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Reducing Subspaces and Sarason Toeplitz Product Problem with Theorem of Brown–Halmos on Fock and Bergman Spaces

الفضاءات الجزئية المختزلة ومسألة ناتج ساراسون تبوليتز مع مبرهنة بروين – هالموس على فضاءات فوك – بيرجمان

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family

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Abstract

The classification of reducing subspaces of a class of multiplication operators, analytic multipliers for a class of Toeplitz operators, tensor products of weighted shifts and a class of non-analytic Toeplitz operators on the Bergman space by the Hardy space of the Bidisk and polydisk are considered. We show the products products of Hankel and Toeplitz operators with Sarason's Toeplitz product problem on the Bergman and a class of Fock spaces. We study the finite rank commentators, perturbation and semicommutators of Toeplitz operators with harmonic symbols and Bergman space. We give a theorem of Brown-Halmos type for Bergman space of Toeplitz operators and modulo finite rank operators.

الخلاصة

قمنا باعتبار التصنيف للفضاءات الجزئية المختزلة الى عائلة المؤثرات الضربية والمضاريب التحليلية لأجل عائلة مؤثرات تبوليتز و نواتج تنسور للإزاحات المرجحة وعائلة مؤثرات تبوليتز غير التحليلية على فضاء ببرجمان بواسطة فضاء هاردي إلى القرص الثنائي والقرص المتعدد. تم توضيح نواتج مؤثرات هانكل وتبوليتز مع مسالة ناتج تبوليتز ساراسون على فضاءات ببرجمان وعائلة فضاءات فوك. قمنا بدراسة مبدلات الرتبة المنتهية والارتجاج وشبه المبدلات لمؤثرات تبوليتز مع الرموز التوافقية وفضاء بيرجمان. قمنا بإعطاء المبرهنة نوع براون –هالموس لأجل فضاء بيرجمان لمؤثرات تبوليتز وبمقياس مؤثرات الرتبة المنتهية.

Introduction

In Douglas et al. (2011) [4] some incisive results are obtained on the structure of the reducing subspaces for the multiplication operator M_φ by a finite Blaschke product φ on the Bergman space on the unit disk. In particular, the linear dimension of the commutant, $A_\varphi = \{M_\varphi, M_\varphi^*\}$, is shown to equal the number of connected components of the Riemann surface, $\varphi^{-1} \circ \varphi$. Using techniques from Douglas et al. (2011) [4] and a uniformization result that expresses φ as a holomorphic covering map in a neighborhood of the boundary of the disk. We completely characterize the reducing subspaces of $T_{z_1^N z_2^M}$ on $A_\alpha^2(D^2)$ where $\alpha > -1$ and N, M are positive integers with $N \neq M$, and show that the minimal reducing subspaces of $T_{z_1^N z_2^M}$ on the unweighted Bergman space and on the weighted Bergman space are different.

We consider the question for which square integrable analytic functions f and g on the unit disk the densely defined products $T_f T_g$ are bounded on the Bergman space. We show results analogous to those obtained by [17] for such Toeplitz products on the Hardy space. Let $m(t)dt$ be a positive measure on $\mathbb{R}^+$. We investigate the relations among the growth of γ_n, the growth of its moment sequence $\{\gamma_n\}$, the growth of its Bergman kernel function $K(x) = \sum \gamma_n^{-1}$, and the growth of the kernel function associated to the measure $K(t)^{-1} m(t)dt$. We consider Hankel operators $H_{\overline{f}}$ with antiholomorphic symbol $\overline{f}$ on the generalized Fock space $A^2(\mu_m)$, where μ_m is the measure with weight $e^{-|z|^m}$, $m > 0$ with respect to the Lebesgue measure in $\mathbb{C}^n$. We show that $H_{\overline{f}}$ is bounded if and only if f is a polynomial of degree at most $\frac{m}{2}$.

We consider the question of when the semi-commutator $T_{fg} - T_f T_g$ on the Bergman space with bounded harmonic symbols is compact. Several conditions equivalent to compactness of $T_{fg} - T_f T_g$ are given. We study the analogues of the Brown–Halmos theorem for Toeplitz operators on the Bergman space. We show that for f and g harmonic, $T_f T_g = T_h$ only in the tribyl case, provided that h is of class C^2 with the invariant laplacian bounded. Here the tribyl cases are $\overline{f}$ or g holomorphic. From this we conclude that the zeroproduct problem for harmonic symbols has only the tribyl solution. We completely characterize finite rank semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols on the Bergman space.

A unilateral weighted shift A is said to be simple if its weight sequence $\{\alpha_n\}$ satisfies $\nabla^3(\alpha_n^2) \neq 0$ for all $n \geq 2$. We show that if A and B are two simple unilateral weighted shifts, then $A \otimes I + I \otimes B$ is reducible if and only if A and B are unitarily equivalent. We completely characterize all the reducing subspaces for a class of non-analytic Toeplitz operators with symbol $\phi(z,w) = \alpha z^k + \beta \overline{w}^l$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 0$.

Hankel operators with anti-holomorphic symbols are studied for a large class of weighted Fock spaces on $\mathbb{C}^n$. The weights defining these Hilbert spaces are radial and subject to a mild smoothness condition. In addition, it is assumed that the weights decay at least as fast as the classical Gaussian weight. The main result says that a Hankel operator on such a Fock space is bounded if and only if the symbol belongs to a certain BMOA space, defined by the Berezin transform. The latter space coincides with a corresponding Bloch space which is defined by means of the Bergman metric. Sarason's Toeplitz product problem asks when the operator $T_u T_{\bar{v}}$ is bounded on various Hilbert spaces of analytic functions, where u and v are analytic. The problem is highly nontribyl for Toeplitz operators on the Hardy space and the Bergman space (even in the case of the unit disk).

Given a complex Borel measure μ with compact support in the complex plane C the sesquilinear form defined on analytic polynomials f and g by $B_\mu(f, g) = \int f\, g d\bar{\mu}$, determines an operator T_μ from the space of such polynomials P to the space of linear functionals on P. We study the product problem of Toeplitz operators on the Bergman space of the unit disk. We characterize when the product of two Toeplitz operators $T_f T_g$ is a finite rank perturbation of another Toeplitz operator T_h, with f, g bounded harmonic and h in C^2 class with invariant Laplacian in L^1.

The Contents

Chapter 1
Classification of Reducing Subspaces

We obtain a complete description of nontribyl minimal reducing subspaces of the multiplication operator by a Blaschke product with four zeros on the Bergman space of the unit disk by the Hardy space of the bidisk. We show that A_φ is commutative, and moreover, that the minimal reducing subspaces are pairwise orthogonal. Finally, an analytic/arithmetic description of the minimal reducing subspaces is also provided, along with the taxonomy of the possible structures of the reducing subspaces in case φ has eight zeros. These results have implications in both operator theory and the geometry of finite Blaschke products.

Section (1.1): A Class of Multiplication Operators on the Bergman Space by the Hardy Space of the Bidisk

For $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space L^2 $(\mathbb{D}, \mathrm{d}A)$ of square integrable functions on $\mathbb{D}$. For a bounded analytic function ϕ on the unit disk, the multiplication operator M_ϕ with symbol ϕ is defined on the Bergman space L_a^2 given by

$$M_\phi h \ = \ \phi h$$

for $h \ \in \ L_a^2$. On the basis $\{e_n\}_{n=0}^{\infty}$, where en is equal to $\sqrt{n \ + \ 1}z^n$, the multiplication operator M_z by z is a weighted shift operator, said to be the Bergman shift:

$$M_z e_n \ = \ \sqrt{\frac{n \ + \ 1}{n \ + \ 2}} \ e_{n+1}.$$

A reducing subspace M for an operator T on a Hilbert space H is a subspace M of H such that $TM \ \subset \ M$ and $T^*M \ \subset \ M$. A reducing subspace M of T is called minimal if M does not have any nontribyl subspaces which are reducing subspaces. We classify reducing subspaces of M_ϕ for the Blaschke product ϕ with four zeros by identifying its minimal reducing subspaces. We lift the Bergman shift up as a compression of a commuting pair of isometries on a nice subspace of the Hardy space of the bidisk. This idea was used in studying the Hilbert modules by R. Douglas and V. Paulsen [6], operator theory in the Hardy space over the bidisk by R. Dougals and R. Yang [7], [19], [20] and [21]; the higher-order Hankel forms by S. Ferguson and R. Rochberg [8] and [9] and the lattice of the invariant subspaces of the Bergman shift by S. Richter [13].

On the Hardy space of the unit disk, for an inner function ϕ, the multiplication operator by ϕ is a pure isometry. So its reducing subspaces are in one-to-one correspondence with the closed subspaces of $H^2 \ \ominus \phi H^2$ [5], [11]. Therefore, it has infinitely many reducing subspaces provided that ϕ is any inner function other than a Mobius function. Many have studied the problem of determining reducing subspaces of a multiplication operator on the Hardy space of the unit circle [2], [3] and [12]. The multiplication operators on the Bergman space possess a very rich structure theory. Even the lattice of the invariant subspaces of the Bergman shift M_z is huge [4]. But the lattice of reducing subspaces of the multiplication operator by a finite Blaschke on the Bergman space seems to be simple. On the Bergman space, Zhu [22] showed that for a Blaschke product ϕ with two zeros, the multiplication operator M_ϕ has exact two nontribyl reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$. The restriction of the multiplication operator on $\mathcal{M}_0$ is unitarily equivalent to the Bergman shift. Using the Hardy space of bidisk in [10], we show that the multiplication operator with a finite Blaschke product ϕ has a unique reducing subspace $\mathcal{M}_0$ (ϕ), on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift and if a multiplication operator has a such reducing subspace, then its symbol must be a finite Blaschke product. The space $\mathcal{M}_0$ (ϕ) is called the distinguished reducing subspace of M_ϕ and is equal to

$$\bigvee \{\phi_0\phi_n : n = 0, 1, \cdots, m, \cdots\}$$

if ϕ vanishes at 0 in [16], i.e,

$$\phi(z) = cz \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z},$$

for some points $\{\alpha_k\}$ in the unit disk and a unimodular constant c. The space has played an important role in classifying reducing subspaces of M_ϕ. In [10], we have shown that for a Blaschke product ϕ of the third order, except for a scalar multiple of the third power of a Mobius transform, M_ϕ has exactly two nontribyl minimal reducing subspaces $\mathcal{M}_0$ (ϕ) and $\mathcal{M}_0(\Phi)^\perp$. The study on reducing subspaces of the multiplication operators M_ϕ on the Bergman space in [10] by using the Hardy space of the bidisk. We will obtain a complete description of nontribyl minimal reducing subspaces of M_ϕ for the fourth order Blaschke product ϕ.

We introduce some notation to lift the Bergman shift as the compression of some isometry on a subspace of the Hardy space of the bidisk and state some theorems in [10]. We state the main result and present its proof. Since the proof is long, two difficult cases in the proof are considered.

For $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$. Let dσ be the rotation invariant Lebesgue measure on $\mathbb{T}^2$. The Hardy space H^2 $(\mathbb{T}^2)$ is the subspace of L^2 $(\mathbb{T}^2, d\sigma)$, where functions in H^2 $(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in the bidisc $\mathbb{D}^2$ with the square summable Fourier coefficients. The Toeplitz operator on H^2 $(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by

$$Tf(h) = P(fh),$$

for $h \in H^2$ $(\mathbb{T}^2)$ where P is the orthogonal projection from L^2 $(\mathbb{T}^2, d\sigma)$ onto H^2 $(\mathbb{T}^2)$. For each integer $n \geq 0$, let

$$p_n(z, w) = \sum_{i=0}^{n} z^i w^{n-i}.$$

Let H be the subspace of H^2 $(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^\infty$. Thus

$$H^2\ (\mathbb{T}^2) = H \oplus cl\{(z - w)H^2(\mathbb{T}^2)\}.$$

Let

$$\mathcal{B} = P_H T_z|_{\mathcal{H}} = P_H T_w|_{\mathcal{H}}$$

where $P_{\mathcal{H}}$ is the orthogonal projection from L^2 $(\mathbb{T}^2, d\sigma)$ onto $\mathcal{H}$. So $\mathcal{B}$ is unitarily equivalent to the Bergman shift M_z on the Bergman space L_a^2 by the following unitary operator $U : L_a^2\ (\mathbb{D}) \rightarrow \mathcal{H}$,

$$Uz^n = \frac{p_n(z, w)}{n + 1}.$$

This implies that the Bergman shift is lifted up as the compression of an isometry on a nice subspace of H^2 $(\mathbb{T}^2)$. Indeed, for each finite Blaschke product $\phi(z)$, the multiplication operator M_φ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

Let L_0 be $\ker T_\phi^*(z) \cap \ker T_\phi^*(w) \cap \mathcal{H}$. In [10], for each $e \in L_0$, we construct functions $\{d_e^k\}$ and d_e^0 such that for each $l \geq 1$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k\big(\phi(z), \phi(w)\big)d_e^{l-k} \in \mathcal{H}$$

and

$$p_l\big(\phi(z), \phi(w)\big)e + p^{l-1}\big(\phi(z), \phi(w)\big)d_e^0 \in \mathcal{H}.$$

On one hand, we have a precise formula of d_e^0 :

$$d_e^0\,(z, w) = we(0, w)e_0(z, w) - w\phi_0(w)e(z, w), \tag{1}$$

where e_0 is the function $\frac{\phi(z)-\phi(w)}{z-w}$. On the other hand, d_e^k is orthogonal to $\ker T_\phi^*(z) \cap \ker T_\phi^*(w) \cap \mathcal{H}$, and for a reducing subspace $\mathcal{M}$ and $e \in \mathcal{M}$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k\big(\phi(z), \phi(w)\big)d_e^{l-k} \in \mathcal{M}.$$

Moreover, the relation between d_e^1 and d_e^0 is given by Theorem 1 in [10] as follows:

Theorem (1.1.1)[1]: If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each $e \in \mathcal{M} \cap L_0$, then there is an element $\tilde{e} \in \mathcal{M} \cap L_0$ and a number λ such that

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0. \tag{2}$$

Since for Blaschke products with smaller order, it is not difficult to calculate $\tilde{e}$ and λ precisely, we are able to classify minimal reducing subspaces of a multiplication operator by a Blaschke product of the fourth order. Main ideas in the proof of Theorems (1.1.7) and (1.1.8) are that by complicated computations we use (2) to derive conditions on zeros of the Blaschke product of the fourth order.

We often in [10] stated as follows.

Theorem (1.1.2)[1]: There is a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift. In fact,

$$\mathcal{M}_0 = \bigvee_{l \geq 0} \{p_l\big(\phi(z), \phi(w)\big)e_0\},$$

and $\left\{\frac{p_l(\phi\,(z),\phi(w))e_0}{\sqrt{l+1}\|e_0\|}\right\}_0^\infty$ form an orthonormal basis of $\mathcal{M}_0$.

We call $\mathcal{M}_0$ to be the distinguished reducing subspace for $\phi(\mathcal{B})$. $\mathcal{M}_0$ is unitarily equivalent to a reducing subspace of M_ϕ contained in the Bergman space, denoted by $\mathcal{M}_0(\phi)$. The space plays an important role in classifying the minimal reducing subspaces of M_ϕ in Theorem (1.1.6).

In [10] we showed that for a nontribyl minimal reducing subspace Ω for $\phi(\mathcal{B})$, either Ω equals $\mathcal{M}_0$ or Ω is a subspace of $\mathcal{M}_0^\perp$. The condition in the following theorem is natural.

Theorem (1.1.3)[1]: Suppose that Ω, M and N are three distinct nontribyl minimal reducing subspaces for $\phi(\mathcal{B})$ and

$$\Omega \subset M \oplus N\,.$$

If they are contained in $\mathcal{M}_0^\perp$, then there is a unitary operator $U : M \to N$ such that U commutes with $\phi(\mathcal{B})$ and $\phi(\mathcal{B})^*$.

Let ϕ be a Blaschke product with four zeros. We will obtain a complete description of minimal reducing subspaces of the multiplication operator M_ϕ. First observe that the multiplication operator M_{z^4} is a weighted shift with multiplicity 4:

$$M_{z^4}\, e_n = \sqrt{\frac{n+1}{n+5}}\, e_{n+4}$$

where e_n equals $\sqrt{n+1}z^n$. By Theorem B [15], M_{z^4} has exact four nontribyl minimal reducing subspaces:

$$M_j = \bigvee \{z^n : n \equiv j \bmod 4\}$$

for $j = 1, 2, 3, 4$. Before stating the main result. It is not difficult to see that the set of finite Blaschke products forms a semigroup under composition of two functions. For a finite Blaschke product ϕ we say that ϕ is decomposable if there are two Blaschke products ψ_1 and ψ_2 with orders greater than 1 such that

$$\phi(z) = \psi_1 \circ \psi_2(z).$$

For each λ in $\mathbb{D}$, let ϕ_λ denote the Mobius transform:

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Define the operator U_λ on the Bergman space as follows:

$$U_\lambda f = f \circ \phi_\lambda k_\lambda$$

for f in L_a^2 where k_λ is the normalized reproducing kernel $\frac{(1-|\lambda|^2)}{(1-\lambda z)^2}$. Clearly, U_λ is a selfadjoint unitary operator on the Bergman space. Using the unitary operator U_λ we have

$$\mathcal{M}_0(\phi) = U_\lambda \mathcal{M}_0(\phi \circ \phi_\lambda)$$

where λ is a zero of the finite Blaschke product ϕ. This easily follows from that $\phi \circ \phi_\lambda$ vanishes at 0 and

$$U_\lambda^* M_\phi U_\lambda = M_\phi \circ \phi_\lambda .$$

We say that two Blaschke products ϕ_1 and ϕ_2 are equivalent if there is a complex number λ in $\mathbb{D}$ such that

$$\phi_1 = \phi_\lambda \circ \phi_2.$$

For two equivalent Blaschke products ϕ_1 and ϕ_2, M_{ϕ_1} and M_{ϕ_2} are mutually analytic function calculus of each other and hence share reducing subspaces. The following main result gives a complete description of minimal reducing subspaces.

To prove the above theorem we need the following two lemmas which tell us when a Blaschke product with order 4 is decomposable.

Lemma (1.1.4)[1]: If a Blaschke product ϕ with order four is decomposable, then the numerator of the rational function $\phi(z) - \phi(w)$ has at least three irreducible factors.

Proof. Suppose that ϕ is the Blaschke product with order four. Let $f(z, w)$ be the numerator of the rational function $\phi(z) - \phi(w)$. If ϕ is decomposable, then $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with order two. Let $g(z, w)$ be the numerator of the rational function $\psi_1(z) - \psi_1(w)$. Clearly, $z - w$ is a factor of $g(z, w)$. Thus we can write

$$g(z, w) = (z - w)p(z, w)$$

for some polynomial $p(z, w)$ of z and w to get

$$g\big(\psi_2(z), \psi_2(w)\big) = \big(\psi_2(z) - \psi_2(w)\big)p\big(\psi_2(z), \psi_2(w)\big).$$

On the other hand, we also have

$$\psi_2(z) - \psi_2(w) = \frac{(z - w)p_2(z, w)}{q_2(z, w)}$$

for two polynomials $p_2(z,w)$ and $q_2(z,w)$ which $p_2(z,w)$ and $q_2(z,w)$ do not have common factor. In fact, $q_2(z,w)$ and the numerator of the rational function $p\big(\psi_2(z),\psi_2(w)\big)$ do not have common factor also. So we obtain

$$g\big(\psi_2(z),\psi_2(w)\big) = \frac{(z-w)p_2(z,w)}{q_2(z,w)}\, p\big(\psi_2(z),\psi_2(w)\big).$$

Since $f(z,w)$ is the numerator of the rational function $g(\psi_2(z),\psi_2(w))$, this gives that $f(z,w)$ has at least three factors. This completes the proof.

For $\alpha,\beta \in \mathbb{D}$, define

$$f_{\alpha,\beta}(w,z) = w^2(w-\alpha)(w-\beta)(1-\overline{\alpha}z)\big(1-\overline{\beta}z\big) - z^2(z-\alpha)(z-\beta)(1-\overline{\alpha}w)\big(1-\overline{\beta}w\big).$$

It is easy to see that $f_{\alpha,\beta}(w,z)$ is the numerator of $z^2\phi_\alpha(z)\phi_\beta(z) - w^2\phi_\alpha(w)\phi_\beta(w)$. The following lemma gives a criteria when the Blaschke product $z^2\phi_\alpha(z)\phi_\beta(z)$ is decomposable.

Lemma (1.1.5)[1]: For α and β in $\mathbb{D}$, one of the following holds.

(i) If both α and β equal zero, then

$$f_{\alpha,\beta}(w,z) = (w-z)(w+z)(w-iz)(w+iz).$$

(ii) If α does not equal either β or $-\beta$, then

$$f_{\alpha,\beta}(w,z) = (w-z)p(w,z)$$

for some irreducible polynomial $p(w,z)$.

(iii) If α equals either β or $-\beta$ but it does not equal zero, then

$$f_{\alpha,\beta}(w,z) = (w-z)p(w,z)q(w,z)$$

for two irreducible distinct polynomials $p(w,z)$ and $q(w,z)$.

Proof. Clearly, (i) holds. To prove (ii), by the example on page 6 of [14] we may assume that none of α and β equals 0. First observe that $(w-z)$ is a factor of the polynomial $f_{\alpha,\beta}(w,z)$. Taking a long division gives

$$f_{\alpha,\beta}(w,z) = (w-z)g_{\alpha,\beta}(w,z)$$

where

$$g_{\alpha,\beta}(w,z) = (1-\overline{\alpha}z)\big(1-\overline{\beta}z\big)w^3 + \big(z-(\alpha+\beta)\big)(1-\overline{\alpha}z)\big(1-\overline{\beta}z\big)w^2 + (z-\alpha)(z-\beta)\big(1-\big(\overline{\alpha}+\overline{\beta}\big)z\big)w + z(z-\alpha)(z-\beta).$$

Next we will show that $g_{\alpha,\beta}(w,z)$ is irreducible. To do this, we assume that $g_{\alpha,\beta}(w,z)$ is reducible to derive a contradiction.

Assuming that $g_{\alpha,\beta}(w,z)$ is reducible, we can factor $g_{\alpha,\beta}(w,z)$ as the product of two polynomials $p(w,z)$ and $q(w,z)$ of z and w with degree of w greater than or equal one. Write

$$p(w,z) = a_1(z)w + a_0(z)$$
$$q(w,z) = b_2(z)w^2 + b_1(z)w + b_0(z)$$

where $a_j(z)$ and $b_j(z)$ are polynomials of z. Since $g_{\alpha,\beta}(w,z)$ equals the product of p(w, z) and q(w, z), taking the product and comparing coefficients of w^k give

$$a_1(z)b_2(z) = (1-\overline{\alpha}z)\big(1-\overline{\beta}z\big), \tag{3}$$
$$a_1(z)b_1(z) + a_0(z)b_2(z) = \big(z-(\alpha+\beta)\big)(1-\overline{\alpha}z)\big(1-\overline{\beta}z\big), \tag{4}$$
$$a_1(z)b_0(z) + a_0(z)b_1(z) = (z-\alpha)(z-\beta)\big(1-\big(\overline{\alpha}+\overline{\beta}\big)z\big), \tag{5}$$
$$a_0(z)b_0(z) = z(z-\alpha)(z-\beta). \tag{6}$$

Equation (3) gives that either

$$a_1(z) = (1 - \overline{\alpha}z) \text{ or}$$
$$a_1(z) = (1 - \overline{\alpha}z)(1 - \overline{\beta}z) \text{ or}$$
$$a_1(z) = 1.$$

In the first case that $a_1(z) = (1 - \overline{\alpha}z)$, (3) gives $b_2(z) = (1 - \overline{\beta}z)$. Thus by Equation (4), we have

$$a_0(z)(1 - \overline{\beta}z) = (1 - \overline{\alpha}z)\left[\left(z - (\alpha + \beta)\right)(1 - \overline{\beta}z) - b_1(z)\right],$$

to get that $(1 - \overline{\alpha}z)$ is a factor of $a_0(z)$, and hence is also a factor of a factor $z(z-\alpha)(z-\beta)$ by (6). This implies that α must equal 0. It is a contradiction.

In the second case that $a_1(z) = (1 - \overline{\alpha}z)(1 - \overline{\beta}z)$, we have that $b_2(z) = 1$ to get that either the degree of $b_1(z)$ or the degree of $b_0(z)$ must be one while the degrees of $b_1(z)$ and $b_0(z)$ are at most one. So the degree of $a_0(z)$ is at most two. Also $a_0(z)$ does not equal zero. Equation (4) gives

$$(1 - \overline{\alpha}z)(1 - \overline{\beta}z)b_1(z) + a_0(z) = (z - (\alpha + \beta))(1 - \overline{\alpha}z)(1 - \overline{\beta}z).$$

Thus $a_0(z) = c_1(1 - \overline{\alpha}z)(1 - \overline{\beta}z)$ for some constant c_1. But Equation (6) gives

$$c_1(1 - \overline{\alpha}z)(1 - \overline{\beta}z)b_0(z) = z(z - \alpha)(z - \beta).$$

Either $c_1 = 0$ or $(1 - \overline{\alpha}z)(1 - \overline{\beta}z)$ is a factor of $z(z - \alpha)(z - \beta)$. This is impossible.

In the third case that $a_1(z) = 1$, then $b_2(z) = (1 - \overline{\alpha}z)(1 - \overline{\beta}z)$. Since the root w of $f_{\alpha,\beta}(w, z)$ is a nonconstant function of z, the degree of $a_0(z)$ must be one. Thus the degrees of $b_1(z)$ and $b_0(z)$ are at most two. By Equation (4) we have

$$(1 - \overline{\alpha}z)(1 - \overline{\beta}z)a_0(z) + b_1(z) = \left(z - (\alpha + \beta)\right)(1 - \overline{\alpha}z)(1 - \overline{\beta}z),$$

to get

$$b_1(z) = (1 - \overline{\alpha}z)(1 - \overline{\beta}z)\left[\left(z - (\alpha + \beta)\right) - a_0(z)\right].$$

Since the degree of $b_1(z)$ is at most two, we have

$$a_0(z) = \left(z - (\alpha + \beta)\right) - c_0;$$
$$b_1(z) = c_0(1 - \overline{\alpha}z)(1 - \overline{\beta}z).$$

Equations (6) and (5) give

$$\left[\left(z - (\alpha + \beta)\right) - c_0\right]b_0(z) = z(z - \alpha)(z - \beta)$$

and

$$b_1(z)\left[\left(z - (\alpha + \beta)\right) - c_0\right] + b_0(z)$$
$$= (z - \alpha)(z - \beta)\left(1 - (\overline{\alpha} + \overline{\beta})z\right).$$

Multiplying the both sides of the last equality by $\left[\left(z - (\alpha + \beta)\right) - c_0\right]$ gives

$$b_1(z)\left[\left(z - (\alpha + \beta)\right) - c_0\right] 2 + z(z - \alpha)(z - \beta)$$
$$= \left[\left(z - (\alpha + \beta)\right) - c_0\right](z - \alpha)(z - \beta)\left(1 - (\overline{\alpha} + \overline{\beta})z\right).$$

This leads to

$$c_0(1 - \overline{\alpha}z)(1 - \overline{\beta}z)\left[\left(z - (\alpha + \beta)\right) - c_0\right] 2 + z(z - \alpha)(z - \beta)$$
$$= \left[\left(z - (\alpha + \beta)\right) - c_0\right](z - \alpha)(z - \beta)\left(1 - (\overline{\alpha} + \overline{\beta})z\right).$$

If $c_0 \neq 0$, then the above equality gives that $(z-\alpha)(z-\beta)$ is a factor of $\left[\left(z-(\alpha+\beta)\right) - c_0\right]^2$. This is impossible.

If $c_0 = 0$, then we have

$$z(z - \alpha)(z - \beta) = \left[\left(z - (\alpha + \beta)\right)\right](z - \alpha)(z - \beta)\left(1 - (\overline{\alpha} + \overline{\beta})z\right).$$

to get $\overline{\alpha} + \overline{\beta} = 0$ and hence $\alpha = -\beta$. It is also a contradiction. This completes the proof that $g_{\alpha,\beta}(w,z)$ is irreducible.

To prove (iii), we note that if α equals β, an easy computation gives

$$f_{\alpha,\beta}(w,z) = (w - z)[((1 - \overline{\alpha}z)w + (z - \alpha)] \times [w(w - \alpha)(1 - \overline{\alpha}z) + z(z - \alpha)(1 - \overline{\alpha}w)].$$

If $\alpha = -\beta$, we also have

$$f_{\alpha,\beta}(w,z) = (w - z)(w + z)[(1 - \overline{\alpha}^2 z^2)w^2 + (z^2 - \alpha^2)].$$

This completes the proof.

Theorem (1.1.6)[1]: Let ϕ be a Blaschke product with four zeros. One of the following holds.

(i) If ϕ is equivalent to z^4, i.e., ϕ is a scalar multiple of the fourth power ϕ_c^4 of the Mobius transform ϕ_c for some complex number c in the unit disk, M_ϕ has exact four nontribyl minimal reducing subspaces

$$\{U_c\mathcal{M}_1, U_c\mathcal{M}_2, U_c\mathcal{M}_3, U_c\mathcal{M}_4\}.$$

(ii) If ϕ is is decomposable but not equivalent to z^4, i.e, $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with orders 2 but not both of ψ_1 and ψ_2 are a scalar multiple of z^2, then M_ϕ has exact three nontribyl minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2)^\perp\}.$$

(iii) If ϕ is not decomposable, then M_ϕ has exact two nontribyl minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}.$$

Proof. Assume that ϕ is a Blaschke product with the fourth order. By the Bochner Theorem [18], ϕ has a critical point c in the unit disk. Le$t\ \lambda = \phi(c)$ be the critical value of ϕ. Then there are two points α and β in the unit disk such that

$$\phi_\lambda \circ \phi \circ \phi_c(z) = \eta z^2 \phi_\alpha \phi_\beta$$

where η is a unimodule constant. Let ψ be $z^2\phi_\alpha\phi_\beta$. Since $\phi\circ\phi_c$ and ψ are mutually analytic function calculus of each other, both $M_{\phi\circ\phi_c}$ and M_ψ share reducing subspaces.

(i) If ϕ is equivalent to z^4, then ψ must equal a scalar multiple of z^4. By Theorem B in [15], M_ψ has exact four nontribyl minimal reducing subspaces

$$\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$$

where

$$\mathcal{M}_j = \bigvee \{z^n : n \equiv j \bmod 4\}$$

for $j = 1, 2, 3, 4$. The four spaces above are also reducing subspaces for $M_{\phi\circ\phi_c}$. Noting

$$U_c^* M_{\phi\circ\phi_c} U_c = M_\phi,$$

we have that M_ϕ has exact four nontribyl minimal reducing subspaces

$$\{U_c\mathcal{M}_1, U_c\mathcal{M}_2, U_c\mathcal{M}_3, U_c\mathcal{M}_4\}.$$

(ii) If ϕ is decomposable but equivalent to z^4, i.e, $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with degrees two and not both ψ_1 and ψ_2 are scalar multiples of z^2, by Lemmas (1.1.4) and (1.1.5), then α equals either β or $-\beta$ but does not equal 0. By Theorem (1.1.2), the restriction of M_{ψ_2} on $\mathcal{M}_0(\psi_2)$ is unitarily equivalent to the Bergman shift. Thus $\mathcal{M}_0(\psi_2)$ is also a reducing subspace of M_ϕ and the restriction of $M_\phi = M_{\psi_1\circ\psi_2}$ on $\mathcal{M}_0(\psi_2)$ is unitarily equivalent to M_{ψ_1} on the Bergman space. By Theorem (1.1.2) again, there is a unique reducing subspace $\mathcal{M}_0(\psi_1)$ on which the restriction M_{ψ_1} is unitarily equivalent to the Bergman shift. Thus there is a subspace of $\mathcal{M}_0(\psi_2)$ on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift. Theorem (1.1.2) implies that $\mathcal{M}_0(\phi)$ is contained in $\mathcal{M}_0(\psi_2)$. Therefore $\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)$ is also a minimal reducing subspace of M_ϕ and

$$L_a^2 = \mathcal{M}_0(\phi) \oplus [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By Theorems (1.1.7) in [17], $\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$ are nontribyl minimal reducing subspaces of M_ϕ. We will show that they are exact nontribyl minimal reducing subspaces of M_ϕ. If this is not true, then there is another minimal reducing subspace Ω of M_ϕ. By Theorem 38 [10], we have

$$\Omega \subset [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By Theorem (1.1.3), there is a unitary operator

$$U : [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)] \to [\mathcal{M}_0(\psi_2)]^\perp$$

which commutes with both M_ϕ and M_ϕ^* . But

$$\dim\ker M_\phi^* \cap [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)] = 1$$

and

$$\dim\ker M_\phi^* \cap [\mathcal{M}_0(\psi_2)]^\perp = 2.$$

This is a contradiction. Thus $\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2)\mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$ are exact nontribyl minimal reducing subspaces of M_ϕ.

(iii) If ϕ is not decomposable, by Lemma (1.1.5), then ϕ equals $z^3\phi_\alpha$ or $z^2\phi_\alpha\phi_\beta$ for two nonzero points $\alpha\ \beta$ in $\mathbb{D}$ and α does not equal β or $-\beta$. By Theorems (1.1.7) and (1.1.8), M_ϕ has exact two nontribyl minimal reducing subspaces $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

We will study reducing subspaces of $M_{z^3\phi_\alpha}$ for a nonzero point $\alpha \in \mathbb{D}$. Recall that $\mathcal{M}_0$ is the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem (1.1.2).

Theorem (1.1.7)[1]: Let $z^3\phi_\alpha$ for a nonzero point $\alpha \in \mathbb{D}$. Then $\phi(\mathcal{B})$ has exact two nontribyl reducing subspaces $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.

Proof. Let $\mathcal{M}_0$ be the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem (1.1.2). By Theorem (1.1.3), we only need to show that $\mathcal{M}_0^\perp$ is a minimal reducing subspace for $\phi(\mathcal{B})$.

Assume that $\mathcal{M}_0^\perp$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. Then by Theorem (1.1.7) in [17] we may assume

$$\mathcal{H} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a nontribyl reducing subspace for $\phi(\mathcal{B})$, $\mathcal{M}_0^\perp = M_0$ is the distinguished reducing subspace for $\phi(\mathcal{B})$ and

$$\mathcal{M}_0^\perp = M_1 \oplus M_2.$$

Recall that

$$\phi_0 = z^2\phi_\alpha,$$
$$L_0 = \operatorname{span}\{1, p_1, p_2, k_\alpha(z)k_\alpha(w)\},$$

and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

We further assume that

$$\dim(M_1 \cap L_0) = 1$$

and

$$\dim(M_2 \cap L_0) = 2.$$

Take $0 \neq e_1 \in M_1 \cap L_0$, $e_2, e_3 \in M_2 \cap L_0$ such that $\{e_2, e_3\}$ are a basis for $M_2 \cap L_0$, then

$$L_0 = span\{e_0, e_1, e_2, e_3\}$$

By (1), we have

$$d^0_{e_j} = we_j\,(0,w)e_0 - \phi(w)e_j$$

and direct computations show that

$$\begin{aligned}\langle d^0_{e_j}, p_k\rangle &= \langle we_j\,(0,w)e_0 - \phi(w)e_j, p_k\rangle\\ &= \langle we_j\,(0,w)e_0, p_k\rangle \quad (\text{by } T*\phi(w)p_k = 0)\\ &= \langle we_j\,(0,w)e_0(w,w), p_k(0,w)\rangle\\ &= \langle we_j\,(0,w)\phi'\,(w), w_k\rangle\\ &= \langle w^3_{e_j}\,(0,w)\big(w\phi'_\alpha\,(w) + 3\phi_\alpha(w)\big), w_k\rangle\\ &= \langle w^{3-k}_{e_j}\,(0,w)\big(w\phi'_\alpha\,(w) + 3\phi_\alpha(w)\big), 1\rangle = 0\end{aligned}$$

for $0 \le k \le 2$, and

$$\langle d^0_{e_j}, k_\alpha(z)k_\alpha(w)\rangle = \alpha e_j\,(0,\alpha)e_0(\alpha,\alpha) = \alpha e_j\,(0,\alpha)\frac{\alpha^3}{1-|\alpha|^2}.$$

This implies that those functions $d^0_{e_j}$ are orthogonal to $\{1, p_1, p_2\}$. Simple calculations give

$$\langle e_0, p_k\rangle = 0$$

for $0 \le k \le 1$,

$$\langle e_0, p_2\rangle = \langle e_0(0,w), p_2(w,w)\rangle = \frac{3}{2}\,\phi''_0\,(0) = -3\alpha \neq 0$$

and

$$\langle e_0, k_\alpha(z)k_\alpha(w)\rangle = e_0(\alpha,\alpha) = \phi'\,(\alpha) = \frac{\alpha^3}{1-|\alpha|^2} \neq 0$$

By Theorem (1.1.1), there are numbers μ, λ_j such that

$$\begin{aligned}d^1_{e_1} &= d^0_{e_1} + \mu e_1 + \lambda_1 e_0\\ d^1_{e_2} &= d^0_{e_2} + \tilde{e}_2 + \lambda_2 e_0\\ d^1_{e_3} &= d^0_{e_3} + \tilde{e}_3 + \lambda_3 e_0\end{aligned}$$

where $\tilde{e}_2, \tilde{e}_3 \in M_2 \cap L_0$.

Now we consider two cases. In each case we will derive a contradiction.

Case 1. $\mu \neq 0$. In this case, we get that e_1 is orthogonal to $\{1, p_1\}$. So $\{1, p_1, e_0, e_1\}$ form an orthogonal basis for L_0.

First we show that $\tilde{e}_2 = 0$. If $\tilde{e}_2 \neq 0$, then we get that $\{1, p_1, e_0, \tilde{e}_2\}$ are also an orthogonal basis for L_0. Thus $\tilde{e}_2 = ce_1$ for some nonzero number c. However, $\tilde{e}_2$ is orthogonal to e_1 since $\tilde{e}_2 \in M_2$ and $e_1 \in M_1$. This is a contradiction. Thus

$$d^1_{e_2} = d^0_{e_2} + \lambda_2 e_0.$$

Since both $d^1_{e_2}$ and $d^0_{e_2}$ are orthogonal to p_2 and

$$\langle e_0, p_2\rangle = -3\alpha \neq 0,$$

we have that $\lambda_2 = 0$ to get that $d^0_{e_2} = d^1_{e_2}$ is orthogonal to L_0. On the other hand,

$$\langle d^0_{e_2}, k_\alpha(z)k_\alpha(w)\rangle = \alpha e_2(0,\alpha)\frac{\alpha^3}{1-|\alpha|^2}.$$

Thus

$$e_2(0,\alpha) = 0.$$

Similarly we get that

$$e_3(0,\alpha) = 0.$$

Moreover, since e_2 and e_3 are orthogonal to $\{e_0, e_1\}$, write

$$e_2 = c_{11} + c_{12}p_1,$$

$$e_3 = c_{21} + c_{22}p_1.$$

Thus we have

$$e_2(0,\alpha) = c_{11} + c_{12}\alpha = 0,$$
$$e_3(0,\alpha) = c_{21} + c_{22}\alpha = 0,$$

to get that e_2 and e_3 are linearly dependent. This leads to a contradiction in this case.

Case 2. $\mu = 0$. In this case we have

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0.$$

Similarly to the proof in Case 1 we get that $\lambda_1 = 0$,

$$d_{e_1}^1 = d_{e_1}^0 \perp L_0 \tag{7}$$

and $e_1(0,\alpha) = 0$. Theorem (1.1.4) in [17] gives that at least one $\tilde{e}_j$, say $\tilde{e}_2$ does not equal 0. Assume that $\tilde{e}_2 \neq 0$, write

$$\tilde{e}_2 = d_{e_2}^1 - d_{e_2}^0 - \lambda_2 e_0.$$

Note that we have shown above that both $d_{e_2}^0$ and e_0 are orthogonal to both 1 and p_1. Thus

$$\tilde{e}_2 \perp \{1, p_1\}$$

and

$$L_0 = span\{1, p_1, e_0, \tilde{e}_2\}.$$

Since e_1 is orthogonal to $\{e_0, \tilde{e}_2\}$ we have

$$e_1 = c_1 + c_2 p_1.$$

Noting that $e_1(0,\alpha) = c_1 + c_2\alpha = 0$ we get

$$e_1 = c_2(-\alpha + p_1).$$

Without loss of generality we assume that

$$e_1 = -\alpha + p_1. \tag{8}$$

Letting e be in $M_2 \cap L_0$ such that e is a nonzero function orthogonal to $\tilde{e}_2$, we have that e is orthogonal to $\{e_0, \tilde{e}_2\}$. Thus e must be in the subspace span$\{1, p_1\}$. So there are two constants b_1 and b_2 such that

$$e = b_1 + b_2 p_1.$$

Noting

$$0 = \langle e, e_1 \rangle = -b_1\overline{\alpha} + 2b_2$$

we have

$$e = \frac{b_1}{2}(2 + \overline{\alpha}p_1).$$

Hence we may assume that

$$e = 2 + \overline{\alpha}p_1. \tag{9}$$

By Theorem (1.1.1) we have

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0$$

for some number λ and $\tilde{e} \in M_2 \cap L_0$. Thus

$$0 = \langle d_{e_1}^1, d_e^1 \rangle = \langle d_{e_1}^1, d_e^0 + \tilde{e} + \lambda e_0 \rangle = \langle d_{e_1}^1, d_e^0 \rangle = \langle d_{e_1}^0, d_e^0 \rangle \qquad (By\,(7)).$$

However, a simple computation gives

$$\begin{aligned}\langle d_{e_1}^0, d_e^0 \rangle &= \langle d_{e_1}^0, we(0,w)e_0 - \phi(w)e \rangle \\ &= \langle d_{e_1}^0, we(0,w)e_0 \rangle \left(\text{by } T_\phi^*(w)d_{e_1}^0 = 0\right) \\ &= \langle we_1(0,w)e_0 - \phi(w)e_1, we(0,w)e_0 \rangle \\ &= \langle we_1(0,w)e_0, we(0,w)e_0 \rangle - \langle \phi(w)e_1, we(0,w)e_0 \rangle.\end{aligned}$$

We need to calculate two terms in the right hand of the above equality. By (8) and (9), the first term becomes

$$\langle we_1(0,w)e_0, we(0,w)e_0\rangle = \langle w(-\alpha + w)e_0, w(2 + \overline{\alpha}w)e_0\rangle$$
$$= \langle(-\alpha + w)e_0, (2 + \overline{\alpha}w)e_0\rangle$$
$$= \langle -\alpha e_0, 2e_0\rangle + \langle we_0, 2e_0\rangle + \langle -\alpha e_0, \alpha w\tilde{e}_0\rangle + \langle we_0, \alpha w\tilde{e}_0\rangle$$
$$= -\alpha\langle e_0, e_0\rangle + 2\langle we_0, e_0\rangle - \alpha^2 \langle e_0, we_0\rangle.$$

The first term in right hand of the last equality is

$$\langle e_0, e_0\rangle = \langle e_0(w,w), e_0(0,w)\rangle = \langle w\phi_0' + \phi_0, \phi_0\rangle$$
$$= \langle w(2w\phi_\alpha + w^2\phi_\alpha'), w^2\phi_\alpha\rangle + \langle\phi_0, \phi_0\rangle = 2 + \langle w\phi_\alpha', \phi_\alpha\rangle + 1 = 4.$$

The last equality follows from

$$\phi_\alpha = -\frac{1}{\overline{\overline{\alpha}}} + \frac{\frac{1}{\overline{\overline{\alpha}}} - \alpha}{1 - \overline{\alpha}w} = -\frac{1}{\overline{\overline{\alpha}}} + \left(\frac{1}{\overline{\overline{\alpha}}} - \alpha\right)K_\alpha(w).$$

Similarly, we have

$$\langle we_0, e_0\rangle = \langle we_0(w,w), e_0(0,w)\rangle = \langle w(w\phi_0' + \phi_0), \phi_0\rangle = \alpha.$$

This gives

$$\langle we_1(0,w)e_0, we(0,w)e_0\rangle = \langle e_1(0,w)e_0, e(0,w)e_0\rangle = \langle(-\alpha + w)e_0, (2 + \overline{\alpha}w)e_0\rangle$$
$$= -2\alpha\langle e_0, e_0\rangle - \alpha^2 \langle e_0, we_0\rangle + 2\langle we_0, e_0\rangle + \alpha\langle we_0, we_0\rangle$$
$$= -8\alpha - \alpha|\alpha|^2 + 2\alpha + 4\alpha = -2\alpha - \alpha|\alpha|^2$$

A simple calculation gives that the second term becomes

$$\langle\phi(w)e_1, we(0,w)e_0\rangle = \langle\phi_0(w)e_1, (2 + \overline{\alpha}w)e_0\rangle$$
$$= \langle\phi_0(w)e_1, 2e_0\rangle + \langle\phi_0(w)e_1, \alpha w\tilde{e}_0\rangle$$
$$= 2\langle\phi_0(w)e_1(w,w), e_0(0,w)\rangle + \alpha\langle\phi_0(w)e_1(w,w), we_0(0,w)\rangle$$
$$= 2\langle e_1(w,w), 1\rangle + \alpha\langle e_1(w,w), w\rangle$$
$$= 2\langle -\alpha + 2w, 1\rangle + \alpha\langle -\alpha + 2w, w\rangle = -2\alpha + 2\alpha = 0.$$

Thus we conclude

$$\langle d_{e_1}^0, d_e^0\rangle = \langle we_1(0,w)e_0, we(0,w)e_0\rangle - \langle\phi(w)e_1, we(0,w)e_0\rangle = -2\alpha - \alpha|\alpha|^2$$
$$= -\alpha(2 + |\alpha|^2) \neq 0$$

to get a contradiction in this case. This completes the proof.

We will classify minimal reducing subspaces of $M_{z^2\phi_\alpha\phi_\beta}$ for two nonzero points α and β in $\mathbb{D}$ and with $\alpha \neq \beta$.

Theorem (1.1.8)[1]: Let ϕ be the Blaschke product $z^2\phi_\alpha\phi_\beta$ for two nonzero points α and β in $\mathbb{D}$. If α does not equal either β or $-\beta$, then $\phi(\mathcal{B})$ has exact two nontribyl reducing subspaces $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.

Proof. By Theorem 27 in [10], if $\mathcal{N}$ is a nontribyl minimal reducing subspace of $\phi(\mathcal{B})$ which is not equal to $\mathcal{M}_0$ then $\mathcal{N}$ is a subspace of $\mathcal{M}_0^\perp$, so we only need to show that $\mathcal{M}_0^\perp$ is a minimal reducing subspace for $\phi(\mathcal{B})$ unless $\alpha = -\beta$.

Assume that $\mathcal{M}_0^\perp$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. By Theorem (1.1.7) in [17], we may assume

$$\mathcal{H} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a reducing subspace for $\phi(\mathcal{B}), M_0 = \mathcal{M}_0$ is the distinguished reducing subspace for $\phi(\mathcal{B})$ and

$$M_1 \oplus M_2 = \mathcal{M}_0^\perp.$$

Recall that

$$\phi_0 = z\phi_\alpha\phi_\beta,$$
$$L_0 = span\{1, p_1, e_\alpha, e_\beta\},$$

with $e_\alpha = k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w)$ and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

So we further assume that the dimension of $M_1 \cap L_0$ is one and the dimension of $M_2 \cap L_0$ is two. Take a nonzero element e_1 in $M_1 \cap L_0$, then by Theorem (1.1.1), there are numbers μ_1, λ_1 such that

$$d_{e_1}^1 = d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0. \qquad (10)$$

We only need to consider two possibilities, μ_1 is zero or nonzero.
If μ_1 is zero, then (10) becomes

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0. \qquad (11)$$

In this case, simple calculations give

$$\begin{aligned}\langle d_{e_1}^0, p_1\rangle &= \langle we_1(0,w)e_0(z,w) - w\phi_0(w)e_1(z,w), p_1(z,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(z,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(0,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), w\rangle \\ &= \langle e_1(0,w)e_0(w,w) - \phi_0(w)e_1(w,w), 1\rangle \\ &= e_1(0,0)e_0(0,0) - \phi_0(0)e_1(0,0) = 0,\end{aligned}$$

and

$$\begin{aligned}\langle e_0, p_1\rangle = \langle e_0(z,w), p_1(z,w)\rangle = \langle e_0(z,w), p_1(w,w)\rangle = \langle e_0(0,w), 2w\rangle = \langle \phi_0(w), 2w\rangle \\ = 2\langle w\phi_\alpha(w)\phi_\beta(w), w\rangle = 2\phi_\alpha(0)\phi_\beta(0) = 2\alpha\beta \neq 0.\end{aligned}$$

Noting that $d_{e_1}^1$ is orthogonal to L_0, by (11) we have that $\lambda_1 = 0$, and hence

$$d_{e_1}^0 = d_{e_1}^1 \perp L_0.$$

So

$$\langle d_{e_1}^0, e_\alpha\rangle = 0 = \langle d_{e_1}^0, e_\beta\rangle.$$

On the other hand,

$$\langle d_{e_1}^0, e_\alpha\rangle = \alpha e_1(0,\alpha)e_0(\alpha,\alpha) - \alpha\phi_0(\alpha)e_1(\alpha,\alpha) = \alpha e_1(0,\alpha)e_0(\alpha,\alpha)$$

and

$$\begin{aligned}\langle d_{e_1}^0, e_\beta\rangle &= \beta e_1(0,\beta)e_0(\beta,\beta) - \beta\phi_0(\beta)e_1(\beta,\beta) \\ &= \beta e_1(0,\beta)e_0(\beta,\beta).\end{aligned}$$

Consequently

$$e_1(0,\alpha) = e_1(0,\beta) = 0. \qquad (12)$$

Observe that e_0, e_1 and 1 are linearly independent. If this is not so, then $1 = ae_0 + be_1$ for some numbers a, b. But $e_1(0,\alpha) = 0$ and $e_0(0,\alpha) = 0$. This forces that $1 = 0$ and leads to a contradiction. By Theorem (1.1.1), we can take an element $e \in M_2 \cap L_0$ such that

$$d_e^1 = d_e^0 + e_2 + \mu e_0$$

with $e_2 \neq 0$ and $e_2 \in M_2 \cap L_0$. Thus we have that e_2 is orthogonal to 1 and so e_2 is in $\{1, e_0, e_1\}^\perp$ and $\{1, e_0, e_1, e_2\}$ form a basis for L_0. Moreover for any $f \in M_2 \cap L_0$,

$$d_f^1 = d_f^0 + g + \lambda e_0$$

for some number λ and $g \in M_2 \cap L_0$. If g does not equal 0 then g is orthogonal to 1. Thus g is in $\{1, e_0, e_1\}^\perp$ and hence

$$g = ce_2$$

for some number c. Therefore taking a nonzero element $e_3 \in M_2 \cap L_0$ which is orthogonal to e_2, we have

$$\begin{aligned}d_{e_2}^1 &= d_{e_2}^0 + \mu_2 e_2 + \lambda_2 e_0, \\ d_{e_3}^1 &= d_{e_3}^0 + \mu_3 e_2 + \lambda_3 e_0,\end{aligned}$$

and $\{e_0, e_1, e_2, e_3\}$ is an orthogonal basis for L_0.

If $\mu_2 = 0$, then by the same reason as before we get

$$\lambda_2 = 0,$$
$$d^0_{e_2} = d^1_{e_2} \perp L_0$$
$$e_2(0,\alpha) = e_2(0,\beta) = 0.$$

So using

$$p_1 \in L_0 = \text{span}\{1, e_0, e_1, e_2\}$$

we have

$$\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta,$$

which contradicts our assumption that $\alpha \neq \beta$. Hence $\mu_2 \neq 0$.

Observe that 1 is in $L_0 = span\{e_0, e_1, e_2, e_3\}$ and orthogonal to both e_0 and e_2. Thus

$$1 = c_1 e_1 + c_3 e_3$$

for some numbers c_1 and c_3. So

$$1 = c_1 e_1(0,\alpha) + c_3 e_3(0,\alpha)$$
$$= c_1 e_1(0,\beta) + c_3 e_3(0,\beta).$$

By (12), we have

$$1 = c_3 e_3(0,\alpha) = c_3 e_3(0,\beta),$$

to obtain that $c_3 \neq 0$ and

$$e_3(0,\alpha) = e_3(0,\beta) = 1/c_3.$$

If $\mu_3 = 0$, then by the same reason as before we get $e_3(0,\alpha) = e_3(0,\beta) = 0$. Hence $\mu_3 \neq 0$. Now by the linearality of $d^1_{(\cdot)}$ and $d^0_{(\cdot)}$ we have

$$d^1_{\mu_3 e_2 - \mu_2 e_3} = d^0_{\mu_3 e_2 - \mu_2 e_3} + (\mu_3 \lambda_2 - \mu_2 \lambda_3) e_0.$$

By the same reason as before we get

$$\mu_3 \lambda_2 - \mu_2 \lambda_3 = 0$$

and

$$d^0_{\mu_3 e_2 - \mu_2 e_3} = d^1_{\mu_3 e_2 - \mu_2 e_3} \perp L_0$$

and therefore

$$\mu_3 e_2(0,\alpha) - \mu_2 e_3(0,\alpha) = \mu_3 e_2(0,\beta) - \mu_2 e_3(0,\beta) = 0.$$

So we get

$$e_2(0,\alpha) = \mu_2/\mu_3 c_3 = e_2(0,\beta).$$

Hence

$$p_1 \in L_0 = span\{1, e_0, e_1, e_2\}.$$

This implies that

$$\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta$$

which again contradicts our assumption that $\alpha \neq \beta$.

Another case is that μ_1 is not equal to 0. In this case, (10) can be rewritten as

$$e_1 = \frac{1}{\mu_1} d^1_{e_1} - \frac{1}{\mu_1} d^0_{e_1} - \frac{\lambda_1}{\mu_1} e_0,$$

and we have that e_1 is orthogonal to 1 since $d^1_{e_1}, d^0_{e_1}$ and e_0 are orthogonal to 1. Thus 1 is in $M_2 \cap L_0$.

By Theorem (1.1.1), there is an element $e \in M_2 \cap L_0$ and a number λ_0 such that

$$d^1_1 = d^0_1 + e + \lambda_0 e_0. \tag{13}$$

If $e = 0$ then $\lambda_0 = 0$, and hence $d^0_1 \perp L_0$ and

$$1 = 1(0,\alpha) = 1(0,\beta).$$

So $e \neq 0$.

Since d^1_1 is in $L_0^{\perp}$, d^1_1 is orthogonal to 1. Noting that d^0_1 and e_0 are orthogonal to 1, we have that $e \perp 1$. Hence we get an orthogonal basis $\{e_0, e_1, 1, e\}$ of L_0.

Claim.

$$e(0,\alpha) - e(0,\beta) = 0.$$

Proof of the claim. Using Theorem (1.1.1) again, we have that

$$d_e^1 = d_e^0 + g + \lambda e_0$$

for some $g \in L_0 \cap M_2$. If $g \neq 0$, we have that $g \perp 1$ since d_e^1, d_e^0, and e_0 are orthogonal to 1. Thus we have that $g = \mu e$ for some number μ to obtain

$$d_e^1 = d_e^0 + \mu e + \lambda e_0.$$

Furthermore by the linearality of $d_{(\cdot)}^1$ and $d_{(\cdot)}^0$ we have that

$$d_{e-\mu_1}^1 = d_{e-\mu_1}^0 + (\lambda - \mu\lambda_0)e_0.$$

By the same reason (namely $d_{e-\mu_1}^1 \perp L_0, d_{e-\mu_1}^0 \perp 1$ and $\langle e_0, 1\rangle \neq 0$) we have that

$$\lambda - \mu\lambda_0 = 0,$$

$$d_{e-\mu_1}^0 = d_{e-\mu_1}^1 \perp L_0$$

and

$$(e - \mu_1)(0,\alpha) = (e - \mu_1)(0,\beta) = 0.$$

Hence we have

$$e(0,\alpha) - e(0,\beta) = \mu - \mu = 0,$$

to complete the proof of the claim.

Let us find the value of λ_0 in (13) which will be used to make the coefficients symmetric with respect to α and β. To do this, we first state a technical lemma which will be used in several other places in the sequel.

Lemma (1.1.9)[1]: If g is in H^2 ($\mathbb{T}$), then

$$\langle wg\phi_0', \phi_0\rangle = g(0) + g(\alpha) + g(\beta).$$

Proof. Since ϕ_0 equals $z\phi_\alpha\phi_\beta$, simple calculations give

$$\begin{aligned}\langle wg\phi_0', \phi_0\rangle &= \langle wg(w\phi_\alpha\phi_\beta)', w\phi_\alpha\phi_\beta\rangle = \langle g(w\phi_\alpha\phi_\beta)', \phi_\alpha\phi_\beta\rangle \\ &= \langle g(\phi_\alpha\phi_\beta + w\phi_\alpha'\phi_\beta + w\phi_\alpha\phi_\beta'), \phi_\alpha\phi_\beta\rangle \\ &= \langle g, 1\rangle + \langle wg\phi_\alpha, \phi_\alpha\rangle + \langle wg\phi_\beta', \phi_\beta\rangle \\ &= g(0) + \langle wg\phi_\alpha', \phi_\alpha\rangle + \langle wg\phi_\beta', \phi_\beta\rangle\end{aligned}$$

Writing ϕ_α as

$$\phi_\alpha = -\frac{1}{\overline{\alpha}} + \frac{\frac{1}{\overline{\alpha}} - \alpha}{1 - \overline{\alpha}w} = -\frac{1}{\overline{\alpha}} + \frac{1 - |\alpha|^2}{\overline{\alpha}} k_\alpha(w),$$

we have

$$\langle wg\phi_\alpha', \phi_\alpha\rangle = \frac{1 - |\alpha|^2}{\alpha} (wg\phi_\alpha')(\alpha) = g(\alpha).$$

The first equality follows from $\langle wg\phi_\alpha', 1\rangle$ equals 0 and the second equality follows from

$$\phi_\alpha'(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

By the symmetry of α and β, similar computations lead to

$$\langle wg\phi_\beta', \phi_\beta\rangle = g(\beta)$$

and the proof is finished.

We state the values of λ_0 and $\langle e_0, e_0\rangle$ as a lemma.

Lemma (1.1.10)[1]:

$$\lambda_0 = -\frac{\alpha + \beta}{4} \tag{14}$$

$$\langle e_0, e_0\rangle = 4 \tag{15}$$

Proof. Since d_1^1 is orthogonal to L_0, e_0 is in L_0, and e is orthogonal to e_0, (13) gives

$$0 = \langle d_1^1, e_0\rangle = \langle d_1^0 + e + \lambda_0 e_0, e_0\rangle = \langle d_1^0, e_0\rangle + \lambda_0\langle e_0, e_0\rangle.$$

We need to compute $\langle d_1^0, e_0\rangle$ and $\langle e_0, e_0\rangle$ respectively.

$$\begin{aligned}\langle d_1^0, e_0\rangle &= \langle -\phi(w) + we_0, e_0\rangle = \langle we_0, e_0\rangle = \langle we_0(w,w), e_0(0,w)\rangle \\ &= \langle w(w\phi_0' + \phi_0), \phi_0\rangle = \langle w^2\phi_0', \phi_0\rangle + \langle w\phi_0, \phi_0\rangle = \langle w^2\phi_0', \phi_0\rangle \\ &= \alpha + \beta.\end{aligned}$$

The last equality follows from Lemma (1.1.9) with $g = w$.

$$\begin{aligned}\langle e_0, e_0\rangle &= \langle e_0(w,w), e_0(0,w)\rangle = \langle w\phi_0' + \phi_0, \phi_0\rangle = \langle w\phi_0', \phi_0\rangle + \langle \phi_0, \phi_0\rangle \\ &= \langle w\phi_0', \phi_0\rangle + 1 = 4,\end{aligned}$$

where the last equality follows from Lemma (1.1.9) with $g = 1$. Hence

$$\alpha + \beta + 4\lambda_0 = 0$$

and

$$\lambda_0 = -\frac{\alpha + \beta}{4}.$$

Let PL_0 denote the projection of $H^2(\mathbb{T}^2)$ onto L_0. The element $PL_0\left(k_\alpha(w) - k_\beta(w)\right)$ has the property that for any $g \in L_0$,

$$\begin{aligned}\langle g, PL_0\left(k_\alpha(w) - k_\beta(w)\right)\rangle &= \langle g, k_\alpha(w) - k_\beta(w)\rangle \\ &= g(0,\alpha) - g(0,\beta).\end{aligned}$$

Thus $PL_0\left(k_\alpha(w) - k_\beta(w)\right)$ is orthogonal to g for $g \in L_0$ with

$$g(0,\alpha) = g(0,\beta).$$

So $PL_0\left(k_\alpha(w) - k_\beta(w)\right)$ is orthogonal to $e_0, 1, e$. On the other hand,

$$\langle p_1, PL_0\left(k_\alpha(w) - k_\beta(w)\right)\rangle = \alpha - \beta \neq 0.$$

This gives that the element $PL_0\left(k_\alpha(w) - k_\beta(w)\right)$ is a nonzero element. Therefore there exists a nonzero number b such that

$$PL_0\left(k_\alpha(w) - k_\beta(w)\right) = be_1.$$

Without loss of generality we assume that

$$e_1 = PL_0\left(k_\alpha(w) - k_\beta(w)\right).$$

Observe that

$$\begin{gathered}p_1\big(\phi(z), \phi(w)\big)e_1 + d_{e_1}^1 \in M_1, \\ p_1\big(\phi(z), \phi(w)\big) + d_1^1 \in M_2, \\ M_1 \perp M_2,\end{gathered}$$

to get

$$\langle p_1\big(\phi(z), \phi(w)\big)e_1 + d_{e_1}^1, p_1\big(\phi(z), \phi(w)\big) + d_1^1\rangle = 0.$$

Thus we have

$$\begin{aligned}0 &= \langle p_1\big(\phi(z), \phi(w)\big)e_1 + d_{e_1}^1, p_1\big(\phi(z), \phi(w)\big) + d_1^1\rangle \\ &= \langle\big(\phi(z) + \phi(w)\big)e_1, \phi(z) + \phi(w)\rangle + \langle d_{e_1}^1, d_1^1\rangle \\ &= \langle d_{e_1}^1, d_1^1\rangle. \qquad (16)\end{aligned}$$

The second equality follows from

$$d_{e_1}^1, d_1^1 \in \ker T_\phi^*(z) \cap \ker T_\phi^*(z).$$

The last equality follows from

$$e_1 \perp 1$$

and

$$e_1, 1 \in \ker T_\phi^*(z) \cap \ker T_\phi^*(z).$$

Substituting (13) into Equation (16), we have

$$0 = \langle d_{e_1}^1, d_1^0 + e + \lambda_0 e_0\rangle = \langle d_{e_1}^1, d_1^0 \rangle = \langle d_{e_1}^1, -\phi(w) + we_0\rangle = \langle d_{e_1}^1, we_0\rangle$$
$$= \langle d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0, we_0\rangle = \langle d_{e_1}^0, we_0\rangle + \mu_1\langle e_1, we_0\rangle + \lambda_1\langle e_0, we_0\rangle.$$

The second equation comes from that $d_{e_1}^1$ is orthogonal to L_0 and both e and e_0 are in L_0. The third equation follows from the definition of d_1^0 and the forth equation follows from that $d_{e_1}^1$ is in $\ker T_\phi^*(z) \cap \ker T_\phi^*(w)$. We need to calculate $\langle d_{e_1}^0, we_0\rangle, \langle e_1, we_0\rangle$, and $\langle e_0, we_0\rangle$ separately.

To get $\langle d_{e_1}^0, we_0\rangle$, by the definition of $d_{e_1}^0$, we have

$$\langle d_{e_1}^0, we_0\rangle = \langle -\phi(w)e_1 + we_1(0,w)e_0, we_0\rangle$$
$$= \langle -\phi(w)e_1, we_0\rangle + \langle we_1(0,w)e_0, we_0\rangle$$

Thus we need to compute $\langle -\phi(w)e_1, we_0\rangle$ and $\langle we_1(0,w)e_0, we_0\rangle$ one by one. The equality

$$\langle -\phi(w)e_1, we_0\rangle = 0$$

follows from the following computations.

$$\langle -\phi(w)e_1, we_0\rangle = \langle -w\phi_0(w)e_1, we_0\rangle = -\langle \phi_0(w)e_1, e_0\rangle$$
$$= -\langle \phi_0(w)e_1(w,w), e_0(0,w)\rangle = -\langle \phi_0(w)e_1(w,w), \phi_0(w)\rangle$$
$$= -\langle e_1(w,w), 1\rangle = -\langle e_1, 1\rangle = 0.$$

To get $\langle we_1(0,w)e_0, we_0\rangle$, we continue as follows.

$$\langle we_1(0,w)e_0, we_0\rangle = \langle e_1(0,w)e_0, e_0\rangle$$
$$= \langle e_1(0,w)e_0(w,w), e_0(0,w)\rangle$$
$$= \langle e_1(0,w)e_0(w,w), \phi_0(w)\rangle$$
$$= \langle e_1(0,w)\big(\phi_0(w) + w\phi_0'(w)\big), \phi_0(w)\rangle$$
$$= \langle e_1(0,w)\phi_0(w), \phi_0(w)\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle$$
$$= \langle e_1(0,w), 1\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle$$
$$= e_1(0,0) + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle$$
$$= \langle e_1, 1\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle$$
$$= \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle$$
$$= e_1(0,\alpha) + e_1(0,\beta).$$

The last equality follows from Lemma (1.1.9) and

$$e_1(0,0) = \langle e_1, 1\rangle = 0.$$

Hence

$$\langle d_{e_1}^0, we_0\rangle = e_1(0,\alpha) + e_1(0,\beta)$$

Recall that

$$d_1^1 = d_1^0 + e + \lambda_0 e_0$$

is orthogonal to L_0 and e_1 is orthogonal to both e, and e_0. Thus

$$0 = \langle e_1, d_1^0 + e + \lambda_0 e_0\rangle = \langle e_1, -\phi(w) + we_0\rangle = \langle e_1, we_0\rangle.$$

From the computation of $\langle d_1^0, e_0\rangle$ in the proof of Lemma (1.1.10) we have showed that

$$\langle we_0, e_0\rangle = \alpha + \beta.$$

Therefore we have that

$$e_1(0,\alpha) + e_1(0,\beta) + \lambda_1\big(\overline{\alpha} + \overline{\beta}\big) = 0. \tag{17}$$

On the other hand,

$$0 = \langle d_{e_1}^1, e_0\rangle = \langle d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0, e_0\rangle = \langle d_{e_1}^0, e_0\rangle + 4\lambda_1$$

and

$$\begin{aligned}\langle d^0_{e_1}, e_0\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, e_0\rangle = \langle we_1(0,w)e_0, e_0\rangle \\ &= \langle we_1(0,w)e_0(w,w), e_0(0,w)\rangle = \langle we_1(0,w)(\phi_0(w) + w\phi_0'), \phi_0(w)\rangle \\ &= \langle w^2 e_1(0,w)\phi_0', \phi_0(w)\rangle = \alpha e_1(0,\alpha) + \beta e_1(0,\beta).\end{aligned}$$

The last equality follows from Lemma (1.1.9) with $g = we_1(0,w)$. Thus

$$\alpha e_1(0,\alpha) + \beta e_1(0,\beta) + 4\lambda_1 = 0.$$

So

$$\lambda_1 = -\frac{\alpha}{4} e_1(0,\alpha) - \frac{\beta}{4} e_1(0,\beta). \tag{18}$$

Substituting (18) into (17), we have

$$\left[1 - \frac{\alpha(\overline{\alpha} + \overline{\beta})}{4}\right] e_1(0,\alpha) + \left[1 - \frac{\beta(\overline{\alpha} + \overline{\beta})}{4}\right] e_1(0,\beta) = 0.$$

Recall that

$$\lambda_0 = -\frac{\alpha + \beta}{4},$$

to get

$$\left(1 + \overline{\lambda}_0\alpha\right)e_1(0,\alpha) + \left(1 + \overline{\lambda}_0\beta\right)e_1(0,\beta) = 0. \tag{19}$$

We are going to draw another equation about $e_1(0,\alpha)$ and $e_1(0,\beta)$ from the property that $d^1_{e_1}$ is orthogonal to L_0. To do this, recall that

$$\begin{aligned}e_1 &= PL_0\left(k_\alpha(w) - k_\beta(w)\right) \in M_1 \cap L_0, \\ d^1_{e_1} &= d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0 \perp L_0, \\ L_0 &= span\{1, p_1, e_\alpha, e_\beta\}, \\ e_\alpha &= k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w).\end{aligned}$$

Thus $d^1_{e_1}$ is orthogonal to p_1, e_α and e_β.

Since $d^1_{e_1}$ is orthogonal to p_1 we have

$$\langle d^0_{e_1}, p_1\rangle + \mu_1\langle e_1, p_1\rangle + \lambda_1\langle e_0, p_1\rangle = 0.$$

Noting

$$\begin{aligned}\langle d^0_{e_1}, p_1\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, p_1\rangle = \langle we_1(0,w)e_0, p_1\rangle \\ &= \langle we_1(0,w)e_0(w,w), w\rangle = \langle e_1(0,w)e_0(w,w), 1\rangle = 0, \\ \langle e_1, p_1\rangle &= \langle PL_0\left(K_\alpha(w) - K_\beta(w)\right), p_1\rangle = \langle K_\alpha(w) - K_\beta(w), p_1\rangle = \overline{\alpha} - \overline{\beta},\end{aligned}$$

and

$$\begin{aligned}\langle e_0, p_1\rangle &= \langle e_0(0,w), p_1(w,w)\rangle = \langle \phi_0(w), 2w\rangle = \langle w\phi_\alpha\phi_\beta, 2w\rangle = 2\langle \phi_\alpha\phi_\beta, 1\rangle \\ &= 2\phi_\alpha(0)\phi_\beta(0) = 2\alpha\beta,\end{aligned}$$

we have

$$\left(\tilde{\alpha} - \overline{\beta}\right)\mu_1 + 2\alpha\beta\lambda_1 = 0,$$

to obtain

$$\lambda_1 = -\mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta}. \tag{20}$$

Since $d^1_{e_1} \perp e_\alpha$, we have

$$\langle d^0_{e_1}, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle + \lambda_1\langle e_0, e_\alpha\rangle = 0,$$

to get

$$\langle d_{e_1}^0, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta}\langle e_0, e_\alpha\rangle = 0. \qquad (21)$$

We need to calculate $\langle d_{e_1}^0, e_\alpha\rangle, \langle e_1, e_\alpha\rangle$ and $\langle e_0, e_\alpha\rangle$. Simple calculations show that

$$\langle d_{e_1}^0, e_\alpha\rangle = \langle -\phi(w)e_1 + we_1(0,w)e_0, e_\alpha\rangle = \langle we_1(0,w)e_0, e_\alpha\rangle = \alpha e_1(0,\alpha)e_0(\alpha,\alpha),$$

$$\langle e_1, e_\alpha\rangle = e_1(\alpha,\alpha) = \langle PL_0\left(k_\alpha(w) - k_\beta(w)\right), e_\alpha\rangle = \langle k_\alpha(w) - k_\beta(w), e_\alpha\rangle$$

$$= \frac{1}{1 - |\alpha|^2} - \frac{1}{1 - \alpha\overline{\beta}}$$

$$= \frac{\alpha(\overline{\alpha} - \overline{\beta})}{(1 - |\alpha|^2)(1 - \alpha\overline{\beta})}, \qquad (22)$$

and

$$\langle e_0, e_\alpha\rangle = e_0(\alpha,\alpha) = \alpha\phi_0'(\alpha) + \phi_0(\alpha)$$

$$= \alpha^2 \frac{1}{1 - |\alpha|^2}\frac{\alpha - \beta}{1 - \alpha\overline{\beta}}. \qquad (23)$$

Thus (22) and (23) give

$$\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)} = \frac{\overline{\alpha} - \overline{\beta}}{\alpha(\alpha - \beta)}.$$

Substituting the above equality in Equation (21) leads to

$$\alpha e_1(0,\alpha)e_0(\alpha,\alpha) + \mu_1 e_1(\alpha,\alpha) - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} e_0(\alpha,\alpha) = 0.$$

Dividing the both sides of the above equality by $e_0(\alpha,\alpha)$ gives

$$\alpha e_1(0,\alpha) + \mu_1\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)} - \mu_1\frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} = 0.$$

Hence we have

$$\alpha e_1(0,\alpha) + \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{\alpha(\alpha - \beta)} - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} = 0,$$

to obtain

$$\alpha e_1(0,\alpha) + (\beta + \lambda_0)\frac{2\mu_1(\overline{\alpha} - \overline{\beta})}{\alpha\beta(\alpha - \beta)} = 0. \qquad (24)$$

Similarly, since $d_{e_1}^1$ is orthogonal to e_β, we have

$$\langle d_{e_1}^0, e_\beta\rangle + \mu_1\langle e_1, e_\beta\rangle + \lambda_1\langle e_0, e_\beta\rangle = 0,$$

to obtain

$$\langle d_{e_1}^0, e_\beta\rangle + \mu_1\langle e_1, e_\beta\rangle - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta}\langle e_0, e_\beta\rangle = 0. \qquad (25)$$

We need to calculate $\langle d_{e_1}^0, e_\beta\rangle, \langle e_1, e_\beta\rangle$ and $\langle e_0, e_\beta\rangle$. Simple calculations as above show that

$$\langle d_{e_1}^0, e_\beta\rangle = \langle -\phi(w)e_1 + we_1(0,w)e_0, e_\beta\rangle = \langle we_1(0,w)e_0, e_\beta\rangle$$

$$= \beta e_1(0,\beta)e_0(\beta,\beta),$$

$$\langle e_1, e_\beta\rangle = e_1(\beta,\beta) = \langle PL_0\left(k_\alpha(w) - k_\beta(w)\right), e_\beta\rangle = \langle k_\alpha(w) - k_\beta(w), e_\beta\rangle$$

$$= \frac{1}{1 - \alpha\overline{\beta}} - \frac{1}{1 - |\beta|^2}$$

$$= \frac{\beta(\overline{\alpha} - \overline{\beta})}{(1 - \alpha\overline{\beta})(1 - |\beta|^2)} \tag{26}$$

$$\langle e_0, e_\beta \rangle = e_0(\beta, \beta) = \beta\phi_0'(\beta) + \phi_0(\beta)$$

$$= \beta^2 \frac{\beta - \alpha}{1 - \alpha\overline{\beta}} \frac{1}{1 - |\beta|^2} \tag{27}$$

Combining (26) with (27) gives

$$\frac{e_1(\beta, \beta)}{e_0(\beta, \beta)} = -\frac{\overline{\alpha} - \overline{\beta}}{\beta(\alpha - \beta)}.$$

Substituting the above equality in (25) gives

$$\beta e_1(0, \beta)e_0(\beta, \beta) + \mu_1 e_1(\beta, \beta) - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} e_0(\beta, \beta) = 0.$$

Dividing both sides of the above equality by $e_0(\beta, \beta)$ gives

$$\beta e_1(0, \beta) + \mu_1 \frac{e_1(\beta, \beta)}{e_0(\beta, \beta)} - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} = 0$$

Hence we have

$$\beta e_1(0, \beta) - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{\beta(\alpha - \beta)} - \mu_1 \frac{\overline{\alpha} - \overline{\beta}}{2\alpha\beta} = 0,$$

to get

$$\beta e_1(0, \beta) - (\alpha + \lambda_0)\frac{2\mu_1(\overline{\alpha} - \overline{\beta})}{\alpha\beta(\alpha - \beta)} = 0. \tag{28}$$

Eliminating $\frac{2\mu_1(\overline{\alpha}-\overline{\beta})}{\alpha\beta(\alpha-\beta)}$ from (24) and (28) gives

$$\alpha(\alpha + \lambda_0)e_1(0, \alpha) + \beta(\beta + \lambda_0)e_1(0, \beta) = 0. \tag{29}$$

Now combining (19) and (29), we have the following linear system of equations about $e_1(0, \alpha)$ and

$$\begin{aligned} e_1(0, \beta)(1 + \overline{\lambda}_0\alpha)e_1(0, \alpha) + (1 + \overline{\lambda}_0\beta)e_1(0, \beta) = 0 \\ \alpha(\alpha + \lambda_0)e_1(0, \alpha) + \beta(\beta + \lambda_0)e_1(0, \beta) = 0. \end{aligned} \tag{30}$$

If

$$e_1(0, \alpha) = e_1(0, \beta) = 0,$$

then p_1 is in $L_0 = \text{span}\{e_0, e_1, 1, e\}$. But noting

$$e_0(0, \alpha) = e_0(0, \beta)$$

and

$$e(0, \alpha) = e(0, \beta)$$

we have

$$p_1(0, \alpha) = p_1(0, \beta),$$

which contradicts the assumption that $\alpha \neq \beta$. So at least one of $e_1(0, \alpha)$ and $e_1(0, \beta)$ is nonzero. Then the determinant of the coefficient matrix of System (30) has to be zero. This implies

$$\begin{vmatrix} 1 + \overline{\lambda}_0\alpha & 1 + \overline{\lambda}_0\beta \\ \alpha(\alpha + \lambda_0) & \beta(\beta + \lambda_0) \end{vmatrix} = 0$$

Making elementary row reductions on the above the determinant, we get

$$\begin{vmatrix} (\alpha - \beta)\overline{\lambda}_0 & 1 + \overline{\lambda}_0\beta \\ (\alpha - \beta)(\alpha + \beta + \lambda_0) & \beta(\beta + \lambda_0) \end{vmatrix} = 0.$$

Since

$$\alpha + \beta = -4\lambda_0$$

and

$$\alpha - \beta \neq 0,$$

we have

$$\begin{vmatrix} \overline{\lambda}_0 & 1 + \overline{\lambda}_0\beta \\ -3\lambda_0 & \beta(\beta + \lambda_0) \end{vmatrix} = 0.$$

Expanding this determinant we have

$$0 = \overline{\lambda}_0(\beta^2 + \beta\lambda_0) + 3\lambda_0\left(1 + \overline{\lambda}_0\beta\right) = \overline{\lambda}_0(\beta^2 + \beta\lambda_0 + 3\beta\lambda_0) + 3\lambda_0$$

$$= \overline{\lambda}_0(\beta^2 + 4\beta\lambda_0) + 3\lambda_0 = \overline{\lambda}_0(-\alpha\beta) + 3\lambda_0$$

Taking absolute value on both sides of the above equation, we have

$$0 = \left|\overline{\lambda}_0(-\alpha\beta) + 3\lambda_0\right| \geq |\lambda_0|(3 - |\alpha\beta|) \geq 2|\lambda_0|,$$

to get

$$\lambda_0 = 0.$$

This implies

$$\alpha + \beta = 0,$$

to complete the proof.

Section (1.2): Analytic Multipliers of the Bergman Space

The present is a continuation of [27] and a series of recent related works, such as [28], [29], [10]. We classify the reducing subspaces of analytic Toeplitz operators with a rational, inner symbol acting on the Bergman space of the unit disk. While a similar study in the case of the Hardy space was completed a long time ago (see [26], [33], [34]), investigation of the Bergman space setting was started only a few years ago. The structure and relative position of these reducing subspaces in the Bergman space reveal a rich geometric (Riemann surface) picture directly dependent on the rational symbol of the Toeplitz operator.

The Bergman space $L_a^2(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$ which are square-integrable with respect to the Lebesgue measure dm on $\mathbb{D}$. For a bounded holomorphic function ϕ on the unit disk, the multiplication operator, $M_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$, is defined by

$$M_\phi(h) = \phi h, \qquad h \in L_a^2(\mathbb{D}).$$

The Toeplitz operator T_ϕ on $L_a^2(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$ acts as

$$T_\phi(h) = P(\phi h), \qquad h \in L_a^2,$$

where P is the orthogonal projection from $L^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$. Note that $T_\phi = M_\phi$ whenever ϕ is holomorphic.

An invariant subspace $\mathcal{M}$ for M_ϕ is a closed subspace of $L_a^2(\mathbb{D})$ satisfying $\phi\mathcal{M} \subseteq \mathcal{M}$. If, in addition, $M_\phi^*\mathcal{M} \subseteq \mathcal{M}$, we call $\mathcal{M}$ a reducing subspace of M_ϕ. We say $\mathcal{M}$ is a minimal reducing subspace if there is no nontribyl reducing subspace for M_ϕ contained in $\mathcal{M}$. The study of invariant subspaces and reducing subspaces for various classes of linear operators has inspired much deep research and prompted many interesting problems. Even for the multiplication operator M_z, the lattice of invariant subspaces of $L_a^2(\mathbb{D})$ is huge and its order structure remains a mystery. Progress in understanding the lattice of reducing subspaces of M_ϕ was only recently made, and only in the case of inner function symbols [27]–[10], [32], [1], [22].

For $\{M_\phi\}' = \{X \in \mathcal{L}\ (L_a^2(\mathbb{D})): M_\phi X = XM_\phi\}$ be the commutant algebra of M_ϕ. The problem of classifying the reducing subspaces of M_ϕ is equivalent to finding the projections in $\{M_\phi\}'$. This classification problem in the case of the Hardy space was the motivation of the highly original works by Thomson and Cowen (see [26],[33],[34]). They used the Riemann surface of $\phi^{-1} \circ \phi$ as a basis for the description of the commutant of M_ϕ acting on the Hardy space. We study that inner function symbols played a dominant role in their studies. In complete analogy, in the Bergman space $L_a^2(\mathbb{D})$ framework, one can essentially use the same proof to show that for a "nice" analytic function f, there exists a finite Blaschke product ϕ such that $\{M_f\}' = \{M_\phi\}'$. Therefore, the structure of the reducing subspaces of the multiplier M_f on the Bergman space of the disk is the same as that for M_ϕ.

Zhu showed in [22] that for each Blaschke product of order 2, there exist exactly 2 different minimal reducing subspaces of M_ϕ. This result also appeared in [32]. Zhu also conjectured in [22] that M_ϕ has exactly n distinct minimal reducing subspaces for a Blaschke product ϕ of order n. The results in [10] disproved Zhu's conjecture, they raised a modification in which M_ϕ was conjectured to have at most n distinct minimal reducing subspaces for a Blaschke product ϕ of order n. Some partial results on this conjecture were obtained in [28],[10],[1]. They proved the finiteness result in case $n \leq 6$, each using a different method. A notable result for the general case [10] is that there always exists a nontribyl minimal reducing subspace $\mathcal{M}$, named the "distinguish subspace", on which the action of M_ϕ is unitarily equivalent to the action of M_z on the Bergman space $L_a^2(\mathbb{D})$. Guo and Huang also revealed in [29] an interesting connection between the structure of the lattice of reducing subspaces of M_ϕ and an isomorphism problem in abstract von Neumann algebras.

The general case was recently studied by Sun and Zheng [27] using a systematic analysis of the local inverses of the ramified finite fibration $\phi^{-1} \circ \phi$ over the disk. They proved that the linear dimension of the commutant $A_\phi = \{M_\phi, M_\phi^*\}'$ is finite. To give a glimpse into the reasoning culminating with the finite dimensionality of the von Neumann algebra A_ϕ we recall that M_ϕ is an operator belonging to the Cowen–Douglas class, that is, the iso-dimensional family of kernels $\ker(M_\phi^* - \bar{z}), \bar{z} \in \mathbb{D},$ is an anti-holomorphic hermitian vector bundle E_ϕ on the disk. An operator X commuting with T_ϕ^* leaves these kernels invariant: $X\left(\ker\left(M_\phi^* - \bar{z}\right)\right) \subset \ker\left(M_\phi^* - \bar{z}\right)$, whence it defines an anti-holomorphic bundle map $X : E_\phi \to E_\phi$. Moreover, if X commutes in addition with M_ϕ, then X is also holomorphic, that is X is an endomorphism of the space of E_ϕ. Thus, the fiber $X(z_0)$ at a prescribed point $z_0 \in \mathbb{D}$ determines the full operator X, and consequently the algebra A_ϕ is finite dimensional. Then the geometry of the branched covering map ϕ takes over, implying, by arguments of the theory of subnormal operators, that $\dim_{\mathbb{C}} A_\phi$ equals the number of connected components of the Riemann surface $\phi^{-1} \circ \phi$. In particular, the number of pairwise orthogonal reducing subspaces of M_ϕ is finite. Furthermore, they raised the following question in [27], whose validity they have established in degree $n \leq 8$.

For a Blaschke product ϕ of finite order, the double commutant algebra A_ϕ is abelian.

Several notable corollaries would follow once the conjecture is verified. For instance, the commutativity of the algebra A_ϕ implies that, for every finite Blaschke product ϕ, the minimal reducing subspaces of M_ϕ are mutually orthogonal; in addition, their number is equal to the number q of connected components of the Riemann surface of $\phi^{-1} \circ \phi$.

The main result offers an affirmative answer to the above problem.

Theorem (1.2.1)[23]: Let ϕ be a finite Blaschke product of order n. Then the von Neumann algebra $A_\phi = \{M_\phi, M_\phi^*\}'$ is commutative of dimension q, and hence $A_\phi \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{q}$, where q is the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$.

The key observation for the proof is that there is an invertible holomorphic function u such that $\phi = u^n$ on Ω, where Ω is a domain in $\mathbb{D}$ including an annulus of all points sufficiently close to the boundary $\mathbb{T}$. This representation provides a canonical ordered set of local inverses which implies that the local inverses for $\phi^{-1} \circ \phi$ commute under composition on Ω.

It also allows us to provide an indirect description of the reducing subspaces. Following [27], there is a partition $\{G_1, \ldots, G_q\}$ of the local inverses for $\phi^{-1} \circ \phi$. We now define a dual partition as follows. For two integers $0 \le j_1, j_2 \le n-1$, write $j_1 \sim j_2$ if

$$\sum_{\rho_k \in G_i} \zeta^{kj_1} = \sum_{\rho_k \in G_i} \zeta^{kj_2} \text{ for any } 1 \le i \le q. \quad (31)$$

Observing that $\sim$ is an equivalence relation, we partition the set $\{0, 1, \ldots, n-1\}$ into equivalence classes $\{G'_1, \ldots, G'_p\}$. Some information on the Riemann surface of $\phi^{-1} \circ \phi$ is given by the following corollary.

Corollary (1.2.2)[23]: The number of components in the dual partition is also equal to q, the number of connected components of the Riemann surface for $\phi^{-1} \circ \phi$

We obtain the following characterization for the minimal reducing subspace of automorphic type. Here $\mathcal{O}(\mathbb{D})$ denotes the space of holomorphic functions on $\mathbb{D}$.

Theorem (1.2.3)[23]: Let ϕ be a finite Blaschke product and $\{G'_1, \ldots, G'_q\}$ be the dual partition for ϕ. Then the multiplication operator M_ϕ has exactly q nontribyl minimal reducing subspaces $\{\mathcal{M}_1, \ldots, \mathcal{M}_q\}$, and for any $1 \le j \le q$

$$\mathcal{M}_j = \{f \in \mathcal{O}(\mathbb{D}): f|_\Omega \in \mathcal{L}_j^\Omega\},$$

where $\mathcal{L}_j^\Omega$ is a subspace of $L^2(\Omega)$ with the orthogonal basis $\{u^i u' : i + 1 \pmod{n} \in G'_j\}$.

Note the $\mathcal{M}_{n-1}$ coincides with the distinguished reducing subspace for M_ϕ shown to exist in [10]. The latter theorem provides a possible way to calculate the reducing subspace if one knows the partition of the family of local inverses. The above corollary hints that the possible partitions are very restricted.

We list some algebraic conditions for the partitions, which offer an arithmetic path towards the classification of finite Blaschke products. The idea is displayed by the classification for the Blaschke products of order 8. In a similar way one can also explain the classifications of the Blaschke products of order 3 or 4 in [10], [1], which have been established by identifying the Bergman space of the disk with the restriction of the Hardy space of the bidisk to the diagonal. We point out that these results and examples provide some very detailed information about the branch covering space defined by a finite Blaschke product.

The notation below is borrowed from [27]. Accordingly, ϕ is a finite Blaschke product having n zeros taking multiplicity into account. The finite set $E' = \phi^{-1}(\phi(\{\beta \in \mathbb{D}: \phi'(\beta) = 0\}))$ denotes the branch points of $\phi, E = \mathbb{D} \backslash E'$ is its complement in $\mathbb{D}$ and let Γ be a choice of curves passing through all points of E' and a fixed point on the unit circle β_0 such that $\mathbb{D}\backslash\Gamma$ is a simply connected region contained in E. Indeed, to be precise, one can construct Γ as follows: order E' as $\{\beta_1, \beta_2, \ldots, \beta_s\}$ such that $k \le j$ iff $Re\beta_k < Re\beta_j$ or $Re\beta_k = Re\beta_j$ and $Im\beta_k \le Im\beta_j$, and set $\beta_0 = Re\beta_1 - i\sqrt{1-(Re\beta_1)^2}$. Letting Γ_k, $0 \le k \le s-1$, be the line segment between β_k and β_{k+1}, we define

$$\Gamma = \bigcup_{0\le k\le s-1} \Gamma_k. \tag{32}$$

By an observation made in [27], the family of analytic local inverses $\{\rho_0, \ldots, \rho_{n-1}\}$ for $\phi^{-1} \circ \phi$ is well defined on $\mathbb{D}\backslash\Gamma$. That is, each ρ_j is a holomorphic function on $\mathbb{D}\backslash\Gamma$ which satisfies $\phi(\rho_j(z)) = \phi(z)$ for $z \in \mathbb{D}\backslash\Gamma$. We define the equivalence relation on the set of local inverseso that $\rho_i \sim \rho_j$ if there exists an arc γ in E such that ρ_i and ρ_j are analytic continuations of each other along γ. The resulting equivalence classes are denoted $\{G_1, \ldots, G_q\}$. For each G_k, $1 \le k \le q$, define the map $\mathcal{E}_k$:

$$(\mathcal{E}_k f)(z) = \sum_{\rho\in G_k} f\,\rho(z)\,\rho'(z), \qquad f \text{ holomorphic on } \mathbb{D}\backslash\Gamma, z \in \mathbb{D}\backslash\Gamma.$$

The central result in [27] asserts that the operators $\{\mathcal{E}_1, \ldots, \mathcal{E}_q\}$ can naturally be extended to bounded operators on the Bergman space $L_a^2(\mathbb{D})$ which are linearly independent, and the double commutant algebra A_ϕ is linearly generated by these operators; that is,

$$A_\phi = \{M_\phi, M_\phi^*\}' = \operatorname{span}\{\mathcal{E}_1, \ldots, \mathcal{E}_q\}.$$

We prove that the von Neumann algebra A_ϕ is commutative.

To accomplish this, we extend the given family of analytic local inverses on $\mathbb{D}\backslash\Gamma$ to a larger region and prove that they commute under composition near the boundary of $\mathbb{D}$. The key observation for the proof of the following lemma is that $\sqrt[n]{(z-a_1)\cdots(z-a_n)}$ is a single-valued holomorphic function on $\mathbb{C}\backslash L$, where L is a curve drawn through the zero set $\{a_1, a_2, \ldots, a_n\}$. One can construct an L and verify the above assertion as follows. Notice that $\sqrt[n]{z+1}$ is holomorphic outside any smooth simple curve connecting -1 and ∞. By changing variables, we observe that, for each $2 \le i \le n$, the function

$$\sqrt[n]{\frac{z-a_i}{z-a_1}} = \sqrt[n]{\frac{a_1-a_i}{z-a_1}+1}$$

is holomorphic outside the line segment connecting a_1 and a_i. Therefore,

$$\sqrt[n]{(z-a_1)\cdots(z-a_n)} = (z-a_1)\sqrt[n]{\frac{z-a_2}{z-a_1}} \cdots \sqrt[n]{\frac{z-a_n}{z-a_1}}$$

is holomorphic outside the arc which consists of the line segments connecting a_1 and a_i for $2 \le i \le n$. See [31] for a complete argument.

Hereafter, let us set $A_r = \{z \in \mathbb{C}: r < |z| < 1\}$ for any $0 < r < 1$, and let $\zeta = e^{\frac{2i\pi}{n}}$ be a primitive n-th root of unity.

Lemma (1.2.4)[23]: For a finite Blaschke product ϕ of order n, there exists a holomorphic function u on a neighborhood of $\mathbb{D}\backslash L$ such that $\phi = u^n$, where L is an arc inside $\mathbb{D}$ containing the zero set of ϕ. Moreover, there exists $0 < r < 1$ such that $\overline{A_r}$ is contained in the image of u and $u: u^{-1}(\overline{Ar}) \to \overline{Ar}$ is invertible.

Proof. Suppose $a_1, \dots, a_n$ are the zeros of ϕ in $\mathbb{D}$ (taking multiplicity into account). Choose an analytic branch for $w = \sqrt[n]{z}$. By [31], $w = \sqrt[n]{(z - a_1) \cdots (z - a_n)}$ is a singlevalued holomorphic function on $\mathbb{C}\backslash L$, where L is a curve drawn through the zero set. If we set

$$u(z) = \frac{\sqrt[n]{(z - a_1) \cdots (z - a_n)}}{\sqrt[n]{(z - \overline{a_1}z) \cdots (z - \overline{a_n}z)}},$$

then $u(z)$ is holomorphic on a neighborhood of $\mathbb{D}\backslash L$ and $u^n = \phi$.

Additionally, one sees that $|u|^n = |\phi|$ on $\mathbb{D}\backslash L$ and hence $u(\mathbb{T}) \subseteq \mathbb{T}$. We claim that $u(\mathbb{T}) = \mathbb{T}$. Indeed, if $u(\mathbb{T}) \neq \mathbb{T}$, then $u : \mathbb{T} \to \mathbb{T}$ is homotopic to a constant map on $\mathbb{T}$. That is, there exists $u(\theta, t) \in C(\mathbb{T} \times [0,1], \mathbb{T})$ such that $u(\theta, 0) = u(\theta)$ and $u(\theta, 1) = 1$. This implies that $\phi = u^n : \mathbb{T} \to \mathbb{T}$ is also homotopic to the constant map by the path $t \to u^n(\cdot, t)$. If we extend each $u(\cdot, t)$ to be a continuous function $\widetilde{u}(\cdot, t)$ on $\overline{\mathbb{D}}$, then by [25] each Toeplitz operator $T_{\widetilde{u}^n(\cdot,t)}$ is Fredholm. Furthermore, using [25] one sees that $t \to \operatorname{Ind}(T_{\widetilde{u}^n(\cdot,t)})$ is a continuous map from [0, 1] to $\mathbb{Z}$. This implies that it is a constant map, which leads to a contradiction since $-n = \operatorname{Ind}(M_\phi) = \operatorname{Ind}(T_{\widetilde{u}^n(\cdot,0)}) = \operatorname{Ind}(T_{\widetilde{u}^n(\cdot,1)}) = \operatorname{Ind}(M_1) = 0$. Therefore, we have that $u(\mathbb{T}) = \mathbb{T}$.

By the open mapping theorem, the image of u is an open subset of $\mathbb{C}$ including $\mathbb{T}$. Therefore, there exists $0 < r < 1$ such that $\overline{A_r} \subseteq u(\overline{\mathbb{D}}\backslash L)$. Now we only need to prove that the map $u : u^{-1}(\overline{A_r}) \to \overline{A_r}$ is injective. In fact, for any $w \in \overline{A_r}$, since $\phi\left(u^{-1}(\zeta^k w)\right) = w^n$ for $0 \leq k \leq n - 1$, we have that

$$\bigcup_{0 \leq k \leq n-1} u^{-1}(\{\zeta^k w\}) \subseteq \phi^{-1}(\{w^n\}).$$

Remarking that the set $\phi^{-1}(\{w^n\})$ includes at most n points and each set $u^{-1}(\{\zeta^k w\})$ is nonempty, one sees that each $u^{-1}(\{\zeta^k w\})$ is a singleton. This means that u is one to one on $u^{-1}(\overline{A_r})$. Therefore, $u : u^{-1}(\overline{A_r}) \to \overline{A_r}$ is invertible, completing the proof.

The above lemma allows us to extend local inverses as follows. We denote $\Omega = u^{-1}(A_r)$, where A_r is the annulus appearing in Lemma (1.2.4). On the connected domain Ω, define $\tilde{\rho}_k(z) = u^{-1}\left(\zeta^k u(z)\right)$ for each $0 \leq k \leq n - 1$. Note that $\tilde{\rho}_k$ is holomorphic and $\phi(\tilde{\rho}_k(z)) = \phi(z)$ for $z \in \Omega$. This means that $\{\tilde{\rho}_k\}_k$ is also the family of local inverses on Ω for $\phi^{-1} \circ \phi$. It follows that $\rho_k = \tilde{\rho}_{i_k}$ for some i_k on $\Omega \cap [\mathbb{D}\backslash\Gamma]$. Matching the maps $\tilde{\rho}_{i_k}$ and ρ_k , respectively, we obtain the family of local inverses on a larger domain $\Omega \cup [\mathbb{D}\backslash\Gamma]$. Furthermore, we can prove the following lemma.

Lemma (1.2.5)[23]: For a finite Blaschke product ϕ, there exists a family of local inverses for $\phi^{-1} \circ \phi$ on the domain $\mathbb{D}\backslash\Gamma$, where $\Gamma' = \bigcup_{1 \leq k \leq s-1} \Gamma_i$ is a proper subset of Γ appearing in (32), which just consists of the set of line segments passing through all critical points E' of ϕ.

Proof. It suffices to show that the family of local inverses $\{\rho_0, \rho_1, \dots, \rho_{n-1}\}$ can analytically be continued across the interior point set $\dot{\Gamma}_0 = \{t\beta_0 + (1 - t)\beta_1 : 0 < t < 1\}$.

To start, we prove that analytic continuation is possible when the points in $\dot{\Gamma}_0$ are close enough to the boundary $\mathbb{T}$. By the continuity of u and the construction of Γ , we can choose a number r close to 1 such that $u(A_{r'}) \subset A_r$ and $A_{r'} \cap \Gamma' = \emptyset$. For each $0 \leq k \leq n - 1$, let $\tilde{\rho}_k(z) = u^{-1}\left(\zeta^k u(z)\right)$ when $z \in A_{r'} \left(\subseteq u^{-1}(A_r)\right)$. Fix a point $z_0 \in A_{r'} \cap [\mathbb{D}\backslash\Gamma]$, and let U be a small open disk containing z_0. Notice that both $\{\rho_0, \rho_1, \dots, \rho_{n-1}\}$ and

$\{\tilde{\rho}_0, \tilde{\rho}_1, \ldots, \tilde{\rho}_{n-1}\}$are local inverses of $\phi^{-1} \circ \phi$ on U. So, after renumbering the local inverses if necessary, we can suppose that $\rho_i = \tilde{\rho}_i$ on U. Since the domain $A_{r'} \cap [\mathbb{D}\backslash\Gamma] = A_{r'}\backslash\Gamma_0$ is connected and includes U, one sees that $\rho_i = \tilde{\rho}_i$ on this domain. Therefore, the family of analytic functions $\{\rho_i \cup \tilde{\rho}_i\}$ defined as

$$[\rho_i \cup \tilde{\rho}_i](x) = \begin{cases} \rho_i(x) & \text{if } x \in \mathbb{D}\backslash\Gamma, \\ \tilde{\rho}_i(x) & \text{if } x \in A_{r'} \end{cases}$$

are local inverses on $A_{r'} \cup [\mathbb{D}\backslash\Gamma']$. We still denote them by $\{\rho_i\}_i$ whenever no confusion arises.

Now let S be a maximal subset of $\dot{\Gamma}_0$ on which these local inverses can't be analytically continued across. That is, $\{\rho_i\}_i$ are holomorphic on the domain $\mathbb{D}\backslash(\Gamma' \cup S)$, and can't be analytically continued across each point in S. We prove by contradiction that the set S is empty. Indeed, assume S is nonempty and let

$$s = \inf\{t:\ t\beta_0 + (1-t)\beta_1 \in S\}.$$

Then S is contained in the line segment from $z_0 = s\beta_0 + (1-s)\beta_1$ to β_1. Since $S \cap A_{r'} = \emptyset$, one sees that $0 < s$ and z_0 is inside $\mathbb{D}$. This means that one can analytically extend the local inverses across $t\beta_0 + (1-t)\beta_1 : t < s$,
and the process stops at z_0. But, since z_0 is a regular point of ϕ, there exists an open disk $V = \{z: |z - z_0| < r_0\}$ with a small r_0, with a small r_0, such that $V \cap \Gamma' = \emptyset$ and $\phi^{-1} \circ \phi$ has n analytic branches on V . Notice that

$$V \cap [\mathbb{D}\backslash (\Gamma' \cup S)] = V\backslash S \supseteq V\backslash L,$$

where L is a line segment from the center z_0 to the boundary of the disk V . It follows that $V \cap [\mathbb{D}\backslash(\Gamma' \cup S)]$ is a connected domain. An argument similar to that in the preceding paragraph shows that the local inverses are holomorphic on $V \cup [\mathbb{D}\backslash(\Gamma' \cup S)]$. By the maximality of S, we have that $V \cap S = \emptyset$, which leads to a contradiction since $z_0 \in \bar{S}$. Therefore, S is empty and the local inverses are holomorphic on $\mathbb{D}\backslash\Gamma'$, completing the proof.

From the proof of the above lemma one derives an intrinsic order for the local inverses. Specifically, we label the local inverses $\{\rho_k(z)\}_{k=0}^{n-1}$ such that $\rho_k(z) = u^{-1}(\zeta^k u(z))$ on Ω for $0 \le k \le n-1$. By a routine argument, we have that each ρ_k is invertible on Ω, and for any pair ρ_k, ρ_k and $z \in \Omega$, we have

$$\rho_k \circ \rho_{k'}(z) = \rho_{k+k' \bmod n}(z).$$

Moreover, with little extra effort, one sees that each ρ_k can also be analytically continued across the boundary T. We prove the main result.

Theorem (1.2.6)[23]: Let ϕ be a finite Blaschke product of order n. Then the von Neumann algebra $A_\phi = \{M_\phi, M_\phi^*\}'$ is commutative of dimension q, and hence $A_\phi \cong \underbrace{C \oplus \cdots \oplus C}_{q}$, where q is the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$.

Proof. It suffices to show that $\mathcal{E}_j \mathcal{E}_i = \mathcal{E}_i\mathcal{E}_j$ for each $1 \le i, j \le q$. Indeed, for any $0 \le k, k' \le n-1$, we have that

$$\rho_k \circ \rho_{k'}(z) = \rho_k \circ \rho_{k'}(z) = \rho_{k+k' \bmod n}(z), \qquad z \in \Omega.$$

Therefore, for any $f \in L_a^2(\mathbb{D})$ and $z \in \Omega$, we have

$$\begin{aligned}(\mathcal{E}_i\mathcal{E}_j f)(z) &= \sum_{\rho \in G_i} \sum_{\tilde{\rho} \in G_j} f(\tilde{\rho}(\rho(z))\tilde{\rho}'(\rho(z))\rho'(z) \\ &= \sum_{\tilde{\rho} \in G_j} \sum_{\rho \in G_i} f\left(\rho(\tilde{\rho}(z))\right)\rho'(\tilde{\rho}(z))\tilde{\rho}'(z) = (\mathcal{E}_j \mathcal{E}_i f)(z).\end{aligned}$$

This implies that $\mathcal{E}_j \mathcal{E}_i(f) = \mathcal{E}_j \mathcal{E}_i(f)$ for any $f \in L_a^2(\mathbb{D})$, completing the proof.

By the final argument in the proof of [27], the statement that A_ϕ is commutative is equivalent to the statement that the minimal reducing subspaces for M_ϕ are pairwise orthogonal. This also means that the number of distinct minimal reducing subspaces of M_ϕ is equal to the dimension of A_ϕ. Hence, one derives the following corollary giving the structure of the reducing subspaces.

Corollary (1.2.7)[23]: Let ϕ be a finite Blaschke product. Then the multiplication operator M_ϕ on the Bergman space $L_a^2(\mathbb{D})$ has exactly q nontribyl minimal reducing subspaces $\{\mathcal{M}_1, \dots, \mathcal{M}_q\}$, and $L_a^2(\mathbb{D}) = \oplus_{k=1}^{q} \mathcal{M}_k$, where q is the number of connected components of the Riemann surface $\phi^{-1} \circ \phi$.

In order to facilitate the comprehension of the rather involved computations included, we analyze first a simple, transparent example. If $\phi = z^n$, then the family of local inverses is $\{\rho_k(z) = \zeta^k z: 0 \le k \le n - 1\}$, and we infer without difficulty that

$$\mathcal{M}_j = \overline{\text{span}}\{z^i : i \ge 0, i \equiv j \,(\text{mod}\, n), 1 \le j \le n,$$

are the minimal reducing subspaces of M_{z^n}. However, such a simple argument is not available in the general case, so we prefer to explain the above description of the $\mathcal{M}_j$ in a less direct way, as follows. Recall that for $\phi = z^n$, we have that

$$(\mathcal{E}_k f)(z) = f(\rho_k(z)\rho_k'(z) = \zeta^k f(\zeta^k z), \qquad 1 \le k \le n.$$

One verifies then that $\mathcal{M}_j$ is the joint eigenspace for the $\mathcal{E}_k$'s corresponding to the eigenvalues $\zeta^{k(j+1)}$. Therefore, every $\mathcal{M}_j$ is a reducing subspace since the $\{\mathcal{E}_k\}$ are normal operators and $A_\phi = \text{span}\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$.

There is a second, more geometric description of $\mathcal{M}_j$ which emerges from this simple example. Let F_j be the flat bundle on $\mathbb{D}_0 = \mathbb{D}\backslash\{0\}$ with respect to the jump ζ^j (see [24] for the precise definition). We cut $\mathbb{D}_0$ along the line $(0,1)$ in $\mathbb{D}_0$, put the rank-one tribyl holomorphic bundle over it, and identify the vector v on the lower copy of $(0,1)$ with the vector $\zeta^j\, v$ on the above copy of $(0,1)$. Then F_j is just the quotient space obtained from this process. One can easily see that the F_j's are all the flat line bundles whose pullback bundle to $\mathbb{D}_0$ induced by the map $z^n : \mathbb{D}_0 \to \mathbb{D}_0$ is the tribyl bundle. This means that each holomorphic on F_j yields a holomorphic function on $\mathbb{D}_0$ by the induced composition. Let

$$L_a^2(F_j) = \left\{\text{holomorphic } s : \mathbb{D}_0 \to F_j : \int_{\mathbb{D}_0} |s|^2\, dm < \infty\right\},$$

and let M_z be the corresponding bundle shift on $L_a^2(F_j)$. Note that $|s|$ is well defined on $\mathbb{D}_0$. Then the operator $U_j : L_a^2(F_j) \to \mathcal{M}_j \,[\subseteq L_a^2(\mathbb{D})]$ defined by $(U_j f)(z) = nz^{n-1} f(z^n)$ is a unitary map, which intertwines $(L_a^2(F_j), M_z)$ and $(\mathcal{M}_j, \mathcal{M}_{z^n})$. In this way flat line bundles provide a natural model for the action of M_{z^n} on the minimal reducing subspaces of M_{z^n}. It is conceivable that some analogous geometric description exists for the action of M_ϕ on the minimal reducing subspaces in general, but, if so, we do not know how to describe it. Thus we follow a different path below.

Returning to the general case of a finite Blaschke product ϕ, we will prove the following theorem. Recall that the dual partition for ϕ is the partition of the set $\{0, 1, \dots, n - 1\}$ corresponding to the equivalence relation defined in (31). We will prove later that the number of components in the dual partition is also equal to q, the number of connected components of the Riemann surface for $\phi^{-1} \circ \phi$.

The remainder is devoted to the proof of this theorem. We begin with a characterization of the $\mathcal{M}_j$'s in term of eigenvalues and eigenspaces of the $\mathcal{E}_k$'s. Adapting, step by step, the proof of [27], we infer that

$$A_\phi = \{M_\phi, M_\phi^*\}' = \text{span}\{\mathcal{E}_1, \dots, \mathcal{E}_q\} = \text{span}\{P_{\mathcal{M}_1}, \dots, P_{\mathcal{M}_q}\},$$

where $P_{\mathcal{M}_k}$ is the projection onto $\mathcal{M}_k$ for $1 \le k \le q$. This means that there are unique constants $\{c_{k_j}, 1 \le j, k \le q\}$ such that

$$\mathcal{E}_k = \sum_{1 \le j \le q} c_{k_j} P_{\mathcal{M}_j}. \tag{33}$$

On the other hand, by a dimension argument, the constant matrix $[c_{k_j}]$ turns out to be invertible. Since the rows of $[c_{k_j}]$ are linearly independent, it follows that $c_{k_{j}1} = c_{k_{j}2}$ for each k if and only if $j_1 = j_2$.

For each tuple $\{c_{k_j}\}_k$, let $\widetilde{\mathcal{M}}_j = \{f \in L_a^2(\mathbb{D}): \mathcal{E}_k f = c_{k_j} f, 1 \le k \le q\}$ be the corresponding common eigenspace for $\{\mathcal{E}_1, \dots, \mathcal{E}_q\}$. As shown in Theorem (1.2.6), each $\mathcal{E}_k$ is a normal operator. By spectral theory, $\widetilde{\mathcal{M}}_{j_1} \perp \widetilde{M}_{j_2}$ if $j_1 \ne j_2$. Since $\mathcal{M}_j \subseteq \widetilde{\mathcal{M}}_j$ for each j, we obtain $\widetilde{\mathcal{M}}_j \perp \mathcal{M}_k$ for $j \ne k$. Noticing that $L_a^2(\mathbb{D}) = \oplus_k \mathcal{M}_k$, one sees that $\mathcal{M}_j = \widetilde{\mathcal{M}}_j$. That is,

$$\mathcal{M}_j = \{f \in L_a^2(\mathbb{D}): \mathcal{E}_k f = c_{k_j} f, \quad 1 \le k \le q\}. \tag{34}$$

We also need the following lemmas concerning the domain $\Omega = u^{-1}(A_r)$. Let $L_a^2(\Omega)$ be the Bergman space which consists of the holomorphic functions in $L^2(\Omega)$, and let $L_{a,p}^2(\Omega)$ be the subspace of $L_a^2(\Omega)$ which is the closure of the polynomial ring in $L_a^2(\Omega)$. Note that since $z^{-1} \in L_a^2(\Omega)$, we have $L_{a,p}^2(\Omega) \ne L_a^2(\Omega)$. Recall that $\mathcal{O}(\mathbb{D})$ denotes the space of holomorphic functions on $\mathbb{D}$.

Lemma (1.2.8)[23]: The restriction operator $i_\Omega : L_a^2(\mathbb{D}) \to L_{a,p}^2(\Omega)$ defined by $i_\Omega(f) = f|_\Omega$ is invertible. Furthermore, $L_a^2(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}): f|_\Omega \in L_a^2(\Omega)\}$.

Proof. As shown in the proof of Lemma (1.2.5), there exists $r > 0$ such that $A_r \subseteq \Omega$. It's well known that there exists a positive constant $C_{r'}$ such that for any polynomial f

$$\|f\|_{L_a^2(\mathbb{D})} \le C_{r'} \|f\|_{L^2(A_{r'})}.$$

This implies for any polynomial f that

$$\|f\|_{L^2(\mathbb{D})} \le C_{r'} \|f\|_{L^2(A_{r'})} \le C_{r'} \|f\|_{L^2(\Omega)} \le C_{r'} \|f\|_{L^2(\mathbb{D})}.$$

Because the polynomial ring is dense in both of the two Hilbert spaces $L_a^2(\mathbb{D})$ and $L_{a,p}^2(\Omega)$, one finds that i_Ω is invertible.

In addition, we have that

$$L_a^2(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}): f|_\Omega \in L_{a,p}^2(\Omega)\} \subseteq \{f \in \mathcal{O}(\mathbb{D}): f|_\Omega \in L_a^2(\Omega)\}.$$

It remains to show that, if $f \in \mathcal{O}(\mathbb{D})$ and $f|_\Omega \in L_a^2(\Omega)$, then $f \in L_a^2(\mathbb{D})$. Indeed, since $A_{r'} \subseteq \Omega$, one sees that $f|_{A_{r'}} \in L_a^2(A_{r'})$. Let $f = \sum_{k=0}^{\infty} a_k z^k$ be the Taylor series expansion of f on $\mathbb{D}$. Since the vectors $\{z^k\}k$ are pairwise orthogonal in $L_a^2(A_{r'})$, we have that the polynomial $p_n = \sum_{k=0}^{n} a_k z^k$ tends to f in the norm of $L_a^2(A_{r'})$ and hence $f \in L_{a,p}^2(A_{r'})$. Therefore, by the argument in the preceding paragraph, there exists $g \in L_a^2(\mathbb{D})$ such that $f|A_{r'} = gA_{r'}$. This means that $f = g \in L_a^2(\mathbb{D})$, as desired.

Now we introduce operators on $L_a^2(\Omega)$ and $L_{a,p}^2(\Omega)$ corresponding to $\{\mathcal{E}_i\}$. We also let M_ϕ denote the multiplication operator on $L_a^2(\Omega)$ or $L_{a,p}^2(\Omega)$ with the bounded analytic symbol ϕ. Recall that each $\rho \in \{\rho_j\}_{j=0}^{n-1}$ is invertible on Ω. Hence, the operator U_ρ^Ω : $L_a^2(\Omega) \to L_a^2(\Omega)$ defined by $U_\rho^\Omega(f) = (f \circ \rho)\rho'$ is a unitary operator with the inverse $U_{\rho-1}^\Omega$. Similarly, for each $1 \le k \le q$, define a linear operator $\mathcal{E}_k^\Omega : L_a^2(\Omega) \to L_a^2(\Omega)$ as

$$\mathcal{E}_k^\Omega(f) = \sum_{\rho \in G_k} U_\rho^\Omega(f) = \sum_{\rho \in G_k} (f \circ \rho)\rho', \qquad f \in L_a^2(\Omega).$$

Moreover, for each $f \in L_{a,p}^2(\Omega)$, there exists some $g \in L_a^2(\mathbb{D})$ such that $g|_\Omega = f$. A direct computation shows that $\mathcal{E}_k(g)|_\Omega = \mathcal{E}_k^\Omega(f)$. Hence, one sees that $\mathcal{E}_k^\Omega(f) \in L_{a,p}^2(\Omega)$. This means that $\mathcal{E}_k^\Omega$ is also a bounded operator on $L_{a,p}^2(\Omega)$ and $i_\Omega \mathcal{E}_k = \mathcal{E}_k^\Omega i_\Omega$. Combining this identity with formula (33) we obtain

$$\mathcal{E}_k^\Omega(f) = \sum_{1 \le j \le q} c_{kj}\, i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}(f), \qquad f \in L_{a,p}^2(\Omega). \tag{35}$$

Furthermore, by [27], for each $1 \le k \le q$ there is an integer k^- with $1 \le k^- \le q$ such that

$$G_{k^-} = G_k^- = \{\rho^{-1} : \rho \in G_k\}.$$

Similar to that used in the proof of [27], we infer that $\mathcal{E}_{k^-}^\Omega = \mathcal{E}_k^{\Omega^*}$. Therefore, $L_{a,p}^2(\Omega)$ is a common reducing subspace of $\{\mathcal{E}_k^\Omega\}$ and each $\mathcal{E}_k^\Omega$ is a normal operator on $L_{a,p}^2(\Omega)$.

For every $1 \le j \le q$, let

$$\mathcal{M}_j^\Omega = i_\Omega(\mathcal{M}_j) = \{f|_\Omega : f \in \mathcal{M}_j\}.$$

We claim that $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1} = P_{\mathcal{M}_j^\Omega}$. Since the range of $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is equal to $\mathcal{M}_j^\Omega$, it suffices to show that $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is a projection. Indeed, a direct computation shows that $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is an idempotent. Furthermore, combining formula (35) and the fact that $[c_{kj}]$ is invertible, every $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is a linear combination of $\{\mathcal{E}_k^\Omega\}$. It follows that every $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is a normal operator. Therefore, $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1}$ is a projection and $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1} = P_{\mathcal{M}}\Omega_j$.

We summarize the consequences of the above argument as follows.

Proposition (1.2.9)[23]: Using the notation above, $L_{a,p}^2(\Omega) = \oplus_{j=1}^q \mathcal{M}_j^\Omega$, and

$$\mathcal{M}_j^\Omega = \{f \in L_{a,p}^2(\mathbb{D}) : \mathcal{E}_k^\Omega f = c_{kj} f,\ 1 \le k \le q\}. \tag{36}$$

In addition, one has

$$\mathcal{E}_k^\Omega(f) = \sum_{1 \le j \le q} c_{kj} P_{\mathcal{M}_j}^\Omega(f), \qquad f \in L_{a,p}^2(\Omega). \tag{37}$$

Proof. Eq. (37) follows from formula (35) and the fact that $i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1} = P_{\mathcal{M}_j^\Omega}$. Combining this with the same argument at the beginning, one sees (36).

Moreover, since

$$P_{\mathcal{M}_i^\Omega} P_{\mathcal{M}_j^\Omega} = i_\Omega P_{\mathcal{M}_i} P_{\mathcal{M}_j} i_\Omega^{-1} = 0$$

if $i \ne j$ and

$$\sum_{j=1}^q P_{\mathcal{M}_j^\Omega} = \sum_{j=1}^q i_\Omega P_{\mathcal{M}_j} i_\Omega^{-1} = I,$$

we have that $L_{a,p}^2(\Omega) = \oplus_j \mathcal{M}_j^\Omega$, completing the proof.

Since ρ_1 is invertible and $\rho_1^n = 1$ on Ω, the operator $U_{\rho_1}^{\Omega} : L_a^2(\Omega) \to L_a^2(\Omega)$ is unitary and $\left(U_{\rho_1}^{\Omega}\right)^n = 1$. By the spectral theory for unitary operators, the $\{\zeta^i\}_{i=0}^{n-1}$ are possible eigenvalues of $U_{\rho_1}^{\Omega}$, and $U_{\rho_1}^{\Omega} = \sum_{i=0}^{n-1} \zeta^i P_{\mathcal{N}_i^{\Omega}}$, where $P_{\mathcal{N}_i^{\Omega}}$ is the projection from $L_a^2(\Omega)$ onto the eigenvector subspace

$$\mathcal{N}_i^{\Omega} = \{f \in L_a^2(\Omega): U_{\rho_1}^{\Omega}(f) = \zeta^i f\}.$$

It follows that $U_{\rho_j}^{\Omega} = \left(U_{\rho_1}^{\Omega}\right)^j = \sum_{i=0}^{n-1} \zeta^{ij} P_{\mathcal{N}_i^{\Omega}}$, and

$$\mathcal{E}_k^{\Omega}(f) = \sum_{\rho_j \in G_k} \sum_{i=0}^{n-1} \zeta^{ij} P_{\mathcal{N}_i^{\Omega}}(f), \qquad f \in L_a^2(\Omega). \tag{38}$$

Furthermore, we have the following lemma. Recall that $u : \Omega = u^{-1}(A_r) \to A_r$ is invertible as shown in Lemma (1.2.4).

Lemma (1.2.10)[23]: $\mathcal{N}_i^{\Omega} = \overline{\operatorname{span}}\{u^k u' : k \in \mathbb{Z},\ k + 1 \equiv i \bmod n\}$.

Proof. Since $u \circ \rho_1 = \zeta u$ on Ω, it is easy to check that

$$U_{\rho_1}(u^k u') = \zeta^i u^k u', \qquad \text{for } k + 1 \equiv i \bmod n.$$

That is, $\mathcal{N}_i^{\Omega}$ is contained in the eigenspace of U_{ρ_1} associated to the eigenvalue ζ^i . It remains to show that $\oplus_i \mathcal{N}_i^{\Omega} = L_a^2(\Omega)$. In fact, we will prove that $\{u^k u' : k \in \mathbb{Z}\}$ is a complete orthogonal system for $L_a^2(\Omega)$.

Define the pull-back operator $C_u : L_a^2(A_r) \to L_a^2(\Omega)$ by

$$C_u f = (f \circ u)u'.$$

Since $u : \Omega \to A_r$ is invertible, C_u is unitary. Noticing that $\{z^k : k \in \mathbb{Z}\}$ is a complete orthogonal basis for $L_a^2(A_r)$, one sees that $\{u^k u' = C_u(z^k): k \in \mathbb{Z}\}$ is a complete orthogonal basis for $L_a^2(\Omega)$, as desired.

Recall that for the partition $\{G_1, \ldots, G_q\}$ of local inverses for $\phi^{-1} \circ \phi$, we say $j_1 \sim j_2$ in the dual partition for two integers $0 \le j_1, j_2 \le n - 1$, if

$$\sum_{\rho_k \in G_i} \zeta^{k j_1} = \sum_{\rho_k \in G_i} \zeta^{k j_2} \quad \text{for any } 1 \le i \le q.$$

The above relation partitions the set $\{0, 1, \ldots, n - 1\}$ into equivalence classes $\{G_1', \ldots, G_p'\}$. For each G_j' in the dual partition, let $\mathcal{L}_j^{\Omega} = \oplus_{i \in G_j'} \mathcal{N}_i^{\Omega}$; that is,

$$\mathcal{L}_j^{\Omega} = \overline{\operatorname{span}}\{u^i u' : i \in \mathbb{Z},\ i + 1 \,(\operatorname{mod} n) \in G_j'\}.$$

Then $\oplus_{j=1}^{p} \mathcal{L}_j^{\Omega} = L_a^2(\Omega)$. From formula (38)

$$\mathcal{E}_k^{\Omega}(f) = \sum_{1 \le j \le p} c_{kj}' P_{\mathcal{L}_j^{\Omega}}(f), \qquad f \in L_a^2(\Omega), \tag{39}$$

where $c_{kj}' = \sum_{\rho_i \in G_k} \zeta^{il}$ for any $l \in G_j'$. By the equivalent condition for the dual partition, $c_{kj_1}' = c_{kj_2}'$ for each k if and only if $j_1 = j_2$. Comparing formulas (36) and (39) yields the following result.

Proposition (1.2.11)[23]: For each $\mathcal{M}_j^{\Omega}$, there exists $1 \le k \le p$ such that $\mathcal{M}_j^{\Omega} = \mathcal{L}_k^{\Omega} \cap L_{a,p}^2(\Omega)$.

Proof. For each $0 \ne f \in \mathcal{M}_j^{\Omega} \subseteq \oplus_k \mathcal{L}_k^{\Omega} = L_a^2(\Omega)$, there exists at least one d_f such that $1 \le d_f \le p$ and the projection of f on $\mathcal{L}_{d_f}^{\Omega}$ is nonzero. We claim that d_f is unique. Indeed, suppose for $k_1 \ne k_2, P_{\mathcal{L}_{k_1}^{\Omega}}(f)$ and $P_{\mathcal{L}_{k_2}^{\Omega}}(f)$ are nonzero. By formula (36), one sees for each $1 \le i \le n$ that,

$$\left[P_{\mathcal{L}_{k_1}} + P_{\mathcal{L}_{k_2}}\right]\mathcal{E}_i^{\Omega}(f) = c_{ij}P_{\mathcal{L}_{k_1}}(f) + c_{ij}P_{\mathcal{L}_{k_2}}(f).$$

Moreover, by formula (39),

$$\left[P_{\mathcal{L}_{k_1}} + P_{\mathcal{L}_{k_2}}\right]\mathcal{E}_i^{\Omega}(f) = c'_{ik_1}P_{\mathcal{L}_{k_1}}(f) + c'_{ik_2}P_{\mathcal{L}_{k_2}}(f).$$

Hence $c'_{ij} = c'_{ik_1} = c'_{ik_2}$ for each i. This leads to a contradiction since $k_1 = k_2$. Therefore, there exists only one integer df such that $P\_(\mathcal{L}_{df}^{\Omega}(f) \neq 0$.

We now prove that d_f is independent of f. Otherwise, there exist $k_1 \neq k_2$ and $f_1, f_2 \in \mathcal{M}_j$ such that both $P_{\mathcal{L}_{k_1}^{\Omega}}(f_1)$ and $P_{\mathcal{L}_{k_2}^{\Omega}}(f_2)$ are nonzero. By the uniqueness proved in the preceding paragraph, we have that $P_{\mathcal{L}_{k_1}^{\Omega}}(f_2) = P_{\mathcal{L}_{k_2}^{\Omega}}(f_1) = 0$. However, this means that both $P_{\mathcal{L}_{k_1}^{\Omega}}(f_1 + f_2)$ and $P_{\mathcal{L}_{k_2}^{\Omega}}(f_1 + f_2)$ are nonzero, which contradicts the uniqueness of $d_{f_1+f_2}$. Therefore, there exists only one integer k such that $P_{\mathcal{L}_k^{\Omega}}\mathcal{M}_j^{\Omega} = \{0\}$. Moreover, we have that $c_{ij} = c'_{ik}$ for each i. Combining this fact with formulas (36) and (39), one sees that

$$\mathcal{M}_j^{\Omega} = \mathcal{L}_k^{\Omega} \cap L^2_{a,p}(\Omega) = \left\{f \in L^2_{a,p}(\mathbb{D}): \mathcal{E}_i^{\Omega} f = c_{ij}f, \quad 1 \leq i \leq q\right\},$$

completing the proof.

We will prove the converse of the above proposition. We begin with some lemmas.

Lemma (1.2.12)[23]: Let f be a function holomorphic on a neighborhood of $\overline{A_r}$. Then for any $k \in \mathbb{Z}, f \perp \overline{z^k}$ in $L^2_a(A_r)$ if and only if $\int_{z\in\mathbb{T}} f(z)\overline{z^k}\, dm(z) = 0$.

Proof. Let a_k be the coefficient for z^k in the Laurent series expansion of f on A_r. Observe that $\{z^k\}_{k=-\infty}^{+\infty}$ is a complete orthogonal basis for both of $L^2_a(A_r)$ and $L^2(\mathbb{T})$. A direct computation shows that $\langle f, z^k\rangle_{L^2_a(A_r)} = a_k\|z^k\|_{L^2_a(A_r)}$ and $\langle f, z^k\rangle_{L^2(\mathbb{T})} = a_k\|z^k\|_{L^2(\mathbb{T})}$, which leads to the desired result.

We also need the following transformation formula.

Lemma (1.2.13)[23]: Let $s : \mathbb{T} \to \mathbb{T}$ be an invertible differentiable map. Then there exists aconstant $s = 1$ or -1, such that for any $f \in C(\mathbb{T})$

$$\int_{\mathbb{T}} f(\theta)dm(\theta) = \epsilon_s \int_{\mathbb{T}} f\big(s(\theta)\big)\frac{s'(\theta)}{is(\theta)}\, dm(\theta).$$

If, in addition, s is holomorphic on a neighborhood of $\mathbb{T}$, then

$$\int_{\mathbb{T}} f(z)\, dm(z) = \epsilon_s \int_{\mathbb{T}} f\big(s(z)\big)\frac{zs'(z)}{s(z)}\, dm(z).$$

Proof. It is sufficient to verify only the first equation. Indeed, the latter equation follows from the former equation and that

$$s'(\theta) = s'(z)\frac{dz}{d\theta} = ie^{i\theta}s'(z) = izs'(z), z \in \mathbb{T}.$$

Without loss of generality, we can suppose that $s(1) = 1$. Then there exists $\tilde{s}: (0, 2\pi) \to (0, 2\pi)$ such that $s(\theta) = e^{i\tilde{s}(\theta)}$. An elementary calculus argument shows that

$$\int_{\mathbb{T}} f(\theta)dm(\theta) = \int_{\mathbb{T}} f\big(s(\theta)\big)|\tilde{s}'(\theta)|dm(\theta).$$

Since s is invertible on $\mathbb{T}$, one has that $\tilde{s} : (0, 2\pi) \to (0, 2\pi)$ is a monotonic function. Therefore, we can choose a constant $\epsilon_s = 1$ or -1 such that $|\tilde{s}'| = \epsilon_s\tilde{s}'$. Moreover, differentiating the equation $s(\theta) = e^{i\tilde{s}(\theta)}$, one sees that $s'(\theta) = ie^{i\tilde{s}(\theta)}\tilde{s}'(\theta) = is(\theta)\tilde{s}'(\theta)$. This implies that $|\tilde{s}'(\theta)| = \frac{\epsilon_s s'(\theta)}{is(\theta)}$, completing the proof.

Lemma (1.2.14)[23]: For any integer $k \geq 0$, there exists some integer $i \geq 0$ such that $\langle z^i, u^k u' \rangle_{L^2(\Omega)} \neq 0$. Therefore, $P_-(L^2_{a,p(\Omega)} \mathcal{N}_k^{\Omega} = \{0\}$ for all $0 \leq k \leq n-1$.

Proof. We prove the statement by contradiction. Suppose that for some $k \geq 0$,

$$\langle z^i, u^k u' \rangle_{L^2(\Omega)} = 0, \qquad \forall i \geq 0.$$

Since the operator $C_u : L^2(A_r) \to L^2(\Omega)$, which appears in Lemma (1.2.10), is unitary, the above equation is equivalent to

$$\langle (u^{-1})^i (u^{-1})', z^k \rangle_{L^2(A_r)} = 0, \qquad \forall i \geq 0.$$

Using Lemma (1.2.12), it follows that foreachinteger$i \geq 0$

$$\langle (u^{-1})^i (u^{-1})', z^k \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} (u^{-1})^i (u^{-1})' \overline{z^k}\, dm(z) = 0.$$

By Lemma (1.2.13), Lemma (1.2.4) and the fact that $|u(z)| = 1$ for $z \in \mathbb{T}$, we have for each integer $i \geq 0$:

$$0 = \int_{\mathbb{T}} z^i (u^{-1})' \circ u(z) \overline{u^k} \frac{z u'(z)}{u(z)} dm(z) = \int_{\mathbb{T}} z^{i+1} \overline{u^{k+1}}\, dm(z) = \langle z^{i+1}, u^{k+1} \rangle_{L^2(\mathbb{T})}.$$

This means that $u^{k+1} \in \overline{H_2(\mathbb{T})}$ and hence $\phi^{k+1} = u^{n(k+1)} \in \overline{H_2(\mathbb{T})}$. Because ϕ^{k+1} is holomorphic on $\mathbb{D}$, we deduce that ϕ^{k+1} is a constant. This leads to a contradiction since ϕ is a nontribyl Blaschke product, completing the proof.

Summarizing the above results, we obtain the converse of Proposition (1.2.11).

Proposition (1.2.15)[23]: For each k, there exists a unique j such that $\mathcal{M}_j^{\Omega} = \mathcal{L}_k^{\Omega} \cap L^2_{a,p}(\Omega)$; that is,

$$L^2_{a,p}(\Omega) = \bigoplus_k \left[\mathcal{L}_k^{\Omega} \cap L^2_{a,p}(\Omega)\right].$$

Proof. From Proposition (1.2.11), for each $1 \leq j \leq q$, there exists only one $1 \leq k_j \leq p$ such that $\mathcal{M}_j^{\Omega} = \mathcal{L}_{k_j}^{\Omega} \cap L^2_{a,p}(\Omega)$. Hence,

$$L^2_{a,p}(\Omega) = \bigoplus_j \left[\mathcal{L}_{k_j}^{\Omega} \cap L^2_{a,p}(\Omega)\right].$$

We claim that the set $\{k_1, \ldots, k_q\}$ is just $\{1, \ldots, p\}$. Indeed, if there exists k such that $1 \leq k \leq p$ but k is not in the set $\{k_1, \ldots, k_q\}$, then $\mathcal{L}_k^{\Omega} \perp \oplus_{k_j} \mathcal{L}_{k_j}^{\Omega}$. This means that $P_{L^2_{a,p}(\Omega)} \mathcal{L}_j^{\Omega} = \{0\}$, which leads to a contradiction, since $\mathcal{L}_k^{\Omega} = \oplus_{j \in G'_k} \mathcal{N}_j^{\Omega}$ and by Lemma (1.2.14) we have that $P_{L^2_{a,p}(\Omega)} \mathcal{N}_j^{\Omega} \neq \{0\}$ for each j. Therefore, the set $\{k_1, \ldots, k_q\}$ includes all integers between 1 and p. It follows that $p = q$ and

$$L^2_{a,p}(\Omega) = \bigoplus_{k=1}^{q} \left[\mathcal{L}_k^{\Omega} \cap L^2_{a,p}(\Omega)\right],$$

as desired.

In the proof of Proposition (1.2.15), one identifies the following intrinsic property of the partition for a finite Blaschke product.

Corollary (1.2.16)[23]: The number of components in the dual partition is also equal to q, the number of connected components of the Riemann surface for $\phi^{-1} \circ \phi$.

Combining Lemma (1.2.8) with Propositions (1.2.11) and (1.2.15), we derive the main result.

Theorem (1.2.17)[23]: Let ϕ be a finite Blaschke product, and $\{G_1',\dots,G_q'\}$ be the dual partition for ϕ. Then the multiplication operator M_ϕ has exactly q nontribyl minimal reducing subspaces $\{\mathcal{M}_1,\dots,\mathcal{M}_q\}$, and for any $1 \le j \le q$

$$\mathcal{M}_j = \{f \in \mathcal{O}(\mathbb{D}): f\,|\Omega \in L_j^{\Omega}\},$$

where $\Omega = u^{-1}(A_r)$ is defined in Lemma (1.2.4), and $\mathcal{L}_j^{\Omega}$ is a subspace of $L^2(\Omega)$ with orthogonal basis $\{u^i u' : i + 1 \pmod n \in G_j'\}$.

Proof. Combining Propositions (1.2.11) and (1.2.15), after renumbering if necessary, we have for each $1 \le j \le q$ that,

$$\mathcal{M}_j^{\Omega} = \mathcal{L}_j^{\Omega} \cap L^2_{(a,p)}(\Omega).$$

Since i_Ω is invertible, we have that

$$\mathcal{M}_j = \{f \in L_a^2(\mathbb{D}): f|_{\Omega} \in \mathcal{M}_j^{\Omega}\} = \{f \in L_a^2(\mathbb{D}): f\,|_{\Omega} \in \mathcal{L}_j^{\Omega}\}.$$

Combining this formula with Lemma (1.2.8), we conclude that

$$\mathcal{M}_j = \{f \in \mathcal{O}(\mathbb{D}): f_\Omega \in \mathcal{L}_j^{\Omega}\},$$

completing the proof of the theorem.

[10], [1] obtained a classification of the structure of the finite Blaschke product ϕ in case ϕ has order 3 or 4. We sketch an arithmetic way towards the classification of finite Blaschke products, displaying the details in the degree 8 case.

Following [27] we define an equivalence relation among finite Blaschke products so that $\phi_1 \sim \phi_2$, if there exist Möbius transformations $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ and $\varphi_b(z) = \frac{b-z}{1-\bar{b}z}$ with $a,b \in \mathbb{D}$ such that $\phi_1 = \varphi_a \circ \varphi_2 \circ \varphi_b$. A finite Blaschke ϕ is called reducible if there exist two nontribyl finite Blaschke products φ_1,φ_2 such that $\phi \sim \varphi_1 \circ \varphi_2$, and ϕ is irreducible if ϕ is not reducible.

For a finite Blaschke product ϕ of order n, let $G_1,\dots,G_q$ be the partition defined by the family of local inverses $\{\rho_0,\dots,\rho_n\}$ for $\phi^{-1} \circ \phi$. When no confusion arises, we write $i \in G_k$ if $\rho_i \in G_k$, and $G_k = \{i_1,i_2,\dots,i_j\}$ if $G_k = \{\rho_{i_1},\rho_{i_2},\dots,\rho_{i_j}\}$. In view of the above notations, $\{G_1,\dots,G_q\}$ is a partition of the additive group $\mathbb{Z}_n = \{0,1,\dots,n-1\}$. One can immediately verify that, if $\phi_1 \sim \phi_2$, then ϕ_1,ϕ_2 yield identical partitions.

Corollary (1.2.16) hints that there should exist some internal algebraic and combinatorial structures for the partitions arising from finite Blaschke products. Although we don't understand these properties completely, we list a few necessary conditions:

(α_0) $\{0\}$ is a singleton in the partition, since $\rho_0(z) = z$ is holomorphic on $\mathbb{D}$.

(α_1) For any pair Gi and Gj , there exist some $G_{k_1},\dots,G_{k_m}$ such that

$$G_i + G_j = G_{k_1} \cup\cdots\cup G_{k_m} \text{ (counting multiplicities on both sides)},$$

where " $+$ " is defined using the addition of $\mathbb{Z}_n$. (This is a consequence of the fact that the product $\mathcal{E}_i\mathcal{E}_j$ is a linear combination of some $\mathcal{E}_k$'s.) (α_2) By [27], for each $G_i = \{i_1,\dots,i_k\}$, there exists j such that

$$G_j = G_i^{-1} = \{n - i_1,\dots,n - i_k\}.$$

(α_3) By Corollary (1.2.16), the number of elements in the dual partition is also q.

We also need the following generalization of [27]. Note that the additive structure for elements in G_k's coincides with compositions near the boundary $\mathbb{T}$.

Lemma (1.2.18)[23]: For a finite Blaschke product ϕ of order n, ϕ is reducible if and only if $G_{k_1} \cup\cdots\cup G_{k_m}$ forms a nontribyl proper subgroup of $\mathbb{Z}_n$, for some subset $G_{k_1},\dots,G_{k_m}$ of the partition arising from ϕ.

Proof. Assume that ϕ is reducible. Without loss of generality, suppose that $\phi = \varphi_1 \circ \varphi_2$ for two nontribyl finite Blaschke products φ_1, φ_2. Since the family of local inverses $\varphi_2^{-1} \circ \varphi_2$ is a cyclic group under compositions near the boundary $\mathbb{T}$, and it is contained in the local inverses of $\phi^{-1} \circ \phi$, the set of the local inverses for $\varphi_2^{-1} \circ \varphi_2$ forms a nontribyl proper subgroup of $\phi^{-1} \circ \phi$.

On the other hand, suppose that $G = G_{k_1} \cup \cdots \cup G_{k_m}$ is a nontribyl proper subgroup of $\mathbb{Z}_n$ for some $G_{k_1}, \ldots, G_{k_m}$. For each $G_{k_i} = \left\{\rho_{i_1}, \ldots, \rho_{i_j}\right\}$, by [27] there exists a polynomial $f_i(w,z)$ of degree j such that $\left\{\rho_{i_1}(z), \ldots, \rho_{i_j}(z)\right\}$ are solutions of fi$(w,z) = 0$. This implies that $\prod_{\rho \in G_{k_i}} \rho(z) = \frac{p_i(z)}{q_i(z)}$ is a quotient of two polynomials $p_i(z), q_i(z)$ of degree at most j . So, if we define

$$\varphi_2(z) = \prod_{\rho \in G} \rho(z) = \prod_{i=1}^{m} \prod_{\rho \in G_{k_i}} \rho(z) = \prod_{i=1}^{m} \frac{p_i(z)}{q_i(z)} ,$$

then $\varphi_2(z)$ is a rational function of degree at most G; here G denotes the number of elements in G. It follows that $\varphi_2(z)$ is holomorphic outside a finite point set S of $\mathbb{D}$. Since each local inverse is bounded by 1 on $\mathbb{D}\backslash\Gamma'$ and $\mathbb{D}\backslash\Gamma'$ is dense in $\mathbb{D}$, we have that φ_2 is also bounded on $\mathbb{D}\backslash S$ and hence it can analytically be continued across S. This means that φ_2 is a bounded holomorphic function on $\mathbb{D}$. By a similar argument involving local inverses, one sees that φ_2 is also continuous on $\mathbb{T}$ and $|\varphi_2(z)| = 1$ whenever $z \in \mathbb{T}$. That implies φ_2 is a finite Blaschke product of order G.

Furthermore, by the group structure of $G, \varphi_2(\rho_i(z)) = \varphi_2(z)$ for each $\rho_i \in G$ if z is close enough to the boundary $\mathbb{T}$. Since $\mathbb{D}\backslash\Gamma'$ is a connected domain including Ω, the equation still holds whenever $z \in \mathbb{D}\backslash\Gamma'$. In other words, the family of local inverses of $\varphi_2^{-1} \circ \varphi_2$ is just, G, a subset in that of $\phi^{-1} \circ \phi$. Consequently, $\phi(z_1) = \phi(z_2)$ if $\varphi_2(z_1) = \varphi_2(z_2)$ and z_1, z_2 are regular points of φ. Hence, if we define

$$\varphi_1(w) = \phi(z) \quad \text{for } w = \varphi_2(z),$$

then φ_1 is well defined outside some finite set of points in $\mathbb{D}$. By a similar argument for φ_2, one sees that φ_1 is also a finite Blaschke product, which satisfies $\phi = \varphi_1 \circ \varphi_2$, completing the proof of the lemma.

By the above proof, one sees that if ϕ is reducible, then some of the local inverses can be analytically continued across some critical points of ϕ. But it is not clear that this is a sufficient condition for ϕ to be reducible.

Based on the above lemma, we explain the classification for a general Blaschke product of order four.

Let ϕ be a Blaschke product of order 4. One of the following scenarios holds.

(i) The partition of ϕ is $\{\{0\},\{1\},\{2\},\{3\}\}$; equivalently, $\phi \sim z^4$.

(ii) The partition of ϕ is $\{\{0\},\{2\},\{1,3\}\}$; equivalently, $\phi \sim \phi_a^2(z^2)$, where $\phi_a = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation with $a \neq 0$.

(iii) The partition of ϕ is $\{\{0\},\{1,2,3\}\}$; equivalently, ϕ is not reducible.

All possibilities above occur for some ϕ, by computations due to Sun, Zheng and Zhong in [1].

We now classify, using purely arithmetical considerations, the possible structure for a finite Blaschke product of order eight.

Theorem (1.2.19)[23]: Let ϕ be a Blaschke product of order 8. One of the following scenarios holds.

(i) The partition of ϕ is $\{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$; equivalently, $\phi \sim z^8$.

(ii) The partition of ϕ is $\{\{0\},\{2\},\{4\},\{6\},\{1,5\},\{3,7\}\}$; equivalently, $\phi \sim \phi_a^2(z^4)$, where $\phi_a = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation with $a \neq 0$.

(iii) The partition of ϕ is $\{\{0\},\{4\},\{1,2,3,5,6,7\}\}$; equivalently, $\phi \sim (z^2)$, where φ is an irreducible Blaschke product of order 4.

(iv) The partition of ϕ is one of $\{\{0\},\{4\},\{2,6\},\{1,3,5,7\}\}$, $\{\{0\},\{4\},\{2,6\},\{1,3\},\{5,7\}\}$, $\{\{0\},\{4\},\{2,6\},\{1,5\},\{3,7\}\}$ or $\{\{0\},\{4\},\{2,6\},\{1,7\},\{3,5\}\}$; equivalently, $\phi \sim \psi(\varphi_a^2(z^2))$, where ψ is a Blaschke product of order 2 and $\phi_a = \frac{a-z}{1-a_z}$ is a Möbius transformation with $a \neq 0$.

(v) The partition of ϕ is $\{\{0\},\{2,4,6\},\{1,3,5,7\}\}$; equivalently, $\phi \sim \psi \circ \varphi$, where ψ is a Blaschke product of order 2 and φ is an irreducible Blaschke product of order 4.

(vi) The partition of ϕ is $\{\{0\},\{1,2,3,4,5,6,7\}\}$; equivalently, ϕ is not reducible.

A similar approach would work for Blaschke products of arbitrary order. However, it seems difficult to decide whether a partition satisfying conditions $(\alpha_0),(\alpha_1),(\alpha_2)$ and (α_3) arises from a finite Blaschke product. For example, we cannot exhibit examples for each partition in case (4) in Theorem (1.2.19), although it is likely that they exist.

Proof . By condition (α_0), $\{0\}$ is a singleton in the partition for ϕ. Without loss of generality, suppose that $G_1 = \{0\}$. We list all possibilities by the minimal number $s = \min\{G_2,\dots,G_q\}$, where G_k is the number of elements in G_k . Clearly $s \neq 4,5,6$.

(I) Case $s = 1$. Suppose without loss of generality that G_2 is also a singleton.

Subcase (A): Suppose G_2 consists of one of the primitive elements $\{1,3,5,7\}$ in $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is generated by any element in $\{1,3,5,7\}$, by conditions (α_1) and (α_2), each G_k is a singleton. That is, the partition is just $\{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$. By [27], one sees that this is equivalent to $\phi \sim z^8$.

Subcase (B): Suppose (A) does not hold and G_2 consists of 2 or 6. By condition (α_1), the partition contains the singletons $\{2\},\{4\},\{6\}$. We list all possible partitions as follows:

(B1) $\{\{0\},\{2\},\{4\},\{6\},\{1,5,3,7\}\}$;

(B2) $\{\{0\},\{2\},\{4\},\{6\},\{1,3\},\{5,7\}\}$;

(B3) $\{\{0\},\{2\},\{4\},\{6\},\{1,5\},\{3,7\}\}$;

(B4) $\{\{0\},\{2\},\{4\},\{6\},\{1,7\},\{3,5\}\}$.

Case $(B2)$ is excluded by condition (α_1), since $\{2\}+\{1,3\}=\{3,5\}$ is not a union of some G_k in $(B2)$. One can get rid of $(B4)$ in a similar way. The remaining cases, $(B1)$ and $(B3)$, satisfy $(\alpha_0),(\alpha_1)$ and (α_2). But, by a direct computation they have the same dual partition $\{\{0\},\{2\},\{4\},\{6\},\{1,5\},\{3,7\}\}$. Using condition (α_3), we have that $\{\{0\},\{2\},\{4\},\{6\},\{1,5\},\{3,7\}\}$ is the unique choice. In this case, by Lemma (1.2.18), there exist a finite Blaschke product φ_1 of order 4 and a finite Blaschke product φ_2 of order 2 such that $\phi = \varphi_2 \circ \varphi_1$. Moreover, by the proof of Lemma (1.2.18), local inverses for φ_1 are $\rho_0,\rho_2,\rho_4,\rho_6$ in the family of local inverses of ϕ. By [27], one sees that this condition is equivalent to $\varphi \sim z^4$. This means that $\phi \sim \psi(z^4)$ for some Blaschke product ψ of order 2. Observe that two local inverses for ψ are holomorphic on $\mathbb{D}$, since one of them, $\rho_0(z) =$

z, is holomorphic. By [27], $\psi = \phi_b \circ z^2 \circ \phi_a$ for some Möbius transforms ϕ_a, ϕ_b. This implies that $\phi \sim \phi_a^2 (z^4)$, and $a \neq 0$, since it would degenerate to subcase (A) if $a = 0$.

We now consider the most complicated case in which $G_2 = \{4\}$ is the unique singleton other than G_1. We divide it into several distinct subcases looking again at the minimal number $t = \min\{G_3, \dots, G_q\}$. Clearly $2 \leq t \leq 5$ and $t \neq 4$. So, t is $2, 3$, or 5.

Subcase (C): $G_1 = \{0\}, G_2 = \{4\}$ and $t = 5$.

The only possibility is the partition $\{\{0\},\{4\},\{1,2,3,5,6,7\}\}$. By Lemma (1.2.18) and the observation that $\psi \sim z^2$ for each Blaschke product ψ of order 2, one sees that there exists a Blaschke product ϕ of order 4 such that $\phi \sim \varphi(z^2)$. We prove that ϕ is not reducible by contradiction. Otherwise, $\phi \sim \varphi_1 \circ \varphi_2$, where φ_1, φ_1 are Blaschke products of order 2. This implies that $\phi \sim \varphi_1 \circ B$ for a Blaschke product B of order 4, which leads to a contraction since by Lemma (1.2.18) $B^{-1} \circ B$ forms a subgroup of order 4 in $\phi - 1 \circ \phi$, as desired.

Subcase (D): $G_1 = \{0\}, G_2 = \{4\}$ and $t = 3$. Then the partition consists of G_1, G_2, G_3, G_4 with $G_3 = G_4 = 3$. Considering condition (α_2) and observing that 4 is the unique element other than 0 for which its inverse is itself, one sees that $G_4^{-1} = G_3$. The following partitions are all possible choices at this point:

(D1) $\{\{0\},\{4\},\{1,2,3\},\{7,6,5\}\}$;

(D2) $\{\{0\},\{4\},\{1,2,5\},\{7,6,3\}\}$;

(D3) $\{\{0\},\{4\},\{1,6,3\},\{7,2,5\}\}$;

(D4) $\{\{0\},\{4\},\{1,6,5\},\{7,2,3\}\}$.

Case (D_1) is impossible by condition (α_1), since

$$\{1,2,3\} + \{7,6,5\} = \{0,7,6,1,0,7,2,1,0\}$$

is not a union of some subsets in (D_1). One can prove similarly that $(D_2), (D_3)$ and (D_4) don't satisfy condition (α_1).

Subcase (E): $G_1 = \{0\}, G_2 = \{4\}$ and $t = 2$.

One possibility is that the partition consists of G_1, G_2, G_3, G_4 with $G_3 = 2$ and $G_4 = 4$. By condition (α_2), we have $G_k^{-1} = G_k$ for each G_k. So, the only possibilities are:

(E1) $\{\{0\},\{4\},\{1,7\},\{2,3,5,6\}\}$;

(E2) $\{\{0\},\{4\},\{2,6\},\{1,3,5,7\}\}$;

(E3) $\{\{0\},\{4\},\{3,5\},\{1,2,6,7\}\}$.

One excludes case (E1) by

$$\{4\} + \{1,7\} = \{5,3\},$$

and case (E3) by

$$\{4\} + \{3,5\} = \{7,1\}.$$

Another possibility is that $G_k = 2$ for any G_k in the partition other than G_1, G_2. There exist $C_6^2 C_4^2 C_2^2 / A_3^3 = 15$ choices:

(E4) $\{\{0\},\{4\},\{1,2\},\{3,5\},\{6,7\}\}$; (E5) $\{\{0\},\{4\},\{1,2\},\{3,6\},\{5,7\}\}$;

(E6) $\{\{0\},\{4\},\{1,2\},\{3,7\},\{5,6\}\}$; (E7) $\{\{0\},\{4\},\{1,3\},\{2,5\},\{6,7\}\}$;

(E8) $\{\{0\},\{4\},\{1,3\},\{2,6\},\{5,7\}\}$; (E9) $\{\{0\},\{4\},\{1,3\},\{2,7\},\{5,6\}\}$;

(E10) $\{\{0\},\{4\},\{1,5\},\{2,3\},\{6,7\}\}$;

(E11) $\{\{0\},\{4\},\{1,5\},\{2,6\},\{3,7\}\}$;

(E12) $\{\{0\},\{4\},\{1,5\},\{2,7\},\{5,6\}\}$;

(E13) $\{\{0\},\{4\},\{1,6\},\{2,3\},\{5,7\}\}$;

(E14) $\{\{0\},\{4\},\{1,6\},\{2,5\},\{3,7\}\}$;
(E15) $\{\{0\},\{4\},\{1,6\},\{2,7\},\{3,5\}\}$;
(E16) $\{\{0\},\{4\},\{1,7\},\{2,3\},\{5,6\}\}$;
(E17) $\{\{0\},\{4\},\{1,7\},\{2,5\},\{3,6\}\}$;
(E18) $\{\{0\},\{4\},\{1,7\},\{2,6\},\{3,5\}\}$.

One excludes most of them by the following observation: if $\{a,b\}$ is included in one of the above partitions, then one of the equations $a + b = 0, a + b = 4$ and $a = 4 + b$ holds. Indeed, by condition (α_1),

$$\{a,b\} + \{a,b\} = \{2a, a + b, a + b, 2b\}$$

is a union of some G_k*'s*. If $\{a + b\}$ is a singleton, then $a + b = 0$ or $a + b = 4$. Otherwise, $a + b$ is including in some G_k satisfying $G_k > 1$. Noticing that each element of G_k is included in $\{a,b\} + \{a,b\}$, one sees that $G_k \neq 3$. It's easy to verify that Gk = 3 since we assume that the singleton $\{a + b\}$ is not in the partition. So, $G_k = 2$ and

$$G_k = \{2a, a + b\} = \{a + b, 2b\}.$$

That is, $2a = 2b$. This means that $a = 4 + b$. Furthermore, noticing that both $2a$ and $a + b = 2a + 4$ are even in that case, one sees that $G_k = \{2,6\}$.

By this observation, all the partitions other than $(E8), (E11)$ and $(E18)$ are excluded. By a direct computation, one sees that $(E8), (E11)$ and $(E18)$ satisfy the other conditions, too.

Moreover, the above argument shows that $(E2), (E8), (E11)$ and $(E18)$ are all the possible partitions that include the sets $\{0\},\{4\},\{2,6\}$. By Lemma (1.2.18) and [1], there exist a Blaschke product ψ of order 2 and a Blaschke product φ of order 4, such that $\phi = \psi \circ \varphi$ and φ is included in case 2 in [1]. This implies that ϕ has the desired decomposition.

We now turn to the cases $s > 1$. Firstly, by condition (α_2), 4 is not included in any G_k for which G_k is even. Otherwise, if $4 \in G_k$, then $G_k^{-1} = G_k$ since 4 is the unique element other than 0 for which its inverse is itself. Therefore,

$$G_k = \{4, k_1, \ldots, k_i, 8 - k_1, \ldots, 8 - k_i\}$$

for some $k_1, \ldots, k_i$. This contradicts the fact that G_k is even. So, $4 \notin G_k$ if G_k is even.

Secondly, the argument used in analyzing subcase (E) is still valid. Hence, if $\{a,b\}$ is in the partition, then $a + b = 0$ or $a = 4 + b$. In the latter case, $\{2,6\}$ is in the partition. Moreover, since $\{a,b\} + \{a,b\}$ is a union of some G_k*'s* satisfying $G_k \leq 2$, and 4 is not included in any such G_k, we have that $4 \neq 2a, 2b, 2(a + b)$. Therefore, neither 2 nor 6 can be included in any G_k when the partition satisfies $s > 1$ and $G_k = 2$. It also implies that $a + b = 0$ if $\{a,b\}$ is in the partition.

(II) Case $s = 2$.

One possibility is that the partition consists of G_1, G_2, G_3 satisfying $G_2 = 2$ and $G_3 = 5$. By the above observation, the partition must be one of the following: (II1) $\{\{0\},\{1,7\},\{2,3,4,5,6\}\}$;
(II2) $\{\{0\},\{3,5\},\{1,2,4,6,7\}\}$.

Obviously, none of them satisfies condition (α_1).

Another scenario is that the partition consists of G_1, G_2, G_3, G_4 satisfying $G_2 = G_3 = 2$ and $G_4 = 3$. By the above argument, $G_4 = \{2,4,6\}$. So, all the possibilities are listed below:

(II3) $\{\{0\},\{1,3\},\{5,7\},\{2,4,6\}\}$;
(II4) $\{\{0\},\{1,5\},\{3,7\},\{2,4,6\}\}$;

(II5) $\{\{0\},\{1,7\},\{3,5\},\{2,4,6\}\}$.
None of them satisfies condition (α_1).
(III) Case $s = 3$.

In this case, the partition consists of G_1, G_2, G_3 satisfying $G_2 = 3$ and $G_3 = 4$. By the above argument and condition (α_2), one sees that $G_2^{-1} = G_2, G_3^{-1} = G_3$ and $4 \in G_2$. So, the partition is one of the following:
$(III1)$ $\{\{0\},\{1,4,7\},\{2,3,5,6\}\}$;
$(III2)$ $\{\{0\},\{2,4,6\},\{1,3,5,7\}\}$;
$(III3)$ $\{\{0\},\{3,4,5\},\{1,2,6,7\}\}$.
Both (III1) and (III2) are excluded by condition (α_1), since $\{1,4,7\}+\{1,4,7\}$ and $\{3,4,5\} + \{3,4,5\}$ are not unions of some subsets in the partitions, respectively. For the finial possibility $\{\{0\},\{2,4,6\},\{1,3,5,7\}\}$, using an argument similar to the above, one sees that it is equivalent to the condition that $\phi \sim \psi \circ \phi$, where ψ is a Blaschke product of order 2 and φ is a Blaschke product of order 4, and ϕ is included in case 3 in [1].
(IV) Case $s = 7$.

The only choice is $\{\{0\},\{1,2,3,4,5,6,7\}\}$. By Lemma (1.2.18), ϕ is not reducible in this case.

We conclude with the following corollary which follows after one summarizes all the possibilities listed above.

Corollary (1.2.20)[23]: Let ϕ be a finite Blaschke product of order 8. Then M_ϕ has exactly 2 nontribyl minimal reducing subspaces if and only if ϕ is not reducible.

It is natural to ask if this result extends to the general case. One can obtain a similar result for order 6 by the above arithmetic way. But, the calculation for order 5 or 7 suggests that some counterexample may exist. A possible guess may be that the result holds whenever the order of ϕ is not prime.

Section (1.3): Toeplitz Operators on the Polydisk

For D denote the open unit disk in the complex plane. For $-1 < \alpha < +\infty$, $L^2(D, dA_\alpha)$ is the space of functions on D which are square integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$, where dA denotes the normalized Lebesgue area measure on $D. L^2(D, dA_\alpha)$ is a Hilbert space with the inner product $\langle f, g\rangle_\alpha = \int_D f(z) g\overline{(z)} dA_\alpha$. The weighted Bergman space A_α^2 is the closed subspace of $L^2(D, dA_\alpha)$ consisting of analytic functions on D. If $\alpha = 0, A_0^2$ is the Bergman space. We write $A^2 = A_0^2$ It is known that $\{\frac{z^n}{||z^n||_\alpha}\}_{n=0}^{+\infty}$ is an orthogonal basis of $A_\alpha^2(D)$. Let $\gamma_n = ||z^n||_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}}$ for n = 0, 1, 2, ... Therefore,

$$||f||_\alpha^2 = \sum_{n=0}^{+\infty} \gamma_n^2 |a_n|^2 < \infty$$

with $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A_\alpha^2(D)$ Denote the unit polydisk by D^n. The weighted Bergman space $A_\alpha^2(D^n)$ is then the space of all holomorphic functions on $L^2(D^n, dv_\alpha)$, where $dv_\alpha(z) = dA_\alpha(z_1) \ldots dA_\alpha(z_n)$. For multi-index $\beta = (\beta_1, \ldots, \beta_n), \beta \geq 0$ means that $\beta_i \geq 0$for any $i \geq 0$. Denote by $z_1^{\beta_1} z_1^{\beta_2} \ldots z_1^{\beta_n}$ and

$$e_\beta = \frac{z^\beta}{\gamma_{\beta_1} \cdots \gamma_{\beta_n}},$$

then $\{e_\beta\}_\beta$ is an orthogonal basis in $A^2_\alpha(D^n)$ Let P be the Bergman orthogonal projection from $L^2(D^n)$ onto $A^2_\alpha(D^2)$.

For a bounded measurable function $f \in L^\infty(D^n)$, the Toeplitz operator with symbol f is defined by $T_f h = P(fh)$ for every $h \in A^2_\alpha(D^n)$.

Recall that in a Hilbert space $\mathcal{H}, a$ (closed) subspace$\mathcal{M}$is called a reducing subspace of the operator T if $T(\mathcal{M}) \subseteq M$ and $T^*(\mathcal{M}) \subseteq M$. A nontribyl reducing subspace M is said to be minimal if the only reducing subspaces contained in M are M and $\{0\}$. On the Bergman space $A^2_\alpha(D^2)$, the reducing subspaces of the Toeplitz operators with finite Blaschke product simples are well studied (see [28], [10], [22] for example). On $A^2_\alpha(D^2)$, Y. Lu and X. Zhou [37] characterized the reducing subspaces of Toeplitz operators $T_{z_1^N z_1^N,} T_{z_1^N}$

We consider the reducing subspaces of the Toeplitz operators $T_{z_1^N z_2^M} on\ A^2_\alpha(D^2)$ and $T_{z_i^N z_j^M}$ on $A^2_\alpha(D^n)$, where$N, M \geq 1$ are integers and
$1 \leq i \leq j \leq n$. Usually, the Toeplitz operators on the unweighted Bergman space and the weighted Bergman space have similar properties (see [38], [39], [40], [41] for example). However, we obtain that the minimal reducing subspaces of $T_{z_1^N z_2^N}$with $N \neq M$ on $A^2_\alpha(D^2)(\alpha \neq 0)$ are less then that on $A^2(D^2)$(see Theorem (1.3.4) and Theorem (1.3.6)).

Let M, N be integers with $M, N \geq 1$ and $M \neq N$. We consider the minimal reducing subspace of $T_{z_1^N z_2^M}$ on $A^2(D^2)$. Here $\gamma_k = \left\|z^k\right\|_0 = \sqrt{\frac{1}{k+1}}$. Let $\rho_1(k) = \frac{(k+1)N}{M} - 1$ $\rho_2(k) = \frac{(k+1)M}{N} - 1$. Let $\mathcal{H}_{nm}$ = Span$\{z_1^n, z_2^m, z_1^{\rho_1(m)}, z_2^{\rho_2(n)}\}$ and P_{nm} be the orthogonal projection from $A^2_\alpha(D^2)$ onto $\mathcal{H}_{nm}$.

Lemma (1.3.1)[35]: Let n, m, h be nonnegative integers. Then the following statements hold:

(a) if $\rho_1(m)$ is an integer, then $\rho_1(m + hM) = \rho_1(m) + hN$ is an integer for every $h \geq 0$;

(b) if $\rho_2(n)$is an integer, then $\rho_2(n + hN) = \rho_2(n) + hM$ is an integer for every $h \geq 0$;

(c) if $\rho_1(m)$ and $\rho_2(n)$ are positive integers, then $\gamma_{\rho_1(m)}\gamma_{\rho_2(n)} = \gamma_m \gamma_n$;

(d) $\rho_1\big(\rho_2(n)\big) = n$ and $\rho_2\big(\rho_1(m)\big) = m$

Proof. Notice that if $\rho_1(m)$ and $\rho_2(n)$ are positive integers, then $\gamma_{\rho_{1(m)}} = \sqrt{\frac{M}{N}}\gamma_m$ and

$\gamma_{\rho_{2(n)}} = \sqrt{\frac{N}{M}}\gamma_n$.

So (c) holds. By the direct calculation, (a), (b) and (d) are obvious.

Theorem (1.3.2)[35]: Let n, m be integers such that $0 \leq n \leq N - 1\ or\ 0 \leq m \leq M - 1$, and both of $\rho_1(m)$ and $\rho_2(n)$ are integers. Then for $a, b \in \mathbb{C}$, $\mathcal{M}$= Span$\{a z_1^{n+hN} z_2^{m+hM} + b z_1^{\rho_1(m+hM)} z_2^{\rho_2(n+hN)}; h = 0,1,2 \ldots\}$ is a minimal reducing subspace of $T_{z_1^N z_2^M}$ on the polydisk.

Proof. By Lemma (1.3.1)(a) and (b), it is easy to check that $T_{z_1^N z_2^M}(\mathcal{M}) \subseteq \mathcal{M}$.

On the other hand,

$$T^*_{z_1^N z_2^M}\big(z_1^k z_2^l\big) = \sum_{\beta \geq 0} \langle T^*_{z_1^N z_2^M} z_1^N z_2^l, e^\beta \rangle\, e^\beta = \begin{cases} \dfrac{\gamma_k^2 \gamma_l^2}{\gamma_{k-N}^2 \gamma_{l-M}^2} z_1^{k-N} z_2^{l-M}, if \quad k \geq N, l \geq M \\ \quad 0\,, \qquad if \quad others \end{cases}$$

For each $h \geq 1$,

$$T^*_{z_1^N z_2^M}\left(z_1^{n+hN} z_2^{m+hM}\right) = \frac{\gamma^2_{n+hN}\gamma^2_{m+hM}}{\gamma^2_{n+(h-1)N}\gamma^2_{m+(h-1)M}} z^{n+(h-1)N} z_2^{m+(h-1)M}$$

$$T^*_{z_1^N z_2^M}\left(z_1^{\rho_1(m+hM)} z_2^{\rho_2(n+hN)}\right) = \mu(a z_1^{n+hN-N} z_2^{m+hM-M} + b z_1^{\rho_1(m+hM-M)} z_2^{\rho_2(n+hN-N)}) \in \mathcal{M}$$

Where

$$\mu = \frac{\gamma^2_{n+hN}\gamma^2_{m} + hM}{\gamma^2_{n+(h-1)N}\,\gamma^2_{m+(h-1)M}} = \frac{\gamma^2_{\rho_1(m+hM)}\gamma^2_{\rho_2(n+hN)}}{\gamma^2_{\rho_1(m+hM)-N}\gamma^2_{\rho_2(n+hN)-M}}$$

Since $0 \leq n \leq N-1 (or\ 0 \leq m \leq M-1)$, we get $\rho_2(n) < M (or \rho_1(m) < N$, respectively). Therefore, $T^*_{z_1^N z_2^M}(a z_1^N z_2^M + b z_1^{\rho_1(m)} z_2^{\rho_2(n)}) = 0 \in \mathcal{M}, T^*_{z_1^N z_2^M}(\ \mathcal{M}\ ) \in \mathcal{M}\mathcal{M}$. So, which finishes the proof.

Lemma (1.3.3)[35]: Suppose $\mathcal{M} \neq 0$ is a reducing subspace of $T_{z_1^N z_2^M}$ in $A^2(D^2)$. Let $f = \sum_{(k,l)\geq 0} a_{k,l} z_1^k z_2^l \in \mathcal{M}$. For each nonnegative integers n, m with $a_{nm} \neq 0$, the following statements hold:

(I) $if\, \rho_1(m), \rho_2(n)$ are integers and $a_{\rho_1(m)\rho_2(n)} \neq 0$ then

$$a_{nm} z_1^n z_2^m + a_{\rho_1(m)\rho_2(n) z_1^{\rho_1(m)} z_2^{\rho_2(n)}} \in \mathcal{M}$$

(II) if at least one of $\rho_1(m), \rho_2(n)$ is not an integer, $or\ a_{\rho_1(m)\rho_2(n)} = 0$,

Proof. For every integer $h \geq 0$, denote by $T_h = T_{z_1^{hN} z_2^{hM}}$. Notice that

$$T_h^* T_h(z_1^n z_2^m) = \frac{\gamma^2_{hN} + n\gamma^2_{hM} + m}{\gamma_n^2 \gamma_m^2} z_1^n z_2^m \in \mathcal{M}\ , \forall n, m \geq 0 \tag{40}$$

Let P_M be the orthogonal projection from $A^2_\alpha(D)$ onto$M$, then for nonnegative integers m, n, k, l,

$$\langle P_M T_h^* T_h z_1^n z_2^m, z_1^k z_2^l \rangle = \langle T_h^* T_h P_M z_1^n z_2^m, z_1^k z_2^l \rangle = \langle P_M z_1^n z_2^m, T_h^* T_h z_1^k z_2^l \rangle$$

thus$\frac{\gamma^2_{hN+k}\gamma^2_{hM+1}}{\gamma_k^2\gamma_l^2} = \frac{\gamma_h^2 N + n\gamma^2 M + m}{\gamma_n^2 \gamma_m^2}$. Equivalently,

$$\frac{(k+1)(l+1)}{(n+1)(m+1)} = \frac{(k+hN+1)(l+hM+1)}{(n+hN+1)(m+hM+1)}, h \geq 0 \tag{41}$$

We claim that (k, l) = (n, m) or (k, l) = $(\rho_1(m), \rho_2(n))$. In fact, let $h \geq +\infty$, then

$$(k+1)(l+1) = (n+1)(m+1). \tag{42}$$

It follows that $(k+hN+1)(l+hM+1) = (n+hN+1)(m+hM+1)$. Since $g(\lambda) = (k+\lambda N+1)(l+\lambda M+1) - (n+\lambda M+1)$ is an analytic polynormal on $\mathbb{C}, g(\lambda) = 0$ for any $\lambda \in \mathbb{C}$. The coefficient of λ must be zero.

We get

$$M(n-k) = N(l-m) \tag{43}$$

This together with (42) implies the claim.

Therefore, $P_M(z_1^n z_2^m) \in \mathcal{H}_{nm}$. Hence,

$$P_{nm} P_M(z_1^n z_2^m) = P_M(z_1^n z_2^m)$$

Since $P_M f$ for every $f \in M$, we arrive to

$$\langle P_M P_{nm} f, z_1^{\rho_1(m)} z_2^{\rho_2(n)} \rangle = \langle P_{nm} f, z_1^{\rho_1(m)}\, z_2^{\rho_2(n)} \rangle$$

Moreover, $\langle P_M P_{nm} f, z_1^k z_2^l \rangle = \langle P_{nm} f = P_M P_{nm}(f) \in M \rangle$. So we get the result.

Theorem (1.3.4)[35]: Suppose $M \neq \{0\}$ is a reducing subspace of $T_{z_1^N z_2^M}$ in the Bergman space $A^2(D^2)$. Then there exist $a, b \in \mathbb{C}$ and nonnegative integers m, n with $0 \leq n \leq N-1$ *or* $0 \leq m \leq M-1$, such that M contains a reducing subspace as follows

$$M_{n,m,a,b} = \text{ Span}\left\{a z_1^{hN+n} z_2^{hM+m} + b z_1^{\rho_1(m+hN)} z_2^{\rho_2(n+hM)} : h = 0,1,2,\ldots\right\},$$

where $\rho_1(m+hN) = \frac{(m+hN+1)M}{N} - 1$ and $\rho_2(n+hM) = \frac{(n+hM+1)N}{M} - 1$. In particular, if $\rho_1(m)(or \rho_2(n))$ is not a positive integer, then b = 0. Moreover, M is minimal if and only if $\mathcal{M} = \mathcal{M}_{n,m,a,b}$.

Proof. (I) If $\mathcal{M} \neq 0$, there exist nonzero function $f \in \mathcal{M}$ and k, l, such that $P_{kl} f \neq 0$. Lemma (1.3.3) implies that

$$gkl = P_{kl} f = a z_1^k z_2^l + b z_1^{\rho_1(l)} z_2^{\rho_2(k)} \in \mathcal{M}$$

Observe that there is a positive integer h_0 such that $a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)} = (T^* z_1^N z_2^M)^{h_0}(g_{kl}) \neq 0, \left(T^*_{z_1^N z_2^M}\right)^{h_0+1}(g_{kl}) = 0$

where $= k - h_0 N$, $m = 1 - h_0 M$.

Clearly, $0 \leq n \leq N - 1 or\ 0 \leq m \leq M - 1$. So Theorem (1.3.2) shows that

$$a z_1^n z_2^m + b z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in M_{n,m,a,b} \subseteq M$$

(II) Suppose $\mathcal{M}$ is minimal. As in (I), there is a nonzero function $a z_1^n z_2^m + b z_1^{\rho_{1(m)}} z_2^{\rho_2(n)} \in M$, then . Then the following statements hold:

(a) if $z_1^n z_2^m \in \mathcal{M}$ *then* $\mathcal{M} = span\{z_1^{n+hN} z_2^{m+hM}, h \geq 0\}$;

(b) if $\rho_1(m), \rho_2(n)$ are integers, and $z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}$, then

$$\mathcal{M} = span\left\{z_1^{\rho_1(m)+hN} z_2^{\rho_2(n)+hM}, h \geq 0\right\};$$

(c) if none of $z_1^n z_2^m$ and $z_1^{\rho_1(m)} z_2^{\rho_2(n)}$ is in $\mathcal{M}$, then $\mathcal{M} = \mathcal{M}_{n,m,a,b}$ with $ab \neq 0$.

So we finish the proof.

Let $-1 < \alpha < +\infty$ with $\alpha \neq 0$. We consider the reducing subspace of $T_{z_1^N z_2^N}$ on the weighted Bergman Space $A_\alpha^2(D)$.

Here $\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}}$ We begin with a useful lemma.

Lemma (1.3.5)[35]: Let M, N, n, m, k, l be nonnegative integers with $1 > m, n > k$ and $M, N \geq 1$. If

$$\gamma_{hN+k}^2 \gamma_{hM+l}^2 = \gamma_{hN+n}^2 \gamma_{hM+m}^2, h \geq 0 \tag{44}$$

then $N = M$, $l = n$ and $m = k$.

Proof. First, note that the equality (44) holds if and only if for any $\curlywedge \in \mathbb{C}$ the following equality holds:

$$\prod_{j=1}^{n-k} (\curlywedge N + j + k) \prod_{j=1}^{l-m} (\curlywedge M + 2 + \alpha + l - j)$$

$$\prod_{j=1}^{n-k}(\lambda N+2+\alpha+n-j)\prod_{j=1}^{l-m}(\lambda M+j+m) \tag{45}$$

By computing the coefficient of $\lambda^{n-k+l-m-1}$ in the equality (45), we obtain

$$M\sum_{j=1}^{n-k}(j+k)+N\sum_{j=1}^{l-m}(2+\alpha+l-j)=M\sum_{j=1}^{n-k}(2+\alpha+n-j)+N\sum_{j=1}^{l=m}(j+m)$$

It follows that $M(n-k)=N(l-m)$.

Second, we prove that if α is not an integer, then the following statements hold:

$$(m+1)N=(k+1)M \text{ and } (l+1+\alpha)N=(n+1+\alpha)M. \tag{46}$$

(a) Let $\lambda_1=\frac{k+1}{N}$. Then $\lambda_1 N+k+1=0$ and $\lambda_1 N+2+\alpha+n-j\neq 0$ for any $1\le j\le n-k$, because$\lambda_1 M+2+\alpha+n-j$ is not an integer. Therefore, the equality (45) implies that $\prod_{j=1}^{l-m}(\lambda_1 M+j+m)=0$. That is, there exists $1\le h_1\le l-m$ such that $\lambda_1 M+m+h_1=0$. So, $h_1=\frac{k+1}{N}M-m\ge 1$ follows that $(m+1)N\le(k+1)M$.

(b) Let $\lambda_2=-\frac{m+1}{M}$ Then $\lambda_2 M+m+1=0$. Similarly, we can get an integer h_2 such that $1\le h_2\le l-m$ and $\lambda_2 N+k+h_2=0$, which implies that $h_2=\frac{m+1}{M}N-k\ge 1$. Thus $(m+1)N\ge(k+1)M$.

Comparing (a) with (b), we arrive at $(m+1)N\ge(k+1)M$.

(c) Let $\mu_1=-\frac{n+1+\alpha}{M}$. Then $\mu_2 M+l+1+\alpha=0$, $\mu_1 N+k+j\neq 0$ for any $1\le j\le n-k$. Therefore, $\prod_{j=1}^{l=m}(\mu_1 M+2+\alpha+l-j)=0$. That is, there exists $1\le h_3\le l-m$ such that $\mu_2 M+2+\alpha+l-h_3=0$. So, $h_3=-\frac{l+1+\alpha}{N}M+(2+\alpha+l)\ge 1$, i.e., $(l+1+\alpha)N\ge(n+1+\alpha)M$.

(d) Let $\mu_2=-\frac{l+1+\alpha}{M}$. Then $\mu_2 M+l+1+\alpha=0$. As in (c), there exists $1\le h_4\le n-k$ such that $\mu_2 N+\alpha+2+n-h_4=0$. So, $1\le h_4=-\frac{l+1+\alpha}{M}N+(2+\alpha+n)\le n-k$ and $(l+1+\alpha)N\le(n+1+\alpha)M$.

Comparing (c) with (d), we arrive at $(l+1+\alpha)N=(n+1+\alpha)M$.

Third, we prove that if α is an positive integer, then (46) holds. In fact, if $1+\alpha\ge 2$ is an integer, then (45) can be simplified into

$$\prod_{j=1}^{k_1}(\lambda N+j+k)\prod_{j=1}^{m_1}(\lambda M+2+\alpha+l-j)$$
$$=\prod_{j=1}^{k_1}(\lambda N+2+\alpha+n-j)\prod_{j=1}^{m_1}(\lambda M+j+m),\forall\lambda\in\mathbb{C}$$

where $2\le k_1\le n-k, 2\le m_1\le l-m, 2+\alpha+n-k_1>k_1+k$ and $2+\alpha+l-m_1>m_1+m$. By the same technique as in second part of the proof, we can get the equalities in (46).

Finally, combining the equalities (46) with $M(n-k)=N(l-m)$, it is easy to get $\alpha N=\alpha M$. Since $\alpha\neq 0$, we have $N=M, l=n, k=m$.

Theorem (1.3.6)[35]: Let $\alpha\neq 0$ $M,N\ge 1$ with $M\neq N$. Suppose $\mathcal{M}\neq\{0\}$is a reducing subspace of $T_{z_1^N z_2^M}$in the weighted $Bergman\ space$ $A_\alpha^2(D^2)$ then there exist nonnegative integers n,m with $0\le n\le N-1$ or $0\le m\le M-1$ such that

$$\mathcal{M}_{nm} = span\{z_1^{hN+n} z_2^{hM+m}: h = 0,1,2, \dots\} \subseteq \mathcal{M}$$

In particular, $\mathcal{M}$ is minimal if and only if there exist n, m as in assumption such that $\mathcal{M}$ = $\mathcal{M}_{mn}$.

Proof. Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace. As in the proof of Lemma (1.3.3), there exist integers n, m such that $P_{\mathcal{M}}(z_1^n z_2^m) \neq 0$ and

$$\frac{\gamma_{hN+k}^2 \gamma_{hM+l}^2}{\gamma_k^2 \gamma_l^2} = \frac{\gamma_{hN+n}^2 \gamma_{hM+m}^2}{\gamma_n^2 \gamma_m^2}, \forall h \geq 0$$

whenever $\langle P_{\mathcal{M}}(z_1^n z_2^m), z_1^k z_2^l \rangle \neq 0$. Considering that $\{\gamma_j\}_{j=0}^{+\infty}$ is strictly decreasing $\frac{\gamma_{hN+k}^2 \gamma_{hM+l}^2}{\gamma_{hN+n}^2 \gamma_{hM+m}^2} \to 1$ as $h \to +\infty$ [36], we obtain that $\gamma_k^2 \gamma_l^2 = \gamma_n^2 \gamma_m^2$ and

$$\gamma_{hN+k}^2 \gamma_{hM+l}^2 = \gamma_{hN+n}^2 \gamma_{hM+m}^2, h \geq 0.$$

This means that one of the following statements holds:

(i) $l = m, n = k$;

(ii) $> m \; and \; n > k$;

(iii)$l < m$ and $n < k$.

Since $N \neq M$, Lemma (1.3.5) implies that (ii) does not hold. By the same technique, (iii) does not hold. So, (i) holds, that is, there exists $c_{nm} \in \mathbb{C}$ such that $P_{\mathcal{M}}(z_1^n z_2^m) = c_{nm} z_1^n z_2^m$

For $f = \sum_{(k,l)\geq 0} a_{kl} \, z_1^k z_2^l \in \mathcal{M}$, we claim that if $a_{nm} \neq 0$, then $c_{nm} \neq 0$. In fact,

$$Q_{nm} f = Q_{nm} P_{\mathcal{M}}(f) = Q_{nm}\Big(\sum_{(k,l)\geq 0} P_{\mathcal{M}}(a_{kl} \, z_1^k z_2^l)\Big)$$
$$= c_{nm} a_{nm} z_1^n z_2^m = c_{nm} Q_{nm} f$$

where Q_{nm} is the orthogonal projection from $A_\alpha^2(D^2)$onto Span$\{z_1^n z_2^m\}$.

Therefore, $c_{nm} = 1 \neq 0$.

Hence $z_1^n z_2^m \in \mathcal{M}$. Choose an integer h_0 such that $0 \leq n - h_0 N \leq N - 1$, $m - h_0 M \geq 0 \; or \; 0 \leq m - h_0 M \leq M - 1, n - h_0 N \geq 0$. As in the proof of Theorem (1.3.4), Span $\{z_1^{n+(h-h_0)N} z^{m+(h-h_0)M} \;\; h$ = 0, 1, 2, . . .$\} \subseteq \mathcal{M}$is a minimal reducing subspace of $T_{z_1^N z_2^M}$. The proof is complete.

Theorem (1.3.7)[35]: Let $N, M \geq 1$ and $N \neq M$. Every nonzero reducing subspace $\mathcal{M}$ of $T_{z_1^N z_2^M}$ in $A_\alpha^2(D^2)$ for every $\alpha > -1$ is a direct (orthogonal) sum of some minimal reducing subspaces.

Proof. We prove the theorem in two cases.

Case one: $\alpha \neq 0$. Let us denote

$\mathcal{M}_{nm}$ = Span$\{z_1^{hN+n} \, z_2^{hM+m}: h = 0,1,2..\}$

where $0 \leq n \leq N - 1 \; or \; 0 \leq m \leq M - 1$. By Lemma (1.3.5), we have $\mathcal{M}_{nm} \subseteq \mathcal{M}$ if and only if there exist some $f \in \mathcal{M}$with $\langle f, z_1^n z_2^m \rangle \neq 0$. Let $E_1 = \{(n, m)\} \geq 0$; $n \leq N - 1 \; or \; 0 \leq m \leq M - 1$,$\langle f, z_1^n z_2^m \rangle \neq 0$ for some $f \in \mathcal{M}\}$ 2 M}. Then $\mathcal{M} = \bigoplus_{n,m \in E_1} \mathcal{M}_{nm}$.

Case two: $\alpha = 0$. For $n, m \geq 0$, there exist $a, b \in \mathbb{C}$ such that $\mathcal{M}$ contains the minimal reducing subspace of $T_{z_1^n z_2^m}$defined by $\mathcal{M}_{n,m,a,b} = span\,\{a z_1^{hN+n} z_2^{hM+m} + b z_1^{\rho_1(m+hN)} z_2^{\rho_2(n+hM)}$:h = 0, 1, 2, . . .}.

In fact,

(i) If $z_1^n z_2^m \in \mathcal{M}, then \; \mathcal{M}_{n,m,1,0} = \mathcal{M}_{nm}$

(ii) If $z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}$ then $\mathcal{M}_{n,m,0,1}$=$\mathcal{M}_{\rho_1(m)\rho_2(n)}$.

(iii) If neither $z_1^n z_n^m$ nor $z_1^{\rho_1(m)} z_2^{\rho_2(n)}$ are in $\mathcal{M}$, and there exists $f \in \mathcal{M}$

such that $P_{nm}f \neq 0$, then Theorem (1.3.4) implies that $\mathcal{M}_{n,m,a,b} \subseteq \mathcal{M}$ is a minimal reducing subspace of $T_{z_1^N z_2^M}$ where $P_{nm}f = az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)}$. It follows that $P_{nm}g = \lambda(az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)})$ forevery $g \in \mathcal{M}$ with $P_{nmg} \neq 0$.

(iv) If $P_{nm}f = 0$ for any $f \in \mathcal{M}$, then $\mathcal{M}_{n,m,a,b} \subseteq \mathcal{M}$ if and only if a = 0, b = 0, i.e.,$\mathcal{M}_{n,m0,0} = \{0\}$.

Let $\mathcal{M}' = \mathcal{M} \ominus \mathcal{M}_{n,m,a,b}$. Then $\mathcal{M}'$ is a reducing subspace. Continuing this process, sin $A^2(D^2) = \oplus_{n,m\geq 0} z_1^n z_2^m$, it is not different to prove that $\mathcal{M}$ is the direct (orthogonal) sum of some minimal reducing subspaces as$\mathcal{M}_{n,m,a,b}$.

In [22], Kehe Zhu shows that a reducing subspace of T_{z^N} on A^2(D) is the direct (orthogonal) sum of at most N minimal reducing subspaces. However, the reducing subspace of $T_{z_1^N z_2^M}$on $A^2(D^2)$ may be the direct (orthogonal) sum of infinity numbers of minimal reducing subspaces. For example, $\mathcal{M}$ =Span$\{z_1^{1+2h} f(z_2); f \in A_\alpha^2(D)$, h = 0, 1, 2, . . .$\}$ is a reducing subspace of $T_{z_1^2 z_2^3}$ and $\mathcal{M} = \oplus_{n=0}^{+\infty} \mathcal{M}_n$, where $\mathcal{M}_n$ = Span$\{z_1^{1+2h} z_2^{n+3h}; h = 0,1,2 ...\}$

We consider the reducing subspace of $T_{z_i^N z_j^M}(N, M \geq 1, N \neq M, i \neq j)$ in the weighted Bergman space $A_\alpha^2(D^2)$ with $N \neq M$.

Theorem (1.3.8)[35]: Suppose $\mathcal{M} \neq \{0\}$ is a reducing subspace of $T_{z_j^N z_j^M}(N, M \geq 1, N \neq M, i \neq j$ in weighted Bergman space.

Then the following statements hold:

(a) if $\alpha = 0$, then there exist functions $g_1, g_2 \in A_\alpha^2(D^{n-2})$ and integers l, m with $0 \leq l \leq N-1$ or $0 \leq m \leq M-1$, such that M contains the reducing subspace
$\mathcal{M}'$ = Span$\{\left(g_1(z') z_1^{hN+l} z_2^{hM+m} + g_2(z') z_1^{\rho_1(l+hN)} z_2^{\rho_2(m+hM)}\right); h \geq 0\}$;

(b) if $\alpha \neq 0$, then there exist a function $g \in A_\alpha^2(D^{n-2})$ and integers l, m with $0 \leq l \leq N-1$ or $0 \leq m \leq M-1$ such that $\mathcal{M}$ contains the reducing subspace

$$\mathcal{M})_{lmg} = span\{z_i^{hN+l} z_j^{hM+m} g(z'): h = 0,1,2, ...\}$$

where $z' = (z_1, ..., z_{i-1,} z_{i+1}, ..., z_{j-1}, z_{j+1}, ..., z_n)$.

Moreover, $\mathcal{M}'$ is the only minimal reducing subspace of $T_{z_i^N z_j^M}$on $A^2(D^2)$ and $\mathcal{M}_{lmg}$ is the only minimal reducing subspace of $T_{z_i^N z_j^M}$on A_α^2 with $\alpha \neq 0$.

Proof. Without loss of generality, let $i = 1$ and $j = 2$. Denote by PM the orthogonal projection from $A_\alpha^2(D^2)$ onto $\mathcal{M}$. Let $z^k = z_1^{k_1} z_2^{k_2} ... z_n^{k_n}$ with $P_\mathcal{M}(z^k) \neq 0$. Let $T_h = T_{z_1^{hN} z_2^{hM}}$. Then $\langle T_h^* T_h P_\mathcal{M} z^k, z^L\rangle = \langle P_\mathcal{M} T_h^* T_h z^K, z^L\rangle$ for any $z^L = z_1^{l_1} z_2^{l_2} ... z_n^{l_n}$ Observe that

$$\langle P_\mathcal{M} z^K, T_h^* T_h z^L\rangle = \frac{\gamma_{hN+l_1}^2 \gamma_{hM+l_2}^2}{\gamma_{l_1}^2 \gamma_{l_2}^2} \langle P_\mathcal{M} z^K, z^L\rangle$$

and

$$\langle T_h^* T_h z^K, P_\mathcal{M} z^L\rangle = \frac{\gamma_{hN+l_1}^2 \gamma_{hM+K_2}^2}{\gamma_{K_1}^2 \gamma_{k_2}^2} \langle z^k, P_\mathcal{M} z^L\rangle$$

Therefore

$$\frac{\gamma_{hN+k_1}^2 \gamma_{hM+k_2}^2}{\gamma_{k_1}^2 \gamma_{k_2}^2} = \frac{\gamma_{hN+l_l}^2 \gamma_{hM+l_2}^2}{\gamma_{l_1}^2 \gamma_{l_2}^2}, \forall h \geq 0$$

Whenever $\langle P_\mathcal{M} z^K, z^L\rangle \neq 0$.

If $\alpha = 0$, then as in Lemma (1.3.3) we have $(l_1, l_2) = (k_1, k_2)$ or $(l_1, l_2) = (\rho_1(k_2), \rho_2(k_1)$ where $\rho_1(k_2), \rho_2(k_1)$ are integers. Thus$P_{\mathcal{M}} z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} z'^{k'}$ and $P_{\mathcal{M}} z^k$are in $z_1^{k_1} z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} A^2(D^{n-2})$, where $z' = (z_3, \dots, z_n)$, and $K' = (k_3, \dots, k_n)$. Let$P_{k_1 k_2}$ be the orthogonal projection from $A^2(D^n)$ onto

$$\text{span}\left\{z_1^{k_1} z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)} z_2^{\rho_2(k_1)} A^2(D^{n-2}); h = 0,1,2 \dots\right\}.$$

Then $P_{k_1 k_2} P_{\mathcal{M}} z^K = P_{\mathcal{M}} P_{k_1 k_2} z^k$. For each $f \in \mathcal{M}$ with $f \neq 0$, there are integers $l, m \geq 0$ such that $P_{lm} f \neq 0$. By the similar technique, we can proof that $\langle P_{\mathcal{M}} P_{ml} f, z^K \rangle = \langle P_{ml} f, z^K \rangle$ for any $K \succeq 0$, i.e.,$P_{\mathcal{M}} P_{ml} f = P_{ml} f$. So, there exist $f_1(z') and\ g_2(z') \in A^2(D^{n-2})$ such that $P_{ml} f = g_1(z') z_1^m z_2^l + g_2(z') z_1^{\rho_1(l)} z_2^{\rho_2(m)} \in \mathcal{M}$, which implies that (a) holds.
If $\alpha \neq 0$, then we arrive at $P_{\mathcal{M}} z^K \in z_1^{k_1} z_2^{k_3} A_\alpha^2(D^{n-2})$. Denote by $P'_{k_1 k_2}$ theorthogonal projection from $A_\alpha^2(D^n)$ onto

$$\text{Span}\{z_1^{k_1} z_2^{k_2} A^2(D^{n-2}); h = 0,1,2, \dots\}$$

Then $P'_{k_1 k_2}(f) = P'_{k_1 k_2} P_{\mathcal{M}}(f) = P_{\mathcal{M}} P'_{k_1 k_2}(f) \in \mathcal{M} for\ each\ f \in \mathcal{M}$ Hence (b) holds. The rest of the proof is obvious.

We obtain similar results for Hankel products $H_f H_g^*$, where f and g are square integrable on the unit disk, and for the mixed Haplitz products $H_f T_g$ and $T_g H_f^*$, where f and g are square integrable on the unit disk and g is analytic. For a large class of measures, we find that these quantities satisfy asymptotic relations similar to the simple exact relations which hold in the model case $m(t) = e^{-t}$. We show that $H_{\overline{f}}$ is compact if and only if f is a polynomial of degree strictly smaller that $\frac{m}{2}$. We also establish that $H_{\overline{f}}$ is in the Schatten class S_p if and only if $p > 2n$ and f is a polynomial of degree strictly smaller than $m\frac{(p-2n)}{2p}$.

Section (2.1): Products of Hankel and Toeplitz Operators

For dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $f \in L^2(\mathbb{D}, dA)$, the Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined densely on the Bergman space L_a^2 by $T_f(h) = P(fh)$ and $Hf(h = (1 - P)(\, fh)$ for all polynomials h, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2 .

The techniques required to solve problems in the Bergman space setting may be very different from those that work in the Hardy space setting.
Often one sees similarities in the theorems, but not the proofs (although in both cases the proofs usually feature an interplay between function theory and operator theory).

On the Hardy space H^2, bounded Toeplitz operators arise only from bounded symbols. In [53] Sarason posed the problem for which f and g in H^2 the densely defined operator $T_f T_{\overline{g}}$ is bounded on H^2. Sarason [53] conjectured that a necessary condition obtained by S. Treil is also sufficient for boundedness of such Toeplitz products. Cruz-Uribe [48] characterized the outer functions f and g for which the Toeplitz product $T_f T_{\overline{g}}$ is bounded and invertible on H^2, providing support for Sarason's conjecture. [59] obtained a partial answer to Sarason's problem by showing that a condition slightly stronger than the one in Sarason's conjecture is sufficient for boundedness of these Toeplitz products on the Hardy space.

On the Bergman space, there are unbounded symbols that induce bounded Toeplitz operators. A Toeplitz operator with analytic symbol is, however, bounded if and only if its symbol is bounded on the unit disk.

Sarason [53] also asked for which analytic functions f and g in L_a^2 the densely defined product $T_f T_{\overline{g}}$ is bounded on L_a^2 . We will obtain a partial answer to this question and prove results analogous to those obtained by [59] for such Toeplitz products on the Hardy space.

On the Bergman space, Luecking [51] has obtained complete characterizations of compactness and boundedness of Hankel operators with symbol in $L^2(\mathbb{D}, dA)$. Little is known concerning the products $H_f^* Hg$ or $H_f H_g^*$ for $f, g \in L^2(\mathbb{D}, dA)$. Even on the Hardy space, problems concerning the products of Toeplitz operators or Hankel operators are much harder than those dealing with a single operator; see [44], [46], [52], [53], [57] and [59]. Many interesting questions concerning products of Toeplitz operators or Hankel operators either on the Hardy space or the Bergman space still remain open. Using the beautiful theory of Hoffman [50] describing the maximal ideal space of $H^\infty(\mathbb{D})$, [58] proved that if f and g are bounded harmonic functions on the unit disk $\mathbb{D}$, then $T_{\bar{f}}T_g - T_{\bar{f}g}$ is compact if and only if $(1-|z|^2)\min\{|\partial f/\partial\bar{z}|, |\partial g/\partial\bar{z}|\} \to 0$, as $|z| \to 1^-$, which is analogous to the results on the Hardy space ([44], [57]). For symbols f and g in $L^2(\mathbb{D}, dA)$, the problems on the product are subtle. In addition to boundedness results for the Toeplitz products discussed in the previous paragraph, we obtain similar results for Hankel products $H_f H_g^*$, where f and g are in $L^2(\mathbb{D}, dA)$, and for the mixed Haplitz products $H_f T_g$ and $T_g H_f^*$, where $f \in L^2(\mathbb{D}, dA)$ and $g \in L_a^2$.

The Bergman space L_a^2 has reproducing kernels K_w given by

$$K_w(z) = \frac{1}{(1-\bar{w}z)^2} \quad ,$$

for $z, w \in \mathbb{D}$: for every $h \in L_a^2$ we have $(h, K_w) = h(w)$, for all $w \in \mathbb{D}$. In particular, we have the following formula for the projection P:

$$Pu(w) = \int_{\mathbb{D}} \frac{u(z)}{(1-\bar{w}z)^2}\, dA(z),$$

for $u\# L^2(\mathbb{D}, dA)$ and $w \in \mathbb{D}$.

We will first discuss how the various Haplitz products are to be defined.

First we consider Toeplitz products. If g is a bounded analytic function on $\mathbb{D}$, then

$$\big(T_{\bar{g}}\, h\big)(w) = \langle T_{\bar{g}}\, h, K_w\rangle = \langle h, gK_w\rangle = \int_{\mathbb{D}} \frac{\overline{g(z)}h(z)}{(1-w\bar{z})^2} \quad dA(z),$$

for all $h \in L_a^2$ and $w \in \mathbb{D}$. If $g \in L_a^2$ and $h \in L_a^2$, we define T_gh by the latter integral:

$$(T_{\bar{g}}h)(w) = \int_{\mathbb{D}} \frac{\overline{g(z)}h(z)}{(1-w\bar{z})^2} \quad dA(z)$$

for $w \in \mathbb{D}$. If f is furthermore in L_a^2 , then the meaning of $T_f T_{\bar{g}}$h is clear: it is the analytic function $fT_{\bar{g}}h$. We will be concerned with the question for which f and g in L_a^2 the operator $T_f T_{\bar{g}}$ is bounded on L_a^2 .

Next we consider Hankel products. If f is bounded and $h \in L_a^2$, then

$$\big(H_f h\big)(w) = f(w)h(w) - P(fh)(w) = \int_{\mathbb{D}} \frac{\big(f(w)-f(z)\big)h(z)}{(1-w\bar{z})^2} \quad dA(z),$$

for all $w \in \mathbb{D}$. The latter formula is to be used to define H_f densely on L_a^2 if $f \in L^2(\mathbb{D}, dA)$. If g is bounded and $u \in (L_a^2)^\perp$, then

$$H_g^*\, u(w) = \langle H_g^*\, u, K_w\rangle = \langle u, H_g K_w\rangle = \langle u, gK_w\rangle,$$

for all $w \in \mathbb{D}$. Since K_w is bounded, the latter formula makes sense for all $g \in L^2(\mathbb{D}, dA)$, and we use it to define the operator H_g^* densely on $(L_a^2)^\perp$.

Note that the star need no longer be the adjoint (but would of course coincide with the adjoint in case the operator H_g is itself bounded).

By Lemma 1 in [51] the set of smooth functions with compact support in $\mathbb{D}$ is dense in $(L_a^2)\perp$, so certainly $C_c(\mathbb{D}) \cap (L_a^2)^\perp$, the set of compactlysupported functions in $(L_a^2)^\perp$ is dense in $(L_a^2)^\perp$. If f, g $\in L^2(\mathbb{D}, dA)$ and $u \in C_c(\mathbb{D}) \cap (L_a^2)^\perp$, then H_g^*u is bounded, and the meaning of $H_f H_g^* u$ is clear: it is the function $H_f(H_g^* u)$. This defines the Hankel product $H_f H_g^*$ on a dense subset of $(L_a^2)^\perp$, namely $C_c(\mathbb{D}) \cap (L_a^2)^\perp$.

The mixed Haplitz operators are defined as follows. For $f \in L_a^2$, $g \in L^2(\mathbb{D}, dA)$ and $u \in C_c(\mathbb{D}) \cap (L_a^2)^\perp$, $T_f H_g^*$u is the analytic function $f(H_g^* u)$.

If $h \in H^\infty$, then $T_g \in L_a^2$, and we define $H_f T_{\bar{g}}\, h$ to be the function $H_f(T_{\bar{g}}\, h)$.

For $w \in \mathbb{D}$, the fractional linear transformation φ_w defined by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}$$

is an automorphism of the unit disk; in fact, the mappings are involutions: $\varphi_w^{-1} = \varphi_w$ The real Jacobian for the change of variable $\xi = \varphi_w(z)$ is equal to $|\varphi_w'(z)|^2 = (1 - |w|^2)^2/|1 - \bar{w}z|^4$, thus we have the change-of-variable formula

$$\int_{\mathbb{D}} h(\varphi_w(z))dA(z) = \int_{\mathbb{D}} \frac{h(z)(1 - |w|^2)^2}{|1 - \bar{w}z|^4}\, dA(z),$$

where h is a positive measurable or integrable function on $\mathbb{D}$. The functions

$$k_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^2}$$

are the normalized reproducing kernels for L_a^2 . The change-of-variable formula can be written as

$$\int_{\mathbb{D}} h(\varphi_w(z))dA(z) = \int_{\mathbb{D}} h(\varphi_w(z))dA(z), \qquad (1)$$

where h is a positive measurable or integrable function on $\mathbb{D}$.

For $w \in \mathbb{D}$ the operator U_w on $L^2(\mathbb{D}, dA)$ is defined by

$$U_w f = (f \circ \varphi_w)k_w.$$

It is easy to see that U_w is a unitary operator which commutes with the Bergman projection. In particular, $T_f U_w = U_w T_{f\circ\varphi_w}$.

The Berezin transform of a function $f \in L^2(\mathbb{D}, dA)$ is the function $\tilde{f}$ defined on $\mathbb{D}$ by

$$\tilde{f}(w) = \int_{\mathbb{D}} f(z)|k_w(z)|^2\, dA(z).$$

In particular, it follows from change-of-variable formula (1) that $|\widetilde{f}|^2(w) = \|f \circ \varphi_w\|_2^2$, for every $f \in L^2(\mathbb{D}, dA)$and $w \in \mathbb{D}$.

It is well-known ([43], [60]) that $\|f\|^2$ is equivalent to $\|(1 - |z|^2)f'\|_2$ for f in the Bergman space L_a^2 with $f(0) = 0$. The following lemma for the inner product in the Bergman space in terms of derivatives of functions will be needed.

Lemma (2.1.1)[42]: If F and G are in L_a^2 , then

$$\int_{\mathbb{D}} F(z)\overline{G(z)}\, dA(z) = 3\int_{\mathbb{D}} (1 - |z|^2)^2\, F(z)\, \overline{G(z)}\, dA(z)$$
$$+\frac{1}{2}\int_{\mathbb{D}} (1 - |z|^2)^2\, F'(z)\, \overline{G'(z)}dA(z)$$
$$+\frac{1}{3}\int_{\mathbb{D}} (1 - |z|^2)^3\, F'(z)\, \overline{G'(z)}\, dA(z).$$

Proof. Using power series it is sufficient to show the identity for $F(z) = G(z) = z^n$. This is a standard calculation using $\int_{\mathbb{D}}(1 - |z|^2)^n\, |z^{2m}\, dA(z) = n!\ m!/(n + m + 1)!$.

We will give estimates on the Toeplitz and Hankel operators that will be used in our sufficiency results for boundedness of certain products of these operators.

Lemma (2.1.2)[42]: Let $f \in L^2(\mathbb{D}, dA)$. Then

$$|(T_{\bar{f}}h)(w)| \le \frac{1}{1-|w|^2}\ \|h\|_2 |\widetilde{f}|^2(w)^{1/2}\ ,$$

and

$$|(H_f^* u)(w)| \le \frac{1}{1-|w|^2}\ \|u\|_2\ \|f \circ \varphi_w P(\,f \circ \varphi_w)\|_2\, ,$$

for all $h \in L_a^2$, $u \in L^2(\mathbb{D}, dA), and\ w \in \mathbb{D}$.

Proof. If $w \in \mathbb{D}$ and $h \in L_a^2$, then

$$(T_{\bar f}h)(w) = \langle T_{\bar f}h, K_w\rangle = \langle h, fK_w\rangle = \frac{1}{1-|w|^2}\langle h, fk_w\rangle.$$

By the Cauchy-Schwarz inequality, $|\langle h, fk_w\rangle| \le \|h\|_2 \| fk_w\|_2$, thus

$$|(T_{\bar f}h)(w)| \le \frac{1}{1-|w|^2}\ \|h\|_2\ \|fk_w\|_2 = \frac{1}{1-|w|^2}\ \|h\|_2\ |\widetilde{f}\ |^2(w)^{1/2},$$

proving the estimate for $T\bar f h$.

Using $H_f k_w = (f - P(\ f\circ\varphi_w)\circ\varphi_w)\ k_w$ (see [56]) we have

$$H_f^*\ u(w) = \frac{1}{1-|w|^2}\ \langle u, H_f\ k_w\rangle = \frac{1}{1-|w|^2}\langle\ u, (\ f - P(\ f\circ\varphi_w)\circ\varphi_w)k_w\rangle.$$

By change-of-variable formula (1) we have$\|\ (\ f - P(\ f\circ\varphi_w)\circ\varphi_w)k_w\ \|_2 = \|\ f\circ\varphi_w - P(\ f\circ\varphi_w)\|_2$, so applying the inequality of Cauchy_Schwarz we get

$$|\langle u, (f - P(\ f\circ\varphi_w)\circ\varphi_w)k_w\rangle| \le \|u\|_2\ \|f\circ\varphi_w - P(\ f\circ\varphi_w)\|_2 \quad .$$

In the following we write P_0 for the integral operator on $L^2(\mathbb{D}, dA)$ with kernel $1/|1-\bar w z|^2$. It is well-known that P_0 is L^p-bounded for $1 < p < \infty$
(see [43] or [60]).

Lemma (2.1.3)[42]: Let $\varepsilon > 0$ and let $\delta = (2+\varepsilon)/(1+\varepsilon)$.

(i) For every $f \in L_a^2$ and $h \in L_a^2$ *:*

$$|(T_{\bar f}h)'(w)| \le \frac{4}{1-|w|^2}\ |\widetilde{f}\ |^{2+\varepsilon}(w)^{1/(2+\varepsilon)}\ P_0[|h|^\delta](w)^{1/\delta},$$

for all $w\ \in \mathbb{D}$.

(ii) For every $g \in L^2(\mathbb{D}, dA)$ and $u \in (L_a^2)^\perp$*:*

$$|(H_g^*u)'\ (w)| \le |\frac{4}{1-|w|^2}\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon}\ P_0[|u|^\delta](w)^{1/\delta},$$

for all $w\ \in \mathbb{D}$.

Proof. Let $\varepsilon > 0$. Note that $\delta = (2+\varepsilon)/(1+\varepsilon)$ is the conjugate index of $2+\varepsilon$.

(i) For $f \in L_a^2$ and $h \in L_a^2$ we have

$$(T_{\bar f}h)(w) = \int_{\mathbb{D}} \frac{f(z)h(z)}{(1-\bar z w)^2}\ \ dA(z),$$

for $w \in \mathbb{D}$. Thus

$$(T_{\bar f}h)'(w) = 2\int_{\mathbb{D}} \frac{\overline{zf(z)}h(z)}{(1-\bar z w)^3}\ dA(z),$$

for $w \in \mathbb{D}$. Applying Hölder's inequality we have

$$\begin{aligned}
&|(T_f^* h)'(w)| \\
&\le 2\int_{\mathbb{D}} \frac{|f(z)|\ |h(z)|}{|1-\bar z w|^3}\ \ dA(z) = 2\ \int_{\mathbb{D}} \frac{|f(z)||h(z)||1-\bar z w|}{|1-\bar z w|^4}\ dA(z) \\
&\le 2\left(\int_{\mathbb{D}} \frac{|f(z)|^{2+\varepsilon}}{|1-\bar z\, w|^4}\ dA(z)\right)^{1/(2+\varepsilon)} \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta\ |1-\bar z\, w|^\delta}{|1-\bar z\, w|^4}\ \ dA(z)\right)^{1/\delta} \\
&= \frac{2\ |f|^{2+\varepsilon}(w)^{1/(2+\varepsilon)}}{1-|w|^2}\ \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-\bar z w|^2}\frac{(1-|w|^2)^{\varepsilon/1+\varepsilon}}{|1-\bar z w|^{\varepsilon/1+\varepsilon}}\ dA(z)\right)^{1/\delta},
\end{aligned}$$

and the inequality follows, since $(1-|w|^2)\ |\ |1-\bar z w| < 2\ and\ 2^{\varepsilon/2+\varepsilon} < 2$.

(ii) For $u \in (L_a^2)^\perp$ we have

$$(H_g^*u)(w) = \langle H_g^*u, K_w\rangle = \langle u, H_g K_w\rangle = \int_{\mathbb{D}} \frac{u(z)\overline{g(z)}}{(1-\bar z\, w)^2}dA(z)$$

.

Thus

$$\left(H_g^*u\right)'(w) = 2\int_{\mathbb{D}} \frac{u(z)\overline{zg(z)}}{(1-\bar{z}w)^3}\, dA(z).$$

Letting G_w denote $P(g\circ\varphi_w)\circ\varphi_w$, the function $z\mapsto zG_w(z)/(1-\overline{w}z)^3$ is in L_a^2 , and since $u\in(L_a^2)^\perp$ we have

$$\int_{\mathbb{D}} \frac{u(z)\overline{zG_w(z)}}{(1-\bar{z}w)^3}\, dA(z) = 0.$$

Thus

$$\left(H_g^*u\right)'(w) = 2\int_{\mathbb{D}} \frac{u(z)\overline{z\left(g(z)-G_w(z)\right)}}{(1-\bar{z}w)^3}\, dA(z).$$

Using the same argument as above, applying Hölder's inequality and change-of-variable formula (1) we have

$$|\left(H_g^*u\right)'(w)| \le 2\left(\int_{\mathbb{D}} \frac{|g(z)-Gw(z)|^{2+\varepsilon}}{|1-\bar{z}w|^4}\, dA(z)\right)^{1/2+\varepsilon}\left(\int_{\mathbb{D}} \frac{|u(z)|^\delta}{|1-\bar{z}w|^{4-\delta}}\, dA(z)\right)^{1/\delta}$$

$$\le \frac{4}{1-|w|^2}\,\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon}\left(\int_{\mathbb{D}} \frac{|u(z)|^\delta}{|1-\bar{z}w|^2}\, dA(z)\right)^{1/\delta}$$

,

as desired.

We discuss several basic identities and inequalities needed to prove necessary conditions for boundedness and compactness of Haplitz products.

For f and g in $L^2(\mathbb{D},dA)$ let $f\otimes g$ be the rank one operator defined by

$$(f\otimes g)\,h = \langle h,g\rangle\, f,$$

for $h\in L^2(\mathbb{D},dA)$. It is easily verified that the norm of $f\otimes g$ is $\|f\|_2\,\|g\|_2$.

If T and S are bounded linear operators, then $T(f\otimes g)S^* = (Tf)\otimes(Sg)$.

Proposition (2.1.4)[42]: On L_a^2 we have

$$k_w\otimes k_w = I - 2T_{\varphi_w}T_{\overline{\varphi}\,w} + T_{\varphi_w}^2 T_{\overline{\varphi}\,w}^2 ,$$

for all $w\in\mathbb{D}$.

Proof. Let $e_n(z) = (n+1)^{1/2}\, z^n$. Then $\{e_n\}$ is a basis of the Bergman space. On this basis, T_z is a weighted shift operator, the so-called Bergman shift. More precisely,

$$T_z e_n = \left(\frac{n+1}{n+2}\right)^{1/2} e_n + 1 \quad and \qquad T\bar{z}\, e_n = T_z^*\, e_n = \left(\frac{n}{n+1}\right)^{1/2} e_{n-1},$$

for n>0, and $T_z^*\, e_0 = 0$. Thus

$$T_zT_{\bar{z}}\, e_n = \frac{n}{n+1} e_n \quad and \quad T_z^2T_{\bar{z}}^2\, e_n = \frac{n-1}{n+1}\, e_n\ ,$$

for $n>0$, and hence

$$(I-2T_zT_{\bar{z}} + T_z^2\, T_{\bar{z}}^2)e_n = \left\{1-\frac{2n}{n+1}+\frac{n-1}{n+1}\right\}\, e_n = 0,$$

for all $n>0$. It follows that

$$I-2T_zT_{\bar{z}} + T_z^2T_{\bar{z}}^2 = e_0\otimes e_0\ .$$

For $w\in\mathbb{D}$ we apply the unitary operator U_w to obtain

$$k_w\otimes k_w = (U_we_0)\otimes(U_we_0) = U_w(e_0\otimes e_0)U_w^*$$
$$= U_w(I-2T_zT_{\bar{z}} + T_z^2\, T_{\bar{z}}^2)U_w^*$$

$$= I - 2T_{\varphi w}T_{\overline{\varphi}\, w} + T^2_{\varphi w}T^2_{\overline{\varphi}\, w} \;,$$

as desired.

Proposition (2.1.5)[42]: If $f, g \in L^2_a$, then

$$\left|\widetilde{f}\right|^2 (w)^{1/2}\, |\widetilde{g}|^2\, (w)^{1/2} \leq 2\, x\left\|T_f T_{\bar g} T_{\varphi w}\left(T_f T_{\bar g}\right)T_{\overline{\varphi}\, w}\right\|,$$

for all $w \in \mathbb{D}$.

Proof. Using the fact that both f and g are analytic, we have $T_f T_{\varphi w} = T_{\varphi w}T_f$ and $T_{\overline{\varphi}\, w}T_{\bar g} = T_{\bar g}\, T_{\overline{\varphi}\, w}$, so by Proposition *(2.1.4)*,

$$T_f\, (k_w \otimes k_w)T_{\bar g} = T_f T_{\bar g} - 2T_{\varphi w}T_f T_{\bar g}\, T_{\overline{\varphi}\, w} + T^2_{\varphi w}T_f T_{\bar g}\, T^2_{\overline{\varphi}\, w}$$

$$= T_f T_{\bar g} - T_{\varphi w}T_f T_{\bar g}\, T_{\overline{\varphi}\, w} - T_{\varphi w}(T_f T_{\bar g} - T_{\varphi w}T_f T_{\bar g}\, T_{\overline{\varphi}\, w})\, T_{\overline{\varphi}\, w}\,.$$

The triangle inequality, the fact that also here $T_f(k_w \otimes k_w)\, T_{\bar g} = (T_f\, k_w) \otimes (T_g k_w)$, and the estimate$\left\|T_{\varphi w}\right\| \leq 1$ imply that

$$\left\|(T_f\, k_w) \otimes (T_g k_w)\right\| \leq 2\, \left\|T_f T_{\bar g} - T_{\varphi w}(T_f T_{\bar g})\, T_{\overline{\varphi}\, w}\right\|.$$

Using change-of-variable formula (1) we have

$$\left\|(T_f\, k_w) \otimes (T_g k_w)\right\| = \|f k_w\|_2\, \|g k_w\|_2 = \left|\widetilde{f}\right|^2 (w)^{1/2}\, |\widetilde{g}|^2\, (w)^{1/2},$$

and the stated result follows.

To deal with products involving Hankel operators, we introduce dual Toeplitz operators. The orthogonal complement $(L^2_a)^{\perp}$ of L^2_a in $L^2(\mathbb{D}, dA)$ is much larger than $\overline{zL^2_a}$. Under the decomposition $L^2(\mathbb{D}, dA) = L^2_a \otimes (L^2_a)^{\perp}$, for $f \in L^{\infty}(\mathbb{D})$ the multiplication operator M_f is represented as

$$M_f = \begin{bmatrix} T_f & H^*_{\bar f} \\ H_f & S_f \end{bmatrix}.$$

The operator S_f is an operator on $(L^2_a)^{\perp}$ we call S_f *t*he dual Toeplitz operator with symbol f. Although these operators differ in many ways froms Toeplitz operators, they do have some of the same basic algebraic properties. We have*:* $S^*_f = S_{\bar f}$ *and* $S_{\alpha f + \beta g} = \alpha S_f + \beta S_g$ *, for* $f, g \in L^{\infty}(\mathbb{D})$, and $\alpha, \beta \in \mathbb{C}$ The identity $M_{fg} = M_f M_g$ implies the following basic algebraic relations between these operators*:*

$$T_{fg} = T_f T_g + H^*_{\bar f} H_g \;, \tag{2}$$

$$S_{fg} = S_f\, S_g + H_f H^*_{\bar g}\,, \tag{3}$$

$$H_{fg} = H_f T_g + S_f H_g\,. \tag{4}$$

Suppose $\varphi \in H^{\infty}$ and $\psi \in L^{\infty}(\mathbb{D})$. If we take $f = \varphi$ and $g = \psi$ in (4) we get $H_{\varphi\psi} = S_{\varphi}T_{\psi}$, since $H_{\varphi} = 0$; on the other hand, taking $f = \psi$ and $g = \varphi$ in(4) gives $H_{\psi\varphi} = H_{\psi}H_{\varphi}$. Thus, if $\varphi \in H^{\infty}$ and $\psi \in L^{\infty}(\mathbb{D})$, then

$$H_{\psi}T_{\varphi} = S_{\varphi}H_{\psi}, \tag{5}$$

and, by taking adjoints,

$$T_{\overline{\varphi}}\, H^*_{\psi} = H^*_{\psi}\, H_{\overline{\varphi}}. \tag{6}$$

For $f \in L^2(\mathbb{D}, dA)$ we extend the dual Toeplitz operator S_f by defining
$S_f u = (I - P)(fu)$, for $u \in C_c(\mathbb{D}) \cap (L^2_a)^{\perp}$.

We will show that identities (5) and (6) also hold if $\varphi \in H^{\infty}$ and $\psi \in L^2(\mathbb{D}, dA)$. For a polynomial h we have $P(\varphi H_{\psi}h) = P(\varphi\psi h - \varphi P(\psi h)) = P(\varphi\psi h) - \varphi P(\psi h)$, thus

$$S\varphi H_{\psi}h = \varphi\big(\psi h - P(\psi h)\big) - P\big(\varphi H_{\psi}h\big) = \varphi\psi h - P(\varphi\psi h) = H_{\psi}T_{\varphi}h,$$

so that (5) also holds if .$\varphi \in H^{\infty}$ and $\psi \in L^2(\mathbb{D}, dA)$.

For $\varphi \in H^\infty$, $\psi \in L^2(\mathbb{D}, dA)$, $u \in C_c(\mathbb{D}) \cap (L_a^2)^\perp$, and $w \in \mathbb{D}$ we have $(T_{\overline{\varphi}} H_\psi^* u)(w) = \langle T_{\overline{\varphi}} H_\psi^* u, K_w \rangle = \langle H_\psi^* u \varphi K_w \rangle$. Using the definition of $H_\psi^* u$ as well as Fubini's Theorem, it is easily verified that

$$\langle H_\psi^* u, \varphi K_w \rangle = \int_{\mathbb{D}} u(z)\psi(z)\varphi(z)K_w(z)dA(z).$$

On the other hand,

$$H_\psi^* S_{\overline{\varphi}} u(w) = \langle S_{\overline{\varphi}} u, \psi K_w \rangle = \langle u, S_\varphi \psi K_w \rangle = \langle u, (I-P)(\varphi\psi K_w) \rangle$$
$$= \langle (I-P)u, \varphi\, \psi K_w \rangle = \langle u, \varphi\, \psi K_w \rangle.$$

Thus we have $T_{\overline{\varphi}} H_\psi^* u = H_\psi^*\, S_{\overline{\varphi}} u$, so that also (6) holds if $\varphi \in H^\infty$ and $\psi \in L^2(\mathbb{D}, dA)$.

Proposition (2.1.6)[42]: If f, g $\in L^2(\mathbb{D}, dA)$., then

$$\| f \circ \varphi_w - P(f \circ \varphi_w)\|_2\, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2$$
$$\leq 2\left\| H_f H_g^* - S_{\varphi w}\left(H_f H_g^*\right) S_{\overline{\varphi} w}\right\|$$

for all $w \in \mathbb{D}$.

Proof. Using Proposition *(2.1.4)* and identities (5) and (6), we have

$$H_f(k_w \otimes k_w)H_g^*$$
$$= H_f H_g^* - 2H_f T_{\varphi w} T_{\overline{\varphi} w} H_g^* + H_f T_{\varphi w}^2 T_f T_{\overline{\varphi}\, w}^2 H_g^*$$
$$= H_f H_g^* - 2S_{\varphi w} H_f T_{\overline{\varphi} w} S_{\overline{\varphi} w} + S_{\varphi w}^2 H_f H_g^* S_{\overline{\varphi} w}$$
$$= H_f H_g^* - S_{\varphi w} H_f H_g^* S_{\overline{\varphi} w} - S_{\varphi w}(H_f H_g^* - S_{\varphi w} H_f H_g^* S_{\overline{\varphi} w})\, S_{\overline{\varphi} w}\,,$$

and, because $H_f(k_w \otimes k_w)H_g^* = (H_f k_w) \otimes (H_g k_w)$, and

$$\left\|(H_f k_w) \otimes (H_g k_w)\right\| = \left\|H_f k_w\right\|_2 \left\|H_g k_w\right\|_2$$
$$= \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2\, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2\,,$$

the stated result follows.

The following proposition shows that the estimates for the Toeplitz products and the Hankel products have their analogues for the mixed products.

Proposition (2.1.7)[42]: If $f \in L_a^2$ and $g \in L^2(\mathbb{D}, dA)$, then

$$\left|\widetilde{f}\right|^2(w)^{1/2}\, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 \leq 2\left\|T_f H_g^* - T_{\varphi w}\left(T_f H_g^*\right) S_{\overline{\varphi}\, w}\right\|,$$

and

$$\left|\widetilde{f}\right|^2(w)^{1/2}\, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 \leq 2\left\|H_g T_{\bar{f}} - S_{\varphi w}(H_g T_{\bar{f}})\, T_{\overline{\varphi}\, w}\right\|,$$

for all $w \in \mathbb{D}$.

Proof. To prove the first inequality we use the identity

$$T_f(k_w \otimes k_w)H_g^* = T_f H_g^* - 2T_{\varphi w} T_f H_g^*\, S_{\overline{\varphi}\, w} + T_{\varphi w}^2 T_f H_g^* S_{\overline{\varphi}\, w}$$

The second inequality follows from an analogous identity. K

We end this with an algebraic result for dual Toeplitz operators.

If f is analytic or $\bar{g}$ is analytic, then $H_f H_{\bar{g}}^* = 0$, and by (3), $S_f S_g = S_{fg}$. The following proposition shows that the converse holds.

Proposition (2.1.8)[42]: Let f and g be $L^\infty(\mathbb{D}, dA)$. If $S_f S_g = S_{fg}$, then either f or $\overline{g}$ is in H^∞.

Proof. If $S_f S_g = S_{fg}$, then by (3), $H_f H_{\bar{g}}^* = 0$, and by Proposition *(2.1.6)*,

$$\|f \circ \varphi_w - P(f \circ \varphi_w)\|_2\, \bar{g} \circ \varphi_w - \|P(\overline{g} \circ \varphi_w)\|_2 = 0,$$

for all $w \in \mathbb{D}$, so the stated result follows.

We give conditions for boundedness of the various Haplitz products.

Theorem (2.1.9)[42]: Let f and g be in L_a^2. If $T_f T_{\bar{g}}$ is bounded, then

$$\sup_{w \in \mathbb{D}} \left|\widetilde{f}\right|^2(w)\, \widetilde{|g|^2}\,(w) < \infty.$$

Proof. Suppose $T_f T_{\bar{g}}$ is bounded.
It follows from Proposition *(2.1.5)* that

$$\left|\widetilde{f}\right|^2(w)^{1/2} \, |\widetilde{g}|^2(w)^{1/2} \leq \|4\, T_f T_{\bar{g}}\|,$$

for all $w \in \mathbb{D}$.

Although we are not able to prove the converse of Theorem *(2.1.9)*, we have the following result.

Theorem (2.1.10)[42]: Let f and g be in L_a^2 . If there is a positive constant εsuch that

$$\sup_{w\in\mathbb{D}} \left| \widetilde{f\,|^{2+\varepsilon}}\,(w)\right| \widetilde{g|^{2+\varepsilon}}(w) < \infty,$$

then the product $T_f T_{\bar{g}}$ is bounded.

Proof. Let u and v be in L_a^2 . To show that the product $T_f T_{\bar{g}}$ is bounded we will estimate $(T_f T_{\bar{g}} u, v)$ using Lemma *(2.1.1)* and Lemmas *(2.1.2)*, *(2.1.3)*. It follows from the inner product formula (Lemma *(2.1.1)*) that

$$\langle T_f T_{\bar{g}} u, v\rangle = (T_{\bar{g}} u, T_{\bar{f}} v) = I + II + III,$$

where

$$I = 3\int_{\mathbb{D}} (1-|w|^2)^2\, (T_{\bar{g}} u)(w)\overline{(T_{\bar{f}}\, v)(w)}\, dA(w),$$
$$II = \frac{1}{2}\int_{\mathbb{D}} (1-|w|^2)^2\, \left(T_{\bar{g}} u\right)'(w)\overline{(T_{\bar{f}} v)'(w)}\, dA(w),$$
$$III = \frac{1}{3}\int_{\mathbb{D}} (1-|w|^2)^3\, \left(T_{\bar{g}} u\right)'(w)\overline{(T_{\bar{f}} v)'(w)} dA(w).$$

It follows from Lemma *(2.1.2)* that

$$|I| \leq 3\int_{\mathbb{D}} \left[\, |\widetilde{f}\,|^2(w)|\widetilde{g}|^2(w)\right]^{1/2} \|u\|_2\, \|v\|_2\, dA(w)$$
$$\leq 3\sup_{w\in\mathbb{D}} \left[\, \left|\widetilde{f}\right|^2(w)|\widetilde{g}|^2(w)\right]^{1/2} \|u\|_2\|v\|_2 \,.$$

Using Lemma *(2.1.3)* we have

$$|II| \leq \frac{4}{2}\int_{\mathbb{D}} [\, |\, \widetilde{f\,|^{2+\varepsilon}}\,(w)|\, \widetilde{g|^{2+\varepsilon}}(w)]^{1/(2+\varepsilon)}$$
$$\times P_0\left[|u|^{\delta}\right](w)^{(1/\delta)}\, P_0\left[|v|^{\delta}\right](w)^{1/\delta} dA(w)$$
$$\leq 2\sup_{w\in\mathbb{D}} [\, |\widetilde{f\,|^{2+\varepsilon}}(w)|\, \widetilde{g|^{2+\varepsilon}}(w)]^{1/(2+\varepsilon)}$$
$$\leq \int_{\mathbb{D}} P_0\left[|u|^{\delta}\right](w)^{1/\delta}\; P_0\left[|v|^{\delta}\right](w)^{1/\delta} dA(w).$$

Since $p = 2\,/\,\delta > 1$ and P_0 is L^{p-}bounded, there exists a constant C such that

$$\int_{\mathbb{D}} P_0\left[|u|^{\delta}\right](w)^{2/\delta}\; dA(w) \leq C\; \int_{\mathbb{D}} P_0\left[|u|^{\delta}\right](w)^{2/\delta}\, x dA(w) = C\|u\|_2^2 \,.$$

By the Cauchy Schwarz inequality,

$$\int_{\mathbb{D}} P_0\left[|u|^{\delta}\right](w)^{1/\delta}\; P_0\left[|v|^{\delta}\right](w)^{(1/\delta)} dA(w) \leq C\, \|u\|_2\|v\|_2 \,,$$

and thus

$$|II| \leq 2C\sup_{w\in\mathbb{D}} [\, |\widetilde{f\,|^{2+\varepsilon}}(w)|\widetilde{g|^{2+\varepsilon}}(w)]^{1/(2+\varepsilon)}\; \|u\|_2\|v\|_2 \,.$$

Term III is estimated similar to II. From the estimates of the three terms I, II, and III, we obtain

$$\left|T_f T_{\bar{g}} u, v\right| \leq M \sup_{w\in\mathbb{D}} [\, |\widetilde{f\,|^{2+\varepsilon}}(w)|\, \widetilde{g|^{2+\varepsilon}}(w)]^{1/(2+\varepsilon)}\; \|u\|_2\|v\|_2 \,,$$

for some constant $M > 0$. So the product $T_f T_{\bar{g}}$ is bounded, as desired.

Using Proposition *(2.1.6)* we obtain a necessary condition on Boundedness of the product $H_f H_g^*$.

Theorem (2.1.11)[42]: Let f and g be in $L^2(\mathbb{D}, dA)$. If $H_f H_g^*$ is bounded, then

$$\sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2 \ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 < \infty.$$

We have not been able to prove the converse of the above theorem. We do however have the following result.

Theorem (2.1.12)[42]: Let f and g be in $L^2(\mathbb{D}, dA)$. If there is a positive constant ε such that

$$\sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon} \ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} < \infty,$$

then the product $H_f H_g^*$ is bounded.

Proof. Let $u, v \in C_c(\mathbb{D}) \cap (L_a^2)^{\perp}$. Using the definitions of $H_g^* u$ and $H_f^* v$, and Fubini's Theorem, we have

$$\langle H_g^* u, H_f^* v\rangle = \int_{\mathbb{D}} \left\{\int_{\mathbb{D}} \frac{\overline{g(z)}u(z)}{(1-w\bar{z})^2}\, dA(z)\right\} = \left\{\int_{\mathbb{D}} \frac{f(\lambda)\overline{v(\lambda)}}{(1-\lambda\bar{w})^2}\, dA(\lambda)\right\} dA(w)$$
$$= \int_{\mathbb{D}} f(\lambda) H_g^* u(\lambda)\overline{v(\lambda)}\, dA(\lambda) = \langle f H_g^* u, v\rangle = \langle H_f H_g^* u, v\rangle.$$

Thus, by Lemma *(2.1.1)* we have

$$(H_f H_g^* u, v) = (H_g^* u, H_f^* v) = I + II + III,$$

where

$$I = 3\int_{\mathbb{D}} (1-|w|^2)^2\, (H_g^* u)(w)\overline{(H_f^* v)(w)}\, dA(w),$$
$$II = \frac{1}{2}\int_{\mathbb{D}} (1-|w|^2)^2\, (H_g^* u)'(w)\overline{(H_f^* v)'(w)}\, dA(w),$$
$$III = \frac{1}{3}\int_{\mathbb{D}} (1-|w|^2)^3\, (H_g^* u)'(w)\overline{(H_f^* v)'(w)}\, dA(w).$$

It follows from Lemma *(2.1.2)* that

$$|I| 3 \sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2 \ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 \|u\|_2 \|v\|_2 .$$

Using Lemma *(2.1.3)* and the Lp-boundedness of operator P_0 we have

$$|II| \le 2C sup_{w\in\mathbb{D}} \|f\circ\varphi_w P(f\, b.w)\|_{2+\varepsilon}$$
$$\times \|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} \|v\|_2 \|u\|_2 .$$

Term III is estimated similar to II, and combining the estimates we get

$$|(H_f H_g^* u, v)| \le M \sup_{w\in\mathbb{D}} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon}$$
$$\times \|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} \|v\|_2 \|u\|_2 ,$$

for some constant $M > 0$. So the product $H_f H_g^*$ is bounded, as desired. K

Analogous to the necessary conditions for boundedness of Toeplitz and Hankel products, Proposition *(2.1.7)* gives necessary conditions for boundedness of the mixed Haplitz products.

Theorem (2.1.13)[42]: Let $f \in L_a^2$ and $g \in L^2(\mathbb{D}, dA)$. If $T_f H_g^*$ *or* $H_g T_{\bar{f}}$ is bounded, then

$$\sup_{w\in\mathbb{D}} |\widetilde{f}|^2(w)^{1/2}\ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 < \infty.$$

We have not been able to prove the converse of the above theorem, but we have the following result, which is proved similarly to Theorems *(2.1.10)* and *(2.1.12)*.

Theorem (2.1.14)[42]: Let $f \in L_a^2$ and $g \in l^2(\mathbb{D}, dA)$ If for a constant $\varepsilon > 0$

$$\sup_{w\in\mathbb{D}} |\, f\, |^{2+\varepsilon}(w)^{1/(2+\varepsilon)} \|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} < \infty,$$

then $T_f H_g^*$ and $H_g T_{\bar{f}}$ are bounded.

We discuss conditions for compactness of the various Haplitz products. The following lemma gives necessary conditions for compactness of operators on L_a^2 , operators on $(L_2^a)^{\perp}$, or operators between these spaces.

Lemma (2.1.15)[42]: If $A: L_a^2 \longrightarrow L_a^2$, $B: L_a^2 \longrightarrow (L_a^2)^\perp$, $C: (L_a^2)^\perp \to L_a^2$ and $D: (L_a^2)^\perp \to (L_a^2)^\perp$ are compact operators, then

$$\|A - T_{\varphi w} A T_{\overline{\varphi} w}\| \to 0,$$
$$\|B - S_{\varphi w} B T_{\overline{\varphi} w}\| \to 0,$$
$$\|C - T_{\varphi w} C S_{\overline{\varphi} w}\| \to 0,$$
$$\|D - S_{\varphi w} D S_{\overline{\varphi} w}\| \to 0,$$

as $|w| \to 1^-$.

Proof. If H_1 and H_2 are Hilbert spaces and $S: H_1 \to H_2$ is a compact operator, then, since operators of finite rank are dense in the set of compact operators, given $\varepsilon > 0$ there exist $f_1, \dots, f_n \in H_1$ and $g_1, \dots, g_n \in H_2$ sothat

$$\|S - \sum_{i=1}^n f_i \otimes g_i\| < \varepsilon.$$

Thus the above statements follow once we prove them for operators of rank one.

If $f \in L^2(\mathbb{D}, dA)$ as $|w| \to 1^-$, then for every $z \in \mathbb{D}$ we have $w - \varphi_w(z) = (1 - |w|^2)z/1 - \overline{w}z \to 0$, so by the Lebesgue Dominated Convergence Theorem, $\|wf - \varphi_w f\|_2 \to 0$ as $|w| \to 1^-$. It follows that $\|\xi f - \varphi_w f\|_2 \to 0$, if $w \in \mathbb{D}$ tends to $\xi \in \partial\mathbb{D}$.

If $f \in L_a^2$, we apply P to obtain

$$\|\xi f - T_{\varphi w} f\|_2 = \|\xi f - P(\varphi_w f)\|_2 \to 0,$$

as w in $\mathbb{D}$ tends to $\xi \in \partial\mathbb{D}$. If f, $g \in L_a^2$, then writing

$$\begin{aligned}
&\| f \otimes g - T\varphi_w (f \otimes g) T_{\overline{\varphi} w}\| \\
&= \|(\xi f) \otimes (\xi g) - (T_{\varphi w} f) \otimes (T_{\varphi w} g)\| \\
&\le \|(\xi f - T_{\varphi w} f) \otimes (\xi g)\| + \|(T_{\varphi w} f) \otimes (\xi g - T_{\varphi w} g)\| \\
&\le \|\xi f - T_{\varphi w} f\|_2 \|g\|_2 \, \|f\|_2 \, \|\xi g - T_{\varphi w} g\|_2 ,
\end{aligned}$$

we see that

$$\|f \otimes g - T_{\varphi w}(f \otimes g) T_{\overline{\varphi} w}\| \to 0$$

as w in $\mathbb{D}$tends to $\xi \in \partial\mathbb{D}$. This proves the statement for operator A.

Suppose $f \in (L_a^2)^\perp$, then $(I - P)(\xi f) = \xi f$, so that

$$\|\xi f - S_{\varphi w} f\|_2 = \|(I - P)(\xi f - \varphi_w f)\|_2 \to 0,$$

as w in $\mathbb{D}$ tends to $\xi \in \partial\mathbb{D}$. If $f, g \in (L_a^2)^\perp$ then writing

$$\begin{aligned}
&\|f \otimes g - S_{\varphi w}(f \otimes g) S_{\overline{\varphi} w}\| \\
&= \|(\xi f) \otimes (\xi g) - (S_{\varphi w} f) \otimes (S_{\varphi w} g)\| \\
&\le \|(\xi f - S_{\varphi w} f) \otimes (\xi g)\| + \|(S_{\varphi w} f) \otimes (\xi g - S_{\varphi w} g)\| \\
&\le \|\xi f - S_{\varphi w} f\|_2 \|g\|_2 + \|f\|_2 \|\xi g - S_{\varphi w} g\|_2 ,
\end{aligned}$$

we get

$$\|f \otimes g - S_{\varphi w}(f \otimes g) S_{\overline{\varphi} w}\| \to 0$$

as w in $\mathbb{D}$ tends to $\in \partial\mathbb{D}$. This proves the statement for operator D.

If $f \in L_a^2$ and $g \in (L_a^2)^\perp$, and $w \in \mathbb{D}$ tends to $\xi \in \partial\mathbb{D}$, then $\|\xi f - T_{\varphi w} f\|_2 \to 0$ and $\|\xi g - S_{\varphi w} g\|_2 \to 0$ imply that $\|f \otimes g - T_{\varphi w}(f \otimes g) S_{\overline{\varphi} w}\| \to 0$ as $|w| \to 1^-$. This proves the statement for operator B.

The statement for operator C is proved similarly.

Theorem (2.1.16)[42]: Let f and g be in H^∞. Then $T_f T_{\overline{g}}$ is compact if and only if $f \equiv 0$ or $g \equiv 0$.

Proof. If $T_f T_{\bar{g}}$ is compact, then by Lemma *(2.1.15)*, $\left\|T_f T_{\bar{g}} - T_{\varphi w} T_f T_{\bar{g}} T_{\overline{\varphi} w}\right\| \to 0$ as $|w| \to 1^-$. Using Lemma *(2.1.5)* it follows that $\left|\widetilde{f}\right|^2(w)^{1/2} \left|\widetilde{g}\right|^2(w)^{1/2} \to 0$as $|w| \to 1^-$. Since $|f(w)|^2 \le |\widetilde{f}|^2(w)$ and $|g(w)|^2 \le |\widetilde{g}|^2(w)$ we obtain $|f(w)g(w)| \to 0$ as $|w| \to 1^-$, and by the Maximum Modulus Principle,$fg \equiv 0, thus\ f \equiv 0\ or\ g \equiv 0$.

Theorem (2.1.17)[42]: Let f and g be in $L \in (\mathbb{D}, dA)$. Then $H_f H_g^*$ is compact if and only if

$$\lim_{|w|\to 1^-} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 \, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 = 0.$$

Proof. First we show the ``if part.'' If $H_f H_g^*$ is compact, then by Lemma *(2.1.15)*, $\left\|H_f H_g^* - S_{\varphi w} H_f H_g^* S_{\overline{\varphi} w}\right\| \to 0$ as $|w| \to 1^-$. Using Lemma (2.1.6) it follows that

$$\|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 - \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 \to 0$$

as $|w| \to 1^-$.

Now we turn to the "only if " part. For $u, v \in C_c(\mathbb{D}) \cap (L_a^2)^\perp$we have

$$\langle H_f H_g^* u, v\rangle = \langle H_g^* u, H_f^* v\rangle = I + II + III,$$

where I, II, and III are as in the proof of Theorem (2.1.12). For $0 < s < 1$ we write $I = I_s + I_s'$, $II = II_s + II_s'$, and III=$III_s + III_s'$, where

$$I_s = 3\int_{s<|w|<1} (1-|w|^2)^2\, (w)(H_f^* v)(w)\, dA(w),$$

$$II_s = \frac{1}{2}\int_{s<|w|<1} (1-|w|^2)^2\, (H_g^* u)'(w)\overline{(H_f^* v)'(w)}\, dA(w),$$

$$III_s = \frac{1}{3}\int_{s<|w|<1} (1-|w|^2)^3\, (H_g^* u)'(w)\overline{(H_f^* v)'(w)} dA(w).$$

It is easy to see that there exist compact operators K_s^I , K_s^{II} and K_s^{III} on$(L_a^2)^\perp$ such that $\langle K_s^I u, v\rangle = I_s'$, $\langle K_s^{II} u, v\rangle = II_s'$ and $\langle K_s^{III} u, v\rangle = III_s'$. Theoperator $K_s = K_s^I + K_s^{II} + K_s^{III}$ is compact, and $\langle (H_f H_g^* - K_s)u, v\rangle = I_s + II_s + III_s$. We will estimate each of the terms Is, IIs and IIIs .It follows from Lemma *(2.1.2)* that

$$|I_s| \le 3 \sup_{s<|w|<1} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 - \|u\|_2 \|v\|_2 .$$

Using Lemma *(2.1.3)* and the L^{p-}boundedness of operator P_0 we have

$$|II_s| \le 2C \sup_{s<|w|<1} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_{2+\varepsilon}$$

$$|\times \|g \circ \varphi_w - P(g \circ \varphi_w)\|_{2+\varepsilon} \|u\|_2 \|v\|_2 .$$

Term III_s is estimated similar to I_s , and we obtain

$$\left|\langle (H_f H_g^* - K_s)u, v\rangle\right| \le C \sup_{s<|w|<1} \|f \circ \varphi_w - P(f \circ w)\|_{2+\varepsilon}$$

$$\times \|g \circ \varphi_w - P(g \circ \varphi_w)\|_{2+\varepsilon} \|u\|_2 \|v\|_2$$

for some constant $C > 0$. Since P is $L^{2+2\varepsilon}$-bounded, there exists a constant C_ε such that

$$\|f \circ \varphi_w - P(f \circ \varphi_w)\|_{2+\varepsilon} \le C\|f\|_\infty^{1+\varepsilon/2+\varepsilon}\ f \circ \varphi_w - P(f \circ \varphi_w)_2^{1/2+\varepsilon} .$$

A similar inequality holds for $\|g \circ \varphi_w - P(g \circ \varphi_w)\|_{2+\varepsilon}$. Thus there exists a constant C' such that

$$|\langle (H_f H_g^* - K_s)u, v\rangle| \le C' sup_{s<|w|<1} \|(f \circ \varphi_w - P(f \circ \varphi_w)\|_2$$

$$\times \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2^{\frac{1}{1+\epsilon}} \|u\|_2 \|v\|_2$$

from which we conclude that

$$H_f H_g^* - K_s \le C'\ sup_{s<|w|<1} \|(f \circ \varphi_w - P(f \circ \varphi_w)\|_2^{1/2+\varepsilon}$$

$$\times \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2^{1/2+\varepsilon} .$$

So if $\|(f \circ \varphi_w - P(f \circ \varphi_w)\|_2 \, \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 \to 0$ as $|w| \to 1^-$, then it follows from the above inequality that $K_s \to H_f H_g^*$ in operator norm, and since each of the K_s is compact, we conclude that operator $H_f H_g^*$ is compact. K Analogous to Theorems *(2.1.16)* and *(2.1.17)* we have the following result for the mixed Haplitz products.

Theorem (2.1.18)[42]: Let $f \in H^\infty$ and $g \in L^\infty(\mathbb{D}, dA)$. Then $T_f H_g^*$ is compact if and only if $H_g T_{\bar{f}}$ is compact if and only if

$$\lim_{|w| \to 1^-} \widetilde{|f|}^2(w)^{1/2} \|g \circ \varphi_w - P(g \circ \varphi_w\|_2 = 0.$$

We discuss compactness of the various Haplitz products with symbols in the maximal ideal space. We first recall the definition and Hoffman's beautiful description of the maximal ideal space.

The maximal ideal space of H^∞ is the set M of multiplicative linear maps from H^∞ onto the field of complex numbers. The Gelfand transform allows us to think of H^∞ as a subalgebra of $C(M)$, the algebra of continuous complex-valued functions on M. By the Stone-Weierstrass theorem, the set of finite sums of functions of the form $f_{\bar{g}}$, with $f, g \in H^\infty$, is dense in C(M), where C(M) is endowed with the usual supremum norm. Thus we can identify C(M) with the closed subspace of $L^\infty(\mathbb{D}, dA)$ generated by functions of the form $f_{\bar{g}}$, with $f, g \in H^\infty$. With this viewpoint, C(M) is the C*-subalgebra of $L^\infty(\mathbb{D}, dA)$ generated by H^∞. For $m \in M$, let $\varphi_m: \mathbb{D} \to M$ denote the Hoffman map. This map is defined by setting

$$\varphi_m(w) = \lim_{z \to m} \varphi_z(w)$$

for $w \epsilon \mathbb{D}$; here we are taking a limit in M. The existence of this limit, as well as many other deep properties of φ_m, was proved by Hoffman [50]. An exposition of Hoffman's results can also be found in [49]. We shall use, without further comment, Hoffman's result that φ_m is a continuous mapping of $\mathbb{D}$ into M. Note that $\varphi_m(0) = m$.

Theorem (2.1.19)[42]: Let f and g be in C(M). Then the product $H_f H_g^*$ is compact if and only if $f \circ \varphi_m$ or $g \circ \varphi_m$ is in H^∞ for every m in $M/\mathbb{D}$.

Proof. By Theorem (2.1.17) it suffices to show that f $\circ\, \varphi_m$ or $g \circ \varphi_m$ is in H^∞, for all m in $M/\mathbb{D}$, is equivalent to

$$\lim_{|w| \to 1^-} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 = 0.$$

If m is in $M/\mathbb{D}$, and (w_j) is a net in $\mathbb{D}$ converging to m, then it is easily seen that $f \circ \varphi_{wj} \to f \circ \varphi_m$ pointwise on $\mathbb{D}$. We claim that in fact $f \circ \varphi_{wj} \to f \circ \varphi_m$ in $L^2(\mathbb{D}, dA)$. Some care needs to be taken to prove this claim, since the bounded convergence theorem does not hold for nets, as opposed to sequences. A standard density argument shows that $f \circ \varphi_{wj} \to f \circ \varphi_m$ uniformly on compact subsets of $\mathbb{D}$ (see [54],). Using that

$$\left\|f \circ \varphi_{wj} - f \circ \varphi_m\right\|_2^2 = \int_{\mathbb{D}} \left|f \circ \varphi_{wj}(z) - f \circ \varphi_m(z)\right|^2 \, dA$$
$$+ \int_{\mathbb{D}/r\bar{D}} \left|f \circ \varphi_{wj} - f \circ \varphi_m\right|^2 dA$$
$$\leq \sup_{|z| \leq r} \left| f \circ \varphi_{wj} - f \circ \varphi_m\right|^2 + 4(1 - r^2)\|f\|_\infty^2,$$

for all $0 < r < 1$, we conclude that indeed $f \circ \varphi_{wj} \to f \circ \varphi_m \; in \; L^2(\mathbb{D}, dA)$. It follows that $(I - P)(f \circ \varphi_{wj}) \to (I - P)(f \circ \varphi_m)$. Consequently,

$$\lim_{w \to m} \left\|f \circ \varphi_w - P\big(f \circ \varphi_{wj}\big)\right\|_2 = \|f \circ \varphi_m - P(f \circ \varphi_m)\|_2 .$$

So $f \circ \varphi_m$ is in H^∞ if and only if

$$\lim_{w \to m} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 = 0.$$

Hence the condition that $f \circ \varphi_m$ or $g \circ \varphi_m$ is in H^∞ is equivalent to

$$\lim_{w\to m}\|f\circ\varphi_w - P(f\circ\varphi_w)\|_2\ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 = 0.$$

This completes the proof.

The above theorem should be compared with the following result.

Theorem (2.1.20)[42]: Let $f, g \in C(M)$. The following statements are equivalent*:*

(i) $H_f^* H_g$ is compact*;*

(ii) $(I-P)(f\circ\varphi_m)\perp = (I-P)(g\circ\varphi_m)$, for all $m\in M\backslash\mathbb{D}$*;*

(iii) $H^*_{f\circ\varphi_m}H_{g\circ\varphi_m} = 0$, for all $m\in M\backslash\mathbb{D}$.

Proof. (i) (ii)*:* If w in $\mathbb{D}$ converges to $m\in M\backslash\mathbb{D}$ then

$$Lim_{w\to m}\langle H_f^* H_g k_w, k_w\rangle = \lim_{w\to m}\langle (I-P)(g\circ\varphi_w), (I-P)(f\circ\varphi_w)\rangle$$
$$= ((I-P)(g\circ\varphi_m), (I-P)(f\circ\varphi_m)).$$

By Theorem (2.1.1) in [46], $H_f^*\ H_g$ is compact if and only if $(H_f^* H_g k_w, k_w)\to 0$ as $|w|\to 1^-$, so $H_f^* H_g$ is compact if and only if

$$\langle (I-P)(g\circ\varphi_m), (I-P)(f\circ\varphi_m)\rangle = 0$$

for all $m\in M\backslash\mathbb{D}$.

(ii) ⇔ (iii)*:* If w in $\mathbb{D}$converges to $m\in M\backslash\mathbb{D}$, then by Lemma 2.8 in [45], we have

$$U_w H_f^* H_g U_w \to H^*_{f\circ\varphi_m}H_{g\circ\varphi_m},$$

where the limit is taken in the strong operator topology. For fixed $z\in\mathbb{D}$, using that $U_w k_z = \xi k_{\varphi_w(z)}$ for unimodular ξ it follows that

$$\langle H^*_{f\circ\varphi_m}H_{g\circ\varphi_m}k_z, k_z\rangle = \lim_{w\to m}\langle U_w H_f^* H_g U_w k_z, k_z\rangle$$
$$= \lim_{w\to m}\langle H_f^* H_g k_{\varphi_w(z)}, k_{\varphi_w(z)}\rangle$$
$$= \langle (I-P)(g\circ\varphi_{m_z}), (I-P)(f\circ\varphi_{m_z})\rangle,$$

where $m=\varphi_m(z)\in M\backslash\mathbb{D}$. Thus (ii) is equivalent to $\langle H^*_{f\circ\varphi_m}H_{g\circ\varphi_m}k_z, k_z\rangle = 0$, for all $z\in\mathbb{D}$, which, by a result of Berezin (see, for example, [55],), is equivalent to $H^*_{f\circ\varphi_m}H_{g\circ\varphi_m} = 0$.

On the Bergman space, it is not clear that $H_f H_g^*$ is compact if and only if $H_f^* H_g$ is compact because we don't know when the product $H_f^* H_g$ is zero even if f and g are in $C(M)$. However, when f and g are bounded harmonic functions on $\mathbb{D}$, combining a theorem in [58] with Theorem *(2.1.19)* yields the following result.

Theorem (2.1.21)[42]: Let f and g be bounded harmonic functions on the unit disk. Then $H_f H_g^*$ is compact if and only if $H_f^* H_g$ is compact.

For mixed Haplitz products we have the following characterization of compactness.

Theorem (2.1.22)[42]: Let $f\in H^\infty$ and $g\in C(M)$. Then $T_f H_g^*$ is compact if and only if $f\circ\varphi_m = 0$ or $g\circ\varphi_m$ is in H^∞ for every m in $M\backslash\mathbb{D}$.

Proof. If m is in $M\backslash\mathbb{D}$, then

$$\|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 \to \|g\circ\varphi_m - P(g\circ\varphi_m)\|_2$$

as $w\to m$. Likewise,

$$|\widetilde{f\,|^2}(w)^{1/2} = \|f\circ\varphi_w\|_2 \to \|f\circ\varphi_m\|_2,$$

as $w\to m$. So

$$|\widetilde{f\,|^2}(w)^{1/2}\ \|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 \to \|f\circ\varphi_m\|_2\|g\circ\varphi_m - P(g\circ\varphi_m)\|_2$$

as $w\to m$. The condition $\widetilde{|f|^2}(w)^{1/2}\|g\circ\varphi_m - P(g\circ\varphi_m)\|_2 \to 0\ as\ |w|\to 1^-$ is therefore equivalent to the condition $\|f\circ\varphi_m\|_2\|g\circ\varphi_m - p(g\circ\varphi_m)\|_2 = 0$, for all $m\in M\backslash\mathbb{D}$, which is equivalent to $f\circ\varphi_m = 0$ or $g\circ\varphi_m = P(g\circ\varphi_m)$, for all $m\in M\backslash\mathbb{D}$, that is, $g\circ\varphi_m = 0$ or $g\circ\varphi_m$ is analytic, for all $m\in M\backslash\mathbb{D}$. K Similarly, Theorem *(2.1.18)* implies*:*

Theorem (2.1.23)[42]: Let $f \in H^{\infty}$ and $g \in C(M)$. Then $T_f H_{\bar{g}}$ is compact if and only if $H_g T_{\bar{f}}$ is compact if and only if $f \circ \varphi_m = 0$ or $g \circ \varphi_m$ is in H^{∞} for every m in $M \backslash \mathbb{D}$.

Based on Theorems *(2.1.9)* and *(2.1.10)* we make the following conjecture, analogous to Sarason's conjecture [53] on the Hardy space.

Conjecture (2.1.24)[42]: Let f and g be in L_a^2 . Then*:*

(i) $T_f T_{\bar{g}}$ is bounded if and only if $\sup_{w \in \mathbb{D}} \left| \widetilde{f} \right|^2(w) \left| \widetilde{g} \right|^2(w) < \infty$.

(ii) $T_f T_{\bar{g}}$ is compact if and only if $\lim_{|w| \to 1^-} \left| \widetilde{f} \right|^2(w) \left| \widetilde{g} \right|^2(w) = 0$.

Theorems (2.1.11), (2.1.12), (2.1.17) and *(2.1.21)* provide support for the following conjecture.

Conjecture (2.1.25)[42]: Let f and g be in $L^2(\mathbb{D}, dA)$. Then*:*

(i) $H_f H_g^*$ is bounded if and only if $H_f^* H_g$ is bounded if and only if

$$\sup_{w \in \mathbb{D}} \| f \circ \varphi_w - P(f \circ \varphi_w) \|_2 \, \| g \circ \varphi_w - P(g \circ \varphi_w) \|_2 < \infty.$$

(ii) $H_f H_g^*$ is compact if and only if $H_f H_g^*$is compact if and only if

$$\lim_{|w| \to 1^-} \| f \circ \varphi_w - P(f \circ \varphi_w) \|_2 \, \| g \circ \varphi_w - P(g \circ \varphi_w) \|_2 = 0.$$

If $H_f H_g^*$ is compact, then by Theorem *(2.1.17),*

$$\| f \circ \varphi_w - P(f \circ \varphi_w) \|_2 \, \| g \circ \varphi_w - P(g \circ \varphi_w) \|_2 \to 0$$

as $|w| \to 1^-$, thus $(H_f^* H_g k_w, k_w) \to 0$ as $|w| \to 1^-$, and by Theorem *(2.1.1)* in [46], $H_f^* H_g$ is compact. K Based on Theorems *(2.1.13), (2.1.14)* and *(2.1.18)* we furthermore make the following conjecture.

Conjecture (2.1.26)[42]: Let f be in $L_2(\mathbb{D}, dA)$ and $g \in L_a^2$. Then

(i) $T_f H_g^*$ is bounded if and only if $H_g T_{\bar{f}}$ is bounded if and only if

$$\sup_{w \in \mathbb{D}} \widetilde{f \, |}^2(w)^{1/2} \| g \circ \varphi_w - P(g \circ \varphi_w) \|_2 \, | < \infty.$$

(ii) $T_f H_g^*$ is compact if and only if $H_g T_{\bar{f}}$ is compact if and only if

$$\lim_{|w| \to 1^-} \left| \tilde{f} \right|^2 (w)^{1/2} \| g \circ \varphi_w - P(g \circ \varphi_w) \|_2 = 0.$$

Section (2.2): Bergman Kernel Asymptotics

Given a positive measure $m(t)dt$ on $\boldsymbol{R}^+$ and its moment sequence $\gamma_n = \int_0^\infty t^n m(t) dt$, $n = 0, 1, 2, \ldots$, we form the associated Bergrnan kernel function, $K_m(x) = \sum \gamma_n^{-1} x^n$. We also formlthe new measure $\left(K_m(t)\right)^{-1} m(t) dt$ and its kernel function, $K_{(K_m)^{-1} m}$ .If we start with $m(t) = e^{-1}$ and do the computations, we find three striking facts: for all $t \in \boldsymbol{R}^+$ and all $a \in \mathbb{C}$,

$$m(t) K_m(t) = 1, \tag{A}$$

$$K_{(K_m)^{-1} m}(t) = 2 K_m^2(t), \tag{B}$$

and

$$\int \int \left| K_{(K_m)^{-1} m}(\bar{a} z) \right| m(|z|^2) \frac{dxdy}{\pi} = 2 K_m(|a|^2). \tag{C}$$

We were doing operator theory on the Fock space, the Hilbert space of entire functions square integrable with respect to the Gaussian density. We wanted to know if similar relations or useful substitutes held in Bergman spaces of entire functions square integrable with respect to other radial measures, $\pi^{-1} m(|z|^2) \, dxdy$. However, although operator theoretic issues influence our discussion of the consequences of our main results, neither our results here nor our methods involve operator theory. See [72].

We collect background information about conjugate functions of convex functions (in the sense of Fenchel, Legendre, and Young) which arises both as we pass from the density m to its moments γ_n, and as we pass from the coefficients of K_m to its values. Informal statements and proof outlines for our two main technical results: Theorem (2.2.2)-which shows how the growth of the density function controls the asymptotic growth of the moment sequence-and Theorem (2.2.3)-which shows how the growth of the coefficient sequence controls the growth of $K_m(re^{i\theta})$ for large r. The next have the statements and proofs of Theorems (2.2.2) and (2.2.3). The basic approach for Theorem (2.2.2) is Laplace's method for asymptotic estimation of integrals which depend on a parameter. To prove Theorem (2.2.3), we join Laplace's method with Poisson summation.

We combine Theorem (2.2.2) and Theorem (2.2.3) to give Theorem (2.2.4), our estimates for the Bergman kernel functions. A consequence of that Theorem (2.2.2) Corollary (2.2.17), which includes the result that, as $r \to \infty$,

$$m(r)K(r) \sim \frac{-\left(r\frac{d}{dr}\right)^2 \log m(r)}{r}.$$

In particular, if $m(r) \sim ar^b e^{-cr^4}$, with $a, b, c, d > 0$, we have

$$m(r)K(r) \sim cd^2 r^{d-1},$$

which is a version of (A). If we take the estimates for K in terms of m and then use Theorem (2.2.2) and Theorem (2.2.3) again to estimate $K_{(K_m)^{-1}m}$, we find that the two expressions in (B) are asymptotically equal. In fact, as is suggested by the example of the exponential density, we see in Theorem (2.2.18) that

$$K_{(K_m)^{-\alpha}m} \sim (1+\alpha)(K_m)^{1+\alpha} \tag{7}$$

for $\alpha > 0$. We also show that the Berezin transform for these Bergrnan spaces is given asymptotically by integration against a Gaussian density. This and (7) are then used to give an asymptotic version of (C) in Corollary (2.2.20).

A summary of these and related results along with some discussion of the operator theory is in [72].

Suppose $A(s)$ is a convex function defined on an interval $I \subset \boldsymbol{R}$. (When convenient, we set $A(s) = +\infty$ for $s \notin I$.) We recall the definition of the conjugate function of A.

$$A^*(x) = \sup_{s \in \boldsymbol{R}} \{xs - A(s)\}. \tag{8}$$

This transformation occurs in various contexts, at times associated with the names Fenchel, Legendre, or Young.

Lemma (2.2.1)[61]: Suppose A is smooth and $A, A', A'' > 0$. Set $s(x) = A'^{-1}(x)$ and $x(s) = A'(s)$. Then $A^*, A^{*\prime}, A^{*\prime\prime} > 0$ and we have, for all s, x,

(i) $s(x(s)) = s, x(s(x)) = x,$

(ii) $sx < A(s) + A^*(x),$

(iii) $A^*(z) = xA'^{-1}(x) - A(A'^{-1}(x)) = xs(x) - A(s(x)),$

(iv) $s(x) = {A^*}'(x) = A'^{-1}(x),$

(v) $A^{**}(s) = A(s),$

(vi) ${A^*}''(x) = A''(s(x))^{-1},$

(vii) $A^{*(3)}(x){A^*}''(x)^{-\frac{3}{2}} = -A^{(3)}(s(x))A''(s(x))^{-\frac{3}{2}},$

(viii) $A^{*(4)}(x){A^*}''(x)^{-2} = -A^{(4)}(s(x))A''(s(x))^{-2} + 3A^{(3)}(s(x))^2 A''(s(x))^{-3}.$

We are only interested in asymptotic behavior for large s and large x. Hence, if necessary to insure that the hypotheses are satisfied, we can first restrict A to an interval (M, ∞) and then set $A = +\infty$ on $(-\infty, M]$. In that case, the conclusions of the lemma hold for all sufficiently large x, s.

Proof. The proof of related results under minimal smoothness assumptions requires care, but here there is no problem. The first statement follows from the definitions, as does the second, which is often called Young's conjugate function inequality. Our assumptions insure that the supremum in (8) is attained at the unique critical point of $xs - A(s)$. This gives the formula for A^*. The first equality in (iv) follows from differentiating (iii). The relation (v) comes from (iii) and (iv). Formula (vi) follows from differentiating (v). Equality (vii) follows from differentiating (vi) and noting that $s'(x) = {A^*}''(x) = A''\big(s(x)\big)^{-1}$. Formula (viii) follows from differentiating (vii), using $s'(x) = A''(x) = A''\big(s(x)\big)^{-1}$, and then using (vii).

The model pair for what we do later is

$$A(s) = e^s - s,$$
$$A^*(x) = (x+1)\log(x+1) - (x+1),$$

which corresponds to $m(t) = exp(-t)$. More generally, for $m(t) = exp\big(-t^\beta\big)$, we have

$$A(s) = e^{\beta s} - s, \tag{9}$$
$$A^*(x) = \left(\frac{x+1}{\beta}\right)\log\left(\frac{x+1}{\beta}\right) - \left(\frac{x+1}{\beta}\right).$$

The theorems and proofs have substantial technical details. However, the basic ideas are quite straightforward. We present the ideas.

Given a positive function $a(s)$ defined on $\boldsymbol{R}^+$, set

$$A(s) = -\log a(e^*) - s. \tag{10}$$

We suppose that for all large s

$$A(s), A''(s), A''(s), A^{(3)}(s), A^{(4)}(s) > 0. \tag{11}$$

Set $s_z = s(x) = A'^{-1}(x)$. Suppose b is a positive function which varies slowly compared to a and set $B(s) = \log b(e^*)$. Let γ_n be the moments of the measure $a(t)b(t)dt; \gamma_n = \int_0^\infty t^n a(t)b(t)dt.$

Theorem (2.2.2)[61]: (informal). As $n \to \infty$, we have

$$\gamma_n \sim e\wedge(A^{*(n)} \frac{\sqrt{2\pi}}{\sqrt{A''(s_n)}} e^{B(s_n)}.$$

In the simplest case, when $a(t) = e^{-t}$ and $b(t) = 1$, this is Stirling's formula. Now suppose $c(x)$ is a positive function on $\boldsymbol{R}^+$. Set

$$\Gamma(x) = \log c(x). \tag{12}$$

Suppose that for all large x

$$\Gamma(x), \Gamma'(x), \Gamma''(x) > 0. \tag{13}$$

However, in contrast to the previous theorem, we now require that as $x \to \infty$

$$\Gamma'(x) \to \infty, \qquad \Gamma''(x), \Gamma^{(3)}(x), \Gamma^{(4)}(x) \to 0. \tag{14}$$

Let Γ^* be the conjugate function of Γ and set $x_s = x(s) = \Gamma'^{-1}(s)$. Suppose that d is a positive function which varies slowly compared to c. Let f be the holomorphic function

$$f(z) = \sum_0^\infty d(n)c(n)^{-1}z^n$$

Theorem (2.2.3)[61]: (informal). f is entire. For small θ we have as $s \to \infty$

$$f\left(e^{s+i\theta}\right) \sim e^{\Gamma^*(s)} \frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x_s)}} d(x_s) e^{ix_s\theta} e^{\frac{\theta^2}{2\Gamma''(x_s)}}.$$

Our main kernel estimate, Theorem (2.2.4), follows quickly from these two results. First we apply Theorem (2.2.2) with the choices $a(t) = m(t^2), b(t) = I$ and then Theorem (2.2.3) with the choices $c(x) = A^*(x), d(x) = c(x)/\gamma_x$. Because we are able to put some of the behavior of the moments into the correction term d, we obtain kernel estimates whose main term involves A^{**}. We then use the fact that $A^{**} = A$. To get estimates for $K_{K_m^{-}\alpha_m}$, we repeat the cycle, using as our new starting choice for a the square of the function used the first time. This forces a nonconstant choice for b. However, b turns out to be slowly varying, so again the main term of the estimate involves $A^{**} = A$.

To prove Theorem (2.2.2), we use Laplace's method for asymptotic evaluation of integrals as it adapts to our situation. We want to estimate

$$\gamma_n = \int_0^\infty t^n a(t) b(t) dt = \int_0^\infty e^{n\log t + \log \alpha(t)} b(t) dt$$

$$= \int_{-\infty}^\infty e^{ns + \log \alpha(e^*) + s} b(e^*) ds = \int_0^\infty e^{ns - A(s)} e^{B(s)} ds. \qquad (15)$$

The hypotheses insure that, for fixed large n, the function ns $-A(s)$ has a maximum value at the point $s_n = A'^{-1}(n)$. The value is $A^*(n) = ns_n - A(s_n)$. We now expand $ns - A(s)$ in a Taylor series about its critical point s_n:

$$n_s - A(s) = A^*(n) - \frac{1}{2} A''(s_n)(s - s_n) + \mathrm{R}.$$

Here R is the remainder tem. If we could drop R and replace $B(s)$, which is built from a slowly varying function, by $B(s_n)$ then we could evaluate the integral and would have γ_n equal to the desired estimate. The technical details of the proof involve estimating the errors that result from dropping R and replacing $B(s)$ by $B(s_n)$.

Introduce the new integration variable $u = s - s_n$. Using $A'(s_n) = n$, we have

$$ns - A(s) = A^*(n) - [A(u + s_n) - A(s_n) - A'(s_n)u].$$

We need to estimate

$$\gamma_n = e^{A^*(n)} \int_{-\infty}^\infty e^{-[A(s_n+u) - A(s_n) - A'(s_n)u]} e^{B(s_n+u)} du. \qquad (16)$$

To do this we select a positive function $\delta = \delta(n)$ and split the integral as

$$\int_{-\infty}^\infty \dots du = \int_{u<-\delta} \dots du + \int_{|u|<\delta} \dots du + \int_{u>\delta} \dots du = L + C + R$$

To estimate C, we want to know that, uniformly in $\{u: |u| \le \delta\}$, we have for some appropriate small K

$$A(s_n + u) = A(s_n) + A'(s_n)u + A''(s_n)u^2/2 + O(\mathrm{K}),$$
$$B(s_n + u) = B(s_n) + O(\mathrm{K}).$$

Those estimates follow from the hypotheses on a and b and Taylor's theorem. Using them, we have

$$e^{A^*(n)} C = e^{A^*(n)} \int_{|u|<\delta} e^{-\frac{A''(s_n)u^2}{2}} e^{B(s_n)} [1 + O(\mathrm{K})] du.$$

Introducing the new variable $v = u\sqrt{A''(s_n)}$, we find that

$$e^{A^*(n)} C = \frac{e^{A^*(n)} e^{B(s_n)}}{\sqrt{A''(s_n)}} \int_{|v|<\delta\sqrt{A''(\delta_n)}} e^{-\frac{v^2}{2}} [1 + O(\mathrm{K})] dv.$$

If we know $\delta^2(n)A''(s_n) \to \infty$, we can conclude that

$$e^{A^*(n)}C = e^{A^*(n)}\frac{\sqrt{2\pi}}{\sqrt{A''(s_n)}}e^{B(s_n)}[1 + O(\mathrm{K})].$$

The tails, L and R, can be estimated by tails of Gaussian integrals and are seen to be $O\left(e^{\frac{\delta^2(n)A''(s_n)}{10}}\right)$. Combining the estimates for L, C, and R gives Theorem (2.2.2).

In the second theorem, we can pass from the sum to the corresponding integral and use a similar argument to get the estimates on the positive real axis. However, that approach doesn't capture the cancellation which occurs off the axis. Hence we split the sum into three terms and estimate the main term, the central one, using Poisson summation.

In Theorem (2.2.3), we show that if we are given the moments $\{\gamma_n\}$ of a density, then the asymptotic growth of the kernel function is given by

$$f(e^s) \sim \sqrt{2\pi}e^{(\log\gamma)^*(s)}, \qquad s \to \infty.$$

Rewriting this in terms of the Taylor coefficients $a_n (= \gamma_n^{-1})$ off, we have

$$a_n \sim \frac{1}{\sqrt{2\pi}}e^{(\log f(e^*))^*(n)}, \qquad n \to \infty.$$

In the other direction, one can ask whether, given an entire function which satisfies appropriate conditions, we can conclude this sort of asymptotic growth for the coefficients. That such estimates do, in fact, hold for a large class of entire functions is a result of Hayman [71].

Suppose $f(z) = \sum_0^\infty a_n z^n$ is an entire function with positive coefficients. Set

$$F(s) = \log f(e^s).$$

We say that f is admissible if $F''(e^s) \to \infty$ as $s \to \infty$ and there is a positive function $\delta(r)$, defined for all sufficiently large r, such that $0 < \delta(r) < \pi$,

$$f(re^{i\theta}) \sim f(r)e^{iF'(\log r)\theta}e^{-\frac{1}{2}F''(\log r)\theta^2} \quad as\ r \to \infty$$

uniformly for $|\theta| \le \delta(r)$, and

$$f(re^{i\theta}) = o(1)\frac{f(r)}{\sqrt{F''(\log r)}},$$

uniformly for $\delta(r) \le |\theta| \le \pi$.

Corollary II of [71] is

Theorem (2.2.5)[61]: If $f(z)$ is admissible, then as $n \to \infty$

$$a_{n+1} \sim \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{F''(F'^{-1}(n))}}e^{-F^*(n)}.$$

In fact, this follows quite easily from the admissibility of f. Most of the work in [71] is in establishing that a substantial number of functions are admissible, in showing that the class of admissible functions has interesting closure properties, and in deriving further consequences of admissibility. Our Theorem (2.2.3) insures that the kernel functions we construct are admissible.

In Hayman's theorem as well as Theorems (2.2.2) and (2.2.3), we see that the transforms have asymptotics described to leading order using the conjugate function. That is in keeping with the heuristic "principle of duality of phases", for describing the asymptotic behavior of Fourier (and related) transforms ([80], p. 358). The principle has a long tradition. Hayman's results are related to earlier results of Wiener and Martin [82], [83] and still earlier results of Hardy and Fejér, both of whom attribute the basic insight to Riemann. (For this see the discussion in [70].) See Evgrafov [69] and Bemdtsson [63].

Related questions have been considered for measures and kernel functions defined on the unit disk. The work goes back to Trent [81], Kriete and MacCluer [76] and Kriete [74]. Here are two of the results of [74].

Suppose that we are working on the Bergman space of the disk with radial weight $(2\pi)^{-1}\, w(r)dx, dy$. Thus $K(x) = \sum \gamma_n^{-1} x^n$ with $\gamma_n = \int_0^1 r^{2n+1} w(r)dr$. Set $\mathrm{A}(s) = \log 2 - \log w\left(e^{-\frac{s}{2}}\right) + s$. Under appropriate conditions on w, a result analogous to Theorem (2.2.2) is obtained.

Theorem (2.2.6)[61]: As $r \to \infty, \gamma_n \sim \sqrt{\pi} \sqrt{\mathrm{A}^{*''}(-n)} e^{\mathrm{A}^*(-n)}$.

This Theorem (2.2.2)s used in the proof of the following quantitative alternative to (A), which plays a major technical role in [74]:

Theorem (2.2.7)[61]: As $r \sim 1^-, m(r)K(r) \nearrow \infty$.

While preparing this, we learned that Kriete has taken his work further and obtained rather comprehensive results on the unit disk [75]. Although the detailed formalism of [74] and [75] differ, there is certainly a similarity between those methods.

Related questions have been studied for nonradial weights using a variety of function theoretic techniques. For instance, it is shown in [77] that under some regularity conditions on the function $w(z) \geq O$, and with the assumption that $-\log w$ is subharmonic, the Bergman kernel $K(z, \zeta)$ for the space $L^2(\mathbb{D}, w(z)dx\, dy) \cap Hol$ satisfies

Proposition (2.2.8)[61]: There are positive constants C_1 and C_2 so that

$$C_1 < \frac{K(z,z)w(z)}{\Delta \log w(z)} < C_2.$$

Similar techniques produce an analogous result for Bergman spaces on the plane. These should be compared with Theorem (2.2.16), which deals with smooth radial weights (on the plane). That result states that, as $z \to \infty$,

$$\frac{K(z,z)w(z)}{-\Delta \log w(z)} = 1 + o(1).$$

Christ, Berndtsson, Ortega-Cerd/L and Seip, Delin, and others have obtained refined estimates on Bergman kernel functions, including estimates off the diagonal, using $\bar{\partial}$ techniques. Those results have a different focus from ours and we merely give [65], [64], [79], and [66].

It is a theorem of Miles and Williamson [78], which proved a conjecture of Renyi and Vincze, that $m(t) = e^{-t}$ is essentially the only function which satisfies (A). It would be interesting to know if there were analogous uniqueness results related to (C).

We shall prove that for fixed α, as $t \to \infty$,

$$K_{(K_m)^{-\alpha} m} \sim (1+\alpha) K_m^{1+\alpha}(\mathrm{t}) \qquad (17)$$

In his interesting study of Berezin quantization, Englis [67], [68] shows that in certain cases, for fixed t, (17) holds as $\alpha \to \infty$. His methods and viewpoint are quite different. We discuss briefly the possibility of obtaining asymptotics as $\alpha \to \infty$ by our methods.

Suppose $m(t)dt$ is a positive measure on $[0, \infty)$. For $x \geq 0$, set $\gamma_x = \gamma(x) = \int_0^\infty t^x m(t)dt$. We assume that m does not have compact support and that $\gamma(x)$ is finite for all x. We write $m(t) = a(t)b(t)$ with a as the main term and b as a slowly varying correction. Although $m = ab$ is the object of interest, most of our computations, and hence also the hypotheses, are in terms of auxiliary functions, A and B, defined by

$$A(s) = -\log a(e^s) - s, \quad (18)$$

$$B(s) = \log b(e^s). \quad (19)$$

Set $s_x = s(x) = A'^{-1}(x)$. Fix $\varepsilon, 1/4 < \varepsilon < 1/2$. We suppose that for all sufficiently large x

$$A^{(i)}(x) > 0, \qquad i = 0, \dots, 4, \quad (20)$$

$$A'''(x) = O\left(A''^{\frac{3}{2}-\varepsilon}(x)\right), \quad (21)$$

$$A^{(4)}(x) = O\left(A''^{\,2-2\varepsilon}(x)\right). \quad (22)$$

The core hypothesis for the proof of Theorem (2.2.2) is that we can find an auxiliary positive function δ such that $\delta^2(x)A''(s_x) \to \infty$ and $\delta^3(x)A'''(sx) \to 0$ as $x \to \infty$. We surrendered a slight amount of generality by assuming (21), but that lets us make a simple choice for δ. Select a with $0 < \alpha < \varepsilon/2 - 1/8$ and set

$$\delta(x) = A''(s_x)^{-\frac{1}{2\alpha}}. \quad (23)$$

The model case for the hypotheses is $A(s) = e^{\beta s} - s$. In that case, (21) and (22) hold with any $\varepsilon < 1/2$. The same is true for $A(s) = e^{h(s)}$ with any function h of regular and modest growth. Hence, for the examples we have in mind, we could restrict attention to A which satisfy (21) and (22) for all e up to $1/2$. In fact, suppose A were to fail (21) for a fixed e because there is some $\varphi < 1/2 - \varepsilon$ so that $A''' \geq CA''^{\,1+\varphi}$. In such a case, we could compare A'' with the exact solution of $f' = Cf^{1+\varphi}$ and conclude that $A''(s)$ cannot be finite for all $s > 0$. Such A are not of interest here. However, we carry the extra generality of allowing (21) and (22) to fail for some $\varepsilon < 1/2$ because it may be useful in some other context. We should note that in the following discussion it may be convenient to think of the model case $\varepsilon = \left(\frac{1}{2}\right)^{-}, \alpha = 0^{+}$.

The estimate on the derivatives of A imply interval estimates.

Lemma (2.2.9)[61]: If we have (21), (22), and (23), then we also have

$$\sup_{|t|<\delta} A''(s_x + t) = \big(1 + o(1)\big)A''(s_x), \quad (24)$$

$$\sup_{|t|<\delta} A'''(s_x + t) = O\left(A''(s_x)^{\frac{3}{2}-\varepsilon}\right), \quad (25)$$

$$\sup_{|t|<\delta} A^{(4)}(s_x + t) = O(A''(s_x)^{2-2\varepsilon}), \quad (26)$$

Proof. Set $g(t) = A''(s_x + t)$. By (21), $g' = O(g^{\frac{3}{2}-\varepsilon})$ and hence $g^{-\frac{3}{2}+\varepsilon}g' = O(1)$. Pick and fix some $t_0, |t_0| < \delta$. Integrating, we find

$$\left|g^{-\frac{1}{2}+\varepsilon}(t_0) - g^{-\frac{1}{2}-\varepsilon}(0)\right| = O(\delta).$$

Hence, recalling the definitions of g and δ, we have

$$\frac{g^{-\frac{1}{2}+\varepsilon}(t_0)}{g^{\frac{1}{2}+\varepsilon}(0)} = 1 + \left|g^{\frac{1}{2}-\varepsilon}(0)\right| O\left(g(0)^{-\frac{1}{2}+\alpha}\right) = 1 + O(g(0)^{\alpha-\varepsilon}),$$

as required for (24). For (25), note that by (21) we have

$$A'''(s_x + t) = O\left(A''^{\frac{3}{2}-\varepsilon}(s_x + t)\right),,$$

and by (24) we can replace $A''^{\frac{3}{2}-\varepsilon}(s_x + t)$ by $A''^{\frac{3}{2}-\varepsilon}(s_x)$. We obtain (26) by the same reasoning.

We say that a positive function b is slowly varying in the first sense with respect to a, ε and a and write $b \in SVI(a, \varepsilon, \alpha)$ if, for B given by (19),

$$B' = O(\delta^2 A''') = O\left(A'^{\frac{1}{2}-\varepsilon+2\alpha}\right), \quad (27)$$

$$B'' = O\left(\left(A''^{\frac{1}{2}-\varepsilon+2\alpha}\right)^2\right). \quad (28)$$

Note that $0 < 1/2 - \varepsilon + 2\alpha < 1/4$. We know from the previous lemma that A'' and A''' satisfy interval estimates. Hence so do B' and B''.

We use the following. Write $X = X(x) = O(\mathcal{E})$ if there is a positive c such that $X = O\big(exp(-A''(s_x)^C)\big)$.

Theorem (2.2.10)[61]: Suppose a and b are positive functions on $\boldsymbol{R}^+$, A and B are defined by (18) and (19), and δ is given by (23). Suppose A satisfies (20), (21), (22), and hence also (24), (25), and (26). Suppose $b \in SVI(a, \varepsilon, \alpha)$. Let

$$\gamma(x) = \int_0^\infty t^x a(t) b(t) dt.$$

As $x \to \infty$, we have

$$\gamma(x) = e^{A^*(x)} \frac{\sqrt{2\pi}}{\sqrt{A''(s_x)}} e^{B(s_x)}\big(1 + O(A''(s_x)^{6\alpha-2\varepsilon}) + O(\mathcal{E})\big). \quad (29)$$

Furthermore, as $x \to \infty$

$$(\log \gamma)'(x) \to \infty \quad (30)$$

$$A^{*\prime}(x) - (\log \gamma)'(x) = O\left(A''^{\,-\frac{1}{2}+2\alpha-\varepsilon}(x)\right), \quad (31)$$

$$A^{*''}(x) - (\log \gamma)''(x) = O\big(A''^{\,-1+6\alpha-2\varepsilon}(x)\big). \quad (32)$$

Notes: (i)

$$6\alpha - 2\varepsilon < -\frac{1}{4}, \qquad -\frac{1}{2} + 2\alpha - \varepsilon < -\frac{3}{4}, \qquad -1 + 6\alpha - 2\varepsilon < -5/4.$$

(ii) The formulation of (29) is redundant, as $O(\mathcal{E})$ is smaller than $A''^{\,6\alpha-2\varepsilon}$. We include it separately because, while the error term $O(A''^{\,6\alpha-2\varepsilon})$ can obviously be refined by straightforward (but lengthy) analysis, the exponential error term appears to be intrinsic to the method.

(iii) (30), (31), and (32) are technical estimates we shall use when we use the output from this theorem as input for Theorem (2.2.3).

Proof. Fix x large. We write δ for $\delta(x)$. Set

$$\alpha_i = A^{(i)}(s_x), \qquad i = 0,1, \ldots,$$
$$B_i = B^{(i)}(s_x), \qquad i = 0,1, \ldots,$$

We saw at (1O) in the proof outline that

$$\gamma(x) = e^{A^*(x)} e^{\beta_0} \int_{-\infty}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0} du.$$

We need to estimate

$$I_j = \int_{-\infty}^{\infty} u^j e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0} du$$

For $j = 0, 1, 2$. The analysis of the tails, L and R, is the same for $j = 0, 1$, and 2; we present the discussion only for $j = 0$. We have

$$I_0 = \int_{u<-\delta} \dots du + \int_{|u|<\delta} \dots du + \int_{u>\delta} \dots du = L + C + R.$$

We first estimate L. Integration by parts gives

$$A(s_x + u) - \alpha_0 - \alpha_1 u = \int_u^0 (r-u)A''(s_x + r)dr.$$

In the integral defining $L, u < -\delta < 0$; thus the integrand in the previous integral is positive. Using this and the monotonicity of A'', we continue with

$$A(s_x + u) - \alpha_0 - \alpha_1 u \geq \int_{-\delta}^0 (r-u)A''(s_x + r)dr$$

$$\geq A''(s_x - \delta)\int_{-\delta}^0 (r-u)\, dr = -\frac{1}{2}(2u+\delta)\delta A''(s_x - \delta)$$

Thus

$$L \leq \int_{-\infty}^{\infty} e^{\frac{1}{2}(2u+\delta)\delta A''(s_{x\delta}) + B(s_x+u) - \beta_0} du = e^{\frac{1}{2}\delta^2 A''(s_x-\delta)} \int_{-\infty}^{\delta} e^{u\delta A''(s_x-\delta) + B(s_x+u) - \beta_0} du$$

Now

$$e^{B(s_x+u)-\beta_0} = \exp \int_0^u B'(s_x + t)dt.$$

Using (27) and recalling that A'' is monotone, we find

$$e^{B(s_x+u)-\beta_0} = \exp\left(O(1)|u|\alpha_2^{\theta}\right),$$

Where $\theta = \frac{1}{2} - \varepsilon + 2\alpha$ is between 0 and $1/4$. Hence

$$L < e^{\frac{1}{2}\delta^2 A''(s_x-\delta)} \int_{-\infty}^{-\delta} \exp\left(u\delta A''(s_x - \delta) + O(1)|u|\alpha_2^{\theta}\right) du.$$

Taking into account (23), and recalling that u is negative in the region of integration, we have

$$u\delta A''(s_x - \delta) + O(1)|u|\alpha_2^{\theta} = u\delta\left(\alpha 2\left(1 + o(1)\right) + O(1)\delta^{-1}a_2^{\theta}\right)$$
$$= u\delta\left(\alpha_2\left(1+o(1)\right) + O(1)\alpha^{1+\alpha-\varepsilon}\right) = u\delta\alpha_2\left(1+o(1)\right).$$

Thus we can continue with

$$L < e^{\frac{1}{2}\delta^2 A''(s_x-\delta)} \int_{-\infty}^{-\delta} \exp\left(u\delta\alpha_2\left(1-o(1)\right)\right) du$$

$$= \frac{1}{\left(1-o(1)\right)^{\delta\alpha_2}} e^{\frac{1}{2}\delta^2\alpha_2\left(1+o(1)\right)} e^{-\delta^2\alpha_2\left(1-o(I)\right)} \leq \frac{\left(1+o(1)\right)}{\delta\alpha_2} e^{-\left(\frac{1}{2}-o(1)\right)\delta^2\alpha_2}$$

$$= O(\mathcal{E}).$$

Hence also $a_2^{\frac{1}{2}} L = O(\mathcal{E})$, which is what we require.

We now look at R. We need to estimate

$$\int_{\delta}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0} du.$$

By Taylor's theorem,

$$A(s_x + u) - \alpha_0 - \alpha_1 u - [B(s_x+u) - \beta_0] = -\beta_1 u + \frac{1}{2}\left(A''(s_x+\xi) - B''(s_x+\xi)\right)u^2$$

for some $\xi \in (0,u)$. Taking into account (27), (28), and the motonicity of A'', we continue with

$$\begin{aligned} A(s_x+u)-\alpha_0-\alpha_1 u-[B(s_x+u)-\beta_0] \\ &= o\left(\alpha_2^{\frac{1}{2}}\right)u+\left(A''(s_x+\xi)+o\big(A''(s_x+\xi)\big)\right) \\ &\geq o\left(\alpha_2^{\frac{1}{2}}\right)u+\left(\frac{1}{2}+o(1)\right)\alpha_2 u^2. \end{aligned}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]}e^{B(s_x+u)-\beta_0}du &\leq \int_{\delta}^{\infty} e^{-\left[o\left(\alpha_2^{\frac{1}{2}}\right)u+\left(\frac{1}{2}+o(1)\right)\alpha_2 u^2\right]}du \\ &\leq \alpha_2^{-\frac{1}{2}}\int_{\delta\alpha_2^{\frac{1}{2}}}^{\infty} e^{-\left[o(1)v+\left(\frac{1}{2}+o(1)\right)v^2\right]}dv = O(\mathcal{E}). \end{aligned}$$

Hence $\alpha_2^{\frac{1}{2}}R = O(\mathcal{E})$, which is what we needed.
We now estimate C. For $j = 0, 1, 2,$ we need to estimate

$$C_j = \int_{|u|<\delta} u^j e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]+B(s_x+u)-\beta_0}du.$$

We now need to take the Taylor series analysis given in the proof outline one step further. By Taylor's theorem, we have

$$\begin{aligned} -[A(s_x+u)-\alpha_0-\alpha_1 u]+\ B(s_x+u)-\beta_0 \\ &= -\frac{1}{2}\alpha_2 u^2-\frac{1}{6}\alpha_3 u^3-\frac{1}{24}\tilde{\alpha}_4 u^4+\beta_1 u+\frac{1}{2}\tilde{\beta}_2 u^2. \end{aligned}$$

Here we use decoration to indicate terms which must be evaluated away from s_x: $\tilde{\alpha}_4 = A^{(4)}(w)$ for some $w, |s_x-w|<\delta, \tilde{\beta}_2 =\ B''(w')$ for some $w', |s_x-w'|<\delta$. We separate the main quadratic term, the odd powers, and the error term. We set $D = -\frac{1}{6}\alpha_3 u^3+\beta_1 u$ and $= \frac{1}{24}\tilde{\alpha}_4 u^4+\frac{1}{2}\tilde{\beta}_2 u^2$, Set $A = \alpha_2^{3\alpha-\varepsilon}$ and note that $3\alpha-\varepsilon < -1/8$. Noting (21), (22), (27), and (28) and Lemma (2.2.9), we have $D = O(A), E = O(A^2)$, and $D^2 = O(A^2)$. Taking into account that $A \to 0$, we have

$$\begin{aligned} C_j = \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}e^D e^E du &= \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}\big(1+D+O(D^2)\big)\big(1+O(E)\big)du \\ &= \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}(1+D+O\big((A^2)\big)du \\ &= \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}(1+D)du+O(A^2)\int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}du \\ &= \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}(1+D)du+O(A^2)\alpha_2^{-\frac{j+1}{2}}. \end{aligned}$$

For $j = 0, 2, \int_{|u|<\delta} u^j e^{-\frac{\alpha_2 u^2}{2}}du$ is the integral of an odd function over a symmetric interval and hence we can drop D from the expressions for c_0 and c_2. We next pass to integrals over the entire line. This introduces an error of $O(\mathcal{E})$, which we absorb into the larger error terms. We have

$$C_0 = \int_{-\infty}^{\infty} e^{-\frac{\alpha_2 u^2}{2}} du + O(\Lambda^2)\alpha_2^{-\frac{1}{2}} = \alpha_2^{-\frac{1}{2}}\left(\sqrt{2\pi} + O(\Lambda^2)\right);$$
$$C_2 = \int_{-\infty}^{\infty} e^{-\frac{\alpha_2 u^2}{2}} du + O(\Lambda^2)\alpha_2^{-\frac{3}{2}} = \alpha_2^{-\frac{3}{2}}\left(\sqrt{2\pi} + O(\Lambda^2)\right).$$

For $j = 1$ the integral involving $u^j e^{-\frac{\alpha_2 u^2}{2}}$ vanishes; and we have, using (21), (23), and (27),

$$C_1 = \int_{-\infty}^{\infty} e^{-\frac{\alpha_2 u^2}{2}} du + O(\Lambda^2)\alpha_2^{-\frac{1}{2}} = O\left(\alpha_2^{-\frac{3}{2}}\beta_1\right) + O\left(\alpha_2^{\frac{5}{2}}\alpha_3\right) + O(\Lambda^2)\alpha_2^{-1}$$
$$= \alpha_2^{-1} O(\alpha_2^{-\varepsilon+2\alpha} + \alpha_2^{-\epsilon} + \Lambda^2) = \alpha_2^{-1}(\alpha_2^{-\epsilon+2\alpha}).$$

Hence

$$e^{A^*(x)} e^{\beta_0} I_0 = e^{A^*(x)} e^{\beta_0} \alpha_2^{-\frac{1}{2}}\left(\sqrt{2\pi} + O(\Lambda^2)\right), \tag{33}$$
$$e^{A^*(x)} e^{\beta_0} I_1 = e^{A^*(x)} e^{\beta_0} \alpha_2^{-1} O(\alpha_2^{-\epsilon+2\alpha}), \tag{34}$$
$$e^{A^*(x)} e^{\beta_0} I_2 = e^{A^*(x)} e^{\beta_0} \alpha_2^{-\frac{3}{2}}\left(\sqrt{2\pi} + O(\Lambda^2)\right), \tag{35}$$

This gives us (29).

We now proceed to verify (30), (31), and (32). For appropriate K, we have

$$\gamma(x) = e^{A^*(x)} e^{\beta_0} \int_{-\infty}^{\infty} K(s_x + u) du.$$

If we differentiate (21) and then follow the same pattern of analysis we find

$$\gamma(j)(x) = e^{A^*(x)} e^{\beta_0} \int_{-\infty}^{\infty} (s_x + u)^j K(s_x + u) du$$

for $j = 1, 2$. If we set

$$J_j = s\int_{-\infty}^{\infty} u^j K(s_x + u) du\,, \qquad j = 0,1,\ldots,$$

then we can write

$$\gamma = e^{A^*} e^{\beta_0} J_0$$
$$\gamma' = e^{A^*} e^{\beta_0} (s_x J_0 + J_1),$$
$$\gamma'' = e^{A^*} e^{\beta_0} (s_x^2 J_0 + 2 s_x J_1 + J_2) \tag{36}$$

From Lemma (2.2.1), we know that $A^{*'}(x) = s_x$ and $A^{*''}(x) = A''(x)^{-1}$. quantities we want to estimate are

$$(\log\gamma)' = s_x + \frac{J_1}{J_0}, \qquad A^{*'} - (\log\gamma)' = \frac{J_1}{J_0}$$
$$A^{*''}(\log\gamma) = A''^{-1} - \frac{J_2}{J_0} + \left(\frac{J_1}{J_0}\right)^2.$$

From (33) and (34), we know that

$$-\frac{J_1}{J_0} = \frac{e^{A^*(x)} e^{\beta_0} \alpha_2^{-1} O(\alpha_2^{2\alpha-\epsilon})}{e^{A^*(x)} e^{\beta_0} \alpha_2^{-\frac{1}{2}}\left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\epsilon})\right)} = \alpha_2^{-\frac{1}{2}} O(\alpha_2^{2\alpha-\epsilon}) = O\left(\alpha_2^{-\frac{1}{2}+2\alpha-\epsilon}\right).$$

This gives (31). Also, noting that $s_x \to \infty$, we have (30). required estimate for $\left(\frac{J_1}{J_0}\right)^2$ in (31). We complete (31) by noting

$$A''^{-1}(s_x) - \frac{J_2}{J_0} = \alpha_2^{-1} - \frac{e^{A^*(x)} e^{\beta_0} \alpha_2^{-\frac{3}{2}} \left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\epsilon})\right)}{e^{A^*(x)} e^{\beta_0} \alpha_2^{-\frac{1}{2}} \left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\epsilon})\right)}$$
$$= \alpha_2^{-1} - \alpha_2^{-1}\left(1 + O(\alpha^{6\alpha-2\epsilon})\right) = O(\alpha^{-1+6\alpha-2\epsilon}),$$
as required.

We start with positive functions c and d defined on $\boldsymbol{R}^+$. We want to estimate $f(z) = \sum c(n)^{-1} d(n) z^n$. Here c will be our main term with d a slowly varying correction. We will do our computational work with the auxiliary functions

$$\Gamma(x) = \log c(x), \tag{37}$$
$$\Delta(x) = \log d(x). \tag{38}$$

Let Γ^* be the conjugate function to Γ and set $x_x = x(s) = \Gamma'^{-1}(s) = \Gamma^{*\,\prime}(s)$. We suppose that $1/4 < \epsilon < 1/2$ is such that as $x \to \infty$

$$\Gamma(x), \Gamma'(x), \Gamma''(x), -\Gamma^{(3)}(x) > 0, \tag{39}$$
$$\Gamma(x), \Gamma'(x) \to \infty, \tag{40}$$
$$\Gamma''(\mathrm{x}), \Gamma^{(3)}(x), \Gamma^{(4)}(x) \to \infty, \tag{41}$$
$$\Gamma'''(x) = O\left(\Gamma''^{\frac{3}{2}+\epsilon}(x)\right), \tag{42}$$
$$\Gamma^{(4)}(x) = O\left(z''^{\,2+2\epsilon}(x)\right). \tag{43}$$

Note that if this holds for e, then it also holds for any ϵ' such that $1/4 < \epsilon' < \epsilon$.

In analogy to the previous theorem, the core hypothesis for the proof is now that we can find an auxiliary function λ such that $\lambda^2\Gamma'' \to \infty$ and $\lambda\Gamma''' \to 0$.
Assumption (42) allows us to use

$$\lambda(s) = \Gamma''(x_s)^{-\frac{1}{2}-\beta} \tag{44}$$

for some $\beta, 0 < \beta < \epsilon/3 - 1/12$, which we now regard as selected and fixed. As before, a convenient choice to keep in mind is $\epsilon = \left(\frac{1}{2}\right)^-, \beta = 0^+$.

Lemma (2.2.11)[61]: lf we have (42), (43), and (14), then we also have (45)

$$\sup_{|\theta|<\lambda} |\Gamma''(x_s + \theta)| = \left(1 + o(1)\right)|\Gamma''(x_s)|, \tag{45}$$
$$\sup_{|\theta|<\lambda} |\Gamma'''(x_s + \theta)| = O\left(\Gamma'''(x_s)^{\frac{3}{2}+\epsilon}\right), \tag{46}$$
$$\sup_{|\theta|<\lambda} \left|\Gamma^{(4)}(x_s + \theta)\right| = O\left(\Gamma^{(4)}(x_s)^{2+2\epsilon}\right), \tag{47}$$

Proof. The proof is the natural modification of the proof of Lemma (2.2.9).

Suppose that d is a positive C^2 function. We say that d is slowly varying in the second sense with respect to c, ϵ, and β, and write $d \in SVII(c, \epsilon, \beta)$, if, as $s \to \infty$

$$\Gamma'(s) - \Delta'(s) \to \infty, \tag{48}$$
$$\Delta' = O\left(\Gamma''^{\frac{1}{2}+\epsilon-\beta}\right), \tag{49}$$
$$\Delta'' = O\left(\left(\Gamma''^{\frac{1}{2}+\epsilon-\beta}\right)^2\right). \tag{50}$$

Note that the assumptions imply

$$\frac{3}{4} < \frac{1}{2} + \epsilon - \beta < 1.$$

In analogy with A and B, these estimates on Γ imply interval estimates for Δ.

For these hypotheses the model case is

$$\Gamma(x)=\frac{x+1}{\gamma}\log\frac{x+1}{\gamma}-\frac{x+1}{\gamma},\qquad \Gamma'(x)=\frac{1}{\gamma}\log\frac{x+1}{\gamma},\qquad \Gamma''(x)=\frac{1}{\gamma(x+1)},$$

and ϵ can be chosen as close as desired to $1/2$.

Set

$$c_i=\Gamma^{(i)}(x_s),\qquad i=0,1,2,\ldots$$
$$d_i=\Delta^{(i)}(x_s),\qquad i=0,1,2,\ldots$$
$$\sigma=x_s-[x_s].$$

Define the scaled parameters

$$C_3=c_3c_2^{-\frac{3}{2}},$$
$$E=d_1c_2^{-\frac{1}{2}}-\frac{1}{2}c_3C_2^{-\frac{3}{2}},$$
$$\Theta=\Theta(\theta,0)=\theta c_2^{-\frac{1}{2}},$$
$$\Theta(n)=\Theta(\theta,n)=(\theta+2\pi n)c_2^{-\frac{1}{2}}.$$

Note that the previous assumptions insure that $C_3,E=o(1)$.

We write $Y=Y(x_s)=O(\mathcal{F})$ if for some $c>0$, $Y=O\big(exp(-\Gamma''(x_s)^{-c})\big)$.

Set

$$\tau=2\epsilon-6\beta-\frac{1}{2}. \tag{51}$$

The assumptions on ε and β insure that $0<\tau<1/2$.

Theorem (2.2.12)[61]: Suppose Γ satisfies (39), (40), (41), (42), (43), and hence also (45), (46), and (47) and that $d\in SVII(c,\epsilon,\beta)$. Set $f(z)=\sum_0^\infty d(n)c(n)^{-1}z^n$. Then f is entire and, as $s\to\infty$, has the asymptotic growth

$$f\big(e^{s+i\theta}\big)=e^{\Gamma^*(s)}\frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x_s)}}d(x_s)e^{i\theta x_s}\left[e^{-\frac{1}{2}\theta^2}(1+i\Theta E+i\Theta^3C_3)+O\left(c_2^{\frac{1}{2}+\tau}\right)\right.$$
$$\left.+\,O(\mathcal{F})\right] \tag{52}$$

In particular, for $c_2^{\frac{1}{2}-\beta}<|\theta|<\pi$.

$$\big|f\big(e^{s+i\theta}\big)\big|=f(e^s)O\left(c_2^{\frac{1}{2}+\tau}\right). \tag{53}$$

As with the previous theorem, the formulation in (46) is redundant. The $O(\mathcal{F})$ error term, which is smaller, is intrinsic to the method; the other could be mechanically refined. Hence we present both.

We use the results of this theorem as input for Theorem (2.2.2). To do that, we require certain estimates on the derivatives of f on the axis. We present those estimates as a lemma now because it is convenient to include their proof along with the proof of Theorem (2.2.3).

Lemma (2.2.13)[61]: In the situation of Theorem (2.2.18), we have the following additional estimates:

$$z\frac{d}{dz}f(e^s)=x_sL_0+L_1, \tag{54}$$

$$\left(z\frac{d}{dz}\right)^2f(e^s)=x_s^2L_0+2x_sL_1+L_2, \tag{55}$$

where for $j = 0,1,2$

$$L_j = e^{\Gamma^*(x)\Delta(x_s)}\sqrt{\frac{2\pi}{c_2}}\left[\sqrt{\frac{c_2}{2\pi}}J_j + O\left(c_2^{\frac{1}{2}+\tau}\right) + O(\mathcal{F})\right] \quad (56)$$

with

$$J_0 = \sqrt{2\pi}\frac{1}{\sqrt{c_2}}, \qquad J_1 = \sqrt{2\pi}\frac{-c_3 + 2d_1c_3}{2(\sqrt{c_2})^5}, \qquad J_2 = \sqrt{2\pi}\frac{1}{(\sqrt{c_2})^3}. \quad (57)$$

Proof. The hypothesis (48) insures that $\lim(d(n)c(n)^{-1})^{\frac{1}{n}} = 0$. Thus f is entire.
We need to estimate

$$\left(z\frac{d}{dz}\right)^i f(z) = \sum_0^\infty n^i d(n)c(n)^{-1}z^n, \qquad i = 0,1,2.$$

We split the sum into a central part and tails. The tails will be estimated by the corresponding integrals, using analysis similar to that in the previous proof. In order to capture the cancellation when $\theta \neq 0$, we treat the central part differently.

The analysis of the tail terms is not essentially changed by the factors n^i; hence we present the estimates only for $i = 0$. We have

$$f(x) = \sum d(n)c(n)^{-1}x^n = \sum e^{n\log x - \Gamma(n) + \Delta(n)}.$$

Writing $z = e^s$, we have $f\left(e^{s+i\theta}\right) = \sum exp(ns - \Gamma(n) + \Delta(n) + in\theta)$. Fix s large. For this $s, xs - \Gamma(x)$ has its maximum at x_s. Set $u = n - x_s$. Thus $e^{in\theta} = e^{ix_s\theta}e^{iu\theta}$. For typographic convenience, we set

$$\Omega = -(\Gamma(x_s - u) - c_0 - c_1u) + \Delta(x_s + u) - d_0.$$

Bringing a factor of $e^{\Gamma^*(n)}e^{ix_s\theta}$ outside the sum, recalling that $c_1 = \Gamma'(x_s) = s$, and doing a bit of rearranging, we find

$$f\left(e^{s+i\theta}\right) = e^{\Gamma^*(s)+\Delta(x_s)+i\theta x_s}\sum_{-x_s}^{\infty} e^{\Omega + iu\theta}$$

We need to show

$$\sqrt{\frac{c_2}{2\pi}}\sum_{-x_s}^{\infty} e^{\Omega+iu\theta} = e^{-\frac{1}{2}\Theta^2}(1 + i\Theta E + i\Theta^3 C_3) + O\left(c_2^{\frac{1}{2}+\tau}\right) + O(\mathcal{F}). \quad (58)$$

We use $\lambda = \lambda(s)$ as given by (44) to split the range of summation into three parts, again L, C, and R. We start the analysis with L. We drop the unimodular factor and dominate the sum by the corresponding integral. That is,

$$\sqrt{\frac{c_2}{2\pi}}L = O(1)\sqrt{\frac{c_2}{2\pi}}\int_{x_s}^{-\lambda} e^{\Omega}du. \quad (59)$$

We now estimate the integrand. We have

$$\Gamma(x_s + u) - c_0 - c_1u = \int_u^0 (r - u)\Gamma''(x_s + r)dr.$$

By the monotonicity of Γ'' we see that if $u < -\lambda$ then

$$\Gamma(x_s + u) - c_0 - c_1u \geq \int_{-\lambda}^0 (r - u)\Gamma''(x_s + r)dr$$

$$\geq \Gamma''(x_s)\int_{-\lambda}^0 (r - u)dr = -\frac{1}{2}\Gamma''(x_s)\lambda(2u + \lambda).$$

Thus, we continue (53) with

$$\sqrt{\frac{c_2}{2\pi}}L = O(1)\sqrt{\frac{c_2}{2\pi}}e^{\frac{\lambda^2 c_2}{2}}\int_{-x^s}^{-\lambda} e^{c_2 u\lambda+\Delta(x_s+u)-d_0}du$$

$$= O(1)e^{\frac{\lambda^2 c_2}{2}}\int_{-\infty}^{-\lambda} e^{w\lambda+\left[\Delta\left(x_s+\frac{w}{\sqrt{c_2}}\right)-d_0\right]}dw$$

We need to estimate the integral. Using (43) to estimate $\left[\Delta\left(x_s+\frac{w}{\sqrt{c_2}}\right)-d_0\right]$ in the integral, we find that, for some positive K,

$$\sqrt{\frac{c_2}{2\pi}}\int_{-\infty}^{-\lambda\sqrt{c_2}} \ldots dw\, w = 0(1)e^{\frac{\lambda^2 c_2}{2}}\int_{-\infty}^{-\lambda\sqrt[3]{c_2}} e^{w\lambda\sqrt{c_2}}e^{-Kw}$$

$$= O(1)\frac{e^{\frac{\lambda^2 c_2}{2}}e^{(-\lambda^2 c_2+K\lambda\sqrt{c_2})}}{\lambda\sqrt{c_2}-K}$$

$$= O(1)\frac{e^{-\frac{\lambda^2 c_2}{2}+K\lambda\sqrt{c_2}}}{\lambda\sqrt{c_2}-K}$$

$$= O(1)\, O\left(exp\left(-\frac{\lambda^2 c_2}{2}+\lambda\sqrt{c_2}K\right)\right) = O(\mathcal{F}),$$

as required. We now look at R. If $u \geq \lambda$, then

$$\Gamma(\mathrm{x_s}+u)-c_0-c_1 u = \int_0^u (u-r)\Gamma''(x_s+r)dr$$

$$\geq \int_0^\lambda (u-r)\Gamma''(x_s+r)dr$$

$$\geq \Gamma''(x_s+\lambda)\int_0^\lambda (u-r)dr$$

$$= \frac{1}{2}\lambda\Gamma''(x_s+\lambda)(2u-\lambda)$$

Set $\tilde{c}_2 = \Gamma''(x_s+\lambda)$. Then

$$R = O(1)\int_\lambda^\infty e^{-\frac{1}{2}\lambda\tilde{c}_2(2u-\lambda)+\Delta(x_s+u)-d_0}du.$$

Thus

$$\sqrt{\frac{c_2}{2\pi}}R = O(1)\,\sqrt{c_2}\,e^{\frac{1}{2}\lambda^2\tilde{c}_2}\int_\lambda^\infty e^{-\lambda\tilde{c}_2 u+\Delta(x_s+u)-d_0}du.$$

Lemma (2.2.11) insures $\tilde{c}_2 \sim c_2$. The hypothesis (43) and the monotonicity of Γ'' insure $\Delta(x_s+u)-d_0 = O(1)c_2^{\frac{1}{2}}u$. Thus we need to estimate

$$I = O(1)\sqrt{c_2}\exp\left(\left(1+\frac{o(1)c_2\lambda^2}{2}\right)\right)\int_\lambda^\infty \left(-\lambda\big(1+\ o(1)\big)c_2 u\ +O(1)c_2^{\frac{1}{2}}u\right)du$$

$$= O(1)\sqrt{c_2}\exp\left(\left(1+\frac{o(1)c_2\lambda^2}{2}\right)\right)\int_\lambda^\infty \left(\left[-\big(1+\ o(1)\big)\lambda^2 c_2 + O(1)c_2^{\frac{1}{2}}\right]\frac{u}{\lambda}\right)du.$$

We know $\lambda c_2^{\frac{1}{2}} \to \infty$. Hence, for large s,

$$\left[-\big(1+o(1)\big)\lambda^2 c_2 + O(1)\lambda c_2^{\frac{1}{2}}\right]\frac{u}{\lambda} \le \left[-\frac{2}{3}\lambda^2 c_2\right]\frac{u}{\lambda} = \frac{2}{3}\lambda c_2 u.$$

Thus

$$I = O(1)\sqrt{c_2}\exp\left(\left(1+\frac{o(1)c_2\lambda^2}{2}\right)\right)\int_\lambda^\infty \exp(-\frac{2\lambda c_2 u}{3})\,du$$

$$= O(1)\sqrt{c_2}\exp\left(\frac{\big(1+\ o(1)\big)c_2\lambda^2}{2}\right)(c_2\lambda)^{-1}\ exp\left(-\frac{2\lambda^2 c_2}{3}\right)$$

$$= O(1)\frac{1}{\sqrt{c_2}\lambda^{-1}}exp\left(\left(\frac{1}{2}-\frac{2}{3}+\ o(1)\right)c_2\lambda^2\right) = o(\mathcal{F}),$$

as required.

We now need to estimate the center term,

$$\sum_C = e^{\Gamma^*(s)+\Delta(x_s)+i\theta x_s}\sum_{|[u]|<\lambda} e^{\Omega+iu\theta}.$$

By Taylor's theorem, we have, for $|u|<\lambda$,

$$\Omega = -\frac{1}{2}c_2u^2 - \frac{1}{6}c_3u^3 + d_1u - \frac{1}{24}\tilde{c}_4 + \frac{1}{2}\tilde{d}_2u^2.$$

Here $\tilde{c}_4 = \Gamma^{(4)}(w)$ and $\tilde{d}_2 = \Delta''(w')$, with $w, w'\Delta(x_s-\lambda, x_s+\lambda)$. Using (42), (43), (49), and (50), we find

$$|\tilde{c}_4u^4| + \left|\hat{d}_2u^2\right| = O\left(c_2^{\frac{1}{2}+\tau}\right),$$

$$(|c_3u^3| + |d_1u|)^2 = O\left(c_2^{\frac{1}{2}+\tau}\right).$$

We have

$$\sum_{|[u]|<\lambda} e^{\Omega+iu\theta} = \sum\left(1+d_1u-\frac{c_3u^3}{6}+O\left(c_2^{\frac{1}{2}+\tau}\right)\right)\exp\left(-\frac{c_2u^2}{2}+iu\theta\right)$$

$$= \sum\left(1+d_1u-\frac{c_3u^3}{6}\right)exp\left(-\frac{c_2u^2}{2}+iu\theta\right)$$

$$+\sum O\left(e_2^{\frac{1}{2}+\tau}\right)\exp\left(-\frac{c_2u^2}{2}+iu\theta\right)$$

$$= \sum_1 + \sum_2 . \tag{60}$$

We estimate $\sum_2$ by passing to absolute values, estimating the truncated Gaussian sum by the corresponding Gaussian integral over the entire line, and then evaluating the integral. This gives $\sum_2 = O\left(c_2^{\frac{1}{2}+\tau}\right)O\left(c_2^{-\frac{1}{2}}\right) = O(c_2^\tau)$, which is what we needed. (Recall from (58) that we pick up an additional factor of $O\left(c_2^{\frac{1}{2}}\right)$ outside of the sum $\sum_{[|u|]<\lambda}$.)

Write $u = k - \sigma$ with $k \in Z$ and $\sigma = x_s - [x_s]$. Now $\sum_1$ is a sum with the range $|k| < \lambda$. However, the natural estimates show that we change things only by $O(\mathcal{F})$ if we replace that with the sum over all integers. We do that and thus now need to estimate

$$\sum_{-\infty}^{\infty}\left(1 + d_1(k-\sigma) - \frac{c_3(k-\sigma)^3}{6}\right)\exp\left(-\frac{c_2(k-\sigma)^2}{2} + i(k-\sigma)^\theta\right).$$

By the Poisson summation formula ([62]), this equals $\sum_{-\infty}^{\infty} h(n)$, where

$$h(n) = \int_{-\infty}^{\infty}\left(1 + d_1(x-\sigma) - \frac{c_3(x-\sigma)^3}{6}\right)$$
$$\times\, exp\left(-\frac{c_2(x_2-\sigma)^2}{2} + i(x\ -\sigma)\theta\right)e^{2\pi inx}dx$$
$$=\ e^{2i\pi n\sigma}\int_{-\infty}^{\infty}\left(1 + d_1 y - \frac{c_3 y^3}{6}\right)exp\ \left(-\frac{c_2 y^2}{2} + (2\pi n\ +\theta)iy\right)dy.$$

Starting with the formula for $\int_{-\infty}^{\infty} e^{-ty^2+2xy}dy$ ([62],) and differentiating with respect to s, t, and then both, we find

$$\int_{-\infty}^{\infty} e^{-ty^2+2xy}dy = \frac{\sqrt{\pi}}{\sqrt{t}}e^{\frac{s^2}{t}}, \tag{61}$$

$$\int_{-\infty}^{\infty} ye^{ty^2+2sy}dy = \frac{s}{t}\frac{\sqrt{\pi}}{\sqrt{t}}e^{\frac{s^2}{t}}, \tag{62}$$

$$\int_{-\infty}^{\infty} y^2e^{-ty^2+2sy}dy = \left[\frac{1}{2t} + \left(\frac{s}{t}\right)^2\right]\frac{\sqrt{\pi}}{\sqrt{t}}e^{\frac{s^2}{t}}, \tag{63}$$

$$\int_{-\infty}^{\infty} y^3e^{-ty^2+2sy}dy = \left[\frac{3s}{2t^2} + \left(\frac{s}{t}\right)^3\right]\frac{\sqrt{\pi}}{\sqrt{t}}e^{\frac{s^2}{t}}. \tag{64}$$

For $n = 0$, direct computation gives

$$h(0) = e^{-\frac{\theta^2}{2c_2}}\left(1 + i\theta\left(\frac{d_1}{c_2} - \frac{1}{2}\left(\frac{c_3}{c_2^2}\right)\right) + i\frac{\theta^3 c_3}{c_2^3}\right)\frac{\sqrt{2\pi}}{\sqrt{c_2}}$$

or, in terms of the scaled parameters,

$$h(0) = e\hat{}(-\frac{1}{2}\Theta^2(1 + i\Theta E + i\Theta^3 C_3)\frac{\sqrt{2\pi}}{\sqrt{c_3}}.$$

In general,

$$h(n)\ e\hat{}(-\frac{1}{2}\Theta^2(1 + i\Theta(n)E + i\Theta(n)^3 C_3)\frac{\sqrt{2\pi}}{\sqrt{c_2}}e^{\pi in\sigma}.$$

In general,$\sum_{n\neq 0}|h(n)|$ is dominated by a geometric series which is dominated by $h(0)O(\mathcal{F})$. However, this fails to be uniform in θ; in fact, $\Theta(-\pi, 1) = O(\pi, 0)$. However, this is only an issue if $n = \pm 1$ and $e^{i\theta}$ is near the negative real axis. In that case, however, both terms are $O(e^{-c_2^{-1}} = O(\mathcal{F})$. Hence all the $h(n)$, for $n > 0$, can be absorbed into the various error terms. Finally, notice that when $c_2^{\frac{1}{2}-\beta} < |\theta|, h(0) = O(\mathcal{F})$. Hence the main term is the contribution associated to $\sum_2$ in (54), which we saw was $O\left(c_2^{\frac{1}{2}+\tau}\right)$. Thus (52) and (53) are done.

We now proceed to the proof of the lemma. For $j = 1, 2$ we want to estimate

$$\left(z\frac{d}{dz}\right)^j f(e^s) = e^{\Gamma^*(s)} d(x_s) \sum_{-x_s}^{\infty} (x_s+u)^j e^{\Omega}.$$

Straightforward manipulation shows that (54) and (55) hold with

$$L_j = e^{\Gamma^*(s)} d(x_s) d(x_s) \sum_{-x_s}^{\infty} u^j e^{\Omega}$$

for $j = 0, 1, 2$. We estimate L_1 and L_2 using the same type of analysis as in the proof of the theorem (which, in fact, treated L_0). That is, the tails contribute an error that is $O(\mathcal{F})$, and the central part of the sum is analyzed using Poisson summation. The situation here is slightly easier because we only want estimates on the positive real axis. Hence the terms in the Poisson summation corresponding to $n \neq 0$ contribute a total error which is $O(\mathcal{F})$. This gives, up to an error term of $O(e_2^{\tau}) + O(\mathcal{F}), L_j = J_j$ This gives us (56) with

$$J_j = \int_{-\infty}^{\infty} \left(1 + d_1 y - \frac{c_3 y^3}{6}\right) exp\left(-\frac{c_2 y^2}{2}\right) dy$$

for $j = 0, 1, 2$. Evaluating those integrals using (61)-(64) then produces the statements in the lemma.

We shall use the output of Theorem (2.2.2) as input for Theorem (2.2.12) and then use the output from Theorem (2.2.12) as input for Theorem (2.2.2). Here we collect the bookkeeping lemmas which show that the functions which arise in this process satisfy the required hypotheses.

First, suppose that we have a and b which satisfy the hypotheses of Theorem (2.2.2), that A and B are given by (18) and (19), and that $\{\Gamma(n)\}$ are the associated moments. We want to use $\{\gamma(n)^{-1}\}$ as power series coefficients in a way which keeps the focus on a as the primary term. We define c, d by $c(x) = exp(A^*(x)), d(x) = c(x)\gamma(x)^{-1}$ and define Γ and Δ by (37) and (38).

Lemma (2.2.14)[61]: Suppose a, ϵ, and α satisfy the hypotheses of Theorem (2.2.2) and $b \in SVI\left(a, \epsilon, \frac{\alpha}{3}\right)$. Then, with the same ϵ, with $\beta = \alpha$, and with $\lambda = \Gamma''^{-\frac{1}{2}-\beta}$, the data $\varepsilon, \beta, \Gamma$, and Δ satisfy the hypotheses of Theorem (2.2.2)1. That is, with the same ε, Γ satisfies (39), (40), (41), (42), and (43), and $d \in SVII(c, \varepsilon, \beta)$.

Proof. The statements about F follow from the hypotheses on A, the fact that $\Gamma = A^*$, and Lemma (2.2.1).

To see that $d \in SVII(c, \varepsilon, \beta)$, note that

$$\Delta = \log c - \log\gamma = \Gamma - \log\gamma = A^* - \log\gamma.$$

Hence, by (30), (31), and (32), A satisfies (49) and (50).

Suppose, now, that we had a, that $b = 1$, that we had a choice of α, and that we then invoked Theorem (2.2.2) with the choice $\alpha^* = \alpha/3$. Of course, for b a constant function, $b \in SVI(\alpha, \varepsilon, \alpha^*)$. Noting the previous lemma, we can then apply Theorem (2.2.12) to the functions c and d just described. That will produce an entire function f. Suppose we have a fixed $\sigma > -1$. We want to apply Theorem (2.2.2) to the functions a_σ and b_σ, selected so that $a_\sigma b_\sigma = af^{-\sigma}$. We set $a_\sigma = ae^{-\sigma A}, b_\sigma = e^{\sigma A} f^{-\sigma}$ and $A_\sigma(s) = -\log a_\sigma(e^z) - s, B_\sigma(s) = \log b_\sigma(e^*)$.

Lemma (2.2.15)[61]: Using a new smaller α, we can apply Theorem (2.2.2) to the functions a_σ and b_σ. That is, for a smaller α, A_σ satisfies (20), (21), (22), (25), and (26). Furthermore, $b_\sigma \in SVI(a_\sigma, \varepsilon, \alpha)$.

Proof. Our choice $c(x) = exp(A^*(x))$ in Theorem (2.2.12) gives $\Gamma = A^*$ and hence $\Gamma^* = A^{**} = A$. Thus

$$A_\sigma(s) = -\log a_\sigma(e^s) - s = -\log a(e^s) - s + \sigma A(s) = (1+\sigma)A(s),$$

and the conclusions for A_σ are immediate. We have $B_\sigma(s) = \ln b_\sigma(e^s) = \sigma(\Gamma^*(s) - \log f(e^s))$. Hence we need estimates for $(\Gamma^* - \log f)'$ and $(F^* - \log f)''$. Set $\mathrm{D} = zd/dz$. Direct computation yields

$$(\log f(e^s))'(e^s) = \frac{\mathrm{D}f(e^s)}{f(e^s)},$$

$$(\log f(e^s))''(e^s) = \frac{\mathrm{D}^2 f(e^s)}{f(e^s)} - \left(\frac{\mathrm{D}f(e^s)}{f(e^s)}\right)^2.$$

Recall that $\Gamma^{*'}(s) = x_s = A'(s)$ and, by Lemma (2.2.1), $\Gamma^{*''} = \Gamma''^{-1} = c_2^{-1}$. Thus, using (54) and (55), we have

$$\begin{aligned}(\Gamma^* - \log f)' &= \Gamma^{*'}(s) - \frac{\mathrm{D}f(e^s)}{f(e^s)} \\ &= x_s - \left(\frac{x_s L_0 + L_1}{L_0}\right)^2 \\ &= -\frac{L_1}{L_0}. \\ &= \frac{\sqrt{c_2/2\pi}\, J_1 + O\left(c_2^{1/2+\tau}\right)}{\sqrt{c_2/2\pi}\, J_0 + O\left(c_2^{1/2+\tau}\right)}.\end{aligned}$$

The last equality follows by using (56) and absorbing $O(\mathcal{F})$ into the other, larger, error term. Using the values of J_0 and J_1, we continue with

$$\begin{aligned}&(\Gamma^* - \log f)' \\ &= \frac{\dfrac{-c_3 + 2d_1c_2}{2c_2^2} + O\left(c_2^{\frac{1}{2}+\tau}\right)}{1 + O\left(c_2^{\frac{1}{2}+\tau}\right)} \\ &- \left(\frac{-c_3 + 2d_1c_2}{c_2^2} + O\left(c_2^{\frac{1}{2}+\tau}\right)\right)\left(1 + O\left(c_2^{\frac{1}{2}+\tau}\right)\right) \\ &= \left(O(c_3c_2^{-2}) + O(d_1c_2^{-1})O\left(c_2^{\frac{1}{2}+\tau}\right)\right)\left(1 + O\left(c_2^{\frac{1}{2}+\tau}\right)\right) \\ &= \left(O\left(c_2^{-\frac{1}{2}+\varepsilon}\right) + O\left(c_2^{-\frac{1}{2}+\varepsilon-\beta}\right) + O\left(c_2^{\frac{1}{2}+\tau}\right)\right)\left(1 + O\left(c_2^{\frac{1}{2}+\tau}\right)\right) \\ &= O(\left(c^{-\frac{1}{2}+\varepsilon-\beta}\right) = O\left(A''^{\frac{1}{2}-\varepsilon+\beta}\right).\end{aligned}$$

Using this estimate for L_1/L_0, we analyze the second derivative by

$$\begin{aligned}(\Gamma^* - \log f)'' &= \Gamma^{*''}(s) - \frac{\mathrm{D}^2 f(e^s)}{f(e^s)} + \left(\frac{\mathrm{D}f(e^s)}{f(e^s)}\right)^2 \\ &= \Gamma''^{-1}(x_s) - \left(\frac{x_s^2 L_0 + 2x_s L_1 + L_2}{L_0}\right) + \left(\frac{x_s L_0 + L_1}{L_0}\right)^2 = c_2^{-1} - \frac{L_2}{L_0} + \left(\frac{L_1}{L_0}\right)^2\end{aligned}$$

$$= c_2^{-1} - \frac{\sqrt{\frac{c_2}{2\pi}} J_2 + O\left(c_2^{\frac{1}{2}+\tau}\right)}{\sqrt{\frac{c_2}{2\pi}} J_0 + O\left(c_2^{\frac{1}{2}+\tau}\right)} + (O\left(A''^{\frac{1}{2}-\varepsilon+\beta}\right)^2$$

$$= c_2^{-1} - \frac{c_2^{-1} + O\left(c_2^{\frac{1}{2}+\tau}\right)}{1 + O\left(c_2^{\frac{1}{2}+\tau}\right)} + \left(O\left(A''^{\frac{1}{2}-\varepsilon+\beta}\right)\right)^2$$

$$= c_2^{-1} - c_2^{-1} + c_2^{-1} O\left(c_2^{\frac{1}{2}+\tau}\right) + \left(O\left(A''^{\frac{1}{2}-\varepsilon+\beta}\right)\right)^2 = O\left(c_2^{-\frac{1}{2}+\tau}\right) + \left(O\left(A''^{\frac{1}{2}-\varepsilon+\beta}\right)\right)^2$$

$$= O\left(A''^{\,1-2\varepsilon+2\beta}\right).$$

This gives the required estimates for B' and B''.

We suppose that m is given and fixed and that $A(s) = -\log m(e^s) - s$ satisfies the hypotheses of Theorem (2.2.2) for some selected ε, α. We use the notation of Theorem (2.2.2) and its proof and of Theorem (2.2.12) and its proof with the choice $\Gamma = A^*$. In particular, we denote the derivatives of A by σ's and of Γ by c's.

Many of our estimates will be in terms of the function A''. We would like to be able to relate those estimates both to the starting function m and to the function φ defined by $m(|z|^2) = exp\left(-2\varphi(z)\right)$, which is often used as a parameterization in this context. By straightforward calculation, we have

$$A''(\log x^2) = -\left(x\frac{d}{dx}\right)^2 (\log m)(x^2) = x^2(\Delta\varphi)(x).$$

Let H_m be the weighted Bergman space,

$$H_m = L^2\left(\mathbb{C}, m(r^2)\frac{\tau dr d\theta}{\pi}\right) \cap Hol.$$

For each $w \in \mathbb{C}$, there is a Bergman kernel function $k_w = k_{m,w}$ which is characterized as that element of H_m which satisfies $f(w) = \langle f, k_w\rangle$ for all f in H_m. Because the monomials are an orthogonal basis of $H_m, k_w(z) = \sum_{n=0}^{\infty} \|z^n\|^{-2}(\overline{w}z)^n$. Thus, setting $\gamma = \int_0^\infty x^n m(x)dx$ and $k_w(z) = K_m(z) = \sum_{n=0}^{\infty} \gamma_n^{-1} z^n$, we have $k_w(z) = K(\overline{w}z)$.We are interested in estimating kw and related objects. Our approach is to start with m, use Theorem (2.2.2) to estimate the γ's in terms of m, and then use those estimates in Theorem (2.2.12) to estimate K. There is no loss of generality in assuming that w is real and positive, and we make that assumption for the rest.

In describing various small quantities, we use the shorthand

$$S(x) = A''(\log x)^{-1}.$$

Here is our main estimate for the Bergman kernel.

Theorem (2.2.16)[61]: As $r \to \infty$, for $|\theta| \le S(wr)^{\frac{1}{2}-6\alpha}$,

$$k_w\left(re^{i\theta}\right) = e^{A(\log wr)} A''(\log wr) e^{i\theta A'(\log wr)} (e^{-\frac{A''(\log wr)\theta^2}{2}} + O\left(S(wr)^{\frac{1}{2}+\tau}\right) \quad (65)$$

and thus

$$k_w\left(re^{i\theta}\right) m(wr) \sim \frac{A''(\log wr)}{wr}\, e^{\frac{A''(\log wr)\theta^2}{2}} \quad (66)$$

On the diagonal,

$$k_w(w) = e^{A(2\log w)}A''(2\log w)\left(1 + O\left(S(w^2)^{\frac{1}{2}+\tau}\right)\right). \quad (67)$$

Far from the axis, $|e| > S(wr)^\gamma$ for (any)fixed $\gamma > O$,

$$k_w\left(re^{i\theta}\right) = k_w(r)O\left(S(w^2)^{\frac{1}{2}+\tau}\right). \quad (68)$$

Note. Recall from Theorem (2.2.12) that $\tau = 2\varepsilon - 6\beta - 1/2$ and $0 < \tau < 1/2$.

Proof. We apply Theorem (2.2.2) with the choice $A(s) = -\log m(e^s) - s\,, B = 0$. Let γ be the moment function we obtain. Lemma (2.2.14) insures that we can then use Theorem (2.2.12) with the choices $c = exp(A^*), d = exp(A^*)\gamma^{-1}$ (and thus $c^{-1}d = \gamma^{-1}$). Theorem (2.2.12) shows that on the positive axis

$$f(e^s)\sim e^{\Gamma^*(s)}\,\frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x_s)}}d(x_s)(1\ + O\left(S(wr)^{\frac{1}{2}+\tau}\right).$$

We have $\Gamma = A^*$ and hence $\Gamma^* = A^{**} = A$, the last by Lemma (2.2.1). We also know, from that lemma, that $\Gamma'' = A^{*''} = A''^{-1}$ and hence $c_2 = A''^{-1}$. Finally, $d = \sqrt{A''}exp(B)/\sqrt{2\pi}$. In this case, $B = 0$ and hence $f(e^s) \sim e^A A''$. From the definitions, we have $e^{A(\log t)} = \frac{1}{tm(t)}$. Recalling that $K = f$ gives (59). The other estimates follow by restricting to appropriate θ.

From this theorem, we get an asymptotic version of (A)

Corollary (2.2.17)[61]:

$$m(r^2)k_r(r)\sim\frac{A''(\log r^2)}{r^2}.$$

In particular, if $m(r)\sim ar^b e^{-cr^d}s(r)$, where $a,b,c,d > 0$ and $s \in SVI\left(ar^b e^{-cr^d},\varepsilon,\alpha\right)$ for ε,α allowed in Theorem (2.2.2), then

$$m(r^2)k_r(r) \sim cd^2r^{2d-2}.$$

Theorem (2.2.16) is not enough to give a version of (B). It shows that log $K_m = (-\log m)\big(1 + o(1)\big)$; but to get to a version of(B), we need to know that a similar estimate holds after we apply $\left(\frac{xd}{dx}\right)^2$ to each side. For that reason, we need to invoke (15) and Theorem (2.2.2) again.

Theorem (2.2.18)[61]: Fix $\sigma > 0$ and set $K_{\sigma w} = k_{mK_m^{-\sigma},w}$. As $r \to \infty$, for $|\theta| \leq\ S(r)^{\frac{1}{2}-6\alpha}$,

$$K_{\sigma,w}\left(re^{i\theta}\right)\sim(1+\sigma)K_{m,w}\left(re^{i\theta}\right)^{1+\sigma}.$$

for $\pi \geq |\theta| > S(r)^{\frac{1}{2}-6\alpha}$

$$K_{\sigma,w}\left(re^{i\theta}\right)\sim K_{\sigma,w}(r)O(S(r)^{\frac{1}{2}+\tau}.$$

Proof. We have $A(s) = -\log m(e^s) - s$. We want to apply Theorem (2.2.2) with the choices $a_\sigma = me^{-\sigma A}$ and $b_\sigma = e^{\sigma A}K_m(x)^{-\sigma}$. The associated function A_σ is

$$A_\sigma(s) = -\log a_\sigma\,(e^s) - s = -\log m\,(e^s) - s - \sigma A(s) = (1+a)A(s).$$

We saw in the proof of Theorem (2.2.16) that $K_m(e^s)\sim e^A A''b = e^A A''$. The same argument applied to a_σ and b_σ gives

$$\begin{aligned}K_{\sigma,w}(x)&\sim e^{A_\sigma}A''_\sigma b_\sigma\\ &= e^{(1+\sigma)A}(1+\sigma)A''b_\sigma\\ &= e(1+\sigma)A\ (1+\sigma)A''\big(e^A K_{m,w}(x)^{-1}\big)^\sigma\end{aligned}$$

$$\sim e^{(1+\sigma)A}(1+\sigma)A''(A'')^{\sigma}$$
$$= (1+\sigma)(e^{A}A'')^{1+\sigma}$$
$$\sim (1+\sigma)K_{m,w}(x)^{1+\sigma}.$$
For small θ, the proof of Theorem (2.2.12) goes through with $\Theta(n)^2$ increased by a factor of $1+\Theta$. For large θ, the argument in that proof gives the required estimates.
Rather than integrate these estimates to get an asymptotic version of (C), we do a slightly more general computation in the following.

The Berezin transform B_m is a valuable tool for studying Toeplitz operators on H_m. For a smooth function $F, B_m(F)$ is defined by
$$B_mF(w) = \langle F\frac{k_w}{\|k_w\|}, \frac{k_w}{\|k_w\|}\rangle = \int\int_C F(z)\frac{|K(\overline{w}z)|^2}{K(|w|^2)}m(|z|^2)\frac{dxdy}{\pi} \quad (69)$$
If we look at the Fock spaces, $m_\sigma(|z|^2) = exp(-(1+\sigma)|z|^2)$, then we have
$$B_{m_\sigma}F(w) = \int\int_C F(z)e^{-(1+\sigma)|z-w|^2}\frac{dxdy}{\pi} = F(w) + \frac{1}{4}\frac{1}{(1+\sigma)}\Delta F(w) + O\left(\frac{1}{(1+\sigma)^2}\right).$$
We would like analogues of these formulas for our more general weights. The general theory of reproducing kernels insures that the Berezin measure
$$d\mu = \frac{|K(\overline{w}z)|^2}{K(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}$$
is always a probability measure. We now want to study d_μ using our asymptotic estimates on the kernel function. First, however, we introduce a further restriction on a, which we formulate in terms of the auxiliary function A of (22). We require $A''(s)$ to be dominated by $exp(s^2)$ in a controlled way. Suppose, therefore, that there exists constants $C > 0, a_0 > 0$ such that for $\alpha > \alpha_0$ and $t > A''(\alpha_0)^{\frac{1}{2}}$ we have
$$\log\frac{A''(\alpha+t)}{A''(\alpha)} < -Ct^2. \quad (70)$$
For context, note that in the model case $A(t) = e^{\beta t} - t$ the left-hand side equals β. For smooth functions F defined on $\mathbb{C}$, set $\|F\| = \sum_{n\le 3}\sup|\nabla^n F|$. In addition to rectangular coordinates on $\mathbb{C}$ we will use coordinates (s,θ) where $w = e^W$ and $z = e^{w+s+i\theta}$ and also use the scaled coordinates (S,Θ) where $S = \sqrt{A''(2w)s}$ and $\Theta = \sqrt{A''(2w)\theta}$. We continue to work of the previous. In particular, we still have the hypotheses and conclusions of Theorem (2.2.16) and Theorem (2.2.18).

Theorem (2.2.19)[61]: In addition to the hypotheses of the previous, suppose that (70) holds. Given F with $\|F\| < \infty$, we have
$$B_mF(w) = \int\int_C F(z)e^{-(S^2+\Theta^2)}\frac{dSd\Theta}{\pi} + O(1)\|F\|S(w^2)^{\epsilon-\alpha}.$$
Proof. We start from (69). We first estimate the integral over the unit disk. On $\mathbb{D}$ we can bound F by $\|F\|$ and $|K(wz)|$ by $K(w)$. Thus we have
$$\left|\int\int_{\mathbb{D}} F(z)\frac{|K(wz)|^2}{K(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}\right| \le C_m\|F\|\frac{K(w)^2}{K(w^2)}.$$
To show that this can be absorbed into the error term, we need to control $K(w)^2/K(w^2)$ for large w. Recall that $\omega = \log w$. By Theorem (2.2.16), it is enough to show that
$$\frac{e^{2A(\omega)}A''(\omega)^2}{e^{A(2\omega)}A''(2\omega)} = O(A''(2\omega)^{-1}).$$
Hence it suffices to show that

$$j(\omega) = 2A(\omega) - A(2\omega) + 2\log A''(\omega)$$
is bounded above. We compute
$$j'(\omega) = 2A'(\omega) - 2A'(2\omega) + \frac{2A''(\omega)}{A''(\omega)}$$
and use the intermediate value theorem on the first pair of terms and the hypothesis (20) on the third. It follows that for some $\widetilde{\omega} \in (\omega, 2\omega)$
$$j'(\omega) \leq -2\omega A''(\overline{\omega}) + 2A''(\omega)^{\frac{1}{2}}.$$
Recalling that A'' is monotone increasing and unbounded, we see that j' is negative for all large ω, which gives what we need.

We now pass to coordinates (s, θ), where $\omega = e^{\omega}$ and $z = e^{\omega+s+i\theta}$ and so $dx\,dy = e^{2(\omega+s)}\,dsd\theta$. By definition, $m(|z|^2) = m\left(e^{2(\omega+s)}\right) = e^{-2(\omega+s)}e^{-A(2\omega+2s)}$. Hence $\pi^{-1}m(|z|^2)dxdy = \pi^{-1}e^{-A(z\omega+2s)}\,ds\,d\theta$.
Set $R = \left\{(s,\theta): |\theta| < O\left(S(wr)^{\frac{1}{2}-6\alpha}\right)\right\}$. In R we use the asymptotic estimates for K given in Theorem (2.2.16). This lets us estimate the integrand by
$$F(z)\frac{\left|e^{A(2\omega+s)}A''(2\omega+s)e^{-\frac{A''(2\omega+s)\theta^2}{2}}\left(1+O\left(S(wr)^{\frac{1}{2}+r}\right)\right)\right|^2}{e^{A(2\omega)}A''(2\omega)\left(1+O\left(S(wr)^{\frac{1}{2}+r}\right)\right)}e^{-A(2\omega+2s)}.$$
Note that $\frac{\left(1+O\left(S(wr)^{\frac{1}{2}+r}\right)\right)^2}{1+O\left(S(wr)^{\frac{1}{2}+r}\right)} = -\left(1+O\left(S(wr)^{\frac{1}{2}+r}\right)\right)$. We shall see that our approximations to the Berezin measure converge to a probability measure; in the course of that analysis, it will be clear that the norms of the approximations are uniformly bounded. Hence the error made by dropping the factors $\left(1+O\left(S(wr)^{\frac{1}{2}+r}\right)\right)$ in the integral can be safely absorbed into the error term. Thus, in R, the integrand can be estimated by
$$F(z)e^{2A(2\omega+s)-A(2\omega)-A(2\omega+2s)}\frac{A''(2\omega+s)^2}{A''(2\omega)}e^{A''(2\omega+s)\theta^2}. \qquad (71)$$
We have estimated the integral over the unit disk, i.e., $s < -\omega$. We now consider the region where $-\omega < s < -\delta$. Set
$$h(s) = 2A(2\delta+s) - A(2\omega) - A(2\omega+2s)$$
and put $\delta = \delta(2\omega)$. We dominate $F(z)$ by $\|F\|$ and first do the integral in θ. Near the axis, we use the Gaussian estimate of (71). Integrating that gives $\left(\sqrt{\pi}+o(1)\right)A''(2\omega+s)^{-\frac{1}{2}}$. Using the estimate in (68) away from the axis, we get a further contribution of $o\left(A''(2\omega+s)^{-\frac{1}{2}}\right)$. Thus, integrating in θ contributes a factor of $O(1)A''(2\omega+s)^{-\frac{1}{2}}$. Hence we must estimate
$$\int_{-\omega}^{-\delta} e^{h(s)}\frac{A''(2\omega+s)^2}{A''(2\omega)A''(2\omega+s)^{\frac{1}{2}}}ds.$$
Now A'' is increasing; hence the fraction in the integrand is at most 1. We need to estimate $\int_{-\omega}^{-\delta} e^{h(s)}ds$. We have $h'(s) = 2A'(2\omega+s) - 2A'(2\omega+2s)$. Since A' is increasing and s is negative, h' is positive and thus h is increasing. Thus the integral is dominated by $we^{h(-\delta)}$.

To estimate $h(-\delta)$, we compute $h''(s) = 2A''(2\omega + s) - 4A''(2\omega + 2s)$ and take note of (18). We find that, on $(-\delta, 0), h''(s) = -2A''(2\omega)\big(1 + o(1)\big)$. Noting that $h(0) = h'(0) = 0$ and integrating twice gives $h(-\delta) = -A''(2\omega)\delta^2\big(1 + o(1)\big)$. Recalling that $A''^{\frac{1}{2}}\delta = A''^k$ for some $k > 0$, we conclude that the integral is dominated by any negative power of A'', a better estimate than needed.

Now we consider the integral over the region where $s > \delta$. Note that $h(0) = h'(0) = 0$ and $h''(0) = -2A''(2\omega)$. Hence, by Taylor's theorem,

$$h(s) = -A''(2\omega)s^2 + \frac{1}{6}h'''(s^*)s^3,$$

with s^* between 0 and s. Again we dominate $F(z)$ by $\|F\|$ and first do the integral in θ, making the same estimates as in the previous case. We are reduced to estimating

$$\int_{\delta}^{\infty} e^{-A''(2\omega)s^2 + \frac{1}{6}h'''(s^*)s^3} \frac{A''(2\omega + s)^2}{A''(2\omega)A''(2\omega + s)^{\frac{1}{2}}} dS.$$

We make the change of variables $s = S/\sqrt{A''(2\omega)}$ and introduce the shorthand φ. We then need to estimate

$$\int_{\delta\sqrt{A''(2\omega)}}^{\infty} e^{-S^2 + \varphi s^3} \frac{A''\left(2\omega + SA''^{-\frac{1}{2}}(2\omega)\right)^{3/2}}{A''(2\omega)^{\frac{3}{2}}} dS.$$

In the region of integration, s is positive and hence s^* is positive. We compute $h''(t) = 2A'''(2\omega + t) - 8A'''(2\omega + 2t)$. Recalling that A''' is positive and increasing, we conclude that φ is negative. Hence we make the integral larger by dropping φS^3. We thus need to estimate

$$\int_{\delta\sqrt{A''(2\omega)}}^{\infty} e^{-S^2} \frac{A''\left(2\omega + SA''^{-\frac{1}{2}}(2\omega)\right)^{3/2}}{A''(2\omega)^{\frac{3}{2}}} dS$$

The estimate (70) insures that the fraction in the integral is dominated by $exp(C'S^2A''^{-1}(2\omega))$. This insures that the integral is $O(e^\wedge - A''^{\theta}$ for some $\theta > 0$, which is more than we need.

Now we look at the range $|s| < \delta$ First we consider the part of that region outside of R. Using (68), we see that, for fixed s, the integration in θ (outside of R) yields an integrand of the form

$$O(1)\|F\|\, e^{-A''(2\omega)s^2 + \frac{1}{6}h'''(s^*)s^3}\, A''(2\omega + s)^2/A''(2\omega)^{\frac{3}{2}+r}$$

In $|s| < \delta$ the hypotheses on A insure that the quotient in this expression is $O\left(A''(2\omega)^{\frac{1}{2}-\tau}\right)$ and that $h'''(s^*)s^3$ is $O(1)$. Thus we must estimate the integral of $O(1)\|F\|e^{-A''(2\omega)(1+\sigma(1))s^2}A''(2\omega)^{\frac{1}{2}-\tau}$. Doing the s integration gives $O(1)\, \|F\|\, A''(2\omega)^{-\tau}$, which is an acceptable error term.

What remains is the main contribution, the integral over the region where s and θ are both small. In that region, we first note that the hypotheses on A and standard Taylor estimates insure that

$$e^{-A''(2\omega)s^2 + \frac{h'''(s^*)s^3}{6}} e^{-A''(2\omega + s)\theta^2} = e^{A''(2\omega)(s^2+\theta^2)}\big(1 + O(S(w^2)^{\varepsilon})\big).$$

Hence, making the change of variable $(S,\Theta)=\left(\sqrt{A''(2\omega)}s,\sqrt{A''(2\omega)}\theta\right)$, we obtain, up to a term which can be safely absorbed into the error term,

$$\int_{|\Theta|<A''(2\omega)^{6\omega}}\int_{|S|<\delta A''(2\omega)^{\frac{1}{2}}}F(z)e^{-(S^2+\Theta^2)}\frac{A''(2\omega+s)^2}{A''(2\omega)^2}\frac{dSd\Theta}{\pi}.$$

Using the Taylor expansion of $A''(2\omega+s)$ about $s=0$, (23), and (25), we see that $A''(2\omega+s)^2/A''(2\omega)^2=1+O\big(A''(2\omega)\big)^{\alpha-\varepsilon}$. It remains only to note that the passage from $\int_{|\Theta|<A''(2\omega)^{6\omega}}\int_{|S|<\delta A''(2\omega)^{\frac{1}{2}}}$ to $\int\int_C$ introduces an error which, in the notation of Theorem (2.2.12), is $O(\mathcal{F})$.

This estimate gives our asymptotic version of (C):

Corollary (2.2.20)[61]: As $w\to\infty$,

$$\int\int_C\left|K_{K_m^{-1}m}(\overline{w}z)\right|m(|z|^2)\frac{dxdy}{\pi}\sim 2K_m(|w|^2). \tag{72}$$

Proof. We use the notation of Theorem (2.2.18), that is, $K_0=K_m$ and $K_1=K_{K_m^{-1}m}$. We want to estimate

$$I=\int\int_C\frac{|K_1(\overline{w}z)|}{K_0(|wz|^2)}m(|z|^2)\frac{dxdy}{\pi}.$$

The same arguments as those in the proof of Theorem (2.2.18) insure that

$$I\sim\int\int_R\frac{|K_1(\overline{w}z)|}{K_0(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}$$

Where

$$R=\left\{(s,\theta):|s|<\delta,|\theta|<\frac{S(2w+r)^{\frac{1}{2}-6\alpha}}{10}\right\}.$$

We rewrite this as

$$I\sim\int\int_R\frac{|K_1(\overline{w}z)|}{|K_0(\overline{w}z)|^2}\frac{|K_0(\overline{w}z)|^2}{K_0(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}\sim\int\int_R F_w(z)\frac{|K_0(\overline{w}z)|^2}{K_0(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}$$

where $F_w=|K_1(\overline{w}z)|/|K_0(\overline{w}z)|^2$ on R. Theorem (2.2.18) insures that $F_w\sim 2$ on R. There is no problem extending F_w to the entire plane with $\|F_w\|$ bounded independently of w. We now apply the previous theorem with $F=F_w$ and find that

$$I=\int\int_C F_w(z)e^{-(s^2+\Theta^2)}\frac{dSd\Theta}{\pi}+o(1).$$

Recalling that $F_w\sim 2$, we obtain $I\sim 2$, which is the desired conclusion.

Our results are estimates in a fixed Bergman space which are asymptotic as $|z|\to\infty$. However, instead of a fixed density m, we could look at the family of densities $m_\sigma=K_m^{-1-\sigma}$ and investigate the asymptotic behavior of the kernel function and Berezin transform for fixed z and as $\sigma\to\infty$. Such questions are of interest in quantization, with $(1+\sigma)^{-1}$ playing the role of Planck's constant. See [67] and [68] for instances of such estimates as well as further discussion. Here we discuss briefly the type of results that could perhaps be obtained by the methods, and why we have not yet obtained them. First, consider Theorem (2.2.18). We have $K_\sigma(re^{i\theta})\sim(1+\sigma)K_m(re^{i\theta})^{1+\sigma}$. There may be a more refined result such as

$$K_\sigma=(1+\sigma)K_m^{1+\sigma}+(\text{something})+\frac{1}{(\sigma+1)}(\text{something})+O\left(\frac{1}{\sigma^2}\right).$$

However, the proof which we give fails to produce such a result. That proof gives

$$K_\sigma(1+\sigma)K_m^{1+\sigma}\left(1+O\left(\frac{1}{A''\beta}\right)\right)^\sigma$$

for some positive β. This is fine for fixed σ and large r, but not for fixed r and large σ. The fact that the right-hand side involves a factor $(1+small)^\sigma$ seems to be intrinsic to the structure of our proof.

It also seems plausible that more is true in Theorem (2.2.19). We can estimate the Gaussian integral by writing F near $z=2\omega$ as a Taylor polynomial of degree 2 in the variables s and θ. The integral of the Taylor remainder gives a contribution smaller than the error term. The polynomial-times-Gaussian can be integrated explicitly, and we obtain

$$B_mF(w)=F(w)+\frac{w^2}{4A''(2\omega)}\Delta F(w)+O(1)\|F\|A''(2\omega)^{\alpha-\varepsilon}.$$

However, this presentation is misleading. We do not know that the third term on the right is smaller than the middle one. The difficulty is not in the estimation of the Gaussian integral, which produces an error that is $O(A''^{\,-2})$. The problem is the error terms on the estimates which led to the Gaussian integral. If it were known that the error terms resulting from that analysis were $O(A''^{-2})$, then we would in fact have

$$B_mF(w)=F(w)+\frac{w^2}{4A''(2\omega)}\Delta F(w)+O(1)\|F\|A''(2\omega)^{\alpha-\varepsilon}.$$

We can carry the speculation a step further. If, instead of fixed m, we now look at the family of densities $m_\sigma=K_m^{-1-\sigma}$ and write B_σ for the corresponding Berezin transforms, we would have

$$B_\sigma F(w)\sim F(w)+\frac{w^2}{4A''_\sigma(2\omega)}\Delta F(w)+O(A''^{-2}_\omega)\|F\|.$$

Now recall from the proof of Theorem (2.2.18) that $A_\sigma=(1+\sigma)A_0$. We could next regard $m, F,$ and w as fixed and let σ grow. That would give, as $\sigma\to\infty$

$$B_\sigma F(w)\sim F(w)+\frac{1}{1+\sigma}\frac{w^2}{4A''_0(2\omega)}\Delta F(w)+O\left(\frac{1}{\sigma^2}\right).$$

Estimates such as this, even with an error term $\sigma\left(\frac{1}{\sigma}\right)$, would be sufficient to give a correspondence principle for Berezin quantization schemes; see the Introduction of [67].

It may be that the methods here can be developed to obtain such estimates for large σ. However, it appears that doing this by direct estimation would be quite awkward. Hence we defer further analysis in the hope of finding a more effective way to organize the ideas.

Section (2.3): Hankel Operators

We consider the Fock type space $A^2(\mu_m)$ consisting of those holomorphic functions which are square integrable with respect to the measure $d\mu_m(z)=e^{-|z|^m}dV(z)$, where $dV(z)$ is the Lebesgue measure on $\mathbb{C}^n$ and $m>0$ is a positive parameter.

When $m=2$ the space $A^2(\mu_2)$ is the Fock space, called also the Segal-Bargmann space. Let I be the identity operator and P_2 is the orthogonal projection from $L^2(\mu_2)$ onto $A^2(\mu_2)$. Let $T(\mathbb{C}^n)$ be the subspace of $L^2(\mu_2)$ consisting of those functions f that satisfy $f(\cdot+a)\in L^2(\mu_2)$ for all $a\in\mathbb{C}^n$.We recall that if $f\in T(\mathbb{C}^n)$, then the Hankel operator H_f with symbol f is defined by

$$H_f(\varphi)=(I-P_2)(f\varphi),$$

for all φ in the dense subspace of $A^2(\mu_2)$ spanned by $\{K_2(\cdot,a), a\in\mathbb{C}^n\}$ where $K_2(z,a):=e^{\langle z,a\rangle}, z,a\in\mathbb{C}^n$, is the Bergman kernel. In this case, the study of compactness of Hankel operators with bounded symbols was considered in the works of Berger and Coburn [134] and Stroethoff [142]. In the more general case $f \in T(\mathbb{C}^n)$, the simultaneous membership of H_f and $H_{\bar{f}}$ to the Schatten classes was characterized by Xia and Zheng [143] and by Bauer [130] in the Hilbert-Schmidt setting. A necessary and sufficient condition for simultaneous boundedness of H_f and $H_{\bar{f}}$ was given recently by Bauer [131]. The tools used in these works use heavily the translation action of the group $\mathbb{C}^n$ and related properties to the Bergman kernel.

We also mention that in the one dimensional case $n = 1$ the study of Hankel operators in the setting $m > 0$ was considered by Schneider $[Sc]$ when the symbolis a monomial. His method is direct and relies on an approximation process.

We consider the general case $m > 0$. We begin by clarifying the appropriate definition of densely defined Hankel operators. Indeed, if $f \in L^2(\mu_m)$ is a function of polynomial growth, then the Hankel operator H_f with symbol f is defined by $H_f(\varphi) = (I - P_m)(f\varphi)$, where P_m is the orthogonal projection from $L^2(\mu_m)$ $onto$ $A^2(\mu_m)$ given by

$$P_m(g)(z) := \int_{\mathbb{C}^n} K_m(z,w)g(w)d\mu_m(w), for\ g \in L^2(\mu_m)$$

where K_m is the Bergman kernel given. We shall show that the righthand side of the latter equality is well-defined for functions g of the form $g = f\varphi$ for all $f \in L^2(\mu_m)$ and φ in the space P of holomorphic polynomials.This allows us to extend the definition of P_m on such functions and, using this, we see that H_f is defined on holomorphic polynomials. In particular, it is densely defined.

We first point out that the techniques used the case $m = 2$ to study Hankel operators do not apply to the case $m \neq 2$. Our goal herein is to develop new methods which are adequate to the setting $m > 0$ in the case of anti-analytic symbols f.

The first main result is the following

We observe that when m is odd, then all bounded Hankel operators with anti-analytic symbols are also compact. This is not the case for m even.

In the particular case $m = 2$, Theorem (2.3.22) was established in a recent work by [131] using a technique which does not work at all when $m \neq 2$.

We recall that an operator T is in the Schatten class $S_p(A^2(\mu_m), L^2(\mu_m))$ if $(T * T)^{\frac{p}{2}}$is in the trace class of $A^2(\mu_m)$. Our second result characterizes such a class of operators.

We discovered recently that a weaker version of our results was established by Knirsch and Schneider [89] in the one dimensional particular case.

We finally mention in passing that Hankel operators with antiholomorphic symbols are intimately related to the $\bar{\partial}$-canonical solution operator (see [85], [86] and [87]).

We recall some facts about Hankel operators with respect to certain rotation invariant measures, see [181]. Let Ω be a rotation invariant open set in $\mathbb{C}^n$ and let μ be a rotation invariant measure on Ω. We suppose that μ has moments of every order; that is,

$$m_k = \int_{\Omega} |z|^{2k} d\mu(z) < +\infty, for\ all\ k \in \mathbb{N}_0.$$

We consider the Hilbert space $L^2(\Omega,\mu)$ of square integrable complex-valued functions on Ω with respect to the measure μ and $A^2(\Omega,\mu)$ its subspace consisting of holomorphic elements. We assume that for each set compact $K \subset \Omega$ there exists $C = C(K) > 0$ such that

$$\sup_{z\in K}|f(z)| \leq C\ \|f\|_{L^2(\Omega,\mu)}$$

for all $f \in A^2(\Omega,\mu)$. Thus $A^2(\Omega,\mu)$ is a closed space of $L^2(\Omega,\mu)$. The corresponding orthogonal projection P_μ will be called the Bergman projection. We also assume that the subspace consisting of all holomorphic polynomials is dense in $A^2(\Omega,\mu)$.Therefore, if $f \in A^2(\Omega,\mu)$ has polynomial growth, then the Hankel operator $H_{\bar f}$ given by

$$H_{\bar f}(\varphi) = (I - P_\mu)(\bar f\varphi)$$

is well defined for all holomorphic polynomials φ In particular, $H_{\bar f}$ is densely defined.

Let $\mathbb{N}_0^n$ denote the set of all n-tuples with components in the set $\mathbb{N}_0$ of all nonnegative integers. If $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}_0^n$, welet $|\alpha| := \alpha_1 + \cdots + \alpha_n$ denote the length of α. If $\beta = (\beta_1,\cdots,\beta_n)\mathbb{N}_0^n$ satisfies $\alpha_j \geq \beta_j$ for all $j = 1,\cdots,n$, then we write $\alpha \geq \beta$. Otherwise, set $\alpha \ngeq \beta$.

The space of polynomials P is endowed with the Fischer inner product [182] $\langle\, ,\rangle_F$, defined on the monomials by

$$\langle z^\alpha, z^\beta\rangle_F = \begin{cases} \alpha!\ if\ \alpha = \beta \\ 0\ if\ \alpha \neq \beta. \end{cases}$$

Finally, if A and B are two quantities, we use the symbol $A \approx B$ whenever $A \leq C_1 B$ and $B \leq C_2 A$, where C_1 and C_2 are positive constants independent of the varying parameters.

We shall express the operators $H_{\bar z^k}$ and $H^*_{\bar z^k} H_{\bar z^l}$ on holomorphic homogeneous polynomials.

Lemma (2.3.1)[84]: Suppose that $\beta, k \in \mathbb{N}_0^n$ and $d \in \mathbb{N}_0$. Then

$$\left(H_{\bar z^k} f\right)(\xi) = \bar\xi^k f(\xi) - \frac{m_d}{m_d - |k|}\frac{\Gamma(n+d-|k|)}{\Gamma(n+d)}\frac{\partial^{|k|}}{\partial\xi^k} f(\xi)$$

for all holomorphic polynomials f of degree d. In particular, if $f = \xi^\alpha$, then

$$\left(H_{\bar z^k} f\right)(\xi) = \begin{cases} \bar\xi^k \xi^\alpha - \dfrac{m_{|\alpha|}}{m_{|\alpha|-|k|}}\dfrac{\Gamma(n+|\alpha|-|k|)}{\Gamma(n+|\alpha|)}\dfrac{\alpha!}{(\alpha-k)!}\,\xi^{\alpha-k}, if\ \alpha \geq k \\ \bar\xi^k\xi^\alpha\ otherwise. \end{cases}$$

Proof. It suffices to prove the lemma for $f(\xi) = \xi^\alpha$, where $\alpha \in \mathbb{N}_0^n$. Let g be a homogeneous polynomial in P. If g is a monomial of the form $g(\xi) = \xi^\beta$, where $\beta \in \mathbb{N}_0^n$, then using the properties of P_μ, we see that

$$\langle P_\mu(\bar z^k f), g\rangle_{L^2(\Omega,\mu)} = \langle f, z^k g\rangle_{L_2(\Omega,\mu)}$$

and hence $\langle P_\mu(\bar z^k f), g\rangle_{L^2(\Omega,\mu)} = 0$ as long as $\alpha \neq k + \beta$. Now let $\alpha = k + \beta$.By Lemma (2.3.1) in [181], we have the following identities

$$\int_\Omega z^\alpha \bar z^\alpha d\mu(z) = \frac{(n-1)!\,m_{|\alpha|}\alpha!}{(n+|\alpha|-1)!} \quad and\ \langle z^\alpha, z^\alpha\rangle_F = \alpha!\,, \tag{72}$$

fromwhich we obtain

$$\langle P_\mu(\bar z^k f), g\rangle_{L^2(\Omega,\mu)} = \frac{(n-1)!\,m_{|\alpha|}}{(n+|\alpha|-1)!}\langle f, z^k g\rangle_F.$$

Since the multiplication operator and the corresponding differentiation operator are adjoint to each other with respect to the Fischer inner product, this implies that

$$\langle P_\mu(\bar z^k f), g\rangle_{L^2(\Omega,\mu)} = \frac{m_{|\beta|+|k|}}{m_{|\beta|}}\frac{(n-1+|\beta|)!}{(n-1+|\beta|+|k|)!}\langle\frac{\partial^{|k|}}{\partial z^k} f, g\rangle_{L^2(\Omega,\mu)}$$

for all holomorphic homogeneous polynomials g of degree $|\beta|$. Therefore, if f is a holomorphic homogeneous polynomial of degree d, we have

$$P_\mu(\bar{z}^k f) = \frac{m_d}{m_{d-|k|/}} \frac{(n-1+d-|k|)!}{(n-1+d)!} \frac{\partial^{|k|}}{\partial z^k} f.$$

This completes the proof of the lemma.

Lemma (2.3.2)[84]: The domain $\mathrm{Dom}(H^*_{\bar{z}^k})$ of $H^*_{\bar{z}^k}$ contains all polynomials in w and $\bar{w}$.

Proof. It suffices to show that, if α and β are fixed in $\mathbb{N}_0^n$, then the linear functional

$$g \mapsto \langle H_{\bar{z}^k}(g), z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)}$$

is bounded on $A^2(\Omega,\mu)$. To do so, choose an integer $d \geq |\alpha| + |\beta| + 2|k|$ and consider the subspace N_d of $A^2(\Omega,\mu)$ consisting of polynomials with degree smaller than or equal to d.We denote by π_d the orthogonal projection from $A^2(\Omega,\mu)$ onto N_d. If $g \in P$, then $(I-\pi_d)g$ is a sum of holomorphic homogeneous polynomials in P with degree at least $d+1$. In view of Lemma (2.3.1), we can write

$$H_{\bar{z}^k} \circ (I-\pi_d)g = \bar{z}^k f + h$$

where f is a sum of holomorphic homogeneous polynomials of degree at least $d+1$ and h is a sum of holomorphic homogeneous polynomials of degree at least $d+1-|k|$. Therefore,

$$\begin{aligned}\langle H_{\bar{z}^k} \circ (I-\pi_d)g, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)} &= \langle \bar{z}^k f, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)} + \langle h, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)} \\ &= \langle f z^\beta, z^{\alpha+k} \rangle_{L^2(\Omega,\mu)} + \langle z^\beta h, z^\alpha \rangle_{L^2(\Omega,\mu)}.\end{aligned}$$

Since $d+1+|\beta| \geq 1+|\alpha|+2|\beta|+2|k| > |\alpha|+|k|$, it follows that $\langle f z^\beta, z^{\alpha+k} \rangle_{L^2(\Omega,\mu)} = 0$. Also, due to the fact that the degree of $z^\beta f$ is greater than $|\alpha|$ we see that $\langle z^\beta f, z^\alpha \rangle_{L^2(\Omega,\mu)} = 0$. Thus $\langle H_{\bar{z}^k} \circ (I-\pi_d)g, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)} = 0$ for all $g \in P$ and consequently

$$\langle H_{\bar{z}^k} g, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)} = \langle H_{\bar{z}^k} \circ (I-\pi_d)g, z^\alpha \bar{z}^\beta \rangle_{L^2(\Omega,\mu)}.$$

The lemma now follows from the fact that$H_{\bar{z}^k} \circ \pi_d$ is of finite rank and hence bounded.

We observe by Lemmas (2.3.1) and (2.3.2) that P is contained in the domain of the operator $H^*_{\bar{v}} H_{\bar{u}}$ for all holomorphic polynomials u and v.

Lemma (2.3.3)[84]: Suppose that u, v and f are holomorphic polynomials. Then

$$H^*_{\bar{v}} H_{\bar{u}} f = P_\mu(v\bar{u}f) - vP_\mu(\bar{u}f).$$

Proof. A little computing shows that for all $g \in A^2(\Omega,\mu)$

$$\begin{aligned}\langle H_{\bar{u}} f, H_{\bar{v}} g \rangle_{L^2(\Omega,\mu)} &= \langle \bar{u}f - P_\mu(\bar{u}f), \bar{v}g - P_\mu(\bar{v}g) \rangle_{L^2(\Omega,\mu)} \\ &= \langle v\bar{u}f, g \rangle_{L^2(\Omega,\mu)} - \langle P_\mu(\bar{u}f), \bar{v}g \rangle_{L^2(\Omega,\mu)} \\ &\quad + \langle (P_\mu - I)(\bar{u}f), P_\mu(\bar{v}g) \rangle_{L^2(\Omega,\mu)} \\ &= \langle v\bar{u}f, g \rangle_{L^2(\Omega,\mu)} - \langle P_\mu(\bar{u}f), \bar{v}g \rangle_{L^2(\Omega,\mu)}\end{aligned}$$

where the latter equality holds since $P_\mu(\bar{v}g) \in A^2(\Omega,\mu)$ and $(P_\mu - I)(\bar{u}f)$ is orthogonal to $A^2(\Omega,\mu)$. This completes the proof.

Lemma (2.3.4)[84]: Assume that k and l are elements of $\mathbb{N}_0^n$. If f is a holomorphic homogeneous polynomial of degree d, then

$$P_\mu(z^l \bar{z}^k f) = \frac{m_{d+|l|}}{m_{d-|k|+|l|}} \frac{\Gamma(d+n-|k|+|l|)}{\Gamma(d+n+|l|)} \frac{\partial^{|k|}}{\partial z^k} f.$$

Proof. It is sufficient to establish the lemma for monomials $f(z) = z^\alpha$. If β is an arbitrary element of $\mathbb{N}_0^n$, then, due to the properties of the Fischer product and (72), we have

$$\begin{aligned}\langle P_\mu(z^l \bar{z}^k f), z^\beta \rangle_{L^2(\Omega,\mu)} &= \langle z^{l+\alpha}, z^{k+\beta} \rangle_{L^2(\Omega,\mu)} \\ &= \frac{(n-1)!\, m_{|\alpha|+|l|}}{(n+|\alpha|+|l|-1)!} \langle \frac{\partial^{|k|}}{\partial z^k}(z^{l+\alpha}), z^\beta \rangle_F\end{aligned}$$

$$= \frac{m_{|\alpha|+|l|}}{m_{|\alpha|+|l|-|k|}} \frac{\Gamma(n+|\alpha|+|l|-|k|)}{\Gamma(n+|\alpha|+|l|)} \langle \frac{\partial^{|k|}}{\partial z^k}(z^l f), z^\beta \rangle_{L^2(\Omega,\mu)}$$

This completes the proof.

In what follows we shall compute c for a holomorphic homogeneous polynomial f.

Lemma (2.3.5)[84]: Suppose that k and l are in $\mathbb{N}_0^n$. If f is a holomorphic homogeneous polynomial of degree d, then

$$H_{\bar{z}^l}^* H_{\bar{z}^k} f = \frac{m_{d+|l|}}{m_{d+|l|-|k|}} \frac{\Gamma(n+d+|l|-|k|)}{\Gamma(n+d+|l|)} \frac{\partial^{|k|}}{\partial z^k}(z^l f) - \frac{m_d}{m_{d-|k|}} \frac{\Gamma(d+n-|k|)}{\Gamma(d+n)} z^l \frac{\partial^{|k|}}{\partial z^k} f$$

In particular, $H_{\bar{z}^l}^* H_{\bar{z}^k} f$ is a holomorphic homogeneous polynomial of degree $d+|l|-|k|$.

Proof. Follows from Lemmas (2.3.1) and (2.3.4).

An immediate consequence of Lemma (2.3.5) gives the following

Proposition (2.3.6)[84]: For each α in $\mathbb{N}_0^n$, the monomial z^α is an eigenvector for the operator $H_{\bar{z}^k}^* H_{\bar{z}^k}$ and the corresponding eigenvalue λ_α is given by

$$\lambda_\alpha = \frac{m_{|\alpha|+|k|}}{m_{|\alpha|}} \frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+|k|)} \frac{(\alpha+k)!}{\alpha!} - \frac{m_{|\alpha|}}{m_{|\alpha|-|k|}} \frac{\Gamma(|\alpha|+n-|k|)}{\Gamma(|\alpha|+n)} \frac{\alpha!}{(\alpha-k)!}$$

if $\alpha \geq k$ and

$$\lambda_\alpha = \frac{m_{|\alpha|+|k|}}{m_{|\alpha|}} \frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+|k|)} \frac{(\alpha+k)!}{\alpha!},$$

otherwise.

We consider the Fock space $A^2(\mu_m)$, for $m > 0$. In this case, the moments of the measure $d\mu_m(z) := e^{-|z|^m} dv(z)$ are given by

$$m_s = \int_{\mathbb{C}^n} |z|^{2s} e^{-|z|^m} dv(z) = \frac{1}{m} \Gamma(\frac{2s+2n}{m}). \quad (73)$$

If k is a multi-index we set $T = H_{\bar{z}^k}^* H_{\bar{z}^k}$. Then T is defined on the dense subspace P of $A^2(\mu_m)$. For each multi-index α, the eigenvalue λ_α of T corresponding to the eigenvector ξ^α is given by Proposition (2.3.6). In what follows we shall study the asymptotic of these eigenvalues. We distinguish the two cases $m = 2$ and $m \neq 2$.

Lemma (2.3.7)[84]: Suppose $m = 2$. Then for each $j = 1, \cdots, n$, the operator $H_{\bar{z}_j}$ is bounded but not compact on $A^2(\mu_m)$. If $|k| \geq 2$, $H_{\bar{z}^k}$ is unbounded on $A^2(\mu_m)$.

Proof. In this case, μ_2 is the Gaussian measure on $\mathbb{C}^n$. Its moments reduce to $m_s = \Gamma(s+n)$. Moreover, if $\alpha \in \mathbb{N}_0^n$,

$$\lambda_\alpha = \begin{cases} \frac{(\alpha+k)!}{\alpha!} - \frac{\alpha!}{(\alpha-k)!} & if\ \alpha \geq k, \\ \frac{(\alpha+k)!}{\alpha!} & if\ \alpha \ngeq k. \end{cases}$$

We first observe that if $|k| = 1$, then the eigenvalues of T are all equal to 1. Therefore, T is bounded but not compact on $A^2(\mu_m)$. This proves the first part ofthe lemma.

Suppose now that $|k| \geq 2$. Choose j_0 in $[1,n]$ so that $k_{j_0} = \max_j k_j$. If d is a nonnegative integer, set $\alpha(k,d) = (k_1, \cdots, k_{j_0-1}, k_{j_0} + d, k_{j_0+1}, \cdots, k_n)$. Then

$$\lambda_{\alpha(k,d)} = \left(\prod_{j\neq j_0}^{n} \frac{(2k_j)!}{(k_j)!}\right) [(d+k_{j_0}+1)\cdots(d+2k_{j_0}] - (d\ +\ 1)\ \cdots\ (d+k_{j_0})$$

Therefore, $\lim_{d\to+\infty} \lambda_{\alpha(k,d)} = +\infty$, showing that T is unbounded on $A^2(\mu_m)$. Thisimplies that $H_{\bar{z}^k}$ is also unbounded.

Henceforth, we assume that $m \neq 2, m > 0$. From Proposition (2.3.6) and (73) we see that if $\alpha \in \mathbb{N}_0^n$, then the eigenvalue λ_α can be written in the form

$$\lambda_\alpha = \begin{cases} A_{|\alpha|}\dfrac{(\alpha\ +\ k)!}{\alpha!} -\ B_{|\alpha|}\dfrac{\alpha!}{(\alpha\ -k)!} if\ \alpha\ \geq\ k \\ A_{|\alpha|}\dfrac{(\alpha\ +\ k)!}{\alpha!}\ if\ \alpha\ \ngeq\ k \end{cases} \quad (74)$$

where, for a nonnegative integer d,

$$\begin{cases} A_d := \dfrac{\Gamma\left(\dfrac{2d+\ 2n}{m} + \dfrac{2|k|}{m}\right)}{\Gamma\left(\dfrac{2d+\ 2n}{m}\right)} \dfrac{\Gamma(d\ +n)}{\Gamma(d+n+|k|)}, \\ B_d := \dfrac{\Gamma\left(\dfrac{2d+\ 2n}{m}\right)}{\Gamma\left(\dfrac{2d+\ 2n}{m} - \dfrac{2|K|}{m}\right)} \dfrac{\Gamma(d+n-|k|)}{\Gamma(d\ +n)}\ . \end{cases} \quad (75)$$

The asymptotic behaviour of the eigenvalues $\{\lambda_\alpha\}$ when $|\alpha| = d \mapsto +\infty$ is given by the following

Lemma (2.3.8)[84]: The sequences (A_d) and (B_d) given by (75) have the asymptotic behavior

$$A_d = \left(\frac{2}{m}\right)^{\frac{2|k|}{m}} (d+n)^{|k|\left(\frac{2}{m}-1\right)} \left[1 - \frac{|k|^2(m-2)}{2m(d+n)} + O\left(\frac{1}{(d+n)^2}\right)\right]$$

$$B_d = \left(\frac{2}{m}\right)^{\frac{2|k|}{m}} (d+n)^{|k|\left(\frac{2}{m}-1\right)} \left[1 + \frac{|k|^2(m-2)}{2m(d+n)} + O\left(\frac{1}{(d+n)^2}\right)\right]$$

$as\ d \mapsto +\infty$.

Proof. Follows from the property of the Gamma function [159]

$$\frac{\Gamma(x+y)}{\Gamma(x+z)} = x^{y-z}\left(1 + \frac{(y-z)(y\ +\ z-1)}{2x} + O(\frac{1}{x^2})\right) as\ x\ \mapsto +\infty, \quad (76)$$

where y and z are real numbers.

Lemma (2.3.9)[84]: The eigenvalues λ_α have the form

$$\lambda_\alpha = \left(\frac{2}{m}\right)^{2\frac{|k|}{m}} (d+n)^{2\frac{|k|}{m}-1} \left(f_n\left(\frac{\alpha_1+1}{d\ +n}, \cdots, \frac{\alpha_n+1}{d+n}\right) + \varepsilon(\alpha)\right),$$

where $\varepsilon(\alpha) = O\left(\frac{1}{d}\right)$and

$$f_n(t_1, \cdots, t_n) := -(m-2)\frac{|k|^2}{m} t^k + \sum_{j=1}^{n} k_j^2 \frac{t^k}{t_j}$$

when $\alpha \geq k$ and $d = |\alpha| \mapsto +\infty$.

Proof. We recall by (74) that if $\alpha \geq k$, then

$$\lambda_\alpha = A_{|\alpha|}\frac{(\alpha+k)!}{\alpha!} - B_{|\alpha|}\frac{\alpha!}{(\alpha-k)!},$$

where (A_d) and (B_d) are given by (75). On the other hand, by (76) we see that

$$\frac{(\alpha_j+k_j)!}{\alpha_j!} = (1+\alpha_j)^{k_j} + k_j(k_j-1)(1+\alpha_j)^{k_j-1} + q_j(1+\alpha_j)$$

$$\frac{\alpha_j!}{(\alpha_j-k_j)!} = (1+\alpha_j)^{k_j} - k_j(k_j+1)(1+\alpha_j)^{k_j-1} + r_j(1+\alpha_j)$$

where q_j and r_j are one variable polynomials of degree at most k_j-2. This implies that

$$\frac{(\alpha+k)!}{\alpha!} = \prod_{j=1}^{n}(1+\alpha_j)^{k_j} + \sum_{j=1}^{n} k_j(k_j-1)(1+\alpha_j)^{k_j-1}\prod_{l\neq j}(1+\alpha_l)^{k_l} + q(\alpha)$$

$$\frac{\alpha!}{(\alpha-k)!} = \prod_{j=1}^{n}(1+\alpha_j)^{k_j} - \sum_{j=1}^{n} k_j(k_j+1)(1+\alpha_j)^{k_j-1}\prod_{l\neq j}(1+\alpha_l)^{k_l} + r(\alpha)$$

where q and r are polynomials of degree at most $|k|-2$. These equalities, combined with Lemma (2.3.8), give the lemma.

Lemma (2.3.10)[84]: The eigenvalues λ_α have the estimate

$$\lambda_\alpha = \left(O(d+n)^{2\frac{|k|}{m}-k_{j0}}\right),$$

as long as $\alpha_{j0} < k_{j0}$ and $d = |\alpha| \mapsto +\infty$.

Proof. Let $j_0 = 1,\ldots,n$ and suppose that $k_{j0} \geq 1$. We recall by (74) that if $\alpha_{j0} < k_{j0}$, then

$$\lambda_\alpha = \frac{A_{|\alpha|}(\alpha+k)!}{\alpha!},$$

where (A_d) is as before. Set $\alpha' = (\alpha_1,\cdots,\alpha_{j_0-1},0,\alpha_{j_0+1},\cdots,\alpha_n)$ and $k' = (k_1,\cdots,k_{j_0-1}0,k_{j_0+1},\cdots,k_n)$. Arguing as in the proof of Lemma (2.3.9) we have

$$\frac{(\alpha+k)!}{\alpha!} \leq (2k_{j_0})!\frac{(\alpha'+k')!}{\alpha'!}$$

$$= (2k_{j_0})!\left[\prod_{j=1,j\neq j_0}^{n}(1+\alpha_j)^{k_j} + \sum_{j=1,j\neq j_0}^{n} k_j(k_j-1)(1+\alpha_j)^{k_j-1}\prod_{s\neq j,l}(1+\alpha_s)^{k_s}\right] + q(\alpha'),$$

where q is a polynomial of degree at most $|k'|-2$. These estimates, combined with Lemma (2.3.8), give the lemma.

Theorem (2.3.11)[84]: The operator $H_{\bar{z}^k} * H_{\bar{z}^k}$ is bounded if and only if $2\frac{|k|}{m} - 1 \leq 0$ and compact if and only if $2\frac{|k|}{m} - 1 < 0$.

Proof. Let Σ_n be the simplex consisting of those $t = (t_1,\cdots,t_n) \in \mathbb{R}^n$ such $t_j \geq 0$ and $t_1 + \cdots + t_n = 1$. By Lemmas (2.3.9) and (2.3.10) we see that

$$\sup_{|\alpha|=d} |\lambda_\alpha \approx (d+n)^{2\frac{|k|}{m}-1} \sup_{|\alpha|=d}\left|f_n\left(\frac{\alpha_1+1}{d+n},\cdots,\frac{\alpha_n+1}{d+n}\right)\right|$$

$$\approx (d+n)^{2\frac{|k|}{m}-1} \sup_{t\in\Sigma_n} |f_n(t)|$$

as $d=|\alpha|\to+\infty$. Now the lemma follows since the operator $H_{\bar{z}^k}*H_{\bar{z}^k}$ is boundedif and only if the sequence $\sup_{|\alpha|=d}|\lambda_\alpha|$ is bounded and $H_{\bar{z}^k}*H_{\bar{z}^k}$ is compact ifand only if the sequence $\sup_{|\alpha|=d}|\lambda_\alpha|$ tends to 0 as $d=|\alpha|\to+\infty$. _

Theorem (2.3.12)[84]: Let $k\in\mathbb{N}_0^n$ and m be a positive real number.

(i) The Hankel operator $H_{\bar{z}^k}$ is bounded on the Fock space $A^2(\mu_m)$ if and only if $m\geq 2|k|$.

(ii) The Hankel operator $H_{\bar{z}^k}$ is compact on the Fock space $A^2(\mu_m)$ if and only if $m>2|k|$.

Proof. We use that the operator $H_{\bar{z}^k}$is bounded if and only if $T=H_{\bar{z}^k}*H_{\bar{z}^k}$ is bounded and $H_{\bar{z}^k}$ is compact if and only if $T=H_{\bar{z}^k}*H_{\bar{z}^k}$ is compact.

Next, assume that $2\frac{|k|}{m}-1<0$ and let $p>0$. We shall investigate the membership of the operator T to a Schatten class S_p . Recall that T is in S_p if and only if the series$\sum\lambda_\alpha^p$ is convergent.

Let d be an integer. We shall estimate the *sum* $s_d=\sum_{|\alpha|=d}\lambda_\alpha^p$, when $d\to+\infty$. The calculations above lead to study the cases $\alpha\geq k$ and its opposite case separately. Let $B_d:=\{\alpha\in\mathbb{N}_0^n,|\alpha|=d\}$. We partition $B_d=B'd\cup B''_d$, where$B'_d=\{\alpha\in B:\alpha\geq k\}$ and $B''_d=B_d\setminus B'_d$. Thus s_d can be written in the form $s_d=s'_d+s''_d$, where $s_d=\sum_{\alpha\in B'_d}\lambda_\alpha^p$ and $s''_d=\sum_{\alpha\in B''_d}\lambda_\alpha^p$.

We need to compare the cardinalities $\#B_d,\#B'_d, and\ \#B''_d$ of these sets.

Lemma (2.3.13)[84]: We have the estimates $\#B_d\approx\#B'_d\approx\frac{d^{n-1}}{(n-1)!}$and $\#B''_d\approx d^{n-2}$ as $d\to+\infty$.

Proof. Let $P_{n,d}$ the space of n variables holomorphic polynomials of degree d. We have $\#B_d=\dim P_{n,d}=\binom{n-1+d}{d}=\frac{(d+n-1)!}{(n-1)!d!}$. Therefore, $\#B_d\sim\frac{1}{(n-1)!}d^{n-1}$as $d\to+\infty$.

On the other hand, for $j=1,\cdots,n$, let $B'_{d,j}=\{\alpha\in B_d,\alpha_j<kj\}$. Since $B'_d=\cup_{1\leq j\leq n}B'_{d,j}$, we see that $\#B'_d\leq\sum_{j=1}^n\#B'_{d,j}$.

If $k_j\geq1$, then

$$B'_{d,j}=\cup_{l=0}^{k_j-1}\{\alpha=(\alpha_1,\cdots,\alpha_{j-1},l,\alpha_{j+1},\cdots,\alpha_n),|\alpha|=d\}$$
$$=\cup_{l=0}^{k_j-1}\{\alpha=(\alpha_1,\cdots,\alpha_{j-1},l,\alpha_{j+1},\cdots,\alpha_n)\sum_{i\neq j}\alpha_i=d-l\}.$$

Therefore, $B'_{d,j}=\sum_{l=0}^{k_j-1}dim\ P_{n-1,d-l}$, and when $d\to+\infty$, $\#B'_{d,j}\sim k_j\frac{d^{n-2}}{(n-2)!}$.

This shows that, when $B'_d\approx d^{n-2}$ as $d\to+\infty$. The lemma now follows from theobservation $\#B_d=\#B'_d+\#B''_d$.

Lemma (2.3.14)[84]: Suppose that $n\geq2$ and g is a continuous function on $\mathbb{R}^n-1$. Consider the open set $\Omega:=\{(t_1,\cdots,t_{n-1}\in\mathbb{R}_+^{n-1},\sum_{j=1}^{n-1}t_j<1\}$. For a multi-index$\gamma=(\gamma_1,\cdots,\gamma_{n-1}$ in $\mathbb{N}_0^{n-1}$, set

$$c_{\gamma,d}:=\left(\frac{\gamma_1+1}{d},\cdots,\frac{\gamma_{n-1}+1}{d}\right)$$
$$\mathbb{J}_d:=\left\{\gamma\in\mathbb{N}_0^{n-1}:\prod_{j=1}^{n-1}\frac{\gamma_j}{d},\frac{\gamma_j+1}{d}\subset\Omega\right\}.$$

Then $\lim_{d\to+\infty}\frac{1}{d^{n-1}}\sum_{\gamma\in\mathbb{J}_d} g(c_{\gamma,d}) = \int_{\Omega} g(t)dt$.

Proof. For $d\in\mathbb{N}_0$, let $\Omega_d = \cup_{\gamma\in\mathbb{J}_d}\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d},\frac{\gamma_j+1}{d}\right]$. It is clear that $\Omega_d\subset\Omega$.Next, we show that $\lim_{d\to+\infty}\chi_{\Omega_d} = \chi_{\Omega}$. If s is a real number, let $[s]$ denote the largest integer smaller than or equal to s. If $t=(t_1,\cdots,t_{n-1})\in\Omega$ and$1\le j\le n-1$, then$\frac{[dt_j]}{d}\le t_j<\frac{[dt_j]}{d}+\frac{1}{d}$. Therefore,

$$\sum_{j=1}^{n-1}\frac{[dt_j]}{d}\le n\sum_{j=1}^{n-1}t_j<\sum_{j=1}^{n-1}\frac{[dt_j]}{d}+\frac{n-1}{d}.$$

Since$\sum_{j=1}^{n-1}t_j<1$, there is an integer d_0 such that for all $d>d_0$ we have $\sum_{j=1}^{n-1}\frac{[dt_j]}{d}+\frac{n-1}{d}<1$. Thus, $t\in\prod_{j=1}^{n-1}\left[\frac{[dt_j]}{d},\frac{[dt_j+1]}{d}\right]$and hence $t\in\Omega d$for all $d>d_0$. Thus $\lim_{d\to+\infty}\chi_{\Omega_d}=\chi_{\Omega}$. Therefore,

$$\frac{1}{d^{(n-1)}}\sum_{\gamma\in\mathbb{J}_d} g(c_\gamma,d)-\int_{\Omega} g(t)dt=\sum_{\gamma\in\mathbb{J}_d}\left[\frac{1}{d^{n-1}}\,g(c_\gamma,d)-\int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d},\frac{\gamma_j+1}{d}\right]} g(t)dt\right]+\int_{\Omega_d} g(t)dt-\int_{\Omega} g(t)dt$$

Since g is a bounded continuous function on the compact set Ω, we have, by Lebesgue's theorem, $\lim_{d\to+\infty}\int_{\Omega_d} g(t)dt = \int_{\Omega} g(t)dt$. On the other hand, by continuityof g on the compact set Ω, we see that

$$\sum_{\gamma\in\mathbb{J}_d}\left[\frac{1}{d^{n-1}}g(c_\gamma,d)-\int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d},\frac{\gamma_j+1}{d}\right]} g(t)dt\right]=\sum_{\gamma\in\mathbb{J}_d}\int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d},\frac{\gamma_j+1}{d}\right]}[g(c_\gamma,d)-g(t)dt]$$

also tends to 0 as $d\to+\infty$. This shows that

$$\lim_{d\to+\infty}\frac{1}{d^{n-1}}\sum_{\gamma\in\mathbb{J}_d} g(c_\gamma,d)=\int_{\Omega} g(t)dt.$$

The above result enables us to estimate s_d when $d=|\alpha|\to+\infty$.

Lemma (2.3.15)[84]: If $p\ge 1$, then

$$s_d\approx d^{n-1}d^p\left(2\frac{|k|}{m}-1\right).$$

Proof. Recall that $s_d=\sum_{\alpha\in B_d'}\lambda_\alpha^p$. By Lemma (2.3.9), we know that the sequence $\{\lambda_\alpha\}_\alpha\in B_d'$ has the following expansion when $d\to+\infty$

$$\lambda_\alpha=\left(\frac{2}{m}\right)^{2\frac{|k|}{m}}(d+n)^{2\frac{|k|}{m}-1}\left(f_n\left(\frac{\alpha_1+1}{d+n},\cdots,\frac{\alpha_n+1}{d+n}\right)+\varepsilon(\alpha)\right),$$

where $\varepsilon(\alpha)=O\left(\frac{1}{d}\right)$and

$$f_n(t_1,\cdots,t_n):=-(m-2)\frac{|k|^2}{m}t^k+\sum_{j=1}^{n}k_j^2\frac{t^k}{t_j}$$

Using the properties of the function $x\to x^p$, we see that there exists a constant $M>0$, such that

$$\left|\left|f_n\left(\frac{\alpha_1+1}{d+n},\cdots,\frac{\alpha_n+1}{d+n}\right)+\varepsilon(\alpha)\right|^p-\left|f_n\left(\frac{\alpha_1+1}{d+n},\cdots,\frac{\alpha_n+1}{d+n}\right)\right|^p\right|\le\frac{M}{d}.$$

Therefore,

$$\lambda_\alpha \approx \left(\frac{2}{m}\right)^{2\frac{|k|}{m}} (d + n)^{2\frac{|k|}{m}-1} \left(f_n\left(\frac{\alpha_1+1}{d + n}, \cdots, \frac{\alpha_n+1}{d + n}\right)\right),$$

as $d = |\alpha| \to +\infty$. Applying Lemmas (2.3.13) we see that

$$s_d \approx \left(\frac{2}{m}\right)^{2p\frac{|k|}{m}} d^{p(2\frac{|k|}{m}-1)} \sum_{\alpha\in B_d} \left|f_n\left(\frac{\alpha_1+1}{d + n}, \cdots, \frac{\alpha_n+1}{d + n}\right)\right|^p$$

$$\approx \left(\frac{2}{m}\right)^{2p\frac{|k|}{m}} d^{n-1} d^{p(2\frac{|k|}{m}-1)} \int_\Omega |f_n(t)|^p dt$$

so that the lemma follows from Lemma (2.3.14).

We recall that an operator T is in the Schatten class $S_p(A^2(\mu_m), L^2(\mu_m))$ if $(T * T)^{\frac{p}{2}}$is in the trace class of $A^2(\mu_m)$. Our second result characterizes such aclass of operators.

Theorem (2.3.16)[84]: Let $k \in \mathbb{N}_0^n$ and m be a positive real number. Then the Hankel operator $H_{\bar{z}^k}$ is in the Schatten class $S_p(A^2(\mu_m), L^2(\mu_m))$if and only if $p > 2n$ and $m(p - 2n) > 2p|k|$.

Proof. We use that the operator $H_{\bar{z}^k}$ is in$S_p(A^2(\mu_m), L^2(\mu_m))$if and only if $T = H_{\bar{z}^k} * H_{\bar{z}^k}$ is $S_{\frac{p}{2}}(A^2(\mu_m))$. Therefore, the theorem follows from Lemma (2.3.15).

We first study the behavior of the Bergman kernel $K_m(z,w)$ corresponding to$A^2(\mu_m)$. Let $E_{\frac{2}{m},\frac{2n}{m}}$be the generalized Mittag-Leffler's function. This is the entirefunction defined by

$$E_{\frac{2}{m},\frac{2n}{m}}(\lambda) := \sum_{d=0}^{+\infty} \frac{\lambda^d}{\Gamma\left(\frac{2d}{m}+\frac{2n}{m}\right)}, \lambda \in \mathbb{C}.$$

We shall express the Bergman kernel in terms of this function. Namely,

Lemma (2.3.17)[84]: The Bergman kernel $K_m(z,w)$ of $A^2(\mu_m)$ is given by

$K_m(z,w) = \frac{m}{(n-1)!} E_{\frac{2}{m},\frac{2n}{m}}^{n-1}(\langle z,w\rangle)$,

where $E_{\frac{2}{m},\frac{2n}{m}}^{n-1}$is the derivatives of $E_{\frac{2}{m},\frac{2n}{m}}$ with order $n-1$.

Proof. The monomials $z^\alpha, \alpha \in \mathbb{N}_0^n$, form an orthogonal basis of $A^2(\mu_m)$.Since

$$\|z^\alpha\|_{L^2(\mu_m)}^2 = \frac{(n-1)!}{m} \frac{\alpha!}{(|\alpha|+n-1)!} \Gamma\left(\frac{2|\alpha|+2n}{m}\right)$$

it follows that the Bergman kernel is

$$K_m(z,w) = \sum_{\alpha\in\mathbb{N}_0^n} \frac{z^\alpha}{\|z^\alpha\|_{L^2(\mu_m)}} \frac{w^\alpha}{\|w^\alpha\|_{L^2(\mu_m)}}$$

$$= \frac{m}{(n-1)!} \sum_{d=0}^{+\infty} \frac{(d+n-1)!}{d!\,\Gamma\left(\frac{2d}{m}+\frac{2n}{m}\right)} (\langle z,w\rangle)^d$$

$$= \frac{m}{(n-1)!} E_{\frac{2}{m},\frac{2n}{m}}^{n-1}(\langle z,w\rangle\&) .$$

This completes the proof of the lemma.

The Bergman projection P_m is given by

$$P_m(f)(z) := \int_{\mathbb{C}^n} K_m(z,w)f(w)d\mu_m(w), \text{ for } f \in L^2(\mu_m). \quad (77)$$

This definition can be extended to functions of the form fg where $f \in L^2(\mu_m)$ and $g \in P$. Indeed,

Lemma (2.3.18)[84]: If $g \in P$ and $z \in \mathbb{C}^n$, then $gK_m(z,\cdot)$ is in $L^2(\mu_m)$.

Proof. It follows from Theorem 2, p. 6 in [90] that the generalized Mittag-Leffler's function is $E_{\frac{2}{m},\frac{2n}{m}}$is an entire function of finite order $\frac{m}{2}$ andtype 1. Therefore$E^{n-1}_{\frac{2}{m},\frac{2n}{m}}$ is also an entire function of finite order $\frac{m}{2}$ and type 1 and hence for any$\varrho > 0$, there is a positive constant C that

$$\left|E^{n-1}_{\frac{2}{m},\frac{2n}{m}}(\lambda)\right| \le Ce^{|\lambda|^{\frac{m+\varrho}{2}}}, \lambda \in \mathbb{C}.$$

This shows that for all $z, w \in \mathbb{C}^n$,

$$|K_m(z,w)| \le Ce^{|\langle z,w\rangle|^{\frac{m+\varrho}{2}}} \le Ce(|z||w|)^{\frac{m+\varrho}{2}},$$

showing that for all $g \in P$ and z fixed in $\mathbb{C}^n$, the function $w \mapsto g(w)K_m(z,w)$ is in $L^2(\mu_m)$ as long as $0 < \varrho < m$.

It follows from Lemma (2.3.18) that if $f \in L^2(\mu_m)$, then the Hankel operator $H_{\bar{f}}$ with symbol$\bar{f}$ is well-defined on P by

$$H_f(g)(z) := \int_{\mathbb{C}^n} \big(f(z) - f(w)\big)K_m(z,w)g(w)d\mu_m(w), g \in P .$$

We point out that the measurable function $z \mapsto H_{\bar{f}}(g)(z)$ is not necessarly an element of $L^2(\mu_m)$.

Denote by M the subspace of those functions $f \in A^2(\mu_m)$ such that $H_{\bar{f}}(g) \in L^2(\mu_m)$ for all $g \in P$, and the densely defined operator $H_{\bar{f}}$ is bounded on $A^2(\mu_m)$. We equip M with seminorm

$$\|f\| := \|H_{\bar{f}}\| + |f(0)|.$$

The subspace of M consisting of functions f such that $H_{\bar{f}}$ is a compact operator will be denoted by M_∞. Then is not hard to see that M_∞ is a closed subspace of M.

If $p \ge 1$, we denote by M_p the subspace of those functions $f \in M$ such that the Hankel operator $H_{\bar{f}}$ is the Schatten class $S_p(A^2(\mu_m), L^2(\mu_m))$. We equip M_p with seminorm

$$\|f\| := \|H_{\bar{f}}\|_{S_p} + |f(0)|.$$

Lemma (2.3.19)[84]: The spaces M and M_p are Banach spaces.

Proof. We prove the lemma for M, the proof for M_p is similar. Let $(f_n)_{n\in\mathbb{N}_0}$ be a Cauchy sequence in M. Without loss of generality we may assume that $f_n(0) = 0$ for all n. The sequence $\left(H_{\bar{f}_n}\right)_{n\in\mathbb{N}_0}$ is a Cauchy sequence of bounded operators on $A^2(\mu_m)$. Therefore, there is an operator T in $A^2(\mu_m)$ such that $\left(H_{\bar{f}_n}\right)_{n\in\mathbb{N}_0}$ converges to T in the norm operator. Let $f := \overline{T(1)}$ be the conjugate of the image $T(1)$ of the constant function 1 under T. Since $H_{\bar{f}_n}(1) = \bar{f}_n$, it follows that

$$\|f_n - f\|_{L^2(\mu_m)} = \|\bar{f}_n - T(1)\|_{L^2(\mu_m)} = \|H_{\bar{f}_n}(1) - T(1)\|_{L^2(\mu_m)} \le \|H_{\bar{f}_n} - T\|$$

showing that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2(\mu_m)} = 0. \quad (78)$$

Thus $f \in A^2(\mu_m)$. We shall show that the Hankel operator $H_{\bar{f}}$ with symbol f is bounded. It is well defined on P. We shall prove that $H_{\bar{f}}$ is equal to T on P. Let g be a holomorphic polynomial. We first observe by (77), (78) and Lemma (2.3.18) that for all $z \in \mathbb{C}^n$ we have

$$\left|\left(P(\bar{f} - \bar{f_n})g\right)(z)\right| \leq \|f_n - f\|_{L^2(\mu_m)} \|gK_m(z,\cdot)\|_{L^2(\mu_m)}$$

showing that $\lim_{n\to+\infty} P((\bar{f} - \bar{f_n})g)(z) = 0$. Since again by (78) we have that $\lim_{n\to+\infty} \left(\bar{f} - \bar{f_n}\right)g\,(z) = 0$, it follows that

$$\lim_{n\to+\infty} H_{\overline{f_n}} - H_{\bar{f}}(g)(z) = 0\,.$$

This proves that $T_g = H_{\bar{f}}(g)$ and hence $T = H_{\bar{f}}$. Therefore M is a Banach space. The proof of that M_p is a Banach space is similar.

For $\theta = (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n$, let R_θ be the unitary linear transformation in $\mathbb{C}^n$ defined by $R_\theta(z) = (e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n)$, for all $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$.

Lemma (2.3.20)[84]: Let $\theta \in \mathbb{R}^n$. Then the operator $R_\theta f := f \circ R_\theta$ is a unitary isometry from $L^2(\mu_m)$ onto itself and from $A^2(\mu_m)$ onto itself. Moreover the following assertions hold.

(i) If $f \in M$ then $R_\theta f \in M$ and $\|R_\theta f\|_M = \|f\|_M$.

(ii) If $f \in M_\infty$, then $R_\theta f \in M_\infty$.

(iii) If $f \in M_p$, then $R_\theta f \in M_p$ and

$$\|R_\theta f\|_{M_p} = \|f\|_{M_p}$$

Proof. It is clear that the operator R_θ is a unitary isometry from $L^2(\mu_m)$ onto itself and from $A^2(\mu_m)$ onto itself. Let f be in M and $\theta \in \mathbb{R}^n$. Then $R_\theta f$ is clearly in $A_2(\mu_m)$. Moveover, if g is an element of P, then by a change of variable we see that

$$\begin{aligned} H_{\overline{R_\theta f}}(g)(z) &= \int_{\mathbb{C}^n} K_m(R_\theta z, w) g(R_{-\theta} w)\left[\overline{R_\theta f}(z) - \bar{f}(w)\right] d\mu_m(w) \\ &\int_{\mathbb{C}^n} K_m(R_\theta z, w)(R_{-\theta} g)\,(w)\left[\bar{f}(R_\theta z) - \bar{f}(w)\right] d\mu_m(w) \\ &= H_{\bar{f}}(R_{-\theta} g)(R_\theta z) \\ &= (R_\theta H_{\bar{f}} R_{-\theta})(g)(z). \end{aligned}$$

Since the adjoint of $R_{-\theta}$ is $R_{-\theta}^* = R_\theta$, it follows that

$$\left\|H_{\overline{R_\theta f}}\right\| = \left\|H_{\bar{f}}\right\|,$$

showing that

$$\|f \circ R_\theta\|_M = \|f\|_M.$$

This proves part (i) of the lemma. The proof of parts (ii) and (iii) of the lemma are similar.

Lemma (2.3.21)[84]: Let $f \in A^2(\mu_m)$.

(i) If $f \in M$, then for any multi-index k that satisfi$\frac{\partial^k f}{\partial z^k}(0) \neq 0$,es the monomial z^k is in M.

(ii) If $f \in M_\infty$, then for any multi-index k that satisfies $\frac{\partial^k f}{\partial z^k}(0) \neq 0$, the monomial z^k is in M_∞.

(iii) If $p \geq 1$ and $f \in M_p$, then for any multi-index k that satisfies $\frac{\partial^k f}{\partial z^k}(0) \neq 0$,the monomial z^k is in M_p.

Proof. To prove (i), suppose that $f \in M$. By the Cauchy formula we have

$$\frac{\partial^k f}{\partial z^k}(0)z^k = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(R_\theta z)\,/e^{ik_1\theta_1\cdots eik_n\theta_n}$$
$$d\theta,$$
where $d\theta := d\theta_1 \cdots d\theta_n$ for $\theta = (\theta_1,\cdots,\theta_n)$. By Lemmas (2.3.19) and (2.3.20) we see that $\frac{\partial^k f}{\partial z^k}(0)z^k \in M$. Therefore, $z^k \in M$ as long as $\frac{\partial^k f}{\partial z^k}(0) \neq 0$, The proof of the remaining statements of the lemma is similar.

Theorem (2.3.22)[84]: Let f be an entire function in $A^2(\mu_m)$, where m is a positive real number.

(i) Then the Hankel operator $H_{\bar f}$ is bounded on the Fock space $A^2(\mu_m)$ if and only if f is a polynomial of degree at most $\frac{m}{2}$.

(ii) The Hankel operator $H_{\bar f}$ is compact on the Fock space $A^2(\mu_m)$ if and only if f is a polynomial of degree smaller than $\frac{m}{2}$.

Theorem (2.3.23)[84]: Let f be an entire function in $A^2(\mu_m)$, where m is a positive real number. Then the Hankel operator $H_{\bar f}$ is in the Schatten class $S_p(A^2(\mu_m), L^2(\mu_m))$ if and only if $p > 2n$ and f is a polynomial of degree smaller than $\frac{m(p-2n)}{2p}$.

Proof of Theorems (2.3.22) and (2.3.23). We first prove Theorem (2.3.22). Let $f \in A^2(\mu_m)$. Suppose that $H_{\bar f}$ is bounded and let k be a multi-index that satisfies $\frac{\partial^k f}{\partial z^k}(0) \neq 0$, By Lemma (2.3.21) we see that the monomial z^k is in M. Now Theorem (2.3.12) implies that $m \geq 2|k|$. Hence f is a polynomial of degree at most $\frac{m}{2}$.

If $H_{\bar f}$ is compact then a similar argument shows that f is a polynomial of degree strictly smaller than $\frac{m}{2}$. The converse follows from Theorem (2.3.12).

The proof of Theorem (2.3.23) is similar to that of Theorem (2.3.22).

Corollary (2.3.24)[185]: Suppose that $\beta, k \in \mathbb{N}_0^n$ and $d \in \mathbb{N}_0$. Then
$$\sum_r \left(H_{\bar z^k} f_r\right)(\xi_r) = \bar\xi_r^k \sum_r f_r(\xi_r) - \frac{(1+\epsilon)_d}{(1+\epsilon)_d - |k|}\frac{\Gamma(n+d-|k|)}{\Gamma(n+d)}\frac{\partial^{|k|}}{\partial \xi_r^k}\sum_r f_r(\xi)$$
for all holomorphic polynomials f_r of degree d. In particular, if $f_r = \xi^{k+\epsilon}$, then
$$\sum_r \left(H_{\bar z^k} f_r\right)(\xi)$$
$$= \begin{cases} \sum_r \bar\xi_r^k \xi_r^{k+\epsilon} - \frac{(1+\epsilon)_{|k+\epsilon|}}{(1+\epsilon)_{|k+\epsilon|-|k|}}\frac{\Gamma(n+|k+\epsilon|-|k|)}{\Gamma(n+|k+\epsilon|)}\frac{(k+\epsilon)!}{(\epsilon)!}\sum_r \xi_r^\epsilon, & \text{if } \epsilon \geq 0 \\ \sum_r \bar\xi_r^k \xi_r^{k+\epsilon} & \text{otherwise.} \end{cases}$$

Proof. It suffices to prove the corollary for $f_r(\xi_r) = \xi_r^{k+\epsilon}$, where $k+\epsilon \in \mathbb{N}_0^n$. Let g_r be a homogeneous polynomial in P. If g_r is a monomial of the form $g_r(\xi_r) = \xi_r^\beta$, where $\beta \in \mathbb{N}_0^n$, then using the properties of P_μ, we see that
$$\sum_r \langle P_\mu(\bar z^k f_r), g_r\rangle_{L^2(\Omega,\mu)} = \sum_r \langle f_r, z^k g_r\rangle_{L_2(\Omega,\mu)}$$
and hence $\sum_r \langle P_\mu(\bar z^k f_r), g_r\rangle_{L^2(\Omega,\mu)} = 0$ as long as $k+\epsilon \neq k+\beta$. Now let $k+\epsilon = k+\beta$. By Lemma 2.1 in [181], we have the following identities

$$\int_{\Omega} z^{k+\epsilon}\bar{z}^{k+\epsilon}d\mu(z)=\frac{(n-1)!\,(1+\epsilon)_{|k+\epsilon|}(k+\epsilon)!}{(n+|k+\epsilon|-1)!}\quad \text{and}\ \langle z^{k+\epsilon},z^{k+\epsilon}\rangle_F=(k+\epsilon)!\,,(79)$$

from which we obtain

$$\sum_r \langle P_\mu(\bar{z}^k f_r),g_r\rangle_{L^2(\Omega,\mu)}=\frac{(n-1)!\,(1+\epsilon)_{|k+\epsilon|}}{(n+|k+\epsilon|-1)!}\sum_r \langle f_r,z^k g_r\rangle_F.$$

Since the multiplication operator and the corresponding differentiation operator are adjoint to each other with respect to the Fischer inner product, this implies that

$$\sum_r \langle P_\mu(\bar{z}^k f_r),g_r\rangle_{L^2(\Omega,\mu)}=\frac{(1+\epsilon)_{|\beta|+|k|}}{(1+\epsilon)_{|\beta|}}\frac{(n-1+|\beta|)!}{(n-1+|\beta|+|k|)!}\sum_r \langle\frac{\partial^{|k|}}{\partial z^k}f_r,g_r\rangle_{L^2(\Omega,\mu)}$$

for all holomorphic homogeneous polynomials g_r of degree $|\beta|$. Therefore, if f_r is a holomorphic homogeneous polynomial of degree d, we have

$$\sum_r P_\mu(\bar{z}^k f_r)=\frac{(1+\epsilon)_d}{(1+\epsilon)_{d-|k|/}}\frac{(n-1+d-|k|)!}{(n-1+d)!}\frac{\partial^{|k|}}{\partial z^k}\sum_r f_r.$$

This completes the proof of the corollary.

Corollary (2.3.25)[185]: The domain $\mathrm{Dom}(H^*_{\bar{z}^k})$ of $H^*_{\bar{z}^k}$ contains all polynomials in w and $\bar{w}$.

Proof. It suffices to show that, if $k+\epsilon$ and β are fixed in $\mathbb{N}_0^n$, then the linear functional

$$g_r\mapsto\langle H_{\bar{z}^k}(g_r),z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}$$

is bounded on $A^2(\Omega,\mu)$. To do so, choose an integer $d\geq|k+\epsilon|+|\beta|+2|k|$ and consider the subspace N_d of $A^2(\Omega,\mu)$ consisting of polynomials with degree smaller than or equal to d.We denote by π_d the orthogonal projection from $A^2(\Omega,\mu)$ onto N_d. If $g_r\in P$, then $(I-\pi_d)g_r$ is a sum of holomorphic homogeneous polynomials in P with degree at least $d+1$. In view of Corollary (2.3.24), we can write

$$\sum_r H_{\bar{z}^k}\circ(I-\pi_d)g_r=\sum_r(\bar{z}^k f_r+h_r)$$

where f_r is a sum of holomorphic homogeneous polynomials of degree at least $d+1$ and h_r is a sum of holomorphic homogeneous polynomials of degree at least $d+1-|k|$. Therefore,

$$\begin{aligned}\sum_r &\langle H_{\bar{z}^k}\circ(I-\pi_d)g_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}\\ &=\sum_r\langle\bar{z}^k f_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}+\sum_r\langle h_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}\\ &=\sum_r\langle f_r z^\beta,z^{k+\epsilon+k}\rangle_{L^2(\Omega,\mu)}+\sum_r\langle z^\beta h_r,z^{k+\epsilon}\rangle_{L^2(\Omega,\mu)}.\end{aligned}$$

Since $d+1+|\beta|\geq1+|k+\epsilon|+2|\beta|+2|k|>|k+\epsilon|+|k|$, it follows that $\sum_r\langle f_r z^\beta,z^{2k+\epsilon}\rangle_{L^2(\Omega,\mu)}=0$. Also, due to the fact that the degree of $z^\beta f_r$ is greater than $|k+\epsilon|$ we see that $\sum_r\langle z^\beta f_r,z^{k+\epsilon}\rangle_{L^2(\Omega,\mu)}=0$. Thus $\langle H_{\bar{z}^k}\circ(I-\pi_d)g_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}=0$ for all $g_r\in P$ and consequently

$$\sum_r\langle H_{\bar{z}^k}g_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}=\sum_r\langle H_{\bar{z}^k}\circ(I-\pi_d)g_r,z^{k+\epsilon}\bar{z}^\beta\rangle_{L^2(\Omega,\mu)}.$$

The corollary now follows from the fact that $H_{\bar{z}^k}\circ\pi_d$ is of finite rank and hence bounded.

Corollary (2.3.26)[185]: Suppose that u,v and f_r are holomorphic polynomials. Then

$$\sum_r H_{\bar{v}}^* H_{\bar{u}} f_r = \sum_r P_\mu(v\bar{u}f_r) - \sum_r vP_\mu(\bar{u}f_r).$$

Proof. A little computing shows that for all $g_r \in A^2(\Omega,\mu)$

$$\sum_r \langle H_{\bar{u}}f_r, H_{\bar{v}}g_r\rangle_{L^2(\Omega,\mu)} = \sum_r \langle \bar{u}f_r - P_\mu(\bar{u}f_r), \bar{v}g_r - P_\mu(\bar{v}g_r)\rangle_{L^2(\Omega,\mu)}$$
$$= \sum_r \langle v\bar{u}f_r, g_r\rangle_{L^2(\Omega,\mu)} - \sum_r \langle P_\mu(\bar{u}f_r), \bar{v}g_r\rangle_{L^2(\Omega,\mu)}$$
$$+ \sum_r \langle (P_\mu - I)(\bar{u}f_r), P_\mu(\bar{v}g_r)\rangle_{L^2(\Omega,\mu)}$$
$$= \sum_r \langle v\bar{u}f_r, g_r\rangle_{L^2(\Omega,\mu)} - \sum_r \langle P_\mu(\bar{u}f_r), \bar{v}g_r\rangle_{L^2(\Omega,\mu)}$$

where the latter equality holds since $P_\mu(\bar{v}g_r) \in A^2(\Omega,\mu)$ and $(P_\mu - I)(\bar{u}f_r)$ is orthogonal to $A^2(\Omega,\mu)$. This completes the proof.

Corollary (2.3.27)[185]: Assume that k and l are elements of $\mathbb{N}_0^n$. If f_r is a holomorphic homogeneous polynomial of degree d, then

$$\sum_r P_\mu(z^l\bar{z}^k f_r) = \frac{(1+\epsilon)_{d+|l|}}{(1+\epsilon)_{d-|k|+|l|}} \frac{\Gamma(d + n - |k| + |l|)}{\Gamma(d+n+|l|)} \frac{\partial^{|k|}}{\partial z^k} \sum_r f_r.$$

Proof. It is sufficient to establish the corollary for monomials $f_r(z) = z^{k+\epsilon}$. If β is an arbitrary element of $\mathbb{N}_0^n$, then, due to the properties of the Fischer product and (79), we have

$$\sum_r \langle P_\mu(z^l\bar{z}^k f_r), z^\beta\rangle_{L^2(\Omega,\mu)} = \langle z^{l+k+\epsilon}, z^{k+\beta}\rangle_{L^2(\Omega,\mu)}$$
$$= \frac{(n-1)!\,(1+\epsilon)_{|k+\epsilon|+|l|}}{(n+|k+\epsilon|+|l|-1)!} \langle \frac{\partial^{|k|}}{\partial z^k}(z^{l+k+\epsilon}), z^\beta\rangle_F$$
$$= \frac{(1+\epsilon)_{|k+\epsilon|+|l|}}{(1+\epsilon)_{|k+\epsilon|+|l|-|k|}} \frac{\Gamma(n+|k+\epsilon|+|l|-|k|)}{\Gamma(n+|k+\epsilon|+|l|)} \sum_r \langle \frac{\partial^{|k|}}{\partial z^k}(z^l f_r), z^\beta\rangle_{L^2(\Omega,\mu)}$$

This completes the proof.

Corollary (2.3.28)[185]: Suppose $\epsilon = 1$. Then for each $j = 1, \cdots, n$, the operator $H_{\bar{z}_j}$ is bounded but not compact on $A^2(\mu_{1+\epsilon})$. If $|k| \geq 2$, $H_{\bar{z}^k}$ is unbounded on $A^2(\mu_{1+\epsilon})$.

Proof. In this case, μ_2 is the Gaussian measure on $\mathbb{C}^n$. Its moments reduce to $(1+\epsilon)_s = \Gamma(s+n)$. Moreover, if $k+\epsilon \in \mathbb{N}_0^n$,

$$\lambda_{k+\epsilon}^r = \begin{cases} \dfrac{(2k+\epsilon)!}{(k+\epsilon)!} - \dfrac{(k+\epsilon)!}{(\epsilon)!} & \text{if } \epsilon \geq 0, \\ \dfrac{(2k+\epsilon)!}{(k+\epsilon)!} & \text{if } \epsilon \ngeq 0. \end{cases}$$

We first observe that if $|k| = 1$, then the eigenvalues of T are all equal to 1. Therefore, T is bounded but not compact on $A^2(\mu_{1+\epsilon})$. This proves the first part of the corollary.

Suppose now that $|k| \geq 2$. Choose j_0 in $[1,n]$ so that $k_{j_0} = \max_j k_j$. If d is a nonnegative integer, set $(k+\epsilon)(k,d) = (k_1, \cdots, k_{j_0-1}, k_{j_0} + d, k_{j_0+1}, \cdots, k_n)$. Then

$$\lambda_{(k+\epsilon)(k,d)}^r = \left(\prod_{j \neq j_0}^n \frac{(2k_j)!}{(k_j)!}\right) [(d + k_{j_0} + 1) \cdots (d + 2k_{j_0}] - (d+1) \cdots (d + k_{j_0})$$

Therefore, $\lim_{d\to+\infty} \lambda^r_{(k+\epsilon)(k,d)} = +\infty$, showing that T is unbounded on $A^2(\mu_{1+\epsilon})$. This implies that $H_{\bar{z}^k}$ is also unbounded.

Henceforth, we assume that $\epsilon \neq 1, \epsilon \geq 0$. From Proposition (2.3.6) and (73) we see that if $k+\epsilon \in \mathbb{N}_0^n$, then the eigenvalue $\lambda^r_{k+\epsilon}$ can be written in the form

$$\lambda^r_{k+\epsilon} = \begin{cases} A_{|k+\epsilon|} \dfrac{(2k+\epsilon)!}{(k+\epsilon)!} - B_{|k+\epsilon|} \dfrac{(k+\epsilon)!}{(\epsilon)!} & \text{if } \epsilon \geq 0 \\ A_{|k+\epsilon|} \dfrac{(2k+\epsilon)!}{(k+\epsilon)!} & \text{if } \epsilon \ngeq 0 \end{cases} \tag{80}$$

where, for a nonnegative integer d,

$$\begin{cases} A_d := \dfrac{\Gamma\left(\frac{2d+2n}{1+\epsilon} + \frac{2|k|}{1+\epsilon}\right)}{\Gamma\left(\frac{2d+2n}{1+\epsilon}\right)} \dfrac{\Gamma(d+n)}{\Gamma(d+n+|k|)}, \\ B_d := \dfrac{\Gamma\left(\frac{2d+2n}{1+\epsilon}\right)}{\Gamma\left(\frac{2d+2n}{1+\epsilon} - \frac{2|K|}{1+\epsilon}\right)} \dfrac{\Gamma(d+n-|k|)}{\Gamma(d+n)}. \end{cases} \tag{81}$$

The asymptotic behaviour of the eigenvalues $\{\lambda^r_{k+\epsilon}\}$ when $|k+\epsilon| = d \mapsto +\infty$ is given by the following (see [84]).

Corollary (2.3.29)[185]: The sequences (A_d) and (B_d) given by (81) have the asymptotic behavior

$$A_d = \left(\frac{2}{1+\epsilon}\right)^{\frac{2|k|}{1+\epsilon}} (d+n)^{|k|\left(\frac{1-\epsilon}{1+\epsilon}\right)} \left[1 - \frac{|k|^2(\epsilon-1)}{2(1+\epsilon)(d+n)} + O\left(\frac{1}{(d+n)^2}\right)\right]$$

$$B_d = \left(\frac{2}{1+\epsilon}\right)^{\frac{2|k|}{1+\epsilon}} (d+n)^{|k|\left(\frac{1-\epsilon}{1+\epsilon}\right)} \left[1 + \frac{|k|^2(\epsilon-1)}{2(1+\epsilon)(d+n)} + O\left(\frac{1}{(d+n)^2}\right)\right]$$

as $d \mapsto +\infty$.

Proof. Follows from the property of the Gamma function [159]

$$\frac{\Gamma(x+y)}{\Gamma(x+z)} = x^{y-z}\left(1 + \frac{(y-z)(y+z-1)}{2x} + O\left(\frac{1}{x^2}\right)\right) \text{ as } x \mapsto +\infty, \tag{82}$$

where y and z are real numbers.

Corollary (2.3.30)[185]: The eigenvalues $\lambda^r_{k+\epsilon}$ have the form

$$\sum_r \lambda^r_{k+\epsilon} = \left(\frac{2}{1+\epsilon}\right)^{2\frac{|k|}{1+\epsilon}} (d+n)^{2\frac{|k|}{1+\epsilon}-1} \sum_r \left((f_r)_n\left(\frac{k_1+\epsilon+1}{d+n}, \cdots, \frac{k_n+\epsilon+1}{d+n}\right) + \varepsilon(k+\epsilon)\right),$$

where $\varepsilon(k+\epsilon) = O\left(\frac{1}{d}\right)$ and

$$(f_r)_n(t_1, \cdots, t_n) := -(\epsilon-1)\frac{|k|^2}{1+\epsilon} t^k + \sum_{j=1}^n k_j^2 \frac{t^k}{t_j}$$

when $\epsilon \geq 0$ and $d = |k+\epsilon| \mapsto +\infty$.

Proof. We recall by (80) that if $\epsilon \geq 0$, then

$$\lambda_{k+\epsilon}^{r}=A_{|k+\epsilon|}\frac{(2k+\epsilon)!}{(k+\epsilon)!}-B_{|k+\epsilon|}\frac{(k+\epsilon)!}{(\epsilon)!}\ ,$$

where (A_d) and (B_d) are given by (81). On the other hand, by (82) we see that

$$\frac{(2k_j+\epsilon)!}{(k_j+\epsilon)!}=\left(1+k_j+\epsilon\right)^{k_j}+k_j\left(k_j-1\right)\left(1+k_j+\epsilon\right)^{k_j-1}+q_j(1\ +k_j+\epsilon)$$

$$\frac{(k_j+\epsilon)!}{(\epsilon)!}=\ \left(1+k_j+\epsilon\right)^{k_j}-k_j\left(k_j+1\right)\left(1+k_j+\epsilon\right)^{k_j-1}+r_j(1+k_j+\epsilon)$$

where q_j and r_j are one variable polynomials of degree at most k_j-2. This implies that

$$\frac{(2k+\epsilon)!}{(k+\epsilon)!}=\prod_{j=1}^{n}\left(1+k_j+\epsilon\right)^{k_j}$$

$$+\sum_{j=1}^{n}k_j\left(k_j-1\right)\left(1+k_j+\epsilon\right)^{k_j-1}\prod_{l\neq j}(1+k_l+\epsilon)^{k_l}+q(k+\epsilon)$$

$$\frac{(k+\epsilon)!}{(\epsilon)!}=\prod_{j=1}^{n}\left(1+k_j+\epsilon\right)^{k_j}-\sum_{j=1}^{n}k_j\left(k_j+1\right)\left(1+k_j+\epsilon\right)^{k_j-1}\prod_{l\neq j}(1+k_l+\epsilon)^{k_l}+r(k+\epsilon)$$

where q and r are polynomials of degree at most $|k|-2$. These equalities, combined with Corollary (2.3.29), give the corollary.

Corollary (2.3.31)[185]: The eigenvalues $\lambda_{k+\epsilon}^{r}$ have the estimate

$$\lambda_{k+\epsilon}^{r}=\ \left(O(d+n)^{2\frac{|k|}{1+\epsilon}-k_{j0}}\right),$$

as long as $\epsilon>0$ and $d=|k_{j_0}-\epsilon|\mapsto+\infty$.

Proof. Let $j_0=1,\ldots,n$ and suppose that $k_{j0}\geq 1$. We recall by (80) that if $\epsilon>0$, then

$$\lambda_{k+\epsilon}^{r}=\frac{A_{|k+\epsilon|}(2k+\epsilon)!}{(k+\epsilon)!},$$

where (A_d) is as before. Set $(k+\epsilon)'=(k_1+\epsilon,\cdots,k_{j_0-1}+\epsilon,0,k_{j_0+1}+\epsilon,\cdots,k_n+\epsilon)$ and $k'=(k_1,\cdots,k_{j_0-1}0,k_{j_0+1},\cdots,k_n)$. Arguing as in the proof of Corollary (2.3.30) we have

$$\frac{(2k+\epsilon)!}{(k+\epsilon)!}\leq\left(2k_{j_0}\right)!\frac{((k+\epsilon)'+k')!}{(k+\epsilon)'!}$$

$$=\left(2k_{j_0}\right)!\left[\prod_{j=1,j\neq j_0}^{n}\left(1+k_j+\epsilon\right)^{k_j}\right.$$

$$\left.+\sum_{j=1,j\neq j_0}^{n}k_j\left(k_j-1\right)\left(1+k_j+\epsilon\right)^{k_j-1}\prod_{s\neq j,l}(1\ +k_s+\epsilon)^{k_s}\right]+q((k+\epsilon)'),$$

where q is a polynomial of degree at most $|k'|-2$. These estimates, combined with Corollary (2.3.29), give the corollary.

Corollary (2.3.32)[185]: The operator $H_{\bar{z}^k}*H_{\bar{z}^k}$ is bounded if and only if $2\frac{|k|}{1+\epsilon}-1\leq 0$ and compact if and only if $2\frac{|k|}{1+\epsilon}-1<0$.

Proof. Let Σ_n be the simplex consisting of those $t=(t_1,\cdots,t_n)\in\mathbb{R}^n$ such $t_j\geq 0$ and $t_1+\cdots+t_n=1$. By Corollaries (2.3.30) and (2.3.31) we see that

$$\sup_{|k+\epsilon|=d}\sum_r |\lambda^r_{k+\epsilon}| \approx (d+n)^{2\frac{|k|}{1+\epsilon}-1}\sup_{|k+\epsilon|=d}\sum_r \left|(f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)\right|$$
$$\approx (d+n)^{2\frac{|k|}{1+\epsilon}-1}\sup_{t\in\Sigma_n}\sum_r |(f_r)_n(t)|$$

as $d=|k+\epsilon|\to+\infty$. Now the corollary follows since the operator $H_{\bar{z}^k} * H_{\bar{z}^k}$ is bounded if and only if the sequence $\sup_{|k+\epsilon|=d}|\lambda^r_{k+\epsilon}|$ is bounded and $H_{\bar{z}^k} * H_{\bar{z}^k}$ is compact if and only if the sequence $\sup_{|k+\epsilon|=d}|\lambda^r_{k+\epsilon}|$ tends to 0 as $d=|k+\epsilon|\to+\infty$.

Corollary (2.3.33)[185]: Let $k\in\mathbb{N}_0^n$ and $1+\epsilon$ be a positive real number.

(i) The Hankel operator $H_{\bar{z}^k}$ is bounded on the Fock space $A^2(\mu_{1+\epsilon})$ if and only if $1+\epsilon\geq 2|k|$.

(ii) The Hankel operator $H_{\bar{z}^k}$ is compact on the Fock space $A^2(\mu_{1+\epsilon})$ if and only if $1+\epsilon> 2|k|$.

Proof. We use that the operator $H_{\bar{z}^k}$is bounded if and only if $T=H_{\bar{z}^k} * H_{\bar{z}^k}$ is bounded and $H_{\bar{z}^k}$ is compact if and only if $T=H_{\bar{z}^k} * H_{\bar{z}^k}$ is compact.

Next, assume that $2\frac{|k|}{1+\epsilon}-1<0$ and let $\epsilon\geq 0$. We shall investigate the membership of the operator T to a Schatten class $S_{1+\epsilon}$. Recall that T is in $S_{1+\epsilon}$ if and only if the series$\sum_r \lambda^{r(1+\epsilon)}_{k+\epsilon}$ is convergent.

Let d be an integer. We shall estimate the sum $s_d=\sum_{|k+\epsilon|=d}\lambda^{r(1+\epsilon)}_{k+\epsilon}$, when $d\to+\infty$. The calculations above lead to study the cases $\epsilon\geq 0$ and its opposite case separately. Let $B_d:=\{k+\epsilon\in\mathbb{N}_0^n, |k+\epsilon|=d\}$. We partition $B_d=B'd\cup B''_d$, where$B'_d=\{k+\epsilon\in B:\epsilon\geq 0\}$ and $B''_d=B_d\setminus B'_d$. Thus s_d can be written in the form $s_d=s'_d+s''_d$, where $s_d=\sum_{k+\epsilon\in B'_d}\sum_r\ \lambda^{r(1+\epsilon)}_{k+\epsilon}$ and $s''_d=\sum_{k+\epsilon\in B''_d}\sum_r\ \lambda^{r(1+\epsilon)}_{k+\epsilon}$.

Corollary (2.3.34)[185]: We have the estimates $\#B_d\approx\#B'_d\approx\frac{d^{n-1}}{(n-1)!}$and $\#B''_d\approx d^{n-2}$ as $d\to+\infty$.

Proof. Let $P_{n,d}$ the space of n variables holomorphic polynomials of degree d. We have $\#B_d=\dim P_{n,d}=\binom{n-1+d}{d}=\frac{(d+n-1)!}{(n-1)!d!}$. Therefore, $\#B_d\sim\frac{1}{(n-1)!}d^{n-1}$as $d\to+\infty$. On the other hand, for $j=1,\cdots,n$, let $B'_{d,j}=\{k+\epsilon\in B_d,\epsilon>0\}$. Since $B'_d=\cup_{1\leq j\leq n}B'_{d,j}$, we see that $\#B'_d\leq\sum_{j=1}^n\#B'_{d,j}$.

If $k_j\geq 1$, then

$$B'_{d,j}=\bigcup_{l=0}^{k_j-1}\{k+\epsilon=(k_1+\epsilon,\cdots,k_{j-1}+\epsilon,l,k_{j+1}+\epsilon,\cdots,k_n+\epsilon),|k+\epsilon|=d\}$$
$$=\bigcup_{l=0}^{k_j-1}\left\{k+\epsilon\right.$$
$$\left.=(k_1+\epsilon,\cdots,k_{j-1}+\epsilon,l,k_{j+1}+\epsilon,\cdots,k_n+\epsilon)\sum_{i\neq j}k_i+\epsilon=d-l\right\}.$$

Therefore, $B'_{d,j}=\sum_{l=0}^{k_j-1}\dim P_{n-1,d-l}$, and when $d\to+\infty$, $\#B'_{d,j}\sim k_j\frac{d^{n-2}}{(n-2)!}$.

This shows that, when $B'_d \approx d^{n-2}$ as $d \to +\infty$. The corollary now follows from the observation $\#B_d = \#B'_d + \#B''_d$.

Corollary (2.3.35)[185]: Suppose that $n \geq 2$ and g_r is a continuous function on $\mathbb{R}^n - 1$. Consider the open set $\Omega := \{(t_1, \cdots, t_{n-1} \in \mathbb{R}_+^{n-1}, \sum_{j=1}^{n-1} t_j < 1\}$. For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_{n-1}$ in $\mathbb{N}_0^{n-1}$, set

$$c_{\gamma,d} := \left(\frac{\gamma_1 + 1}{d}, \cdots, \frac{\gamma_{n-1} + 1}{d}\right)$$

$$\mathbb{J}_d := \left\{\gamma \in \mathbb{N}_0^{n-1} : \prod_{j=1}^{n-1} \frac{\gamma_j}{d}, \frac{\gamma_j + 1}{d} \subset \Omega\right\}.$$

Then $\lim_{d\to+\infty} \frac{1}{d^{n-1}} \sum_{\gamma\in\mathbb{J}_d} \sum_r g_r(c_{\gamma,d}) = \int_\Omega \sum_r g_r(t)dt.$

Proof. For $d \in \mathbb{N}_0$, let $\Omega_d = \cup_{\gamma\in\mathbb{J}_d} \prod_{j=1}^{n-1} \left[\frac{\gamma_j}{d}, \frac{\gamma_j+1}{d}\right]$. It is clear that $\Omega_d \subset \Omega$.Next, we show that $\lim_{d\to+\infty} \chi_{\Omega_d} = \chi_\Omega$. If s is a real number, let $[s]$ denote the largest integer smaller than or equal to s. If $t = (t_1, \cdots, t_{n-1}) \in \Omega$ and$1 \leq j \leq n-1$, then$\frac{[dt_j]}{d} \leq t_j < \frac{[dt_j]}{d} + \frac{1}{d}$. Therefore,

$$\sum_{j=1}^{n-1} \frac{[dt_j]}{d} \leq n \sum_{j=1}^{n-1} t_j < \sum_{j=1}^{n-1} \frac{[dt_j]}{d} + \frac{n-1}{d}.$$

Since$\sum_{j=1}^{n-1} t_j < 1,$ there is an integer d_0 such that for all $d > d_0$ we have $\sum_{j=1}^{n-1} \frac{[dt_j]}{d} + \frac{n-1}{d} < 1$. Thus, $t \in \prod_{j=1}^{n-1} \left[\frac{[dt_j]}{d}, \frac{[dt_j+1]}{d}\right]$and hence $t \in \Omega$dfor all $d > d_0$. Thus $\lim_{d\to+\infty} \chi_{\Omega_d} = \chi_\Omega$. Therefore,

$$\frac{1}{d^{(n-1)}} \sum_{\gamma\in\mathbb{J}_d} \sum_r g_r(c_\gamma, d) - \int_\Omega \sum_r g_r(t)dt$$
$$= \sum_{\gamma\in\mathbb{J}_d} \sum_r \left[\frac{1}{d^{n-1}} g_r(c_\gamma, d) - \int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d}, \frac{\gamma_j+1}{d}\right]} g_r(t)dt\right]$$
$$+ \int_{\Omega_d} \sum_r g_r(t)dt - \int_\Omega \sum_r g_r(t)dt$$

Since g_r is a bounded continuous function on the compact set Ω, we have, by Lebesgue's theorem, $\lim_{d\to+\infty} \int_{\Omega_d} \sum_r g_r(t)dt = \int_\Omega \sum_r g_r(t)dt$. On the other hand, by continuity of g_r on the compact set Ω, we see that

$$\sum_{\gamma\in\mathbb{J}_d} \sum_r \left[\frac{1}{d^{n-1}} g_r(c_\gamma, d) - \int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d}, \frac{\gamma_j+1}{d}\right]} g_r(t)dt\right]$$
$$= \sum_{\gamma\in\mathbb{J}_d} \int_{\prod_{j=1}^{n-1}\left[\frac{\gamma_j}{d}, \frac{\gamma_j+1}{d}\right]} \sum_r \left[g_r(c_\gamma, d) - g_r(t)dt\right]$$

also tends to 0 as $d \to +\infty$. This shows that

$$\lim_{d\to+\infty} \frac{1}{d^{n-1}} \sum_{\gamma\in\mathbb{J}_d} \sum_r g_r(c_\gamma, d) = \int_\Omega \sum_r g_r(t)dt.$$

Corollary (2.3.36)[185]: If $\epsilon \geq 0$, then

$$s_d \approx d^{n-1}d^{1+\epsilon}\left(2\frac{|k|}{1+\epsilon}-1\right).$$

Proof. Recall that $s_d=\sum_{k+\epsilon\in B'_d}\sum_r \lambda_{k+\epsilon}^{r(1+\epsilon)}$. By Corollary (2.3.30), we know that the sequence $\{\lambda_{k+\epsilon}^r\}_{k+\epsilon}\in B'_d$ has the following expansion when $d\to+\infty$

$$\sum_r \lambda_{k+\epsilon}^r=\left(\frac{2}{1+\epsilon}\right)^{2\frac{|k|}{1+\epsilon}}(d+n)^{2\frac{|k|}{1+\epsilon}-1}\sum_r\left((f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)\right.$$
$$\left.+\varepsilon(k+\epsilon)\right),$$

where $\varepsilon(k+\epsilon)=O\left(\frac{1}{d}\right)$and

$$(f_r)_n(t_1,\cdots,t_n):=-(\epsilon-1)\frac{|k|^2}{1+\epsilon}t^k+\sum_{j=1}^n k_j^2\frac{t^k}{t_j}$$

Using the properties of the function $x\to x^{1+\epsilon}$, we see that there exists a constant $M>0$, such that

$$\sum_r\left\|(f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)+\varepsilon(k+\epsilon)\right\|^{1+\epsilon}$$
$$-\sum_r\left\|(f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)\right|^{1+\epsilon}\Bigg|\le\frac{M}{d}.$$

Therefore,

$$\sum_r \lambda_{k+\epsilon}^r\approx\left(\frac{2}{1+\epsilon}\right)^{2\frac{|k|}{1+\epsilon}}(d+n)^{2\frac{|k|}{1+\epsilon}-1}\sum_r\left((f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)\right),$$

as $d=|k+\epsilon|\to+\infty$. Applying Corollary (2.3.34) we see that

$$s_d\approx\left(\frac{2}{1+\epsilon}\right)^{2|k|}d^{(1+\epsilon)\left(2\frac{|k|}{1+\epsilon}-1\right)}\sum_{k+\epsilon\in B_d}\sum_r\left|(f_r)_n\left(\frac{k_1+\epsilon+1}{d+n},\cdots,\frac{k_n+\epsilon+1}{d+n}\right)\right|^{1+\epsilon}$$
$$\approx\left(\frac{2}{1+\epsilon}\right)^{2|k|}d^{n-1}d^{(1+\epsilon)\left(2\frac{|k|}{1+\epsilon}-1\right)}\int_\Omega\sum_r|(f_r)_n(t)|^{1+\epsilon}dt$$

so that the corollary follows from Corollary (2.3.35).

Corollary (2.3.37)[185]: Let $k\in\mathbb{N}_0^n$and $1+\epsilon$ be a positive real number. Then the Hankel operator $H_{\bar{z}^k}$ is in the Schatten class $S_{2n+\epsilon}(A^2(\mu_{1+\epsilon}),L^2(\mu_{1+\epsilon}))$if and only if $\epsilon>0$ and $\epsilon(1+\epsilon)>2(2n+\epsilon)|k|$.

Proof. We use that the operator $H_{\bar{z}^k}$ is in $S_{2n+\epsilon}(A^2(\mu_{1+\epsilon}),L^2(\mu_{1+\epsilon}))$if and only if $T=H_{\bar{z}^k}*H_{\bar{z}^k}$ is $S_{\frac{2n+\epsilon}{2}}(A^2(\mu_{1+\epsilon}))$. Therefore, the corollary follows from Corollary (2.3.36)

Corollary (2.3.38)[185]: The Bergman kernel $K_{1+\epsilon}(z,w)$ of $A^2(\mu_{1+\epsilon})$ is given by

$$K_{1+\epsilon}(z,w)=\frac{1+\epsilon}{(n-1)!}E^{n-1}_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}(\langle z,w\rangle),$$

where $E^{n-1}_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}$is the derivatives of $E_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}$ with order $n-1$.

Proof. The monomials $z^{k+\epsilon},k+\epsilon\in\mathbb{N}_0^n$, form an orthogonal basis of $A^2(\mu_{1+\epsilon})$.Since

$$\left\|z^{k+\epsilon}\right\|^2_{L^2(\mu_{1+\epsilon})}=\frac{(n-1)!}{1+\epsilon}\frac{(k+\epsilon)!}{(|k+\epsilon|+n-1)!}\Gamma\left(\frac{2|k+\epsilon|+2n}{1+\epsilon}\right)\qquad(83)$$

it follows that the Bergman kernel is

$$K_{1+\epsilon}(z,w) = \sum_{k+\epsilon\in\mathbb{N}_0^n} \frac{z^{k+\epsilon}}{\|z^{k+\epsilon}\|_{L^2(\mu_{1+\epsilon})}} \frac{w^{k+\epsilon}}{\|w^{k+\epsilon}\|_{L^2(\mu_{1+\epsilon})}}$$

$$= \frac{1+\epsilon}{(n-1)!}\sum_{d=0}^{+\infty} \frac{(d+n-1)!}{d!\,\Gamma\left(\frac{2d}{1+\epsilon}+\frac{2n}{1+\epsilon}\right)} (\langle z,w\rangle)^d = \frac{1+\epsilon}{(n-1)!} E^{n-1}_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}(\langle z,w\rangle)\,.$$

This completes the proof of the corollary.

Corollary (2.3.39)[185]: If $g_r \in P$ and $z \in \mathbb{C}^n$, then $g_r K_{1+\epsilon}(z,\cdot)$ is in $L^2(\mu_{1+\epsilon})$.

Proof. It follows from Theorem 2, p. 6 in [90] that the generalized Mittag-Leffler's function is $E_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}$ is an entire function of finite order $\frac{1+\epsilon}{2}$ and type 1. Therefore $E^{n-1}_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}$ is also an entire function of finite order $\frac{1+\epsilon}{2}$ and type 1 and hence for any $\varrho > 0$, there is a positive constant C that

$$\sum_r \left|E^{n-1}_{\frac{2}{1+\epsilon},\frac{2n}{1+\epsilon}}(\lambda^r)\right| \le C\sum_r e^{|\lambda^r|^{\frac{1+\epsilon+\varrho}{2}}}, \qquad \lambda^r \in \mathbb{C}.$$

This shows that for all $z, w \in \mathbb{C}^n$,

$$|K_{1+\epsilon}(z,w)| \le Ce^{|\langle z,w\rangle|^{\frac{1+\epsilon+\varrho}{2}}} \le Ce(|z||w|)^{\frac{1+\epsilon+\varrho}{2}},$$

showing that for all $g_r \in P$ and z fixed in $\mathbb{C}^n$, the function $w \mapsto g_r(w)K_{1+\epsilon}(z,w)$ is in $L^2(\mu_{1+\epsilon})$ as long as $0 < \varrho < 1+\epsilon$.

Corollary (2.3.40)[185]: The spaces M and $M_{1+\epsilon}$ are Banach spaces.

Proof. We prove the corollary for M, the proof for $M_{1+\epsilon}$ is similar. Let $((f_r)_n)_{n\in\mathbb{N}_0}$ be a Cauchy sequence in M. Without loss of generality we may assume that $(f_r)_n(0) = 0$ for all n. The sequence $\left(H_{\sum_r \overline{(f_r)_n}}\right)_{n\in\mathbb{N}_0}$ is a Cauchy sequence of bounded operators on $A^2(\mu_{1+\epsilon})$. Therefore, there is an operator T in $A^2(\mu_{1+\epsilon})$ such that $\left(H_{\sum_r \overline{(f_r)_n}}\right)_{n\in\mathbb{N}_0}$ converges to T in the norm operator. Let $f_r := \overline{T(\imath)}$ be the conjugate of the image $T(i)$ of the constant function 1 under T. Since $H_{\sum_r \overline{(f_r)_n}}$
$(i) = \overline{(f_r)_n}$, it follows that

$$\sum_r \|(f_r)_n - f_r\|_{L^2(\mu_{1+\epsilon})} = \sum_r \left\|\overline{(f_r)_n} - T\,(1)\right\|_{L^2(\mu_{1+\epsilon})} = \left\|H_{\sum_r\overline{(f_r)_n}}(1) - T(1)\right\|_{L^2(\mu_{1+\epsilon})}$$
$$\le \left\|H_{\sum_r\overline{(f_r)_n}} - T\right\|$$

showing that

$$\lim_{n\to\infty}\sum_r \|(f_r)_n - f_r\|_{L^2(\mu_{1+\epsilon})} = 0. \tag{84}$$

Thus $f_r \in A^2(\mu_{1+\epsilon})$. We shall show that the Hankel operator $H_{\sum_r \overline{f_r}}$ with symbol $\sum_r f_r$ is bounded. It is well defined on P. We shall prove that $H_{\sum_r \overline{f_r}}$ is equal to T on P. Let g_r be a holomorphic polynomial. We first observe by (83), (84) and Corollary (2.3.39) that for all $z \in \mathbb{C}^n$ we have

$$\sum_r \left|\left(P(\overline{f_r} - \overline{(f_r)_n})g_r\right)(z)\right| \le \sum_r \|(f_r)_n - f_r\|_{L^2(\mu_{1+\epsilon})}\|g_r K_{1+\epsilon}(z,\cdot)\|_{L^2(\mu_{1+\epsilon})}$$

showing that $\lim_{n\to+\infty}\sum_r P((\overline{f_r} - \overline{(f_r)_n})g_r)(z) = 0$. Since again by (84) we have that $\lim_{n\to+\infty}\sum_r \left(\overline{f_r} - \overline{(f_r)_n}\right)g_r(z) = 0$, it follows that

$$\lim_{n\to+\infty} H_{\sum_r \overline{(f_r)}_n} - H_{\sum_r \bar{f}_r}(g_r)(z) = 0\,.$$

This proves that $T_{g_r} = H_{\sum_r \bar{f}_r}(g_r)$ and hence $T = H_{\sum_r \bar{f}_r}$. Therefore M is a Banach space. The proof of that $M_{1+\epsilon}$ is a Banach space is similar.

Corollary (2.3.41)[185]: Let $\theta \in \mathbb{R}^n$. Then the operator $R_\theta f_r := f_r \circ R_\theta$ is a unitary isometry from $L^2(\mu_{1+\epsilon})$ onto itself and from $A^2(\mu_{1+\epsilon})$ onto itself. Moreover the following assertions hold.

(i) If $f_r \in M$ then $R_\theta f_r \in M$ and $\sum_r \|R_\theta f_r\|_M = \sum_r \|f_r\|_M$.

(ii) If $f_r \in M_\infty$, then $R_\theta f_r \in M_\infty$.

(iii) If $f_r \in M_{1+\epsilon}$, then $R_\theta f_r \in M_{1+\epsilon}$ and

$$\sum_r \|R_\theta f_r\|_{M_{1+\epsilon}} = \sum_r \|f_r\|_{M_{1+\epsilon}}$$

Proof. It is clear that the operator R_θ is a unitary isometry from $L^2(\mu_{1+\epsilon})$ onto itself and from $A^2(\mu_{1+\epsilon})$ onto itself. Let f_r be in M and $\theta \in \mathbb{R}^n$. Then $R_\theta f_r$ is clearly in $A_2(\mu_{1+\epsilon})$. Moveover, if g_r is an element of P, then by a change of variable we see that

$$H_{\sum_r \overline{R_\theta f_r}}(g_r)(z) = \int_{\mathbb{C}^n} \sum_r K_{1+\epsilon}(R_\theta z, w) g_r(R_{-\theta} w)\left[\overline{R_\theta f_r}(z) - \bar{f}_r(w)\right] d\mu_{1+\epsilon}(w)$$

$$\int_{\mathbb{C}^n} \sum_r K_{1+\epsilon}(R_\theta z, w)(R_{-\theta} g_r)(w)\left[\bar{f}_r(R_\theta z) - \bar{f}_r(w)\right] d\mu_{1+\epsilon}(w) = H_{\sum_r \bar{f}_r}(R_{-\theta} g_r)(R_\theta z)$$

$$= \left(R_\theta H_{\sum_r \bar{f}_r} R_{-\theta}\right) \sum_r (g_r)(z).$$

Since the adjoint of $R_{-\theta}$ is $R_{-\theta}^* = R_\theta$, it follows that

$$\left\|H_{\sum_r \overline{R_\theta f_r}}\right\| = \left\|H_{\sum_r \bar{f}_r}\right\|,$$

showing that

$$\sum_r \|f_r \circ R_\theta\|_M = \sum_r \|f_r\|_M.$$

This proves part (i) of the corollary. The proof of parts (ii) and (iii) of the corollary are similar.

Corollary (2.3.42)[185]: Let $f_r \in A^2(\mu_{1+\epsilon})$.

(i) If $f_r \in M$, then for any multi-index k that satisfies $\sum_r \frac{\partial^k f_r}{\partial z^k}(0) \neq 0$, the monomial z^k is in M.

(ii) If $f_r \in M_\infty$, then for any multi-index k that satisfies $\sum_r \frac{\partial^k f_r}{\partial z^k}(0) \neq 0$, the monomial z^k is in M_∞.

(iii) If $\epsilon \geq 0$ and $f_r \in M_{1+\epsilon}$, then for any multi-index k that satisfies $\sum_r \frac{\partial^k f_r}{\partial z^k}(0) \neq 0$,the monomial z^k is in $M_{1+\epsilon}$.

Proof. To prove (i), suppose that $f_r \in M$. By the Cauchy formula we have

$$\sum_r \frac{\partial^k f_r}{\partial z^k}(0) z^k = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \sum_r f_r(R_\theta z) \,/ e^{ik_1\theta_1 \cdots eik_n\theta_n} d\theta,$$

where $d\theta := d\theta_1 \cdots d\theta_n$ for $\theta = (\theta_1, \cdots, \theta_n)$. By Corollaries (2.3.40) and (2.3.41) we see that $\sum_r \frac{\partial^k f_r}{\partial z^k}(0) z^k \in M$. Therefore, $z^k \in M$ as long as $\sum_r \frac{\partial^k f_r}{\partial z^k}(0) \neq 0$, The proof of the remaining statements of the corollary is similar.

Chapter 3
Toeplitz Operators on Bergman Space and Finite Rank Commutators

We show a conjecture of Axler that for bounded analytic functions f and g on the unit disk, $T_f^*T_g - T_gT_f^*$ is compact iff either f or g is constant on each Gleason part $P(m)$ except D. We provide examples that show that the Brown–Halmos theorem fails for general symbols, even for symbols continuous up to the boundary. We show that if the product of two Toeplitz operators with bounded harmonic symbols has finite rank, then one of the Toeplitz operators must be zero.

Section (3.1): Hankel Operators

We consider the question of when the semi-commutator $T_{f_g} - T_fT_g$ on the Bergman space with bounded harmonic symbols is compact. Several conditions equivalent to compactness of $T_{f_g} - T_fT_g$ are given. As a consequence we show a conjecture of Axler that for bounded analytic functions f and g on the unit disk, $T_f^*T_g - T_gT_f^*$ is compact iff either f or g is constant on each Gleason part $P(m)$ except D. 1989 Academic Press, Inc.

We consider the question of when the product $H_f^*H_g$ of two Hankel operators on the Bergman space with bounded harmonic symbols is compact. The product $H_f^*H_g$ is equal to the semi-commutator $T_{\bar{f}g} - T_{\bar{f}}T_g$. Several conditions equivalent to compactness of $H_f^*H_g$ are given. Consequently we prove Axler's conjecture [92].

As is well known, for f and g in $L^\infty(\partial D)$, Axler, Chang, and Sarason [93] and Volberg [57] have shown that $H_f^*H_g$ on the Hardy space is compact iff $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C(\partial D)$. By means of the theorem of Axler and Shields [95], we also obtain that $H_f^*H_g$ is compact iff

$$H^\infty[f] \cap H^\infty[g] \subset \{u \in \mathbb{C}(\mathcal{M}): u|_{P(m)} \in H^\infty|_{P(m)}\ for\ thin\ part\ P(m) in\ \mathcal{M}\}$$

for bounded harmonic functions f and g.

For$D$ denote the open unit disk in the complex plane $\mathbb{C}$, and dA the usual normalized area measure on D. The Bergman space L_a^2 is the Hilbert space of analytic functions $g: D \to \mathbb{C}$ with inner product given by

$$\langle f, g\rangle = \int_D f(z)\bar{g}(z)dA(z).$$

As usual, $L^\infty(D)$ denotes the set of bounded measurable functions on D, and $H^\infty(D)$ is the set of bounded analytic functions on D. Let P denote the orthogonal projection of $L^2(D, dA)$ onto $L_a^2(D)$. For $f \in L^\infty(D)$, the Hankel operator $H_f: L_a^2 + \left(L_a^2(D)\right)^\perp$ and the Toeplitz operator $T_f: L_a^2 \to L_a^2$ are defined by $H_f(h) = (I - P)(fh)$ and $T_f(h) = P(fh)$, respectively.

Let $\mathcal{M}$ be the maximal ideal space of $H^\infty(D)$. The Gleason part $P(m)$ corresponding to m is the equivalence class of a point m in $\mathcal{M}$,

$$P(m) = (m_1 \in \mathcal{M}: \rho(m, m_1) < 1\},$$

where $p(m, m_1)$ is the pseduo-hyperbolic distance from m_1 to m defined by

$$p(m, m_1) = \sup\{|f(m_1)|; f \in H^\infty(D), \|f\|_\infty \leq 1 \text{ and } f(m) = 0\}.$$

If m and m_1 are in the usual disk, the pseduo-hyperbolic distance is given by

$$\rho(m, m_1) = \left|\frac{m_1 - m}{1 - \bar{m}_1 m}\right|.$$

If a is a point of D, let $L_\alpha(z)$ be the linear fractional map

$$L_\alpha(z) = \frac{z+\alpha}{1+\bar{\alpha}z}.$$

Call a sequence (z_n) in D thin if $\lim_{n\to\infty} \pi_{k\neq n}|z_n - z_k|/|1-\bar{z}_n z_k| = 1$ and a part $P(m)$ thin if m is in the closure of some thin sequence.

Now we state some of Hoffman's results [50].

Hl. Let m be any point of $\mathcal{M}\backslash D$. There exists a sequence $\{\beta_n\}$ in D such that $\{\beta_n\}$ has no accumulation points in D, m is in the closure of $\{\beta_n\}$, the coresponding maps L_{β_n} converge pointwise to L_m, a map from D into $\mathcal{M}$, and for any bounded analytic function $h, h\circ L_{\beta_n}$ converges to $h\circ L_m$, uniformly on compacta, so that

$$(h\circ L'_m)(0) = \lim_{\beta_n\to m}(1-|\beta_n|^2)h'(\beta_n).$$

H2. If the Gleason part $P(m)$ contains at least two points, $P(m)$ is an analytic disk, and L_m is a one-to-one analytic map from D onto the Gleason part $P(m)$.

The map L_m plays an important role. The Gleason part does the same job on the Bergman space as the support set on the Hardy space.

For f analytic on D, the Bloch norm $\|f\|_\beta$ of f is defined by

$$\|f\|_\beta = \sup((1-|\lambda|^2)|f'(\lambda)| : \lambda\in D.$$

The Bloch space β is the set of analytic functions f on D such that $\|f\|_\beta < \infty$.
$D(z,r)$ will denote the pseudo-hyperbolic disc $\{w\in D: \rho(w,z) < r\}$ for $z\in D$ and $0 < r < 1$, and k_z is the normalized Bergman reproducing kernel

$$\frac{1-|z|^2}{(1-\bar{z}w)^2}$$

Theorem (3.1.1)[58]: Suppose f and g are bounded harmonic functions on D. Then the following conditions are equivalent:

(a) $H_f^* H_g$ is compact;

(b) $T_{\bar{f}} T_g - T_{\bar{f}g}$ is compact;

(c) For each thin part $P(m)$ except D, either $f|_{P(m)} \in H^\infty(D)|_{P(m)}$ or $g|_{P(m)} \in H^\infty(D)|_{P(m)}$;

(d) For m in $\mathcal{M}\backslash D$, either $f\circ L_m \in H^\infty$ or $g\circ L_m\in H^\infty$;

(e) $H^\infty(D)[f]\cap H^\infty(D)[g] \subset \{u\in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ for each thin part $P(m)$ except $D\}$;

(f) $\lim_{|z|\to 1} \min\{(1-|z|^2)|(\partial f/\partial\bar{z})(z)|, (1-|z|^2)|(\partial g/\partial\bar{z})(z)|\} = 0.$

The following theorem, which was conjectured by Axler [92], is valid.

Theorem (3.1.2)[58]: Suppose f and g are bounded analytic functions on D. Then the following conditions are equivalent:

(a') $H_f^* H_{\bar{g}}$ is compact;

(b') $T_f T_g^* - T_{\bar{g}f}$ is compact;

(c') For each thin part $P(m)$ except D, either $f|_{P(m)}$ or $g|_{P(m)}$ is constant;

(d') For each Gleason part $P(m)$ except D, either $f|_{P(m)}$ or $g|_{P(m)}$ is constant;

(e') $H^\infty(D)[\bar{f}]\cap H^\infty(D)[\bar{g}] \subset \{u\in C(\mathcal{M}): u|_{P(m)}\in H^\infty(D)|_{P(m)}$ *for each thin part except* $D\}$;

(f') $\lim_{|z|\to 1}\min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\} = 0.$

We shall prove $(d)\Rightarrow(f)$ and $(f)\Rightarrow(a)$ and $(a)\Rightarrow(d)\Rightarrow(c)\Leftrightarrow(e)$ and $(c)\Rightarrow(a)$. It is easy to show that

$$T_{\bar{f}}T_g - T_{\bar{f}g} = -H_f^* H_g.$$

Hence the equivalence of (a) and (b) is true. The equivalence $(f') \Leftrightarrow (c')$ that may have been known before will be proved. We show that if f and g are bounded harmonic functions and $H_f^* H_g = 0$, then either f or g is in $H^\infty(D)$. This means that $T_f^* T_g = T_{\bar{f}g}$ if f either f or g is analytic.

In fact, it is natural that the thin part plays the special role since there is a function 4 in H^∞ such that $\phi \circ L_m(z) = z$ for the thin part m, which is not true for all Gleason parts.

At the same time that the results were obtained, S. Axler and P. Gorkin [94] proved the same result as in Theorem (3.1.20), using methods different from ours.

We describe the function properties of a bounded analytic function f if f is constant on some Gleason part $P(m)$. We assume $m \notin D$ from now on.

Lemma (3.1.3)[58]: If f is H^∞ and constant on $P(m)$, then for fixed $0 < r < 1$

$$\lim_{z\to m} \max_{w\in \bar{D}(z,r)} (1-|w|^2) f'(w) = 0.$$

Proof: Suppose that there is a net (z_α) in D such that $z_\alpha \to m$. Then $f \circ L_{z_\alpha} \to f \circ L_m$ uniformly on compacta from Hl. Since $f \circ L_m$ is constant, $f \circ L_m'(z) = \lim (f \circ L_{z_\alpha})'(z) = 0$.

Since for w in $\bar{D}(z_\alpha, r)$ there is a z in $\bar{D}(0,r)$ such that $w - L_{z_\alpha}(z)$, we get

$$\max_{\bar{D}(z_\alpha,r)} f'(w)(1-|w|^2) = \max_{\bar{D}(0,r)} \left|(f\circ L_{z_\alpha})'(z)\right| (1-|z|^2) \to 0.$$

Thus we have proved the lemma.

Lemma (3.1.4)[58]: Let f and g be in $H^\infty(D)$. If either f or g is constant on each thin part then for all $0 < r < 1$

$$\lim_{|z|\to 1} \min\{\max_{s\in\bar{D}(z,r)} (1-|s|^2)|f'(s)|, \max_{t\in\bar{D}(z,r)} (1-|t|^2)|g'(t)|\} = 0.$$

Proof. Suppose either f or g is constant on each thin part, but there are points (z_n) in D with $|z_n| \to 1$ such that for some $\varepsilon > 0$ and fixed $0 < r < 1$

$$\min\{\max_{s\in\bar{D}(z_n,r)} (1-|s|^2)|f'(s)|, \max_{t\in\bar{D}(z_n,r)} (1-|t|^2)|g'(t)|\} \geq \varepsilon \ \forall n. \qquad (*)$$

Clearly z_n may be chosen so that $\{z_n\}$ is a thin sequence.

Let m be in the closure of $\{z_n\}$ in $\mathcal{M}$. There is a subnet $\{z_{n_k}\}$ of $\{z_n\}$ converging to m in $\mathcal{M}$; then m is a thin part. Without loss of generality we may assume that f is constant on $P(m)$. It follows from Lemma (3.1.3) that

$$\lim_{z_{n_k}\to m} \max_{s\in\bar{D}(z_{n_k},r)} (1-|s|^2) f'(s) = 0.$$

The above equation contradicts $(*)$, so the proof is complete.

Theorem (3.1.5)[58]: Suppose f and g are in $H^\infty(D)$. Then the following are equivalent :

(c') either f or g is constant on each thin part P(m);

(f') $\lim_{|z|\to 1} \min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\} = 0.$

Proof: In fact Lemma (3.1.4) implies that

$$\lim_{|z|\to 1} \min\{(1-|z|^2)|f'(z)|, (1-|z|^2)|g'(z)|\} = 0,$$

provided that either f or g is constant on each thin part $P(m)$ except D. Suppose that a thin part $m \in \mathcal{M}\backslash D$ and neither f nor g is constant on $P(m)$. From Hl we may assume that $f \circ L_m'(0) \neq 0$ and $g \circ L_m'(0) \neq 0$. Thus there is a net $\{z_n\}$ in D coverging to m such that

$$\lim_{z_n\to m} (1-|z_n|^2) f'(z_n) = f\circ L_m'(0)$$

$$\lim_{z_n\to m} (1-|z_n|^2) g'(z_n) = g\circ L_m'(0)$$

Clearly z_n may be chosen so that $\{z_n\}$ is a thin part.

So

$$\lim_{|z|\to 1} \min\{(1-|z|^2)|f'(z)|,\ \ (1-|z|^2)|g'(z)|\}$$
$$= \min\{|f\circ L'_m(0)|,\ \ |g\circ L'_m(0)|\} > 0.$$

The above contradiction completes the proof

Lemma (3.1.6)[58]: Let $0 < r < 1$ and let f and g be functions in the Bloch space. If

$$\lim_{|z|\to 1} \min\{\max_{s\in \overline{D}(z,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(z,r)} (1-|t|^2)|g'(t)|\} = 0$$

then

$$\lim_{|z|\to 1} \int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w) = 0.$$

Proof. For $w \in D(z,r)$ we have

$$f(w)-\ f(z) = \int_0^1 f'[tw+(1-t)z](w-z)dt\,.$$

Thus

$$|f(w)-\ f(z)| \le |w-z| \int_0^1 |f'[tw+(1-t)z]|dt \le \max_{\overline{D}(z,r)} |f'(s)||w-z|. \qquad (1)$$

Because

$$|w-z| \le \operatorname{diam} D(z,r) \le C \inf_{D(z,r)} (1-|S|^2)\,|f'(s)| \qquad (2)$$

it follows from (1) that

$$|f(w)-f(z)| \le\ C \max_{\overline{D}(z,r)} (1-|S|^2)\,|f'(s)|$$

In fact the above inequality is also true if f is replaced by g. Thus

$$\int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w)$$
$$\le C^2 \int_{D(z,r)} [\max_{s\in \overline{D}(2,r)} (1-|s|^2)|f'(s)|][\max_{t\in \overline{D}(2,r)} (1-|t|^2)|g'(t)|]|k_2|^2 dA$$
$$\le C^2 \min\{\max_{s\in \overline{D}(z,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(Z,r)} (1-|t|^2)|g'(t)|\}$$
$$\times \max\{\max_{s\in \overline{D}(z,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(z,r)} (1-|t|^2)|g'(t)|\}.$$

Since f and g are in the Bloch space, there is a constant $B > 0$ such that

$$\max\{\max_{s\in \overline{D}(z,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(z,r)} (1-|t|^2)|g'(t)|\} \le B.$$

Therefore

$$\int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w)$$
$$\le BC^2 \min\{\max_{s\in \overline{D}(z,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(z,r)} (1-|t|^2)|g'(t)|\}.$$

By the hypothesis we obtain

$$\lim_{|z|\to 1} \int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w) = 0,$$

completing the proof.

Lemma (3.1.7)[58]: If f and g are in the Bloch space and for all $0 < r < 1$

$$\min\{\max_{s\in \overline{D}(2,r)} (1-|s|^2)|f'(s)|\,,\ \max_{t\in \overline{D}(z,r)} (1-|t|^2)|g'(t)|\} \to 0,$$

as $|z| \to 1$, then

$$\lim_{|z|\to 1}\int_{\overline{D}} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w) = 0.$$

Proof. Now we estimate the following integral for fixed $0 < r < 1$:

$$\int_{D\setminus D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA$$

$$\leq \left(\int_{D\setminus D(z,r)} |f(w)-f(z)|^2 |g(w)-g(z)|^2 |k_2|^2 dA\right)^{1/2}$$

$$\times \left(\int_{D\setminus D(z,r)} |k_2|^2 dA\right)^{1/2}.$$

It follows from [92] that there is a constant $C > 0$ such that

$$\left(\int_D |f(w)-f(z)|^4 |k_2|^2 dA\right)^{1/4} \leq C\|f\|_\beta \leq \left(\int_\beta |f(w)-f(z)|^4 |k_2|^2 dA\right)^{1/4}$$

$$\times \left(\int_\beta |g(w)-g(z)|^4 |k_2|^2\right)^{1/4} (1-r^2)^{1/2} \tag{3}$$

and

$$\left(\int_D |g(w)-g(z)|^4 |k_2|^2 dA\right)^{1/4} \simeq C\|g\|_\beta.$$

For any $\varepsilon > 0$ we may choose

$$\delta = \frac{\varepsilon^2}{(1+r)C^4\left(\|f\|_\beta + 1\right)^2\left(\|g\|_\beta + 1\right)^2}.$$

If $1 - r < \delta$ the inequality (3) implies

$$\int_{D\setminus D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA \leq C^2\|f\|_\beta\|g\|_\beta(1-r^2)^{1/2}$$

$$\leq C\|f\|_\beta\|g\|_\beta \frac{\varepsilon}{C\left(\|f\|_\beta+1\right)\left(\|g\|_\beta+1\right)} < \varepsilon.$$

Thus

$$\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA$$

$$\leq \int_{D\setminus D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA$$

$$+ \int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA$$

$$\leq \varepsilon + \int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA$$

If $1 - r < \delta$. Lemma (3.1.6) says

$$\lim_{|z|\to 1}\int_{D(z,r)} |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA\,(w) = 0$$

for fixed $0 < r < 1$. The above inequality gives

$$\lim_{|z|\to 1}\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA\,(w) \leq \varepsilon.$$

so

$$\lim_{|z|\to 1}\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w)=0$$

since ε is arbitrary. The proof is finished.

We state the following lemma which is proved in [92] and will be used in the proof of Lemma (3.1.9).

Lemma (3.1.8)[58]: Let

$$K=\sup\left\{\int_D |1-z\alpha|^{-6/5}(1-|\alpha|)^{-3/5} dA(\alpha): z\in D\right\}$$

Then $K<\infty$.

The following lemma will be used twice in the proof of Theorem (3.1.10).

Lemma (3.1.9)[58]: There is a constant $C>0$ such that

$$\int_D \frac{|f(w)-f(z)||g(w)-g(z)|}{|1-zw|^2\sqrt{1-|w|^2}} dA(w)$$
$$\leq \frac{C}{\sqrt{1-|z|^2}}\left[\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w)\right]^{1/12}$$

For all $z\in D$.

Proof: Fix $z\in D$, and make the change of variables by $\lambda=\phi_z(w)$ to get

$$\int_D \frac{|f(w)-f(z)||g(w)-g(z)|}{|1-zw|^2\sqrt{1-|w|^2}} dA(w)$$
$$=\frac{1}{\sqrt{1-|z|^2}}\left[\int_D \frac{|f\circ\phi_2(\lambda)-f(z)|^{\delta}|g\circ\phi_2(\lambda)-g(z)|^{\delta}}{|1-\bar{z}\lambda|\sqrt{1-|\lambda|^2}} dA(\lambda)\right]$$
$$\leq \frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{\delta}|g\circ\phi(\lambda)-g(z)|^{\delta} dA(\lambda)\right]^{1/6}$$
$$\times\left[\int_D |1-\bar{z}\lambda|^{-6/5}(1-|\lambda|^2)^{-3/5} dA(\lambda)\right]^{5/6}.$$

It follows from Lemma (3.1.8) that

$$\int_D \frac{|f(w)-f(z)||g(w)-g(z)|}{|1-zw|^2\sqrt{1-|w|^2}} dA(w)$$
$$\leq \frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{\delta}|g\circ\phi_2(\lambda)-g(z)|^{\delta} dA(\lambda)\right]^{1/6} K^{5/6}$$
$$\leq \frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)||g\circ\phi_2(\lambda)-g(z)| dA(\lambda)\right]^{1/12} K^{5/6}$$
$$\times\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{11}|g\circ\phi_2(\lambda)-g(z)|^{11} dA(\lambda)\right]^{1/12}$$

(by Cauchy-Schwarz inequality)

$$\leq \frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{\delta}|g\circ\phi_2(\lambda)-g(z)|^{\delta} dA(\lambda)\right]^{1/6} K^{5/6}$$
$$\times\left[\int_D |f\circ\phi_2(\lambda)-f(z)|^{22} dA(\lambda)\right]^{1/66}$$

$$\times\left[\int_D |g\circ\phi_2(\lambda)-g(z)|^{22}dA(\lambda)\right]^{1/66}$$

(by Cauchy-Schwarz inequality)

$$\le \frac{1}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)||g\circ\phi_2(\lambda)-g(z)|dA(\lambda)\right]^{1/12}$$

$$\times K^{5/6}\|f\|_\beta^{1/\delta}\|g\|_\beta^{1/\delta}$$

(this inequality comes from [92])

$$\le \frac{C}{\sqrt{1-|z|^2}}\left[\int_D |f\circ\phi_2(\lambda)-f(z)||g\circ\phi_2(\lambda)-g(z)|dA(\lambda)\right]^{1/12}.$$

The proof is complete.

Theorem (3.1.10)[58]: If f and g are in the Bloch space and

$$\lim_{|z|\to 1}\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2 dA(w)=0,$$

then $H_{\bar f}^* H_{\bar g}$ is compact.

Proof: For any $h\in L_a^2(D)$ and $z\in D$ we have

$$\left(H_f^* H_g h\right)(z)=\frac{1}{1-|z|^2}\langle H_f^* H_g h, k_z\rangle$$

$$=\frac{1}{1-|z|^2}\langle H_{\bar g}h, H_{\bar f}k_z\rangle$$

$$=\frac{1}{1-|z|^2}\langle\left(\bar g-\bar g(z)\right)h,\left(\bar f-\bar f(z)\right)k_z\rangle$$

$$=\int_D \frac{\left(f(w)-f(z)\right)\left(\bar g(w)-\bar g(z)\right)}{(1-z\bar w)^2}h(w)dA(w)$$

It is obvious that for fixed $0<r<1$ the operator S_r defined by

$$S_r h(z)=\int_D \frac{\left(f(w)-f(z)\right)\left(\bar g(w)-\bar g(z)\right)}{(1-z\bar w)^2}h(w)\chi_{D(0,r)}(z)dA(w)$$

is a compact operator from $L_a^2(D)$ to $L^2(D)$. In fact, S_r is a Hilbert-Schmidt operator because

$$\frac{\left(f(w)-f(z)\right)\left(\bar g(w)-\bar g(z)\right)}{(1-z\bar w)^2}\chi_{D(0,r)}(z)$$

is in $L^2(D\times D)$.

For any $h\in L_a^2(D)$ and $z\in D$ we have

$$\le\int_D \frac{\chi_{D\setminus rD}(w)|f(w)-f(z)||g(w)-g(z)||h(w)|}{|1-z\bar w|^2}dA(w)$$

$$\le\left[\int_D \frac{\chi_{D\setminus rD}(z)|f(w)-f(z)||g(w)-g(z)|}{|1-z\bar w|^2\sqrt{1-|w|^2}}dA(w)\right]^{1/2}$$

$$\times\left[\int_D \frac{\chi_{D\setminus rD}(z)|f(w)-f(z)||g(w)-g(z)|\sqrt{1-|w|^2}}{|1-z\bar w|^2}|h(w)|^2dA(w)\right]^{1/2} \quad (4)$$

Combining (4) and Lemma (3.1.9) gives

$$\left\|\left(H_{\bar f}^* H_{\bar g}-S_r\right)h\right\|^2$$

$$= \int_D \left|\left[\left(H_f^* H_g - S_r\right)h\right](z)\right|^2 dA(z)$$

$$\leq \int_D \left[\int_D \frac{\chi_{D\setminus rD}(z)|f(w)-f(z)||g(w)-g(z)|}{|1-z\bar{w}|^2\sqrt{1-|w|^2}} dA(w)\right]$$

$$\times \left[\int_D \frac{|f(w)-f(z)||g(w)-g(z)| - \sqrt{1-|w|^2}}{|1-z\bar{w}|^2} |h(w)|^2 dA(w)\right] dA(z)$$

(by Cauchy-Schwarz inequality)

$$\leq \int_{D\setminus rD} \frac{C}{\sqrt{1-|z|^2}} \left[\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2\, dA(w)\right]^{1/2}$$

$$\times \left[\int_D \int_D \frac{|f(w)-f(z)||g(w)-g(z)|\sqrt{1-|w|^2}}{|1-\bar{z}w|^2\sqrt{1-|z|^2}} |h(w)|^2 dA(w)\right] dA(z)$$

(by Lemma (3.1.9))

$$\leq C \sup_{z\in D\setminus rD} \left[\int_D |f(w)-f(z)||g(w)-g(z)||k_2|^2\, dA(w)\right]^{1/2}$$

$$\times \int_D \int_D \frac{|f(w)-f(z)||g(w)-g(z)|\sqrt{1-|w|^2}}{|1-\bar{z}w|^2\sqrt{1-|z|^2}} |h(w)|^2 dA(w) dA(z)$$

(by Lemma (3.1.9) again)

$$\leq C \sup_{u\in D\setminus rD} \left[\int_D |f(w)-f(u)||g(w)-g(u)||k_u|^2\, dA(w)\right]^{1/2}$$

$$\times \int_D |h(w)|^2 \frac{C\sqrt{1-|w|^2}}{\sqrt{1-|w|^2}} \left[\int_D |f(w)-f(z)||g(w)-g(z)||k_w|^2\, dA(z)\right]^{1/12} dA(w)$$

$$\leq C \sup_{u\in D\setminus rD} \left[\int_D |f(w)-f(u)||g(w)-g(u)||k_u|^2\, dA(w)\right]^{1/12}$$

$$\times C \sup_{u\in D} \left[\int_D |f(w)-f(u)||g(w)-g(u)||k_u|^2\, dA(w)\right]^{1/12}$$

It is easy to verify that

$$C \sup_{u\in D} \left[\int_D |f(w)-f(u)||g(w)-g(u)||k_u|^2\, dA(w)\right]^{1/2} = \mu^2$$

is bounded since f and g are in Bloch space. Thus

$$\left\|H_{\bar{f}}^* H_{\bar{g}} - S_r\right\| \leq \mu\, C \sup_{u\in D\setminus rD} \left[\int_D |f(w)-f(u)||g(w)-g(u)||k_u|^2\, dA(w)\right]^{1/24}.$$

Since

$$\lim_{|z|\to 1} \int_D |f(w)-f(z)||g(w)-g(z)||k_u|^2\, dA(w) = 0,$$

we have

$$\lim_{r\to 1} \left\|H_{\bar{f}}^* H_{\bar{g}} - S_r\right\| = 0;$$

so $H_{\bar{f}}^* H_{\bar{g}}$ is compact since S_r is compact for any $0 < r < 1$, completing the proof.

Although Theorem (3.1.11) is a corollary of Theorem (3.1.12), we give a proof of Theorem (3.1.11) by combining with the lemmas and Theorem (3.1.10).

Theorem (3.1.11)[58]: If f and g are in $H^{\infty}(D)$ and either f or g is constant on each Gleason part $P(m)$ of $\mathcal{M}$, then $H_{\bar{f}}^{*}H_{\bar{g}}$ is compact.

Proof. From Theorem (3.1.10) it suffices to prove

$$\lim_{|z|\to 1}\int_{D}|f(w)-f(z)||g(w)-g(z)||k_u|^2\,dA(w)=0. \quad (5)$$

Combining Lemmas (3.1.4) and (3.1.6) with Lemma (3.1.7) implies that the above equation (5) holds. So $H_{\bar{f}}^{*}H_{\bar{g}}$ is compact.

Theorem (3.1.12)[58]: If f and g are bounded harmonic functions on D and for each thin Part $P(m)$ of $\mathcal{M}$ either $f|_{P(m)}\in H^{\infty}|_{P(m)}$ or $g|_{P(m)}\in H^{\infty}|_{P(m)}$, then

(a) $H_f^{*}\,H_g$ is compact;

(f) $\lim\limits_{|z|\to 1}\min\{(1-|z|^2)|(\partial f/\partial\bar{z})(z)\},\{(1-|z|^2)|(\partial g/\partial\bar{z})(z)\}\ =0$

Proof. Since f and g are bounded harmonic functions on D, there are functions f_1, f_2, g_1, and g_2 in the Bloch space such that $f\ =\ f_1\ +\bar{f}_2$ and $g=g_1\ +\bar{g}_2$. Thus

$$H_f^{*}\,H_g=H_{f_2}^{*}\,H_{\bar{g}_2}$$

and $(\partial f/\partial\bar{z})(z)=\bar{f}_2'(z),(\partial g/\partial\bar{z})(z)=\bar{g}_2'(z)$.

Combining Lemmas (3.1.4)-(3.1.7) with Theorem (3.1.10) shows that it is sufficient to prove that for fixed $0\ <\ r\ <\ 1$

$$\lim_{|z|\to 1}\min\left\{\max_{s\in\bar{D}(z,r)}(1-|s|^2)|f_2(s)|\,,\max_{t\in\bar{D}(z,r)}(1-|t|^2)|g_2'(t)|\right\}=0. \quad (6)$$

Suppose that (6) does not hold. There are points $\{z_n\}\subset D$ and $\varepsilon>\ 0$ such that

$$\min\left\{\max_{s\in\bar{D}(z_n,r)}(1-|s|^2)|f_2'(s)|\,,\max_{t\in\bar{D}(z_n,r)}(1-|t|^2)|g_2'(t)|\right\}\geq\varepsilon$$

and $\{z_n\}$ has no accumulation points in D. Since there is a thin sub-sequence of $\{z_n\}$, we may assume that $\{z_n\}$ is thin. Let m be in the closure $\{z_n\}$ in $\mathcal{M}$. Without loss of generality we may assume $f|_{P(m)}\in H^{\infty}|_{P(m)}$ and $\{z_n\}$ converges to m. Let $\{w_n\}$ be points in D satisfying

$$1.\, w_n\in D(z_n,r)$$
$$2.\,(1-|w_n|^2|f'(w_n)|=\max_{s\in\bar{D}(z_n,r)}(1-|s|^2)|f_2'(s)|. \quad (7)$$

Since $\mathcal{M}$ is compact, there is a subnet of $\{w_n\}$ converging to some m_1. For convenience we may assume that $\{w_n\}$ converges to m_1. Since $\rho(z_n,w_n)\leq r$, then m_1 is in $P(m)$.
Since $f|_{P(m)}\in H^{\infty}|_{P(m)}$,we have

$$f|_{P(m_1)}\in H^{*}|_{P(m_1)}$$

Thus

$$\frac{\partial}{\partial\bar{z}}f\circ L_{m_1}(0)=0.$$

On the other hand

$$\frac{\partial}{\partial\bar{z}}f\circ L_{m_1}(0)=\lim_{w_n\to m_1}\frac{\partial}{\partial\bar{z}}f\circ L_{w_n}(0)=\lim_{w_n\to m_1}\frac{\partial}{\partial\bar{z}}(f_1\circ L_{w_n}+\bar{f}_2\circ L_{w_n})(0)$$
$$=\lim_{w_n\to m_1}(1-|w_n|2)\bar{f}_2'(w_n).$$

This contradicts (7). The proof is finished.

We first consider compactness of $H_{\bar{f}}^{*}H_{\bar{g}}$ for f and g in $H^{\infty}(D)$ and make use of the maps L_m to turn the compactness of $H_{\bar{f}}^{*}H_{\bar{g}}$ into the condition

$$H_{\bar{f}\circ L_m}^{*}H_{\bar{g}\circ L_m}=0$$

The following Lemma (3.1.13) is the partial result of [49].

Lemma (3.1.13)[58]: Let $m(z) = L_m(z)$ be in $P(m)$ for some z in D. Then there is a constant $c, |c| = 1$ such that

$$L_{m(z)}(cw) = L_m \circ \phi_z(w).$$

Proof. It follows from H^2 that L_m and $L_{m(z)}$ are one-to-one analytic maps from D onto the Gleason part $P(m)$. Thus $L_{m(z)}^{-1} \circ L_m \circ \phi_z: D \to D$ is an onto, one-to-one, analytic function, and $L_{m(z)}^{-1} \circ L_m \circ \phi_z(0) = 0$. It is well known that there is a constant c such that $|c| = 1$ and

$$L_{m(z)}^{-1} \circ L_m \circ \phi_z(w) = cw.$$

so $L_{m(z)}(cw) = L_m \circ \phi_z(w)$.

Although Proposition (3.1.14) is a corollary of Proposition (3.1.16), we give its proof since the proof is also interesting.

Proposition (3.1.14)[58]: If $H_{\bar{f}}^* H_{\bar{g}}$ is compact for f and g in $H^\infty(D)$, then

$$H_{\bar{f}\circ L_m}^* H_{\bar{g}\circ L_m} = 0$$

For all m in $\mathcal{M}\backslash D$.

Proof: For any $m \in \mathcal{M}\backslash D$ there is a net $\{z_n\} \subset D$ converging to m (Corona Theorem), so

$$f \circ L_{2_n}(w) - f \circ L_{2_n}(0) \to f \circ L_m(w) - f \circ L_m(0) \text{ pointwise,}$$
$$g \circ L_{2_n}(w) - g \circ L_{2_n}(0) \to g \circ L_m(w) - g \circ L_m(0) \text{ pointwise.}$$

In addition $f \circ L_m(z)$ and $g \circ L_m(z)$ are bounded on D. So for any bounded analytic function h

$$\lim_{2_n \to m} \int_D h(f \circ L_{2_n} - f \circ L_{2_n}(0))(\bar{g} \circ L_{2_n} - \bar{g} \circ L_{2_n}(0))\, dA$$
$$= \int_D h(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))\, dA.$$

On the other hand,

$$\int_D h(f \circ L_{z_n} - f \circ L_{z_n}(0))(\bar{g} \circ L_{z_n} - \bar{g} \circ L_{z_n}(0))\, dA$$
$$= \int_D (f - f \circ L_{z_n}(0))(\bar{g} - \bar{g} \circ L_{z_n}(0))\left|k_{z_n}\right|^2 h \circ L_{z_n}\, dA$$
$$= \langle H_{\bar{g}} h \circ L_{z_n} k_{z_n}, H_j k_{z_n}\rangle = \langle h \circ L_{z_n} k_{z_n}, H_{\bar{g}}^* H_j k_{z_n}\rangle. \quad (8)$$

Since $H_{\bar{f}}^* H_{\bar{g}}$ is compact

$$\left\|H_{\bar{g}}^* H_{\bar{f}} k_{z_n}\right\| \to 0 \text{ as } z_n \to m$$

Therefore

$$\langle h \circ L_{z_n} k_{z_n}, H_{\bar{g}}^* H_{\bar{f}} k_{z_n}\rangle \le \left\|h \circ L_{z_n} k_{z_n}\right\|\left\|H_{\bar{g}}^* H_{\bar{f}} k_{z_n}\right\|$$
$$\le \|h\|_m \left\|H_{\bar{g}}^* H_{\bar{f}} k_{z_n}\right\| \to 0 \text{ as } z_n \to m. \quad (9)$$

Combining (8) and (9) we get that

$$\lim_{z_n \to m} \int_D h(f \circ L_{z_n} - f \circ L_{z_n}(0))(\bar{g} \circ L_{z_n} - \bar{g} \circ L_{z_n}(0))\, dA = 0.$$

This implies that

$$\int_D h(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))\, dA = 0 \quad (10)$$

for all bounded analytic functions h. Replacing m by $m(z)$ in (10) we obtain

$$\int_D h(f \circ L_{m(z)} - f \circ L_{m(z)}(0))(\bar{g} \circ L_{m(z)} - \bar{g} \circ L_{m(z)}(0))\, dA = 0.$$

The above equality combined with Lemma (3.1.13) implies

$$\int_D h(f \circ L_m \circ \phi_z - f \circ L_m(0))(\bar{g} \circ L_m \circ \phi_z - \bar{g} \circ L_m(0))\, dA = 0.$$

Changing the variables by $\lambda = \phi_z(w)$ gives

$$\int_D h \circ \phi_z(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))|k_2|^2\, dA = 0.$$

We may subsitute $h \circ \phi_z$ for h to change the above equality into

$$\int_D h(f \circ L_m - f \circ L_m(0))(\bar{g} \circ L_m - \bar{g} \circ L_m(0))|k_2|^2\, dA = 0.$$

Thus

$$\langle h, H^*_{\bar{g}\circ L_m} H_{\bar{f}\circ L_m} k_2 \rangle = 0.$$

We know that $H^\infty(D)$ is dense in $L^2_a(D)$. So

$$H^*_{\bar{f}\circ L_m} H_{\bar{g}\circ L_m} k_2 = 0,$$

which implies that

$$H^*_{\bar{f}\circ L_m} H_{\bar{g}\circ L_m} = 0.$$

Before we generalize the above proposition for bounded harmonic functions f and g, we need the following lemma which is proved in [91].

Lemma (3.1.15)[58]: Let ϕ be a Mobius transformation from D onto D and define an operator U_ϕ on $L^2(D)$ by

$$U_\phi g(z) = g[\phi(z)][\phi'(z)].$$

Then

(a) U_ϕ is unitary,

(b) $PU_\phi = U_\phi P$.

Proposition (3.1.16)[58]: If f and g are bounded harmonic functions and $H^*_f H_g$ is compact, then

$$H^*_{f\circ L_m} H_{g\circ L_m} = 0$$

for all m in $\mathcal{M}\backslash D$.

Proof. Since f and g are harmonic functions, there are Bloch functions f_1, f_2, g_1, and g_2 such that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$. For any m in $\mathcal{M}\backslash D$ there is a net $\{z_n\}$ converging to m. Then

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}\left[f \circ L_{z_n}(z)\right] &\to \frac{\partial}{\partial \bar{z}}[f \circ L_m(z)] \quad \text{pointwise} \\ \frac{\partial}{\partial \bar{z}}\left[g \circ L_{z_n}(z)\right] &\to \frac{\partial}{\partial \bar{z}}[g \circ L_m(z)] \quad \text{pointwise.} \end{aligned} \tag{11}$$

Since $f \circ L_m$ and $g \circ L_m$ are bounded harmonic on D, there are Bloch functions f_3, f_4, g_1, and g_2 such that

$$f \circ L_m = f_3 + \bar{f}_4 \text{ and } \frac{\partial}{\partial \bar{z}}(f \circ L_m) = \bar{f}'_4,$$

$$g \circ L_m = g_3 + \bar{g}_4 \text{ and } \frac{\partial}{\partial \bar{z}}(g \circ L_m) = \bar{g}'_4.$$

So

$$\begin{aligned} \bar{f}_4(w) - \bar{f}_4(0) &= \int_0^1 \frac{\partial}{\partial \bar{z}}[f \circ L_m(tw)]dt \\ \bar{g}_4(w) - \bar{g}_4(0) &= \int_0^1 \frac{\partial}{\partial \bar{z}}[g \circ L_m(tw)]dt\,. \end{aligned} \tag{12}$$

Now

$$\langle H^*_{f d_m} H_{g d_m} k_0, h k_0\rangle = \int_D (f_4(w) - f_4(0))(\bar{g}_4(w) - \bar{g}_4(0))\, h dA(w). \quad (13)$$

For fixed h in $H^\infty(D)$ and any $\varepsilon > 0$ there is a r_0 in $(0,1)$ such that if $l > r > r_0$ then

$$\int_{D\backslash rD} |f_4(w) - f_4(0)||g_4(w) - g_4(0)|\,|h| dA(w) < \varepsilon$$

$$\int_{D\backslash rD} \left|f_2 \circ \phi_{z_n}(w) - f_2 \circ \phi_{z_n}(0)\right|\left|g_2 \circ \phi_{z_n}(w) - g_2 \circ \phi_{z_n}(0)\right||h| dA(w) < \varepsilon$$

for all z_n in D. Combining (11) with (12) implies

$$\left|\int_D [f_4(w) - f_4(0)][\bar{g}_4(w) - \bar{g}_4(0)]\, \bar{h} dA(w)\right|$$

$$= \left|\int_D |w|^2 \int_0^1 \int_0^1 \frac{\partial}{\partial \bar{z}} g \circ L_m(tw) \frac{\partial}{\partial z} \bar{f} \circ L_m(sw) dt\, ds\, \bar{h}(w)\, dA(w)\right|$$

$$\leq \varepsilon + \overline{\lim_{z_n \to m}} \left|\int_{rD} |w|^2 \int_0^1 \int_0^1 \frac{\partial}{\partial \bar{z}} g \circ L_{z_n}(tw) \frac{\partial}{\partial z} \bar{f} \circ L_{z_n}(sw) dt\, ds\, \bar{h}(w)\, dA(w)\right|$$

(by Fatou's lemma and (11))

$$\leq 2\varepsilon + \overline{\lim_{z_n \to m}} \left|\int_D \left[f_2 \circ L_{z_n}(w) - f_2(z_n)\right]\left[\bar{g}_2 \circ L_{z_n}(w) - \bar{g}_2(z_n)\right] dA(w)\right|$$

$$\leq 2\varepsilon + \overline{\lim_{z_n \to m}} \left|\langle H^*_f H_g l_{z_n}, h \circ \phi_{z_n} k_{z_n}\rangle\right|$$

$\leq 2\varepsilon$.

Since ε is arbitrary

$$\langle H^*_{f\circ L_m} H_{g\circ L_m} k_0, h k_0\rangle = 0.$$

Substituting $m(z)$ for m we have

$$\langle H^*_{f\circ L_{m(z)}} H_{g\circ L_{m(z)}} k_0, h k_0\rangle = 0.$$

Using Lemmas (3.1.13) and (3.1.15) we obtain

$$\langle H^*_{f\circ L_m} H_{g\circ L_m} k_z, h \circ \phi_z k_z\rangle = 0.$$

This implies

$$H^*_{f\circ L_m} H_{g\circ L_m} = 0.$$

Lemma (3.1.17)[58]: Suppose that f and g are bounded harmonic functions. If $H^*_f H_g = 0$ then for all $Z \in D$ and $\xi \in \partial D$

$$H^*_{f\circ\phi_z} H_{g\circ\phi_z} = 0 \quad \text{and} \quad H^*_{f_\xi} H_{g_\xi} = 0,$$

where $f_\xi(w) = f(\xi w)$.

Proof: Let ϕ be a Mobius function mapping D onto D. From Lemma (3.1.15) it is easy to verify that

$$U_\phi H^*_f H_g U^*_\phi = H^*_{f\circ\phi} H^*_{g\circ\phi}.$$

So this implies the lemma if $\phi(w)$ is replaced by $\phi_z(w)$ or ξw, respectively.

We comment on some facts that will be used in the proof of Theorem (3.1.19). Suppose that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$, where f_1 and g_1 are in Bloch space and Hardy space H^2. If $H^*_f H_g = 0$, then

$$\int_D [f_2 \circ \phi_z(w) - f_2(z)][\bar{g}_2 \circ \phi_z(w) - \bar{g}_2(z)] dA(w) = 0.$$

This is equivalent to

$$\int_D f_2(w)\bar{g}_2(w)|k_2|^2 dA(w) = f_2(z)\bar{g}_2(z). \quad (14)$$

Replacing f_2 and g_2 by $f_2 \circ \phi_\lambda$ and $g_2 \circ \phi_\lambda$ or $f_{2\xi}$ and $g_{2\xi}$ respectively, by Lemma (3.1.17) we have

$$\int_D f_2 \circ \phi_\lambda(w)\bar{g}_2 \circ \phi_\lambda(w)|k_2|^2 dA(w) = f_2 \circ \phi_\lambda(z)\bar{g}_2 \circ \phi_\lambda(z).$$

and

$$\int_D f_2(\xi w)\bar{g}_2(\xi w)|k_2|^2 dA(w) = f_2(\xi z)\bar{g}_2(\xi z),$$

where $z \in D$ and $\xi \in \partial D$.

We state the following lemma which is the special case of [91].

Lemma (3.1.18)[58]: Let f be a continuous function on the closed unit disk $\overline{D}$.

Then the following are equivalent:

(a) f is harmonic on D;

(b) for each z in D

$$f(z) = \int_D |k_2|^2 f(\xi) dA(\xi).$$

Theorem (3.1.19)[58]: Suppose f and g are bounded harmonic functions on D. If $H_f^* H_g = 0$, then either f or g is in $H^\infty(D)$.

Proof. Let $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$ where f_i and g_i are in the Bloch space and H^2. Then $H_f^* H_g = 0$ implies

$$H_{\bar{f}_2}^* H_{\bar{g}_2} = 0.$$

The remark after Lemma (3.1.17) gives

$$\int_D f_2 \circ \phi_\lambda(w)\bar{g}_2 \circ \phi_\lambda(w)|k_2|^2 dA(w) = f_2 \circ \phi_\lambda(z)\bar{g}_2 \circ \phi_\lambda(z)$$

and

$$\int_D f_2(\xi w)\bar{g}_2(\xi w)|k_2|^2 dA(w) = f_2(\xi z)\bar{g}_2(\xi z).$$

Set

$$G(z) = \int_{\xi \in \partial D} f_2(\xi z)\bar{g}_2(\xi z) d\theta/2\pi$$

and suppose $f_2 \circ \phi_\lambda(w) = \sum_{n=0}^{\infty} a_n(\lambda) w^n$ and $g_2 \circ \phi_\lambda(w) = \sum_{n=0}^{\infty} b_n(\lambda) w^n$. Then $\sum_{n=0}^{\infty} |a_n(\lambda)|^2 < \infty$ and $\sum_{n=0}^{\infty} |b_n(\lambda)|^2 < \infty$. Thus

$$G(z) = \sum_{n=0}^{\infty} a_n(0)\bar{b}_n(0)|z|^{2n}.$$

Since $\sum_{n=0}^{\infty} a_n(0)\bar{b}_n(0)|z|^{2n}$ converges unformly on $\overline{D}$, the function G is continuous. By (14) we get

$$\int_D G(w)|k_z(w)|^2 dA(w) = G(z).$$

It follows from Lemma (3.1.3) that $G(z)$ is harmonic. Let Δ denote the Laplace operator. It is easy to verify that

$$\Delta G(z) = 4 \sum_{n=0}^{\infty} n^2 a_n(0)\bar{b}_n(0)|z|^{2(n-1)}.$$

So $\Delta G(z) = 0$ implies that $a_n(0)\bar{b}_n(0) = 0, n > 1$. Similarily we can prove that $a_n(z)\bar{b}_n(z) = 0, n > 1$. Now we consider only $a_1(z)b_1(z) = 0$. Without loss of generality we may assume that there are points $\{w_n\}$ in $D(0,r)$ for some $0 < r < 1$ and $\{w_n\}$ has at least one accumulation in $D(0,r)$ such that $a_1(w_n) = 0$ for all n. In fact $a_1(z) = [f_2 \circ \phi_z]'(0) = (1-|z|^2)f_2'(z)$. Thus $f_2'(w_n) = 0$. Therefore $f_2'(w) = 0$ for all w in D. This means that f is constant. The proof is complete.

Considering Toeplitz operators on the Bergman space we interpret the theorem to mean that $T_{\bar{f}}^* T_g = T_{\bar{f}g}$ for bounded harmonic functions f and g if f either f or g is in $H^\infty(D)$. On the Hardy space the above result is true for all f and g in $L^\infty(\partial D)$. But on the Bergman space we do not know when $T_{\bar{f}}^* T_g = T_{\bar{f}g}$ is true for f and g in $L^\infty(\bar{D})$.

Now we turn to the proof of the following theorem.

Theorem (3.1.20)[58]: Suppose f and g are bounded harmonic functions on D. If $H_{\bar{f}}^* H_g$ is compact, then either $f \circ L_m$, or $g \circ L_m$ is in $H^\infty(D)$ for m in $\mathcal{M}\backslash D$.

Proof. Proposition (3.1.16) says that

$$H_{f\circ L_m}^* H_{g\circ L_m} = 0$$

for all m in $\mathcal{M}\backslash D$ if $H_{\bar{f}}^* H_g$ is compact. It follows from Theorem (3.1.19) that either $f \circ L_m$ or $g \circ L_m$ is in $H^\infty(D)$.

Theorem (3.1.20) gives that either f or g is in $H^\infty(D)|_{P(m)}$ on each thin part $P(m)$ since $b \circ L_m(z) = z$ for some b in $H^\infty(D)$. So far we have proved that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (f)$. We will prove $(d) \Leftrightarrow (e)$. The theorem of Axler and Shields makes the proof of the following theorem possible.

Theorem (3.1.21)[58]: Let f and g be bounded harmonic fuctions on D. The following are equivalent:

(c) For each thin part $P(m)$ except D, either $f|_{P(m)} \in H^\infty(D)|_{P(m)}$ or $g|_{P(m)} \in H^\infty(D)|_{P(m)}$;

(e) $H^\infty(D)[f] \cap H^\infty(D)[g] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ for each thin part $P(m)\}$.

Proof. That (c) implies (e) is obvious. Now we prove that (e) implies (c). Let$m$ in $\mathcal{M}\backslash D$ and $P(m)$ be an analytic disk. That $H^\infty(D)[f] \cap H^\infty(D)[g] \subset \{u \in C(\mathcal{M}): u|_{P(m)} \in H^\infty(D)$ for thin part $P(m)\}$ means that

$$H^\infty(D)[f] \cap H^\infty(D)[g]|_{P(m)} \subset H^\infty(D)|_{P(m)}, \tag{15}$$

In fact $H^\infty(D)[f] \cap H^\infty(D)[g] \circ L_m = H^\infty \circ L_m(D)(f \circ L_m) \cap H^\infty L_m(D)[g \circ L_m]$ and $H^\infty \circ L_m(D) = H^\infty$ since $P(m)$ is thin. Thus

$$H^\infty(D)[f] \cap H^\infty(D)[g] \circ L_m = H^\infty(D)[f \circ L_m] \cap H^\infty(D)[g \circ L_m], \tag{16}$$

The theorem of Axler and Shields says that if u and v are bounded harmonic but not analytic on D, then $H^\infty(D) + C(\bar{D}) \subset H^\infty(D)[u] \cap H^\infty(D)[v]$. Equations (15) and (16) imply

$$H^\infty(D)[f \circ L_m] \cap H^\infty(D)[g \circ L_m] \subset H^\infty(D).$$

Thus either $f \circ L_m$ or $g \circ L_m$ is in $H^\infty(D)$. We have finished the proof.

(i) From the proof of Theorem (3.1.11) we see that the theorem is also valid if D is replaced by the unit ball B_n in C^n. It is natural to ask if Theorem (3.1.20) is true on B_n. But no one knows what the Gleason parts of the maximal ideal space of $H^\infty(B_n)$ look like. In fact whether the Corona Theorem is valid on $\mathcal{M}(H^\infty(B_n))$ is unknown.

(ii) Looking at our proof carefully we observe that the main results are also valid on the weighted Bergman space using the following Proposition (3.1.22) instead of Lemma (3.1.8) in the above process.

Proposition (3.1.22)[58]: Let $\alpha, \beta > -1$. Then if $[\frac{1}{2} - (\alpha/2 - \beta/q)] < 1/p < 1$ and $1/q + 1/p = 1$, there is an $M > 0$ such that for all z in D

$$\int_D \frac{(1-|w|^2)^{P(\alpha/2-\beta/q)}}{|1-\bar{z}w|^2(1-|w|^2)^{P/2}} dA(w) < M.$$

Indeed we can obtain the result on the weigted Bergman space analogous to that on the Bergman space in [58] by means of Proposition (3.1.22). Before we state the following theorem, we define the weighted Bergman space and $\mathrm{VMO}_\partial(D)$. The weighted Bergman space $A_{2\alpha}(\alpha > -1)$ is defined by $\{f: f$ is analytic on the unit disk D and $\int |f(z)|^2(1-|z|^2)^\alpha dA(z) < \infty\}$, and $\mathrm{VMO}_\partial(D)$ is the following set

$$\{f \in L^1(D): \int \left|\tilde{f}(z) - f \circ \phi_z(u)\right| dA(u) \to 0 \text{ as } |z| \to 1\},$$

Where $\tilde{f}(z) = \int f(u)|k_z(u)|dA(u)$. Roughly speaking $\mathrm{VMO}_\partial(D)$ is the space of intergrable functions on D with vanishing mean oscillation near the boundary of D.

Theorem (3.1.23)[58]: Let $\alpha > -1$ and f be in $L^\infty(D)$. Then H_f and H_f are compact on weighted Bergman space $A_{2\alpha}$ iff f is in $\mathrm{VMO}_\partial(D)$.

(iii) From many recent results on Hankel operators and Toeplitz operators on the Bergman space it seems more natural to deal with the Toeplitz operators and Hankel operators with bounded harmonic symbols than with the symbols in $L^\infty(D)$.

Section (3.2): A Theorem of Brown–Halmos Type

For D denote the open unit disc in the complex plane. By L^2 we mean the Lebesgue space with respect to the normalized Lebesgue measure $dA = \frac{1}{\pi}\, dx\, dy$ on D. For f and g in L^2, $\langle f, g, \rangle$ will denote the usual L^2 inner product and $\left||f|\right|_2$ will denote the norm in L^2. By B^2, the Bergman space, we mean the subspace of L^2 consisting of the holomorphic functions on D. For a bounded function u on D we have the Toeplitz operator $T_u : B^2\ Q\ B^2$ given by $T_u f = P(uf)$ where $P: L^2\ Q\ B^2$ is the orthogonal projection. The function u is called the symbol of T_u. An operator that will arise in our study of Toeplitz operators is the Berezin transform, defined for any integrable function f on D by the formula

$$Bf(z) \int_D f \frac{1\, z - \zeta}{1-\bar{z}\zeta}\ dA(z).$$

If we make a change of variables we see that

$$Bf(z) = (1 - |z|^2)^2\ F \int_D \frac{f(\zeta)}{|1 - z\bar{\zeta}|^4}\ dA(\zeta).$$

It is well known that $Bu = u$ for any harmonic function u. However, B is not a projection onto the harmonic functions, that is, Bu is not always harmonic. In fact, if $v = Bu$ is harmonic then $B(u - v) = 0$ since B reproduces harmonic functions. It is easily seen that the Berezin transform is injective and hence $u = v$. In other words Bu is harmonic if and only if u is harmonic. We also have the kernel functions k_w for each $w \in D$ defined by $k_w(z) = 1/(1 - z\overline{w})^2$. The relation of these kernel functions with the projection P is the following: if $f \in L^2$ then $(Pf)(z) = Of\langle, kzP. \rangle$ In particular, if $f \in B^2$ then $f(z) = \langle f, k_z \rangle$. We shall denote the laplacian by $\varDelta = \partial^2/(\partial z\, \partial\bar{z})$ and the invariant laplacian by $\widetilde{\varDelta} = (1-|z|^2)^2 \varDelta$.

There is an extensive literature on Toeplitz operators on the Hardy space H^2. (See [98] for the definitions of the Hardy space and their Toeplitz operators.) In the Hardy space (and in the Bergman space as well, see property 2, below) it is routine that if $\overline{u}$ or v is holomorphic then $T_u T_v = T_{uv}$. I_n [98] it was shown by A. Brown and P. Halmos that, in the Hardy space case, the converse is true. That is, if $T_u T_v = T_w$ then one of the two symbols $\overline{u}$ or v must be holomorphic and in this case $w = uv$. From this they easily deduce that if $T_u T_v = 0$ then one of the symbols u or v must be identically zero. There are many other interesting corollaries to their result. Returning to the Bergman space case, it has been an open problem for some time to determine if there is a theorem of Brown–Halmos type for Toeplitz operators. We show that in general there is not. We show that there are functions u, v and w which are continuous on the closed unit disc with $T_u T_v = T_w$ but neither $\bar{u}$ nor v is holomorphic. This example will be given in Some Examples. We do have a theorem of Brown–Halmos type if we put some restrictions on the symbols. We show, in Corollary (3.2.12), that if f and g are bounded harmonic functions and $T_f T_g = 0$ then one of the two symbols f or g is identically zero. This ‘‘zero product’’ problem for arbitrary bounded symbols f and g is still open.

Next we describe the results.

Theorem (3.2.1)[96]: Suppose f and g are bounded harmonic functions and that h is a bounded C^2 function with the property that $\widetilde{\Delta}\, h$ is also bounded in D. Assume that $T_f T_g = T_h$, then one of the following holds:

f is conjugate holomorphic g is holomorphic and in either case $h = fg$.

We will make two comments on Theorem (3.2.1). There is a general feeling that if a theorem from the Hardy space theory of Toeplitz operators is not true for all Bergman space Toeplitz operators then it should be true for those operators whose symbols come from the algebra $\mathcal{U}$. Here $\mathcal{U}$ is the uniform closure of the algebra generated by the bounded harmonic functions. This is not the case for Theorem (3.2.1) because in our counterexample, alluded to above, the symbols are continuous on the closed disc and hence belong to $\mathcal{u}$. The other comment has to do with the fact that the function h in Theorem (3.2.1) is required to satisfy a much weaker condition than is required of f and g. This leads us to ask if Theorem (3.2.1) remains true if we just require that the functions f, g and h all have their invariant laplacians bounded in D. The answer is no. We will give an example of functions f, g and h all of which are of class C^2 up to the boundary of D and such that $T_f T_g = T_h$ but neither $\check{f}$ nor g is holomorphic.

From Theorem (3.2.1) we get the following results on products of Toeplitz operators, some of which are parallel to results in [98].

The next rephrasing of Corollary (3.2.12) is a cancellation law for Toeplitz operators.

Corollary (3.2.2)[96]: If f, g and h are bounded harmonic symbols such that $T_f T_g = T_f T_h$ and f is not identically 0 then $g = h$.

The next corollary says that if a Toeplitz operator with bounded harmonic symbol has an inverse of the same type then this can only happen in the most tribyl way

Corollary (3.2.3)[96]: If f and g are bounded and harmonic and $T_f T_g = I$ then either f and g are both holomorphic or they are both conjugate holomorphic and in either case $f = \frac{1}{g}$.

The next corollary says there are no idempotent Toeplitz operators with bounded harmonic symbol other than the obvious ones.

Corollary (3.2.4)[96]: If f is bounded and harmonic and $T_f^2 = T_f$ then $f — 0$ or $f — 1$.

Our last corollary was proved by Zheng in [58] Theorem 5, by a different method.

Corollary (3.2.5)[96]: If f and g are bounded harmonic symbols and $T_f T_g = T_{fg}$ then either g is holomorphic or f is conjugate holomorphic.

We would like to point out that our method, when applied to the Hardy space case, gives a simple ‘‘function theoretic’’ proof of the Brown–Halmos theorem. This proof will be given in the paragraph following the proof of Proposition (3.2.6) This proof clarifies, for us, the differences between the Hardy space and Bergman space cases.

In Proposition (3.2.6) we give a pair of function theoretic identities involving f, g and h that are equivalent to $T_f T_g = T_h$, in the case that f and g are bounded harmonic functions and h is only assumed to be nearly bounded in D (see below for the definition of nearly bounded). The proof of Theorem (3.2.1) is based on an analysis of these identities.

We list here some well known and easy properties of Toeplitz operators:

(i) If $T_u = 0$ then $u = 0$ almost everywhere.

(ii) If f is holomorphic then $T_u T_f = T_{uf}$, and $T_{\overline{f}} T_u = T_{\overline{f}u}$ for any u.

(iii) $T_u^* = T_{\overline{u}}$.

(iv) If f is holomorphic and not identically zero then T_f is one to one.

(v) If g $\in B$ and $w \in D$ then $P(\overline{\mathrm{g}}k_w) = \overline{\mathrm{g}}(w)k_w$.

A good reference for (ii) through (v) above is Axler's survey [97]. Property (i) does not seem to be stated specifically but it is very easy: $T_u = 0$ implies that u is orthogonal to all polynomials (in z and $\overline{z}$) and hence $u = 0$ almost everywhere since such polynomials are dense in L^2 . Before turning to the proofs of our results we need to say a few words about Toeplitz operators with unbounded symbols. Even though we are interested primarily in operators with bounded symbol, operators with unbounded symbol arise naturally. In contrast to the Hardy space case, unbounded symbols can give rise to bounded operators on the Bergman space. For example if $F \in L^1(D)$ and has compact support K in D then we can define $T_F f(z) = \int_D (F(\zeta) f(\zeta)/(1 - z\overline{\zeta})^2) \, dA(\zeta)$. Then

$$|T_F f(z)| \le C \int_K |F| dA(\sup K \ |f|) \le C_1 \int_K |F| \, dA \, ||f||_2 .$$

Here the last inequality follows from the fact that for $f \in B^2$ the L^2 norm dominates the sup norm over any compact set. This says that the sup norm of T_F f is dominated by a constant times the L^2 norm of f and hence $||T_F f||_2 \le C \, ||f||_2$ for some constant C. More generally, $if\ F \in L^1(D)$ and there is an $r < 1$such that f is bounded on $z\, r < |z| < 1\}$ then T_F is bounded on B^2 because F can be written as an L^1 function with compact support plus a bounded function. Such a function will be called ''nearly bounded''. A function of this kind will arise in the construction of our counterexamples. The basic properties listed above for bounded symbols continue to be true for nearly bounded symbols but the only one we will use is the following: if u is nearly bounded and g is bounded and holomorphic then $T_u T_g = T_{u\mathrm{g}}$, but this is obvious.

We prove the identities on which our other results are based. If f is a bounded complex valued harmonic function defined in D then there are holomorphic functions f_1 and f_2 such that $f = f_1 + f_2$. This decomposition is unique if we require $f_2(0) = 0$. Of course, f_1 and f_2 are not necessarily bounded but they are certainly Bloch functions.

Proposition (3.2.6)[96]: Suppose that $f = f_1 + \overline{f_2}$, $\mathrm{g} = \mathrm{g}_1 + \overline{\mathrm{g}_2}$ are bounded harmonic functions with f_i, g_i holomorphic and h is nearly bounded in D. Then the following are equivalent.

(i) $T_f T_g = T_h$.

(ii) For all $z \in D\ we$ have

$$f_1(z)\mathrm{g}_1(z) + \overline{f}_2(z)\overline{g}_2(z) + f_1(z)_2(z) = B(h - \overline{f}_2\, g_1)(z).$$

(iii) For all $(z, w) \in D \times D$ we have

$$f_1(z)\mathrm{g}_1(z) + \overline{f_2}(\overline{w})g_2(\overline{w}) + f_1(z)\mathrm{g}_2(\text{-}\overline{w})$$

$$= (1 - zw)^2 \int \frac{\int_D \left(h(\zeta) - f_2(\zeta) g_1(\zeta) \right)}{\left(1 - \bar{\zeta} z\right)^2 (1 - \zeta w)^2} \, dA(\zeta).$$

Proof. Now $T_f T_g = T_h$ if and only if $T_f T_g k_w = T_h k_w$ for all $w \in D$. Using Property 5 above we see that

$$T_g k_w = P(g_1 k_w + \bar{g}_2 k_w) = g_1 k_w + \bar{g}_2(w) k_w.$$

It now follows from another application of Property 5 that

$$T_f T_g k_w = P\left((f_1 + \bar{f}_2)(g_1 k_w + \bar{g}_2(w) k_w) \right)$$
$$= f_1 g_1 k_w + \bar{g}_2(w) f_1 k_w + \bar{g}_2(w) \bar{f}_2(w) k_w + P(\bar{f}_2 g_1 k_w).$$

So we see that $T_f T_g = T_h$ is equivalent to

$$f_1(z) g_1(z) + \bar{f}_2(w) \bar{g}_2(w) + f_1(z) \bar{g}_2(w) + \frac{1}{k_w(z)} P\left(\bar{f}_2 g_1 k_w\right)(z)$$
$$= \frac{1}{k_w(z)} P(h k_w)(z),$$

for all z, w in D. But this is just Eq. (iii) with w replaced by $\bar{w}$. This shows that (i) and (iii) are equivalent. If we let $w = \bar{z}$ in (iii) we get (ii). It remains to show that (ii) implies (iii). Both sides of Eq. (iii) are holomorphic in (z, w) in the bidisc $D \times D$. Assuming (ii) they are equal on $\{(z, w) : w = \bar{z}\}$ and hence they are equal on the bidisc. This finishes the proof of the proposition. Before continuing with the proof of Theorem (3.2.1) we will discuss what happens when we apply our method to the Hardy space case. If f, g and h are L^∞. functions on the circle, then we can write $f = f_1 + \bar{f}_2$ where f_1 and f_2 are in $H^2 \cap BMO$ and similarly for g and h. If we let $S_z(e^{i\theta}) = 1/(1 - \bar{z} e^{i\theta})$ be the Szego kernel and P_S the Szego projection of L^2 onto H^2 then application of the method of Proposition (3.2.6) leads to:

$$f_1(z) g_1(z) + \bar{f}_2(z) \bar{g}_2(z) + f_1(z) \bar{g}_2(z) + \frac{1}{S_z(z)} P_S\left(\bar{f}_2 g_1 S_z\right)(z) = h(z).$$

But now we see that $\left(\frac{1}{S_z}(z)\right) P_S(u S_z)(z)$ is Poisson integral of u for any u which is integrable on the circle. So we see that every term in the above display with the exception of $f_1 \bar{g}_2$ is obviously harmonic. It follows that $f_1 \bar{g}_2$ is harmonic from which it follows that f_1 or g_2 is constant. This is the same as saying that f is conjugate holomorphic or that g is holomorphic. In the Bergman space case the Berezin transform appears rather than the Poisson integral and since the Berezin transform does not always yield harmonic functions, we have some more work to do.

Now assume that the hypotheses of Theorem (3.2.1) hold. From Proposition (3.2.6)(ii) we know that

$$f_1 g_1 + \bar{f}_2 \bar{g}_2 + f_1 \bar{g}_2 = B\left(h - \bar{f}_2 g_1\right).$$

Since B reproduces harmonic functions we see that

$$f_1 \bar{g}_2 = B(u),$$

where $u = h - \bar{f}_2 g_1 - f_1 g_1 - \bar{f}_2 \bar{g}_2$.

Notice that $\tilde{\Delta} u = \tilde{\Delta} h - \tilde{\Delta} \bar{f}_2 g_1$ is bounded.

This is so because $\tilde{\Delta} h$ is bounded by assumption and

$$\tilde{\Delta} \bar{f}_2 g_1(z) = (1 - |z|^2)^2 \bar{f}_2'(z) g_1'(z)$$

is bounded since f_2 and g_1 are Bloch functions. We want to conclude that f_1 or g_2 is constant. This will follow from the following

Proposition (3.2.7)[96]: Suppose that f and g are holomorphic in D and $f\bar{g} = Bu$ where $u \in L^1(D) \cap C^2(D)$ and $\widetilde{\Delta}u \in L^\infty.(D)$ then either f is constant or g is constant.

In the proof of Proposition (3.2.7) we will want to use the fact that the invariant laplacian commutes with the Berezin transform. This last fact has been known for some time, (see the discussion in [99]); however, the proofs we have been able to find are given only for functions of compact support or are based on the fact that B is in some sense a function of $D\widetilde{\Delta}$. Since this fact is crucial to our argument we have decided, with no claim to originality, to include a simple direct proof of what we need.

Lemma (3.2.8)[96]: Suppose that u is twice continuously differentiable in D and u and $\widetilde{\Delta}u$ are in $L^1(D)$ then $\widetilde{\Delta}Bu = B(\widetilde{\Delta}u)$.

Proof. We fix $0 < r < 1$ and $z \in D$ and consider

$$\frac{\int_{D_r} \left((r^2 - |w|^2)^2 \Delta u(w)\right)}{|1 - \overline{w}\ z|^4}\ dA(w),$$

where D_r is the disc of radius r centered at the origin. By Green's theoremthis is equal to

$$\int_{D_r} u(w)\Delta_w \frac{(r^2 - |w|^2)}{|1 - \bar{w}z|^4}\ dA(w)$$

(the boundary terms are 0 because $(r^2 - |w|^2)^2$ and its normal derivative both vanish on the boundary).

Now $|(r^2 - |w|^2)^2 \Delta u| \leq |(1 - |w|^2)^2 \Delta u|$ in D_r so we may take the limit as $r \to 1$ under the integral sign in the first integral.

Now for each fixed $z \in D, \Delta_w((r^2 - |w|^2)\text{^}2\ /|1 - \bar{w}z|^4)$ converges pointwise and boundedly to $\Delta_w((1 - |w|^2)^2/|1 - \bar{w}z|^4)$ as $r \to 1$. So we obtain that

$$\int_D u(w)\Delta_w \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^4}\ dA(w) = \int_D \frac{\widetilde{\Delta}u(w)}{|1 - \bar{w}z|^4}\ dA(w).$$

In the first integral we now use the remarkable but easily verified identity

$$\Delta_w \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^4} = \Delta_z \frac{(1 - |z|^2)^2}{|1 - \bar{w}z|^4}.$$

If we multiply the resulting equation by $(1 - |z|^2)^2$ we see that the lemma is proved.

Now we turn to the proof of Proposition (3.2.7). We are assuming that $f\bar{g} = Bu$. We take the invariant laplacian of both sides of this identity and we arrive at (after dividing by $(1 - |z|^2)^2$),

$$f'(z)\bar{g}'(z) = \int_D \frac{\bar{\Delta}u(\xi)}{\left|1 - \bar{\xi}z\right|^4}\ dA(\xi).$$

Next we ''complexify'' this identity.

Lemma (3.2.9)[96]: For all $z, w \in D$ we have

$$f'(z)\bar{g}'(\bar{w}) = \int_D \frac{\Delta u(\xi)}{\left(1 - \bar{\xi}z\right)^2 (1 - \xi w)^2}\ dA(\xi).$$

Proof. The functions on either side of the displayed equation are holomorphic in the bidisc $\{(z, w): |z| < 1, |w| < 1\}$ and they are equal on the subset $\{(z, \bar{z})\}$ and hence are equal on the whole bidisc.

Proceeding with the proof of Proposition (3.2.7) we take the identity of the Lemma (3.2.9) and we differentiate k times with respect to w and then let $w = 0$. We arrive at

$$\int_D \frac{\xi^k \sigma(\xi)}{\left(1 - \bar{\xi} z\right)^2} \, dA(\xi) = C_k f'(z),$$

for some constants $C_k, k = 1, 2, \ldots$, where $\sigma(\xi) = \widetilde{\Delta} u(\xi)$.

Consider now the Toeplitz operator T_σ with the possibly non-harmonic bounded symbol $\sigma(\xi)$. The above display tells us that $T_\sigma(\xi^k)$ is a multiple of f' for all non-negative integers k. There are two possibilities, $T\sigma(\xi^k) = 0$ for all k, or not. If the first holds then $T_\sigma p = 0$ for all polynomials and hence $T\sigma = 0$ on the Bergman space and hence $\sigma = 0$. In the other case $T_\sigma(\xi^k) \neq 0$ for some k. That is, some $C_k \neq 0$. This means that $f' \in B^2$ since it is a multiple of $T_\sigma(\xi^k)$ for some k. So if s were not zero then Ts would be a rank one operator. So we need to know that there are no rank one Toeplitz operators with bounded symbol.

Lemma (3.2.10)[96]: If σ is a bounded (not necessarily harmonic) function in D and dim $T_\sigma B^2$ [1 then $\sigma \equiv 0$.

Proof. The proof depends on the following idea due to R. Rochberg valid for any bounded function σ: $If\ w = u + iv$ and $\hat{\sigma}$ denotes the Fourier transform of s then

$$\hat{\sigma}(v, u) = \int_D e_{-\frac{w}{2}} \, \overline{e_{\frac{w}{2}} \sigma} \, dA = \langle T_\sigma e_{-\frac{w}{2}}, e_{\frac{w}{2}} \rangle,$$

where $e_w(z) = e^{wz}$. Now the hypothesis of this lemma implies that

$$T_\sigma f = \langle f, \emptyset \rangle F$$

for some $\emptyset, F \in B$ so we see that

$\hat{\sigma}(v, u) = \langle e_{-\frac{w}{2}}, \emptyset \rangle \langle \boldsymbol{F}, e_{\frac{w}{2}} \rangle = G(w) \bar{H}(w),$

where G, H are entire functions. So we have $|\hat{\sigma}| = |G\bar{H}| = |GH|$. But $\hat{\sigma}$ is continuous and goes to 0 at ∞. and GH is entire. It follows that GH and hence $\hat{\sigma}$ is identically 0 from which it follows that $\sigma \equiv 0$.

So we see that in any case $\sigma \equiv 0$. This means that $\Delta u \equiv 0$ and hence that $f' \bar{g}' \equiv 0$ which implies that f is constant or g is constant. This finishes the proof of Proposition (3.2.7) and hence the proof of Theorem (3.2.1).

It is natural to ask if the hypothesis on $\widetilde{\Delta} u$ in Proposition (3.2.7) is necessary. Pursuing this question will lead us to the examples mentioned. The simplest question one could ask is: does there exist a function $u \in L^1(D)$ such that $z\bar{z} = Bu(z)$? The answer is yes, with $u(\xi) = 1 - log\ 1/|\xi|^2$. To see this we need to show that if $v(\xi) = log\ 1/|\xi|^2$ then $Bv(z) = 1 - |z|^2$. If we use the second of the two formulas given for Bv we want to show that

$$(1 - |z|^2)^2 \int_D \frac{log\ |\xi|^2}{\left|1 - \bar{\xi} z\right|^4} \, dA(\xi) = |z|^2 - 1.$$

Since $\left|1 - \bar{\xi} z\right|^{-4} \sum_{n,k} (n+1)(k+1)\left(\bar{\xi} z\right) n \, (\xi \bar{z})^k$ and since v is radial we see that

$$\int_D \frac{log\ |\xi|^2}{\left|1 - \bar{\xi} z\right|^4} \, dA(\xi) = \sum_n (n+1)^2 \int_D |\xi|^{2n} \log|\xi|^2 \, dA(\xi) |z|^{2n}.$$

Now the integral in the right hand side of the above expression is easily calculated to be $-1/(n+1)^2$. So the sum on the right hand side of the above display is $-1/1 - |z|^2$. Multiplying by $(1 - |z|^2)^2$ we see that $Bv(z) = 1 - |z|^2$, as claimed.

But now if we recall the equivalence of (i) and (ii) of Proposition (3.2.6) we see that we have

$$T_z T\bar{z} = T_{u(z)}$$

where $u(z) = 1 - log\, 1/|z|^2$. Of course u is not bounded but it is nearly bounded so Proposition (3.2.6) applies. Now if we compose both sides of the above display on the right by T_z we get

$$T_z T_{|z|^2} = T_{zu(z)}.$$

This equation is of the form $T_f T_g = T_h$ where f, g and h are continuous on the closed disc but neither $\bar{f}$ nor g is holomorphic. If we compose on the right again by T_{z^2} we get

$$T_z T_{\bar{z}z^3} = T_{z^3}\, u(z).$$

In this equation of the form $T_f T_g = T_h$, all three symbols have bounded invariant laplacian since they are all of class C^2 in the closed disc but neither $\bar{f}$ nor g is holomorphic.

Next we discuss the proofs of the corollaries.

Corollary (3.2.11)[96]: $If\ f, g$ and h are bounded harmonic functions and $T_f T_g = T_h$ then one of the following holds:

(i) f and g are holomorphic

(ii) f and g are conjugate holomorphic

(iii) f is constant

(iv) g is constant.

Proof. Theorem (3.2.1) tells us that f is conjugate holomorphic or g is holomorphic. Suppose g is holomorphic then $fg = h$. In particular, fg is harmonic. $\Delta fg = \frac{\sigma f}{\sigma \bar{z}}\, g'$. It follows that f is holomorphic as well or that g is constant. If f is conjugate holomorphic the argument is similar.

Corollary (3.2.12)[96]: If f and g are bounded harmonic functions and $T_f T_g = 0$ then either $f —\ 0$ or $g —\ 0$.

Proof. By Theorem (3.2.1) $fg = 0$ in D and since f and g are harmonic one of them is identically 0.

Corollary (3.2.2) follows from Corollary (3.2.12) since the hypothesis of Corollary (3.2.2) implies that $T_f T_{g-h} = 0$. Corollary (3.2.3) follows by observing that $I = T_h$ where h is the constant function 1 and then applying Corollary (3.2.12). The proof of Corollary (3.2.4) is similar. To prove Corollary (3.2.5) we need only check that $\widetilde{\Delta} fg \in L^\infty$. but this follows since f and g are bounded and harmonic.

Section (3.3): Semicommutators of Toeplitz Operators with Harmonic Symbols

For dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on D that are also in $L^2(D, dA)$. For $z \in D$, the Bergman reproducing kernel is the function $K_z \in L_a^2$ such that

$$h(z) = \langle h, K_z \rangle$$

for every $h \in L_a^2$. The normalized Bergman reproducing kernel k_z is the function $K_z/\|K_z\|_2$. Here the norm $\|\ \|_2$ and the inner product $\langle\ ,\ \rangle$ are taken in the space $L^2(D, dA)$.

For $f \in L^\infty(D, dA)$, the Toeplitz operator T_f with symbol f is the operator on L_a^2 defined by $T_f h = P(fh)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2. We denote the semicommutator and commutator of two Toeplitz operators T_f and T_g by

$$(T_f, T_g] = T_{fg} - T_f T_g$$

and

$$[T_f, T_g] = T_f T_g - T_g T_f$$
respectively. Note that if $g \in H^\infty(D)$ (the set of bounded analytic functions on D), then T_g is just the operator of multiplication by g on L^2_a and hence $(T_f, T_g] = 0$ for any $f \in L^\infty(D, dA)$.

For a bounded operator S on L^2_a, the Berezin transform of S is the function $B(S)$ on D defined by
$$B(S)(z) = \langle Sk_z, k_z \rangle.$$
The Berezin transform $B(u)(z)$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u. In other words,
$$B(u)(z) = B(T_u)(z) = \int_D u\left(\frac{z-w}{1-\bar{z}w}\right) dA(w).$$
The last equality follows from the change of variable in the definition of the Berezin transform. The above integral formula extends the Berezin tranform to $L^1(D, dA)$ and clearly gives
$$B(u)(z) = u(z) \tag{17}$$
for any harmonic function $u \in L^1(D, dA)$.

Let Δ denote the Laplace operator $4\frac{\partial^2}{\partial z \partial z}$. A function h on D is harmonic if $\Delta h(z) \equiv 0$ on D. We use $\tilde{\Delta}$ to denote the invariant Laplace operator $(1 - |z|^2)^2\, 4\frac{\partial^2}{\partial z \partial \bar{z}}$. The invariant Laplace operator commutes with the Berezin transform [96], [103], which is useful in studying Toeplitz operators on the Bergman space [96].

An operator A on a Hilbert space H is said to have finite rank if the closure of Ran (A) of the range $A(H)$ of the operator has finite dimension. For a bounded operator A on H, define $rank(A) = dim\, Ran(A)$. If A has finite rank, then $rank(A) < \infty$.

We study the problem for which bounded harmonic functions f, g on the unit disk, the semicommutator $(T_f, T_g]$ or commutator $[T_f, T_g]$ has finite rank on the Bergman space. The analogous problem on the Hardy space has been completely solved in [93], [102]. We will reduce the problem to the problem of when a Toeplitz operator has finite rank. Although the problem on finite rank Toeplitz operators remains open, Ahern and Cuckovic [96] have shown that for $u \in L^\infty(D)$, if T_u has rank one then $u = 0$. One naturally conjectures that for $u \in L^\infty(D)$, if T_u has finite rank, then $u = 0$. We will show that this conjecture is true provided that u is a finite sum of products of an analytic function and a co-analytic function in $L^2(D, dA)$. Using the result we shall completely characterize finite rank semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols. The zero semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols has been completely characterized in [101] and [105]. In fact, we shall show that if the semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols has finite rank, then it must be zero. This is not the case on the Hardy space [93], [102]. Moreover on the Bergman space there exist nonzero compact semicommutators or commutators of two Toeplitz operators with bounded harmonic symbols [54], [105]. We will show that for two bounded harmonic functions f, g, if the product $T_f\, T_g$ has finite rank, then either f or g equals 0, which extends the result on the zero products of Toeplitz operators in [96].

We study Toeplitz operators with finite rank. For a family $\{A_n\}$ of operators on the Hilbert space H and an operator A on H, we say that A_n converges to A in weak operator topology, if for each $x, y \in H$,

$$\lim_{n\to}\langle A_n x, y\rangle = \langle Ax, y\rangle.$$

The following result is implicitly contained in [104]. We include a proof for completeness.

Lemma (3.3.1)[100]: Suppose that A_n and A are bounded operators on the Hilbert space H. If A_n converges to A in the weak operator topology, then

$$rank(A) \le \liminf_{n\to\infty} rank(A_n).$$

Proof. Let l denote $\liminf_{n\to\infty} rank(A_n)$. We need only consider the case $l < \infty$. We claim that $rank(A) \le l$. If this is false, we may assume that $rank(A) \ge l + 1$. Thus there are $(l + 1)$ elements $\{x_j\}_{j=1}^{l+1}$ in H such that $\{Ax_j\}_{j=1}^{l+1}$ are linearly independent and so

$$det\left[\langle Ax_i, Ax_j\rangle\right]_{(l+1)\times(l+1)} \neq 0$$

where $det\left[\langle Ax_i, Ax_j\rangle\right]_{(l+1)\times(l+1)}$ denotes the determinant of the $(l+1)\times(l+1)$ matrix $\left[\langle Ax_i, Ax_j\rangle\right]_{(l+1)\times(l+1)}$. Since A_n converges to A in the weak operator topology, for each i, j,

$$\lim_{n\to\infty}\langle A_n x_i, Ax_j\rangle = \langle Ax_i, Ax_j\rangle.$$

This gives

$$\lim_{n\to\infty} det\left[\langle A_n x_i, Ax_j\rangle\right]_{(l+1)\times(l+1)} = det\left[\langle Ax_i, Ax_j\rangle\right]_{(l+1)\times(l+1)}.$$

Thus for some large N,

$$det\left[\langle A_N x_i, Ax_j\rangle\right]_{(l+1)\times(l+1)} \neq 0, \tag{18}$$

but

$$rank(A_N) \le l. \tag{19}$$

So (19) gives that there are constants c_i with $\sum_{i=1}^{l+1}|c_i| \neq 0$ such that

$$\sum_{i=1}^{l+1} c_i A_N x_i = 0$$

Hence

$$c\left[\langle A_N x_i, Ax_j\rangle\right]_{(l+1)\times(l+1)} = 0$$

where c $= (c_1, \cdots, c_{l+1})$. This implies

$$\left[\langle A_N x_i, Ax_j\rangle\right]_{(l+1)\times(l+1)} = 0.$$

It contradicts (18) to complete the proof.

Theorem (3.3.2)[100]: Suppose that f is in $L^\infty(D)$ and equal to $\sum_{j=1}^{l} f_j(z)\overline{g_j(z)}$ for finitely many functions $f_j(z)$ and $g_j(z)$ analytic on the unit disk D. If T_f has finite rank, then $f = 0$.

Proof. First we will show that $T_{|f|^2}$ has finite rank. To do so, for each $0 < r < 1$, define $f_r(z) = f(rz)$. Let $g_r = \overline{f_r}$. Since

$$f(z) = \sum_{j=1}^{l} f_j(z)\overline{g_j(z)}$$

for finitely many functions $f_j(z)$ and $g_j(z)$ in L_a^2, we have

$$T_{fg_r} = T_{f\overline{\left(\sum_{j=1}^{l} f_j(rz)\overline{g_j(rz)}\right)}} = \sum_{j=1}^{l} T_{f\overline{f_j(rz)}g_j(rz)} = \sum_{j=1}^{l} T_{\overline{f_j(rz)}} T_f T_{g_j(rz)}.$$

The last equality follows from the basic properties of Toeplitz operators [43]

$$T_{\overline{h}} T_f = T_{\overline{h}f}$$

and

$$T_f\,T_h = T_{fh},$$

for $f \in L^\infty(D, dA)$ and $h \in H^\infty(D)$. If T_f has finite rank and $rank(T_f) = N$, then for each $0 < r < 1$,

$$rank(T_{fg_r}) \le Nl.$$

Thus

$$\limsup_{r\to 1} rank(T_{fg_r}) \le Nl.$$

Next we shall show that T_{fg_r} converges to $T_{|f|^2}$ in the weak operator topology. To do this, we observe that for each $z \in D$,

$$|f(z)g_r(z)| = |f(z)f(rz)| \le \|f\|_\infty^2,$$

and

$$\lim_{r\to 1^-} f(z)g_r(z) = |f(z)|^2.$$

By the dominant convergence theorem we have that for $h_1, h_2 \in L_a^2$,

$$\lim_{r\to 1^-}\int_D f(z)g_r(z)h_1(z)\overline{h_2(z)}dA(z) = \int_D |f(z)|^2 h_1(z)\overline{h_2(z)}dA(z),$$

to obtain

$$\begin{aligned}\lim_{r\to 1^-}\langle T_{fg_r}h_1, h_2\rangle &= \lim_{r\to 1^-}\langle fg_r h_1, h_2\rangle \\ &= \lim_{r\to 1^-}\int_D f(z)g_r(z)h_1(z)\overline{h_2(z)}dA(z) \\ &= \int_D |f(z)|^2 h_1(z)\overline{h_2(z)}dA(z) \\ &= \langle T_{|f|^2}h_1, h_2\rangle.\end{aligned}$$

This means that T_{fg_r} converges to $T_{|f|^2}$ in weak operator topology. By Lemma (3.3.1), we have that the Toeplitz operator $T_{|f|^2}$ with nonnegative function symbol has finite rank and its rank is at most Nl.

To finish the proof we need to prove that if the Toeplitz operator with nonnegative function symbol has finite rank, it must be zero. This was well known. For completeness, we include a proof here. Since $T_{|f|^2}$ has finite rank, the kernel of $T_{|f|^2}$ contains a nonzero function $h \in L_a^2$. Thus

$$\begin{aligned}0 &= \langle T_{|f|^2}h, h\rangle \\ &= \langle |f|^2 h, h\rangle \\ &= \int_D |f(z)|^2|h(z)|^2 dA(z)\end{aligned}$$

and so

$$|f(z)|^2|h(z)|^2 = 0$$

for a.e. $z \in D$. Noting that $h(z)$ is in the Bergman space, we conclude that $f = 0$ in $L^\infty(D, dA)$ to complete the proof.

For $f \in L^\infty(D, dA)$, the Hankel operator H_f with symbol f is the operator on L_a^2 defined by $H_f h = (I - P)(fh)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2. The relation between Toeplitz operators and Hankel operators is established by the following well-known identity:

$$(T_f, T_g] = H_{\bar f}^* H_g.$$

We shall reduce the problem of when a finite sum of products of two Hankel operators has finite rank to the problem of when a Toeplitz operator has finite rank.

For each bounded harmonic function f on the unit disk, f can be written uniquely as a sum of an analytic function and a co-analytic function on the unit disk D up to a constant. Let f_+ denote the analytic part and f_- the co-analytic part with $f_-(0) = 0$. In fact, both f_+ and $\overline{f_-}$ are in both the Hardy space H^2 and the Bloch space [43], [49].

For bounded harmonic functions f_i and g_i on the unit disk for $i = 1, \cdots, k$, define

$$\sigma(f_1, \cdots, f_k;\ g_1 \cdots, g_k) = \tilde{\Delta}[\sum_{i=1}^{k} (f_i)_- (g_i)_+].$$

For two bounded harmonic functions f and g on the unit disk, let $\sigma_{sc}(f, g)$ denote $\sigma(g;\ f)$ and $\sigma_c(f, g)$ denote $\sigma(f, -g;\ g, f)$. Easy calculations give

$$\sigma(f_1, \cdots, f_k;\ g_1 \cdots, g_k) = (1 - |z|^2)^2 \sum_{i=1}^{k} (f_i)'_- (g_i)'_+ \tag{20}$$

where $(f_i)'_- = \partial_{\bar{z}} f_i$. Hence

$$\begin{aligned}
\sigma_{sc}(f, g) &= \tilde{\Delta}(f_+ g_-) \\
&= (1 - |z|^2)^2 (\partial_z f)(\partial_{\bar{z}} g) \\
&= (1 - |z|^2) f'_+(z)(1 - |z|^2) g'_-(z), \\
\sigma_c(f, g) &= \tilde{\Delta}[f_- g_+ - f_+ g_-] \\
&= (1 - |z|^2)^2 [(\partial_{\bar{z}} f)(\partial_z g) - (\partial_z f)(\partial_{\bar{z}} g)] \\
&= (1 - |z|^2) f'_-(z)(1 - |z|^2) g'_+(z) - (1 - |z|^2) f'_+(z)(1 - |z|^2) g'_-(z).
\end{aligned}$$

Lemma (3.3.3)[100]: Suppose that f_i and g_i are bounded harmonic functions on the unit disk for $i = 1, \cdots, k$. Then $\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k)$ is in $L^\infty(D, dA)$.

Proof. Since f_i and g_i are bounded harmonic functions on the unit disk, $(f_i)_+, \overline{(f_i)_-}, (g_i)_+$ and $\overline{(g_i)_-}$ are in the Bloch space

$$B = \{h :\ h \text{ analytic on } D, \sup_{z \in D} (1 - |z|^2)|h'(z)| < \infty\}$$

(see [43]). (20) gives that $\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k)$ is in $L^\infty(D, dA)$.

Proposition (3.3.4)[100]: Suppose that f_i and g_i are bounded harmonic functions on D for $i = 1, \cdots, k$. If the finite sum $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$ of products of Hankel operators has finite rank, then $T_{\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k)}$ has finite rank.

Proof. For these bounded harmonic functions f_i, g_i on the unit disk, write

$$f_i = (f_i)_+ + (f_i)_-$$

and

$$g_i = (g_i)_+ + (g_i)_-,$$

where $(f_i)_+, (g_i)_+, \overline{(f_i)_-}$, and $\overline{(g_i)_-}$ are in the Hardy space H^2. By Lemma (3.3.3), $\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k)(z)$ is in $L^\infty(D, dA)$. Thus $T_{\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k)}$ is bounded on the Bergman space L^2_a.

We shall get the Berezin transform of $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$. First we calculate the Berezin transform of $B((T_f, T_g])(z)$ of the semicommutator $(T_f, T_g]$. By the basic properties of Toeplitz operators on the Bergman space [43], [14], we have

$$T_f k_z = \big(f_+ + f_-(z)\big) k_z,$$

for $z \in D$. Since f is harmonic in the unit disk, we also have

$$B(f)(z) = f(z).$$

For two bounded harmonic functions f, g on D, easy calculations give

$$
\begin{aligned}
B((T_f, T_g])(z) &= B(T_{fg} - T_f T_g)(z) \\
&= \langle fgk_z, k_z\rangle - \langle (g_+ + g_-(z))k_z, \bar{f}k_z\rangle \\
&= \langle [fg - f(g_+ + g_-(z))]k_z, k_z\rangle \\
&= \langle [f(g_- - g_-(z))]k_z, k_z\rangle \\
&= \langle [f_+g_- + f_-g_- - fg_-(z)]k_z, k_z\rangle \\
&= \langle f_+g_-k_z, k_z\rangle + \langle f_-g_-k_z, k_z\rangle - g_-(z)\langle fk_z, k_z\rangle \\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)B(f)(z) \\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)f(z) \\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)(f_+(z) + f_-(z)) \\
&= B(f_+g_-)(z) - f_+(z)g_-(z)
\end{aligned}
$$

for all $z \in D$. Noting

$$(T_f, T_g] = H_{\bar{f}}^* H_g,$$

we have

$$B(H_{\bar{f}}^* H_g)(z) = B(f_+g_-)(z) - f_+(z)g_-(z).$$

Thus

$$B(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j})(z) = B(\sum_{j=1}^{k} (g_j)_+(f_j)_-)(z) - \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).$$

Applying the invariant Laplace operator $\tilde{\Delta}$ to both sides of the above equation gives

$$
\begin{aligned}
&\tilde{\Delta}B(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j})(z) \\
&= [\tilde{\Delta}B(\sum_{j=1}^{k} (g_j)_+(f_j)_-)](z) - [\tilde{\Delta}\sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z)].
\end{aligned}
$$

Since the invariant Laplace operator commutes with the Berezin transform [96], we have

$$B(\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k))(z) = (1 - |z|^2)^2[\sum_{j=1}^{k} (g_j)'_+(f_j)'_-(z)] + \tilde{\Delta}B(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j})(z)$$

In other words, the above equality becomes

$$
\begin{aligned}
\langle T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)}k_z, k_z\rangle &= B(\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k))(z) \\
&= (1 - |z|^2)^2[\sum_{j=1}^{k} (g_j)'_+(z)(f_j)'_-(z)] + \tilde{\Delta}B(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j})(z).
\end{aligned}
$$

For two functions x and y in L_a^2, define the operator $x \otimes y$ of rank one to be

$$(x \otimes y)f = \langle f, y\rangle x$$

for $f \in L_a^2$. Then it is easy to verify

$$
\begin{aligned}
B(x \otimes y)(z) &= \langle (x \otimes y)k_z, k_z\rangle \\
&= (1 - |z|^2)^2\langle (x \otimes y)k_z, k_z\rangle \\
&= (1 - |z|^2)^2\langle k_z, y\rangle\langle x, k_z\rangle \\
&= (1 - |z|^2)^2 x(z)\overline{y(z)},
\end{aligned}
$$

for $z \in D$. If the semicommutator $\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j}$ has finite $rank\ N$, then there exist functions x_j and y_j in L_a^2 for $j = 1, \ldots, N$ such that

$$\sum_{j=1}^{k} H_{\overline{g_j}}^{*} H_{f_j} = \sum_{j=1}^{N} x_j \otimes y_j.$$

Thus

$$B(\sum_{j=1}^{k} H_{\overline{g_j}}^{*} H_{f_j})\,(z) \;=\; (1-|z|^2)^2(\sum_{j=1}^{N} x_j(z)\overline{y_j(z)}).$$

Observe

$$(1-|z|^2)^2(\sum_{j=1}^{N} x_j(z)\overline{y_j(z)}) = (\sum_{j=1}^{3N} \hat{x}_j(z)\overline{\hat{y}_j(z)})$$

where $\hat{x}_j$ and $\hat{y}_j$ are in the Bergman space L_a^2 . So

$$\langle T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} k_z, k_z\rangle$$

$$= (1-|z|^2)^2[\sum_{j=1}^{k} (g_j)'_+(z)(f_j)'_-(z)] + (1-|z|^2)^2(\sum_{j=1}^{3N} \hat{x}'_j(z)\overline{\hat{y}'_j(z)})$$

Dividing by $(1-|z|^2)^2$, we obtain

$$\langle T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} K_z, K_z\rangle = \sum_{j=1}^{k} (g_j)'_+(z)(f_j)'_-(z) + (\sum_{j=1}^{3N} \hat{x}'_j(z)\overline{\hat{y}'_j(z)}). \tag{21}$$

As in [96] we complexify the above identity. Write the left hand side as an integral as in [96] to get

$$\langle T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} K_z, K_z\rangle = \int_D \sigma(f_1,\cdots,f_k;\; g_1,\cdots,g_k)(\lambda)\frac{1}{|1-\tilde{z}\lambda|^4}\,dA(\lambda).$$

Since the right hand side of (21) and the above integral are real analytic functions of z and $\bar{z}$ we obtain

$$\langle T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} K_w, K_z\rangle = \sum_{j=1}^{k} (g_j)'_+(z)(f_j)'_-(w) + (\sum_{j=1}^{3N} \hat{x}'_j(z)\overline{\hat{y}'_j(w)}).$$

Differentiating both sides of the above equation l times with respect to $\bar{w}$ and then letting $w \;=\; 0$ give

$$T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} z^l \;= \sum_{j=1}^{k} a_{lj}(g_j)'_+(z) + \sum_{j=1}^{3N} b_{lj}\hat{x}'_j(z) \tag{22}$$

for some constants a_{lj}, b_{lj}.

Although some of $(g_j)'_+$ and $\hat{x}'_j$ may not be in L_a^2, we observe that for each $0 \;<\; r \;<\; 1$, all of $\left(g_j\right)'_+|_{rD}$ for $j \;=\; 1,\ldots,k$ and $\hat{x}'_j|_{rD}$ for $j \;=\; 1,\ldots,3N$ are in $L_a^2(rD, dA)$.

$$T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)}\; \mathit{has\ finite\ rank\ on\ the\ Bergman\ space}\ L_a^2.$$

If this claim is false, we may assume that there are $3N + k + 1$ linearly independent functions $\{\phi_\mu\}_{\mu=1}^{3N+k+1}$ in the range of $T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)}$. Thus for each $0 < r < 1, \{\phi_\mu\big|_{rD}\}_{\mu=1}^{3N+k+1}$ are also linearly independent in the space $L_a^2(rD, dA)$. Since analytic polynomials are dense in L_a^2, for each μ, there are analytic polynomials $p_{\mu l}$ such that $T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} p_{\mu l}$ converges to ϕ_μ. Thus $T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)} p_{\mu l}$ converges uniformly to ϕ_μ on each compact subset of the unit disk D. Noting that rD is contained in a compact subset of the unit disk, we have

$$\lim_{l\to\infty}\int_{rD}\left|T_{\sigma(f_1,\cdots,f_k;\,g_1,\cdots,g_k)}p_{\mu l}(z)-\phi_\mu(z)\right|^2 dA(z) = 0.$$

On the other hand, (22) gives that $T_{\sigma(f_1,\cdots,f_k;\,g_1,\cdots,g_k)}p_{\mu l}\big|_{rD}$ is contained in the subspace spanned by $\left(g_j\right)'_+|_{rD}$ and $\hat{x}'_j|_{rD}$ of $L^2_a(rD,dA)$. But the subspace has dimension at most $3N + k$. This contradicts that $\{\phi_\mu\big|_{rD}\}_{\mu=1}^{3N+k+1}$ are also linearly independent and hence gives that $T_{\sigma(f_1,\cdots,f_k;\,g_1,\cdots,g_k)}$ has finite rank to complete the proof.

Theorem (3.3.5)[100]: Suppose that f and g are bounded harmonic functions on the unit disk. The semicommutator $(T_f, T_g]$ has finite rank if and only if either $\bar{f}$ or g is analytic on the unit disk.

Proof. If either $\bar{f}$ or g is analytic on the unit disk, then $T_f T_g = T_{fg}$ and so the semi commutator $(T_f, T_g]$ equals 0.

If the semicommutator $(T_f, T_g]$ has finite rank, noting

$$(T_f, T_g] = H^*_{\bar{f}} H_g$$

by Proposition (3.3.4), the Toeplitz operator $T\_\sigma_{sc}(f,g)$ has finite rank. Since

$$\sigma_{sc}(f,g)(z) = (1-|z|^2)^2 f'_+(z)g'_-(z)$$
$$= f'_+(z)g'_-(z) - 2zf'_+(z)g'_-(z)\bar{z} + z^2 f'_+(z)g'_-(z)\bar{z}^2,$$

Theorem (3.3.2) gives that for $z \in D$,

$$\sigma_{sc}(f,g)(z) = (1-|z|^2)^2 f'_+(z)g'_-(z) \equiv 0.$$

This implies

$$f'_+(z)g'_-(z) \equiv 0$$

on D. Thus either f_+ or g_- is constant on D. So we conclude that either $\bar{f}$ or g is analytic on D to complete the proof.

Theorem (3.3.6)[100]: Suppose that f and g are bounded harmonic functions on the unit disk. The commutator $[T_f, T_g]$ has finite rank if and only if f and g are both analytic on D, or $\bar{f}$ and $\bar{g}$ are both analytic on D, or there are constants c_1, c_2, not both 0 such that $c_1 f + c_2 g$ is constant on D.

Proof. If f and g are both analytic on D, both T_f and T_g are multiplication operators on the Bergman space and then they are commuting. Hence the commutator $[T_f, T_g]$ equals 0.

If $\bar{f}$ and $\bar{g}$ are both analytic on D, both T_f and T_g are adjoints of multiplication operators on the Bergman space and then they are commuting. Hence the commutator $[T_f, T_g]$ equals 0.

If there are constants c_1, c_2, not both 0 such that $c_1 f + c_2 g$ is constant on D, noting that the Toeplitz operator with constant symbol commutes with any bounded operator on the Bergman space, we have that T_f commutes with T_g to obtain that the commutator $\left[T_f, T_g\right]$ equals 0.

Conversely, if the commutator $\left[T_f, T_g\right]$ has finite rank, noting

$$\left[T_f, T_g\right] = T_f T_g - T_g T_f$$
$$= (T_{gf} - T_g T_f) - (T_{fg} - T_f T_g)$$
$$= (T_g, T_f] - (T_f, T_g]$$
$$= H^*_{\bar{g}} H_f - H^*_{\bar{f}} H_g,$$

we have that $H^*_{\bar{g}} H_f - H^*_{\bar{f}} H_g$ has also finite rank. Lemma (3.3.3) gives that $\sigma_c(f,g)$ is bounded on D, and easy calculations give

$$\sigma_c(f,g)(z) = (1-|z|^2)^2[f'_-(z)g'_+(z) - f'_+(z)g'_-(z)]$$

$$= f'_-(z)g'_+(z) - f'_+(z)g'_-(z) - 2\bar{z}f'_-(z)g'_+(z)z$$
$$+2zf'_+(z)g'_-(z)\bar{z} + \bar{z}^2 f'_-(z)g'_+(z)z^2 - z^2 f'_+(z)g'_-(z)\bar{z}^2.$$
Thus Theorem (3.3.2) and Proposition (3.3.4) give that $\sigma_c(f,g)(z) \equiv 0$ on the unit disk.

Let $u = g_+ + ig_-$ and $v = if_+ + f_-$. Clearly, u and v are harmonic on D.
An easy calculation gives
$$\tilde{\Delta}(uv) = \tilde{\Delta}[g_+f_- - f_+g_- + ig_+f_+ + ig_-f_-]\tilde{\Delta}[g_+f_- - f_+g_-]$$
$$= (1 - |z|^2)^2[f'_-(z)g'_+(z) - f'_+(z)g'_-(z)]$$
$$= \sigma_c(f,g)(z).$$
Thus uv is also harmonic on D. By Lemma 4.2 [45], we have that at least one of the following conditions holds

(i) u and v are both analytic on D;

(ii) $\bar{u}$ and $\bar{v}$ are both analytic on D;

(iii) there exist complex numbers α, β, not both 0, such that $\alpha u + \beta v$ and $\bar{\alpha}\bar{u} - \bar{\beta}\bar{v}$ are both analytic on D.

Condition (i) gives that f and g are both analytic on D. Condition (ii) gives that $\bar{f}$ and $\bar{g}$ are analytic on D. Condition (iii) gives that $\alpha(g_+ + ig_-) + \beta(if_+ + f_-)$ and $\bar{\alpha}\overline{(g_+ + ig_-)} - \bar{\beta}\overline{(if_+ + f_-)}$ are both analytic on D. Thus $\alpha i g_- + \beta f_-$ and $\bar{\alpha}\overline{g_+} - \bar{\beta}\overline{if_+}$ are constants on D, and so $\alpha g_- - \beta i f_-$and $\alpha g_+ - \beta i f_+$ are constants on D. Hence we conclude
$$\alpha g - i\beta f = (\alpha g_- - i\beta f_-) + (\alpha g_+ - \beta i f_+)$$
is constant on D. This completes the proof.

Theorem (3.3.7)[100]: Suppose that f and g are bounded harmonic functions on the unit disk. T_fT_g has finite rank if and only if either f or g equals 0.

Proof. It is clear that if either f or g equals 0, then $T_fT_g = 0$.
Conversely, if T_fT_g has finite rank, we shall show that either f or g equals 0. An easy calculation gives
$$B(T_fT_g)(z) = B(fg)(z) - B(f_+g_-)(z) + f_+(z)g_-(z). \tag{23}$$
Applying the invariant Laplace operator $\widetilde{\Delta}$ to both sides of the above equation gives
$$[\widetilde{\Delta}B(T_fT_g)](z) = \widetilde{\Delta}B(fg - f_+g_-)(z) + \widetilde{\Delta}[f_+(z)g_-(z)].$$
Since the invariant Laplace operator commutes with the Berezin transform (Lemma (3.3.1), [96]), we have
$$B(\widetilde{\Delta}(fg - f_+g_-))(z) = [\widetilde{\Delta}B(T_fT_g)](z) - \widetilde{\Delta}[f_+(z)g_-(z)].$$
As in the proof of Proposition (3.3.4), the Toeplitz operator $T_{\widetilde{\Delta}(fg - f_+g_-)}$ has finite rank. Theorem (3.3.2) gives that $\widetilde{\Delta}(fg - f_+g_-) \equiv 0$. This implies that $fg - f_+g_-$ is harmonic and $f'_-(z)g'_+(z) = 0$ on D. Thus either f_- or g_+ is constant and hence either f or $\bar{g}$ is analytic on D.
On the other hand, since $fg - f_+g_-$ is harmonic (23) gives
$$B(T_fT_g)(z) = f(z)g(z).$$
By the main result of [46],
$$\lim_{|z|\to 1} B(T_fT_g)(z) = 0 .$$
Because the radial limits of both f and g exist on the unit circle, we have that $f(z)g(z) \equiv 0$ on the unit circle and then either f or g equals 0 on the unit circle. Hence f or g equals 0 on the unit disk. This completes the proof.

Theorem (3.3.8)[100]: Suppose that f_i and g_i are bounded harmonic functions on D for $i = 1, \dots, k$. The following are equivalent

(i) $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$ has finite rank.

(ii) $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j} = 0.$

(iii) $\sigma(f_1, \dots, f_k;\ g_1, \dots, g_k) \equiv 0$

Proof. It is clear that (ii) implies (i).

First we prove that (i) implies (iii). Proposition (3.3.4) immediately gives that $T_{\sigma(f_1,\cdots,f_k;\, g_1,\cdots,g_k)}$ has finite rank. Theorem (3.3.2) gives that

$$\sigma(f_1, \cdots, f_k;\ g_1, \cdots, g_k) \equiv 0.$$

To prove that (iii) implies (ii), we need the following equality obtained in the proof of Proposition (3.3.4)

$$B(\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j})(z) = B(\sum_{j=1}^{k} (g_j)_+ (f_j)_-)(z) - \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).$$

(iii) implies that the function $\sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z)$ is harmonic and hence

$$B(\sum_{j=1}^{k} (g_j)_+ (f_j)_-)(z) = \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).$$

Therefore

$$B(\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j})(z) = 0.$$

By the injection of the Berezin transform [55], we conclude that the operator $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$ must equal 0 to complete the proof.

Chapter 4
Reducing Subspaces

We characterize the nontribyl reducing subspaces of the Toeplitz operator $T_{z_1^N z_2^M}$ on the Bergman space $A^2(\mathbb{D}^2)$, where N and M are positive integers. We study the reducing subspaces of $A^k \otimes I + I \otimes B^l$ and give some examples. As an application, we study the reducing subspaces of multiplication operators $M_{z^k+\alpha w^l}$ on function spaces. We show that the von Neumann algebra $\mathcal{V}^*(\phi) = \{T_\phi, T_\phi^*\}$ is abelian.

Section (4.1): A Class of Toeplitz Operators on the Bergman Space of the Bidisk

For $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $-1 < \alpha < \infty$, let $L^2(\mathbb{D}, dA_\alpha)$ be the Hilbert space of square integrable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} |f(z)\overline{g(z)}dA_\alpha(z), f, g \in A_\alpha^2(\mathbb{D}),$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

and dA is the normalized area measure on $\mathbb{D}$.

The weighted Bergman space $A_\alpha^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of all the analytic functions in $\mathbb{D}$. We denote

$$\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+\alpha+2)}}$$

for $n = 0, 1, 2, \ldots$. Therefore,

$$\|f\|_\alpha^2 = \sum_{n=0}^{+\infty} \gamma_n^2 |a_n|^2 < \infty,$$

where $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A_\alpha^2(\mathbb{D})$. Especially when $\alpha = 0$, we write $A^2(\mathbb{D}) = A_0^2(\mathbb{D})$. In this case, $\gamma_n = \sqrt{\frac{1}{n+1}}$. .

Denote by $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ the bidisk. The Bergman space $A^2(\mathbb{D}^2)$ is the space of all holomorphic functions in $L^2(D^2, d\mu)$ where $d\mu(z) = dA(z_1)dA(z_2)$. For multi-index $\beta = (\beta_1, \beta_2)$, denote $z^\beta = z_1^{\beta_1} z_2^{\beta_2}$ and

$$e_\beta = \frac{z^\beta}{\gamma_{\beta_1}\gamma_{\beta_2}}.$$

Then $\{e_\beta\}_{\beta \geq 0}$ ($\beta \geq 0$ means that $\beta_1 \geq 0$ and $\beta_2 \geq 0$) is an orthogonal basis in $A^2(\mathbb{D}^2)$. For a bounded measurable function $f \in L^\infty(\mathbb{D}^2)$, the Toeplitz operator with symbol f is defined by $T_f h = P(fh)$ for every $h \in A^2(\mathbb{D}^2)$, where P is the Bergman orthogonal projection from $L^2(\mathbb{D}^2, d\mu)$ onto $A^2(\mathbb{D}^2)$.

Recall that for a bounded linear operator T on a Hilbert space H, a closed subspace $\mathcal{M}$ is called a reducing subspace of the operator T, if $T(\mathcal{M}) \subset \mathcal{M}$ and $T^*(\mathcal{M}) \subset \mathcal{M}$. A reducing subspace $\mathcal{M}$ is said to be minimal if there is no nonzero reducing subspace $\mathcal{N}$ such that $\mathcal{N}$ is properly contained in $\mathcal{M}$.

On the Bergman space over $\mathbb{D}$, it is proved that T_B has just two non-tribyl reducing subspaces [32], [22], where B is the product of two Blaschke factors.

In [110], M. Stessin and K. Zhu gave a complete description of the reducing subspaces of weighted unilateral shift operators of finite multiplicity. In particular, T_{z^n} has n distinct minimal reducing subspaces. If B is a finite Blaschke product (order $n \geq 2$), the number of nontribyl minimal reducing subspaces of T_B equals the number of connected components of the Riemann surface of $B^{-1} \circ B$ over $\mathbb{D}$ (see [23], [27], [28], [10], [109], [1]). Further, if B is an infinite Blaschke product or a covering map, the relative research can be founded in [29], [30], [108].

On the Bergman space of bidisk, Y. Lu and X. Zhou [37] characterized the reducing subspaces of $T_{z_1^N z_2^N}$, $T_{z_1^N}$ and $T_{z_2^N}$, respectively. The reducing sub-spaces of $T_{z_1^N z_2^M}$on the weighted Bergman space $A_\alpha^2(\mathbb{D}^2)$ have been completely described in [35]. For $p = \alpha z^k + \beta w^l$, the minimal reducing subspaces of T_p on $A^2(\mathbb{D}^2)$ and the commutant algebra $\mathcal{V}^*(p) = \{T_p, T_p^*\}'$ was described in [107], [111]. We mainly consider the reducing subspaces for the Toeplitz operator $T_{z_1^N z_2^{-M}}$on the Bergman space $A^2(\mathbb{D}^2)$, where N and M are positive integers.

We will give a complete characterization of the reducing subspaces of $T_{z_1^N z_2^{-M}}$.

Through-out, denote $T = T_{z_1^N z_2^{-M}}$, where N and M are positive integers. Denote by $[f]$ the reducing subspace of T generated by $f \in A^2(\mathbb{D}^2)$.
Let $\mathbb{N}$ be the set of all the nonnegative integers.
By direct calculation, we know that

$$T^h(z_1^k z_2^l) = \begin{cases} \frac{\gamma_l^2}{\gamma_l^2 - hM} z_1^{k+hN} z_2^{l-hM}, & if\ l \geq hM \\ 0, & if\ l < hM \end{cases};$$

$$T^{*h}(z_1^k z_2^l) = \begin{cases} \frac{\gamma_k^2}{\gamma_k^2 - hN} z_1^{k-hN} z_2^{l+hM}, & if\ k \geq hN \\ 0, & if\ l < hN \end{cases}$$

for $k, l, h \in \mathbb{N}$. Set

$$E_0 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: 0 \leq k < N, 0 \leq l < M\},$$
$$E_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: k \geq 2N\},$$
$$E_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: l \geq 2M, 0 \leq k < 2N\},$$
$$E_3 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: N \leq k < 2N, M \leq l < 2M\},$$
$$E_4 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: 0 \leq k < N, M \leq l < 2M\},$$
$$E_5 = \{(k,l) \in \mathbb{N} \times \mathbb{N}: 0 \leq l < M, N \leq k < 2N\}.$$

Clearly,

$$A^2(\mathbb{D}^2) = \overset{5}{\underset{i=0}{\oplus}} \overline{\text{span}}\{z_1^p z_2^q : (p,q) \in E_i\}.$$

Notice that $\mathcal{M}_0 = \text{span}\{z_1^p z_2^q : (p,q) \in E_0\}$ is a reducing subspace of T. To find other reducing subspaces, we first study the orthogonal decomposition of $z_1^k z_2^l$ with respect to $\mathcal{M}$.

Lemma (4.1.1)[106]: Suppose $\mathcal{M} \subset \mathcal{M}_0^\perp$ is a reducing subspace of T. Let $P_\mathcal{M}$ be the orthogonal projection from $A^2(\mathbb{D}^2)$ onto$M$.

(i) If $(k,l) \in E_1 \cup E_2 \cup E_3$, then $P_\mathcal{M} z_1^k z_2^l = \lambda z_1^k z_2^l$ with some $\lambda \in \mathbb{C}$.

(ii) If $(k,l) \in E_4$, then

$$P_\mathcal{M} z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n,m) \in E_4\}.$$

(iii) If $(k,l) \in E_5$, then

$$P_\mathcal{M} z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n,m) \in E_5\}.$$

Proof. Let $k, l \in \mathbb{N}$. Since $\mathcal{M} \perp \mathcal{M}_0, \langle P_{\mathcal{M}}(z_1^k z_2^l)\, z_1^p z_2^q \rangle = 0$ for $(p,q) \in E_0$.
In the following, we consider the inner product $\langle P_{\mathcal{M}}(z_1^k z_2^l)\, z_1^p z_2^q \rangle = 0$ for $(p,q) \in \cup_{i=1}^5 E_i$.
For every nonnegative integer h satisfying $l \geq hM$,

$$T^{h*}\, T^h(z_1^k\, z_2^l) = \frac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} z_1^k z_2^l. \tag{1}$$

By computation,

$$\begin{aligned}\frac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} \langle P_{\mathcal{M}}(z_1^k z_2^l)\, z_1^p z_2^q \rangle &= \langle P_{\mathcal{M}} T^{h*}\, T^h(z_1^k z_2^l)\, z_1^p z_2^q \rangle \\ &= \langle P_{\mathcal{M}}(z_1^k z_2^l) T^{h*}\, T^h\, (z_1^p z_2^q) \rangle\end{aligned}$$

$$T^h(z_1^k\, z_2^l) = \begin{cases} \dfrac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} \langle P_{\mathcal{M}}(z_1^k z_2^l)\, z_1^p z_2^q \rangle & q \geq hM \\ 0, & q < hM \end{cases}.$$

Recall that $[s] = \max\{n \in \mathbb{Z}: n \leq s\}$ for real number s. By above equality,
we get that if $\langle P_{\mathcal{M}}(z_1^k z_2^l)\, z_1^p z_2^q \rangle \neq 0$, then

$$\frac{\gamma_l^2 \gamma_{k+hN}^2}{\gamma_{l-hM}^2 \gamma_k^2} = \frac{\gamma_q^2 \gamma_{p+hN}^2}{\gamma_{l-hM}^2 \gamma_p^2} \tag{2}$$

for $0 \leq h \leq \left[\frac{l}{M}\right], q \geq \left[\frac{l}{M}\right] M$.
Equivalently,

$$\frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1+hN)(q+1-hM)}{(p+1+hN)(l+1-M)} \tag{3}$$

for $0 \leq h \leq \left[\frac{l}{M}\right], q \geq \left[\frac{l}{M}\right] M$.
(i) If $(k,l) \in E_1 \cup E_2 \cup E_3$, we will show that the equality (2) holds if and only if $p = k$ and $q = l$.
Case one: $l \geq 2M$.

Let $g_1(\lambda) = (k+1)(q+1)(p+1+\lambda N)(l+1-\lambda M), g_2(\lambda) = (p+1)(l+1)(k+1+\lambda N)(q+1-\lambda M)$ and $g(\lambda) = g_1(\lambda) - g_2(\lambda)$.
Since $l \geq 2M$, we have $g(0) = g(1) = g(2) = 0$. Considering $g(\lambda)$ is a quadratic polynomial, we have $g(\lambda) \equiv 0$ on $\mathbb{C}$. Therefore, g_1 and g_2 have the same zeros,i.e.,

$$\begin{cases} (k+1)(q+1)NM = (p+1)(l+1)NM \\ (k+1)(q+1)\frac{p+1}{N} = (p+1)(l+1)\frac{k+1}{N} \\ (k+1)(q+1)\frac{l+1}{M} = (p+1)(l+1)\frac{q+1}{M} \end{cases}.$$

It follows that $p = k$ and $q = l$.
Case two: $k \geq 2N$.

Replacing $T*T$ by $T\,T^*$ in Case one, we can get the desire result. The details are listed as follows.

Since

$$T^h\, T^{h*}(\, z_1^k\, z_2^l) = \frac{\gamma_k^2 \gamma_{l+hM}^2}{\gamma_{k-hN}^2 \gamma_l^2} z_1^k\, z_l^2, \forall\, 0 \leq h \leq \left[\frac{k}{N}\right],$$

we know that

$$\frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2}\langle PM(z_1^k\ z_l^2), z_1^p\ z_2^q\rangle = \langle PMT^h\ T^{h*}(z_1^k\ z_l^2), z_1^p\ z_2^q\rangle$$

$$= \langle PM(z_1^k\ z_l^2), T^h\ T^{h*}\ (z_1^p\ z_2^q)\rangle = \begin{cases}\frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2}\langle PM(z_1^k\ z_l^2), z_1^p\ z_2^q\rangle \text{if } p \geq hN \\ 0 \quad \text{if } p < hN.\end{cases}$$

Therefore, $\langle PM(z_1^k\ z_l^2), z_1^p\ z_2^q\rangle \neq 0$ will give that

$$\frac{\gamma_k^2\gamma_{l+hM}^2}{\gamma_{k-hN}^2\gamma_l^2} = \frac{\gamma_p^2\gamma_{q+hM}^2}{\gamma_{p-hN}^2\gamma_q^2} \tag{4}$$

for $0 \leq h \leq \left[\frac{k}{N}\right]$ and $p \geq \left[\frac{k}{N}\right]N$. Equivalently,

$$\frac{(k+1)(q+1)}{(p+1)(l\ +\ 1)} = \frac{(k+1-hN)(q+1+hM)}{(p+1-hN)(l\ +\ 1\ +\ hM)} \tag{5}$$

for $0 \leq h \leq \left[\frac{k}{N}\right]$ and $p \geq \left[\frac{k}{N}\right]N$. So when $k \geq 2N$, the above equality follows for $h = 0, 1, 2$. In this case we will get $p = k$ and $q = l$ by the same arguments as the case $l \geq 2M$ has done.

Case three: $(k, l) \in E_3 = \{(n, m) \in \mathbb{N}^2 : N \leq n < 2N, M \leq m < 2M\}$.

In this case, $\left[\frac{k}{N}\right] \geq 1$ and $\left[\frac{l}{M}\right] \geq 1$. Then equalities (3) and (5) hold for $h = 0, 1$. Recall that $g(\lambda) = g_1(\lambda) - g_2(\lambda)$, where $g_1(\lambda) = (k+1)(q+1)(p+1+\lambda N)(l\ +\ 1-\lambda M)$ and $g_2(\lambda) = (p\ +1)(l\ +\ 1)(k\ +\ 1+\lambda N)(q\ +1-\lambda M)$. We get $g(0) = g(1) = g(-1) = 0$. Therefore, we obtain that $p = k$ and $q = l$.

(ii) Suppose that $(k, l) \in E_4$. We need only prove that

$$P_{\mathcal{M}}(z_1^k\ z_2^l) \perp \overline{\operatorname{span}}\left\{ z_1^n z_2^m : (n, m) \in \left(\bigcup_{i=1}^{3} E_i\right) \bigcup E_5 \right\}.$$

If $(n, m) \in E_1 \cup E_2 \cup E_3$, the conclusion (i) implies that $P_{\mathcal{M}} z_1^n z_2^m = \lambda z_1^n z_2^m$ for some $\lambda \in \mathbb{C}$. Thus

$$\langle P_{\mathcal{M}} z_1^k\ z_l^2, z_1^n\ z_2^m\rangle = \langle z_1^k\ z_l^2, P_{\mathcal{M}} z_1^n\ z_2^m\rangle = \bar{\lambda}\langle z_1^k\ z_l^2, z_1^n\ z_2^m\rangle = 0.$$

That is, $P_{\mathcal{M}} z_1^k z_2^l \perp \overline{\operatorname{span}}\{zp1 z_2^q : (p, q) \in E_1 \cup E_2 \cup E_3\}$.

If $(n, m) \in E_5 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \leq l < M, N \leq k < 2N\}$,

$$\langle P_{\mathcal{M}} z_1^k\ z_2^l, z_1^n\ z_2^m\rangle = \frac{\gamma_{l-M}^2\gamma_l^2}{\gamma_l^2\gamma_{k+N}^2}\langle P_{\mathcal{M}} T^*\ T z_1^k\ z_2^l, z_1^n\ z_2^m\rangle = \frac{\gamma_{l-M}^2\gamma_l^2}{\gamma_l^2\gamma_{k+N}^2}\langle T P_{\mathcal{M}} z_1^k\ z_2^l, T\ z_1^n\ z_2^m\rangle = 0,$$

where the last equality comes from $\operatorname{span}\{z_1^p z_2^q : (p, q) \in E_5\} \subseteq KerT$. Thus

$$P_{\mathcal{M}} z_1^k z_2^l \perp \operatorname{span}\{z_1^p z_2^q : (p, q) \in E_5\}.$$

(iii) Replacing T^*T by $T\ T^*$ in (ii), we get the desired result.

Remark (4.1.2)[106]: Let $\mathcal{M} \subset \mathcal{M}_0^\perp$ is a nonzero reducing subspace of . In (i) of Lemma (4.1.1), we indeed get that $\lambda = 0$ or 1, that is $z_1^k\ z_2^l \in \mathcal{M} or\ z_1^k\ z_2^l \in \mathcal{M}^\perp$ for each $(k, l) \in E_1 \cup E_2 \cup E_3$.

If $z_1^k\ z_2^l \in \mathcal{M}$, then

$$\left[[z_1^k\ z_2^l\right] = \operatorname{span}\{z_1^{k-hN}\ z_2^{l+hM} : k - hN \geq 0, l\ + hM \geq 0, h \in \mathbb{Z}\} \tag{6}$$

is a minimal reducing subspace of T, containing in$\mathcal{M}$. Moreover, if $z_1^k\ z_2^l, z_1^p\ z_2^q \in \mathcal{M}$, and $(k, l), (p, q) \in E_1 \cup E_2 \cup E_3$, then it's clear that either $\left[z_1^k z_2^l\right] \perp \left[z_1^p\ z_2^q\right] or \left[z_1^k\ z_2^l\right] = \left[z_1^p\ z_2^q\right]$. So for any non-zero function $(z) = \sum_{(k,l) \in E_1 \cup E_2 \cup E_3} a_k\ , z_1^k\ z_2^l$,

$[f]$ is the direct sum of some minimal reducing subspace as (6).

We define two equivalences on E_4 and E_5 respectively by:

(i) for $(p,q),(k,l) \in E_4, (p,q) \sim 1(k,l) \Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1+N)(q+1-M)}{(p+1+N)(l+1-M)}$;

(ii) for $(p,q),(k,l) \in E_5, (p,q) \sim 2\ (k,l) \Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1-N)(q+1+M)}{(p+1-N)(l+1+M)}$.

It is easy to check that

(i) $(p,q) \in E_4 \Leftrightarrow (p+N, q-M) \in E_5$;

(ii) for$(p,q),(k,l) \in E_4, (p,q) \sim 1(k,l) \Leftrightarrow (p+N, q-M) \sim 2(k+N, l-M)$;

(iii) for $(p,q),(k,l) \in E_5, (p,q) \sim 2(k,l) \Leftrightarrow (p-N, q+M) \sim 1\ (k-N, l+M)$.

For $(n,m) \in E_4$ and $(k,l) \in E_5$, let

$$P_{n,m} : A^2(\mathbb{D}^2) \to \operatorname{span}\{z_1^p z_2^q : (p,q) \sim 1\ (n,m), (p,q) \in E_4\},$$
$$Q_{k,l} : A^2(\mathbb{D}^2) \to \operatorname{span}\{z_1^k\ z_2^l : (p,q) \sim 2(k,l), (p,q)) \in E_5\}$$

be two orthogonal projections. For $f \in A^2(\mathbb{D}^2)$ and $P_{n,m}f \neq 0$, we have

$$[P_{n,m}f] = \operatorname{span}\{P_{n,m}f, TP_{n,m}f\} \qquad (7),$$

since $T^* P_{n,m}f = 0, T^2 P_{n,m}f = 0$ and $T^* TP_{n,m}f = \frac{\gamma_m^2 \gamma_{n+N}^2}{\gamma_{m-M}^2 \gamma_n^2} P_{n,m}f$. Similarly, if $f \in \mathcal{M}$ and $Q_{k,l}f \neq 0$, then

$$[Q_{k,l}f] = \operatorname{span}\{Q_{k,l}f, T^*\ Q_{k,l}f\}. \qquad (8)$$

Lemma (4.1.3)[106]: Let $\mathcal{M} \subset \mathcal{M}_0^\perp$ be a reducing subspace of T and $(n,m) \in E_4$.

Then the following statements hold.

(a) If $f \in \mathcal{M}$, then $[P_{n,m}f] \subset \mathcal{M}$ and $[Q_{n+N,m-M}f] \subset \mathcal{M}$.

(b) If $f_1, f_2 \in P_{n,m}\mathcal{M}$ and $f_1\ \perp f_2$, then $[f_1] \perp [\,f_2]$.

(c) $P_{n,m}T^*f = T^*Q_{n+N,m-M}f$ and $TP_{n,m}f = Q_{n+N,m-M}T\ f, \forall f \in \mathcal{M}$.

(d) If $f \in \mathcal{M}$, then $[P_{n,m}f] = [Q_{n+N,m-M}Tf]$ and $[Q_{n+N,m-M}f] = [P_{n,m}T^*f]$.

(e) $P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M} \subset \mathcal{M}$ is a reducing subspace of T .

Proof. (a) For every $f \in \mathcal{M}$, we know that $P_{\mathcal{M}}P_{n,m}f = P_{n,m}f$, since $P_{\mathcal{M}}P_{n,m} = P_{n,m}P_{\mathcal{M}}$, which obtained by the following simple facts:

(i) if $(k,l) \in E_4$, then $P_{\mathcal{M}}z_1^k\ z_2^l \in \operatorname{span}\{z_1^p z_2^q : (p,q) \in E_4\}$;

(ii) if $(k,l) \notin\ E_4$, then $P_{\mathcal{M}}z_1^k\ z_2^l \perp \operatorname{span}\{z_1^p z_2^q : (p,q) \in E_4\}$;

So $P_{n,m}f \in \mathcal{M}$, which implies that $[P_{n,m}f] \subset \mathcal{M}$.

Similarly, we have $P_{\mathcal{M}}Q_{n+N,m-M}f = Q_{n+N,m-M}f$, which shows that $Q_{n+N,m-M}f \in \mathcal{M}$. Thus $\left[Q_{n+N,m-M}f\right] \subset \mathcal{M}$.

(b) It is clear that $Tf_1, Tf_2 \in \operatorname{span}\{z_1^k z_2^l : (k,l) \in E_5\}$; and

$$\langle Tf_1, Tf_2\rangle = \langle T^*T\,f_1, f_2\rangle = \frac{\gamma_{n+N}^2 \gamma_m^2}{\gamma_n^2 \gamma_{m-M}^2} \langle f_1, f_2\rangle = 0.$$

Equality (7) shows that

$$[f_1] = \operatorname{span}\{f_1, T\,f_1\}, [f_2] = \operatorname{span}\{\,f_2, T\,f_2\}.$$

So $[f_1] \perp [\,f_2]$.

(c) For every $(n,m) \in E_4$, let

$$\mathcal{M}_{n,m} = \operatorname{span}\{z_1^k\ z_2^l : (k,l) \sim 1(n,m), (k,l) \in E_4\};$$
$$\mathcal{M}_{n+N,m-M} = \operatorname{span}\{z_1^k\ z_2^l : (k,l) \sim 2(n+N, m-M), (k,l) \in E_5\}.$$

Then$\mathcal{M}_{n,m}$ and $\mathcal{M}_{n+N,m-M}$ are finite dimension, and the following statements hold:

(i) $T\mathcal{M}_{n,m} = \mathcal{M}_{n+N,m-M}$ and $T^*\ \mathcal{M}_{n+N,m-M} = \mathcal{M}_{n,m}$;

(ii) $T\left(\mathcal{M}_{n,m}^\perp\right) \subset \mathcal{M}_{n+N,m-M}^\perp$ and $T^*\left(\mathcal{M}_{n+N,m-M}^\perp\right) \subset \mathcal{M}_{n,m}^\perp$.

Therefore, $TP_{n,m}f = Q_{n+N,m-M}Tf$ and $P_{n,m}T^{*}f = T^{*}Q_{n+N,m-M}f$ for any $f \in \mathcal{M}$.

(d) By equality (7), conclusion (c) and

$$T^{*}TP_{n,m}f = \frac{\gamma_{n+N}^{2}\gamma_{m}^{2}}{\gamma_{n}^{2}\gamma_{m-M}^{2}}P_{n,m}f, \tag{9}$$

we have

$$\begin{aligned}\left[Q_{n+N,m-M}Tf\right] &= \text{span}\left\{Q_{n+N,m-M}Tf, T^{*}Q_{n+N,m-M}Tf\right\}\\ &= \text{span}\left\{TP_{n,m}f\right\} \oplus \text{span}\left\{T^{*}TP_{n,m}f\right\}\\ &= \text{span}\{TP_{n,m}f\} \oplus \text{span}\{P_{n,m}f\} = [P_{n,m}f].\end{aligned}$$

Similarly, $[Q_{n+N,m-M}f] = [P_{n,m}T^{*}f]$ comes from equality (8), conclusion (c)And

$$T\,T^{*}Q_{n+N,m-M}f = \frac{\gamma_{n+N}^{2}\gamma_{m}^{2}}{\gamma_{n}^{2}\gamma_{m-M}^{2}}\,Q_{n+N,m-M}f. \tag{10}$$

(e) By equalities (9), (10) and conclusion (c), we have

$$Q_{n+N,m-M}\mathcal{M} = TT^{*}\left(Q_{n+N,m-M}\mathcal{M}\right) = TP_{n,m}T^{*}\mathcal{M}. \tag{11}$$

$$P_{n,m}\mathcal{M} = T^{*}T\,(P_{n,m}\mathcal{M}) = T^{*}Q_{n+N,m-M}T\mathcal{M}.$$

Therefore, we only need to show that $P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}$ is an invariant subspace of $T$and T^{*}. In fact

$$T\left(P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}\right) = TP_{n,m}\mathcal{M} = Q_{n+N,m-M}\mathcal{M},$$

where the last equality comes from $TP_{n,m}f = Q_{n+N,m-M}Tf \in Q_{n+N,m-M}\mathcal{M}$ and $Q_{n+N,m-M}f \in TP_{n,m}T^{*}\mathcal{M} \subset TP_{n,m}\mathcal{M}$ for all $f \in \mathcal{M}$. Therefore,

$$T\left(P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}\right) \subset P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}.$$

Similarly, we can prove that

$$T^{*}(P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}) = T^{*}Q_{n+N,m-M}\mathcal{M} = P_{n,m}\mathcal{M}.$$

So we finish the proof.

Theorem (4.1.4)[106]: Let $\mathcal{M} \subset \mathcal{M}_{0}^{\perp}$be a non-zero reducing subspace of T on the bidisk. Then $\mathcal{M} = \mathcal{M}_{1} \oplus \mathcal{M}_{2}$, where

(i) $\mathcal{M}_{1}$ is a direct sum of minimal reducing subspace $\left[z_{1}^{p}z_{2}^{q}\right]$ with $z_{1}^{p}z_{2}^{q} \in \mathcal{M}$for some $(p;q) \in E_{1} \cup E_{2} \cup E_{3}$;

(ii) (ii) $\mathcal{M}_{2}$ is a direct sum of minimal reducing subspace $[f]$ with $f \in P_{n,m}\mathcal{M}$ for some $(n;m) \in E_{4}$.

Proof. Firstly, we prove that

$$\mathcal{M} = \mathcal{M}_{1} \oplus \oplus_{(n;m)\in E}\,(P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}. \tag{12}$$

where$\mathcal{M}_{1} = \oplus_{(p;q)\in\Lambda}\,[z_{1}^{p}z_{2}^{q}]$ with $\Lambda = \left\{(p;\, q) \in E_{1} \cup E_{2} \cup E_{3} : z_{1}^{p}z_{2}^{q} \in \mathcal{M}\right\}$, and E is the partition of E_{4} by the equivalence ~ 1. Set$\mathcal{H}_{n;m} = P_{n,m}\mathcal{M} \oplus Q_{n+N,m-M}\mathcal{M}$.

On the one hand, $\mathcal{M}_{1} \oplus \oplus_{(n;m)\in E}\,\mathcal{H}_{n,m} \subset \mathcal{M}$, since $\mathcal{M}_{1} \subset \mathcal{M}$ is a reducing subspace of T, and conclusion (e) in Lemma (4.1.3) implies that $\oplus_{(n;m)\in E}\,\mathcal{H}_{n;m} \subset \mathcal{M}$. On the other hand, for $g = g_{1} + g_{2} \in \mathcal{M}$ with

$$g_{1}(z) = \sum_{(p;q)\in E_{1}\cup E_{2}\cup E_{3}} a_{p;q}z_{1}^{p}z_{2}^{q}\,, g_{2}(z) = \sum_{(p;q)\in E_{4}\cup E_{5}} a_{p;q}z_{1}^{p}z_{2}^{q}. \tag{13}$$

Remark (4.1.2) shows that $g_{1} \in \mathcal{M}_{1} \subset \mathcal{M}$, which implies that $g_{2} = g - g_{1} \in \mathcal{M}$. Therefore, $g_{2} = \sum_{(n,m)\in E}(P_{n,m}g_{2} + Q_{n+N,m-M}g_{2}) = \in \oplus_{(n;m)\in E}\,\mathcal{H}_{n;m}$. It follows that $\mathcal{M}$ is in the direct sum of $\mathcal{M}_{1}$ and $\left\{\mathcal{H}_{n,m}\right\}$ with $(n;m) \in E$. So we have equality (12) holds.

Secondly, for each $(n,m) \in E_4$, we prove that $\mathcal{H}_{n,m}$ is the direct sum of minimal reducing subspaces as $[f] = \text{span}\{f, Tf\}$ with $f \in P_{n,m}\mathcal{M}$. There are some steps in the proof.

Step 1. Take $0 \neq f_1 \in P_{n,m}\mathcal{M}$. Then $[f_1] = \text{span}\{f_1, Tf_1\} \subset \mathcal{H}_{n,m}$.

Step 2. If $P_{n,m}\mathcal{M} \neq \mathbb{C}f_1$, take $0 \neq f_2 \in P_{n,m}\mathcal{M} \ominus \mathbb{C}f_1$. Then $[f_2] = \text{span}\{f_2, T f_2\} \subset \mathcal{H}_{n,m} \ominus [f_1]$:

Step 3. If $P_{n,m}\mathcal{M} \neq \text{span}\{f_1, f_2\}$, take $0 \neq f_3 \in P_{n,m}\mathcal{M} \ominus \text{span}\{f_1, f_2\}$. Then

$$[f_3] = \text{span}\{f_3, T f_3\} \subset \mathcal{H}_{n,m} \ominus [f_1] \ominus [f_2].$$

If $P_{n,m}\mathcal{M} \neq \text{span}\{f_1, f_2, f_3\}$,, continue this process. This process will stop in finite steps, since the dimension of $\mathcal{H}_{n,m}$is finite. Thus, we finish the proof.

By conclusions (a) and (d) in Lemma (4.1.3) and equalities in (11), we get

$$\begin{aligned}\left[P_{n,mg}, Q_{n+N,m-Mg}\right] &= \left[P_{n,mg}, P_{n,mg}T^*g\right] \\ &= \text{span}\{P_{n,mg}, P_{n,mg}T^*g\} \oplus \text{span}\{Q_{n+N,m-Mg}Q_{n+N,m-M}Tg\}.\end{aligned}$$

Notice that $\text{span}\{P_{n,mg}, P_{n,mg}T^*g\}$ has an orthonormal basis $\{e_1, \dots, e_k\}$, since the dimension of $\text{span}\{P_{n,mg}, P_{n,mg}T^*g\}$ is finite. Conclusion (b) in Lemma (4.1.3) shows that $[e_i] \perp [e_j]$ for $i \neq j$. Then we get

$$[P_{n,mg}, P_{n,m}T^*g] = \oplus_{j=1}^{k} [e_j] = \oplus_{j=1}^{k} \text{span}\{e_j, Te_j\}.$$

Similarly, we can prove that

$$[g_2] = \oplus_{(n;m)\in E} [Q_{n+N,m-Mg}, Q_{n+N,m-M}Tg],$$

And

$$[Q_{n+N,m-Mg}, Q_{n+N,m-M}Tg] = \oplus_{j=1}^{l} [h_j] = \oplus_{j=1}^{l} \text{span}\{h_j, T^*h_j\},$$

where $\{h_1, \dots, h_l\}$ is an orthonormal basis of

$$\text{span}\{Q_{n+N,m-Mg}, Q_{n+N,m-M}Tg\}.$$

In the last part, we give some examples of the reducing sub- spaces of $T_{z_1^N \bar{z}_2^M}$ for the case that $N = M$ and $N \neq M$, respectively.

Example (4.1.5)[106]: Fix $a, b, c, d, e \in \mathbb{C}$ with $e \neq 0$. Let

$$f(z_1, z_2) = az_1^9 z_2^{14} + bz_1^7 z_2^{15} + cz_1^5 z_2^{17} + dz_1^4 z_2^{19} + ez_1^{11} z_2^{12},$$

and $[f]$ be the reducing subspace of $T_{z_1^{10} \bar{z}_2^{10}}$generated by f. Then

$$[f] = \text{span}\{f_1, f_2\} \oplus \text{span}\{z_1^{11+10h} z_2^{12-10h} : h = -1, 0, 1\}.$$

Where

$$\begin{aligned} f_1(z_1, z_2) &= az_1^9 z_2^{14} + bz_1^7 z_2^{15} + cz_1^5 z_2^{17} + dz_1^4 z_2^{19}, \\ f_2(z_1, z_2) &= \frac{a}{3} z_1^{19} z_2^4 + \frac{3b}{8} z_1^{17} z_2^5 + \frac{4c}{9} z_1^{15} z_2^7 + \frac{d}{2} z_1^{14} z_2^9.\end{aligned}$$

Proof. Notice that $(11, 12) \in E_3$ and$(9, 14) \in E_4$. A direct computation shows that $(9, 14) \sim 1(7, 15) \sim 1(5, 17) \sim 1(4, 19)$. Remark (4.1.2) implies that $f_1 = P_{4,19}f$ and $z_1^{11} z_2^{12}$ are in$\mathcal{M}$. There is $\text{span}\{P_{4,19}f, P_{4,19}T^*f\} = [f_1] = \text{span}\{f_1, f_2\}$. Therefore we get the desired result.

Example (4.1.6)[106]: Let $f(z_1, z_2) = z_1^4 z_2^{14} + z_1^7 z_2^7 + z_1^3 z_2^{15}$ and $[f]$ be the reducing subspace of $T_{z_1^5 \bar{z}_2^{10}}$generated by f. Then

$$[f] = \text{span}\left\{z_1^4 z_2^{14} + z_1^3 z_2^{15}, \frac{1}{3} z_1^9 z_2^4 + \frac{3}{8} z_1^8 z_2^5\right\} \oplus \text{span}\{\{z_1^7 z_2^7, z_1^2 z_2^{17}\}.$$

Proof. Notice that $(7, 7) \in E_5, (4, 14), (3, 15) \in E_4$ and $(4, 14) \sim 1(3, 15)$. Let $f_1 = P_{4,14}f = z_1^4 z_2^{14} + z_1^3 z_2^{15}$ and $f_2 = Q_{7,7}f = z_1^7 z_2^7$. Then $[P_{4,14}f, P_{4,14}T^*f] = [f_1] =$

span$\left\{z_1^4 z_2^{14} + z_1^3 z_2^{15}, \frac{1}{3}\ z_1^9 z_2^4 + \frac{3}{8}\ z_1^8 z_2^5\right\}$, $[P_{2,17}f, P_{2,17}T^*f] = [Q_{7,7}f, Q_{7,7}Tf] = [f_2] =$ span$\{z_1^7 z_2^7, z_1^2 z_2^{17}\}$. Then we finish the proof.

Example (4.1.7)[106]: Let $f(z_1, z_2) = z_1^3 z_2^8 + z_1^7 z_2^3$, and $[f]$ be the reducing subspace of $T_{z_1^4 \bar{z}_2^5}$generated by f. Then

$$[f] = \text{span}\{z_1^3 z_2^8, z_1^7 z_2^3\}$$

Proof. Notice that $(3,8) \in E_4, (7,3) \in E_5$. It is easy to check that $T_{z_1^4 \bar{z}_2^5} z_1^4 z_2^5 z_1^3 z_2^8 = \frac{4}{9}\ z_1^7 z_2^3$ and $T^*_{z_1^4 \bar{z}_2^5}\ z_1^7 z_2^3 = \frac{1}{2}\ z_1^3 z_2^8$. So $[z_1^3 z_2^8] = [z_1^7 z_2^3] =$ span$\{z_1^3 z_2^8, z_1^7 z_2^3\}$. It meansthat $[f] =$ span$\{z_1^3 z_2^8, z_1^7 z_2^3\}$.

Example (4.1.8)[106]: Let $f(z_1, z_2) = z_1^2 z_2^{17} + z_1^4 z_2^{14} + z_1^9 z_2^4 + z_1^3 z_2^{15} + z_1^8 z_2^5$ and $[f]$ be the reducing subspace of $T_{z_1^5 \bar{z}_2^{10}}$ generated by f. Then

$$\begin{aligned}[f] &= [z_1^2 z_2^{17}] \oplus [z_1^4 z_2^{14}] \oplus [z_1^3 z_2^{15}] \\ &= [z_1^2 z_2^{17}] \oplus [z_1^4 z_2^{14} + z_1^3 z_2^{15}] \oplus \left[z_1^4 z_2^{14} - \frac{64}{75}\ z_1^3 z_2^{15}\right] \\ &= [z_1^7 z_2^7] \oplus [z_1^9 z_2^4 + z_1^8 z_2^5\ ] \oplus \left[z_1^9 z_2^4 - \frac{27}{25}\ z_1^8 z_2^5\right].\end{aligned}$$

Proof. Notice that $(2,17), (4,14), (3,15) \in E_4, (9,4), (8,5) \in E_5$ and $(4,14) \sim 1(3,15), (9,4) \sim 2\ (8,5)$.

(i) Since $P_{4,14}T^*f = T^*\left(z_1^9 z_2^4 + z_1^8 z_2^5\right) = \frac{1}{2}\ z_1^4 z_2^{14} + \frac{4}{9}\ z_1^3 z_2^{15}$, we have

$$\text{span}\{P_{4,14}f, P_{4,14}T^*f\} = \text{span}\{z_1^4 z_2^{14}, z_1^3 z_2^{15}\}$$

Therefore, $[f] = [z_1^2 z_2^{17}] \oplus [z_1^4 z_2^{14}] \oplus [z_1^3 z_2^{15}] =$ span$[z_1^2 z_2^{17}, z_1^7 z_2^7] \oplus [z_1^4 z_2^{14}] \oplus$ span$\{z_1^4 z_2^{14}, z_1^9 z_2^4\} \oplus$ span$\{z_1^3 z_2^{15}, z_1^8 z_2^5\}$.

(ii) It is easy to check that $\langle z_1^4 z_2^{14} - \frac{64}{75}\ z_1^3 z_2^{15}, z_1^4 z_2^{14} + z_1^3 z_2^{15}\rangle = 0$

And span$\{P_{4,14}f, P_{4,14}T^*f\} =$ span$\left\{z_1^4 z_2^{14} + z_1^3 z_2^{15}, z_1^4 z_2^{14} - \frac{64}{75}\ z_1^3 z_2^{15}\right\}$.

So $[f] = [z_1^4 z_2^{14} + z_1^3 z_2^{15}] \oplus \left[z_1^4 z_2^{14} - \frac{64}{75}\ z_1^3 z_2^{15}\right] \oplus [z_1^2 z_2^{17}]$.

(iii) Notice that

$$\begin{aligned}\text{Span}\{Q_{9,4}f, Q_{9,4}Tf\} &= \text{span}\left\{z_1^9 z_2^4 + z_1^8 z_2^5, \frac{1}{3}\ z_1^9 z_2^4 + \frac{3}{8} z_1^8 z_2^5\right\} \\ &= \text{span}\left\{z_1^9 z_2^4 + z_1^8 z_2^5, z_1^9 z_2^4 - \frac{27}{25}\ z_1^8 z_2^5\right\},\end{aligned}$$

Where $z_1^9 z_2^4 - \frac{27}{25} z_1^8 z_2^5 \perp Q_{9,4}f$. Then

$$[f] = [z_1^7 z_2^7] \oplus [z_1^9 z_2^4 + z_1^8 z_2^5] \oplus \left[z_1^9 z_2^4 - \frac{27}{25}\ z_1^8 z_2^5\right].$$

For the case that $a = \frac{9}{8}$, we have $\left[g + \frac{9}{8}\ z_1^8 z_2^5\right] =$ span$\left\{z_1^4 z_2^{14} + z_1^3 z_2^{15}, z_1^9 z_2^4 + \frac{9}{8}\ z_1^8 z_2^5\right\} = [z_1^4 z_2^{14} + z_1^9 z_2^4]$since $T^*\left(g + \frac{9}{8}\ z_1^8 z_2^5\right) = \frac{1}{2} P_{4,14} g$.

Section (4.2): Tensor Products of Weighted Shifts

Suppose that H and K are two separable Hilbert spaces. If $A \in B(H)$ and $B \in B(K)$, then $M = A \otimes I + I \otimes B$ is a bounded operator on the Hilbert space $H \otimes K$. If both H and K are of finite dimensions, then M is related to the famous Sylvester equation [113]. The Sylvester equation is a matrix equation of the form

$$AX + XB = C,$$

where A, B and C are given $n \times n$ matrices. This equation has a unique solution X for any C if and only if $A \otimes I + I \otimes B^T$ is invertible.

In the case of finite dimensions, the Jordan decomposition of $M = A \otimes I + I \otimes B$ has been completely described [118], [119], [122]. It is proved that if A and B are both Jordan blocks, then M is not a single Jordan block unless H or K is of dimension one.

We consider the reducing subspaces of M. Recall that a closed subspace $X \subseteq H$ is a reducing subspace of A if $AX \subseteq X$ and $A * X \subseteq X$. Denote $\mathcal{V}^*(A) = \{A^*\}$. It is easy to see that $\mathcal{V}^*(A)$ is equal to the commutant of the von Neumann algebra generated by A. Then X reduces A if and only if the projection P_X from H onto X is in $\mathcal{V}^*(A)$. A is called irreducible if the only reducing subspaces are 0 and H. Obviously, A is irreducible if and only if $\mathcal{V}^*(A) = \mathbb{C}$.

It is easy to verify that $\mathcal{V}^*(M) \supseteq \mathcal{V}^*(A) \otimes \mathcal{V}^*(B)$. Thus, it is natural to ask when the equality holds. If we choose A and B to be both irreducible, then $\mathcal{V}^*(A) \otimes \mathcal{V}^*(B) \cong \mathbb{C}$. In this case, if M is irreducible, then the equality holds.

We prove that M is not irreducible if A is unitarily equivalent to B, see Proposition (4.2.13). However, we also show that there exists a class $\mathcal{F}$ such that if A and B are both in $\mathcal{F}$, then M is irreducible if and only if A and B are not unitarily equivalent.

We study reducing subspaces of multiplication operators on function spaces. This topic began with [10],[110],[12],[111], [32], [17], [1], [22], and several brilliant results are obtained in [4], [5], [7]–[109]. This has already attracted a lot of attention and it is an opportunity to study the case where the underlying function space is defined on a higher-dimensional domain [108], [37], [35], [112]. One can see [108], [112]. Furthermore in [108], [112], the research objects can be recognized by $M_z^k \otimes I + \alpha I \otimes M_w^l$, where M_z and M_w are multiplication operators on the Bergman space $L^2(\mathbb{D})$. In that case, A and B are unilateral weighted shifts of finite multiplicity. It is well known that unilateral weighted shifts are always irreducible, hence it is natural to consider that $M = A \otimes I + I \otimes B$ where A and B are unilateral weighted shifts. See [114], [117], [121] for more on unilateral weighted shifts.

For $\mathbb{Z}_+$ denote the set of all non-negative integers. Let $\{e_n\}$_($n \in \mathbb{Z}_+$ (resp. $\{f_m\}_{m\in\mathbb{Z}_+}$) be orthonormal basis for H (resp. K), and $Ae_n = \alpha_n e_{n+1}$ (resp. $Bf_m = \beta_m f_{m+1}$) for $n \in \mathbb{Z}_+$ (resp. $m \in \mathbb{Z}_+$). Here $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ (resp. $\{\beta_m\}$_($m \in \mathbb{Z}_+$) is the weight sequence of A (resp. B). Note that $\|A\| = \sup_n |\alpha_n| < \infty$ and $\|B\| = \sup_m |\beta_m| < \infty$. Then $\{e_n \otimes f_m\}_{n,m\in\mathbb{Z}_+}$ is an orthonormal basis for $H \otimes K$ and we have

$$Me_n \otimes f_m = \alpha_n e_{n+1} \otimes f_m + \beta_m e_n \otimes f_{m+1}, \qquad n,m \in \mathbb{Z}_+. \qquad (14)$$

A unilateral weighted shift A is said to be simple if $\nabla^3[|\alpha|^2](n) = 0$ whenever n^2, where ∇ is the backward difference operator defined by $\nabla[f](n) = f(n) - f(n-1)$. It is easy to check that the multiplication operators M_z are simple on both Dirichlet space and Bergman space.

Let X be a reducing subspace of A. Then X is minimal if there is no nonzero reducing subspace Y properly contained in X.

Theorem (4.2.1)[112]: If $A \in B(H)$ and $B \in B(K)$ are two simple unilateral weighted shifts, then $A \otimes I + I \otimes B$ is reducible if and only if A and B are unitarily equivalent. In this case, $H \otimes K$ is the direct sum of two minimal reducing subspaces.

Based on this theorem, we will classify $\mathcal{V}^*(A \otimes I + I \otimes B)$. We find that there are only two types: C and $C \oplus C$.

At last, we point out that although the reducing subspaces of A_k are completed solved in [111], it remains unclear in our setting $A_k \otimes I + I \otimes B_l$. We will also study the reducing subspaces of $A_k \otimes I + I \otimes B_l$.

Let H' be a Hilbert space with an orthonormal basis $\{e_{n,m}\}_{n,m\in\mathbb{Z}_+}$. We use the assumption of (14), and define an operator M acting on the Hilbert space H as follows:

$$M' e_{n,m} = \alpha_n e_{n+1,m} + \beta_m e_{n,m+1}, n, m \in \mathbb{Z}_+,$$

then M' is bounded. Taking $H_1 = \overline{\text{span}}\{e_{n,0}\}\_(n \in \mathbb{Z}_+$ and $H_2 = \overline{\text{span}}\{e_{0,m}\}_{m\in\mathbb{Z}_+}$, and defining A' on H_1 by $A'e_{n,0} = \alpha_n e_{n+1,0}$, and B' on H_2 by $B'e_{0,m} = \beta_m e_{0,m+1}$, then there exists a unitary equivalence between H' and $H_1 \otimes H_2$. Furthermore, M' is unitarily equivalent to $A' \otimes I + I \otimes B'$ in this case. Keeping this unitary equivalence in mind, we will suppress the tensor product symbol and write enfm for $e_n \otimes f_m$. If $A \in B(H)$ (resp. $B \in B(K)$) is unitarily equivalent to $A' \in B(H)$ (resp. $B' \in B(K)$) by a unitary $U \in B(H)$ (resp. $V \in B(K)$), then $A \otimes I + I \otimes B$ is unitarily equivalent to $A' \otimes I + I \otimes B'$ by the unitary $U \otimes V$. For unilateral weighted shifts A and A' with weight sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ and $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, respectively, we have that A and A' are unitarily equivalent if and only if $|\alpha_n| = |\alpha'_n|$ for all $n \in \mathbb{Z}_+$.

Thus in (14), we can assume that α_n and β_m are all strictly positive. Then A is simple if and only if $\nabla^3[\alpha_2](n) \neq 0$ whenever $n \geq 2$.

A unilateral weighted shift can be represented by a multiplication operator acting on an analytic function space. We will use this systematically because of its convenience for computation. Let $\omega = \{\omega_0, \omega_1, \ldots, \omega_n, \ldots\}$ be a sequence of positive numbers. Let $H_2(\omega)$ be the Hilbert space consisting of analytic functions

$$f(z) = \sum_{k=0}^{\infty} a_k z_k$$

such that

$$\|f\|^2 = \|f\|_\omega^2 = \sum_{k=0}^{\infty} \omega_k |a_k|^2 < \infty.$$

Then $\|z_n\|^2 = \omega_n$, and $\left\{\frac{z_n}{\sqrt{\omega_n}}\right\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis for $H_2(\omega)$. It is well known that the multiplication operator M_z is unitarily equivalent to a unilateral weighted shift A with the weight sequence

$$\left\{\alpha_n = \sqrt{\frac{\omega_{n+1}}{\omega_n}}\right\}_{n\in\mathbb{Z}_+} .$$

To ensure that M_z is bounded, we always assume supn $\frac{\omega_{n+1}}{\omega_n} < \infty$. Similarly, we denote by $H_2(\delta)$ the Hilbert space consisting of analytic functions $g(w)$ such that $\|g\|_\delta < \infty$. Then the multiplication operator M_w is unitarily equivalent to a unilateral weighted shift B with weight sequence

$$\left\{\beta_m = \sqrt{\frac{\delta_{m+1}}{\delta_m}}\right\}_{m\in\mathbb{Z}_+} .$$

Also, $\sup_m \frac{\delta_{m+1}}{\delta_m} < \infty$.

Then we realize the tensor product $H^2(\omega) \otimes H^2(\delta)$ as the Hilbert space consisting of analytic functions

$$f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$$

such that

$$\|f\|^2 = \|f\|_{\omega,\delta}^2 = \sum_{k,l=0}^{\infty} \omega_k \delta_l |a_{kl}|^2 < \infty.$$

Under these notation, $M = A \otimes I + I \otimes B$ is unitarily equivalent to M_{z+w} on $H^2(\omega) \otimes H^2(\delta)$. It is not necessary to distinguish M_z from $M_z \otimes I$. We use M_z to represent multiplication operators both on $H^2(\omega)$ and $H^2(\omega) \otimes H^2(\delta)$. It is similar for M_w. In this case, $M_{z+w} = M_z + M_w$. From now on, $H^2(\omega) \otimes H^2(\delta)$ is denoted by $H^2(\omega,\delta)$.

For further simplicity of notation, we can assume $\nabla[f](0) = f(0)$.

By the above simplification, we can reduce the study of reducing subspaces of $A \otimes I + I \otimes B$ to that of M_{z+w}. Firstly, we start with several definitions and lemmas, many of which originate from [108], [112]. We define $T = [M_{z+w}^*, M_{z+w}] = M_{z+w}^* M_{z+w} - M_{z+w} M_{z+w}^*$. Set $\phi(n) = \nabla\left[\frac{\omega_{\cdot+1}}{\omega_{\cdot}}\right](n)$ and $\psi(m) = \nabla\left[\frac{\delta_{\cdot+1}}{\delta_{\cdot}}\right](m)$. Then a routine computation gives that

$$T z^n w^m = \big(\phi(n) + \psi(m)\big) z^n w^m, \qquad n,m \in \mathbb{Z}_+.$$

Define an equivalence relation $\sim$ on $\mathbb{Z}_+^2$ by

$$(n,m) \sim (n,m) \Leftrightarrow \phi(n) + \psi(m) = \phi(n) + \psi(m).$$

Since T is diagonal with respect to the bases $\{z^n w^m\}_{n,m\in\mathbb{Z}_+}$, there is a spectral decomposition $H^2(\omega,\delta) = \bigoplus Q_d$ such that $z^n w^m$ and $z^{n'} w^{m'}$ belong to the same Q_d if and only if $(n,m) \sim (n',m')$. It is easy to see that for each monomial znwm, the projection Q_d maps it either to 0 or to itself. Let $\mathcal{A}$ be a collection of bounded operators on a Hilbert space H. When $F \subseteq H$, we define

$$\mathcal{A}F = \overline{\operatorname{span}}\{A\,h : A \in \mathcal{A}, h \in F\},$$

And

$$\tilde{\mathcal{A}}^F = \{T \in B(H) : T\,F \subseteq \mathcal{A}F\}.$$

Then define $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}} = \bigcap_{F \subseteq H} \tilde{\mathcal{A}}^F .$$

Thus for all $B \subseteq \tilde{\mathcal{A}}$ and $F \subseteq H$, we have $\mathcal{B}F \subseteq \mathcal{A}F$.

In fact, there is a related concept. If $\mathcal{B}$ is any linear subspace (not necessarily closed) of $B(H)$, then the attached space [114] for $\mathcal{B}$ is defined as

$$\operatorname{Ref}\mathcal{B} = \{T \in B(H) : T\,h \in \overline{\operatorname{span}}\{\mathcal{B}h\} \text{ for all } h \in H\}.$$

Actually we have $\tilde{\mathcal{A}} = \operatorname{Ref}(\operatorname{span}\mathcal{A})$, where $\operatorname{span}\mathcal{A}$ is the linear subspace spanned by $\mathcal{A}$. The fact that Ref $\mathcal{B}$ is always strongly closed implies that $\tilde{\mathcal{A}}$ contains the SOT closure of $\operatorname{span}\mathcal{A}$, where SOT means the strong operator topology.

In our concrete case, for each $n \in \mathbb{Z}_+$, define

$$\mathcal{S}^n = \left\{ \prod_{k=1}^{m} M_{z+w}^{i_k} M_{z+w}^{*j_k}, i_k, j_k \in \mathbb{Z}_+ : \sum_{k=1}^{m} (i_k - j_k) = n \right\}$$

and

$$\mathcal{S}^{*n} = \left\{ \prod_{k=1}^{m} M_{z+w}^{i_k} M_{z+w}^{*jk}, i_k, j_k \in \mathbb{Z}_+ : \sum_{k=1}^{m} (i_k - j_k) = -n \right\}.$$

Note that $\mathcal{S}^m(\mathcal{S}^n F) = \mathcal{S}^{m+n}F$, for any $F \subseteq H^2(\omega,\delta)$ and $n, m \in \mathbb{Z}_+$. For simplicity, write $\mathcal{S}^n = \mathcal{S}^{*-n}$ and $\mathcal{S}^{*n} = \mathcal{S}^{-n}$ if $n < 0$. Then

$$\mathcal{S}^m(\mathcal{S}^n F) \subseteq \mathcal{S}^{m+n}F, n, m \in \mathbb{Z},$$

where $F \subseteq H^2(\omega,\delta)$.

It is easy to see that span$\mathcal{S}^0$ contains the linear span of all T^n's whenever $n \in \mathbb{Z}_+$. By the spectral decomposition $H^2(\omega,\delta) = \bigoplus Q_d$ with respect to T, since Q_d is a Borel functional calculus of $T \in$ span$\mathcal{S}^0$, and $\mathcal{S}^0$ contains the SOT closure of span$\mathcal{S}^0$, we get that $Q_d \in \mathcal{S}^0$. Thus, for any $F \subseteq H^2(\omega,\delta)$ and $n \in \mathbb{Z}$, we obtain the following inequality:

$$Q_d \mathcal{S}^n F \subseteq \mathcal{S}^0(\mathcal{S}^n F) \subseteq \mathcal{S}^n F. \tag{15}$$

For each $r \in \mathbb{Z}_+$, we define

$$E_r = \text{span}\{z^n w^m : n + m = r\}.$$

It is easy to show that $\dim E_r = r + 1$ and $H^2(\omega,\delta) = \bigoplus_{r=0}^{\infty} E_r$. Furthermore, $\mathcal{S}^n E_r \subseteq E_{r+n}$, for every $n \in \mathbb{Z}_+$. Actually, one of the main purposes is to determine whether $\mathcal{S}^n E_r = E_{r+n}$ whenever r and n are given.

The following two lemmas generalize the corresponding results of [112].

Lemma (4.2.2)[112]: Suppose that $r \in \mathbb{Z}_+$. If the statement

$$(r+1, 0) \sim (r, 1) \sim \cdots \sim (0, r+1) \tag{16}$$

does not hold, then $\mathcal{S}^1 E_r = E_{r+1}$.

Proof. The proof comes from [108].

Suppose (16) is false for $r \in \mathbb{Z}_+$. It is clear that \a$S^1 E_r \subseteq E_{r+1}$. For the inverse inclusion, it suffices to show that

$$\dim \ \mathcal{S}^1 E_r \geq r + 2 = \dim \ E_{r+1}.$$

To see this, we first show that $TM_{z+w}(E_r) \subseteq M_{z+w}(E_r)$. Otherwise, for all $j = 0, 1, \ldots, r$,

$$T((z+w) z^{r-j} w^j) \in M_{z+w}(E_r).$$

Recall that

$$T z^n w^m = \big(\phi(n) + \varphi(m)\big) z^n w^m, \ n, m \in \mathbb{Z}_+,$$

and we will find that

$$(r - j + 1, j) \sim (r - j, j + 1), 0 \, j \, r.$$

Thus,

$$(r+1, 0) \sim (r, 1) \sim \cdots \sim (0, r+1),$$

which leads to a contradiction. Therefore, $TM_{z+w}(E_r) \subseteq M_{z+w}(E_r)$. Hence,

$$\dim \ (\mathcal{S}^1 E_r) \geq \dim \ \big(M_{z+w}(E_r)\big) + 1 = r + 2,$$

as desired. The proof is finished.

Lemma (4.2.3)[112]: $\mathcal{S}^{*1} E_{r+1} = E_r$ for each $r \in \mathbb{Z}_+$. Consequently, if $m \geq n \geq 0$, then $\mathcal{S}^{*n} E_m = E_{m-n}$.

Proof. The proof comes from [112].

It suffices to prove that $\mathcal{S}^{*1} E_{r+1} = E_r$. Since it is clear that $\mathcal{S}^{*1} E_{r+1} \subseteq E_r$, it remains to show that $\mathcal{S}^{*1} E_{r+1} \supseteq E_r$.

By direct computation we get

$$M_{z+w}^{*}z^{r+1} = \frac{\omega_{r+1}}{\omega_r} zr,$$

which yields that $z_r \in \mathcal{S}^{*1}E_{r+1}$. By induction, we assume that for some integer $j < r$,

$$z^{r-i}w^i \in \mathcal{S}^{*1}E_{r+1}, 0 \le i \le j.$$

By simple calculations, we obtain that

$$M_{z+w}^{*}z^{r-j}w^{j+1} = \frac{\omega_{r-j}}{\omega_{r-j-1}} z^{r-j-1}w^{j+1} + \frac{\delta_{j+1}}{\delta_j} z^{r-j}w_j ,$$

which yields $z^{r-j-1}w^{j+1} \in \mathcal{S}^{*1}E_{r+1}$. Therefore, $\mathcal{S}^{*1}E_{r+1} = E_r$.

Finally, if K is a reducing subspace of M_{z+w} and dim $(K/\mathcal{S}K) = 1$, then K is minimal. For details, we refer the reader to [112].

Recall that a weighted shift A is simple if $\nabla^3[\alpha^2](n) = 0$ whenever $n \ge 2$. For M_z it is equivalent to saying that

$$\varphi(n + 1) - \varphi(n) \neq \varphi(n) - \varphi(n - 1) \text{ for all } n \ge 1,$$

where $\varphi(n) = \nabla\left[\frac{\omega_{\cdot+1}}{\omega_{\cdot}}\right](n)$. The statement for M_w is similar. Henceforth, all weighted shifts are assumed to be simple.

The following lemma weakens the assumptions of the corresponding result of [112].

Lemma (4.2.4)[112]: *Let* $n, m \in \mathbb{Z}_+$ and $n \ge 1$. If $(n, m) \sim (n - 1, m + 1)$, then $(n + 1, m) \sim (n, m + 1)$ and $(n, m + 1) \sim (n - 1, m + 2)$.

Proof. Assume conversely that $(n + 1, m) \sim (n, m + 1)$, i.e.,

$$\varphi(n + 1) + \psi(m) = \varphi(n) + \psi(m + 1).$$

Combining this with $(n, m) \sim (n - 1, m + 1)$, i.e.,

$$\varphi(n) + \psi(m) = \varphi(n - 1) + \psi(m + 1),$$

yields

$$\varphi(n + 1) - \varphi(n) = \varphi(n) - \varphi(n - 1).$$

Since M_z is simple and $n \ge 1$, this leads to a contradiction. Hence $(n + 1, m) \nsim (n, m + 1)$. Similarly, $(n, m + 1) \nsim (n - 1, m + 2)$.

Now we can prove the following result.

Proposition (4.2.5)[112]: $\mathcal{S}^1E_r = E_{r+1}$ for each integer $r \ge 1$.

Proof. The idea of this proof comes from [112]. If the statement was false for some $r \ge 1$, then by Lemma (4.2.2),

$$(r + 1, 0) \sim (r, 1) \sim \cdots \sim (0, r + 1).$$

By Lemma (4.2.4), since $(r + 1, 0) \sim (r, 1)$, we have $(r + 2, 0) \nsim (r + 1, 1)$ and $(r + 1, 1) \nsim (r, 2)$. Using the spectral decomposition for , there is a spectral projection $Q \in \widetilde{\mathcal{S}^0}$ such that $Qz^{r+2} = Qz^r w^2 = 0, Qz^{r+1}w = z^{r+1}w$.

Note that $M_{z+w}^2 z^r = z^{r+2} + 2z^{r+1}w + z^r w^2 \in \mathcal{S}^2E_r$, thus we obtain $z^{r+1}w = \frac{1}{2}QM_{z+w}^2 z^r \in \mathcal{S}^2E_r$ by (15).

Since $(r, 1) \sim (r - 1, 2)$, applying Lemma (4.2.4) again we have $(r + 1, 1) \nsim (r, 2)$ and $(r, 2) \nsim (r - 1, 3)$.

Note that $M_{z+w}^2 z^{r-1}w = z^{r+1}w + 2z^r w^2 + z^{r-1}w^3$. Using the same argument above, we can show that $z^r w^2, z^{r-1}w^3 \in \mathcal{S}^2E_r$. Furthermore, $z^{r+2} = M_{z+w}^2 z^r - 2z^{r+1}w - z^r w^2 \in \mathcal{S}^2E_r$. This induction will lead to $\mathcal{S}^2E_r = E_{r+2}$. Therefore, $\mathcal{S}_1E_r \supseteq \mathcal{S}^{*1}\mathcal{S}^2E_r = \mathcal{S}^{*1}E_{r+2} = E_{r+1}$, where the last identity follows from Lemma (4.2.3). This leads to a contradiction.

Based on the above lemma, we see that

$$\dim\left(\frac{H^2(\omega,\delta)}{\mathcal{S}^1H^2(\omega,\delta)}\right)=\dim\left(\frac{(E_0+E_1)}{\mathcal{S}^1E_0}\right)\leq 2.$$

Due to this inequality, we obtain the following corollary:

Corollary (4.2.6)[112]: Any nontribyl reducing subspace of $H^2(\omega,\delta)$ is all minimal. Hence $H^2(\omega,\delta)$ is either minimal or a direct sum of two minimal reducing subspaces.

Proof. If $H^2(\omega,\delta)$ is not minimal, let $H = 0$ be a nontribyl reducing subspace of $H^2(\omega,\delta)$. Since $\mathcal{S}$ increases degree by one, $\mathcal{S}^1H \subsetneqq H$ and $\mathcal{S}^1H^\perp \subsetneqq H^\perp$. Writing $H^2(\omega,\delta) = H \oplus H^\perp$, then we see that

$$\dim\left(\frac{H^2(\omega,\delta)}{\mathcal{S}^1H^2(\omega,\delta)}\right)=\dim\left(\frac{H}{\mathcal{S}^1H}\right)+\dim\left(\frac{H^\perp}{\mathcal{S}^1H^\perp}\right)\leq 2.$$

It yields

$$\dim\left(\frac{H}{\mathcal{S}^1H}\right) = 1 \text{ and} \dim\left(\frac{H^\perp}{\mathcal{S}^1H^\perp}\right)=1.$$

This means that both H and $H^\perp$ are minimal.

In fact, we claim that if $H^2(\omega,\delta)$ is not minimal, then there are only two minimal reducing subspaces. However, the proof of this assertion requires more. We have to postpone.

We will find a method to judge whether

$$\varphi(n+1)-\varphi(n)\neq\varphi(n)-\varphi(n-1) \text{for all } n\geq 1, \tag{17}$$

whenever φ is given and $n\in\mathbb{Z}_+$. These results will be useful.

Suppose that n varies in $\mathbb{Z}_+$. Let $\mathbb{R}_+$ denotes the set of all non-negative real numbers. In many examples, there is always a sufficiently smooth function f defined on $\mathbb{R}_+$ such that $\varphi(n) = \nabla[f](n)$ whenever $n\geq 1$ and $\varphi(0) = f(0)$. In this case, (17) is equivalent to

$$\nabla^3[f](n) \neq 0 \text{ for all } n\geq 3 \tag{18}$$

and

$$f(1)-2f(0)\neq f(2)-2f(1)+f(0). \tag{19}$$

Such a function is called to be simple.

Lemma (4.2.7)[112]: Let $f\in C^3(\mathbb{R}^+)$. If $f(x) = 0$ on $\mathbb{R}_+$, then $\nabla^2[f](x)$ is strictly monotone for all $x\geq 2$, thus (18) is true.

Proof. If $x\geq 2$, by the differential mean value theorem,

$$\begin{aligned}(\nabla^2[f])(x) &= \nabla^2[f'](x)=[f'(x)-f'(x-1)]-[f'(x-1)-f'(x-2)]\\ &= f''(\xi_1)-f''(\xi_2)=f'''(\xi_3)(\xi_1-\xi_2),\end{aligned}$$

where $\xi_1\in(x-1,x),\xi_2\in(x-2,x-1),\xi_3\in(x-2,x)$.

It yields that $(\nabla^2[f])(x) = 0$ for all $x\geq 2$. The proof is complete by Darboux theorem.

Thus, to verify that a function $f\in C^3(\mathbb{R}_+)$ is simple, we will show $f(x) = 0$ on $\mathbb{R}_+$ first, which implies (18). Then we turn to check (19).

Next, some simple functions will be given. For further discussion, we introduce a new concept. Let $f\in C^3(\mathbb{R}_+)$. If $f>0, f''<0, f'''>0, f''''<0$, then f is called to be strongly simple. For this we have the following key lemma.

Lemma (4.2.8)[112]: A strongly simple function is simple.

Proof. Let $f\in C^3(\mathbb{R}_+)$ be strongly simple. Since $f'''<0$, (18) is true by Lemma (4.2.7). Next we check (19),

$$f(1)-2f(0)\neq f(2)-2f(1)+f(0).$$

Put

$$g(x) = f(x+1)-2f(x), x\in\mathbb{R}_+.$$

Taking differentiation, we get

$$g'(x) = f'(x+1) - 2f'(x), \qquad x \in \mathbb{R}_+.$$

Since $f'' > 0, f'$ is strictly increasing. It follows that $0 > f'(x+1) > f'(x)$ for $x \in \mathbb{R}_+$, thus

$$f'(x+1) > 2f'(x), x \in \mathbb{R}_+.$$

We obtain the inequality $g > 0$, i.e., g is strictly increasing, hence we have

$$g(0) < g(1) < g(1) + f(0),$$

which is (19). This completes the proof. The following lemma tells us that the set of all strongly simple functions forms a semigroup.

Lemma (4.2.9)[112]: The product of finitely many strongly simple functions is strongly simple.

Proof. It suffices to prove that if $f_1, f_2 \in C^3(\mathbb{R}_+)$ are both strongly simple, then $f = f_1 f_2$ is strongly simple. Of course, $f > 0$. By Leibniz formula,

$$f^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f_1^{(j)} f_2^{(k-j)}, \qquad k = 1,2,3,$$

where

$$\binom{k}{j} = \frac{k!}{j!\,(k-j)!}.$$

Since each f_i is strongly simple, we can see that each term in f', f'' and f''' is strictly negative, strictly positive and strictly negative, respectively. Thus f is strongly simple.

There are many strongly simple functions.

Lemma (4.2.10)[112]: $f(x) = \left(\frac{s+x}{t+x}\right)^{\gamma}$, where $0 < s < 0$, is strongly simple.

Proof. The proof is straightforward.

Suppose that H is a Hilbert space, and $M \in B(H)$. When $F \subseteq H$, we denote by $[F]M$ the reducing subspace of M generated by F.

In what follows, set $H = H^2(\omega, \delta)$ and $M = M_{z+w}$, then $[F]_M$ will be written as $[F]$.

Proposition (4.2.11)[112]: $H^2(\omega, \delta)$ is not minimal if and only if $\mathcal{S}^1 E_0 \neq E_1$. In this case, $H^2(\omega, \delta) = [1] \oplus [z - w]$.

Proof. If $\mathcal{S}^1 E_0 = E_1$, then $H^2(\omega, \delta)$ is obviously minimal. Conversely, assume $\mathcal{S}^1 E_0 \neq E_1$, then $\mathcal{S}^1 E_0$ is a dimension one subspace. Since $z + w \in \mathcal{S}_1 E_0$, it must be that $\mathcal{S}^1 E_0 = C\{z + w\}$, i.e., for all $\mathcal{S} \in \mathcal{S}^1$, there exists a complex number λ such that $\mathcal{S}^1 = \lambda(z + w)$. Solving the equation on $\beta, i.e.,$

$$\langle z + w, z + \beta w \rangle = 0,$$

we get

$$\bar{\beta} = -\frac{\delta_0 \omega_1}{\delta_1 \omega_0}. \tag{20}$$

Notice $[113] = \bigoplus_{r=0}^{\infty} S_r E_0$, hence it yields that $[113] \perp (z + \beta w)$. Furthermore, we have $[113] = H^2(\omega, \delta)$. According to Corollary (4.2.6), we see that [113] and [113]⊥ are both minimal. Thus $[113]\perp = [z + \beta w]$. Next we show that $\beta = -1$.

Since $\mathcal{S}^1 E_0 \neq E_1$, it follows that $(1,0) \sim (0,1)$. By Lemma (4.2.4), we obtain $(2,0) \nsim (1,1)$ and $(1,1) \nsim (0,2)$. So we can select a spectral projection Q from $\widetilde{\mathcal{S}^0}$ such that $Qz^2 = Qw^2 = 0$ and $Qzw = zw$.

Using (15), we have

$$\mathcal{S}^{-1}(Q\mathcal{S}^2E_0) \subseteq \mathcal{S}^{-1}(\mathcal{S}^2E_0) \subseteq \mathcal{S}^1E_0.$$

Thus we will find that there exists a complex number λ such that

$$M^*_{z+w}T\, QM^2_{z+w}1 = \lambda(z + w). \tag{21}$$

On the other hand,

$$\begin{aligned} M^*_{z+w}T\, QM^2_{z+w}1 &= M^*_{z+w}T\, Q(z^2 + 2zw + w^2) \\ &= 2M^*_{z+w}Tzw \\ &= 2\mu M^*_{z+w}zw \\ &= 2\mu\left(\frac{\omega_1}{\omega_0}w + \frac{\delta_1}{\delta_0}z\right), \end{aligned} \tag{22}$$

where $\mu = \varphi(1) + \psi(1)$. Comparing the corresponding coefficients of (21) and (22) gives

$$\frac{\omega_1}{\omega_0} = \frac{\delta_1}{\delta_0},$$

thus $\beta = -1$ by (20).

The following gives a necessary condition for $\mathcal{S}^1E_0 = E_1$. The sufficiency part will be shown later.

Proposition (4.2.12)[112]: If $\mathcal{S}^1E_0 \neq E_1$, then

$$\frac{\omega_{i+1}}{\omega_i} = \frac{\delta_{i+1}}{\delta_i}, \qquad i \in \mathbb{Z}_+,$$

i.e., M_z and M_w are unitarily equivalent.

Proof. We have already shown that it is true for $i = 0$. By induction, assume it is true for all $i < n-1$, we will prove it is true for $i = n - 1$.

It follows from Proposition (4.2.11) that [113] $\perp$ $[z - w]$. We get

$$\begin{aligned} 0 &= \langle (z + w)^n, (z - w)(z + w)^{n-1}\rangle \\ &= \langle\left(\sum_{i=0}^{n}\binom{n}{i}z^{n-i}w^i\right), (z - w)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}z^{n-1-i}w^i\right)\rangle \\ &= \langle\left(\sum_{i=0}^{n}\binom{n}{i}z^{n-i}w^i\right), \left(\sum_{i=0}^{n-1}\binom{n-1}{i}z^{n-1-i}w^i\right) - \left(\sum_{i=0}^{n-1}\binom{n-1}{i}z^{n-1-i}w^i\right)\rangle \\ &= \langle z^n + w^m, z^n - w^m\rangle + \sum_{i=1}^{n-1}\langle\binom{n}{i}z^{n-i}w^i, [\binom{n-1}{i}\binom{n-1}{i-1}z^{n-i}w^i\rangle. \end{aligned}$$

But

$$\begin{aligned} &\sum_{i=1}^{n-1}\langle\binom{n}{i}z^{n-i}w^i, \left[\binom{n-1}{i} - \binom{n-1}{i-1}\right]z^{n-i}w^i\rangle \\ &= \sum_{i=1}^{n-1}\binom{n}{i}\left[\binom{n-1}{i} - \binom{n-1}{i-1}\right]\langle z^{n-i}w^i, z^{n-i}w^i\rangle \\ &= \sum_{i=1}^{\theta(n)}\binom{n}{i}[\binom{n-1}{i} - \binom{n-1}{i-1}(\omega_{n-i}\delta_i - \omega_i\delta_{n-i}) = 0, \end{aligned}$$

where

$$\theta(n) = \begin{cases} \frac{n-1}{2}, \text{for n odd,} \\ \frac{n-2}{2}, \text{for n even} \end{cases}$$

Hence,

$$0 = \langle z^n + w^n, z^n - w^n \rangle = \omega_n \delta_0 - \omega_0 \delta_n.$$

This together with the assumption gives that $\frac{\omega_n}{\omega_{n-1}} = \delta_n/\delta_{n-1}$, and the proof is finished.

For sufficiency, there is a general statement.

Proposition (4.2.13)[112]: Suppose that H and K both are of dimensions at least two. Let $A \in B(H)$ and $B \in B(K)$. If A and B are unitarily equivalent, then $A \otimes I + I \otimes B$ is reducible.

Proof. If A and B are unitarily equivalent, suppose $U \in B(H,K)$ is unitary such that $UA = BU, AU^* = U^*B$. Then U can be used to define a self-adjoint unitary V on $H \otimes K$ by $V(f \otimes g) = U^*g \otimes Uf$. Note that $V = I$ is impossible by dimensions of H and K. Since V is self-adjoint, $\frac{I \pm V}{2}$ constitute a complete projection system. We have a decomposition as follows:

$$H \otimes K = (I + V)(H \otimes K) \oplus (I - V)(H \otimes K).$$

Furthermore, it is easy to verify that $(A \otimes I + I \otimes B)V = V(A \otimes I + I \otimes B)$. Indeed,

$$\begin{aligned}(A \otimes I + I \otimes B)V(f \otimes g) &= (A \otimes I + I \otimes B)(U^*g \otimes Uf) \\ &= AU^*g \otimes Uf + U^*g \otimes BUf \\ &= U^*Bg \otimes Uf + U^*g \otimes UAf \\ &= V(A \otimes I + I \otimes B)(f \otimes g).\end{aligned}$$

Hence $\frac{I \pm V}{2}(H \otimes V)$ are nontribyl reducing subspaces of $A \otimes I + I \otimes B$. This leads to the reducibility of $A \otimes I + I \otimes B$.

Combining Propositions (4.2.11)–(4.2.13), we get the main result.

Theorem (4.2.14)[112]: If A and B are two simple unilateral weighted shifts on separable Hilbert spaces H and K, respectively, then

(i) $M = A \otimes I + I \otimes B$ is reducible if and only if A and B are unitarily equivalent. In this case, $H \otimes K$ is the direct sum of two minimal reducing subspaces of M.

(ii) If $Ae_n = \alpha_n e_{n+1}, Bf_m = \beta_m f_{m+1}$, and $\alpha_n = \beta_n$ for all $n \in \mathbb{Z}_+$, then $H \otimes K = [e_0 f_0]M \oplus [e_1 f_0 - e_0 f_1]_M$, where the summands are the minimal reducing subspaces of M.

In fact, adopting the same argument as shown before, we can carry out the proof of the following.

Corollary (4.2.15)[112]: If A and B are two unilateral weighted shifts such that $\mathcal{S}^1 E_r = E_{r+1}$ for all $r \geq 1$, together with the property: $(1,0) \sim (0,1)$ implies $(2,0) \nsim (1,1)$ and $(1,1) \nsim (0,2)$, then the conclusions in Theorem (4.2.14) are still true.

According to Corollary (4.2.6), $H^2(\omega,\delta)$ is either minimal or a direct sum of two minimal reducing subspaces. If $H^2(\omega,\delta)$ is minimal, then $\mathcal{V}^*(M_{z+w}) = \mathbb{C}$. When M_z and M_w are unitarily equivalent, we will prove that M_{z+w} has exactly 2 minimal reducing subspaces.

Proposition (4.2.16)[112]: If M_z and M_w are unitarily equivalent, then $\mathcal{V}^*(M_{z+w}) \cong \mathbb{C} \oplus \mathbb{C}$.

Proof. Due to Proposition (4.2.11), it follows that $H^2(\omega,\delta) = [1] \oplus [z - w]$, and $\mathcal{S}^1 E_0 = C\{z + w\}$. Since both [113] and $[z - w]$ are minimal, it is easy to verify that

$$[1] = \bigoplus_{r=0}^{\infty} \mathcal{S}^r E_0, \qquad [z - w] = \bigoplus_{r=0}^{\infty} \mathcal{S}^r\{z - w\}. \tag{23}$$

Recalling that $H^2(\omega,\delta) = \oplus E_r$, we have

$$\mathcal{S}^{r+1}E_0 \oplus \mathcal{S}^r\{z - w\} = E_{r+1}, r \in \mathbb{Z}_+$$

In especial, $\mathcal{S}^1 E_0 \oplus \mathcal{S}^0\{z - w\} = E_1$. It leads to $\mathcal{S}^0\{z - w\} = \mathbb{C}\{z - w\}$.

Let P be the projection from $H^2(\omega,\delta)$ onto [113], and $Q = I - P$. Clearly, Q is the projection from $H^2(\omega,\delta)$ onto [z − w], and both P and Q are in $\mathcal{V}^*(M_{z+w})$. We claim that P and Q are not equivalent in $\mathcal{V}^*(M_{z+w})$. If otherwise, then there is a partial isometry $U \in \mathcal{V}_*(M_{z+w})$ with the initial space [113] and final space $[z - w]$. Using (23), we get

$$\begin{aligned} U(\mathcal{S}^r E_0) &= U(\mathcal{S}^r[1] \ominus \mathcal{S}^{r+1}[1]) \\ &= \mathcal{S}^r U[1] \ \mathcal{S}^{r+1} U[1] \\ &= \mathcal{S}^r[z - w] \ominus \mathcal{S}^{r+1}[z - w] \\ &= \mathcal{S}^r\{z - w\}, \end{aligned}$$

for $r \in \mathbb{Z}_+$. Thus we obtain $\dim\ (\mathcal{S}^r E_0) = \dim\ (\mathcal{S}^r\{z - w\})$. Consequently, we infer that

$$\dim\ (\mathcal{S}^1\{z - w\}) = \dim\ (\mathcal{S}^1 E_0) = 1.$$

Thus $\mathcal{S}^1\{z - w\} = \mathbb{C}\{z^2 - w^2\}$. In summary, we have two equalities:

$$UE_0 = \mathbb{C}\{z - w\} \text{and } U(C\{z + w\}) = \mathbb{C}\{z^2 - w^2\}.$$

It tells us that there exist two nonzero numbers c_0 and c_1 such that

$$U_1 = c_0(z - w) \text{ and } U(z + w) = c_1(z_2 - w_2).$$

Since $U \in \mathcal{V}^*(M_{z+w})$, we have $UT = TU$. Notice M_z and M_w are unitarily equivalent, i.e., $\varphi = \psi$. By straightforward computation, we get

$$UT1 = \big(\varphi(0) + \varphi(0)\big)U1 = \big(\varphi(0) + \varphi(0)\big)c_0(z - w),$$

and

$$TU1 = c_0 T\,(z - w) = \big(\varphi(1) + \varphi(0)\big)c_0(z - w).$$

Thus

$$\varphi(0) = \varphi(1). \tag{24}$$

Also,

$$UT\,(z + w) = (\varphi(1) + \varphi(0))U(z + w) = \big(\varphi(1) + \varphi(0)\big)c_1(z^2 - w^2),$$

and

$$TU(z + w) = c_1 T\,(z^2 - w^2) = \big(\varphi(2) + \varphi(0)\big)c_1(z^2 - w^2).$$

Thus

$$\varphi(2) = \varphi(1).$$

But, this together with (24), implies that

$$0 = \varphi(2) - \varphi(1) = \varphi(1) - \varphi(0).$$

This contradicts the fact that M_z is simple. Hence, P and Q are not equivalent in $\mathcal{V}^*(M_{z+w})$.

We are now in a position to prove that there are only two nontribyl projections in $\mathcal{V}^*(M_{z+w})$. Let $R \in \mathcal{V}^*(M_{z+w})$ be another nontribyl projection and R will be minimal by Corollary (4.2.6). Since $I = P + Q$, we have $R = P\,R + QR$. If $PR = 0$, then $QR = R$. It yields that $R \subseteq Q$. Thus $R = Q$ by minimality. Similarly, if $QR = 0$ then $R = P$. The remaining case is that both PR and QR are not zero. Then R is equivalent to P and Q simultaneously in $\mathcal{V}^*(M_{z+w})$ by the theory of von Neumann algebra. This contradicts the fact that P and Q are not equivalent in $\mathcal{V}^*(M_{z+w})$.

The above reasoning shows that $\mathcal{V}^*(M_{z+w})$ only contains 2 nontribyl projections P and Q, and hence $\mathcal{V}^*(M_{z+w}) \cong \mathbb{C} \oplus \mathbb{C}$.

Theorem (4.2.17)[112]: Let A and B be two simple unilateral weighted shifts on separable Hilbert spaces H and K, respectively. Then we have

(i) $\mathcal{V}^*(A \otimes I + I \otimes B) \cong \mathbb{C}$, if A and B are not unitarily equivalent;

(ii) $\mathcal{V}^*(A \otimes I + I \otimes B) \cong \mathbb{C} \oplus \mathbb{C}$, if A and B are unitarily equivalent.

Theorems (4.2.14) and (4.2.17) have many applications. They can be used to investigate the reducing subspaces of $N = A^k \otimes I + \alpha I \otimes B^l$ where $\alpha \in C \setminus \{0\}$. They also can be used to compute reducing subspaces of multiplication operators $M_{z^k+\alpha w^l}$ on some familiar function spaces such as weighted Dirichlet spaces over the bidisk.

In convention, an operator is called a standard model if it is unitarily equivalent to some $A \otimes I + I \otimes B$ where A and B are unilateral weighted shifts. Of course, $A \otimes I + \alpha I \otimes B$ is a standard model in this case. We will show that in fact, N is a direct sum of finite standard models. This idea comes from [108]. Suppose that

$$\Omega \triangleq \{(a,b) \in \mathbb{Z}_+^2 : 0\, a \leq k-1, 0 \leq b \leq l-1\}.$$

Then $|\Omega| = k_l$. For each $(a,b) \in \Omega$, we define

$$H_a = \overline{\text{span}}\{e_{a+nk} : n \in \mathbb{Z}_+\}, K_b = \overline{\text{span}}\{f_{b+ml} : m \in \mathbb{Z}_+\},$$

And

$$H_{a,b} = \overline{\text{span}}\{e_{a+nk} f_{b+ml} : n,m \in \mathbb{Z}_+\} = H_a \otimes K_b.$$

Then we have

$$H = \bigoplus_{0 \leq a \leq k-1} H_a, \qquad K = \bigoplus_{0 \leq b \leq l-1} K_b,$$

And

$$H \otimes K = \bigoplus_{(a,b)\in\Omega} H_{a,b} = \bigoplus_{(a,b)\in\Omega} (H_a \otimes K_b).$$

If k is a positive integer, we denote

$$\alpha_n^{[k]} = \alpha_n \alpha_{n+1} \cdots \alpha_{n+k-1}, \qquad n \in \mathbb{Z}_+$$

and if $k \leq n$ then

$$\alpha_n^{[-k]} = \overline{\alpha_{n-1}\alpha_{n-2} \cdots \alpha_{n-k}} = \alpha_{n-k}^{[k]}, \qquad n \in \mathbb{Z}_+.$$

Proposition (4.2.18)[112]: N is a direct sum of kl standard models.

Proof. For a and A^k, we have

$$A^k e_{a+nk} = \alpha_{a+nk}^{[k]} e_{a+(n+1)k}, A^{k*} e_{a+(n+1)k} = \alpha_{a+nk}^{[-k]} e_{a+nk}, \qquad n \in \mathbb{Z}_+.$$

and

$$A^{k*} e_a = 0.$$

Thus H_a reduces A^k. Similarly, K_b reduces .

Denote by $A_a = A^k|H_a, B_b = B^l\big|_{K_b}$ and $N_{a,b} = N|_{H_{a,b}}$, then $N_{a,b} = A_a \otimes I + \alpha I \otimes B_b$. Furthermore, $N = \oplus_{(a,b)\in\Omega} N_{a,b}$.

In fact, A_a is a unilateral weighted shift $A' : H_a$ has the weight sequence $\left\{\alpha'_n = \alpha_{a+nk}^{[k]}\right\}_{n\in\mathbb{Z}_+}$ with respect to the orthonormal basis $\{e'_n = e_{a+nk}\}_{n\in\mathbb{Z}_+}$. B_b is a unilateral weighted shift $B' : K_b$ has the weight sequence $\left\{\beta'_m = \beta_{b+ml}^{[l]}\right\}_{m\in\mathbb{Z}_+}$ with respect to the orthonormal basis $\{f'_m = f_{b+ml}\}_{m\in\mathbb{Z}_+}$. Thus $N_{a,b}$ is a standard model and N is a direct sum of kl standard models.

There are two natural questions: When $H_{a,b}$ is minimal and when they are not unitary equivalent? We will solve the first question in many cases and leave the second one to further consideration. Now let us translate them into function models. Then N will be translated into $M_{z^k+\alpha w^l}$. For each $(a,b)\ \in \Omega$, we define

$$H^2(\omega)_a = \overline{\operatorname{span}}\{z^{a+nk} : n \in \mathbb{Z}_+\}, H^2(\delta)_b = \overline{\operatorname{span}}\{w^{b+ml} : m \in \mathbb{Z}_+\},$$

and

$$H^2(\omega,\delta)_{a,b} = \overline{\operatorname{span}}\{z^{a+nk}w^{b+ml} : n,m \in \mathbb{Z}_+\} = H^2(\omega)_a \otimes H^2(\delta)_b.$$

According to the proof of Proposition (4.2.18), a routine computation gives rise to the following statements:

(i) $H^2(\omega,\delta)_{a,b}$ is a reducing subspace;

(ii) $M_{z^k}\big|_{H^2}(\omega)_a$ is a unilateral weighted shift $A' : H^2(\omega)a$ has the weight sequence $\left\{\alpha_n' = \sqrt{\frac{\omega_{a+(n+1)k}}{\omega_{a+nk}}}\right\}_{n\in\mathbb{Z}_+}$ with respect to the orthonormal basis $\left\{e_n' = \frac{z^{a+nk}}{\sqrt{\omega_{a+nk}}}\right\}_{n\in\mathbb{Z}_+}$.

$\alpha M_{w^l}\big|_{H^2}(\delta)_b$ is a unilateral weighted shift $B' : H^2(\delta)_b$ has the weight sequence $\left\{\beta_m' = \alpha\sqrt{\frac{\delta_{b+(m+1)l}}{\delta_{b+ml}}}\right\}_{m\in\mathbb{Z}_+}$ with respect to the orthonormal basis $\left\{f_m' = \frac{w_{b+ml}}{\sqrt{\delta_{b+ml}}}\right\}_{m\in\mathbb{Z}_+}$.

Henceforth, we denote by $[F]$ the reducing subspace of $M_{z^k+\alpha w^l}$ generated by F.

By Theorems (4.2.14) and (4.2.17), we obtain the following proposition:

Proposition (4.2.19)[112]: If $M_{z^k}\big|_{H^2}(\omega)_a$ and $M_{w^l}\big|_{H^2}(\delta)_b$ are simple, then $H^2(\omega,\delta)_{a,b}$ is not minimal for $M_{z^k+\alpha w^l}$ if and only if $M_{z^k}\big|_{H^2}(\omega)_a \cong \alpha M_{w^l}\big|_{H^2}(\delta)_b$. In this case, $H^2(\omega,\delta)_{a,b} = [z^a w^b] \oplus \left[z^a w^b\left(z^k - |\alpha|_{w^l}\right)\right]$, where the summands are the only nontribyl reducing subspaces of $M_{z^k+\alpha w^l}$.

Proof. The first statement is given by Theorem (4.2.14). Next, we need to show that the decomposition of $H^2(\omega,\delta)_{a,b}$ has the disired form. If we see A' and B' in (ii) above, then

$$H^2(\omega,\delta)_{a,b} = [e_0'f_0'] \oplus [e_1'f_0' - e_0'f_1'] = [z^a w^b] \oplus \left[\frac{z^{a+k}w^b}{\sqrt{\omega^{a+k}\delta_b}} - \frac{z^a w^{b+l}}{\omega^a\delta^{b+l}}\right]$$
$$= [z^a w^b] \oplus [z^a w^b\,(z^k - |\alpha| w^l\,)],$$

where

$$|\alpha| = \frac{\omega_{a+k}\delta_b}{\sqrt{\omega_a\delta_{b+l}}}$$

is given by $M_{z^k}\big|_{H^2(\omega)^a} \cong \alpha M_{w^l}\big|_{H^2(\delta)^b}$. Then the proof is completed by Theorem (4.2.17).

In what follows, we will check some classical function spaces.

For each standard model $M_{z^k+\alpha w^l}\big|\_(H^2(\omega,\delta)_{a,b}$, denote

$$f_a(n) = \frac{\omega_{a+(n+1)k}}{\omega_{a+nk}}, \qquad n \in \mathbb{Z}_+,$$

and

$$g_b(m) = \frac{\delta_{b+(m+1)l}}{\delta_{b+ml}}, \qquad m \in \mathbb{Z}_+.$$

Then $M_{z^k}\big|_{H^2(\omega)_a}$ (resp. $M_{w^l}\big|_{H^2(\delta)_b}$) is simple if and only if f_a (resp. g_b) is simple.

Let $H = D_\beta \otimes D_\gamma$ be the tensor product of two weighted Dirichlet spaces, where $\beta,\gamma \in \mathbb{R}$. Then $H = H^2_{(\omega,\delta)}$, where $\omega_n = (n+1)^\beta, \delta_m = (m+1)^\gamma$. For $p_\alpha = z^k + \alpha w^l$ and $(a,b) \in \Omega$, we have

$$f_a(n) = \left(\frac{a + (n + 1)k + 1}{a + nk + 1}\right)^{\beta},$$
$$g_b(m) = \left(\frac{b + (m + 1)l + 1}{b + ml + 1}\right)^{\gamma}.$$
Denote $(s,t) = \left(\frac{a+1}{k}, \frac{b+1}{l}\right)$, then we obtain
$$f_a(n) = \left(\frac{s + n + 1}{s + n}\right)^{\beta} \text{ and } g_b(m) = \left(\frac{t + m + 1}{t + m}\right)^{\gamma}.$$
Proposition (4.2.20)[112]: If $H = D_\beta \otimes D_\gamma$ and $\beta, \gamma \in (0, +\infty)$, then

(i) $M_{z^k}\big|_{D_{\beta_a}}$ and $M_{w^l}\big|_{D_{\gamma\, b}}$ are simple for all admissible k, l, a, b;

(ii) $M_{z^k}\big|_{D_{\beta_a}}$ and $\alpha M_{w^l}\big|_{D^{\gamma}{b}}$ are unitarily equivalent if and only if $|\alpha| = 1, \beta = \gamma, s = t$;

(iii) in (ii), $H_{a,b} = [z^a w^b] \oplus [z^a w^b(z^k - w^l)]$, where the summands are the only nontribyl reducing subspaces of $M_{z^k+w^l}$;

(iv) in other cases, $H_{a,b}$'s are minimal reducing subspaces of $M_{z^k+\alpha w^l}$.

Proof. By Lemma (4.2.10), if $\beta, \gamma \in (0, +\infty)$, then $M_{z^k}\big|_{D_{\beta_a}}$ and $M_{w^l}\big|_{D_{\gamma_b}}$ are all simple for all admissible k, l, a, b. This proves (i).

Next, $M_{z^k}\big|_{D_{\beta_a}}$ and $\alpha M_{w^l}\big|_{D_{\gamma_b}}$ b are unitarily equivalent if and only if
$$\sqrt{\frac{\omega_{a+(n+1)k}}{\omega_{a+nk}}} = |\alpha| \frac{\delta_{b+(n+1)l}}{\delta_{b+nl}}$$
holds for all $n \in Z+$, i.e.,
$$\left(\frac{s + n + 1}{s + n}\right)^{\beta} = |\alpha|^2 \left(\frac{t + n + 1}{t + n}\right)^{\gamma}, \quad n \in \mathbb{Z}_+.$$
First letting n tend to infinity, we get $|\alpha| = 1$. Then taking differentiation, we get $-\beta\left(\frac{s+n+1}{s+n}\right)^{\beta-1} \frac{1}{(s+n)^2} = -\gamma\left(\frac{(t+n+1)}{t+n}\right)^{\gamma-1} \frac{1}{(t+n)^2}$, $n \in \mathbb{Z}_+$,
i.e.,
$$\frac{\beta}{\gamma} = \left(\frac{t + n + 1}{t + n}\right)^{\gamma-1} \left(\frac{s + n}{s + n + 1}\right)^{\beta-1} \left(\frac{s + n}{t + n}\right)^2, \quad n \in \mathbb{Z}_+.$$
Letting n tend to infinity again, we get $\beta = \gamma$. Now we have
$$\frac{s + n + 1}{s + n} = \frac{t + n + 1}{t + n}, \quad n \in \mathbb{Z}_+.$$
This will lead to $s = t$. This completes (ii).

(iii) and (iv) are given by Proposition (4.2.19).

If $\beta = \gamma = 1$, then $H = D_\beta \otimes D_\gamma$ is the Dirichlet space over the bidisk. For $\beta < 0$ or $\gamma < 0$, it need to check the simple condition. If $\beta = \gamma = -1$, then $H = D_\beta \otimes D_\gamma$ is the Bergman space $L^2_a(\mathbb{D}^2)$ over the bidisk. Some similar and tedious manipulations still yield that f_a and g_b are simple functions, hence we obtain many results again in [108], [112]. This provides a good explanation for their different behavior on α. If $\beta = \gamma = 0$, then $H = H^2(\mathbb{D}^2)$ is the Hardy space over the bidisk. Each $H_{a,b}$ is minimal if and only if $|\alpha|^2 = 1$. If $|\alpha|^2 = 1$, then each $H_{a,b}$ has the same structure as in Proposition (4.2.20)(iii). Dan [108] also considered this problem in a different way. In particular, the reducing subspaces of multiplication operators $M_{z+\alpha w}$ on the Hardy space, Dirichlet space and Bergman space over the bidisk all have the same structure. In other words, their $\mathcal{V}^*(M_{z+\alpha w})$'s are isomorphic. For more examples, we can introduce a new class of function spaces. We call them ultra-weighted Bergman spaces. Let $A^2_{\alpha,\beta} = H^2(\omega)$, where $\alpha > -1, \beta \in R, \omega_n = \left(\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}\right)^\beta$. It is obvious that $A^2_{0,\beta} = D_{-\beta}$ and $A^2_{\alpha,1}$ are the usual weighted Bergman spaces.

Let $H = A^2_{\beta,\gamma} \otimes A^2_{\delta,\lambda}$ be the tensor product of two ultra-weighted Bergman spaces, where $\beta,\delta > -1, \gamma,\lambda \in \mathbb{R}$. Then $H = H^2_{(\omega,\delta)}$, where $\omega_n = \left(\frac{n!\Gamma(2+\beta)}{\Gamma(2+\beta+n)}\right)^\gamma$, $\delta_m = \left(\frac{m!\Gamma(2+\delta)}{\Gamma(2+\delta+m)}\right)^\lambda$. For $p_\alpha = z^k + \alpha w^l$ and $(a,b) \in \Omega$, we have

$$f_a(n) = \left(\frac{(a + (n+1)k)!\,\Gamma(2+\beta+a+nk)}{(a+nk)!\,\Gamma(2+\beta+a+(n+1)k)}\right)^\gamma,$$

and

$$g_b(m) = \left(\frac{(b + (m+1)l)!\,\Gamma(2+\delta+b+ml)}{(b+ml)!\,\Gamma(2+\delta+b+(m+1)l)}\right)^\lambda.$$

Denote $(s,t) = (a+1\,k, b+1\,l)$, then

$$f_a(n) = \left(\frac{\left(s+n+1-\frac{1}{k}\right)\left(s+n+1-\frac{2}{k}\right)\cdots\left(s+n+1-\frac{k}{k}\right)}{\left(s+n+1+\frac{1+\beta}{k}-\frac{1}{k}\right)\left(s+n+1+\frac{1+\beta}{k}-\frac{2}{k}\right)\cdots\left(s+n+1+\frac{1+\beta}{k}-\frac{k}{k}\right)}\right.$$

and g_b has the similar form.

Now we can state the following proposition:

Proposition (4.2.21)[112]: If $H = A^2_{\beta,\gamma} \otimes A^2_{\delta,\lambda}$, where $\beta,\delta > -1$ and $\gamma,\lambda < 0$, then

(i) $M_{z^k}\big|_{(A^2_{\beta,\gamma})_a}$ and $M_{w^l}\big|_{(A^2_{(\delta,\lambda)_b})}$ are simple for all admissible k,l,a,b;

(ii) if $|\alpha| = 1$ and $\gamma(1+\beta) = \lambda(1+\delta)$, then there may be many (a,b)'s which make $M_{z^k}\big|_{(A^2_{\beta,\gamma})_a}$ and $\alpha M_{w^l}\big|_{(A^2_{\delta,\lambda})_b}$ unitarily equivalent;

(iii) in (ii), $H_{a,b} = [z^a w^b] \oplus \left[z^{aw^b(z^k - w^l)}\right]$, where the summands are the only nontribyl reducing subspaces of $M_{z^k+w^l}$;

(iv) in other cases, $H_{a,b}$'s are minimal reducing subspaces of $M_{z^k+\alpha w^l}$.

Proof. By Lemmas (4.2.10) and (4.2.9), we find that $M_{z^k}\big|_{(A^2_{\beta,\gamma})_a}$ and $M_{w^l}\big|_{(A^2_{\delta,\lambda})_b}$ are all simple for all admissible k,l,a,b. This proves (i).

For unitarily equivalent, consider

$$\left(\frac{\left(s+n+1-\frac{1}{k}\right)\left(s+n+1-\frac{2}{k}\right)\cdots\left(s+n+1-\frac{k}{k}\right)}{\left(s+n+1+\frac{1+\beta}{k}-\frac{1}{k}\right)\left(s+n+1+\frac{1+\beta}{k}-\frac{2}{k}\right)\cdots\left(s+n+1+\frac{1+\beta}{k}-\frac{k}{k}\right)}\right)^\gamma$$

$$= |\alpha|^2 \left(\frac{\left(t+n+1-\frac{1}{l}\right)\left(t+n+1-\frac{2}{l}\right)\cdots\left(t+n+1-\frac{l}{l}\right)}{\left(t+n+1+\frac{1+\delta}{l}-\frac{1}{l}\right)\left(t+n+1+\frac{1+\delta}{l}-\frac{2}{l}\right)\cdots\left(t+n+1+\frac{1+\delta}{l}-\frac{l}{l}\right)}\right.$$

for all $n \in \mathbb{Z}_+$. First letting n tend to infinity, we get $|\alpha| = 1$. Set

$$h_i(n) = \frac{s + n + 1 - \frac{i}{k}}{s + n + 1 + \frac{1+\beta}{k} - \frac{i}{k}}, \qquad i = 1, \ldots, k, n \in \mathbb{Z}_+,$$

and

$$l_j(n) = \frac{t + n + 1 - \frac{j}{l}}{t + n + 1 + \frac{1+\delta}{l} - \frac{j}{l}}, \qquad j = 1, \ldots, l, n \in \mathbb{Z}_+,$$

then the above equation becomes

$$\left(h_1(n) \cdots h_k(n)\right)^{\gamma} = \left(l_1(n) \cdots l_l(n)\right)^{\lambda}, \quad n \in \mathbb{Z}_+. \tag{25}$$

Note that, $\lim_{n\to\infty} h_i(n) = 1,\ i = 1, \ldots, k$, and

$$\lim_{n\to\infty} n^2 \frac{h_i'(n)}{h_i(n)} = \frac{1 + \beta}{k}, \qquad i = 1, \ldots, k.$$

The similar result holds also for $l_j, j = 1, \ldots, l$.

Taking differentiation to (25), we get

$$\gamma\left(h_1(n) \cdots h_k(n)\right)^{\gamma}\left(\frac{h_1'(n)}{h_{1(n)}} + \cdots + \frac{h_k'(n)}{h_{k(n)}}\right)$$
$$= \lambda\left(l_1(n) \cdots lk(n)\right)^{\lambda}\left(\frac{l_1'(n)}{l_1(n)} + \cdots + \frac{l_l'(n)}{l_l(n)}\right), \qquad n \in \mathbb{Z}_+.$$

Multiplying both sides by n^2, then letting n go to infinity again, we get $\gamma(1 + \beta) = \lambda(1 + \delta)$. This completes (ii).

(iii) and (iv) are given by Proposition (4.2.19).

In conclusion, we see that if $|\alpha| \neq 1$, then $H_{a,b}$ must be a minimal reducing subspace of $M_{z^k+\alpha w^l}$ in many cases.

Section (4.3): A Class of Non-Analytic Toeplitz Operators on the Bidisk

For $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and $dA(z)$ denote the normalized area measure over $\mathbb{D}$. Let $A^2(\mathbb{D}^2)$ denote the Bergman space consisting of all holomorphic functions over $\mathbb{D}^2$, which are square integrable with respect to the normalized volume measure $dA(z)dA(w)$. Then $A^2(\mathbb{D}^2)$ is a Hilbert space with inner product $\langle f, g\rangle = \int_{\mathbb{D}^2} f\bar{g}\,dA(z)dA(w)$. Given an essentially bounded function ϕ, the Toeplitz operator T_ϕ is defined by $T_\phi f = P(\phi f)$ for $f \in A^2(\mathbb{D}^2)$. Put $\mathcal{V}^*(\phi) = \{T_\phi, T_\phi^*\}'$, the commutant algebra of the C^*–algebra generated by T_ϕ in $B(A^2(\mathbb{D}^2))$. As is given in [115], $\mathcal{V}^*(\phi)$ is a von Neumann algebra and is the norm closed linear span of its projections.

For a bounded linear operator S on a Hilbert space $\mathcal{H}$, a closed subspace $\mathcal{M}$ is called a reducing subspace for S if $S\mathcal{M} \subseteq \mathcal{M}$ and $S\mathcal{M}^{\perp} \subseteq \mathcal{M}^{\perp}$. In addition, $\mathcal{M}$ is called minimal if there is no nonzero reducing subspace $\mathcal{N}$ satisfying $\mathcal{N} \subsetneq \mathcal{M}$. It is well known that $\mathcal{M}$ is a reducing subspace for S if and only if $SP_{\mathcal{M}} = P_{\mathcal{M}}S$, where $P_{\mathcal{M}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. In this way, the range of projections in $\mathcal{V}^*(\phi)$ and the reducing subspaces for T_ϕ are in one-to-one correspondence. Therefore, in some sense, studying the structure of von Neumann algebra $\mathcal{V}^*(\phi)$ is equivalent to investigating the structure of the reducing subspaces for T_ϕ.

For B_N denote a Blaschke product of finite order N on $\mathbb{D}$. In 2009, Zhu [128] proved that a multiplication operator M_{B_2} on $L_a^2(\mathbb{D})$ has two distinct nontrival minimal reducing subspaces, and conjectured M_{B_N} has exactly N distinct nontrival minimal reducing subspaces. In particular, if $B_N(z) = z^N, M_{z^N}$ is a weighted unilateral shift operator of finite multiplicity on a weighted sequence space. Stessin and Zhu [111] showed that every reducing subspace for M_{z^N} contains a minimal reducing subspace as $X_n = \overline{\text{span}\{z^{n+kN} : k = 0,1,2,\cdots\}}$ with $0 \le n \le N - 1$. What is worth mentioning, Hardy spaces, Bergman spaces and Dirichlet Spaces are three particular cases of the weighted sequence spaces.

Further, Douglas and Kim [125], Li, Lan and Liu [127] generalized the results to some weighted unilateral shift operators on $L_a^2(A_r)$ and $F_\alpha^2(\alpha > 0)$ (the square integrable analytic functions on the annulus A_r with respect to the normalized measure $dA(z)$, and the square integrable entire functions on the whole complex plane $\mathbb{C}$ with respect to the Gaussian measure, respectively). In 2004, Hu, Sun, Xu and Yu [126] proved that there is always a nontribyl reducing subspace for M_{B_N}. In 2009, Guo, Sun, Zheng and Zhong [10] disproved Zhu's conjecture and proposed the modified conjecture that M_{B_N} has at most N distinct nontrival minimal reducing subspaces. On the basis of ([27], [28], [10], [1], [17], etal.) by Guo, Huang, Sun, Zheng and Zhong, et al., Douglas, Putinar and Wang [23] obtained that the number of nontrival minimal reducing subspaces for M_{B_N} equals the number of connected components of the Riemann surface $B_N^{-1} \circ B_N$ on the unit disk. As verified in [27], [28], this result is equivalent to the assertion that $\mathcal{V}^*(B_N)$ is abelian. For infinite Blaschke products, Guo and Huang [8] proved that for "most" thin Blaschke products B, M_B has no nontribyl reducing subspace.

For high-dimensional domains, research on reducing-subspace problems began with some special monomial symbols. Lu and Zhou [37] completely characterized the structure of the reducing subspaces for $M_{z^k w^k}$ on the weighted Bergman spaces over $\mathbb{D}^2$. Shi and Lu [35] found all the minimal reducing subspaces for $M_{z^k w^l}(k \neq l)$ on $A_\alpha^2(\mathbb{D}^n)(\alpha > -1)$ and showed that the un-weighted case has more minimal reducing subspaces than the weighted case. Guo and Huang [30] gave the direct decompose of the reducing subspaces for M_{z^a} with $a \in \mathbb{Z}_+^d$ on a multi-dimensional separable Hilbert space by a different approach. For the case that p is a polynomial, the reducing subspaces for $T_{\alpha z^k + \beta w^l}$ $(\alpha, \beta \in \mathbb{C})$ and the structure of $\mathcal{V}^*(\alpha z^k + \beta w^l)$ are investigated in [108], [112]. More generally, Guo and Wang [113] studied the reducing subspaces for $A^k \otimes I + I \otimes B^l$ where $A \in B(H), B \in B(K)$ are two simple unilateral weighted shifts.

Motivated by the research of multiplication operators, we wonder what the results about the Toeplitz operator with non-analytic symbols look like. Compared with the analytic conditions, the tools for the Toeplitz operators with general non-analytic symbols seem far fewer at present. Albaseer, Shi and Lu [107] characterized the reducing subspaces for $T_{z^k \overline{w}^l}$ on $A^2(\mathbb{D}^2)$. Let $\varphi(z,w) = \alpha z^k + \beta \overline{w}^l$ where α and β are nonzero complex numbers. We find all the minimal reducing subspaces for the Toeplitz operator T_φ on $A^2(\mathbb{D}^2)$, and consider the algebraic structure of $\mathcal{V}^*(\varphi)$. Unlike the analytic condition, we obtain that $\mathcal{V}^*(\varphi)$ is always abelian for every $\alpha\beta \neq 0$. The following theorem is the main result.

Theorem (4.3.1)[124]: Let $\varphi(z,w) = \alpha z^k + \beta \overline{w}^l$, where α, β are nonzero complex numbers and k, l are positive integers. Then

$$L_{a,b} = \overline{\text{span}\{z^{a+nk} w^{b+ml} \mid n, m \in \mathbb{Z}_+\}} (0 \le a \le k - 1, 0 \le b \le l - 1)$$

are exactly all the minimal reducing subspaces for T_φ. Furthermore, $\mathcal{V}^*(\varphi)$ is $*$ −isomorphic to

$$\bigoplus_{i=1}^{kl} \mathbb{C},$$

and then $\mathcal{V}^*(\varphi)$ is abelian.

Since $\alpha \neq 0$, the operators $T_{\alpha z^k+\beta \bar{w}^l}$ and $T_{z^k+\frac{\beta}{\alpha}\bar{w}^l}$ have the same reducing subspaces. For each $c \in \mathbb{C}-\{0\}, T_{z^k+c\bar{w}^l}, T_{z^k+|c|\bar{w}^l}$ and $T_{w^k+|c|\bar{z}^l} = T^*_{|c|\left(z^l+\frac{1}{|c|}\bar{w}^k\right)}$ are unitarily equivalent to each other. Therefore, we only need to prove the result under the assumption $Tz^k + \alpha\bar{w}^l$ with $0 < \alpha \leq 1$.

We determine all the minimal reducing subspaces for T_φ in Theorem (4.3.1). We prove that $\mathcal{V}^*(\varphi)$ is abelian.

For $\mathbb{Z}$ denote all integers, $\mathbb{Z}_+$ denote all nonnegative integers and $\mathbb{Q}$ denote the set of rational numbers. For positive integers k, l, define

$$\Omega = \{(a,b) \in \mathbb{Z}_+^2 |\, 0 \leq a \leq k-1, 0 \leq b \leq l-1\}.$$

For each $(a,b) \in \Omega$, put $s = \frac{a+1}{k}$, $t = \frac{b+1}{l}$. Then $s,t \in (0,1] \cap \mathbb{Q}$. For $\alpha \in (0,1]$, divide Ω into two parts:

$$\Omega_{\alpha,1} = \{(a,b) \in \Omega |\, \frac{s}{s+1} - \alpha^2 \frac{t}{t+1} \neq 0\},$$

and

$$\Omega_{\alpha,2} = \{(a,b) \in \Omega |\, \frac{s}{s+1} - \alpha^2 \frac{t}{t+1} = 0\}.$$

Let $p_\alpha(z,w) = z^k + \alpha\bar{w}^l (0 < \alpha \leq 1)$. Obviously, the subspace

$$L_{a,b} = \overline{\operatorname{span}\{z^{a+nk} w^{b+ml} |\, n,m \in \mathbb{Z}_+\}}$$

is a reducing subspace for T_{p_α}, and $A^2(\mathbb{D}^2) = \bigoplus_{(a,b)\in\Omega} L_{a,b}$. Denote by $T_\alpha = T^*_{p_\alpha} T_{p_\alpha} - T_{p_\alpha} T^*_{p_\alpha}$, then

$$T_\alpha = (T^*_{z^k} T_{z^k} - T_{z^k} T^*_{z^k}) - \alpha^2 (T^*_{w^l} T_{w^l} - T_{w^l} T^*_{w^l})$$

and

$$T_{\alpha} z^{a+nk} w^{b+ml} = \big(\phi(s,n) - \alpha^2 \phi(t,m)\big) z^{a+nk} w^{b+ml}, \qquad (26)$$

where

$$\phi(u,p) = \begin{cases} \dfrac{1}{(u+p)(u+p+1)}, & p > 0, \\ \dfrac{u}{u+1}, & p = 0. \end{cases} \qquad (27)$$

Set

$$\lambda_\alpha(a,b,n,m) = \phi(s,n) - \alpha^2 \phi(t,m). \qquad (28)$$

For $(a,b),(a',b') \in \Omega$, define an equivalence on $\mathbb{Z}_+^2$ by

$$(a,b,n,m) \sim (a',b',n',m') \Leftrightarrow \lambda_\alpha(a,b,n,m) = \lambda_\alpha(a',b',n',m'). \qquad (29)$$

If $(a,b) = (a',b')$, this notation can be simplified as

$$(n,m) \sim_{a,b} (n',m') \Leftrightarrow \lambda_\alpha(a,b,n,m) = \lambda_\alpha(a,b,n',m').$$

Define

$$\Delta'_{n,m} = \{(n',m') : (a,b,n,m) \sim (a',b',n',m'), (n',m') \in \mathbb{Z}_+^2\}$$

and

$$\Delta_{n,m} = \{(n',m') : (n,m) \sim_{a,b} (n',m'), (n',m') \in \mathbb{Z}_+^2\}.$$

Concerning $\mathbb{Z}_+^2$, define the partial order $\geq$ by setting

$$(n_1,m_1) \geq (n_2,m_2) \text{ if } n_1 \geq n_2 \text{ and } m_1 \geq m_2.$$

Put $\Gamma = \{(n,m) \in \mathbb{Z}_+^2 | (n,m) \geq (1,1)\}$ and $\Gamma^c = \mathbb{Z}_+^2 \backslash \Gamma$. For each $f \in A^2(\mathbb{D}^2), [f]_\alpha$ denotes the reducing subspace for T_{p_α} generated by f, i.e., the smallest reducing subspace for T_{p_α} containing f.

We provide some useful lemmas.

Lemma (4.3.2)[124]: If $0 < \alpha \leq 1, (a,b) \in \Omega, n, n', m, m' \in \mathbb{Z}_+$, then

$(i)(n,m) \sim_{a,b} (n,m\_)$ *if and only if* $m = m'$;

$(ii)(n,m) \sim_{a,b} (n',m)$ *if and only if* $n = n'$;

(iii) *if* $(a,b) \in \Omega_{1,1} \cup \Omega_{\alpha,2}(\alpha \in (0,1))$, *then* $(0,0) \nsim_{a,b} (n,n)$ *for* $n \geq 1$.

Proof. (i) Obviously, we only need to prove the necessity. By (27), (28) and (29), we know that $\phi(t,m) = \phi(t,m')$.

If $m, m' \geq 1$, then $(t+m)(t+m+1) = (t+m')(t+m'+1)$. It follows that $m = m'$.

If one of m, m' is 0, without loss of generality, assume $m \geq 1$ and $m' = 0$. Then

$$\frac{1}{(t+m)(t+m+1)} = \frac{t}{t+1}.$$

That is, $F(t) = t(t+m)(t+m+1) - (t+1) = 0$, where F is a polynomial with integral coefficients, with leading coefficient equal to 1 and with degree at least 3. By the theory of algebra, all rational roots of F are integers. Since $t \in (0,1] \cap \mathbb{Q}$, we have $t = 1$ and $m = 0$. This leads to a contradiction.

(ii) By the symmetry of n and m in the proof of (i), we have (ii) holds.

(iii) The assumption $(a,b) \in \Omega_{1,1} \cup \Omega_{\alpha,2}(\alpha \in (0,1))$ shows that $s \neq t$. Notice that $(0,0) \sim_{a,b} (n,n)$ for some $n \geq 1$ if and only if

$$\frac{s}{s+1} - \frac{1}{(s+n)(s+n+1)} = \frac{\alpha^2 t}{t+1} - \frac{\alpha^2}{(t+n)(t+n+1)}.$$

If $(a,b) \in \Omega_{1,1}$, consider the function $G(x) = \frac{x}{x+1} - \frac{1}{(x+n)(x+n+1)}, n \geq 1$.

Obviously, $G(x)$ is strictly increasing on $(0,+\infty)$. Then $G(s) \neq G(t)$, i.e., $(0,0) \nsim_{a,b} (n,n)$.

If $(a,b) \in \Omega_{\alpha,2}(0 < \alpha < 1)$, we have $\frac{s}{s+1} = \alpha^2 \frac{t}{t+1} < \frac{t}{t+1}$. Then $t > s$. It follows that $\frac{1}{(s+n)(s+n+1)} > \frac{1}{(t+n)(t+n+1)} > \frac{\alpha^2}{(t+n)(t+n+1)}$. Hence, $(0,0) \nsim_{a,b} (n,n)$.

Lemma (4.3.3)[124]: If $(a,b) \in \Omega_{1,1} \cup \Omega_{\alpha,2}(0 < \alpha < 1), (a',b') \in \Omega$ and $(n,m) \in \Delta'_{0,0} \cap \Gamma$, then $(n \pm 1, m \pm 1) \notin \Delta'_{0,0} \cap \Gamma$.

Proof. Suppose $(n+1, m+1) \in \Delta'_{0,0}$. For $0 < \alpha \leq 1$, we get two equations as follows:

$$\frac{s}{s+1} - \frac{\alpha^2 t}{t+1} = \frac{1}{(s'+n)(s'+n+1)} - \frac{\alpha^2}{(t'+m)(t'+m+1)}, \quad (30)$$

and

$$\frac{s}{s+1} - \frac{\alpha^2 t}{t+1} = \frac{1}{(s'+n+1)(s'+n+2)} - \frac{\alpha^2}{(t'+m+1)(t'+m+2)}. \quad (31)$$

If $(a,b) \in \Omega_{\alpha,2}$ $(0 < \alpha < 1)$, Eq. (30) and Eq. (31) imply that

$$\alpha^2 = \frac{(t'+m)(t'+m+1)}{(s'+n)(s'+n+1)} = \frac{(t'+m+1)(t'+m+2)}{(s'+n+1)(s'+n+2)}.$$

Therefore, $t' + m = s' + n$ and $\alpha = 1$, which is a contradiction.

If $(a,b) \in \Omega_{1,1}$, then $t' + m = s' + n$. Eq. (30) implies that $(a,b) \in \Omega_{1,2}$, which is also a contradiction. So $(n + 1, m + 1) \notin \Delta'_{0,0} \cap \Gamma$.
Replace (n,m) by $(n - 1, m - 1)$. Assume $(n - 1, m - 1) \in \Delta'_{0,0} \cap \Gamma$. As the proof above, we have $(n,m) = (n - 1 + 1, m - 1 + 1) \notin \Delta'_{0,0}$, which is a contradiction. The proof is finished.
Lemma (4.3.4)[124]: If $(a,b) \in \Omega_{1,2}$ and $n,m \in \mathbb{Z}_+$, then the following statements hold:
(i) $(n,m) \in \Delta_{0,0}$ *if and only if* $n = m$.
(ii) $(n' + 1, n') \in \Delta_{n+1,n}$ *if and only if* $n = n'$.
Proof. (i) It is easy to see that $(n,n) \in \Delta_{0,0}$ for $n \in \mathbb{Z}_+$, since $\lambda_1(a,b,0,0) = \lambda_1(a,b,n,n) = 0$. On the other hand, assume $(n,m) \in \Delta_{0,0}$. If $n = 0$, Lemma (4.3.2)(i) shows that $m = 0$. If $n \geq 1$, Eq. (27) and Eq. (28) imply that $\phi(s,m) = \frac{1}{(s+n)(s+n+1)}$.
Since $F(s) = s(s+n)(s+n+1) - (s+1) = 0$ has no rational roots in $(0,1]$, we have $m \geq 1$ and $\phi(s,m) = \frac{1}{(s+m)(s+m+1)} = \frac{1}{(s+n)(s+n+1)}$. Hence $m = n$.
(ii) We only need to prove the necessity. If $n = 0$ and $n' \geq 1$, then $(n+1,n) \sim_{a,b} (n' + 1, n')$ indicates

$$\frac{1}{(s + 1)(s + 2)} - \frac{s}{s + 1} = \frac{-2}{(s + n')(s + n' + 1)(s + n' + 2)}.$$

By the theory of algebra, we obtain $s = 1$, and then $n' = 0$, which is a contradiction with $n' \geq 1$.
If $n \geq 1$, by the argument above, we have $n' \geq 1$. Therefore, $(n + 1, n) \sim_{a,b} (n' + 1, n')$ gives

$$\frac{2}{(s + n)(s + n + 1)(s + n + 2)} = \frac{2}{(s + n')(s + n' + 1)(s + n' + 2)}.$$

It follows that $n = n'$.
The following lemmas hold for every $0 < \alpha \leq 1$.
Lemma (4.3.5)[124]: If $(a,b),(a',b') \in \Omega$ and $(n_0,m_0) \in \Delta'_{n,m}$, then the following conclusions hold:

(i) *if* $(n,m),(n_0,m_0) \geq (1,0)$ *and* $n \neq n_0$, *then* $(n_0 + 1, m_0) \notin \Delta'_{n+1,m}$;
(ii) *if* $(n,m),(n_0,m_0) \geq (0,1)$ *and* $m \neq m_0$, *then* $(n_0, m_0 + 1) \notin \Delta'_{n,m+1}$;
(iii) *if* $(a,b) = (a',b'), n = 0$ *and* $n_0 \geq 1$, *then* $(n_0 + 1, m_0) \notin \Delta_{1,m}$.

Proof. (i) If $(n_0,m_0) \in \Delta'_{n,m}$ and $(n_0 + 1, m_0) \in \Delta'_{n+1,m}$, then

$$\frac{1}{(s' + n_0)(s' + n_0 + 1)} - \alpha^2\phi(t',m_0) = \frac{1}{(s + n)(s + n + 1)} - \alpha^2\phi(t,m),$$

and

$$\frac{1}{(s' + n_0 + 1)(s' + n_0 + 2)} - \alpha^2\phi(t',m_0) = \frac{1}{(s + n + 1)(s + n + 2)} - \alpha^2\phi(t,m).$$

It follows that

$$\frac{1}{(s' + n_0)(s' + n_0 + 1)(s' + n_0 + 2)} = \frac{1}{(s + n)(s + n + 1)(s + n + 2)}.$$

Thus $s' + n_0 = s + n$. Since $-1 \leq -s' < n_0 - n = s - s' < s \leq 1$, we have $n_0 = n$. This is a contradiction. So (i) holds.
(ii) Since the proof of this case is similar to the proof of (i), we omit the details.
(iii) Assume $(n_0 + 1, m_0) \in \Delta_{1,m}$. A computation shows that

$$\frac{s}{s+1}-\frac{1}{(s+1)(s+2)}=\frac{2}{(s+n_0)(s+n_0+1)(s+n_0+2)}$$

As in the proof of Lemma (4.3.4) (ii), there is no solution on $(0,1]$ for any $n_0 \geq 1$. Therefore, the assumption is not true. We finish the proof.

Lemma (4.3.6)[124]: If $(a,b) \in \Omega$, then $[z^a w^b]_\alpha = L_{a,b}$.

Proof. Obviously, we only need to show $z^{a+nk}w^{b+ml} \in [z^a w^b]_\alpha$ for $n,m \geq 0$. Notice that

$$T_{p_\alpha}^n z^a w^b = z^{a+nk}w^b \in [z^a w^b]_\alpha, \qquad T_{p_\alpha}^{*m} z^a w^b = \alpha^m z^a w^{b+ml} \in [z^a w^b]_\alpha.$$

Denote by $\gamma_k = \|z^k\|^2 = \frac{1}{k+1}$. Then we have

$$T_{p_\alpha}^* z^{a+nk}w^b = \alpha z^{a+nk}w^{b+l} + \frac{\gamma_{a+nk}}{\gamma_{a+(n-1)k}} z^{a+(n-1)k}w^b \in [z^a w^b]_\alpha.$$

It follows that $z^{a+nk}w^{b+l} \in [z^a w^b]_\alpha$ for all $n \geq 1$. For every $q \geq 1$, since

$$\alpha z^{a+nk}w^{b+(q+1)l} = T_{p_\alpha}^* z^{a+nk}w^{b+ql} - \frac{\gamma_{a+nk}}{\gamma_{a+(n-1)k}} z^{a+(n-1)k}w^{b+ql},$$

it is easy to get the desired result by induction.

We will prove that $L_{a,b}((a,b) \in \Omega)$ are the minimal reducing subspaces for T_{p_α} and any two distinct parameter-pairs (a,b) generate nonequivalent reducing subspaces $L_{a,b}$.

Theorem (4.3.7)[124]: If $(a,b) \in \Omega$, then $L_{a,b}$ is a minimal reducing subspace for T_{p_α} for any $0 < \alpha \leq 1$.

Proof. Obviously, $L_{a,b}$ is a reducing subspace for T_{p_α} for any $0 < \alpha \leq 1$. We only need to prove that $L_{a,b}$ is minimal.

Suppose there is a reducing subspace M_1 included in $L_{a,b}$. Denote by P_1 the orthogonal projection from $A^2(\mathbb{D}^2)$ onto M_1. Recall that $T_\alpha = T_{p_\alpha}^* T_{p_\alpha} - T_{p_\alpha} T_{p_\alpha}^*$. By (26) and

$$T_\alpha P_1 z^{a+nk}w^{b+ml} = P_1 T_\alpha z^{a+nk}w^{b+ml} = \lambda_\alpha(a,b,n,m) z^{a+nk}w^{b+ml},$$

we get $P_1 z^{a+nk}w^{b+ml} \in \operatorname{span}\{z^{a+pk}w^{b+ql} : (p,q) \in \Delta_{n,m}\}$. Further, Lemma (4.3.2) (i) and (ii) deduce that

$$f_1 = P_1 z^a w^b = a_{00} z^a w^b + \sum_{(n,m)\in\Delta_{0,0}\cap\Gamma\mathrm{b}} a_{nm} z^{a+nk}w^{b+ml}.$$

Then

$$T_{p_\alpha} f_1 = a_{00} z^{a+k}w^b + \sum_{(n,m)\in\Delta_{0,0}\cap\Gamma} a_{nm}\left[z^{a+(n+1)k}w^{b+ml} + \frac{\alpha\gamma_{b+ml}}{\gamma_{b+(m-1)l}} z^{a+nk}w^{b+(m-1)l}\right]. \quad (32)$$

Notice that

$$T_{p_\alpha} f_1 = P_1 T_{p_\alpha} z^a w^b = P_1 z^{a+k}w^b \in \operatorname{span}\{z^{a+pk}w^{b+ql} : (p,q) \in \Delta_{1,0}\}. \quad (33)$$

Claim (4.3.8)[124]: $f_1 = a_{00} z^a w^b$.

The proof of the above Claim (4.3.8) will be divided into three cases: $(a,b) \in \Omega_{1,1} \cup \Omega_{\alpha,2}(0 < \alpha < 1), (a,b) \in \Omega_{1,2}$ and $(a,b) \in \Omega_{\alpha,1}(0 < \alpha < 1)$.

Case (1): $(a,b) \in \Omega_{1,1} \cup \Omega_{\alpha,2}(0 < \alpha < 1)$. Without confusion, the range of α is $(0,1]$. Lemma (4.3.3) shows that $(n \pm 1, m \pm 1) \notin \Delta_{0,0} \cap \Gamma$, since $(n,m) \in \Delta_{0,0} \cap \Gamma$.

Thus the coefficient of $z^{a+(n+1)k}w^{b+ml}$ in Eq. (32) is a_{nm}. Associated with Lemma (4.3.5) (iii) and Eq. (32), we deduce that $a_{nm} = 0$ for $(n,m) \in \Gamma$. Hence, $f_1 = a_{00} z^a w^b$.

Case (2): $(a,b) \in \Omega_{1,2}$. Let $a_n = a_{nn}$ for convenience. Lemma (4.3.4) deduces that

$$f_1 = \sum_{n\in\mathbb{Z}_+} a_n z^{a+nk} w^{b+nl}.$$

By simple computations, Eq. (32) becomes

$$T_{p_\alpha} f_1 = \left(a_0 + a_1 \frac{\gamma_{b+l}}{\gamma_b}\right) z^{a+k} w^b + \sum_{n\geq 1}\left(a_n + a_{n+1}\frac{\gamma_{b+(n+1)l}}{\gamma_{b+nl}}\right) z^{a+(n+1)k} w^{b+nl}. \quad (34)$$

Lemma (4.3.5)(iii), along with Eq. (33) and (34), shows that

$$a_n + a_{n+1}\frac{\gamma_{b+(n+1)l}}{\gamma_{b+nl}} = 0, \qquad \forall n \geq 1.$$

It follows that $a_n = (-1)^{n-1}\frac{\gamma_{b+l}}{\gamma_{b+nl}} a_1$. Thus

$$|a_n|^2 \left\|z^{a+nk} w^{b+nl}\right\|^2 = \frac{(b + nl + 1)}{(a + nk + 1)(b + l + 1)^2}|a_1|^2.$$

Since $f_1 \in A^2(\mathbb{D}^2)$, we have that $|a_n|^2\left\|z^{a+nk}w^{b+nl}\right\|^2 \to 0$ as $n \to \infty$. Thus $a_1 = 0$, which indicates that $a_n = 0$ for $n \geq 1$. Hence, $f_1 = a_{00} z^a w^b$.

Case (3): $(a, b) \in \Omega_{\alpha,1}$ $(0 < \alpha < 1)$. Firstly, we prove that $\Delta_{0,0}$ is a finite set. Otherwise, by Lemma (4.3.2) (i) and (ii), there exists $(n_k, m_k) \geq (k, k)$ satisfying $(n_k, m_k) \in \Delta_{0,0} \cap \Gamma$ for every $k \geq 1$. That is, $\frac{s}{s+1} - \alpha^2\frac{t}{t+1} = \frac{1}{(s+n_k)(s+n_k+1)} - \alpha^2\frac{1}{(t+m_k)(t+m_k+1)}$. Letting $k \to +\infty$ shows that $\frac{s}{s+1} - \alpha^2\frac{t}{t+1} = 0$, which is a contradiction. Therefore,

$$f_1 = a_{00} z^a w^b + \sum_{j=1}^{N} b_j z^{a+p_jk} w^{b+q_jl},$$

where N is a positive integer and $(p_j, q_j) \in \Delta_{0,0} \cap \Gamma$. Reset $\{(p_i, q_i)\}(1 \leq i \leq N)$ as $1 \leq p_1 < p_2 < \cdots < p_N$. Thus,

$$T_{p_\alpha} f_1 = a_{00} z^{a+k} w^b + \sum_{j=1}^{N} b_j\left(z^{a+(p_j+1)k} w^{b+q_jl} + \alpha\frac{\gamma_{b+q_jl}}{\gamma_{b+(q_j-1)l}} z^{a+p_jk} w^{b+(q_j-1)l}\right).$$

By Lemma (4.3.5)(iii) again, we have $(p_N + 1, q_N) \notin \Delta_{1,0}$. Then Eq. (33) shows that $b_N = 0$. Repeat this process until we get $b_j = 0, j = 1, \cdots, N$. Therefore, $f_1 = a_{00} z^a w^b$. So we get the desired result.

According to the above Claim (4.3.8), we know that either $z^a w^b \in M_1$ or $z^a w^b \perp M_1$.

Consequently, using Lemma (4.3.6), we obtain that either $M_1 = L_{a,b}$ or $M_1 = \{0\}$. So $L_{a,b}$ is minimal.

As in [30], we say that two reducing subspaces M_1 and M_2 of T_φ are called unitarily equivalent if there exists a unitary operator U from M onto N and U commutes with T_φ. One can show that M_1 is unitarily equivalent to M_2 if and only if P_{M_1} and P_{M_2} are equivalent in $\mathcal{V}^*(\varphi)$, that is, there is a partial isometry V in $\mathcal{V}^*(\varphi)$ such that

$$V^*V = P_{M_1}, \qquad VV^* = P_{M_2}.$$

Now, we are ready to prove that $L_{a,b}$ and $L_{a',b'}$ are not unitarily equivalent if $(a, b) \neq (a', b')$. For $0 < \alpha < 1$, our proof is divided into two parts:

(i) $(a, b) \in \Omega_{\alpha,2}, (a', b') \in \Omega$;

(ii) $(a, b), (a', b') \in \Omega_{\alpha,1}$.

For $\alpha = 1$, our proof is also divided into two parts:

(iii) $(a,b) \in \Omega_{1,1}, (a',b') \in \Omega$;

(iv) $(a,b),(a',b') \in \Omega_{1,2}$.

Since the proof of (i) and (iii) are similar, we write the proof of them together.

Theorem (4.3.9)[124]: If $(a,b) \in \Omega_{\alpha,2}(0 < \alpha < 1)$(or $(a,b) \in \Omega_{1,1}$), $(a',b') \in \Omega$ and
$(a,b) \neq (a',b')$, then $L_{a,b}$ and $L_{a',b'}$ are not unitarily equivalent reducing subspaces for $T_{p_\alpha}(or\ T_{p_1})$.

Proof. Without confusion, the range of α is $(0,1]$. Assume conversely that $L_{a,b}$ and $L_{a',b'}$ are unitarily equivalent, then there is a partial isometry $U \in \mathcal{V}^*(p_\alpha)$ such that $U|_{L_{a,b}}$ is a unitary operator from $L_{a,b}$ onto $L_{a'b'}$.

Recall that $T_\alpha = T_{p_\alpha}^* T_{p_\alpha} - T_{p_\alpha} T_{p_\alpha}^*$. For $(n,m) \in \mathbb{Z}_+^2$, suppose

$$Uz^{a+nk}w^{b+ml} = \sum_{(p,q)\in\mathbb{Z}_+^2} a_{pq} z^{a'+pk} w^{b'+ql},$$

where $a_{pq} \in \mathbb{C}$. Moreover, we have

$$UT_\alpha z^{a+nk}w^{b+ml} = \sum_{(p,q)\in\mathbb{Z}_+^2} a_{pq}\lambda_\alpha(a,b,n,m) z^{a'+pk} w^{b'+ql},$$

and

$$T_\alpha U z^{a+nk}w^{b+ml} = \sum_{(p,q)\in\mathbb{Z}_+^2} a_{pq}\lambda_\alpha(a',b',p,q) z^{a'+pk} w^{b'+ql}.$$

Thus, $UT_\alpha = T_\alpha U$ indicates that

$$Uz^{a+nk}w^{b+ml} \in \overline{\operatorname{span}\{z^{a'+pk}w^{b'+ql} : (p,q) \in \Delta'_{n,m}\}}. \tag{35}$$

By Lemma (4.3.2)(i) and (ii), we know that there is at most one $n_0 \geq 1$ satisfying $(n_0, 0) \in \Delta'_{0,0}$, and there is at most one $m_0 \geq 1$ satisfying $(0, m_0) \in \Delta'_{0,0}$. So

$$Uz^a w^b = a_{00} z^{a'} w^{b'} + a_{n_0,0} z^{a'+n_0 k} w^{b'}$$
$$+ a_{0,m_0} z^{a'} w^{b'+m_0 l} + \sum_{(n,m)\in\Delta'_{0,0}\cap\Gamma} a_{nm} z^{a'+nk} w^{b'+ml}.$$

By simple computations, we have

$$UT_{p_\alpha}^* T_{p_\alpha} z^a w^b = \alpha U z^{a+k} w^{b+l} + \frac{\gamma_{a+k}}{\gamma_a} U z^a w^b, \tag{36}$$

and

$$T_{p_\alpha}^* T_{p_\alpha} U z^a w^b = a_{00}\left(\alpha z^{a'+k} w^{b'+l} + \frac{\gamma_{a'+k}}{\gamma_{a'}} z^{a'} w^{b'}\right)$$
$$+ a_{n_0,0}\left(\alpha z^{a'+(n_0+1)k} w^{b'+l} + \frac{\gamma_{a'+(n_0+1)k}}{\gamma_{a'+n_0 k}} z^{a'+n_0 k} w^{b'}\right)$$
$$+ a_{0,m_0}\left[\alpha z^{a'+k} w^{b'+(m_0+1)l} + \left(\frac{\gamma_{a'+k}}{\gamma_{a'}} + \alpha^2 \frac{\gamma_{b'+m_0 l}}{\gamma_{b'+(m_0-1)l}}\right) z^{a'} w^{b'+m_0 l}\right]$$

$$+ \sum_{(n,m)\in\Delta'_{0,0}\cap\Gamma} a_{nm} \Bigg[\alpha z^{a'+(n+1)k} w^{b'+(m+1)l}$$
$$+ \left(\frac{\gamma_{a'+(n+1)k}}{\gamma_{a'+nk}} + \alpha^2 \frac{\gamma_{b'+ml}}{\gamma_{b'+(m-1)l}} \right) z^{a'+nk} w^{b'+ml}$$
$$+ \alpha \frac{\gamma_{b'+ml}}{\gamma_{b'+(m-1)l}} \frac{\gamma_{a'+nk}}{\gamma_{a'+(n-1)k}} z^{a'+(n-1)k} w^{b'+(m-1)l} \Bigg]. \tag{37}$$

Compare the coefficients of $z^{a'+nk}w^{b'+ml}$ for $(n,m) \in \Delta'_{0,0} \cap \Gamma$, respectively. If $(n,m) \in \Gamma \setminus \{(1,1),(n_0+1,1),(1,m_0+1)\}$, along with (35), we obtain that

$$\left(\frac{\gamma_{a'+(n+1)k}}{\gamma_{a'+nk}} + \alpha^2 \frac{\gamma_{b'+ml}}{\gamma_{b'+(m-1)l}} \right) a_{nm} = \frac{\gamma_{a+k}}{\gamma_a} a_{nm},$$

since $\Delta'_{0,0} \cap \Delta'_{1,1} = \emptyset$ (by Lemma (4.3.2)(iii)), $(n \pm 1, m \pm 1) \notin \Delta'_{0,0} \cap \Gamma$ (by Lemma 2.2), and $(n_0,0),(0,m_0) \notin \Gamma$. However, $\frac{\gamma_{a'+(n+1)k}}{\gamma_{a'+nk}} + \alpha^2 \frac{\gamma_{b'+ml}}{\gamma_{b'+(m-1)l}} \geq \frac{\gamma_{a'+(n+1)k}}{\gamma_{a'+nk}} > \frac{\gamma_{a+k}}{\gamma_a}$. So $a_{nm} = 0$. It means that

$$Uz^a w^b = c_0 z^{a'} w^{b'} + c_1 z^{a'+k} w^{b'+l} + c_2 z^{a'+n_0 k} w^{b'} + c_3 z^{a'+(n_0+1)k} w^{b'+l} + c_4 z^{a'} w^{b'+m_0 l} + c_5 z^{a'+k} w^{b'+(m_0+1)l}.$$

Some simple computations show that

$$\begin{aligned} Uz^{a+k} w^b &= T_{p_\alpha} U z^a w^b \\ &= \left(c_0 + \alpha c_1 \frac{\gamma_{b'+l}}{\gamma_{b'}} \right) z^{a'+k} w^{b'} + c_1 z^{a'+2k} w^{b'+l} \\ &+ \left(c_2 + \alpha c_3 \frac{\gamma_{b'+k}}{\gamma_{b'}} \right) z^{a'+(n_0+1)k} w^{b'} + c_3 z^{a'+(n_0+2)k} w^{b'+l} + h_1(z,w), \end{aligned}$$

and

$$\begin{aligned} Uz^{a+2k} w^b &= T_{p_\alpha} U z^{a+k} w^b \\ &= \left(c_0 + 2\alpha c_1 \frac{\gamma_{b'+l}}{\gamma_{b'}} \right) z^{a'+2k} w^{b'} + c_1 z^{a'+3k} w^{b'+l} \\ &+ \left(c_2 + 2\alpha c_3 \frac{\gamma_{b'+k}}{\gamma_{b'}} \right) z^{a'+(n_0+2)k} w^{b'} + c_3 z^{a'+(n_0+3)k} w^{b'+l} + T_{p_\alpha} h_1, \end{aligned}$$

where

$$h_1 \perp \{z^{a'+k} w^{b'}, z^{a'+2k} w^{b'+l}, z^{a'+(n_0+1)k} w^{b'}, z^{a'+(n_0+2)k} w^{b'+l}\}$$

and

$$T_{p_\alpha} h_1 \perp \{z^{a'+2k} w^{b'+l}, z^{a'+3k} w^{b'+l}, z^{a'+(n_0+2)k} w^{b'}, z^{a'+(n_0+3)k} w^{b'+l}\}.$$

By Lemma (4.3.5)(i), we obtain that (2, 1) $\in \Delta'_{1,0}$ and (3, 1) $\in \Delta'_{2,0}$ are incompatible. So $c_1 = 0$. Moreover, the incompatibility of $(n_0+2,1) \in \Delta'_{1,0}$ and $(n_0+3,1) \in \Delta'_{2,0}$ shows that $c_3 = 0$; the incompatibility of $(n_0+1,0) \in \Delta'_{1,0}$ and $(n_0+2,0) \in \Delta'_{2,0}$ implies $c_2 = 0$.

Symmetrically, we consider $Uz^a w^{b+l}$ and $Uz^a w^{b+2l}$. In a similar way, Lemma (4.3.5) (ii) implies that $c_4 = c_5 = 0$. So

$$Uz^a w^b = c_0 z^{a'} w^{b'}, \qquad Uz^{a+k} w^b = c_0 z^{a'+k} w^{b'}, \qquad Uz^a w^{b+l} = c_0 z^{a'} w^{b'+l}.$$

Since $U|_{L_{a,b}}$ is a unitary operator, we get $c_0 \neq 0$. Then $(0,0) \in \Delta'_{0,0}, (1,0) \in \Delta'_{1,0}$ and $(0,1) \in \Delta'_{0,1}$. By some calculations, it is easy to check that

$$\frac{1}{(s+1)(s+2)}-\frac{s}{s+1}=\frac{1}{(s'+1)(s'+2)}-\frac{s'}{s'+1};$$
$$\frac{1}{(t+1)(t+2)}-\frac{t}{t+1}=\frac{1}{(t'+1)(t'+2)}-\frac{t'}{t'+1}.$$
It follows that $(a,b)=(a',b')$, which is a contradiction. Hence, we complete the proof.

Theorem (4.3.10)[124]: If $(a,b),(a',b')\in\Omega_{\alpha,1}(0<\alpha<1)$ and $(a,b)\neq(a',b')$, then $L_{a,b}$ and $L_{a',b'}$ are not unitarily equivalent.

Proof. Firstly, by the analogous proof in Theorem (4.3.7), we get that $\Delta'_{0,0}$ is a finite set. Secondly, assume conversely $L_{a,b}$ and $L_{a',b'}$ are unitarily equivalent, then there is a partial isometry $U\in\mathcal{V}^*(p_\alpha)$ such that $U|_{L_{a,b}}$ is a unitary operator from $L_{a,b}$ onto $L_{a',b'}$. As the analysis above, set
$$Uz^aw^b=a_0z^{a'}w^{b'}+u_{n_0}z^{a'+n_0k}w^{b'}+u_{m_0}z^{a'}w^{b'+m_0l}+\sum_{i=1}^{N}a_iz^{a'+n_ik}w^{b'+m_il},$$
where N is a positive integer, $(n_i,m_i)\in\Delta'_{0,0}\cap\Gamma, n_i\neq n_0, m_i\neq m_0$ and $a_i,u_{n_0},u_{m_0}\in\mathbb{C}$. Rearrange $\{(n_i,m_i)\}(1\le i\le N)$ as $1\le n_1<n_2<\cdots<n_N$.

By $U\in\mathcal{V}^*(p_\alpha)$, we have
$$Uz^{a+k}w^b=a_Nz^{a'+(n_N+1)k}w^{b'+m_Nl}+g_1(z,w),$$
and
$$Uz^{a+2k}w^b=a_Nz^{a'+(n_N+2)k}w^{b'+m_Nl}+\alpha a_N\frac{\gamma_{b'+m_Nl}}{\gamma_{b'+(m_N-1)l}}z^{a'+(n_N+1)k}w^{b'+(m_N-1)l}+T_{p_\alpha}g_1(z,w),$$
where $g_1\perp\{z^{a'+(n_N+1)k}w^{b'+m_Nl}\}, T_{p_\alpha}g_1\perp\{z^{a'+(n_N+2)k}w^{b'+m_Nl}\}$.

By Lemma (4.3.5)(i), $(n_N+1,m_N)\in\Delta'_{1,0}$ and $(n_N+2,m_N)\in\Delta'_{2,0}$ are incompatible. Since
$$Uz^{a+2k}w^b\subseteq\overline{\operatorname{span}\{z^{a'+pk}w^{b'+ql}:(p,q)\in\Delta'_{2,0}\}}$$
and
$$Uz^{a+k}w^b\subseteq\overline{\operatorname{span}\{z^{a'+pk}w^{b'+ql}:(p,q)\in\Delta'_{1,0}\}},$$
we have $a_N=0$. Repeat this process until we get $a_i=0$ for $i=1,\cdots,N$.

Therefore, we deduce that
$$Uz^aw^b=a_0z^{a'}w^{b'}+u_{n_0}z^{a'+n_0k}w^{b'}+u_{m_0}z^{a'}w^{b'+m_0l}.$$
By similar way as above, it is easy to get $u_{n_0}=0$.

On the other hand, we have
$$Uz^aw^{b+l}=\alpha a_0z^{a'}w^{b'+l}+\alpha u_{m_0}z^{a'}w^{b'+(m_0+1)l},$$
and
$$Uz^aw^{b+2l}=\alpha a_0z^{a'}w^{b'+2l}+\alpha u_{m_0}z^{a'}w^{b'+(m_0+2)l}.$$
By Lemma (4.3.5)(ii), the incompatibility of $(0,m_0+1)\in\Delta'_{0,1}$ and $(0,m_0+2)\in\Delta'_{0,2}$ shows that $u_{m_0}=0$.

Since $U|_{L_{a,b}}$ is unitary, we have $a_0\neq0$. It follows that $(0,0)\in\Delta'_{0,0}$ and $(0,1)\in\Delta'_{0,1}$. A simple calculation implies that $t=t'$ and $s=s'$, i.e., $(a,b)=(a',b')$. Thus the assumption is false and we finish the proof.

Theorem (4.3.11)[124]: If $(a,b),(a',b')\in\Omega_{1,2}$ and $(a,b)\neq(a',b')$, then $L_{a,b}$ and $L_{a',b'}$ are not unitarily equivalent.

Proof. Assume conversely that $L_{a,b}$ and $L_{a',b'}$ are unitarily equivalent, then there is a partial isometry $U \in \mathcal{V}^*(p_1)$ such that $U|_{L_{a,b}}$ is a unitary operator from $L_{a,b}$ onto $L_{a',b'}$. By the analogous argument in Theorem (4.3.9) and by Lemma (4.3.4), we have

$$Uz^a w^b = \sum_{n\in\mathbb{Z}_+} a_n z^{a'+nk} w^{b'+nl}.$$

Moreover,

$$Uz^{a+k}w^b = T_p Uz^a w^b = \left(a_0 + a_1 \frac{\gamma_{b'+l}}{\gamma_{b'}}\right) z^{a'+k} w^{b'} + \sum_{n\geq 1}\left(a_n + a_{n+1}\frac{\gamma_{b'+(n+1)l}}{\gamma_{b'+nl}}\right) z^{a'+(n+1)k} w^{b'+nl}.$$

By $(a,b),(a',b') \in \Omega_{1,2}$, we have $s = t, s' = t'$ and $(0,0) \in \Delta'_{0,0}$. Obviously, $(a,b) \neq (a',b')$ implies that $s \neq s'$, forcing $(1,0) \notin \Delta'_{1,0}$. Thus, $a_0 + a_1 \frac{\gamma_{b'+l}}{\gamma_{b'}} = 0$.
Lemma (4.3.4)(ii) shows that $(q+1,q) \in \Delta'_{1,0}$ for at most one $q \geq 1$. Therefore, we deduce that

$$Uz^{a+k}w^b = cz^{a'+(q+1)k}w^{b'+ql},$$

where $c \in \mathbb{C}$. Moreover,

$$Uz^{a+2k}w^l = cz^{a'+(q+2)k}w^{b'+ql} + c\frac{\gamma_{b'+ql}}{\gamma_{b'+(q-1)l}} z^{a'+(q+1)k}w^{b'+(q-1)l}.$$

By Lemma (4.3.5)(i), $(q+1,q) \in \Delta'_{1,0}$ and $(q+2,q) \in \Delta'_{2,0}$ are incompatible.
So $c = 0$, which contradicts with $U \neq 0$. Hence, we arrive at the desired conclusion.

We consider the structure of the von Neumann algebra $\mathcal{V}^*(p_\alpha)$ for $0 < \alpha \leq 1$. Rewrite the minimal reducing subspaces $L_{a,b}((a,b) \in \Omega)$ by $M_1, M_2, \cdots, M_r$, where $r = kl$. As mentioned, M_1 is unitarily equivalent to M_2 if and only if P_{M_1} and P_{M_2} are equivalent in $\mathcal{V}^*(p_\alpha)$, thus we get any two of the minimal projections $P_{M_i}, i = 1, \cdots, r$, are not equivalent.

As in [112], there are no other minimal projections in $\mathcal{V}^*(p_\alpha)$ except $\{P_{M_i} | \, i = 1, \cdots, r\}$. We briefly recall the main ideas of the proof. Let Q be a minimal projection, which is distinct from all $P_{M_i}(1 \leq i \leq r)$. Since the direct sum of $P_{M_i}(1 \leq i \leq r)$ is the whole space $A^2(\mathbb{D}^2)$, there are at least two minimal projections P_{M_i} and P_{M_j} such that $P_{M_i}Q \neq 0$ and $P_{M_j}Q \neq 0$. Then Q is equivalent to P_{M_j} and P_{M_i}, which is a contradiction.
The following proposition comes from [30]:
Proposition (4.3.12)[124]: Let ε denote the set of all minimal projections in a von Neumann algebra $\mathcal{A}$ and suppose

$$\bigvee_{E\in\varepsilon} E = I.$$

Then there is a family of $\{\Lambda_i\}$ of subsets of ε such that

$$\sum_i \sum_{E\in\Lambda_i} = I,$$

(i) each $\{\Lambda_i\}$ consists of pairwisely orthogonal, mutually equivalent projections in $\mathcal{A}$;
(ii) if E', E'' lie in different $\{\Lambda_i\}$, then E' is not equivalent to E'';
Consequently, the von Neumann algebra $\mathcal{A}$ is $*$- isomorphic to

$$\oplus_i M_{n_i}(\mathbb{C}),$$

where n_i denotes the cardinality of $\{\Lambda_i\}$, allowed to be infinity.

Associated with $\bigvee_{i=1}^{r} P_{M_i} = I$, we have the following result.

Theorem (4.3.13)[124]: If $0 < \alpha \leq 1$, then $\mathcal{V}^*(p_\alpha)$ is $*$- isomorphic to

$$\bigoplus_{i=1}^{kl} \mathbb{C}.$$

Moreover, $\mathcal{V}^*(p_\alpha)$ is abelian.

Chapter 5
Hankel Operators and Products of Toeplitz Operators

We show the characterization of boundedness relies on certain precise estimates for the Bergman kernel and the Bergman metric. Characterizations of compact Hankel operators and Schatten class Hankel operators are also given. In the latter case, results on Carleson measures and Toeplitz operators along with Hörmander's L^2 estimates for the $\overline{\partial}$ operator are key ingredients in the proof. We show that the product $T_f T_g$ of Toeplitz operators on the Fock F_α^2 of $\mathbb{C}^n$ is bounded if and only if $f(z) = e^{q(z)}$ and $g(z) = ce^{-q(z)}$, where c is a nonzero constant and q is a linear polynomial. We provide a complete solution to the problem for a class of Fock spaces on the complex plane. In particular, this generalizes an earlier result of Cho, Park, and Zhu.

Section (5.1): Fock Spaces and Related Bergman Kernel Estimates

The basics of Hankel operators with anti-holomorphic symbols for a large class of weighted Fock spaces are presented. Thus certain natural analogues of BMOA, the Bloch space, the little Bloch space, and the Besov spaces are identified and shown to play similar roles as their classical counterparts do. We will see that these spaces contain all holomorphic polynomials and are infinite-dimensional whenever the weight decays so fast that there exist functions of infinite order belonging to the Fock space.

The setting is the following. Consider C^3-function $\Psi : [0,+\infty[\rightarrow [0,+\infty[$ such that

$$\Psi'(x) > 0, \qquad \Psi''(x) \geq 0, \qquad \text{and } \Psi'''(x) \geq 0. \tag{1}$$

We will refer to such a function as a logarithmic growth function. Note that (1) effectively says that should grow at least as a linear function. Set

$$d\mu_\Psi(z) := e^{-\Psi}(|z|^2)dV(z),$$

where dV denotes Lebesgue measure on $\mathbb{C}^n$, and let $A^2(\Psi)$ be the Fock space defined as the closure of the set of holomorphic polynomials in $L^2(\mu_\Psi)$. We observe that $A^2(\Psi)$ coincides with the classical Fock space when is a suitably normalized linear function.

It is immediate that

$$s_d := \int_0^{+\infty} x^d e^{-\Psi(x)} dx < +\infty$$

for all nonnegative integers d. As shown in [137], the series

$$F_s(\zeta) := \sum_{d=0}^{+\infty} \frac{\zeta^d}{s_d}, \quad \zeta \in \mathbb{C}$$

has an infinite radius of convergence and $A^2(\Psi)$ is a reproducing kernel Hilbert space with reproducing kernel

$$K_\Psi(z,w) = \frac{1}{(n-1)}!\ F_s^{(n-1)}(\langle z,w\rangle), (z,w) \in \mathbb{C}^n.$$

This implies that the orthogonal projection P_Ψ from $L^2(\mu)$ onto A^2 (Ψ) can be expressed as

$$(P_\Psi g)(z) = \int_{\mathbb{C}^n} K_\Psi(z,w)g(w)d\mu_\Psi(w), z \in \mathbb{C}^n,$$

for every function g in $L^2(\mu)$. The domain of this integral operator can be extended to include functions g that satisfy $K_\Psi(z,\cdot)g \in L^1(\mu)$ for every z in $\mathbb{C}^n$. This extension allows us to define (big) Hankel operators. To do so, denote by $\mathcal{T}(\Psi)$ the class of all f in

$L^2(\mu)$ such that $f\phi K_\Psi(z,\cdot) \in L^1(\mu_\Psi)$ for all holomorphic polynomials φ and z in $\mathbb{C}$^n and the function

$$H_f\,(\varphi)(z) := \int_{\mathbb{C}^n} K_\Psi(z,w)\varphi(w)[f\,(z) - f\,(w)]\,d\mu_\Psi(w), \quad z \in \mathbb{C}^n$$

is in L^2 (μ_Ψ). This is a densely defined operator from $A^2(\Psi)$ into $L^2(\mu_\Psi)$ which will be called the Hankel operator H_f with symbol f . It can be written in the form

$$H_f\,(\varphi) = (I - P_\Psi)(f\,\varphi)$$

for all holomorphic polynomials φ. It is clear that the class T (Ψ) contains all holomorphic polynomials.

The main theorem involves the analogues in our setting of the space BMOA and the Bloch space. The analogue of BMOA is most conveniently defined by the Berezin transform, which for a linear operator T on $A^2(\Psi)$ is the function $\tilde{T}$ defined on $\mathbb{C}^n$ by

$$\tilde{T}\,(z) := \frac{\langle T\,K_\Psi(\cdot,z), K_\Psi(\cdot,z)\rangle}{K_\Psi(z,z)}.$$

If $T = M_f$ is the operator of multiplication by the function f , then we just set $\widetilde{M_f} = \tilde{f}$. We set

$$\|f\,\|_{BMO} := \sup_{(z\in\mathbb{C}^n)} (MOf\,)(z),$$

Where

$$(MOf\,)(z) := \sqrt{|\,\widetilde{f}\,|^2(z) - |\tilde{f}\,(z)|^2},$$

and define BMO(Ψ) as the set of functions f on $\mathbb{C}^n$ for which $|\widetilde{f}\,|2(z)$ is finite for every z and $\|f\|$ BMO $< \infty$. It is plain that BMO(Ψ) is a subset of $\mathcal{T}$ (Ψ). The space BMOA(Ψ) is the subspace of BMO(Ψ)consisting of analytic elements; this space is in turn a subset of $T\,(\Psi) \cap A^2(\Psi)$.

We next introduce the Bergman metric associated withΨ. To this end, set $(z) =$ log$K_\Psi(z,z)$and

$$\beta^2(z,\xi) := \sum_{j,k=1}^{n} \frac{\partial^2\Lambda_\Psi(z)}{\partial z_j\,\partial \bar{z}_k}\,\xi j\,\bar{\xi}\,k$$

for arbitrary vectors $z = (z_1,\ldots,z_n)$ and $\xi = (\xi_1,\ldots,\xi_n)$ in $\mathbb{C}^n$. The corresponding distance is given by

$$\varrho(z,w) := \inf_\gamma \int_0^1 \beta\big(\gamma(t),\gamma'(t)\big)dt, \tag{2}$$

where the infimum is taken over all piecewise C^1-smooth curves $\gamma : [0,1] \to \mathbb{C}^n$ such that $\gamma(0) = z$ and $\gamma(1) = w$. We define the Bloch space $B(\Psi)$ to be the space of all entire functions f such that

$$f\,B(\Psi) := \sup_{z\in\mathbb{C}^n}\left[\sup_{\xi\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle(\nabla f\,)(z),\bar{\xi}\rangle|}{\beta(z,\xi)}\right] < +\infty. \tag{3}$$

In what follows, the function

$$\Phi(x) := x\Psi'(x)$$

will play a central role. By (1), we have that both $\Phi'(x) > 0$ and $\Phi''(x) > 0$, and it may be checked that $\Phi'(|z|^2)$ coincides with the Laplacian of $(|z|^2)$ when $n = 1$ and in general is bounded below and above by positive constants times this Laplacian for arbitrary $n > 1$.

We state the main result.

Theorem (5.1.1)[129]: LetΨbe a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that

$$\Phi''(t) = O\left(t^{-\frac{1}{2}}\,[\Phi'(t)]^{1+\eta}\right) \text{ when } t \to \infty. \tag{4}$$

If f is an entire function on $\mathbb{C}^n$, then the following statements are equivalent:

(i) The function f belongs to $T\ (\Psi)$ and the Hankel operator $H_{\bar{f}}$ on $A^2(\Psi)$ is bounded;

(ii) The function f belongs to $BMOA(\Psi)$; (iii) The function f belongs to $B(\Psi)$.

Note that the additional assumption (4) is just a mild smoothness condition, which holds whenever is a nontribyl polynomial or a reasonably well-behaved function of super-polynomial growth. As part of the proof of Theorem (5.1.1), we will perform a precise computation of the asymptotic behavior of $\beta(z,\xi)$ when $|z| \to \infty$. We state this result as a separate theorem.

We observe that for the classical Fock space (Ψ a linear function) we have $\Psi''(x) \equiv 0$, and so the "directional" term in $\beta(z,\xi)$ is not present. Note also that $B(\Psi)$ contains all polynomials and is infinite-dimensional whenever the growth of $\Psi'\ (x)$ is super-polynomial. In the language of entire functions, this means that $A^2(\Psi)$ contains functions of infinite order. When $n = 1, \beta^2(v,\xi)$ can be replaced by $\Phi'(|z|^2)|\xi\ |^2$. The same is also true when is a polynomial, because then Ψ' and Φ' have the same asymptotic behavior. In the latter case, our two theorems give the following precise result: If $\Psi$is a polynomial of degree d, then $B(\Psi)$ consists of all holomorphic polynomials of degree at most d; cf. Theorem (5.1.1) in [137].

The implication (i) ⇒ (ii) in Theorem (5.1.1) is standard; it follows from general arguments for reproducing kernels. Likewise, the implication (ii) ⇒ (iii) can be established by a well-known argument concerning the Bergman metric. The proof of Theorem (5.1.1) therefore deals mainly with the implication (iii) ⇒ (i). The crucial technical ingredients in the proof of this result are certain estimates for the Bergman kernel $K_\Psi(z,w)$. Such estimates have previously been obtained by F. Holland and R. Rochberg in [62]. The results of [62] are not directly applicable because we need more precise off-diagonal estimates for the kernel than those given. Our method of proof is similar to that of [62], but our approach highlights more explicitly the interplay between the smoothness of and theΨoff-diagonal decay of the Bergman kernel. This is where the additional smoothness condition (4) comes into play; many of our estimates can be performed with sufficient precision without the assumption that (4) holds, but some condition of this kind seems to be needed for our off-diagonal estimates.

The fact that the Bergman metric is the notion used to define the Bloch space $B(\Psi\ )$ suggests that Theorem (5.1.1) should be extendable beyond the case of radial weights. To obtain such an extension, one would need a replacement of our Fourier-analytic approach, which relies crucially on the representation of the Bergman kernel as a power series.

The machinery developed to prove Theorem (5.1.1) leads with little extra effort to a characterization of compact Hankel operators in terms of the obvious counterparts to VMOA and the little Bloch space; for details. In our study of Schatten class Hankel operators, however, some additional techniques will be used. We will need more precise local information about the Bergman metric, namely that balls of fixed radius in the Bergman metric are effectively certain ellipsoids in the Euclidean metric of $\mathbb{C}^n$. These results appear to be of independent interest; in particular, they lead to a characterization of Carleson measures and in turn to a characterization of the spectral properties of Toeplitz operators. Building on these results and using L^2 estimates for the $\overline{\partial}$ operator, we obtain a characterization of Schatten class Hankel operators.

To place the present investigation, we close this introduction with a few words on the literature. Boundedness and compactness of Hankel operators with arbitrary symbols have previously been considered only for the classical Fock space (Ψ a linear function); see, [130], [131], [134], [135], [142], [143]. The methods, relying on the transitive self-action of the group $\mathbb{C}^n$, cannot be extended beyond this special case. Hankel operators with anti-holomorphic symbols defined on more general weighted Fock spaces were studied recently in [137] and [84], where it was shown that anti-holomorphic polynomials do not automatically induce bounded Hankel operators. For Bergman kernel estimates in similar settings, see [140] and [141]. We finally mention [73] and [132]; the first focuses on small Hankel operators and the Heisenberg group action, while the second deals with Hankel operators for the Bergman projection on smoothly bounded pseudoconvex domains in $\mathbb{C}^n$.

The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, then we write $U(z) \simeq V(z)$.

The following standard argument shows that (i) implies (ii) in Theorem (5.1.1). To begin with, we note that if f is in $A^2b(\psi)$, then $\tilde{f} \quad = f$. Moreover, by the definition of the reproducing kernel, a computation shows that

$$\left|\widetilde{f}\right|^2(z) - |f(z)|^2 = \int_{\mathbb{C}^n} |f(\xi) - f(z)^2 \frac{\left|K_\Psi(\xi, z)\right|^2}{K_\Psi(z,z)}\, d\mu(\xi) = \frac{\left\| H_{\bar{f}} K_\Psi(\cdot, z)\right\|^2}{K(z,z)}. \quad (5)$$

Hence, if $H_{\bar{f}}$ is bounded, then $\|f\|_{BMO} < +\infty$. The implication (ii) ⇒ (iii) is a consequence of the following lemma, the proof of which is exactly as the proof of Corollary 1 in [133].

Lemma (5.1.2)[129]: Suppose that f is in $BMOA(\Psi)$. Then for every piecewise C^1-smooth curve : $[0,1] \to \mathbb{C}^n$ we have

$$\left|\frac{d}{dt}(f \circ \gamma)(t)\right| \leq 2\sqrt{2}\beta\big(\gamma(t), \gamma'(t)\big)(MOf)\big(\gamma(t)\big).$$

If we choose $\gamma(t) = z + t\xi$, then we obtain

$$\frac{\left|\langle (\nabla f)(z), \overline{\xi}\rangle\right|}{\beta(z,\xi)} \leq 2\sqrt{2}(MOf)(z) \quad (6)$$

for all z in $\mathbb{C}^n$ and ξ in $\mathbb{C}^n$ $\{0\}$.

This a some what elaborate preparation for the proof of Theorem (5.1.9) and also the proof of the implication (iii) ⇒ (i) in Theorem (5.1.1).

Set

$$\theta_0(r) := [r\Phi'(r)]^{-\frac{1}{2}}.$$

The key estimates for the Bergman kernel are the following.

Lemma (5.1.3)[129]: Let η be as in Theorem (5.1.1). Then, for any fixed $\alpha > \eta$, we have

$$\sup_{|\tau| \leq t^{\frac{1}{2}}[\phi'(t)]^{-\alpha}} \phi(t + \tau) = \big(1 + o(1)\big)\phi'(t)$$

when $t \to \infty$.

Proof. The proof is similar to the proof of Lemma 6 in [62]. By (4), $[\phi'(x)]^{-1} - \eta \times \phi'(x) = O(x^{-\frac{1}{2}})$ when $x \to \infty$, which implies that

$$|[\phi'(t + \tau)]^{-\eta} - [\phi'(t)]^{-\eta}| = |\tau| O(t^{-\frac{1}{2}}\tau)$$

when $t \to \infty$. The result follows from this relation.

In order to estimate $|K_\Psi(z, w)|$, we need precise information about the moments s_d. To this end, note that the integrand of

$$\int_0^\infty x^t e^{-\Psi(x)} dx$$

attains its maximum $at\ x = \phi^{-1}(t)$. Set

$$h_t(x) = -t \log x + \Psi(x) - (-t\, log\phi^{-1}(t) + (\phi^{-1}(t))$$

and

$$I(t) = \int_0^\infty e^{-h_t(x)} dx;$$

we may then write

$$s_d = e^{d\, log\phi^{-1}(d) - \Psi\left(\phi^{-1}(d)\right)} I(d).$$

We have the following precise estimate for $I(t)$.

Lemma (5.1.4)[129]: For the function I(t), we have

$$I(t) = \left(\sqrt{2\pi} + o(1)\right)\left[\frac{\phi^{-1}(t)}{\phi\left(\phi' - 1(t)\right)}\right]^{\frac{1}{2}}$$

when $t \to \infty$.

Proof. Set $\tau(x) = \sqrt{x}[\phi'(x)]^{-\alpha}$, where $\eta < \alpha < 1/2$.Since

$$h_t''(x) = \frac{\phi(x)}{x} + \frac{t}{x^2} b - \frac{\phi'(x)}{x^2} = \frac{\phi'(x)}{x} + \frac{1}{x^2}\left[\phi\left(\phi^{-1}(t)\right) - \phi(x)\right],$$

we have, by Lemma (5.1.3),

$$h_t''(x) = h_t''(\phi^{-1}(t))(1 + o(1))$$

when $|x - \phi^{-1}(t)| \le \tau(\phi^{-1}(t))$. On the other hand, by the convexity of ht , we then have $|h_t(x)| \ge \frac{1}{2}(h_t''(\phi^{-1}(t)) + o(1))\tau(\phi^{-1}(t))|x - \phi^{-1}(t)|$ for $|x - \phi^{-1}(t)| \ge \tau(\phi^{-1}(t))$. Setting for simplicity

$$c = h_t''\left(\phi^{-1}(t)\right) = \frac{\phi'\left(\phi^{-1}(t)\right)}{\phi^{-1}(t)},$$

we then get

$$I(t) = \int_{|x| \le \tau\left(\phi^{-1}(t)\right)} e^{-\frac{1}{2}(c+o(1))x^2} dx + E(t), \tag{7}$$

Where

$$|E(t)| \le 2\int_{x \ge \tau\left(\phi^{-1}(t)\right)} e^{-\frac{1}{2}(c+o(1))\tau\left(\phi^{-1}(t)\right)x} dx.$$

Thus the result follows, since the integral in (7) can be estimated by the corresponding Gaussian integral from $-\infty$ to ∞.

We will estimate a number of integrals in a similar fashion, using Lemma (5.1.3) to split the domain of integration. The integrands will be of the type $e^{-g_t(x)} S_t(x)$ and satisfy the following:

(I) gt attains its minimum at a point $x_0 = x_0(t) \to \infty$ with $g_t''(x) = \left(1 + o(1)\right)c$ for $|x - x_0| \le \tau$ and $\frac{1}{\tau} = o(c)$ when $t \to \infty$.

(II) For $|x - x_0| \leq \tau$, $S_t(x)$ can be estimated by a constant C times $|x - x_0|^m$ for some positive integer m.

(III) When $|x - x_0| \geq \tau$ and $|x - x_0|$ grows, the function $e^{-g_t(x)S_t}(x)$ decays so fast that

$$\int_0^\infty e^{g_t(x)} |S_t(x)|dx = (1 + o(1)) \int_{|x-x_0|\leq\tau} e^{-g_t(x)}|S_t(x)|dx.$$

Taking into account the formula

$$\int_0^\infty x^m e^{-\frac{1}{2}cx^2} dx = \left(\frac{c}{2}\right)^{-\frac{m+1}{2}} \int_0^\infty x^m e^{-x^2} dx, \quad (8)$$

we then arrive at the estimate

$$\int_0^\infty e^{-h_t(x)} S_t(x)dx = O\left(Cc^{-\frac{m+1}{2}}\right) \quad (9)$$

when $t \to \infty$.

We will at one point encounter a slightly different variant of this scheme, obtained by replacing (II) by the following:

(II') *For* $|x - x_0| \leq \tau$, we have $S(x) = (1 + o(1))(x - x_0)$ when $t \to \infty$.

In this case, because of the symmetry around the point x_0, we get the slightly better estimate

$$\int_0^\infty e^{-h_t(x)}S(x)dx = o(c^{-1}) \quad (10)$$

when $t \to \infty$.

In the following we will omit most of the details of such calculus arguments. We will briefly state that conditions $(I), (II), (III)$ (or, respectively, $(I), (II'), (III)$) are satisfied and conclude that this leads to the estimate (9) (or, respectively, (10)).

In the proof of the next lemma, we will use this scheme three times.

Lemma (5.1.5)[129]: We have

$$I'(t) = O\left(\left[\Phi^{-1}(t)\Phi'\left(\Phi^{-1}(t)\right)\right]^{-\frac{1}{2}} I(t)\right);$$

$$I''(t) = O\left(\left[\Phi^{-1}(t)\Phi'\left(\Phi^{-1}(t)\right)\right]^{-1} I(t)\right);$$

$$I''(t) = O\left(\left[\Phi^{-1}(t)\Phi'\left(\Phi^{-1}(t)\right)\right]^{-\frac{3}{2}} I(t)\right)$$

when $t \to \infty$.

Proof. We begin by noting that I' can be computed in the following painless way:

$$I'(t) = \int_0^\infty log \frac{x}{\Phi^{-1}(t)} e^{-h_t(x)}dx; \quad (11)$$

this holds because $h_t'(\Phi^{-1}(t)) = 0$. For the same reason, we get

$$I''(t) = \int_0^\infty - \left[\frac{\left(\Phi^{(-1)}\right)'(t)}{\Phi^{-1}(t)} + log\left(\frac{x}{\Phi^{-1}(t)}\right)^2\right] e^{-h_t(x)}dx \quad (12)$$

and

$$I''(t)=\int_0^\infty\left[-\left[\frac{\left(\Phi^{(-1)}\ \right)(t)}{\Phi^{-1}(t)}\right]'-3\frac{\left(\Phi^{(-1)}\ \right)'(t)}{\Phi^{-1}(t)}\ log\frac{x}{\Phi^{-1}(t)}\right.$$

$$\left.+\left(log\frac{x}{\Phi^{-1}(t)}\right)^3\right]e^{-h_t(x)}dx. \quad (13)$$

We use that $[\Phi^{-1}]'\ (t)\ =\ 1/\Phi'\ (\Phi^{-1}(t))$, and then in (13) we also use the fact that

$$\left[\frac{1}{\Phi'\big((\Phi^{-1})(t)\big)\Phi^{-1}(t)}\right]'\ =\ -\frac{\Phi''\big(\Phi^{-1}(t)\big)}{\left[\Phi'\big(\Phi^{-1}(t)\big)\right]^3\Phi^{-1}(t)}-\frac{1}{\left[\Phi'\big(\Phi^{-1}(t)\big)\Phi^{-1}(t)\right]^2}\ ; (14)$$

we apply condition (4) to the first term on the right-hand side. When we estimate the integrals in (11), (12), and (13), we use that

$$\left|log\frac{x}{\Phi^{-1}(t)}\right|\ \leq\ e\frac{|x-\Phi^{-1}(t)|}{-\Phi^{-1}(t)}$$

for $x\ \geq\ e^{-1}\Phi^{-1}(t)$ and that, say,

$$\left|log\frac{x}{\Phi^{-1}(t)}\right|\leq \log\ \frac{1}{\Phi^{-1}(t)}$$

when $1\ \leq\ x<e^{-1}\Phi^{-1}(t)$. In each case, the integrand satisfies conditions $(I),(II),(III)$ with $g_t\ =\ h_t$, so that we may use (9). The desired results for I',I'',I''' nowfollow from (9).

We will need similar estimates for the function

$$L_r(t)=\exp\left(t\log r\ -\ t\ log\Phi^{-1}(t)\ +\Psi\left(\Phi^{-1}(t)\right)\right),$$

where r is a positive are a meter.

Lemma (5.1.6)[129]: We have

$$L_r'\ (t)=-\log\ \frac{\Phi^{-1}\ (t)}{r}\ \ L_r(t);$$

$$L_r''\ (t)=\left[\left(\log\ \ \frac{\Phi^{-1}(t)}{r}\ \right)^2-\frac{1}{\Phi'\big(\Phi^{-1}(t)\big)\Phi^{-1}(t)}\right]Lr(t);$$

$$L_r'''\ (t)=\left[\left(-\log\ \ \frac{\Phi^{-1}(t)}{r}\right)^3+\frac{3\ log\frac{\Phi^{-1}(t)}{r}}{\Phi'\big(\Phi^{-1}(t)\big)\Phi^{-1}(t)}\right.$$

$$\left.+\left(\left[O\Phi'\big(\Phi^{-1}(t)\big)\Phi^{-1}(t)\right]^{-\frac{3}{2}}\right)\right]L_r(t)$$

when $t\ \to\ \infty$.

Proof. The first and the second of these formulas are obtained by direct computation.We arrive at the estimate for the third derivative by again using (14) and then applying condition (4).

Lemma (5.1.7)[129]: Suppose that (4) holds. Let z and w be arbitrary points in $\mathbb{C}^n$ such that $\langle z,w\rangle\neq\ 0$, and write $\langle z,w\rangle\ =\ re^{i\theta}$, where $r\ >\ 0$ and $-\pi\ \theta_0(r)$. Moreover, there exists a positive constant c such that if $\theta\ <\ c\theta_0(r)$, then
$|K_\Psi(\ z,w)|\gtrsim\phi'(r)[\ \Psi'(r)]^{n-1}\ e^\Psi(r)$.

Proof. We begin by recalling that

$$K_\Psi(z,w) = k(\langle z,w\rangle),$$

Where

$$k(\zeta) := \frac{1}{(n-1)!} \sum_{d=n-1}^{\infty} \frac{d(d-1)\cdots(d-n+2)}{sd} \zeta^{-n+1}.$$

We set $\langle z,w\rangle = re^{i\theta}$ and assume that $r > 0$ and $|\theta| \leq \pi$. We may then write

$$\frac{(\langle z,w\rangle^d)}{s_d} = \frac{L_r(d)}{I(d)} \exp(id\theta)$$

and hence

$$\langle z,w\rangle^{n-1} K_\Psi(z,w) = r^{n-1} exp(i(n-1)\theta) k\left(re^{i\theta}\right)$$

$$= \frac{1}{(n-1)!} \sum_{d=n-1}^{\infty} d(d-1)\cdots(d-n+2) \frac{L_r(d)}{I(d)} \exp(id\theta).$$

Let $\Omega(t)$ be a function in $C^3(\mathbb{R})$ so that

$$\Omega(t) = \frac{1}{(n-1)!} \frac{t(t-1)\cdots(t-n+2)L_r(t)}{I(t)}$$

For $t \geq n-1$ and $\Omega(t) = 0$ for $t \leq n-2$. Then the Poisson summation formula gives

$$r^{n-1} \exp(i(n-1)\theta) k\left(re^{i\theta}\right) = \sum_{j=-\infty}^{\infty} \widetilde{\Omega}(j),$$

where

$$\widetilde{\Omega}(j) \int_{-\infty}^{\infty} \Omega(t) e^{i(2\pi j+\theta)t} = dt.$$

Integrating by parts, we obtain

$$r^{n-1}\left|k\left(re^{i\theta}\right)\right| \leq \left|\widetilde{\Omega}(0)\right| + \|\Omega'''\|1 \sum_{j=1}^{\infty} \frac{2}{(2\pi)^3\left(j-\frac{1}{2}\right)^3}.$$

Since

$$\left|\widetilde{\Omega}(0)\right| \leq \min(\|\Omega\|\,1, |\theta|^{-3}\ \|\Omega'''\|1),$$

the proof of the first part of the lemma is complete if we can prove that

$$\|\Omega\|1 \lesssim \left(\Phi(r)\right)^{n-1} \Phi'(r) e^{\Psi(r)} \tag{15}$$

and

$$\|\Omega'''\|_1 \lesssim \left(\Phi(r)\right)^{n-1} \frac{e^{\Psi(r)}}{r^{\frac{3}{2}}\sqrt{\Phi'(r)}}. \tag{16}$$

We first estimate $\|\Omega\|_1$. We write $L_r(t) = xp(-g_r(t))$ and claim that conditions $(I), (II), (III)$ above hold. To see this, we observe that, by the first formula of Lemma (5.1.6), L_r attains its maximum at $t = \Phi(r)$. Moreover, g_r is a convex function and

$$g_r''(t) = \frac{1}{\Phi'\left(\Phi^{-1}(t)\right)\Phi^{-1}(t)}.$$

Lemma (5.1.3) implies that

$$g_r''(t) = \left(1 + o(1)\right) g_r''\left(\Phi(r)\right)$$

When $|t - \Phi(r)| \leq \sqrt{r}[\Phi'(r)]^{1-2\alpha}$. The remaining details are carried out as in the proof of Lemma (5.1.4). Using (9) with m = 0 and Lemma (5.1.4), we therefore get

$$\|\Omega\|_1 = |\Phi(r)(\Phi(r)-1)\ \cdots\ (\Phi(r)-n+2)|\frac{L_r(\Phi(r))}{I(\Phi(r))}\ \left(\sqrt{2\pi} + o(1)\right)[\Phi'(r)r]^{\frac{1}{2}}$$
$$= \left(1 + o(1)\right)\left(\Phi(r)\right)^{n-1}\Phi'(r)e^{\Psi(r)},$$
which shows that (15) holds.

To arrive at (16), we need a pointwise estimate for Ω'''. To simplify the writing, we set
$$a = \left|log\frac{\Phi^{-1}(t)}{r}\right|\ and\quad b = \Phi'\left(\Phi^{-1}(t)\right)\Phi^{-1}(t)^{-\frac{1}{2}}.$$
Then using the Leibniz rule along with Lemmas (5.1.5) and (5.1.6), we get
$$|\Omega'''(t)| \lesssim (a^3 + a^2b + ab^2 + b^3)\Omega(t).$$
By a straightforward calculus argument, we verify that each of the terms in this expression satisfies (I), (II), and III) above, again with
$$x_0 = \Phi(r)\ \tau = \sqrt{r}[\Phi'(r)]^{1-2\alpha}.$$
We now use (9) to achieve he desired estimate for each of the terms in this $a^m b^{3-m}\Omega(t)$.

The previous proof also ives the second estimate when $\theta = 0$, because then $\widetilde{\Omega}(0) = \|\Omega\|_1$. To prove it in eneral, we need to check that $k(r) \backsimeq |k(r)e^{i\theta}\ )|$ when $|\theta| \leq c[r\Phi'(r)]^{-\frac{1}{2}}$. To this nd, note that
$$\widetilde{\Omega}(0) = e^{i\theta(r)}\int_{-\infty}^{\infty}\Omega(t)e^{i\theta(t-(r))}dt,$$
which implies that
$$\left|\widetilde{\Omega}(0)\right| \geq \|\Omega\|_1 - \int_{-\infty}^{\infty}\Omega(t)|\theta||t - \Phi(r)|dt\ .$$
The integral on the right is computed using (5.1.4)) with $m = 1$, and so we get
$$\left|\widetilde{\Omega}(0)\right| \geq \|\Omega\|_1 \left(1 - C|\theta|[r\,\Phi'(r)]^{\frac{1}{2}}\right).$$
Thus the second estimate in Lemma (5.1.7) holds for c sufficiently small.

We close by roving some estimates for another function that will be important later. Set
$$Q_x(r) = \frac{1}{2}\left(\Psi(r^2) + \Psi(x^2)\right) - \Psi(xr). \qquad (17)$$
Lemma (5.1.8)[129]: Let α be a positive number such that $\eta < \alpha < 1/2$, let x_1 and x_2 be the two points such that $x_1 < x < x_2$ and
$$|x - x_1| = |x - x_2| = [\Phi(x)]^{-\alpha},$$
and set $c = \Psi'(0)$. When r → ∞, we have
$$Q_x''(r) = \left(1 + o(1)\right)\Phi'(x^2),\quad x_1 \leq r \leq x_2; \qquad (18)$$
$$Q_x(r) \geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(x^2)]^{1-2\alpha},\ \ r < x_1; \qquad (19)$$
$$Q_x(r) \geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(x^2)]^{1-2\alpha},\ \ r > x_2. \qquad (20)$$
Proof. We begin by noting that
$$Q_x'(r) = r\Psi'(r^2) - x\Psi'(xr)$$
and
$$Q_x''(r) = \Psi'(r^2) + 2r^2\Psi''(r^2) - x^2\Psi''(xr).$$
We observe that for $x_1 \leq r \leq x_2$ Lemma (5.1.3) applies:
$$Q_x''(r) = \Psi'(r^2) + r^2\Psi'(r^2) - x^2\Psi''(xr) = \left(1 + o(1)\right)\Psi'(x^2),$$

and so have established (18). For $r < x_1$, we use the following estimate:

$$Qx\ (r) \geq \frac{1}{2}\int_r^x \ \Psi'(s^2)(s - x)ds\ \frac{1}{2}\int_{x-[\Phi'(\,r^2)]^{-\alpha}}^x \ \int_{x-[\Phi'(r^2)]^{-\alpha}}^t \ Q_x''\ (u)dudt$$

$$\geq \frac{c}{4}\ (x\ -\ r)^2\ + \left(\frac{1}{4}\ +\ o(1)\right)\ [\Phi'(r^2)]^{1-2\alpha},$$

where Lemma (5.1.3) is applied once more. Hence (20) also holds.

Theorem (5.1.9)[129]: Let be a logarithmic growth function, and suppose that there exists a real number $\eta\ <\ 1/2$ such that (4) holds. Then we have, uniformly in ξ , that $\beta^2(z,\xi) =\ \big(1\ +\ o(1)\big)\ |\xi|^2\ (|z|^2) +\ |z,\xi\ |^2\ \ (|z|^2)$ when $|z| \to \infty$.

Proof. Computation of the Bergman Metric We begin by recalling that

$$K_\psi(z,z) =\ k(r^2),$$

Where

$$k(r) = \sum_{n=0}^{\infty} c_d r^d\ \ ,$$

and

$$cd := ((d\ +\ 1) \cdots \frac{d\ +\ n\ -\ 1}{(n\ -\ 1)!\, s_{d+n-1}}.$$

A computation shows that

$$\beta^2(z,\xi) :=\ |\xi|^2(\frac{k'(|z|^2)}{k(|z|^2)}\ \ +\ |\langle z,\xi\ \rangle|^2 \left[\frac{k''\big((|z|^2\ )\big)}{k\ \big((|z|^2)\big)} - \left(\frac{k'(|z|^2)}{k(|z|^2)}\right)^2\ \right].$$

Thus Theorem (5.1.9) is a consequence of the following lemma.

The proof of this lemma relies on the following estimates.

Lemma (5.1.10)[129]: Suppose that (4) holds and let the coefficients cd be as defined above. Then we have

$$\sum_{d=1}^{\infty} c_d\ \big(d\ - \Phi\ (r)\big)\, r^d\ =\ o\left([r\Phi'(r)]^{\frac{1}{2}}k(r)\right), \tag{21}$$

$$\sum_{d=1}^{\infty} c_d\ \big(d\ -\ \Phi(r)\big)^2 r^d\ =\ \big(1\ +\ o(1)\big)r\Phi'(r)k(r)$$

when r → ∞.

Proof. The proof is essentially the same as the proof for the diagonal estimates in Lemma (5.1.7). The only difference is that we replace the function $\Omega(t)\ by\ (t\ - \Phi\ (r))\Omega(t)$ and $\big(t\ -\ \Phi(r)\big)^2\Omega(t)$, respectively. In the first case, we have a function that satisfies condition (II'). This means that we may use (10) to arrive at (21). To establish (4.2), we may apply (8) with m = 2

and take into account that we have the explicit factor $\big(t\ - \Phi(r)\big)^2\ in\ front\ of\ \Omega(t)$.

Lemma (5.1.11)[129]: Suppose that (4) holds. Then we have

$$\frac{k'(r)}{k(r)}\ \ \big(1\ +\ o(1)\big)\psi'(r),$$

$$\left(\frac{k'(r)}{k(r)}\right)'\ \ =\ \big(1\ +\ o(1)\big)\psi''(r) +\ o(1)\,\frac{\psi'(r)}{r}$$

when r → ∞.

Proof. We write

$$k'(r) = \frac{\Phi(r)}{r}\left(k(r) + O(1)\right) + \frac{1}{r}\sum_{\mathrm{d}=1}^{\infty} c_d\left(d - \Phi(r)\right)r^d;$$

using Lemma (5.1.10), we obtain

$$\frac{k'(r)}{k(r)} = \left(1 + o(1)\right)\Psi'(r) + o\left(\left[\frac{\Phi'(r)}{r}\right]^{\frac{1}{2}}\right).$$

The desired estimate for k''/k follows because, in view of emma (5.1.3), we have

$$\Phi(r) \geq \int_{r-r1/2[\Phi'(r)]^{-\alpha}}^{r} \Phi'(t)dt = \left(1 + o(1)\right)r^{\frac{1}{2}}[\Phi'(r)]^{1-\alpha}$$

for some $\alpha < 1/2$.

To arrive at the second estimate, we first observe that

$$k''(r) = \frac{\Phi(r) - 1}{r}\left(k'(r) + O(1)\right) + \frac{1}{r}\sum_{d=2}^{\infty} c_d d\left(d - \Phi(r)\right)r^{d-1}$$

$$= \frac{\Phi(r) - 1}{r}\left(k'(r) + O(1)\right) + \frac{\Phi(r)}{r^2}\sum_{d=2}^{\infty} c_d\left(d - \Phi(r)\right)r^d$$

$$+ \frac{1}{r^2}\sum_{d=2}^{\infty} c_d\left(d - \Phi(r)\right)^2 r^d.$$

Combining our expressions for k' and k'', we find that

$$k''(r)k(r) - \left(k'(r)\right)^2 = \frac{k(r)}{r^2}\sum_{d=2}^{\infty} c_d\left(d - \Phi(r)\right)^2 r^d$$

$$-\frac{1}{r^2}\left[\sum_{d=2}^{\infty} c_d\left(d - \Phi(r)\right)r^d\right]^2$$

$$-\frac{k(r)k'(r)}{r} + \Psi'(r)O\left(k(r) + k'(r)\right).$$

Using again Lemma (5.1.10) and the estimate already obtained for k'/k, we get

$$\left(\frac{k'(r)}{k(r)}\right)' = \left(1 + o(1)\right)\frac{\Phi'(r)}{r} - \left(1 + o(1)\right)\frac{\Phi(r)}{r^2}$$

from which the second estimate in Lemma (5.1.11) follows.

We finally turn to the roof hat (iii) implies (i) in Theorem (5.1.1). A different proof, using L^2 estimates for the $\bar{\partial}$ perator, will be given, subject to anadditional mild smoothness condition on. The proof gives a ore nformative norm estimate, which will be crucial in our study of Schatten class ankel operators. The proof to be given below has the advantage that it does not require f to be holomorphic.

Using the reproducing formula, we find that

$$H_{\bar{f}g}(z) = \int_{\mathbb{C}^n} \left(\overline{f(z)} - \overline{f(w)}\right)K_{\psi}(z,w)g(w)d\mu\psi(w).$$

Therefore, by the definition of B(Ψ), we have

$$\left|H_{\bar{f}}g(z)\right| \leq \|f\|\, B(\Psi)\int_{\mathbb{C}^n} \varrho(z,w)K_\Psi(z,w)g(w)d\mu_\Psi(w).$$

Thus it suffices to prove that the operator A defined as

$$Ag(z) = \int_{\mathbb{C}_n} \varrho(z,w)K_\Psi(z,w)g(w)d\mu_\Psi(w)$$

is bounded on $L^2(\mu_\Psi)$.

We shall use a standard technique known as Schur's test [106]. Set

$$H(z,w) = \varrho(z,w)|K_\Psi(z,w)|e^{-\frac{1}{2}\left(\Psi(|z|^2)+(|w|^2)\right)}.$$

By the Cauchy–Schwarz inequality, we obtain

$$|(Ag)(z)|^2 e^{-\Psi(|z|^2)} \lesssim \int_{\mathbb{C}^n} H(z,\zeta)dV(\zeta)\int_{\mathbb{C}_n} H(z,w)|g(w)|^2 e^{-(|w|2)}\, dV(w).$$

this means that the operator A is bounded on $L^2(\mu_\Psi)$ if

$$\sup_z \int_{\mathbb{C}^n} H(z,\zeta)dV(\zeta) < \infty. \tag{22}$$

We therefore to establish (22).

Without loss of generality, we may assume that $z = (x,0,\ldots,0)$ with $x > 0$. We begin by estimating (z,w). To this end, write $w = (w_1,\xi)$ with ξ a vector in $\mathbb{C}^{n-1}$ and $w_1 = re^{i\theta}$ when $n > 1$. Set $e_1 = (1,0,\ldots,0)$ and consider the three curves

$$\gamma_1(t) = xe^{it}\, e_1, 0 \leq t \leq \theta,$$
$$\gamma_2(t) = \big(x + t(r-x)\big)e^{i\theta}\, e_1, 0 \leq t \leq 1,$$
$$\gamma_3(t) = \big(re^{i\theta}, t\xi\big), 0 \leq t \leq 1,$$

which together constitute a piecewise smooth curve from z to w. (When $n = 1, \gamma_3$ does not appear and can be neglected.) Note that

$$|\langle\gamma_1(t),\gamma_1'(t)\rangle| = |\gamma_1'(t)||\gamma\ 1(t)| = x^2,$$
$$|\langle\gamma_2(t),\gamma_2'(t)\rangle| = |\gamma_2(t)||\gamma_2'(t)| = \big(x + t(r-x)\big)|x-r|,$$
$$|\langle\gamma_3(t),\gamma_3'(t)\rangle| = t|\xi|^2\ .$$

By these observations and Theorem (5.1.9), we get the following estimate:

$$\varrho(z,w) \lesssim x|\theta|[\Phi'(x^2)]^{\frac{1}{2}} + \big[\Phi'\big(max(x^2,r^2)\big)\big]^{\frac{1}{2}}|x-r|$$
$$+ |\xi|\,[\Psi'(r^2+|\xi|^2\ )]^{\frac{1}{2}} + |\xi|^2[\Psi''(r^2 + |\xi|^2)]^{\frac{1}{2}}.$$

When estimating the last term on the right-hand side of his nequality, we will use that

$$[\Psi'(y)]^2 \gtrsim (y), \tag{23}$$

which is a consequence of our assumptions (1) and (4). Indeed, assuming $\Psi'' > 0$, we have $\Psi''(y) \asymp \Phi'(y)$ ince Ψ'' is a nondecreasing function. Thus (23) is equivalent to the following:

$$\Phi(t) \gtrsim t^{\frac{1}{2}}[\Phi'(t)]^{\frac{1}{2}}.$$

We arrive at this estimate because

$$\Phi(t) = \Phi(0) + \int_0^t \Phi'\ \ (\tau)d\tau \geq \Phi(0)\big(1 + o(1)\big)t^{\frac{1}{2}}[\Phi'(t)]^{\frac{1}{2}},$$

where in the second step we used Lemma (5.1.3) with $\alpha = 1/2$. For $\zeta = |\zeta|e^{i\theta}$, we set

$$h(\zeta) = \begin{cases} \Phi'(|\zeta|), |\theta| \leq \theta_0(|\zeta|), \\ |\zeta|^{-\frac{3}{2}}[\Phi'(|\zeta|)]^{-\frac{1}{2}}|\theta|^{-3}, |\theta| > \theta_0(|\zeta|). \end{cases}$$

Using this notation and Lemma (5.1.7), we hen obtain

$$H(z,w) \lesssim (x,w)h\left(xre^{i\theta}\right)[\Psi'(xr)]^{n-1}e^{-\frac{1}{2}}\left((x^2)+(r^2+|\xi\,|2)\right)-(xr).$$

By Fubini's Theorem, we may compute the integral in (22) by first integrating with respect to he vector ξ over $\mathbb{C}^n-1$ and then taking an area integral with respect to the omplex variable w_1 over $\mathbb{C}$. Since $y\,\Psi' \to (r^2 + y^2)$ attains its maximum at $y = 0$ nd has a second derivative larger than $2\Psi'\,(r^2)$, we have that $\Psi(r^2 + y^2) - \Psi(r^2) \geq \Psi'(r^2)y^2$. Using spherical coordinates along with this fact, we find that

$$\int_{\mathbb{C}^{n-1}} e^{-(r^2+|\xi\,|^2\,)}\,dV_{n-1}(\xi) \lesssim e^{\Psi'-(r^2)}\,[\Psi'(r^2)]^{-n+1}.$$

Similarly, again using spherical coordinates, we get

$$\int_{\mathbb{C}^{n-1}} \Theta(r,|\xi\,|)e^{\Psi-(r^2+|\xi\,|^2)}\,dV_{n-1}(\xi) = C\int_0^\infty \Theta\,(r,y)y^{2n-2}e^{\Psi-(r^2+y^2)}\,dy,$$

where C is the surface area of the unit sphere in $\mathbb{C}^{n-1}\,nd$ Θ is any suitable function of two variables. From the estimate for $\varrho(z,w))$ we see that we are interested in the following two choices: (1) $\Theta(r,y) = y[\Psi'(r^2 + y^2)]^{\frac{1}{2}}$ and (2) $\Theta(r,y) = y^2\Psi(r^2 + y^2)$. In case (1), we use the Cauchy–Schwarz inequality, so that we get

$$\int_{\mathbb{C}^{n-1}} |\xi\,|[\Psi'(r^2 + |\xi\,|^2\,)]^{\frac{1}{2}}\,e\text{^}(\Psi-(r^2+|\xi\,|^2\,)dV_{n-1}(\xi)$$

$$\lesssim e^{-\Psi(r^2)}\left[\int_0^\infty y^{4n-3}e^{-\left(\Psi(r^2+y^2)-\Psi(r^2)\right)}dy\right]^{\frac{1}{2}}.$$

Estimating $\Psi(r^2 + y^2) - \Psi(r^2)$ as above, we therefore get

$$\int_{\mathbb{C}^{n-1}} |\xi\,|[\Psi'(r^2 + |\xi\,|^2)]^{\frac{1}{2}}e^{-\Psi(r^2+|\xi\,|^2\,)}\,dV_{n-1}(\xi) \lesssim e^{-\Psi(r^2)}\,[\Psi'(r^2)]^{-n+1}\,.$$

In case (2), we integrate by parts and get

$$\int_{\mathbb{C}^{n-1}}|\xi\,|^2\Psi'\,(r^2 + |\xi\,|^2)e^{-\Psi(r^2+|\xi\,|^2\,)}\,dV_{n-1}(\xi) \lesssim \int_0^\infty y^{2n\,-1}e(-\Psi(r^2+y^2)\,dy\,.$$

We proceed as above and obtain

$$\int_{\mathbb{C}^{n-1}}|\xi\,|^2\Psi'\,(r^2 + |\xi\,|^2)e^{-\Psi(r^2+|\xi\,|^2\,)}\,dV_{n-1}(\xi) \lesssim e^{-\Psi(r^2)}\,[\Psi'(r^2)]^{-n+1}.$$

With σ denoting Lebesgue measure on $\mathbb{C}$, we therefore get

$$\int_{\mathbb{C}^n} H(z,w)dV\,(w) \lesssim \int_{\mathbb{C}} G(x,r,\theta)\left[\frac{\Psi'(rx)}{\Psi'(r^2)}\right]^{n-1} h\left(xre^{i\theta}\,\right)e^{-Q_x\,(r)}d\sigma\left(re^{i\theta}\,\right),$$

Where

$$G(x,r,\theta) = x|\theta|[\Phi'(x^2)]^{\frac{1}{2}} + \left[\Phi'\left(max(x^2,r^2)\right)\right]^{\frac{1}{2}}|x - r| + 1$$

and Q_x is as efined by (17). We now resort to polar coordinates; simple calculations show that

$\int_{-\pi}^{\pi} h\left(xre^{i\theta}\,\right)d\theta \lesssim \left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}}$ and $\int_{-\pi}^{\pi}|\theta|h\left(xre^{i\theta}\,\right)d\theta \lesssim \frac{1}{xr}$

so that

$$\int_{\mathbb{C}^n} (z,w)dV\,(w) \lesssim \int_0^\infty \left(S_x\,(r) + T_x\,(r)\right)e^{-Q_x\,(r)}rdr,$$

Where

$$S_x(r) = \left(\frac{[\Phi'(x^2)]^{\frac{1}{2}}}{r} + \left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}}\right)\left[\frac{\Psi'(rx)}{\Psi'(r^2)}\right]^{n-1}$$

and

$$T_x(r) = \varphi(\max(x^2, r^2))|x - r| \left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}}\left[\frac{\Psi'(rx)}{\Psi'(r^2)}\right]^{n-1}.$$

By Lemma (5.1.8) and a straightforward argument, we find that both $S_x e^{-Q_x}$ and $T_x e^{-Q_x}$ satisfy conditions $(I), (II), (III)$ (with $x = t, Q_x = g_t, x_0 = x$, and $\tau = [\Phi'(x)]^{-\alpha}$). Hence (9) applies with $m = 0$ and $m = 1$ for the respective integrands, so that we get

$$\sup_{x>0} \int_0^\infty S_x(r) e^{-Q_x(r)} r dr < \infty$$

And

$$\sup_{x>0} \int_0^\infty T_x(r) e^{-Q_x(r)} r dr < \infty .$$

We may therefore conclude that (22) holds.

We study the relation between the spectral properties of Hankel operators and the asymptotic behavior of their symbols. We begin with the case of compact Hankel operators.

An entire function is said to be of vanishing mean oscillation with respect toΨif $(MOf)(z) = o(1)$ as $|z| \to +\infty$. Entire functions of vanishing mean oscillation form a closed subspace of $BMOA(\Psi)$ which we will denote by $VMOA(\Psi)$. In accordance with our preceding discussion, we define the little Bloch space $B_0(\Psi)$ as the collection of functions f in $B(\Psi)$ for which

$$\operatorname{Sup}_{\xi\in\mathbb{C}^n\{0\}} \frac{\left|\langle\nabla f(z), \bar{\xi}\rangle\right|}{\beta(z,\xi)} = o(1) \text{when } |z| \to +\infty.$$

Our proof of Theorem (5.1.14) requires the following two lemmas.

Lemma (5.1.12)[129]: The normalized Bergman kernels $\frac{K_\Psi(\cdot,z)}{\sqrt{K(z,z)}}$ converge weakly to 0 in $A^2(\Psi)$ when $|z| \to +\infty$.

Proof. Since the holomorphic polynomials are dense in $A^2(\Psi)$, it suffices to show that for any non-negative integer m, we have

$$\frac{|z|^m}{\sqrt{K(z,z)}} \to 0$$

as $|z| \to +\infty$. But this holds tribylly because $K(z,z)$ is an infinite power series in $|z|^2$ with positive coefficients.

Lemma (5.1.13)[129]: Let $:\mathbb{C}^n \to \mathbb{C}$ be a function for which there exist positive numbers R and ε such that

$$|f(z) - f(w)| \leq \varepsilon(z,w)$$

whenever $|z| \geq R$. Then there exists a function $f_0 : \mathbb{C}^n \to \mathbb{C}$ such that $f(z) = f_0(z)$ for $|z| \geq R$ and

$$|f_0(z) - f_0(w)| \leq \varepsilon(z,w)$$

for all points z and w in $\mathbb{C}^n$.

Proof. We argue as in the proof of Lemma 5.1 in [130]. We assume without loss of generality that f is real-valued and set

$$f_0(z) := \inf_{w\in\mathbb{C}^n}\{f(w) + \varepsilon\varrho(z,w)\} .$$

Then a straightforward argument using the triangle inequality for the Bergman metric shows that f_0 has the desired properties.

Theorem (5.1.14)[129]: Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that (4) holds. *If* f is an entire function on $\mathbb{C}^n$, then the following statements are equivalent:

(i) The function f belongs to $\mathcal{T}(\Psi)$ and the Hankel operator $H_{\bar{f}}$ on $A^2(\Psi)$ is compact;

(ii) The function f belongs to $VMOA(\Psi)$;

(iii) The function f belongs to $B_0(\Psi)$.

Proof. We first prove the implication (i) ⇒ (ii). Assuming that $H_{\bar{f}}$ is compact, we obtain, using Lemma (5.1.12), that

$$[(MOf)(z)]^2 = \frac{\left\|H_{\bar{f}}K_\Psi(\cdot,z)\right\|^2}{K(z,z)} \to 0$$

when $|z| \to +\infty$. This gives the desired conclusion.

We next note that the implication (ii) ⇒ (iii) is immediate from (6). Finally, to prove that (iii) implies (i), in view of Theorem (5.1.1), we only need to prove that the bounded Hankel operator $H_{\bar{f}}$ is compact whenever (iii) is satisfied. To see that this holds, we choose an arbitrary positive ε. Assuming (iii), we may find a positive R_0 such that

$$\left|\langle(\nabla f)(z),\bar{\xi}\rangle\right| \le \frac{\varepsilon}{2}\,\beta(z,\xi)$$

whenever $|z| \ge R_0$ and ξ is in $\mathbb{C}^n \setminus \{0\}$. Then for some $R > R_0$ we have

$$|f(z) - f(w)| \le \varepsilon\varrho(z,w)$$

as long as $|z| \ge R$. Indeed, this follows because $\beta(z,\xi)/|\xi| \to \infty$ when $|z| \to \infty$ so that, whenever |z| is sufficiently large, (z,w) is "essentially" determined by the contribution to the integral in (2) from the points that lie outside the ball of radius R_0 centered at 0. Now let f_0 be the function obtained from Lemma (5.1.13). We write

$$H_{\bar{f}} = H_{\bar{f}-\bar{f_0}} + H_{\bar{f_0}}$$

and observe that $\bar{f} - \bar{f}0$ is a compactly supported continuous function on $\mathbb{C}^n$. Hence $H_{\bar{f}-\bar{f}0}$ is compact. On the other hand, if g is a holomorphic polynomial, then

$$H\bar{f}_0\, g(z) \lesssim \int_{\mathbb{C}_n} \left|\bar{f}_0(w) - \bar{f}_0(z)\right| |K_\Psi(z,w)g(w)|d\mu_\Psi(w)$$

$$\le \varepsilon \int_{\mathbb{C}_n} \beta(z,\xi)|K_\Psi(z,w)g(w)|d\mu_\Psi(w)$$

so that, by the proof of Theorem (5.1.1), we see that $\left\|H\bar{f}_0\right\| \lesssim \varepsilon$. The implication (iii) ⇒ (i) follows because ε can be chosen arbitrarily small.

In what follows, we will need the analogue of Lemma (5.1.3) for the function when $n > 1$. We will therefore assumΨe that

$$\Psi''(t) = O(t^{-\frac{1}{2}}\,[\Psi'(t)]^{1+\eta})\; when\; t \to \infty \tag{24}$$

for some $\eta < 1/2$ whenever $n > 1$. This is again a mild smoothness condition on Ψ.

Lemma (5.1.15)[129]: Assume that (24) holds for some $\eta < 1/2$. Then, for any fixed $\alpha > \eta$, we have

$$\sup_{|t|\le t^{\frac{1}{2}}[\Psi'(t)]^{-\alpha}} \Psi'(t + \tau) = \big(1 + o(1)\big)\Psi'(t)$$

when $t \to \infty$.

We are interested in describing geometrically the Bergman ball

$$B(z,a) = \{w : \varrho(z,w) < a\}.$$

Let P_z denote the orthogonal projection in $\mathbb{C}^n$ onto the complex line $\{\zeta z : \zeta \in \mathbb{C}\}$, where z is an arbitrary point in $\mathbb{C}^n \setminus \{0\}$. It will be convenient to let P_0 denote the identity map. We use the notation

$$D(z,a) = \left\{w : |z - P_z w| \le a[\Phi'(|z|^2)]^{-\frac{1}{2}}, |w - P_z w| \le a[\Psi'(|z|^2)]^{-\frac{1}{2}}\right\}.$$

Then we have the following result.

Lemma (5.1.16)[129]: Suppose that there exists a real number η < 1/2 such that (4) holds and that (24) holds if $n > 1$. Then, for every positive number a, there exist two positive numbers m and M such that

$$D(z,m) \subset B(z,a) \subset D(z,M)$$

$for\ every\ z\ in\ \mathbb{C}^n$.

Proof. It suffices to prove that

$$\varrho(z,w) \simeq |z - P_z w|[[\Phi'(|z|^2)]^{\frac{1}{2}} + |w - P_z w|[\Psi'(|z|2)]^{\frac{1}{2}} \quad (25)$$

for w in $D(z,M)$ for any fixed positive number M. (The latter term vanishes and can be disregarded when $n = 1$.) To begin with, we note that heorem B gives that

$$\varrho(z,w) \simeq \inf_{\gamma} \int_0^1 \left(|\gamma'(t)|[\Psi'(|\gamma(t)|^2)]^{\frac{1}{2}} + |\langle\gamma(t),\gamma'\ t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}\right) dt, \quad (26)$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0,1] \to \mathbb{C}^n$ such that $\gamma(0) = z$ and $\gamma(1) = w$. If we choose γ to be the line segment from $z\ to\ P_z w$ followed by the line segment from $P_z w$ to w and use that $\Psi''(x) = o([\Psi'(x)]^{\frac{1}{2}})$ on the latter part of γ , we get from (26) that

$$\varrho(z,w) \lesssim |z - P_z w|[\Phi'(|z|^2)]^{\frac{1}{2}} + |P_z w - w|[\Psi'(|z|^2)]^{\frac{1}{2}} + |P_z w - w|^2 o(\Psi'(|z|^2)).$$

this gives the desired bound from above because, by assumption, $|P_z w - w| \le M[\Psi'|z|^2)]^{-\frac{1}{2}}$.

To prove the bound from below, we argue in the following way. Let $\ell(\gamma)$ denote the Euclidean length of γ . Set

$$\varrho_\gamma^*(z,w) = \int_0^1 |\gamma'(t)|[\Psi'(|\gamma(t)|^2)]^{\frac{1}{2}} + |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}\ dt$$

and $\varrho^*(z,w) = inf_y\ \varrho_\gamma^*(z,w)$.We observe hat (26)) implies that

$$\varrho(z,w) \gtrsim \inf_t [\Psi'(|\gamma(t)|^2)]^{\frac{1}{2}}\ell(\gamma) \quad (27)$$

whenever, say, $\varrho_\gamma^*(z,w) \le 2\varrho * (z,w)$. Since we know by the first part of the proof that $\varrho(z,w) \lesssim 1$, this implies that

$$\ell(\gamma) \lesssim \inf_t [\Psi'(|\gamma(t)|^2)]^{-\frac{1}{2}}.$$

By Lemma (5.1.15), we therefore have

$$\ell(\gamma) \lesssim [\Psi'(|z|^2)]^{\frac{1}{2}},$$

which, in view of (27), in turn gives

$$\ell(\gamma) \lesssim [\Psi'(|z|^2)]^{-\frac{1}{2}}\varrho(z,w). \quad (28)$$

Now et γ be any curve such that $\varrho_\gamma^*(z,w) \le 2\varrho^*(z,w)$. We then get from (26) that

$$\varrho(z,w) \gtrsim |z - w|[\Psi'(|z|^2)]^{\frac{1}{2}} + \int_0^1 |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)2)]|^{\frac{1}{2}}dt. \quad (29)$$

Set $\gamma_0(t) = P_z(\gamma(t))$ and $\gamma_1(t) = \gamma(t) - \gamma_0(t)$. Note that $\gamma_1(0) = 0$ and that $\ell(\gamma_1) \leq \ell(\gamma)$. By orthogonality and the triangle inequality, we get

$$\int_0^1 |\langle\gamma(t), \gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}dt \geq \int_0^1 |\gamma_0(t)||\gamma_0'(t)|[\Psi''(|\gamma_0(t)|^2)]^{\frac{1}{2}}dt$$

$$- \int_0^1 |\langle\gamma_1(t), \gamma_1'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}dt.$$

Let t_1 be the smallest t such that $|z - \gamma_0(t)| = |z - P_z w|$. Using that $\Psi''(x) = o([\Psi'(x)]^2)$ and (28), we then get

$$\int_0^1 |\langle\gamma(t), \gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}dt$$

$$\geq \big(1 + o(1)\big)\int_0^{t_1} |z||\gamma_0'(t)|[\Psi''(|z|^2)]^{\frac{1}{2}}dt - [\ell(\gamma)]^2 o\big(\Psi'(|z|^2)\big)$$

$$\gtrsim |z - P_z w||z|[\Psi''(|z|^2)]^{\frac{1}{2}} - o(1)\varrho(z,w)$$

when $|z| \to \infty$. Plugging this estimate into (29), we obtain the desired bound from below.

It follows from the previous lemma that the Euclidean volume of $B(z,r)$ can be estimated as

$$|B(z,r)| \simeq [\Phi'(|z|^2)]^{-\frac{1}{2}}[\Psi'(|z|^2)]^{\frac{n-1}{2}} \tag{30}$$

when r is a fixed positive number. We will now use this fact to establish two covering lemmas.

Lemma (5.1.17)[129]: Suppose that there exists a real number $\eta < \frac{1}{2}$ such that (4) holds and that (24) holds if $n > 1$. Let R be a positive number and m a positive integer. Then there exists a positive integer N such that every Bergman ball $B(a,r)$with $r \leq R$ can be covered by N Bergman balls $B\left(a_k, \frac{r}{m}\right)$.

Proof. Fix a ball $B(a,r)$. Choose $a_0 := a$ and let a_1 be a point in $\mathbb{C}^n$ such that $\varrho(a, a_1) = r/m$. Now iterate so that in the $k-th$ step a_k is chosen as a point in the complement of $\bigcup_{j=1}^{k-1} B(aj, r/m)$ minimizing the distance from a, and let J be the smallest k such that $\varrho(a, a_k) \geq r$. Then the balls $B(a_0, r/m), \ldots, B(a_{J-1}, r/m)$ constitute a covering of $B(a,r)$. By the triangle inequality, we see that the sets $B(a_j, r/(2m))$ are mutually disjoint, and they are all contained in $B(a, r + r/(2m))$ when $j < J$ .Hence

$$\sum_{j=0}^{J-1}(a_j, r/(2m))\ |B(a, r + r/(2m))|.$$

On the other hand, by (30), it follows that there is a positive number C depending on R and m but not on a such that

$$\frac{1}{C}\left|B\left(a, r + \frac{r}{2m}\right)\right| \leq \left|B\left(a_j, \frac{r}{2m}\right)\right|$$

for every j . We observe that it suffices to take N to be the smallest positive integer larger than or equal to C.

Inspired by the construction in the previous lemma, we introduce the following notion. We say that a sequence of distinct points (a_k) in $\mathbb{C}^n$ is a -lattice if there exists a positive number r such that the balls $B(a_k, r)$ constitute a covering of $\mathbb{C}^n$ and the balls $B\left(a_k, \frac{r}{2}\right)$ are mutually disjoint. Replacing a by, say, 0, and $\frac{r}{m}$by r in the previous proof, we have a straightforward way of constructing a -lattice. Note that since the balls $B\left(a_k,\frac{r}{2}\right)$ are mutually disjoint, we must have $\left(a_k,a_j\right)\geq r$ when $k\neq j$. The number r, which may fail to be unique, is called a covering radius for the -lattice (a_k). The supremum of all the covering radii is again a covering radius; it will be called the maximal covering radius for (a_k).

Lemma (5.1.18)[129]: Suppose that there exists a real number $\eta<\frac{1}{2}$ such that (4) holds and that (24) holds $if\ n>1$, and let R be a positive number. Then there exists a positive integer N such that if (a_k) is a -lattice with maximal covering radius $r\leq\frac{R}{2}$, then every point z in $\mathbb{C}^n$ belongs to at most N of the sets $B(a_k,2r)$.

Proof. Let N be the integer obtained from Lemma (5.1.17) for the given R when $m=4$ and assume that $z\in\bigcap_{j=1}^{N+1}B(a_{kj},2r)$. Then a_{kj} is in $B(z,2r)$ for every $j=1,\dots,N+1$. If the sets $B(z_1,r/2),\dots,B(z_N,r/2)$ constitute a covering of $B(z,2r)$, the existence of which is guaranteed by Lemma (5.1.17), then at least one of the sets $B(z_k,r/2)$ must contain two of the points $a_{kj}, j=1,\dots,N+1$. On the other hand, by the triangle inequality, we have reached a contradiction because the minimal distance between any two points in the sequence (a_k) cannot be smaller than r.

For a nonnegative Borel measure ν on $\mathbb{C}^n$, we set

$$d\nu_\Psi(z)=e^{-\Psi(|z|^2)}\,d\nu(z).$$

Such a measure ν is called a Carleson measure for $A^2(\Psi)$ if there is a positive constant C such that

$$\int_{\mathbb{C}^n}|f(z)|^2d\nu_\Psi(z)\leq C\int_{\mathbb{C}^n}|f(z)|^2d\mu_\Psi(z)$$

for every function f in $A^2(\Psi)$. Thus ν is a Carleson measure for $A^2(\Psi)$ if and only if the embedding E_ν of $A^2(\Psi)$into the space $L^2(\nu_\Psi)$ is bounded.

Lemma (5.1.19)[129]: Suppose that there exists a real number $\eta<1/2$ such that (4) holds and that (24) holds if $n>1$. Then there exists a positive number r_0 such that

$$|K_\Psi(z,w)|^2\backsimeq K(z,z)K(w,w)$$

holds for z and w whenever $(z,w)\leq r_0$.

Proof. The lemma follows from Lemma (5.1.7) along with Lemma (5.1.16).

Lemma (5.1.20)[129]: Suppose that there exists a real number $\eta<1/2$ such that (4) holds and that (24) holds if $n>1$, and let r_0 be the constant from Lemma (5.1.19). Then there is a constant C such that

$$|f(z)|^2e^{-\Psi(|z|^2)}\leq\frac{C}{|B(z,r)|}\int_{B(z,r)}|f(w)|^2d\mu_\Psi(w)$$

for every entire $f$unction f on $\mathbb{C}^n$ and every z in $\mathbb{C}^n$.

Proof. By Lemma (5.1.19), the holomorphic function $w\to K(z,w)$ does not vanish at any point in $B(z,r)$. Thus the function $w\to|f(w)|^2|K_\Psi(z,w)|^{-2}$ is subharmonic in $B(z,r)$. Choosing m as in Lemma (5.1.16), we therefore get

$$|f(z)|^2|K(z,z)|^{-2}\lesssim\frac{1}{|D(z,m)|}\int_{D(z,m)}|f(w)|^2|K_\Psi(z,w)|^{-2}dV(w)$$
$$\lesssim\frac{1}{|B(z,r)|}\int_{B(z,r)}|f(w)|^2|K_\Psi(z,w)|^{-2}dV(w).$$

Applying Lemma (5.1.19) to the integrand to the left and then Lemma (5.1.7) to each side, we arrive at the desired estimate.

Note that, by (30), the lemma is valid for all positive r, with the additional proviso that C depends on r.

Theorem (5.1.21)[129]: Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that (4) holds and that (24) holds if $n > 1$. If ν is a nonnegative Borel measure on $\mathbb{C}^n$, then the following statements are equivalent:

(i) ν is a Carleson measure for $A^2(\Psi)$;

(ii) There is a constant $C > 0$ such that

$$\int_{\mathbb{C}^n} \frac{|K_\Psi(w,z)|^2}{K(z,z)}\, d\nu_\Psi(w) \le C$$

for every z in $\mathbb{C}^n$;

(iii) For every positive number r, there is a positive number C such that

$$\nu\big(B(z,r)\big) \le C|B(z,r)|$$

for every z in $\mathbb{C}^n$;

(iv) There exist $a\Psi$ -lattice (a_k) and a positive number C such that

$$\nu\big(B(a_k,r)\big) \le C|B(a_k,r)|$$

for every point k, where r is the maximal covering radius for (a_k). We prepare for the proof of Theorem (5.1.21) by establishing the following two lemmas.

Proof. We begin by noting that the implication (i) ⇒ (ii) is tribyl because it is just the statement that the Carleson measure condition holds for the functions $K(\cdot,z)$. To prove that (ii) implies (iii), we assume that (ii) holds and consider a ball $B(z,r)$ where r is a fixed positive number. Then, by Lemma (5.1.19) and (30), we have

$$\frac{1}{|B(z,r)|} \lesssim \frac{|K_\Psi(z,w)|^2}{K_\Psi(z,z)}\, e^{-\Psi(|w|^2)}$$

when $(z,w) \le r_0$, and therefore we obtain

$$\frac{\nu\big(B(z,r)\big)}{|B(z,r)|} \lesssim \int_{\mathbb{C}^n} \frac{|K_\Psi(z,w)|^2}{K(z,z)}\, e^{-\Psi(|w|^2)}\, d\nu(w) \le C.$$

The implication (iii) ⇒ (iv) is tribyl (modulo the existence of Ψ-lattices), and we are therefore done if we can prove that (iv) implies (i). To this end, assume that (iv) holds, and let (a_k) be aΨ-lattice with maximal covering radius $r. By$ Lemma (5.1.20), we see that

$$\sup_{z\in B(a_k,r)} |f(z)|^2 e^{-\Psi(|z|^2)} \lesssim \frac{1}{|B(a_k,2r)|} \int_{B(a_k,2r)} |f(w)|^2 d\mu_\Psi(z)$$

for every k. We therefore get

$$\int_{\mathbb{C}^n} |f(z)|^2 d\nu_\Psi(z) \lesssim \sum_k \int_{B(a_k,2r)} |f(w)|^2 d\mu_\Psi(w) \lesssim \int_{\mathbb{C}^n} |f(w)|^2 d\mu_\Psi(w),$$

where the latter inequality holds by Lemma (5.1.18).

For ν a nonnegative Borel measure on $\mathbb{C}^n$, we define the Toeplitz operator T_ν on $A^2(\Psi)$ in the following way:

$$(T_\nu f)(z) := \int_{\mathbb{C}^n} f(w) K_\Psi(z,w) e^{-\Psi(|w|^2)}\, d\nu(w)\,.$$

A computation shows that $E_\nu^* E_\nu = T_\nu$. Thus Theorem (5.1.21) characterizes bounded Toeplitz operators. Compact Toeplitz operators can likewise be characterized by socalled vanishing Carleson measures; an obvious and straightforward modification of Theorem

(5.1.21) gives a description of such measures. Toeplitz operators belonging to the Schatten classes S_p are characterized by the following theorem.

Lemma (5.1.22)[129]: Suppose that (e_j) is an orthonormal basis for $A^2(\Psi)$ and that (a_j) is a Ψ -lattice. Then the operator J on $A^2(\Psi)$ defined by

$$Je_j(z) := \frac{K_\Psi(z, a_j)}{\sqrt{K_\Psi(a_j, a_j)}}$$

is bounded.

Proof. For two arbitrary functions $f = \sum_j cj\, ej$ and g in $A^2(\Psi)$, the reproducing formula and the Cauchy–Schwarz inequality give

$$|\langle Jf, g\rangle|^2 = \left|\sum_j Cj \frac{g(a_j)}{\sqrt{K_\Psi(a_j, a_j)}}\right|^2 \leq \left(\sum_j |cj|^2\right) \sum_k \frac{|g(a_k)|^2}{K_\Psi(a_k, a_k)} .$$

If we set

$$\nu := \sum_k \frac{e^{\Psi(|a_j|^2)}}{K_\Psi(a_j, a_j)} \delta a_j ,$$

then we may write this estimate as

$$|Jf, g|^2 \leq \|f\|^2_{A^2(\Psi)} \int_{\mathbb{C}^n} \mathbb{C}^n |g(z)|2 d\nu_\Psi(z).$$

By Theorem (5.1.21), we see that ν is a Carleson measure, which implies that J is a bounded operator on $A^2(\Psi)$.

Lemma (5.1.23)[129]: Suppose that T is a positive operator on $A^2(\Psi)$. Then the trace of T can be computed as

$$Tr(T) = \int_{\mathbb{C}^n} \widetilde{T}(z) K_\Psi(z, z) d\mu(z).$$

Proof. We write $K_\Psi(z, w) = \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)}$, where (e_k) is an orthonormal basis for $A^2(\Psi)$. The lemma is then proved by means of the following computation:

$$Tr(T) \sum_{k=0}^{\infty} \langle Tf_k, f_k\rangle A^2(\Psi) = \mathbb{C}^n\, T\, K_\Psi(\cdot, z), K_\Psi(\cdot, z) A^2(\Psi) d\mu_\Psi(z) .$$

Theorem (5.1.24)[129]: Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that (4) holds and that (24) holds if $n > 1. If\ \nu$ is a nonnegative Borel measure on $\mathbb{C}^n$ and $p \geq 1$, then the following statements are equivalent:

(i) The Toeplitz operator T_ν on $A^2(\Psi)$ belongs to the Schatten class S_p ;

(ii) There exists a Ψ -lattice (a_k) such that

$$\sum_{k=1}^{\infty} \left(\frac{\nu(B(a_k, r))}{|B(a_k, r)|}\right)^p < +\infty,$$

where r is the maximal covering radius for (a_k).

Proof. We begin by assuming that T_ν is in S_p. Pick aΨ-lattice (a_j) and let r be its maximal covering radius. By (30) and Lemma (5.1.19), we have

$$\sum_k \left(\frac{\nu(B(a_k, r))}{|B(a_k, r)|}\right)^p \backsimeq \sum_k \left(\int_{B(a_k, r)} K_\Psi(w, w) d\nu_\Psi(w)\right)^p$$

$$\asymp \sum_k \left(\int_{B(a_k,r)} \frac{|K_\Psi(a_k,w)|^2}{K_\Psi(a_k,a_k)} \, d\nu(w) \right)^p .$$

By Lemma (5.1.18) and our assumption on ν, this gives

$$\sum_k \left(\frac{\nu\big(B(a_k,r)\big)}{|B(a_k,r)|} \right)^p \lesssim \sum_k \left(\int_{\mathbb{C}^n} \frac{|K_\Psi(a_k,w)|^2}{K_\Psi(a_k,a_k)} \, d\mu_\Psi(w) \right)^p .$$

If we construct J as in Lemma (5.1.22), then the right-hand side equals $\sum_k |\langle J * T_\nu J e_k, e_k\rangle|^p$. Since J is a bounded operator, $J^* T_\nu J$ also belongs to S_p, and so the latter sum converges. We conclude that (i) implies (ii). We will use an interpolation argument to prove that (ii) implies (i). We already know from Theorem (5.1.21) that T_ν is in the Schatten class S_∞ whenever $\nu(B(a_k,r)) \le C|B(a_k,r)|$ for some positive constant C. Suppose now that

$$\sum_k \frac{\nu\big(B(a_k,r)\big)}{|B(a_k,r)|} < +\infty,$$

and let (e_j) be an orthonormal basis for $A^2(\Psi)$. By the reproducing formula, we have

$$\langle T_\nu \, e_j \, , e_j \rangle = \int_{\mathbb{C}^n} \left| e_j \, (w) \right|^2 d\nu_\Psi(w),$$

which implies that

$$\sum_j \left| \langle T_\nu \, e_j \, , e_j \rangle \right| = \int_{\mathbb{C}^n} K_\Psi(w,w) d\nu_\Psi(w) \le \sum_k \int_{B(a_k,r)} K(w,w) d\nu_\Psi(w).$$

Again using Lemma (5.1.7), we then get

$$\sum_j \left| \langle T_\nu \, e_j \, , e_j \rangle \right| \lesssim \sum_k \frac{\nu\big(B(a_k,r)\big)}{|B(a_k,r)|} < +\infty,$$

which means that T_ν belongs to S_1. By interpolation, we conclude that (ii) implies (i).

We remark that the theorems proved generalize results for the classical Fock space when $n = 1$ obtained recently in [139]. It may be noted that Theorem (5.1.21) above could be elaborated to include two additional conditions for membership in S_p, in accordance with Theorem 4.4 in [139]. The proof would be essentially the same as the proof of the latter theorem. Note that [139] also treats Schatten class membership of Toeplitz operators for $p < 1$.

We suggest two possible definitions of Besov spaces, in accordance with our respective definitions of $BMOA(\Psi)$ and $B(\Psi)$. We let $B_m^p(\Psi)$ denote the set of entire functions f such that

$$\int_{\mathbb{C}^n} [MOf \, (z)] p K_\Psi(z,z) d\mu_\Psi(z) < \infty;$$

for a function h : $\mathbb{C}^n \to \mathbb{C}^n b$, we set

$$|h(z)|_\beta = \sup_{\xi \in \mathbb{C}^n \setminus \{0\}} \frac{\left| \langle h(z), \bar{\xi} \rangle \right|}{\beta(z,\xi)},$$

and we let $B_d^p \, (\Psi)$ be the set of entire functions f for which

$$\int_{\mathbb{C}^n} |\nabla f \, (z)|_\beta^p K_\Psi(z,z) d\mu_\Psi(z) < \infty.$$

These definitions are in line with those of K. Zhu for Hankel operators on the Bergman space of the unit ball in $\mathbb{C}^n$ [144].

It is immediate from (6) that $B_m^p(\Psi) \subset B_d^p(\Psi)$. The basic question is whether these spaces coincide and in fact characterize Schatten class Hankel operators with anti-holomorphic symbols. The following theorem gives an affirmative answer to this question.

Theorem (5.1.25)[129]: Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that (4) holds and that (24) holds if $n > 1$. If f is an entire function on $\mathbb{C}^n$ and $p \geq 2$, then the following statements are equivalent:

(i) The function f belongs to $\mathcal{T}(\Psi)$ and the Hankel operator $H_{\bar{f}}$ on $A^2(\Psi)$ is in the Schatten class S_p;

(ii) The function f belongs to $B_m^p(\Psi)$;

(iii) The function f belongs to $B_d^p(\Psi)$.

Proof. We have already observed that the implication (ii) ⇒ (iii) is an immediate consequence of (6). The implication (i) ⇒ (ii) relies on the following general Hilbert space argument. If (i) holds, then the operator $\left[H_{\bar{f}}^* H_{\bar{f}}\right]^{\frac{p}{2}}$ is in the trace class S_1. Applying Lemma (5.1.23) and using the spectral Theorem (5.1.1)long with Hölder's inequality, we obtain

$$Tr\left(\left[H_{\bar{f}}^* H_{\bar{f}}\right]^{\frac{p}{2}}\right) = \int_{\mathbb{C}^n} \langle\left[H_{\bar{f}}^* H_{\bar{f}}\right]^{\frac{p}{2}} K(\cdot,z), K_\Psi(\cdot,z)\rangle\, d\mu_\Psi(z)$$

$$\gtrsim \int_{\mathbb{C}^n}\left[\frac{\left\|H_{\bar{f}} K_\Psi(\cdot,z)\right\|^2}{K_\Psi(z,z)}\right]^{\frac{p}{2}} K_\Psi(z,z)\, d\mu_\Psi(z).$$

Recalling the computation made in (5), we arrive at (ii).

Our proof of the implication (iii) ⇒ (i) will use a version of L. Hörmander's L^2 estimates for the $\bar{\partial}$ operator. To this end, write $\triangle_\Psi(z) = (|z|^2)$ and observe that

$$\alpha^2(z,\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 \triangle_\Psi(z)}{\partial z_j\, \partial \bar{z}_k}\, \xi_j\, \bar{\xi}_k = |\xi|^2 \Psi'(|z|^2) + |\langle z, \xi\rangle|^2 \Psi''(|z|^2)$$

for arbitrary vectors $z = (z_1, \ldots, z_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$. By Theorem (5.1.9), we therefore have $\alpha(z,\xi) \asymp \beta(z,\xi)$. Now let $L_\beta^2(\mu_\Psi)$ be the space of vector-valued functions $h = (h_1, \ldots, h_n)$, identified with the corresponding (0, 1)-forms $h_1 d\bar{z}1 + \cdots + h_n d\bar{z}_n$ such that

$$\|h\|_{L_\beta^2(\mu)}^2 := \int_{\mathbb{C}^n} |h(z)|_\beta^2\, d\mu_\Psi(z) < \infty.$$

It follows from Theorem 2.2 in [136] (a special case of a theorem proved by $J.-P.\ De$ mailly in [138]) that the operator S giving the canonical solution to the $\bar{\partial}$-problem is bounded from $L_\beta^2(\mu_\Psi)$ into $L^2(\mu)$.

Since f is holomorphic, we have

$$\tilde{\partial}\left(H_{\bar{f}} g\right) = \overline{\nabla f}_g$$

when g is in $A^2(\Psi)$, whence $H_{\bar{f}} g = S(\overline{\nabla f} g)$. Thus it follows that

$$\left\|H_{\bar{f}} g\right\|_{L^2(\mu_\Psi)} \lesssim \int_{\mathbb{C}^n} |\nabla f(z)|_\beta^2 |g(z)|^2 d\mu_\Psi(z). \tag{31}$$

If we set $dv(z) = |\nabla f(z)|_\beta^2\, dV(z)$, this may be written as

$$H_{\bar{f}}^* H_{\bar{f}}\, M_{|\nabla f|_\beta}^* M_{|\nabla f|_\beta} = T\nu,$$

where as before M_h denotes the operator of multiplication by h from $A^2(\Psi)$ into $L^2(\mu_\Psi). By$ Theorem (5.1.24), it remains to verify that (iii) implies that for some Ψ-lattice (a_k) we have

$$\sum_{k=1}^{\infty}\left(\frac{\nu\big(B(a_k,r)\big)}{|B(a_k,r)|}\right)^{\frac{p}{2}} < +\infty, \tag{32}$$

where r is the maximal covering radius for (a_k). To this end, we first observe that Hölder's inequality gives that

$$\left(\frac{\nu\big(B(z,r)\big)}{|B(z,r)|}\right)^{\frac{p}{2}} \lesssim \frac{1}{|B(z,r)|}\int_{B(z,r)} \left|\nabla f\left(B(z,r)\right)\right|_\beta^p dV(w).$$

Hence, using (30) and Lemma (5.1.7), we obtain

$$\left(\frac{\nu\big(B(z,r)\big)}{|B(z,r)|}\right)^{\frac{p}{2}} \lesssim \int_{B(z,r)} |\nabla f(z)|_\beta^p K(z,z) dV(w).$$

Now choosing anyΨ-lattice (a_k) and using Lemma (5.1.18), we arrive at (32). Several remarks are in order. First, note that (31) gives another proof of the implication (iii) ⇒ (i) in Theorem (5.1.1), subject to the additional smoothness condition (24). Second, as shown in [137], there are nontribyl Hankel operators in S_p only when $p > 2n$. This fact is easy to see from Theorem (5.1.25) when $n = 1$, because then

$$|\nabla f(z)|_\beta \asymp |f'(z)|[\Phi'(|z|^2)]^{-\frac{1}{2}},$$

whence f is in $B_d^p(\Psi)$ if and only if

$$\int_{\mathbb{C}} |f'(z)|^p[\Phi'(|z|^2)]^{1-\frac{p}{2}} dV(z) < \infty. \tag{33}$$

When $n > 1$, the computation of $|\nabla f(z)|_\beta$ is less straightforward, but we always have

$$|\nabla f(z)|[\Phi'(|z|^2)]^{-\frac{1}{2}} \lesssim |\nabla f(z)|\beta\, |\nabla \lesssim f(z)|[\Psi'(|z|^2)]^{-\frac{1}{2}}.$$

The estimate from above shows that the condition

$$\int_{\mathbb{C}^n} |\nabla f(z)|^p\, \Phi'(|z|^2)[\Psi'(|z|^2)]^{n-1-\frac{p}{2}} dV(z) < \infty \tag{34}$$

is sufficient for f to belong to $B_d^p(\Psi)$, and the estimate from below shows thatthis is also necessary when Φ'/Ψ' is a bounded function. We conclude from (33) and (34) that if the growth of Ψ is super-polynomial, then $B_d^p(\Psi)$ is infinite-dimensional and contains all polynomials if and only if $p > 2n$. This is immediate when $n = 1$, and it follows also when $n > 1$ because

$$\int_0^\infty \frac{\Psi''(t)}{[\Psi'(t)]^{1+\delta}} dt \le (1)/(\delta[\Psi'(0)]^\delta < \infty$$

for every $\delta > 0$. If, on the other hand, is a polynomial, then Φ'/Ψ' is a bounded function, and one may use (34) and Theorem (5.1.25) to deduce Theorem (5.1.9) in [84]. It is not hard to check that if f is a monomial and $n > 1$, then

$$|\nabla f(z)|_\beta \asymp |\nabla f(z)||[\Psi'(|z|^2)]^{-\frac{1}{2}}$$

for z belonging to a set of infinite volume measure. By Lemma (5.1.2) in [84] and Theorem (5.1.25) above, one may therefore conclude as in [84] that $B_d^p(\Psi)$ is nontribyl only if $p > 2n$.

Section (5.2): The Fock Space

For $\mathbb{C}^n$ be the complex n-space. For points $z = (z_1,\cdot\cdot\cdot, z_n)$ and $w = (w_1,\cdot\cdot\cdot, w_n)$ in $\mathbb{C}^n$ we write

$$z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j , \quad |z| = \sqrt{z \cdot \bar{z}}.$$

Let dv be ordinary volume measure on $\mathbb{C}^n$. For any positive parameter α we consider the Gaussian measure

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z).$$

The Fock space F_α^2 is the closed subspace of entire functions in $L^2(\mathbb{C}^n, d\lambda_\alpha)$. The orthog1onal projection $P : L^2(\mathbb{C}^n, d\lambda^\alpha) \to F_\alpha^2$ is given by

$$P f(z) = \int_{\mathbb{C}^n} K(z,w) f(w) d\lambda_\alpha(w),$$

where $K(z,w) = e^{\alpha z\cdot\bar{w}}$ is the reproducing kernel of F_α^2 .

We say that f satisfies Condition (G) if the function $z \mapsto f(z)e^{\alpha z\cdot\bar{w}}$ belongs to $L^1(\mathbb{C}^n, d\lambda_\alpha)$ for every $w \in \mathbb{C}^n$. Equivalently, f satisfies Condition (G) if every translate of $f\ z \mapsto f(z + w)$, belongs to $L^1(\mathbb{C}^n, d\lambda_\alpha)$. If $f \in F_\alpha^2$, then there exists a constant $C > 0$ such that

$$|f(z)| \le C e^{\frac{\alpha}{2}|z|^2}, z \in \mathbb{C}^n .$$

This clearly implies that f satisfies Condition (G).

If f satisfies Condition (G), we can define a linear operator T^f on F_α^2 by $T_{fg} = P(fg)$, where

$$g(z) = \sum_{k=1}^{N} c_k K(z, w_k)$$

is any finite linear combination of kernel functions. It is easy to see that the set of all finite linear combinations of kernel functions is dense in F_α^2 . Here $P(fg)$ is to be interpreted as the following integral:

$$T_f\, g(z) = \int_{Cn} f(w) g(w) e^{\alpha z\cdot\bar{w}} d\lambda_\alpha(w), z \in \mathbb{C}^n .$$

Therefore, for g in a dense subset of F_α^2 ,T_{fg} is a well-defined entire function (not necessarily in F_α^2 though). We study the Toeplitz product $T_f T_g$, where f and g are functions in F_α^2. Such a product is well defined on the set of finite linear combinations of kernel functions. The main concern is the following: what conditions on f and g will ensure that the Toeplitz product $T_f T_g$ extends to a bounded (or compact) operator on F_α^2 ?

This problem was first raised by Sarason in [53] in the context of Hardy and Bergman spaces. It was partially solved for Toeplitz operators on the Hardy space of the unit circle in [59], on the Bergman space of the unit disk in [42], on the Bergman space of the polydisk in [149], and on the Bergman space of the unit ball in [148], [150]. In all these cases, the necessary and/or sufficient condition for $T_f T_g$ to be bounded is

$$\operatorname{Sup}_{z\in\Omega} \widetilde{|f|^{2+\varepsilon}}(z) \widetilde{|g|^{2+\varepsilon}}(z) < \infty,$$

where ε is any positive number and $\tilde{f}$ denotes the Berezin transform of f.

Note that in the Hardy space case, the Berezin transform is nothing but the classical Poisson transform.

We obtain a much more explicit characterization for $T_f T_{\bar{g}}$ to be bounded on the Fock space.

Main Theorem (5.2.1)[145]: Let f and g be functions in F_α^2, not identically zero. Then $T_f T_{\bar{g}}$ is bounded on F_α^2 if and only if $f = e^q$ and $g = ce^{-q}$, where c is a nonzero complex constant and q is a complex linear polynomial.

Furthermore, our proof reveals that when $T_f T_{\bar{g}}$ is bounded, it must be a constant times a unitary operator. Consequently, $T_f T_{\bar{g}}$ is never compact unless it is the zero operator.

As another by-product of our analysis, we will construct a class of unbounded, densely defined, operators on the Fock space whose Berezin transform is bounded. It has been known that such operators exist, but our examples are very simple products of Toeplitz operators.

Proof: For any point $a \in \mathbb{C}^n$ we consider the operator $U_a : F_\alpha^2 \rightarrow F_\alpha^2$ defined by

$$U_a f(z) = f(z - a)k_a(z),$$

Where

$$k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = e^{\alpha z \cdot \bar{a} - \frac{\alpha}{2}|a|^2}$$

is the normalized reproducing kernel of F_α^2 at a. It follows from a change of variables that each Ua is a unitary operator on F_α^2.

We begin with the very special case of Toeplitz operators induced by kernel functions.

Lemma (5.2.2)[145]: Let $a \in \mathbb{C}^n, f(z) = e^{\alpha z \cdot \bar{a}}, and\ g(z) = e^{\alpha z \cdot \bar{a}}$. We have

$$T_f T_{\bar{g}} = e^{\frac{\alpha}{2}|a|^2} U_a.$$

In particular, $T_f T_{\bar{g}}$ is bounded on F_α^2.

Proof. To avoid tribylity we assume that a is nonzero. The Toeplitz operator T_f is just multiplication by f, as a densely defined unbounded linear operator. So we focus on the operator $T_{\bar{g}}$.

Given any function $h \in F_\alpha^2$, we have

$$T_{\bar{g}} h(z) = \int_{\mathbb{C}^n} \overline{g(w)} h(w) K(z,w) d\lambda_\alpha(w) = \int_{\mathbb{C}^n} h(w) e^{\alpha(z-a)\cdot \bar{w}} d\lambda_\alpha(w) = h(z - a).$$

Therefore, the Toeplitz operator $T_{\bar{g}}$ is an operator of translation, and

$$T_f T_{\bar{g}} h(z) = e^{\alpha z \cdot \bar{a}} h(z - a) = e^{\frac{\alpha}{2}|a|^2} U_a h(z).$$

This proves the desired result.

An immediate consequence of Lemma (5.2.2) is that if $f = C_1 e^q\ and\ g = C_2 e^{-q}$, where C_1 and C_2 are complex constants and q is a complex linear polynomial, then there exists a complex constant c and a unitary operator U such that $T_f T_{\bar{g}} = cU$.

To deal with more general symbol functions, we need the following characterization of nonvanishing functions in F_α^2.

Lemma (5.2.3)[145]: If f is a nonvanishing function in F_α^2, then there exists a complex polynomial q, with $\deg(q) \leq 2$, such that $f = e^q$.

Proof. In the case when the dimension $n = 1$, the Weierstrass factorization of functions in the Fock space F_α^2 takes the form $f(z) = P(z)e^{q(z)}$, where P is the canonical Weierstrass product associated to the zero sequence of f, and $q(z) = az^2 + bz + c$ is a quadratic polynomial with $|a| < \frac{\alpha}{2}$. In particular, if f is zero-free, then $f = e^q$ for some quadratic polynomial. See [151].

When $n > 1$, we no longer have such a nice factorization. But the absence of zeros makes a special version of the factorization above still valid. More specifically, if f is any function in $F_\alpha^2 = F_\alpha^2(\mathbb{C}^n)$ and f is nonvanishing, then the function $z_1 \mapsto f(z_1, \cdots, z_n)$ is in $F_\alpha^2(\mathbb{C})$, so by the factorization theorem stated in the previous paragraph,

$$f(z_1, \cdots, z_n) = e^{az_1^2 + bz_1 + c},$$

where a, b, and c are holomorphic functions of $z_2, \cdots, z_n$. Repeat this for every independent variable, we conclude that $f = e^q$ for some polynomial of degree $2n$ or less.

Recall that every function $f \in F_\alpha^2$ satisfies the pointwise estimate

$$|f(z)| \leq Ce^{\frac{\alpha}{2}|z|^2}, z \in \mathbb{C}^n.$$

If q is a polynomial of degree N and $N > 2$, then for any fixed $\zeta = (\zeta_1, \cdots, \zeta_n)$ on the unit sphere of $\mathbb{C}^n$ with each $\zeta k \neq 0$, and for $z = r\zeta$, where $r > 0$, we have $q(z) \sim r^N$ as $r \to \infty$, which shows that the estimate $|f(z)| \leq Ce^{\frac{\alpha}{2}|z|^2}$ is impossible to hold. This shows that the degree of q is less than or equal to 2.

We can now prove the main result, which we restate as follows.

Theorem (5.2.4)[145]: Suppose f and g are functions in F_α^2. Then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on F_α^2 if and only if one of the following two conditions holds:

(a) At least one of f and g is identically zero.

(b) There exists a linear polynomial q and a nonzero constant c such that $f = e^q$ and $g = ce^{-q}$.

Proof. If condition (a) holds, then the Toeplitz product $T_f T_{\bar{g}}$ is 0. If condition (b) holds, the boundedness of $T_f T_{\bar{g}}$ follows from Lemma (5.2.2).

Next assume that $T = T_f T_{\bar{g}}$ is bounded on F_α^2. Then the Berezin transform $\tilde{T}$ is a bounded function on $\mathbb{C}^n$, where

$$\tilde{T}(z) = \left(T_f T_{\bar{g}} k_z, k_z\right) \quad z \in \mathbb{C}^n.$$

It follows from the integral representation of $T_{\bar{g}}$ and the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$ that $T_{\bar{g}} k_z = \overline{g(z)}\, k_z$. Therefore,

$$\tilde{T}(z) = \overline{g(z)}(f k_z, k_z), \qquad z \in \mathbb{C}^n.$$

Write the inner product above as an integral and apply the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$ one more time. We obtain $\tilde{T}(z) = f(z)\overline{g(z)}$. It follows that $|f(z)\overline{g(z)}| \leq \|T\|$ for all $z \in \mathbb{C}^n$. But fg is entire, so by Liouville theorem, there is a constant c such that $fg = c$.

If $c = 0$, then at least one of f and g must be identically zero, so condition (a) holds.

If $c \neq 0$, then both f and g are nonvanishing. By Lemma (5.2.3), there exists a complex polynomial q, with $\deg(q) \leq 2$, such that $f = e^q$ and $g = ce^{-q}$.

It remains for us to show that $\deg(q) \leq 1$. Let us assume $\deg(q) = 2$, in the hope of reaching a contradition, and write $q = q_2 + q_1$, where q_1 is linear and q_2 is a homogeneous polynomial of degree 2. By the boundedness of $T = T_f T_{\bar{g}}$ on F_α^2, the function

$$T(z, w) = \langle T_f T_{\bar{g}} k_z, k_w \rangle, \qquad z \in \mathbb{C}^n, w \in \mathbb{C}^n,$$

is bounded on $\mathbb{C}^n \times \mathbb{C}^n$. We proceed to show that this is impossible unless $q_2 = 0$.

Again, by the integral representation for Toeplitz operators and the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$, it is easy to obtain that

$$T(z, w) = f(w)\overline{g(z)}e^{-\frac{\alpha}{2}|z|^2 + \alpha w \cdot \bar{z} - \frac{\alpha}{2}|w|^2}.$$

It follows that

$$|T(z,w)| = |f(w)g(z)|e^{-\frac{\alpha}{2}|z-w|^2}$$
for all $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Using the explict form of f and g, we can write
$$|T(z,w)| = \left|c \exp\big(q_2(w) - q_2(z) + q_1(w) - q_1(z)\big)\right|e^{-\frac{\alpha}{2}|z-w|^2}.$$
Since q_1 is linear, it is easy to see that there is a point $a \in \mathbb{C}^n$ such that
$$q_1(w) - q_1(z) = (w - z) \cdot \bar{a}$$
for all z and w.

For the second-degree homogeneous polynomial q_2 we can find a complex matrix $A = A_{n\times n}$, symmetric in the real sense, such that $q_2(z) = \langle Az, z\rangle$, where $\langle\, ,\rangle$ is the real inner product. Fix two points u and v in $\mathbb{C}^n$ such that $\mathrm{Re}\langle Au, v\rangle \neq 0$. This is possible as long as $A \neq 0$. Now let $z = ru$ and $w = ru + v$, where r is any real number. We have
$$q_2(w) - q_2(z) = q_2(z + v) - q_2(z) = \langle A(z + v), z + v\rangle - \langle Az, z\rangle$$
$$= \langle Az, v\rangle + \langle Av, z\rangle + \langle Av, v\rangle = 2r\langle Au, v\rangle + \langle Av, v\rangle.$$
It follows that there exists a positive constant $M = M(u, v)$ such that
$$|T(z,w)| = M|\exp(2r\langle Au, v\rangle)| = M\exp(2rRe\langle Au, v\rangle).$$
Since $\mathrm{Re}\langle Au, v\rangle \neq 0$, this shows that $T(z, w)$ cannot be a bounded function on $\mathbb{C}^n \times \mathbb{C}^n$. This contradition shows that $A = 0$ and the polynomial q must be linear.

As a consequence of the analysis above, we obtain an interesting class of unbounded operators on F_α^2 whose Berezin transforms are bounded.

Corollary (5.2.5)[145]: Suppose $f(z) = e^q$ and $g = e^{-q}$, where q is any second-degree homogeneous polynomial whose coefficients are small enough so that f and g belong to F_α^2. Then the Toeplitz product $T_f T_{\bar{g}}$ is unbounded on F_α^2, but its Berezin transform is bounded.

Proof. By Theorem (5.2.4), the operator $T_f T_{\bar{g}}$ is unbounded. On the other hand, by the proof of Theorem (5.2.4), the Berezin transform of $T = T_f T_{\bar{g}}$ is given by
$$\tilde{T}(z) = f(z)\overline{g(z)}, \qquad z \in \mathbb{C}^n.$$
It follows that $|T(z)| = |f(z)g(z)| = 1$ for all $z \in \mathbb{C}^n$.

Another consequence of the earlier analysis is the following.

Corollary (5.2.6)[145]: If f and g are functions in F_α^2, then the following conditions are equivalent:

(a) $T_f T_{\bar{g}}$ is compact.

(b) $T_f T_{\bar{g}} = 0$.

(c) $f = 0$ or $g = 0$.

Proof. Combining Lemma (5.2.2) and Theorem (5.2.4), we see that whenever $T_f T_{\bar{g}}$ is compact on F_α^2, we must have $f = 0$ or $g = 0$. This clearly gives the desired result.

For any $0 < p \leq \infty$ let F_α^p denote the Fock space consisting of entire functions f such that the function $f(z)e^{-\frac{\alpha}{2}|z|^2}$ belongs to $L^p(\mathbb{C}^n, dv)$. When $0 < p < \infty$, the norm in F_α^p is defined by
$$\|f\|_{p,\alpha} = \left[\left(\frac{p\alpha}{2\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dv(z)\right]^{\frac{1}{p}}.$$
For $p = \infty$, the norm in F_α^∞ is defined by
$$\|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}^n} |f(z)|e^{-\frac{\alpha}{2}|z|^2}.$$
It is easy to check that the normalized reproducing kernel

$$k_a(z) = e^{\alpha z\cdot\bar{a}-\frac{\alpha}{2}|a|^2}$$
is a unit vector in each F_α^p , where $0 < p \le \infty$. Also, it can be shown that the set of functions of the form

$$f(z) = \sum_{k=1}^{N} c_k K(z, a_k) = \sum_{k=1}^{N} c_k e^{\alpha z\cdot\bar{a}_k}$$

is dense in each F_α^p , where $0 < p < \infty$. See [151].

Therefore, if $0 < p < \infty$ and f satisfies Condition (G), we can consider the action of the Toeplitz operator T_f on F_α^p . Also, if $f \in F_\alpha^p$, then it satisfies the pointwise estimate $|f(z)| \le Ce^{\frac{\alpha}{2}|z|^2}$, which implies that f satisfies Condition (G).

When $1 < p < \infty$ and $1/p + 1/q = 1$, the dual space of F_α^p can be identified with F_α^q under the integral pairing

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\, d\lambda_\alpha(z).$$

When $0 < p \le 1$, the dual space of F_α^p can be identified with F_α^∞ under the same integral pairing above. See [73], [151].

Thus for functions f and g in F_α^p , if the Toeplitz product $T = T_f T_{\bar{g}}$ is bounded on F_α^p, we can still consider the function

$$T(z, w) = \langle T_f T_{\bar{g}} k_z, k_w\rangle_\alpha$$

on $\mathbb{C}^n \times \mathbb{C}^n$. Exactly the same arguments will yield the following result.

Theorem (5.2.7)[145]: Suppose $0 < p < \infty$. If f and g are functions in F_α^p, not identically zero, then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on F_α^p if and only if $f = e^q$ and $g = ce^{-q}$, where c is a nonzero complex contant and q is a complex linear polynomial.

We extend the results here to more general Fock-type spaces. In particular, generalization to the Fock-Sobolev spaces studied in [146], [147] should be possible.

We take a second look at the original Hardy space setting. More specifically, if f and g are functions in the Hardy space H^2 (of the unit disk, for example), the the boundedness of the Toeplitz product $T_f T_{\bar{g}}$ on H^2 implies that the product function fg is in H^∞. Is it possible to derive more detailed information about f and g, say in terms of inner and outer functions? A more explicit condition on f and g (as opposed to the condition $\widetilde{|f|^{2+\varepsilon}}\widetilde{|g|^{2+\varepsilon}} \in L^\infty$) would certainly be more desirable.

We hope that will generate some further interest in this subject.

Section (5.3): Sarason's Toeplitz Product Problem

For $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. The Hardy space H^2 consists of functions $f \in L^2(\mathbb{T})$ such that its Fourier coefficients satisfy $\hat{f}_n = 0$ for all $n < 0$. Given a function $\varphi \in L^2(\mathbb{T})$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is densely defined by $T_\varphi f = P(\varphi f)$, where $P : L^2(\mathbb{T}) \to H^2$ is the Riesz-Szego projection.

The original problem that Sarason proposed in [53] was this: characterize the pairs of outer functions u and v in H^2 such that the operator $T_u T_v$ is bounded on H^2. Inner factors can easily be disposed of, so it was only necessary to consider outer functions in the Hardy space case. It was further observed in [53] that a necessary condition for the boundedness of $T_u T_v$ on H^2 is that

$$\sup_{w\in\mathbb{D}} P_w(|u|^2)P_w(|v|^2) < \infty,$$

where $P_w(f)$ means the Poisson transform of f at $w \in \mathbb{D}$. In fact, the arguments in [53] show that

$$\sup_{w\in\mathbb{D}} P_w(|u|^2)P_w(|v|^2) \le 4\|T_uT_v\|^2 . \qquad (35)$$

For A^2 denote the Bergman space consisting of analytic functions in $L^2(\mathbb{D}, dA)$, where dA is ordinary area measure on the unit disk. If $P : L^2(\mathbb{D}, dA) \to A^2$ is the Bergman projection, then Toeplitz operators T_φ on A^2 are defined by $T_\varphi f = P(\varphi f)$. Sarason also posed a similar problem in [53] for the Bergman space: characterize functions u and v in A^2 such that the Toeplitz product T_uT_v is bounded on A^2. It was shown in [42] that

$$\sup_{w\in\mathbb{D}} \widetilde{|u|^2}(w)\widetilde{|v|^2}(w) \le 16\|T_uT_{\bar{v}}\|^2 \qquad (36)$$

for all functions u and v in the Bergman space A^2, where $\widetilde{f}(w)$ is the socalled Berezin transform of f at w. This provides a necessary condition for the boundedness of T_uT_v on A^2 in terms of the Berezin transform.

The Berezin transform is well defined in many other different contexts. In particular, the classical Poisson transform is the Berezin transform of the Hardy space H^2. So the estimates in (35) and (36) are in exactly the same spirit. Sarason stated in [53] that "it is tempting to conjecture that" T_uT_v is bounded on H^2 or A^2 if and only if $\widetilde{|u|^2}(w)\widetilde{|v|^2}(w)$ is a bounded function on $\mathbb{D}$. It has by now become standard to call this "Sarason's conjecture for Toeplitz products".

It turns out that Sarason's conjecture is false for both the Hardy space and the Bergman space of the unit disk, and the conjecture fails in a big way. See [153], [160] for counter-examples. In these cases, Sarason's problem is naturally connected to certain two-weight norm inequalities in harmonic analysis, and counter-examples for Sarason's conjecture were constructed by means of the dyadic model approach in harmonic analysis.

Another setting where Toeplitz operators have been widely studied is the Fock space. More specifically, we let $\mathcal{F}^2$ be the space of all entire functions f on $\mathbb{C}$ that are square-integrable with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dA(z).$$

The function

$$K(z,w) = e^{z\overline{w}}, z, w \in \mathbb{C},$$

is the reproducing kernel of $\mathcal{F}^2$ and the orthogonal projection P from $L^2(\mathbb{C}, d\lambda)$ onto $\mathcal{F}^2$ is the integral operator defined by

$$P f(z) = \int_{\mathbb{C}} K(z,w)f(w)d\lambda(w), \qquad z \in \mathbb{C}.$$

If φ is in $L^2(\mathbb{C}, d\lambda)$ such that the function $z \to \varphi(z)K(z,w)$ belongs to $L^1(\mathbb{C}, d\lambda)$ for any $w \in \mathbb{C}$, we can define the Toeplitz operator T_φ with symbol φ by $T_\varphi f = P(\varphi f)$, or

$$T\varphi f(z) = \int_{\mathbb{C}} K(z,w)\varphi(w)f(w)d\lambda(w), \qquad z \in \mathbb{C},$$

When

$$f(w) = \sum_{k=1}^{N} c_k K(w, c_k)$$

is a finite linear combination of kernel functions. Since the set of all finite linear combinations of kernel functions is dense in $\mathcal{F}^2$, the operator T_φ is densely defined and $T_\varphi f$ is an entire function. See [152] for basic information about the Fock space and Toeplitz operators on it.

In [146], Cho, Park and Zhu solved Sarason's problem for the Fock space. More specifically, they obtained the following simple characterization for $T_u T_v$ to be bounded on $\mathcal{F}^2$: if u and v are functions in $\mathcal{F}^2$, not identically zero, then $T_u T_{\bar{v}}$ is bounded on $\mathcal{F}^2$ if and only if $u = e^q$ and $v = ce^{-q}$, where c is a nonzero constant and q is a complex linear polynomial. As a consequence of this, it can be shown that Sarason's conjecture is actually true for Toeplitz products on $\mathcal{F}^2$.

We consider the weighted Fock space $\mathcal{F}_m^2$, consisting of all entire functions in $L^2(\mathbb{C}, d\lambda_m)$, where $d\lambda_m$ are the generalized Gaussian measure defined by

$$d\lambda_m(z) = e^{-|z|^{2m}}\, dA(z), \qquad m \geq 1.$$

Toeplitz operators on $\mathcal{F}_m^2$ are defined exactly the same as the cases above, using the orthogonal projection $P : L^2(\mathbb{C}, d\lambda_m) \to \mathcal{F}_m^2$.

We will solve Sarason's problem and prove Sarason's conjecture for the weighted Fock spaces $\mathcal{F}_m^2$. The main result can be stated as follows.

Main Theorem (5.3.1)[152]: Let u and v be in $\mathcal{F}_m^2$, not identically zero. The following conditions are equivalent:

(i) The product $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$.

(ii) There exist a polynomial g of degree at most m and a nonzero complex constant c such that $u(z) = e^{g(z)}\, and\, v(z) = ce^{-g(z)}$.

(iii) The product $\widetilde{|u|}^2(z)\widetilde{|v|}^2(z)$ is a bounded function on $\mathbb{C}$.

Furthermore, in the affirmative case, we have the following estimate of the norm:

$$\|T\| \leq C_1 e^{C_2\|g\|_{H^2}^2},$$

where $\|g\|_{H^2}$ is the norm in the Hardy space of the unit disc, and C_1 and C_2 are positive constants independent of g.

Let us mention that [158] contains partial results related to Sarason's conjecture on the Fock space. The arguments in [146] depend on the explicit form of the reproducing kernel and the Weyl operators induced by translations of the complex plane. Both of these are no longer available for the spaces $\mathcal{F}_m^2$: there is no simple formula for the reproducing kernel of $\mathcal{F}_m^2$ and the translations on the complex plane do not induce nice operators on $\mathcal{F}_m^2$. Therefore, we need to develop new techniques to tackle the problem.

We recall some properties of the Hilbert space $\mathcal{F}_m^2$. It was shown in [84] that the reproducing kernel of $\mathcal{F}_m^2$ is given by the formula

$$K_m(z,w) = \frac{m}{\pi} \sum_{k=0}^{+\infty} \frac{(z\bar{w})^k}{\Gamma\left(\frac{k+1}{m}\right)}. \tag{37}$$

In terms of the Mittag-Leffler function

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \qquad \gamma, \beta > 0,$$

we can also write

$$K_m(z,w) = \frac{m}{\pi} E_{\frac{1}{m},\frac{1}{m}}(z\bar{w}). \tag{38}$$

Recall that the asymptotics of the Mittag-Leffler function $E_{\frac{1}{m},\frac{1}{m}}(z)$ as $|z| \to +\infty$ are given by

$$E_{\frac{1}{m},\frac{1}{m}}(z) = \begin{cases} mz^{m-1}e^{z^m}\left(1 + o(1)\right), & |\arg z| \le \frac{\pi}{2m}, \\ O\left(\frac{1}{z}\right), & \frac{\pi}{2m} < |\arg z| \le \pi \end{cases} \quad (39)$$

for $m > \frac{1}{2}$, and by

$$E_{\frac{1}{m},\frac{1}{m}}(z) = m \sum_{j=-N}^{N} z^{m-1} e^{2\pi i j(m-1)} e^{z^m e^{2\pi i j m}} + O\left(\frac{1}{z}\right), \quad -\pi < \arg z \le \pi,$$

for $0 < m \le \frac{1}{2}$, where N is the integer satisfying $N < \frac{1}{2m} \le N + 1$ and the powers z^{m-1} and z^m are the principal branches. See, for example, Bateman and Erdelyi [155], vol. III, 18.1, formulas (55)–(56).

The asymptotic estimates of the Mittag-Leffler function $E_{\frac{1}{m},\frac{1}{m}}$ provide the following estimates for the reproducing kernel $K_m(z,w)$, which is a consequence of the results in [84] and Lemma (5.3.4) in [130].

Lemma (5.3.2)[152]: For arbitrary points $x, r \in (0,+\infty)$ and $\theta \in (-\pi,\pi)$ we have

$$\left|K_m\left(x, re^{i\theta}\right)\right| \lesssim \begin{cases} (xr)^{m-1} e^{(xr)^m \cos(m\theta)} & |\theta| \le \frac{\pi}{2m} \\ O\left(\frac{1}{xr}\right), & \frac{\pi}{2m} \le |\theta| < \pi \end{cases}$$

as $xr \to +\infty$. Moreover, there is a constant $c > 0$ such that for all $|\theta| \le c\theta_0(xr)$ we have

$$|K_m(x, re^{i\theta})| \gtrsim (xr)^{m-1} e^{(xr)m}$$

as $xr \to +\infty$, where $\theta_0(r) = r^{-\frac{m}{2}}/m$.

On several occasions later on we will need to know the maximum order of a function in $\mathcal{F}_m^2$. For example, if we have a non-vanishing function f in $\mathcal{F}_m^2$ and if we know that the order of f is finite, then we can write $f = e^q$ with q being a polynomial. The following estimate allows us to do this.

Lemma (5.3.3)[152]: If $f \in \mathcal{F}_m^2$, there is a constant $C > 0$ such that

$$|f(z)| \le C\,|z|^{m-1} e^{\frac{1}{2}|z|^{2m}}, \quad z \in \mathbb{C}.$$

Consequently, the order of every function in $\mathcal{F}_m^2$ is at most $2m$.

Proof. By the reproducing property and Cauchy-Schwartz inequality, we have

$$|f(z)| = \left|\int_{\mathbb{C}} f(w)K_m(z,w)d\lambda_m(w)\right| \le \|f\|\, K_m(z,z)^{½}$$

for all $f \in \mathcal{F}_m^2$ and all $z \in \mathbb{C}$. The desired estimate then follows from Lemma (5.3.2). See [156].

Another consequence of the above lemma is that, for any function $u \in \mathcal{F}_m^2$, the Toeplitz operators T_u and $T_{\bar{u}}$ are both densely defined on $\mathcal{F}_m^2$.

We prove the equivalence of conditions (35) and (36) in the Main Theorem (5.3.1) stated, which provides a simple and complete solution to Sarason's problem for Toeplitz products on the Fock space $\mathcal{F}_m^2$. We break the proof into several lemmas.

Lemma (5.3.4)[152]: Suppose that u and v are functions in $\mathcal{F}_m^2$, each not identically zero, and that the operator $T = T_uT_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$. Then there exists a polynomial g of degree at most m and a nonzero complex constant c such that $u(z) = e^{g(z)}$ and $v(z) = ce^{-g(z)}$.

Proof. If $T = T_u T_v$ is bounded on $\mathcal{F}_m^2$, then the Berezin transform T is bounded, where

$$T(z) = \langle T_u T_{\bar{v}} k_z, k_z \rangle, \qquad z \in \mathbb{C}.$$

By the reproducing property of the kernel functions, it is easy to see that

$$T(z) = u(z)\overline{v(z)}.$$

Since each k_z is a unit vector, it follows from the Cauchy-Schwarz inequality that

$$|u(z)v(z)| = |\tilde{T}(z)| \leq \|T\|$$

for all $z \in \mathbb{C}$. This together with Liouville's theorem shows that there exist a constant c such that $uv = c$. Since neither u nor v is identically zero, we have $c \neq 0$. Consequently, both u and v are non-vanishing.

Recall from Lemma (5.3.3) that the order of functions in $\mathcal{F}_m^2$ is at most $2m$, so there is a polynomial of degree d,

$$g(z) = \sum_{k=0}^{d} a_k z^k, \qquad d \leq [2m],$$

such that $u = e^g$ and $v = ce^{-g}$. It remains to show that $d \leq m$.

Since T is bounded on $\mathcal{F}_m^2$, the function

$$F(z,w) = \frac{\langle T\left(K_m(\cdot,w)\right), K_m(\cdot,z)\rangle}{\sqrt{K_m(z,z)}\sqrt{K_m(w,w)}}$$

must be bounded on $\mathbb{C}^2$. On general reproducing Hilbert spaces, we always have

$$\langle T_u T_{\bar{v}} K_w, K_z \rangle = \langle T_{\bar{v}} K_w, T_u K_z \rangle = \langle \bar{v}(w) K_w, u(z) K_z \rangle = u(z)\bar{v}(w) K(z,w).$$

It follows that

$$F(z,w) = \bar{c} e^{g(z)-\overline{g(w)}} \frac{K_m(z,w)}{\sqrt{K_m(z,z)}\sqrt{K_m(w,w)}}.$$

From Lemma (5.3.2) we deduce that

$$|F(z,w)| \gtrsim e^{Re\left(g(z)-g(w)\right)} e^{-\frac{1}{2}(|z|^m - |w|^m)^2} \tag{40}$$

for all $|\arg(z\bar{w})| \leq c\theta_0(|zw|)$ as $|zw|$ grows to infinity. Choose $x > 0$ sufficiently large and set

$$z(x) = xe^{i\frac{\pi}{2d}} e^{-i\frac{\arg(a_d)}{d}},$$

and

$$w(x) = xe^{i\frac{\pi}{2d}} e^{-i\frac{\arg(a_d)+\frac{c}{2mx^m}}{d}}.$$

Since

$$\theta_0(|z(x)w(x)|) = \frac{1}{mx^m},$$

we can apply (40) to $z(x)$ and $w(x)$ to get

$$e^{Re\left(g(z(x))-g(w(x))\right)} \lesssim \sup_{(z,w)\in\mathbb{C}^2} |F(z,w)| < \infty \tag{41}$$

as x grows to infinity. On the other hand, a few computations show that

$$Re\left(g(z(x)) - g(w(x))\right) = \sum_{j=0}^{d} x^j \, Re\left(a_j e^{ij\frac{\pi}{2d} - i\frac{j}{d} arg[a_d)} \left(1 - e^{-i\frac{c_j}{2mdx^m}}\right)\right)$$

$$\left(1 - e^{-i\frac{c_j}{2mdx^m}}\right)$$

$$= |a_d|\, x^d \sin\left(\frac{c}{2mx^m}\right) + g_{d-1}(x),$$

Where

$$g_{d-1}(x) = \sum_{j=0}^{d-1} x^j Re\left(a_j e^{i\frac{j\pi}{2d} - i\frac{j}{d}\arg(a_d)} \left(1 - e - i\frac{c_j}{2mdx^m}\right)\right)$$

$$= -\sum_{j=0}^{d-1} |a_j| x^j \sin\left(\frac{j\pi}{2d} + arg\, a_j - \frac{j}{d}\, arg\,(a_d)\right) \sin\frac{c_j}{2mdx^m}$$

$$+ \sum_{j=0}^{d-1} |a_j|\, x^j \cos\left[\frac{j\pi}{2d} + arg\, a_j - \frac{j}{d}\, arg\,(a_d)\right]\left[1 - cos\frac{c_j}{2mdx^m}\right]$$

$$\lesssim x^{d-1-m}.$$

Therefore, there exist some $x_0 > 0$ and $\delta > 0$ such that

$$Re\left(g(z(x)) - g(w(x))\right) \geq \frac{\delta |a_d| x^d}{x^m}$$

for all $x \geq x_0$. Since $a_d \neq 0$, it follows from (41) that $d \leq m$.

On several occasions later on we will need to estimate the integral

$$I(a) = \int_0^\infty e^{-\frac{1}{2}r^{2m} + ar^d}\, r^N\, dr,$$

where $m > 0, 0 \leq d \leq m, N > -1,$ and $a \geq 0.$

First, suppose $a > 1$. By various changes of variables, we have

$$I(a) = \int_0^1 e^{-\frac{1}{2}r^{2m} + ar^d}\, r^N\, dr + \int_1^\infty e^{-\frac{1}{2}r^{2m} + ar^d}\, r^N\, dr$$

$$\leq e^a \int_0^1 r^N dr + \int_1^\infty e^{-\frac{1}{2}r^{2m} + ar^m}\, r^N\, dr$$

$$= \frac{e^a}{N+1} + e^{\frac{a^2}{2}} \int_1^\infty e^{-\frac{1}{2}(r^m - a)^2} r^N\, dr$$

$$= \frac{e^a}{N+1} + \frac{e^{\frac{a^2}{2}}}{m} \int_1^\infty e\hat{}(-\frac{1}{2}(t-a)^2\, t^{\frac{N+1}{m} - 1}\, dt.$$

If $\frac{N+1}{m} - 1 \leq 0$, then

$$I(a) \leq \frac{e^a}{N+1} + \frac{\sqrt{2\pi}}{m} e^{\frac{a^2}{2}} \leq \left(\frac{\sqrt{e}}{N+1} + \frac{\sqrt{2\pi}}{m}\right) e^{\frac{a^2}{2}}.$$

Otherwise, we have $\frac{N+1}{m} - 1 > 0$. Using the fact that $u \longmapsto u^{\frac{N+1}{m} - 1}$ is increasing, we see that

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m} - 1} dt \leq \left(\frac{3a}{2}\right)^{\frac{N+1}{m} - 1} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\frac{t^2}{2}}\, dt \leq \sqrt{2\pi}\, \left(\frac{3a}{2}\right)^{\frac{N+1}{m} - 1}.$$

For the same reason we also have

$$\int_{\frac{a}{2}}^{+\infty} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m} - 1}\, dt \leq \int_{\frac{a}{2}}^{+\infty} e^{-\frac{t^2}{2}} (3t)^{\frac{N+1}{m} - 1} dt$$

$$\leq 3^{\frac{N+1}{m}-1}\int_0^{+\infty} t^{\frac{N+1}{m}-1}e^{-\frac{t^2}{2}}\,dt$$
$$= \frac{\sqrt{2}}{2}\left(3\sqrt{2}\right)^{\frac{N+1}{m}-1}\int_0^{+\infty} u^{\frac{N+1}{m}-1}e^{-u}\,dt$$
$$= \frac{\sqrt{2}}{2}\left(3\sqrt{2}\right)^{\frac{N+1}{m}-1}\Gamma\left(\frac{N+1}{2m}\right).$$

In the case when $1 - a < -a\,2$ (or equivalently $a > 2$),

$$\int_{1-a}^{-\frac{a}{2}} e^{-\frac{t^2}{2}}(t+a)^{\frac{N+1}{m}-1}\,dt \leq \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1}\int_{1-a}^{-\frac{a}{2}} e^{-\frac{t^2}{2}}\,dt$$
$$\leq \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1}\int_{1-a}^{-\frac{a}{2}} e^{\frac{at}{4}}dt \leq \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1}\frac{4}{a}e^{-\frac{a^2}{8}} \leq 2\left(\frac{a}{2}\right)^{\frac{N+1}{m}-1}.$$

It follows that there exists a constant $C = C(m,N) > 0$ such that

$$\int_1^{\infty} e^{-\frac{1}{2}(t-a)^2}t^{\frac{N+1}{m}-1}dt = \int_{1-a}^{\infty} e^{-\frac{t^2}{2}}(t+a)^{\frac{N+1}{m}-1}dt \leq C\,(1+a)^{\frac{N+1}{m}-1}$$

for $\frac{N+1}{m} - 1 > 0$. It is then easy to find another positive constant $C = C(m,N)$, independent of a, such that

$$I(a) \leq C\,(1+a)^{\frac{N+1}{m}-1}e^{\frac{a^2}{2}}$$

for all $a \geq 1$ and $\frac{N+1}{m} - 1 > 0$. Therefore,

$$\int_0^{\infty} e^{-\frac{1}{2}r^{2m}+ar^d}\,r^N\,dr \leq C\,(1+a)^{\max\left(0,\frac{N+1}{m}-1\right)}\,e^{\frac{a^2}{2}} \qquad (42)$$

for all $a \geq 1$. Since $I(a)$ is increasing in a, the estimate above holds for $0 \leq a \leq 1$ as well.

Lemma (5.3.5)[152]: For any $m > 0, \delta > 0, R \geq 1, N > -1$, and $p \geq 0$, we can find a constant $C > 0$ (depending on R, δ, p, N, m but not on a, d, x) such that

$$x^{N+1-p}\int_{\frac{R}{x^2}}^{+\infty} e^{-\frac{x^{2m}}{2}(1+r^{2m})+ax^d(1+\delta r^d)}\,r^N\,dr \leq C\,(1+a)^{\max\left(0,\frac{N+p+1}{m}-1\right)}e^{\frac{1+\delta^2}{2}a^2}$$

and

$$x^m\int_{\frac{R}{x^2}}^{+\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2+ax^d(1-r^d)}r^{\frac{m}{2}}\,dr \leq C(1+a)e^{\frac{a^2}{2}}$$

for all $x > 0, a > 0$, and $0 \leq d \leq m$.

Proof. Let $I = I(m,N,p,R,x,a,d)$ denote the first integral that we are trying to estimate. If $x \geq 1$, we have

$$I \;=\; x^{N+1-p}\, e^{-\frac{x^{2m}}{2}+ax^d} \int_{\frac{R}{x^2}}^{\infty} e^{-\frac{(xr)^{2m}}{2}+a\delta(xr)^d}\, r^N\, dr$$
$$\le\; x^{-p}\, e^{-\frac{x^{2m}}{2}+ax^m} \int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d}\, r^N dr$$
$$\le\; e^{-\frac{1}{2}(x^m-a)^2+\frac{a^2}{2}}\, \frac{\int_{\frac{R}{x}}^{\infty} r^p}{R^p}\, e^{-\frac{1}{2}r^{2m}+a\delta r^d}\, r^N\, dr$$
$$\le \frac{e^{\frac{a^2}{2}}}{R^p} \int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d}\, r^{N+p} dr.$$

The desired result then follows from (42).

If $0 < x < 1$, we have

$$I \;=\; x^{N+1-p}\, e^{-\frac{x^{2m}}{2}+ax^d} \int_{\frac{R}{x^2}}^{\infty} e^{-\frac{(xr)^{2m}}{2}+a\delta(xr)^d}\, r^N\, dr \le\; e^a x^{-p} \int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d}\, r^N dr$$
$$\le \frac{e^{\frac{a^2}{2}+1}}{R^p} \int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d}\, r^{N+p} dr.$$

The desired estimate follows from (42) again.

To prove the second part of the lemma, denote by $J = J(m,d,R,x,a)$ the second integral that we are trying to estimate. Then it is clear from a change of variables that for $0 < x < 1$ we have

$$J(m,d,R,x,a) \;=\; x^{\frac{m}{2}-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m-r^m)^2+a\left(x^d-r^d\right)} r^{\frac{m}{2}}\, dr$$
$$\le \frac{e^a}{R}\, x^{\frac{m}{2}} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}\left(x^{2m}-2(xr)^m+r^{2m}\right)} r^{\frac{m}{2}+1}\, dr$$
$$\le \frac{e^a}{R} \int_0^{+\infty} e^{-\frac{r^{2m}}{2}+r^m}\, r^{\frac{m}{2}+1}\, dr = \; Ce^a$$
$$\le\; C'(1\;+\;a)e^{\frac{a^2}{2}},$$

where the constants C and C only depend on R and m.

Next assume that $x \;\ge\; 1$. In case $R \;\le\; x^2$ we write $J \;=\; J_1 \;+\; J_2$, where

$$J_1 \;=\; J_1(m,d,R,x,a) \;=\; x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2+ax^d(1-r^d)}\, r^{\frac{m}{2}}\, dr,$$

And

$$J_2 \;=\; J_2(m,d,R,x,a) \;=\; x^m \int_1^{\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2+ax^d(1-r^d)}\, r^{\frac{m}{2}}\, dr.$$

Otherwise we just use $J \;\le\; J_2$. So it suffices to estimate the two integrals above.

To handle $J_1(m,d,R,x,a)$, we fix $\varepsilon > 0$ and consider two cases. In the case $x^m \le a(1\;+\;\varepsilon)$, we have

$$J_1(m,d,R,x,a) \;\le\; x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2+ax^d\left(1-r^d\right)}\, r^{\frac{m}{2}}\, dr$$

$$\leq a(1+\varepsilon)e^{\frac{a^2}{2}}\int_{\frac{R}{x^2}}^{1} e^{-\frac{1}{2}(x^m(1-r^m)-a)^2} r^{\frac{m}{2}}\, dr \leq a(1+\varepsilon)e^{\frac{a^2}{2}}.$$

When $x^m \geq a(1+\varepsilon)$, we set $y = x^m$ and $\tau = (y-a)/2$. Then we have

$$\tau \geq \frac{\varepsilon}{2(1+\varepsilon)}\, y \to +\infty$$

as $y \to +\infty$. By successive changes of variables we see that

$$J_1(m,d,R,x,a) \leq x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2 + ax^m(1-r^m)} r^{\frac{m}{2}}\, dr$$

$$= \frac{y}{m}\int_0^{1-\frac{R^m}{y^2}} (1-r)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{y^2r^2}{2}+ayr}\, dr$$

$$= \frac{1}{m}\int_0^{y-\frac{R^m}{y}} \left(1-\frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}+ar}\, dr$$

$$= \frac{e^{\frac{a^2}{2}}}{m}\int_{-a}^{y-a-\frac{R^m}{y}} \left(1-\frac{a}{y}-\frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr.$$

This shows that for $1 \leq m \leq 2$ we have

$$J_1 \leq \frac{e^{\frac{a^2}{2}}}{m}\int_{-a}^{y-a-\frac{R^m}{y}} e^{-\frac{r^2}{2}}\, dr \leq \frac{\sqrt{2\pi}}{m}\, e^{\frac{a^2}{2}}.$$

Thus we suppose that $m > 2$. Then

$$\int_{-\tau}^{\tau}\left(1-\frac{a}{y}-\frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr \leq \left(1-\frac{a}{y}-\frac{\tau}{y}\right)^{\frac{1}{m}-\frac{1}{2}}\int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr$$

$$= \left(\frac{\tau}{2y}\right)^{\frac{1}{m}-\frac{1}{2}}\int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr \leq \sqrt{2\pi}\left(\frac{\varepsilon}{4(1+\varepsilon)}\right)^{\frac{1}{m}-\frac{1}{2}}.$$

Moreover, in case $-a < -\tau$, we have

$$\int_{-a}^{-\tau}\left(1-\frac{a}{y}-\frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dv \leq \left(1-\frac{a}{y}+\frac{\tau}{y}\right)^{\frac{1}{m}-\frac{1}{2}}\int_{-a}^{-\tau} e^{-\frac{\tau|r|}{2}}\, dr$$

$$\leq 2\left(\frac{3\varepsilon}{2(1+\varepsilon)}\right)^{\frac{1}{m}-\frac{1}{2}}\frac{e^{-\frac{\tau^2}{2}}}{\tau}$$

$$\leq 4\left(\frac{3}{2}\right)^{\frac{1}{m}-\frac{1}{2}}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{1}{m}-\frac{3}{2}} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.$$

Similarly, in case $y - a - \frac{R^m}{y} \geq \tau$, we have

$$\int_{\tau}^{y-a-\frac{R^m}{y}}\left[1-\frac{a}{y}-\frac{r}{y}\right]^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr \leq \left[\frac{R^m}{y^2}\right]^{\frac{1}{m}-\frac{1}{2}}\int_{\tau}^{y-a-\frac{R^m}{y}} e^{-\frac{\tau r}{2}}\, dr$$

$$\leq 2R^{1-\frac{m}{2}}\left[\frac{\varepsilon}{2(1+\varepsilon)}\right]^{\frac{2}{m}-1}\tau^{-\frac{2}{m}} e^{-\frac{\tau^2}{2}}\left(since\ \tau \geq \frac{\varepsilon}{2(1+\varepsilon)}\right)$$

$$\leq 4R^{1-\frac{m}{2}}\frac{1+\varepsilon}{\varepsilon} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.$$

The last three estimates yield

$$J_1 \leq C(1 + a)e^{\frac{a^2}{2}}$$

for some C > 0 that is independent of x and a.

To establish the estimate for J_2, we perform a change of variables to obtain

$$J_2 \leq x^m \int_1^{+\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2} r^{\frac{m}{2}} dr = \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}} \left(\frac{r}{x^m} + 1\right)^{\frac{1}{m}-\frac{1}{2}} dr.$$

If $m \geq 2$, we have

$$J_2 \leq \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}} dr,$$

and if $1 \leq m < 2$, we have

$$J_2 \leq \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}}(r + 1)^{\frac{1}{m}-\frac{1}{2}} dr.$$

Therefore, $J_2 \leq C$ for some $C > 0$ that is independent of x and a. This completes the proof of the lemma.

In the proof of the Main Theorem (5.3.1), we will have to estimate the following two integrals:

$$I(x,r) = \int_{|\theta|\leq\frac{\pi}{2m}} e^{-(xr)^m+2ar^d\sin^2\left(\frac{\theta d}{2}\right)}\left|K_m\left(x,re^{i\theta}\right)\right| d\theta,$$

and

$$J(x,r) = \int_{|\theta|\geq\frac{\pi}{2m}} e^{-(xr)^m+a\left(x^d+r^d\right)} \left|K_m\left(x,re^{i\theta}\right)\right| d\theta,$$

where $x, r, a \in (0,+\infty)$ and $0 \leq d \leq m$.

Lemma (5.3.6)[152]: For any $m > 0$ there exist positive constants $C = C(m)$ and $R = R(m)$ such that

$$I(x,r) \leq C(xr)^{m-1} \int_0^1 e^{-\left((xr)^m-ar^d\right)t^2} dt$$

And

$$J(x,r) \leq \frac{Ce^{-(xr)^m+a\left(x^d+r^d\right)}}{xr}$$

for all $a > 0, 0 \leq d \leq m$, and $x > 0$ with $xr > R$.

Proof. It follows from Lemma (5.3.2) that there exist positive constants $C = C(m)$ and $R = R(m)$ such that for all $a > 0$ and $xr > R$ we have

$$\begin{aligned}
I(x,r) &\leq C(xr)^{m-1}\int_{|\theta|\leq\frac{\pi}{2m}} e^{-(xr)^m+(xr)^m\cos(m\theta)+2ar^d\sin^2\left(\frac{\theta d}{2}\right)} d\theta \\
&= 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2(xr)^m\sin^2\left(\frac{m\theta}{2}\right)+2ar^d\sin^2\left(\frac{\theta d}{2}\right)} d\theta \\
&\leq 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2(xr)^m\sin^2\left(\frac{m\theta}{2}\right)+2ar^d\sin^2\left(\frac{m\theta}{2}\right)} d\theta \\
&\leq 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2\left((xr)^m-ar^d\right)\sin^2\left(\frac{m\theta}{2}\right)} d\theta
\end{aligned}$$

$$= \frac{4C}{m} (xr)^{m-1} \int_0^{\frac{\sqrt{2}}{2}} e^{-2((xr)^m - ar^d)t^2} \frac{dt}{\sqrt{1-t^2}}$$
$$\leq \frac{4\sqrt{2}C}{m} (xr)^{m-1} \int_0^{\frac{\sqrt{2}}{2}} e^{-2((xr)^m - ar^d)t^2}\, dt$$
$$\leq \frac{4\sqrt{2}C}{m} (xr)^{m-1} \int_0^{1} e^{-((xr)^m - ar^d)t^2}\, dt.$$

The estimate

$$J(x,r) \leq \frac{Ce^{-(xr)^m + a(x^d + r^d)}}{xr}, \qquad xr > R,$$

also follows from Lemma (5.3.2).

Lemma (5.3.7)[152]: For any $m \geq 1$ there exist constants $R = R(m) > 1$ and $C = C(m) > 0$ such that

$$\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} I(x,r) r\, dr \leq C\,(1 + a)^{\frac{1}{m}-1} e^{a^2}$$

and

$$\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2} J(x,r) r\, dr \leq C\,(1 + a)^{\max\left(0, \frac{2}{m}-1\right)} e^{a^2}$$

for all $x > 0, a > 0$, and $0 \leq d \leq m$.

Proof. For convenience we write

$$A_I(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} I(x,r) r,$$

and

$$A_J(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2} J(x,r) r.$$

Let R and C be the constants from Lemma (5.3.6). In the integrands we have $r > R/x$, or $x_r > R$, so according to Lemma (5.3.6),

$$I(x,r) \leq C(xr)^{m-1} \int_0^1 e^{-(xr)^m t^2 + ar^d t^2}\, dt.$$

If, in addition, $x \leq 1$, then

$$I(x,r) \leq Cr^{m-1} e^{ar^d},$$

and

$$A_I(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2}\, e^{ax^d - ar^d} I(x,r) r \leq Cr^m e^a e^{-\frac{1}{2}(x^m - r^m)^2}.$$

It follows that

$$\int_{\frac{R}{x}}^{\infty} A_I(x,r)\, dr \leq Ce^a \int_{\frac{R}{x}}^{\infty} r^m e^{-\frac{1}{2}(x^m - r^m)^2}\, dr$$
$$\leq Ce^a \int_0^{\infty} r^m e^{-\frac{1}{2}x^{2m} + x^m r^m - \frac{1}{2} r^{2m}}\, dr$$
$$\leq Ce^a \int_0^{\infty} r^m e^{r^m - \frac{1}{2} r^{2m}}\, dr$$
$$\leq C\,(1 + a)^{\frac{1}{m}-1} e^{a^2}.$$

for all $a > 0$ and $0 < x \leq 1$.

Similarly, if $x \leq 1$ (and $xr > R$), we deduce from Lemma (5.3.6) and (42) that

$$\int_{\frac{R}{x}}^{\infty} A_J(x,r)\,dr \le \frac{C}{R}\int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}(x^m-r^m)^2}\, e^{-(x^r)^m+ax^d+ar^d}\, r\, dr$$

$$\le \frac{Ce^a}{R}\int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}r^{2m}+ar^d}\, r\, dr$$

$$\le C'(1+a)\max\left(0,\frac{2}{m}-1\right)e^{a^2}.$$

Suppose now that $x \ge 1$ and $rx > R$. By Lemma (5.3.6) again,

$$A_I(x,r) \le Cr(xr)^{m-1}\, e^{-\frac{1}{2}(x^m-r^m)^2+a(x^d-r^d)}\int_0^1 e^{-t^2((xr)^m-ar^d)}\,dt.$$

Fix a sufficiently small $\varepsilon \in (0,1)$. If $(xr)m \ge ar^d(1+\varepsilon)$, then

$$\int_0^1 e^{-t^2((xr)^m-ar^d)}\,dt = \frac{1}{\sqrt{(xr)^m - ar^d}}\int_0^{\sqrt{(xr)^m-ar^d}} e^{-s^2}\,ds$$

$$\le \frac{1}{\sqrt{(xr)^m - ar^d}}\int_0^{\infty} e^{-s^2}\,ds$$

$$= \frac{\sqrt{\pi}}{2}\frac{(xr)^{-\frac{m}{2}}}{\sqrt{1-\left(\frac{ar^d}{(xr)^m}\right)}}$$

$$\le \sqrt{\frac{\pi(1+\varepsilon)}{4\varepsilon}}\,(xr)^{-\frac{m}{2}},$$

so there exists a constant $C = C(m)$ such that

$$A_I(x,r) \le Cr(xr)^{\frac{m}{2}-1}\, e^{-\frac{1}{2}(x^m-r^m)^2+a(x^d-r^d)}.$$

If $(xr)^m \le ar^d(1+\varepsilon)$, we have

$$A_I(x,r)\, a^{\frac{m-1}{m}}\, r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m}+r^{2m})+ax^d}\int_0^1 e^{(1-t^2)((xr)^m-ar^d)}\,dt$$

$$\le a^{\frac{m-1}{m}}\, r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m}+r^{2m})+a(x^d+\varepsilon r^d)}.$$

It follows that

$$\int_{\frac{R}{x}}^{+\infty} A_I(x,r)\,dr \lesssim x^{\frac{m}{2}-1}\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m-r^m)^2+a(x^d-r^d)} r^{\frac{m}{2}}\,dr$$

$$+a^{\frac{m-1}{m}}\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^{2m}+r^{2m})+a(x^d+\varepsilon r^d)}dr.$$

The change of variables $r \longmapsto xr$ along with the second part of Lemma (5.3.5) shows that

$$x^{\frac{m}{2}-1}\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m-r^m)^2+a(x^d-r^d)}r^{\frac{m}{2}}\,dr \le C(1+a)e^{\frac{a^2}{2}}.$$

Similarly, the change of variables $r \longmapsto xr$ together with the first part Lemma (5.3.5) shows that

$$\int_{\frac{R}{x}}^{+\infty} r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m}+r^{2m})+a(x^d+\varepsilon r^d)}\,dr \le C(1+a)^{\frac{d(m-1)+m}{m}} e^{\frac{1+\varepsilon^2}{2}a^2}.$$

We may assume that $\varepsilon < 1$. Then we can find a positive constant C such that

$$a^{\frac{m-1}{m}} \int_{\frac{R}{x}}^{+\infty} r^{\frac{d(m-1)+m}{m}} e^{-\frac{1}{2}(x^{2m}+r^{2m})+a(x^d+\varepsilon r^d)} dr \leq C(1+a)^{\frac{1}{m}-1} e^{a^2}.$$

It follows that

$$\int_{\frac{R}{x}}^{+\infty} A_I(x,r)\, dr \leq C(1+a)^{\frac{1}{m}-1} e^{a^2}$$

for some other positive constant C that is independent of a and x. This proves the first estimate of the lemma.

To establish the second estimate of the lemma, we use Lemma (5.3.6) to get

$$xA_J(x,xr) = x^2 re^{-\frac{x^{2m}}{2}(1-r^m)^2} J(x,xr) \leq Ce^{-\frac{x^{2m}}{2}(1+r^{2m})+ax^d(1+r^d)}.$$

It follows from this and Lemma (5.3.5) that

$$\int_{\frac{R}{x}}^{+\infty} A_J(x,r)\, dr = x \int_{\frac{R}{x^2}}^{+\infty} A_J(x,xr)\, dr \leq C(1+a)^{\max\left(0,\frac{2}{m}-1\right)} e^{a^2}.$$

This completes the proof of the lemma.

Lemma (5.3.8)[152]: If $u(z) = e^{g(z)}$ and $v(z) = e^{-g(z)}$, where g is a polynomial of degree at most m, then the operator $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$.

Proof. To prove the boundedness of $T = T_u T_{\bar{v}}$, we shall use a standard technique known as Schur's test [162]. Since

$$T f(z) = \int_{\mathbb{C}} K_m(z,w) e^{g(z)-\overline{g(w)}} f(w) e^{-|w|^{2m}} dA(w),$$

we have

$$|T f(z)| e^{-\frac{1}{2}|z|^{2m}} \leq \int_{\mathbb{C}} H_g(z,w) |f(w)| e^{-\frac{1}{2}|w|^{2m}} dA(w),$$

where

$$H_g(z,w) := |K_m(z,w)| e^{-\frac{1}{2}(|z|^{2m}+|w|^{2m})+Re\left(g(z)-\overline{g(w)}\right)}.$$

Thus T will be bounded on $\mathcal{F}_m^2$ if the integral operator S_g defined by

$$S_g f(z) = \int_{\mathbb{C}} \left(H_g(z,w) + H_g(w,z)\right) f(w)\, dA(w)$$

is bounded on $L^2(\mathbb{C}, dA)$. Let

$$H_g(z) = \int_{\mathbb{C}} H_g(z,w)\, dA(w), \qquad z \in \mathbb{C}.$$

Since

$$H_{-g}(z) = \int_{\backslash calcolus} \mathbb{C}\, H_g(w,z)\, dA(w),$$

for all $z \in \mathbb{C}$, by Schur's test, the operator S_g is bounded on $L^2(\mathbb{C}, dA)$ if we can find a positive constant C such that

$$H_{g(z)} + H_{-g(z)} \leq C, \qquad z \in \mathbb{C}.$$

By the Cauchy-Schwarz inequality, we have

$$H_{g_1+g_2}(z) \leq \sqrt{H_{2g_1}(z) H_{2g_2}(z)}$$

for all $z \in \mathbb{C}$ and holomorphic polynomials g_1 and g_2. Moreover, if

$$U_\theta(z) = e^{i\theta} z, \qquad z \in \mathbb{C}, \theta \in [-\pi,\pi],$$

Then

$$H_{goU_\theta} = H_g \, o \, U_\theta$$

for all $z \in \mathbb{C}, \theta \in [-\pi, \pi]$, and holomorphic polynomials g. Therefore, we only need prove the theorem for $g(z) = az^d$ with some $a > 0$ and $d \leq m$ and establish that

$$\sup_{x \geq 0} H_g(x) \leq C_1 e^{C_2 a^2}, \tag{43}$$

where C_k are positive constants independent of a and d (but dependent on m). We will see that C_2 can be chosen as any constant greater than 1.

It is also easy to see that we only need to prove (43) for $x \geq 1$. This will allow us to use the inequality $x^d \leq x^m$ for the rest of this proof.

For $R > 0$ sufficiently large (we will specify the requirement on R later) we write

$$H_g(x) = \int_{|xw| \leq R} H_g(x,w) \, dA(w) + \int_{|xw| \geq R} H_g(x,w) \, dA(w).$$

We will show that both integrals are, up to a multiplicative constant, bounded above by $e^{(1+\varepsilon)a^2}$.

By properties of the Mittag-Leffler function, we have

$$|K_m(x,w)| \leq \frac{m}{\pi} E_{\frac{1}{m},\frac{1}{m}}(R) := C_R, \qquad |xw| \leq R.$$

It follows that the integral

$$I_1 = \int_{x|w| \leq R} H_g(x,w) \, dA(w)$$

Satisfies

$$\begin{aligned} I_1 &= \int_{x|w| \leq R} |K_m(z,w)| e^{-\frac{1}{2}(|z|^{2m} + |w|^{2m}) + aRe\,(x^d - w^d)} \, dA(w) \\ &\leq C_R \int_{x|w| \leq R} e^{-\frac{1}{2}(x^{2m} + |w|^{2m}) + aRe(x^d - w^d)} \, dA(w) \\ &\leq C_R e^{-\frac{1}{2}x^{2m} + ax^d} \int_{x|w| \leq R} e^{-\frac{|w|^{2m}}{2} + a|w|^d} \, dA(w) \\ &\leq 2\pi C_R e^{-\frac{1}{2}x^{2m} + ax^m} \int_0^{+\infty} e^{-\frac{r^{2m}}{2} + ar^d} \, r \, dr \\ &\leq 2\pi C_R e^{\frac{a^2}{2}} \int_0^{+\infty} e^{-\frac{r^{2m}}{2} + ar^d} \, r \, dr \\ &\leq C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}, \end{aligned}$$

where the last inequality follows from (42).

We now focus on the integral

$$I_2 = \int_{x|w| \geq R} H_g(x,w) \, dA(w).$$

Observe that for all x, r, and θ we have

$$\begin{aligned} Re\left(x^d - r^d e^{id\theta}\right) = x^d - r^d \cos(d\theta) &= x^d - r^d + r^d \left(1 - \cos(d\theta)\right) \\ &= x^d - r^d + 2r^d \sin^2 \left(\frac{d\theta}{2}\right). \end{aligned}$$

It follows from polar coordinates that

$$I_2 = \int_{\frac{R}{x}}^{+\infty} \int_{-\pi}^{\pi} H_g\left(x, re^{i\theta}\right) r \, d\theta \, dr$$

$$= \int_{\frac{R}{x}}^{+\infty} \int_{-\pi}^{\pi} e^{-\frac{1}{2}(x^{2m}+r^{2m})+a\left(x^d - r^d \cos(d\theta)\right)} \left|K_m\left(x, re^{i\theta}\right)\right| r\, d\theta\, dr$$

$$= \int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a\left(x^d - r^d\right) - (xr)^m}\, r\, dr \int_{-\pi}^{\pi} e^{2ar^d} \sin^2\left(\frac{d\theta}{2}\right) \left|K_m\left(x, re^{i\theta}\right)\right| d\theta$$

$$\le \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2} \left(e^{a\left(x^d - r^d\right)} I(x,r) + J(x,r)\right) r\, dr,$$

Where

$$I(x,r) = \int_{|\theta| \le \frac{\pi}{2m}} e^{-(xr)^m + 2ar^d \sin^2\left(\frac{d\theta}{2}\right)} \left|K_m\left(x, re^{i\theta}\right)\right| d\theta,$$

and

$$J(x,r) = \int_{|\theta| \ge \frac{\pi}{2m}} e^{-(xr)^m + a\left(x^d + r^d\right)} \left|K_m\left(x, re^{i\theta}\right)\right| d\theta.$$

By Lemma (5.3.7), there exists another constant $C > 0$ such that

$$I_2 \le C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}.$$

Therefore,

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_g(z,w)\, dA(w) \le C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}$$

for yet another constant C that is independent of a and d. Similarly, we also have

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_{-g}(z,w)\, dA(w) \le C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}$$

This yields (43) and proves the lemma.

We show that Sarason's conjecture is true for Toeplitz products on the Fock type space $\mathcal{F}_m^2$. We will prove that condition (37) in the Main Theorem (5.3.1) stated in the introduction is equivalent to conditions (35) and (36). Again we will break the proof down into several lemmas.

Lemma (5.3.9)[152]: Suppose u and v are functions in $\mathcal{F}_m^2$, not identically zero, such that the operator $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$. Then the function $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on the complex plane.

Proof. Since $T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$, the operator $(T_u T_{\bar{v}})^* = T_u T_{\bar{v}}$ and the products $(T_u T_{\bar{v}})^* T_u T_{\bar{v}}$ and $(T_v T_{\bar{u}})^* T_v T_{\bar{u}}$ are also bounded on $\mathcal{F}_m^2$. Consequently, their Berezin transforms are all bounded functions on $\mathbb{C}$.

For any $z \in \mathbb{C}$ we let k_z denote the normalized reproducing kernel of $\mathcal{F}_m^2$ at z. Then $\langle (T_u T_{\bar{v}})^* T_u T_{\bar{v}} k_z, k_z \rangle = \langle T_u T_{\bar{v}} k_z, T_u T_{\bar{v}} k_z \rangle = \langle u\overline{v(z)} k_z, u\overline{v(z)} k_z \rangle = |v(z)|^2 \widetilde{|u|^2}(z)$ is bounded on $\mathbb{C}$. Similarly $|u(z)|^2 \widetilde{|v|^2}(z)$ is bounded on $\mathbb{C}$. By the proof of Lemma (5.3.4), the product uv is a non-zero complex constant, say, $u(z)v(z) = C$. It follows that the function

$$\widetilde{|v|^2}(z)\widetilde{|u|^2}(z) = |u(z)|^2 \widetilde{|v|^2}(z) |v(z)|^2 \widetilde{|u|^2}(z) \frac{1}{|C|^2}$$

is bounded as well.

To complete the proof of Sarason's conjecture, we will need to find a lower bound for the function

$$\mathcal{B}(z) = \widetilde{|v|^2}(z) |u(z)|^2,$$

where $u = e^g, v = e^{-g}$, and g is a polynomial of degree d. We write

$$g(z) = a_d z^d + g_{d-1}(z),$$
where
$$a_d = ae^{i\alpha_d}, \qquad a > 0,$$
and
$$g_{d-1}(z) = \sum_{l=0}^{d-1} a_l z^l .$$
In the remainder, we will have to handle several integrals of the form
$$I(x) = \int_J S_x(r) e^{-g_x(r)} \, dr,$$
where S_x and g_x are C^3-functions on the interval J, and the real number x tends to $+\infty$. We will make use of the following variant of the Laplace method (see [130]).

Lemma (5.3.10)[152]: Suppose that

(a) g_x attains its minimum at a point r_x, which tends to $+\infty$ as x tends to $+\infty$, with $c_x = g_x''(r_x) > 0$;

(b) there exists τ_x such that for $|r - r_x| < \tau_x, g_x''(r) = c_x\big(1 + o(1)\big)$ as x ten ds to $+\infty$;

(c) for $|r - r_x| < \tau_x, S_x(r) \sim S_x(r_x)$;

(d) we have
$$\int_J S_x(r) e^{-g_x(r)} \, dr = \big(1 + o(1)\big) \int_{|r-r_x|<\tau_x} S_x(r) e^{-g_x(r)} \, dr$$
Then we have the following estimate
$$I(x) = \left(\sqrt{2\pi} + o(1)\right) [c_x]^{-1/2} S_x(r_x) e^{-g_x(r_x)}, \qquad x \to +\infty. \tag{44}$$
The computations in [130] ensure that, under the assumptions on g_x and S_x, we have
$$\int_{|r-r_x|>\tau_x} S_x(r) e^{-g_x(r)} \, dr \, (c_x\tau_x)^{-1} \int_{|t|>\tau_x} e^{-\frac{1}{3}\tau_x c_x t} \, dt. \tag{45}$$
In particular, if one of the two conditions $c_x\tau_x^2 \to +\infty$ and $c_x\tau_x \to +\infty$ is satisfied, then hypothesis (d) in Lemma (5.3.10) holds.

The study of $\mathcal{B}(z)$ will require some additional technical lemmas.

Lemma (5.3.11)[152]: For $z = xe^{i\phi}$, with $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have
$$\mathcal{B}(z) \gtrsim \int_0^{+\infty} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r)} dr$$
as $x \to +\infty$, where
$$h_x(r) = (r^m - x^m)^2 - 2a(x^d - r^d) + C(r^{d-1} + x^{d-1} + 1), \tag{46}$$
for some positive constant C.

Proof. It is easy to see that
$$\mathcal{B}(z) = \int_{\mathbb{C}} |K_m(w,z)|^2 \, e^{2Re\left(g(z)-g(w)\right)} \, [K_m(z,z)]^{-1} \, e^{-|w|^{2m}} \, dA(w),$$
which, in terms of polar coordinates, can be rewritten as
$$\int_0^{+\infty} \int_\pi^{-\pi} \left|K_m\left(re^{i\theta}, z\right)\right|^2 e^{2Re\left(g(z)-g\left(re^{i\theta}\right)\right)} \, [K_m(x,x)]^{-1} \, e^{-r^{2m}} r \, dr \, d\theta.$$
By Lemma (5.3.2), B(z) is greater than or equal to

$$\int_0^{+\infty}\int_{|\theta-\phi|\le c\theta_0(rx)} \left|K_m(re^{i\theta},z)\right|^2 e^{2Re\left(g(z)-g(re^{i\theta})\right)}\,[K_m(x,x)]^{-1}\,e^{-r2m}\,rdr\,d\theta.$$

This together with Lemma (5.3.2) shows that

$$\mathcal{B}(z)\int_0^{+\infty} r^{2(m-1)}e^{-(r^m-x^m)^2}\,I(r,z)rdr,$$

where

$$I(r,z)\;=\int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left(g(z)-g(re^{i\theta})\right)}\,d\theta.$$

Note that

$$\begin{aligned}I(r,z)\;&=\int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left[ae^{i\alpha_d}\,(x^de^{id\phi}-r^de^{id\theta})\right]+2Re\left[g_{d-1}(z)-g_{d-1}(re^{i\theta})\right]}d\theta\\ &=\int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left[ae^{i(\alpha_d+d\phi)}(x^d-r^de^{id(\theta-\phi)})\right]+2Re\left[g_{d-1}(z)-g_{d-1}(re^{i\theta})\right]}d\theta.\end{aligned}$$

The condition on ϕ yields

$$I(r,z)\;=\int_{|\theta|\le c\theta_0(rx)} e^{2aRe\left[a(x^d-r^de^{id\theta})\right]+2Re\left[g_{d-1}\,(z)-g_{d-1}(re^{(i\theta+\phi)})\right]}\,d\theta.$$

Since

$$g_{d-1}(z)\;-\;g_{d-1}\left(re^{i(\theta+\phi)}\right)\;=\sum_{l=0}^{d-1} a_l\left(x^l\,e^{il\phi}\;-\;r^l\,e^{il(\theta+\phi)}\right),$$

we have

$$Re\left[g_{d-1}(z)-\;g_{d-1}\left(re^{i(\theta+\phi)}\right)\right]\ge\;-C(r^{d-1}\;+\;x^{d-1}\;+\;1)$$

for some constant C. It follows that

$$I(r,z)\ge\;e^{-C(r^{d-1}+x^{d-1}+1)}\int_{|\theta|\le c\theta_0(rx)} e^{2aRe\left[(x^d-r^de^{id\theta})\right]}d\theta.$$

For the integral we have

$$\begin{aligned}J(r,z):=\;\int_{|\theta|\le c\theta_0(rx)} e^{2aRe\left[(x^d-r^de^{id\theta})\right]}d\theta\;&=\int_{|\theta|\le c\theta_0(rx)} e^{2a\left(x^d-r^d\,cos(d\theta)\right)}\,d\theta\\ &=\int_{|\theta|\le c\theta_0(rx)} e^{2a\left(x^d-r^d+(-\,cos(d\theta)+1)r^d\right)}d\theta\\ &=\int_{|\theta|\le c\theta_0(rx)} e^{2a\left(x^d-r^d+2\left(sin\left(\frac{d\theta}{2}\right)^2\right)r^d\right)}d\theta\\ &\ge\;e^{2a(x^d-r^d)}\int_{|\theta|\le c\theta_0(rx)} e^{4|a_d|\,sin\left(\frac{d\theta}{2}\right)^2\,r^d}d\theta\\ &\ge e^{2a(x^d-r^d)}\int_{|\theta|\le c\theta_0(rx)} d\theta\\ &\gtrsim e^{2a(x^d-r^d)}\,(rx)^{-\frac{m}{2}}\,,\end{aligned}$$

which completes the proof of the lemma.

Lemma (5.3.12)[152]: Assume $d=2m$. For $z=xe^{i\phi}$, where $x\;>\;0$ and $e^{i(\alpha_d+d\phi)}=1$, we have

$$\mathcal{B}(z)e^{(1+o(1))\frac{2a}{(1+2a)}x^{2m}}\,,\qquad x\;\to\;+\infty.$$

Proof. For x large enough, the function h_x defined in (46) is convex on some interval $[M_x, +\infty)$ and attains its minimum at some point r_x. In order to bound $\mathcal{B}(z)$ from below, we shall use the modified Laplace method from Lemma (5.3.10). Since

$$h'_x(r) = 2mr^{m-1}(r^m - x^m) + 2adr^{d-1} + C(d-1)r^{d-2}, \quad (47)$$

we have

$$h'_x(r) = 2m(1+2a)r^{2m-1} - 2mx^m r^{m-1} + C(d-1)r^{d-2},$$

and

$$h''_x(r) = 2m(2m-1)(1+2a)r^{2m-2} - 2m(m-1)x^m r^{m-2} + C(d-1)(d-2)r^{d-3}.$$

Writing $h'_x(r_x) = 0$ and letting x tend to $+\infty$, we obtain

$$m(1+2a)(r_x)^{2m-1} \sim mx^m r_x^{m-1},$$

or

$$r_x \sim (1+2a)^{-\frac{1}{m}} x. \quad (48)$$

Thus there exists ρ_x, which tends to 0 as x tends to $+\infty$, such that

$$r_x = (1+2a)^{-\frac{1}{m}} x(1+\rho_x). \quad (49)$$

When x tends to $+\infty$, we have

$$\begin{aligned} h_x(r_x) &\sim (r_x^m - x^m)^2 + 2a(r_x^{2m} - x^{2m}) \\ &\sim (r_x^m - x^m)\left[(r_x^m - x^m) + 2a(r_x^m + x^m)\right] \\ &\sim x^{2m}[(1+2a)^{-1}(1+\rho_x)^m - 1][(1+2a)^{-1}(1+\rho_x)^m - 1 \\ &\quad + 2a\left((1+2a)^{-1}(1+\rho_x)^m + 1\right)] \sim -x^{2m}\frac{2a}{(1+2a)}, \end{aligned}$$

or

$$-h_x(r_x) \sim x^{2m}\frac{2a}{(1+2a)}. \quad (50)$$

In order to estimate $c_x := h''_x(r_x)$, we compute that

$$h''_x(r_x) \sim 2m^2(1+2a)^{-1+\frac{2}{m}} x^{2m-2}.$$

Thus we get

$$c_x \approx x^{2m-2}. \quad (51)$$

For r in a neighborhood of r_x we set $r = (1+\sigma_x)r_x$, where $\sigma_x = \sigma_x(r) \to 0$ as $x \to +\infty$; a little computation shows that

$$h''_x(r) \sim h''_x(r_x)$$

as $x \to +\infty$. Taking $\tau_x = r_x^{1/2}$ and $|r - r_x| < \tau_x$, we have $h''_x(r) = \big(1 + o(1)\big)c_x$, so

$$h_x(r) - h_x(r_x) = \frac{1}{2} c_x(r - r_x)^2\big(1 + o(1)\big).$$

Thus

$$\begin{aligned} \int_{|r-r_x|<\tau_x} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))} dr &= \int_{|t|<\tau_x} e^{-\frac{1}{2}c_x t^2(1+o(1))} dt \\ &\sim \frac{1}{\sqrt{c_x}} \int_{|y|<\tau_x\sqrt{c_x}} e^{-\frac{1}{2}y^2} dy \approx \frac{1}{\sqrt{c_x}}, \end{aligned}$$

because $c_x \tau_x^2 \approx r_x^{2m-1}$ tends to $+\infty$ as x tends to $+\infty$. Finally, the estimates

$$\begin{aligned} \mathcal{B}(z) &\gtrsim \int_{|r-r_x|<\tau_x} (r_x)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r)} dr \\ &= \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r_x)} e^{-[h_x(r)-h_x(r_x)]} dr \end{aligned}$$

$$= e^{-h_x(r_x)} \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))}\, dr$$
$$\sim e^{-h_x(r_x)} r_x^{\frac{3}{2}m-1} x^{-\frac{m}{2}} \int_{|r-r_x|<\tau_x} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))}\, dr$$
$$\approx e^{-h_x(r_x)} r_x^{\frac{3}{2}m-1} x^{-\frac{m}{2}} \frac{1}{\sqrt{c_x}}$$
along with (48), (50), and (51) give the lemma.

Lemma (5.3.13)[152]: Assume $d < 2m$. For $z = xe^{i\phi}$, with $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have

$$\mathcal{B}(z) \gtrsim e^{(1+o(1))\frac{a^2d^2}{m^2}x^{2d-2m} - Cx^{d-1-m}}, \qquad x \to +\infty$$

for some positive constant C.

Proof. Let $\tau_x = o(x)$ be a positive real number that will be specified later. As in the proof of Lemma (5.3.11) we have

$$\mathcal{B}(z) \gtrsim \int_0^{+\infty} r^{2(m-1)} e^{-(r^m - x^m)^2} I(r,z) r\, dr$$
$$\gtrsim \int_{|r-x|\le\tau_x} r^{2(m-1)} e - (r^m - x^m)^2\, I(r,z) r\, dr,$$

where

$$I(r,z) = \int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left(g(z)-g(re^{i\theta})\right)}\, d\theta.$$

There exists $c' > 0$ such that for $|r - x| \le \tau_x$ we have

$$I(r,z) \ge \int_{|\theta-\phi|\le c'\theta_0(x^2)} e^{2Re\left(g(z)-g(re^{i\theta})\right)}\, d\theta$$
$$= \int_{|\theta|\le c'\theta_0(x^2)} e^{2aRe\left(x^d - r^d e^{id\theta}\right) + 2Re\left[g_{d-1}(z) - g_{d-1}(re^{i\theta})\right]} d\theta$$
$$= \int_{|\theta|\le c'\theta_0(x^2)} e^{2aRe\left(x^d - r^d e^{id\theta}\right)} - 2\sum_{l=0}^{d-1} |a_l|\left|x^l - r^l e^{il\theta}\right|\, d\theta.$$

Now for $|r - x| \le \tau_x$, we write $r = (1 + \sigma)x$, where σ tends to 0 as $x \to +\infty$. Thus for $0 \le l \le d-1$ and $|\theta| \le c'\theta_0(x^2)$, we obtain

$$\left|x^l - r^l e^{il\theta}\right|^2 = x^{2l}[1 - 2(1+\sigma)^l \cos(l\theta) + (1+\sigma)^{2l}]$$
$$= x^{2l}\left[1 - 2\left(1 + l\sigma + O(\sigma^2)\right) \cos(l\theta) + 1 + 2l\sigma + O(\sigma^2)\right]$$
$$= x^{2l}\left[2\left(1 - \cos(l\theta)\right)(1 + l\sigma) + O(\sigma^2)\right]$$
$$\lesssim x^{2l}\left[\sin^2\left(\frac{l\theta}{2}\right) + \sigma^2\right] \lesssim x^{2l}[\theta^2 + \sigma^2].$$

Next choosing $|\sigma| \le x^{-m}$, we get

$$\left|x^l - r^l e^{il\theta}\right| \lesssim x^{2l} x^{-2m} \lesssim x^{2(d-1)-2m}$$

or

$$\left|x^l - r^l e^{il\theta}\right| \lesssim x^{d-1-m}.$$

Thus there exists a positive constant C such that for $|r - x| \le \tau_x$ and $|\theta| \le c'\theta_0(x^2)$,

$$2\sum_{l=0}^{d-1} |a_l|\left|x^l - r^l e^{il\theta}\right| \le Cx^{d-1-m}.$$

It follows that

$$I(r,z) \ge \int_{|\theta|\le c'\theta_0(x^2)} e^{2aRe\left(x^d - r^d e^{id\theta}\right)} - Cx^{d-1-m} d\theta$$
$$\gtrsim x^{-m} e^{2aRe\left(x^d - r^d e^{id\theta}\right) - Cx^{d-1-m}}.$$

Then

$$\mathcal{B}(z)\int_{|r-x|\le\tau_x} r^{2m-1} e^{-(r^m - x^m)^2}\, x^{-m} e^{2a\left(x^d - r^d\right) - Cx^{d-1-m}}\, dr$$
$$= x^{-m} e^{-Cx^{d-1-m}} \int_{|r-x|\le\tau} r^{2m-1}\, e^{-h_x(r)} dr,$$

where

$$h_x(r) = (r^m - x^m)^2 - 2a(x^d - r^d).$$

It is easy to see that h_x attains its minimum at r_x with $r_x \sim x$ as $x \to +\infty$. Again we write

$$r_x = x(1 + \rho_x), \tag{52}$$

where ρ_x tends to 0 as $x \to +\infty$. Using the fact that $h_x'(r_x) = 0$, we have

$$2mx^{2m-1}(1 + \rho_x)^{m-1}[(1 + \rho_x)^m - 1] \sim -2adx^{d-1}(1 + \rho_x)^{d-1},$$

and

$$2mx^{2m-1} m\rho_x \sim -2adx^{d-1}.$$

Therefore,

$$\rho_x \sim -\frac{ad}{m^2} x^{d-2m}. \tag{53}$$

Since

$$h_x''(r) = 2m(2m - 1)r^{2m-2} - 2m(m - 1)x^m r^{m-2} + 2ad(d - 1)r^{d-2}$$

and $d < 2m$, we get

$$h_x''(r_x) \sim 2mx^{2m-2}[(2m - 1)(1 + \rho_x)^{2m-2} - (m - 1)(1 + \rho_x)^{m-2}]$$
$$\sim 2m^2 x^{2m-2}.$$

also,

$$h_x(r_x) \sim x^{2m}[(1 + \rho_x)^m - 1]^2 + 2ax^d [(1 + \rho_x)^d - 1]$$
$$+ C(x^{d-1} + r_x^{d-1} + 1) \sim m^2 \rho_x^2 x^{2m} + 2ax^d d\rho_x$$

It follows that

$$c_x \sim 2m^2 x^{2m-2}, \tag{54}$$

and

$$-h_x(r_x) \sim \frac{a^2 d^2}{m^2} x^{2d-2m}. \tag{55}$$

Reasoning as in the proof of Lemma (5.3.12), we arrive at

$$\mathcal{B}(z) \gtrsim\gtrsim x^{-m} e^{-Cx^{d-1-m}}\, e^{-h_x(r_x)}\, x^{2m-1}\frac{1}{\sqrt{c_x}}.$$

The desired estimate then follows from (55), and (54).

Lemma (5.3.14)[152]: Suppose u and v are functions in $\mathcal{F}_m^2$, not identically zero, such that $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on the complex plane. Then there exists a nonzero constant C and a polynomial g of degree at most m such that $u(z) = eg(z)\ and\ v(z) = Ce^{-g(z)}$.

Proof. It is easy to check that for $u \in \mathcal{F}_m^2$ we have

$$u(z) = \int_{\mathbb{C}} u(x)|k_z(x)|^2\, d\lambda_m(x) = \tilde{u}(z).$$

Also, it follows from the Cauchy-Schwarz inequality that $|u(z)|^2 \le \widetilde{|u|^2}(z)$. So if $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on $\mathbb{C}$, then $\mathcal{B}(z)$ and $|u(z)v(z)|^2$ are also bounded. Consequently, uv is a constant, there is a non-zero constant C and a polynomial g such that $u = e^g$ and $v = Ce^{-g}$. The condition $u \in \mathcal{F}_m^2$ implies that the degree d of g is at most $2m$; see Lemma (5.3.3).

We shall consider the case where $u(z) = e^{g(z)}$ and $v(z) = e^{-g(z)}$. We will show that that the boundedness of $\mathcal{B}(z)$ implies $d \le m$. If $2m$ is an integer, Lemma (5.3.12) shows that we must have $d < 2m$.

Thus, in any case ($2m$ being an integer or not), a necessary condition is $d < 2m$. The desired result now follows from Lemma (5.3.13).

We specialize to the case $m = 1$ and make several additional remarks. For convenience we will alter notation somewhat here.

Thus for any $\alpha > 0$ we let $\mathcal{F}_\alpha^2$ denote the Fock space of entire functions f on the complex plane $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2\, d\lambda_\alpha(z) < \infty,$$

where

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z).$$

Toeplitz operators on $\mathcal{F}_\alpha^2$ are defined exactly the same as before using the orthogonal projection $P_\alpha : L^2(\mathbb{C}, d\lambda_\alpha) \to \mathcal{F}_\alpha^2$.

Suppose u and v are functions in $\mathcal{F}_\alpha^2$, not identically zero. It was proved in [146] that $T_uT_{\bar{v}}$ is bounded on the Fock space $\mathcal{F}_\alpha^2$ if and only if there is a point $a \in \mathbb{C}$ such that

$$u(z) = be^{\alpha\bar{a}z}, \qquad v(z) = ce^{-\alpha\bar{a}z}, \tag{56}$$

where b and c are nonzero constants. This certainly solves Sarason's problem for Toeplitz products on the space $\mathcal{F}_\alpha^2$. But [146] somehow did not address Sarason's conjecture, which now of course follows from our main result.

We want to make two points here. First, the proof of Sarason's conjecture for $\mathcal{F}_\alpha^2$ is relatively simple after Sarason's problem is solved. Second, Sarason's conjecture holds for the Fock space $\mathcal{F}_\alpha^2$ for completely different reasons than was originally thought, namely, the motivation for Sarason's conjecture provided in [53] for the cases of Hardy and Bergman spaces is no longer valid for the Fock space. It is therefore somewhat amusing that Sarason's conjecture turns out to be true for the Fock space but fails for the Hardy and Bergman spaces.

Suppose u and v are given by (56). We have

$$\begin{aligned}\widetilde{|u|^2}(z) = \|f\, k_z\|^2 &= \int_{\mathbb{C}} |f(w)e^{\alpha w\bar{z}} - (\alpha 2)|z|^2|^2\, d\lambda_\alpha(w)\\ &= |b|^2 e^{-\alpha|z|^2} \int_{\mathbb{C}} \left|e^{\alpha w(\bar{a}+\bar{z})}\right|^2 d\lambda_\alpha(w)\\ &= |b|^2 e^{-\alpha|z|^2+\alpha|a+z|^2}\\ &= |b|^2 e^{\alpha(|a|^2+\bar{a}z+a\bar{z})}.\end{aligned}$$

Similarly,

$$\widetilde{|v|^2}(z) = |c|^2 e^{\alpha(|a|^2-\bar{a}z-a\bar{z})}.$$

It follows that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) = |bc|^2 e^{2\alpha|a|^2}$$

is a constant and hence a bounded function on $\mathbb{C}$.

On the other hand, it follows from Hölder's inequality that we always have

$$|u(z)|^2 \le \widetilde{|u|^2}(z), \qquad u \in \mathcal{F}_\alpha^2, z \in \mathbb{C}.$$

Therefore, if $\widetilde{|u|^2}\widetilde{|v|^2}$ is a bounded function on $\mathbb{C}$, then there exists a positive constant M such that

$$|u(z)v(z)|^2 \le \widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \le M$$

for all $z \in \mathbb{C}$. Thus, as a bounded entire function, uv must be constant, say $u(z)v(z) = \mathbb{C}$ for all $z \in \mathbb{C}$. Since u and v are not identically zero, we must have $C \neq 0$. Since functions in $\mathcal{F}_\alpha^2$ must have order less than or equal to 2, we can write $u(z) = e^{p(z)}$, where

$$p(z) = az^2 + bz + c$$

is a polynomial of degree less than or equal to 2. But $u(z)v(z)$ is constant, so $v(z) = e^{q(z)}$, where

$$q(z) = -az^2 - bz + d$$

is another polynomial of degree less than or equal to 2.

We will show that $a = 0$. To do this, we will estimate the Berezin transform $\widetilde{|u|^2}$ when u is a quadratic exponential function as given above. More specifically, for $C_1 = |e^c|^2$, we have

$$\begin{aligned}\widetilde{|u|^2}(z) &= C_1 \int_{\mathbb{C}} \left|e^{a(z+w)^2+b(z+w)}\right|^2 d\lambda_\alpha(w)\\ &= C_1 \left|e^{az^2+bz}\right|^2 \int_{\mathbb{C}} \left|e^{aw^2+(b+2az)w}\right|^2 d\lambda_\alpha(w).\end{aligned}$$

Write $b + 2az = \alpha\bar{\zeta}$. Then it follows from the inequality $\widetilde{|F|^2} \ge |\widetilde{F}|^2$ for $F \in F_\alpha^2$ again that

$$\begin{aligned}\widetilde{|u|^2}(z) &= C_1 \left|e^{az^2+bz}\right|^2 e^{\alpha|\zeta|^2} \int_{\mathbb{C}} \left|e^{aw^2} k_\zeta(w)\right|^2 d\lambda_\alpha(w)\\ &\ge C_1\left|e^{az^2+bz}\right|^2 e^{\alpha|\zeta|^2} \left|e^{a\zeta^2}\right|^2.\end{aligned}$$

If we do the same estimate for the function v, the result is

$$\widetilde{|v|^2}(z) \ge C_2 \left|e^{-az^2-bz}\right|^2 e^{\alpha|\zeta|^2} \left|e^{-a\zeta^2}\right|^2,$$

where ζ is the same as before and $C_2 = \left|e^d\right|^2$. It follows that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \ge C_1C_2e^{2\alpha|\zeta|^2} = C_1C_2e^{2|b+2az|^2/\alpha}.$$

This shows that $\widetilde{|u|^2}\widetilde{|v|^2}$ is unbounded unless $a = 0$. Therefore, the boundedness of $\widetilde{|u|^2}\widetilde{|v|^2}$ implies that

$$u(z) = e^{bz+c}, \qquad v(z) = e^{-bz+d}.$$

By [146], the product T_uT_v is bounded on F_α^2. In fact, T_uT_v is a constant times a unitary operator.

Combining the arguments above and the main result of [146] we have actually proved that the following conditions are equivalent for u and v in F_α^2:

(a) $T_uT_{\bar{v}}$ is bounded on F_α^2.
(b) $T_uT_{\bar{v}}$ is a constant multiple of a unitary operator.
(b) $\widetilde{|u|^2}\widetilde{|v|^2}$ is bounded on $\mathbb{C}$.
(c) $\widetilde{|u|^2}\widetilde{|v|^2}$ is constant on $\mathbb{C}$.

Recall that in the case of Hardy and Bergman spaces, there is actually an absolute constant C (4 for the Hardy space and 16 for the Bergman space) such that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \le C\|T_u T_{\bar{v}}\|^2$$

for all $u, v,$ and z. We now show that such an estimate is not possible for the Fock space. To see this, consider the functions

$$u(z) = e^{\alpha \bar{a} z}, \qquad v(z) = e^{-\alpha \bar{a} z}.$$

By calculations done in [146], we have

$$T_u T_{\bar{v}} = e^{\alpha |a|^2/2} W_a,$$

where W_a is the Weyl unitary operator defined by $W_a f(z) = f(z-a)k_a(z)$. On the other hand, by calculations done earlier, we have

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) = e^{2\alpha|a|^2}.$$

It is then clear that there is NO constant C such that

$$e^{2\alpha|a|^2} \le Ce^{\alpha|a|^2/2}$$

for all $a \in \mathbb{C}$. Therefore, there is NO constant C such that

$$\sup_{z\in\mathbb{C}} \widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \le C\|T_u T_{\bar{v}}\|^2$$

for all u and v. In other words, the easy direction for Sarason's conjecture in the cases of Hardy and Bergman spaces becomes difficult for Fock spaces.

Corollary (5.3.15)[185]: If $f_s \in \mathcal{F}^2_{\frac{1}{2}-\epsilon}$, there is a constant $C > 0$ such that

$$|f_s(z_n)| \le C\,|z_n|^{-\left(\frac{1}{2}+\epsilon\right)} e^{\frac{1}{2}|z_n|^{1-2\epsilon}}, \qquad z_n \in \mathbb{C}.$$

Consequently, the order of every function in $\mathcal{F}^2_{\frac{1}{2}-\epsilon}$ is at most $(1-2\epsilon)$.

Proof. By the reproducing property and Cauchy-Schwartz inequality, we have

$$|f_s(z_n)| = \left|\int_{\mathbb{C}} \sum_s f_s(w_n) K_{\frac{1}{2}-\epsilon}(z_n, w_n)\, d\lambda_{\frac{1}{2}-\epsilon}(w_n)\right| \le \sum_s \|f_s\|\, K_{\frac{1}{2}-\epsilon}(z_n, z_n)^{1/2}$$

for all $f_s \in \mathcal{F}^2_{\frac{1}{2}-\epsilon}$ and all $z_n \in \mathbb{C}$. The desired estimate then follows from Lemma (5.3.2). See [156] for more details.

Corollary (5.3.16)[185]: Suppose that u^2 and v^2 are functions in $\mathcal{F}^2_{\frac{1}{2}-\epsilon}$, each not identically zero, and that the operator $T = T_{u^2}T_{\bar{v}^2}$ is bounded on $\mathcal{F}^2_{\frac{1}{2}-\epsilon}$. Then there exists a polynomial g of degree at most $(\frac{1}{2}-\epsilon)$ and a nonzero complex constant c such that $u^2(z_n) = e^{g(z_n)}$ and $v^2(z_n) = ce^{-g(z_n)}$.

Proof. If $T = T_{u^2}T_{v^2}$ is bounded on $\mathcal{F}^2_{\frac{1}{2}-\epsilon}$, then the Berezin transform T is bounded, where

$$T(z_n) = \langle T_{u^2}T_{\bar{v}^2}k_{z_n}, k_{z_n}\rangle, \qquad z_n \in \mathbb{C}.$$

By the reproducing property of the kernel functions, it is easy to see that

$$T(z_n) = u^2(z_n)\overline{v^2(z_n)}.$$

Since each k_{z_n} is a unit vector, it follows from the Cauchy-Schwarz inequality that

$$|u^2(z_n)v^2(z_n)| = \left|\tilde{T}(z_n)\right| \le \|T\|$$

for all $z_n \in \mathbb{C}$. This together with Liouville's theorem shows that there exist a constant c such that $u^2v^2 = c$. Since neither u^2 nor v^2 is identically zero, we have $c \ne 0$. Consequently, both u^2 and v^2 are non-vanishing.

Recall from Corollary (5.3.15) that the order of functions in $\mathcal{F}^2_{d+\epsilon}$ is at most $2(d+\epsilon)$, so there is a polynomial of degree d,

$$g(z_n)=\sum_{k=0}^{d}(1+\epsilon)_k z_n^k, \qquad d \le [2(d+\epsilon)],$$

such that $u^2 = e^g$ and $v^2 = ce^{-g}$. It remains to show that $\epsilon \ge 0$.

Since T is bounded on $\mathcal{F}^2_{d+\epsilon}$, the function

$$F(z_n,w_n)=\frac{\langle T\left(K_{d+\epsilon}(\cdot,w_n)\right),K_{d+\epsilon}(\cdot,z_n)\rangle}{\sqrt{K_{d+\epsilon}(z_n,z_n)}\sqrt{K_{d+\epsilon}(w_n,w_n)}}$$

must be bounded on $\mathbb{C}^2$. On general reproducing Hilbert spaces, we always have

$$\langle T_{u^2}T_{\bar{v}^2}K_{w_n},K_{z_n}\rangle = \langle T_{\bar{v}^2}K_{w_n},T_{u^2}K_{z_n}\rangle = \langle \bar{v}^2(w_n)K_{w_n},u^2(z_n)K_{z_n}\rangle$$
$$= u^2(z_n)\bar{v}^2(w_n)K(z_n,w_n).$$

It follows that

$$F(z_n,w_n)= \bar{c}e^{g(z_n)-\overline{g(w_n)}}\frac{K_{d+\epsilon}(z_n,w_n)}{\sqrt{K_{d+\epsilon}(z_n,z_n)}\sqrt{K_{d+\epsilon}(w_n,w_n)}}.$$

From Lemma (5.3.2) we deduce that

$$|F(z_n,w_n)| \gtrsim e^{Re\left(g(z_n)-g(w_n)\right)}e^{-\frac{1}{2}\left(|z_n|^{d+\epsilon}-|w_n|^{d+\epsilon}\right)^2} \tag{57}$$

for all $|\arg(z_n\overline{w_n})| \le c\theta_0(|z_nw_n|)$ as $|z_nw_n|$ grows to infinity. Choose $x_n > 0$ sufficiently large and set

$$z_n(x_n) = x_n e^{i\frac{\pi}{2d}}e^{-i\frac{\arg((1+\epsilon)_d)}{d}},$$

and

$$w_n(x_n) = x_n e^{i\frac{\pi}{2d}}e^{-i\frac{\arg((1+\epsilon)_d)+\frac{c}{2(d+\epsilon)x_n^{d+\epsilon}}}{d}}.$$

Since

$$\theta_0(|z_n(x_n)w_n(x_n)|) = \frac{1}{(d+\epsilon)x_n^{d+\epsilon}},$$

we can apply (57) to $z_n(x_n)$ and $w_n(x_n)$ to get

$$e^{Re\left(g(z_n(x_n))-g(w_n(x_n))\right)} \lesssim \sup_{(z_n,w_n)\in\mathbb{C}^2}|F(z_n,w_n)| < \infty \tag{58}$$

as x_n grows to infinity. On the other hand, a few computations show that

$$Re\ \left(g\left(z_n(x_n)\right) - g\left(w_n(x_n)\right)\right)$$
$$=\sum_{j=0}^{d} x_n^j\, Re\left((1+\epsilon)_j e^{ij\frac{\pi}{2d}-i\frac{j}{d}\arg[(1+\epsilon)_d)}\left(1 - e^{-i\frac{cj}{2(d+\epsilon)dx_n^{d+\epsilon}}}\right)\right.$$
$$\left(1 - e^{-i\frac{cj}{2(d+\epsilon)dx_n^{d+\epsilon}}}\right) = |(1+\epsilon)_d|\, x_n^d \sin\left(\frac{c}{2(d+\epsilon)x_n^{d+\epsilon}}\right) + g_{d-1}(x_n),$$
$$= |(1+\epsilon)_d|\, x_n^d \sin\left(\frac{c}{2(d+\epsilon)x_n^{d+\epsilon}}\right) + g_{d-1}(x_n),\Bigg)$$

where

$$g_{d-1}(x_n) = \sum_{j=0}^{d-1} x_n^j Re\left((1+\epsilon)_j e^{i\frac{j\pi}{2d} - i\frac{j}{d}\arg((1+\epsilon)_d)} \left(1 - e - i\frac{cj}{2(d+\epsilon)dx_n^{d+\epsilon}}\right)\right)$$

$$= -\sum_{j=0}^{d-1} \left|(1+\epsilon)_j\right| x_n^j \sin\left(\frac{j\pi}{2d} + \arg(1+\epsilon)_j\right.$$

$$\left. - \frac{j}{d}\arg((1+\epsilon)_d)\right) \sin\frac{cj}{2(d+\epsilon)dx_n^{d+\epsilon}}$$

$$+ \sum_{j=0}^{d-1} |(1+\epsilon)_j| x_n^j \cos\left[\frac{j\pi}{2d} + \arg(1+\epsilon)_j - \frac{j}{d}\arg((1+\epsilon)_d)\right]\left[1\right.$$

$$\left. - \cos\frac{cj}{2(d+\epsilon)dx_n^{d+\epsilon}}\right] \lesssim x_n^{\epsilon-1}.$$

Therefore, there exist some $(x_n)_0 > 0$ and $\delta > 0$ such that

$$Re\left(g(z_n(x_n)) - g(w_n(x_n))\right) \geq \frac{\delta|(1+\epsilon)_d|x_n^d}{x_n^{d+\epsilon}}$$

for all $x_n \geq (x_n)_0$. Since $(1+\epsilon)_d \neq 0$, it follows from (58) that $\epsilon \geq 0$.
On several occasions later on we will need to estimate the integral

$$I(1+\epsilon) = \int_0^\infty e^{-\frac{1}{2}r^{2(d+\epsilon)}+(1+\epsilon)r^d} r^{\epsilon-1}\, dr,$$

where $\epsilon \geq 0$.

First, suppose $\epsilon > 0$. By various changes of variables, we have

$$I(1+\epsilon) = \int_0^1 e^{-\frac{1}{2}r^{2(d+\epsilon)}+(1+\epsilon)r^d} r^{\epsilon-1}\, dr + \int_1^\infty e^{-\frac{1}{2}r^{2(d+\epsilon)}+(1+\epsilon)r^d} r^{\epsilon-1}\, dr$$

$$\leq e^{1+\epsilon}\int_0^1 r^{\epsilon-1}dr + \int_1^\infty e^{-\frac{1}{2}r^{2(d+\epsilon)}+(1+\epsilon)r^{d+\epsilon}\, r^{\epsilon-1}}\, dr$$

$$= \frac{e^{1+\epsilon}}{\epsilon} + e^{\frac{(1+\epsilon)^2}{2}}\int_1^\infty e^{-\frac{1}{2}\left(r^{d+\epsilon}-(1+\epsilon)\right)^2} r^{\epsilon-1}\, dr$$

$$= \frac{e^{1+\epsilon}}{\epsilon} + \frac{e^{\frac{(1+\epsilon)^2}{2}}}{d+\epsilon}\int_1^\infty e^{-\frac{1}{2}(t-1+\epsilon)^2}\, t^{\frac{-d}{d+\epsilon}}\, dt.$$

If $\frac{\epsilon}{d+\epsilon} - 1 \leq 0$, then

$$I(1+\epsilon) \leq \frac{e^{1+\epsilon}}{\epsilon} + \frac{\sqrt{2\pi}}{d+\epsilon}e^{\frac{(1+\epsilon)^2}{2}} \leq \left(\frac{\sqrt{e}}{\epsilon} + \frac{\sqrt{2\pi}}{d+\epsilon}\right)e^{\frac{(1+\epsilon)^2}{2}}.$$

Otherwise, we have $\frac{\epsilon}{d+\epsilon} - 1 > 0$. Using the fact that $u^2 \longmapsto u^{\frac{-2d}{d+\epsilon}}$ is increasing, we see that

$$\int_{-\frac{1+\epsilon}{2}}^{\frac{1+\epsilon}{2}} e^{-\frac{t^2}{2}}(t+1+\epsilon)^{\frac{-d}{d+\epsilon}}dt \leq \left(\frac{3(1+\epsilon)}{2}\right)^{\frac{-d}{d+\epsilon}}\int_{-\frac{1+\epsilon}{2}}^{\frac{1+\epsilon}{2}} e^{-\frac{t^2}{2}}\, dt \leq \sqrt{2\pi}\left(\frac{3(1+\epsilon)}{2}\right)^{\frac{-d}{d+\epsilon}}.$$

For the same reason we also have

$$\int_{\frac{1+\epsilon}{2}}^{+\infty} e^{-\frac{t^2}{2}} (t + 1+\epsilon)^{\frac{-d}{d+\epsilon}} dt \leq \int_{\frac{1+\epsilon}{2}}^{+\infty} e^{-\frac{t^2}{2}} (3t)^{\frac{-d}{d+\epsilon}} dt \leq 3^{\frac{-d}{d+\epsilon}} \int_0^{+\infty} t^{\frac{-d}{d+\epsilon}} e^{-\frac{t^2}{2}} dt$$

$$= \frac{\sqrt{2}}{2}(3\sqrt{2})^{\frac{-d}{d+\epsilon}} \int_0^{+\infty} u^{\frac{-2d}{d+\epsilon}} e^{-u^2} dt = \frac{\sqrt{2}}{2}(3\sqrt{2})^{\frac{-d}{d+\epsilon}} \Gamma\left(\frac{\epsilon-1}{2(d+\epsilon)}\right).$$

In the case when $\epsilon > 0$ (or equivalently $\epsilon > 0$),

$$\int_{-(1+\epsilon)}^{-\frac{2+\epsilon}{2}} e^{-\frac{t^2}{2}} (t + 2+\epsilon)^{\frac{-d}{d+\epsilon}} dt \leq \left(\frac{2+\epsilon}{2}\right)^{\frac{-d}{d+\epsilon}} \int_{-(1+\epsilon)}^{-\frac{2+\epsilon}{2}} e^{-\frac{t^2}{2}} dt$$

$$\leq \left(\frac{2+\epsilon}{2}\right)^{\frac{-d}{d+\epsilon}} \int_{-(1+\epsilon)}^{-\frac{2+\epsilon}{2}} e^{\frac{(2+\epsilon)t}{4}} dt \leq \left(\frac{2+\epsilon}{2}\right)^{\frac{-d}{d+\epsilon}} \frac{4}{2+\epsilon} e^{-\frac{(2+\epsilon)^2}{8}}$$

$$\leq 2\left(\frac{2+\epsilon}{2}\right)^{\frac{-d}{d+\epsilon}}.$$

It follows that there exists a constant $C = C(d+\epsilon, \epsilon-1) > 0$ such that

$$\int_1^{\infty} e^{-\frac{1}{2}(t-(2+\epsilon))^2} t^{\frac{-d}{d+\epsilon}} dt = \int_{-(1+\epsilon)}^{\infty} e^{-\frac{t^2}{2}} (t + 2+\epsilon)^{\frac{-d}{d+\epsilon}} dt \leq C\,(1 + 2+\epsilon)^{\frac{-d}{d+\epsilon}}$$

for $d < 0$. It is then easy to find another positive constant $C = C(d+\epsilon, \epsilon-1)$, independent of $(1+\epsilon)$, such that

$$I(1+\epsilon) \leq C\,(2+\epsilon)^{\frac{-d}{d+\epsilon}} e^{\frac{(1+\epsilon)^2}{2}}$$

for all $\epsilon \geq 0$ and $d < 0$. Therefore,

$$\int_0^{\infty} e^{-\frac{1}{2}r^{2(d+\epsilon)}+(1+\epsilon)r^d} r^{\epsilon-1} dr \leq C\,(2+\epsilon)^{\max\left(0,\frac{-d}{d+\epsilon}\right)} e^{\frac{(1+\epsilon)^2}{2}} \tag{59}$$

for all $\epsilon \geq 0$. Since $I(1+\epsilon)$ is increasing in $(1+\epsilon)$, the estimate above holds for $0 \leq \epsilon \leq 1$ as well.

Corollary (5.3.17)[185]: For any $\delta > 0, \epsilon > 0$, we can find a constant $C > 0$ (depending on $1+\epsilon, \delta, 1+\epsilon, \epsilon-1, 1+\epsilon$ but not on $1+\epsilon, d, x_n$) such that

$$x_n^{-1} \int_{\frac{1+\epsilon}{x_n^2}}^{+\infty} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1+r^{2(1+\epsilon)})+(1+\epsilon)x_n^d(1+\delta r^d)} r^{\epsilon-1} dr \leq C\,(2+\epsilon)^{\max\left(0,\frac{\epsilon}{1+\epsilon}\right)} e^{\frac{1+\delta^2}{2}(1+\epsilon)^2}$$

and

$$x_n^{1+\epsilon} \int_{\frac{1+\epsilon}{x_n^2}}^{+\infty} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1-r^{1+\epsilon})^2+(1+\epsilon)x_n^d(1-r^d)} r^{\frac{1+\epsilon}{2}} dr \leq C(2+\epsilon) e^{\frac{(1+\epsilon)^2}{2}}$$

for all $x_n > 0$, $\epsilon \geq 0$.

Proof. Let $I = I(1+\epsilon, \epsilon-1, 1+\epsilon, 1+\epsilon, x_n, 1+\epsilon, d)$ denote the first integral that we are trying to estimate. If $x_n \geq 1$, we have

$$I = x_n^{-1} e^{-\frac{x_n^{2(1+\epsilon)}}{2}+(1+\epsilon)x_n^d} \int_{\frac{1+\epsilon}{x_n^2}}^{\infty} e^{-\frac{(x_n r)^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta(x_n r)^d} r^{\epsilon-1} dr$$

$$\leq x_n^{-(1+\epsilon)} e^{-\frac{x_n^{2(1+\epsilon)}}{2}+(1+\epsilon)x_n^{1+\epsilon}} \int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{r^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta r^d} r^{\epsilon-1} dr$$

$$\leq e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-(1+\epsilon)\right)^2+\frac{(1+\epsilon)^2}{2}} \frac{\int_{\frac{1+\epsilon}{x_n}}^{\infty} r^{1+\epsilon}}{(1+\epsilon)^{1+\epsilon}} e^{-\frac{1}{2}r^{2(1+\epsilon)}+(1+\epsilon)\delta r^d} r^{\epsilon-1}\, dr$$

$$\leq \frac{e^{\frac{(1+\epsilon)^2}{2}}}{(1+\epsilon)^{1+\epsilon}} \int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{r^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta r^d} r^{2\epsilon} dr.$$

The desired result then follows from (59).

If $0 < x_n < 1$, we have

$$I = x_n^{-1} e^{-\frac{x_n^{2(1+\epsilon)}}{2}+(1+\epsilon)x_n^d} \int_{\frac{1+\epsilon}{x_n^2}}^{\infty} e^{-\frac{(x_n r)^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta(x_n r)^d} r^{\epsilon-1}\, dr$$

$$\leq e^{1+\epsilon} x_n^{-(1+\epsilon)} \int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{r^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta r^d} r^{\epsilon-1} dr$$

$$\leq \frac{e^{\frac{(1+\epsilon)^2}{2}+1}}{(1+\epsilon)^{1+\epsilon}} \int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{r^{2(1+\epsilon)}}{2}+(1+\epsilon)\delta r^d} r^{2\epsilon} dr.$$

The desired estimate follows from (59) again.

To prove the second part of the corollary, denote by $J = J(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon)$ the second integral that we are trying to estimate. Then it is clear from a change of variables that for $0 < x_n < 1$ we have

$$J(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon) = x_n^{\frac{\epsilon-1}{2}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)\left(x_n^d-r^d\right)} r^{\frac{1+\epsilon}{2}}\, dr$$

$$\leq \frac{e^{1+\epsilon}}{1+\epsilon} x_n^{\frac{1+\epsilon}{2}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{2(1+\epsilon)}-2(x_n r)^{1+\epsilon}+r^{2(1+\epsilon)}\right)} r^{\frac{3+\epsilon}{2}}\, dr$$

$$\leq \frac{e^{1+\epsilon}}{1+\epsilon} \int_0^{+\infty} e^{-\frac{r^{2(1+\epsilon)}}{2}+r^{1+\epsilon}} r^{\frac{3+\epsilon}{2}}\, dr = Ce^{1+\epsilon} \leq C'(2+\epsilon)e^{\frac{(1+\epsilon)^2}{2}},$$

where the constants C and C only depend on $1+\epsilon$.

Next assume that $x_n \geq 1$. In case $1+\epsilon \leq x_n^2$ we write $J = J_1 + J_2$, where

$$J_1 = J_1(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon)$$

$$= x_n^{1+\epsilon} \int_{\frac{1+\epsilon}{x_n^2}}^{1} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1-r^{1+\epsilon})^2+(1+\epsilon)x_n^d(1-r^d)} r^{\frac{1+\epsilon}{2}}\, dr,$$

And

$$J_2 = J_2(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon)$$

$$= x_n^{1+\epsilon} \int_1^{\infty} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1-r^{1+\epsilon})^2+(1+\epsilon)x_n^d(1-r^d)} r^{\frac{1+\epsilon}{2}}\, dr.$$

Otherwise we just use $J_1 \leq J_2$. So it suffices to estimate the two integrals above.

To handle $J_1(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon)$, we fix $\varepsilon > 0$ and consider two cases. In the case $x_n^{1+\epsilon} \leq (1+\epsilon)^2$, we have

$$J_1(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon) \le x_n^{1+\epsilon} \int_{\frac{1+\epsilon}{x_n^2}}^{1} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1-r^{1+\epsilon})^2 + (1+\epsilon)x_n^d(1-r^d)}\, r^{\frac{1+\epsilon}{2}}\, dr$$

$$\le (1+\epsilon)^2 e^{\frac{(1+\epsilon)^2}{2}} \int_{\frac{1+\epsilon}{x_n^2}}^{1} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}(1-r^{1+\epsilon})-(1+\epsilon)\right)^2}\, r^{\frac{1+\epsilon}{2}}\, dr$$

$$\le (1+\varepsilon)^2 e^{\frac{(1+\epsilon)^2}{2}}.$$

When $x_n^{1+\epsilon} \ge (1+\varepsilon)^2$, we set $y_n = x_n^{1+\epsilon}$ and $\tau = (y_n - (1+\epsilon))/2$. Then we have

$$\tau \ge \frac{\varepsilon}{2(1+\varepsilon)}\, y_n \to +\infty$$

as $y_n \to +\infty$. By successive changes of variables we see that

$$J_1(1+\epsilon, d, 1+\epsilon, x_n, 1+\epsilon) \le x_n^{1+\epsilon} \int_{\frac{1+\epsilon}{x_n^2}}^{1} e^{-\frac{x_n^{2(1+\epsilon)}}{2}(1-r^{1+\epsilon})^2 + (1+\epsilon)x_n^{1+\epsilon}(1-r^{1+\epsilon})} r^{\frac{1+\epsilon}{2}}\, dr$$

$$= \frac{y_n}{1+\epsilon} \int_0^{1-\frac{(1+\epsilon)^{1+\epsilon}}{y_n^2}} (1-r)^{\frac{1}{1+\epsilon}-\frac{1}{2}}\, e^{-\frac{y_n^2 r^2}{2} + (1+\epsilon)y_n r}\, dr$$

$$= \frac{1}{1+\epsilon} \int_0^{y_n - \frac{(1+\epsilon)^{1+\epsilon}}{y_n}} \left(1 - \frac{r}{y_n}\right)^{\frac{1-\epsilon}{2(1+\epsilon)}} e^{-\frac{r^2}{2} + (1+\epsilon) r}\, dr$$

$$= \frac{e^{\frac{(1+\epsilon)^2}{2}}}{1+\epsilon} \int_{-(1+\epsilon)}^{y_n - (1+\epsilon) - \frac{(1+\epsilon)^{1+\epsilon}}{y_n}} \left(1 - \frac{1+\epsilon}{y_n} - \frac{r}{y_n}\right)^{\frac{1}{1+\epsilon}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr.$$

This shows that for $0 \le \epsilon \le 1$ we have

$$J_1 \le \frac{e^{\frac{(1+\epsilon)^2}{2}}}{1+\epsilon} \int_{-(1+\epsilon)}^{y_n - (1+\epsilon) - \frac{(1+\epsilon)^{1+\epsilon}}{y_n}} e^{-\frac{r^2}{2}}\, dr \le \frac{\sqrt{2\pi}}{1+\epsilon}\, e^{\frac{(1+\epsilon)^2}{2}}.$$

Thus we suppose that $\epsilon > 0$. Then

$$\int_{-\tau}^{\tau} \left(1 - \frac{1+\epsilon}{y_n} - \frac{r}{y_n}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} e^{-\frac{r^2}{2}}\, dr \le \left(1 - \frac{1+\epsilon}{y_n} - \frac{\tau}{y_n}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr$$

$$= \left(\frac{\tau}{2y_n}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr \le \sqrt{2\pi} \left(\frac{\varepsilon}{4(1+\varepsilon)}\right)^{-\frac{\epsilon}{2(2+\epsilon)}}.$$

Moreover, in case $-(1+\epsilon) < -\tau$, we have

$$\int_{-(1+\epsilon)}^{-\tau} \left(1 - \frac{1+\epsilon}{y_n} - \frac{r}{y_n}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} e^{-\frac{r^2}{2}}\, dv^2$$

$$\le \left(1 - \frac{1+\epsilon}{y_n} + \frac{\tau}{y_n}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} \int_{-(1+\epsilon)}^{-\tau} e^{-\frac{\tau|r|}{2}}\, dr$$

$$\le 2\left(\frac{3\varepsilon}{2(1+\varepsilon)}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} \frac{e^{-\frac{\tau^2}{2}}}{\tau} \le 4\left(\frac{3}{2}\right)^{-\frac{\epsilon}{2(2+\epsilon)}} \left(\frac{\varepsilon}{1+\varepsilon}\right)^{-\frac{5+\epsilon}{2(2+\epsilon)}} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.$$

Similarly, in case $y_n - (1+\epsilon) - \frac{(1+\epsilon)^{2+\epsilon}}{y_n} \ge \tau$, we have

$$\int_{\tau}^{y_n-(1+\epsilon)-\frac{(1+\epsilon)^{2+\epsilon}}{y_n}}\left[1-\frac{1+\epsilon}{y_n}-\frac{r}{y_n}\right]^{-\frac{\epsilon}{2(2+\epsilon)}} e^{-\frac{r^2}{2}}\,dr$$
$$\le\left[\frac{(1+\epsilon)^{2+\epsilon}}{y_n^2}\right]^{-\frac{\epsilon}{2(2+\epsilon)}}\int_{\tau}^{y_n-(1+\epsilon)-\frac{(1+\epsilon)^{2+\epsilon}}{y_n}} e^{-\frac{\tau r}{2}}\,dr$$
$$\le\ 2(1+\epsilon)^{-\frac{\epsilon}{2}}\left[\frac{\varepsilon}{2(1+\varepsilon)}\right]^{-\frac{\epsilon}{2+\epsilon}}\tau^{-\frac{2}{2+\epsilon}}\,e^{-\frac{\tau^2}{2}}\left(since\ \tau\ \ge\frac{\varepsilon}{2(1+\varepsilon)}\right)$$
$$\le\ 4(1+\epsilon)^{-\frac{\epsilon}{2}}\,\frac{1+\varepsilon}{\varepsilon}\,e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}\ .$$

The last three estimates yield

$$J_1\ \le\ C(2+\epsilon)e^{\frac{(1+\epsilon)^2}{2}}$$

for some $C > 0$ that is independent of x_n and $(1+\epsilon)$.

To establish the estimate for J_2, we perform a change of variables to obtain

$$J_2\ \le\ x_n^{2+\epsilon}\int_1^{+\infty} e^{-\frac{x_n^{2(2+\epsilon)}}{2}(1-r^{2+\epsilon})^2}\, r^{\frac{2+\epsilon}{2}}\,dr\ =\frac{1}{2+\epsilon}\int_0^{+\infty} e^{-\frac{r^2}{2}}\left(\frac{r}{x_n^{2+\epsilon}}+1\right)^{-\frac{\epsilon}{2(2+\epsilon)}}dr.$$

If $\epsilon\ \ge 0$, we have

$$J_2\ \le\frac{1}{2+\epsilon}\int_0^{+\infty} e^{-\frac{r^2}{2}}\,dr,$$

and if $0\le\epsilon\ <1$, we have

$$J_2\ \le\frac{1}{1+\epsilon}\int_0^{+\infty} e^{-\frac{r^2}{2}}(r\ +\ 1)^{\frac{1-\epsilon}{2(1+\epsilon)}}\,dr.$$

Therefore, $J_2\ \le\ C$ for some $C\ >\ 0$ that is independent of x_n and $(1+\epsilon)$. This completes the proof of the corollary.

In the proof of the main theorem, we will have to estimate the following two integrals:

$$I(x_n,r)\ =\int_{|\theta|\le\frac{\pi}{2(1+\epsilon)}} e^{-(x_nr)^{1+\epsilon}+2(1+\epsilon)r^d\sin^2\left(\frac{\theta d}{2}\right)}\left|K_{1+\epsilon}\left(x_n,re^{i\theta}\right)\right|\,d\theta,$$

and

$$J(x_n,r)=\int_{|\theta|\ge\frac{\pi}{2(1+\epsilon)}} e^{-(x_nr)^{1+\epsilon}+(1+\epsilon)\left(x_n^d+r^d\right)}\left|K_{1+\epsilon}\left(x_n,re^{i\theta}\right)\right|\,d\theta,$$

where $x_n,r,1+\epsilon\ \in\ (0,+\infty)$ and $\epsilon\ge 0$.

Corollary (5.3.18)[185]: For any $\epsilon>-1$ there exist positive constants $C\ =\ C(1+\epsilon)$ and $\epsilon=-1$ such that

$$I(x_n,r)\ \le\ C(x_nr)^{\epsilon}\int_0^1 e^{-\left((x_nr)^{1+\epsilon}-(1+\epsilon)r^d\right)t^2}\,dt$$

and

$$J(x_n,r)\ \le\frac{Ce^{-(x_nr)^{1+\epsilon}+(1+\epsilon)\left(x_n^d+r^d\right)}}{x_nr}$$

for all $\epsilon\ge 0$, and $x_n\ >\ 0$ with $x_nr\ >\ 1+\epsilon$.

Proof. It follows from Lemma (5.3.2) that there exist positive constants $C\ =\ C(1+\epsilon)$ and $\epsilon=-1$ such that for all $\epsilon\ge\ 0$ and $x_nr\ >\ 1+\epsilon$ we have

$$I(x_n,r) \le C(x_nr)^{\epsilon}\int_{|\theta|\le\frac{\pi}{2(1+\epsilon)}} e^{-(x_nr)^{1+\epsilon}+(x_nr)^{1+\epsilon}\cos((1+\epsilon)\theta)+2(1+\epsilon)r^d\sin^2\left(\frac{\theta d}{2}\right)}\,d\theta$$
$$= 2C(x_nr)^{\epsilon}\int_0^{\frac{\pi}{2(1+\epsilon)}} e^{-2(x_nr)^{1+\epsilon}\sin^2\left(\frac{(1+\epsilon)\theta}{2}\right)+2(1+\epsilon)r^d\sin^2\left(\frac{\theta d}{2}\right)}\,d\theta$$
$$\le 2C(x_nr)^{\epsilon}\int_0^{\frac{\pi}{2(1+\epsilon)}} e^{-2(x_nr)^{1+\epsilon}\sin^2\left(\frac{(1+\epsilon)\theta}{2}\right)+2(1+\epsilon)r^d\sin^2\left(\frac{(1+\epsilon)\theta}{2}\right)}\,d\theta$$
$$\le 2C(x_nr)^{\epsilon}\int_0^{\frac{\pi}{2(1+\epsilon)}} e^{-2\left((x_nr)^{1+\epsilon}-(1+\epsilon)r^d\right)\sin^2\left(\frac{(1+\epsilon)\theta}{2}\right)}\,d\theta$$
$$=\frac{4C}{1+\epsilon}(x_nr)^{\epsilon}\int_0^{\frac{\sqrt{2}}{2}} e^{-2\left((x_nr)^{1+\epsilon}-(1+\epsilon)r^d\right)t^2}\frac{dt}{\sqrt{1-t^2}}$$
$$\le\frac{4\sqrt{2}C}{1+\epsilon}(x_nr)^{\epsilon}\int_0^{\frac{\sqrt{2}}{2}} e^{-2\left((x_nr)^{1+\epsilon}-(1+\epsilon)r^d\right)t^2}\,dt$$
$$\le\frac{4\sqrt{2}C}{1+\epsilon}(x_nr)^{\epsilon}\int_0^{1} e^{-\left((x_nr)^{1+\epsilon}-(1+\epsilon)r^d\right)t^2}\,dt.$$

The estimate
$$J(x_n,r) \le \frac{Ce^{-(x_nr)^{1+\epsilon}+(1+\epsilon)\left(x_n^d+r^d\right)}}{x_nr}, \qquad x_nr > 1+\epsilon,$$
also follows from Lemma (5.3.2).

Corollary (5.3.19)[185]: For any $\epsilon \ge 0$ there exist constants $\epsilon > 0$ and $C = C(1+\epsilon) > 0$ such that
$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)(x_n^d-r^d)}\, I(x_n,r)r\,dr \le C\,(2+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}\,e^{(1+\epsilon)^2}$$
and
$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2} J(x_n,r)r\,dr \le C\,(2+\epsilon)^{\max\left(0,-\frac{\epsilon}{1+\epsilon}\right)}e^{(1+\epsilon)^2}$$
for all $x_n > 0,\ \epsilon \ge 0$.

Proof. For convenience we write x_n
$$A_I\,(x_n,r) = e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)(x_n^d-r^d)}\, I(x_n,r)r,$$
and
$$A_J\,(x_n,r) = e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2} J(x_n,r)r.$$
Let $(1+\epsilon)$ and C be the constants from Corollary (5.3.18). In the integrands we have $r > 1+\epsilon/x_n$, or $x_nr > 1+\epsilon$, so according to Corollary (5.3.18),
$$I(x_n,r) \le C(x_nr)^{\epsilon}\int_0^1 e^{-(x_nr)^{1+\epsilon}t^2+(1+\epsilon)r^dt^2}\,dt.$$
If, in addition, $x_n \le 1$, then
$$I(x_n,r) \le Cr^{\epsilon}e^{(1+\epsilon)r^d},$$
and

$$A_I\,(x_n,r) = \, e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2}\, e^{(1+\epsilon)x_n^d-(1+\epsilon)r^d}\, I(x_n,r)r \, \leq \, Cr^{1+\epsilon}e^{1+\epsilon}e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2}.$$

It follows that

$$\int_{\frac{1+\epsilon}{x_n}}^{\infty} A_I\,(x_n,r)\,dr \,\leq\, Ce^{1+\epsilon}\int_{\frac{1+\epsilon}{x_n}}^{\infty} r^{1+\epsilon}e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2}\,dr$$

$$\leq\, Ce^{1+\epsilon}\int_0^{\infty} r^{1+\epsilon}e^{-\frac{1}{2}x_n^{2(1+\epsilon)}+x_n^{1+\epsilon}r^{1+\epsilon}-\frac{1}{2}r^{2(1+\epsilon)}}\,dr$$

$$\leq\, Ce^{1+\epsilon}\int_0^{\infty} r^{1+\epsilon}e^{r^{1+\epsilon-\frac{1}{2}r^{2(1+\epsilon)}}}\,dr \leq\, C\,(1\,+\,1+\epsilon)^{-\frac{\epsilon}{1+\epsilon}}e^{(1+\epsilon)^2}.$$

for all $\epsilon \geq\, 0$ and $0\, <\, x_n\, \leq\, 1$.

Similarly, if $x_n\, \leq\, 1$ (and $x_n r\, >\, 1+\epsilon$), we deduce from Corollary (5.3.18) and (59) that

$$\int_{\frac{1+\epsilon}{x_n}}^{\infty} A_J\,(x_n,r)\,dr \,\leq \frac{C}{1+\epsilon}\int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2}\, e^{-(x_n^r)^{1+\epsilon}+(1+\epsilon)x_n^d+(1+\epsilon)r^d\,r}\,dr$$

$$\leq \frac{Ce^{1+\epsilon}}{1+\epsilon}\int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{1}{2}r^{2(1+\epsilon)}+(1+\epsilon)r^d}\,r\,dr$$

$$\leq\, C'(2+\epsilon)\max\left(0,\frac{1-\epsilon}{1+\epsilon}\right)e^{(1+\epsilon)^2}.$$

Suppose now that $x_n\, \geq\, 1$ and $rx_n\, >\, 1+\epsilon$. By Corollary (5.3.18) again,

$$A_I\,(x_n,r)\,\leq\, Cr(x_n r)^{\epsilon}\, e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)\left(x_n^d-r^d\right)}\int_0^1 e^{-t^2\left((x_n r)^{1+\epsilon}-(1+\epsilon)r^d\right)}\,dt.$$

Fix a sufficiently small $\varepsilon\, \in\, (0,1)$. If $(x_n r)^{1+\epsilon}\, \geq\, (1+\epsilon)r^d(1\, +\, \varepsilon)$, then

$$\int_0^1 e^{-t^2\left((x_n r)^{1+\epsilon}-(1+\epsilon)r^d\right)}\,dt \,=\, \frac{1}{\sqrt{(x_n r)^{1+\epsilon}\,-\,(1+\epsilon)r^d}}\int_0^{\sqrt{(x_n r)^{1+\epsilon}\,-\,(1+\epsilon)r^d}} e^{-s^2}\,ds$$

$$\leq\, \frac{1}{\sqrt{(x_n r)^{1+\epsilon}\,-\,(1+\epsilon)r^d}}\int_0^{\infty} e^{-s^2}\,ds \,=\frac{\sqrt{\pi}}{2}\frac{(x_n r)^{-\frac{1+\epsilon}{2}}}{\sqrt{1\,-\left(\frac{(1+\epsilon)r^d}{(x_n r)^{1+\epsilon}}\right)}}$$

$$\leq\, \sqrt{\frac{\pi(1\,+\,\varepsilon)}{4\varepsilon}}\,(x_n r)^{-\frac{1+\epsilon}{2}},$$

so there exists a constant $C\, =\, C(1+\epsilon)$ such that

$$A_I\,(x_n,r)\,\leq\, Cr(x_n r)^{\frac{\epsilon-1}{2}}\, e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)(x_n^d-r^d)}.$$

If $(x_n r)^{1+\epsilon}\, \leq\, r^d(1\, +\epsilon)^2$, we have

$$A_I(x_n,r)\,(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}r^{\frac{d(\epsilon)+1+\epsilon}{1+\epsilon}}\, e^{-\frac{1}{2}\left(x_n^{2(1x_n+\epsilon)}+r^{2(1+\epsilon)}\right)+(1+\epsilon)x_n^d}\int_0^1 e^{(1-t^2)\left((x_n r)^{1+\epsilon}-(1+\epsilon)r^d\right)}\,dt$$

$$\leq\, (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}r^{\frac{d(\epsilon)+1+\epsilon}{1+\epsilon}}\, e^{-\frac{1}{2}\left(x_n^{2(1+\epsilon)}+r^{2(1+\epsilon)}\right)+(1+\epsilon)\left(x_n^d+\varepsilon r^d\right)}.$$

It follows that

$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} A_I\,(x_n,r)\,dr$$
$$\lesssim x_n^{\frac{\epsilon-1}{2}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)\left(x_n^d-r^d\right)} r^{\frac{1+\epsilon}{2}}\,dr$$
$$+ (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{2(1+\epsilon)}+r^{2(1+\epsilon)}\right)+(1+\epsilon)\left(x_n^d+\varepsilon r^d\right)} dr.$$
The change of variables $r \longmapsto x_n r$ along with the second part of Corollary (5.3.17) shows that
$$x_n^{\frac{\epsilon-1}{2}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{1+\epsilon}-r^{1+\epsilon}\right)^2+(1+\epsilon)\left(x_n^d-r^d\right)} r^{\frac{1+\epsilon}{2}}\,dr \le C(2+\epsilon)e^{\frac{(1+\epsilon)^2}{2}}.$$
Similarly, the change of variables $r \longmapsto x_n r$ together with the first part Corollary (5.3.17) shows that
$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} r^{\frac{d(\epsilon)+1+\epsilon}{1+\epsilon}} e^{-\frac{1}{2}\left(x_n^{2(1+\epsilon)}+r^{2(1+\epsilon)}\right)+(1+\epsilon)\left(x_n^d+\varepsilon r^d\right)}\,dr \le C(2+\epsilon)^{\frac{d(\epsilon)+1+\epsilon}{1+\epsilon}} e^{\frac{1+\varepsilon^2}{2}(1+\epsilon)^2}.$$
We may assume that $\varepsilon < 1$. Then we can find a positive constant C such that
$$(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}} \int_{\frac{1+\epsilon}{x_n}}^{+\infty} r^{\frac{d(\epsilon)+1+\epsilon}{1+\epsilon}} e^{-\frac{1}{2}\left(x_n^{2(1+\epsilon)}+r^{2(1+\epsilon)}\right)+(1+\epsilon)\left(x_n^d+\varepsilon r^d\right)} dr$$
$$\le C(2+\epsilon)^{-\frac{\epsilon}{1+\epsilon}} e^{(1+\epsilon)^2}.$$

It follows that
$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} A_I\,(x_n,r)\,dr \le C\,(2+\epsilon)^{-\frac{\epsilon}{1+\epsilon}} e^{(1+\epsilon)^2}$$
for some other positive constant C that is independent of $(1+\epsilon)$ and x_n. This proves the first estimate of the corollary.

To establish the second estimate of the corollary, we use Corollary (5.3.18) to get
$$x_n A_J\,(x_n,x_n r) = x_n^2\, r e^{-\frac{x_n^{2(1+\epsilon)}}{2}\left(1-r^{1+\epsilon}\right)^2} J(x_n,x_n r)$$
$$\le Ce^{-\frac{x_n^{2(1+\epsilon)}}{2}\left(1+r^{2(1+\epsilon)}\right)+(1+\epsilon)x_n^d\left(1+r^d\right)}.$$
It follows from this and Corollary (5.3.17) that
$$\int_{\frac{1+\epsilon}{x_n}}^{+\infty} A_J\,(x_n,r)\,dr = x_n \int_{\frac{1+\epsilon}{x_n^2}}^{+\infty} A_J\,(x_n,x_n r)\,dr \le C(2+\epsilon)^{\max\left(0,\frac{1-\epsilon}{1+\epsilon}\right)} e^{(1+\epsilon)^2}.$$
This completes the proof of the corollary.

Corollary (5.3.20)[185]: If $u^2(z_n) = e^{g(z_n)}$ and $v^2(z_n) = e^{-g(z_n)}$, where g is a polynomial of degree at most $1+\epsilon$, then the operator $T = T_{u^2}T_{\bar{v}^2}$ is bounded on $\mathcal{F}^2_{1+\epsilon}$.

Proof. To prove the boundedness of $T = T_{u^2}T_{\bar{v}^2}$, we shall use a standard technique known as Schur's test [162]. Since
$$T\left(\sum_s f_s\,(z_n)\right) = \int_{\mathbb{C}} \sum_s K_{1+\epsilon}(z_n,w_n) e^{g(z_n)-\overline{g(w_n)}} f_s(w_n) e^{-|w_n|^{2(1+\epsilon)}}\,dA(w_n),$$

we have

$$\left|T\left(\sum_s f_s\,(z_n)\right)\right| e^{-\frac{1}{2}|z_n|^{2(1+\epsilon)}} \leq \int_{\mathbb{C}} \sum_s \; H_g(z_n, w_n)|f_s(w_n)|e^{-\frac{1}{2}|w_n|^{2(1+\epsilon)}} dA(w_n),$$

where

$$H_g(z_n, w_n) := |K_{1+\epsilon}(z_n, w_n)| e^{-\frac{1}{2}\left(|z_n|^{2(1+\epsilon)}+|w_n|^{2(1+\epsilon)}\right)+Re\left(g(z_n)-\overline{g(w_n)}\right)}.$$

Thus T will be bounded on $\mathcal{F}^2_{1+\epsilon}$ if the integral operator S_g defined by

$$S_g\left(\sum_s f_s\,(z_n)\right) = \int_{\mathbb{C}} \sum_s \; \left(H_g(z_n, w_n) + H_g(w_n, z_n)\right)\, f_s(w_n)\, dA(w_n)$$

is bounded on $L^2(\mathbb{C}, dA)$. Let

$$H_g(z_n) = \int_{\mathbb{C}} H_g(z_n, w_n)\, dA(w_n), \qquad z_n \in \mathbb{C}.$$

Since

$$H_{-g}(z_n) = \int_{\mathbb{C}} \mathbb{C}\, H_g(w_n, z_n)\, dA(w_n),$$

for all $z_n \in \mathbb{C}$, by Schur's test, the operator S_g is bounded on $L^2(\mathbb{C}, dA)$ if we can find a positive constant C such that

$$H_{g(z_n)} + H_{-g(z_n)} \leq C, \qquad z_n \in \mathbb{C}.$$

By the Cauchy-Schwarz inequality, we have

$$H_{g_1+g_2}\,(z_n) \leq \sqrt{H_{2g_1}\,(z_n) H_{2g_2}\,(z_n)}$$

for all $z_n \in \mathbb{C}$ and holomorphic polynomials g_1 and g_2. Moreover, if

$$U_\theta(z_n) = e^{i\theta} z_n, \qquad z_n \in \mathbb{C}, \theta \in [-\pi, \pi],$$

then

$$H_{goU_\theta} = H_g\; o\; U_\theta$$

for all $z_n \in \mathbb{C}, \theta \in [-\pi, \pi]$, and holomorphic polynomials g. Therefore, we only need prove the theorem for $g(z_n) = (1+\epsilon) z_n^d$ with some $\epsilon \geq 0$ and establish that

$$\sup_{x_n \geq 0} H_g(x_n) \leq C_1 e^{C_2(1+\epsilon)^2}, \qquad (60)$$

where C_k are positive constants independent of $(1+\epsilon)$ and d (but dependent on $d+\epsilon$). We will see that C_2 can be chosen as any constant greater than 1.

It is also easy to see that we only need to prove (60) for $x_n \geq 1$. This will allow us to use the inequality $x_n^d \leq x_n^{d+\epsilon}$ for the rest of this proof.

For $\epsilon \geq 0$ sufficiently large (we will specify the requirement on $(1+\epsilon)$ later) we write

$$H_g(x_n) = \int_{|x_n w_n| \leq 1+\epsilon} H_g(x_n, w_n)\, dA(w_n) + \int_{|x_n w_n| \geq 1+\epsilon} H_g(x_n, w_n)\, dA(w_n).$$

We will show that both integrals are, up to a multiplicative constant, bounded above by $e^{(1+\varepsilon)(1+\epsilon)^2}$.

By properties of the Mittag-Leffler function, we have

$$|K_{d+\epsilon}(x_n, w_n)| \leq \frac{d+\epsilon}{\pi} E_{\frac{1}{d+\epsilon}, \frac{1}{d+\epsilon}}(1+\epsilon) := C_{1+\epsilon}, \qquad |x_n w_n| \leq 1+\epsilon.$$

It follows that the integral

$$I_1 = \int_{x_n|w_n|\le 1+\epsilon} H_g(x_n, w_n)\, dA(w_n)$$

Satisfies

$$\begin{aligned} I_1 &= \int_{x_n|w_n|\le 1+\epsilon} |K_{d+\epsilon}(z_n, w_n)| e^{-\frac{1}{2}\left(|z_n|^{2(d+\epsilon)}+|w_n|^{2(d+\epsilon)}\right)+(1+\epsilon)Re\,\left(x_n^d - w_n^d\right)}\, dA(w_n) \\ &\le C_{1+\epsilon}\int_{x_n|w_n|\le 1+\epsilon} e^{-\frac{1}{2}\left(x_n^{2(d+\epsilon)}+|w_n|^{2(d+\epsilon)}\right)+(1+\epsilon)Re\left(x_n^d - w_n^d\right)}\, dA(w_n) \\ &\le C_{1+\epsilon} e^{-\frac{1}{2}x_n^{2(d+\epsilon)}+(1+\epsilon)x_n^d}\int_{x_n|w_n|\le 1+\epsilon} e^{-\frac{|w_n|^{2(d+\epsilon)}}{2}+(1+\epsilon)|w_n|^d}\, dA(w_n) \\ &\le 2\pi C_{1+\epsilon} e^{-\frac{1}{2}x_n^{2(d+\epsilon)}+(1+\epsilon)x_n^{d+\epsilon}}\int_0^{+\infty} e^{-\frac{r^{2(d+\epsilon)}}{2}+(1+\epsilon)r^d}\, r\, dr \\ &\le 2\pi C_{1+\epsilon} e^{\frac{(1+\epsilon)^2}{2}}\int_0^{+\infty} e^{-\frac{r^{2(d+\epsilon)}}{2}+(1+\epsilon)r^d}\, r\, dr \\ &\le C(2+\epsilon)^{\max\left(0,\frac{2-d-\epsilon}{d+\epsilon}\right)} e^{(1+\epsilon)^2}, \end{aligned}$$

where the last inequality follows from (59).

We now focus on the integral

$$I_2 = \int_{x_n|w_n|\ge 1+\epsilon} H_g(x_n, w_n)\, dA(w_n).$$

Observe that for all $x_n, r,$ and θ we have

$$\begin{aligned} Re\left(x_n^d - r^d e^{id\theta}\right) &= x_n^d - r^d\cos(d\theta) = x_n^d - r^d\left(2 - \cos(d\theta)\right) \\ &= x_n^d + r^d \sin^2\left(\frac{d\theta}{2}\right). \end{aligned}$$

It follows from polar coordinates that

$$\begin{aligned} I_2 &= \int_{\frac{1+\epsilon}{x_n}}^{+\infty}\int_{-\pi}^{\pi} H_g\left(x_n, re^{i\theta}\right) r\, d\theta\, dr \\ &= \int_{\frac{1+\epsilon}{x_n}}^{+\infty}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\left(x_n^{2(d+\epsilon)}+r^{2(d+\epsilon)}\right)+(1+\epsilon)\left(x_n^d - r^d\cos(d\theta)\right)}\left|K_{d+\epsilon}\left(x_n, re^{i\theta}\right)\right| r\, d\theta\, dr \\ &= \int_{\frac{1+\epsilon}{x_n}}^{\infty} e^{-\frac{1}{2}\left(x_n^{d+\epsilon}-r^{d+\epsilon}\right)^2+(1+\epsilon)(x_n^d - r^d)-(x_n r)^{d+\epsilon}}\, r\, dr\int_{-\pi}^{\pi} e^{2(1+\epsilon)r^d}\sin^2\left(\frac{d\theta}{2}\right)\left|K_{d+\epsilon}\left(x_n, re^{i\theta}\right)\right| \\ &\le \int_{\frac{1+\epsilon}{x_n}}^{+\infty} e^{-\frac{1}{2}\left(x_n^{d+\epsilon}-r^{d+\epsilon}\right)^2}\left(e^{(1+\epsilon)\left(x_n^d - r^d\right)} I(x_n, r) + J(x_n, r)\right) r\, dr, \end{aligned}$$

Where

$$I(x_n, r) = \int_{|\theta|\le\frac{\pi}{2(d+\epsilon)}} e^{-(x_n r)^{d+\epsilon}+2(1+\epsilon)r^d\sin^2\left(\frac{d\theta}{2}\right)}\left|K_{d+\epsilon}\left(x_n, re^{i\theta}\right)\right| d\theta,$$

and

$$J(x_n, r) = \int_{|\theta|\ge\frac{\pi}{2(d+\epsilon)}} e^{-(x_n r)^{d+\epsilon}+(1+\epsilon)\left(x_n^d + r^d\right)}\left|K_{d+\epsilon}\left(x_n, re^{i\theta}\right)\right| d\theta.$$

By Corollary (5.3.19), there exists another constant $C > 0$ such that

$$I_2 \le C(2+\epsilon)^{\max\left(0,\frac{2-d-\epsilon}{d+\epsilon}\right)} e^{(1+\epsilon)^2}.$$

Therefore,

$$\sup_{z_n\in\mathbb{C}} \int_{\mathbb{C}} H_g(z_n, w_n)\, dA(w_n) \le C(2+\epsilon)^{\max\left(0, \frac{2-d-\epsilon}{d+\epsilon}\right)} e^{(1+\epsilon)^2}$$

for yet another constant C that is independent of $(1+\epsilon)$ and d. Similarly, we also have

$$\sup_{z_n\in\mathbb{C}} \int_{\mathbb{C}} H_{-g}(z_n, w_n)\, dA(w_n) \le C(2+\epsilon)^{\max\left(0, \frac{2-d-\epsilon}{d+\epsilon}\right)} e^{(1+\epsilon)^2}$$

This yields (60) and proves the corollary.

Corollary (5.3.21)[185]: Suppose u^2 and v^2 are functions in $\mathcal{F}^2_{d+\epsilon}$, not identically zero, such that the operator $T = T_{u^2}T_{\bar{v}^2}$ is bounded on $\mathcal{F}^2_{d+\epsilon}$. Then the function $\widetilde{|u^2|^2}(z_n)\widetilde{|v^2|^2}(z_n)$ is bounded on the complex plane.

Proof. Since $T_{u^2}T_{\bar{v}^2}$ is bounded on $\mathcal{F}^2_{d+\epsilon}$, the operator $(T_{u^2}T_{\bar{v}^2})^* = T_{u^2}T_{\bar{v}^2}$ and the products $(T_{u^2}T_{\bar{v}^2})^* T_{u^2}T_{\bar{v}^2}$ and $\left(T_{v^2}T_{\overline{u^2}}\right)^* T_{v^2}T_{\bar{u}^2}$ are also bounded on $\mathcal{F}^2_{d+\epsilon}$. Consequently, their Berezin transforms are all bounded functions on $\mathbb{C}$.

For any $z_n \in \mathbb{C}$ we let k_{z_n} denote the normalized reproducing kernel of $\mathcal{F}^2_{d+\epsilon}$ at z_n. Then

$$\begin{aligned}\langle (T_{u^2}T_{\bar{v}^2})^* T_{u^2}T_{\bar{v}^2}k_{z_n}, k_{z_n}\rangle &= \langle T_{u^2}T_{\bar{v}^2}k_{z_n}, T_{u^2}T_{\bar{v}^2}k_{z_n}\rangle \\ &= \langle u^2\overline{v^2(z_n)}k_{z_n}, u^2\overline{v^2(z_n)}k_{z_n}\rangle = |v^2(z_n)|^2\, \widetilde{|u^2|^2}(z_n)\end{aligned}$$

is bounded on $\mathbb{C}$. Similarly $|u^2(z_n)|^2\, \widetilde{|v^2|^2}(z_n)$ is bounded on $\mathbb{C}$. By the proof of Corollary (5.3.16), the product u^2v^2 is a non-zero complex constant, say, $u^2(z_n)v^2(z_n) = C$. It follows that the function

$$\widetilde{|v^2|^2}(z_n)\widetilde{|u^2|^2}(z_n) = |u^2(z_n)|^2\, \widetilde{|v^2|^2}(z_n)|v^2(z_n)|^2\, \widetilde{|u^2|^2}(z_n)\frac{1}{|C|^2}$$

is bounded as well.

To complete the proof of Sarason's conjecture, we will need to find a lower bound for the function

$$\mathcal{B}(z_n) = \widetilde{|v^2|^2}(z_n)|u^2(z_n)|^2,$$

where $u^2 = e^g, v^2 = e^{-g}$, and g is a polynomial of degree d. We write

$$g(z_n) = (1+\epsilon)_d z_n^d + g_{d-1}(z_n),$$

where

$$(1+\epsilon)_d = (1+\epsilon)e^{i(1+\epsilon)_d}, \qquad \epsilon > 0,$$

and

$$g_{d-1}(z_n) = \sum_{l=0}^{d-1} (1+\epsilon)_l z_n^l.$$

In the remainder, we will have to handle several integrals of the form

$$I(x_n) = \int_J S_{x_n}(r)e^{-g_{x_n}(r)}\, dr,$$

where S_{x_n} and g_{x_n} are C^3-functions on the interval J, and the real number x_n tends to $+\infty$. We will make use of the following variant of the Laplace method (see [130]).

Corollary (5.3.22)[185]: For $z_n = x_n e^{i\phi}$, with $x_n > 0$ and $e^{i((1+\epsilon)_d + d\phi)} = 1$, we have

$$\mathcal{B}(z_n) \gtrsim \int_0^{+\infty} (rx_n)^{-\frac{d+\epsilon}{2}} r^{2(d+\epsilon)-1} e^{-h_{x_n}(r)} dr$$

as $x_n \to +\infty$, where

$$h_{x_n}(r) = (r^{d+\epsilon} - x_n^{d+\epsilon})^2 - 2(1+\epsilon)(x_n^d - r^d) + C(r^{d-1} + x_n^{d-1} + 1), \quad (61)$$

for some positive constant C.

Proof. It is easy to see that

$$\mathcal{B}(z_n) = \int_{\mathbb{C}} |K_{d+\epsilon}(w_n, z_n)|^2 \, e^{2Re(g(z_n)-g(w_n))} \, [K_{d+\epsilon}(z_n, z_n)]^{-1} \, e^{-|w_n|^{2(d+\epsilon)}} \, dA(w_n),$$

which, in terms of polar coordinates, can be rewritten as

$$\int_0^{+\infty} \int_{\pi}^{-\pi} \left|K_{d+\epsilon}\left(re^{i\theta}, z_n\right)\right|^2 e^{2Re\left(g(z_n)-g\left(re^{i\theta}\right)\right)} \, [K_{d+\epsilon}(x_n, x_n)]^{-1} \, e^{-r^{2(d+\epsilon)}} r \, dr \, d\theta.$$

By Lemma (5.3.2), $\mathcal{B}(z_n)$ is greater than or equal to

$$\int_0^{+\infty} \int_{|\theta-\phi|\le c\theta_0(rx_n)} \left|K_{d+\epsilon}\left(re^{i\theta}, z_n\right)\right|^2 e^{2Re\left(g(z_n)-g\left(re^{i\theta}\right)\right)} \, [K_{d+\epsilon}(x_n, x_n)]^{-1} \, e^{-r2(d+\epsilon)} \, rdr \, d\theta.$$

This together with Lemma (5.3.2) shows that

$$\mathcal{B}(z_n) \int_0^{+\infty} r^{2(d+\epsilon-1)} e^{-\left(r^{d+\epsilon}-x_n^{d+\epsilon}\right)^2} \, I(r, z_n) r dr,$$

where

$$I(r, z_n) = \int_{|\theta-\phi|\le c\theta_0(rx_n)} e^{2Re\left(g(z_n)-g\left(re^{i\theta}\right)\right)} \, d\theta.$$

Note that

$$I(r, z_n) = \int_{|\theta-\phi|\le c\theta_0(rx_n)} e^{2Re\left[(1+\epsilon)e^{i(1+\epsilon)_d} \left(x_n^d e^{id\phi} - r^d e^{id\theta}\right)\right] + 2Re\left[g_{d-1}(z_n) - g_{d-1}\left(re^{i\theta}\right)\right]} d\theta$$

$$= \int_{|\theta-\phi|\le c\theta_0(rx_n)} e^{2Re\left[(1+\epsilon)e^{i\left((1+\epsilon)_d + d\phi\right)}\left(x_n^d - r^d e^{id(\theta-\phi)}\right)\right] + 2Re\left[g_{d-1}(z_n) - g_{d-1}\left(re^{i\theta}\right)\right]} d\theta.$$

The condition on ϕ yields

$$I(r, z_n) = \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)Re\left[(1+\epsilon)\left(x_n^d - r^d e^{id\theta}\right)\right] + 2Re\left[g_{d-1}(z_n) - g_{d-1}\left(re^{(i\theta+\phi)}\right)\right]} \, d\theta.$$

Since

$$g_{d-1}(z_n) - g_{d-1}\left(re^{i(\theta+\phi)}\right) = \sum_{l=0}^{d-1} (1+\epsilon)_l \left(x_n^l \, e^{il\phi} - r^l \, e^{il(\theta+\phi)}\right),$$

we have

$$Re\left[g_{d-1}(z_n) - g_{d-1}\left(re^{i(\theta+\phi)}\right)\right] \ge -C\left(r^{d-1} + x_n^{d-1} + 1\right)$$

for some constant C. It follows that

$$I(r, z_n) \ge e^{-C\left(r^{d-1}+x_n^{d-1}+1\right)} \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)Re\left[\left(x_n^d - r^d e^{id\theta}\right)\right]} d\theta.$$

For the integral we have

$$J(r,z_n) := \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)Re[(x_n^d - r^d e^{id\theta})]} d\theta$$
$$= \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)\left(x_n^d - r^d \cos(d\theta)\right)} d\theta$$
$$= \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)\left(x_n^d - r^d + (-\cos(d\theta)+1)r^d\right)} d\theta$$
$$= \int_{|\theta|\le c\theta_0(rx_n)} e^{2(1+\epsilon)\left(x_n^d - r^d + 2\left(\sin\left(\frac{d\theta}{2}\right)^2\right) r^d\right)} d\theta$$
$$\ge e^{2(1+\epsilon)(x_n^d - r^d)} \int_{|\theta|\le c\theta_0(rx_n)} e^{4|(1+\epsilon)_d| \sin\left(\frac{d\theta}{2}\right)^2 r^d} d\theta$$
$$\ge e^{2(1+\epsilon)(x_n^d - r^d)} \int_{|\theta|\le c\theta_0(rx_n)} d\theta \gtrsim e^{2(1+\epsilon)(x_n^d - r^d)} (rx_n)^{-\frac{d+\epsilon}{2}} ,$$

which completes the proof of the corollary.

Corollary (5.3.23)[185]: Assume $\epsilon = 0$. For $z_n = x_n e^{i\phi}$,where $x_n > 0$ and $e^{i((1+\epsilon)_d + d\phi)} = 1$, we have

$$\mathcal{B}(z_n) e^{(1+o(1))\frac{2(1+\epsilon)}{(1+2(1+\epsilon))} x_n^{d+2\epsilon}}, \qquad x_n \to +\infty.$$

Proof. For x_n large enough, the function h_{x_n} defined in (61) is convex on some interval $[M_{x_n}, +\infty)$ and attains its minimum at some point r_{x_n}. In order to bound $\mathcal{B}(z_n)$ from below, we shall use the modified Laplace method from Lemma (5.3.10). Since

$$h'_{x_n}(r) = (d+2\epsilon) r^{\frac{d+2\epsilon-2}{2}} \left(r^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right) + 2(1+\epsilon) d r^{d-1} + C\,(d-1) r^{d-2}, \tag{62}$$

we have

$$h'_{x_n}(r) = (d+2\epsilon)(1 + 2(1+\epsilon)) r^{d+2\epsilon-1} - (d+2\epsilon) x_n^{\frac{d}{2}+\epsilon} r^{\frac{d+2\epsilon-2}{2}} + C(d-1) r^{d-2},$$

and

$$h''_{x_n}(r) = (d+2\epsilon)(d+2\epsilon-1)(1+2(1+\epsilon)) r^{d+2\epsilon-2} - (d + 2\epsilon)\left(\frac{d+2\epsilon-2}{2}\right) x_n^{\frac{d}{2}+\epsilon} r^{\frac{d}{2}+\epsilon-2} + C(d-1)(d-2) r^{d-3}.$$

Writing $h'_{x_n}(r_{x_n}) = 0$ and letting x_n tend to $+\infty$, we obtain

$$\left(\frac{d}{2}+\epsilon\right)(1 + 2(1+\epsilon))(r_{x_n})^{d+2\epsilon-1} \sim \left(\frac{d}{2}+\epsilon\right) x_n^{\frac{d}{2}+\epsilon} r_{x_n}^{\frac{d+2\epsilon-2}{2}},$$

or

$$r_{x_n} \sim (1 + 2(1+\epsilon))^{-\frac{1}{\frac{d}{2}+\epsilon}} x_n. \tag{63}$$

Thus there exists ρ_{x_n}, which tends to 0 as x_n tends to $+\infty$, such that

$$r_{x_n} = (1 + 2(1+\epsilon))^{-\frac{1}{\frac{d}{2}+\epsilon}} x_n (1 + \rho_{x_n}). \tag{64}$$

When x_n tends to $+\infty$, we have

$$h_{x_n}(r_{x_n}) \sim \left(r_{x_n}^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right)^2 + 2(1+\epsilon)\left(r_{x_n}^{d+2\epsilon} - x_n^{d+2\epsilon}\right)$$
$$\sim \left(r_{x_n}^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right)\left[\left(r_{x_n}^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right) + 2(1+\epsilon)\left(r_{x_n}^{\frac{d}{2}+\epsilon} + x_n^{\frac{d}{2}+\epsilon}\right)\right]$$
$$\sim x_n^{d+2\epsilon}\left[(1 + 2(1+\epsilon))^{-1}\left(1 + \rho_{x_n}\right)^{\frac{d}{2}+\epsilon} - 1\right]\left[(1 + 2(1+\epsilon))^{-1}\left(1 + \rho_{x_n}\right)^{\frac{d}{2}+\epsilon} - 1 + 2(1 + \epsilon)\left((1 + 2(1+\epsilon))^{-1}\left(1 + \rho_{x_n}\right)^{\frac{d}{2}+\epsilon} + 1\right)\right]$$
$$\sim -x_n^{d+2\epsilon}\frac{2(1+\epsilon)}{(1 + 2(1+\epsilon))},$$

or

$$-h_{x_n}(r_{x_n}) \sim x_n^{d+2\epsilon}\frac{2(1+\epsilon)}{(1 + 2(1+\epsilon))}. \qquad (65)$$

In order to estimate $c_{x_n} := h''_{x_n}(r_{x_n})$, we compute that

$$h''_{x_n}(r_{x_n}) \sim 2\left(\frac{d}{2}+\epsilon\right)^2 (1 + 2(1+\epsilon))^{\frac{4-d-2\epsilon}{d+2\epsilon}} x_n^{d+2\epsilon-2}.$$

Thus we get

$$c_{x_n} \approx x_n^{d+2\epsilon-2}. \qquad (66)$$

For r in a neighborhood of r_{x_n} we set $r = (1+\sigma_{x_n})r_{x_n}$, where $\sigma_{x_n} = \sigma_{x_n}(r) \to 0$ as $x_n \to +\infty$; a little computation shows that

$$h''_{x_n}(r) \sim h''_{x_n}(r_{x_n})$$

as $x_n \to +\infty$. Taking $\tau_{x_n} = r_{x_n}^{1/2}$ and $|r - r_{x_n}| < \tau_{x_n}$, we have $h''_{x_n}(r) = (1 + o(1))c_{x_n}$, so

$$h_{x_n}(r) - h_{x_n}(r_{x_n}) = \frac{1}{2}c_{x_n}(r - r_{x_n})^2(1 + o(1)).$$

Thus

$$\int_{|r-r_{x_n}|<\tau_{x_n}} e^{-\frac{1}{2}c_{x_n}(r-r_{x_n})^2(1+o(1))}dr = \int_{|t|<\tau_{x_n}} e^{-\frac{1}{2}c_{x_n}t^2(1+o(1))}dt$$
$$\sim \frac{1}{\sqrt{c_{x_n}}}\int_{|y_n|<\tau_{x_n}\sqrt{c_{x_n}}} e^{-\frac{1}{2}y_n^2}dy_n \approx \frac{1}{\sqrt{c_{x_n}}},$$

because $c_{x_n}\tau_{x_n}^2 \approx r_{x_n}^{d+2\epsilon-1}$ tends to $+\infty$ as x_n tends to $+\infty$. Finally, the estimates

$$\mathcal{B}(z_n) \gtrsim \int_{|r-r_{x_n}|<\tau_{x_n}} \left(r_{x_n}\right)^{-(d+2\epsilon)} r^{d+2\epsilon-1} e^{-h_{x_n}(r)}\, dr$$
$$= \int_{|r-r_{x_n}|<\tau_{x_n}} (rx_n)^{-(d+2\epsilon)} r^{d+2\epsilon-1} e^{-h_{x_n}(r_{x_n})} e^{-[h_{x_n}(r)-h_{x_n}(r_{x_n})]}\, dr$$
$$= e^{-h_{x_n}(r_{x_n})} \int_{|r-r_{x_n}|<\tau_{x_n}} (rx_n)^{-(d+2\epsilon)} r^{d+2\epsilon-1} e^{-\frac{1}{2}c_{x_n}(r-r_{x_n})^2(1+o(1))}\, dr$$
$$\sim e^{-h_{x_n}(r_{x_n})} r_{x_n}^{\frac{3}{2}\left(\frac{d}{2}+\epsilon\right)-1} x_n^{-(d+2\epsilon)} \int_{|r-r_{x_n}|<\tau_{x_n}} e^{-\frac{1}{2}c_{x_n}(r-r_{x_n})^2(1+o(1))}\, dr$$
$$\approx e^{-h_{x_n}(r_{x_n})} r_{x_n}^{\frac{3}{2}\left(\frac{d}{2}+\epsilon\right)-1} x_n^{-(d+2\epsilon)} \frac{1}{\sqrt{c_{x_n}}}$$

along with (63), (65), and (66) give the corollary

Corollary (5.3.24)[185]: Assume $\epsilon > 0$. For $z_n = x_n e^{i\phi}$, with $x_n > 0$ and $e^{i((1+\epsilon)_d+d\phi)} = 1$, we have

$$\mathcal{B}(z_n) \gtrsim e^{(1+o(1))\frac{(1+\epsilon)^2 d^2}{\left(\frac{d}{2}+\epsilon\right)^2} x_n^{d-2\epsilon} - C x_n^{\frac{d}{2}-\epsilon-1}}, \qquad x_n \to +\infty$$

for some positive constant C

Proof. Let $\tau_{x_n} = o(x_n)$ be a positive real number that will be specified later. As in the proof of Corollary (5.3.22) we have

$$\mathcal{B}(z_n) \gtrsim \int_0^{+\infty} r^{2\left(\frac{d}{2}+\epsilon-1\right)} e^{-\left(r^{\frac{d}{2}+\epsilon}-x_n^{\frac{d}{2}+\epsilon}\right)^2} I(r,z_n) r\, dr$$
$$\gtrsim \int_{|r-x_n|\le\tau_{x_n}} r^{2\left(\frac{d}{2}+\epsilon-1\right)} e - \left(r^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right)^2 I(r,z_n) r\, dr,$$

where

$$I(r,z_n) = \int_{|\theta-\phi|\le c\theta_0(rx_n)} e^{2Re\left(g(z_n)-g(re^{i\theta})\right)}\, d\theta.$$

There exists $c' > 0$ such that for $|r - x_n| \le \tau_{x_n}$ we have

$$I(r,z_n) \ge \int_{|\theta-\phi|\le c'\theta_0(x_n^2)} e^{2Re\left(g(z_n)-g(re^{i\theta})\right)}\, d\theta$$
$$= \int_{|\theta|\le c'\theta_0(x_n^2)} e^{2(1+\epsilon)Re\left(x_n^d - r^d e^{id\theta}\right) + 2Re\left[g_{d-1}(z_n)-g_{d-1}(re^{i\theta})\right]} d\theta$$
$$= \int_{|\theta|\le c'\theta_0(x_n^2)} e^{2(1+\epsilon)Re\left(x_n^d - r^d e^{id\theta}\right)} - 2\sum_{l=0}^{d-1} |(1+\epsilon)_l| \left|x_n^l - r^l e^{il\theta}\right| d\theta.$$

Now for $|r - x_n| \le \tau_{x_n}$, we write $r = (1+\sigma)x_n$, where σ tends to 0 as $x_n \to +\infty$. Thus for $0 \le l \le d-1$ and $|\theta| \le c'\theta_0(x_n^2)$, we obtain

$$\left|x_n^l - r^l e^{il\theta}\right|^2 = x_n^{2l}[1 - 2(1+\sigma)^l \cos(l\theta) + (1+\sigma)^{2l}]$$
$$= x_n^{2l}\left[1 - 2\left(1 + l\sigma + O(\sigma^2)\right) \cos(l\theta) + 1 + 2l\sigma + O(\sigma^2)\right]$$
$$= x_n^{2l}\left[2\left(1 - \cos(l\theta)\right)(1 + l\sigma) + O(\sigma^2)\right]$$
$$\lesssim x_n^{2l}\left[\sin^2\left(\frac{l\theta}{2}\right) + \sigma^2\right] \qquad \lesssim x_n^{2l}[\theta^2 + \sigma^2].$$

Next choosing $|\sigma| \le x_n^{-\left(\frac{d}{2}+\epsilon\right)}$, we get

$$\left|x_n^l - r^l e^{il\theta}\right| \lesssim x_n^{2l} x_n^{-(d+2\epsilon)} \lesssim x_n^{d-2(1+\epsilon)}$$

or

$$\left|x_n^l - r^l e^{il\theta}\right| \lesssim x_n^{\frac{d}{2}-(1+\epsilon)}.$$

Thus there exists a positive constant C such that for $|r - x_n| \le \tau_{x_n}$ and $|\theta| \le c'\theta_0(x_n^2)$,

$$2\sum_{l=0}^{d-1} |(1+\epsilon)_l|\left|x_n^l - r^l e^{il\theta}\right| \le Cx_n^{\frac{d}{2}-(1+\epsilon)}.$$

It follows that

$$I(r, z_n) \ge \int_{|\theta|\le c'\theta_0(x_n^2)} e^{2(1+\epsilon)Re\left(x_n^d - r^d e^{id\theta}\right) - Cx_n^{\frac{d}{2}-(1+\epsilon)}} d\theta$$

$$\gtrsim x_n^{-\left(\frac{d}{2}+\epsilon\right)} e^{2(1+\epsilon)Re\left(x_n^d - r^d e^{id\theta}\right) - Cx_n^{\frac{d}{2}-(1+\epsilon)}}.$$

Then

$$\mathcal{B}(z_n)\int_{|r-x_n|\le\tau_{x_n}} r^{d+2\epsilon-1} e^{-\left(r^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right)^2} x_n^{-\left(\frac{d}{2}+\epsilon\right)} e^{2(1+\epsilon)\left(x_n^d - r^d\right) - Cx_n^{\frac{d}{2}-(1+\epsilon)}} dr$$

$$= x_n^{-\left(\frac{d}{2}+\epsilon\right)} e^{-Cx_n^{\frac{d}{2}-(1+\epsilon)}} \int_{|r-x_n|\le\tau} r^{d+2\epsilon-1} e^{-h_{x_n}(r)} dr,$$

where

$$h_{x_n}(r) = \left(r^{\frac{d}{2}+\epsilon} - x_n^{\frac{d}{2}+\epsilon}\right)^2 - 2(1+\epsilon)(x_n^d - r^d).$$

It is easy to see that h_{x_n} attains its minimum at r_{x_n} with $r_{x_n} \sim x_n$ as $x_n \to +\infty$. Again we write

$$r_{x_n} = x_n\left(1 + \rho_{x_n}\right), \tag{67}$$

where ρ_{x_n} tends to 0 as $x_n \to +\infty$. Using the fact that $h'_{x_n}\left(r_{x_n}\right) = 0$, we have $(d + 2\epsilon)x_n^{d+2\epsilon-1}\left(1 + \rho_{x_n}\right)^{\frac{d}{2}+\epsilon-1}\left[\left(1 + \rho_{x_n}\right)^{\frac{d}{2}+\epsilon} - 1\right] \sim -2(1+\epsilon)dx_n^{d-1}\left(1 + \rho_{x_n}\right)^{d-1}$, and

$$(d + 2\epsilon)x_n^{d+2\epsilon-1}\left(\frac{d}{2}+\epsilon\right)\rho_{x_n} \sim -2(1+\epsilon)dx_n^{d-1}.$$

Therefore,

$$\rho_{x_n} \sim -\frac{(1+\epsilon)d}{\left(\frac{d}{2}+\epsilon\right)^2} x_n^{-2\epsilon}. \tag{68}$$

Since

$$h''_{x_n}(r) = (d+2\epsilon)(d+2\epsilon - 1)r^{d+2\epsilon-2} - (d+2\epsilon)\left(\frac{d}{2}+\epsilon - 1\right)x_n^{\frac{d}{2}+\epsilon} r^{\frac{d}{2}+\epsilon-2} + 2(1+\epsilon)d(d-1)r^{d-2}$$

and $\epsilon > 0$, we get

$$h''_{x_n}(r_{x_n}) \sim (d + 2\epsilon)x_n^{d+2\epsilon-2}\left[(d+2\epsilon-1)(1+\rho_{x_n})^{d+2\epsilon-2} - \left(\frac{d}{2}+\epsilon-1\right)(1+\rho_{x_n})^{\frac{d}{2}+\epsilon-2}\right] \sim 2\left(\frac{d}{2}+\epsilon\right)^2 x_n^{d+2\epsilon-2}.$$

Also,

$$h_{x_n}(r_{x_n}) \sim x_n^{d+2\epsilon}\left[(1+\rho_{x_n})^{\frac{d}{2}+\epsilon} - 1\right]^2 + 2(1+\epsilon)x_n^d\left[(1+\rho_{x_n})^d - 1\right] + C\left(x_n^{d-1} + r_{x_n}^{d-1} + 1\right) \sim \left(\frac{d}{2}+\epsilon\right)^2 \rho_{x_n}^2 x_n^{d+2\epsilon} + 2(1+\epsilon)x_n^d\, d\rho_{x_n}$$

It follows that

$$c_{x_n} \sim 2\left(\frac{d}{2}+\epsilon\right)^2 x_n^{d+2\epsilon-2}, \tag{69}$$

and

$$-h_{x_n}(r_{x_n}) \sim \frac{(1+\epsilon)^2 d^2}{\left(\frac{d}{2}+\epsilon\right)^2} x_n^{d-2\epsilon}. \tag{70}$$

Reasoning as in the proof of Corollary (5.3.23), we arrive at

$$\mathcal{B}(z_n) \gtrsim\gtrsim x_n^{-\left(\frac{d}{2}+\epsilon\right)} e^{-Cx_n^{\frac{d}{2}-(1+\epsilon)}} e^{-h_{x_n}(r_{x_n})} x_n^{d+2\epsilon-1}\frac{1}{\sqrt{c_{x_n}}}.$$

The desired estimate then follows from (70), and (69).

Corollary (5.3.25)[185]: Suppose u^2 and v^2 are functions in $\mathcal{F}^2_{\frac{d}{2}+\epsilon}$, not identically zero, such that $|\widetilde{u^2|}^2(z_n)|\widetilde{v^2|}^2(z_n)$ is bounded on the complex plane. Then there exists a nonzero constant C and a polynomial g of degree at most $\left(\frac{d}{2}+\epsilon\right)$ such that $u^2(z_n) = eg(z_n)$ and $v^2(z_n) = Ce^{-g(z_n)}$.

Proof. It is easy to check that for $u^2 \in \mathcal{F}^2_{\frac{d}{2}+\epsilon}$ we have

$$u^2(z_n) = \int_{\mathbb{C}} u^2(x_n)\left|k_{z_n}(x_n)\right|^2 d\lambda_{\frac{d}{2}+\epsilon}(x_n) = \tilde{u}^2(z_n).$$

Also, it follows from the Cauchy-Schwarz inequality that $|u^2(z_n)|^2 \le |\widetilde{u^2|}^2(z_n)$. So if $|\widetilde{u^2|}^2(z_n)|\widetilde{v^2|}^2(z_n)$ is bounded on $\mathbb{C}$, then $\mathcal{B}(z_n)$ and $|u^2(z_n)v^2(z_n)|^2$ are also bounded. Consequently, u^2v^2 is a constant, there is a non-zero constant C and a polynomial g such that $u^2 = e^g$ and $v^2 = Ce^{-g}$. The condition $u^2 \in \mathcal{F}^2_{\frac{d}{2}+\epsilon}$ implies that the degree d of g is at most $(d+2\epsilon)$; see Corollary (5.3.15).

Without loss of generality we shall consider the case where $u^2(z_n) = e^{g(z_n)}$ and $v^2(z_n) = e^{-g(z_n)}$. We will show that that the boundedness of $\mathcal{B}(z_n)$ implies $\epsilon \ge 0$. If $(d+\epsilon)$ is an integer, Corollary (5.3.23) shows that we must have $\epsilon > 0$.
Thus, in any case ($(d+2\epsilon)$ being an integer or not), a necessary condition is $\epsilon > 0$. The desired result now follows from Corollary (5.3.24).

Chapter 6
Finite Rank Perturbations and Theorem of Brown–Halmos Type

We study finite rank perturbations of the Brown-Halmos type results involving products of Toeplitz operators acting on the Bergman space. We show that operator is called the Toeplitz operator with symbol μ. We show that T_μ has finite rank if and only if μ is a finite linear combination of point masses. Application to Toeplitz operators on the Bergman space is immediate. We show that there is no nontribyl rank one perturbation. However, in the case rank $m \geq 2$, we construct an example that shows there are bounded harmonic functions f, g and h such that $T_f T_g - T_h$ has rank exactly m.

Section (6.1): Perturbations of Toeplitz Operators

Ahern and Cuckovic [101] proved an analogue of the well-known Brown- Halmos theorem for the Bergman space Toeplitz operators with harmonic symbols. To state the result, we introduce the notation. Let $\mathbb{D}$ denote the open unit disk in the complex plane and let dA denote the normalized Lebesgue area measure on $\mathbb{D}$. As usual, $L^2(\mathbb{D})$ is the space of measurable complex valued functions f on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty$. The Bergman space $L_a^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ consisting of the analytic functions on $\mathbb{D}$. Let $P : L^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ denote the orthogonal projection. For a bounded function u on $\mathbb{D}$ we have the Toeplitz operator $T_u: L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ given by $T_u f = P(uf)$. We denote the Laplacian $\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}$ and the invariant Laplacian by $\tilde{\Delta} = (1 - |z|^2)^2 \Delta$. We can now state the above mentioned theorem.

Theorem (6.1.1)[163]: Suppose f and g are bounded harmonic functions and h is a bounded C^2 function such that Δh is bounded on D. If $T_f T_g = T_h$, then either f is conjugate analytic or g is analytic. In either case, $h = fg$.

Later on, Ahern [164] removed the assumptions on h and showed the theorem is true for $h \in L^\infty(\mathbb{D})$. From Theorem (6.1.1), Ahern and ˇCuˇckovi´c obtained a sequence of results on products of Toeplitz operators that are parallel to the corollaries of the Brown-Halmos theorem for the Hardy space obtained in [98]. We list some of them.

Corollary (6.1.2)[163]: If f, g and h are bounded harmonic functions and $T_f T_g = T_h$, then one of the following holds:

(i) f and g are analytic.
(ii) f and g are conjugate analytic.
(iii) f is constant.
(iv) g is constant.

The next one resolved an open problem about zero products.

Corollary (6.1.3)[163]: If f and g are bounded harmonic functions and $T_f T_g = 0$, then either $f = 0$ or $g = 0$.

Corollary (6.1.4)[163]: If f and g are bounded and harmonic and $T_f T_g = I$, then either f and g are both analytic or they are both conjugate analytic. In either case $fg = 1$.

Corollary (6.1.5)[163]: If f is bounded and harmonic and $T_f^2 = T_f$, then $f \equiv 0$ or $f \equiv 1$.

Corollary (6.1.6)[163]: If f and g are bounded harmonic and $T_f T_g = T_f g$, then either g is analytic or f is conjugate analytic.

This last corollary was proved earlier by Zheng [58] using a different method. We point out that in [165] we constructed examples of Toeplitz operators with radial symbols that show that some of these corollaries do not hold in general. One of the main steps in the proof of Theorem (6.1.1) is the study of the range of the Berezin transform. For any integrable function f on $\mathbb{D}$, the Berezin transform is defined by

$$Bf(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - z\bar{w}|^4} \, dA(w).$$

Let $K_z(w) = \frac{1}{(1-w\bar{z})^2}$ denote the Bergman kernel for $z \in \mathbb{D}$. Then $k_z(w)$ denotes the normalized Bergman kernel:

$$k_z(w) = \frac{1 - |z|^2}{(1 - w\bar{z})^2} \ , w \in \mathbb{D}.$$

The Berezin transform can then be expressed as $Bf(z) = \langle fk_z, k_z\rangle$, where $\langle \cdot , \rangle$ denotes the $L^2(\mathbb{D})$ inner product. We can also define the Berezin transform of any bounded operator S as $B(S)(z) = Sk_z, k_z$, for $z \in \mathbb{D}$. Another important step in the proof of Theorem (6.1.1) is the proof of the fact that a rank 1 Toeplitz operator on $L_a^2(\mathbb{D})$ must be 0. For any operator A, rank (A) = dim Ran(A). Compact Toeplitz operators on $L_a^2(\mathbb{D})$ have been characterized by Axler and Zheng using the Berezin transform (see [46]). Surprisingly characterizing finite rank Toeplitz operators on $L_a^2(\mathbb{D})$ is still an open problem. The common conjecture among the experts is that a finite rank Toeplitz operator on $L_a^2(\mathbb{D})$ must be 0. In Guo, Sun and Zheng [100] have proved this conjecture in a special case.

Theorem (6.1.7)[163]: Suppose that f is in $L^\infty(D)$ and $f = \sum_{j=1}^{l} f_j\ (z)\ \overline{(g_j(z))}$ for finitely many functions f_j and g_j analytic on D. If T_f has finite rank, then $f = 0$.

Using this theorem, they obtained an extension of Corollary(6.1.2) on the zero products of Toeplitz operators. More specifically, they proved that for two bounded harmonic functions f and g, if the product T_fT_g has finite rank, then either $f = 0$ or $g = 0$. We think of this product as $T_fT_g = 0 + F, F$ finite rank, so product is a finite rank perturbation of 0. Similarly, they also obtained a result characterizing finite rank semicommutators $T_{fg} - T_fT_g$ of Toeplitz operators with harmonic symbols. This is a finite rank perturbation extension of Corollary (6.1.5), where we considered the case $T_{fg} - T_fT_g = 0$. Inspired by these results of Guo, Sun and Zheng, we want to obtain results on finite rank perturbations of the products in the other corollaries listed above. They will follow from the following result. Before we state it, we recall two known results. First, an operator F of finite rank N can be written as$\sum_{j=1}^{N} x_j \otimes y_j$, for some functions x_j , y_j in $L_a^2(\mathbb{D})\ for\ j = 1,\ldots,N$. Here $x \otimes y$ is the rank one operator defined by $(x \otimes y)h = h, yx$, where x, y, h are in $L_a^2(\mathbb{D})$. Second, if f is a bounded harmonic function on $\mathbb{D}$, f can be written as $f_1 + f_2$, where f_1 and f_2 are analytic functions that belong to the Bloch space $B = \{f : f\ analytic\ on\ \mathbb{D}\ and \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty\}$.

Theorem (6.1.8)[163]: Suppose $f = f_1 + \overline{f_2}, g = g_1 + \overline{g_2}$ and $h = h_1 + \overline{h_2}$ are bounded harmonic functions on $\mathbb{D}$ such that $h_1, h_2 \in H^\infty(\mathbb{D})$. Suppose that $T_fT_g = T_{h^n} + F$, where $F = \sum_{J=1}^{N} x_j \otimes y_j$ is of finite rank $N, x_j , y_j \in L_a^2(\mathbb{D})$for $j = 1,\ldots,N$ and $N, n \in \mathbb{N}$. Then:

(i)$g_1(z)\overline{f_2}(z) - h^n(z)$ is harmonic,

(ii)$f(z)g(z) = h^n(z) + (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}\ , for\ z \in \mathbb{D}$.

Conversely, suppose $f = f_1 + \overline{f_2}, g = g_1 + \overline{g_2}$ and $h = h_1 + \overline{h_2}$ are bounded harmonic functions on $\mathbb{D}$ such that (i) holds on $\mathbb{D}$. If there exist nonzero vectors

$x_1, \ldots, x_N, y_1, \ldots, y_N$ in $L_a^2(\mathbb{D})$, such that (ii) holds for $z \in \mathbb{D}$, then $T_f T_g = T_{h^n} + F$, where $F = \sum_{j=1}^{N} x_j \otimes y_j$ is a finite rank operator.
In particular, condition (ii) implies that $fg = h^n$ $a.e. on$ $\partial\mathbb{D}$. If in addition, $T_{h^n} + F$ is an isometry, then

$$\|h\|_{L^\infty(\mathbb{D})} = \|h\|_{L^\infty(\partial\mathbb{D})} = \|fg\|_{L^\infty(\partial\mathbb{D})} = 1.$$

Proof. Suppose that $T_f T_g = T_{h^n} + F$. Then we have

$$B(T_f T_g) = B(h^n) + B(F). \quad (1)$$

As in [101],

$$B(T_f T_g)(z) = f_1(z)g_1(z) + f_1(z)\overline{g_2(z)} + \overline{f_2(z)}\,\overline{g_2(z)} + B(\bar{f}_2 g_1)(z),$$ for $z \in \mathbb{D}$. It is also easy to show that

$$B(F)(z) = \sum_{j=1}^{N} B(x_j \otimes y_j)(z) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}.$$

Thus (1) can be written as

$$f_1(z)g_1(z) + f_1(z)\overline{g_2(z)} + \overline{f_2(z)}\,\overline{g_2(z)} + B(\bar{f}_2 g_1) - B(h^n) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} \quad (2)$$

for $z \in D$. It is well known that the Berezin transform fixes L^1-harmonic functions, i.e., $B(u) = u$ if u is harmonic. Thus (2) can be written as

$$B(f_1 g_1 + \bar{f}_2 \bar{g}_2 + \bar{f}_2 g_1 - h^n)(z) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} - f_1(z)\overline{g_2(z)}.$$

Apply the invariant Laplacian $\tilde{\Delta}$ to both sides and use the fact that $\tilde{\Delta}$ commutes with B (see [101]), to obtain

$$B\left(\tilde{\Delta}(\bar{f}_2 g_1 - h^n)\right)(z) = \tilde{\Delta}\left[(1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}\right] - \tilde{\Delta}\left[f_1(z)\overline{g_2(z)}\right] \quad (3)$$

for $z \in \mathbb{D}$. Let $\sigma = \tilde{\Delta}(\bar{f}_2 g_1 - h_n)$. After cancelling $(1 - |z|^2)^2$ on both sides of (3) we have

$$\int_{\mathbb{D}} \frac{\sigma(\xi)}{|1 - \bar{\xi}z|^4}\, dA(\xi) = \Delta\left[(1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}\right] - f'_1(z)\overline{g'_2(z)}. \quad (4)$$

Notice that $(1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} = \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} - 2\sum_{j=1}^{N} z\, x_j(z)\overline{z y_j(z)} + \sum_{j=1}^{N} z^2 x_j(z)\overline{z^2 y_j(z)}$ for $z \in \mathbb{D}$, which can be written as $3\sum_{j=1}^{N} \tilde{x}_j(z)\overline{\tilde{y}_j(z)}$ with $\tilde{x}_j, \tilde{y}_j \in L_a^2(\mathbb{D})$. With this in mind, we can complexify (4) as was done in Lemma 2 of [101] to obtain

$$\frac{\sigma(\xi)}{(1 - \bar{\xi}z)^2 (1 - \xi w)^2}\, dA(\xi) = \sum_{j=1}^{3N} \tilde{x}_j(z)\overline{\tilde{y}'_j(\overline{w})} - f_1'(z) g_2'(\overline{w}) \quad (5)$$

for all $z, w \in \mathbb{D}$. If we differentiate (5) k times with respect to w and then let $w = 0$, we get

$$\int_{\mathbb{D}} \frac{\xi^k \sigma(\xi)}{(1 - \bar{\xi}z)^2}\, dA(\xi) = \sum_{j=1}^{3N} a_{k_j} \tilde{x}_j'(z) - c_k f_1'(z) \quad (6)$$

for some constants a_{kj}, c_k, $k = 1, 2, \ldots$. Then (6) tells us that for any $k \in \mathbb{N}$, we have

$$T_\sigma(\xi^k) = \int_{\mathbb{D}} \frac{\xi^k \sigma(\xi)}{(1 - \bar{\xi} z)^2} dA(\xi) = \sum_{j=1}^{3N} a_{kj} \tilde{x}_j'(z) - c_k f_1'(z)$$

Using the argument of Proposition 4 in [100] we have that T_σ has finite rank.

Notice that $\tilde{\Delta}(\overline{f_2} g_1) = (1 - |z|^2)^2 \bar{f}'_2(z) g_1{}'(z)$ is bounded since f_2 and g_1 belong to the Bloch space. If $n = 1$, then $\Delta h = 0$. For $n > 1$,

$$h^n = (h_1 + h_2)^n = \sum_{k=0}^{n} \binom{n}{k} h_1^k \cdot h_2^{-n-k}$$

so that

$$\tilde{\Delta}(h^n) = \sum_{k=1}^{n} \binom{n}{k} (1 - |z|^2)^2 k h_1^{k-1} \cdot h_1' \cdot (n - k) \bar{h}_2^{(n-k-1)} \bar{h}_2'$$

which is also bounded, since h_1 and h_2 are bounded by the assumption and they also belong to the Bloch space.

Thus $\sigma(z)$ is in $L^\infty(\mathbb{D})$ and it is of the form $\sum_{j=1}^{3n+3} F_j(z)\overline{G_j(z)}$ for some analytic functions F_j and $G_j, j = 1, \dots, n$. By Theorem $B, \sigma \equiv 0$, and hence $\bar{f}_2 g_1 - h^n$ is a harmonic function. Thus (i) holds. Now (2) gives

$$f_1(z)g_1(z) + f_1(z)\overline{g_2}(z) + \overline{f_2(z)\, g_2(z)}$$
$$+\bar{f}_2(z)g_1(z) - h^n(z) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}$$

for all $z \in \mathbb{D}$. In other words $(fg)(z) - h^n(z) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}$ which gives (ii). The expression on the right-hand side is equal to $B\left(\sum_{j=1}^{n} x_j \otimes y_j\right)(z)$ which goes to 0 as $|z| \to 1$, since $\sum_{j=1}^{N} x_j \otimes y_j$ is a finite rank operator and therefore compact. Hence $fg = h^n$ a.e. on $\partial\mathbb{D}$. Also notice that (3) implies that $f_1(z)\overline{g_2(z)} = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} + u(z)$ for some harmonic function u. If the operator $F = 0$, then this would imply $f_1'(z)\overline{g_2'(z)} = 0$ on $\mathbb{D}$. This means that f_1 is constant or g_2 is constant. In other words, f is conjugate analytic or g is analytic which is consistent with Theorem (6.1.1) from [101].

Conversely, suppose that $g_1(z)\bar{f}_2(z) - h^n(z)$ is harmonic on $\mathbb{D}$, and $f(z)g(z) = h^n(z) + (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}$. As calculated earlier,

$$\begin{aligned} &B\left(T_f T_g - T_{h^n}\right)(z) \\ &= f_1(z)g_1(z) + f_1(z)\overline{g_2(z)} + \overline{f_2(z)g_2(z)} + B\left(\bar{f}_2\, g_1 - h^n\right)(z) \\ &= (fg)(z) - h^n(z) = (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)} \\ &= B\left(\sum_{j=1}^{N} x_j \otimes y_j\right). \end{aligned}$$

Since the Berezin transform is one-to-one, it follows that $T_f T_g = T_{h^n} + \sum_{j=1}^{N} x_j \otimes y_j$ and the converse is proved.

Assume, in addition, that $T_{h^n} + F$ is an isometry. Then $(T_{h^n} + F) * (T_{h^n} + F) = I$ or

$$T_{\bar{h}^n} T_{h^n} + F^* T_{h^n} + T_{\bar{h}^n} F + F^* F = I. \quad (7)$$

We will recall some classical results about the algebra of bounded analytic functions on $\mathbb{D}$, denoted by H^∞. Let M denote the maximal ideal space of H^∞. Hoffman ([50], Lemma 4.4) has proved that the algebra $C(M)$ is identical to the sup norm closure of the algebra generated by the bounded harmonic functions. Thus $h^n \in C(M)$. On the ideal M we can introduce an equivalence relation: $m_1 \sim m_2$ if and only if $\rho(m_1, m_2) < 1$, where

$$\rho(m_1, m_2) = \sup \left\{ \left|\hat{f}(m_2)\right| : f \in H^\infty, \|f\| \le 1, \hat{f}(m_1) = 0 \right\}.$$

Here $\hat{f}$ is the Gelfand transform of f defined by $\hat{f}(m) = m(f), m \in M$. The equivalence classes are called Gleason parts. Let M_1 denote the set of one-point parts in M, and

$$J = \{\varphi \in C(M) : \varphi = 0 \text{ on } M_1\}.$$

Let $\tau\big(C(M)\big)$ be the closed subalgebra of the algebra of all bounded linear operators on $L_a^2(\mathbb{D})$ generated by $\{T_\varphi : \varphi \in C(M)\}$ and let $\mathcal{C}$ be the commutator ideal of $\tau\big(C(M)\big)$. McDonald and Sundberg [167] have proved that $C(M)/J$ is isomorphic to $\tau(C(M))/\mathcal{C}$ with the isomorphism

$$\varphi + J \hookrightarrow T_\varphi + \mathcal{C}.$$

It is also well known that $\mathcal{C}$ contains all compact operators. Let $\Pi: \tau\big(C(M)\big) \to \tau(C(M))/\mathcal{C}$ be the quotient map. Apply Π to the equation (7) and notice that F is finite rank and hence F is compact. Therefore F^* is also compact so (7) becomes

$$\Pi(T_{\bar{h}^n})\Pi(T_{h^n}) = \Pi(I).$$

Applying the isomorphism above, we obtain

$$(\bar{h}^n + J)(h^n + J) = 1 + J$$

or $\bar{h}^n \cdot h^n - 1 \in J$. This means $\bar{h}^n \cdot h^n - 1 = 0$ on M_1. But the maximal ideal space of $L^\infty(\partial\mathbb{D})$ is a subset of M_1. Hence

$$\varphi(\bar{h}^n) \cdot \varphi(\bar{h}^n) = 1$$

or

$$|\varphi(h)| = 1 \text{ for all } \varphi \in M(L^\infty(\partial\mathbb{D})).$$

Since h is a bounded harmonic function on $\mathbb{D}$, we can identify it with its boundary value function, which we denote by h again. By Hoffman [166], p. 170 the Gelfand transform maps $L^\infty(\partial\mathbb{D})$ isometrically and isomorphically onto $C(M(L^\infty))$. Thus we have

$$\|h\|_{L^\infty(\mathbb{D})} = \|h\|_{L^\infty(\partial\mathbb{D})} = \left\|\hat{h}\right\|_{C(M(L^\infty))} = sup\{|\varphi(h)| : \varphi \in M\big(L^\infty(\partial\mathbb{D})\big)\} = 1.$$

Then clearly $\left|\left|h^n\right|\right|_{L^\infty(\partial\mathbb{D})} = 1$ and since $fg = h^n$ a.e. on $\partial\mathbb{D}$, we have

$$\|h\|_{L^\infty(\mathbb{D})} = \|h^n\|_{L^\infty(\partial\mathbb{D})} = \|fg\|_{L^\infty(\partial\mathbb{D})} = 1.$$

Corollary (6.1.9)[163]: Suppose $f = f_1 + \overline{f_2}, g = g_1 + \overline{g}_2$ and $h = h_1 + \bar{h}_2$ are bounded harmonic functions on $\mathbb{D}$ and $x_1, \dots, x_N, y_1, \dots, y_N$ are in $L_a^2(\mathbb{D})$. Then $T_f T_g = T_h + \sum_{j=1}^N x_j \otimes y_j$ if and only if the following two conditions hold: (i) either f is analytic or g is conjugate analytic,

(ii) $(fg)(z) = h(z) + (1 - |z|^2)^2 \sum_{j=1}^N x_j(z)\overline{y_j(z)}$, for $z \in \mathbb{D}$.

Proof. Apply Theorem (6.1.8) with $n = 1$. Then $T_f T_g = T_h + F$ implies that $g_1 \overline{f_2}$ is harmonic on $\mathbb{D}$, so that $\Delta\big(g_1 \bar{f_2}\big)(z) = g_1'(z)\bar{f}_2'(z) = 0$. Hence $g_1 =$ constant or f_2 is constant on $\mathbb{D}$ which means that either f is analytic or g is conjugate analytic. The other statements follow immediately from Theorem (6.1.8). If $F = 0$, then (3) implies $f_1'(z)\bar{g}_2'(z) = 0$ and hence either f is conjugate analytic or g is analytic. If f is analytic and f is conjugate analytic, then clearly f is constant. The same situation for g leads to the conclusion that g is constant. Otherwise, both f and g are analytic on $\mathbb{D}$ or both f and g are conjugate analytic and Corollary (6.1.2) follows.

Conversely, if f is analytic, then f_2 is constant so that $g_1\bar{f_2} - h$ is harmonic on $\mathbb{D}$. Apply Theorem (6.1.8) now with $n = 1$ and the converse follows. Similarly, the statement follows if g is conjugate analytic.

A finite rank perturbation version of Corollary (6.1.3) is contained in the following corollary.

Corollary (6.1.10)[163]: Suppose f and g are bounded and harmonic on $\mathbb{D}$. Then $T_fT_g = I + \sum_{j=1}^{N} x_j \otimes y_j$ and $x_1, \ldots, x_N, y_1, \ldots, y_N$ are in $L_a^2(\mathbb{D})$ if and only if the following two conditions hold:

(i) either f is analytic or g is conjugate analytic,

(ii) $f(z)g(z) = 1 + (1 - |z|^2)^2 \sum_{j=1}^{N} x_j(z)\overline{y_j(z)}$, for $z \in \mathbb{D}$.

Corollary (6.1.11)[163]: If f is bounded and harmonic and $T_f^2 = T_f + F$, then $f = 0$ or $f = 1$ on $\mathbb{D}$.

Proof. By Corollary (6.1.9), f is analytic or f is conjugate analytic and $f^2 = f$ a.e. on $\partial\mathbb{D}$. This means that $f(f - 1) = 0$ a.e. on $\partial\mathbb{D}$. If f is analytic, then either $f = 0$ on $\partial\mathbb{D}$ (and hence $f \equiv 0$ on $\mathbb{D}$) or $f = 1$ on $\partial\mathbb{D}$ (and hence $f \equiv 1$ on $\mathbb{D}$). The same conclusion follows if f is conjugate analytic.

If we slightly modify the argument in the proof of Theorem (6.1.7), we get the following proposition.

Proposition (6.1.12)[163]: Suppose $E \subset \mathbb{D}$ is a starlike with respect to 0 compact set. Let

$$f(z) = \chi_E(z) \sum_{j=1}^{\ell} f_j(z)\overline{g_j(z)},$$

with f_j, g_j analytic on $\mathbb{D}$ for $j = 1, \ldots, \ell$ If T_f has finite rank N, then $f = 0$.

Proof. Clearly f is bounded. As in the proof of Theorem (6.1.7), for $0 < r < 1$, define $f_r(z) = f(rz)$ and let $g_r(z) = \overline{f}_r$. Then

$$T_{fg_r} = T_{f\chi_E(rz)\sum_{j=1}^{\ell} \overline{f_j(rz)}g_j(rz)} = \sum_{j=1}^{\ell} T_{\overline{f_j(rz)}} T_{f\chi_E(rz)} T_{g_j(rz)}. \qquad (8)$$

But notice $f(z)\chi_E(rz) = \chi_E(z)\chi_E(rz)\sum_{j=1}^{\ell} f_j(z)\overline{g_j(z)}$.

If $z \in E$, then $rz \in E$ too since E is starlike. Thus $\chi_E(z)\chi_E(rz) = 1$.

If $z \notin E, \chi_E(z) = 0$. Hence in both cases $\chi_E(z)\chi_E(rz) = \chi_E(z)$. Thus $f(z)\chi_E(rz) = f(z)$. Now (8) gives that $T_{fg_r} = \sum_{j=1}^{\ell} T_{\overline{f_j(rz)}} T_{f(z)} T_{g_j(rz)}$ and consequently rank $T_{fg_r} \le N\ell$, for all r. Thus $\limsup_{r\to 1}$ rank $T_{fg_r} \le N\ell$. We continue as in Theorem (6.1.7) and conclude that $T_{|f|^2}$ has finite rank and therefore $f \equiv 0$ see [100].

Finally we would like to prove another zero product result involving two Toeplitz operators.

Proposition (6.1.13)[163]: Suppose $\mathbb{D}_r = r\mathbb{D}$ for some $r \in (0,1), h$ is an analytic function on $\mathbb{D}$ and $g = g_1 + \bar{g}_2$ is a bounded harmonic function. If $f = \chi_{\mathbb{D}_r} h$ and $T_fT_g = 0$, then either $f = 0$ or $g = 0$.

Proof. Suppose $f \in L^\infty(\mathbb{D}), g = g_1 + \overline{g_2}$ is a bounded and harmonic function, and $T_fT_g = 0$. Then

$$B\big(T_fT_g\big)(z) = \langle T_fT_gk_z,k_z\rangle = (1-|z|^2)^2\langle T_fP(g_1+\overline{g_2})K_z,K_z\rangle$$
$$= (1-|z|^2)^2\{\langle fg_1K_z,K_z\rangle + \langle fP(\overline{g_2}K_z),K_z\rangle\}$$
$$= B(fg_1)(z) + \overline{g_2(z)}(Bf)(z) = 0. \tag{9}$$
Suppose now that $f = \chi_{\mathbb{D}_r} h$, where h is analytic. Then (9) means
$$\int_{\mathbb{D}_r} \frac{h(\xi)g_1(\xi)}{\left|1-z\bar{\xi}\right|^4}\, dA(\xi) + g_2(z)\int_{\mathbb{D}_r} \frac{h(\xi)}{\left|1-z\bar{\xi}\right|^4}\, dA(\xi) = 0. \tag{10}$$
Let $w = \frac{\xi}{r}$; then (10) becomes
$$r^2\int_{\mathbb{D}} \frac{h(wr)g_1(wr)}{|1-z\bar{w}r|^4}\, dA(w) + r^2\overline{g_2(z)}\int_{\mathbb{D}} \frac{h(wr)}{|1-z\bar{w}r|^4}\, dA(w) = 0$$
or
$$(1-r^2|z|^2)^2\int_{\mathbb{D}} \frac{h(wr)g_1(wr)}{|1-\bar{w}zr|^4} dA(w) + \overline{g_2(z)}(1-r^2|z|^2)^2\int_{\mathbb{D}} \frac{h(wr)}{|1-\bar{w}zr|^4} dA(w) = 0$$
so that
$$B[(hg_1)_r](rz) + \overline{g_2(z)}B(h_r)(rz) = 0.$$
Since the Berezin transform fixes analytic functions, we have
$$h(r^2z)g_1(r^2z) + \overline{g_2(z)}h(r^2z) = 0, \qquad \text{for } z \in \mathbb{D}$$
which implies
$$h(r^2z)\left[g_1(r^2z) + \overline{g_2(z)}\,\right] = 0.$$
Then either h = 0 or $g_1(r^2z) = \overline{-g_2(z)}$ for $z \in \mathbb{D}$.

If $h = 0$, then $f = 0$.

In the second case, an analytic function g_1r^2 is equal to the conjugate analytic function, so they both are constant functions; i.e., g_1 = constant and g_2 = constant.

If g = constant, then $T_fT_g = 0$ implies g = constant = 0 or $f = 0$. Thus we have proved the proposition.

Section (6.2): Finite Rank Toeplitz Operators

In classical function theory of the unit disk, Toeplitz operators were defined on the Hardy space H^2 by $T_\phi f = P(\phi f)$, where ϕ is a bounded measurable function on the unit circle $\mathbb{T} = \partial\mathbb{D}$ and P is the Szeg″o projection from L^2 (of the unit circle) to H^2. McDonald and Sundberg [167] defined Toeplitz operators on the Bergman space A^2 analogously: ϕ is a function on the interior of the disk and P is the Bergman projection from $L^2(dA)$ (dA being area measure) to A^2.

In the Bergman space one can have $\phi f \in L^2$ for all $f \in A^2$ even if ϕ is unbounded. Moreover, the formula for the Bergman projection as an integral can be applied even when the product ϕf is only in L^1. Given that, one quickly realizes that the formula for the Toeplitz operator
$$P(\phi f)(z) = \int \frac{\phi(w)f(w)}{(1-\bar{w}z)^2}\, dA(w)$$
allows one to extend the notion of Toeplitz operators to symbols that are measures (or even compactly supported distributions): simply replace $\phi\, dA$ with $d\mu$ in the formula (or apply the distribution to the appropriate product). [171] determined necessary and sufficient conditions on a positive measure μ for T_μ to belong to the Schatten classes S_p. For complex measures the conditions were only sufficient. The same is true for the characterization of finite rank operators T_μ: necessary and sufficient conditions for positive measures were obtained, only sufficient for complex measures.

The characterization obtained here (that μ must be a finite sum of point masses) proves the conjecture that for $\phi \in L^\infty, T_\phi$ has finite rank only if it is 0. After was submitted, I learned of partial results on this conjecture in a preprint by A. Pushnitski, G. Rozenblum and N. Shirokov. They imposed some extra conditions on ϕ. In addition, Namita Das communicated some incomplete work on the same conjecture.

Initially, let μ be any complex regular Borel measure on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Integration with respect to area measure is denoted with dA.
The set of all analytic functions on $\mathbb{D}$ will be denoted $\mathcal{H}(\mathbb{D})$ or simply $\mathcal{H}$.

The Bergman space A^2 of the unit disk is the space of all functions analytic in $\mathbb{D}$ which belong to $L^2 = L^2(dA)$, that is, $A^2 = L^2 \cap \mathcal{H}$. The inner product in L^2 is denoted $\langle f, g\rangle = \frac{1}{\pi}\int f(z)\overline{g(z)}\, dA(z)$ and the corresponding norm is denoted $\|f\| = \langle f, f\rangle^{1/2}$. The Bergman kernel is the function $K(z,w) = K_w(z) = (1-\bar{w}z)^{-2}$. It satisfies $Pf(w) = \langle f, K_w\rangle$ for all $f \in L^2$ where P is the orthogonal projection from L^2 to A^2. In particular, if $f \in A^2$, then $f(w) = \langle f, K_w\rangle$.

The Toeplitz operator on A^2 with symbol μ is denoted T_μ and is formally defined by

$$T_\mu(f)(w) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(z)}{(1-z\bar{w})^2}\, d\mu(z). \tag{11}$$

If μ has the form $\phi\, dA$ for some bounded measurable function ϕ, then T_μ is denoted T_ϕ and satisfies $T_\phi f = P(\phi f), f \in A^2$. For arbitrary measures on $\mathbb{D}$, T_μ may be only densely defined because the integral (11) can only be guaranteed to converge for bounded f. Even if it converges, the result need not be in A^2. We will view T_μ as an operator defined on the dense subset of polynomials with range in the set of all analytic functions on $\mathbb{D}$. The question of when T_μ extends to A^2 or has values in A^2 will not be considered here. However, we note that if $|\mu|$ is a Carleson measure for A^2, then it is always true that T_μ is bounded from A^2 to A^2. In particular this is true for measures μ whose support is a compact subset of $\mathbb{D}$ as well as for measures of the form $\phi\, dA$ with ϕ bounded.

The following is the main theorem, whose proof will occupy the majority.

Theorem (6.2.1)[168]: The rank of T_μ is finite if and only if μ is a finite linear combination of point masses.

The Bergman space setting is completely unnecessary and we will actually prove a Theorem (6.2.1)bout operators on the space of analytic polynomials. Moreover, a large part of our proof does not require μ to be a measure. We have, formally, $\langle T_\mu f, g\rangle = \int f\bar{g}\,\mu$. For this to be true in the strict sense of the definition of $\langle\cdot,\cdot\rangle$, we would need to justify the implied exchange of integrals. What is clear, however, is that if μ is a measure on $\mathbb{D}$, then $T_\mu f$ will always produce an analytic function in $\mathbb{D}$. If μ is a measure on any disk, then we can use the same formula for $T_\mu f$ and obtain a function analytic in some neighborhood of 0. The coefficients of any formal power series determine a linear functional on the space of polynomials in $\bar{z}$ in a standard way. If we interpret $T_\mu f$ in this way it is easy to prove that $T_\mu f(\bar{g}) = \int f\bar{g}\, d\mu$.

Thus T_μ can always be seen as taking polynomials to linear functionals on the conjugate analytic polynomials and μ can be seen as a linear functional on the space of polynomials in z and $\bar{z}$. Moreover, these two objects determine each other.

We now generalize these observations.

Let $\mathcal{P}$ denote the algebra of complex polynomials over $\mathbb{C}$ in the variable z and let $\bar{\mathcal{P}}$ denote the polynomials in $\bar{z}$. Both are subalgebras of $\mathbb{C}[z,\bar{z}]$, the polynomials in both variables. Let μ be a linear functional on $\mathbb{C}[z,\bar{z}]$ and let $B_\mu(f,g) = \mu(f\bar{g})$.

Let $T_\mu f$ denote the linear functional on $\bar{\mathcal{P}}$ defined by $T_\mu f(\bar{g}) = B_\mu(f,g) = \mu(f\bar{g})$.

One can determine the nature of μ by defining a topology on $\mathbb{C}[z,\bar{z}]$ and requiring that μ be continuous in that topology. For example, if $\mathbb{C}[z,\bar{z}]$ is given the topology of uniform convergence on compact sets, then a continuous μ can be identified with a complex measure with compact support. Compactly supported distributions come from the topology of uniform convergence on compact sets of all derivatives. Continuity in the $L^1(\mathbb{D},dA)$ norm implies a bounded measurable function. We will need the exact nature of μ only in the last stages of our proof.

If the operator T_μ has rank less than N, then if we select N polynomials f_j , there will exist a nontribyl linear relation

$$\sum_{j=1}^{N} c_j T_\mu f_j = 0. \tag{12}$$

If we apply these functionals $T_\mu f_j$ to polynomials $\bar{g}_i, 1 \le i \le N$, we obtain a set of column vectors in $\mathbb{C}^N$ that satisfies a linear relation with the same constants as (12). Thus, the matrix whose i,j entry is $\mu(f_j\bar{g}_i)$ has a determinant equal to 0.

The determinant is linear in each column and μ is a linear functional, so we can write

$$\mu\left(f_1(z) \cdot \begin{vmatrix} \overline{g_1(z)} & \mu(f_2\bar{g}_1) & \cdots & \mu(f_N\bar{g}_1) \\ \overline{g_2(z)} & \mu(f_2\bar{g}_2) & \cdots & \mu(f_N\bar{g}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{g_N(z)} & \mu(f_2\bar{g}_N) & \cdots & \mu(f_N\bar{g}_N) \end{vmatrix} \right) = 0.$$

Let us introduce the variable z_1 in place of z above and use μ_1 for μ acting in the variable z_1. Now we repeat this process in each column (using the variable z_j in column j and the notation μ_j for μ acting in z_j) to obtain

$$\mu_1\left(\mu_2\left(\dots \mu_N\left(\prod_{k=1}^{N} f_k(z_k) \det\left(\overline{g_i(z_j)} \right) \right) \dots \right) \right) = 0. \tag{13}$$

We now specialize to the case where each g_i has the form $g_i(z) = z^{k_i}$ with $k_1 < k_2 < \cdots < k_N$. Let $J = (k_i)$ denote any such increasing N-tuple of nonnegative integers. Write Z for the N-tuple $(z_1, z_2, \dots, z_N)$ and write $V_J(Z)$ for the determinant $\det\left(z_j^{k_i}\right)$. Taking finite sums of equations (13), we get for any polynomial $F(Z)$ in N variables:

$$\mu^N\left(F(Z)\overline{V_J(Z)}\right) = 0 \tag{14}$$

where μ^N is our abbrebytion for successive applications of μ in each variable.

We now determine what one gets when we take linear combinations of V_J with varying J in this equation. We claim one gets

$$\mu^N\left(F(Z)\overline{G(Z)}\right) = 0 \tag{15}$$

for all polynomials F and all antisymmetric polynomials G. We now digress for a short discussion of symmetric and antisymmetric polynomials.

A polynomial $F(Z)$ is called symmetric if it is invariant under permutations of the variables z_j , that is, $F(\pi(Z)) = F(Z)$, where $\pi(Z)$ is the N-tuple consisting of the permutation of the coordinates of Z. We call $G(Z)$ antisymmetric (or alternating) if it changes sign with each transposition of coordinates. That is, $G(\pi(Z)) = \epsilon_\pi G(Z)$, where ϵ_π is $+1$ for an even permutation π and -1 for an odd π.

Denote by $SF(Z)$ and $AF(Z)$ the symmetric and antisymmetric projections of a function F. That is,

$$SF(Z) = \frac{1}{N!}\sum_{\pi} F(\pi(Z)) \text{ and } AF(Z) = \frac{1}{N!}\sum_{\pi} \epsilon_\pi F(\pi(Z)), \quad (16)$$

where each sum is over all permutations. For any polynomial F, SF is symmetric and AF is antisymmetric. If F is symmetric and G is antisymmetric, then $SF = F, AF = 0, SG = 0$ and $AG = G$.

We observe that the vector space of all antisymmetric polynomials is the range of A and is therefore the span of the images of all monomials. If $G(Z) = Z^J = z_1^{k_1} z_2^{k_2} \ldots z_N^{k_N}$ is a monomial, then $AG(Z)$ is easily seen to be 0 if any of the exponents are equal. Moreover, if the monomial G' is obtained from the monomial G by a permutation of the exponents, then $AG'(Z) = \pm AG(Z)$. Thus, the set of antisymmetric polynomials is spanned by $A(Z^J)$ as J varies over increasing N-tuples of nonnegative integers. It follows easily from the formula for the determinant as a signed sum of products that $A(Z^J) = V_J(Z)/N!$. Thus, summing equations (14) produces equation (15), as claimed.

If $J = (0, 1, 2, \ldots, N-1)$, then $V_J(Z) = V(Z)$ is the Vandermonde determinant. Clearly the product of a symmetric polynomial and $V(Z)$ is antisymmetric. Ultimately, we will only need the fact that these products are in the range of A.
However, the argument of the following paragraph shows that every antisymmetric polynomial is in fact the product of a symmetric polynomial and $V(Z)$.

The Vandermonde determinant $V(Z)$ is the minimal-degree polynomial $G(Z)$ vanishing on all the varieties $\mathcal{V}_{i,j} = \{Z : z_i = z_j\}$ for all pairs of indices (i,j) with $i \neq j$. Therefore the ideal generated by $V(Z)$ is a radical ideal, and, by the Hilbert Nullstellensatz (see for example [170]), any other polynomial vanishing on $\bigcup_{(i,j)} \mathcal{V}_{i,j}$ is divisible by $V(Z)$. It is clear that any antisymmetric polynomial $G(Z)$ vanishes on $\bigcup_{(i,j)} \mathcal{V}_{i,j}$ and hence $G(Z)$ is divisible by $V(Z)$.

This fact that every antisymmetric polynomial is divisible by $V(Z)$ is known. It is stated in the Encyclopedic Dictionary of Mathematics [169], but it has been hard to find a published proof.

Let us recall that the following equation:

$$\mu^N\left(F(Z)\overline{V_J(Z)}\right) = 0 \quad (17)$$

implies that for all polynomials F,

$$\mu^N\left(F(Z)\overline{G(Z)}\right) = 0 \quad \text{for all antisymmetric } G, \text{and so}$$

$$\mu^N\left(F(Z)\overline{H(Z)V(Z)}\right) = 0 \quad \text{for all symmetric } H.$$

Specializing to F of the form $F_1 V$ with F_1 symmetric gives us

$$\mu^N\left(F_1(Z)\overline{F_2(Z)}|V(Z)|^2\right) = 0 \quad \text{for all symmetric polynomials } F_1 \text{ and } F_2. \quad (18)$$

Now is the time to use the fact that μ is a measure, and to require that it have compact support. Let us restate our main theorem in the form of the ideas we have been using. In this form it is actually more general.

Theorem (6.2.1)[168]: (Restated). Let μ be a measure on $\mathbb{C}$ with compact support. Let T_μ be the operator from $\mathcal{P}$ to linear functionals on $\bar{\mathcal{P}}$ by $T_\mu f(\bar{g}) = \int f\bar{g}\, d\mu$. Then T_μ has finite rank if and only if the support of μ is finite.

In the present case, μ^N is just a product measure on $\mathbb{C}^N$. We formally state our conclusions thus far in the language of measures and integration:

Proposition (6.2.2)[168]: If T_μ has rank less than N, then for all symmetric polynomials F_1 and F_2

$$\int_{\mathbb{C}^N} F_1(Z)\overline{F_2(Z)}|V(Z)|^2\, d\mu^N(Z) = 0. \tag{19}$$

It is clear that finite sums of products of the form $F_1(Z)\overline{F_2(Z)}$ (with F_1 and F_2 symmetric) form an algebra $\mathcal{A}$ of functions on $\mathbb{C}$ which contains the constants and is closed under conjugation. It doesn't separate points because each element is constant on sets of points that are permutations of one another. Define an equivalence relation $\sim$ on $\mathbb{C}^N$ by the fact that $Z_1 \sim Z_2$ if and only if $Z_2 = \pi(Z_1)$ for some permutation π. Let $Z = (z_1, \ldots, z_N)$ and $W = (w_1, \ldots, w_N)$. If $Z \nsim W$, then the polynomials $p(t) = \prod (t - z_j)$ and $q(t) = \prod (t - w_j)$ have different zeros (or the same zeros with different multiplicities). This implies that the coefficient of some power of t in $p(t)$ differs from the corresponding coefficient in $q(t)$. Thus there is an elementary symmetric function that differs at Z and W. Consequently, $\mathcal{A}$ separates equivalence classes.

Let us give the quotient space $\mathbb{C}^N/\sim$ the standard quotient space topology. If K is any compact set in $\mathbb{C}^N$ that is invariant with respect to $\sim$, then $K/\sim$ is compact and Hausdorff. Also, any symmetric continuous function on $\mathbb{C}^N$ induces a continuous function on $\mathbb{C}/\sim$ (and conversely). Thus we can apply the Stone-Weierstrass theorem (on $K/\sim$) to conclude that $\mathcal{A}$ is dense in the space of continuous symmetric functions, in the topology of uniform convergence on any compact set. Therefore, for any continuous symmetric function $f(Z)$

$$\int_{\mathbb{C}^N} f(Z)|V(Z)|^2\, d\mu^N(Z) = 0. \tag{20}$$

If f is an arbitrary continuous funtion, the above integral will be the same as the corresponding integral with Sf replacing f. This is because the function $|V(Z)|^2$ and the product measure μ^N are both invariant under permutations of the coordinates. We conclude that this integral vanishes for any continuous f and so the measure $|V(Z)|^2\, d\mu^N(Z)$ must be zero. Thus, μ^N is supported on the set where V vanishes.

This means μ must have fewer than N points in its support: for if z_j are N distinct points in the support of μ, then the point $Z = (z_1, \ldots, z_N)$ is in the support of μ^N but $V(Z) \neq 0$, a contradiction.

In fact, when the number of points in the support is finite, it is precisely the rank of T_μ: if the support of μ is $\{z_1, z_2, \ldots, z_M\}$, then the range of T_μ contains the M independent evaluation functionals.

Note that the rank zero case has been known for at least a century: $\int f\bar{g}\, d\mu = 0$ for all polynomials f and g clearly implies $\mu = 0$ by the Stone-Weierstrass theorem.

Let X be any subspace of A^2 with finite codimension. Let S be the closure of $\{\sum f_j\bar{g}_j : f_j \in X, g_j \in A^2\}$ in the topology of uniform convergence on compact sets. Suppose S is not all of $C(\mathbb{D})$; then there exists a measure μ with compact support in $\mathbb{D}$ such that $\int f\bar{g}\, d\mu = 0$ for all $f \in S$ and all $g \in A^2$. That is, the range Y_μ of T_μ is contained in $X^\perp$, a finite dimensional set. This implies that μ is a finite sum of point masses and so Y_μ is spanned by the set of K_a for a in the support of μ. If we repeat this for all possible measures that annihilate S we get a set E of all such points a. Then E is finite because the corresponding Ka are all independent and in $X^{\perp}$. Therefore, S has finite codimension and contains all functions that vanish on E.

Also, $f \in X$ implies $\langle f, K_a\rangle = 0$ for all $a \in E$ so all the functions in S vanish on E. Thus S is the ideal of all functions vanishing on E. This gives us the following corollary.

Corollary (6.2.3)[168]: If X is a subspace of A^2 with finite codimension, then the closure of the span of $X\overline{A^2}$ in the topology of uniform convergence on compact sets is an ideal in $C(\mathbb{D})$ with a finite zero set. If X has no common zeros, it is all of $C(\mathbb{D})$.

Note that it is not clear a priori that the closed span of $X\overline{A^2}$ is even closed under multiplication.

One can define operators to which our results apply that seem to have little to do with Toeplitz operators and nothing to do with Bergman spaces. For example, let μ be a measure on $\overline{\mathbb{D}}$ and define an operator from (say) the disk algebra to entire functions by $S_\mu(f)(w) = \int \exp(\bar{z}w)f(z)\, d\mu(z)$. Then, since $\exp(\bar{z}w)$ is a reproducing kernel for some appropriate normalization of the Fock space, one obtains $\langle S_\mu(f), g\rangle = \int f\bar{g}\, d\mu$ for all polynomials f and g. If S_μ has finite rank, then μ must have finite support.

Section (6.3): Bergman Space Modulo Finite Rank Operators

Many algebraic properties of Toeplitz operators on analytic function spaces have been studied. We are concerned with the problem of when the product of two Toeplitz operators $T_f T_g$ is a finite perturbation of another Toeplitz operator T_h. We take the Bergman space as the domain and study the question for f, g bounded harmonic and h in C^2 class with the invariant Laplacian in L^1.

Let dA denote the Lebesgue area measure on the unit disk D in the complex plane, normalized so that the measure of the disk D is 1. The Bergman space L^2 a is the Hilbert space consisting of analytic functions on D that are square integrable with respect to the measure dA. For $\varphi \in L^2(D, \mathrm{d}A)$, the Toeplitz operator T_φ with symbol φ is defined densely on L_a^2 by

$$T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection from $L^2(D, \mathrm{d}A)$ to L_a^2.

For general operator S on a Hilbert space, the rank(S) is defined as the dimension of closure of the range of S. S is called finite rank operator with rank r if it is bounded and rank$(S) = r < \infty$. On the Bergman space, the rank r operator has the expression

$$S = \sum_{i=1}^{r} x_i \otimes y_i,$$

where $\{x_i\}_{i=1}^r, \{y_i\}_{i=1}^r$ are two sets of linearly independent functions in L_a^2 and we use the standard notation for rank-one operators in the Hilbert space: $x \otimes y : h \to \langle h, y\rangle x$. A tool that arises in the study of the Bergman space is the Berezin transform. Given an (possibly unbounded) operator S on L_a^2, with its domain containing all the normalized reproducing kernels $k_z(w) = \frac{(1-|z|^2)^2}{(1-\bar{z}w)^2}$, the Berezin transform of S is the function

$$B[S](z) = \langle Sk_z, k_z\rangle, \qquad z \in D,$$

where $\langle , \rangle$ is the inner product in L_a^2. It was proved that the Berezin transform is injective [55], which means $B[S](z) = B[T](z)$ will imply $S = T$ for two operators S, T on L_a^2. For an integrable function f on D, the Berezin transform of f is the function

$$B[f](z) = \langle fk_z, k_z\rangle$$
For $u \in L^1(D)$, it was shown in [173], [176] that $B(u) = u$ if and only if u is harmonic. We shall denote the Laplacian by $\Delta = \frac{\partial^2}{\partial_z\bar{\partial}_z}$ and the invariant Laplacian by $\tilde{\Delta} = (1 - |z|^2)^2\Delta$.

Since the Bergman projection maps $L^\infty(D)$ to the Bloch space [43] and the Szego projection maps $L^\infty(\partial D)$ into BMOA [177], we note that for a bounded harmonic function ϕ on D, ϕ can be written uniquely as a sum of an analytic function ϕ_+ and a conjugate analytic function ϕ_- with

$$\phi_-(0) = 0: \phi = \phi_+ + \phi_-,$$

where ϕ_+ and ϕ_- are in the Bloch space and BMOA. We say that ϕ_+ and ϕ_- are the analytic part of φ and the conjugate analytic part of φ respectively.

The earliest characterization of product problem is on the Hardy space of the unit circle ∂D by Brown and Halmos [98], said that for $f, g \in L^\infty, T_fT_g = T_h$ if and only if either $\overline{f}$ or g is analytic and $fg = h$ a.e. ∂D. One would expect if similar result holds on the Bergman space. The situation is more complicated on the Bergman space. For bounded harmonic functions f and g on D, the third author [58] proved that $T_fT_g = T_{fg}$ on the Bergman space if either $\overline{f}$ or g is analytic. Then Ahern and Čučković [96] obtained the characterization analogues to the Brown–Halmos theorem for the Bergman space Toeplitz operators with harmonic symbols.

Theorem (6.3.1)[172]: Suppose f and g are bounded harmonic functions on the unit disk and h is a bounded C^2 that $\tilde{\Delta}\, h \in L^\infty(D)$. If $T_fT_g = T_h$, then either f is conjugate analytic or g is analytic. In either case, $h = fg$.

Later in [165], Ahern removed the assumption on h and showed that the above theorem is true for the function h bounded on the unit disk. For more general symbols, surprisingly, Ahern [165] had the following example,

$$T_zT_{\overline{z}^2z} = T_{2z\overline{z}-1}. \tag{21}$$

It means even if $T_fT_g = T_h, h$ does not have to be equal to fg. More interesting examples are shown in [166].

Inspired by (21), we construct the following examples showing that the product of two Toeplitz operators can be a nonzero finite rank perturbation of another Toeplitz operator.

Example (6.3.2)[172]:

$$T_{z^2}\,T_{z^3} - T_{3\overline{z}-\frac{2}{z}} = 1 \otimes z,$$

$$T_{2z^2}\,T_{\overline{z}^3z} - T_{(6|z|^2-4)} = 1 \otimes 1.$$

To get the first equation, applying Berezin transform to right hand side for each $w \in D$,

$$\langle(1 \otimes z)k_w, k_w\rangle = (1 - |w|^2)^2\overline{w}$$

Also by simple calculation

$$\langle\left(T_z - 2T_zT_{\overline{z}^2} + T_{z^2}\,T_{\overline{z}^3}\right)k_w, k_w\rangle = (1 - |w|^2)^2\overline{w}.$$

This is $B[1 \otimes z](w) = B\left[T_z - 2T_zT_{\overline{z}^2} + T_{z^2}\,T_{z^3}\right](w)$. Since the Berezin transform is injective, it follows

$$1 \otimes z = T_{\overline{z}} - 2T_zT_{\overline{z}^2} + T_{z^2}\,T_{\overline{z}^3}\,.$$

Combining with Identity (21), we obtain the first example

$$T_{z^2}\,T_{\overline{z}^3} - T_{3\overline{z}-\frac{2}{z}} = 1 \otimes z.$$

To get the second equation, by using above we have

$$\left[T_{z^2}\, T_{\bar{z}^3} \; - \; T_{3\bar{z}-\frac{2}{z}}\right] T_z \; = \; [1 \; \otimes \; z] T_z \; = \; 1 \; \otimes \; (T_z^*\, z) = 1 \; \otimes \frac{1}{2}.$$

Combining (21) with the above identity implies

$$T_{2z^2}\, T_{\bar{z}^3 z} \; - \; T_{(6|z|^2-4)} \; = \; 1 \; \otimes \; 1.$$

Note that in above examples, the two symbols are either not harmonic or unbounded. Naturally, one may ask the following question.

Question (6.3.3)[172]: Can $T_f T_g \; - \; T_h$ be a nonzero finite rank operator on L_a^2 if f, g are bounded harmonic functions and $h \; \in \; L^\infty(D)$?

On the Hardy space, the symbol mapping [175] said fg must be equal to h. Axler, Chang and Sarason gave an affirmative answer to Question (6.3.3) in [93] as the following theorem.

Theorem (6.3.4)[172]: (The variant theorem of Axler–Chang–Sarason). Suppose $f, \bar{g} \; \in \; BMOA$. Then the semicommutator $\hat{T}_{fg} \; - \; \hat{T}_f \hat{T}_g \left(= \; \hat{H}_{\bar{f}}^*\, \hat{H}_g\right)$ is a finite rank operator if and only if either $\hat{H}_{\bar{f}}$ or $\hat{H}_g$ is a finite rank operator. Here $\hat{T}_f$ and $\hat{H}_f$ denote the Toeplitz operator and Hankel operator on the Hardy space respectively.

In [179], Richman obtained a formula on the ranks.

Theorem (6.3.5)[172]: Suppose that $f, \bar{g} \; \in \; BMOA$

$$\operatorname{rank} \hat{H}_{\bar{f}}^*\, \hat{H}_g \; = \; \min\left\{\operatorname{rank} \hat{H}_{\bar{f}}, \operatorname{rank} \hat{H}_g\right\}.$$

On the Bergman space, Guo, Sun and the third author [100] showed that for bounded harmonic functions f, g and $h \; = \; fg, T_f T_g \; - T_h$ is a finite rank operator on L_a^2 if and only if $T_f T_g \; - \; T_h \; = \; 0$. Hence by Ahern and Čučković theorem [96], either f or g is analytic. As Luecking [169] showed that there is no non-tribyl finite rank Toeplitz operator with bounded symbol on the Bergman space, one may expect that the answer to Question (6.3.3) should be analogous to Ahern and Čučković's Theorem (Theorem (6.3.1)). Indeed, Čučković [164] has studied this question and obtained:

Theorem (6.3.6)[172]: Suppose f, g and h are bounded harmonic functions and h_+ and $\overline{h_-}$ are in $H^\infty(D)$. Then $T_f T_g \; - T_{h^n} \; = \; \sum_{j=1}^{r} \; x_j \; \otimes y_j$, where $x_j, y_j \; \in \; L_a^2$, if and only if the following conditions hold:

(a) $f_- g_+ \; - \; h^n$ is harmonic,

(b) $f(z)g(z) \; = \; h^n(z) \; + \; (1 \; - \; |z|^2)^2 \; \sum_{j=1}^{r} \; x_j\, (z) y_j\, (z)$, for $z \; \in \; D$.

Question (6.3.3) for other function spaces has been studied as well. In [174], Choe, Koo and Lee got a result similar to the above theorem for pluriharmonic functions f, g and n-harmonic function h on the polydisk.

Our first result is to give a negative answer to the question for the perturbation of rank one operators.

The above theorem may be viewed as the version of the Ahern–Čučković theorem (Theorem (6.3.1))–the Brown–Halmos type theorem for the Bergman Toeplitz operators modulo rank one operators. The main ideas of the proof are to use the Berezin transform on the Bergman space and the Hardy space, to exchange Toeplitz operators identities on the Bergman space to Toeplitz operators identities on the Hardy space, and to use the Bochner theorem on critical points of rational functions [180].

We get an affirmative answer to Question (6.3.3) for a perturbation of operators with higher rank.

Theorem (6.3.7)[172]: For each $m \geq 2$, there exist rational functions f, g, h_+ and $\overline{h_-}$ in $H^\infty(D)$ such that $T_f T_{\overline{g}} - T_{h_+} + h_-$ has finite rank and its rank equals m.

In general, we characterize when $T_f T_g - T_h$ has finite rank if f and g are bounded harmonic functions and h is a bounded C^2 function such that $\tilde{\Delta} h \in L^1(D)$. For our purpose, let us introduce some notations. For each polynomial $P(z)$ of z with degree N,

$$P(z) = P_0 + P_1 z + \cdots P_N z^N .$$

Denote

$$\tilde{P}(z) = z^N \overline{P\left(\frac{1}{\overline{z}}\right)} = P_0 z^N + P_1 z^{N-1} + \cdots + P_N .$$

For $m \geq N$, we define

$$\tilde{P}_m(z) = \tilde{P}(z) z^{m-N} .$$

The above theorem is analogous to the Axler, Chang and Sarason theorem (Theorem (6.3.4)) for finite rank perturbation on the Hardy space [93], where the third condition should be changed into that either f_+ or $\overline{g_-}$ is a rational function but the last two conditions are not required and the second condition is replaced by

$$h = fg$$

on ∂D.

We will extend the Čučković theorem (Theorem (6.3.6)) to obtain a necessary and sufficient condition for $T_f T_g - T_h$ to have finite rank for more general h by using Luecking's theorem on the finite rank Toeplitz operators [169]. Next by the injective property of the Berezin transform on bounded operators on the Bergman space or the Hardy space, we will show that f_+, g_-, h_+ and h_- are rational functions. By the Bochner's theorem [180] for rational functions and deriving some functions identities on f, g and h, we will prove Theorem (6.3.13). Then computing the action of the product of two Toeplitz operators on the orthogonal basis $\{(k + 1) z_k\}_{k=0}^{\infty}$, we get some identities on these symbols of Toeplitz operators to prove Theorem (6.3.16). Using these identities in Theorem (6.3.16), we will prove Theorem (6.3.7) by constructing examples.

Using the Luecking theorem on the finite rank Toeplitz operators [169], we get the following theorem which extends the Čučković theorem (Theorem (6.3.6)).

Theorem (6.3.8)[172]: Suppose that f and g are bounded harmonic functions,

$$h \in \cap_{q>1} L^q(D) \cap C^2, \tilde{\Delta} h \in L^1(D).$$

Then $T_f T_g - T_h$ has finite rank on L_a^2 if and only if the following conditions hold:

(a) $f_- g_+ - h$ is harmonic.

(b) There exist nonzero vectors $x_1, \ldots, x_r, y_1, \ldots, y_r$ in L_a^2, such that

$$f(z) g(z) = h(z) + (1 - |z|^2)^2 \sum_{j=1}^{r} x_j(z) y_j(z) \tag{22}$$

for $z \in D$.

We should point out that the idea in the proof is the combination of the proof of Theorem 1 in [164], Luecking Theorem on finite rank Toeplitz operators [169] and the proof of Proportion 4 in [100].

For operators S_1 and S_2 that are densely defined on L_a^2 or $H^2(D)$, we say that $S_1 = S_2$ if $S_1 p = S_2 p$ for each analytic polynomial p.

Proof. Suppose

$$T_f T_g - T_h = F,$$

where $F = \sum_{j=1}^{r} x_j \otimes y_j$ is a finite rank operator on L_a^2 for two sets of linearly independent functions $x_j, y_j \in L_a^2, j = 1, 2, \ldots, r$. Taking the Berezin transform both sides of the above equation gives

$$B[T_f T_g](z) = B[h](z) + B[F](z) \tag{23}$$

for each z in D. By hypothesis, f and g are bounded harmonic functions on the unit disk, we can write

$$f = f_+ + f_-, \qquad g = g_+ + g_-,$$

where f+, f–, g+, and g– are in the Bergman space L_a^2 and in the Bloch space contained in $\bigcap_{p>1} L^p(D)$. Using

$$T_g - k_z = g_-(z)k_z, \qquad T_{f_+}^* k_z = \overline{f_+(z)}k_z,$$

we show

$$B[T_f T_g](z) = B[f_- g_+](z) + f_+(z)g_+(z) + f_-(z)g_-(z) + f_+(z)g_-(z), \tag{24}$$

$$B[F](z) = \sum_{j=1}^{r} \langle k_z, y_j\rangle \langle x_j, k_z\rangle = (1 - |z|^2)^2 \sum_{j=1}^{r} x_j(z)y_j(z). \tag{25}$$

That is

$$B(f_- g_+ + f_- g_- + f_+ g_+ - h)(z) = (1 - |z|^2)^2 \sum_{j=1}^{r} x_j(z)y_j(z) - f_+(z)g_-(z).$$

Further expanding the right side,

$$(1 - |z|^2)^2 x_j(z)y_j(z) = x_j(z)\overline{y_j(z)} - 2z x_j(z) z\overline{y_j(z)} + z^2 x_j(z) z^2 \overline{y_j(z)}.$$

Then we obtain

$$B(f_- g_+ + f_- g_- + f_+ g_+ - h)(z) = 3\sum_{j=1}^{r} X_j(z)Y_j(z) - f_+(z)g_-(z) \tag{26}$$

where $X_j, Y_j \in L_a^2$. Notice that f_- and g_+ are in the algebra $\bigcap_{p>1} L^p(D)$, we have $f_- g_+ \in \bigcap_{1<q<\infty}, L^q(D)$

$$\tilde{\Delta}[f_-(z)g_+(z)] = [(1 - |z|^2)f_-'(z)][(1 - |z|^2)g_+'(z)] \in L_1(D)$$

where $f_-'(z) = \frac{\partial f_-}{\partial \bar{z}}$ and $g_+'(z) = \frac{\partial g_+}{\partial z}$ Therefore

$$\tilde{\Delta}[f_-(z)g_+(z) + f_-(z)g_-(z) + f_+(z)g_+(z) - h(z)] = \tilde{\Delta}[f_-(z)g_+(z) - h(z)]$$

is in $L^1(D)$. By Lemma 1 in [96], the invariant Laplace operator commutes with the Berezin transform:

$$\tilde{\Delta}\{B[f_- g_+ - h](z)\} = B[\tilde{\Delta}(f_- g_+ - h)](z),$$

applying the invariant Laplacian $\tilde{\Delta}$ to Equation (26), we get

$$B[\tilde{\Delta}(f_- g_+ - h)](z) = \tilde{\Delta}\left[\sum_{j=1}^{3r} X_j(z)\overline{Y_j(z)}\right] - \tilde{\Delta}[f_+(z)g_-(z)]$$

$$= (1 - |z|^2)^2 \left[\sum_{j=1}^{3r} X_j'(z)\overline{Y_j'(z)} - f_+'(z)g_-'(z)\right]$$

for $z \in D$. Canceling the factor $(1 - |z|^2)^2$ in both sides of above equation we obtain

$$\langle \tilde{\Delta}(f_- g_+ - h)K_z, K_z\rangle = \sum_{j=1}^{3r} X_j'(z)\overline{Y_j'(z)} - f_+'(z)g_-'(z) \tag{27}$$

Now one can "complexify" above equation to get

$$\langle \tilde{\Delta}(f_{-}g_{+} - h)K_z, K_w\rangle = \sum_{j=1}^{3r} X_j'(z)\overline{Y_j'(w)} - f_+'(z)g_-'(w) \qquad (28)$$

since the above equation holds on the bidisc as both sides of the above equation are holomorphic in the bidisc $\{(z,w): |z| < 1, |w| < 1\}$ and are equal on $\{(z,w): |z| < 1, w = \overline{z}\}$. Next we take the kth derivative to both sides of Equation (28) with respect to w and then evaluate the values at $w = 0$, it follows

$$\langle \tilde{\Delta}[f_{-}g_{+} - h]\xi^k, K_z(\xi)\rangle = \sum_{j=1}^{3r} b_{j,k}X_j'(z) + a_k f_+'(z)$$

for some constants $b_{j,k}, a_k$.

Although some of the X_j' and f_+' may not be in L_a^2, we observe that for each $0 < s < 1$, all of X_j' and f_+' are bounded and analytic on sD. Hence

$$\left\{T_{\tilde{\Delta}[f_{-}g_{+}-h]}\,\xi^k\right\}(sz) = \sum_{j=1}^{3r} b_{j,k}X_j'(rz) + a_k f_+'(sz).$$

We claim that $T_{\tilde{\Delta}[f_{-}g_{+}-h]}$ has finite rank on the Bergman space L_a^2. If the claim is false, we may assume that there are $3r+2$ linearly independent functions $\{u_l\}_{l=1}^{3r+2}$ in the range of $T_{\tilde{\Delta}[f_{-}g_{+}-h]}$. Then for each $0 < s < 1$, $\{u_l|sD\}_{l=1}^{3r+2}$ are also linearly independent in the space $L_a^2(sD)$. Since analytic polynomials are dense in $L_a^2(sD)$, for each l there are analytic polynomials p_{lj} such that $T_{\tilde{\Delta}[f_{-}g_{+}-h]}\,p_{lj}$ converges to u_l as $(j \to \infty)$. Thus $T_{\tilde{\Delta}[f_{-}g_{+}-h]}\,p_{lj}$ converges uniformly to u_l on every compact subset of the unit disk D. Note that sD is contained in a compact subset of the unit disk, we have

$$\lim_{j\to\infty} \int_{sD} \left|\{T_{\tilde{\Delta}[f_{-}g_{+}-h]}\,p_{lj} - u_l\right|^2 \mathrm{d}A(z) = 0.$$

By the above formula, $T_{\tilde{\Delta}[f_{-}g_{+}-h]}\,p_{lj}\,|sD$ is contained in the subspace $\operatorname{span}\left\{X_j'(sz), f_+'(sz)\right\}_{j=1}^{3r}$, so

$$u_l(sz) \in \operatorname{span}\left\{X_j'(sz), f_+'(sz)\right\}_{j=1}^{3r}$$

for $l = 1,2,\ldots,3r+2$. But this contradicts that $\{u_l(sz)\}_{l=1}^{3r+2}$ are linearly independent, and hence the claim follows.

By Luecking's Theorem in [169], which says that there is no nonzero finite rank Toeplitz operators on the Bergman space with symbol in $L^1(D)$, we have

$$\tilde{\Delta}[f_{-}(z)g_{+}(z) - h(z)] = 0$$

on D. This implies that $f_{-}g_{+} - h$ is a harmonic function and hence

$$B[f_{-}g_{+} - h](z) = f_{-}(z)g_{+}(z) - h(z).$$

So Equation (23) gives

$$\begin{aligned} &f_{-}(z)g_{+}(z) - h(z)\\ &\quad= (1-|z|^2)^2 \sum_{j=1}^{r} x_j(z)\overline{y_j(z)} - f_{-}(z)g_{-}(z) - f_{+}(z)g_{+}(z)\\ &\quad - f_{+}(z)g_{-}(z). \end{aligned}$$

Therefore we obtain

$$f(z)g(z) = h(z) + (1 - |z|^2)^2 \sum_{j=1}^{r} x\_j \; (z)\overline{y_j \,(z)}.$$

Conversely, suppose that $f_- g_+ - h$ is harmonic on D, and

$$f(z)g(z) = h(z) + (1 - |z|^2)^2 \sum_{j=1}^{r} x_j \,(z)\overline{y_j \,(z)}.$$

It follows from (24) and (25) that

$$B\big[T_f T_g - T_h\big](z) = f(z)g(z) - h(z) = B\left[\sum_{j=1}^{r} x_j \otimes y_j\right](z).$$

Since the Berezin transform is injective [55], we conclude that

$$T_f T_g - T_h = \sum_{j=1}^{r} x_j \otimes y_j$$

The proof is complete.

The Hardy space H^2 is the subspace of analytic functions on D whose Taylor coefficients are square summable. It can be also identified (by radial limits) with the subspace of $L^2(\partial D)$ of functions whose negative Fourier coefficients vanish. For $p \geq 1$, the classical Hardy space H^p is the subspace of $L^p(\partial D)$ consisting of those functions whose negative Fourier coefficients vanish. Let $\hat{P}$ denote the Szego projection: the orthogonal projection from $L^2(\partial D)$ onto H^2. Since $\hat{P}$ is a bounded projection from $L^q(\partial D)$ onto H^q for $q > 1$, for each bounded harmonic function ϕ on D, we have that both ϕ_+ and ϕ_- are in $\bigcap_{q>1} H^q$. We let $\hat{T}_f$ denote the Toeplitz operator on the Hardy space H^2 and $\hat{H}_f$ the Hankel operator on H^2 which are defined by

$$\hat{T}_f h = \hat{P}(fh),$$
$$\hat{H}_f h = \big(I - \hat{P}\big)(fh)$$

for $h \in H^2$. There is an extensive literature on Toeplitz operators on the Hardy space H^2 [175]. We will give a characterization when f and g are holomorphic functions in $BMOA$. The main idea is to exchange an identity of the Toeplitz operators on the Bergman space to an identity of Toeplitz operators on Hardy space by the Berezin transforms.

For an operator (possibly unbounded) S on H^2, define the Berezin transform

$$B[S](z) = \frac{\langle S\hat{k}_z, \hat{k}_z\rangle}{H^2}$$

if the domain of the operator S contains all the normalized reproducing kernels ˆ

$$k_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)}$$

of H^2. The Berezin transform is also injective on the Hardy space.

A simple calculation gives that for two nonnegative integers k and l,

$$T_{\bar{z}^k}\, z^l = 0$$

if $k > l$ and

$$T_{\bar{z}^k}\, z^l = \frac{l - k + 1}{l + 1}\, z^{l-k}$$

if $k \leq l$. This immediately leads to the following lemma.

Lemma (6.3.9)[172]: If $u(z) = u_0 + u_1 z + \cdots + u_{m-1}z^m + \cdots$ is in L_a^2, then for each $m \geq 0$,

$$T_{\overline{u}}z^{m+k} = \frac{(m+k+1)\overline{u}_0 z^{m+k} + (m+k)\overline{u}_1 z^{m+k-1} + \cdots + (k+1)\overline{u}_m + \cdots + \overline{u}_{m+k}}{m+k+1},$$

for $k \geq -m$. We will prove the following theorem.

Theorem (6.3.10)[172]: Suppose that f and g are two nonconstant functions in $BMOA$ and h is in $L^q(D)C^2(D)$ with $\tilde{\Delta}h \in L^1(D)$. If there are two families $\{x_j\}_{j=1}^r$ and $\{y_j\}_{j=1}^r$ of linearly independent functions in L_a^2 such that

$$T_f T_g - T_h = \sum_{j=1}^{r} x_j \otimes y_j ,$$

where $r \geq 1$, then h is harmonic function, and

$$h = h_+ + h_-,$$

where h_+ and $\overline{h_-}$ are analytic functions, f, g, h_+, and $\overline{h}_-$ and all x_i, y_i are rational functions $(i = 1,2,\ldots,r, r > 0)$. Moreover there are analytic polynomials $q(z), p(z), F(z), G(z), b(z), B(z), c_1(z), \cdots, c_r(z), d_1(z), \cdots, d_r(z)$ such that

$$f(z) = \frac{F(z)}{q(z)}, \quad g(z) = \frac{G(z)}{p(z)}, h_+(z) = \frac{b(z)}{q(z)}, \overline{h}_-(z) = \frac{B(z)}{p(z)} \tag{29}$$

$$x_j(z) = \frac{c_j(z)}{q(z)}, y_j(z) = \frac{d_j(z)}{p(z)} \tag{30}$$

Where

$$\deg c_j \leq \max\{deg\, F, deg\, q\} - 2,$$

and

$$deg\, d_j \leq \max\{deg\, G, deg\, p\} - 2,$$

for $j = 1,2,\ldots,r$. In fact,

$$\text{rank}\left(\hat{T}_f \hat{T}_{\overline{g}} - \hat{T}_h\right) = \min\{\max\{deg\, B, deg\, p\}, \max\{deg\, F, deg\, q\}\}. \tag{31}$$

If

$$\min\{\max\{deg\, B, deg\, p\}, \max\{deg\, F, deg\, q\}\} < 2,$$

then

$$T_f T_g - T_h = 0.$$

Proof. Suppose

$$T_f T_{\overline{g}} - T_h = \sum_{j=1}^{r} x_j \otimes y_j \tag{32}$$

where x_j, y_j are in L_a^2 such that $x_1, x_2, \ldots, x_r$ are linearly independent and $y_1, y_2, \ldots, y_r$ are linearly independent.

By Theorem (6.3.8), since f and g are analytic, we have that $f_-(\overline{g})_+ - h = -h$ is harmonic. Thus h is harmonic. So we can write $h = h_+ + h_-$, where $h_+, \overline{h_-}$ are functions in H^2.

First we show that x_i, y_i are in H^2, for $i = 1,2,\ldots,r$. By Theorem (6.3.8), we have that

$$f(z)\overline{g(z)} - h(z) = (1 - |z|^2)^2 \sum_{i=1}^{r} x_i(z)\overline{y_i(z)} \tag{33}$$

By the "complexify" argument used in the proof of Theorem (6.3.8), we obtain

$$f(z)\overline{g(\overline{w})} - h_+(z) - h_-(\overline{w}) = (1 - zw)^2 \sum_{i=1}^{r} x_i(z)\overline{y_i(\overline{w})} \quad (34)$$

For a fixed t in (0, 1), we have

$$f(z)\overline{g(t\overline{w})} - h_+(z) - h_-(t\overline{w}) = (1 - tzw)^2 \sum_{i=1}^{r} x_i(z)\overline{y_i(t\overline{w})}, \quad (35)$$

to get

$$\sum_{i=1}^{r} x_i(z)y_i(t\overline{w}) = \frac{1}{(1 - tzw)^2}\left[f(z)\overline{g(t\overline{w})} - h_+(z) - h_-(t\overline{w})\right].$$

For each j, pairing both sides of the above equation with $y_j(tw)$ and then integrating about w, we have

$$\sum_{i=1}^{r}\left[\int_D y_j(tw)\overline{y_i(t\overline{w})}dA(w)\right]x_i(z)$$
$$= \int_D \frac{y_j(tw)}{(1 - tzw)^2}\left[f(z)\overline{g(t\overline{w})} - h_+(z) - h_-(t\overline{w})\right]dA(w) = g_j(z),$$

where

$$g_j(z) = \int_D \frac{y_j(tw)}{(1 - tzw)^2}\left[f(z)\overline{g(t\overline{w})} - h_+(z) - h_-(t\overline{w})\right]dA(w)$$

is in H^2 since f and h_+ are in H^2. The coefficient matrix of the above system is given by

$$\left[\int_D y_j(tw)\overline{y_i(t\overline{w})}dA(w)\right]_{i,j},$$

and is invertible since $\{y_1, y_2, \cdots, y_r\}$ are linearly independent in L_a^2. Cramer's rule gives

$$x_i(z) = \sum_{i=1}^{r} b_{ij}g_j(z)$$

for each z in D and some constants b_{ij}. This gives that each x_i is in H^2 since gj is in H^2. Similarly we can show that each y_i is also in H^2.

Next we will show that both f and g are rational functions. Rewriting Equation (33), for every $z \in D$, we have

$$f(z)\overline{g(z)} - h(z) = (1 - |z|^2)\left\{\sum_{i=1}^{r} x_i(z)\overline{y_i(z)} - \sum_{i=1}^{r} zx_i(z)\overline{zy_i(z)}\right\} \quad (36)$$

Simple computations give

$$\langle[\hat{T}_f\hat{T}_g - \hat{T}_h]\hat{k}_z, \hat{k}_z\rangle = f(z)\overline{g(z)} - h(z),$$
$$\langle\left[\sum_{i=1}^{r} x_i \otimes y_i - \sum_{i=1}^{r} \varsigma x_i \otimes \varsigma y_i\right]\hat{k}_z, \hat{k}_z\rangle$$
$$= (1 - |z|^2)\left\{\sum_{i=1}^{r} x_i(z)\overline{y_i(z)} - \sum_{i=1}^{r} zx_i(z)\overline{zy_i(z)}\right\}.$$

Then Equation (36) implies that

$$\langle [\hat{T}_f \hat{T}_g - \hat{T}_h] \hat{k}_z, \hat{k}_z \rangle = \langle \left[\sum_{i=1}^{r} x_i \otimes y_i - \sum_{i=1}^{r} \varsigma x_i \otimes \varsigma y_i \right] \hat{k}_z, \hat{k}_z \rangle. \quad (37)$$

Since the Berezin transform is one-to-one from the algebra of bounded operators on the Hardy space to $L^\infty(D)$, we have

$$\hat{T}_f \hat{T}_g - \hat{T}_h = \sum_{i=1}^{r} x_i \otimes y_i - \sum_{i=1}^{r} \varsigma x_i \otimes \varsigma y_i. \quad (38)$$

Thus $\hat{T}_f \hat{T}_g - \hat{T}_h$ is an operator at most $2r$ rank and the well-defined symbol map on the Toeplitz algebra on the Hardy space [175] gives

$$f(z)g(z) - h(z) = 0$$

on the unit circle ∂D. Thus we have

$$\hat{T}_f \hat{T}_g - \hat{T}_{f\overline{g}} = \sum_{i=1}^{r} x_i \otimes y_i - \sum_{i=1}^{r} \varsigma x_i \otimes \varsigma y_i,$$

to get

$$-\hat{H}^*_{\overline{f}} \hat{H}_{\overline{g}} = \sum_{i=1}^{r} x_i \otimes y_i - \sum_{i=1}^{r} \varsigma x_i \otimes \varsigma y_i$$

is a finite rank operator as

$$\hat{T}_f \hat{T}_g - \hat{T}_{f\overline{g}} = -\hat{H}^*_{\overline{f}} \hat{H}_{\overline{g}}.$$

By Theorem (6.3.4), either $\hat{H}_{\overline{f}}$ or $\hat{H}_{\overline{g}}$ has finite rank. Let

$$n = \text{rank}\left(\hat{T}_f \hat{T}_g - \hat{T}_{f\overline{g}}\right).$$

By Theorem (6.3.5), either rank $\hat{H}_{\overline{f}}$ or rank $\hat{H}_{\overline{g}}$ equals n. By the Kronecker theorem (p. 21, [178]), we have that either f or g is a rational function with degree n. We may assume that the degree of f equals n and

$$f(z) = \frac{F(z)}{q(z)},$$

where $F(z)$ and $q(z)$ are polynomials with degrees at most n and do not have any common factors.

Next we will show that h_+ and $x_i, i = 1, 2, \ldots, r$ are all rational functions with the same denominator $q(z)$. Moreover we can write

$$h_+(z) = \frac{b(z)}{q(z)}$$

and

$$x_j(z) = \frac{c_j(z)}{q(z)},$$

where $b(z)$ is an analytic polynomial with degree at most n and $c_j(z)$ is an analytic polynomial with degree less than $n - 2$, for $j = 1, 2, \ldots, r$.

Since $f\overline{g} = h$ on $\partial D, qf\overline{g} = qh_+ + qh_-$ and both q and $F = qf$ are analytic polynomials with degree at most n, we have

$$b = qh_+ = \hat{P}[(qf)g] - \hat{P}[qh_-] = \hat{P}[Fg] - \hat{P}[qh_-]$$

is an analytic polynomial with degree at most $\max\{deg\, F, deg\, q\} = deg\, f$. Therefore h_+ is a rational function:

$$h_+(z) = \frac{b(z)}{q(z)}.$$

Using Identity (32) and $(z) = \frac{F(z)}{q(z)}$, we have

$$T_F\, T_{\overline{g}} - T_q T_{h_-} - T_b = T_q\left[T_f T_{\overline{g}} - T_h\right] = T_{qf} T_{\overline{g}} - T_q T_{h_-} - T_{qh^+}$$

$$\sum_{i=1}^{r} (qx_i) \otimes y_i = \sum_{i=1}^{r} c_i \otimes y_i \qquad (39)$$

where $c_i = qx_i$.

Let P_N be the projection from L_a^2 onto the subspace spanned by $\{1, z, \cdots, z_N\}$ which consists of polynomials of z with degree at most N. Since $\{y_i\}_{i=1}^{r}$ are linearly independent in L_a^2, we see that for some sufficient large $N, \{P_N(y_i)\}_{i=1}^{N}$ are also linearly independent. Applying the operator identity (39) to each $P_N(y_j)$ gives the following system of functions

$$F(z) T_{\overline{g}} P_N(y_j) - q(z) T_{h_-} P_N(y_j) - b(z) P_N(y_j) = \sum_{i=1}^{r} \langle P_N(y_j), y_i \rangle_{c_i}$$

$$= \sum_{i=1}^{r} \langle P_N(y_j), P_N(y_i) \rangle_{c_i}$$

for $j = 1, \cdots, r$. By Lemma (6.3.9), we have that $T_{\overline{g}} P_N(y_j)$ and $T_{h_-} P_N(y_j)$ are polynomials of z. This gives that the left hand side of each equation in the above system is a polynomial of z. Since the $r \times r$ matrix $\left(\langle P_N(y_j), P_N(y_i) \rangle\right)_{i,j}$ is invertible, solving the above system for c_i we have that c_i is a polynomial of z.

Next we will show that $g, \overline{h_-}$ and $y_i, i = 1, 2, \ldots, r$ are all rational functions with the same denominator.

Taking partial derivative both sides of Equation (32) about z gives

$$f(z)\overline{g}(z) - h_+'(z)$$

$$= (1 - |z|^2)\left\{\sum_{i=1}^{r} \left[(1 - |z|^2) x_i'(z)\overline{y_i(z)} - 2\overline{z} x_i(z)\overline{y_i(z)}\right]\right\} \qquad (40)$$

Thus for each $\varsigma \in \partial D$, we have

$$f'(\varsigma)\overline{g(\varsigma)} = h_+'(\varsigma), \qquad (41)$$

to get that $g(\varsigma)$ is a rational function on the unit circle as $f'(\varsigma)$ and $h_+'(\varsigma)$ are rational functions on the unit circle. This gives that $g(z)$ is a rational function on the unit disk. Let

$$g(z) = \frac{G(z)}{p(z)}$$

for two polynomials $G(z)$ and $p(z)$ with degree at most degree m of g. Using

$$T_g T_{\overline{f}} - T_h = \sum_{i=1}^{r} y_i \otimes x_i,$$

and repeating the above argument, we can get that h_- and all y_i are rational functions:

$$\overline{h_-(z)} = \frac{B(z)}{p(z)}, \qquad y_i(z) = \frac{d_i(z)}{p(z)}$$

for some polynomials $B(z), d_1(z), \cdots, d_r(z)$ with degree at most m.

Last we will obtain the remaining result of the theorem.

Theorem (6.3.5) implies

$$n = \operatorname{rank} \widehat{H}_{\overline{f}}^{*} \widehat{H}_{\overline{g}} = \min\left\{\operatorname{rank} \widehat{H}_{\overline{f}}, \operatorname{rank} \widehat{H}_{\overline{g}}\right\} = \min\{n, m\}.$$

By the Kronecker theorem [178], we have

$$\operatorname{rank} \widehat{H}_{\overline{f}} = \max\{deg\, F, deg\, q\}.$$

Thus we obtain (31). To complete the proof we need only to show

$$deg\, c_i(z) \le n - 2, \qquad deg\, d_i(z) \le m - 2.$$

To do this, using (29), (30) and (34) we have

$$F(z)\overline{G(\overline{w})} - b(z)\overline{p(\overline{w})} - q(z)\overline{B(\overline{w})} = (1 - zw)^2 \sum_{i=1}^{r} c_i(z)\overline{d_i(\overline{w})}.$$

Since the left hand side of the above equation is a polynomial of z with degree at most n and is also a polynomial of w with degree at most m and the degree of $(1 - zw)^2$ about z or w is 2, we conclude that the degree of each $c_i(z)$ is at most $n - 2$ and the degree of each $d_i(w)$ is at most $m - 2$. If either n or m is less than 2, we have that

$$\sum_{i=1}^{r} c_i(z)\overline{d_i(\overline{w})} = 0.$$

This gives that

$$\sum_{i=1}^{r} x_i \otimes y_i = 0,$$

and hence

$$T_f T_g - T_h = 0.$$

This completes the proof.

We will prove Theorems (6.3.13), (6.3.7), and (6.3.16). We need some notation. For each α in the unit disk, define a unitary operator U_α on $L^2(D)$:

$$U_\alpha \phi = \phi(\phi_\alpha(z))k_\alpha(z)$$

for $\varphi \in L^2(D)$, where $\phi_\alpha(z)$ is the Mobius transform $\frac{\alpha - z}{1 - \alpha\overline{z}}$. As pointed out in [46], $U_\alpha U_\alpha = I$ and

$$U_\alpha T_f U_\alpha = T_{f\circ\phi_\alpha}.$$

Using the above properties one can easily get the following useful and simple lemma which we will use in the proof of Theorem (6.3.13). We omit its proof.

Lemma (6.3.11)[172]: For any $\alpha \in D$, if

$$T_f T_{\overline{g}} - T_h = \sum_{i=1}^{r} x_i \otimes y_i,$$

then

$$T_{f\circ\phi_\alpha} T_{\overline{g}\circ\varphi_\alpha} - T_{h\circ\phi_\alpha} = \sum_{i=1}^{r} (U_\alpha x_i) \otimes (U_\alpha y_i).$$

In the proof of Theorem (6.3.13) we will use the following corollary of Bochner's theorem on critical points of a rational function [180].

Theorem (6.3.12)[172]: If the circular regions $|z| \le r_1$ and $|z| \ge r_2 (> r_1)$ contain respectively the zeros and poles of a rational function $R(z)$ of degree n, those regions contain all the critical points of $R(z)$, and the former region contains precisely $n - 1$ critical points.

Now we are ready to present the proof of Theorem (6.3.13).

Theorem (6.3.13)[172]: Suppose f and g are bounded harmonic functions and h is a bounded C^2 function such that $\tilde{\Delta}\, h \in L^1(D)$. If $T_f T_g - T_h$ has the rank at most one, then either f is conjugate analytic or g is analytic. In either case, $h = fg$.

Proof. First we reduce Theorem (6.3.13) to the special case that f and $\overline{g}$ are in $BMOA$ and h is in $L^q(D)C^2(D)$ for any $q > 1$ with

$$\tilde{\Delta}\, h \in L^1(D).$$

To do so, write $f = f_+ + f_-$ and $g = g_+ + g_-$. We have

$$T_f T_g - T_h = T_{f_+} T_{g_-} - T_{h_- f_- g_+ - f_- g_- - f_+ g_+}.$$

Let $G = h - f_- g_+ - f_- g_- - f_+ g_+$. Then $T_{f_+} T_{g_-} - T_G$ has rank at most one also. By Theorem (6.3.10), G is harmonic on the unit disk. So we may consider the finite rank operator $T_f T_g - T_h$ with rank at most one where $f = f_+, g = g_-$, and $h = G$. We will show that either f or g is constant and $T_f T_{\overline{g}} - T_h = 0$. This gives that $h = f\overline{g}$.

Since $T_f T_{\overline{g}} - T_h$ has rank at the most one, there are two functions x_1 and y_1 in L^2 a such that

$$T_f T_{\overline{g}} - T_h = x_1 \otimes y_1. \tag{42}$$

By Theorem (6.3.10), h is harmonic on D and $x_1 \in H^2$ and $y_1 \in H^2$. We write $h = h_+ + h_-$ where h_+ is analytic part of h and h_- is conjugate analytic part of h.

By Theorem (6.3.10), we may assume that $\hat{T}_f \hat{T}_{\overline{g}} - \hat{T}_h$ is a finite rank operator with rank $n = 2$ and there are analytic polynomials $q(z), p(z), F(z), G(z), b(z), B(z), c_1(z), d_1(z)$ such that

$$f(z) = \frac{F(z)}{q(z)}, \qquad g(z) = \frac{G(z)}{p(z)}, \qquad h_+(z) = \frac{b(z)}{q(z)}, \qquad \overline{h}_-(z) = \frac{B(z)}{p(z)}$$

and

$$x_1(z) = \frac{c_1(z)}{q(z)}, \qquad y_1(z) = \frac{d_1(z)}{p(z)}.$$

As for α in D except for one point, the degree of the denominator of $R \circ \phi_\alpha$ is greater than or equal to the degree of the numerator of $R \circ \phi_\alpha$ for a rational function R, by Lemma (6.3.11), we may assume $deg\, q = 2 \le m = deg\, p, deg\, F \le 2$ and $deg\, G \le m$. Then $deg\, F \le 2, deg\, G \le m, deg\, b \le 2, deg\, B \le m,$

$$deg\, c_1 = 0,$$

and

$$deg\, d_1 \le m - 2.$$

By Theorem (6.3.8) we have

$$f(z)\overline{g(z)} = h(z) + (1 - |z|^2)^2 x_1(z)\overline{y_1(z)} \tag{43}$$

holds for $z \in D$. Complexify (43) to get

$$f(z)\overline{g(w)} = h_+(z) + h_-(w) + (1 - z\overline{w})^2 x_1(z)\overline{y_1(w)}$$

holds for z and w in D. Since these functions in the above equation are rational functions we have

$$f(z)\overline{g(w)} = h_+(z) + h_-(w) + (1 - z\overline{w})^2 x_1(z)\overline{y_1(w)} \tag{44}$$

for z and w in the complex plane C except for finitely many points.

Taking partial derivative both sides of (44) with respect to z gives

$$f\,(z)\overline{g(w)} = h'_+(z) - 2\overline{w}(1 - z\overline{w})x(z)\overline{y(w)} + (1 - z\overline{w})^2 x\,(z)\overline{y(w)}.$$

Letting $w = \frac{1}{\overline{z}}$ in the above equation, we have

$$f\ (z)g\left(\frac{1}{\overline{z}}\right) = h'_+(z),$$

to get

$$f'(z)\frac{\tilde{G}_m(z)}{\tilde{p}(z)} = h'_+(z).$$

Noting that zeros of $p(z)$ are outside of the unit disk, we see that zeros of $\tilde{p}(z)$ are in the unit disk. In addition, poles of f lie outside of the unit disk. Observe that f equals zero at the zero set of $\tilde{p}(z)$ in the unit disk which has two points α_1 and α_2 with multiplicity. Thus f has two critical points α_1 and α_2 with multiplicity in the unit disk. So $f - f(\alpha_1)$ has two critical points α_1 and α_2 with multiplicity in the unit disk. On the other hand, $f - f(\alpha_1)$ has only one repeated zero α_1 in the unit disk. Thus the unit circle separates the all zeros of $f - f(\alpha_1)$ from all poles of $f - f(\alpha_1)$. Theorem (6.3.12) says that $f - f(\alpha_1)$ has only one critical point in the unit disk. This contracts that $f - f(\alpha_1)$ has two critical points α1 and α2 with multiplicity in the unit disk. This implies that f is a constant.

For each polynomial $P(z)$ of z with degree N,

$$P(z) = P_0 + P_1 z + \cdots P_N z^N,$$

recall

$$\begin{aligned}\tilde{P}(z) &= z^N P\left(\frac{1}{\overline{z}}\right)\\ &= P_0 z^N + P_1 z^{N-1} + \cdots + P_N.\end{aligned}$$

For $m \geq N$, recall

$$\tilde{P}_m(z) = \tilde{P}(z) z^{m-N}.$$

To get Theorem (6.3.16) we need only the following theorem.

Theorem (6.3.14)[172]: Suppose that F, G, q, p, b and B are polynomials of z and the degree of p equals m, degrees of G and B are at most m. $T_F T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}}$ has finite rank if and only if

$$F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) = 0, \quad (45)$$

$$F(z)\tilde{G}'_m(z) - b(z)\tilde{p}'_m(z) - q(z)\tilde{B}'_m(z) = 0. \quad (46)$$

Proof. Let $e_k = (k+1)z_k$. Then $\{e_k\}_{k=0}^{\infty}$ is an orthogonal basis of the Bergman space L_a^2. Using Lemma (6.3.9) first we calculate $\left[T_F T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}}\right] e_{m+k-1}$. Writing

$$G(z) = u_0 + u_1 z + \cdots + u_{m_G} z^{m_G}, \qquad B(z) = v_0 + v_1 z + \cdots + v_{m_B} z^{m_B},$$

$$p(z) = p_0 + p_1 z + \cdots + p_m z^m, \qquad \max\{m_G, m_B\} \leq m.$$

By Lemma (6.3.9), we have that for $k \geq 1$,

$$\begin{aligned}&\left[T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}\right] e_{m+k-1}\\ &= F(z)\left[(m+k)\overline{u}_0 z^{m+k-1} + \cdots + (k+m-m_G)\overline{u}_{m_G} z^{m-m_G+k-1}\right]\\ &\quad -q(z)\left[(m+k)\overline{v}_0 z^{m+k-1} + \cdots + (k+m-mB)\overline{v}_{m_B} z^{m-m_B+k-1}\right]\\ &\quad -b(z)\left[(m+k)\overline{p}_0 z^{m+k-1} + \cdots + k\overline{p}_m z^{k-1}\right)\\ &= F(z)\left(\tilde{G}_m(z) z^k\right) - q(z)\left(\tilde{B}_m(z) z^k\right) - b(z)(\tilde{p}_m(z) z^k)\\ &= \left[F(z)\tilde{G}_m(z) - q(z)\tilde{B}_m(z) - b(z)\tilde{p}_m(z)\right] k z^{k-1}\\ &\quad + \left[F(z)\tilde{G}_m(z) - q(z)\tilde{B}_m(z) - b(z)\tilde{p}_m(z)\right] z^k. \quad (47)\end{aligned}$$

If (45) and (46) hold, then the above equalities give that $T_F T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}}$ vanishes on $\{e_l\}_{l=m}^{\infty}$. This gives that $T_F T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}}$ has finite rank.

Conversely suppose that $T_F T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}}$ has finite rank. We may assume that

$$T_F\, T_{\tilde{G}} - T_b T_{\overline{p}} - T_q T_{\tilde{B}} = \sum_{i=1}^{r} c_i \otimes d_i. \qquad (48)$$

Thus taking the Berezin transform both sides of the above equality gives

$$F(z)\overline{G(z)} - b(z)\overline{p(z)} - q(z)\overline{B(z)} = (1 - |z|^2)^2 \sum_{i=1}^{r} c_i(z) d_i(z)$$

for z in D. Complexify the above equation to obtain

$$F(z)\overline{G(w)} - b(z)\overline{p(w)} - q(z)\overline{B(w)} = (1 - |z|^2)^2 \sum_{i=1}^{r} c_i(z) d_i(w). \qquad (49)$$

By Theorem (6.3.10), we notice that the both sides of the above equation are polynomials of both z and w. Letting $w = \frac{1}{z}$ in (49) and then multiplying both sides of (49) give

$$F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) = 0, \qquad (50)$$

which is (45). Thus (47) becomes

$$\left[T_F\, T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}\right] e_{m+k-1} = \left[F(z)\tilde{G}'_m(z) - q(z)\tilde{B}'_m(z) - b(z)\tilde{p}'_m(z)\right] z^k$$

for $k \geq 0$. Since $T_F\, T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}$ has finite rank, we have that the dimension of the range of $T_F\, T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}$ is of finite dimension. On the hand, its range contains $\left\{\left[F(z)\tilde{G}'_m(z) - q(z)\tilde{B}'_m(z) - b(z)\tilde{p}'_m(z)\right] z^k\right\}_{k=0}^{\infty}$. Thus

$$F(z)\tilde{G}'_m(z) - q(z)\tilde{B}'_m(z) - b(z)\tilde{p}'_m(z) = 0,$$

which is (46). This completes the proof.

Solving for F and b in (45) and (46) gives the following theorem if F and q do not have any nontribyl common factors.

Theorem (6.3.15)[172]: Suppose that F, G, q, p, b and B are polynomials of z and degree of p equals m, degrees of G and B are at most m. If F and q do not have any nontribyl common factors and either $\frac{G}{p}$ or $\frac{B}{p}$ is not a constant, then (45) and (46) are equivalent to the existence of a nonzero polynomial P of z such that

$$\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m = qP, \qquad (51)$$
$$\tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m = FP, \qquad (52)$$
$$\tilde{B}_m\tilde{G}'_m - \tilde{B}'_m\tilde{G}_m = bP. \qquad (53)$$

Proof. Suppose that (51), (52) and (53) hold for a nonzero polynomial P. Then we have

$$P\left[F\tilde{G}_m - b\tilde{p}_m - q\tilde{B}_m\right] = \tilde{G}_m\left(\tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m\right) - \tilde{p}_m\left(\tilde{B}_m\tilde{G}'_m - \tilde{B}'_m\tilde{G}_m\right) - \tilde{B}_m\left(\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m\right) = 0,$$

and

$$P\left[F\tilde{G}'_m - b\tilde{p}'_m - q\tilde{B}'_m\right] = \tilde{G}'_m\left(\tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m\right) - \tilde{p}'_m\left(\tilde{B}_m\tilde{G}'_m - \tilde{B}'_m\tilde{G}_m\right) - \tilde{B}'_m\left(\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m\right) = 0.$$

This gives (45) and (46). Conversely, suppose that (45) and (46) hold.

Then we have the following system

$$\begin{bmatrix} \tilde{G}_m & \tilde{p}_m \\ \tilde{G}'_m & \tilde{p}'_m \end{bmatrix} \begin{bmatrix} F \\ -b \end{bmatrix} = q \begin{bmatrix} \tilde{B}_m \\ \tilde{B}'_m \end{bmatrix}.$$

Multiplying both sides of the above system by

$$\begin{bmatrix} \tilde{p}'_m & -\tilde{p}_m \\ -\tilde{G}'_m & \tilde{G}_m \end{bmatrix}$$

gives

$$\left(\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m\right)\begin{bmatrix} F \\ -b \end{bmatrix} = q\begin{bmatrix} \tilde{p}'_m & -\tilde{p}_m \\ -\tilde{G}'_m & \tilde{G}_m \end{bmatrix}\begin{bmatrix} \tilde{B}_m \\ \tilde{B}'_m \end{bmatrix} = q\begin{bmatrix} \tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m \\ -\tilde{B}_m\tilde{G}'_m + \tilde{B}'_m\tilde{G}_m \end{bmatrix}.$$

Since F and q do not have any common factors, the first equation in the above system gives that there is a polynomial P such that

$$\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m = qP,$$
$$\tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m = FP.$$

The second equation the above system gives

$$\tilde{B}_m\tilde{G}'_m - \tilde{B}'_m\tilde{G}_m = bP.$$

If P equals 0, we have

$$\tilde{G}_m\tilde{p}'_m - \tilde{G}'_m\tilde{p}_m = \tilde{B}_m\tilde{p}'_m - \tilde{B}'_m\tilde{p}_m = \tilde{B}_m\tilde{G}'_m - \tilde{B}'_m\tilde{G}_m = 0,$$

to get

$$\tilde{G}'_m\,\tilde{G}_m = \tilde{p}'_m\,\tilde{p}_m = \tilde{B}'_m\,\tilde{B}_m\,.$$

Solving the above differential equations gives

$$\tilde{G}_m = \lambda\tilde{p}_m = \lambda\tilde{B}_m,$$

for some constant λ and hence

$$G = \lambda\overline{p} = \lambda\overline{B}.$$

Since either $\frac{G}{p}$ or $\frac{B}{p}$ is not a constant, we have that P is a nonzero polynomial. This completes the proof.

Theorem (6.3.16)[172]: Suppose f and g are bounded harmonic functions, h is a bounded C^2 function such that $\tilde{\Delta}\, h \in L^1(D)$. $T_fT_g - T_h$ has finite rank greater than one if and only if

(i) $\overline{f_-}$ and g_+ are in $H^\infty(D)$;

(ii) $H = h_-(f_-g_+ + f_-g_- + f_+g_+)$ is harmonic on D;

(iii) f_+ is a rational function $\frac{F}{q}$ with degree $n > 2$, $\overline{g_-}$ is a rational function $\frac{G}{p}$ with degree $m > 2$, H_+ is a rational function $\frac{b}{q}$ with degree n and $\overline{H_-}$ is a rational function $\frac{B}{p}$ with degree m;

(iv)

$$F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) = 0;$$

(v)

$$F(z)\tilde{G}'_m(z) - b(z)\tilde{p}'_m(z) - q(z)\tilde{B}'_m(z) = 0.$$

Proof. Since f and g are bounded and harmonic on D, first we write

$$f = f_+ + f_-;$$

and

$$g = g_+ + g_-$$

for $f_+, \overline{f_-}, g_+$ and $\overline{g_-}$ in $BMOA$.

Suppose that $T_fT_g - T_h$ has finite rank on L^2_a. By Theorem (6.3.8), we have that $H = h_-\,(f_-g_+ + f_-g_- + f_+g_+)$ is harmonic on D, which is Condition (ii) Also we have

$$T_{f_+}\,T_{g_-} - T_H = T_fT_g - T_h$$

has finite rank. Theorem (6.3.10) gives that there are analytic polynomials $q(z), p(z), F(z), G(z), b(z), B(z)$ such that

$$f_+(z)=\frac{F(z)}{q(z)}\,, \qquad \overline{g_-(z)}=\frac{G(z)}{p(z)}\,, H_+(z)=\frac{b(z)}{q(z)}\,, \overline{H}_-(z)=\frac{B(z)}{p(z)}.$$

This gives Condition (i). Let

$$\min\{m = \max\{deg\, B, deg\, p\}, n = \max\{deg\, F, deg\, q\}\} > 2.$$

Theorem (6.3.10) gives Condition (iii). By Theorem (6.3.14), we have

$$F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) = 0;$$
$$F(z)\tilde{G}'_m(z) - b(z)\tilde{p}'_m(z) - q(z)\tilde{B}'_m(z) = 0,$$

which are Conditions (iv) and (v).

Conversely, suppose

(i) $\overline{f_-}$ and g_+ are in $H^\infty(D)$;

(ii) $H = h - (f_-g_+ + f_-g_- + f_+g_+)$ is harmonic on D;

(iii) f_+ is a rational function $\frac{F}{q}$ with degree $n > 2, \overline{g_-}$ is a rational function $\frac{G}{p}$ with degree $m > 2, H_+$ is a rational function $\frac{b}{q}$ with degree n and $\overline{H_-}$ is a rational function $\frac{B}{p}$ with degree m;

(iv)

$$F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) = 0;$$

(v)

$$F(z)\tilde{G}'_m(z) - b(z)\tilde{p}'_m(z) - q(z)\tilde{B}'_m(z) = 0,$$

By Theorem (6.3.14), using Conditions (iv) and (v) we have that $T_F\, T_{\tilde{G}} - T_bT_{\overline{p}} - T_qT_{\tilde{B}}$ has finite rank:

$$T_F\, T_{\tilde{G}} - T_bT_{\overline{p}} - T_qT_{\tilde{B}} = \sum_{i=1}^{r} c_i \otimes d_i.$$

Thus

$$T_{f_+}\, T_{\overline{g_-}} - T_H = T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{b}{q}+\frac{\tilde{B}}{\overline{p}}} = T_{\frac{1}{q}}\left[T_F\, T_{\tilde{G}} - T_bT_{\overline{p}} - T_qT_{\tilde{B}}\right]T_{\frac{1}{p}}$$
$$= T_{\frac{1}{q}}\left[\sum_{i=1}^{m-1} c_i \otimes d_i\right]T_{\frac{1}{p}} = \sum_{i=1}^{m-1} \frac{c_i}{q} \otimes \frac{d_i}{p}$$

has finite rank. So

$$T_fT_g - T_h = T_{f_+}\, T_{g_-} - T_H$$

has finite rank. This completes the proof of Theorem (6.3.16).

Using Theorem (6.3.16) and Theorem (6.3.15) we have the following corollary.

Corollary (6.3.17)[172]: Suppose f and g are bounded harmonic functions, h is a bounded C^2 function such that $\tilde{\Delta} h \in L^1(D)$. If $T_fT_{\overline{g}} - T_h$ has finite rank and either f or g is a polynomial of z or h is analytic or co-analytic and both f and g are analytic, then either f or g is constant and

$$f\overline{g} = h.$$

Proof. Let $H = h - (f_-g_- + f_-g_+ + f_+g_-)$. By Theorem (6.3.16), H is harmonic in the unit disk and there are analytic polynomials $q(z), p(z), F(z), G(z), b(z), B(z)$ such that $H = H_+ + H_-$,

$$f_+(z)=\frac{F(z)}{q(z)}\,, \qquad \overline{g_+(z)}=\frac{G(z)}{p(z)}\,, \qquad H_+(z)=\frac{b(z)}{q(z)}\,, \qquad \overline{H}_-(z)=\frac{B(z)}{p(z)}.$$

By Theorem (6.3.13), we may assume that

$$\min\{m = \max\{deg\, B, deg\, p\}, n = \max\{deg\, F, deg\, q\}\} > 2.$$

We may assume that $m \geq n$ and $G(0) = 0$ and $B(0) = 0$. As the degree of $f = \frac{F}{q}$ is equal to n, Theorem (6.3.10) gives that F and q do not have any common factors. Also both the degree of $\tilde{G}_m$ and the degree of $\tilde{B}_m$ are less than m as $G(0) = 0$ and $B(0) = 0$. Since $T_f T_{\overline{g}} - T_h = T_{f_+} T_{\overline{g_+}} - T_H$ has finite rank, Theorem (6.3.15) gives

$$\tilde{G}_m \tilde{p}'_m - \tilde{G}'_m \tilde{p}_m = qP, \tag{54}$$

$$\tilde{B}_m \tilde{p}'_m - \tilde{B}'_m \tilde{p}_m = FP, \tag{55}$$

$$\tilde{B}_m \tilde{G}'_m - \tilde{B}'_m \tilde{G}_m = bP. \tag{56}$$

Suppose that either f or g is a polynomial. In the first case that f is a polynomial, we have that $f_- = 0$ and $q = 1$. Noting that the degree of $\tilde{p}_m$ equals m, we have that the degree of $\tilde{G}_m \tilde{p}'_m - \tilde{G}'_m \tilde{p}_m$ equals deg $\tilde{G}_m + m - 1.$ (54) gives

$$deg\, P = deg\, \tilde{G}_m + m - 1.$$

On the other hand, using (56) we have

$$deg\, b + deg\, P \leq deg\, \tilde{B}_m + deg\, \tilde{G}_m - 1 \leq m - 1 + deg\, \tilde{G}_m - 1.$$

Thus b must equal 0 identically and so $\tilde{B}_m$ equals $\lambda \tilde{G}_m$ for some constant λ. Using (55) we have that

$$deg\, F + deg\, P = deg\, \tilde{G}_m + m - 1,$$

to obtain that F is a constant. Thus f_+ is constant and so is f. In the case that g is a polynomial, we have that $g_- = 0$ and $p = 1$. Thus

$$\tilde{p}_m = z^m, \qquad \tilde{p}'_m = mz^{m-1}.$$

(54) and (55) give

$$\tilde{G}_m m z^{m-1} - \tilde{G}'_m z^m = qP,$$

$$\tilde{B}_m m z^{m-1} - \tilde{B}'_m z^m = FP.$$

So we have

$$f_+ = \frac{F}{q} = \frac{FP}{qP} = \frac{\tilde{B}_m m z^{m-1} - \tilde{B}'_m z^m}{\tilde{G}_m m z^{m-1} - \tilde{G}'_m z^m} = \frac{\tilde{B}_m m - \tilde{B}'_m z}{\tilde{G}_m m - \tilde{G}'_m z}$$

to get that the degree n of f_+ must equal m. Repeating the argument in the first case we have that g must be a constant.

If h is co-analytic and both f and g are analytic, then $f_- = g_- = 0$. Since $H = h - (f_- g_- + f_- \overline{g_+} + f_+ \overline{g_-} = h$ is co-analytic, we have that b equals 0 identically. (56) gives

$$\tilde{B}_m \tilde{G}'_m - \tilde{B}'_m \tilde{G}_m = 0.$$

Thus

$$\frac{\tilde{G}'_m}{\tilde{G}_m} = \frac{\tilde{B}'_m}{\tilde{B}_m}.$$

Integrating both sides of above equation gives that

$$\tilde{G}_m = \lambda^{-1} \tilde{B}_m,$$

for some constant λ. Thus we have

$$\overline{B} = \lambda \overline{G},$$

to get

$$H_-(z) = \frac{\overline{B(z)}}{p(z)} = \frac{\lambda \overline{G(z)}}{p(z)} = \lambda \overline{g_+}.$$

This gives that

$$T_{f_+} T_{\overline{g_+}} - T_{H_+ + H_-} = T_{f_+} T_{\overline{g_+}} - T_{H_-} = T_{f_+ - \lambda} T_{\overline{g_+}}$$

has finite rank and hence is compact. By the main theorem in [46], we have that the Berezin transform of $T_{f_+ - \lambda} T_{\overline{g_+}}$ vanishes on the unit circle. On the other hand, the Berezin transform of $T_{f_+ - \lambda} T_{\overline{g_+}}$ equals $(f_+ - \lambda)\overline{g_+}$ and hence $(f_+ - \lambda)\overline{g_+}$ vanishes on the unit circle and so does $(f_+ - \lambda) g_+$. We conclude that either f_+ or g_+ is a constant. Thus either f or g is constant.

If h is co-analytic and both f and g are analytic, then $f_- = g_- = 0$. Since $H = h - (f_- g_- + f_- g_+ + f_+ g_-) = h$ is analytic, we have that B equals 0 identically. In this case we consider that $T_{\overline{g}} T_{\overline{f}} - T_{\overline{H_+} + \overline{H_-}}$ has finite rank. Similarly we have that either f or g is a constant. This completes the proof.

We have the following theorem that implies Theorem (6.3.7).

Theorem (6.3.18)[172]: For each $m \geq 3$, if three nonzero real numbers α, β and γ satisfy that $|\beta| > 1, \alpha \neq \gamma$,

$$|\alpha|^{1/(m-1)} = |\beta|^{1/m},$$

and

$$\frac{m - 1}{|\beta|} - 1 > \frac{m}{|\alpha|},$$

then $T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{b}{q} + \frac{\tilde{B}}{\tilde{p}}}$ has finite rank and its rank equals $m - 1$ where

$$G(z) = \alpha z + z^m,$$
$$p(z) = \beta + z^m,$$
$$B(z) = \gamma z + z^m,$$
$$F(z) = \beta\gamma\left(z^m + \frac{m}{\gamma} z - \frac{m - 1}{\beta}\right),$$
$$q(z) = \beta\alpha\left(z^m + \frac{m}{\alpha} z - \frac{m - 1}{\beta}\right),$$
$$b(z) = (m - 1)(\alpha - \gamma).$$

Proof. Let polynomials F, q, b, G, B and p be given in the theorem. Simple calculation gives

$$\tilde{G}_m(z) = \alpha z^{m-1} + 1,$$
$$\tilde{p}_m(z) = \beta z^m + 1,$$
$$\tilde{B}_m(z) = \gamma z^{m-1} + 1,$$
$$\tilde{G}'_m(z) = (m - 1)\alpha z^{m-2},$$
$$\tilde{p}'_m(z) = m\beta z^{m-1},$$
$$\tilde{B}'_m(z) = (m - 1)\gamma z^{m-2}.$$

Thus we have

$$\begin{aligned} F(z)\tilde{G}_m(z) - b(z)\tilde{p}_m(z) - q(z)\tilde{B}_m(z) \\ &= \beta\gamma\left(z^m + \frac{m}{\gamma} z - \frac{m - 1}{\beta}\right)(\alpha z^{m-1} + 1) \\ &- (m - 1)(\alpha - \gamma)(\beta z^m + 1) \\ &- \beta\alpha\left(z^m + \frac{m}{\alpha} z - \frac{m - 1}{\beta}\right)(\gamma z^{m-1} + 1) = 0, \end{aligned}$$

$$F(z)\tilde{G}'_m(z) - b(z)\tilde{p}'_m(z) - q(z)\tilde{B}'_m(z)$$

$$= \beta\gamma\left(z^m + \frac{m}{\gamma} z - \frac{m-1}{\beta}\right)(m-1)\alpha z^{m-2} - (m-1)(\alpha-\gamma)m\beta z^{m-1}$$

$$-\beta\alpha\left(z^m + \frac{m}{\alpha} z - \frac{m-1}{\beta}\right)(m-1)\gamma z^{m-2} = 0.$$

Theorem (6.3.14) gives that $T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{b}{q}+\frac{\tilde{B}}{\tilde{p}}}$ has finite rank. Let $e_k = (k+1)z^k$. Then $\{e_k\}_{k=0}^{\infty}$ is an orthogonal basis of the Bergman space L_a^2. Simple calculation gives that for $1 \le k \le m-1$,

$$\left[T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}\right] e_k = F(z)\alpha k z^{k-1} - q(z)\gamma k z^{k-1} - b(z)\beta(k+1)z^k$$

$$= \beta\gamma\left(z^m + \frac{m}{\gamma} z - \frac{m-1}{\beta}\right)\alpha k z^{k-1}$$

$$-\beta\alpha\left(z^m + \frac{m}{\alpha} z - \frac{m-1}{\beta}\right)\gamma k z^{k-1} - (m-1)(\alpha-\gamma)\beta(k+1)z^k$$

$$= kz^{k-1}\beta\alpha\gamma\left(\frac{m}{\gamma} z - \frac{m}{\alpha} z\right) - (m-1)(\alpha-\gamma)\beta(k+1)z^k$$

$$= (k+1-m)\beta(\alpha-\gamma)z^k.$$

From (47) in the proof of Theorem (6.3.14) we also have that for $l \ge 1$,

$$\left[T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}\right] e_{m+l-1} = \left[F(z)\tilde{G}_m'(z) - q(z)\tilde{B}_m'(z) - b(z)\tilde{p}_m'(z)\right]z^l$$

$$+\left[F(z)\tilde{G}_m(z) - q(z)\tilde{B}_m(z) - b(z)\tilde{p}_m(z)\right]lz^{l-1} = 0.$$

Thus the range of $T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}$ is spanned by $\{1, z, \cdots, z^{m-2}\}$. So $T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}$ has finite rank and its rank is m − 1. Now we can write

$$T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}} = \sum_{i=1}^{m-1} c_i \otimes d_i$$

for some polynomials c_i and d_i.

Since $m-1 \ge 2$, for any two nonzero real numbers α and β such that $|\beta| > 1$ and

$$\frac{m-1}{|\beta|} - 1 > \frac{m}{|\alpha|},$$

we have

$$\frac{|q(z)|}{|\alpha\beta|} \ge \frac{m-1}{|\beta|} - |z|^m - \frac{m}{|\alpha|}|z| \ge \frac{m-1}{|\beta|} - 1 - \frac{m}{|\alpha|} > 0,$$

for z in the closure of the unit disk. Thus q does not have any zero in the closure of the unit disk. The condition on α, β, and γ in the theorem leads that $F(z)$ and $q(z)$ do not have any common factors and $G(z)$ and $p(z)$ do not have any common factors. Since

$$T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{\tilde{B}}{\tilde{p}}} - T_{\frac{b}{q}} = T_{\frac{1}{q}}\left[T_F T_{\tilde{G}} - T_q T_{\tilde{B}} - T_b T_{\overline{p}}\right] T_{\frac{1}{p}} = T_{\frac{1}{q}}\left[\sum_{i=1}^{m-1} c_i \otimes d_i\right] T_{\frac{1}{\tilde{p}}}$$

$$= \sum_{i=1}^{m-1}\left(\frac{c_i}{q}\right) \otimes \left(\frac{d_i}{p}\right),$$

we conclude that $T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{\tilde{B}}{\tilde{p}}} - T_{\frac{b}{q}}$ has finite rank and its rank is also equal to $m-1$ to complete the proof.

Letting $m = 3, \alpha = 10, \beta = \frac{3}{2}$ and $\gamma = 1$ in the above theorem suggests the following concrete example.

Example (6.3.19)[172]: Let $F(z) = \frac{3}{2}\left(z^3 + 3z - \frac{4}{3}\right)$, $q(z) = 15\left(z^3 + \frac{3}{10}z - \frac{4}{3}\right)$, $b(z) = 18$, $G(z) = 10z + z^3$, $B(z) = z + z^3$, and $p(z) = \frac{3}{2} + z^3$. The above theorem and some calculations used in the proof of the above theorem give

$$T_{\frac{F}{q}} T_{\frac{\tilde{G}}{p}} - T_{\frac{b}{q}+\frac{\tilde{B}}{\tilde{p}}} = -27\left[\frac{1}{q} \otimes \frac{1}{p} + \frac{1}{2}\frac{z}{q} \otimes \frac{z}{p}\right].$$

List of Symbols

Symbol	Page
L_a^p : Bergman space	1
L^2 : Hilbert space	1
H^2 : Hardy space	1
$\ominus$: Direct difference	1
L^∞ : Essential Lebesgue space	2
$\oplus$: Orthogonal sum	2
cl : closure	2
Ker : Kernel	3
Mod : Modular	4
dim : dimension	8
Re : Real	23
Im : Imaginary	23
Ind : Index	24
inf : infimum	25
$L_{a,p}^2$: Bergman space	27
min : minimal	34
A_∞^2 : Bergman space	37
H^∞ : Essential Hardy space	45
$\otimes$: Tensor product	49
sup : Supremum	52
ℓ^2 : space of sequences	53
Hol : Holomorphic	63
$A^2(\mu_m)$: Fock space	83
H_f : Hankel operators	84
max : maximum	108
diam : diameter	109
VMO : Vanishing mean Oscillation	120
L^1 : Lebesgue on the real line	120
$L_{\mathcal{M}}$: Toepliz operator	120
BMO : Bounded mean Oscillation	123
ran : rang	127
det : determinant	128
A_α^2 : Bergman space	136
SOT : Strong Operator Topology	146
BMOA : The space of analytic Bounded mean Oscillation	169
$A^2(\Psi)$: Besov space	169
VMOA : The space of analytic functions of Vanishing mean Oscillation	171
F_α^2 : Fock space	192
deg : degree	194
$\mathcal{F}_m^2$: Fock space	198

arg	: argument	199
a. e	: almost everywhere	240
L^q	: Dual of Lebesgue space	251
L^p	: Lebesgue space	252
H^p	: Hardy space	254
H^q	: Dual of Hardy space	254

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