e.$ is needed in the proof (see Proposition (4.1.1)) that $\rho_{p[\cdot],\delta(\cdot)}$ is a Banach function norm, namely, when we show that $\rho_{p[\cdot],\delta(\cdot)}(f) = 0$ implies $f = 0$ ($if \delta = 0$ in a set $E, |\ E\ | > 0$, then it would be $\rho_{p[\cdot],\delta(\cdot)}(\chi_E) = 0$).

It is also needed that δ must be bounded, again in the proof of Proposition (4.1.1) (see Property 7). The assumption $\|\delta\|_\infty \leq 1$ is made in view of the following reason. Suppose for the moment that the simbol $\rho_{p[\cdot],\delta(\cdot)}$ has been defined assuming only δ bounded. It is easy, in this case, for a given δ such that $\|\delta\|_\infty > 1$, to construct a function $\bar{\delta}$ which gives an equivalent function norm and whose *esssup i*s bounded by 1: in fact, setting

$$\bar{\delta}(x) = \frac{\delta(x)}{\|\delta\|_\infty},$$

since

$$\delta(x) = \frac{\delta(x)}{\|\delta\|_\infty} \|\delta\|_\infty \leq \|\delta\|_\infty \bar{\delta}(x)$$

and

$$\bar{\delta}(x) = \frac{\delta(x)}{\|\delta\|_\infty} \leq \delta(x),$$

we have

$$\rho_{p[\cdot],\delta(\cdot)}(f) \le \rho_{p[\cdot],\delta(\cdot)}(f) \le \|\delta\|_\infty \rho_{p[\cdot],\delta(\cdot)}(f)$$

Now we spend few words about the symbol $\rho_{p[\cdot],\delta(\cdot)}(f)$, namely, on the square brackets inp $[\cdot]$. It would be natural to write $p(\cdot)$, because the modular depends on the variable exponent $p\,(\cdot)$, but the same symbol is already well known at least in a couple of contexts in the theory of variable Lebesgue spaces with a different meaning: it could be confused with the weighted modular associated with $p(\cdot)$ or with the modular of the variable Lorentz space (see [230]). Moreover, note that in the definition of $\rho_{p[\cdot],\delta(\cdot)}(f)$there is the letter f on the left hand side and there is $f(\cdot)$ on the right hand side: we add $(\cdot)$ inside the Banach function norm of a Lebesgue space. This choice is made because in such situation there are either x, either the variable of f to deal with, and it should be clear that the norm must not computed with respect to the x variable.

We exhibit now a few interesting particular cases. The simbol $\rho_q, 1 \le q \le \infty$, implicit in (3), stands for the usual Banach function norm of the Lebesgue space $L^q(I)$.

We begin by observing that if $p(\cdot) \equiv \rho_\infty(p(\cdot))$ on a set of positive measure, it is $\rho_{p[\cdot],\delta(\cdot)}(f) \approx \rho_{\rho_\infty(p(\cdot))}(f(\cdot))$ (the symbol $A \approx B$, where$A$ and B depend on $f \in M_0^+$, means, here and in the following, that there exist positive constants c_1, c_2, independent of f, such that $c_1 A \le B \le c_2 A$) and therefore $\rho_{p[\cdot],\delta(\cdot)}$reduces to the Banach function norm of a Lebesgue space. In fact, it is

$$\rho_{p[\cdot],\delta(\cdot)}(f) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x)}\,(\delta(x) f(\cdot))$$

$$\le \operatorname*{ess\,sup}_{x\in I} \left(\delta(x)\rho_{\rho_\infty(p(\cdot))}\big(f(\cdot)\big)\right) \le \rho_{\rho_\infty(p(\cdot))}\big(f(\cdot)\big) \operatorname*{ess\,sup}_{x\in I} \delta(x) \le \rho_{\rho_\infty(p(\cdot))}(f(\cdot))$$

On the other hand, if $p(\cdot) \equiv \rho_\infty(p(\cdot))$ in $E, |\,E\,| > 0$ (the symbol $|E|$ will denote the Lebesgue measure of E), we can consider a set $E' \subset E$ such that $|E'| > 0$ and

$$\delta(x) \ge \frac{\|\delta\|_{L^\infty(E)}}{2} \;\; \forall x \in E' \; a.e.$$

Then

$$\rho_{\rho_\infty(p(\cdot))}(f(\cdot)) \le \delta(x)^{-1} \operatorname*{ess\,sup}_{x\in E'} \delta(x)\, \rho_{\rho_\infty(p(\cdot))}(f(\cdot)) \;\forall x \in E' \; a.e.$$

hence

$$\rho_{\rho_\infty(p(\cdot))}\big(f(\cdot)\big) \le \frac{2}{\|\delta\|_{L^\infty(E')}}\; \rho_{p[\cdot],\delta(\cdot)}(f)$$

The (standard) grand Lebesgue spaces $L^{p)}(I)$ can be immediately obtained from (3) setting $\delta\,(x) = x$ and $p(x) = p - x$, where $1 < p < \infty$. The generalized grand Lebesgue spaces considered in [227] are evidently included in the family of the function norm (3): the function $\delta(\epsilon)^{\frac{1}{p-\epsilon}}$ in [227] corresponds to the function δ in our notation.

Setting in (3) $p(x) = 1/x$ and $\delta(x) = x^\alpha, \alpha > 0$, we get a norm of the Orlicz space $EXP_{1/\alpha}$ (see e.g. [228, 229]), which is the space of the functions such that $exp((\lambda f)^{1/\alpha}) \in L^1(I)$ for some $\lambda > 0$: this can be easily seen setting $p(x) = 1/x$ in Lemma 2.1 in [230] (see also [231]). Note also that setting in (3) $p(x)=x$ and choosing certain continuous functions δ one gets the so-called bilateral grand Lebesgue spaces studied in [232]. We are going to show that the functional $\rho_{p[\cdot],\delta(\cdot)}$ is a Banach function norm, i.e. (see e.g. [233])

$(i).\ \rho_{p[\cdot],\delta(\cdot)}(f)\geq 0$

$(ii).\ \rho_{p[\cdot],\delta(\cdot)}(f)=0$ iff $f=0$ in $I\ a.e.$

$(iii).\ \rho_{p[\cdot],\delta(\cdot)}(\lambda f)=\lambda\rho_{p[\cdot],\delta(\cdot)}(f)\ \forall\ \lambda>0$

$(iv).\ \rho_{p[\cdot],\delta(\cdot)}(f+g)\leq\rho_{p[\cdot],\delta(\cdot)}(f)+\rho_{p[\cdot],\delta(\cdot)}(g)$

$(v).\ if\ 0\leq g\leq f$ in $I\ a.e.$, then $\rho_{p[\cdot],\delta(\cdot)}(g)\leq\rho_{p[\cdot],\delta(\cdot)}(f)$

$(vi).\ if\ 0\leq f_n\uparrow f$ in $I\ a.e.$, then $\rho_{p[\cdot],\delta(\cdot)}(f_n)\uparrow\rho_{p[\cdot],\delta(\cdot)}(f)$

$(vii)\ \rho_{p[\cdot],\delta(\cdot)}(\chi_E)<+\infty\ \forall E\subset I$

$(viii).\ \int_E f dx\leq C(p,\delta,E)\rho_{p[\cdot],\delta(\cdot)}(f)\forall\ E\subset I$

Proposition(4.1.1)[222]: $.\rho_{p[\cdot],\delta(\cdot)}$ is a Banach function norm.

Proof: (i). It is obvious.

(ii). It is

$$f=0\ a.e.\Rightarrow\rho_{p(x)}(\delta(x)f(\cdot))=0,\forall x\in I\ a.e.\Rightarrow\rho_{p[\cdot],\delta(\cdot)}(f)=0$$

and, being $\delta>0\ a.e.$,

$$\rho_{p[\cdot],\delta(\cdot)}(f)=0\ \Rightarrow\rho_{p(x)}\big(\delta(x)f(\cdot)\big)=0,\qquad\forall x\in I\ a.e.\ \Rightarrow\delta(x)\rho_{p(x)}(f(\cdot))=0,\quad\forall x\in I\ a.e.$$
$$\Rightarrow\rho_{p(x)}\big(f(\cdot)\big)=0,\qquad\forall x\in I\ a.e.\Rightarrow f=0\ inI\ a.e.$$

(iii). It follows immediately from the homogeneity of the modular $\rho_{p(x)}\ \forall x\in I\ a.e.$

(iv). It is consequence of the corresponding property for the norm of the Lebesgue spaces:

$$\rho_{p(x)}\big(\delta(x)(f+g)(\cdot)\big)\leq\rho_{p(x)}\big(\delta(x)f(\cdot)\big)+\rho_{p(x)}\big(\delta(x)g(\cdot)\big)\forall x\in I\ a.e.$$

(v). As above, it is consequence of the order-preserving property of the modular $\rho_{p(x)}\ \forall x\in I\ a.e.$

(vi). If $0\leq f_n\uparrow f\ in\ I\ a.e.$,

$$\rho_{p[\cdot],\delta(\cdot)}(f_n)\uparrow\sup_n\rho_{p[\cdot],\delta(\cdot)}(f_n)\ =\sup_n\operatorname*{ess\,sup}_{x\in I}\rho_{p(x)}\big(\delta(x)f_n(\cdot)\big)$$

$$=\sup_n\rho_\infty\Big(\rho_{p(x)}\big(\delta(x)f_n(\cdot)\big)\Big)=\rho_\infty\Big(\sup_n\rho_{p(x)}\big(\delta(x)f_n(\cdot)\big)\Big)$$
$$=\operatorname*{ess\,sup}_{x\in I}\rho_{p(x)}(\delta(x)f(\cdot))=\rho_{p[\cdot],\delta(\cdot)}(f)$$

(vii). Let $E\subset I$

$$\rho_{p[\cdot],\delta(\cdot)}(\chi_E)=\operatorname*{ess\,sup}_{x\in I}\delta(x)\rho_{p(x)}\big(\chi_E(\cdot)\big)\leq\operatorname*{ess\,sup}_{x\in I}\delta(x)<\infty$$

$(viii)$. Let $E_\delta\subset I,|E_\delta|>0$ be such that

$$\delta(x)>\frac{\|\delta\|_\infty}{2},\qquad\forall x\in E_\delta a.e.$$

We have

$$\delta(x)^{-1} < \frac{2}{\|\delta\|_\infty}, \qquad \forall x \in E_\delta \ a.e.$$

and therefore, by Hölder's inequality, for $x \in E_\delta a.e.$

$$\int_E f dx = \int_0^1 f\chi_E dx \le \rho_{p(x)}(f(\cdot))\rho_{p'(x)}(\chi_E(\cdot))$$
$$= \delta(x)\rho_{p(x)}(f(\cdot))\delta(x)^{-1}\rho_{p'(x)}(\chi_E(\cdot))$$
$$\le \operatorname*{ess\,sup}_{x\in E_\delta} \delta(x)\rho_{p(x)}(f(\cdot)) \cdot \operatorname*{ess\,sup}_{x\in E_\delta} \delta(x)^{-1}\rho_{p'(x)}(\chi_E(\cdot))$$
$$\le \rho_{p[\cdot],\delta(\cdot)}(f) \cdot \operatorname*{ess\,sup}_{x\in E_\delta} \delta(x)^{-1} \le \frac{2}{\|\delta\|_\infty}\rho_{p[\cdot],\delta(\cdot)}(f)$$

We are now in position to make the following

Definition $(\mathbf{4.1.2})[\mathbf{222}]$: Let $p(\cdot) \in M, p(\cdot) \ge 1 \ a.e.$ and $\delta \in L^\infty(I), \delta > 0 \ a.e., 0 < \|\delta\|_\infty \le 1$. The Banach function space

$$L^{\rho_{p[\cdot],\delta(\cdot)}}(I) = \{f \in M_0 : \|f\|_{\rho_{p[\cdot],\delta(\cdot)}} = \rho_{p[\cdot],\delta(\cdot)}(|f|) < \infty\}$$

is called fully measurable grand Lebesgue space.

We remark that the fully measurable grand Lebesgue spaces are rearrangement-invariant Banach function spaces. We observe also that these spaces cannot be considered in the framework of the so-called mixed norm spaces. In fact, the mixed norm spaces are defined starting from two Banach function spaces X and Y, and considering the set $X[Y]$of all functions $f(x,t)$such that $x \to \|f(x,\cdot)\|_Y$ belongs to X; the norm of f is then written $\|f\|_{X[Y]} = \|\|f(x,\cdot)\|_Y\|_X$ (see details in [230], which gives a definition introduced in [236], going back to [233]). In our case $\|f\|_{p[\cdot],\delta(\cdot)}$ is of the type $\|\|f(x,\cdot)\|_Y\|_X$ where X is a weighted L^∞ space but Y depends on x (in fact, $Y = L^{p(x)}$) . We recall that [232] introduced the composed grand Lebesgue spaces, which are of the type $\|f\|_{X[Y]} = \|\|f(x,\cdot)\|_Y\|_X$ where X is a rearrangement-invariant Banach function space, while in our case it is not. In conclusion, our spaces escape from both categories of spaces. The norm of grand Lebesgue spaces has been of interest in the framework of extrapolation theory (see [226]). Variants of the norm of grand Lebesgue spaces are of recent interest, too. Grand Bochner–Lebesgue spaces have been considered in [227]. We mention the [231], where Herz-Morrey spaces are considered; the norm of these spaces, even if different from that one we consider, has connections either with the grand Lebesgue spaces, either with the variable exponent Lebesgue spaces. Grand Lebesgue spaces over sets of infinite measure have been recently considered in [236]. Finally, we recall another variant recently appeared in [231], where a composition of norms of grand Lebesgue spaces has been investigated.

Generalizations of the norm of the grand Lebesgue spaces with δ different from $\delta(x) = x$ are of interest in Applications: for instance, in PDFs we mention [233], in Harmonic Analysis we mention [226]. We believe that the new context of the measurable data will be fruitful also for the study of the small Lebesgue spaces, defined as the associate spaces

of the grand Lebesgue spaces. Very recently they appeared of crucial importance in the question of the dimension-free Sobolev embedding theorem ([229]).
We begin by a remark on the norm of the grand Lebesgue spaces, given by

$$\|f\|_{p)} = \operatorname*{esssup}_{0<\epsilon<p-1} \epsilon^{\frac{1}{p^{-\epsilon}}} \left(\int_1 f(t)^{p^{-\epsilon}} \, dt \right)^{\frac{1}{p^{-\epsilon}}} \quad (1 < p < \infty) \qquad (4)$$

It is known that the "influent" values of ϵ in such norm are the "small" ones: in fact in [236] it has been used the fact that the supremum over $\epsilon \in (0, \sigma)$ plays the same role as the supremumover all the interval $(0, p-1)$. The same phenomenon has been used in [230], where the interval$(0, (p-1)/2)$ has been considered in the associate spaces of the grand Lebesgue spaces. This remark has been formalized and generalized in the recent [233], where the more general framework of the grand grand Morrey spaces has been considered. Note that the small values of ϵ correspond to the large values of the exponent $p^{-\epsilon}$, which appears in the norm (4).
The next two results show that this phenomenon holds also for the fully measurable grand Lebesgue spaces; namely, the norm is essentially given by the supremum over the x's where the exponent $p(\cdot)$ is large. In Theorem (4.1.3) we show that if the supremum is considered over any level set of the exponent $p(\cdot)$, then one gets an equivalent norm. In Theorem (4.1.4) we will show that two functions δ coinciding on all of the level sets sufficiently high of the exponent $p(\cdot)$ generate the same Banach function norm. This result can be combined with Theorem (4.1.7), where for any given δ it is constructed a special $\bar{\delta}$ satisfying the assumption of Theorem (4.1.4). This will lead to a kind of regularity result: in the case $p(\cdot)$upper semicontinuous, without loss of generality one may assume δ lower semicontinuous, and the "esssup" defining the norm of the fully measurable grand Lebesgue spaces can be written as " sup".
The results are the typical ones to be obtained after a generic definition like Definition (4.1.2): different preliminary data may determine the same set of functions. This situation appears, for instance, for Orlicz spaces (see e.g. [230, Theorem 3.4 p. 18, see also p. 22]); in the case of weighted Orlicz spaces, where different nontrivial weights may give the same space [234]; for variable Lebesgue spaces, see [238]; for the generalized grand Lebesgue spaces, see [227]. We call these results reduction theorems because the class of the original data can be reduced, without loss of Banach function spaces.
Theorem (4.1.3)[222]: Let $p(\cdot) \in M, p(\cdot) \geq 1 \ a.e., p(\cdot) \not\equiv 1$ and $\delta \in L^\infty(I), \delta > 0 \ a.e., 0 < \|\delta\|_\infty \leq 1$. For $f \in M^+$ we have

$$\rho_{p[\cdot],\delta(\cdot)}(f) \approx \operatorname*{esssup}_{x \in p^{-1}([\tau, \rho_\infty(p(\cdot))])} \rho_{p(x)}\big(\delta(x) f(\cdot)\big) \qquad \forall \tau \in [1, \rho_\infty(p(\cdot))[$$

Proof: If $\tau = 1$ or if $\tau \in]1, \rho_\infty(p(\cdot))[$is such that $|\, p^{-1}([1, \tau[\,)\,| = 0$ then

$$\rho_{p[\cdot],\delta(\cdot)}(f) = \operatorname*{esssup}_{x \in p^{-1}([\tau, \rho_\infty(p(\cdot))])} \rho_{p(x)}\big(\delta(x) f(\cdot)\big)$$

and therefore there is nothing to prove. Otherwise it is $\tau \in]1, \rho_\infty(p(\cdot))[$ and $|\, p^{-1}([1, \tau[\,)\,| > 0$. Setting

$$E_1^\tau = p^{-1}\big(\big[\tau, \rho_\infty\big(p(\cdot)\big)\big]\big), \qquad T_1^\tau = \operatorname*{esssup}_{x\in E_1^\tau} \rho_{p(x)}\big(\delta(x) f(\cdot)\big)$$

and

$$E_2^\tau = I \setminus E_1^\tau = p^{-1}([1,\tau[\,), \qquad T_2^\tau = \operatorname*{esssup}_{x\in E_2^\tau} \rho_{p(x)}\big(\delta(x) f(\cdot)\big)$$

we have, by Hölder's inequality,

$$\rho_{p(x)}\big(f(\cdot)\big) \le \rho_\tau\big(f(\cdot)\big) \quad \forall x \in E_2^\tau \ a.e.$$

And

$$\rho_\tau\big(f(\cdot)\big) \le \rho_{p(x)}\big(f(\cdot)\big) \quad \forall x \in E_1^\tau \ a.e.$$

Then for $x \in E_1^\tau \ a.e.$

$$\operatorname*{esssup}_{x\in E_2^\tau} \rho_{p(x)}\big(\delta(x) f(\cdot)\big) \le \operatorname*{esssup}_{x\in E_2^\tau} \rho_\tau\big(\delta(x) f(\cdot)\big) = \left(\operatorname*{esssup}_{x\in E_2^\tau} \delta(x)\right) \rho_\tau(f(\cdot))$$

$$\le \left(\operatorname*{esssup}_{x\in E_2^\tau} \delta(x)\right) \rho_{p(x)}(f(\cdot))$$

If $E_\delta^\tau \subset E_1^\tau$ is such that $| E_\delta^\tau | > 0$ and

$$\delta(x) > \frac{\|\delta\|_{L^\infty(E_1^\tau)}}{2} \quad \forall x \in E_\delta^\tau \ a.e.$$

it is, for $x \in E_\delta^\tau \ a.e.$,

$$T_2^\tau \le \left(\operatorname*{esssup}_{x\in E_2^\tau} \delta(x)\right) \delta(x)^{-1} \delta(x) \rho_{p(x)}(f(\cdot))$$

and therefore

$$T_2^\tau \le \operatorname*{esssup}_{x\in E_2^\tau} \delta(x) \cdot \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}} \cdot \operatorname*{esssup}_{x\in E_1^\tau} \delta(x) \rho_{p(x)}\big(f(\cdot)\big) \le \left(\operatorname*{esssup}_{x\in E_2^\tau} \delta(x) \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}}\right) T_1^\tau$$

Therefore, being

$$\rho_{p[\cdot],\delta(\cdot)}(f) = \max\{T_1^\tau, T_2^\tau\}$$

we have

$$\rho_{p[\cdot],\delta(\cdot)}(f) \le \max\left\{1, \operatorname*{esssup}_{x\in E_2^\tau} \delta(x) \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}}\right\} T_1^\tau \qquad (5)$$

Finally, we observe that trivially $T_1^\tau \le \rho_{p[\cdot],\delta(\cdot)}(f)$.

Theorem (4.1.4)[222]: Let $p(\cdot) \in M, 1 \le p(\cdot) < \rho_\infty(p(\cdot)) a.e., \tau_0 \in [1, \rho_\infty(p(\cdot))[$ and $\delta_1, \delta_2 \in L^\infty(I), \delta_1, \delta_2 > 0 \ a.e., 0 < \|\delta_1\|_\infty \le 1, 0 < \|\delta_2\|_\infty \le 1$. If

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta_1(x) = \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta_2(x) \quad \forall \tau \in [\tau_0, \rho_\infty(p(\cdot))[\ (6)$$

then for every $f \in M_0^+$

$$\rho_{p[\cdot],\delta_1(\cdot)}(f) = \rho_{p[\cdot],\delta_2(\cdot)}(f) \qquad (7)$$

Proof: It is sufficient to prove the inequality $\le$ in (7), excluding the case $f \equiv 0$. Let $0 < \sigma < \eta < \rho_{p[\cdot],\delta_1(\cdot)}(f)$. There exists $T_\eta \subset I, | T_\eta | > 0$, such that

$$\rho_{p(x)}\big(\delta_1(x)f(\cdot)\big) > \eta \qquad \forall x \in T_\eta \ a.e. \tag{8}$$

We may assume, without loss of generality, that $\rho_{p(x)}(\delta_1(x)f(\cdot)) < \infty \forall x \in T_\eta \ a.e.$, otherwise the assertion is trivially true (because in this case it would be $\rho_{p(x)}(f(\cdot)) = \infty$ and both sides of (7) would be infinite); we may therefore write $\|f\|_{p(x)}$ in place of $\rho_{p(x)}(f(\cdot))$. For any $\tau, 1 \leq \tau < \rho_\infty(p(\cdot))$, let $F_{\tau,\eta}$ be the set

$$F_{\tau,\eta} = p^{-1}([1,\tau]) \cap T_\eta$$

We observe that

$$\bigcup_{\tau \in [\tau_0, \rho_\infty(p(\cdot))[} p^{-1}([1,\tau]) = I$$

Put

$$\tau_{*,\eta} = \inf\{\tau' \in [\,\tau_0, \rho_\infty(p(\cdot))[: | \ F_{\tau,\eta} \ | > 0 \forall \tau \ \in [\,\tau', \rho_\infty(p(\cdot))[\,\}$$

We claim that

$$T_\eta \subset p^{-1}([\tau_{*,\eta}, \rho_\infty(p(\cdot))[) \tag{9}$$

In fact, if $\tau_{*,\eta} > 1$ (if $\tau_{*,\eta} = 1$ then (9) it is trivially true), we have

$$| \ F_{\tau,\eta} \ | = | \ p^{-1}([1,\tau]) \cap T_\eta \ | = 0 \qquad \forall \tau \ \in [1, \tau_{*,\eta}[$$

and therefore

$$T_\eta \subset p^{-1}(\,]\tau, \rho_\infty\big(p(\cdot)\big)[\,) \qquad \forall \tau \ \in [1, \tau_{*,\eta}[$$

from which (9) follows. By (8)

$$\delta_1(x) > \frac{\eta}{\|f\|_{p(x)}} \ \geq \frac{\eta}{\|f\|_\tau} \qquad \forall x \in F_{\tau,\eta} \ a.e., \qquad \forall \tau \ \in]\tau_{*,\eta}, \rho_\infty(p(\cdot))[\tag{10}$$

and therefore by (9)

$$\operatorname*{esssup}_{x \in p^{-1}(\,]\tau_{*,\eta}, \rho_\infty(p(\cdot))[\,)} \delta_1(x) \ \geq \operatorname*{esssup}_{x \in T_\eta} \delta_1(x) \geq \operatorname*{esssup}_{x \in F_{\tau,\eta}} \delta_1(x) > \frac{\eta}{\|f\|_\tau} \quad \forall \tau \in]\tau_{*,\eta}, \rho_\infty(p(\cdot))[$$

Passing, on the right hand side, on the supremum over $\tau, \tau_{*,\eta} < \tau \ < \rho_\infty(p(\cdot))$

$$\operatorname*{esssup}_{x \in p^{-1}(\,[\tau_{*,\eta}, \rho_\infty(p(\cdot))[\,)} \delta_1(x) \geq \frac{\eta}{\|f\|_{\tau_{*,\eta}}}$$

and therefore, by our assumption (6), being $\sigma < \eta$,

$$\operatorname*{esssup}_{x \in p^{-1}(\,[\tau_{*,\eta}, \rho_\infty(p(\cdot))[\,)} \delta_2(x) > \frac{\sigma}{\|f\|_{\tau_{*,\eta}}} \tag{11}$$

We deduce that there exists $T'_\eta \ \subset p^{-1}([\tau_{*,\eta}, \rho_\infty(p(\cdot))[\,), | \ T'_\eta \ | > 0$, such that

$$\delta_2(x) > \frac{\sigma}{\|f\|_{\tau_{*,\eta}}} \qquad \forall x \in T'_\eta \ a.e.$$

and therefore by (11)

$$\rho_{p[\cdot],\delta_2(\cdot)}(f) \geq \operatorname*{esssup}_{x\in T'_\eta} \delta_2(x)\|f\|_{p(x)} \geq \operatorname*{esssup}_{x\in T'_\eta} \delta_2(x)\|f\|_{\tau_{*,\eta}} > \sigma$$

Since this is true for all $\sigma < \rho_{p[\cdot],\delta_1(\cdot)}(f)$, we may conclude that $\rho_{p[\cdot],\delta_2(\cdot)}(f) \geq \rho_{p[\cdot],\delta_1(\cdot)}(f)$.

Corollary (4.1.5)[222]: Let $p(\cdot) \in M, p(\cdot) \geq 1\ a.e., p(\cdot) \not\equiv 1, \tau_0 \in [1, \rho_\infty(p(\cdot))[$ and $\delta_1, \delta_2 \in L^\infty(I), \delta_1, \delta_2 > 0\ a.e., 0 < \|\delta_1\|_\infty \leq 1, 0 < \|\delta_2\|_\infty \leq 1$. If

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))[)} \delta_1(x) = \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))[)} \delta_2(x)\ \forall\, \tau \in [\tau_0, \rho_\infty(p(\cdot))[\ (12)$$

Then

$$\rho_{p[\cdot],\delta_1(\cdot)}(f) \approx \rho_{p[\cdot],\delta_2(\cdot)}(\cdot)(f) \qquad (13)$$

Proof: If $p(\cdot) < \rho_\infty(p(\cdot))\ a.e.$, by Theorem (4.1.4) we know that equality holds in (13). Otherwise, we recall that we showed that both sides of (13) are equivalent to the function norm $\rho\rho_\infty(p(\cdot))$.

Example (4.1.6)[222]: We remark that in general equality in (13) does not hold when $p(\cdot) = \rho_\infty(p(\cdot))$ on a set of positive measure. It is sufficient to consider $p_0 \in]1,\infty], p(x) = p_0\chi_{(0,1/2)}(x) + \chi_{(1/2,1)}(x), \delta_0 \in]0,1[, \alpha_i \in]0,1[, \delta_0 < \alpha_i, \delta_i(x) = \delta_0\,\chi_{(0,1/2)}(x) + \alpha_i\chi_{(1/2,1)}(x), i = 1,2$. It is $\rho_{p[\cdot],\delta_i(\cdot)}(f) = \max\{\delta_0\,\rho_{p_0}(f(\cdot)), \alpha_i\rho_1(f(\cdot))\}$, and therefore if $f \equiv 1$ it is $\rho_{p[\cdot],\delta_i(\cdot)}(f) = \alpha_i, i = 1,2$. In next result we will show the coincidence of the essential supremum of two measurable functions. The fact that

$$x \in I \to \operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y)$$

is a measurable function has been shown.

Theorem (4.1.7)[222]: If $p(\cdot) \in M, p(\cdot) \geq 1\ a.e., p(\cdot) \not\equiv 1$ and $\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \leq 1$, then

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(x) = \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y)\right]\ \forall\tau \in [1, \rho_\infty(p(\cdot))[$$

Proof: Since for every $\tau \in [1, \rho_\infty(p(\cdot))[$ it is $\{y \in I : p(y) \geq p(x)\} \subset p^{-1}([\tau, \rho_\infty(p(\cdot))])$ for $a.e. x \in p^{-1}([\tau, \rho_\infty(p(\cdot))])$, we have

$$\operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y) \leq \operatorname*{esssup}_{z\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(z)$$

and therefore, since the right hand side does not depend on x,

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y)\right] \leq \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(x)\ \forall\,\tau \in [1, \rho_\infty(p(\cdot))[$$

We need to prove

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(x) \leq \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y)\right]\ \forall\tau \in [1, \rho_\infty(p(\cdot))[$$

If, on the contrary, there exists $\tau \in [1, \rho_\infty(p(\cdot))[$ such that

$$\operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(x) > \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y)\right]$$

then

$$E_{p,\tau} := \left\{ x \in p^{-1}([\tau, \rho_\infty(p(\cdot))] : \delta(x) > \operatorname*{esssup}_{x \in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right] \right\}$$

has a positive measure. There are two cases:

(i) $\operatorname*{essinf}_{x \in E_{p,\tau}} p(x) = \operatorname*{esssup}_{x \in E_{p,\tau}} p(x)$

(ii) $\operatorname*{essinf}_{x \in E_{p,\tau}} p(x) < \operatorname*{esssup}_{x \in E_{p,\tau}} p(x)$

In the case (i) $p(\cdot)$is constant $a.e.$ in $E_{p,\tau}$, and

$$\operatorname*{esssup}_{x \in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right] \geq \operatorname*{esssup}_{x \in E_{p,\tau}} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right]$$

$$\geq \operatorname*{esssup}_{x \in E_{p,\tau}} \left[\operatorname*{esssup}_{y \in E_{p,\tau}} \delta(y) \right] = \operatorname*{esssup}_{x \in E_{p,\tau}} \delta(x) > \operatorname*{esssup}_{x \in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right]$$

which is absurd. In the case (ii), let

$$\alpha \in \left] \operatorname*{essinf}_{x \in E_{p,\tau}} p(x), \operatorname*{esssup}_{x \in E_{p,\tau}} p(x) \right[$$

and set

$$F_{p,\tau} := p^{-1}\left(\left[\operatorname*{essinf}_{x \in E_{p,\tau}} p(x), \alpha \right] \right) \cap E_{p,\tau}$$

$$G_{p,\tau} := p^{-1}\left(\left] \alpha, \operatorname*{esssup}_{x \in E_{p,\tau}} p(x) \right] \right) \cap E_{p,\tau}$$

and observe that both such sets have positive measure.

We have

$$\operatorname*{esssup}_{x \in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right] \geq \operatorname*{esssup}_{x \in G_{p,\tau}} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right] \geq \operatorname*{esssup}_{x \in G_{p,\tau}} \left[\operatorname*{esssup}_{y \in F_{p,\tau}} \delta(y) \right]$$

$$= \operatorname*{esssup}_{x \in G_{p,\tau}} \delta(x) > \operatorname*{esssup}_{x \in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \left[\operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y) \right]$$

which is absurd.

Theorem (4.1.7), combined with Theorem (4.1.4), tells that a function δ can be substituted by

$$x \in I \to \bar{\delta}(x) = \operatorname*{esssup}_{y: p(y) \geq p(x)} \delta(y)$$

without changing the space. In the case considered in [237] where the exponent is decreasing, this means that one can consider the generalized grand Lebesgue spaces only for the δ's increasing (see Proposition (4.1.3) therein). In our case we can assert that the meaningful δ's are those ones which have "in some sense" the opposite monotonicity with respect to $p(\cdot)$: if $p(\cdot)$ is defined pointwise,

$$p(x_1) \leq p(x_2) \Rightarrow \bar{\delta}(x_1) \geq \bar{\delta}(x_2) \qquad (14)$$

The notion of monotonicity of a function equal or opposite with respect to another is known in literature: see e.g. the equally ordered functions in [235] or [241]. As consequence of (14), we can show that if $p(\cdot)$ is an $u.s.c.$ exponent, then it is not restrictive to make the extra assumption $\delta\ l.s.c.$

Proposition $(\mathbf{4.1.8})[\mathbf{222}]$: If $p(\cdot) \in M, p(\cdot) \geq 1\ a.e., p(\cdot)$ has no flat zone, $p\ u.s.c.$, and $\delta \in L^{\infty}(I), \delta > 0\ a.e., 0 < \|\delta\|_{\infty} \leq 1$, then there exists $\bar{\delta} \in L^{\infty}(I), \bar{\delta} > 0\ a.e., 0 < \|\bar{\delta}\|_{\infty} \leq 1, \bar{\delta}\ l.s.c.$, such that for every $f \in M_0^+$

$$\rho_{p[\cdot],\delta(\cdot)}(f) \approx \rho_{p[\cdot],\bar{\delta}(\cdot)}(f) \qquad (15)$$

Proof: Let $x_0 \in I$, and let $x_n \in I, x_n \to x_0$. By our assumption

$$p(x_0) \geq \limsup_n p(x_n)$$

If $p(\cdot) = \rho_{\infty}(p(\cdot))a.e.$, the assert is trivial (we can take for instance $\bar{\delta} \equiv 1$). Otherwise, setting

$$\bar{\delta}(x) = \operatorname*{esssup}_{y:p(y)\geq p(x)} \delta(y) \quad \forall x \in I$$

the equivalence (15) follows from Corollary (4.1.5) and Theorem (4.1.7). We need only to show that $\bar{\delta}$ is $l.s.c., i.e.$

$$\bar{\delta}(x_0) \leq \liminf_n \bar{\delta}(x_n) \qquad (16)$$

Let us set

$$A = \{n \in \mathbb{N}\ :\ p(x_n) \leq p(x_0)\}$$
$$B = \{n \in \mathbb{N}\ :\ p(x_n) > p(x_0)\}$$

so that $\mathbb{N} = A \cup B$. If $n \in A$, it is $p(x_n) \leq p(x_0)$and therefore, by (14), $\bar{\delta}(x_n) \geq \bar{\delta}(x_0)$. If B is finite,the assertion is trivially true. Otherwise, let $B = \{k_n\}$and set $t_n = x_{k_n}$. It is

$$p(t_n) > p(x_0)\ \ \forall\, n \in \mathbb{N}$$
$$p(x_0) = \limsup_n p(t_n)$$

and therefore

$$p(x_0) \leq \liminf_n p(t_n) \leq \limsup_n p(t_n) = p(x_0)$$

from which

$$\lim_n p(t_n) = p(x_0)$$

Since $p(\cdot)$ has no flat zone,

$$\bigcup_n \{y\ :\ p(y) \geq p(t_n)\}\ = \{y\ :\ p(y) \geq p(x_0)\}$$

and therefore

$$\lim_n \bar{\delta}(t_n)\ = \bar{\delta}(x_0)$$

from which (16) follows.

After Proposition (4.1.8) we know that when the exponent is $u.s.c.$, then it is possible to write equivalently the expression of the norm in the fully measurable Lebesgue spaces by using "sup" instead of "esssup" (because the measurable function δ can be changed into a

$l.s.c.$ function, defined pointwise). Since the exponent $p(x) = p - x$ has no flat zone, this result generalizes the analogous reduction shown in [237]

Theorem $(\mathbf{4.1.9})[\mathbf{222}]$: Let $p(\cdot) \in M, p(\cdot) > 1\ a.e., \delta \in L^{\infty}(I), \delta > 0\ a.e., 0 < \|\delta\|_{\infty} \leq 1$ and for every $f \in M_0^+$ let

$$F(x) = \int_0^x f ds \quad \in [0, \infty]\ \forall x \in I$$

There exists a constant $c(p(\cdot), \delta) > 1$ such that the following inequality holds

$$\rho_{p[\cdot],\delta(\cdot)}(F) \leq c(p(\cdot), \delta)\rho_{p[\cdot],\delta(\cdot)}(f) \quad \forall f \in M_0^+ \tag{17}$$

Proof: Fix $\tau \in]1, \rho_{\infty}(p(\cdot))[, |\ p^{-1}([1, \tau[\ )\ | > 0$. Applying the classical Hardy's inequality with the exponent $p = p(x)$ and multiplying both sides by $\delta(x)$, we get

$$\left(\int_0^1 \left(\delta(x) \int_0^t f\, ds \right)^{p(x)} dt \right)^{\frac{1}{p(x)}} \leq \frac{p(x)}{p(x) - 1} \left(\int_0^1 (\delta(x) f) p(x)\, dt \right)^{\frac{1}{p(x)}} \quad \forall x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))]) a.e.$$

Passing to the esssup over x in both sides, the previous inequality becomes

$$\underset{x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))])}{\text{esssup}} \left(\int_0^1 \left(\delta(x) \int_0^t f\, ds \right)^{p(x)} dt \right)^{\frac{1}{p(x)}}$$

$$\leq \underset{x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))])}{\text{esssup}} \frac{p(x)}{p(x) - 1} \underset{x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))])}{\text{esssup}} \left(\int_0^1 (\delta(x) f) p(x)\, dt \right)^{\frac{1}{p(x)}}$$

and therefore, by Theorem (4.1.3) (see (5))

$$\rho_{p[\cdot],\delta(\cdot)}(F) \leq \max\left\{ 1, \underset{x \in I \setminus p^{-1}([\tau, \rho_{\infty}(p(\cdot))]\,)}{\text{esssup}} \delta(x) \frac{2}{\|\delta\|_{L^{\infty}(p^{-1}([\tau, \rho_{\infty}(p(\cdot))]\,))}} \right\} \cdot$$

$$\cdot \underset{p^{-1}([\tau, \rho_{\infty}(p(\cdot))]\,)}{\text{esssup}} \delta(x) \rho_{p(x)}(F(\cdot))$$

$$\leq \max\left\{ 1, \underset{x \in I \setminus p^{-1}([\tau, \rho_{\infty}(p(\cdot))]\,)}{\text{esssup}} \delta(x) \frac{2}{\|\delta\|_{L^{\infty}(p^{-1}([\tau, \rho_{\infty}(p(\cdot))]\,))}} \right\} \cdot$$

$$\cdot \underset{x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))])}{\text{esssup}} \frac{p(x)}{p(x) - 1} \underset{x \in p^{-1}([\tau, \rho_{\infty}(p(\cdot))])}{\text{esssup}} \left(\int_0^1 (\delta(x) f) p(x)\, dt \right)^{\frac{1}{p(x)}}$$

Setting

$$c(p(\cdot),\delta) := \inf_{\tau} \max\left\{1, \operatorname*{esssup}_{x\in I\setminus p^{-1}([\tau,\rho_\infty(p(\cdot))])} \delta(x) \frac{2}{\|\delta\|_{L^\infty(p^{-1}([\tau,\rho_\infty(p(\cdot))]))}}\right\} \cdot$$
$$\cdot \operatorname*{esssup}_{x\in p^{-1}([\tau,\rho_\infty(p(\cdot))])} \frac{p(x)}{p(x)-1} > 1$$
we get the desidered inequality (17).

As consequence of Theorem (4.1.9) we can get the boundedness result for the Hardy-Littlewood maximal operator, defined by

$$Mf(t) = \sup_{I\supset J\ni t} \frac{1}{|J|} \int_J |f|\, ds \qquad \forall t \in I, \quad f \in L^1(I)$$

where the supremum extends over all nondegenerate intervals, contained in I and containing t. We omit the (very short) proof, which is the same of Corollary 2.3 in [236]: the argument, inspired by [234, Thm 3.10, $p.$ 125], uses the property of the space to be rearrangement-invariant.

Corollary (4.1.10)[222]: Let $p(\cdot) \in M, p(\cdot) > 1\ a.e.$ and $\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \le 1$. There exists aconstant $c(p(\cdot),\delta) > 1$ such that the following inequality holds

$$\|Mf\|_{p[\cdot],\delta(\cdot)} \le c(p(\cdot),\delta)\|f\|_{p[\cdot],\delta(\cdot)} \qquad \forall f \in L^1(I)$$

Section (4.2): Weighted Fully Measurable Grand Lebesgue Spaces

The grand Lebesgue spaces $L^{p)}(1 < p < \infty)$ were introduced by Iwaniec and Sbordone in [261], in the framework of the study of Jacobian determinant. These spaces are Banach function spaces and, when considered over (0,1), are defined as

$$L^{p)}(0,1) = \left\{f : (0,1) \to \mathbb{R}\ measurable : \|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(t)|^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}} < \infty\right\}. \qquad (18)$$

Since then, these spaces attract interest because of their essential role and applications in various fields, such as in $PDE's$ theory (see e.g. [262]), in function spaces theory (see e.g. [263]) and in interpolation-extrapolation theory (see e.g. [264]). They have been widely investigated and several variations have been studied (see e.g. [265]).

In [266] (see also [267]) the authors introduced the weighted grand Lebesgue spaces $L_w^{p)}(0,1)$ equipped with the norm

$$\|f\|_{p),w} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(t)|^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}}, \qquad (1 < p < \infty), \qquad (19)$$

where w is a weight on (0,1) and they studied the boundedness of the Hardy– Littlewood maximal operator, defined by

$$Mf(x) = \sup_{x\in J\subset(0,1)} \frac{1}{|J|}\int_J |f|\ dt\ , \qquad x\ \in\ (0,1), \tag{20}$$

where the supremum extends over all nondegenerate intervals J, contained in $(0,1)$ and containing x, and $|J|$ denotes the Lebesgue measure of J. In the framework of the standard Lebesgue spaces, it is well known that, when $1\ <\ p\ <\ \infty$,

$$\|Mf\|_{p,w}\ \le c\|f\|_{p,w} \tag{21}$$

if and only if w satisfies the A_p condition of Muckenoupt ($w\ \in\ A_p$)

$$\sup_J\left(\frac{1}{|J|}\int_J w\ dt\right)\left(\frac{1}{|J|}\int_J w^{-\frac{1}{p-1}}\,dt\right)^{p-1} =:A_p(w)\ <\ \infty, \tag{22}$$

where the supremum in (22) extends over all intervals $J\ \subset\ (0,1)$, and $\|\cdot\|_{p,w}$ in (21) denotes the norm in the weighted Lebesgue spaces L_w^p , given by

$$\|f\|_{p,w}\ = \left(\int_0^1 |f|^p\ w\ dt\right)^{\frac{1}{p}} \quad .$$

In $[262]$ the authors characterized the weights for which the inequality

$$\|Mf\|_{p),w}\ \le c\|f\|_{p),w} \tag{23}$$

holds, where c is a constant independent of f, namely, they proved that Condition (22) is necessary and sufficient for the validity of inequality (23), too.

Since then, boundedness properties of operators of various type have been investigated in these spaces and their generalizations. We recall some of these results. In $[262]$ an analogous result to that one in $[262]$ has been proved for the onedimensional singular Hilbert operator. In $[264]$ boundedness of weighted singular integral operators in grand Lebesgue spaces we restudied . In $[269]$ families of weighted grand Lebesgue spaces which generalize weighted grand Lebesgue spaces were introduced and boundedness results of the Hardy–Littlewood maximal operator and the Calderón–Zygmund singular operators were established.

$[267]$ introduced the weighted grand space $L_w^{\infty)}$ $(0,1)$, which is equivalent to the weighted space EXP and boundedness results for the Hardy–Littlewood maximal operator and the Hilbert transform were given.

In $[269]$ weighted strong and weak-type norm inequalities for the Hardy–Littlewood maximal operator on the variable Lebesgue space $L^{p(\cdot)}$ were proved. In the same direction of the results in $[262]$, inspired by the new spaces introduced in $[270]$, We consider the weighted fully measurable grand Lebesgue spaces and we establish the maximal Theorem of Hardy–Littlewood, when $p^+\ = \|p\|_\infty <\ +\infty$. In the unweighted case, werecover the boundedness result proven in $[\ 271]$.

Let M be the set of all Lebesgue measurable functions in $I\ =\ (0,1)\ \subset\ \mathbb{R}$ with values in $[-\infty,+\infty]$, M^+ the subset of the nonnegative functions, M_0 the subset of the real valued functions, and M_0^+ the subset of the real valued, nonnegative functions.

Let $p(\cdot)\ \in M_0, p(\cdot)\ \ge 1\ a.e.$ and $\delta\ \in\ L^\infty(I), \delta > 0\ a.e., 0\ < \|\delta\|_\infty\ \le 1.$ In $[266]$ the following generalization of the grand Lebesgue spaces was considered

$$L^{p),\delta}(I) = \left\{ f \in M_0 : \|f\|_{p),\delta} = \sup_{0<\varepsilon<p-1} \left(\delta(\varepsilon) \int_I |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} < \infty \right\}. \quad (24)$$

In [272], using variable exponents, the authors built a new class of Banach function spaces, in a framework different than the variable Lebesgue spaces See [278]), called the fully measurable grand Lebesgue spaces

$$L^{p[\cdot],\delta(\cdot)}(I) = \{ f \in M_0 : \|f\|_{p[\cdot],\delta(\cdot)} = \rho_{p[\cdot],\delta(\cdot)}(|f|) < \infty \}, \quad (25)$$

where

$$\rho_{p[\cdot],\delta(\cdot)}(f) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x)} \left(\delta(x) f(\cdot)\right) (f \in M_0^+) \quad (26)$$

and

$$\rho_{p(x)}\left(\delta(x) f(\cdot)\right) = \left(\int_I \left(\delta(x) f(t)\right)^{p(x)} dt \right)^{\frac{1}{p(x)}} if\ 1 \le p(x) < \infty. \quad (27)$$

Remark $(\mathbf{4.2.1})[\mathbf{260}]$: In [274] $\rho_{p(x)}(\delta(x)\, f(\cdot))$ is defined also for $p(x) = \infty$, and it is proved that if $p(x) = \infty$in a set of positive measure, then $\rho_{p[\cdot],\delta(\cdot)}(f) \approx \|f\|_\infty$.

For suitable choices of $p(x)$ and $\delta(x)$ these spaces reduce to the classical and grand Lebesgue spaces, to the generalized grand spaces $L^{p),\delta}(I)$ defined in (24) and to the Orliczspace $EXP\ \alpha(\alpha > 0)$(see [274]). In [273] the fully measurable small Lebesgue spaces have been introduced as a generalization of the small Lebesgue spaces given in [275]. Let w be a weight on I, i.e. an a.e. positive, integrable function on I. We set

$$\rho_{p[\cdot],\delta(\cdot),w}(f) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x),w} \left(\delta(x) f(\cdot)\right)\ (f \in M_0^+), \quad (28)$$

where

$$\rho_{p(x),w}\left(\delta(x) f(\cdot)\right) = \left(\int_I \left(\delta(x) f(t)\right)^{p(x)} w(t) dt \right)^{\frac{1}{p(x)}}, 1 \le p(x) < \infty. \quad (29)$$

Arguing as in [275] we note that it makes sense to consider the function

$$x \in I \to \|f\|_{p(x),w}$$

which is measurable. Then the functional in (29) can be written equivalently as

$$\rho_{p(x),w}(\delta(x)\, f(\cdot)) = \delta(x) \rho_{p(x),w}(f(\cdot)) = \delta(x) \|f\|_{p(x),w},$$

where $\|f\|_{p(x),w}$denotes the norm in the weighted Lebesgue space $L_w^{p(x)}$ (for the sake of clarity, this is the classical weighted Lebesgue space, with weight w and constant exponent $p(x)$: here x plays the role of a parameter).

We note that if $p(\cdot) \equiv p \in [1, \infty[$, then (28) is equivalent to the norm in the classical weighted Lebesgue space $L_w^p(I)$, since

$$\rho_{p[\cdot],\delta(\cdot),w}(f) = \|\delta\|_\infty \|f\|_{p,w}.$$

Next, we remark that the fully measurable grand Lebesgue spaces are not included in the class of the weighted Banach function spaces considered in [280].
We now prove that in the following it will suffice to consider only $p(\cdot) \in M_0^+ , p(\cdot) > 1$: for the esssup in (28) only the points x where $p(x) > 1$ really matter.
Proposition $(\mathbf{4.2.2})[\mathbf{260}]$: Let w be a weight on $I, p(\cdot) \in M_0^+ , p(\cdot) \geq 1\ a.e.$ and $\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \leq 1.$ For $f \in M_0^+$, if $\tau \in [p^-, p^+]$ and $|p^{-1}([\tau, p^+])| > 0$, then

$$\rho_{p[\cdot],\delta(\cdot),w}(f) \approx \operatorname*{ess\,sup}_{x \in p^{-1}([\tau,p^+])} \rho_{p(x),w}(\delta(x) f(\cdot)), \tag{30}$$

where $p^- = \text{ess inf}_{x\in I}\ p(x)\ \ and\ p^+ = \text{ess sup}_{x\in I}\ p(x)\ \ a.$
Proof : Setting

$$E_1^\tau = p^{-1}([\tau, p^+]), \qquad E_2^\tau = p^{-1}([p^-, \tau[),$$

if $|E_2| = 0$ then (30) is trivially true. Otherwise, putting

$$T_1^\tau = \operatorname*{ess\,sup}_{x\in E_1^\tau} \delta(x) \left(\int_I f(t)^{p(x)} w(t) \right)^{\frac{1}{p(x)}},$$

$$T_2^\tau = \operatorname*{ess\,sup}_{x\in E_2^\tau} \delta(x) \left(\int_I f(t)^{p(x)} w(t) \right)^{\frac{1}{p(x)}},$$

we have

$$\rho_{p[\cdot],\delta(\cdot),w}(f) = \max\{T_1^\tau, T_2^\tau\}.$$

For $x \in E_2^\tau\ a.e.$ we obtain

$$\left(\int_I f(t)^{p(x)} w(t) \right)^{\frac{1}{p(x)}} \leq \left(\int_I f(t)^\tau w(t) \right)^{\frac{1}{\tau}} \left(1 + \int_I w(t) dt \right)^{\frac{\tau - p^-}{p - \tau}}. \tag{31}$$

In fact by Hölder's inequality with exponents $\frac{\tau}{p(x)}$ and $\frac{\tau}{\tau - p(x)}$, we get

$$\begin{aligned} \left(\int_I f(t)^{p(x)} w(t) \right)^{\frac{1}{p(x)}} &= \left(\int_I f(t)^{p(x)} w(t)^{\frac{p(x)}{\tau}} w(t)^{\frac{\tau - p(x)}{\tau}} \right)^{\frac{1}{p(x)}} dt \\ &\leq \left(\int_I f(t)^\tau w(t) \right)^{\frac{1}{\tau}} \left(\int_I w(t) dt \right)^{\frac{\tau - p(x)}{p(x)\tau}} \\ &\leq \left(\int_I f(t)^\tau w(t) \right)^{\frac{1}{\tau}} \left(1 + \int_I w(t) dt \right)^{\frac{\tau - p^-}{p - \tau}}. \end{aligned}$$

Hence for $x \in E_2^\tau\ a.e.$

$$\delta(x) \left(\int_I f(t)^{p(x)} w(t) \right)^{\frac{1}{p(x)}} \leq \|\delta\|_\infty \left(\int_I f(t)^\tau w(t) \right)^{\frac{1}{\tau}} \left(1 + \int_I w(t) dt \right)^{\frac{\tau - p^-}{p - \tau}},$$

that implies

$$T_2^\tau \leq \|\delta\|_\infty \left(\int_I f(t)^\tau w(t)\right)^{\frac{1}{\tau}} \left(1 + \int_I w(t)dt\right)^{\frac{\tau - p^-}{p-\tau}}. \tag{32}$$

Similarly, for $x \in E_1^\tau \ a.e.$, we get

$$\left(\int_I f(t)^\tau w(t)\right)^{\frac{1}{\tau}} \leq \left(\int_I f(t)^{p(x)} w(t)\right)^{\frac{1}{p(x)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - \tau}{p-\tau}} \tag{33}$$

using Hölder's inequality with exponents $\frac{p(x)}{\tau}$ and $\frac{p(x)}{p(x)-\tau}$ if $p(x) > \tau$.

If $E_\delta^\tau \subset E_1^\tau$ is such that $|E_\delta^\tau| > 0$ and

$$\delta(x) > \frac{\|\delta\|_{L^\infty(E_1^\tau)}}{2}, \forall x \in E_\delta^\tau \ a.e.,$$

for $x \in E_\delta^\tau \ a.e.$ we have

$$\left(\int_I f(t)^\tau w(t)\right)^{\frac{1}{\tau}} \leq \delta(x)^{-1}\delta(x) \left(\int_I f(t)^{p(x)} w(t)\right)^{\frac{1}{p(x)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - \tau}{p-\tau}}$$

$$\leq \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}} \operatorname*{ess\,sup}_{x \in E_1^\tau} \delta(x)$$

$$\times \left(\int_I f(t)^{p(x)} w(t)\right)^{\frac{1}{p(x)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - \tau}{p-\tau}}.$$

Hence by (32) we have

$$T_2^\tau \leq \|\delta\|_\infty \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - p^-}{p^-}} T_1^\tau$$

and

$$\rho_{p[\cdot],\delta(\cdot),w}(f) = \max\{T_1, T_2\}$$

$$\leq \max\left\{1, \|\delta\|_\infty \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - p^-}{p^-}} T_1^\tau\right\}$$

$$= \|\delta\|_\infty \frac{2}{\|\delta\|_{L^\infty(E_1^\tau)}} \left(1 + \int_I w(t)dt\right)^{\frac{p^+ - p^-}{p^-}} T_1^\tau.$$

Finally, we observe that

$$T_1^\tau \leq \rho_{p[\cdot],\delta(\cdot),w}(f)$$

and this completes the proof.

Proposition $(\mathbf{4.2.3})[\mathbf{260}]$: The functional $\rho_{p[\cdot],\delta(\cdot),w}$ defined in (28) is a Banach function norm, i.e. for all $f, g, f_n \in M_0^+ (n \in \mathbb{N})$, the following properties hold:

(i) $\rho_{p[\cdot],\delta(\cdot),w}(f) \geq 0$

(ii). $\rho_{p[\cdot],\delta(\cdot),w}(f) = 0$ iff $f = 0$ in I $a.e.$
(iii). $\rho_{p[\cdot],\delta(\cdot),w}(\lambda f) = \lambda\rho_{p[\cdot],\delta(\cdot),w}(f)$ $\forall\lambda > 0$
(iv). $\rho_{p[\cdot],\delta(\cdot),w}(f+g) \le \rho_{p[\cdot],\delta(\cdot),w}(f) + \rho_{p[\cdot],\delta(\cdot),w}(g)$
(v). if $0 \le g \le f$ in I $a.e.$, then $\rho_{p[\cdot],\delta(\cdot),w}(g) \le \rho_{p[\cdot],\delta(\cdot),w}(f)$
(vi). if $0 \le f_n \uparrow f$ in I $a.e.$, then $\rho_{p[\cdot],\delta(\cdot),w}(f_n) \uparrow \rho_{p[\cdot],\delta(\cdot),w}(f)$
(vii). $\rho_{p[\cdot],\delta(\cdot),w}(\chi_E) < +\infty$ $\forall E \subset I$
$(viii)$. $\int_E f\,w dx \le C(p,\delta,E)\rho_{p[\cdot],\delta(\cdot),w}(f)$ $\forall E \subset I$.

Proof: (i) and (iii) are obvious, (iv) and (vi) are consequence of the corresponding property of the norm in the weighted Lebesgue spaces $L_w^{p(x)}$, so we prove only (ii) and (vii)–$(viii)$

2. $f = 0 a.e. \Rightarrow \rho_{p(x),w}(\delta(x) f(\cdot)) = 0, \forall x \in I\ a.e. \Rightarrow \rho_{p[\cdot],\delta(\cdot),w}(f) = 0.$

On the other hand, since $\delta > 0$ $a.e.$ and $w > 0$ $a.e.$,

$$\begin{aligned}\rho_{p[\cdot],\delta(\cdot),w}(f) = 0 &\Rightarrow \rho_{p(x),w}(\delta(x) f(\cdot)) = 0, \forall x \in I\ a.e.\\ &\Rightarrow \delta(x)\rho_{p(x),w}(f(\cdot)) = 0, \forall x \in I\ a.e.\\ &\Rightarrow \rho_{p(x),w}(f(\cdot)) = 0, \forall x \in I\ a.e.|\\ &\Rightarrow f\,w = 0, \forall x \in I\ a.e. \Rightarrow f = 0, \forall x \in I\ a.e.\end{aligned}$$

(vi). If $0 \le f_n \uparrow f$ in I $a.e.$, then

$$\begin{aligned}\rho_{p[\cdot],\delta(\cdot),w}(f_n) &= \operatorname*{ess\,sup}_{x\in I} \delta(x)\,\|f_n\|_{p(x),w} \uparrow \sup_n \operatorname*{ess\,sup}_{x\in I} \delta(x)\,\|f_n\|_{p(x),w}\\ &= \operatorname*{ess\,sup}_{x\in I}\left(\sup_n \delta(x)\|f_n\|_{p(x),w}\right)\\ &= \operatorname*{ess\,sup}_{x\in I}\delta(x)\left(\sup_n\|f_n\|_{p(x),w}\right)\\ &= \operatorname*{ess\,sup}_{x\in I}\delta(x)\|f_n\|_{p(x),w} = \rho_{p[\cdot],\delta(\cdot),w}(f).\end{aligned}$$

(vii). Let $E \subset I$, then, being w integrable on I, it is

$$\begin{aligned}\rho_{p[\cdot],\delta(\cdot),w}(\chi_E) &= \operatorname*{ess\,sup}_{x\in I}\delta(x)\left(\int_E w(t)dt\right)^{\frac{1}{p(x)}}\\ &\le \operatorname*{ess\,sup}_{x\in I}\delta(x)\left(\int_0^1 w(t)dt\right)^{\frac{1}{p(x)}} < +\infty.\end{aligned}$$

$(viii)$. Let $E \subset I$ and let $E_\delta \subset I, |E_\delta| > 0$ be such that

$$\delta(x) > \frac{\|\delta\|_\infty}{2}, \qquad \forall x \in E_\delta\ a.e.$$

By Hölder's inequality with conjugate exponents $p(x)$ and $p'(x)=\frac{p(x)}{p(x)-1}$, for $x \in E_\delta$ $a.e.$, and taking into account that we can consider $p(\cdot)>1$ $a.e.$ by Proposition (4.2.2), we have

$$\int_E w(t)dt = \int_I f\chi_E w dt \;=\int_I f w^{\frac{1}{p(x)}}\chi_E w^{\frac{1}{p'(x)}}\;dt$$

$$\le \left(\int_I f(t)^{p(x)}w(t)dt\right)^{\frac{1}{p(x)}} \left(\int_I \chi_E(t)^{p'(x)}w(t)dt\right)^{\frac{1}{p'(x)}}$$

$$=\;\delta(x)\left(\int_I f(t)^{p(x)}w(t)dt\right)^{\frac{1}{p(x)}}\delta(x)^{-1}\left(\int_I \chi_E(t)^{p'(x)}w(t)dt\right)^{\frac{1}{p'(x)}}$$

$$\le\; \operatorname*{ess\,sup}_{x\in E_\delta}\delta(x)\left(\int_I f(t)^{p(x)}w(t)dt\right)^{\frac{1}{p(x)}}$$

$$\times \operatorname*{ess\,sup}_{x\in E_\delta}\delta(x)^{-1}\left(\int_E w(t)dt\right)^{\frac{1}{p'(x)}}$$

$$\le\; \rho_{p[\cdot],\delta(\cdot),w}(\,f\,)\cdot\frac{2}{\|\delta\|_\infty}\;\operatorname*{ess\,sup}_{x\in I}\left(\int_E w(t)dt\right)^{\frac{1}{p'(x)}}\;.$$

As a consequence of Proposition (4.2.3), the space

$$L^{p[\cdot],\delta(\cdot),w}(I)=\left\{f\;\in M_0:\;\|f\|_{p[\cdot],\delta(\cdot),w}\;=\rho_{p[\cdot],\delta(\cdot),w}(|f|)<\infty\right\} \qquad (34)$$

is a Banach function space (see e.g. [274]), which we will call the weighted fully measurable grand Lebesgue space.

we characterize the weights for which the Hardy–Littlewood maximal operator

$$Mf(x)=\sup_{x\in J\subset I}\frac{1}{|J|}\;\int_J |f|\,dt\,,x\;\in\;I$$

is bounded on the weighted fully measurable grand Lebesgue space $L^{p[\cdot],\delta(\cdot),w}(I)$. The proof of the maximal theorem uses the following well known lemma.

Lemma $(\mathbf{4.2.4})[\mathbf{260}]$: [266, Lemma 5, p 214] If $1<p<\infty$ and $w\in A_p$ on I with constant $A_p(w)$, then there exist constant $\sigma>0$ and $L>0$ such that $w\in A_{p-\varepsilon}$ on I with constant $A_{p-\varepsilon}(w)\le L$, for all $0<\varepsilon<\sigma$.

Theorem $(\mathbf{4.2.5})[\mathbf{260}]$: Let w be a weight on $I, p(\cdot)\in M_0$ be such that $p(\cdot)>1 a.e., p^+<+\infty$ and $\delta\in L^\infty(I), \delta>0$ $a.e.$, $0<\|\delta\|_\infty\le 1$. Then the inequality

$$\|Mf\|_{p[\cdot],\delta(\cdot),w}\le c\|f\|_{p[\cdot],\delta(\cdot),w} \qquad (35)$$

holds if and only if w belongs to the Muckenhoupt class A_{p^+}, where c is a constant independent of f.

Proof: First of all we observe that, without loss of generality, we may assume $p(\cdot) < p^+$ in I a.e.Infact,ifthereexistsasetofpositivemeasure $E_1 \subset I$ such that $p(\cdot) \equiv p^+$, then by Proposition (4.2.2)

$$\|Mf\|_{p[\cdot],\delta(\cdot),w} \approx \|Mf\|_{p^+,w}$$

and our theorem reduces to the maximal theorem in classical weighted Lebesgue spaces $L_w^{p^+}$. We begin to prove the necessary condition, therefore let us assume that the Inequality (35) holds. We have to prove that $w \in A_{p^+}$, that is

$$\sup_J \left(\frac{1}{|J|}\int_J w dt\right)\left(\frac{1}{|J|}\int_J w^{-\frac{1}{p^+-1}}\, dt\right)^{p^+-1} =: A_{p^+}(w) < \infty. \qquad (36)$$

Fix $J \subset I$. By the definition of maximal operator we have

$$\int_J |f|\, dt \le M(f\chi_J)(x), x \in J.$$

By the Assumption (35) we have

$$\|M(f\chi_J)\|_{p[\cdot],\delta(\cdot),w} \le c\|f\chi_J\|_{p[\cdot],\delta(\cdot),w}.$$

Therefore

$$\left(\int_J |f|\, dt\right)\|\chi_J\|_{p[\cdot],\delta(\cdot),w}$$
$$= \left\|\int_J |f|\, dt\chi_J\right\|_{p[\cdot],\delta(\cdot),w}$$
$$\le \|M(f\chi_J)\|_{p[\cdot],\delta(\cdot),w} \le c\|f\chi_J\|_{p[\cdot],\delta(\cdot),w}$$
$$= c \operatorname*{ess\,sup}_{x\in I}\left(\int_J (\delta(x)|f(t)|)^{p(x)}\, w(t)dt\right)^{\frac{1}{p(x)}}$$
$$= c \operatorname*{ess\,sup}_{x\in I}\left(\int_J (\delta(x)|f(t)|)^{p(x)}\, w(t)^{\frac{p(x)}{p^+}}\, w(t)^{\frac{p^+-p(x)}{p^+}}\, dt\right)^{\frac{1}{p(x)}}.$$

Applying Hölder's inequality with exponents $\frac{p^+}{p(x)}$ and $\frac{p^+}{p^+-p(x)}$ we have

$$\left(\int_J |f|\, dt\right)\|\chi_J\|_{p[\cdot],\delta(\cdot),w}$$
$$\le c \operatorname*{ess\,sup}_{x\in I}\left(\int_J (\delta(x)|f(t)|)^{p^+} w(t)dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{\frac{p^+-p(x)}{p^+\cdot p(x)}}$$
$$= c \operatorname*{ess\,sup}_{x\in I}\delta(x)\left(\int_J |f(t)|^{p^+} w(t)dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{\frac{p^+-p(x)}{p^+\cdot p(x)}}$$

$$= c\left(\int_J |f(t)|^{p^+} w(t)dt\right)^{\frac{1}{p^+}} \operatorname*{ess\,sup}_{x\in I} \delta(x)\left(\int_J w(t)dt\right)^{\frac{1}{p(x)}-\frac{1}{p^+}}$$

$$= c\left(\int_J |f(t)|^{p^+} w(t)dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{-\frac{1}{p^+}} \operatorname*{ess\,sup}_{x\in I}\delta(x)\left(\int_J w(t)dt\right)^{\frac{1}{p(x)}}$$

$$= c\left(\int_J |f(t)|^{p^+} w(t)dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{-\frac{1}{p^+}} \|\chi_J\|_{p[\cdot],\delta(\cdot),w}.$$

Hence

$$\left(\int_J |f|\ dt\right) \le c\left(\int_J |f(t)|^{p^+} w(t)dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{-\frac{1}{p^+}}.$$

At this point we show, by contradiction, that we may consider the case $w^{-\frac{1}{p^+-1}} \in L^1(J)$. In fact, if $w^{-\frac{1}{p^+-1}} \notin L^1(J)$, then $w^{-\frac{1}{p^+}} \notin L^{(p^+)'}(J)$. Therefore there exists $g \in L^{p^+}(J)$ such that

$$\int_J g(t)w(t)^{-\frac{1}{p^+}}\ dt = \infty$$

Define $f = gw^{-\frac{1}{p^+}}$, we have

$$Mf(x) = \infty, \forall x \in J, \tag{37}$$

hence

$$\|Mf\|_{p[\cdot],\delta(\cdot),w} = \infty$$

and, by the Assumption (35), we get

$$\|f\|_{p[\cdot],\delta(\cdot),w} = \infty. \tag{38}$$

On the other hand, since $f^{p^+}w = g^{p^+}$ *and* $g^{p^+} \in L^1$, we have $f \in L_w^{p^+}$. By the maximal theorem in the classical Lebesgue spaces it is $Mf \in L_w^{p^+}$, which is in contradiction with (37). Moreover, Hölder's inequality implies $\rho_{p(x),w}(\delta(x)\, f(\cdot)) < \infty, \forall x \in J$, hence $\|f\|_{p[\cdot],\delta(\cdot),w} < \infty$, against (38).

Hence $w^{-\frac{1}{p^+-1}} \in L^1(J)$. Choosing $f = w^{-\frac{1}{p^+-1}}$, we have

$$\int_J w^{-\frac{1}{p^+-1}}(t)\ dt \le c\left(\int_J w^{-\frac{1}{p^+-1}}(t)\,dt\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{-\frac{1}{p^+}}$$

and therefore it is

$$\left(\frac{1}{|J|}\right)^{\frac{1}{p^+}}\left(\int_J w(t)dt\right)^{\frac{1}{p^+}}\left(\frac{1}{|J|}\right)^{1-\frac{1}{p^+}}\left(\int_J w^{-\frac{1}{p^+-1}}(t)\,dt\right)^{1-\frac{1}{p^+}} \le c.$$

Raising to the power p^+

$$\left(\int_J w(t)dt\right)\left(\int_J w^{-\frac{1}{p^+-1}}(t)\,dt\right)^{p^+-1} \leq c$$

and so $w \in A_{p^+}$.

Now we prove the sufficient condition, therefore let us assume $w \in A_{p^+}$. By Lemma (4.2.4) there exists $1 < \tau_0 < p^+$, such that $w \in A_\eta, \forall \eta \in]\tau_0, p^+[$and thereforethere exists $\tau_0 < \tau < p^+$, such that $w \in A_\eta, \forall \eta \in [\tau, p^+[$. Then

$$w \in A_{p(x)} \, \forall x \in p^{-1}([\tau, p^+[) \; a.e.$$

and, by the maximal theorem in the Lebesgue spaces,

$$\|Mf\|_{p(x),w} \leq c\|f\|_{p(x),w}, \qquad x \in p^{-1}([\tau, p^+]) \; a.e. \tag{39}$$

with uniform constant.

By Proposition (4.2.2) we have

$$\|Mf\|_{p[\cdot],\delta(\cdot),w} \leq c\|f\|_{p[\cdot],\delta(\cdot),w}. \tag{40}$$

Remark $(\mathbf{4.2.6})[\mathbf{260}]$: We observe that the sufficient condition of the maximal theorem in the weighted fully measurable grand Lebesgue spaces holds also in the case $p^+ = +\infty$. Namely, recalling that $A_\infty = \cup_{1<p<\infty} A_p$, we have

$$w \in A_\infty \Rightarrow \exists p > 1 : w \in A_p \Rightarrow w \in A_q, \forall q \geq p.$$

By the maximal theorem in the Lebesgue spaces,

$$\|Mf\|_{p(x),w} \leq c\|f\|_{p(x),w}, x \in p^{-1}([p, p^+[) \; a.e. \tag{41}$$

By Proposition (4.2.2) (where $\tau = p$) we have

$$\|Mf\|_{p[\cdot],\delta(\cdot),w} \leq c\|f\|_{p[\cdot],\delta(\cdot),w}. \tag{42}$$

We get back the boundedness result for the Hardy–Littlewood maximal operator established by Gao, Cui, Liang in [277].

Section (4.3): Fully Measurable Small Lebesgue Spaces

In [291] Iwaniec and Sbordone introduced the grand Lebesgue spaces $L^{p)}(\Omega)$ $(1 < p < \infty), \Omega \subset \mathbb{R}^n$ of finite measure, in connection with the study of the integrability properties of the Jacobian determinant. In the case $\Omega = I = (0,1)$ such spaces are defined as the Banach function spaces (see e.g. [292] for the definition) of the measurable functions f on I such that

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1}\left(\epsilon\int_0^1 |f(t)|^{p-\epsilon}\,dt\right)^{\frac{1}{p-\epsilon}} < \infty.$$

Since then the grand Lebesgue spaces play an important role in PDE's theory (see e.g. [295]), in Function Spaces theory (see e.g. [293]) and in interpolation–extrapolation theory (see e.g. [294]). They have been widely investigated and several variations have been studied, among which, in [297], the spaces

$$L^{p),\delta}(I) = \left\{ f : I \to \mathbb{R}\, measurable : \|f\|_{p),\delta} = \sup_{0<\epsilon<p-1} \left(\delta(\epsilon) \int_0^1 |f(t)|^{p-\epsilon}\, dt \right)^{\frac{1}{p-\epsilon}} < \infty \right\} \qquad (43)$$

where δ is a measurable function in I, have been considered. It has been shown that the interesting case is when δ is left continuous, increasing (i.e. $0 < \epsilon_1 < \epsilon_2 < p - 1 \Rightarrow \delta(\epsilon_1) \leq \delta(\epsilon_2)$) such that $\delta(0^+) = 0$ and with values in $]0, 1]$.

Let M be the set of all Lebesgue measurable functions in I with values in $[-\infty, +\infty], M^+$ the subset of the nonnegative functions, M_0 the subset of the finite a.e. functions, and M_0^+ the subset of the finite a.e., nonnegative functions.

Recently in [294] the following further generalization of $\|f\|_{p),\delta}$ was introduced, where in (43) $p - \epsilon$ is changed into a general measurable function.

Definition $(\mathbf{4.3.1})[\mathbf{290}]$: ([294]). Let $p(\cdot) \in M, p(\cdot) \geq 1\, a.e.$ and $\delta \in L^\infty(I), \delta > 0\, a.e., 0 < \|\delta\|_\infty \leq 1$. The Banach function spaces

$$L^{p[\cdot]),\delta(\cdot)}(I) = \{ f \in M_0 : \|f\|_{p[\cdot]),\delta(\cdot)} = \rho_{p[\cdot]),\delta(\cdot)}(|f|) < \infty \}, \qquad (44)$$

where

$$\rho_{p[\cdot]),\delta(\cdot)}(f) = \operatorname*{ess\,sup}_{x\in I} \rho_{p(x)}\big(\delta(x) f(\cdot)\big)\ (f \in M_0^+) \qquad (45)$$

and

$$\rho_{p(x)}\big(\delta(x) f(\cdot)\big) = \begin{cases} \left(\int_I \big(\delta(x) f(t)\big)^{p(x)} dt \right)^{\frac{1}{p(x)}} & if\ 1 \leq p(x) < \infty \\ \operatorname*{ess\,sup}_{t\in I} \big(\delta(x) f(t)\big) & if\ p(x) = \infty \end{cases} \qquad (46)$$

are called fully measurable grand Lebesgue spaces.

We point out that, in the previous definition, we choice the symbol $\rho_{p[\cdot]),\delta(\cdot)}(f)$ with square brackets in $p[\cdot]$ and not the more natural $p(\cdot)$ to avoid confusion since the symbol $p(\cdot)$ is already used in the theory of variable spaces with a different meaning. (See for example the monographs [296]for an exhaustive treatment of the variable exponent Lebesgue spaces.)

The (standard) grand Lebesgue spaces $L^{p)}(I)$ can be immediately obtained from (46) setting $p(x) = p - x, 1 < p < \infty$ and $\delta(x) = x$.

The generalized grand Lebesgue spaces (43) are evidently included in the spaces (44): the function $\delta(\epsilon)^{\frac{1}{p-\epsilon}}$ in (43) corresponds to the function $\delta(\epsilon)$ in (46).

Setting in (46)$p(x) = \frac{1}{x}$ and $\delta(x) = x^\alpha, \alpha > 0$, a norm of the Orlicz space $EXP_{\frac{1}{\alpha}}$ is obtained while, if in (46) $p(x) = x$, suitable continuous functions δ give the so-called bilateral grand Lebesgue spaces (see [296]). For the weighted fully measurable grand Lebesgue spaces see [293].

In [293] Fiorenza introduced an explicit equivalent expression of the norm of the associate space of the grand Lebesgue space $L^{p)}$, denoted by $L^{(p'}, p' = \frac{p}{p-1}$. They are Banach function spaces, defined through the abstract function norm

$$\|f\|_{(p'} = \sup\left\{\int_0^1 fg\,dx : \; g \in M_0^+ \;, \|g\|_{p)} \le 1\right\},$$

and they are called small Lebesgue spaces (see also [295]). It has been proved that

$$\|f\|_{(p'} \approx \inf_{\sum_{k=1}^{f=\infty} f_k} \left\{\sum_{k=1}^{\infty} \inf_{0<\epsilon<p-1} \epsilon^{-\frac{1}{p-\epsilon}} \left(\int_0^1 |f_k|^{(p-\epsilon)'}\right)^{\frac{1}{(p-\epsilon)'}}\right\} \quad , \qquad (47)$$

where $f_k \in M_0, k \in \mathbb{N}$. Some properties of small Lebesgue spaces which follow from their definition, along with some applications, are in [294]. Their role in Calculus of Variations (see [295]), the $G\Gamma$spaces (see [296]). Very recently they appeared of crucial importance in the question of the dimension-free Sobolev embedding theorem [297].

In [298] Fiorenza and Karadzhov found the following equivalent expression for the norm by using deeply extrapolation–interpolation techniques:

$$\|f\|_{(p'} \approx \int_0^1 (1-\log t)^{-\frac{1}{p'}} \left(\int_0^t f^*(s)^{p'}\,ds\right)^{\frac{1}{p'}} \frac{dt}{t}$$

where f^*denotes the decreasing rearrangement of f (see [294]); later a direct proof of such equivalence was given in [296].

Inspired by the norm in (45) of fully measurable grand Lebesgue spaces, We consider a generalization of the norm of small Lebesgue spaces (47) where $p-\epsilon$ is changed into a general measurable function. We give the following

Definition $(4.3.2)$[290]: Let $p(\cdot) \in M$, $p(\cdot) \ge 1\; a.e.$ and $\delta \in L^\infty(I), \delta > 0\; a.e., 0 < \|\delta\|_\infty \le 1$. The spaces

$$L^{(p[\cdot],\delta(\cdot)}(I) = \{f \in M_0 : \rho_{(p[\cdot],\delta(\cdot)}(|f|) < \infty\}, \qquad (48)$$

where

$$\rho_{(p[\cdot],\delta(\cdot)}(f) = \inf_{\sum_{k=1}^{f=\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right) \quad (f, f_k \in M_0^+) \qquad (49)$$

and $\rho_{p(x)}$ is defined through (46), are called fully measurable small Lebesgue spaces.

we prove that $\rho_{(p[\cdot],\delta(\cdot)}$is a Banach function norm and therefore the spaces defined in (48) are Banach function spaces under the norm given by $f \in L^{(p[\cdot],\delta(\cdot)}(I) \to \rho_{(p[\cdot],\delta(\cdot)}(f) = \|f\|_{(p[\cdot],\delta(\cdot)}$. We need to show (as in [293]) that the infimum over I in the norm $\rho_{(p[\cdot],\delta(\cdot)}$ can be computed also in smaller intervals included in I.

Finally, , we prove a Hölder-type inequality of fully measurable small Lebesgue spaces which reduces to the classical Hölder's inequality in the setting of Orlicz spaces $EXP_{1/\alpha}$ and $L(\log L)^\alpha$ $(\alpha > 0)$, for suitable choices of $p(x)$ and $\delta(x)$.

Our first remark is that if $p(\cdot) \equiv p \in [1,\infty]$, the grand space (44) reduces to the classical Lebesgue space $L^p(I)$:

$$\rho_{p[\cdot]),\delta(\cdot)}(f) = \rho_{p,\delta(\cdot)}(f) = \|\delta\|_\infty \|f\|_p.$$

Similarly, if $p(\cdot) \equiv p \in [1,\infty]$, the small space (48) reduces to the classical Lebesgue space $L^p(I)$:

$$\rho_{(p[\cdot],\delta(\cdot)}(f) = \rho_{(p,\delta(\cdot)}(f) = \inf_{\sum_{k=1}^{f=\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \delta(x)^{-1}\rho_{p(x)}(f_k(\cdot)) = \|\delta\|_{\infty}^{-1}\|f\|_p,$$

the last equality due on one hand to the triangle inequality, on the other hand choosing the trivial decomposition $f = f + 0 + \dots + 0 + \dots$. Another remark is of reduction type: in the following it will suffice to consider only $p(\cdot) \in M_0^+, p(\cdot) \geq 1$. Indeed, for any $f \in M_0^+$ and for any decomposition $f = \sum_{k=1}^{\infty} f_k$ with $f_k \in M_0^+$, we show that if $I = I_1 \cup I_2$ is such that $p(x) < \infty$ in I_1 $a.e.$ and $p(x) = \infty$ in I_2 $a.e.$, with $|I_1| > 0, |I_2| > 0$, then

$$\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}(\delta(x)^{-1} f_k(\cdot)) = \operatorname*{ess\,inf}_{x\in I_1} \rho_{p(x)}(\delta(x)^{-1} f_k(\cdot)). \qquad (50)$$

Namely, first we observe that for any $k \in \mathbb{N}$

$$\begin{aligned}
&\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}(\delta(x)^{-1} f_k(\cdot)) \\
&= \min\left\{\operatorname*{ess\,inf}_{x\in I_1} \rho_{p(x)}\big(\delta(x)^{-1} f_k(\cdot)\big), \operatorname*{ess\,inf}_{x\in I_2} \rho_{p(x)}\big(\delta(x)^{-1} f_k(\cdot)\big)\right\} \\
&= \min\left\{\operatorname*{ess\,inf}_{x\in I_1} \delta(x)^{-1}\left(\int_I \big(f_k(t)\big)^{p(x)}\, dt\right)^{\frac{1}{p(x)}}, \operatorname*{ess\,inf}_{x\in I_2} \delta(x)^{-1} \operatorname*{ess\,sup}_{t\in I} \big(f_k(t)\big)\right\} \\
&= \operatorname*{ess\,inf}_{x\in I_1} \rho_{p(x)}\big(\delta(x)^{-1} f_k(\cdot)\big)
\end{aligned}$$

that yields (50).

Now we prove that $L^{(p[\cdot],\delta(\cdot)}(I)$ is a Banach space. We recall the following

Lemma (4.3.3)[290]: ([293, Lemma 2.1]). If $f, g \in M_0^+$ and $g \leq f = \sum_{k=1}^{\infty} f_k$ with $f_k \geq 0, \forall k \in \mathbb{N}$, then the functions

$$h_k = \left[f_k - \max\left(g - \sum_{j=1}^{k-1} f_j, 0\right)\right] \chi_{\left\{\sum_{j=1}^{k} f_j > 0\right\}} \;\; \forall k \in \mathbb{N}$$

are such that

$$0 \leq h_k \leq f_k \;\forall k \in \mathbb{N}$$

and

$$g = \sum_{k=1}^{\infty} (f_k - h_k) .$$

Theorem (4.3.4)[290]: Let $p(\cdot) \in M, p(\cdot) \geq 1$ $a.e.$ and $\delta \in L^{\infty}(I), \delta > 0$ $a.e.$, $0 < \|\delta\|_{\infty} \leq 1$. The space $L^{(p[\cdot],\delta(\cdot)}(I)$ is a Banach space.

Proof: We exclude the known case of $p(\cdot)$ constant, and we may assume, without loss of generality, that $p(\cdot)$ is finite $a.e.$ We will prove the properties of the normed spaces and we will get the completeness through the Riesz–Fischer property: therefore it suffices to prove (see [294, $p.$ 32, $n.$ 11]) that, for all $f, g, f^{(n)}, (n \in \mathbb{N})$, in M_0^+, for all constants $\lambda > 0$, and for all measurable subsets $E \subset I$, the following properties hold

(i). $\rho_{(p[\cdot],\delta(\cdot)}(f) \geq 0$

$(ii). \rho_{(p[\cdot],\delta(\cdot)}(f) = 0 \text{iff } f = 0$ in $I\ a.e.$
$(iii). \rho_{(p[\cdot],\delta(\cdot)}(\lambda f) = \lambda\rho_{(p[\cdot],\delta(\cdot)}(f)\ \forall\lambda > 0$
(iv). if $0 \le g \le f$ in $I\ a.e.$, then $\rho_{(p[\cdot],\delta(\cdot)}(g) \le \rho_{(p[\cdot],\delta(\cdot)}(f)$
$(v). \rho_{(p[\cdot],\delta(\cdot)}\left(\sum_{n=1}^{\infty} f^{(n)}\right) \le \sum_{n=1}^{\infty} \rho_{(p[\cdot],\delta(\cdot)}\big(f^{(n)}\big)$
$(vi). \rho_{(p[\cdot],\delta(\cdot)}(\chi_E) < +\infty\ \forall E \subset I$
$(vii). \int_E f dx \le C(p,\delta,E)\rho_{(p[\cdot],\delta(\cdot)}(f)\ \forall E \subset I.$
(i). It is obvious.
(ii). If $f = 0$ in $I\ a.e.$ it suffices to choose $f_k = 0$ for all $k \in \mathbb{N}$ to get $\rho_{(p[\cdot],\delta(\cdot)}(f) = 0$. Conversely, if $\rho_{(p[\cdot],\delta(\cdot)}(f) = 0$, for any decomposition $f = \sum_{k=1}^{\infty} f_k$ with $f_k \ge 0$, by Fatou's lemma, we have

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in I}\delta^{-1}(x)\int_I f(t)dt &= \operatorname*{ess\,inf}_{x\in I}\delta^{-1}(x)\int_I \sum_{k=1}^{\infty} f_k(t)dt\\ &\le \sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\delta^{-1}(x)\int_I f_k(t)dt\\ &\le \sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\delta^{-1}(x)\left(\int_I f_k(t)^{p(x)}\ dt\right)^{1/p(x)}\end{aligned}$$

from which

$$\operatorname*{ess\,inf}_{x\in I}\delta^{-1}(x)\int_I f(t)dt \le \rho_{(p[\cdot],\delta(\cdot)}(f), \tag{51}$$

hence $f = 0$ in $I\ a.e.$
(iii). If $\lambda f = \sum_{k=1}^{\infty} h_k$ is a decomposition with $h_k \ge 0$ then

$$\begin{aligned}\rho_{(p[\cdot],\delta(\cdot)}(\lambda f) &= \inf_{\lambda f=\sum_{k=1}^{\infty} h_k}\sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\rho_{p(x)}\big(\delta(x)^{-1}h_k(\cdot)\big)\\ &= \inf_{\lambda f=\sum_{k=1}^{\infty} h_k/\lambda}\sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\rho_{p(x)}(\delta(x)^{-1}h_k(\cdot)/\lambda) = \lambda\rho_{(p[\cdot],\delta(\cdot)}(f).\end{aligned}$$

(iv). For any decomposition $f = \sum_{k=1}^{\infty} f_k$ with $f_k \ge 0$, let $h_k, k \in \mathbb{N}$, be given by Lemma (4.3.3) such that
$g = \sum_{k=1}^{\infty}(f_k - h_k), f_k - h_k \ge 0, \forall k \in \mathbb{N}$. We have

$$\begin{aligned}\rho_{(p[\cdot],\delta(\cdot)}(f) &= \inf_{f=\sum_{k=1}^{\infty} f_k}\sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\rho_{p(x)}\big(\delta(x)^{-1}f_k(\cdot)\big)\\ &\ge \inf_{f=\sum_{k=1}^{\infty} f_k}\sum_{k=1}^{\infty}\operatorname*{ess\,inf}_{x\in I}\rho_{p(x)}\big(\delta(x)^{-1}(f_k-h_k)(\cdot)\big)\end{aligned}$$

$$\geq \inf_{g=\sum_{k=1}^{\infty} g_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\big(\delta(x)^{-1} g_k(\cdot)\big) = \rho_{(p[\cdot],\delta(\cdot)}(g).$$

(v). Let us assume that the functions $f^{(n)} \in M_0^+$ are such that $\sum_{n=1}^{\infty} \rho_{(p[\cdot],\delta(\cdot)}\big(f^{(n)}\big) < \infty, \forall n \in \mathbb{N}, otherwise$ the assertion is trivial. Let $\epsilon > 0$ and let $f_k^{(n)} \in M_0^+$ be such that

$$f^{(n)} = \sum_{k=1}^{\infty} f_k^{(n)} \quad \forall n \in \mathbb{N}$$

and

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\left(\delta(x)^{-1} f_k^{(n)}(\cdot)\right) < \rho_{(p[\cdot],\delta(\cdot)}(f(n)) + \frac{\epsilon}{2n} \;\forall n \in \mathbb{N}. \quad (52)$$

We have

$$\rho_{(p[\cdot],\delta(\cdot)}\left(\sum_{n=1}^{\infty} f^{(n)}\right) = \rho_{(p[\cdot],\delta(\cdot)}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_k^{(n)}\right) = \rho_{(p[\cdot],\delta(\cdot)}\left(\sum_{n,k=1}^{\infty} f_k^{(n)}\right)$$

$$\leq \sum_{n,k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\left(\delta(x)^{-1} f_k^{(n)}(\cdot)\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\,(\delta(x)^{-1} f_k^{(n)}(\cdot)).$$

Therefore, by (52), we have

$$\rho_{(p[\cdot],\delta(\cdot)}\left(\sum_{n=1}^{\infty} f^{(n)}\right) \leq \sum_{n=1}^{\infty} \rho_{(p[\cdot],\delta(\cdot)}\big(f^{(n)}\big) + \epsilon \quad \forall\, \epsilon > 0.$$

Since ϵ is arbitrary . Let us consider the trivial decomposition $\chi_E = \chi_E + 0 + \ldots 0 + \ldots$. We have

$$\rho_{(p[\cdot],\delta(\cdot)}(\chi_E) \leq \operatorname*{ess\,inf}_{x\in I} \delta^{-1}(x)|E|^{\frac{1}{p(x)}} .$$

(vii). By (51) ,

$$\int_E f dx = \int_0^1 f\chi_E dx = \|\delta\|_\infty \operatorname*{ess\,inf}_{x\in I} \delta^{-1}(x) \int_I f\chi_E dx \leq \|\delta\|_\infty \rho_{(p[\cdot],\delta(\cdot)}(f).$$

Later we will need the following theorem, where, in the spirit of [292, Theorem 1], it is shown that the essinf can be equivalently computed in the set where $p(\cdot)$ is small.

Theorem (4.3.5)[290]: Let $p(\cdot) \in M_0^+, p(\cdot) \geq 1\, a.e.$ and $\delta \in L^\infty(I), \delta > 0\, a.e., 0 < \|\delta\|_\infty \leq 1.$ For $f \in M_0^+$ the norm $\rho_{(p[\cdot],\delta(\cdot)}(f)$ defined in (49) is equivalent to

$$\tilde{\rho}_{(p[\cdot],\delta(\cdot)}(f) := \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in p^{-1}([1,\tau])} \rho_{p(x)}\big(\delta(x)^{-1} f_k(\cdot)\big) \quad \forall \tau \in]\operatorname*{ess\,inf}_{x\in I} p(x), \infty[\quad (53)$$

Proof: The following inequality

$$\rho_{(p[\cdot],\delta(\cdot)}(f) \leq \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in p^{-1}([1,\tau])} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right)$$

holds trivially for all $\tau \in]\operatorname{ess\,inf}_{x\in I} p(x), \infty[$. Now we have to prove that there exists c_τ such that

$$\rho_{(p[\cdot],\delta(\cdot)}(f) \geq c_\tau \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in p^{-1}([1,\tau])} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right).$$

If $\tau \geq \rho_\infty(p(\cdot))$ then of course

$$\rho_{(p[\cdot],\delta(\cdot)}(f) = \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in p^{-1}([1,\tau])} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right)$$

and therefore there is nothing to prove.

Otherwise, it is $\tau \in]\operatorname*{ess\,inf}_{x\in I} p(x), \rho_\infty(p(\cdot))[$ and therefore $|p^{-1}(\,]\tau, \rho_\infty(p(\cdot))]\,)| > 0.$ Setting

$$E_1^\tau = p^{-1}\left(\left[\tau, \rho_\infty\left(p(\cdot)\right)\right]\right), \qquad T_1^{\tau,k} = \operatorname*{ess\,inf}_{x\in E_1^\tau} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right)$$

and

$$E_2^\tau = I \setminus E_1^\tau = p^{-1}([1,\tau[), \qquad T_2^{\tau,k} = \operatorname*{ess\,inf}_{x\in E_2^\tau} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right),$$

it is $|E_1^\tau| > 0, |E_2^\tau| > 0$ and we have, by Hölder's inequality,

$$\rho_{p(x)}(f_k(\cdot)) \leq \rho_\tau(f_k(\cdot)) \; \forall x \in E_2^\tau a.e.$$

and

$$\rho_\tau\left(f_k(\cdot)\right) \leq \rho_{p(x)}\left(f_k(\cdot)\right) \forall x \in E_1^\tau \; a.e.$$

Then for $x \in E_2^\tau a.e.$

$$\operatorname*{ess\,inf}_{x\in E_1^\tau} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right) \geq \operatorname*{ess\,inf}_{x\in E_1^\tau} \rho_\tau\left(\delta(x)^{-1} f_k(\cdot)\right)$$
$$= \left(\operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1}\right) \rho_\tau\left(f_k(\cdot)\right) \geq \left(\operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1}\right) \rho_{p(x)}(f_k(\cdot)).$$

If $E_\delta^\tau \subset E_2^\tau$ is such that $|E_\delta^\tau| > 0$ and

$$\delta(x) > \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2} \;\; \forall x \in E_\delta^\tau \; a.e.$$

it is, for $x \in E_\delta^\tau \; a.e.$,

$$T_1^{\tau,k} \geq \left(\operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1}\right) \delta(x)\delta(x)^{-1} \rho_{p(x)}(f_k(\cdot))$$
$$\geq \left(\operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1}\right) \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2} \delta(x)^{-1} \rho_{p(x)}(f_k(\cdot))$$

and therefore

$$T_1^{\tau,k} \geq \operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1} \cdot \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2} \cdot \operatorname*{ess\,inf}_{x\in E_1^\tau} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right)$$
$$= \operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1} \cdot \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2} \cdot T_2^{\tau,k}.$$
Therefore, being
$$\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right) = \min\left\{T_1^{\tau,k}, T_2^{\tau,k}\right\}$$
we have
$$\operatorname*{ess\,inf}_{x\in I} \rho_{p(x)}\left(\delta(x)^{-1} f_k(\cdot)\right) \geq \min\left\{1, \operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1} \cdot \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2} T_2^{\tau,k}\right\}. \quad (54)$$
Now, summing over k and passing to the infimum over decomposition $f = \sum_{k=1}^{\infty} f_k$, we get the assertion with
$$c_\tau = \min\left\{1, \operatorname*{ess\,inf}_{x\in E_1^\tau} \delta(x)^{-1} \cdot \frac{\|\delta\|_{L^\infty(E_2^\tau)}}{2}\right\}.$$
In order to prove that the Banach space $L^{(p[\cdot],\delta(\cdot)}(I)$ is a Banach function space it remains to show the validity of the Fatou property
$$0 \leq f^{(n)} \uparrow f\, a.e.\, in\, I \Rightarrow \rho_{(p[\cdot],\delta(\cdot)}\left(f^{(n)}\right) \uparrow \rho_{(p[\cdot],\delta(\cdot)}(f),$$
which is a immediate consequence of Levi’s Theorem .

We first recall the following two lemmas.

Lemma $(\mathbf{4.3.6})[\mathbf{290}]$: If $0 \leq b \leq a, p \geq 1 \Rightarrow a^p - b^p \geq (a-b)^p$.

Lemma $(\mathbf{4.3.7})[\mathbf{290}]$: ([298, Lemma 4]). If $0 \leq b < a, \alpha > 0, a \geq (1+\alpha)b, 0 < \gamma_0 \leq \gamma < 1$, then there exists $c = c(\alpha, \gamma_0)$ such that
$$(a-b)^\gamma \leq c(a^\gamma - b^\gamma),$$
where
$$c = c(\alpha,\gamma_0) = \frac{\alpha^{\gamma_0}}{(1+\alpha)^{\gamma_0} - 1}.$$
Theorem $(\mathbf{4.3.8})[\mathbf{290}]$: (The Levi’s theorem of monotone convergence for fully measurable small Lebesgue spaces). Let $p(\cdot) \in M_0^+, p(\cdot) \geq 1\, a.e.$ and $\delta \in L^\infty(I), \delta > 0\, a.e., 0 < \|\delta\|_\infty \leq 1.$ Let $f^{(n)} \in M_0^+$ be a monotone increasing sequence (i.e. $f^{(n)} \leq f^{(n+1)}$) such that
$$M = \sup_n \rho_{(p[\cdot],\delta(\cdot)}\left(f^{(n)}\right) < +\infty.$$
Then the function $f = \sup_n f^{(n)}$ is such that

$i)\, f \in L^{(p[\cdot],\delta(\cdot)}(I)$
$ii)\, f^{(n)} \nearrow f\, in\, I\, a.e.$
$iii) f^{(n)} \to f\, in L^{(p[\cdot],\delta(\cdot)}(I).$

Proof: Fix $\tau \in]\operatorname{ess\,inf}_{x\in I} p(x), \infty[$. Without loss of generality we may assume that the sequence $\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(n)}\right)$ is convergent, where $\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(n)}\right)$ is the expression equivalent to $\rho_{(p[\cdot],\delta(\cdot)}\left(f^{(n)}\right)$ defined in (53): in fact, if it is not the case, we can extract a suitable subsequence of $f^{(n)}$ and we prove first the theorem for such subsequence. This is sufficient to get the full assertion.

Let $I' = p^{-1}([1,\tau]) \subset I$, so that $|I'| > 0$, and let $\sigma > 0, r \in \mathbb{N}, r > 1$. Let $f^{(r)} = \sum_{k=1}^{\infty} f_k^{(r)}$ be such that

$$\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I'} \rho_{p(x)}\left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) < \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) + \sigma. \qquad (55)$$

For each $k \in \mathbb{N}$ there exists $T_{k,\sigma} \subset I', |T_{k,\sigma}| > 0$, such that

$$\rho_{p(x)}\left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) < \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I'} \rho_{p(x)}\left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) + \frac{\sigma}{2^k} \forall x \in T_{k,\sigma}\ a.e., \forall k \in \mathbb{N},$$

therefore

$$\operatorname*{ess\,inf}_{x\in T_{k,\sigma}}\left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) < \operatorname*{ess\,inf}_{x\in I'} \rho_{p(x)}\left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) + \frac{\sigma}{2^k}\ \forall k \in \mathbb{N}. \qquad (56)$$

On the other hand, let $s \in \mathbb{N}, s < r$. Since $f^{(n)}$ is an increasing sequence we have $f^{(s)} \leq f^{(r)}$, and, by Lemma (4.3.3), there exists $f_k^{(s)}$ such that

$$f^{(s)} = \sum_{k=1}^{\infty} f_k^{(s)}, \qquad f_k^{(s)} \leq f_k^{(r)} \forall k \in \mathbb{N}.$$

Then $f^{(r)} - f^{(s)} = \sum_{k=1}^{\infty}\left(f_k^{(r)} - f_k^{(s)}\right)$ and therefore

$$\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)} - f^{(s)}\right) \leq \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \rho_{p(x)}\left(\delta(x)^{-1}\left(f^{(r)} - f^{(s)}\right)(\cdot)\right).$$

By Lemma (4.3.6)

$$\rho_{p(x)}\left(\delta(x)^{-1}\left(f^{(r)} - f^{(s)}\right)(\cdot)\right) \leq \delta(x)^{-1}\left(\int_I \left(f^{(r)}(t)\right)^{p(x)} dt - \int_I \left(f^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}} \quad \forall x \in T^{k,\sigma}\ a.e.$$

Hence

$$\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)} - f^{(s)}\right) \leq \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \rho_{p(x)}\left(\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt - \int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}\right).$$

Now fix $0 < \alpha < 1$, and let

$$A_\alpha = \left\{k \in \mathbb{N} : \int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt < (1+\alpha) \int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt, \forall x \in T_{k,\sigma}\ a.e\right\}.$$

and

$$B_\alpha = \mathbb{N} \backslash A_\alpha.$$

We have

$$\sum_{k\in A_\alpha} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \delta(x)^{-1}\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt - \int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}$$

$$\leq \sum_{k\in A_\alpha} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \alpha^{\frac{1}{p(x)}}\, \delta(x)^{-1}\left(\int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}$$

$$\leq \sum_{k\in A_\alpha} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}}\, \delta(x)^{-1}\left(\int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}$$

$$\leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}} \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \delta(x)^{-1}\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}$$

$$\leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}}\left(\sigma + \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x\in I'} \delta(x)^{-1}\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}\right)$$

$$\leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}}\left(\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) + 2\sigma\right) \leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}}(M + 2\sigma),$$

where in the last two lines we used (56) and (55) respectively.
On the other hand, by Lemma (4.3.7), there exists a constant

$$c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) = \frac{\alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}}}{(1+\alpha)^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}} - 1}$$

such that

$$\sum_{k\in B_\alpha} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \delta(x)^{-1}\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt - \int_I \left(f_k^{(s)}(t)\right)^{p(x)}\right)^{\frac{1}{p(x)}}$$

$$\leq c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \sum_{k\in B_\alpha} \operatorname*{ess\,inf}_{x\in T_{k,\sigma}} \delta(x)^{-1}\left(\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}} - \left(\int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt\right)^{\frac{1}{p(x)}}\right)$$

$$\leq c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in T_{k,\sigma}} \delta(x)^{-1} \left(\left(\int_I \left(f_k^{(r)}(t)\right)^{p(x)} dt \right)^{\frac{1}{p(x)}} - \left(\int_I \left(f_k^{(s)}(t)\right)^{p(x)} dt \right)^{\frac{1}{p(x)}} \right)$$

$$\leq c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \left(\sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in I'} \rho_{p(x)} \left(\delta(x)^{-1} f_k^{(r)}(\cdot)\right) + \sigma - \sum_{k=1}^{\infty} \operatorname*{ess\,inf}_{x \in I'} \rho_{p(x)} \left(\delta(x)^{-1} f_k^{(s)}(\cdot)\right) \right)$$

$$\leq c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) + 2\sigma - \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(s)}\right) ,$$

where in the last two lines we used again (56) and (55). Then

$$\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)} - f^{(s)}\right) \leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}} (M + 2\sigma) + c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) + 2\sigma - \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(s)}\right).$$

Letting $\sigma \to 0$, we get, for any $r, s \in \mathbb{N}, r > s$,

$$\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)} - f^{(s)}\right) \leq \alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}} + c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) - \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(s)}\right). \quad (57)$$

Let $\epsilon > 0$ and fix α_ϵ such that

$$\alpha^{\frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}} M < \frac{\epsilon}{2}\,. \quad (58)$$

On the other hand, since the sequence $\tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right)$ is convergent, there exists $n_\epsilon \in \mathbb{N}$ such that

$$c\left(\alpha, \frac{1}{\operatorname*{ess\,sup}_{I'} p(x)}\right) \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)}\right) - \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(s)}\right) < \frac{\epsilon}{2} \;\forall r > s > n_\epsilon. \quad (59)$$

By (57), (58) and (59) we have that

$$\forall \epsilon > 0 \;\exists n_\epsilon \in \mathbb{N} : \tilde{\rho}_{(p[\cdot],\delta(\cdot)}\left(f^{(r)} - f^{(s)}\right) < \epsilon \;\forall r > s > n_\epsilon,$$

therefore, by Theorem (4.3.5), $f^{(n)}$ is a Cauchy sequence in $L^{(p[\cdot],\delta(\cdot)}(I)$ and, by Theorem (4.3.4), converges to some function $f \in L^{(p[\cdot],\delta(\cdot)}(I)$.

Since by property (vii). of Theorem(4.3.4)it is $L^{(p[\cdot],\delta(\cdot)}(I) \subset L^1(I)$, it follows that in I a.e. the limit f coincides with $\sup_n f^{(n)}$, which is also the a.e. limit of $f^{(n)}$.

As consequence of Theorem (4.3.4) and Theorem (4.3.8) we can state the following

Corollary $(\mathbf{4.3.9})[\mathbf{290}]$: Let $p(\cdot) \in M_0^+$,$p(\cdot) \geq 1\ a.e.$ and $\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \leq 1$. The space $L^{(p[\cdot],\delta(\cdot)}(I)$is a Banach function space.

we prove a Hölder-type inequality of fully measurable small Lebesgue spaces. The next result is a direct generalization of the Hölder inequality between grand and small Lebesgue spaces proved in [293]. Here we include the details because, as a consequence, it turns out that this argument is also an alternative approach to the duality between $L(\log L)^\alpha(I)$ and $EXP_{1/\alpha}(I)$.

Theorem $(\mathbf{4.3.10})[\mathbf{290}]$: Let $p(\cdot) \in M$,$p \geq 1\ a.e.$ and $\delta \in L^\infty(I), \delta > 0\ a.e., 0 < \|\delta\|_\infty \leq 1$. If $f, g \in M_0^+$, then fg is integrable and

$$\int_I fg dt \leq \rho_{p[\cdot]),\delta(\cdot)}(f)\rho_{(p'[\cdot],\delta(\cdot)}(g), \qquad (60)$$

where $p'(x) = p(x)/(p(x) - 1)$ denotes the conjugate exponent of $p(x)$(we set $1/0 = \infty$).

Proof: Let $g = \sum_{k=1}^\infty g_k$ be any decomposition with $g_k \geq 0$. For each $k \in \mathbb{N}$ we have

$$\int_I f g_k dt \leq \rho_{p(x)}\big(f(\cdot)\big)\rho_{p'(x)}\big(g_k(\cdot)\big)$$

$$= \delta(x)\rho_{p(x)}(f(\cdot))\delta(x)^{-1}\rho_{p'(x)}\big(g_k(\cdot)\big)dx$$

$$\leq \left(\operatorname*{ess\,sup}_{x\in I} \rho_{p(x)}\big(\delta(x)f(\cdot)\big)\right)\rho_{p'(x)}\big(\delta(x)^{-1}g_k(\cdot)\big)$$

$$= \rho_{p[\cdot]),\delta(\cdot)}(f)\rho_{p'(x)}\big(\delta(x)^{-1}g_k(\cdot)\big)$$

and therefore

$$\int_I f g_k dt \leq \rho_{p[\cdot]),\delta(\cdot)}(f) \operatorname*{ess\,inf}_{x\in I} \rho_{p'(x)}\big(\delta(x)^{-1}g_k(\cdot)\big) \ .$$

In conclusion

$$\int_I f g\, dt = \int_I f \sum_{k=1}^\infty g_k\, dt \leq \sum_{k=1}^\infty \int_I f g_k dt$$

$$\leq \sum_{k=1}^\infty \rho_{p[\cdot]),\delta(\cdot)}(f) \operatorname*{ess\,inf}_{x\in I} \rho_{p'(x)}\big(\delta(x)^{-1}g_k(\cdot)\big)$$

so that

$$\int_I f g\, dt \leq \rho_{p[\cdot]),\delta(\cdot)}(f) \inf_{g=\sum_{k=1}^\infty g_k} \sum_{k=1}^\infty \operatorname*{ess\,inf}_{x\in I} \rho_{p'(x)}\big(\delta(x)^{-1}g_k(\cdot)\big)$$

as desired.

Next theorem provides a norm equivalent to the norm in the Orlicz–Zygmund space $L(\log L)^\alpha(I)(\alpha > 0)$. We recall that $L(\log L)^\alpha(I)$ is the Orlicz space generated by the function $\Phi(t) = t\log^\alpha(e + t)$. The dual space and the associate space of $L(\log L)^\alpha(I)$

coincides with the Orlicz space $EXP_{1/\alpha}(I)$, generated by the function $\Phi(t) = exp(t^{1/\alpha}) - 1$, consisting of all measurable functions f on I such that $exp((\lambda f)^{1/\alpha}) \in L^1(I)$ for some $\lambda > 0$. The associate space of $EXP_{1/\alpha}(I)$ is $L(\log L)^\alpha(I)$, while the dual space of $EXP_{1/\alpha}(I)$ includes $L(\log L)^\alpha(I)$. A decomposition formula of the dual of $EXP_{1/\alpha}(I)$ has been given in [296].

Theorem $(4.3.11)$[290]: Let $I = (0,1), \alpha > 0$ and $f \in M^+$. The following equivalence

$$\inf_{g=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_{t\in I} t^{-\alpha} \|f_k\|_{L^{\frac{1}{1-t}}(I)} \approx \|f\|_{L(\log L)^\alpha(I)} \tag{61}$$

holds.

Proof: First we note that obviously it is

$$\inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_{t\in I} t^{-\alpha} \|f_k\|_{L^{\left(\frac{1}{t}\right)'}(I)} \le \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^{-\alpha} \|f_k\|_{(2^k)'}, \tag{62}$$

where $\left(\frac{1}{t}\right)' = \frac{1}{1-t}$ $and (2^k)' = \frac{2^k}{2^k - 1}$.

We recall the formulas in [11, Theorem $2, p. 72$] (see also $[16, p. 273]$) for the norm in $L^r(\log L)^\alpha$: if $1 \le r < \infty, k \in \mathbb{N}$, then the functional

$$\|f\|_{r,\alpha} = \inf_{f=\sum_{k=1}^{\infty} f_k} \left(\sum_{k=1}^{\infty} 2^{k\alpha} \|f_k\|_{r(2^k)'}^r\right)^{1/r} \tag{63}$$

defines a norm in $L^r(\log L)^\alpha$ equivalent to the Luxemburg norm. Then, by (62) and (63), it follows

$$\exists c_1 > 0 : \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_I t^{-\alpha} \|f_k\|_{L^{\frac{1}{1-t}}(I)} \le c_1 \|f\|_{L(\log L)^\alpha(I)}. \tag{64}$$

On the other hand, let $f = \sum_{k=1}^{\infty} f_k$ be any decomposition with $f_k \ge 0$. For each $k \in \mathbb{N}$ we have, by (52) in [296],

$$\exists c > 0: \|f_k\|_{L(\log L)^\alpha(I)} \le c t^{-\alpha} \|f_k\|_{\frac{1}{1-t}},$$

hence

$$\|f_k\|_{L(\log L)^\alpha(I)} \le \inf_{t\in I} c t^{-\alpha} \|f_k\|_{\frac{1}{1-t}}$$

and therefore

$$\|f\|_{L(\log L)^\alpha(I)} \le \sum_{k=1}^{\infty} \|f_k\|_{L(\log L)^\alpha(I)} \le c \sum_{k=1}^{\infty} \inf_{t\in I} t^{-\alpha} \|f_k\|_{\frac{1}{1-t}}.$$

From the previous inequality, passing to the infimum over all decomposition of f, we have

$$\|f\|_{L(\log L)^\alpha(I)} \le c \inf_{f=\sum_{k=1}^{\infty} f_k} \sum_{k=1}^{\infty} \inf_{t\in I} t^{-\alpha} \|f_k\|_{\frac{1}{1-t}}$$

and the assert is proved .

Chapter 5
Direct and Inverse Theorems with Approximation of Functions Problems

We show results obtained in present radically differ from other authors' results on this subject because we don't require from variable exponent $p(x)$ the fulfillment of additional condition $p(x) \geq p > 1$, which is closely related with boundedness of Hardy - Littlewood maximal function $M(f)$ in $L_{2\pi}^{p(x)}$. In the definition of the modulus of continuity of a function $f(x) \in L_{2\pi}^{p(x)}$, we replace the ordinary shift $f^h(x) = f(x+h)$ by an averaged shift determined by Steklov's function $s_h(f)(x) = \frac{1}{h}\int_0^h f(x+t)dt$. We obtain a converse theorem of trigonometric approximation in the weighted Lebesgue spaces and obtain some converse theorems of algebraic polynomial approximation in the weighted Smirnov spaces. Moreover, the constructive characterization problems for the some subclasses are discussed.

Section (5.1): Approximation Theory in Variable Lebesgue and Sobolev Spaces

In 1976, the year when we began studying the topology of space $L^{p(x)}(E)$, there was no theory of variable exponent Lebesgue spaces. There was only example of measurable functions set noted by Orlicz in [325]. Common modular spaces theory was being developed by the Japanese mathematicians ($H.$ Nakano [326], [327]), and functional modular spaces theory - by the Polish mathematicians (J. Musielak and W. Orlicz [328], [330]). Also note the work of Russian mathematician I. V. Tsenov [329].

But in these theories there was no consideration of a special theory of $L^{p(x)}(E)$ spaces. Such spaces were noted only as exotic examples of modular spaces. Spaces of functions integrable with an exponent ceased to play the role of exotic examples of modular spaces and set off on their path of development once the topology of these spaces was shown to be normable, with one of the equivalent norms given by Kolmogorov's well-known theorem on the normability of linear topological spaces having a bounded balanced convex neighbourhood of zero [337]. A.N.Kolmogorov [337] introduced a norm on such spaces by means of the Minkowski functional. In the same direction, the author showed in 1976 (but published [338] only in 1979) that the Lebesgue space $L_\mu^{p(x)}(E)$ with variable exponent $p(x) \geq 1$ (this space consists of measurable functions $f(x)$ on E such that $|f(x)|^{p(x)}$ is integrable on E) is a normed space with the norm of $f \in L_\mu^{p(x)}(E)$ given by

$$\|f\|_{p(\cdot)}(E) = \inf\left\{\alpha > 0 \mid \int_E \left|\frac{f(x)}{\alpha}\right|^{p(x)} \mu(dx) \leq 1\right\}. \qquad (1)$$

For unknown reasons, many authors call such norms Luxemburg norms instead of Kolmogorov norms.

In [328], conditions on variable exponent $p(x)$ for the $L^{p(x)}(E)$ space to be a linear topological space, were found. It was shown that $L^{p(x)}(E)$ will be a linear topological space if and only if $p(x)$ is essentially bounded function, $i.e. 0 < p(x) \le \bar{p}$ for almost every $x \in E$.

The case when $p(x)$ is not essentially bounded was considered in [328]. Such case is arising in the problem of finding conjugate space $[L^{p(x)}(E)]^*$ (space of continuous linear functionals) when essinf $p(x) = 1$. Moreover, there can be cases when $p(x) = \infty$ on set with nonzero measure. In all such cases, the corresponding spaces $[L^{p(x)}(E)]^*$ were found in [328].

Results and methods developed in [328] have been used in the sequel by many authors (quoting or not quoting [328]) and they represent now a kind of folklore in the theory of spaces $L^{p(x)}(E)$.

The next stage in the development of the theory of the spaces $L_\mu^{p(x)}(E)$ was the imposition of stronger conditions on the variable exponent $p(x)$ and obtaining $L_\mu^{p(x)}(E)$ analogues of classical results that were well known in the case of constant $p(x)$. The first step in this direction was made by the author [329] who showed that if μ is the ordinary Lebesgue measure on the line, then Haars system forms a basis for $L^{p(x)}([0,1])$ if and only if the variable exponent $p(x) \ge 1$ satisfies the Dini-Lipschitz condition on [0,1]:

$$|p(x) - p(y)| \log \frac{1}{|x-y|} \le C \; (|x-y| \le \frac{1}{2}).$$

Under the same hypotheses, the author [334] proved that some families of convolution operators are uniformly bounded in $L_\mu^{p(x)}([0,2\pi])$. This covers in particular a large class of classical operators, including the operators of Fejr, de la Valle-Poussin, Abel, Steklov and many others.

Substantial contributions to the theory of the spaces $L_\mu^{p(x)}(E)$ were made by V. V. Zhikov [326]–[328] and L. Diening in [336] − [337]. The best result obtained in [335] − [336] is as follows. Suppose that Ω is a bounded domain in $\mathbb{R}^n, \mu$ is the ordinary Lebesgue measure on $\mathbb{R}^n$, and $p(x)$ is defined on Ω and satisfies the conditions $1 < p_-(\Omega) \le p(x) \le p^-(\Omega) < \infty, |p(x) - p(y)| \log \frac{1}{|x-y|} \le C \; (|x-y| \le \frac{1}{2} \;\; x, y \in \Omega)$. Then the operator $M(f)$ of the Hardy-Littlewood maximal function acts boundedly on $L_\mu^{p(x)}(\Omega)$.

As a corollary, it was shown in [329] that under the same restrictions on $P(x)$ and some additional condition on $P(x)$ outside some ball, the well-known Calderon−Zygmund operators act boundedly in $L_\mu^{p(x)}(\mathbb{R}^n)$. In particular, for $n = 1$ it follows that the Hilbert transform is bounded in $L_\mu^{p(x)}(\mathbb{R})$ provided that $1 < p_1 \le p(x) \le p_2 < \infty, |p(x) - p(y)| \log \frac{1}{|x-y|} \le C \; (|x-y| \le \frac{1}{2} \;\; x, y \in \mathbb{R})$ and $P(x)$ coincides with a constant outside some interval. Thus, the connection between the Dini−Lipschitz condition for the variable exponent $p(x)$ and the uniform boundedness in $L_\mu^{p(x)}(\mathbb{E})$ of families of classical operators, described by the author in [338], [339], turned out to be characteristic in the construction of a deep theory of integral operators in the spaces $Lp(x)\,\mu\,(\mathbb{R})$. Numerous recent results obtained by specialists in the theory of differential equations show that a similar situation arises when constructing a deep theory of differential equations in Sobolev spaces with variable exponent. Many references can be found in the recent monograph [337]. Among them, a special place belongs , where the spaces $L_{\mu}^{p(x)}$ ($\mathbb{E}$) were used for the first time to study problems arising in the multidimensional calculus of variations. The properties of singular integrals in the spaces $L_{\mu}^{p(x)}$ ($\mathbb{E}$)were studied in under the same logarithmic DiniLipschitz condition on the variable exponent $p(x)$.

Here we consider the problem of the approximation of functions by trigonometric polynomials in the metric of $L_{\mu}^{p(x)}$ ($[0,2\pi]$). Suppose that $p = p(x)$ is a measurable 2π-periodic function, $p_- = \inf\{p(x) : x \in \mathbb{R}\}, p^- = \sup\{p(x) : x \in \mathbb{R}\}, 1 \leq p_- \leq p^- < \infty, L_{2\pi}^{p(x)}$ is the space of measurable 2π-periodic functions $f(x)$ with $\int_0^{2\pi} |f(x)|^{p(x)}\, dx < \infty$. Putting

$$\|f\|_{p(\cdot)} = \inf\left\{\alpha > 0 : \int_0^{2\pi} \left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\}, \qquad (2)$$

we turn $L_{2\pi}^{p(x)}$ into a Banach space. We write $P_{2\pi}$ for the set of all 2π-periodic variable exponents $p = p(x) \geq 1$ satisfying the condition

$$|p(x) - p(y)| \ln \frac{2\pi}{|x-y|} \leq d \ (x, y \in [0,2\pi]). \qquad (3)$$

The subclass of all $p = p(x) \in P_{2\pi}$ satisfying the additional condition $p_- > 1$, is denoted by $\hat{P}_{2\pi}$. The author proved [326] that if $p(x) \in \hat{P}_{2\pi}$, then the trigonometric system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ forms a basis for the space $L_{2\pi}^{p(x)}$. In other words, putting

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\, dt, \qquad k \in \mathbb{Z},$$

$$S_n(f) = S_n(f,x) = \sum_{k=-n}^{n} \hat{f}_k\, e^{ikx}, \qquad (4)$$

we have the estimate

$$\|S_n(f)\|_{p(\cdot)} \leq c(p)\|f\|_{p(\cdot)} \ (n = 0,1,\dots). \qquad (5)$$

It follows that the Fourier series of a function $f \in L_{2\pi}^{p(x)}$ converges to it in the norm (2), that is,

$$\|f - S_n(f)\|_{p(\cdot)} \to 0 \ (n \to \infty).$$

Moreover, if $p(x) \in \hat{P}_{2\pi}$, then the order of approximation of $f \in L_{2\pi}^{p(x)}$ by the partial sums (4) in the norm (2) as $n \to \infty$ coincides with the order of best approximation

$$E_n(f)_{p(\cdot)} = \inf_{T_n}\|f - T_n\|_{p(\cdot)}, \qquad (6)$$

where the infimum is taken over all trigonometric polynomials

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}. \tag{7}$$

We may now ask how the rate of decay of $E_n(f)_{p(\cdot)}$ as $n \to \infty$ depends on the properties of $f \in L_{2\pi}^{p(x)}$. In other words, we want to define the modulus of continuity of a function $f \in L_{2\pi}^{p(x)}$ and estimate $E_n(f)_{p(\cdot)}$ in terms of it. As mentioned in [335], the quantity

$$\omega(f,\delta)_{p(\cdot)} = \sup_{0<h\le\delta} \|f - f(* + h)\|_{p(\cdot)}$$

cannot play the role of the modulus of continuity of $f \in L_{2\pi}^{p(x)}$ in the case of a variable exponent $p = p(x)$ because, generally speaking, the equation $\lim_{\delta\to 0} \omega(f,\delta)_{p(\cdot)} = 0$ does not hold for all such f. If $p(x)$ is not equal to a constant almost everywhere on $[0,2\pi]$, then the shift $f_h(x) = f(x + h)$ of a function $f(x)$ in $L_{2\pi}^{p(x)}$ need not belong to $L_{2\pi}^{p(x)}$. Quite the contrary, the integral $\int_0^{2\pi} |f(x + h)|^{p(x)}\, dx$ usually diverges for $h \neq 0$. This was the main obstacle in the way of transferring the main theorems of the theory of approximation by trigonometric polynomials to the case of spaces $L_{2\pi}^{p(x)}$. We give one of the possible ways to overcome this obstacle by using certain types of Steklov functions. We put

$$f_h(x) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x + t)\, dt, \;\; s_h(f)(x) = f_h\left(x + \frac{h}{2}\right)$$

$$= \frac{1}{h} \int_0^h f(x + t)\, dt \tag{8}$$

and consider the quantity

$$\Omega(f,\delta)_{p(\cdot)} = \sup_{0<h\le\delta} \left\| f - f_h(* + \frac{h}{2}) \right\|_{p(\cdot)} = \sup_{0<h\le\delta} \|f - s_h(f)\|_{p(\cdot)}. \tag{9}$$

It follows from the author's results in [326] that if $p(x) \in P_{2\pi}$, then the function $\Omega(f,\delta)_{p(\cdot)}$ is continuous on $[0,\infty)$ and $\lim_{\delta\to 0} \Omega(f,\delta)_{p(\cdot)} = 0$. It also follows from the definition (9) that $\Omega(f,\delta)_{p(\cdot)}$)is a non-decreasing function of δ. We call $\Omega(f,\delta)_{p(\cdot)}$ the modulus of continuity of a function $f \in L_{2\pi}^{p(x)}$.

It was proved in author's works [328] – [330] that if the variable exponent $p(x) \in P_{2\pi}$ and $f \in L_{2\pi}^{p(x)}$, then the following Jackson-type inequality holds:

$$E_n(f)_{p(\cdot)} \le c(p)\Omega\left(f, \frac{1}{n}\right)_{p(\cdot)}. \tag{10}$$

Moreover, if $\Omega(f,\delta)_{p(\cdot)} \leq c\delta^{\alpha}\ (0 < \alpha < 1)$, then the converse assertion holds. Namely, if $E_n(f)_{p(\cdot)} \leq c/n^{\alpha}\ (n = 1,2,\dots)$, then $\Omega(f,\delta)_{p(\cdot)} = O(\delta^{\alpha})$. We note that in [336] we considered the quantity

$$\Omega^{\gamma}(f,0)_{p(\cdot)} = 0, \Omega^{\gamma}(f,0)_{p(\cdot)} = \sup_{\substack{h,\tau \\ |\tau|^{1/\gamma} \leq h \leq \delta}} \|f - f_h(* + \tau)\|_{p(\cdot)}\ , \qquad (11)$$

where $\gamma > 0$. We call it the γ-modulus of continuity of a function $f(x) \in L_{2\pi}^{p(x)}$. It follows from (9) and (11) that

$$\Omega(f,\delta)_{p(\cdot)} = \sup_{0<h\leq\delta} \left\| f - f_h\left(* + \frac{h}{2}\right)\right\|_{p(\cdot)} \leq \Omega^1(f,\delta)_{p(\cdot)}. \qquad (12)$$

On the other hand, the following result was proved in [326]:

Theorem $(\mathbf{5.1.1})[\mathbf{324}]$: If $p(x) \in P_{2\pi}, f(x) \in L_{2\pi}^{p(x)}$, then the function $g(\delta) = \Omega^{\gamma}(f,\delta)_{p(\cdot)}$ is non-decreasing on $[0,\infty]$ and continuous at the point $\delta = 0$. In particular, Theorem (5.1.1) and the estimate (12) yield the equation

$$\lim_{\delta\to 0} \Omega(f,\delta)_{p(\cdot)} = 0, \qquad (13)$$

mentioned above.

The proof of Theorem (5.1.1) is based on the uniform boundedness in $L_{2\pi}^{p(x)}\ 0 < h \leq 1, |\tau| \leq \pi h^{\gamma}$ of the family of shifts of the Steklov functions

$$S_{h,\tau}(f) = S_{h,\tau}(f)(x) = f_{h,\tau}(x) = f_h(x + \tau) = \frac{1}{h}\int_{x+\tau-\frac{h}{2}}^{x+\tau+\frac{h}{2}} f(t)\ dt.$$

Namely, it was proved in [326] that if $p(x) \in P_{2\pi}$, then

$$\left\|S_{h,\tau}(f)\right\|_{p(\cdot)} \leq c(d)(2\pi + 1)^{p^-}\|f\|_{p(\cdot)}\ 0 < h \leq 1, |\tau| \leq \pi h^{\gamma}, \qquad (14)$$

where d is the constant in the inequality (3).

We mention that the direct and inverse theorems of approximation theory in the spaces $L_{2\pi}^{p(x)}$ were obtained in [331][334] under the assumption that $p(x) \in \hat{P}_{2\pi}$. The principal difference between our results and those in [331][334] is that we are able to get rid of the restriction $p_- > 1$ and prove the direct and inverse theorems of approximation theory in $L_{2\pi}^{p(x)}$ under the natural assumption $p_- \geq 1$, where $p_- = inf\{p(x) : x \in \mathbb{R}\}$ (by the definition above). The results in [331][334] were obtained for $p_- > 1$, and we stress that this is not accidental. The methods used in those s to study the direct and inverse problems of approximation theory $L_{2\pi}^{p(x)}$ (and even in the more general weighted spaces $L_{2\pi,\rho}^{p(x)}$ with variable exponent)are based, either directly or indirectly, on the boundedness in $L_{2\pi}^{p(x)}$ of the operator $M(f)$ given by the Hardy-Littlewood maximal function (or of its analogues and generalizations in $L_{2\pi,\rho}^{p(x)}$), and it is well known that this holds only for $p_- > 1$. For example, in [331] the proof of a direct Jackson-type theorem for $L_{2\pi}^{p(x)}$ under the assumption that $p(x) \in \hat{P}_{2\pi}$ is based on the facts that the operator of conjugation (of functions) is bounded in $L_{2\pi}^{p(x)}$ and the trigonometric system forms a basis there. These facts were established by the author [326] using the boundedness in $L_{2\pi}^{p(x)}$ of the Hilbert transform under the assumption that $p(x) \in \hat{P}_{2\pi}$, and this boundedness was deduced in [335] from that of the maximal function, which was proved in [334]. To obtain direct and inverse theorems of approximation theory in $L_{2\pi}^{p(x)}$, where the variable exponent $p(x) \in P_{2\pi}$ satisfies the Dini-Lipschitz condition (3) and may be equal to 1 at some points (that is, $p^- = 1$), it is required to develop essentially new approaches which do not use the properties of the maximal function $M(f)$. In author's works [328] – [330] we make an attempt to solve the part of this problem that concerns Jackson's first theorem. One of the instruments in the proof of Jackson's first theorem in $L_{2\pi}^{p(x)}$ is Jackson's well-known operator (trigonometric polynomial of degree $2n-2$)

$$D_n(f) = D_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)J_n(t)\,dt \quad (n = 1,2,\dots),$$

where

$$J_n(x) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)^4 .$$

We proved in [329], [330] that $D_n(f)(x)$ approximates every $f(x) \in L_{2\pi}^{p(x)}$ with accuracy $O(\Omega(f,\frac{1}{n})_{p(\cdot)})$. In other words, if $f(x) \in L_{2\pi}^{p(x)}$ with $p(x) \in P_{2\pi}$, then

$$\|f - D_n(f)\|_{p(\cdot)} \le c(p)\Omega(f,\frac{1}{n})_{p(\cdot)},$$

which again gives the inequality (10).

The proof of the inequality (analogue of Jackson's second theorem)

$$E_n(f)_{p(\cdot)} \le c(p)\frac{1}{n^r}\Omega\left(f^{(r)},\frac{1}{n}\right)_{p(\cdot)} \tag{15}$$

encounters additional difficulties, and in [328]– [330] we have not been able to overcome them in the general case when $p(x) \in P_{2\pi}$. Therefore in [328] – [330] we only give it for $p(x) \in \hat{P}_{2\pi}$. But in present work we consider the general case when $p(x) \in P_{2\pi}$. We succeeded in proving that the inequality (15) holds for every function $f(x) \in W_{p(\cdot)}^r$, where $p(x) \in P_{2\pi}$, $W_{p(\cdot)}^r$ is the Sobolev space of 2π-periodical functions $f(x)$ such that $f^{(r-1)}(x)$ is absolutely continuous in $[0,2\pi]$ and $f(r)(x) \in L_{2\pi}^{p(x)}$. In the author's works [335] – [336] it is shown that one of instruments in the proof of inequality (15) is the Valle - Poussin's well-known means

$$V_m^n(f) = V_m^n(f,x) = \frac{1}{m+1}\sum_{l=0}^{m} S_{n+l}(f,x),$$

where Fourier sums $S_k(f,x)$ are defined in (4). Namely, in [335]–[336] the following inequality

$$\| f - V_m^n(f)\|_{p(\cdot)} \leq \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)} \qquad (16)$$

is proved, where $p = p(x) \in P_{2\pi}, r \geq 0, f(x) \in W_{p(\cdot)}^r, m \in \{n-1, n\}$. The estimate (15) (analogue of Jackson's second theorem) follows from (10) and (16) as a corollary. The complete proof of inequality (16) .

We will consider in $L^{p(x)}$ Sobolev type classes $W_{p(\cdot)}^r(M)$, which consist of 2π-periodical $r-1$ times continuously differentiable functions $f(x)$, whose derivative $f^{(r-1)}(x)$ is absolutely continuous in $[0,2\pi]$ and $f^{(r)}(x) \in L_{2\pi}^{p(x)}, \left\|f^{(r)}\right\|_{p(\cdot)} \leq M$. Let us assume

$$W_{p(\cdot)}^r = \bigcup_{M>0} W_{p(\cdot)}^r(M), W_{p(\cdot)}^0 = L_{2\pi}^{p(x)}.$$

We can consider the Fourier series for $f \in L_{2\pi}^{p(x)}$:

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \qquad (17)$$

and partial sum of Fourier series

$$S_n(f) = S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx , \qquad (18)$$

Where

$$a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos kt \, dt , \qquad b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin kt \, dt .$$

If $r \geq 1, p(x) \geq 1$ and $f \in W_{p(\cdot)}^r$, then $[337, p. 75]$

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t) B_r(t-x)\, dt, \qquad (19)$$

where

$$B_r(u) = \sum_{k=1}^{\infty} \cos\frac{\left(ku + \frac{\pi r}{2}\right)}{k^r} \qquad (20)$$

is the Bernoulli function. Since $S_n^{(r)}(f,x) = S_n(f^{(r)},x)$, then we conclude from (19) and (20) an equality for $f \in W_{p(\cdot)}^r$

$$f(x) - S_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t) R_{r,n}(t-x)\, dt, \qquad (21)$$

$$R_{r,n}(u) = \sum_{k=1+n}^{\infty} \cos\frac{\left(ku + \frac{\pi r}{2}\right)}{k^r}. \quad (22)$$

We will define Vallee-Poussin means $V_m^n(f) = V_m^n(f,x)$ by equality

$$V_m^n(f) = V_m^n(f,x) = \frac{1}{m+1}\sum_{l=0}^{m} S_{n+l}(f,x). \quad (23)$$

Matching equalities (21) and (22) with (23) we notice

$$f(x) - V_m^n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)\frac{1}{m+1}\sum_{l=0}^{m} R_{r,n+l}(t-x)\ dt. \quad (24)$$

We will assume

$$\mathfrak{K}_{r,m+1}^{n}(u) = (m+1)^{r-1}\sum_{l=0}^{m} R_{r,n+l}(n) \quad (25)$$

and transcribe (24)

$$f(x) - V_m^n\ (f,x) = \frac{1}{\pi(m+1)^r}\int_{-\pi}^{\pi} f^{(r)}(t)\mathfrak{K}_{r,m+1}^{n}(t-x)\,dt. \quad (26)$$

Since, by (25), $\mathfrak{K}_{r,m+1}^{r}(x)$ is orthogonal to all trigonometric polynomials of degree not greater than n, then we obtain from (26)

$$f(x) - V_m^n(f,x)$$
$$= 1\frac{1}{\pi(m+1)^r}\int_{-\pi}^{\pi}\left(f^{(r)}(t) - T_n(t)\right)\mathfrak{K}_{r,m+1}^{n}(t-x)\,dt, \quad (27)$$

where $T_n(x)$ is an arbitrary trigonometric polynomial of degree n. Now we can state the next result.

Theorem $(\mathbf{5.1.2})[\mathbf{324}]$: Let $p = p(x) \in P_{2\pi}, r \geq 0, f(x) \in W_{p(\cdot)}^{r}$. Then the following estimates hold:

$$\|f - V_{n-1}^{n}\ (f)\|_{p(\cdot)} \leq \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)}, \quad (28)$$

$$\|f - V_n^n\ (f)\|_{p(\cdot)} \leq \frac{c(p)}{n^r} E_n(f^{(r)})_{p(\cdot)}. \quad (29)$$

Proof : is based on a number of auxiliary assertions concerning functions $\mathfrak{K}_{r,m+1}^{n}(u)$.

Lemma $(\mathbf{5.1.3})[\mathbf{324}]$: We have the following equalities

$$\kappa_{2s,n}^{n}(u) = (-1)^s n^{2s-1} \sum_{l=0}^{n-2}\sum_{k=0}^{\infty} \frac{\sin\frac{l+1}{2}u \sin\frac{k+1}{2}u \cos(2n+k+l+2)\frac{u}{2}}{2\sin^2\frac{u}{2}} \Delta^2 g_s(n+1+k+l) + (-1)^{s-1} n^{2s-1} \sum_{k=0}^{\infty} \frac{\sin\frac{n}{2}u\ \sin\frac{k+1}{2}u\cos(3n+k+1)}{2\sin^2\frac{u}{2}} \Delta q_s(2n+k), \quad (30)$$

$$\kappa_{2s-1,n}^{n}(u) = (-1)^s n^{2s-2} \sum_{l=0}^{n-2}\sum_{k=0}^{\infty} \frac{\sin\frac{l+1}{2}u \sin\frac{k+1}{2}u \cos(2n+k+l+2)\frac{u}{2}}{2\sin^2\frac{u}{2}} \Delta^2 g_s(n+1+k+l) + (-1)^{s-1} n^{2s-2} \sum_{k=0}^{\infty} \frac{\sin\frac{n}{2}u\ \sin\frac{k+1}{2}u\cos(3n+k+1)}{2\sin^2\frac{u}{2}} \Delta q_s(2n+k), \quad (31)$$

where $g_s(t) = t^{-2s}, q_s(t) = t^{-2s+1}, \Delta\varphi(t) = \varphi(t+1) - \varphi(t), \Delta^2\varphi(t) = \varphi(t+2) - 2\varphi(t+1) + \varphi(t)$.

Proof: From (22) and (25) we have

$$\kappa_{r,m+1}^{n}(u) = (m+1)^{r-1} \sum_{l=0}^{m}\sum_{k=0}^{\infty} \frac{\cos\left[(n+k+l+1)u + \frac{\pi r}{2}\right]}{(n+k+l+1)^r},$$

So, with the help of Abel transform, we can write

$$\kappa_{r,m+1}^{n}(u) = (m+1)^{r-1} \sum_{l=0}^{m}\sum_{k=0}^{\infty} \left[\frac{1}{(n+1+k+l)^r} - \frac{1}{(n+2+k+l)^r}\right] v_{k,l}^{n}(u), \quad (32)$$

where

$$v_{k,l}^{n}(u) = \sum_{j=0}^{k} \cos\left[(n+1+l+j)u + \frac{\pi r}{2}\right]. \quad (33)$$

We will consider the case when $m = n-1$ and the two cases of r, even and odd. If $r = 2s$, then $\cos(\mu u + \frac{\pi r}{2}) = (-1)^s \cos\mu u$. Therefore, (33) takes the form

$$v_{k,l}^{n}(u) = (-1)^{s}\sum_{j=0}^{k}\cos(n+1+l+j)u =$$

$$= (-1)^{s} \cdot \frac{\sin(2(n+1+l+k)+1)\frac{u}{2} - \sin(2(n+l)+1)\frac{u}{2}}{2\sin\frac{u}{2}} \qquad (34)$$

From (32) and (34) we have

$$\kappa_{2s,n}^{n}(u) = (-1)^{s-1}n^{2s-1} \times \sum_{k=0}^{\infty}\sum_{l=0}^{n-1}\Delta g_{s}(n+1+k+l)\frac{\sin(2(n+1+l+k)+1)\frac{u}{2} - \sin(2(n+l)+1)\frac{u}{2}}{2sin\frac{u}{2}}. \qquad (35)$$

We apply Abel transform to the inner sum again. From (35) we get

$$\kappa_{2s,n}^{n}(u) = (-1)^{s-1}n^{2s-1}\sum_{k=0}^{\infty}\sum_{l=0}^{n-1}\Delta g_{s}(n+1+k+l)\frac{W_{k,l}^{n}(u) - W_{0,l}^{n}(u)}{2sin\frac{u}{2}} +$$

$$+(-1)^{s-1}n^{2s-1}\sum_{k=0}^{\infty}\Delta g_{s}(2n+k)\frac{W_{k,n-1}^{n}(u) - W_{0,n-1}^{n}(u)}{2sin\frac{u}{2}}, \qquad (36)$$

Where

$$W_{k,l}^{n}(u) = \sum_{\mu=0}^{l}\sin(2(n+1+\mu+k)+1)\frac{u}{2}$$

$$= \frac{\sin^{2}(n+l+k+2)\frac{u}{2} - \sin^{2}(n+k+1)\frac{u}{2}}{\sin\frac{u}{2}} =$$

$$= \frac{1}{\sin\frac{u}{2}}\left(\sin(n+l+k+2) - \sin(n+k+1)\frac{u}{2}\right)\left(\sin(n+l+k+2) + \sin(n+k+1)\frac{u}{2}\right)$$

$$= \frac{4}{\sin\frac{u}{2}}\ \sin\frac{l+1}{4}u \cdot \cos\left(n+k+1+\frac{l+1}{2}\right)\frac{u}{2} \cdot \sin\left(n+k+1+\frac{l+1}{2}\right)\frac{u}{2}\cos\frac{l+1}{4}. \qquad (37)$$

From (37) we get

$$W_{k,l}^{n}(u) - W_{0,l}^{n}(u) =$$

$$= \frac{4}{\sin\frac{u}{2}} \sin\frac{l+1}{2} \cos\frac{l+1}{2}\Big(\sin(n+k+1+\frac{l+1}{2})\frac{u}{2}\cos(n+k$$

$$+1+\frac{l+1}{2})\frac{u}{2} - -\sin(n+\frac{l+1}{2})\frac{u}{2}\cos(n+\frac{l+1}{2})\frac{u}{2}\Big) =$$

$$\frac{\sin\frac{l+1}{2}}{\sin\frac{u}{2}}\left(\sin(2(n+k+1)+l+1)\frac{u}{2} - \sin(2n+l+1)\frac{u}{2}\right)$$

$$= \frac{1}{\sin\frac{u}{2}} \sin\frac{l+1}{2} u \sin\frac{k+1}{2} u \cos(2n+k+l+2)\frac{u}{2}. \qquad (38)$$

So, the equality (30) follows from (36) and (38). Equality (31) is proved similarly. Lemma (5.1.3) is proved.

Lemma $(\mathbf{5.1.4})[\mathbf{324}]$: Suppose $0 \le k \le l$. Then

$$A_{k,l} = \int_0^{\pi} \frac{|\sin\frac{k+1}{2} u \sin\frac{l+1}{2} u|}{\sin\frac{u}{2}} du \le 2(k+1)(2+\ln\frac{l+1}{k+1}) + \frac{\pi}{3-\frac{\pi^2}{8}}.$$

Proof: We have

$$A_{k,l} = 2\int_0^{\frac{\pi}{2}} \frac{|\sin(k+1)u \sin(l+1)|}{\sin^2 u} du$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{|\sin(k+1)u\sin(l+1)|}{u^2} du$$

$$+ \int_0^{\frac{\pi}{2}} |\sin(k+1)u\sin(l+1)u|\varphi(u)\,du, \qquad (39)$$

where

$$\varphi(u) = \frac{1}{\sin^2 u} - \frac{1}{u^2} = \frac{u^2 - \sin^2 u}{u^2 \sin^2 u}.$$

Suppose $0 < u < \frac{\pi}{2}$. Then

$$\varphi(u) = \frac{(u + \sin u)(u - \sin u)}{u^2 \sin^2 u} = \frac{(2u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots)(\frac{u^3}{3!} - \frac{u^5}{5!} + \dots)}{u^2 \sin^2 u}$$

$$= \frac{(2 - \frac{u^2}{3!} + \frac{u^4}{5!} - \dots)(\frac{1}{3!} - \frac{u^2}{5!} + \dots)}{\left(\frac{\sin u}{u}\right)^2}$$

$$= \frac{(2 - \frac{u^2}{3!} + \frac{u^4}{5!} - \dots)(\frac{1}{3!} - \frac{u^2}{5!} + \dots)}{(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \dots)^2} << \frac{\frac{2}{3!}}{\left(1 - \frac{u^2}{3!}\right)^2} = \frac{1}{3(1 - \frac{\pi^2}{24})}$$

$$= \frac{1}{3 - \frac{\pi^2}{8}}$$

and, therefore,

$$2\int_0^{\frac{\pi}{2}} |\sin(k + 1)u \sin(l + 1)u|\varphi(u)\ du \le \frac{1}{3 - \frac{\pi^2}{8}}. \qquad (40)$$

On the other hand,

$$\int_0^{\frac{\pi}{2}} \frac{|\sin(k + 1)u \sin(l + 1)u|}{u^2}\ du = (k + 1) \int_0^{\frac{\pi}{2}(k+1)} \frac{|\sin u \sin \frac{l+1}{k+1}\ u|}{u^2}\ du \le$$

$$(k+1)\int_0^1 \frac{\left|\sin\frac{l+1}{k+1}u\right|}{u}\,du + (k+1)\int_1^{\frac{\pi}{2}(k+1)} \frac{du}{u^2}$$

$$= (k+1)\int_0^{\frac{l+1}{k+1}} \frac{|\sin u|}{u}\,du + (k+1)$$

$$= (k+1)\left[\int_0^1 \frac{\sin u}{u}du + \int_0^{\frac{l+1}{k+1}} \frac{du}{u} + 1\right]$$

$$< (k+1)\left(2 + \ln\frac{l+1}{k+1}\right). \qquad (41)$$

The statement of the lemma follows from equality (39) and inequalities (40) and (41).

Lemma (5.1.5)[324]: If $r \geq 1$, then

$$\int_{-\pi}^{\pi} |\kappa_{r,n}^{n}|\,du \leq c(r).$$

Proof: Consider the case of even $r = 2s$. Then, from Lemma (5.1.3) we have

$$\int_{-\pi}^{\pi} |\kappa_{r,n}^{n}|\,du$$

$$\leq n^{2s-1}\sum_{l=0}^{n-2}\sum_{k=0}^{\infty}\Delta^2 g_s(n+k+l)\frac{1}{2}\int_{-\pi}^{\pi}\frac{\left|\sin\frac{k+1}{2}u\ \sin\frac{l+1}{2}u\right|}{\sin^2\frac{u}{2}}du$$

$$+ n^{2s-1}\sum_{k=0}^{\infty}\Delta g_s(2n+k)\frac{1}{2}\int_{-\pi}^{\pi}\frac{\left|\sin\frac{n}{2}u\ \sin\frac{k+1}{2}u\right|}{\sin^2\frac{u}{2}}du. \qquad (42)$$

Because of the Lemma (5.1.4),

$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{\left|\sin\frac{k+1}{2}u\ \sin\frac{l+1}{2}u\right|}{\sin^2\frac{u}{2}}du$$

$$= \begin{cases} 2(k+1)\left(2+\ln\frac{l+1}{k+1}\right) + \dfrac{\pi}{3-\frac{\pi^2}{8}}, & k \leq l, \\[2ex] 2(k+1)\left(2+\ln\frac{k+1}{l+1}\right) + \dfrac{\pi}{3-\frac{\pi^2}{8}}, & l < k. \end{cases} \qquad (43)$$

Next, since the $\Delta^2 g_s(t) = g_s''(\bar{t})$ $(t \leq \bar{t} \leq t+2)$, then

$$\Delta^2 g_s(n+1+k+l) = g_s''(\bar{t}) = \frac{2s(2s+1)}{\bar{t}^{2s+2}}$$
$$\leq \frac{2s(2s+1)}{(n+1+k+l)^{2s+2}}, \tag{44}$$
where $n+1+k+l < \bar{t} < n+1+k+l+2$ and, similarly,
$$\Delta g_s(2n+k) \leq \frac{2s}{(2n+k)^{2s+1}}. \tag{45}$$
From (43) and (44) we have ($l \leq n-2$)
$$\sum_{k=l}^{\infty} \Delta^2 g_s(n+1+k+l) \frac{1}{2} \int_{-\pi}^{\pi} \frac{\left|\sin \frac{k+1}{2}u \ \sin \frac{l+1}{2}u\right|}{\sin^2 \frac{u}{2}} du \leq$$

$$\leq \sum_{k=l}^{\infty} \frac{4s(2s+1)\left[2+\ln \frac{k+1}{l+1}\right] + \frac{\pi}{3-\frac{\pi^2}{8}}}{(n+1+k+l)^{2s+2}} \leq$$
$$\leq \sum_{k=l}^{\infty} \frac{4s(2s+1)\left[2(l+1)+k-l+\frac{\pi}{3-\frac{\pi^2}{8}}\right]}{(n+1+k+l)^{2s+2}}$$
$$\leq c(s) n^{-2s}, \tag{46}$$
$$\sum_{k=0}^{l} \frac{2s(2s+1)[(k+1)(2+\ln \frac{l+1}{k+1} + \frac{\pi}{3-\frac{\pi^2}{8}})]}{(n+1+k+l)^{2s+2}} \leq c(s) n^{-2s}, \tag{47}$$
therefore
$$n^{2s-1} X \sum_{l=0}^{n-1} \sum_{k=0}^{\infty} \Delta^2 g_s(n+1+k+l) \frac{1}{2} \int_{-\pi}^{\pi} \frac{\left|\sin \frac{k+1}{2}u \sin \frac{l+1}{2}u\right|}{\sin^2 \frac{u}{2}} du$$
$$\leq c(s) n^{2s-1} \sum_{l=0}^{n-2} n^{-2s} \leq c(s). \tag{48}$$
Next, from (43) and (45) we have
$$n^{2s-1} \sum_{k=0}^{\infty} \Delta g_s(2n+k) \frac{1}{2} \int_{-\pi}^{\pi} \frac{\left|\sin \frac{n}{2}u \ \sin \frac{k+1}{2}u\right|}{\sin^2 \frac{u}{2}} du \leq n^{2s-1}$$

$$\sum_{k=0}^{n} \frac{4s\left[(k + 1)(2 + \ln\frac{n}{k+1}) + \frac{\pi}{3+\frac{\pi^2}{8}}\right]}{(2n + k)^2 s + 1}$$

$$+ n^{2s-1} \sum_{k=n+1}^{\infty} \frac{4s\left[n(\ln\frac{n}{k+1} + 2) + \frac{\pi}{3+\frac{\pi^2}{8}}\right]}{(2n + k)^2 s + 1}$$

$$= n^{2s-1} \sum_{k=n+1}^{\infty} \frac{4s(2n + \frac{\pi}{3+\frac{\pi^2}{8}})^{2s+1}}{2n + k}$$

$$+ n^{2s-1} \sum_{k=n+1}^{\infty} \frac{4sn\ln\frac{k+1}{n}}{(2n + k)^{2s+1}} \leq c_1(s) + c_2(s)n^{2s} \sum_{k=n+1}^{\infty} \ln\frac{(1 + \frac{k-n+1}{n})}{(2n + k)^{2s+1}} =$$

$$c_1(s) + c_2(s)n^{2s}\, X \sum_{j=1}^{\infty} \frac{\ln(1 + \frac{j}{n})}{(3n - 1 + j)^{2s+1}} = c(s) + \frac{c(s)}{n} \sum_{j=1}^{\infty} \frac{\ln(1 + \frac{j}{n})}{(3 - \frac{1}{n} + \frac{j}{n})^{2s+1}}$$

$$\leq c(s)\left(1 + \int_0^{\infty} \frac{\ln(1 + x)dx}{(2 + x)^{2s+1}}\right) \leq c(s). \qquad (49)$$

Comparing (48) and (49) with (42), we complete the proof of Lemma (5.1.5) for even $r \geq 1$. Lemma ((5.1.5)) is proved similarly in case of odd $r = 2s - 1$.

Lemma (5.1.6)[324]: For $n^{-\frac{1}{2}} \leq u \leq 2\pi - n^{-\frac{1}{2}}$ we have inequality

$$\left|\kappa_{r,n}^{n}(u)\right| \leq c(r).$$

Proof: If $n^{-\frac{1}{2}} \leq u \leq 2\pi - n^{-\frac{1}{2}}$, then $1 \sin^2\frac{u}{2} \leq \frac{\pi^2}{4}n$ and, therefore, from Lemma (5.1.3) and inequalities (44) and (45) we have

$$\left|\kappa_{2s,n}^{n}(u)\right| \leq c(s)n^{2s}\left(\sum_{l=0}^{n-2}\sum_{k=0}^{\infty} \frac{1}{(n + 1 + k + l)^{2s+2}} + \sum_{k=0}^{\infty} \frac{1}{(2n + k)^{2s+1}}\right)$$

$$\leq c(s)$$

and, similarly, $|\kappa_{2s-1}^{n}(u)| \leq c(s)$. Lemma (5.1.6) is proved.

Lemma (5.1.7)[324]: We have the estimate

$$\max_u\left|\kappa_{r,n}^{n}(u)\right| \leq c(r)n \quad (n = 1,2,\ldots).$$

Proof: Consider the case $r = 1$. So, from (22) we have $(0 \leq u \leq 2\pi)$

$$R_{1,n}(u) = \sum_{k=n+1}^{\infty} \frac{\sin ku}{k} = \frac{\pi - u}{2} - \sum_{k=1}^{n} \frac{\sin ku}{k}. \qquad (50)$$

It is well known [337, p. 105], that

$$\left|\sum_{k=1}^{n} \frac{\sin ku}{k}\right| \leq c \quad (n = 1,2,\dots),$$

Assertion of Lemma (5.1.7) follows from (25).J Now we need one result established in author's [330]. Define for every $\lambda \geq 1$ a measurable 2π-periodical essentionally confined function (kernel) $\kappa_\lambda = \kappa_\lambda(x)$. Then we can define linear operator

$$\kappa_\lambda(f) = \kappa_\lambda(f)(x) = \int_{-\pi}^{\pi} f(t)\kappa_\lambda(t-x)\,dt, \qquad (51)$$

functional in space $L_{2\pi}^{p(x)}$. We will say that the kernel family $\{\kappa_\lambda(x)\}_{1\leq\lambda<\infty}$ satisfies the conditions $A), B)$ $C)$, respectively, if the following estimates hold:

$$A) \int_{-\pi}^{\pi} |\kappa_\lambda(t)|\, dt \leq c_1,$$

$$B) \sup_x |\kappa_\lambda(x)| \leq c_2\lambda^v,$$

$$C) |\kappa_\lambda(x)| \leq c_3\,\lambda^{-\gamma} \leq |x| \leq \pi,$$

where $v, \gamma, cj > 0$ are independent of λ. The theorem below was proved in [330].

Theorem (5.1.8) [324]: Let $\kappa_\lambda = \kappa_\lambda(x)$ $(1 \leq \lambda < \infty)$ satisfy the conditions $A)$— $C)$. If $p(x) \in P_{2\pi}$, then the operator family convolution $\{\kappa_\lambda(f)\}_{\lambda\geq 1}$, defined by the equality (51), is uniformly bounded in $L_{2\pi}^{p(x)}$.

Now we can formulate the following auxiliary assertion:

Lemma (5.1.9)[324]: Let $p(x) \in P_{2\pi}, f \in L_{2\pi}^{p(x)}$,

$$K_n(f) = K_n(f)(x) = \int_{-\pi}^{\pi} f(t)\kappa_{r,n}^n(t-x)\,dt, \qquad (n = 1,2,\dots). \qquad (52)$$

Then we have the estimate

$$\|K_n(f)\|_{p(\cdot)} \leq c_r(p)\|f\|_{p(\cdot)}.$$

The assertion of this Lemma follows directly from Theorem (5.1.8), because in view of Lemmas (5.1.5)— (5.1.7) the kernel family $\kappa_{r,n}^n(x)$ $(n = 1,2,\dots)$ satisfies the conditions $A)$— $C)$. Let's return to the Proof of Theorem (5.1.3). From the equality (27) and Lemma (5.1.9) we have

$$\|f - V_{n-1}^n(f)\|_{p(\cdot)} \leq \frac{c_r(p)}{n^r} \left\|f^{(r)} - T_n\right\|_{p(\cdot)}, \qquad (53)$$

where $T_n = T_n(x)$ is an arbitrary trigonometric polynomial of degree n. The estimate (28) follows from (53). As for estimate (29), its proof is quite similar. The Theorem (5.1.2) is proved.

Now let's mention the theorem proved in [339]:

Theorem (5.1.10) [324]: Let $p = p(x) \in P_{2\pi}, f(x) \in L_{2\pi}^{p(x)}$. Then the following estimate holds:

$$E_n(f)_{p(\cdot)} \le c(p)\Omega(f, \frac{1}{n})_{p(\cdot)}.$$

Combined, Theorem (5.1.10) and Theorem (5.1.2) make it possible to formulate

Consequence (5.1.11)[324]: Let $p = p(x) \in P_{2\pi}, r \ge 0, f(x) \in W_{p(\cdot)}^r$. Then the following estimates hold:

$$\|f - V_{n-1}^n (f)\|_{p(\cdot)} \le \frac{c(p)}{n^r} \Omega\left(f^{(r)}, \frac{1}{n}\right)_{p(\cdot)}, \tag{54}$$

$$\|f - V_n^n (f)\|_{p(\cdot)} \le \frac{c(p)}{n^r} \Omega\left(f^{(r)}, \frac{1}{n}\right)_{p(\cdot)}. \tag{55}$$

Consequence (5.1.12)[324]: Let $p = p(x) \in P_{2\pi}, r \ge 0, f(x) \in W_{p(\cdot)}^r$. Then the following estimate holds ($m = 1,2,\dots$):

$$E_m(f)_{p(\cdot)} \le \frac{c(p)}{m^r} \Omega\left(f^{(r)}, \frac{2}{m}\right)_{p(\cdot)}. \tag{56}$$

Proof: If $m = 2n$, then estimate (56) follows from (54). If $m = 2n - 1$, then (56) follows from (55). Consequence (5.1.12) is proved .

Section (5.2): Weighted Lebesgue and Smirnov Spaces

For $L^p(T)$ be the Lebesgue space of 2π-periodic real valued functions defined on $T := [-\pi, \pi]$ such that

$$\|f\|_p := \begin{cases} \left(\int_T |f(x)|^p \, dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in T} |f(x)|, & p = \infty, \end{cases}$$

is finite.

A function $\omega : T \to [0, \infty]$ will be called a weight if ω is measurable and almost everywhere $(a.e.)$ positive.

For a weight ω we denote by $L^p(T, \omega)$ the class of measurable functions $f : T \to \mathbb{R}$ such that $\omega f \in L^p(T)$. We set $\|f\|_{p,\omega} := \|\omega f\|_p$.

If $p^{-1} + q^{-1} = 1, 1 < p < \infty, \omega \in L^p(T)$, and $1/\omega \in L^q(T)$ then

$$L^\infty(T) \subset L^p(T, \omega) \subset L^1(T).$$

A 2π-periodic weight function ω belongs to the Muckenhoupt class A_p, if

$$\left(\frac{1}{|J|} \int_J \omega^p(x) \, dx\right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-q}(x) \, dx\right)^{1/q} \le C$$

with a finite constant C independent of J, where J is any subinterval of T and $|J|$ denotes the length of J. Let

$$S\,[f] := \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{57}$$

be the Fourier series of a function $f \in L^1(T)$ with $\int_T f(x)\,dx = 0$; so $c_0 = 0$ in (57). For $\alpha > 0$, the α-th integral of f is defined by

$$I_\alpha(x,f) := \sum_{k\in\mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx} \ ,$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha}\, e^{(-1/2)\pi\, i\alpha\, sign\, k} \ and\ \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}.$$

It is known $[363, V.2, p.134]$ that

$$f_\alpha(x) := I_\alpha(x,f)$$

exists a.e. on T and $f_\alpha \in L^1(T)$. For $\alpha \in (0,1)$ we set

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f)$$

if the right-hand side exists. Then we define

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f),$$

where $r \in \mathbb{Z}^+ := \{1,2,3,\ldots\}$.

Throughout this work by $C(\alpha), c_1, c_2, \ldots, c_i(\alpha, \ldots), c_j(\beta, \ldots), \ldots$ we denote the constants (which can be different in different places) such that they are absolute or depend only on the parameters given in the corresponding brackets.

Let $x, t \in \mathbb{R}, \alpha \in \mathbb{R}^+ := (0,\infty), 1 < p < \infty$. We set

$$\Delta_t^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^\alpha [C_k^\alpha] f(x + (\alpha - k)t) \ , f \in L^p(T,\omega), \tag{58}$$

where $[C_k^\alpha] := \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ for $k > 1$, $[C_k^\alpha] := \alpha$ for $k = 1$ and $[C_k^\alpha] := 1$ for $k = 0$.

Since

$$|[C_k^\alpha]| \le \frac{c_1(\alpha)}{k^{\alpha+1}} \ , for\ k \in \mathbb{Z}^+,$$

we have

$$C(\alpha) := \sum_{k=0}^{\infty} |[C_k^\alpha]| < \infty,$$

and $\Delta_t^\alpha f(x)$ is defined $a.e.$ If $\alpha \in \mathbb{Z}^+$, then the fractional difference $\Delta_t^\alpha f(x)$ coincides with usual forward difference, namely,

$$\Delta_t^\alpha f(x) = \sum_{k=0}^{\alpha} (-1)^\alpha [C_k^\alpha] f(x + (\alpha - k)t)$$

$$= \sum_{k=0}^{\infty} (-1)^{\alpha-k} [C_k^{\alpha}] f (x + kt), \alpha \in \mathbb{Z}^+.$$

We define

$$\sigma_{\delta}^{\alpha} f (x) := \frac{1}{\delta} \int_{\delta}^{0} |\Delta_t^{\alpha} f (x)| dt \ , f \in L^p(T, \omega), 1 < p < \infty.$$

Using the boundedness of the Hardy-Littlewood Maximal function in $L^p(T, \omega), 1 < p < \infty, \omega \in A_p$, we get

$$\|\sigma_{\delta}^{\alpha} f (x)\|_{p,\omega} \leq C (\alpha) c_1(p) \|f\|_{p,\omega} < \infty. \tag{59}$$

Now, if $\alpha \in \mathbb{R}^+$, we define the α-th mean modulus of smoothness of a function $f \in L^p(T, \omega)$, where $1 < p < \infty$ and $\omega \in A_p$, as

$$\Omega_{\alpha}(f, h)_{p,\omega} := \sup_{|\delta| \leq h} \|\sigma_{\delta}^{\alpha} f (x)\|_{p,\omega} \quad .$$

Theorem $(\mathbf{5.2.1})[\mathbf{362}]$: Let $f \in W_p^{\alpha} (T, \omega), \alpha > 0, \omega \in A_p, 1 < p < \infty$. If, for some $T_n \in T_n$

$$\|f - T_n\|_{p,\omega} \leq c(p) E_n (f)_{p,\omega} , \qquad n = 0,1,2, \ldots ,$$

then

$$\left\| f^{(\alpha)} - T_n^{(\alpha)} \right\|_{p,\omega} \leq c(\alpha, p) E_n \left(f^{(\alpha)} \right)_{p,\omega}, \ n = 0,1,2, \ldots .$$

Proof: We put $S_{\nu} f (x) := S_{\nu}(x, f) := \sum_{k=-\nu}^{\nu} c_k e^{ikx}$ for the ν-th partial sum of the Fourier series (57) of $f \in W_p^{\alpha} (T, \omega)$ and $W_n(f) := W_n(x, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_{\nu}(x, f) , n = 0,1,2, \ldots$.

Hence $W_n(x, f(\alpha)) = W_n^{(\alpha)}(x, f)$.

Consequently

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{p,\omega} \leq \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{p,\omega}$$
$$+ \left\| T_n^{(\alpha)} (\cdot, W_n(f)) - T_n^{(\alpha)} (\cdot, f) \right\|_{p,\omega} + \left\| W_n^{(\alpha)} (\cdot, f) - T_n^{(\alpha)} (\cdot, W_n(f)) \right\|_{p,\omega} .$$

We denote by $T_n^* (x, f)$ the best approximating trigonometric polynomial of degree at most n to f in $L^p(T, \omega)$. In this case, using the boundedness of W_n in $L^p(T, \omega)$, we obtain

$$\begin{aligned} &\left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{p,\omega} \\ &\leq \left\| f^{(\alpha)}(\cdot) - T_n^*(\cdot, f^{(\alpha)}) \right\|_{p,\omega} + \left\| T_n^*(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)}) \right\|_{p,\omega} \\ &\leq c(p) E_n\left(f^{(\alpha)}\right)_{p,\omega} + \left\| W_n(\cdot, T_n^*(f(\alpha)) - f^{(\alpha)}) \right\|_{p,\omega} \\ &\leq c_1 (\alpha, p) E_n\left(f^{(\alpha)}\right)_{p,\omega} . \end{aligned}$$

From [365] we get

$$\left\|T_n^{(\alpha)}(\cdot,W_n(f))-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\omega}\le c_2(\alpha,p)n^{\alpha}\|T_n(\cdot,W_n(f))-T_n(\cdot,f)\|_{p,\omega}$$

and

$$\left\|W_n^{(\alpha)}(\cdot,f)-T_n^{(\alpha)}(\cdot,W_n(f))\right\|_{p,\omega}\le c_3(\alpha,p)(2n)^{\alpha}\|W_n(\cdot,f)-T_n(\cdot,W_n(f))\|_{p,\omega}$$
$$\le c_4(\alpha,p)(2n)^{\alpha}E_n\big(W_n(f)\big)_{p,\omega}.$$

Therefore

$$\|T_n(\cdot,f)-T_n(\cdot,W_n(f))\|_{p,\omega}\le\|T_n(\cdot,f)-W_n(\cdot,W_n(f))\|_{p,\omega}$$
$$+\|W_n(\cdot,f)-f(\cdot)\|_{p,\omega}+\|f(\cdot)-T_n(\cdot,f)\|_{p,\omega}$$
$$\le c(p)E_n\big(W_n(f)\big)_{p,\omega}+c_5(p)E_n(f)_{p,\omega}+c(p)E_n(f)_{p,\omega}.$$

Since $E_n\big(W_n(f)\big)_{p,\omega}\le c_6(p)E_n(f)_{p,\omega}$ we get

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\omega}$$
$$\le c_1(\alpha,p)E_n\big(f^{(\alpha)}\big)_{p,\omega}+n^{\alpha}\left\{c_6(\alpha,p)E_n\big(W_n(f)\big)_{p,\omega}+c_7(\alpha,p)E_n(f)_{p,\omega}\right\}$$
$$+c_8(\alpha,p)(2n)^{\alpha}E_n\big(W_n(f)\big)_{p,\omega}$$
$$\le c_1(\alpha,p)E_n\big(f^{(\alpha)}\big)_{p,\omega}+c_9(\alpha,p)n^{\alpha}E_n(f)_{p,\omega}.$$

By [371, $Th.$ 1.1] we have

$$E_n(f)_{p,\omega}\le\frac{c(\alpha,p)}{(n+1)^{\alpha}}E_n\big(f^{(\alpha)}\big)_{p,\omega}, \tag{60}$$

so we finally obtain

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{p,\omega}\le c(\alpha,p)E_n\big(f^{(\alpha)}\big)_{p,\omega}.$$

The next result was proved in [365] for $\omega\equiv1$.

Theorem (5.2.2)[362]: Let $0<\alpha\le1, r=0,1,2,3,\dots,\omega\in A_p, 1<p<\infty$, and $T_n\in T_n, n\ge1$. Then

$$\Omega_{r+\alpha}(T_n,h)_{p,\omega}\le c(p,r)h^{\alpha+r}\left\|T_n^{(\alpha+r)}\right\|_{p,\omega}, 0<h\le\frac{\pi}{n}. \tag{61}$$

Proof: Let

$$F(x):=\Delta_t^{\alpha+r}T_n\left(x-\frac{\alpha+r}{2}t\right)=\sum_{\nu\in\mathbb{Z}_n^*}(2i\sin\nu t/2)^{\alpha+r}c_\nu e^{i\nu x}$$

and

$$f\,(x) := \Delta_t^r\, T_n^{(\alpha)}\left(x - \frac{r}{2}\,t\right) = \sum_{\nu\in\mathbb{Z}_n^*} (2i\sin\nu t/2)^r \;\; (i\nu)^{(\alpha)}\, c_\nu e^{i\nu x}.$$

If we put

$$\varphi(z) := \left(2i\sin\frac{zt}{2}\right)^r \;(iz)^{(\alpha)}, g\,(z) := \left(\frac{2}{z}\sin tz/2\right)^\alpha, |z| \le n, g\,(0) := t^\alpha,$$

we find that

$$f\,(x) = \sum_{\nu\in\mathbb{Z}_n^*} \varphi(\nu)c_\nu e^{i\nu x}, F\,(x) = \sum_{\nu\in\mathbb{Z}_n^*} \varphi(\nu)g\,(\nu)c_\nu e^{i\nu x}.$$

The function g is positive, even and satifies $g'(z) \le 0, g"(z) \le 0$ for $z \in [0,n], 0 < t \le \pi/n.$ Hence

$$g\,(z) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi z/n}$$

uniformly on $[-n,n]$, with $d_0 > 0, (-1)^{k+1} d_k \ge 0, d_{-k} = d_k\ (k = 1,2,\dots)$ (see, [8]). We get that

$$F\,(x) = \sum_{k=-\infty}^{\infty} d_k f\left(x + \frac{k\pi}{n}\right)$$

and therefore

$$\Delta_t^{\alpha+r}\, T_n(\cdot) = \sum_{k=-\infty}^{\infty} d_k \Delta_t^r\, T_n^{(\alpha)}\left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2}t\right).$$

Consequently, we obtain

$$\frac{1}{\delta}\int_\delta^0 \|\Delta_t^{\alpha+r}\, T_n\,(\cdot)dt\|_{p,\omega} = \left\|\frac{1}{\delta}\int_\delta^0 \left|\sum_{k=-\infty}^{\infty} d_k \Delta_t^r T_n^{(\alpha)}\left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2}\,t\right)\right| dt\right\|_{p,\omega}$$

$$\le \sum_{k=-\infty}^{\infty} |d_k|\, \left\|\frac{1}{\delta}\int_\delta^0 \left|\Delta_t^r T_n^{(\alpha)}\left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2}\,t\right)\right| dt\right\|_{p,\omega}.$$

Since

$$\Delta_t^r\, T_n^{(\alpha)}\,(\cdot) = \int_0^t \cdots \int_0^t T_n^{(\alpha+r)}\,(\cdot + t_1 + \ldots + t_r)dt_1 \ldots dt_r,$$

we find

$$\Omega_{r+\alpha}\,(T_n,h)_{p,\omega} \le \sum_{k=-\infty}^{\infty} |d_k| \sup_{|\delta|\le h} \left\|\frac{1}{\delta}\int_0^\delta \left|\sum_{k=-\infty}^{\infty} d_k \Delta_t^r\, T_n^{(\alpha)}\left(\cdot + \frac{k\pi}{n} + \frac{\alpha}{2}\right)t\right| dt\right\|_{p,\omega}$$

$$= \sum_{k=-\infty}^{\infty} |d_k|\,,$$

$$\sup_{|\delta|\le h}\left\|\frac{1}{\delta}\int_0^{\delta}\left|\int_0^{t}\cdots\int_0^{t}T_n^{(\alpha+r)}\left(\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\,t+t_1+\ldots+t_r\right)dt_1\ldots dt_r\right|dt\right\|_{p,\omega}$$

$$\le h^r\sum_{k=-\infty}^{\infty}|d_k|\,,$$

$$\sup_{|\delta|\le h}\left\|\frac{1}{\delta}\int_0^{\delta}\left|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}T_n^{(\alpha+r)}\left(\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\,t+t_1+\ldots+t_r\right)dt_1\ldots dt_r\right|dt\right\|_{p,\omega}$$

$$\le h^r\sum_{k=-\infty}^{\infty}|d_k|\ ,\sup_{|\delta|\le h}$$

$$\left\|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\left\{\frac{1}{\delta}\int_0^{\delta}\left|T_n^{(\alpha+r)}\left(\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\,t+t_1+\ldots+t_r\right)\right|dt\right\}dt_1\ldots dt_r\right\|_{p,\omega}$$

$$\le c_{10}(r,p)h^r\sum_{k=-\infty}^{\infty}|d_k|\ \sup_{|\delta|\le h}\left\|\frac{1}{\delta}\int_0^{\delta}\left|T_n^{(\alpha+r)}\left(\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\,t\right)\right|dt\right\|_{p,\omega}$$

$$\le c_2(r,p)h^r\sup_{|\delta|\le h}\sum_{k=-\infty}^{\infty}|d_k|\left\|\frac{1}{\frac{\alpha}{2}\delta}\int_{\cdot+\frac{k\pi}{n}}^{\cdot+\frac{k\pi}{n}+\frac{\alpha}{2}\delta}\left|T_n^{(\alpha+r)}(u)\right|du\right\|_{p,\omega}.$$

By [368] we have

$$\sum_{k=-\infty}^{\infty}|d_k|<2g(0)=2t^{\alpha},\qquad 0<t\le\pi/n,$$

so

$$\sum_{k=-\infty}^{\infty}|d_k|<2h^{\alpha}$$

for $0<t\le\delta\le h\le\pi/n$. Hence

$$\Omega_{\alpha+r}(T_n,h)_{p,\omega}\le c_{11}(r,p)h^{\alpha+r}\left\|T_n^{(\alpha+r)}\right\|_{p,\omega}.$$

On the other hand, we get, by a similar argument, that the same inequality holds also if $0<-h\le\pi/n$. Thus the proof of the theorem is completed.

The next result is a generalization of Theorem 2 of [364] to the fractional case.

Theorem $(\mathbf{5.2.3})[\mathbf{362}]$: Let $\alpha>0,\omega\in A_p,1<p<\infty$. Then the following inequality holds for $f\in L^p(T,\omega)$

$$\Omega_\alpha\,(f,\pi/(n+1))_{p,\omega} \le \frac{c(\alpha,p)}{(n+1)^\alpha}\sum_{\nu=0}^{n}(\nu+1)^{\alpha-1}\,E_\nu(f)_{p,\omega}\,, n=0,1,2,\ldots.$$

Proof: Let $T_n \in T_n$ be the best approximating polynomial of $f \in L^p(T,\omega)$ and let $m \in \mathbb{Z}^+$. Then by assertion (ii) of by (59) we have

$$\Omega_\alpha\,(f,\pi/(n+1))_{p,\omega} \le \Omega_\alpha(f-T_{2^m},\pi/(n+1))_{p,\omega} + \Omega_\alpha(T_{2^m},\pi/(n+1))_{p,\omega}$$
$$\le c_{12}\,(\alpha,p)E_{2^m}\,(f)_{p,\omega} + \Omega_\alpha(T_{2^m},\pi/(n+1))_{p,\omega}\ .$$

Using Theorem (5.2.1), we get

$$\Omega_\alpha(T_{2^m},\pi/(n+1))_{p,\omega} \le c_{13}\,(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \left\|T_{2^m}^{(\alpha)}\right\|_{p,\omega}, n+1 \ge 2^m.$$

Since

$$T_{2^m}^{(\alpha)}\,(x) = T_1^{(\alpha)}\,(x) + \sum_{\nu=0}^{m-1}\left\{T_{2^{\nu+1}}^{(\alpha)}\,(x) - T_{2^\nu}^{(\alpha)}\,(x)\right\},$$

we obtain

$$\Omega_\alpha(T_{2^m},\pi/(n+1))_{p,\omega}$$
$$\le c_{13}\,(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha\left\{\left\|T_1^{(\alpha)}\right\|_{p,\omega} + \sum_{\nu=0}^{m-1}\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\omega}\right\}.$$

From Bernstein's inequality (see [365]) for fractional derivatives in $L^p\,(T,\omega)$, where $\omega \in A_p$ and $1 < p < \infty$, we have

$$\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\omega} \le c_{14}\,(\alpha,p)2^{\nu\alpha}\left\|T_{2^{\nu+1}} - T_{2^\nu}\right\|_{p,\omega} \le c_{15}\,(\alpha,p)2^{\nu\alpha+1}E_{2^\nu}\,(f)_{p,\omega}$$

and

$$\left\|T_1^{(\alpha)}\right\|_{p,\omega} = \left\|T_1^{(\alpha)} - T_0^{(\alpha)}\right\|_{p,\omega} \le c_{16}\,(\alpha,p)E_0(f)_{p,\omega}\,.$$

Hence

$$\Omega_\alpha(T_{2^m},\pi/(n+1))_{p,\omega}$$
$$\le c_{17}\,(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha\left\{E_0(f)_{p,\omega} + \sum_{\nu=0}^{m-1}2^{(\nu+1)\alpha}E_{2^\nu}\,(f)_{p,\omega}\right\}.$$

It is easily seen that

$$2^{(\nu+1)\alpha}E_{2^\nu}\,(f)_{p,\omega} \le c_{18}(\alpha)\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{\alpha-1}E_\mu(f)_{p,\omega}\,,\qquad \nu=1,2,3,\ldots.$$

Therefore

$$\Omega_\alpha\,(T_{2^m},\pi/(n+1))_{p,\omega}$$
$$\le c_{17}\,(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha$$

$$\left\{E_0(f)_{p,\omega} + 2^\alpha E_1(f)_{p,\omega} + c_{18}(\alpha) \sum_{\nu=1}^{m} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1} E_\mu(f)_{p,\omega}\right\}$$

$$\leq c_{19}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \left\{E_0(f)_{p,\omega} + \sum_{\mu=1}^{2^m} \mu^{\alpha-1} E_\mu(f)_{p,\omega}\right\}$$

$$\leq c_{20}(\alpha,p)\left(\frac{\pi}{n+1}\right)^\alpha \sum_{\nu=0}^{2^m-1} (\nu+1)^{\alpha-1} E_\nu(f)_{p,\omega} .$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_\alpha (T_{2^m}, \pi/(n+1))_{p,\omega} \leq \frac{c_{21}(\alpha,p)}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} \ E_\nu(f)_{p,\omega}$$

and

$$E_{2^m}(f)_{p,\omega} \leq E_{2^{m-1}}(f)_{p,\omega} \leq \frac{c_{22}(\alpha,p)}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} \ E_\nu(f)_{p,\omega}.$$

This finishes the proof.
The next result was proved for $\alpha = 1$ in [364].

Theorem (5.2.4)[362]: If f $\in W_p^{\alpha+r}(T,\omega), 0 < \alpha \leq 1, r = 0,1,2,3,\ldots, \omega \in A_p, 1 < p < \infty$, then

$$\Omega_{r+\alpha}(f,h)_{p,\omega} \leq c(\alpha,r,p) h^{\alpha+r} \left\|f^{(\alpha+r)}\right\|_{p,\omega}, \qquad 0 < h \leq \pi.$$

Proof: Let $T_n \in T_n$ be the trigonometric polynomial of best approximationof f in $L^p(T,\omega)$ metric. By Theorem (5.2.1), and (59) we get

$$\Omega_{\alpha+r}(f,h)_{p,\omega} \leq \Omega_{\alpha+r}(T_n,h)_{p,\omega} + \Omega_{\alpha+r}(f-T_n,h)_{p,\omega}$$

$$\leq c(p,r) h^{\alpha+r} \left\|T_n^{(\alpha+r)}\right\|_{p,\omega} + c_{22}(p,\alpha,r) E_n(f)_{p,\omega}, 0 < h \leq \pi/n.$$

Then, using inequality (10) of [364], (60), and Theorem 2 of [365], we have

$$E_n(f)_{p,\omega} \leq \frac{c(p,\alpha,r)}{(n+1)^\alpha} E_n \left\|f^{(\alpha)}\right\|_{p,\omega} \leq \frac{c_{18}(p,\alpha,r)}{(n+1)^\alpha} \ \Omega_r\left(f^{(\alpha)}, \frac{2\pi}{n+1}\right)_{p,\omega}$$

$$\leq \frac{c_{23}(p,\alpha,r)}{(n+1)^\alpha} \left(\frac{2\pi}{n+1}\right)^r \left\|f^{(\alpha+r)}\right\|_{p,\omega} .$$

By Theorem (5.2.1) we find

$$\left\|T_n^{(\alpha+r)}\right\|_{p,\omega} \leq \left\|T_n^{(\alpha+r)} - f^{(\alpha+r)}\right\|_{p,\omega} + \left\|f^{(\alpha+r)}\right\|_{p,\omega}$$

$$\leq c(p,\alpha,r) E_n\left(f^{(\alpha+r)}\right)_{p,\omega} + \left\|f^{(\alpha+r)}\right\|_{p,\omega} \leq c_{24}(p,\alpha,r)\left\|f^{(\alpha+r)}\right\|_{p,\omega} .$$

Choosing h with $\pi/(n+1) < h \leq \pi/n, n = 1,2,3,\ldots$, we obtain

$$\Omega_{\alpha+r}(f,h)_{p,\omega} \leq c(p,\alpha,r)h^{\alpha+r}\left\|f^{(\alpha+r)}\right\|_{p,\omega}$$

and we are done.

Theorem (5.2.5)[362]: Let $f \in L^p(T,\omega), 1 < p < \infty, \omega \in A_p$. If $\beta \in (0,\infty)$ and

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1}E_\nu(f)_{p,\omega} < \infty,$$

then

$$E_n\left(f^{(\beta)}\right)_{p,\omega} \leq c(p,\beta)\left((n+1)^\beta E_n(f)_{p,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\beta-1}E_\nu(f)_{p,\omega}\right).$$

Proof: Since

$$\left\|f^{(\beta)} - S_n f^{(\beta)}\right\|_{p,\omega}$$

$$\leq \left\|S_{2^{m+2}}f^{(\beta)} - S_n f^{(\beta)}\right\|_{p,\omega} + \sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}f^{(\beta)} - S_{2^k}f^{(\beta)}\right\|_{p,\omega},$$

we have for $2^m < n < 2^{m+1}$

$$\left\|S_{2^{m+2}}f^{(\beta)} - S_n f^{(\beta)}\right\|_{p,\omega}$$

$$\leq c_{25}(p,\beta)2^{(m+2)\beta}E_n(f)_{p,\omega} \leq \frac{c_{26}(p,\beta)}{(n+1)^\beta}E_n(f)_{p,\omega}.$$

On the other hand, we find

$$\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}f^{(\beta)} - S_{2^k}f^{(\beta)}\right\|_{p,\omega} \leq c_{27}(p,\beta)\sum_{k=m+2}^{\infty} 2^{(k+1)\beta}E_{2^k}(f)_{p,\omega}$$

$$= c_{29}(p,\beta)\sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\beta-1}E_\nu(f)_{p,\omega} \leq c_{29}(p,\beta)\sum_{\nu=n+1}^{\infty} \nu^{\beta-1}E_\nu(f)_{p,\omega}$$

which finishes the proof.

Corollary (5.2.6)[362]: Let $f \in W_p^\alpha(T,\omega), (1 < p < \infty), \omega \in A_p, \beta \in (0,\infty)$ and

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1}E_\nu(f)_{p,\omega} < \infty$$

for some $\alpha > 0$. If $n = 0,1,2,\ldots$, then

$$\Omega_\beta\left(f^{(\alpha)},\frac{\pi}{n+1}\right)_{p,\omega}$$

$$\leq \frac{c_{43}(\alpha,p,\beta)}{(n+1)^\beta}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+\beta-1}E_\nu(f)_{p,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1}E_\nu(f)_{p,\omega}.$$

Let Γ be a rectifiable Jordan curve and let $G := int\Gamma, G^- := ext\Gamma, \mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}, \mathbb{T} := \partial\mathbb{D}, \mathbb{D}^- := ext\mathbb{T}$. Without loss of generality we may assume $0 \in G$. We denote by $L^p(\Gamma), 1 \le p < \infty$, the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arclength. By $E_p(G)$ and $E_p(G^-), 0 < p < \infty$, we denote the Smirnov classes of analytic functions in G and G^-, respectively. Let $w = \varphi(z)$ and $w = \varphi_1(z)$ be the conformal mappings of G^- and G onto $\mathbb{D}^-$normalized by the conditions

$$\varphi(\infty) = \infty, \lim_{z\to\infty} \varphi(z)/z > 0 \ and \ \varphi_1(0) = \infty, \qquad \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively. Let $f \in L^1(\Gamma)$. Then

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \qquad z \in G,$$

is analytic on G.

Let ω be a weight function on Γ and let $L^p(\Gamma,\omega)$ be the weighted Lebesgue space on $\Gamma, i.e.$, the space of measurable functions on Γ for which

$$\|f\|_{L^p(\Gamma,\omega)} := \left(\int_\Gamma |f(z)|^p\,\omega^p(z)|dz|\right)^{1/p} < \infty.$$

The weighted Smirnov spaces $E_p(G,\omega)$ and $E_p(G^-,\omega)$ are defined as

$$E_p(G,\omega) := \{f \in E_1(G) : f \in L^p(\Gamma,\omega)\},$$
$$E_p(G^-,\omega) := \{f \in E_1(G^-) : f \in L^p(\Gamma,\omega)\}.$$

We also define the following subspace of $E_p(G^-,\omega)$

$$\tilde{E}_p(G^-,\omega) := \left\{f \in E_p(G^-,\omega) : f(\infty) = 0\right\}.$$

Let $1 < p < \infty, z \in \Gamma, \varepsilon > 0$, and $\Gamma(z,\varepsilon) := \{t \in \Gamma : |t - z| < \varepsilon\}, \frac{1}{p} + \frac{1}{q} = 1$. A weight function ω belongs to the Muckenhoupt class $A_p(\Gamma)$ if the condition

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_{\Gamma(z,\varepsilon)}\omega^p(\tau)|d\tau|\right)^{\frac{1}{p}}\left(\frac{1}{\varepsilon}\int_{\Gamma(z,\varepsilon)}\omega^{-q}(\tau)|d\tau|\right)^{\frac{1}{q}} < \infty,$$

holds.

With every weight function ω on Γ, we associate the other weights on $\mathbb{T}$ by setting $\omega_0 := \omega \circ \psi, \omega_1 := \omega \circ \psi_1$. For an arbitrary $f \in L^p(\Gamma,\omega)$ we set

$$f_0(w) := f(\psi(w)), f_1(w) := f(\psi_1(w)), w \in \mathbb{T}.$$

If Γ is a Dini-smooth curve, then the condition $f \in L^p(\Gamma,\omega)$ implies that $f_0 \in L^p(\mathbb{T},\omega_0)$ and $f_1 \in L^p(\mathbb{T},\omega_1)$. Using the nontangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$ we define for a function $f \in L^p(\Gamma,\omega)$ and $\alpha \in \mathbb{R}^+$

$$\Omega_k(f,\delta)_{\Gamma,p,\omega} := \Omega_k(f_0^+,\delta)_{p,\omega_0}, \delta > 0,$$
$$\widetilde{\Omega}_k(f,\delta)_{\Gamma,p,\omega} := \Omega_k(f_1^+,\delta)_{p,\omega_1}, \delta > 0. \qquad (62)$$

We set

$$E_n(f,G)_{p,\omega} := \inf_{P\in P_n}\|f - P\|_{L^p(\Gamma,\omega)}, \tilde{E}_n(g,G^-)_{p,\omega} := \inf_{R\in R_n}\|g - R\|_{L^p(\Gamma,\omega)},$$

where $f \in E_p(G,\omega), g \in E_p(G^-,\omega), P_n$ is the set of algebraic polynomials of degree not greater than n, and R_n is the set of rational functions of the form $\sum_{k=0}^{n} \frac{a_k}{z^k}$.
Some converse approximation theorems in the weighted Lebesgue spaces $L^p(T,\omega), 1 < p < \infty, \omega \in A_p$ were proved in [371] and [372]. In the weighted Smirnov spaces $E_p(G,\omega), \omega \in A_p(\Gamma), 1 < p < \infty$, the converse approximation theorems were proved in [373] for Butzer-Wehrens modulus of smoothness.
In the following we investigate the approximation problems in the weighted Smirnov spaces in terms of the α-th mean modulus of smoothness. The following converse theorems can be proved by the method given in [372] and [373].
Theorem (5.2.7)[362]: Let G be a finite, simply connected domain with a Dini-smooth boundary Γ. If $\alpha > 0$ and $f \in E_p(G,\omega), \omega \in A_p(\Gamma), 1 < p < \infty$, then

$$\Omega_\alpha(f,1/n)_{\Gamma,p,\omega} \leq \frac{c(\Gamma,p,\alpha)}{n^\alpha} \sum_{k=0}^{n} (k+1)^{\alpha-1} E_k(f,G)_{p,\omega}, n = 1,2,\dots.$$

If $\alpha = 2r, r = 1,2,\dots$, this result was proved in [373] for a different but equivalent modulus of smoothness. The converse theorem for an unbounded domain G^- is also true.
Theorem (5.2.8)[362]: Let Γ be a Dini-smooth curve. If $\alpha > 0, f \in \tilde{E}_p(G^-,\omega)$, and $\omega \in A_p(\Gamma), 1 < p < \infty$, then

$$\widetilde{\Omega}_\alpha(f,1/n)_{\Gamma,p,\omega} \leq \frac{c(\Gamma,p,\alpha)}{n^\alpha} \sum_{k=0}^{n} (k+1)^{\alpha-1} \tilde{E}_k(f,G^-)_{p,\omega}, n = 1,2,3,\dots.$$

Section (5.3): Lebesgue Spaces with Variable Exponent

Let $T := [0,2\pi]$ and let $p(\cdot): T \to [0,\infty)$ be a Lebesgue measurable 2π periodic function such that

$$1 \leq p_- := \operatorname*{ess\,inf}_{x\in T} p(x) \leq \operatorname*{ess\,sup}_{x\in T} p(x) := p_+ < \infty.$$

In addition to this requirement if

$$|p(x)-p(y)| \ln \frac{2\pi}{|x-y|} \leq d, \forall x,y \in [0,2\pi]$$

with a positive constant d, then we say that $p(\cdot) \in P(T)$. We also define $P_0(T) := \{p(\cdot) \in P(T): p_- > 1\}$.

The variable exponent Lebesgue space $L^{p(\cdot)}(T)$ is defined as the set of all the Lebesgue measurable 2π periodic functions f such that

$$\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0: \rho_{p(\cdot)}(f/\lambda) \leq 1\}$$

it becomes a Banach space.

These spaces were introduced by Orlicz in [379]. Interest in the variable exponent Lebesgue spaces has been increased since $1990s$, because of their usage in the different applications problems in mechanic, especially in fluid dynamic for the modelling of electrorheological fluids and also in the study of image processing and some physical problems (see [374]). Nowadays, there are sufficient investigations relating to the fundamental problems of these spaces, in the view of the potential theory, maximal and singular integral operator theory and others. Some of the corresponding results can be found in the monographs mentioned above. However the approximation problems in these spaces were not investigated widely. Meanwhile, some of the fundamental problems of approximation theory in the variable exponent Lebesgue spaces of periodic and non periodic functions which are defined on the intervals of real line, were studied and solved by Sharapudinov see [375]).

One of the main problems observed in the investigations on the approximation theory is the correct definition of the modulus of smoothness that will provide us with a better tool to deal with the rate of the best approximation, inverse theorems and also some other similar problems. The detailed information regarding to the different moduli of the smoothness considered monograph [374] and also [375] and [376]. The classical modulus of the smoothness, which constructed by using the shift operator $f\ (\cdot + h)$, has proved to be very useful tool for solving the above mentioned problems in the classical Lebesgue spaces. However it is a fact that $L^{p(\cdot)}(T)$ is noninvariant with respect to the usual shift operator $f\ (\cdot + h)$, in general [382]. On the other hand, the Steklov mean value operator

$$\sigma_h(f) := \frac{1}{h}\int_0^h f\ (x\ +\ t)dt\,, h\ >\ 0$$

is bounded in $L^{p(\cdot)}\ (T)$, which follows from the boundedness of the maximal function in $L^{p(\cdot)}(T), p(\cdot)\ \in P_0\,(T)$, showed in [373]. By using this result, the first order modulus of smoothness

$$\Omega_{p(\cdot)}\,(f,\delta) := \sup_{0<h\le\delta}\left\|\frac{1}{h}\int_0^h |f\ (\cdot) - f\ (\cdot + t)|dt\right\|_{p(\cdot)}$$

was constructed in [375] and in the term of this modulus were obtained the direct theorem of approximation theory in $L^{p(\cdot)}(T), p(\cdot)\ \in P_0\,(T)$, and also some results on the approximation by the Nörlund means of Fourier series in $L^{p(\cdot)}(T)$. Similar results under the condition of $p(\cdot)\ \in P_0\ (T)$ using some other modulus of smoothness were proved . In the more general case, i.e. in the case of $p(\cdot)\ \in P(T)\ \supset P_0\ (T)$ introducing the modulus

$$\Omega(f,\delta)_{p(\cdot)} := \sup_{0<h\le\delta}\left\|\frac{1}{h}\int_0^h |f\ (\cdot) - f\ (\cdot + t)|dt\right\|_{p(\cdot)}$$

which is more sensitive than $\Omega_{p(\cdot)}\,(f,\delta)$, the direct and inverse theorems were proved by Sharapudinov in [382]. In term of $\Omega(f,\delta)_{p(\cdot)}, p(\cdot)\ \in P(T)$, one general inverse theorem, which generalizes the inverse theorem obtained in [375], was proved in [376]. The

basicity problems of some well known trigonometric systems and the uniform boundedness problems of some families of convolution operators in the weighted variable exponent Lebesgue spaces were studied in [377] and [378], respectively.
We define the r-th ($r = 1, 2, \ldots$) modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot)}$ in $L^{p(\cdot)}(T), p(\cdot) \in P(T)$, and investigate the approximation problems in theterm of this modulus.
Definition $(\mathbf{5.3.1})[\mathbf{372}]$: Let $f \in L^{p(\cdot)}(T)$ with $p(\cdot) \in P(T)$ and let

$$\Delta_t^r f(x) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} f(x + st), \qquad r = 1, 2, \ldots.$$

We define the r-th modulus of smoothness as

$$\Omega_r(f,\delta)_{p(\cdot)} := \sup_{0<h\le\delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f dt \right\|_{p(\cdot)}, \delta > 0.$$

It is easily to show that in the case of $p(\cdot) = \text{const}$ this modulus is equivalent to the classical modulus of smoothness defined as $\sup_{|t|\le\delta} \|\Delta_t^r f(x)\|_p$. For $f \in L^{p(\cdot)}(T)$ we define the best approximation number

$$E_n(f)_{p(\cdot)} := \inf\{\|f - T_n\|_{p(\cdot)} : T_n \in \Pi_n\}$$

in the class Π_n of the trigonometric polynomials of degree not exceeding n. Throughout by $c(\cdot), c_1(\cdot), c_2(\cdot), \ldots, c(\cdot,\cdot), c_1(\cdot,\cdot), c_2(\cdot,\cdot), \ldots$, we denote the constants depending on the parameters which are given in the corresponding brackets.
The main direct and inverse results obtained are as following.
Theorem $(\mathbf{5.3.2})[\mathbf{372}]$: Let $p(\cdot) \in P(T), r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $f \in L^{p(\cdot)}(T)$ and $n \in \mathbb{N}$ the inequality

$$E_n(f)_{p(\cdot)} \le c(p,r)\Omega_r(f, 1/n)_{p(\cdot)}$$

holds.
Theorem (5.3.2) in the case of $r = 1$ was proved in [378].
Theorem $(\mathbf{5.3.3})[\mathbf{372}]$: Let $p(\cdot) \in P(T), r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $f \in L^{p(\cdot)}(T)$ and $n \in \mathbb{N}$ the inequality

$$\Omega_r(f, 1/n)_{p(\cdot)} \le \frac{c(p,r)}{n^r} \sum_{k=0}^{n} (k + 1)^{r-1} E_k(f)_{p(\cdot)}$$

holds.
Theorem (5.3.3) in the case of $r = 1$ was proved in [376]. Denoting by

$$W_k^{p(\cdot)}(T) := \{f : f^{(k-1)} \textit{ is absolutely continuous and } f^{(k)} \in L^{p(\cdot)}(T)\},$$

$k = 1, 2, \ldots$, the variable exponent Sobolev space and using Consequence 2.1 in [379] and also the boundedness of $\Omega(f,\delta)_{p(\cdot)}$ we have the inequality

$$E_n(f)_{p(\cdot)} \le \frac{c(p)}{n^k} \|f^{(k)}\|_{p(\cdot)}, \qquad (63)$$

which implies by using the standard way, the estimation

$$E_n(f)_{p(\cdot)} \le \frac{c(p)}{n^k} E_n(f^{(k)})_{p(\cdot)}.$$

Combining this estimation with Theorem (5.3.2) we have
Corollary (5.3.4)[372]: Let $p(\cdot) \in P(T), k \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $f \in W_k^{p(\cdot)}(T)$ and $n \in \mathbb{N}$ the inequality
$$E_n(f)_{p(\cdot)} \leq \frac{c(p,r)}{n^k}\, \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)}$$
holds.
In the case of $r = 1$
Corollary (5.3.4) was obtained in [373]. Theorem (5.3.3) also implies

Corollary (5.3.5)[372]: If $E_n(f)_{p(\cdot)} = O(n^{-\alpha}), \alpha > 0$, then under the conditions of Theorem (5.3.3)
$$\Omega_r(f,\delta)_{p(\cdot)} = \begin{cases} O(\delta^\alpha) & ,r > \alpha \\ O(\delta^\alpha \log(1/\delta)) & ,r = \alpha \\ O(\delta^r) & ,r < \alpha\,. \end{cases}$$
Hence, if we define a generalized Lipschitz class $Lip_{\alpha,r}^{p(\cdot)}(T)$ for $\alpha > 0$ and $r := [\alpha] + 1$ ($[\alpha]$ is the integer part of α) as
$$Lip_{\alpha,r}^{p(\cdot)}(T) := \left\{f \in L^{p(\cdot)}(T) : \Omega_r(f,\delta)_{p(\cdot)} = O(\delta^\alpha), \delta > 0\right\},$$
then we have
Corollary (5.3.6)[372]: If $E_n(f)_{p(\cdot)} = O(n^{-\alpha}), \alpha > 0$, then under the conditions of Theorem (5.3.3), $f \in Lip_{\alpha,r}^{p(\cdot)}(T)$.
On the other hand, from Theorem (5.3.2) we also get
Corollary (5.3.7)[372]: If $f \in Lip_{\alpha,r}^{p(\cdot)}(T)$ with $p(\cdot) \in P(T)$ and for some $\alpha > 0$, then $E_n(f)_{p(\cdot)} = O(n^{-\alpha})$.
Now Corollaries (5.3.6) and (5.3.7) imply
Theorem (5.3.8)[372]: Let $f \in L^{p(\cdot)}(T), p(\cdot) \in P(T)$, and let $\alpha > 0$. The following statements are equivalent:
$i) f \in Lip_{\alpha,r}^{p(\cdot)}(T)$,
$ii)\ E_n(f)_{p(\cdot)} = O(n^{-\alpha}), n \in \mathbb{N}$.
Note that when $p(\cdot)$ is a constant the classical analogues of Theorems (5.3.2) – (5.3.8),proved in the term of the classical r-th modulus of smoothness constructed via usual shift $f(\cdot + h)$ can be found in the monographs we sometimes use the techniques developed in [378] and [379]. First of all, we obtain some important properties of the modulus $\Omega_r(f,\delta)_{p(\cdot)}$. Subadditivity property of $\Omega_r(f,\delta)_{p(\cdot)}$is immediately follows from Definition (5.3.1); that is for $f, g \in L^{p(\cdot)}(T)$ we have
$$\Omega_r(f + g,\delta)_{p(\cdot)} \leq \Omega_r(f,\delta)_{p(\cdot)} + \Omega_r(g,\delta)_{p(\cdot)}\,. \qquad (64)$$
Let $\lambda, \gamma > 0$, and let $|\tau| \leq \pi/\lambda^\gamma$. We consider the Steklov operator $S_{\lambda,\tau}f$, defined as
$$S_{\lambda,\tau}f := \left(S_{\lambda,\tau}f\right)(x) := \lambda \int_{x+\tau-1/2\lambda}^{x+\tau+1/2\lambda} f(t)\,dt.$$

The following lemma was proved in [384].

Lemma $(\mathbf{5.3.9})[\mathbf{372}]$: Let $p(\cdot) \in P(T)$ and let $0 < \gamma \leq 1$. Then the family of the Steklov operators $S_{\lambda,\tau}(f)$ is uniformly bounded in $L^{p(\cdot)}(T)$ for $1 \leq \lambda < \infty, |\tau| \leq \pi/\lambda^{\gamma}$, i.e., there exists a positive constant $c(p)$ such that

$$\left\|S_{\lambda,\tau} f\right\|_{p(\cdot)} \leq c(p)\|f\|_{p(\cdot)}, 1 \leq \lambda < \infty, |\tau| \leq \pi/\lambda^{\gamma}.$$

The following lemma shows that the modulus $\Omega_r(f,\delta)_{p(\cdot)}$ is well defined.

Lemma $(\mathbf{5.3.10})[\mathbf{372}]$: Let $p(\cdot) \in P(T)$ and $r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$, such that

$$\Omega_r(f,\delta)_{p(\cdot)} \leq c(p,r)\|f\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}(T)$ and $\delta > 0$.

Proof: Since for any positive integer s with $0 < sh \leq 1$

$$\frac{1}{h}\int_0^h f(x+st)dt = \frac{1}{sh}\int_x^{x+sh} f(u)du = \frac{1}{sh}\sum_{x+sh/2-sh/2}^{x+sh/2+sh/2} f(u)du$$
$$= \left(S_{\frac{1}{sh},\frac{sh}{2}} f\right)(x),$$

denoting $\lambda := 1/(sh), \tau := sh/2$ and applying Lemma (5.3.9) we have that

$$\left\|\frac{1}{h}\int_0^h f(.+st)dt\right\|_{p(\cdot)} = \left\|S_{\frac{1}{sh},\frac{sh}{2}} f\right\|_{p(\cdot)} \leq c(p)\|f\|_{p(\cdot)}.$$

Hence

$$\left\|\frac{1}{h}\int_0^h \Delta_t^r f dt\right\|_{p(\cdot)} = \left\|\frac{1}{h}\int_0^h \sum_{s=0}^{r} (-1)^{r+s}\binom{r}{s} f(\cdot+st)dt\right\|_{p(\cdot)}$$
$$\leq c_1(r)\sum_{s=0}^{r}\left\|\frac{1}{h}\int_0^h f(\cdot+st)dt\right\|_{p(\cdot)}$$
$$\leq c(p,r)\|f\|_{p(\cdot)},$$

which implies that

$$\Omega_r(f,\delta)_{p(\cdot)} \leq c(p,r)\|f\|_{p(\cdot)}.$$

Lemma $(\mathbf{5.3.11})[\mathbf{372}]$: If $f \in L^{p(\cdot)}(T), p(\cdot) \in P(T)$, then $\lim_{\delta\to 0}\Omega_r(f,\delta)_{p(\cdot)} = 0$for every positive integer r.

Proof: At first we suppose that $f \in L^{p(\cdot)}(T)$ is a continuous function. Then for any $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that $|f(x+mt) - f(x+(m+1)t)| \leq \varepsilon/\left\{[2\pi]^{1/p} - (2^{r-1})\right\}$for $0 < t \leq h \leq \delta$ and $m = 0,1,2,\ldots,r$. Hence

$$\int_{-\pi}^{\pi}\left|\frac{1}{h}\int_0^h \Delta_t^r f\,(x)dt/\varepsilon\right|^{p(x)} dx = \int_{-\pi}^{\pi}\left|\frac{1}{h}\int_0^h \frac{\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s} f\,(x + st)}{\varepsilon}\,dt\right|^{p(x)} dx$$

$$\leq \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s} f\,(x + st)}{\varepsilon}\,dt\right)^{p(x)} dx$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{\sum_{m=0}^{r-1}(-1)^{r+m}\binom{r-1}{m}[f\,(x + mt) - f\,(x + (m + 1)t)]}{\varepsilon}\,dt\right)^{p(x)} dx$$

$$\leq \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{\sum_{m=0}^{r-1}\binom{r-1}{m}\,|f\,(x + mt) - f\,(x + (m + 1)t)|}{\varepsilon}\,dt\right)^{p(x)} dx$$

$$\leq \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{\varepsilon\sum_{m=0}^{r-1}\binom{r-1}{m}}{[2\pi]^{1/p_-}\,(2^{r-1})\,\varepsilon}\,dt\right)^{p(x)} dx$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{2^{r-1}}{[2\pi]^{1/p_-}\,(2^{r-1})}\,dt\right)^{p(x)} dx$$

$$\leq \int_{-\pi}^{\pi}\left(\frac{1}{h}\int_0^h \frac{dt}{[2\pi]^{1/p_-}}\right)^{p_-} dx = 1.$$

Therefore,

$$\left\|\frac{1}{h}\int_0^h \Delta_t^r(f)\right\|_{p(\cdot)} \leq \varepsilon$$

which implies that $\Omega_r(f,\delta)_{p(\cdot)} \leq \varepsilon$.

If $f \in L^{p(\cdot)}(T)$ is not continuous on T, then by density of the set of continuous functions $[378, pp. 145-146]$ in $L^{p(\cdot)}(T)$, for any $\varepsilon > 0$ there exist a 2π periodic continuous function g with $\|f - g\|_{p(\cdot)} \leq \varepsilon$ and a number $\delta(\varepsilon) > 0$ such that $\Omega_r(g,\delta)_{p(\cdot)} \leq \varepsilon$ for every $\delta < \delta(\varepsilon)$. Hence by (64) and Lemma (5.3.10)

$$\begin{aligned}\Omega_r(f,\delta)_{p(\cdot)} &\leq \Omega_r(f-g,\delta)_{p(\cdot)}) + \Omega_r(g,\delta)_{p(\cdot)}\\ &\leq c(p,r)\|f - g\|_{p(\cdot)} + \varepsilon\\ &\leq [c(p,r) + 1]\varepsilon,\end{aligned}$$

which implies the required relation $\lim_{\delta\to 0}\Omega_r(f,\delta)_{p(\cdot)} = 0$.

Lemma $(\mathbf{5.3.12})[\mathbf{372}]$: Let $p(\cdot) \in P(T)$. Then there exists a positive constant $c(p,r)$, such that for any $r \in \mathbb{N}, \delta > 0$ and for any function $f \in W_r^{p(\cdot)}(T)$ the inequality

$$\Omega_r(f,\delta)_{p(\cdot)} \leq c(p,r)\delta^r \left\|f^{(r)}\right\|_{p(\cdot)}$$

holds .

Proof: Since

$$\Delta_t^r f(x) = \int_0^t \int_0^t \dots \int_0^t f^{(r)}(x + t_1 + \dots + t_r) dt_1 \dots dt_r,$$

applying r times the generalized Minkowski inequality we have

$$\left\| \frac{1}{h} \int_0^h \Delta_t^r(f) dt \right\|_{p(\cdot)} \le c_1(p) \frac{1}{h} \int_0^h \|\Delta_t^r f\|_{p(\cdot)} \, dt$$

$$\le c_1(p) h^r \frac{1}{h^{r+1}} \int_0^h \left\| \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r) dt_1 \dots dt_r \right\|_{p(\cdot)} dt$$

$$= c_1(p) h^r \frac{1}{h} \int_0^h \left\| \frac{1}{h} \int_0^t \left| \frac{1}{h^{r-1}} \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r) dt_1 \dots dt_{r-1} \right| dt_r \right\|_{p(\cdot)} dt$$

$$\le c_1(p) h^r \frac{1}{h} \int_0^h \left\| \frac{1}{h} \int_0^h \left| \frac{1}{h^{r-1}} \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_r) dt_1 \dots dt_{r-1} \right| dt_r \right\|_{p(\cdot)} dt$$

$$\le c_2(p) h^r \frac{1}{h} \int_0^h \left\| \frac{1}{h^{r-1}} \int_0^t \dots \int_0^t f^{(r)}(. + t_1 + \dots + t_{r-1}) dt_1 \dots dt_{r-1} \right\|_{p(\cdot)} dt$$

$$\le \dots \le c_3(p,r) h^r \frac{1}{h} \int_0^h \left\| \left\{ \frac{1}{h} \int_0^h \left| f^{(r)}(\cdot + t_1) \right| dt_1 \right\} \right\|_{p(\cdot)} dt$$

$$\le c_4(p,r) h^r \left\| f^{(r)} \right\|_{p(\cdot)} \frac{1}{h} \int_0^h dt = c_4(p,r) h^r \left\| f^{(r)} \right\|_{p(\cdot)},$$

and taking here the supremum we obtain the inequality

$$\Omega_r(f,\delta)_{p(\cdot)} \le c(p,r) \delta^r \left\| f^{(r)} \right\|_{p(\cdot)}.$$

For $f \in L^{p(\cdot)}(T)$ and $\delta > 0$ we define the Steklov mean value function as

$$f_{r,\delta}(x) := \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\{ \frac{1}{h^r} \sum_{s=0}^{r-1} (-1)^{r+s+1} \binom{r}{s} \int_0^h \dots \dots \int_0^h f\left(x + \frac{r-s}{r} [t_1 + \dots + t_r] \right) dt_1 \dots dt_r \right\} dh, \quad (65)$$

which plays a crucial role in this work.

Lemma $(\mathbf{5.3.13})$[372]: If $f \in L^{p(\cdot)}(T), p(\cdot) \in P(T)$, then $f_{r,\delta} \in W_r^{p(\cdot)}(T)$ for $\delta > 0$ and $r \in \mathbb{N}$.

Proof: Differentiating $r-1$ times the terms under the sum in (65) and setting $t := \frac{r-s}{r} t_r$ we see that

$$\left\{ \int_0^h \dots \int_0^h f\left(x + \frac{r-s}{r} [t_1 + \dots + t_r] \right) dt_1 \dots dt_r \right\}^{(r-1)}$$

$$= \left\{\int_0^h \left(\frac{r}{r-s}\right)^{r-1} \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^{r+m} f\left(x + \frac{r-s}{r} t_r + m\,\frac{r-s}{r}\,h\right)\right\} dt$$

$$= \int_0^h \left(\frac{r}{r-s}\right)^{r-1} \Delta^{r-1}_{\frac{r-s}{r}h} f\left(x + \frac{r-s}{r} t_r\right) dt_r$$

$$= \int_0^{\frac{r-s}{r}h} \left(\frac{r}{r-s}\right)^{r} \Delta^{r-1}_{\frac{r-s}{r}h} f\,(x + t)dt\ ,$$

and then via (65)

$$f^{(r-1)}_{r,\delta}(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\left\{\sum_{s=0}^{r-1} (-1)^{r+s+1}\binom{r}{s}\int_0^{\frac{r-s}{r}h} \left(\frac{r}{r-s}\right)^{r} \Delta^{r-1}_{\frac{r-s}{r}h} f\,(x + t)dt\right\} dh$$

$$= \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\left\{\sum_{s=0}^{r-1} \int_x^{x+\frac{r-s}{r}h} (-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r} \Delta^{r-1}_{\frac{r-s}{r}h} f\,(t)dt\right\} dh. \quad (66)$$

Since the Steklov mean value function $f^{(r-1)}_{r,\delta}$ is an indefinite Lebesgue integral, its absolute continuity on $[0,2\pi]$ can be showed by standard way. It remains to prove the imbedding $f^{(r)}_{r,\delta} \in L^{p(\cdot)}(T)$. Differentiating the relation (66)we obtain

$$f^{(r)}_{r,\delta}(x) = \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{h^r}\left(\sum_{s=0}^{r-1} (-1)^{r+s+1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r} \Delta^{r}_{\frac{r-s}{r}h} f\,(x)\right) dh$$

and denoting $t := \frac{r-s}{r}\,h$ we have

$$\left|f^{(r)}_{r,\delta}, \delta\,(x)\right| \le \frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left|\frac{1}{\delta}\int_{\delta/2}^{\delta} \Delta^{r}_{\frac{r-s}{r}h} f\,(x)dh\right|$$

$$= \frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left|\frac{1}{\frac{r-s}{r}\,\delta}\int_{\frac{r-s}{r}(\delta/2)}^{\frac{r-s}{r}\delta} \Delta^{r}_{t} f\,(x)dt\right|$$

$$\le \frac{2^{r+1}}{\delta^r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left\{\left|\frac{1}{\frac{r-s}{r}\,\delta}\int_0^{\frac{r-s}{r}\delta} \Delta^{r}_{t} f\,(x)dt + \frac{1}{\frac{r-s}{r}\,\delta}\int_0^{\frac{r-s}{r}(\delta/2)} \Delta^{r}_{t} f\,(x)dt\right|\right\},$$

which by Lemma (5.3.10) implies the inequality

$$\left\|f^{(r)}_{r,\delta}\right\|_{p(\cdot)} \le\ 2c(r)\delta^{-r}\Omega_r\,(f,\delta)_{p(\cdot)}\ \le\ c_5(p,r)\|f\|_{p(\cdot)}\,. \quad (67)$$

Since $f\ \in\ L^{p(\cdot)}(T)$ the relation (67) means that $f^{(r)}_{r,\delta} \in L^{p(\cdot)}(T)$.

Let $f\ \in L^{p(\cdot)}(T), p(\cdot)\ \in P(T)$. For $\delta > 0$ and $r\ = 1\,,2,\ldots,$ after some necessary simplifications we have

$$|f_{r,\delta}(x)-f(x)| = \frac{2}{\delta}\left|\int_{\delta/2}^{\delta}\left\{\frac{1}{h^r}\int_0^h \dots \int_0^h \varDelta^r_{\frac{t_1+\dots+t_r}{r}} f(x)dt_1\dots dt_r\right\} dh\right|$$

and then by the generalized Minkowski inequality

$$\begin{aligned}
&\left\|f_{r,\delta}-f\right\|_{p(\cdot)} \\
&\le c_6(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_0^h \dots \int_0^h \left\|\frac{1}{h}\int_0^h \varDelta^r_{\frac{t_1+\dots+t_r}{r}} f(x)dt_1\right\|_{p(\cdot)} dt_2\dots dt_r\right\}dh \\
&= c_6(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_0^h \dots \int_0^h \left\|\frac{1}{h}\int_{t_2+\dots+t_r}^{h+t_2+\dots+t_r} \varDelta^r_{\frac{t}{r}} f dt\right\|_{p(\cdot)} dt_2\dots dt_r\right\} dh. \quad (68)
\end{aligned}$$

Since

$$\begin{aligned}
\left\|\frac{1}{h}\int_{t_2+\dots+t_r}^{h+t_2+\dots+t_r} \varDelta^r_{\frac{t}{r}} f dt\right\|_{p(\cdot)} &= \left\|\frac{1}{h}\left(\int_{t_2+\dots+t_r}^{h+t_2+\dots+t_r} \varDelta^r_{\frac{t}{r}} f dt - \int_0^{t_2+\dots+t_r} \varDelta^r_{\frac{t}{r}} f dt\right)\right\|_{p(\cdot)} \\
&\le \left\|\frac{1}{(h+t_2+\dots+t_r)/r}\int_0^{(h+t_2+\dots+t_r)/r} \varDelta^r_t f dt\right\|_{p(\cdot)} \\
&\quad + \left\|\frac{1}{(t_2+\dots+t_r)/r}\int_0^{(t_2+\dots+t_r)/r} \varDelta^r_t f dt\right\|_{p(\cdot)} \\
&\le \sup_{(h+t_2+\dots+t_r)/r\le\delta}\left\|\frac{1}{(h+t_2+\dots+t_r)/r}\int_0^{(h+t_2+\dots+t_r)/r} \varDelta^r_t f dt\right\|_{p(\cdot)} \\
&\quad + \sup_{(t_2+\dots+t_r)/r\le\delta}\left\|\frac{1}{(t_2+\dots+t_r)/r}\int_0^{(t_2+\dots+t_r)/r} \varDelta^r_t f dt\right\|_{p(\cdot)} \\
&= \varOmega_r(f,\delta)_{p(\cdot)} + \varOmega_r(f,\delta)_{p(\cdot)} = 2\varOmega_r(f,\delta)_{p(\cdot)}, \quad (69)
\end{aligned}$$

combining (68) and (69) we have

$$\begin{aligned}
\left\|f_{r,\delta}-f\right\|_{p(\cdot)} &\le c_7(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{h^{r-1}}\int_0^h \dots \int_0^h \varOmega_r(f,\delta)_{p(\cdot)} dt_2\dots dt_r\right\}dh \\
&\le c_7(p,r)\varOmega_r(f,\delta)_{p(\cdot)}\frac{2}{\delta}\int_{\delta/2}^{\delta} dh = c_7(p,r)\varOmega_r(f,\delta)_{p(\cdot)}. \quad (70)
\end{aligned}$$

Now using the relations (63), (70) and (67), respectively we conclude that

$$\begin{aligned}
E_n(f)_{p(\cdot)} &\le E_n\left(f - f_{r,1/n}\right)_{p(\cdot)} + E_n\left(f_{r,1/n}\right)_{p(\cdot)} \\
&\le \left\|f_{r,1/n}-f\right\|_{p(\cdot)} + \frac{c(p)}{n^r}\left\|f^{(r)}_{r,1/n}\right\|_{p(\cdot)} \\
&\le c_8(p,r)\varOmega_r(f,1/n)_{p(\cdot)} + \frac{c_9(p,r)}{n^r}n^r\varOmega_r(f,1/n)_{p(\cdot)} \\
&\le c(p,r)\varOmega_r(f,1/n)_{p(\cdot)}.
\end{aligned}$$

Let T_n be a best approximation trigonometric polynomial for $f \in L^{p(\cdot)}(T)$, which exists in the case of $p(\cdot) \in P(T)$ and is unique when $p_- > 1$(see, [381, $p.$ 130, Theorems 3.2.1 and 3.2.2] and also: [383, $p.$ 59]). Let also $m \in \mathbb{N}$ be the number, such that $2^m \le n < 2^{m+1}$. Since

$$\Omega_r(f, 1/n)_{p(\cdot)} \le \Omega_r(f - T_{2^{m+1}}, 1/n)_{p(\cdot)} + \Omega_r(T_{2^{m+1}}, 1/n)_{p(\cdot)}, \qquad (71)$$

using the inequality [383, $p.$ 209]

$$2^{(\nu+1)r} E_{2^\nu}(f)_{p(\cdot)} \le 2^{2r} \sum_{k=2^{\nu-1}+1}^{2^\nu} k^{r-1} E_k(f)_{p(\cdot)} \qquad (72)$$

we have

$$\Omega_r(f - T_{2^{m+1}}, 1/n)_{p(\cdot)} \le c(p,r)\|f - T_{2^{m+1}}\|_{p(\cdot)} = c(p,r) E_{2^{m+1}}(f)_{p(\cdot)}$$

$$\le c(p,r)\frac{2^{(m+1)r}}{n^r} E_{2^m}(f)_{p(\cdot)} \le \frac{c(p,r)}{n^r} 2^{2r} \sum_{k=2^{m-1}+1}^{2^m} k^{r-1} E_k(f)_{p(\cdot)}. \qquad (73)$$

On the other hand, applying Lemma (5.3.12) and the Bernstein inequality

$$\|T_n'\|_{p(\cdot)} \le c(p) n \|T_n\|_{p(\cdot)}$$

for trigonometric polynomials, proved in [385], and (72) we get

$$\Omega_r(T_{2^{m+1}}, 1/n)_{p(\cdot)} \le \frac{c(p,r)}{n^r} \left\| T_{2^{m+1}}^{(r)} \right\|_{p(\cdot)}$$

$$= \frac{c(p,r)}{n^r} \left\| T_1^{(r)} + \sum_{\nu=0}^{m} \left(T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right) \right\|_{p(\cdot)}$$

$$\le \frac{c(p,r)}{n^r} \left(\left\| T_1^{(r)} \right\|_{p(\cdot)} + \left\| \sum_{\nu=0}^{m} \left(T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right) \right\|_{p(\cdot)} \right)$$

$$\le \frac{c_{10}(p,r)}{n^r} \left(\left\| T_1^{(r)} \right\|_{p(\cdot)} + \sum_{\nu=0}^{m} 2^{(\nu+1)r} \left\| \left(T_{2^{\nu+1}} - T_{2^\nu} \right) \right\|_{p(\cdot)} \right)$$

$$\le \frac{c_{11}(p,r)}{n^r} \left(E_0(f)_{p(\cdot)} + \sum_{\nu=0}^{m} 2^{(\nu+1)r} E_{2^\nu}(f)_{p(\cdot)} \right)$$

$$= \frac{c_{11}(p,r)}{n^r} \left(E_0(f)_{p(\cdot)} + 2^r E_1(f)_{p(\cdot)} + \sum_{\nu=1}^{m} 2^{(\nu+1)r} E_{2^\nu}(f)_{p(\cdot)} \right)$$

$$\le \frac{c_{11}(p,r)}{n^r} \left(E_0(f)_{p(\cdot)} + 2^r E_0(f)_{p(\cdot)} + \sum_{\nu=1}^{m} \sum_{k=2^{\nu-1}+1}^{2^\nu} k^{r-1} E_k(f)_{p(\cdot)} \right)$$

$$\leq \frac{c_{12}(p,r)}{n^r}\left(E_0(f)_{p(\cdot)} \ + \sum_{k=1}^{2^m} k^{r-1} E_k(f)_{p(\cdot)}\right). \tag{74}$$

Combining (71), (73) and (74), we conclude that

$$\Omega_r\,(f, 1/n)_{p(\cdot)}$$

$$\leq \frac{c_{13}(p,r)}{n^r}\left(\sum_{k=2^{m-1}+1}^{2^m} k^{r-1} E_k(f)_{p(\cdot)} + E_0(f)_{p(\cdot)} + \sum_{k=1}^{2^m} k^{r-1} E_k(f)_{p(\cdot)}\right)$$

$$\leq \frac{c_{14}(p,r)}{n^r}\left(\sum_{k=1}^{2^m} k^{r-1} E_k(f)_{p(\cdot)} + E_0(f)_{p(\cdot)}\right)$$

$$\leq \frac{c\,(p,r)}{n^r}\left(\sum_{k=0}^{n} (k+1)^{r-1}\ E_k(f)_{p(\cdot)}\right).$$

Corollary (5.3.14)[492]: Let $p(\cdot) \in P(T)$ and $r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$, such that

$$\sum \Omega(f^j, \delta)_{p(\cdot)} \leq c(p,r) \sum \left\| f^j \right\|_{p(\cdot)}$$

for every $f^j \in L^{p(\cdot)}(T)$ and $\delta > 0$.

Proof Since for any positive integer $r+\varepsilon$ with $0 < (r+\varepsilon)(1+\varepsilon) \leq 1$

$$\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum f^j\ (x^2 + (r+\varepsilon)t)dt \ = \frac{1}{(r+\varepsilon)(1+\varepsilon)} \int_{x^2}^{x^2+(r+\varepsilon)(1+\varepsilon)} \sum f^j\ (u)du$$

$$= \frac{1}{(r+\varepsilon)(1+\varepsilon)} \sum_{x^2}^{x^2+(r+\varepsilon)(1+\varepsilon)/2+(r+\varepsilon)(1+\varepsilon)/2} \sum f^j\ (u)du$$

$$= \sum \left(S_{\frac{1}{(r+\varepsilon)(1+\varepsilon)}, \frac{(r+\varepsilon)(1+\varepsilon)}{2}}\ f^j\right)(x^2),$$

denoting $\lambda := 1/((r+\varepsilon)(1+\varepsilon)), \tau := (r+\varepsilon)(1+\varepsilon)/2$ and applying ,we have that

$$\left\| \frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} \sum f^j\ (.+(r+\varepsilon)t)dt \right\|_{p(\cdot)} = \sum \left\| S_{\frac{1}{(r+\varepsilon)(1+\varepsilon)}, \frac{(r+\varepsilon)(1+\varepsilon)}{2}}\ f^j \right\|_{p(\cdot)}$$

$$\leq c(p) \sum \left\| f^j \right\|_{p(\cdot)}.$$

Hence

$$\sum\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\Delta_t^r\left(f^j\right)dt\right\|_{p(\cdot)}$$

$$=\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum_{r+\varepsilon=0}^{r}(-1)^{2r+\varepsilon}\binom{r}{r+\varepsilon}f^j\left(\cdot+(r+\varepsilon)t\right)dt\right\|_{p(\cdot)}$$

$$\leq c_1(r)\sum_{r+\varepsilon=0}^{r}\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum f^j\left(\cdot+(1+\varepsilon)t\right)dt\right\|_{p(\cdot)}$$

$$\leq c(p,r)\sum\left\|f^j\right\|_{p(\cdot)},$$

which implies that

$$\sum\Omega(f^j,\delta)_{p(\cdot)}\leq c(p,r)\sum\left\|f^j\right\|_{p(\cdot)}.$$

Corollary (5.3.15)[492]: If $f^j\in L^{p(\cdot)}(T), p(\cdot)\in P(T)$, then $\lim_{\delta\to 0}\Omega_r\left(f^j,\delta\right)_{p(\cdot)}=0$ for every positive integer r.

Proof At first we suppose that $f^j\in L^{p(\cdot)}(T)$ is a continuous function. Then for any $\varepsilon>0$ there is a number $\delta(\varepsilon)>0$ such that $\sum\left|f^j(x+mt)-\sum f^j\left(x^2+(m+1)t\right)\right|\leq\varepsilon/\left\{[2\pi]^{1/p}-\left(2^{r-1}\right)\right\}$ for $0<t\leq 1+\varepsilon\leq\delta$ and $m=0,1,2,\ldots,r$. Hence

$$\int_{-\pi}^{\pi}\left|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum\Delta_t^r\left(f^j\right)\left(x^2\right)dt/\varepsilon\right|^{p(x^2)}dx^2$$

$$=\int_{-\pi}^{\pi}\left|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{\sum_{r+\varepsilon=0}^{r}(-1)^{2r+\varepsilon}\binom{r}{r+\varepsilon}\sum f^j\left(x^2+(r+\varepsilon)t\right)}{\varepsilon}dt\right|^{p(x^2)}dx^2$$

$$\leq\int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{\sum_{r+\varepsilon=0}^{r}(-1)^{2r+\varepsilon}\binom{r}{r+\varepsilon}\sum f^j\left(x^2+(r+\varepsilon)t\right)}{\varepsilon}dt\right)^{p(x^2)}dx^2$$

$$=\int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{\sum_{m=0}^{r-1}(-1)^{r+m}\binom{r-1}{m}\left[\sum f^j\left(x^2+mt\right)-\sum f^j\left(x^2+(m+1)t\right)\right]}{\varepsilon}dt\right)^{p(x^2)}$$

$$=\int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{\sum_{m=0}^{r-1}\binom{r-1}{m}\left|\sum f^j\left(x^2+mt\right)-\sum f^j\left(x^2+(m+1)t\right)\right|}{\varepsilon}dt\right)^{p(x^2)}dx^2$$

$$\leq\int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{\varepsilon\sum_{m=0}^{r-1}\binom{r-1}{m}}{[2\pi]^{1/p_-}\left(2^{r-1}\right)\varepsilon}dt\right)^{p(x^2)}dx^2$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{2^{r-1}}{[2\pi]^{1/p_-}\ (2^{r-1})}dt\right)^{p(x^2)} dx^2$$
$$\leq \int_{-\pi}^{\pi}\left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\frac{dt}{[2\pi]^{1/p_-}}\right)^{p_-} dx^2 = 1.$$

Therefore,

$$\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum \Delta_t^r(f^j)\right\|_{p(\cdot)} \leq \varepsilon$$

which implies that $\sum \Omega(f^j,\delta)_{p(\cdot)} \leq \varepsilon$.

If $f^j \in L^{p(\cdot)}(T)$ is not continuous on T, then by density of the set of continuous functions $[378, pp. 145-146]$ in $L^{p(\cdot)}(T)$, for any $\varepsilon > 0$ there exist a 2π periodic continuous function g^j with $\sum\left\|f^j - g^j\right\|_{p(\cdot)} \leq \varepsilon$ and a number $\delta(\varepsilon) > 0$ such that $\Omega_r(g^j,\delta)_{p(\cdot)} \leq \varepsilon$ for every $\delta < \delta(\varepsilon)$. Hence by (2) and Lemma 2

$$\sum \Omega_r(f^j,\delta)_{p(\cdot)} \leq \sum \Omega_r(f^j - g^j,\delta)_{p(\cdot)}) + \sum \Omega_r(g^j,\delta)_{p(\cdot)}$$
$$\leq c(p,r)\sum\left\|f^j - g^j\right\|_{p(\cdot)} + \varepsilon$$
$$\leq [c(p,r)+1]\varepsilon,$$

which implies the required relation $\lim_{\delta\to 0}\sum \Omega(f^j,\delta)_{p(\cdot)} = 0$.

Corollary (5.3.16)[492]: Let $p(\cdot) \in P(T)$. Then there exists a positive constant $c(p,r)$, such that for any $r \in \mathbb{N}, \delta > 0$ and for any sequence of function $f^j \in W_r^{p(\cdot)}(T)$ the inequality

$$\sum \Omega(f^j,\delta)_{p(\cdot)} \leq c(p,r)\delta^r \sum \left\|{f^j}^{(r)}\right\|_{p(\cdot)}$$

holds .

Proof Since

$$\sum \Delta_t^r(f^j)(x^2) = \int_0^t\int_0^t \cdots \int_0^t \sum (f^j)^{(r)}(x^2 + t_1 + \ldots + t_r)dt_1\ldots dt_r,$$

applying r times the generalized Minkowski inequality we have

$$\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum \Delta_t^r(f^j)\,dt\right\|_{p(\cdot)} \leq c_1(p)\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum\left\|\Delta_t^r f^j\right\|_{p(\cdot)} dt$$
$$\leq c_1(p)(1+\varepsilon)^r\frac{1}{(1+\varepsilon)^{r+1}}\int_0^{1+\varepsilon}\left\|\int_0^t \cdots\int_0^t \sum (f^j)^{(r)}(. + t_1 + \ldots + t_r)dt_1\ldots dt_r\right\|_{p(\cdot)} dt$$
$$= c_1(p)(1+\varepsilon)^r\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left\|\frac{1}{1+\varepsilon}\int_0^t\left|\frac{1}{(1+\varepsilon)^{r-1}}\int_0^t \cdots\int_0^t \sum (f^j)^{(r)}(. + t_1 + \ldots + t_r)dt_1\ldots dt_{r-1}\right|dt_r\right|$$

$$\leq c_1(p)(1 + \varepsilon)^r \frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left|\frac{1}{(1+\varepsilon)^{r-1}}\int_0^t \dots\int_0^t \sum \left(f^j\right)^{(r)} (. + t_1 + \dots + t_r)dt_1 \dots dt_{r-1}\right| dt$$

$$\leq c_2(p)(1 + \varepsilon)^r \frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left\|\frac{1}{(1+\varepsilon)^{r-1}}\int_0^t \dots\int_0^t \sum \left(f^j\right)^{(r)} (. + t_1 + \dots + t_{r-1})dt_1 \dots dt_{r-1}\right\|_{p(\cdot)} dt$$

$$\leq \dots \leq c_3(p,r)(1+\varepsilon)^r \frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left\|\left\{\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\left|\sum \left(f^j\right)^{(r)} (\cdot + t_1)\right| dt_1\right\}\right\|_{p(\cdot)} dt$$

$$\leq c_4(p,r)(1+\varepsilon)^r \sum \left\|\left(f^j\right)^{(r)}\right\|_{p(\cdot)} \frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} dt = c_4(p,r)(1+\varepsilon)^r \left\|\left(f^j\right)^{(r)}\right\|_{p(\cdot)},$$

and taking here the supremum we obtain the inequality

$$\sum \Omega \left(f^j, \delta\right)_{p(\cdot)} \leq c(p,r)\delta^r \sum \left\|\left(f^j\right)^{(r)}\right\|_{p(\cdot)} .$$

For $f^j \in L^{p(\cdot)}(T)$ and $\delta > 0$ we define the Steklov mean value function as $\sum f^j_{r,\delta}\ (x^2)$:

$$= \frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{(1+\varepsilon)^r}\sum_{s=0}^{r-1}(-1)^{2r+\varepsilon+1}\binom{r}{r+\varepsilon}\int_0^{1+\varepsilon}\dots\dots\int_0^{1+\varepsilon}\sum f^j\left(x^2 + \frac{-\varepsilon}{r}[t_1 + \dots + t_r]\right)dt_1 \dots dt_r\right\} d(1+\varepsilon), \qquad (75)$$

which plays a crucial role in this work.

Corollary (5.3.17)[492]: If $f^j \in L^{p(\cdot)}(T), p(\cdot) \in P(T)$, then $f^j_{r,\delta} \in W_r^{p(\cdot)}(T)$ for $\delta > 0$ and $r \in \mathbb{N}$.

Proof Differentiating $r-1$ times the terms under the sum in (75) and setting $t := \frac{-\varepsilon}{r} t_r$ we see that

$$\left\{\int_0^{1+\varepsilon}\dots\int_0^{1+\varepsilon}\sum f^j\left(x^2 + \frac{-\varepsilon}{r}[t_1 + \dots + t_r]\right)dt_1 \dots dt_r\right\}^{(r-1)}$$

$$= \left\{\int_0^h \left(\frac{r}{-\varepsilon}\right)^{r-1}\sum_{m=0}^{r-1}\binom{r-1}{m}(-1)^{r+m}\sum f^j\left(x^2 + \frac{-\varepsilon}{r}t_r + m\frac{-\varepsilon}{r}(1+\varepsilon)\right)\right\} dt$$

$$= \int_0^{1+\varepsilon}(-\varepsilon)^{r-1} \Delta^{r-1}_{\frac{-\varepsilon}{r}(1+\varepsilon)} \sum f^j\left(x^2 + \frac{-\varepsilon}{r}t_r\right)dt_r$$

$$= \int_0^{\frac{-\varepsilon}{r}(1+\varepsilon)}\left(\frac{r}{-\varepsilon}\right)^r \Delta^{r-1}_{\frac{-\varepsilon}{r}(1+\varepsilon)}\sum f^j\ (x^2 + t)dt\ ,$$

and then via (75)

$$\sum {f^j}_{r,\delta}^{(r-1)}\ (x^2)$$
$$= \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{(1+\varepsilon)^r}\left\{\sum_{r+\varepsilon=0}^{r-1}(-1)^{r+s+1}\binom{r}{r+\varepsilon}\int_0^{\frac{r-s}{r}(1+\varepsilon)}\left(\frac{r}{-\varepsilon}\right)^r \Delta_{\frac{-\varepsilon}{r}(1+\varepsilon)}^{r-1}\sum f^j\ (x^2 + t)dt\right\} d(1+\varepsilon)$$
$$= \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{(1+\varepsilon)^r}\left\{\sum_{r+\varepsilon=0}^{r-1}\int_{x^2}^{x^2+\frac{-\varepsilon}{r}(1+\varepsilon)}(-1)^{2r+\varepsilon+1}\binom{r}{r+\varepsilon}\left(\frac{r}{-\varepsilon}\right)^r \Delta_{\frac{-\varepsilon}{r}(r+\varepsilon)}^{r-1}\sum f^j\ (t)dt\right\} d(1+\varepsilon). \quad (76)$$

Since the Steklov mean value functions $(f^j)_{r,\delta}^{(r-1)}$ is an indefinite Lebesgue integral, its absolute continuity on $[0,2\pi]$ can be showed by standard way. It remains to prove the imbedding $\sum(f^j)_{r,\delta}^{(r-1)} \in L^{p(\cdot)}(T)$. Differentiating the relation (76)we obtain

$$\sum (f^j)_{r,\delta}^{(r-1)}(x^2)$$
$$= \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{1}{(1+\varepsilon)^r}\left(\sum_{r+\varepsilon=0}^{r-1}(-1)^{2r+\varepsilon+1}\binom{r}{r+\varepsilon}\left(\frac{r}{-\varepsilon}\right)^r \sum \Delta_{\frac{-\varepsilon}{r}(r+\varepsilon)}^{r}(f^j)\ (x^2)\right) d(1+\varepsilon)$$

and denoting $t := \frac{-\varepsilon}{r}(1+\varepsilon)$ we have

$$\sum \left|(f^j)_{r,\delta'}^{(r)}\delta\ (x^2)\right|$$
$$\leq \frac{2^{r+1}}{\delta^r}\sum_{r+\varepsilon=0}^{r-1}\binom{r}{r+\varepsilon}\left(\frac{r}{-\varepsilon}\right)^r \left|\frac{1}{\delta}\int_{\delta/2}^{\delta}\sum \Delta_{\frac{-\varepsilon}{r}(1+\varepsilon)}^{r}(f^j)\ (x^2) d(1+\varepsilon)\right|$$
$$= \frac{2^{r+1}}{\delta^r}\sum_{r+\varepsilon=0}^{r-1}\binom{r}{r+\varepsilon}\left(\frac{r}{-\varepsilon}\right)^r \left|\frac{1}{\frac{-\varepsilon}{r}\delta}\int_{\frac{-\varepsilon}{r}(\delta/2)}^{\frac{-\varepsilon}{r}\delta}\sum \Delta_t^r(f^j)\ (x^2)dt\right|$$
$$\leq \frac{2^{r+1}}{\delta^r}\sum_{r+\varepsilon=0}^{r-1}\binom{r}{r+\varepsilon}\left(\frac{r}{-\varepsilon}\right)^r \left\{\left|\frac{1}{\frac{-\varepsilon}{r}\delta}\int_0^{\frac{-\varepsilon}{r}\delta}\sum \Delta_t^r(f^j)\ (x^2)dt + \frac{1}{\frac{-\varepsilon}{r}\delta}\int_0^{\frac{-\varepsilon}{r}(\delta/2)}\sum \Delta_t^r(f^j)\ (x^2)dt\right|\right\},$$

which by Lemma (5.3.2)implies the inequality

$$\left\|\sum (f^j)_{r,\delta}^{(r)}\right\|_{p(\cdot)} \leq\ 2c(r)\delta^{-r}\sum \Omega(f^j,\delta)_{p(\cdot)}\ \leq\ c_5(p,r)\sum \left\|f^j\right\|_{p(\cdot)}. \quad (77)$$

Since $f^j \in L^{p(\cdot)}(T)$ the relation (5) means that $\left(f^j\right)_{r,\delta}^{(r)} \in L^{p(\cdot)}(T)$.

Corollary (5.3.18)[492]: Let $p(\cdot) \in P(T), r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every sequence $f^j \in L^{p(\cdot)}(T)$ and $n \in \mathbb{N}$ the inequality

$$E_n\left(\sum f^j\right)_{p(\cdot)} \leq c(p,r)\Omega_r\left(\sum f^j, 1/n\right)_{p(\cdot)}$$

holds.

Theorem (5.3.1) in the case of $r = 1$ was proved in [375].

Proof : Let $f^j \in L^{p(\cdot)}(T), p(\cdot) \in P(T)$. For $\delta > 0$ and $r \in N$, after some necessary simplifications we have

$$\left|\sum f_{r,\delta}^j(x^2) - \sum f^j(x^2)\right| = \frac{2}{\delta}\left|\int_{\delta/2}^{\delta}\left\{\frac{1}{(1+\varepsilon)^r}\int_0^{1+\varepsilon}\cdots\int_0^{1+\varepsilon}\sum \Delta_{\frac{t_1+\ldots+t_r}{r}}^r\left(f^j\right)(x^2)dt_1\ldots dt_r\right\}d(1+\varepsilon)\right|$$

and then by the generalized Minkowski inequality

$$\left\|\sum f_{r,\delta}^j - \sum f^j\right\|_{p(\cdot)} \leq c_6(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{(1+\varepsilon)^{r-1}}\int_0^{1+\varepsilon}\cdots\int_0^{1+\varepsilon}\left\|\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\sum \Delta_{\frac{t_1+\cdots+t_r}{r}}^r\left(f^j\right)(x^2)dt_1\right\|_{p(\cdot)}dt_2\ldots dt_r\right\}d(1+\varepsilon)$$

$$= c_6(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{(1+\varepsilon)^{r-1}}\int_0^{1+\varepsilon}\cdots\int_0^{1+\varepsilon}\left\|\frac{1}{1+\varepsilon}\int_{t_2+\ldots+t_r}^{1+\varepsilon+t_2+\ldots+t_r}\sum \Delta_{\frac{t}{r}}^r\left(f^j\right)dt\right\|_{p(\cdot)}dt_2\ldots dt_r\right\}d(1+\varepsilon).$$

Since

$$\left\|\frac{1}{1+\varepsilon}\int_{t_2+\ldots+t_r}^{1+\varepsilon+t_2+\ldots+t_r}\sum \Delta_{\frac{t}{r}}^r\left(f^j\right)dt\right\|_{p(\cdot)} = \left\|\frac{1}{1+\varepsilon}\left(\int_{t_2+\ldots+t_r}^{1+\varepsilon+t_2+\ldots+t_r}\sum \Delta_{\frac{t}{r}}^r\left(f^j\right)dt - \int_0^{t_2+\ldots+t_r}\sum \Delta_{\frac{t}{r}}^r\left(f^j\right)dt\right)\right\|_{p(\cdot)}$$

$$\leq \left\|\frac{1}{(1+\varepsilon+t_2+\ldots+t_r)/r}\int_0^{(1+\varepsilon+t_2+\ldots+t_r)/r}\sum \Delta_t^r\left(f^j\right)dt\right\|_{p(\cdot)}$$

$$+\left\|\frac{1}{(t_2+\ldots+t_r)/r}\int_0^{(t_2+\ldots+t_r)/r}\sum \Delta_t^r(f^j)\,dt\right\|_{p(\cdot)}$$

$$\leq \sup_{(1+\varepsilon+t_2+\ldots+t_r)/r\leq\delta}\left\|\frac{1}{(1+\varepsilon+t_2+\ldots+t_r)/r}\int_0^{(1+\varepsilon+t_2+\ldots+t_r)/r}\sum \Delta_t^r(f^j)\,dt\right\|_{p(\cdot)}$$

$$+\sup_{(t_2+\ldots+t_r)/r\leq\delta}\left\|\frac{1}{(t_2+\ldots+t_r)/r}\int_0^{(t_2+\ldots+t_r)/r}\sum \Delta_t^r(f^j)\,dt\right\|_{p(\cdot)}$$

$$=\sum \Omega(f^j,\delta)_{p(\cdot)}+\sum \Omega(f^j,\delta)_{p(\cdot)} = 2\sum \Omega(f^j,\delta)_{p(\cdot)}\ ,(78)$$

combining (6) and (7) we have

$$\left\|\sum f_{r,\delta}^j-\sum f^j\right\|_{p(\cdot)}$$

$$\leq c_7(p,r)\frac{2}{\delta}\int_{\delta/2}^{\delta}\left\{\frac{1}{(1+\varepsilon)^{r-1}}\int_0^{1+\varepsilon}\ldots\int_0^{1+\varepsilon}\Omega\sum(f^j,\delta)_{p(\cdot)}dt_2\ldots dt_r\right\}d(1+\varepsilon)$$

$$\leq c_7(p,r)\sum \Omega(f^j,\delta)_{p(\cdot)}\frac{2}{\delta}\int_{\delta/2}^{\delta}d(1+\varepsilon) = c_7(p,r)\Omega\sum(f^j,\delta)_{p(\cdot)}. \quad (79)$$

Now using the relations (1), (8) and (5), respectively we conclude that

$$\sum E_n(f^j)_{p(\cdot)} \leq E_n\left(\sum f^j-\sum f_{r,1/n}^j\right)_{p(\cdot)}+\sum E_n\left(f_{r,1/n}^j\right)_{p(\cdot)}$$

$$\leq \left\|\sum f_{r,1/n}^j-\sum f^j\right\|_{p(\cdot)}+\frac{c(p)}{n^r}\sum\left\|{f^j}_{r,1/n}^{(r)}\right\|_{p(\cdot)}$$

$$\leq c_8(p,r)\sum \Omega\left(f^j,1/n\right)_{p(\cdot)}+\frac{c_9(p,r)}{n^r}n^r\sum \Omega\left(f^j,1/n\right)_{p(\cdot)}$$

$$\leq c(p,r)\sum \Omega\left(f^j,1/n\right)_{p(\cdot)}.$$

Corollary (5.3.19)[492]: Let $p(\cdot)\in P(T), r\in\mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every sequence $f^j\in L^{p(\cdot)}(T)$ and $n\in\mathbb{N}$ the inequality

$$\Omega\left(\sum f^j,1/n\right)_{p(\cdot)}\leq\frac{c(p,r)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k\left(\sum f^j\right)_{p(\cdot)}$$

holds.

In the case of $r=1$ was proved in [376]. Denoting by

$$W_k^{p(\cdot)}(T):=\left\{\sum f^j:\ \left(f^j\right)^{(k-1)}\ is\ absolutely\ continuous\ and\ {f^j}^{(k)}\in L^{p(\cdot)}(T)\right\},$$

$k=1,2,\ldots$, the variable exponent Sobolev space and using Consequence 2.1 in [373] and also the boundedness of $\Omega(f^j,\delta)_{p(\cdot)}$ we have the inequality

$$E_n(\sum f^j)_{p(\cdot)} \leq \frac{c(p)}{n^k} \sum \left\|(f^j)^{(k)}\right\|_{p(\cdot)},$$

which implies by using the standard way, the estimation

$$E_n(\sum f^j)_{p(\cdot)} \leq \frac{c(p)}{n^k} E_n\left((f^j)^{(k)}\right)_{p(\cdot)}.$$

Proof : Let T_n be a best approximation trigonometric polynomial for $f^j \in L^{p(\cdot)}(T)$, which exists in the case of $p(\cdot) \in P(T)$ and is unique when $p_- > 1$(see, [381, $p.$ 130, Theorems 3.2.1 and 3.2.2] and also: [381, $p.$ 59]). Let also $m \in \mathbb{N}$ be the number, such that $2^m \leq n < 2^{m+1}$. Since

$$\sum \Omega\left(f^j, 1/n\right)_{p(\cdot)} \leq \sum \Omega f^j{}_{p(\cdot)} + \Omega(T_{2^{m+1}}, 1/n)_{p(\cdot)}, \qquad (80)$$

using the inequality [378, $p.$ 209]

$$2^{(\nu+1)r} \sum E_{2^\nu}\ (f^j)_{p(\cdot)} \leq 2^{2r} \sum_{k=2^{\nu-1}+1}^{2^\nu} \sum k^{r-1} E_k(f^j)_{p(\cdot)} \quad (81)$$

we have

$$\sum \Omega\left(f^j - T_{2^{m+1}}, 1/n\right)_{p(\cdot)} \leq c(p,r) \sum \left\|f^j - T_{2^{m+1}}\right\|_{p(\cdot)} = c(p,r) \sum E_{2^{m+1}}\left(f^j\right)_{p(\cdot)}$$

$$\leq c(p,r) \frac{2^{(m+1)r}}{n^r} \sum E_{2^m}\left(f^j\right)_{p(\cdot)} \leq \frac{c(p,r)}{n^r} 2^{2r} \sum_{k=2^{m-1}+1}^{2^m} \sum k^{r-1} E_k\,(f^j)_{p(\cdot)} \ . (82)$$

On the other hand, applying Lemma (5.3.4) and the Bernstein inequality

$$\|T_n'\|_{p(\cdot)} \leq c(p) n \|T_n\|_{p(\cdot)}$$

for trigonometric polynomials, proved in [382], and (10) we get

$$\Omega(T_{2^{m+1}}, 1/n)_{p(\cdot)} \leq \frac{c(p,r)}{n^r} \left\|T_{2^{m+1}}^{(r)}\right\|_{p(\cdot)}$$

$$= \frac{c(p,r)}{n^r} \left\|T_1^{(r)} + \sum_{\nu=0}^{m} \left(T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right)\right\|_{p(\cdot)}$$

$$\leq \frac{c(p,r)}{n^r} \left(\left\|T_1^{(r)}\right\|_{p(\cdot)} + \left\|\sum_{\nu=0}^{m} \left(T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}\right)\right\|_{p(\cdot)} \right)$$

$$\leq \frac{c_{10}(p,r)}{n^r} \left(\left\|T_1^{(r)}\right\|_{p(\cdot)} + \sum_{\nu=0}^{m} 2^{(\nu+1)r} \left\|\left(T_{2^{\nu+1}} - T_{2^\nu}\right)\right\|_{p(\cdot)} \right)$$

$$\leq \frac{c_{11}(p,r)}{n^r} \sum \left(E_0(f^j)_{p(\cdot)} + \sum_{\nu=0}^{m} \sum 2^{(\nu+1)r} E_{2^\nu}(f^j)_{p(\cdot)} \right)$$

$$= \frac{c_{11}(p,r)}{n^r} \sum \left(E_0(f^j)_{p(\cdot)} + 2^r E_1(f^j)_{p(\cdot)} + \sum_{\nu=1}^{m} 2^{(\nu+1)r} E_{2^\nu}(f^j)_{p(\cdot)} \right)$$

$$\leq \frac{c_{11}(p,r)}{n^r} \sum \left(E_0(f^j)_{p(\cdot)} + 2^r E_0(f^j)_{p(\cdot)} + \sum_{\nu=1}^{m} \sum_{k=2^{\nu-1}+1}^{2^\nu} k^{r-1} E_k(f^j)_{p(\cdot)} \right)$$

$$\leq \frac{c_{12}(p,r)}{n^r} \sum \left(E_0(f^j)_{p(\cdot)} + \sum_{k=1}^{2^m} k^{r-1} E_k(f^j)_{p(\cdot)} \right). \quad (83)$$

Combining (9), (11) and (12), we conclude that

$$\sum \Omega \ (f^j, 1/n)_{p(\cdot)}$$

$$\leq \frac{c_{13}(p,r)}{n^r} \sum \left(\sum_{k=2^{m-1}+1}^{2^m} k^{r-1} E_k(f^j)_{p(\cdot)} + E_0(f^j)_{p(\cdot)} + \sum_{k=1}^{2^m} k^{r-1} E_k(f^j)_{p(\cdot)} \right)$$

$$\leq \frac{c_{14}(p,r)}{n^r} \sum \left(\sum_{k=1}^{2^m} k^{r-1} E_k(f^j)_{p(\cdot)} + E_0(f^j)_{p(\cdot)} \right)$$

$$\leq \frac{c\,(p,r)}{n^r} \sum \left(\sum_{k=0}^{n} (k+1)^{r-1} E_k(f^j)_{p(\cdot)} \right).$$

Chapter 6
The Maximal Operator in Variable Lebesgue Spaces and Atomic Decompositions

We show a modular inequality with no assumptions on the exponent function, and a strong norm inequality if we assume the exponent function is log- Hölder continuous. As an application of our approach we give another proof of a related endpoint result due to H$\ddot{a}$st$\ddot{o}$. There are many ways to prove such boundedness. For example, the boundedness of commutators can be proved by the sharp maximal inequalities. But here, we propose a different method based upon our atomic decomposition. As a corollary we get sufficient conditions for the modular inequality .

Section (6.1): *L*log *L* Results

The Hardy-Littlewood maximal operator is defined for all $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. (Equivalently, the supremum may be taken over all balls centered at x, or over all cubes Q containing x.)

It is well known that for $1 < p < \infty, M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, but that given any $f \in L^1(\mathbb{R}^n), f \neq 0, Mf \notin L^1(\mathbb{R}^n)$. In fact, Mf need not even be locally integrable. For instance, if we let

$$f(x) = \frac{1}{x\log(x)^2}\, \chi_{(0,1/e)}(x),$$

then on $(0,1/e)$,

$$Mf(x) \approx \frac{1}{x|log(x)|},$$

so Mf is not integrable on any interval containing the origin. Wiener [395] (see also [396]) proved that Mf is locally integrable if f is in $L\log L$. More precisely, he showed that given any ball B,

$$\int_B Mf(x)\, dx \le 2|B| + C\int_{\mathbb{R}^n} |f(x)|\log(e + |f(x)|)\, dx. \qquad (1)$$

The L^p boundedness results, $p > 1$, have been generalized to the variable L^p spaces. Given an open set Ω and a measurable function $p(\cdot)\colon \Omega \to [1,\infty)$, we define the space $L^{p(\cdot)}(\Omega)$ to be the space of functions such that for some $\lambda > 0$,

$$\rho_\lambda(f) = \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space when equipped with the Musielak-Orlicz norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0\colon \rho_\lambda(f) \le 1\}.$$

If the set of λ's such that $\rho_\lambda(f) \le 1$ is empty, then we let $\|f\|_{L^{p(\cdot)}(\Omega)} = +\infty$. If $\Omega = \mathbb{R}^n$, we often write simply $L^{p(\cdot)}$ instead of $L^{p(\cdot)}(\mathbb{R}^n)$. Given a set Ω let

$$p_-(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p_+(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

For brevity we write $p_- = p_-(\mathbb{R}^n)$ and $p_+ = p_+(\mathbb{R}^n)$.

Theorem $(\mathbf{6.1.1})[\mathbf{394}]$: Let $p(\cdot): \mathbb{R}^n \to [1,\infty)$ be such that $1 < p_- \le p_+ < \infty$. Suppose further that $p(\cdot)$ satisfies the log-H¨older continuity conditions

$$|p(x) - p(y)| \le \frac{C}{-log|x-y|}, \qquad x, y \in \mathbb{R}^n, \qquad |x-y| < 1/2, \qquad (2)$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \qquad x, y \in \mathbb{R}^n, \qquad |y| \ge |x|. \qquad (3)$$

Then the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem (6.1.1) was first proved by Diening [396] with the stronger hypothesis that $p(\cdot)$ is constant outside of a large ball. The full result was proved independently by Cruz-Uribe, Fiorenza and Neugebauer [397] and Nekvinda [398]. (Nekvinda did so with the second condition replaced by a somewhat more general condition. The relationship between the two conditions is discussed in [399].) More recently, Diening [400] found a complex necessary and sufficient condition on $p(\cdot)$ for the maximal operator to be bounded.

The assumption that $p_- > 1$ is necessary for Theorem (6.1.1) to be true: in [401] it was shown that if $p_- = 1$ and $p(\cdot)$ is lower semicontinuous, then M cannot be bounded on $L^{p(\cdot)}$. Therefore, in this case we are interested in the local integrability of Mf.

Our first result is a generalization of Wiener's inequality (1) with no assumptions on $p(\cdot)$.

Theorem $(\mathbf{6.1.2})[\mathbf{394}]$: Given $p(\cdot): \mathbb{R}^n \to [1,\infty)$, then for any $\epsilon > 0$, there exists a constant C, depending on $\epsilon$and $p(\cdot)$, such that for any ball B,

$$\int_B Mf(x)\, dx \le 2|B| + C\int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e+|f(x)|)^{q(x)}\, dx, \qquad (4)$$

where $q(x) = \max(\epsilon^{-1}(\epsilon + 1 - p(x)), 0)$.

Remark $(\mathbf{6.1.3})[\mathbf{394}]$: When $p(\cdot) = 1$, (4) reduces to (1). By a straightforward estimate, we could prove (4) with $q(\cdot) = 1$ (and with $2\,|B|$ replaced by $3|B|$) directly from(1): write $f = f_1 + f_2$, where $f_1 = f\chi_{\{f\le 1\}}$; estimate the integral of Mf_1 by $|B|$ and apply Wiener's result to the integral of Mf_2.

However, the central feature of Theorem (6.1.2) is that $q(\cdot)$ decreases in size as $p(\cdot)$ increases, and disappears on the set where $p(x) \ge 1 + \epsilon$. Inequality (1) implies an inequality in the scale of Orlicz spaces:

$$\|Mf\|_{L^1(B)} \le C\|f\|_{L\log L(\mathbb{R}^n)}, \qquad (5)$$

where $L\log L\,(\mathbb{R}^n)$ is the Orlicz space $L^{\Phi}(\mathbb{R}^n)$, $\Phi(t) = t\,log(e+t)$. As a corollary to Theorem (6.1.2) we can prove the corresponding inequality in the scale of variable Orlicz spaces (also known as Musielak-Orlicz spaces). Let $\Phi : \mathbb{R}^n \times \mathbb{R}^+ \to [0,\infty)$ be such that for each $x \in \mathbb{R}^n$, the function $\Phi(x,\cdot)$ is nondecreasing, continuous and convex. Assume that $\Phi(x,0) = 0$, $\Phi(x,t) > 0$ if $t > 0$, and $\Phi(x,t) \to \infty$as $t \to \infty$.

We also assume that for each $t \ge 0$, the function $\Phi(\cdot,t)$ is a measurable function. Define the space $L^{\Phi(,)}(\mathbb{R}^n)$ to be the set of all functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Phi(x, |f(x)|/\lambda)\, dx \;<\; +\infty,$$

equipped with the norm

$$\|f\|_{L^{\Phi(,)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0: \int_{\mathbb{R}^n} \Phi(x, |f(x)|\lambda - 1)\, dx \;\leq 1\right\}.$$

If $\Phi(x,t) = t^{p(x)}$, then $L^{\Phi(,)}$ reduces to the space $L^{p(\cdot)}$. For complete information on these spaces, see Musielak [405].

A special family of the variable Orlicz spaces is the generalization of the Zygmund spaces $L^p(\log L)^q$. If we let $\Phi(x,t) = t^{p(x)} \log(e+t)^{q(x)}$, then Φ satisfies the above hypotheses and we can define the space $L^{\Phi(,)}$. Hereafter we will denote this space by $L^{p(\cdot)}(\log L)^{q(\cdot)}$.

Corollary $(\mathbf{6.1.4})[\mathbf{394}]$: Given $p(\cdot): \mathbb{R}^n \to [1, \infty)$, there exists a constant C depending on $p(\cdot)$, and $|B|$ such that

$$\|Mf\|_{L^1(B)} \leq\; C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)}.$$

We can considerably improve the local integrability of Mf if we assume that $p(\cdot)$ satisfies the log- Hölder's continuity conditions (2) and (3).

Theorem $(\mathbf{6.1.5})[\mathbf{394}]$: Let $p(\cdot): \mathbb{R}^n \to [1, \infty)$ be a function that satisfies (2) and (3). Given, $0 \;< \epsilon < 1$, then there exist continuous functions $r(\cdot)$ and $q(\cdot)$ such that:

(1) $r(\cdot)$ is log-H¨older continuous, $r(x) = p(x)$ whenever $p(\cdot)$ takes on values outside the range $(1{,}1+\epsilon)$, and$1 \;< r\,(x) \;< p\,(x)$ if $p(\cdot)$ takes values in $(1\,{,}1+\epsilon)$.

(2) $0 \leq\; q(x) \leq 1, q(x) = 1$if $p(x) = 1$, and $q(x) = 0$ if $p(x) \geq 1+\epsilon$.

(3) Given a ball B, there is a constant C depending on $|B|$, and $p(\cdot)$ such that

$$\|Mf\|_{L^{r(\cdot)}(B)} \leq\; C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)}. \tag{6}$$

Remark $(\mathbf{6.1.6})[\mathbf{394}]$: In Theorem (6.1.2), $q(\cdot)$ is essentially a linear function of $p(\cdot)$, but in Theorem (6.1.5) it is a considerably more complicated function that is roughly a linear function of $1/p(\cdot)$. It would be interesting to determine the optimal exponent function $q(\cdot)$ in each result. (The authors want to thank P. H$\ddot{a}$st$\ddot{o}$for suggesting this problem.)

The converse of Wiener's inequality. Stein [395] proved that the converse of (1) is also true. More precisely, he showed that given a ball B, if $supp(f) \;\subset\; B$ and $Mf \;\in L^1(B)$, then

$$\int_B |f(x)| \log(e + |f(x)|)\, dx \;<\; \infty. \tag{7}$$

A similar result holds in variable L^p spaces.

Theorem $(\mathbf{6.1.7})[\mathbf{394}]$: Given a function $p(\cdot): \mathbb{R}^n \to [1, \infty)$, let $q(x) = \chi_{\{x: p(x)=1\}}$. Then for any ball B, if $supp(f) \subset\; B$ and $\|Mf\|_{L^{p(\cdot)}(B)} <\; \infty$, then $\|f\|_{L^{p(\cdot)}(log\;L)^{q(\cdot)}(B)} <\; \infty$.

Theorem (6.1.7) seems to be an unsatisfactory converse to Theorems (6.1.2) and (6.1.5) since $q(\cdot)$, the exponent on the logarithm, is not continuous. One might conjecture that a sharper result holds, particularly if $p(\cdot)$ and $q(\cdot)$ satisfy some continuity condition.

But surprisingly, given very reasonable continuity assumptions on these functions, a sharper inequality cannot hold, as the next example shows.

Example $(\mathbf{6.1.8})[\mathbf{394}]$: Given exponent functions $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ and $q(\cdot)\colon \mathbb{R}^n \to [0,1]$, such that $p_+ < \infty, q(x) = 1$ if $p(x) = 1$ and $q(x) < 1$ if $p(x) > 1$, suppose that $p(\cdot)$ satisfies the log- H$ö$lder continuity condition (2) and $q(\cdot)$ satisfies the $log-log$ −H$ö$lder condition

$$|q(x)-q(y)| \le \frac{C}{\log\log(|x-y|^{-1})}, \qquad |x-y| < e^{-e}. \tag{8}$$

Then there exists a function f supported on a ball B contained in the set where $p(x) > 1$ and $0 < q\,(x) < 1$ such that $\|Mf\|_{L^{p(\cdot)}(B)} < \infty$ but $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(B)} = \infty$.

The local integrability of Mf when $p_- = 1$ has also been considered by H$ä$st$ö$ [392]. He considered the case when the set $\{x : p(x) = 1\}$ hasmeasure 0, and the set where $p(x)$ is close to 1 is small. More precisely, he showed that if Ω is a bounded set, and if for some $\epsilon > 0$ and all $s > 0$ sufficiently small,

$|\{x \in \Omega : p(x) \le \lambda_\epsilon(s)\}| \le Cs$, where $\lambda_\epsilon(s) = 1 + (1+\epsilon)\frac{\log\log(1/s)}{\log(1/s)}$,

then $\|Mf\|_{L^1(\Omega)} \le \|f\|_{L^{p(\cdot)}(\Omega)}$. The heart of his proof is to show that given thesehypotheses, $\|f\|_{L\,log\,L(\Omega)} \le \|f\|_{L^{p(\cdot)}(\Omega)}$; the desired conclusion then follows immediately from inequality (5). By combining his ideas with those in the proof of Theorem (6.1.2) we can give a new proof of his result, one which does not pass through the classical inequality.

Theorem $(\mathbf{6.1.9})[\mathbf{394}]$: Given $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$, suppose there exist constants $\epsilon, 0 < \epsilon < 1, K > 0$, and $\delta, 0 < \delta < e^{-e}$, such that for $0 < s \le \delta$,

$$|\{x \in \mathbb{R}^n : p(x) \le \lambda_\epsilon(s)\}| \le Ks.$$

Then given any ball B, there exists a constant C (depending on $|B|, p(\cdot), \epsilon, \delta$ and K) such that

$$\|Mf\|_{L^1(B)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem (6.1.9) is modestly stronger than the original result of H$ä$st$ö$since we can take the domain of f to be unbounded. (If we replace f by f_{χ_Ω}, Ω bounded, we immediately get the same result for any bounded domain.) Further, he assumes that $p_+ < \infty$, whereas we allow unbounded exponents.

Proposition $(\mathbf{6.1.10})[\mathbf{394}]$: Let $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ be a bounded measurable function and let $\delta < e^{-e}, \epsilon, C$, and K be positive constants. If $p_- = 1$, then it is not possible for $p(\cdot)$ to satisfy the log-H$ö$lder continuity condition (2) and satisfy

$$|\{x \in \mathbb{R}^n : p(x) \le \lambda(s)\}| \le Ks, \qquad 0 < s < \delta. \tag{9}$$

A result of the type conjectured above may still be true, but we have no insight on how to prove it without log- H$ö$lder continuity.

We prove Theorem (6.1.2) and Corollary (6.1.4). we prove Theorem (6.1.5). we prove Theorem (6.1.7) and construct Example (6.1.8). Finally, we prove Theorem (6.1.9) and Proposition (6.1.10). In order to emphasize that we are dealing with variable exponents, we will always write $p(\cdot)$ instead of p to denote an exponent function. Unless otherwise specified, C and c will denote positive constants which will depend only on the dimension n, any underlying sets (such as a ball B), and the exponent function $p(\cdot)$.
The proof of Theorem (6.1.2) requires two lemmas. The first is a generalization due to Aguilar Cañestro and Ortega Salvador [396] of a modular inequality in [397]. For completeness we include the short proof.

Lemma (6.1.11)[394]: Given $p(\cdot): \mathbb{R}^n \to [1,\infty)$, then for all $t > 0$,

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le C\{x: |f(x)| > t/2\} \left(\frac{|f(x)|}{t/2}\right)^{p(x)} dx.$$

Proof: Fix $t > 0$, and write $f = f_1 + f_2$, where $f_1(x) = f(x)\chi_{\{x:|f(x)|>t/2\}}(x)$. Then for all x, $Mf_2(x) \le t/2$, so

$$Mf(x) \le Mf_1(x) + Mf_2(x) \le Mf_1(x) + t/2.$$

Therefore, by the weak (1,1) inequality for the maximal operator (see [401]), and since $p(x) \ge 1$,

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le |\{x \in \mathbb{R}^n : Mf_1(x) > t/2\}|$$
$$\le C \int_{\{x:|f(x)|>t/2\}} \frac{|f(x)|}{t/2}\, dx \le C \int_{\{x:|f(x)|>t/2\}} \left(\frac{|f(x)|}{t/2}\right)^{p(x)} dx.$$

To state the second lemma, recall that a function A is log-convex if $\log A$ is convex.

Lemma (6.1.12)[394]: For any $a > 1$, the function

$$A(x) = \begin{cases} \dfrac{a^x - 1}{x} & 0 < x \le 1 \\ \log(a) & x = 0 \end{cases}$$

is log-convex. In particular, given ϵ, $0 < \epsilon \le 1$, for all x, $0 \le x \le \epsilon$,

$$A(x) \le \epsilon^{-1} a^x \log(a)^{1-x/\epsilon}.$$

Proof: We first use the power series expansion of e^x to show that

$$\frac{a^x - 1}{\log(a)} \ge x a^{\frac{x}{2}}. \tag{10}$$

Since $2^n \ge n + 1$, for all $n \ge 0$,

$$x a^{x/2} = \sum_{n=0}^{\infty} \frac{x^{n+1} \log(a)^n}{2^n\, n!} \le \sum_{n=0}^{\infty} \frac{x^{n+1} \log(a)^n}{(n+1)!} = \frac{1}{\log(a)} \sum_{n=0}^{\infty} \frac{x^n \log(a)^n}{n!} = \frac{a^x - 1}{\log(a)}.$$

We now show that $C(x) = \log(A(x))$ is convex by showing that $C''(x) \ge 0$:

$$C''(x) = \frac{1}{x^2} - \frac{a^x \log(a)^2}{(a^x - 1)^2} \ge \frac{1}{x^2} - a^x \left(\frac{1}{x a^{x/2}}\right)^2 = 0.$$

(The middle inequality follows from (10).)

Finally, to establish the desired inequality, we first let $\epsilon = 1$. Then by logconvexity we have that for all $x \leq 1$,

$$A(x) = A(1 \cdot x + 0 \cdot (1-x)) \leq A(1)^x A(0)^{1-x} \leq a^x \log(a)^{1-x}.$$

Now fix $\epsilon < 1$ and $0 < x \leq \epsilon$. Then

$$A(x/\epsilon) = \frac{a^{x/\epsilon} - 1}{x/\epsilon} \leq a^{x/\epsilon} \log(a)^{1-x/\epsilon}.$$

Since this inequality is true for any $a > 1$, replace a by $\bar{a} = a^{1/\epsilon} > 1$; then we get that

$$\frac{\bar{a}^x - 1}{x/\epsilon} \leq \bar{a}^x \log(\bar{a}^\epsilon)^{1-x/\epsilon},$$

which in turn implies

$$\frac{\bar{a}^x - 1}{x} \leq \epsilon^{-x/\epsilon} \bar{a}^x \log(\bar{a})^{1-x\epsilon} \leq \epsilon^{-1-} \bar{a}^x \log(\bar{a})^{1-x/\epsilon}.$$

Given the exponent function $p(\cdot)$, fix a function f such that the right-hand side of (4) is finite. Without loss of generality we may assume that f is non-negative.

Fix $\epsilon > 0$ and define the function $\bar{p}$ by

$$\bar{p}(x) = \begin{cases} \dfrac{p(x)+1}{2} & if\ 1 \leq p(x) < 1+\epsilon \\ p(x) & if\ p(x) \geq 1+\epsilon. \end{cases}$$

For all $x, p(x)/2 \leq \bar{p}(x) \leq p(x)$. Fix a ball B; then

$$\int_B Mf(x)\,dx \leq 2|B| + \int_{\{x\in B: Mf(x)>2\}} Mf(x)\,dx$$

$$= 2\,|B| + \int_2^\infty |\{x \in B : Mf(x) > t\}|\,dt;$$

by Lemma (6.1.11),

$$\leq 2|B| + C\int_2^\infty \int_{\{x:f(x)>t/2\}} \left(\frac{f(x)}{t/2}\right)^{\bar{p}(x)} dxdt;$$

by the change of variables $s = t/2$ this becomes

$$\leq 2|B| + C\int_2^\infty \int_{\{x:f(x)>s\}} \left(\frac{f(x)}{s}\right)^{\bar{p}(x)} dxds;$$

by Fubini's theorem,

$$= 2\,|B| + C\int_{\{x:f(x)>1\}} \int_1^{f(x)} s^{-\bar{p}(x)}\,dsf(x)^{\bar{p}(x)}\ dx.$$

We evaluate the inner integral depending on the size of $\bar{p}(x)$. If $\bar{p}(x) = 1$, then

$$\int_1^{f(x)} s^{-\bar{p}(x)}\,ds = \int_1^{f(x)} s^{-1}\,ds = log(f(x)).$$

If $\bar{p}(x) \geq 1+\epsilon$, then

$$\int_1^{f(x)} s^{-\bar{p}(x)}\, ds \; = \frac{1}{\bar{p}(x)-1}\left(1 - f(x)^{1-\bar{p}(x)}\right) \le 1/\epsilon.$$

If $1 < \bar{p}(x) < 1+\epsilon$, then we actually have that $1 < \bar{p}(x) < 1+\epsilon/2$. Thus, since x is such that $f(x) > 1$,

$$\int_1^{f(x)} s^{-\bar{p}(x)}\, ds \; = \frac{f(x)^{1-\bar{p}(x)}}{\bar{p}(x)-1}\left(f(x)^{1-\bar{p}(x)} - 1\right)$$

$$\le \left(\frac{f(x)^{\bar{p}(x)-1}-1}{\bar{p}(x)-1}\right) \;;$$

by Lemma (6.1.12) with $a = f(x)$, with x replaced by $\bar{p}(x)-1$, and with ϵ replaced by $\epsilon/2$, we have that

$$\le 2\epsilon^{-1}[\log(f(x))]^{1-(\bar{p}(x)-1)/(\epsilon/2)} f(x)^{\bar{p}(x)-1}.$$

Returning now to our original estimate, if we define the sets

$$E_1 = \{x : f(x) > 1\} \cap \{x : \bar{p}(x) = 1\},$$
$$E_2 = \{x : f(x) > 1\} \cap \{x : \bar{p}(x) \ge 1+\epsilon\},$$
$$E_3 = \{x : f(x) > 1\} \cap \{x : 1 < \bar{p}(x) < 1+\epsilon\},$$

then

$$2|B| + C\int_{\{x:f(x)>1\}} \int_1^{f(x)} s^{-\bar{p}(x)}\, ds f(x)^{\bar{p}(x)}\, dx$$
$$= 2\,|B| + C\int_{E_1} f(x)^{\bar{p}(x)} \log(f(x))\, dx$$
$$+C\epsilon^{-1}\int_{E_2} f(x)^{\bar{p}(x)}\, dx$$
$$+C\epsilon^{-1}\int_{E_3} f(x)^{\bar{p}(x)-1}\, [\log(f(x))]^{1-(\bar{p}(x)-1)/(\epsilon/2)}\, dx$$
$$\le 2|B| + C\int_{\{x:f(x)>1\}} f(x)^{p(x)}\left[\log\left(e+f(x)\right)\right]^{q(x)} dx$$
$$\le 2|B| + C\int_{\mathbb{R}^n} f(x)^{p(x)}\left[\log\left(e+f(x)\right)\right]^{q(x)}\, dx.$$

This completes the proof.

Fix a function f and a ball B; then we can apply Theorem (6.1.2) to the family of functions $f/\lambda, \lambda > 0$, to get

$$(2|B| + C)^{-1}\|Mf\|_{L^1(B)}$$
$$= \inf\left\{\lambda > 0 : \int_B M(f/\lambda)(x)\, dx \le 2|B| + C\right\}$$
$$\le \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}[\log(e+|f(x)/\lambda|)]^{q(x)}\, dx \le 1\right\}$$
$$= \|f\|_{L^{p(\cdot)}(log\, L)^{q(\cdot)}(\mathbb{R}^n)}.$$

In our proof we need a few basic facts about variable L^p and Orlicz spaces. For proofs, see Kov$\acute{a}\check{c}$ik and R$\acute{a}$kosn´ık [400] and Musielak [403, Section 8, p. 43].

Lemma (6.1.13)[394]: Given the exponent function $p(\cdot): \Omega \to [1, \infty)$, if $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then

$$\int_\Omega |f(x)|^{p(x)}\, dx \leq \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Conversely, if

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty,$$

then $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$.

Lemma (6.1.14)[394]: Given variable Orlicz spaces $L^{\Phi_1(,)}$ and $L^{\Phi_2(,)}$, if $\Phi_1(x,t) \leq \Phi_2(x,t), x \geq 0, t > 0$, then for all f, $\|f\|_{L^{\Phi_1(,)}(\mathbb{R}^n)} \leq \|f\|_{L^{\Phi_2(,)}(\mathbb{R}^n)}$.

Similarly, given any ball B, if for some $t_0 > 0, \Phi_1(x,t) \leq \Phi_2(x,t), x \geq 0, t > t_0$, then there exists a constant C depending on $|B|$ such that $\|f\|_{L^{\Phi_1(,)}(B)} \leq C\|f\|_{L^{\Phi_2(,)}(B)}$.

The proof also depends on the following lemma due to Capone, Cruz-Uribe and Fiorenza [402], who used it to give a new proof of the boundedness of the maximal operator on variable L^p spaces.

Lemma (6.1.15)[394]: Given $p(\cdot): \mathbb{R}^n \to [1, \infty)$ such that $1 \leq p_- \leq p_+ < \infty$, and such that (2) and (3) hold, then there exists a bounded function S such that if $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then

$$Mf(x)^{p(x)} \leq CM\big(f(\cdot)^{p(\cdot)}\big)(x) + S(x).$$

Lemma (6.1.16)[394]: Let f and g be two functions that satisfy the log- Hölder conditions (2) and (3). Then $max(f, g)$ and $min(f, g)$ also satisfy these conditions. Proof of Theorem (6.1.5). Fix $\epsilon, 0 < \epsilon < 1$. We first define r(x). Let

$$R(x) = p(x) + (p(x) - 1)(p(x) - (1 + \epsilon)) = (p(x) - 1)(p(x) - \epsilon + 1).$$

Since $p(\cdot)$ is log −H¨older continuous, $p_+ < \infty$, so

$$|p(x)^2 - p(y)^2| \leq 2p_+|p(x) - p(y)|;$$

hence, $p(x)^2$ satisfies (2) and (3), and so $R(\cdot)$ does as well. Now let $r(x) = min\, p(x), R(x)$. Then by Lemma (6.1.16), $r(\cdot)$ satisfies (2) and (3). If $p(x) = 1$ or if $p(x) \geq 1 + \epsilon$, then $r(x) = p(x)$. If x is such that $1 < p(x) < 1 + \epsilon$, then $1 < r(x) < p(x)$.

To define $q(\cdot)$ we first modify $r(\cdot)$. Let $F = \{x : p(x) \leq 1 + \epsilon/3\}$ and let

$$r^* = \sup_{x \in F} r(x).$$

(If F is empty, let $r^* = 1 + \epsilon/3$.) Define $\tilde{r}(x) = \min r(x), r^*$, and let

$$q(x) = \max\left(\frac{3}{\epsilon}\left(1 + \frac{\epsilon}{3} - \frac{p(x)}{\tilde{r}(x)}\right), 0\right).$$

By Lemma (6.1.16), $\tilde{r}(\cdot)$ is log −H¨older continuous, so $q(\cdot)$ is continuous. Since $\tilde{r}(x) \leq r(x) \leq p(x)$, we have $0 \leq q(x) \leq 1$. Furthermore, if $x \in F, r(x) <$

$p\ (x)\ \le\ 1+\epsilon/3$,so $r^*\ \le\ 1+\epsilon/3$. Thus for all $x\in\mathbb{R}^n, \tilde{r}(x)\ \le r^*\ \le\ 1+\epsilon/3$. Therefore, if x is suchthat $p(x)\ge 1+$, then

$$\frac{p(x)}{\tilde{r}(x)}\ge \frac{1+\epsilon}{1+\epsilon/3}\ \ge 1+\frac{\epsilon}{3}.$$

Hence, if $p(x)\ge 1+\epsilon, q(x)\ =\ 0$. Similarly, if $p(x)\ =\ 1,\ \tilde{r}(x)=r(x)\ =\ 1$, so $q(x)\ =\ 1$.

Now fix B and $f\ \in\ L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$; by homogeneity we may assume without loss of generality that $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}\ =\ 1$. Then by Lemma (6.1.14), $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\ \le\ 1$ and by Lemma (6.1.13),

$$\int_{\mathbb{R}^n}|f(x)|^{p(x)}\ dx\ \le 1.$$

To complete the proof, again by Lemma (6.1.13) it will suffice to show that

$$\int_B Mf(x)^{r(x)}\,dx\ \le\ C, \tag{11}$$

where C depends only on $|B|, \epsilon$, and $p(\cdot)$.

We define the following sets:

$$B_1=\{x\ \in\ B:\ Mf(x)\le 1, p(x)\ >\ 1+\epsilon/3\},$$
$$B_2=\{x\ \in\ B:\ Mf(x)\ >\ 1, p(x)\ >\ 1+\epsilon/3\},$$
$$B_3=\{x\ \in\ B:\ p(x)\le 1+\epsilon/3\}.$$

Now divide the integral in (11) into three pieces:

$$\int_B Mf(x)^{r(x)}\ dx\ =\int_{B_1} Mf(x)^{r(x)}\ dx\ +\int_{B_2} Mf(x)^{r(x)}\ dx\ +\int_{B_3} Mf(x)^{r(x)}\ dx.$$

We estimate each integral in turn. The first is straightforward:

$$\int_{B_1} Mf(x)^{r(x)}\,dx\ \le |B_1|\le |B|\ <\ \infty.$$

To estimate the second, first note that since $r(x)\le\ p(x)$,

$$\int_{B_2} Mf(x)^{r(x)}\,dx\ \le\int_{B_2} Mf(x)^{p(x)}\,dx.$$

If $x\ \in B_2, p(x)/(1\ +\ \epsilon/3)\ \ge\ 1$ and $p(\cdot)/(1\ +\epsilon/3)$ is log- Hölder continuous, so by Lemma (6.1.15),

$$\le\ C\int_{B_2} M\big(|f(\cdot)|^{p(\cdot)/(1+\epsilon/3)}\big)^{1+\epsilon/3}\ dx\ +C\int_{B_2} S(x)^{1+\epsilon/3}\ dx.$$

Since the maximal operator is bounded on $L^{1+\epsilon/3}$ and S is bounded,

$$\le\ C\int_{\mathbb{R}^n}|f(x)|^{p(x)}\,dx\ +C$$
$$\le\ C.$$

Finally, to estimate the third integral, since$\tilde{r}(\cdot)$ satisfies the hypotheses of Lemma (6.1.15) we have that

$$\int_{B_3} Mf(x)^{r(x)}\,dx = \int_{B_3} Mf(x)^{\tilde{r}(x)}\,dx$$
$$\leq C\int_{B_3} M\big(|f(\cdot)|^{\tilde{r}(\cdot)}\big)\,dx + \int_{B_3} S(x)\,dx$$
$$\leq C\int_{B} M\big(|f(\cdot)|^{\tilde{r}(\cdot)}\big)\,dx + C.$$

Now apply Theorem (6.1.2) with exponent $p(\cdot)/\tilde{r}(\cdot)$ and with ϵreplaced by $\epsilon/3$ to get

$$\leq C|B| + C\int_{\mathbb{R}^n} |f(x)|^{p(x)} \log\big(e + \big|f(x)|^{\tilde{r}(x)}\big)^{q(x)}\,dx$$
$$+C \leq C|B| + C\int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e + |f(x)|\,)^{q(x)}\,dx$$
$$+C \leq C.$$

This completes the proof.

Fix a ball B and f with $supp(f) \subset B$. Without loss of generality we may assume $\|Mf\|_{L^{p(\cdot)}(B)} = 1$, so by Lemma (6.1.14), $\|Mf\|_{L^1(B)} \leq 1$. Hence, by (7), Stein's converse of Wiener's inequality,

$$\int_B |f(x)| \log(e + |f(x)|)\,dx \leq C,$$

where C depends on $|B|$. Therefore,

$$\int_B |f(x)|^{p(x)} \log(e + |f(x)|\,)^{q(x)}\,dx$$
$$= \int_{\{x:p(x)=1\}} |f(x)| \log(e + |f(x)|)\,dx + \int_{\{x:p(x)>1\}} |f(x)|^{p(x)}\,dx$$
$$\leq \int_{\{x:p(x)=1\}} |f(x)| \log(e + |f(x)|)\,dx + \int_{\{x:p(x)>1\}} Mf(x)^{p(x)}\,dx \leq C,$$

so by Lemma (6.1.13), $\|f\|_{L^{p(\cdot)}(log)^{q(\cdot)}(B)}$ is finite.

The construction of Example (6.1.8) requires one lemma whose proof will be given after the construction itself.

Lemma $(\mathbf{6.1.17})$[394]: Let $a(\cdot), b(\cdot) : [0, e^{-e}) \to (0, \infty)$ be such that $a(0) < 1$, $a(\cdot)$ satisfies(2) and $b(\cdot)$ satisfies (8). If we define

$$f(x) = x^{-a(x)} \log(1/x)^{-b(x)} \chi_{(0,e^{-e})},$$

then there exists a constant C such that for all $x \in (0, e^{-e})$, $Mf(x) \leq Cf(x)$.

Proof : Fix $x \in (0, e^{-e})$; since $supp(f)$ is contained in this interval, there exists an interval $I \subset (0, e^{-e})$ containing x such that

$$Mf(x) \leq \frac{2}{|I|}\int_I f(t)\,dt.$$

Therefore,

$$\frac{Mf(x)}{f(x)} \le \frac{2}{|I|}\, x^{a(x)} \log(1/x)^{b(x)} \int_I t^{-a(t)} \log(1/t)^{b(t)}\ dt.$$

We now apply the continuity assumptions on $a(\cdot)$ and $b(\cdot)$. First, from (8) we have that

$$\log(1/x)^{b(x)} = \log(1/x)^{b(0)} \log(1/x)^{b(x)-b(0)}$$
$$\le \log(1/x)^{b(0)} \exp(|b(x) - b(0)| \log\log(1/x)) \le C \log(1/x)^{b(0)}.$$

Similarly, we have for each $t \in I$ that

$$\log(1/t)^{-b(t)} \le C \log(1/t)^{b(0)}.$$

Exactly the same argument using (2) shows that $x^{a(x)} \le Cx^{a(0)}$ and $t^{-a(t)} \le Ct^{-a(0)}$. Hence,

$$\frac{Mf(x)}{f(x)} \le Cx^{a(x)} \log(1/x)^{b(0)} \frac{1}{|I|} \int_I t^{-a(0)} \log(1/t)^{b(0)}\ dt.$$

Since the function $t^{-a(0)} \log(1/t)^{-b(t)}$ is decreasing, we can increase its average by taking the average over the interval $(0, x)$. Thus, since $a(0) < 1$,

$$\frac{Mf(x)}{f(x)} \le Cx^{a(0)-1} \log(1/x)^{b(0)} \int_0^x t^{-a(0)} \log(1/t)^{-b(0)}\, dt \le Cx^{a(0)-1} \int_0^x t^{-a(0)}\ dt$$
$$\le C.$$

This completes the proof.

The proof is initially very similar to the proof of Theorem (6.1.2). Fix a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Without loss of generality we may assume that f is non-negative. We may also assume that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$, so by Lemma (6.1.13),

$$\int_{\mathbb{R}^n} f(x)^{p(x)}\ dx \le 1.$$

Fix a ball B; then again by Lemma (6.1.13) it will suffice to prove

$$\int_B Mf(x)\, dx \le C, \tag{12}$$

where C depends only on $|B|, \epsilon, \delta, K$, and $p(\cdot)$. Fix $\gamma > 1/\epsilon$ and define the function

$$\bar{p}(x) = \begin{cases} \dfrac{\gamma + p(x)}{1+\gamma} & 1 < p(x) < \Lambda \\ p(x) & p(x) \ge \Lambda, \end{cases}$$

where the value Λ will be chosen below and will depend only on δ. Note that $p(x)/(\gamma + 1) \le \bar{p}(x) \le p(x)$ for all x.

We now argue exactly as we did in the proof of Theorem (6.1.2) to get

$$\int_B Mf(x)\, dx \le 2\,|B| + C \int_{\{x: f(x)>1\}} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1}\, f(x)^{\bar{p}(x)}\ dx. \tag{13}$$

To estimate this integral, we decompose the set $E = \{x \in \mathbb{R}^n : f(x) > 1\}$. Choose $k > 1$ so that $e^{-k} \le \delta$. For $j \ge k$, let $\alpha_j = (1+\epsilon)log(j)/j$. Define the sets

$$A_0 = \{x \in E : p(x) > 1 + \alpha_k\},$$
$$A_j = \{x \in E : 1 + \alpha_{j+1} < p(x) \le 1 + \alpha_j\}, j \ge k.$$

Then $E = A_0 \bigcup_{j\geq k} A_j$ and by hypothesis, if $j \geq k, |A_j| \leq Ce^{-j}$.
We further subdivide the sets $A_j, j \geq k$: define

$$B_j = \left\{ x \in A_j : f(x) \geq \left(\frac{\gamma + 1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma \alpha_{j+1}}} \right\},$$

$$C_j = \left\{ x \in A_j : f(x) < \left(\frac{\gamma + 1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma \alpha_{j+1}}} \right\}.$$

Then the right-hand side of (13) is bounded by

$$C \int_{A_0} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} \, dx + C \sum_{j\geq k} \int_{B_j} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} \, dx$$

$$+ C \sum_{j\geq k} \int_{C_j} \frac{1 - f(x)^{1-\bar{p}(x)}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)} \, dx$$

$$= I_1 + I_2 + I_3.$$

We will estimate each term separately. To estimate I_1, we now fix $\Lambda = 1 + \alpha_k$. Then for all $x \in A_0, \bar{p}(x) = p(x) \geq \Lambda$, so

$$I_1 = C \int_{A_0} \frac{1 - f(x)^{1-p(x)}}{p(x) - 1} f(x)^{p(x)} \, dx \leq C(\Lambda - 1)^{-1} \int_{A_0} f(x)^{p(x)} \, dx \leq C.$$

(Here the constant C depends on $p(\cdot), \epsilon$ and δ.) To estimate I_2, first note that the integrand in each term is bounded by

$$\frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} f(x)^{\bar{p}(x)+\gamma(\bar{p}(x)-1)} = \frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} f(x)^{p(x)}.$$

We will now show that the fraction on the right-hand side is bounded by 1. On A_j,

$$1 + \frac{\alpha_{j+1}}{\gamma + 1} < \bar{p}(x) \leq 1 + \frac{\alpha_j}{\gamma + 1}.$$

Therefore, on B_j,

$$\frac{f(x)^{\gamma(1-\bar{p}(x))}}{\bar{p}(x) - 1} \leq \left(\frac{\alpha_{j+1}}{\gamma + 1} \right)^{\gamma \cdot \frac{\gamma+1}{\gamma \alpha_{j+1}} \cdot \frac{\alpha_{j+1}}{\gamma + 1}} \left(\frac{\alpha_{j+1}}{\gamma + 1} \right)^{-1} = 1,$$

so we have that

$$I_2 \leq C \sum_{j\geq k} \int_{B_j} f(x)^{p(x)} \, dx \leq C \int_{\mathbb{R}^n} f(x)^{p(x)} \, dx \leq C.$$

Finally, we estimate I_3. By the definition of C_j,

$$I_3 \leq C \sum_{j\geq k} \int_{C_j} \frac{f(x)^{\bar{p}(x)}}{\bar{p}(x) - 1} dx \leq C \sum_{j\geq k} \left(\frac{\gamma + 1}{\alpha_{j+1}} \right)^{\frac{\gamma+1}{\gamma \alpha_{j+1}} \left(1 + \frac{\alpha_j}{\gamma+1}\right)} \left(\frac{\alpha_{j+1}}{\gamma + 1} \right)^{-1} |C_j|.$$

We estimate the j-th term:

$$\left(\frac{\gamma\ +1}{\alpha_{j+1}}\right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}\left(1+\frac{\alpha_j}{\gamma+1}\right)}\ \left(\frac{\alpha_{j+1}}{\gamma\ +1}\right)^{-1}|C_j|$$

$$\leq\ \left(\frac{\gamma\ +1}{\alpha_{j+1}}\right)^{\frac{\gamma+1}{\gamma\alpha_{j+1}}\left(1+\frac{\alpha_j}{\gamma+1}\right)+1}\ e^{-j}$$

$$\leq\ C\exp\left(\left(\frac{\gamma\ +1}{\alpha_{j+1}}\left(1+\ \frac{\alpha_j}{\gamma+1}\right)+1\right)\log\left(\frac{\gamma\ +1}{\alpha_{j+1}}\right)-j\right)$$

$$=\ C\exp\left(\left(\frac{\gamma\ +1}{\alpha_{j+1}}+\ \frac{\alpha_j}{\gamma+1}+1\right)\log\left(\frac{\gamma\ +1}{\alpha_{j+1}}\right)-j\right);$$

since $\alpha_j/\alpha_{j+1}\ \leq 2$,

$$\leq\ C\exp\left(\left(\frac{\gamma\ +1}{\alpha_{j+1}}+\frac{3}{\gamma}\right)\log\left(\frac{\gamma\ +1}{\alpha_{j+1}}\right)-j\right);$$

by the definition of α_{j+1},

$$=\exp\left(\left(\log\left(\frac{\gamma\ +1}{1+\epsilon}\right)\ +\log\left(\frac{j\ +1}{\log(j\ +1)}\right)\right)\left(\frac{\gamma\ +1}{\gamma(1+\epsilon)}\ \frac{j\ +1}{\log(j\ +1)}+\frac{3}{\gamma}\right)-j\right).$$

Since $\gamma>\frac{1}{\epsilon},\gamma\ +1<\gamma(1+\epsilon)$. Hence, for all j sufficiently large, there exists $\beta>0$ such that the exponent is dominated by $-\beta_j$. Therefore, we have that for some large constant C,

$$I_3\leq\ C\sum_{j\geq k}e^{-\beta_j}<\ \infty.$$

Combining the previous three estimates, we see that (12) holds; this completes our proof.

Assume to the contrary that there exists $p(\cdot)$ that simultaneously satisfies (2) and (9). We will derive a contradiction. There are two cases.

Case I. If there exists a ball B such that $p_-(B)\ =\ 1$, then, since (2) implies that $p(\cdot)$ is continuous, there exists $x_0\in\mathbb{R}^n$ such that $p(x_0)\ =\ 1$.

In this case, we begin by observing that $\lambda_\epsilon:\ (0,e^{-e})\ \rightarrow\ (1{,}1\ +\ (1\ +\epsilon)/e)$ is strictly increasing and so invertible. Therefore, we can rewrite (9) as

$|\{x\ \in\mathbb{R}^n:\ p(x)\leq\ t\}|\ \leq\ K\lambda_\epsilon^{-1}(t)$

for all $t>1$ sufficiently close to 1.

Similarly, if we apply the log-H¨older condition (2) in a neighborhood of x_0, we get that for all x sufficiently close to x_0,

$$|p(x)-1|\leq\frac{C}{-log(|x-x_0|)}\ .$$

Hence, for all s sufficiently small,

$$\{x\ \in\mathbb{R}^n:|x-x_0|<s\}\subset\left\{x\ \in\mathbb{R}^n:\ p(x)\leq1+\frac{C}{log(1/s)}\right\},$$

and, in particular,

$$\omega_n s^n \le \left\{ x \in \mathbb{R}^n : p(x) \le 1 + \frac{C}{\log\left(\frac{1}{s}\right)} \right\}, \tag{14}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Now define $\psi : (0,1) \to (1,\infty)$ by

$$\psi(s) = 1 + \frac{C}{log(1/s)}.$$

Then ψ is also strictly increasing and invertible. Thus we can rewrite (14) as

$$\omega_n \left(\psi^{-1}(t)\right)^n \le |\{x \in \mathbb{R}^n : p(x) \le t\}|$$

for all $t > 1$ sufficiently close to 1.

Therefore, we will get the desired contradiction if we can show that for all such t,

$$K\lambda_\epsilon^{-1}(t) < \omega_n \left(\psi^{-1}(t)\right)^n,$$

or equivalently that

$$t < \lambda_\epsilon\left(\omega_n K^{-1} \left(\psi^{-1}(t)\right)^n\right).$$

To show this, note that if we take t sufficiently close to 1, then we have that

$\log\left(\log(\omega_n K^{-1}) + \frac{nC}{t-1}\right) > \frac{2(nC+1)}{1+\epsilon}$,

$$(t-1)\log(\omega_n^{-1} K) < 1.$$

Thus, for all such t,

$$\begin{aligned} \lambda_\epsilon\left(\omega_n K^{-1}\left(\psi^{-1}(t)\right)^n\right) &= 1 + (1+\epsilon)\frac{\log(\log(\omega_n^{-1} K\psi^{-1}(t)^{-n}))}{\log(\omega_n^{-1} K\psi^{-1}(t)^{-n})} \\ &> 1 + (1+\epsilon)\frac{\frac{2(nC+1)}{1+\epsilon}}{\log(\omega_n^{-1} K) + \frac{nC}{t-1}} \\ &= 1 + (nC+1)\frac{2(t-1)}{(t-1)\log(\omega_n^{-1} K) + nC} \\ &> 1 + 2(t-1) \\ &> t. \end{aligned}$$

This is the desired contradiction, which completes the proof of Case I.

Case II. Suppose that $p_- = 1$ but $p(x) > 1$ for all $x \in \mathbb{R}^n$. Then there exists a sequence $\{x_n\}$ such that $|x_n| \to \infty$ and $p(x_n) \to 1$ as $n \to \infty$. Furthermore, by passing to a subsequence we may assume that for all n and m, $|x_n - x_m| \ge 2$. Now fix any $s > 0$. Then there exists $N > 0$ such that if $n \ge N, p(x_n) < \lambda_\epsilon(s)$. But then by condition (2), there exists $\sigma, 0 < \sigma < 1,$ such that if $|x - x_n| < \sigma$ for some $n \ge N,$ then $p(x) < \lambda_\epsilon(s)$. The balls $B_\sigma(x_n)$ are disjoint, so their union has infinite measure. Therefore, the set $\{x \in \mathbb{R}^n : p(x) \le \lambda_\epsilon(s)\}$ has infinite measure. This contradicts the assumption that (9) holds, and our proof is complete.

Section (6.2): Hardy Spaces with Variable Exponents and its Application to Bounded Linear Operators

We apply the atomic decomposition results of Hardy spaces with variable exponents, which was partly obtained in our earlier [412], to the boundedness of linear operators and to compare it with the atomic decomposition results of classical Hardy spaces.

Before we describe Hardy spaces with variable exponents, let us recall classical Hardy spaces. Let $0 < p < \infty$. The Hardy space $H^p(\mathbb{R}^n)$ is given by the set of all distributions $f \in S'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{H^p} \equiv \|\sup_{t>0}|e^{t\Delta}f|\|_{L^p}$ is finite, where $\{e^{t\Delta}\}_{t>0}$ denotes the heat semigroup.

we replace $L^p(\mathbb{R}^n)$ with $L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(\mathbb{R}^n)$is called variable Lebesgue spaces and initiated by Nakano [412]. As a counterpart for Hardy spaces, we are led to considering Hardy spaces with variable exponents where we work mainly on. Now let us describe Hardy spaces with variable exponents and their decomposition results. Let $p(\cdot): \mathbb{R}^n \to (0,\infty)$ and $f : \mathbb{R}^n \to \mathbb{C}$ be measurable functions. Then define the variable Lebesgue quasi-norm $\|f\|_{L^{p(\cdot)}}$ of f;

$$\|f\|_{L^{p(\cdot)}} \equiv \inf\left\{\lambda > 0: \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}, \tag{15}$$

where $\inf \emptyset \equiv \infty$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$for which the quasi-norm $\|f\|_{L^{p(\cdot)}}$is finite.

Here and below, we shall postulate the following conditions on $p(\cdot)$:

(log −Hölder continuity) $|p(x) - p(y)| \le \frac{C}{\log(1/|x-y|)}$ for $|x - y| \le \frac{1}{2}$, (16)

(decay condition) $|p(x) - p(y)| \le \frac{C}{log(e+|x|)}$ for $|y| \ge |x|$. (17)

The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is given by the set of all distributions $f \in S'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{H^{p(\cdot)}} \equiv \left\|\sup_{t>0}|e^{t\Delta}f|\right\|_{L^{p(\cdot)}} \tag{18}$$

is finite. If we assume $1 < p_- \equiv \inf_{x\in\mathbb{R}^n} p(x) \le p_+ \equiv \sup_{x\in\mathbb{R}^n} p(x) < \infty$, (16) and (17), then, from Proposition (6.2.6) below, the Hardy–Littlewood maximal operator M is known to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and from the reflexivity of $L^{p(\cdot)}(\mathbb{R}^n)$, we can prove $L^{p(\cdot)}(\mathbb{R}^n) = H^{p(\cdot)}(\mathbb{R}^n)$with norm equivalence.

First, denote by $L^q_{comp}(\mathbb{R}^n)$ the set of all $L^q(\mathbb{R}^n)$-functions with compact support. For $L = 0,1,2,\cdots, P_L(\mathbb{R}^n)$ denotes the set of all polynomialswith degree less than or equal to L and $P_{-1}(\mathbb{R}^n) \equiv \{0\}$. The space $P_L(\mathbb{R}^n)^\perp$is the set of all integrable functions f satisfying $\int_{\mathbb{R}^n}(1+|x|)^L|A(x)|\,dx < \infty$and $\int_{\mathbb{R}^n} x^\alpha A(x)\,dx = 0$ for all multiindices α such that $|\alpha| \le L$. By conven-tion, $P_{-1}(\mathbb{R}^n)^\perp$ is the set of all measurable functions. For $L = -1,0,1,\cdots$ we define $L^{q,L}_{comp}(\mathbb{R}^n) \equiv L^q_{comp}(\mathbb{R}^n) \cap P_L(\mathbb{R}^n)^\perp$. If C depends on some parameters such that s, then we write $A \le C_s B$.

We define

$$\underline{p} = \min(p_-, 1), \qquad d_{p(\cdot)} \equiv \max\{[n/p_- - n], -1\}$$
for $p \in (0, \infty)$.

Theorem $(\mathbf{6.2.1})[\mathbf{411}]$: Let $p(\cdot)$ satisfy $0 < p_- \leq p_+ < \infty$ as well as (16) and (17).Let $L \in \mathbb{N} \cup \{0\}$ and $s \in (0, \infty)$.

(i). Let $q > p_+$ when $p_+ \geq 1$ and $q \geq 1$ when $p_+ < 1$. Suppose that we are given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of non-negative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L^{q,d}_{comp}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{\infty}$ such that

$$supp(a_j) \subset Q_j, \qquad \|a_j\|_{L^q} \leq |Q_j|^{1/q} \tag{19}$$

that

$$\left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} < \infty. \tag{20}$$

Then the series $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $H^{p(\cdot)}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{H^{p(\cdot)}} \leq C \left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}}$$

(ii). Let $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then there exists a decomposition

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

in $S'(\mathbb{R}^n)$ by means of countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L^{\infty,L}_{comp}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{\infty}$ such that

$$|a_j| \leq \chi_{Q_j},$$

and that

$$\left\| \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^{s} \right)^{1/s} \right\|_{L^{p(\cdot)}} \leq C_s \|f\|_{H^{p(\cdot)}}.$$

Here and below, we use the following convention about cubes: By a “cube” we mean a closed cube whose edges are parallel to the coordinate axes. Its side length is denoted by $\ell(Q)$ and its center by cQ. For $c > 0, cQ$ denotes a cube concentric to Q with sidelength $c\ell(Q)$.

In [412, Theorems 4.5 and 4.6], the possibility when $d_{p(\cdot)} = -1$ was excluded but actually it is possible by Theorem (6.2.1). Next, we present a decomposition result for compactly supported functions.

Theorem $(\mathbf{6.2.2})$[**411**]: Let $\kappa > 1, s > 0, \max(1, p_+) < q < \infty$ and $L \geq d_{p(\cdot)}$. Suppose $f \in L^{q,L}_{comp}(\mathbb{R}^n)$ is supported on a cube Q. Then there exists a decomposition $f = \sum_{j=1}^{N} \lambda_j a_j$ by means of finite collections of cubes $\{Q_j\}_{j=1}^{N}$, of non-negative numbers $\{\lambda_j\}_{j=1}^{N}$ and of $L^{q,L}_{comp}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{N}$ such that

$$\|a_j\|_{L^q} \leq |Q_j|^{1/q}, \qquad supp(a_j) \subset Q_j \subset \kappa Q \quad (j = 1, 2, \cdots, N)$$

and that

$$\left\| \left(\sum_{j=1}^{\infty} \left(\lambda_{j\chi Q_j}\right)^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} \leq C_\kappa \|f\|_{H^{p(\cdot)}}.$$

For comparison, we dare repeat to state Theorems (6.2.1)and (6.2.2) for $L^p(\mathbb{R}^n)$ spaces as Theorems (6.2.3) and (6.2.4).

Theorem $(\mathbf{6.2.3})$[**411**]: Let $p \in (1, \infty), L \in \mathbb{N} \cup \{0, -1\}$ and $s \in (0, \infty)$. Suppose $p < q \leq \infty$.

(i). Suppose that we are given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L^{Q}_{comp}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{\infty}$ such that

$$supp\ (a_j) \subset Q_j, \qquad \|a_j\|_{L^q} \leq |Q_j|^{1/q}$$

and that

$$\left\| \sum_{j=1}^{\infty} \lambda_{j\chi Q_j} \right\|_{L^p} < \infty.$$

Then the series $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $L^p(\mathbb{R}^n)$ and satisfies

$$\|f\|_{L^p} \leq C \left\| \sum_{j=1}^{\infty} \lambda_{j\chi Q_j} \right\|_{L^p} < \infty.$$

(ii). Let $s > 0$ and $L \geq d_p$. Let $f \in L^p(\mathbb{R}^n)$. Then there exist a decomposition $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $L^p(\mathbb{R}^n)$ by means of countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of non-negative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L^{\infty,L}_{comp}(\mathbb{R}^n)$ $-$functions $\{a_j\}_{j=1}^{\infty}$ such that

$$|a_j| \leq \chi_{Q_j},$$

and that

$$\left\|\left(\sum_{j=1}^{\infty}\left(\lambda_{j\chi Q_j}\right)^s\right)^{1/s}\right\|_{L^p} \leq C_s\|f\|_{L^p}.$$

Theorem (6.2.4)[411]: Let $p, q, s \in (0, \infty)$ and $L \geq d_p$ satisfy $1 < p < q < \infty$. Let $f \in L^{q,L}_{comp}(\mathbb{R}^n)$. Then there exists a decomposition $f = \sum_{j=1}^{N} \lambda_j a_j$ by means of finite collections of cubes $\{Q_j\}_{j=1}^{N}$, of non-negative numbers $\{\lambda_j\}_{j=1}^{N}$ and of $L^{q,L}_{comp}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{N}$ such that

$$|a_j| \leq \chi_{Q_j}$$

and that

$$\left\|\left(\sum_{j=1}^{N}\left(\lambda_{j\chi Q_j}\right)^s\right)^{1/s}\right\|_{L^p} \leq C_s\|f\|_{L^p}$$

Remark that Theorems (6.2.3) and (6.2.4) are already included in [414] and that Theorems (6.2.3)(i) and (6.2.4) with $s = p$ are included in [413, Theorems 2.1 and 2.2]. Let us look back on the history of spaces with variable exponents. It seems that the theory dates back of Orlicz [316]. Later, Nakano and Luxemberg independently considered spaces of variable exponents [417] in $1950's$. Especially, the definition of the variable exponent Lebesgue spaces can be found in [418]. It had been left intact until Kovacik and J. R´akosn´ık investigated Sobolev spaces based on Lebesgue spaces with variable exponents. About the fractional integral operators, much was studied from earlier. From the point of harmonic analysis, Diening paved the theory of the boundedness of the Hardy–Littlewood maximal operator in [416]. Based upon the pioneering [417], many investigated the boundedness of the Hardy–Littlewood maximal operator in [418]. With the boundedness of the Hardy–Littlewood maximal operator, the boundedness of other related operators (see [419] for example) and the theory of function spaces (see [420] for example) are developed rapidly. See also surveys [421]. In [422] variable exponent Campanato spaces are defined in the setting of quasimetric measure spaces. As for Hardy spaces with variable exponents, see [423] as well as [424]. Among others, in addition to the recent development about the spaces with variable exponents, the localization principle proved by Hasto is important [425], which seems to have a connection with the proof of the Hardy–Littlewood maximal operator. For their precise statements of the key facts, which we use, see to Proposition (6.2.5) here.

We learn that spaces with variable exponents are difficult to analyze. The main reason was the difficulty of the proof of the boundedness of the Hardy–Littlewood maximal operator ; Diening works paved the way. See [426] .

Apart from the development of spaces with variable exponents, the classical Hardy space $H^p(\mathbb{R}^n)$ has three different aspects as was described by Stein [427]. When $0 < p < 1$, $H^p(\mathbb{R}^n)$ contains distributions which are not $L^1_{loc}(\mathbb{R}^n)$ functions. When $p = 1$, $H^p(\mathbb{R}^n)$ is strictly embedded into $L^1(\mathbb{R}^n)$. When $1 < p < \infty$, by virtue of reflexivity of $L^p(\mathbb{R}^n)$ and the boundedness of the Hardy–Littlewood maximal operator, $H^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ coincide as a subset of $S'(\mathbb{R}^n)$. To have a unified understanding of this strange but important phenomenon, we can use Lebesgue spaces with variable exponents. Notice that we did not require that $p_+ \leq 1$ nor that $p_- > 1$ in Theorems (6.2.1) and (6.2.2). So, once we propose a framework of Hardy spaces with variable exponents, we can treat them in a unified manner.

Frst, we recall some elementary facts for variable Lebesgue spaces. Then we prove Theorems (6.2.1) and (6.2.2). we shall review some fundamental facts for variable exponent Lebesgue spaces. is intended as a quick review of key inequalities in variable Lebesgue spaces collects a maximal inequality. We recall and supplement some basic facts about Hardy spaces with variable exponents is the heart of the present . Theorem (6.2.1)(i), Theorem (6.2.1)(ii) and Theorem (6.2.2) are proved and we consider applications of Theorems (6.2.1) and (6.2.2). deals with fractional integral operators. is devoted to the review of the definition and the boundedness of the singular integral operators. intends as the definition and the boundedness of commutators. The Fefferman–Phong inequality is considered , where we are convinced that we essentially improve the result of [419]. By the Fefferman–Phong inequality, or the trace inequality, we mean

$$\|g \cdot I_\alpha f\|_X \leq c\|g\|_Y \cdot \|f\|_Z \tag{21}$$

for some Banach spaces X, Y and Z. When X and Z are Morrey spaces, namely, if their norms are given by

$$\|f\|_X = \|f\|_{M^{p_1}_{q_1}} = \sup_{x\in\mathbb{R}^n} |Q|^{\frac{1}{p_1}-\frac{1}{q_1}}\left(\int_Q |f(y)|^{q_1} dy\right)^{1/q_1}$$

and

$$\|h\|_Z = \|h\|_{M^{p_2}_{q_2}} = \sup_{x\in\mathbb{R}^n} |Q|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_Q |h(y)|^{q_2} dy\right)^{1/q_2},$$

then (21) is referred to as the Olsen inequality. In [415], for $m = 1$ Olsen considered (21) to investigate the Schr¨odinger equation with Y being Morrey spaces. Later, many authors considered and sharpened (21) with $m = 1$. see [422] for related results. We take up the Hardy operator . Finally we disprove that the Fourier transform is not bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p'(\cdot)}(\mathbb{R}^n)$ even when the exponent $p(\cdot)$ satisfies $1 < p_- \leq p_+ < \infty$ as well as (16) and (17).

We consider (15) under the conditions (16) and (17).

Note that $p_\infty = \lim_{x\to\infty} p(x)$ exists in view of (17). From $p_+ < \infty$ and (16) it follows that

$$|p(x)-p(y)| \leq C\ \frac{1}{log(e\ +1/|x-y|)}\ for\ all\ x,y\ \in \mathbb{R}^n. \qquad (22)$$

Observe that (17) is equivalent to the following estimate;

$$|p(x)-p_\infty| \leq C\ \frac{1}{log(e\ +|x|)}\ for\ all\ x\ \in \mathbb{R}^n. \qquad (23)$$

Note that (23) is equivalent to $\left|log(e\ +|x|)^{(p(x)-p_\infty)}\right| \leq C$, that is,

$$\frac{(e\ +|x|)^{p(x)}}{(e\ +|x|)^{p_\infty}} \sim 1\ \ for\ all\ x\ \in \mathbb{R}^n. \qquad (24)$$

Among other related in equalities, we recall the following localization principle due to H$\ddot{a}$sto [421]:

Proposition $(\mathbf{6.2.5})[\mathbf{411}]$: Under the conditions (22) and (23), the equivalence

$$\|f\|_{L^{p(\cdot)}} \sim \left(\sum_{m\in\mathbb{Z}^n}\left(\left\|\chi_{m+[0,1]^n}f\right\|_{L^{p(\cdot)}}\right)^{p_\infty}\right)^{1/p_\infty}\ \left(f\ \in L^{p(\cdot)}(\mathbb{R}^n)\right)$$

holds.

we still need the following Fefferman–Stein type inequality for the Hardy–Littlewood maximal operator M, which is given by

$$Mf(x) \equiv \sup_{Q\in Q(x)} \frac{1}{|Q|}\int_Q |f(y)|dy\ . \qquad (25)$$

Here $Q(x)$ denotes the set of all cubes containing x. We invoke the following estimate:

Proposition $(\mathbf{6.2.6})[\mathbf{411}]$: Let $p(\cdot)$ satisfy

$$1\ < p_-\ \leq p_+\ <\ \infty$$

as well as (16) and (17). For every $q\ \in (1,\infty]$,

$$\left\|\left(\sum_{j=1}^{\infty} Mf_j{}^q\right)^{1/q}\right\|_{L^{p(\cdot)}} \leq\ C_{p(\cdot),q}\ \left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{1/q}\right\|_{L^{p(\cdot)}}.$$

Proposition (6.2.6) seems to have been a hint of defining the sequence norm in Theorems (6.2.1) and (6.2.2). An important fact illustrated in $[428, p. 1746]$ was that we can not replace q with variable exponents.

The following results for variable Hardy spaces are known and in the present we take them for granted: first, we recall some of equivalent expression about Hardy spaces with variable exponents. We topologize $S(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N|\partial^\alpha\varphi(x)|$$

for each $N\ \in \mathbb{N}$. Define

$$F_N\ \equiv \{\varphi \in S\ (\mathbb{R}^n): p_N(\varphi) \leq 1\}. \qquad (26)$$

Let $f\ \in S'(\mathbb{R}^n)$. Denote by Mf the grand maximal operator given by

$$Mf(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\ \cdot) *\ f(x)|: t > 0, \psi \in F_N\},$$

where we choose and fix a large integer N. Below we write $B(r) \equiv \{x \in \mathbb{R}^n : |x| \leq r\}$. The Fourier transform and its inverse are defined respectively by

$$Ff(\xi) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) exp(-ix \cdot \xi)\, dx,$$

$$F^{-1}f(x) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi) exp(ix \cdot \xi)\, d\xi.$$

Theorem $(\mathbf{6.2.7})[\mathbf{411}]$: ([421, Chapters 3 and 5]). Let $p(\cdot)$ satisfy $0 < p_- \leq p_+ < \infty$as well as (16) and (17). Let $f \in S'(\mathbb{R}^n)$ and let $\varphi \in S'(\mathbb{R}^n)$ satisfy the non-degenerate condition $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$. Let $\psi \in S'(\mathbb{R}^n)$ be chosen so that $\chi_{B(1)} \leq \psi \leq \chi_{B(2)}$. Define $\Delta_j f(x) \equiv F^{-1}[(\psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot))Ff](x)$.Then the following are equivalent :

(i). $f \in H^{p(\cdot)}(\mathbb{R}^n)$.

(ii). $\|Mf\|_{L^{p(\cdot)}}$ is finite.

(iii). $\|\sup_{t>0}|t^{-n}\varphi(t^{-1}\cdot) * f|\|_{L^{p(\cdot)}}$ is finite.

(iv). $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ holds in $S'(\mathbb{R}^n)$ and $\left\|\left(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}} < \infty$.

If one of these conditions is satisfied, then

$$\|f\|_{H^{p(\cdot)}} \sim \|Mf\|_{L^{p(\cdot)}} \sim \left\|\sup_{t>0}\left|\frac{\varphi(t^{-1}\cdot) * f}{t^n}\right|\right\|_{L^{p(\cdot)}} \sim \left\|\left(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}}$$

holds.

Proof: The conditions (17). are equivalent as we can see [421, Chapter 3]. Assume $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then by [420, Chapter 5.3], we have

$$\|f\|_{H^{p(\cdot)}} \sim \left\|\left(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}} < \infty.$$

Let us show that $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ holds in $S'(\mathbb{R}^n)$. Set $f_N \equiv \sum_{j=-N}^{N} \Delta_j f$ for each $N \in \mathbb{N}$. Then $\{f_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $H^{p(\cdot)}(\mathbb{R}^n)$. Denote by g its limit in $H^{p(\cdot)}(\mathbb{R}^n)$. Since it is established in [421, Remark 3.5] that $H^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$, it follows that $g = \sum_{j=-\infty}^{\infty} \Delta_j f$ holds in $S'(\mathbb{R}^n)$. Since $f - g$ has frequency support in $\{0\}$, $f - g$ agree with a polynomial P in $S'(\mathbb{R}^n)$. Since $P = f - g$ belongs to $H^{p(\cdot)}(\mathbb{R}^n)$, we must have $P = 0$. Thus, it follows that

$$f = g = \sum_{j=-\infty}^{\infty} \Delta_j f$$

holds in $S'(\mathbb{R}^n)$.

then $\left\{\sum_{j=-N}^{N} \Delta_j f\right\}_{N=1}^{\infty}$ is a Cauchy sequence in $H^{p(\cdot)}(\mathbb{R}^n)$ because we know that

$$\|g\|_{H^{p(\cdot)}} \sim \left\| \left(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}}$$

for all $g \in H^{p(\cdot)}(\mathbb{R}^n)$. (See [421, Section 5.3].) Hence $\left\{\sum_{j=-N}^{N} \Delta_j f\right\}_{N=1}^{\infty}$ is convergent to an element $h \in H^{p(\cdot)}(\mathbb{R}^n)$. The convergence takes place in the topology of $S'(\mathbb{R}^n)$ as well. Thus, it follows that

$$f = \lim_{N\to\infty} \sum_{j=-N}^{N} \Delta_j f = h \in H^{p(\cdot)}(\mathbb{R}^n).$$

To state some fundamental embeddings, we define

$$S_\infty(\mathbb{R}^n) = S(\mathbb{R}^n) \cap \left(\bigcap_{L=1}^{\infty} P_L(\mathbb{R}^n)^{\perp} \right).$$

Theorem $(6.2.8)$[411]: Let $p(\cdot)$ satisfy $0 < p_- \le p_+ < \infty$ as well as (16) and (17).
(i). $S_\infty(\mathbb{R}^n) \hookrightarrow H^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ in the sense of continuous embedding.
(ii). $L_{comp}^{p_+ +1, d_{p(\cdot)}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$.

Proof: The inclusion $H^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ is proved in [422, Remark 3.1]. We also know that $L_{comp}^{p_+ +1, d_{p(\cdot)}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$. (See [422, Section 4].) The inclusion $S_\infty(\mathbb{R}^n) \hookrightarrow H^{p(\cdot)}(\mathbb{R}^n)$ holds since $S_\infty(\mathbb{R}^n) \hookrightarrow H^{p_-}(\mathbb{R}^n) \cap H^{p_+}(\mathbb{R}^n) \hookrightarrow H^{p(\cdot)}(\mathbb{R}^n)$.

We modify the proof of our earlier [421]. Actually, the following key lemma is improved:

Lemma $(6.2.9)$[411]: Let

$$q > p_+ \ge 1 \quad or \quad q = 1 > p_+. \tag{27}$$

Suppose that we are given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L_{comp}^{q,d_p}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{\infty}$ such that

$$supp(a_j) \subset Q_j, \|a_j\|_{L^q} \le |Q_j|^{1/q}. \tag{28}$$

Then

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j a_j|^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} \le C \left\| \left(\sum_{j=1}^{\infty} \left(\lambda_{j\chi Q_j}\right)^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} < \infty.$$

Remark that the condition (27) was $q \gg 1$ in our earlier [421, Theorem 4.6].

Proof: As before, we can assume that the sums are essentially finite. Choose a positive function $g \in L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)$ so that

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j a_j|^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} = \left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left|\lambda_j a_j(x)\right|^{\underline{p}} g(x) dx \right)^{\frac{1}{\underline{p}}} .$$

Then by the Hölder inequality, we obtain

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left|\lambda_j a_j(x)\right|^{\underline{p}} g(x) dx = \sum_{j=1}^{\infty} |\lambda_j|^{\underline{p}} \int_{\mathbb{R}^n} \left|a_j(x)\right|^{\underline{p}} g(x) dx$$

$$\leq \sum_{j=1}^{\infty} |\lambda_j|^{\underline{p}} \left(\left\|a_j\right\|_{L^q} \right)^{\underline{p}} \|g\|_{L^{\left(q/\underline{p}\right)'}(Q_j)}.$$

If we invoke (28), then we have

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left|\lambda_j a_j(x)\right|^{\underline{p}} g(x) dx \leq \sum_{j=1}^{\infty} |\lambda_j|^{\underline{p}} \left(\left|Q_j\right|^{1/q} \right)^{\underline{p}} \|g\|_{L^{\left(q/\underline{p}\right)'}(Q_j)}$$

$$\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\lambda_j|^{\underline{p}} \chi_{Q_j}(x) M \left[g^{\left(q/\underline{p}\right)'} \right](x)^{1/\left(q/\underline{p}\right)'} dx.$$

An arithmetic shows

$$\left(\frac{q}{\underline{p}}\right)' < \left\{ \left(\frac{p(x)}{\underline{p}} \right)' \right\}_{-} \Longleftrightarrow \frac{q}{\underline{p}} > \frac{p_+}{\underline{p}} \Longleftrightarrow q > p_+.$$

Thus, we can use Hölder inequality and obtain the desired result.

Now we prove Theorem (6.2.1).

Assume for the time being that $\lambda_j = 0$ with finite number of exception. Fix $\varphi \in S(\mathbb{R}^n)$ satisfying the non-degenerate condition $\int \varphi(x)\ dx \neq 0$. As weshowed in $[422, (5.2)]$, we have

$$M_\varphi a_j(x) \leq C\chi_{3Q_j} M a_j(x) + M\chi_{Q_j}(x)^{\frac{n+d_{p_-}+1}{n}} \quad (x \in \mathbb{R}^n).$$

This pointwise estimate yields

$$\left\| M_\varphi \left(\sum_{j=1}^{\infty} \lambda_j a_j \right) \right\|_{L^{p(\cdot)}}$$

$$\leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} M a_j + \sum_{j=1}^{\infty} \lambda_j \left(M\chi_{Q_j} \right)^{\frac{n+d_{p_-}+1}{n}} \right\|_{L^{p(\cdot)}}$$

$$\leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} M a_j \right\|_{L^{p(\cdot)}} + \left\| \sum_{j=1}^{\infty} \lambda_j \left(M \chi_{Q_j} \right)^{\frac{n+d_{p_-}+1}{n}} \right\|_{L^{p(\cdot)}} .$$

Therefore, by Proposition (6.2.6) and Lemma (6.2.9), we obtain

$$\left\| M_{\varphi} \left(\sum_{j=1}^{\infty} \lambda_j a_j \right) \right\|_{L^{p(\cdot)}}$$

$$\leq C \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \chi_{3Q_j} M a_j \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} + \left\| \sum_{j=1}^{\infty} \lambda_j \left(M \chi_{Q_j} \right)^{\frac{n+d_{p_-}+1}{n}} \right\|_{L^{p(\cdot)}}$$

$$\leq C \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} .$$

In summary, we obtained

$$\left\| M_{\varphi} \left(\sum_{j=1}^{\infty} \lambda_j a_j \right) \right\|_{L^{p(\cdot)}} \leq C \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} . \tag{29}$$

Therefore, the result is proved if λ has only a finite number of non-zero entries.
Suppose that we are given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$, of non-negative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and of $L_{comp}^{q,d_p}(\mathbb{R}^n)$-functions $\{a_j\}_{j=1}^{\infty}$ satisfying (19) and (20). Then from (29), we learn that

$$\left\| M_{\varphi} \left(\sum_{j=N_1}^{N_2} \lambda_j a_j \right) \right\|_{L^{p(\cdot)}} \leq C \left\| \left\{ \sum_{j=N_1}^{N_2} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} \tag{30}$$

for $1 \leq N_1 \leq N_2 < \infty$. Therefore, $\left\{ \sum_{j=1}^{J} \lambda_j a_j \right\}_{J=1}^{\infty}$ is a Cauchy sequence in $H^{p(\cdot)}(\mathbb{R}^n)$ and converges to an element $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Since $H^{p(\cdot)}(\mathbb{R}^n)$ is known to be embedded continuously into $S'(\mathbb{R}^n)$, it follows that the sequence $\left\{\sum_{j=1}^{J} \lambda_j a_j\right\}_{J=1}^{\infty}$ converges to f in $S'(\mathbb{R}^n)$. Note that

$$\lim_{J\to\infty} \left\| \left\{ \sum_{j=1}^{J} \left(\lambda_j \chi_{Q_j}\right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} = \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}\right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}}$$

by the monotone convergence theorem. Consequently, from (30) and the Fatou lemma, we deduce

$$\|f\|_{H^{p(\cdot)}} \leq C \lim_{J\to\infty} \left\| M_\varphi \left(\sum_{j=1}^{J} \lambda_j a_j \right) \right\|_{L^{p(\cdot)}} \leq C \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}\right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}}$$

and Theorem (6.2.1). was proved.

We invoke the following decomposition result from [427]:

Lemma $(6.2.10)$[411]: Let $d \in \{-1,0,1,2,\cdots\}$and $j \in \mathbb{Z}$. Suppose that $f \in L^q(\mathbb{R}^n)$ with $q \geq 1$. Then there exist collections of cubes $\{Q^*_{j,k}\}_{k\in K_j}$ and functions $\{\eta_{j,k}\}_{k\in K_j} \subset C^\infty_{comp}(\mathbb{R}^n)$, and a decomposition $f = g_j + b_j, b = \sum_{k\in K_j} b_{j,k}$, such that

(i) The $\{Q^*_{j,k}\}_{k\in K_j}$have the bounded intersection property, and

$$\bigcup_{k\in K_j} Q^*_{j,k} = \left\{Mf > 2^j\right\} = O_j.$$

(ii) Each function $\eta_{j,k}$ is supported in $Q^*_{j,k}$ and

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\{Mf>2^j\}}, \qquad 0 \leq \eta_{j,k} \leq 1.$$

(iii) The distribution g_j satisfies the inequality:

$$Mg_j(x) \leq C \left(Mf(x)\chi_{\{Mf>2^j\}}(x) + 2^j \sum_k \frac{{\ell_{j,k}}^{n+d+1}}{\left(\ell_{j,k} + |x - x_{j,k}|\right)^{n+d+1}} \right)$$

for $x \in \mathbb{R}^n$.

(iv) Each distribution $b_{j,k}$ is given by $b_{j,k} \equiv \eta_{j,k}(f - c_{j,k})$ with a polynomial $c_{j,k} \in P_d(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} b_{j,k}(x) q(x)\, dx = 0$for all $q \in P_d(\mathbb{R}^n)$, and

$$Mb_{j,k}(x) \leq C \left(Mf(x)\chi_{Q^*_{j,k}}(x) + \frac{2^j {\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n/Q^*_{j,k}(x)} \right) \ (x \in \mathbb{R}^n).$$

In the above, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the side-length of $Q^*_{j,k}$, respectively, and the implicit constants are dependent only on n.
the routine argument described in [427] and the density result obtained in [421], we can assume that $f \in L^{q,d_{p(\cdot)}}(\mathbb{R}^n)$ with $q > max(1, p_+)$. For each $j \in \mathbb{Z}$, consider the level set

$$O_j \equiv \{x \in \mathbb{R}^n : Mf(x) > 2^j\}. \tag{31}$$

Then it follows immediately from the definition that

$$O_{j+1} \subset O_j. \tag{32}$$

If we invoke Lemma (6.2.10), then f can be decomposed;

$$f = g_j + b_j, b_j = \sum_k b_{j,k}, b_{j,k} = \eta_{j,k}(f - c_{j,k})$$

where each $b_{j,k}$ is supported in a cube $Q^*_{j,k}$ as is described in Lemma (6.2.10). We have shown in $[421, p. 3691]$

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j), \tag{33}$$

with the sum converging in the sense of distributions. Here, going through the same argument as the one in $[427, p108–109]$, we have an expression;

$$f = \sum_{j,k} A_{j,k}, g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z}) \tag{34}$$

in the sense of distributions, where each $A_{j,k}$, supported in $Q^*_{j,k}$, satisfies the pointwise estimate $|A_{j,k}(x)| \le C_0 2^j$ for some universal constant C_0 and the moment condition $\int_{\mathbb{R}^n} A_{j,k}(x)\, q(x) dx = 0$ for every $q \in P_d(\mathbb{R}^n)$. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j}, \qquad \kappa_{j,k} \equiv C_0 2^j.$$

Then we automatically obtain that each $a_{j,k}$ satisfies

$$|a_{j,k}| \le \chi_{Q^*_{j,k}}, \qquad a_{j,k} \perp P_L(\mathbb{R}^n)$$

and that

$$f = \sum_{j,k} \kappa_{j,k} a_{j,k}$$

in the topology of $L^q(\mathbb{R}^n) \approx H^q(\mathbb{R}^n)$, since $f \in L^q(\mathbb{R}^n)$. It remains to prove the estimate of coefficients; once this can be achieved, we have only to rearrange $\{A_{j,k}\}_{j,k}$ and $\{\kappa_{j,k}\}_{j,k}$. From the definition we need to estimate

$$\left\| \left(\sum_{j=1}^{\infty} \left| \lambda_j \chi_{Q_j} \right|^s \right)^{1/s} \right\|_{L^{p(\cdot)}}$$

$$= \inf\left\{\lambda > 0\colon \int_{\mathbb{R}^n}\left(\sum_{j,k}\left(\frac{\kappa_{j,k}\chi_{Q^*_{j,k}}(x)}{\lambda\left\|\chi_{Q^*_{j,k}}\right\|_{L^{p(\cdot)}}}\right)^s\right)^{\frac{p(x)}{s}} dx \le 1\right\}.$$

Since $\{Q^*_{j,k}\}_k$ forms a Whitney covering of O_j (see Lemma (6.2.10) (i)), we have

$$\left\|\left(\sum_{j=1}^{\infty}\left|\lambda_j\chi_{Q_j}\right|^s\right)^{1/s}\right\|_{L^{p(\cdot)}}$$

$$\sim \inf\left\{\lambda > 0\colon \int_{\mathbb{R}^n}\left(\sum_{j=-\infty}^{\infty}\left(\frac{2^j\chi_{Q_j}(x)}{\lambda}\right)^s\right)^{\frac{p(x)}{s}} dx \le 1\right\}.$$

Recall that $Q_j \supset Q_{j+1}$ for each $j \in \mathbb{Z}$ (see (32) above). Consequently wehave

$$\sum_{j=-\infty}^{\infty}\left(\frac{2^j\chi_{Q_j}(x)}{\lambda}\right)^s \sim \left(\sum_{j=-\infty}^{\infty}\frac{2^j\chi_{Q_j}(x)}{\lambda}\right)^s \sim \left(\sum_{j=-\infty}^{\infty}\frac{2^j\chi_{Q_j}/Q_{j+1}(x)}{\lambda}\right)^s .$$

Thus, we obtain

$$\left\|\left(\sum_{j=1}^{\infty}\left|\lambda_j\chi_{Q_j}\right|^s\right)^{1/s}\right\|_{L^{p(\cdot)}} \le C\inf\left\{\lambda > 0\colon \int_{\mathbb{R}^n}\left(\sum_{j=-\infty}^{\infty}\frac{2^j\chi_{Q_j}/Q_{j+1}(x)}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

We deduce from (31), the definition of O_j that

$$\int_{\mathbb{R}^n}\left(\sum_{j=-\infty}^{\infty}\frac{2^j\chi_{Q_j}/Q_{j+1}(x)}{\lambda}\right)^{p(x)} dx = \sum_{j=-\infty}^{\infty}\int_{Q_j/Q_{j+1}}\left(\frac{2^j}{\lambda}\right)^{p(x)} dx$$

$$\sim \int_{\mathbb{R}^n}\left(\frac{Mf(x)}{\lambda}\right)^{p(x)} dx.$$

Therefore, we obtain

$$\left\|\left(\sum_{j=1}^{\infty}\left|\lambda_j\chi_{Q_j}\right|^s\right)^{1/s}\right\|_{L^{p(\cdot)}} \le C\inf\left\{\lambda > 0\colon \int_{\mathbb{R}^n}\left(\frac{Mf(x)}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

$$= C\|f\|_{H^{p(\cdot)}}, \tag{35}$$

if $f \in L^{q,d_{p(\cdot)}}_{comp}(\mathbb{R}^n)$.

Under a special setting that $f \in L^{q,L}_{comp}(\mathbb{R}^n)$, we shall reexamine the proof of Theorem (6.2.1) we need to consider the truncation with respect to j. This consists of two steps.
As the first step, we consider a truncation with respect to j. We disregard $j \geq j_0$ for some j_0. According to the proof of Theorem 4.5 in [421], we know the structure of $Mf(x)$. More precisely,

$$Mf(x) \leq C_\kappa \left\{ \frac{1}{|Q|} \left(\frac{\ell(Q)}{\ell(Q) + |x - cQ|} \right)^{-n-d_p-1} \|f\|_{L^1(Q)} + \chi_{\kappa Q}(x) Mf(x) \right\}.$$

Recall that O_j is given by (31) and that we have (32). Therefore, there exists $j_0 \in \mathbb{Z}$ such that

$$O_j \subset \kappa Q \tag{36}$$

for all $j \geq j_0$. Then

$$supp(g_{j_0}) = supp\left(f - \sum_{j \geq j_0} \sum_k A_{j,k} \right) \subset O_{j_0} \subset \kappa Q$$

and

$$|g_{j_0}(x)| \leq C_0 2^{j_0} \chi_{\kappa Q}(x) \leq C_0 \left(\inf_{y \in O_j} Mf(y) \right) \chi_{\kappa Q}(x). \tag{37}$$

Here C_0 is a constant that needs to be specified. Let $\lambda_1 = 2^{j_0} C_0$. Then

$$\|\lambda_1 \chi_{\kappa Q}\|_{L^{p(\cdot)}} \leq C \|Mf\|_{L^{p(\cdot)}} \leq C \|f\|_{H^{p(\cdot)}}. \tag{38}$$

The next step is, roughly speaking, to truncate of j and k such that $j \leq j_0$. Fix $x \in \mathbb{R}^n$ and write $j_1 = j_1(x) = [1 + log_2 Mf(x)]$. Recall that $A_{j,k}$ is supported in $Q^*_{j,k} \subset O_j$. If $j \geq j_1$, then $x \notin O_j$. Thus, in view of the expression (34) and the bounded overlapping property of $Q^*_{j,k}$, we have

$$\sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} |A_{j,k}(x)| = \sum_{j=j_0}^{j_1(x)} \sum_{k \in \mathbb{Z}} |A_{j,k}(x)| \leq C \sum_{j \leq j_1(x)} 2^j \leq C 2^{j_1(x)} \leq C Mf(x).$$

This means that

$$supp\left(\sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} |A_{j,k}(x)| \right) \subset O_{j_0} \text{ and that } \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} |A_{j,k}(x)| \in L^q(\mathbb{R}^n). \tag{39}$$

fg

$$\frac{\|\chi_{\kappa Q}\|_{L^{p(\cdot)}}}{\|f\|_{H^{p(\cdot)}}} \left\| \sum_{(j,k) \in ([j_0,\infty) \cap \mathbb{Z}) \times \mathbb{Z}/F} A_{j,k} \right\|_{L^q} \leq |\kappa Q|^{\frac{1}{q}}. \tag{40}$$

Set

$$h \equiv \frac{\|\chi_{\kappa Q}\|_{L^{p(\cdot)}}}{\|f\|_{H^{p(\cdot)}}} \sum_{(j,k) \in ([j_0,\infty) \cap \mathbb{Z}) \times \mathbb{Z}/F} A_{j,k}.$$

Then, from (35), (37), (38), (39) and (40), we conclude

$$f = \lambda_1 \frac{g_{j_0}}{\lambda_1} + \sum_{(j,k)\in F} \kappa_{j,k} a_{j,k} + \frac{\|f\|_{H^{p(\cdot)}}}{\|\chi_{\kappa Q}\|_{L^{p(\cdot)}}} \cdot h$$

is the desired finite decomposition.

Now we investigate the boundedness of fractional integral operator I_α of order α, which is given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \;(x \in \mathbb{R}^n).$$

Theorem (6.2.11)[411]: Let $p(\cdot)$ satisfy

$$0 < p_- \leq p_+ < \frac{n}{\alpha} \tag{41}$$

as well as (16) and (17). Define an index $q(\cdot)$ by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n} \quad (x \in \mathbb{R}^n). \tag{42}$$

Then Iα, which is defined initially on $L^\infty_{comp}(\mathbb{R}^n) \cap H^{p(\cdot)}(\mathbb{R}^n)$, can be extended to a bounded linear operator from $H^{p(\cdot)}(\mathbb{R}^n)$ to $H^{q(\cdot)}(\mathbb{R}^n)$.

The following lemma, dealing with quantitative information, is necessary for the proof:

Lemma (6.2.12)[411]: Let $\alpha \in (0,n)$. Let $p(\cdot)$ satisfy (41) as well as (16) and (17). Define $q(\cdot)$ by (42). Then, for sequences $\{Q_j\}_{j=1}^\infty$ of cubes and $\{\lambda_j\}_{j=1}^\infty$ of non-negative numbers, we have

$$\left\|\sum_{j=1}^\infty \lambda_j \ell(Q_j)^\alpha \chi_{Q_j}\right\|_{L^{q(\cdot)}} \leq C \left\|\sum_{j=1}^\infty \lambda_j \chi_{Q_j}\right\|_{L^{p(\cdot)}}.$$

Proof: By Proposition (6.2.6), we may assume that each Q_j is a dyadic cube, namely, $Q_j = Q_{\nu_j,m} = 2^{-\nu_j} \prod_{k=1}^n [m_{j,k}, m_{j,k} + 1)$ for

$$(\nu_j, m) = (\nu_j, m_{j,1}, m_{j,2}, \cdots, m_{j,n}) \in \mathbb{Z}^{n+1}.$$

Indeed, if we let $R_j \equiv 2^{-\nu_j} \prod_{k=1}^n [m_{j,k}, m_{j,k} + 1)$ be a (non-unique) dyadic cube such that $10^n|Q_j| \geq |R_j|$ and that the triple

$$3R_j \equiv 2^{-\nu_j} \prod_{k=1}^n [m_{j,k} - 1, m_{j,k} + 2)$$

engulfs Q_j, then we have

$$\left\|\sum_{j=1}^\infty \lambda_j \ell(Q_j)^\alpha \chi_{Q_j}\right\|_{L^{q(\cdot)}} \leq C \left\|\sum_{j=1}^\infty \lambda_j \ell(R_j)^\alpha \left(M\chi_{R_j}\right)^{\frac{2}{min(1,q-)}}\right\|_{L^{q(\cdot)}}.$$

Assuming that the assertion is true for dyadic cubes, we obtain

$$\left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha} \chi_{Q_j} \right\|_{L^{q(\cdot)}} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{R_j} \right\|_{L^{p(\cdot)}}$$
$$\leq C \left\| \sum_{j=1}^{\infty} \lambda_j \left(M\chi_{Q_j} \right)^{\frac{2}{\underline{p}}} \right\|_{L^{p(\cdot)}}$$
$$\leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}$$

and our claim that Q_j is dyadic is justified. Since dyadic cubes form a grid, that is, $Q \cap R$ equal Q or R as long as two dyadic cubes Q and R intersect, we can assume that $Q_j \neq Q_{j'}$ for $1 \leq j < j' < \infty$. With these two reductions in mind, let us prove the lemma. Write

$$F(x) \equiv \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(x), \qquad G(x) \equiv \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha} \chi_{Q_j}(x) .$$

Assuming that $\|F\|_{L^{p(\cdot)}} = 1$, let us prove that $\|G\|_{L^{q(\cdot)}} \leq C$. We distinguish two cases:
Case (1): $|Q_j| \leq 1$ for each j.
Case (2): $|Q_j| \geq 2$ for each j.

First assume Case (1). Fix $x \in \mathbb{R}^n$. Then we have

$$|Q_j|^{\frac{1}{p_-}} \sim |Q_j|^{\frac{1}{p_+}} \sim \left\| \chi_{Q_j} \right\|_{L^{p(\cdot)}} \leq \frac{1}{\lambda_j}$$

by virtue of (22). From this, we have

$$G(x) \leq C \sum_{j=1}^{\infty} \min\left(\ell(Q_j)^{\alpha} F(x), \ell(Q_j)^{\alpha - \frac{n}{p(x)}} \right) .$$

Thus, since we are assuming (41) and (42), we obtain

$$G(x) \leq C F(x)^{\frac{p(x)}{q(x)}} \quad (x \in \mathbb{R}^n)$$

by virtue of our reduction that each $\{Q_j\}_{j=1}^{\infty}$ is a dyadic cube and $Q_j \neq Q_{j'}$ for $1 \leq j < j' < \infty$. Thus,

$$\int_{\mathbb{R}^n} G(x)^{q(x)}\, dx \leq C \int_{\mathbb{R}^n} F(x)^{p(x)}\, dx = C.$$

Next, assume Case (2). Then we have

$$|Q_j|^{\frac{1}{p_\infty}} \sim \left\| \chi_{Q_j} \right\|_{L^{p(\cdot)}} \leq \frac{1}{\lambda_j},$$

which shows $\|G\|_{L^{q(\cdot)}} \leq C$.
Going through the same argument as Case (1), we obtain

$$G(x) \leq CF(x)^{\frac{p_\infty}{q_\infty}} \ (x \in \mathbb{R}^n).$$

Hence, since in each $Q_{0,m}, (m \in \mathbb{Z}^n) F$ and G are constant functions and we are assuming (17), if we denote $v_0 = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right)$, then we obtain

$$\|G\|_{L^{q(\cdot)}} \sim \left(\sum_{m \in \mathbb{Z}^n} G(m + v_0)^{q_\infty}\right)^{\frac{1}{q_\infty}}$$

$$\leq C\left(\sum_{m \in \mathbb{Z}^n} F(m + v_0)^{p_\infty}\right)^{\frac{1}{q_\infty}}$$

$$\sim \|F\|_{L^{p(\cdot)}}{}^{\frac{p_\infty}{q_\infty}} = 1$$

by virtue of the localization principle, Proposition (6.2.5). Thus, in Case (2), the proof is complete as well. Combining Cases (1) and (2), we obtain

$$\|G\|_{L^{q(\cdot)}} \leq C\left(\left\|\sum_{j;\ell(Q_j)\leq 1} \lambda_j (Q_j)^\alpha \chi_{Q_j}(x)\right\|_{L^{q(\cdot)}} + \left\|\sum_{j;\ell(Q_j)>1} \lambda_j (Q_j)^\alpha \chi_{Q_j}(x)\right\|_{L^{q(\cdot)}}\right) \leq C$$

and the proof is therefore complete.

We may assume $f \in L^{\infty,L}_{comp}(\mathbb{R}^n)$ with $L \gg 1$ in view of the density of $L^{\infty,L}_{comp}(\mathbb{R}^n)$ in $H^{p(\cdot)}(\mathbb{R}^n)$. Then we have

$$f = \sum_{j=1}^{N} \lambda_j a_j$$

as we described in Theorem (6.2.2). By virtue of the moment condition

$$|I_\alpha f(x)| \leq C \sum_{j=1}^{N} \frac{\lambda_j \ell(Q_j)^{L+n}}{\left(\ell(Q_j) + |x - c(Q_j)|\right)^{L+n-\alpha}}$$

$$\leq C \sum_{j=1}^{N} \lambda_j \ell(Q_j)^\alpha M\chi_{Q_j}(x)^{\frac{L+n-\alpha}{n}} .$$

Since we can take L large enough, we can assume

$$\frac{L+n-\alpha}{n} > 1.$$

Thus, by Proposition (6.2.6), it follows that

$$\|I_\alpha f\|_{L^{q(\cdot)}} \leq C\left\|\sum_{j=1}^{N} \lambda_j \ell(Q_j)^\alpha M\chi_{Q_j}(x)^{\frac{L+n-\alpha}{n}}\right\|_{L^{q(\cdot)}}$$

$$\leq C \left\| \sum_{j=1}^{N} \lambda_j \ell(Q_j)^{\alpha} \chi_{Q_j} \right\|_{L^{q(\cdot)}} .$$

If we invoke Lemma (6.2.12), then we have

$$\|I_\alpha f\|_{L^{q(\cdot)}} \leq C \left\| \sum_{j=1}^{N} \lambda_j \chi_{Q_j} \right\|_{L^{q(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}.$$

This is the desired inequality.

By a "singular integral operator", we mean an $L^2(\mathbb{R}^n)$-bounded linear operator T equipped with the kernel K satisfying the following properties:

(i) K is a C-valued measurable function on $\mathbb{R}^n \times \mathbb{R}^n \backslash diag$, where diag is a diagonal set given by $diag \equiv \{ (x,x) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathbb{R}^n\}$.

(ii) On $\mathbb{R}^n \times \mathbb{R}^n \backslash diag$, the size estimate

$$|K(x,y)| \leq C|x-y|^{-n}$$

holds.

(iii) If $(x,y),(x,z) \in \mathbb{R}^n \times \mathbb{R}^n \backslash diag$ satisfy $2|y - z| \leq |x - z|$, then the Hölmander estimate

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leq C \frac{|y-z|}{|x-y|^{n+1}}$$

holds.

(iv) If $f \in L^2_{comp}(\mathbb{R}^n)$, then

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

for almost all $x \in \mathbb{R}^n \backslash supp(f)$. A well-known fact in harmonic analysis is that T can be extended to a bounded linear operator on $L^q(\mathbb{R}^n)$ for all $1 < q < \infty$. Thus, with this fact, we tacitly assume that T is defined on $\bigcup_{1<q<\infty} L^q(\mathbb{R}^n)$. In [421, Proposition 5.3], we proved the following result:

Theorem (6.2.13)[411]: Assume that $1 \leq p_- \leq p_+ < \infty$. Given a singular integral operator T above, T, restricted to $H^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, extends to a bounded linear operator from $H^{p(\cdot)}(\mathbb{R}^n)$to $L^{p(\cdot)}(\mathbb{R}^n)$.

we investigate the boundedness of commutators generated by singular integral operators and $BMO(\mathbb{R}^n)$ −functions. Recall first that a locally integrable function b is said to belong to $BMO(\mathbb{R}^n)$, if b satisfies;

$$\|b\|_{BMO} \equiv \sup_{Q \in Q} \frac{1}{|Q|} \int_Q |b(x) - bQ| \, dx < \infty.$$

Here we wrote

$$bQ \equiv \frac{1}{|Q|} \int_Q b(x)\, dx.$$

By the John–Nirenberg inequality, for all $1 < q < \infty$ and for all $b \in BMO(\mathbb{R}^n)$,

$$\sup_{Q\in Q}\left(\frac{1}{|Q|}\int_Q |b(x)-bQ|^q\,dx\right)^{1/q} \le C_q\|b\|_{BMO}. \tag{43}$$

In view of (43), for any $f \in L^{\infty,d}_{comp}(\mathbb{R}^n)$ with $d \ge 0$, we can define $[T,b]f \in L^1_{loc}(\mathbb{R}^n)$ by

$$[T,b]f(x) \equiv T[b\cdot f](x) - b(x)Tf(x). \tag{44}$$

Note that $b\cdot f \in L^q_{comp}(\mathbb{R}^n)$ for all $q \in (1,\infty)$ and hence the definition (44) above makes sense., we prove;

Theorem $(6.2.14)[411]$: Let $p(\cdot)$ satisfy

$$1 < p_- \le p_+ < \infty$$

as well as (16) and (17). Then, if $d \ge max(1, d_{p(\cdot)})$,

$$\|[T,b]f\|_{L^{p(\cdot)}} \le C\|f\|_{H^{p(\cdot)}}$$

for $f \in L^{\infty,d}_{comp}(\mathbb{R}^n)$. In particular, $[T,b]$ extends to a bounded linear operator from $H^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. To prove Theorem (6.2.14), we need the following estimate:

Lemma $(6.2.15)[411]$: Let Q be a cube and A be an $L^\infty(\mathbb{R}^n)$-function such that

$$|A| \le \chi_Q, \qquad \int_Q A(x)\,dx = 0$$

for almost all $x \in \mathbb{R}^n$. Then for any $q \in (1,\infty), k \in \mathbb{N}, b \in BMO(\mathbb{R}^n)$ and singular integral operators T, we have;

$$\left\|\chi_{2^kQ/2^{k-1}Q}([T,b]A - T[(b_Q - b)A])\right\|_{L^q} \le Ck2^{-k(n+1)}|2^kQ|^{1/q}. \tag{45}$$

Proof: For the purpose of proving (45), we can assume that $b_Q = 0$. Notice that, since A has a moment,

$$[T,b]A(x) + T[b\cdot A](x) = b(x)\int_{\mathbb{R}^n}(K(x,y) - K(x,c_Q))A(y)\,dy.$$

Note also that;

$$\left|b(x)\int_{\mathbb{R}^n}\left(K(x,y)-K\left(x,c_Q\right)\right)A(y)\,dy\right| \le |b(x)|\,\frac{\ell(Q)^{n+1}}{|x-y|^{n+1}}\ (x \notin 2Q).$$

Thus, if we use (43), then we have

$$\begin{aligned}\int_{2^kQ\backslash 2k^{-1}Q}\left|b(x)\int_{\mathbb{R}^n}\left(K(x,y)-K\left(x,c_Q\right)\right)A(y)\,dy\right|^q dx \\ \le C2^{-kq(n+1)}\int_{2^kQ}|b(x)|^q\,dx \\ \le Ck^q 2^{-kq(n+1)+kn}\ell(Q)^n,\end{aligned}$$

proving the lemma.

Let $f \in L^{\infty,d}_{comp}(\mathbb{R}^n)$. Then we have

$$f = \sum_{j=1}^{N} \lambda_j a_j$$

as we described in Theorem (6.2.2). Consequently,

$$[T, b]f = \sum_{j=1}^{N} \lambda_j \left[T, b - b_{Q_j}\right] a_j$$

$$= \sum_{j=1}^{N} \lambda_j \chi_{Q_j} \left(b - b_{Q_j}\right) T a_j + \sum_{j=1}^{N} \sum_{k=1}^{\infty} \lambda_j \chi_{2^k Q \setminus 2k^{-1} Q} \left(b - b_{Q_j}\right) T a_j$$

$$-T \left(\sum_{j=1}^{N} \lambda_j \left(b - b_{Q_j}\right) a_j \right).$$

By using Theorem (6.2.1) and the boundedness of T (see Theorem (6.2.13)),we obtain

$$\left\| T \left(\sum_{j=1}^{N} \lambda_j \left(b - b_{Q_j}\right) a_j \right) \right\|_{L^{p(\cdot)}} \le C \left\| \sum_{j=1}^{N} \lambda_j \left(b - b_{Q_j}\right) a_j \right\|_{L^{p(\cdot)}} \le C \|f\|_{L^{p(\cdot)}}$$

Let us prove

$$\left\| \left\{ \sum_{j=1}^{N} \sum_{k=1}^{\infty} \left(k 2^{-k(n+1)} \lambda_j \chi_{2^k Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} \le C \left\| \left\{ \sum_{j=1}^{N} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} . (46)$$

Once this is proved, according to Theorem (6.2.1), we have $[T, b]f \in L^{p(\cdot)}(\mathbb{R}^n)$ and

$$\|[T, b]f\|_{L^{p(\cdot)}} \le C \left\| \left\{ \sum_{j=1}^{N} \left(\lambda_j \chi_{Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}} \le C \|f\|_{L^{p(\cdot)}}.$$

It remains to estimate (46), By using the maximal operator M, we can further proceed and we have

$$\left\| \left\{ \sum_{j=1}^{N} \sum_{k=1}^{\infty} \left(k 2^{-k(n+1)} \lambda_j \chi_{2^k Q_j} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}}$$

$$\leq C\left\|\left\{\sum_{j=1}^{N}\sum_{k=1}^{\infty}\left(\lambda_j k2^{-k(n+1-\varepsilon)+kn/\underline{p}}\left(M\chi_{Q_j}\right)^{\frac{1+\varepsilon}{\underline{p}}}\right)^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}}.$$

Assuming $p_- > \frac{n}{n+1}$, we can choose $\varepsilon > 0$ so that $(n+1-\varepsilon)p_- > n$. Withthis choice, the above series is summable and

$$\left\|\left\{\sum_{j=1}^{N}\sum_{k=1}^{\infty}\left(k2^{-k(n+1)}\lambda_j\chi_{2^kQ_j}\right)^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}} \leq C\left\|\left\{\sum_{j=1}^{N}\sum_{k=1}^{\infty}\left(\lambda_j\left(M\chi_{Q_j}\right)^{\frac{1+\varepsilon}{\underline{p}}}\right)^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}}.$$

Since $\varepsilon > 0$, if we use Proposition (6.2.6) and the John–Nirenberg inequality, then we obtain (45). If we modify the proof, we have a similar assertion when $p_- = 1$; the range space will be replaced by the weak space $w - L^{p(\cdot)}(\mathbb{R}^n)$. We omit the detail.

Applying the improved atomic decomposition, we can prove the following theorem:

Theorem $(\mathbf{6.2.16})[\mathbf{411}]$: Let $0 < \alpha < n$ and $1 \leq q \leq \frac{n}{\alpha}$. Let $p(\cdot)$ satisfy $1 \leq p_- \leq p_+ < \infty, q > p_+$ as well as (16) and (17). Then for $g \in M_q^{n/\alpha}(\mathbb{R}^n)$ and $f \in H^{p(\cdot)}(\mathbb{R}^n)$, we have

$$\|g \cdot I_\alpha f\|_{L^{p(\cdot)}} \leq C\|g\|_{M_q^{n/\alpha}}\|f\|_{H^{p(\cdot)}}.$$

Proof: By density, we can assume that $f \in L_{comp}^{\infty,d}(\mathbb{R}^n)$. Then we have

$$f = \sum_{j=1}^{N}\lambda_j a_j$$

as we described in Theorem (6.2.2). Here we take $d \in \mathbb{N}$ so that $d > \alpha + \frac{n}{p_-}$. With this decomposition,

$$|g \cdot I_\alpha f(x)| \leq C\sum_{k=1}^{\infty}2^{-kL}\left(\sum_{j=1}^{N}|g(x)|\ell(Q_j)^\alpha\lambda_j\chi_{2^kQ_j}(x)\right).$$

Here $L = d - \alpha$. Observe that

$$\left\|\sum_{j=1}^{N}|g|\ell(Q_j)^\alpha\lambda_j\chi_{2^kQ_j}\right\|_{L^{p(\cdot)}} = 2^{-k\alpha}\left\|\sum_{j=1}^{N}|g|\ell(2^kQ_j)^\alpha\lambda_j\chi_{2^kQ_j}\right\|_{L^{p(\cdot)}}$$

$$\leq C\|g\|_{M_q^{n/\alpha}}\left\|\sum_{j=1}^{N}\lambda_j\chi_{2^kQ_j}\right\|_{L^{p(\cdot)}}$$

$$\leq C2^{\frac{kn}{p_-}} \|g\|_{M_q^{n/\alpha}} \left\| \sum_{j=1}^{N} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}$$

in view of Theorem (6.2.1) and Proposition (6.2.6). Since $d > \alpha + \frac{n}{p_-}$, the above estimate is summable over k.

We place ourselves in $\mathbb{R}_+$ and consider the Hardy operator.

$$Hf(t) = \frac{1}{t}\int_0^t f(s)ds \qquad (t \in \mathbb{R}_+).$$

Although H is defined for functions defined on $(0,\infty)$, we shall use the zero extension to define $Hf(t)$ for functions on $\mathbb{R}$. That is, for $f \in L^\infty_{comp}(\mathbb{R})$, we define

$$Hf(t) = \chi_{(0,\infty)}(t) \times \frac{1}{t}\int_0^t f(s)ds\,.$$

Here and below we assume that $p(\cdot)$ satisfies

$$1 \leq p_- \leq p_+ < \infty$$

as well as (16) and (17).

If $a \in L^{\infty,1}_{comp}(0,\infty)$ is supported on a cube Q contained in $(0,\infty)$, then a simple calculation shows $|Ha(t)| \leq \|a\|_{L^\infty}\chi_Q(t)$. Since $L^{\infty,1}_{comp}(0,\infty)$ is dense in $H^{p(\cdot)}(\mathbb{R})$, and any function $f \in L^{\infty,1}_{comp}(0,\infty)$ admits a finite decomposition $f = \sum_{j=1}^N \lambda_j a_j$ in the way described in Theorem (6.2.2), we can recapture [425, (1.1)] with $\alpha = 0$.

Theorem (6.2.17)[411]: Assume that $p : \mathbb{R} \to [1,\infty]$ satisfies

$$1 \leq p_- \leq p_+ < \infty$$

as well as (16) and (17). Then the Hardy operator H is bounded from $H^{p(\cdot)}(\mathbb{R})$ to $L^{p(\cdot)}(\mathbb{R})$.

Here we disprove that a natural extension of the Hausdorff–Young inequality is available for variable Lebesgue spaces.

Proposition (6.2.18)[411]: Let

$$p(x) \equiv \min\left\{\frac{3}{2}, \max\{|x| - 0.1, 1\}\right\} \quad (x \in \mathbb{R}^n).$$

Then there does not exist a constant $C > 0$ such that

$$\|Ff\|_{L^{p'(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}} \tag{47}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Proof: Assume that inequality (47) holds. Let $a \in C^\infty_{comp}(B((2n)^{-1}))$ satisfy $Fa(0) = 1$ and $\kappa = \{\kappa_m\}_{m\in\mathbb{Z}^n}$ be an $\ell^2(\mathbb{Z}^n)$-sequence such that $\kappa_m = 0$ if $|m| \geq 3$. Then

$$F_\kappa(x) \equiv \sum_{m\in\mathbb{Z}^n} \kappa_m a(x-m) \;\; (x \in \mathbb{R}^n)$$

belongs to $L^2(\mathbb{R}^n)$ and

$$\|F_\kappa\|_{L^{p(\cdot)}} \leq C\|a\|_{L^2} \cdot \|\kappa\|_{L^{3/2}(\mathbb{Z}^n)}. \tag{48}$$

On the other hand,

$$|FF_\kappa(0)| = \left|\sum_{m\in\mathbb{Z}^n} \kappa_m\right| \leq \|FF_\kappa\|_{L^{p'(\cdot)}}. \quad (49)$$

If we combine (47), (48) and (49), then

$$\left|\sum_{m\in\mathbb{Z}^n} \kappa_m\right| \leq C\|\kappa\|_{L^{3/2}(\mathbb{Z}^n)}. \quad (50)$$

Since (50) is valid for all $\ell^2(\mathbb{Z}^n)$--sequences $\kappa = \{\kappa_m\}_{m\in\mathbb{Z}^n}$ such that $\kappa_m = 0$ if $|m| \geq 3$, (50) is a contradiction.

Section (6.3): Modular Inequalities for Maximal Operator

The variable Lebesgue spaces are a generalization of the classical Lebesgue spaces, where the constant exponent p is replaced by a variable exponent function $p(\cdot)$. They have been studied extensively for the past twenty years, particularly for their applications to $PDEs$, the calculus of variations [463], but also for their use in a variety of physical and engineering contexts: the modeling of electrorheological fluids [464], the analysis of quasi-Newtonian fluids [465], fluid flow in porous media [466], magnetostatics [467] and image reconstruction [468].

For $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set, $0 < |\Omega| \leqslant \infty$. Given a measurable exponent function $p(\cdot): \Omega \to [1,\infty)$, hereafter denoted by $p(\cdot) \in P(\Omega)$, for any measurable set $E \subset \mathbb{R}^n, |E \cap \Omega| > 0$, we set

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E\cap\Omega} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x\in E\cap\Omega} p(x).$$

For brevity, we set $p_- = p_-(\Omega)$ and $p_+ = p_+(\Omega)$. The space $L^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions f such that for some $\lambda > 0, \rho_{p(\cdot),\Omega}(f/\lambda) < \infty$, where$\rho_{p(\cdot),\Omega}$ is the modular functional defined by

$$\rho_{p(\cdot),\Omega}(f) = \int_\Omega |f(x)|^{p(x)}\, dx .$$

In situations where there is no ambiguity we will simply write $\rho_{p(\cdot)}(f)$ or $\rho(f)$. The space $L^{p(\cdot)}(\Omega)$ is a Banach function space when equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0: \rho_{p(\cdot),\Omega}(f/\lambda) \leqslant 1\}. \quad (51)$$

When $p(\cdot) = p$, a constant, then $L^{p(\cdot)}(\Omega) = L^p(\Omega)$ and (51) reduces to the classical norm on $L^p(\Omega)$. For the properties of these spaces, see [469].

Given a function $f \in L^1_{loc(\mathbb{R}^n)}$, the (uncentered) Hardy–Littlewood maximal function Mf is defined for $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x and whose sides are parallel to the coordinate axes. (See [470].) If $f \in L^1_{loc(\Omega)}$, then we define Mf by extending f to be identically 0 on $\mathbb{R}^n\backslash\Omega$. The following result, proved by Neugebauer and

[471], gives a nearly optimal sufficient condition on the exponent $p(\cdot)$ for the maximal operator to satisfy a norm inequality on $L^{p(\cdot)}(\Omega)$.

Theorem $(6.3.1)$[411]: Given an open set $\Omega \subset \mathbb{R}^n$, let $p(\cdot) \in P(\Omega)$ be such that $1 < p_- \leqslant p_+ < \infty$ and $p(\cdot) \in LH(\Omega), i.e., p(\cdot)$ is log- Hölder continuous both locally and at infinity:

$$|p(x) - p(y)| \leqslant \frac{C_0}{-\log(|x-y|)}, \qquad |x-y| < \frac{1}{2}, x, y \in \Omega,$$

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\log(e+|x|)}, \qquad x \in \Omega.$$

Then M is bounded on $L^{p(\cdot)}(\Omega)$:

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{52}$$

In the constant exponent case, Theorem (6.3.1) reduces to the classical result that the maximal operator is bounded on $L^p(\Omega), 1 < p < \infty$. In this case, the norm inequality is equivalent to the modular inequality

$$\int_\Omega Mf(x)^p \, dx \leqslant C \int_\Omega |f(x)|^p \, dx.$$

Similar modular inequalities hold in the scale of Orlicz spaces :see , [472]. It is therefore natural to consider the analogous question of modular inequalities for the maximal operator on the variable Lebesgue spaces:

$$\int_\Omega Mf(x)^{p(x)} \, dx \leqslant C \int_\Omega |f(x)|^{p(x)} \, dx. \tag{53}$$

Since inequality (53) implies the norm inequality (52), it is clear that stronger hypotheses may be needed on the exponent function $p(\cdot)$ for the modular inequality to hold. The following example from [473] shows that log- Hölder continuity is not sufficient and the modular inequality need not hold even for a smooth exponent function.

Example $(6.3.2)$[411]: Let $p(\cdot) \in P(\mathbb{R})$ be a measurable exponent function which is equal to 2 on the interval [0,1] and equal to 3 on [2,3] (we make no other assumptions on $p(\cdot)$). Define the sequence of functions$\{f_k\}_{k\in\mathbb{N}} = \{k_{\chi[0,1]}\}_{k\in\mathbb{N}}$. Then for any $x \in [2,3]$,

$$Mf_k(x) \geqslant \frac{1}{3}\int_0^3 |f_k(y)|\, dy = \frac{k}{3},$$

so that

$$\rho_{p(\cdot),\mathbb{R}}(Mf_k) \geqslant \int_2^3 \left(\frac{k}{3}\right)^3 dx = \frac{k^3}{27}.$$

On the other hand $\rho_{p(\cdot),\mathbb{R}}(f_k) = k^2$, so (53) cannot hold. In fact, when $\Omega = \mathbb{R}^n$ and $p_+ < \infty$, Lerner [474] showed that inequality (53) never holds unless $p(\cdot)$ is constant.

Theorem $(6.3.3)$[411]: Let $p(\cdot) \in P(\mathbb{R}^n), p_+ < \infty$. Then the modular inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)} dx \leqslant C_{p(\cdot),n} \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx,$$

where $C_{p(\cdot),n}$is a constant depending on $n, p(\cdot)$ but independent of f,
holds if and only if there is a constant $p > 1$ such that $p(\cdot) = p$ almost everywhere.
Theorem $(\mathbf{6.3.4})[\mathbf{411}]$: Given $p(\cdot) \in P(\mathbb{R}^n)$such that $1 < p- \leqslant p+ < \infty$and $p(\cdot) \in LH(\mathbb{R}^n)$, suppose $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\|f\|_{p(\cdot)} \leqslant 1$. Then

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\ dx \leqslant C_{p(\cdot),n} \int_{\mathbb{R}^n} |f(x)|^{p(x)}\ dx + C_{p(\cdot),n} \int_{\mathbb{R}^n} \frac{dx}{(e+|x|)np_-},$$

where the constant $C_{p(\cdot),n}$ depends on $n, p(\cdot)$ but is independent of f.
We give necessary and sufficient conditions for modular inequalities of the form

$$\int_{\Omega} Mf(x)^{p(x)}\ dx \leqslant c_1 \int_{\Omega} |f(x)|^{q(x)} dx + c_2, \tag{54}$$

to hold for all measurable functions f, where $p(\cdot), q(\cdot) \in P(\Omega)$, and $c_1 > 0, c_2 \geqslant 0$ are constants depending on $n, p(\cdot), q(\cdot)$ and $|\Omega|$, but are independent of f. We are interested in the weakest possible conditions on the exponent functions $p(\cdot)$ and $q(\cdot)$ for (54) to hold. In particular, we want to prove modular inequalities without assuming any smoothness conditions on the exponents. we will only consider the case $p(\cdot) \not\equiv 1$. The endpoint case when $p(\cdot) \equiv 1$ is substantially different. If Ω is bounded and $q- > 1$, then (54) always holds: this is an immediate consequence of [475, Theorem 1.2]. If $\Omega = \mathbb{R}^n$, then (54) never holds, since Mf is never in $L^1(\mathbb{R}^n)$ unless $f = 0$ a.e. More generally, given any set Ω with infinite measure, then arguing as in Example (6.3.6), we would have $L^{q(\cdot)}(\Omega) \subset L^1(\Omega)$, which is impossible: see [476, Theorem 2.45]. When $q- = 1$ the problem of characterizing $q(\cdot)$ is open. Some delicate results in [477]show that this problem depends on how quickly $q(\cdot)$ approaches 1.
Our two main results completely characterize the exponents $p(\cdot)$ and $q(\cdot)$ so that the modular inequality holds. Our characterization depends strongly on whether Ω has finite or infinite measure; When Ω has finite measure our result is remarkably simple.
Theorem $(\mathbf{6.3.5})[\mathbf{411}]$: Given a set $\Omega \subseteq \mathbb{R}^n, 0 < |\Omega| < \infty$, let $p(\cdot), q(\cdot) \in P(\Omega), p(\cdot) \not\equiv 1$. Then the modular inequality (54) holds if and only if $p_+(\Omega) \leqslant q_-(\Omega)$.
As our second result below shows, the assumption that $|\Omega| < \infty$ is critical in Theorem (6.3.5). But to motivate this result, we first give the following example.
Example $(\mathbf{6.3.6})[\mathbf{411}]$: If $\Omega \subseteq \mathbb{R}^n, |\Omega| = \infty$, and if $p(\cdot) \in P(\Omega), q(\cdot) \in P(\Omega)$, then the assumption that $p_+(\Omega) \leqslant q_-(\Omega)$ is not sufficient for (54) to be true. We first consider the case $p_+(\Omega) = q_-(\Omega)$. Fix an open set $\Omega, |\Omega| = \infty$, and constants $1 < p < q < \infty$. Define $p(\cdot) \equiv p$ and

$$q(x) = \begin{cases} p & if\ x \in Q \\ q & if\ x \in \Omega \setminus Q, \end{cases}$$

where $Q \subset \Omega$ is a cube. Then $p_+(\Omega) = q_-(\Omega)$. Suppose (54) holds; then we would have

$$\int_{\Omega} |f(x)|^p\, dx \leqslant \int_{\Omega} Mf(x)^p\ dx$$

$$\leqslant c_1 \int_{\Omega} Q|f(x)|^p\ dx + c_1 \int_{\Omega\setminus Q} |f(x)|^q\ dx + c_2.$$

But then, if we let $f := g\chi_{\Omega\setminus Q}$, we would get the embedding $L^q(\Omega\setminus Q) \subset L^p(\Omega\setminus Q)$, which does not hold when $p < q$ since Ω has infinite measure [476].
The case $p_+(\Omega) < q_-(\Omega)$ is obtained from the same argument by taking $Q = \emptyset$.
The problem in Example (6.3.6) arises because the exponents $p(\cdot)$ and $q(\cdot)$ behave differently at infinity. To avoid this, we make the following definition.
Definition $(6.3.7)$[411]: Given a set Ω, $|\Omega| = \infty$, let F_Ω denote the collection of subsets of Ω that have infinite measure. Given $p(\cdot), q(\cdot) \in P(\Omega)$,we say that $p(\cdot)$ and $q(\cdot)$ touch at infinity, and denote this by $p(\cdot) \simeq q(\cdot)$, if for every $E \in F_\Omega$,

$$p_+(E) = p_+(\Omega) = q_-(\Omega) = q_-(E).$$

The exponents in Example (6.3.6) do not touch at infinity. We consider three additional examples.
Example $(6.3.8)$[411]: Let $\Omega = \mathbb{R}$.
(i) The exponents $p(x) = 2 - (1 + x^2)^{-1}, q(x) = 2 + (1 + x^2)^{-1}$ touch at infinity.
(ii) On the other hand, if we let $\tilde{q}(x) = a + (1 + x^2)^{-1}, a > 2$, then $p(\cdot)$ and$\tilde{q}(\cdot)$ do not touch at infinity.
(iii) Finally, if $p(x) \equiv 2$ and $q(x) = 2 + \chi_E$, where E is any bounded measurable set, then $p(\cdot)$ and $q(\cdot)$ touch at infinity.

We can now state our second main result, characterizing the modular inequality on sets Ω with infinite measure.
Theorem $(6.3.9)$[411]: Given a set $\Omega \subseteq \mathbb{R}^n$, $|\Omega| = \infty$, let $p(\cdot), q(\cdot) \in P(\Omega), p(\cdot) \not\equiv 1$. Define $D := \{x \in \Omega : p(x) < q(x)\} \neq \emptyset$. Then the following are equivalent:
(i) The modular inequality (54) holds;
(ii) $p(\cdot) \simeq q(\cdot)$ and $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$;
(iii) $p(\cdot) \simeq q(\cdot)$ and there exists $\lambda > 1$ such that

$$\rho_{r(\cdot),D}(1/\lambda) = \int_D \lambda^{-r(x)}\ dx < \infty, \tag{55}$$

where $r(\cdot)$ is the defect exponent defined by $\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{1}{q(x)}$;
(iv) $p(\cdot) \simeq q(\cdot)$ and there exists a measurable function ω, $0 < \omega(\cdot) \leqslant 1$, such that

$$\rho_{p(\cdot),D}(\omega) = \int_D \omega(x)^{p(x)} dx < \infty \tag{56}$$

and

$$\left\|\omega(\cdot)^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)} \cdot \left\|\omega(\cdot)^{-|q(\cdot) - p_+|}\right\|_{L^\infty(D)} < \infty. \tag{57}$$

Corollary $(6.3.10)$[411]: Given a set Ω and $p(\cdot), q(\cdot) \in P(\Omega)$, suppose that either $|\Omega| < \infty$ and $p_+(\Omega) \leqslant q_-(\Omega)$, or $|\Omega| = \infty, p(\cdot) \simeq q(\cdot)$, and (55) holds. If T is any operator that is bounded on $L^p(\Omega)$ for all $1 < p < \infty$, then

$$\int_\Omega |Tf(x)|^{p(x)}\ dx \leqslant c_1 \int_\Omega |f(x)|^{q(x)}\ dx + c_2,$$

with positive constants c_1, c_2 that depend on $p(\cdot), q(\cdot)$ and T but not on f.

The assumption on the operator T is very general and is satisfied by most of the classical operators of harmonic analysis: for example, it holds for Calder´on–Zygmund singular integral operators and square functions. In fact, a close examination of the proof shows that we can assume less: given fixed $p(\cdot)$ and $q(\cdot)$, we only require that the operator is bounded on $L^{p_+}(\Omega)$. As a consequence, we can prove a modular inequality for the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx$$

on variable Lebesgue spaces, using the Plancherel theorem that $\|\hat{f}\|_2 = \|f\|_2$. The importance of this result follows from the fact that natural generalization of the Hausdorff–Young inequality fails in the variable exponent setting. (See [475, Section 5.6.10]

Corollary $(\mathbf{6.3.11})[\mathbf{411}]$: Given $p(\cdot), q(\cdot) \in P(\mathbb{R}^n), p_+ = 2$, suppose $p(\cdot) \precsim q(\cdot)$, and (55) holds. Then

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^{p(\xi)}\, d\xi \leqslant c_1 \int_{\mathbb{R}^n} |f(x)|^{q(x)}\, dx + c_2,$$

with positive constants c_1, c_2 that depend on $p(\cdot)$ and $q(\cdot)$ but not on f.

Corollary $(\mathbf{6.3.12})[\mathbf{411}]$: Given $\Omega \subset \mathbb{R}^n, suppose\ |\Omega| < \infty$. If $1 < p \leqslant q < \infty$, then the following inequality holds

$$\int_\Omega Mf(x)^p\, dx \leqslant c_1 \int_\Omega |f(x)|^q\, dx + c_2, \tag{58}$$

for every $f \in L^q(\Omega)$ and for some positive constants c_1, c_2 depending on $n, p, q, |\Omega|$, but independent of f.

If $|\Omega| = \infty$, then inequality (58) holds if and only if $1 < p = q$. Moreover, if T is an operator that is bounded on $L^p(\Omega), 1 < p < \infty$, then these conditions are sufficient for T to satisfy the modular inequality

$$\int_\Omega |Tf(x)|^p\, dx \leqslant c_1 \int_\Omega |f(x)|^q\, dx + c_2.$$

To prove Theorems (6.3.5) and (6.3.9), we will first prove the following proposition which establishes a necessary condition which for sets Ω of finite measure is also sufficient.

Proposition $(\mathbf{6.3.13})[\mathbf{411}]$: Given $p(\cdot), q(\cdot) \in P(\Omega)$, if the modular inequality (54) holds, then

$$p_+(\Omega) \leqslant q_-(\Omega). \tag{59}$$

As a corollary to Proposition (6.3.13), together with the classical theorem on the boundedness of the maximal operator on $L^p(\Omega), 1 < p < \infty (cf.$ [415]), we immediately get the following generalization of Theorem (6.3.3) to arbitrary domains and unbounded exponent functions.

Corollary $(\mathbf{6.3.14})[\mathbf{411}]$:. Given an open set Ω and $p(\cdot) \in P(\Omega)$, the modular inequality

$$\int_\Omega Mf(x)^{p(x)}\, dx \leqslant c_1 \int_\Omega |f(x)|^{p(x)} dx + c_2,$$

with positive constants c_1, c_2 depending on $n, p(\cdot), q(\cdot)$ and $|\Omega|$ but independent of f, holds if and only if $p(\cdot)$ equals a constant $p > 1$ almost everywhere.

Lemma $(\mathbf{6.3.15})[\mathbf{411}]$: Given a set $\Omega \subseteq \mathbb{R}^n$, let $p(\cdot) \in P(\Omega), q(\cdot) \in P(\Omega)$. Then the following conditions are equivalent:

(i) $p_+(Q) \leqslant q_-(Q)$ for every $Q \in Q_\Omega$;

(ii) $p_+(\Omega) \leqslant q_-(\Omega)$.

Proof: The fact that (ii) implies (i) is easy: for any $Q \in Q_\Omega$ we have

$$p_+(Q) = p_+(Q \cap \Omega) \leqslant p_+(\Omega) \leqslant q_-(\Omega) \leqslant q_-(Q \cap \Omega) = q_-(Q).$$

In order to prove that (i) implies (ii), let $\{Q_n\}_{n\in\mathbb{N}}$ be a countable cover of Ω by elements of Q_Ω. We then have that if $p_+(Q) \leqslant q_-(Q)$ for every $Q \in Q_\Omega$, then

$$p_+(Q_m) \leqslant q_-(Q_n)\ \forall m, n \in \mathbb{N}. \tag{60}$$

To see this, note that for every $m, n \in \mathbb{N}$, there exists a cube $Q_{m,n} \in Q_\Omega$ such that $Q_m \cup Q_n \subseteq Q_{m,n}$. By hypothesis $p_+(Q_{m,n}) \leqslant q_-(Q_{m,n})$, so

$$p_+(Q_m) \leqslant p_+(Q_{m,n}) \leqslant q_-(Q_{m,n}) \leqslant q_-(Q_n).$$

Now, if we first take the supremum over $m \in \mathbb{N}$ and then take the infimum over $n \in \mathbb{N}$, by (60) we get $\sup_{m\in\mathbb{N}} p_+(Q_m) \leqslant \inf_{n\in\mathbb{N}} q_-(Q_n)$. Therefore,

$$p_+(\Omega) = p_+\left(\bigcup_{m\in\mathbb{N}} Q_m\right) = \sup_{m\in\mathbb{N}} p_+(Q_m)$$

$$\leqslant \inf_{n\in\mathbb{N}} q_-(Q_n) = q_-\left(\bigcup_{n\in\mathbb{N}} Q_n\right) = q_-(\Omega).$$

The following argument is inspired by Example (6.3.2) and is similar to the proof of Theorem 1.3 in [476, Thm. 5.1].

Proof : If (59) does not hold, then by Lemma (6.3.15) there exists a cube $Q \in Q_\Omega$ such that $p_+(Q) > q_-(Q)$. Let α, β be such that

$$q_-(Q) < \alpha < \beta < p_+(Q).$$

Let $E_\beta \subset Q \cap \Omega, |E_\beta| > 0$, be such that $p(x) \geqslant \beta$ for $a.e.\, x \in E_\beta$. Similarly, let $E_\alpha \subset Q \cap \Omega, |E_\alpha| > 0$, be such that $q(x) \leqslant \alpha$ for a.$e.\, x \in E_\alpha$. Define $f = \lambda\chi_{E_\alpha}$, where $\lambda > 1$. Then for all $z \in Q$,

$$Mf(z) \geqslant \frac{1}{|Q|}\int_Q |f(y)|dy = \frac{\lambda|E_\alpha|}{|Q|}.$$

Moreover, if $\lambda > |Q|/|E_\alpha|$, then $(\lambda|E_\alpha|/|Q|)^{p(x)} \geqslant (\lambda|E_\alpha|/|Q|)^\beta$ for every $x \in E_\beta$. Hence,

$$\int_\Omega Mf(x)^{p(x)}dx \geqslant \int_{E_\beta}\left(\frac{\lambda|E_\alpha|}{|Q|}\right)^{p(x)} dx \geqslant |E_\beta|\left(\frac{\lambda|E_\alpha|}{|Q|}\right)^\beta.$$

On the other hand,

$$\int_\Omega |f(x)|^{q(x)}\, dx = \int_{E_\alpha} \lambda^{q(x)}\, dx \leqslant |E_\alpha|\lambda^\alpha.$$

Therefore, if (54) holds, then we must have that

$$|E_\beta|\left(\frac{\lambda|E_\alpha|}{|Q|}\right)^\beta \leqslant c_1|E_\alpha|\lambda^\alpha + c_2$$

for all λ sufficiently large, which is a contradiction since $\alpha < \beta$.
By Proposition (6.3.13) we have that if the modular inequality (54) holds, then $p_+(\Omega) \leqslant q_-(\Omega)$. Therefore, it remains to show that this condition is sufficient.
Fix a set Ω and $p(\cdot), q(\cdot) \in P(\Omega)$ such that $p_+(\Omega) \leqslant q_-(\Omega)$, and fix a function f. Given a set $E \subseteq \Omega$, we define

$$I(E) = \int_E Mf(x)^{p(x)}\, dx\ , \qquad F(E) = \int_E |f(x)|^{p_+}\, dx,$$

and

$$D_1(Mf) = \{x \in \Omega : Mf(x) > 1\}, \qquad D_1(f) = \{x \in \Omega : |f(x)| > 1\}.$$

We now estimate as follows:

$$\int_\Omega Mf(x)dx = I(D_1(Mf)) + I(\Omega \setminus D_1(Mf)).$$

We immediately have that $I(\Omega\setminus D_1(Mf)) \leqslant |\Omega|$. On the other hand, since $p(\cdot) \not\equiv 1, p_+ > 1$, so the maximal operator is bounded on $L^{p_+}(\Omega)$. Hence,

$$I(D_1(Mf)) \leqslant \int_{D_1(Mf)} Mf(x)^{p_+}\, dx \leqslant c_{p_+,n}\int_\Omega |f(x)|^{p_+}\, dx = c_{p_+,n}F(\Omega).$$

To estimate $F(\Omega)$ we argue similarly: since $p_+(\Omega) \leqslant q_-(\Omega)$,

$$F(\Omega) = F(D_1(f)) + F(\Omega \setminus D_1(f)) \leqslant \int_{D_1(f)} |f(x)|^{q(x)}\, dx + |\Omega|.$$

If we combine all of these inequalities, we get

$$\int_\Omega Mf(x)\, dx \leqslant c_{p_+,n}\int_\Omega |f(x)|^{q(x)}\, dx + (c_{p_+,n} + 1)|\Omega|.$$

This completes the proof of sufficiency.
We will prove the following chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

[$(i) \Rightarrow (ii)$] We first prove that if the modular inequality (54) holds, then $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$. Since$L^{p(\cdot)}$is a Banach function space, the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is equivalent (cf. [473, Thm. 1.8]) to the set-theoretical inclusion $L^{q(\cdot)}(\Omega) \subseteq L^{p(\cdot)}(\Omega)$. Since $Mf(x) \geqslant |f(x)|\ a.e.$ in Ω, if (54) holds, then $\rho_{p(\cdot),\Omega}(f) \leqslant c_1\rho_{q(\cdot),\Omega}(f) + c_2$. Fix $f \in L^{q(\cdot)}(\Omega)$; then for some $\lambda > 0, \rho_{q(\cdot),\Omega}(f/\lambda) < \infty$. Therefore,

$$\rho_{p(\cdot),\Omega}(f/\lambda) \leqslant c_1\rho_{q(\cdot),\Omega}(f/\lambda) + c_2 < \infty,$$

and so $f \in L^{p(\cdot)}(\Omega)$. We now prove that if (54) holds, then $p(\cdot) \preccurlyeq q(\cdot)$. Given any measurable set $E \in F_\Omega$ and any measurable function $f : E \subseteq \Omega \to \mathbb{R}$, (54) implies that

$$\int_E |f(x)|^{p(x)}\, dx \leqslant c_1\int_E |f(x)|^{q(x)}\, dx + c_2, \tag{61}$$

with $c_1, c_2 > 0$ the same constants. Fix $E \in F_\Omega$ and define $f(x) = \lambda \cdot \chi_{B_\delta \cap E(x)}$, $0 < \lambda < 1$and $B_\delta = B(0,\delta)$. Since $0 < \lambda < 1$, for $x \in E$, $\lambda^{p_+(E)} \leqslant \lambda^{p(x)}$ and $\lambda^{q(x)} \leqslant \lambda^{q_-(E)}$. Therefore, by (61),

$$|E \cap B_\delta|\lambda^{p_+(E)} \leqslant \int_{E\cap B_\delta} \lambda^{p(x)}\ dx$$

$$\leqslant c_1 \int_{E\cap B_\delta} \lambda^{q(x)}\, dx + c_2 \leqslant c_1|E \cap B_\delta|\lambda^{q_-(E)} + c_2.$$

Since $|E \cap B_\delta| \to \infty$ as $\delta \to \infty$, we get that $\lambda^{p_+(E)}\,(E) \leqslant c_1\lambda^{q_-(E)} + c_2|E \cap B_\delta|^{-1}$ for δ sufficiently large. If we take the limit as $\delta \to \infty$, we get that if (54) holds, then

$$\lambda^{p_+(E)} \leqslant c_1\lambda^{q_-(E)} \quad \forall 0 < \lambda < 1.$$

Since $p_+(E) \leqslant q_-(E)$ we must have that $p_+(E) = q_-(E)$ and $c_1 \geqslant 1$.
Finally, since by Proposition (6.3.13), $p_+(\Omega) \leqslant q_-(\Omega)$, and since $p_+(E) \leqslant p_+(\Omega) \leqslant q_-(\Omega) \leqslant q_-(E)$, we get that $p(\cdot) \,\hat{\approx}\, q(\cdot)$.
$[(ii) \Rightarrow (iii)]$ As noted above, this implication follows from the fact that the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is equivalent to assuming $p(x) \leqslant q(x)$ and (55) holds. (See [472, Thm. 2.45].)
$[(iii) \Rightarrow (iv)]$ We explicitly construct the function ω. Since $p(\cdot) \,\hat{\approx}\, q(\cdot)$, we claim that there exists $\kappa > 1$ such that $|E_{q(\cdot),\kappa}| < \infty$, where $E_{q(\cdot),\kappa} = \{x \in \Omega : q(x) > \kappa\}$. For if not, then for all $\kappa > 1$, $|E_{q(\cdot),\kappa}| = \infty$. In particular, if we set $\kappa = p_+(\Omega) + 1$,then $E_{q(\cdot),\kappa} \in F_\Omega$ and $q_-(E_{q(\cdot),\kappa}) > p_+(\Omega) \geqslant p_+(E_{q(\cdot),\kappa})$, a contradiction.
Fix such a κ and define

$$\omega(x) := \begin{cases} \lambda^{-r(x)/p(x)} & x \in D\backslash E_{q(\cdot),\kappa}, \\ 1 & x \in D \cap E_{q(\cdot),\kappa}, \end{cases}$$

where $r(\cdot)$ is the defect exponent defined by $\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{1}{q(x)}$. Since $\lambda > 1$, we have that $0 < \omega(\cdot) \leqslant 1$ and

$$\omega(\cdot)^{-|p_+ - p(\cdot)|} = \lambda^{\frac{p_+ - p(\cdot)}{q(\cdot)-p(\cdot)} q(\cdot)} \leqslant \lambda^{q(\cdot)} \leqslant \lambda^{\kappa}\ on\ D\backslash E_{q(\cdot),\kappa},$$

$$\omega(\cdot)^{-|q(\cdot) - p_+|} = \lambda^{\frac{q(\cdot) - p_+}{q(\cdot)-p(\cdot)} q(\cdot)} \leqslant \lambda^{q(\cdot)} \leqslant \lambda^{\kappa}\ on\ D\backslash E_{q(\cdot),\kappa}.$$

Moreover, $\omega(\cdot)^{-|p_+ - p(\cdot)|} = \omega(\cdot)^{-|q(\cdot)-p_+|} = 1 \leqslant \lambda^\kappa$ on the set $D \cap E_{q(\cdot),\kappa}$ and therefore (57) holds. Finally, to prove (56) we estimate as follows:

$$\rho_{p(\cdot),D}(\omega) = \int_{D\backslash E_{q(\cdot),\kappa}} \lambda^{-r(x)}\, dx + |E_{q(\cdot),\kappa}| \leqslant \int_D \lambda^{-r(x)}\, dx + |E_{q(\cdot),\kappa}| < \infty.$$

$[(iv) \Rightarrow (i)]$ The proof of this implication is similar to the proof of sufficiency in the proof of Theorem (6.3.5). However, since $|\Omega| = \infty$ we need to introduce ω and use $\rho_{p(\cdot),D}(\omega)$ in place of $|\Omega|$. As before, given a measurable function f and a measurable set $E \subseteq \Omega$, define

$$I(E) = \int_E Mf(x)^{p(x)}\, dx, \qquad F(E) = \int_E |f(x)|^{p_+}\ dx.$$

Recall that $D = \{x \in \Omega : p(x) < q(x)\}$ and write

$$\int_{\Omega} Mf(x)^{p(x)}\, dx = I(D) + I(\Omega \setminus D).$$

Since $p_+ \leqslant q_-$, we have $p(\cdot) = p_+ = q_- = q(\cdot)$ on $\Omega \setminus D$. Therefore, since $p(\cdot) \not\equiv 1, p_+ > 1$, so the maximal operator is bounded on $L^{p_+}(\Omega)$. Hence,

$$I(\Omega \setminus D) = \int_{\Omega\setminus D} Mf(x)^{p_+}\, dx \leqslant c_{p_+,n}\, F(\Omega).$$

To estimate $I(D)$, define $D_\omega(Mf) = \{x \in D : Mf(x) > \omega(x)\}$ where ω is the function from our hypothesis (iv). Then

$$I(D) = \int_{D\setminus D_\omega(Mf)} Mf(x)^{p(x)}\, dx + \int_{D_\omega(Mf)} Mf(x)^{p(x)}\, dx$$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(Mf)} \left(\frac{Mf(x)}{\omega(x)}\right)^{p(x)} \omega(x)^{p(x)}\, dx.$$

Since $Mf(\cdot)/\omega(\cdot) > 1$ on $D_\omega(Mf)$,

$$\leqslant \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(Mf_2)} \left(\frac{Mf(x)}{\omega(x)}\right)^{p_+} \omega(x)^{p(x)} dx$$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)} \int_D (Mf(x))^{p_+}\, dx.$$

Again since M is bounded on $L^{p_+}(\Omega)$,

$$\leqslant \rho_{p(\cdot),D}(\omega) + c_{n,p_+} \cdot \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)} F(\Omega).$$

If we combine the above inequalities we get

$$I(\Omega) \leqslant \left[c_{n,p_+}\left(1 + \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)}\right)\right] F(\Omega) + \rho_{p(\cdot),D}(\omega), \tag{62}$$

so to complete the proof we need to estimate $F(\Omega) = F(D) + F(\Omega \setminus D)$. As before we have $p(\cdot) = p_+ = q_- = q(\cdot)$ on $\Omega \setminus D$, so

$$F(\Omega \setminus D) = \int_{\Omega\setminus D} |f(x)|^{p_+} = \int_{\Omega\setminus D} |f(x)|^{q(x)}\, dx.$$

To estimate $F(D)$, let $D_\omega(f) = \{x \in D : |f(x)| > \omega(x)\}$. Since $0 < \omega \leqslant 1$ and $p_+ \geqslant p(\cdot)$, we have $\rho_{p_+,D}(\omega) \leqslant \rho_{p(\cdot),D}(\omega)$. Therefore,

$$F(D) = \int_{D\setminus D_\omega(f)} |f(x)|^{p_+}\, dx + \int_{D_\omega(f)} |f(x)|^{p_+}\, dx$$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(f)} \left(\frac{|f(x)|}{\omega(x)}\right)^{p_+} \cdot \omega(x)^{p_+} dx.$$

Since $|f(\cdot)|/\omega(\cdot) > 1$ on $D_\omega(f)$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(f)} \left(\frac{|f(x)|}{\omega(x)}\right)^{q(x)} \cdot \omega(x)^{p_+} dx$$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)} \int_D |f(x)|^{q(x)}\,dx.$$

If we combine the previous two estimates, we get

$$F(\Omega) \leqslant \int_{\Omega\setminus D} |f(x)|^{q(x)}\,dx + \rho_{p(\cdot),D}(\omega) + \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)} \int_D |f(x)|^{q(x)}\,dx$$
$$\leqslant \left(1 + \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)}\right) \int_\Omega |f(x)|^{q(x)}\,dx + \rho_{p(\cdot),D}(\omega).$$

Together with inequality (62) this gives us the modular inequality (54). This completes the proof.

Corollary (6.3.16)[492]: Given a set $\Omega \subseteq \mathbb{R}^n$, let $p(\cdot) \in P(\Omega), q(\cdot) \in P(\Omega)$. Then the following conditions are equivalent:

(i) $p_+(Q_r) \leqslant q_-(Q_r)$ for every $Q_r \in (Q_r)_\Omega$;

(ii) $p_+(\Omega) \leqslant q_-(\Omega)$.

Proof. The fact that (ii) implies (i) is easy: for any $Q \in Q_\Omega$ we have

$$p_+(Q_r) = p_+(Q_r \cap \Omega) \leqslant p_+(\Omega) \leqslant q_-(\Omega) \leqslant q_-(Q_r \cap \Omega) = q_-(Q_r).$$

In order to prove that (i) implies (ii), let $\{(Q_r)_n\}_{n\in\mathbb{N}}$ be a countable cover of Ω by elements of $(Q_r)_\Omega$. We then have that if $p_+(Q_r) \leqslant q_-(Q_r)$ for every $Q_r \in (Q_r)_\Omega$, then

$$p_+((Q_r)_m) \leqslant q_-\big((Q_r)_{m,n}\big)\ \forall m,n \in \mathbb{N}.$$

To see this, note that for every $m,n \in \mathbb{N}$, there exists a cube $(Q_r)_{m,n} \in (Q_r)_\Omega$ such that $(Q_r)_m \cup (Q_r)_n \subseteq (Q_r)_{m,n}$. By hypothesis $p_+((Q_r)_{m,n}) \leqslant q_-((Q_r)_{m,n})$, so

$$p_+((Q_r)_m) \leqslant p_+((Q_r)_{m,n}) \leqslant q_-((Q_r)_{m,n}) \leqslant q_-((Q_r)_n).$$

Now, if we first take the supremum over $m \in \mathbb{N}$ and then take the infimum over $n \in \mathbb{N}$, by (50) we get $\sup_{m\in\mathbb{N}} p_+((Q_r)_m) \leqslant \inf_{n\in\mathbb{N}} q_-((Q_r)_n)$. Therefore,

$$p_+(\Omega) = p_+\left(\bigcup_{m\in\mathbb{N}} (Q_r)_m\right) = \sup_{m\in\mathbb{N}} p_+((Q_r)_m)$$
$$\leqslant \inf_{n\in\mathbb{N}} q_-((Q_r)_n) = q_-\left(\bigcup_{n\in\mathbb{N}} (Q_r)_n\right) = q_-(\Omega).$$

The following argument is inspired and is similar to the proof of Theorem 1.3 in [468, Thm. 5.1].

Corollary (6.3.17)[492]: Given $p(\cdot), q(\cdot) \in P(\Omega)$, if the modular inequality holds, then

$$p_+(\Omega) \leqslant q_-(\Omega).$$

As a corollary , together with the classical theorem on the boundedness of the maximal operator on $L^{1+\varepsilon}(\Omega), 0 < \varepsilon < \infty (cf. [465])$, we immediately get the following generalization to arbitrary domains and unbounded exponent functions.

Proof : If (52) does not hold, then there exists a cube $Q_r \in (Q_r)_\Omega$ such that $p_+(Q_r) > q_-(Q_r)$. Let $\alpha, \alpha+\varepsilon$ be such that

$$q_-(Q_r) < \alpha < \alpha + \varepsilon < p_+(Q_r).$$

Let $E_{\alpha+\varepsilon} \subset Q_r \cap \Omega, |E_{\alpha+\varepsilon}| > 0$, be such that $p(x^2) \geqslant \alpha + \varepsilon$ for $a.e.x \in E_{\alpha+\varepsilon}$. Similarly, let $E_\alpha \subset Q_r \cap \Omega, |E_\alpha| > 0$, be such that $q(x^2) \leqslant \alpha$ for a.$e.x \in E_\alpha$. Define $f^r = \lambda\chi_{E_\alpha}$, where $\lambda > 1$. Then for all $x + 2\varepsilon \in Q_r$,

$$M\left(\sum f^r\,(x+2\varepsilon)\right) \geqslant \frac{1}{|Q_r|}\int_{Q_r}\sum |f^r(x+\varepsilon)\,|d(x+\varepsilon) = \frac{\lambda|E_\alpha|}{|Q_r|}\ .$$

Moreover, if $\lambda > |Q_r|/|E_\alpha|$, then $(\lambda|E_\alpha|/|Q_r|)^{p(x^2)} \geqslant (\lambda|E_\alpha|/|Q_r|)^{\beta}$ for every $x \in E_{\alpha+\varepsilon}$. Hence,

$$\int_\Omega M\left(\sum f^r\,(x)^{p(x^2)}\right)dx \ \geqslant \int_{E_{\alpha+\varepsilon}}\left(\frac{\lambda|E_\alpha|}{|Q_r|}\right)^{p(x^2)} dx \ \geqslant |E_{\alpha+\varepsilon}|\left(\frac{\lambda|E_\alpha|}{|Q_r|}\right)^{\alpha+\varepsilon}.$$

On the other hand,

$$\int_\Omega \sum |f^2(x)|^{q(x^2)}\ dx \ = \int_{E_\alpha} \lambda^{q(x^2)}\ dx \ \leqslant |E_\alpha|\lambda^\alpha.$$

Therefore, holds, then we must have that

$$|E_{\alpha+\varepsilon}|\left(\frac{\lambda|E_\alpha|}{|Q_r|}\right)^{\alpha+\varepsilon} \leqslant (1+\varepsilon)|E_\alpha|\lambda^\alpha + 1 + 2\varepsilon$$

for all λ sufficiently large, which is a contradiction since $\varepsilon > 0$.

Corollary (6.3.18)[492]: Given a set $\Omega \subseteq \mathbb{R}^n, 0 < |\Omega| < \infty$, let $p(\cdot), q(\cdot) \in P(\Omega), p(\cdot) \not\equiv 1$. Then the modular inequality holds if and only if $p_+(\Omega) \leqslant q_-(\Omega)$.

We show, the assumption that $|\Omega| < \infty$ is critical . , we first give the following example.

Proof : By Proposition (6.318) we have that if the modular inequality holds, then $p_+(\Omega) \leqslant q_-(\Omega)$. Therefore, it remains to show that this condition is sufficient.

Fix a set Ω and $p(\cdot), q(\cdot) \in P(\Omega)$ such that $p_+(\Omega) \leqslant q_-(\Omega)$, and fix a sequence of functions f^r. Given a set $E \subseteq \Omega$, we define

$$I(E) = \int_E \sum Mf^r(x)^{p(x^2)}\ dx\ , \qquad F(E) = \int_E \sum |f^r(x)|^{p_+}\ dx,$$

and

$$D_1(\left(M\sum f^r\right)) = \{x \in \Omega :\ M\left(\sum f^r\,(x)\right) > 1\},$$

$$D_1(f^r) = \{x \in \Omega : \sum |f^r(x)| > 1\}.$$

We now estimate as follows:

$$\int_\Omega \sum Mf^r(x)dx \ = \sum I(D_1(Mf^r)) + I(\Omega \setminus D_1(M\sum f^r)).$$

We immediately have that $I(\Omega \setminus D_1(M(\sum f^r))) \leqslant |\Omega|$. On the other hand, since $p(\cdot) \not\equiv 1, p_+ > 1$, so the maximal operator is bounded on $L^{p_+}(\Omega)$. Hence,

$$I(D_1(M\left(\sum f^r\right))) \leqslant \int_{D_1(Mf^r)} \sum Mf^r(x)^{p_+}\,dx \leqslant c_{p_+,n}\int_\Omega \sum |f^r(x)|^{p_+}\,dx$$
$$= c_{p_+,n}F(\Omega).$$
To estimate $F(\Omega)$ we argue similarly: since $p_+(\Omega) \leqslant q_-(\Omega)$,
$$F(\Omega) = F(D_1(\sum f^r)) + F(\Omega \backslash D_1(\sum f^r)) \leqslant \int_{D_1(f^r)} \sum |f^r(x)|^{q(x^2)}\,dx + |\Omega|.$$
If we combine all of these inequalities, we get
$$\int_\Omega \sum Mf^r(x)\,dx \leqslant c_{p_+,n}\int_\Omega \sum |f^r(x)|^{q(x^2)}\,dx + (c_{p_+,n}+1)|\Omega|.$$
This completes the proof of sufficiency.

Corollary (6.3.19)[492]: Given a set $\Omega \subseteq \mathbb{R}^n, |\Omega| = \infty$, let $p(\cdot), q(\cdot) \in P(\Omega), p(\cdot) \not\equiv 1.$ Define $D := \{x \in \Omega : p(x^2) < q(x^2)\} \neq \emptyset$. Then the following are equivalent:

(i) The modular inequality holds;

(ii) $p(\cdot) \precsim q(\cdot)$ and $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$;

(iii) $p(\cdot) \precsim q(\cdot)$ and there exists $\lambda > 1$ such that
$$\rho_{r(\cdot),D}(1/\lambda) = \int_D \lambda^{-r(x^2)}\,dx < \infty,$$
where $r(\cdot)$ is the defect exponent defined by $\frac{1}{r(x^2)} = \frac{1}{p(x^2)} - \frac{1}{q(x^2)}$;

(iv) $p(\cdot) \precsim q(\cdot)$ and there exists a measurable function $\omega, 0 < \omega(\cdot) \leqslant 1$, such that
$$\rho_{p(\cdot),D}(\omega) = \int_D \omega(x)^{p(x^2)}dx < \infty$$
and
$$\left\|\omega(\cdot)^{-|p_+-p(\cdot)|}\right\|_{L^\infty(D)} \cdot \left\|\omega(\cdot)^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)} < \infty.$$
Proof : We will prove the following chain of implications:
$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$
$[(i) \Rightarrow (ii)]$ We first prove that if the modular inequality (55) holds, then $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$. Since$L^{p(\cdot)}$is a Banach function space, the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is equivalent (cf. [463, Thm. 1.8]) to the set-theoretical inclusion $L^{q(\cdot)}(\Omega) \subseteq L^{p(\cdot)}(\Omega)$. Since $M(\sum f^r(x)) \geqslant \sum |f^r(x)|$ $a.e.$ in Ω, if holds, then $\rho_{p(\cdot),\Omega}(\sum f^r) \leqslant (1+\varepsilon)\rho_{q(\cdot),\Omega}(\sum f^r) + 1 + 2\varepsilon$. Fix $f^r \in L^{q(\cdot)}(\Omega)$; then for some $\lambda > 0, \rho_{q(\cdot),\Omega}(\sum(f^r/\lambda)) < \infty$. Therefore,
$$\rho_{p(\cdot),\Omega}(\sum f^r v/\lambda) \leqslant (1+\varepsilon)\rho_{q(\cdot),\Omega}(\sum f^r/\lambda) + 1 + 2\varepsilon < \infty,$$
and so $f^r \in L^{p(\cdot)}(\Omega)$. We now prove that holds, then $p(\cdot) \precsim q(\cdot)$. Given any measurable set $E \in F_\Omega$ and any measurable sequence of functions $f^r : E \subseteq \Omega \to \mathbb{R}$, (55) implies that
$$\int_E \sum |f^r(x)|^{p(x^2)}\,dx \leqslant (1+\varepsilon)\int_E \sum |f^r(x)|^{q(x^2)}\,dx + 1 + 2\varepsilon,$$

with $\varepsilon > 0$ the same constant. Fix $E \in F_\Omega$ and define $f^r(x) = \lambda \cdot \chi_{B_\delta \cap E(x)}, 0 < \lambda <$ 1and $B_\delta = B(0,\delta)$. Since $0 < \lambda < 1$, for $x \in E, \lambda^{p_+(E)} \leqslant \lambda^{p(x^2)}$ and $\lambda^{q(x^2)} \leqslant \lambda^{q_-(E)}$. Therefore, ,

$$|E \cap B_\delta| \lambda^{p_+(E)} \leqslant \int_{E\cap B_\delta} \lambda^{p(x^2)}\, dx$$

$$\leqslant (1+\varepsilon)\int_{E\cap B_\delta} \lambda^{q(x^2)}\, dx + 1 + 2\varepsilon \leqslant (1+\varepsilon)|E \cap B_\delta|\lambda^{q_-(E)} + 1 + 2\varepsilon.$$

Since $|E \cap B_\delta| \to \infty$ as $\delta \to \infty$,We get that $\lambda^{p_+(E)}(E) \leqslant (1+\varepsilon)\lambda^{q_-(E)} + (1+2\varepsilon)|E \cap B_\delta|^{-1}$ for δ sufficiently large. If we take the limit as $\delta \to \infty$, we get that holds, then

$$\lambda^{p_+(E)} \leqslant (1+\varepsilon)\lambda^{q_-(E)} \quad \forall 0 < \lambda < 1.$$

Since $p_+(E) \leqslant q_-(E)$ we must have that $p_+(E) = q_-(E)$ and $\varepsilon \geqslant 0$.

Finally, since by Proposition (6.3.13), $p_+(\Omega) \leqslant q_-(\Omega)$, and since $p_+(E) \leqslant p_+(\Omega) \leqslant q_-(\Omega) \leqslant q_-(E)$, we get that $p(\cdot) \precsim q(\cdot)$.

$[(ii) \Rightarrow (iii)]$ As noted above, this implication follows from the fact that the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is equivalent to assuming $p(x^2) \leqslant q(x^2)$ holds. (See [468, Thm. 2.45].)

$[(iii) \Rightarrow (iv)]$ We explicitly construct the function ω. Since $p(\cdot) \precsim q(\cdot)$, we claim that there exists $\varepsilon > 0$ such that $|E_{q(\cdot),1+\varepsilon}| < \infty$, where $E_{q(\cdot),\kappa} = \{x \in \Omega : q(x^2) > 1 + \varepsilon\}$. For if not, then for all $\varepsilon > 0, |E_{q(\cdot),1+\varepsilon}| = \infty$. In particular, if we set $1+\varepsilon = p_+(\Omega) + 1$,then $E_{q(\cdot),\kappa} \in F_\Omega$ and $q_-(E_{q(\cdot),1+\varepsilon}) > p_+(\Omega) \geqslant p_+(E_{q(\cdot),1+\varepsilon})$, a contradiction.

Fix such a $1+\varepsilon$ and define

$$\omega(x) := \begin{cases} \lambda^{-r(x^2)/p(x^2)} & x \in D \backslash E_{q(\cdot),1+\varepsilon}, \\ 1 & x \in D \cap E_{q(\cdot),1+\varepsilon}, \end{cases}$$

where $r(\cdot)$ is the defect exponent defined by $\frac{1}{r(x^2)} = \frac{1}{p(x^2)} - \frac{1}{q(x^2)}$. Since $\lambda > 1$, we have that $0 < \omega(\cdot)$
$\leqslant 1$ and

$$\omega(\cdot)^{-|p_+ - p(\cdot)|} = \lambda^{\frac{p_+ - p(\cdot)}{q(\cdot) - p(\cdot)} q(\cdot)} \leqslant \lambda^{q(\cdot)} \leqslant \lambda^{1+\varepsilon}\ on\ D \backslash E_{q(\cdot),1+\varepsilon},$$

$$\omega(\cdot)^{-|q(\cdot) - p_+|} = \lambda^{\frac{q(\cdot) - p_+}{q(\cdot) - p(\cdot)} q(\cdot)} \leqslant \lambda^{q(\cdot)} \leqslant \lambda^{1+\varepsilon}\ on\ D \backslash E_{q(\cdot),1+\varepsilon}.$$

Moreover, $\omega(\cdot)^{-|p_+ - p(\cdot)|} = \omega(\cdot)^{-|q(\cdot) - p_+|} = 1 \leqslant \lambda^{1+\varepsilon}$ on the set $D \cap E_{q(\cdot),1+\varepsilon}$ and therefore holds. Finally, to prove we estimate as follows:

$$\rho_{p(\cdot),D}(\omega) = \int_{D \backslash E_{q(\cdot),1+\varepsilon}} \lambda^{-r(x^2)}\, dx + |E_{q(\cdot),1+\varepsilon}| \leqslant \int_D \lambda^{-r(x^2)}\, dx + |E_{q(\cdot),1+\varepsilon}| < \infty.$$

$[(iv) \Rightarrow (i)]$ The proof of this implication is similar to the proof of sufficiency . However, since $|\Omega| = \infty$ we need to introduce ω and use $\rho_{p(\cdot),D}(\omega)$ in place of $|\Omega|$. As before, given a measurable sequence of function f^r and a measurable set $E \subseteq \Omega$, define

$$I(E) = \int_E \sum Mf^r(x)^{p(x^2)}\, dx, \qquad F(E) = \int_E \sum |f^r(\, x)|^{p_+}\ dx.$$

Recall that $D\ = \{x\ \in\ \Omega :\ p(x^2)\ <\ q(x^2)\}$ and write

$$\int_\Omega \sum Mf^2(x)^{p(x^2)}\, dx\ =\ I(D) + I(\Omega \setminus D).$$

Since $p_+ \leqslant q_-$, we have $p(\cdot)\ =\ p_+ = q_-\ =\ q(\cdot)$ on $\Omega \setminus D$. Therefore, since $p(\cdot) \not\equiv 1, p_+ >\ 1$, so the maximal operator is bounded on $L^{p_+}(\Omega)$. Hence,

$$I(\Omega \setminus D) = \int_{\Omega \setminus D} \sum Mf^2(x)^{p_+}\ dx\ \leqslant\ c_{p_+,n}\, F(\Omega).$$

To estimate $I(D)$, define $D_\omega(M(\sum f^r)) = \{x\ \in\ D\ :\ M(\sum f^r)(x)\ >\ \omega(x)\}$ where ω is the function from our hypothesis $(\, iv)$. Then

$$I(D) = \int_{D \setminus D_\omega(Mf^r)} \sum Mf^r(x)^{p(x^2)}\ dx + \int_{D_\omega(Mf^r)} \sum Mf^r(x)^{p(x^2)}\ dx$$

$$\leqslant\ \rho_{p(\cdot),D}(\omega) + \int_{D_\omega \sum(Mf^r)} \left(\frac{Mf^r(x)}{\omega(x)}\right)^{p(x^2)}\ \omega(x)^{p(x^2)}\, dx.$$

Since $M(\sum f^r\ (\cdot))/\omega(\cdot)\ >\ 1$ on $D_\omega(\sum(Mf^r))$,

$$\leqslant \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(Mf_2^r)} \sum \left(\frac{Mf^r(x)}{\omega(x)}\right)^{p_+}\ \omega(x)^{p(x^2)} dx$$

$$\leqslant\ \rho_{p(\cdot),D}(\omega) + \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)} \int_D \sum (Mf^r(x))^{p_+}\, dx.$$

Again since M is bounded on $L^{p_+}(\Omega)$,

$$\leqslant\ \rho_{p(\cdot),D}(\omega) + c_{n,p_+} \cdot \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)} F(\Omega).$$

If we combine the above inequalities we get

$$I(\Omega) \leqslant \left[c_{n,p_+}\left(1 + \left\|\omega^{-|p_+ - p(\cdot)|}\right\|_{L^\infty(D)}\right)\right] F(\Omega) + \rho_{p(\cdot),D}(\omega),$$

so to complete the proof we need to estimate $F(\Omega)\ =\ F(D) +\ F(\Omega \setminus D)$. As before we have $p(\cdot)\ =\ p_+\ =\ q_-\ =\ q(\cdot)$ on $\Omega \setminus D$, so

$$F(\Omega \setminus D) = \int_{\Omega \setminus D} \sum |f^r(x)|^{p_+}\ = \int_{\Omega \setminus D} \sum |f^r(x)|^{q(x^2)}\, dx.$$

To estimate $F(D)$, let $D_\omega(f^r)\ =\ \{x\ \in\ D :\ |f^r(x)|\ >\ \omega(x)\}$. Since $0\ <\ \omega\ \leqslant\ 1$ and $p_+ \geqslant\ p(\cdot)$, we have $\rho_{p_+,D}(\omega) \leqslant\ \rho_{p(\cdot),D}(\omega)$. Therefore,

$$F(D) = \int_{D \setminus D_\omega(f^r)} \sum |f^r(x)|^{p_+}\ dx + \int_{D_\omega(f^r)} \sum |f^r(x)|^{p_+}\ dx$$

$$\leqslant\ \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(f^r)} \sum \left(\frac{|f^r(x)|}{\omega(x)}\right)^{p_+} \cdot \omega(x)^{p_+} dx.$$

Since $|f^r(\cdot)|/\omega(\cdot)\ >\ 1$ on $D_\omega(f^r)$

$$\leqslant\ \rho_{p(\cdot),D}(\omega) + \int_{D_\omega(f^r)} \sum \left(\frac{|f^r(x)|}{\omega(x)}\right)^{q(x^2)} \cdot \omega(x)^{p_+} dx$$

$$\leqslant \rho_{p(\cdot),D}(\omega) + \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)} \int_D \sum |f^r(x)|^{q(x^2)}\, dx.$$

If we combine the previous two estimates, we get

$$F(\Omega) \leqslant \int_{\Omega\setminus D} \sum |f^r(x)|^{q(x^2)}\, dx + \rho_{p(\cdot),D}(\omega)$$

$$+ \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)} \int_D \sum |f^r\,(x)|^{q(x^2)}\, dx$$

$$\leqslant \left(1 + \left\|\omega^{-|q(\cdot)-p_+|}\right\|_{L^\infty(D)}\right) \int_\Omega \sum |f^r(x)|^{q(x^2)}\, dx + \rho_{p(\cdot),D}(\omega).$$

Together with inequality this gives us the modular inequality . This completes the proof.

List of Symbols

Symbol	page
L^p: Lebesgue space	1
Meas: measure	1
L^∞: essential lebesgue space	1
$W^{1,N}$:sobolev space	1
Osc: Oscillation	1
$W_0^{1,p}$:sobolev space	1
Sup :Supremum	2
L^1:Lebesgue intergral on real line	4
Ess: essential	4
L^r: Lebesgue space	4
Inf: infimum	4
Loc:locally	4
PSR: Poincar Sobolev rearrangement	5
a.e: almost every where	17
Max: maximum	20
PDES: partial differential equations	20
Lip: Lipschitz	34
$L^{p(x)}$: Lebesgue space with variable exponent	34
AMDS: Almost monotone decreasing sequence	37
AMLS: Almost monotone increasing sequence	37
$W^{p(x)}$: Sobolev space with variable exponent	38
$H^{p(.)}$:Hardy space of variable exponent	50
$L_w^{p(.)}$:Lebesgue Space with avariable exponent with a weight	69
$W_{p(.)w}^a$:Sobolev Space with avariable exponent with a weight	69
Supp: support	96
L^q: Dual of Lebesgue space	119
u.s.c: upper strictly convex	126
l.s.c: lower strictly convex	126
Int : interior	175
Comp: compact	175
Diag: diagonal	209
BMO: Bounded Mean Osciution	226

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