Sudan University of Science and Technology
College of Graduate Studies

Representations of ∗-Semigroups Associated to Invariant Kernels and Seminormed ∗-Subalgebras with Symmetric Generators of C^*-Algebras

تمثيلات شبه زمر - ∗ المشاركة الى النويات اللامتغيرة والجبريات الجزئية – ∗ شبه المنتظمة مع المولدات المتماثلة لجبريات – C^*

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

By
Aboubakr Khalifa Abass Khalifa

Supervisor
Prof. Dr. Shawgy Hussein AbdAlla

2021

Dedication

To my Family.

Acknowledgements

I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide. Prof. Dr. Shawgy Hussein AbdAlla.

Abstract

We show the inverse limits, the positive definite kernels, the dual spaces and the topological representation of C^*-algebras with maps between locally C^*-algebras and seminormed $*$-subalgebras of ℓ^∞. The representations of Hermitian kernels by means of Krein spaces and of $*$-semigroups associated to invariant kernels with values adjointable and application of Jacobi representation convex topological R-algebras and the dilations of some VH-spaces operator valued kernels are considered. We give some new classes, a canonical decomposition, an approximation of unitary equivalence and a C^*-algebra approach to complex and skew symmetric operators. We determine and characterize the C^*-algebras with Hausdorff spectrum and complex symmetric generators of C^* -algebras.

الخلاصة

تم توضيح نهايات الانعكاس والنويات المحددة الموجبة والفضاءات الثنائية والتمثيل التوبولوجي لجبريات C^*– مع الرواسم بين جبريات – C^* الموضعية والجبريات الجزئية – * شبه المنتظمة الى ℓ^∞. قمنا باعتبار التمثيلات لنويات هيرميشان بواسطة الأوساط لفضاءات كيرين وشبه زمر – * المشاركة الى النويات اللامتغيرة مع القيم المساعدة والتطبيق لجبريات – R التبولوجية المحدبة تمثيل جاكوبي والتمددات لبعض نويات قيم مؤثر فضاءات – VH. قمنا بأعطاء بعض العائلات الجديدة والتفكيك القانوني والتقريب للتكافؤ الآحادي ومقاربة جبر – C^* الى موثرات التماثل المركبة والانحرافية. قمنا بتحديد وتشخيص جبريات - C^* مع طيف هاوسدورف ومؤثرات التماثل المركبة لجبريات – C^*.

Introduction

We present a theory of positive definite kernels of Hilbert C*-modules. We study Hermitian kernels invariant under the action of a semigroup with involution. We characterize those Hermitian kernels that realize the given action by bounded operators on a Krĕın space. This is motivated by the GNS representation of $*$-algebras associated to Hermitian functionals, the dilation theory of Hermitian maps on C*-algebras, as well as others.

We show that if $\mathfrak{A}$ is a separable C^*-algebra then $\mathfrak{A}$ is type I if and only if $\mathfrak{A}$ is GCR and $\mathfrak{A}$ is type I if and only if $\mathfrak{A}$ has a smooth dual. We are it always possible to define a natural topology in the set $\mathcal{B} = \bigcup_{P\in\Omega} A(P)$ so that A is represented as the algebra of all continuous cross-sections of $\mathcal{B}$ vanishing at infinity.

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$. We show that binormal operators, operators that are algebraic of degree two (including all idempotents), and large classes of rank-one perturbations of normal operators are complex symmetric. An operator T on a complex Hilbert space $\mathcal{H}$ is said to be complex symmetric if there exists a conjugate-linear, isometric involution $C :$ $\mathcal{H} \to \mathcal{H}$ so that $CTC = T^*$. An operator T on a complex Hilbert space H is called skew symmetric if T can be represented as a skew symmetric matrix relative to some orthonormal basis for $\mathcal{H}$. We study the approximation of skew symmetric operators and provide a C^*-algebra approach to skew symmetric operators.

We investigate VH-spaces (Vector Hilbert spaces, or Loynes spaces) operator valued Hermitian kernels that are invariant under actions of $*$-semigroups from the point of view of generation of $*$-representations, linearizations (Kolmogorov decompositions), and reproducing kernel spaces. We consider positive semidefinite kernels valued in the $*$-algebra of adjointable operators on a VE-space (Vector Euclidean space) and that are invariant under actions of $*$-semigroups. A rather general dilation theorem is stated and proved: for these kind of kernels, representations of the $*$-semigroup on either the VE-spaces of linearisation of the kernels or on their reproducing kernel VE-spaces are obtainable.

By the spectrum of a C*-algebra we mean the set of unitary equivalence classes of irreducible representations equipped with the hull-kernel topology. We are concerned with characterizing the C*-algebras with identity which have Hausdorff spectrum. We characterize the C*-algebras with identity and bounded representation dimension which have Hausdorff spectrum. For A be a commutative unital R-algebra and let ρ be a seminorm on A which satisfies $\rho(ab) \leq \rho(a)\rho(b)$. We apply T. Jacobi's representation theorem [10] to determine the closure of a $\sum A^{2d}$ -module S of A in the topology induced by ρ, for any integer $d \geq 1$. We show that this closure is exactly the set of all elements a ∈ A such that $\alpha(a) \geq 0$ for every ρ-continuous R-algebra homomorphism α : A Đ → R with $\alpha(S) \subseteq [0, \infty)$, and that this result continues to hold when ρ is replaced by any locally multiplicatively convex topology τ on A. Arbitrary representations of an involutive commutative unital Falgebra A as a subalgeba of $\mathbb{F}^X$ are considered, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and X = ∅. The Gelfand spectrum of A is explained as a topological extension of X where a seminorm on the image of A in $\mathbb{F}^X$ is present. It is shown that among all seminorms, the sup-norm is of special importance which reduces $\mathbb{F}^X$ to $\ell^\infty(X)$. The Banach subalgebra of $\ell^\infty(X)$ of all Σ-measurable bounded functions on X, $M_b(X, \Sigma)$, is studied for which Σ is a σ-algebra of subsets of X.

An operator T on a complex Hilbert space H is called a complex symmetric operator if there exists a conjugate-linear, isometric involution $C : H \to H$ so that $CTC = T^*$. We study the approximation of complex symmetric operators. By virtue of an intensive analysis of compact operators in singly generated C^*-algebras, we obtain a complete characterization of norm limits of complex symmetric operators and provide a classification of complex symmetric operators up to approximate unitary equivalence. Certain connections between complex symmetric operators and anti-automorphisms of singly generated C^*-algebras are established. This provides a C^*-algebra approach to the norm closure problem for complex symmetric operators. For $T \in \mathcal{B}(\mathcal{H})$ satisfying $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we give several characterizations for T to be a norm limit of complex symmetric operators. We give necessary and sufficient conditions for an essentially normal operator T to have its C^*-algebra $C^*(T)$ generated by a complex symmetric operator.

The Contents

Chapter 1
Inverse and Positive Definite Representations of Hermitian Kernels

We state develop certain properties of inverse limits of C^*-algebras which are needed for the development of their represent able K-theory. We give, including a representation of a Hilbert C^*-module as a concrete space of operators and a construction of the exterior tensor product of two Hilbert C^*-modules. We explain the key role played by the technique of induced Krĕın spaces and a lifting property associated to them.

Section (1.1): Limits of C^*-Algebras:

We develop certain properties of inverse limits of C^*-algebras which are needed for the development of their represent able K-theory in [31]. The algebras were first systematically studied in [18] as a generalization of C^*-algebras, and were called locally C^*-algebras. (Also see [46]), they have since been studied, under various names in [37], [13], [14], [15] and elsewhere. Voiculescu introduced essentially equivalent objects called pro-C^*-algebras in [41] where he applied them to the construction noncommutative analogs of various classical Lie groups. Countable inverse limits were introduced in [9] under the name of F^*-algebras they were reintroduced by Arveson in [5] as σ-C^*-algebras and were used there for the construction of tangent algebra of a C^*-algebra.

We will follow Voiculescu (approximately) and Arveson, and call the objects we study pro-C^*-algebras and, in the case of countable inverse limits, σ-C^*-algebras. The interest in them stems from the fact that the category of σ- C^*-algebras contains both C^*-algebras and objects corresponding to classifying spaces of compact Lie groups. It is also possible that the noncommutative analogs of loop spaces will be found among the pro-C^*- -algebras. The topics that we treat here are chosen because they are needed for the following application. In [31] and [32] we define representable K-theory for, σ-C^*-algebras, and generalize the Atiyah-Segal completion Theorem [7] to C*-algebras. This Theorem asserts that, if G is a compact Lie group, X is a compact G-space, and the equivariant K-theory $K_G^*(X)$ (defined in [38]) is finitely generated over the representation ring $R(G)$, then a certain completion $K_G^*(X)^\wedge$ is naturally isomorphic to the representable K-theory $RK^*((X \times EG)/G)$. Here EG is a contractible space on which G acts freely, and it cannot be replaced by the algebra of continuous functions vanishing at infinity on any locally compact space. However, a substitute for EG can be chosen in such a way that the analog of the functor $X \longmapsto (X \times EG)/G$ sends C^*-Algebras to σ-C^*-algebras. Thus, we need enough information about to σ-C^*-algebras be able to define their representable K-theory.

The original purpose "or generalizing the Atiyah-Segal completion Theorem was to obtain the following Corollary, not involving σ-C^*-algebras, which will proved in [32]: if $t \mapsto x^{(t)}$ is a homotopy of actions of a compact Lie group G on a C^*-algebras$A$, and if $K_{*}.\left(C^*(\,G, A, \alpha^{(0)}\,)\right)$ and $K_{*}.\left(C^*(\,G, A, \alpha^{(1)}\,)\right)$ are both finitely generated as $R\,(G)$-modules. Then for appropriate completions there is an isomorphism

$$K_{*}.\left(C^*(\,G, A, \alpha^{(0)}\,)\right)^\wedge \cong K_{*}.\left(C^*(\,G, A, \alpha^{(1)}\,)\right)^\wedge.$$

Here $\left(C^*(\,G, A, \alpha^{(t)}\,)\right)$ is the crossed product C^*-algebra and the $R(G)$-module structure is as defined. (The result is false without the completions, as will be shown in [32].) Cur proof makes essential use of representable K-theory of certain o-C*-algebras. (One can obtain a weaker result without explicitly using representable K-theory or σ-C^*-algebras, but the proof is artificial and the result is not strong enough to prove, for instance, the nonexistence of homotopies of actions.) Thus, even in a problem only involving σ-C^*-algebras we are led to introduce σ-C^*-algebras and their represent able K-theory.

We present some basic defines and propositions, and some examples. There is some overlap with the material of [18] and [37]. For completeness we state all of the results, but we give proofs only when they are shorter or when we improve the results.

We give a new characterization of the commutative unital pro-C^*-algebras, and counterexamples to several plausible conjectures related to this characterization.

We devoted to sensor products, limits, approximate identities, and multiplier algebras. Most of the material has not previously appeared, although UN extensive treatment of tensor products from a different point of view is given in [14], and approximate identities are shown to exist in [18].

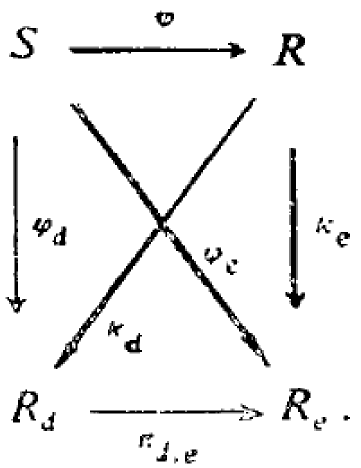
We take up Hilbert modules over inverse limits of C^*-algebra. These have not previously appeared in the literature, and the proofs are not quite straightforward. We restrict ourselves to σ-C^*-algebras, and prove for them several results, such as a stabilization Theorem by generated Hilbert modules, which we were unable to prove for more verse limits.

Recall that an inverse system of rings consists of a directed set D, a ring R_d , for each $d \in D$, and homomorphisms

$$\pi_{d.e}: R_d\text{-}R_e,$$

for all pairs $(d, e) \in D \times D$ such that $d > e$. These homomorphisms are required to satisfy $\pi_{d,d} = id_{R_d}$ and $\pi_{e,f} \circ \pi_{d,e} = \pi_{d,f}$ for $d \geq e \geq f$.

The inverse limit of this inverse system is a ring R equipped with homomorphisms $\aleph_d: R \to R_d$, such that $\pi_{d,e} \circ \aleph_d = \aleph_e$, whenever $d, e \in D$ with $d > e$, and satisfying the following universal property in the category of rings. Given any ring S and homomorphisms $\varphi_d: S \to R_d$, satisfying $\pi_{d,e} \circ \varphi_d = \varphi_e$ for $e \leq d$, there exists a unique homomorphism $\varphi = S \to R$ making the following diagrams commute for $d \geq e$:



The inverse limit R, denoted by $\lim R_d$, can be conveniently obtained as

$$R = \left\{ r \in \prod_{d \in D} R_d : \pi_{d,e}\big(r(d)\big) = r(e) \text{for all } d, e \in D \text{ such that } d \geq e \right\}$$

With this identification, x, simply becomes the restriction to R of the projection from $\prod_{e \in d} R_e$ toR_d. Observe that if each R_d, is a topological ring, and if the maps $\pi_{e,d}$ are all continuous, then R is also a topological ring, with the restriction of the product topology, and the maps $\aleph_{,d}$, are continuous. In fact, this topology on R is the weakest such that the maps $\aleph_{,d}$, are all continuous, and R is the inverse limit of the system $\{R_d\}$ in the category of topological rings.

We will refer to elements of R as coherent sequences $\{r_d\}$ (where $r_d \in R_d$ for $d \in D$) wherever it is convenient to do so.

We will occasionally also take inverse limits of modules, vector spaces and abelian groups. Thus, we note that the results just stated for rings are also valid in these other categories. Furthermore, if $\{R_d\}$ is an inverse system of rings as above, and if $\{M_d\}$ is an inverse system of abelian groups such that each M_d is an M_d-module, and such that the maps $\sigma_d = M_d \to M_e$, satisfy $\sigma_{d,e}(rm) = \pi_{d,e}(r)\sigma_{d,e}(m)$ for $r \in R_d$ and $m \in M_d$, then $\lim_{\rightarrow} M_d$ is a $\left(\lim_{\rightarrow} R_d\right)$-module in a natural way, and the action is continuous.

The following definition is a way of singling out the inverse limits of C^*-algebra without specifying a particular system. (The inverse limit is unchanged. For example, if the directed set is replaced by a cofinal subset.)

Definition (1.1.1)[1]: A pro-C^*-algebra is a complete Hausdorff topological C^*-algebra over C whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 if and only if $p(a_2) \to 0$ for every continuous C^*-seminorm p on A. These objects are called locally C^*-algebras in [18] and LMC^*- algebras in [37]. If the topology is determined by only countably many C^*- seminorms, then we have the σ-C^*-algebras of [5]. Closely related objects were called pro-C^*-algebras in [41]: the exact relation will be explained after Corollary (1.1.13).

If A is a pro-C^*algebras then $S(A)$ denotes the set of all continuous C^*-seminorms on A. For $P \in S(A)$, we let $\mathrm{Ker}(P)$ be the set $\{a \in A: P(a) = 0\}$ which is a closed ideal in A. (This notation is not quite standard because p is not a homomorphism.)

We also let A_p, be the completion of $A/\ker(P)$ in the norm given by p, so that$A$, is a C^*-algebra. (We will see in Corollary (1.1.12) that $\grave{A_p}\ker(P)$ is in fact already complete.) For $a \in A$, we denote its image in A_P, by a_P.

Proposition (1.1.2)[1]: Topological C^*-algebra A is a pro-C^*-algebra if and only if it is the inverse limit, in the sense above, of an inverse system of C^* -algebras and $*$-homomorphisms. In this case. We have $A \cong \lim_{P \in S(A)} A_P$ for the proof, see the remarks following Satz 1.1 in [37]. Note that $S(A)$ is directed with the order $P \leq q$ if $P(X) \leq q(X)$ for all x, and that there is a canonical subjective map $A_q \to A_p$ whenever $P \leq q$. One of the most useful consequences of this Proposition is that every coherent sequence in $\{A_P: P \in S(A)\}$ determines a clement of A.

The homomorphisms of pro-C^* -algebras are of course the continuous $*$-homomorphisms. Since $*$-homomorphisms need not be continuous (see Example (1.1.24)). We adopt the convention, unless otherwise specified, “homomorphism means” “ continu $*$ -homomorphism”.

Examples (1.1.3)[1]: (i) Any C^*- algebra is a pro-C^*-algebra.

(ii) A closed $*$-subalgebra of a pro-C^*-algebra is again a pro- C^*-algebra.

(iii) If X is a compactly generated space ([43]), then $C(X)$, the set of all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets, is a pro-C^*-algebra. (We should point out here that Example 2.1(iii) of [I7] is wrong. Since the algebras considered there need not be complete. See Example (1.1.25).)

(iv) A product of C^*-algebras, with the product topology, is a pro-C^*-algebra.

(v) A σ-C^*-algebra ([5], page 255) is a pro-C^*-algebra. In particular, the tangent algebra denied there is a pro-C^*-algebra.
(vi) Given any sets G of generators and R of relations, as m [8], satisfying the consistency condition but not necessarily the boundedness condition. there is a universal pro-C^*-algebra. which by abuse of notation we write $C^*(G, R)$, generated by the elements of G subject in the relations R. to construct it, let $F(G)$ be the tree associative $*$-algebra on the set G. for any functions $P: G \to L(H)$ where $L(H)$ is the algebra of bounded operators on some hilbert space H, we also let p denote the extension to a $*$-homomorphism from $F(G)$ to $L(H)$ then $C^*(G, R)$ is the Hausdorff completion of $F(G)$ for the topology given by the C^*-seminorms $a \longmapsto \left||P(a)|\right|$ as p runs through all functions from G to $L(H)$ such that the elements $P(G)$ satisfy the relations R in $L(H)$ this procedure can be shown to work for much more general relations than the ones considered in [8]. see [34] for more details.
(vii) associated to every pro-C^*- algebra as in [41] there is an inverse limit of C^*-algebras, and thus a pro-C^*- algebra in our sense. thus, the category of pro-C^*-algebras contain various dual group algebras.
(viii) we consider a specific example similar to but not the same as the examples in [41], namely the noncommutative infinite unitary group $\cup_{r,e}(\infty)$ is the noncommutative analog of $\lim\limits_{\leftarrow} U(n)$. let $U_{r,e}(n)$be the universal unital C^*- algebra generated by $\left\{X_{i_j}\right\}_{i,j=1}^{n}$, subject in the relation that $\left(X_{i_j}\right)$ is a unitary element of $M_n\left(U_{n,e}(n)\right)$. these algebras were first introduced in [10].) define a map $\pi_n: U_{n,e}(n+1) \to U_{n,e}(n)$ by $x_{i_j} \longmapsto x_{i_j}$ for $1 \le i, j \le n$, $x_{n+1,n+1} \longmapsto 1$ and $x_{i_j} \longmapsto 0$ when $j = n+1$ or $j = n+1$ but not both. then $U_{n,e}(\infty)$ true is defined to be $\lim\limits_{\rightarrow} U_{n,e}(n)$.
(ix) The multiplier algebra of the Pedersen ideal of a C^*-algebra (See [23]) is a pro-C^*algebra, See [33] for Details.

We define functional calculus in pro-C^*- Algebras. For this we need the unitization and the Spectrum.

Definition (1.1.4)[1]: ([l7], Theorem 2.3). Let A be a pro-C^*-algebra then its unitization A^* is the vector space $A \oplus C$ topologiezed as the direct sum and with adjoint and multiplication defined as for the unitization of a C^*- algebra. note that A^* is a pro-C^*-algebra, since $A^+ = \lim\limits_{\leftarrow} A_P^+$.

definition (1.1.5)[1]: Let A be a unital pro-C^*-algebra and let $a \in A$. then the spectrum $SP(a)$ or $a \in A$ is the set $\{\lambda \in C: \lambda - a \text{ is not invertible}\}$it is not unital, then the spectrum is taken with respect to A^+.

Unlike in a C^*-algebra, the spectrum need be neither closed nor bounded Indeed, if $S \subset C$ is any nonempty subset, then $C(S)$ is a pro-C^*- Algebra. (Note that S is metrizable, hence compactly generated by [43], L43.) The identity function $z: S \to C$ is an element of $C(S)$ whose spectrum is S. However.spectrum is always nonempty indeed by examining coherent sequences. One obtains the following:

Lemma (1.1.6)[1]: ([26]). Let $A = \lim\limits_{\leftarrow} A_{d,}$ and suppose the maps

$$\pi_{d,e}: A_d \to A_d \text{ are all unital. then for } a \in A \text{ we have } SP(a) = \bigcup_d sp\left(x_d(a)\right),$$

where $x_d: A \to A_d$ is the canonical map

A stronger result is found in Theorem 7.1 of [3] in the case of countable inverse limits a further generalization is given in [4].

Corollary (1.1.7)[1]: ([18] Corollary 2.2 and Proposition 2.1 also see [46] Theorem (1.1.28)) let A be a pro-C^*-algebra, and le $a \in A$. Then :

(i) if a is selfadjoint then $sp(a) \subset R$

(ii) if a has the form b^*h, then $sp(a) \subset [0,\infty)$

(iii) if a is uitary, then $sp(a) \subset \{\lambda \in C: |\lambda = 1|\}$

Proof. this follows immediately from the Lemma and the corresponding facts in C^*-algebras.

Proposition (1.1.8)[1]: ([18], Theorem (1.1.18) and 2.5 ;[46], Theorem (1.1.31)). Let A be pro- C^* -algebra, and let $a \in A$ be normal, that is $a^* a = aa^*$ .then there a unique hommomorphism from the pro-C^*-algebra $\{f \in C(SP((a)): f(0) = 0\}$ to A which series the identity function to A.if A is unital then this map extend uniquely to a hommorphic from C(SP(a)) to A which sends 1 to 1.

Proof. The required map is the one sending f to the coherent sequence

$$\{f(a): p \in S(A)\}.$$

The proof that it satisfied the required properties is easy.

We write of course $f(a)$ for the image of f under this map .

In the same manner. We obtain holomorphic functional calculus for arbitrary elements of a pro-C^*-algebra.for convenience, we state only the unital ease. If $U \subset C$ is open. Then we let $H(U)$denote the set of all holomorphic functions from $U\ to\ C$ with the topology of uniform convergence on compact subsets.

Proposition (1.1.9)[1]: Let A be a unital pro-C^*-algebra, let $a \in A$, and let $U \subset C$ be an open set containing $SP(a)$ then there exists a unique continuous unital homomorphism $f \longmapsto f(a)$ from $H(U)$to A sending the identity function to a . This homomorphism satisfies $SP\big(f(a)\big) = f\big(SP(a)\big)$.

Of course in this situation, $f \longmapsto f(a)$ is not a $*$-homomorphism . also its perfectly permissible to take $U = SP(a)$if $SP(a)$ happens to be open .

Definition (1.1.10)[1]: (Compare [37], satz (1.1.28)) let A be a pro-C^*-algebra. then the set of bounded elements of A is the set

$$b(A) = \{a \in A: \|a\|_\infty = \sup\{p(a): p \in S(A)\} < \infty\}.$$

Proposition (1.1.11)[1]: Let A be a pro-C^*-algebra then:

(i) $b(A)$ is a C^*-algebra in the norm $\|\ \|_\infty$.

(ii) if $a \in A$ is normal and $f \in C\big(SP(a)\big)$ id bounded, then $f(a) \in b(A)$.

(iii) if $a \in A$ is normal then $a \in b(A)$ if and only if $SP(a)$ is bounded.

(iv) $b(A)$ is dense in A.

(v) for $a \in b(A)$, we have $SP_{h(A)}(a) = \overline{SP_A(a)}$.

(vi) if $q \in S(A)$, then the map from $b(A)\ to\ A_q$is surjective.

Proof. (i) see [37], satz (1.1.28).

(ii) we have $p\big(f(a)\big) \le sup_{\lambda \in SP(A)} f|\lambda|$ for all $p \in S(A)$.

(iii)we have $\|a\|_\infty = sup_{\lambda \in SP(A)} p(a) = sup_{\lambda \in SP(A)} |\lambda|$ because $SP(a) = \bigcup_{p \in S(A)} SP\big(a_p\big)$ and each a_p is normal.

(iii) this is [37], satz however, a shorter proof is as follows by considering the decomposition into real and imaginary parts, its enough to prove that the selfajoint part of $b(A)$ is dense in selfadjoint part of A in [37], it is proved that for $x \in A$ selfadjoint, there is a net

$\{x_x\}$ in $b(A) \cap \{x\}^{``}$ (second commutant) converging to $x$ . We produce sequence $\{x_x\}$in$b(A) \cap \{x\}^{``}$ which converges to x, be setting $x_x = f_x(x)_x$ where

$$f_x(\lambda) = \begin{cases} -n & \lambda \leq -n \\ \lambda & -n < \lambda < n \\ n & n \leq \lambda \end{cases}$$

(v) since $SP_{b(A)}(a)$ is closed and contains $SP_A(a)$ the inclusion $\overline{SP_A(a)} \subset SP_{b(A)}(a)$ is immediate. For the reverse inclusion, note that if the distance from λ to $SP_A(a)$ is $\varepsilon > 0$, then $P((\lambda - a)^{-1}) \leq 1\backslash\varepsilon$ for all $P \in S(A)$.

(vi) this following immediately in (iv), as is shown in[37] in the remark after Folgerung (1.1.53). (note that there A_P means the algebra $A/\ker(p)$ before being completed.)

Corollary (1.1.12)[1]: ([37], Folgerung (1.1.53)). For $p \in S(A)$, the map $A \to A_p$ is surjective, that is $A/\ker(p)$ is complete.

Corollary (1.1.13)[1]: (Compare [37], Folgerung (1.1.30)). Let $\varphi: A \to B$ be a $*$-homomorphism (not necessarily continuous) between pro-C^*-algebras. then φ defines a homomorphism from $b(A)\ to\ b(B)$.

Proof. Taking unitizations, we may assume that φ is unital. Then for any $a \in A$ we have $SP(\varphi(a)) \subset SP(a)$. If a is selfadjoint. then so is $\varphi(a)$, is bounded by Proposition (1.1.11) (iii). Now use the decomposition into real and imaginary parts.

We note that this result cannot be used to prove that every homomorphism is continuous, in fact in Example (1.1.24) below, we will produce a discontinuous homomorphism by exhibiting a pro-C°-algebra A such that $b(A) = A$ as sets but he topologies arc different.

We can now explain how our pro-C^*-algebras are equivalent to those of [41]. if A is one of our pro-C^*-algebras. then for any cofinal subset D of $S(A)$, the pair $(b(A), D)$ is a pro-C^*-algebra as in [41]. While if (B, D) is a pro-C^*-algebra as in [41], D being a directed system of C^*-seminorms on B whose supremuir is the norm on B, then

$$A = \frac{\lim}{p \in D} B/\ker(p)$$

is a pro-C^*-algebra in our sense, and satisfies $b(A) = B$. Also, notes that if $\{A_d\}$ is an inverse system of C^*-algebras, then $b(\underline{\lim} A_d)$ is the inverse limit of $\{A_d\}$ in the category of C^*-algebras (as opposed to the inverse limit in the category of topological algebras. which is what we have designated $\underline{\lim} A_d$).

We also note that the term "bounded elements" Is justified by looking at $b(A)$ for some of the examples consider carlier. For example, if X is compactly generated. Then $b(C(X))$ is the algebra $C_b(X)$ of all bounded continuous functions on X. If A is a product $\prod_{i\in I} A_i$, then $b(A)$ is the ι^∞ sum of the A. consisting of all $a \in \prod_{i\in I} A_i$ such that $\sup\{Ia_i: i \in I\} < \infty$ recall into n unital topological algebra is called a Q-algebra if its group of invertible elements are open, and that a nonunital topological algebra is $Q -$ algebra if it is unitization is a Q-algebra. See [24], it is frequently assumed that the pro-C^*-algebras in question are also Q-algebra. Therefore we include the following proposition. (This result has already been noticed by Mallios – see[16].)

Proposition (1.1.14)[1]: A pro-C^*-algebra A is a Q-algebra if and only if it is isomorphic, as a topological $*$-algebra to a C^*-algebra.

Proof. It is well known that Banach algebras are Q-algebra. So let A be a pro-C^*-algebra which also a Q-algebra. We may assume that A is unitl Since the group of invertible elements is open, there is$p \in S(A)$ and $\varepsilon > 0$ such that the set $U = \{a \in A: p(a - 1) < \varepsilon\}$

consist entirely of invertible elements. Let $a \in \ker(p)$, and suppose that $a \neq 0$. then there is $q \in S(A)$ such that $a_q \neq 0$, whence $a_q^* a_q \neq 0$. Using Lemma (1.1.6), we see that there is a positive real number $\lambda \in SP(a^*a)$. Therefore $I - \lambda^{-1}a^*a$ Is not invertible. However, $1 - \lambda^{-1}a^*a \in U$ since $P(\lambda^{-1}a^*a) = 0$. This is a contradiction and it follows that $\ker(p) = \{0\}$. Now let $q \in S(A)$, and suppose that $p \leq q$. Then there is a surjective map $A_q \to A_p$. Since $A \to A_q$, is surjective (by Corollary (1.1.12)), while $A \to A_p$, is injective (because $\ker(p) = \{0\}$), we see that $A_q \to A_p$, is injective as well. Therefore it is an isometry (because A_q, and A_p, are C^*-algebras). whence $q = p$. Since $S(A)$ is directed, we conclude that $q \leq p$ for all $q \in S(A)$. Consequently the map $A \to A_p$ which is already known to be continuous and bijective, has a continuous inverse.

So A is isomorphic, as a topological $*$-algebra. to the C^*-algebra A_p.

It follows that the "complete locally $m-$convex QC^*-algebra of [24] and the Waelbrocclk C^*-algebras" of [25] are exactly the C^*-algebras.

We consider the commutative unital pro-C^*-algebras. The results in [18] and [37] are useful representations commutative pro-C^*-algebras, but they give us no convenient way of determining what all of the commutative pro-C^*-algebras are.Using the notion of a quasitopological space.

Due to Spanier [45], we obtain a much more satisfactory result, namely that a certain functor analogous to $X \mapsto C(X)$ is contravariant category equivalence. We begin by recalling the definition.

Definition (1.1.15)[1]: ([46]). A quasitopolgy on a set X is an assignment to each compact Hausdorff space K or a set $Q(\mathrm{K}, \mathrm{X})$ or functions from K to X such that the following conditions hold:

(i) $Q(K, X)$ contains all constants functions from K to X.

(ii) if $f: K_1 \to K_2$ is continuous and $g \in Q(K, X)$ then $g \circ f \in Q(K, X)$.

(iii) if K is a disjoint union of compact Hausdorff spaces K_1andK_2 then $f \in Q(K, X)$ whenever $f|K_i \in Q(K, X)|$ for $i = 1,2$.

(iv) if $f: K_1 \to K_2$ is continuous and surjective, and if $g: K_2 \to X$ is a function such that $g \circ f \in Q(K_1, X)$ then $g \in Q(K_2, X)$.

If X and Y are quasitopological spaces, that is sets equipped with quasitopologies, then a function $h: X \to Y$ is said to be quasicontinuous if for every compact Hausdorff space K and every $f \in Q(K, X)$. The function $h \circ f$ is in$Q(K, X)$.

Any topological space X can be made into a quasitopological space by letting $Q(K, X)$ be the set or all continuous functions from K to X. Thus. it makes sense to speak of a quasicontinuous function from, for example, a quasitopological space X to a topological space Y. We remark that as observed [39]. The generated compactly spaces then become a full subcategory or the category of quasitopological spices and quasicontinuous functions.

The spaces relevant to study of pro-C^*-algebras are given in the following definition:

Definition (1.1.16)[1]: A (quasi-)topological space X is called completely Huasdorff if for any two distinct points $x, y \in X$ There is a (quasi-)contentious function $f: x \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

This condition is stronger than being Hausdorff and weaker than complete regularity, even among the compactly generated topological spaces - see Examples (1.1.26) and (1.1.27).

Definition (1.1.17)[1]: Let X be a quasitopological space. Then $C(X)$ denotes the $*$-algebra of all quasicontinuous functions $f: X - C$. with the topology determined by the seminorms $\|f_{g,k}\| = \|f \circ g\|_{\infty}$ for K Compact Hausdorff and $g \in Q(K,X)$.

Lemma (1.1.18)[1]: If X is a quasitopological space then $C(X)$ is a pro-C^*-algebra.

Proof. The only issue is completeness. So let $\{f_X\}$ be a Cauchy net in $C(X)$ For each $x \in X$ the inclusion $\{x\} \to X$ IS in $Q(K,X)$ whence f_x converges point wise to a function $f: X \to C$. If now $g \in Q(K,X)$.then $f_a \circ g$ must converge uniformly to some $f^{(g)} \in C(K)$, and clearly $f^{(g)} = f \circ g$ it follows that f is quasicontinuous. And that $f_x \to C(X)$.

Our main result is that $X \mapsto C(X)$, restricted to the full subcategory of completely Hausdorf quasitopological spaces, defines a contravariant category equivalence. It is proving our result. It is useful to introduce the following category or compactly generated spaces with distinguished families of compact subsets. As a byproduct of our proof, we then obtain a more concrete description or the completely Hausdorff quasitopological spaces.

Definition (1.1.19)[1]: Let X be a topological space. A distinguished family of compact subsets of X is a set F of compact subsets of X satisfying the following properties :

(i) every one point subset of X is in F.

(ii) a compact subset of an element of F is in F.

(iii) the union of two elements of F is in F.

(iv) the family F detremines the topology of X, that is .a subset $C \subset X$ is closed if and only if $C \cap K$ is closed for all $K \in F$.

If (X_1, F_1) and (K_2, F_2) are topological spaces with distinguished families of compact subsets , then a morphism from (X_1, F_1) to (K_2, F_2) is a continuous function $f: X_1 \to X_2$ such that $f(K) \in F_2$ for every $K \in K_1$.

Proposition (1.1.20)[1]: The category of completely Hausdorff spaces with distinguished families of compact subsets is equivalent to the category of completely Hausdorff quasitopological spaces, via the functor assigning to (X, F) the quasitopology given by

$$Q_F(K_1, X) = \{f: K \to X: f \text{ } is \text{ } continuous \text{ } and \text{ } f(K) \in F\}.$$

Furthermore, under the correspondence of this functor, a function from X to a topological space is continuous if and only if it is quasicontinuous.

Proof. We first observe that the statement of the Theorem defines a functor.

The sets $Q_F(K_1, X)$ satisfy conditions (l) through (iii) of Definition (1.1.19) by the corresponding conditions of Definition (1.1.19), and they satisfy (iv) for the same reason that the quasitopology defined by a topology satisfies (iv). If $f: X_1 \to X_2$ is continuous and $f(K) \in F_2$ for $K \in F_1$, then it is immediate that f is quasicontinuous.

We now construct an inverse to this functor. Let X be a completely Hausdorff quasitopological space. Define a topology on X by declaring $U \subset X$ to be apen if for every compact Hausdorff space K and every $g \in Q\,(K, X)$, the set $g^{-1}(U)$ is open in K. It is immediate that this does in fact define a topology on X, and that each $Q(K, X)$ consists of functions which are continuous with respect to this topology. Furthermore, it is easily verified that if $f: X \to Y$ is any function to a topological space Y, then f is continuous if and only if f is quasicontinuous. In particular, X is completely Hausdorff in this topology.

We now define F_X to be the set of all ranges of elements of the sets $Q\,(K, X)$ These ranges are all compact because the elements of $Q\,(K, X)$ are continuous Conditions (I) and (iii) of Definition (1.1.19) follow from the corresponding conditions of Definitions (1.1.15), and (1.1.19) (ii) is obtained by using the fact that compact subsets of X are closed and applying Definition (1.1.15)(ii) to an appropriate inclusion map. To verify

Definition (1.1.19) (iv) let $C \subset X$, and suppose that $C \cap g(K)$ is closed whenever $g \in Q\,(K,X)$. Then $g^{-1}(C)$ is closed whenever $g \in Q\,(K,X)$, whence C is closed by the definition of the topology on X This completes the verification that F is a distinguished family of compact subsets of X.

To complete the definition of the inverse factor, we look at morphisms.

Thus let $f: X_1 \to X_2$, be quasicontinuous. Then for $g \in Q(K, X_1)$, the function fog is in $Q(K, X_2)$ and is hence continuous. It follows that$f$ is quasicontinuous whe X_1 is regarded as a quasitopological space and X_2 as a topological space Therefore f is continuous. It is obvious that$f$ sends ranges of elements of $Q(K, X_2)$ to ranges of elements of $Q(K, X_2)$.

It remains to prove that our two functors really are inverse to each other if one starts with a space X with a distintuished family of compact subsets, then the topology and family of compact subsets obtained from the associated quasi topology are the same as the original topology and distinguished family, using Definition (1.1.15)(iv) (for the topology) and the fact that X is Hausdorff.

For the composition in the other order, it is necessary to show that if X is a quasitopolngical space, then $Q_{F_X}\,(K, X_2) = Q(K.X)$ for every compact Hausdorff space K. It is only necessary to prove that $Q_{F_X} = (K, X) \subset Q(K, X)$ So let $g \in Q_{F_X}(K, X)$. Then there is a compact Hausdorff' space L and a function $f \in Q(L, X)$ such that $g(K) = f(L)$.writing $g = i = g_n$ where i is the inclusion of $g(K)$in X.

And $g_\circ : K \to g(K)$ in the obvious surjection, we see by Definition (1.1.15)(ii) that it suffices to show that $i \in Q(g(K), X)$. Thus, we may assume that g is surjective, and in fact a homeomorphism onto its image .Therefore $h = g^{-1} \circ f: L \to K$ is a continuous surjective map such that $g: h \in Q(L, X)$. So $g \in Q(K, X)$ by Definition (1.1.15)(iv), as desired.

As in the preceding proof ,we will write (X, F_X) for the topological space with distinguished family of compact subset determined by the qusitopological space X.

Lemma (1.1.21)[1]: Let X be a completely Hausdorff topological space, and let F be a family of compact subsets of X satisaying the first three conditions of Definition (1.1.15) then for any compact set $L \notin F$, there exista net of continuous functions on X which converges uniformly to 0 in the members of F and does not convert uniformly on L.

Proof. Let ∈. and choose 2 point $x \in L — K$ Because X is completely Hausdorff. there is for every $y \in K$ continuous functions $f_y: X \to [C, 1]$ such that $f_y(x) = 1$ and $f_y(y) = 1$ corsponding f_y with a continuous function from [0,1] to [0,1] which sends 1 to 1 and vanishes on a neighborhood of 0. We may assume that f_y vanishes in the neighborhood of y. Since K is a compact, the infimum of an a appropriate finite subcollection of the functions f will be a continuous function $h_k: X \to [0,1]$ which vanishes in k and is equal to $i\ at\ x \in L$. the set F is directed with respect to the inclusion (by Definition (1.1.15)(iii)), so $\{h_g\}_{k \in F}$is the required net.

Theorem (1.1.22)[1]: The functor $X \mapsto C(X)$ is contravariant category equivalence from the category of completely Hausdorff quasitopological spaces to the category of commutative unital $pro\text{-}C^*\text{-}algebras$ and unital homomorphism.

If $f: X_2 \to X_2$ is quasicontinuous, then $C(f): C(X_2) \to C(X_1)$ is the homomorphism given by $C(f)(h) = h \circ f$.

Proof. Here also we need an inverse functor.it assigns to a commutative unital pro-C^*-algebra A the space $\Phi(A)$ of all (continuons) homomorphism from $A\ to\ C$, if A is a compact Hausdorff space then $Q\big(K, \Phi(\mathrm{A})\big)$ is taken to be the set of all functions $g: K \to \Phi(A)$ such that the formula $\varphi_x(a)(x) = g(x)(a)$, for $a \in A$, and $x \in K$ .defines a (continuous) homomorphism from $A\ to\ C(K)$. Definition (1.1.15)(i) though Definition (1.1.15)(iii) of quasitopology is immediate. For Definition (1.1.15)(iv) $let\ f: K_1 \to K_2\ and\ g: K_2 \to \Phi(A)$ be as in Definition (1.1.15)(iv). Let $\psi: C(K_2) \to C(K_1)$ be given by $\psi(h) = h \circ f$ then ψ is a homomorphism into its image.

The relations$\varphi_{g\circ f} = \psi \circ \varphi_g\ and\ g \circ f\ Q(K_1, X)$ now imply that $g \in Q(K_2, X)$. As desired. Also note that for every $a \in A$. the function $x \mapsto x(a)$ from $\Phi(A)\ to\ C$ is quasicontinuous. It follows that $\Phi(A)$ is completely Hausdorf. To complete the construction, observe that if $\psi: A_1 \to A_2$, is a (continuous) unital homomorphism.then the function $x \mapsto x \circ \psi$, from $\Phi(A_2)\ to\ \Phi(A_1)$. is quasicontinuous.

We now prove that these two functors are inverses or each other. This will be done using Proposition (1.1.20), let A be a commutative unital pro-C^*-algebra. For each continuous C^*-seminorm P on A. let $\Phi(A_P)$ have the usual $weak^*$ topology and identity (ii) in the obvious way with a subset of $\Phi(A)$. Then $\Phi(A) = \bigcup_{P \in s(A)} \Phi(A_P)$.

Let $\Phi(A)$ have the direct limit topology. and set $F = \{\Phi(A_P): P \in S(A)\}$. Then it is easy to show that $(\Phi(A), F) = \left(\Phi(A), F_{\Phi(A)}\right)$, We must therefore prove that the obvious map from A to the continuous functions on $\Phi(A)$, with the topology of uniform convergence on members of F. is an isomorphism of pro-C^*-algeblras. This is equivalent to the assertion that $A \cong \lim\limits_{\to} C\left(\Phi(A_P)\right)$ via the obvious map, and follows from the natural isomorphism $C\left(\Phi(A_P)\right) \cong A_P$, and Proposition (1.1.2). This proves that the composite of our functors in one order is the identity.

For the other order, we let X be a completely Hausdorff quasitopclogical space. Topologies $\Phi\left(C(X)\right)$ in the manner or the previous paragraph. and let F be the corresponding distinguished family of compact sets. We must show that the map sending x to the evaluation $ev_x\ at\ x$determines an isomorphism from $(X, F_X)\ to\ \left(\Phi\left(C(X)\right), F\right)$. The injectivity of $x \mapsto ev_x$follows from the fact that X is completely Hausdorff. For subjectivity. let $\alpha: C(X) \to C$ be a homomorphism. Then there is $K\ and\ g \in Q(K, X)$ such that $|\alpha(h)| \le \|h\|_{k,g} = \|h \circ g\|_\infty$. for all $h \in C(X)$.

It follows that α defines a homomorphism from $C(K)\ to\ C$. which must be ev, for some $y \in K$.Then $\alpha = ev_{g(y)}$.

For $g \in Q(K, X)$. we clearly $\{ev_x: x \in g(K)\} \in F$. Conversely. it $L \subset X$ a compact set not in F_x. then it Follows from Lemma (1.1.21) that $\{ev_x: x \in L\} \notin F$.

Furthermore, if $K \in F_x$, then the relative topology from X is the same as the relative topology from the identification of K with a sub of $Q\left(C(X)\right)$. (K is compact in both topologies.) It follows from Definition (1.1.15)(iv) and the definition of the topology on $\Phi\left(C(X)\right)$ that $x \mapsto ev_x$. is a homomorphism. This completes the proof that $(X, F_x) \cong \left(\Phi\left(c(X)\right), F\right)$.

Corollary (1.1.23)[1]: Let X be a completely Hausdorff quasitopological space . then $C(X) \cong \varprojlim_{K \in F_x} C(K)$.

Example (1.1.24)[1]: (Weidner). We will produce a commutative unital pro-C^*-algebra A which is not isomorphic, as a pro-C^*-algebra, to $C(X)$ for any completely Hausdorff topological space x. Thus, one cannot avoid using quasitopolugics or distinguished families of compact subsets, at least if one insists that the continuous functions separate the points.

Let F be the set of countable closed subsets or [0, 1] possessing only finitely cluster points. Then F satisfies the conditions of Definition (1.1.19) relative to the usual topology on [0, 1]. (for Definition (1.1.19) (iv), note that the sets of the form $\{x_n\} \cup \{x\}$. where $x_n \to x_1$ already determine the topology) Now let A be $C([0,1])$ with the topology of uniform convergence on the members of F. It follows from Lemma (1.1.18) that A is n pro-C^*-algebra.

Suppose that X is a completely Hausdorff topological space such that there is an isomorphism $\varphi: A \to C(X)$. Then for each $x \in X$, the homomorphism $ev_x \circ \varphi$ must be evaluation at some $f(x) \in [0,1]$. (This follows from the proof of Theorem (1.1.22).) Clearly $\varphi(h) = h \circ f$ for every $h \in A$. The function f is injective because X completely Hausdorff, and continuous because the usual topology on [0. l) is the weak topology determined by A. Also, f must have dense range because φ is injective,if now $\iota \notin f(X)$, then the function $h(X) = \left(\iota - f(x)\right)^{-1}$ but not in the range of (φ Thus f is in fact directive.

It follows from Lemma (1.1.18) that $f^{-1}(K)$ is not compact for $K \notin F$ Therefore f is not a homeomorphism. so that there is a set $C \subset [0.1]$ which has a limit point $\iota \notin C$, but such that $f^{-1}(C)$ is closed. Let $\{\iota_n\}$ be a sequence in C which converges to ι, and let $T = \{\iota\} \cup \{t_n : n \in Z^+\}$. Then $\{f^{-1}\{t_n\}\} = f^{-1}(T) \cap f^{-1}(C)$ is a closed subset of X, and it is not compact because its image $T \backslash \{t\}\ in$ [0,1] .is not compact. Consequently $f^{-1}(T)$ is not compact. This contradicts the assumpition that φ is a homeomorphism, by Lemma (1.1.21). Therefore the space X cannot exist.

We also point that Lemma (1.1.21) shows that the identity map from A to $C([0,1])$ (with its usual topology) is a discontinuous directive $*$-homomorphism from a pro-C^*-algebra to a C^*-algebra. Furthermore, note that every element of A is bounded, even 'though A is not a C^*-algebra. Actually, these phenomena can occur even for $A = C(X)$ for an appropriate compactly generated completely Hausdorff space X, for example the set of countable ordinals. (See Proposition 12.2 and the remark following in [26]).

Example (1.1.25)[1]: We will produce a completely regular space X such that $C(X)$ is not algebraically isomorphic to any pro-C^*-algebra, Let Z^+ he the set of positive integers. let βZ^+ be its Stone-Céch compactification, choose $x_0 \in \beta Z^+ \backslash Z^+$,. and let $X = Z^+ \cup \{x_0\}$. Then X is completely regular. since it is a subset of βX^+. and it is real compact ([17]). since it is countable, (See [7], 8.2.)

Suppose that $C(X)$ is algebraically isomorphic to a pro-C^*-algebra Then we must have $C(X) \cong C(Y)$ algebraically for some compactly generated completely Hausdorff space Y, By [l6]. (1.1.34), there is a completely regular space Z and a continuous subjective function $f: Y \to Z$ such that the corresponding map $C(Z) \to C(y)$ is an algebraic isomorphism. since Y is completely Hausdorff must also be injective. Let W be the realcompactification of Z ([17]. 8.4 and 8,5). so that in particular $C(W) \cong C(Z)$ all algebraically. Therefore $C(W) \cong C(X)$ algebraically. so, by [17], 8.3, we have $W \cong X$, Since Z is a subspace of W, this homeomorphism implies that Z is countable and hence already realcompact, that is.$W = Z \cong X$. We thus have a continuous bijective map $f: Y \to X$ such that $h \mapsto h \circ f$ is an algebraic isomorphism $C(X)$ to $C(Y)$. By [26], Example 12.5, every compact subset of X is finite. Therefore every compact subset of Y is finite, and, since Y is compactly generated, Y must be discrete. Since there are no discontinuous functions on Y. but there are discontinuous functions on X (for example,$h = 0\ on\ Z^+\ and\ h(x_0) = 1$), we obtain a contradiction. Thus, there is no topology on $C(X)$ in which it is a pro-C^*-algebra.

Example (1.1.26)[1]: We will product: a completely Hausdorff compactly generated space X which is not completely regular (in fact. not regular). Thus, the topology on X is not the weak topology determined by $C(X)$, and hence differs from the topology used in [2] and [37]. Also, one cannot require the spaces in Definition (1.1.19) to be completely regular. Let Ω be the first uncountable ordinal، let ω to be the first infinite ordinal, set $Y_1 = \{x: x \leq \Omega\}$ and $Y_2 = \{x: x\} \leq \omega$ and let $T = Y_1 \times Y_2 \backslash \{(\Omega, \omega)\}$. Then it IS well ltnown (see [20], Problem 4F) that T is not normal, and in fact that the closed subsets $A = \{\Omega\} \times \{x: x < \omega\}\ and\ B = \{x: X < \Omega\} \times \{\omega\}$ do not have disjoint neighborhoods.

Let X be the space T with the subset A collapsed in a point. with the quotient topology. This is a space or the sort shown in [21]. Problem 4G to by Hausdorff but not regular. (The point A and the closed set B do not have disjoint neighborhoods.) Now $Y_1 \times Y_2$, is compact. so that T is locally compact and hence compactly generated ([43], 1.4.1). it now follows from [39]. (1.1.20), that X is compactly generated.

Furthermore, X is completely Hausdorff let $let\ x, y \in T$ be two points whose images in X are distinct. Then at least one or them, say x, is not in A. Since Its completely regular (being a subspace or the normal space $Y_1 \times Y_2$). there is a continuous function $f: T \to [0,1]$ such that $f(x) = 0$ and $f = 1$ on $\{y\} \cup A$ This function defines continuous function from X to $[0,1]$ taking the values 0 and l on tilted images of x and y respectively.

Example (1.1.27)[1]: We will produce a regular compactly generated space Y which is not completely huasdorff. As a consequence. we obtain an inverse system $\{A_d\}$ such will the maps $A_d \to A_e$, are all surjective but the maps $\lim\limits_{\to} A_d \to A_e$ are not all surjective fndeed have $C(Y) = \lim\limits_{\to} C(K)$as K runs through all compact subsets of $Y$and each restriction map $C(K) \to C(L)$ is surjective but the maps $C(Y) \to C(K)$ are not all surjective (take $K = \{a, b\}$ where $a, b \in Y$ can not be separated by a continuous function.)

The space Y is the space of [12]. it is shown there that Y is regular and not completely Hausdorff, so we need only show that Y is compactly generated. This fact was pointed out to us by Mladen Bestvina.

Let T be as in the previous example, and let $X = Z \times TU\{a, b\}$. Where $Z \times T$ is given the product topology. a neighborhood base α consists of the sets $[n, \infty) \times TU\{\alpha\}$ and a neighborhood base at b consists of the sets $(-\infty, n] \times T \cup \{b\}$.

(The intervals are to be interpreted in Z) Then the space Y is an Identification space of X. from which it follows ([382. (1.1.20)) that it is sufficient to prove that X is completely generated. This is easily seen to follow from the fact that $Z \times T$ is locally compact. and hence compactly generated. together with the fact that $\alpha\ and\ b$ it have countable neighborhood bases.

We generalize to pro-C^*-algebras two standard construction on C^*algebras, namely tensor products and multiplier algebras. We also consider direct and inverse limits. and approximate identities. Tensor products have previously been studied (from the different point of view) in [14], but there is very little overlap and our discussion. Approximate: identities are shown exist in [l7].

We begin with tensor products. Unless otherwise specified all tensor products of C^*-algebras are maximal C^*tensor products. (See [40]. For general information on tensor products of C^*-algebras.) the Topology in the following definition appears in [14], where it is called the projective tensorial l.m.c. C^*-topology

Definition (1.1.28)[1]: Let A and B be pro-C^*-algebras. Their maximal tensor product $A \otimes B$ is the pro-C^*-algebra obtained by completing the algebraic tensor product of $A\ and\ B$for the family of greatest C^*-cross-seminorms $p \otimes q$ determined by p and q, as p runs through $S(A)$ and q runs through $S(B)$.

As an immediate Corollary of the definition, we obtain:

Proposition (1.1.29)[1]: If $A = \varprojlim_{d \in D} A_d$ and $B = \varprojlim_{e \in E} B_e$ $then$ $A \otimes B \cong \varprojlim_{(d,e) \in D \times E} A_d \otimes B_e$.

Of course, in $D \times E$ we have $(d_1, e_1) \le (d_2, e_2)$ exactly when $d_1 \le d_2$ and $e_1 \le e_2$.

Proof. The only nontrivial point is to ensure that if $(d_1, e_1) \le (d_2, e_2)$, $then$ $there$ is in fact an extension of the obvious homomorphism of the algebraic tensor products to a homomorphism $A_{d_2} \otimes B_{e_2} \to A_{d_1} \otimes B_{e_1}$ this follows from [40].

We then obtain the usual universal property.

Proposition (1.1.30)[1]: Let A, B and C be a pro-C^*-algebras, and let $\varphi: A \to C$ and $\psi: B \to C$ be homomorphism whose range commute , then there is a unique homomorphism $\eta: A \otimes B \to C$ such that $\eta(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A, b \in B$.

Proof. since the algebraic tensor product is dense $A \otimes B$, the homomorphism η is unique if it is exist for existence it suffices to find continuous homomorphism $\eta_r: A \otimes B \to C_r$ for $r \in S(C)$ which are coherent in the obvious sense . to define η_r use the continuity of φ and ψ to $find$ $p \in S(A)$ and $q \in S(B)$ $such$ $that$ $p \circ \varphi, q \circ \psi \le r$ then take η_r to the composite $A \otimes B \to B_p \otimes B_p \to C_r$;the first map is continuous by the definition of $A \otimes B$and the second one exist by the corresponding universal property for C^*-algebras it is easily seen that η_r does not depends on the choice of p and q.

The minimal tensor product can be defined in the same way, using the injective tensorial l.m.c C^*-topology as in [14]. Minimal tensor products are also functorial, as can be seen from the corresponding result for C^*algebras,[40] Proposition IV.4.2. see [14] for more in this direction.

For the applications we have in mind. However, at least one of the factors‹ say A. will be nuclear in the sense that A_p, is nuclear for every $P \in S(A)$. In this ease, the minimal and maximal tensor products will agree. (This remark generalizes the comments about type I algebras on page 126 of [l3].) Note that any commutative pro-C^*-algebra (unital or not) is nuclear, and that any nuclear C^*-algebra is nuclear at a pro-C^*-algebra.

We now show that the tensor product of pro-C^*-algebra A with an algebra of the form $C(X)$ is what one expects. or X is qausitopological space (Definition (1.1.15)), then we let $C(X, A)$ be the $*$-algebra of all quasicontinuous functions from X to A, with the topology determined by the C^*-seminorms $\|f\|_{k,gp} = \sup p\big(f \circ g(x)\big)$ $x \in K$.

For K compact Hausdorff $g \in Q(K, X)$, and $p \in S(A)$. equivalently (using Proposition (1.1.20).), $C(X, A)$ is the algebra of all continuous functions from X to A with the topology of uniform convergence on each element of F_X in each continuous C^*-seminorm on A.

Proposition (1.1.31)[1]: Let X be a completely Hausdorff quasitopological space then the obvious map from $C(X) \otimes A$ to $C(X, A)$ is an isomorphism.

Proof. Write $A = \lim A_p$ and $C(X)$=$\varprojlim_{k \in F_x} C(K)$. Now apply Proposition (1.1.29), using the fact that C^*-seminorms $f \mapsto \sup_{x \in K}^{p(f(x))}$ which defines the topology on $C(X, A)$ are exactly the cross-norms $p \otimes \| \, \|_k$where $\|f\|_k = \sup_{x \in k} |f(x)|$.

A similar result holds when $C(X)$ is replaced by the C^*-algebras $C_0(X)$ of continuous complex-valued functions vanishing at infinity on the locally compact space X thus given a pro-C^*algebra A. We let $C_0(X, A)$ be the set of all continuous functions $f: X \to A$ which vanish at infinity in the sense that $p \circ f$ vanishes at infinity for every $p \in S(A)$.

Proposition (1.1.32)[1]: Let A be a pro-C^*-algebra and let X be a locally compact then $C_0(X)\otimes A \cong C_0(X,A)$ via the obvious map.

Proof. By the reasoning of the previous proof we must show that the obvious map from $\lim_{p\in S(A)} C_0\left(X, A_p\right)$ to $C_0(X,A)$ is an isomorphism , this is essential trivial.

Proposition (1.1.33)[1]: Inverse limits exist in the category of pro-C^*-algebras if $\{A_a\}_{a\in I}$ is a direct system of pro-C^*-algebras , with homomorphism $\varphi_{\alpha,\beta}: A_\alpha \to A_\beta$ for $x \leq \beta$, then the direct limit is constructed as follows . Let

$$D = \left\{p \in \prod_{\alpha\in I} S(A_\alpha): P_\alpha \circ \varphi_{\alpha\beta} \text{ for } \alpha \leq \beta\right\},$$

Ordered by $p \leq q$ if $p_\alpha \leq q_\alpha$ for all α. then D is a directed set. For $P \in D$, set $B_P = \lim_{\rightarrow}(A_\alpha)_{P_\alpha}$, and set $B = \lim_{\rightarrow} B_P$ then $\lim A_\alpha$ is the closure of the union of the images of the A_α and B. we omit the details of the proof because direct limits are sufficiently badly behaved that they do not seem to be of much use .indeed ,in the following example , we produce a countable direct system in which every map is injective and no algebra is zero , but for which the direct limit is zero. also we show in example (1.1.59). That a countable direct limit of σ-C^*-algebras is usually not a σ-C^*-algebra .

Example (1.1.34)[1]: Write $Q = \{x_1, x_2, \ldots\}$ and set $X_n = Q\backslash\{x_1, \ldots, x_n\}$.
Set $A_n = C(X_n)$, and let $\varphi_n: A_n \to A_{n+1}$ be the restriction map. Note that φ_n is injctive, since X_{n+1} is dense in X_n. We claim that $\lim_{\rightarrow} A_n = 0$. it suffices to show that, for any sequence $P_2, P_1, \ldots$ of continuous C^*-seminorms $A_1, A_2, \ldots$ satisfying $P_{n+1} \circ \varphi_n \leq P_n$ for all n, we have $\lim_{\rightarrow}(A_n)_{P_n} = 0$. for each n there is a compact set $K_n \subset X_n$ such that $P_n(f) = \sup_{x\in k_n}|f(x)|$ for all $f \in A_n$. the condition $P_{n+1} \circ \varphi_n \leq P_n$ is equivalent to $K_{n+1} \subset K_n$.since $\bigcap_n X_n = \emptyset$, we have $\bigcap_{\text{n}} K_n = \emptyset$, whence $K_m = \emptyset$ for some m. so $P_m = 0$ and $\lim_{\rightarrow}(A_n)_{P_N} = 0$, as desired.

Before turning to multiplier algebras, we need a Lemma to the effect that pro-C^*-algebras have approximate identities. Following [29], 1.4.1, we use the following strong definition of an approximate identity.

Definition (1.1.35)[1]: Let A be a pro-C^*-algebra. Then an approximate identity for A is an increasing net $\{e_\lambda\}$ of positive elements of A such that $\|e_\lambda\|_\infty \leq 1$ for all λ and, for all $a \in A$, we have $e_\lambda a \to a$ and ae_λhere. Of course, x is positive if it has the form y^*y for some $y \in A$;equivalently ,x is normal and $\text{sp}(x) \subset [0.\infty)$.

Proposition (1.1.36)[1]: every approximate identity for $b(A)$ is an approximate identity for A.

Proof. By definition. An increasing net $\{e_\lambda\}$ of positive elements, bounded by 1 , is an approximate identity for A if $p(e_\lambda a - a) \to 0$ and $p(ae_\lambda - a) \to 0$ for all $a \in A$ and $p \in S(A)$. the result not follows from the fact (Proposition (1.1.11) (v)) that the map from $b(A)$ to A_P is surjective.

Corollary (1.1.37)[1]: (Compare [18], Theorem (1.1.20)). Every pro-C^*-algebra A has an approximate identity which is also an approximate identity for $b(A)$.

Definition (1.1.38)[1]: Let A be a pro-C^*-algebra. Then the multiplier algebra of A is the set $M(A)$ of all pairs (l, r) of continuous linear maps from A to$A$ such that l and r are respectively left and right A-module homomorphism, and $r(a)b = al(b)$ for all $a, b \in A$. such a pair is called multiplier . (Compare [29], (1.1.37).1, where such objects are called double centralizers. Since we have no reason to think that such map is automatically continuous, we simply assume it.) addition is defined as usual , multiplication is $(l_1, r_1)(l_2, r_2) = (l_2 l_1, r_2 r_1)$ and adjoint is $(l, r)^* = (l^*, r^*)$, where $r^*(a) = r\,(a^*)^*$ and similarly for l^* for each $p \in S(A)$, we define a C^* -seminorm by $\|l, r\|_{p,e} = \sup\{p(l(a)) : p(a) \leq 1\}$, and a family of seminorms , indexed by $a \in A$, by $\|l, r\|_{p,e} = p(l(a)) + p(r(a))$. (it will be proved in the next Theorem that $\| \ \|_p$ is in fact a C^*-seminorm.) the seminorm topology on $M(A)$ is the one generated by the seminorms $\|.\|_p$ for $p \in S(A)$ and is the analog of the norm topology on the multiplier algebra of the C^* -algebra. The strict topology on $M(A)$ is the one generated by the seminorms $\|.\|_{p,a}$ for $p \in S(A)$ and $a \in A$. finally, we define a map from A to $M(A)$ by $a \mapsto (l_a, r_a)$, where $l_a(a)ab$ and $r_a(b) = ba$ for $a, b \in A$.

Theorem (1.1.39)[1]: Let A be a pro-C^*-algebra. Then:

(i) if $A \cong \varinjlim_{d \in D} A_d$, and the maps $x_d : A \to A_d$ are all surjective then $M(A)$, with it is seminorm topology is isomorphic to $\varprojlim M(A_d)$.

(ii) The isomorphism of (i) identifies the strict topology on $M(M)$ with the topology on $\varprojlim M(A_d)$ obtained by taking the inverse limit for the strict topologies on the $M(A_d)$.

(iii) $M(A)$ is a pro-C^*-algebra in it is seminorm topology.

(iv) $M(A)$ is complete in the strict topology

(v) The map $a \mapsto (l_d, r_d)$ is a homomorphism of A onto a closed (in the seminorm topology) I deal of A.

(vi) The image of A under the map of (v) is dense in $M(A)$ for the strict topology.

Proof.(i) since $x_d : A \to A_d$ is surjective for all d, the maps $A_d \to A_e$, are also all surjective therefore we have maps $M(A_d) \to M(A_e)$ defined as in Theorem (1.1.49) of [2] .(they need not be serjective –see the example following that Theorem).furthermore , if $P_d \in S(A)$ is defined by $P_d(a) = \|x_d(a)\|$, then we have $A_d \cong A_{p_d}$. Therefore the inverse system $\{A_d : d \in D\}$ is cofinal subsystem of the inverse system $\{A_p : P \in S(A)\}$. consequently the inverse system $\{M(A_d) : d \in D\}$ and $\{M(A_p) : p \in S(A)\}$ have the same inverse limit and it is enough to prove the result for $D = S(A)$.

It is clear that every element of $\varinjlim_{p \in S(A)} M(A_p)$ defines a multiplier of A and that the resulting map to $M(A)$ is a homomorphism onto the set of elements $x \in M(A)$ such that $\|x\|_p < \infty$ for all p so we have to product that if $(l, r) \in M(A)$ then $|(l, r)| < \infty$. this will follow if we can show that (l, r) defines a multiplier of A_p since multipliers of C^*-algebras are automatically bounded ([29],3.12.2). So let $a \in \ker(P)$; we have to show that $l(a), r(a) \in \ker(P)$.since $\ker(P)$ is closed subalgebra of A. it's a pro-C^*-algebra and therefore has an approximate identity $\{e_\lambda\}$. Then $r(a) = \lim_\lambda r(e_\lambda a) = \lim_\lambda e_\lambda a(a) \in \ker(p)$, since r is continuous.

Similarly $l(a) \in \ker(P)$. so (l, r) defines an element of $M(A_p)$.

(ii) For all same reason as in (i), its enough to consider the particular inverse system $\{A_P : P \in S(A)\}$.(note that if $B \to C$ is a surjective map of C^*-algebras, then $M(B) \to M(C)$ is strictly continuous) the statement to be proved is now immediate.

(iii) This follows from (i) because there is always at least one inverse system $\{A_d\}$ with inverse limit A such that the maps $A \to A_d$ are surjective, namely $\{A_P : P \in S(A)\}$.

(iv) $M(A_P)$ is complete in the strict topology by ([11], Proposition 3.6), and inverse limits of complete spaces are complete. Now use (ii).

(v) This follows immediately from equation $\|l_a, r_a\|_p = P(a)$.

(vi) Let $\{e_\lambda\}$ be an approximate identity for A, and let $(l,r) \in M(A)$. we claim that $\left(l_{r(e_\lambda)}, r_{r(e_\lambda)}\right) \to (l,r)$ strictly. Now the algebraic properties of multipliers and definition of $\|.\|_{p,a}$ give

$$\left\|(l,r) - \left(l_{r(e_\lambda)}, r_{r(e_\lambda)}\right)\right\|_{p,a} = P\big(l(a) - e_\lambda l(a)\big) + P\big(r(a - ae_\lambda)\big).$$

Since $\{e_\lambda\}$ is an approximate identity and r is continuous, both terms in the right converge to 0.

Using (v) of the previous theorem, we will identify A with the obvious closed subalgebra of $M(A)$.

Multiplier algebras of pro-C^*-algebras have the same kind of functriality as for ordinary C^*-algebras:

Proposition (1.1.40)[1]: (i) let $\varphi: A \to B$ be a homomorphism of pro-C^*-algebras which has a dense range . then φ determines a canonical homomorphism $M(A) \to M(B)$.

(ii) Let B be a pro-C^*-algebra then let A be closed subalgebra of B containing an approximate identity for B then $M(A)$ can be canonically identified with subalgeba of $M(B)$.

Proof. (i) its enough to produce a consistent family of maps from $M(A)$ to $M\big(B_q\big)$ for$q \in S(B)$. so fix q, and note that $q \circ \varphi \in S(A)$. furthermore, the obvious map from $A_{q\circ\varphi}$ to B_q is a homomorphism of C^*-algebras which has dense range and therefore surjective .the required map is then the composite of $M(A) \to M\big(A_{q\circ\varphi}\big)$ and the map $M\left(A_{q\circ\varphi} \to M\big(B_q\big)\right)$ defined in [2], Theorem (1.1.49).

(ii) For $p \in S(B)$,the restriction $p|A$ is in $S(A)$,and A_p is a C^*-subalgebra of B_P containing an approximate identity for B_P. So $M(A_P) \subset M(B_P)$ by [2], Proposition (1.1.20) since A is closed in B, we have $A = \varinjlim_{p\in S(B)} A_P$. Now use the easily verified fact that the inverse limit of injective maps is injective .

For use in [31], we prove here the analogs of two other well known facts about multiplier algebras of C^*-algebras. for the porpose of the next Lemma, a subset S of a pro-C^*-algebra A is bounded if for all $p \in S(A)$ there is a constant $M(P)$ such that $p(a) \leq M(p)$ for all $a \in S$. (this is the usual notion of boundedness in topological vector spaces. Note that any subset of $b(A)$which is bounded for $\|.\|_\infty$ is bound in A, but of course not conversely.)

Proposition (1.1.41)[1]: Multiplication is jointly strictly continuous on bounded subsets of $M(A)$, for any pro-C^*-algebra A.

Proof. let $S, T \subset M(A)$ be bounded , let $p \in S(A)$, and let $M(P)$ be a bound for the values of $\|.\|_P$ on S and T.let $\{x_\lambda\}$ and $\{y_\lambda\}$ be nets in S and T converging to x and y respectivly . then , for all $a \in A$, we have

$$p(x_\lambda y_\lambda - xya) \leq M(P)P(y_\lambda a - ya) + P(x_\lambda ya - xya) \to 0.$$

Similarly $p(ax_\lambda y_\lambda - axy) \to 0$.

Proposition (1.1.42)[1]: Let X be a completely Hausdorff quasitopological space and let A be a pro-C^*-algebra. Then $M(C(X) \otimes A)$ can be a canonically identified with the set of all strictly continuous functions from X to $M(A)$.

Proof. This is true for X compact and A a C^*-algebra by [2], Corollary (1.1.31). the result of the Proposition is obtained by writing $C(X) = \varinjlim_{k \in F_X} C(K)$, and taking inverse limits, using Proposition (1.1.31) and Theorem (1.1.39).

We now define Hilbert modules over pro-C^*-algebras. The results are the obvious generalization of the known results over C^*-algebras, and can be made to follow from them. The proofs, however are not quite as straightforward .Hilbert modules over pro-C^*-algebras do not seem to have previously appeared in the literature, except in [24]. Where the special case of finitely generated projective modules, and where the Hilbert space $l^2(A)$ over A, in the special case in which A is also a Q-algebra, is discussed. (This special case is useless for our applications – see Proposition (1.1.14).)

See [19] and [20] for the standard definitions and the result which we generalize below. (See [35].) We state all the definitions first, and then prove that they make sense afterwards.

Definition (1.1.43)[1]: Let A be a pro-C^*-algebra. And let E be a complex vector space which is also a right A-module , compatibly with the complex algebra structure then E is a pre-Hilbert A-module if its equipped with an A-alued inner product $\langle \ , \ \rangle: E \times E \to A$ which is C-and A-linear in its second variable, is conjugate C-and A-linear in its first variable, satisfies $\langle \xi, \eta \rangle^* = \langle \xi, \eta \rangle$ for $\xi, \eta \in E$, and is positive ($\langle \xi, \eta \rangle \geq 0$ in A for all ξ, and $\langle \xi, \eta \rangle = 0$ only if $\xi = 0$) . we say that E is Hilbert A-module if E is complete in the family of seminorms $\|\xi\|_p = p(\langle \xi, \eta \rangle\ )^{1/2}$ for $[P \in S(A)$.

If E is a Hilbert A-module and $\varphi: A \to B$ is a homomorphism of pro-C^*-algebras, then we construct a Hilbert B-module $\varphi_*(E)$ as follows. First , from the algebraic tensor product $E \otimes_A B$, which is right B-module in the obvious way .(of course, we identify $\lambda\xi \otimes b$ and $\xi \otimes \lambda b$ for $\xi = E, b \in B$, and $\lambda \in C$.) then define a B-module pre-inner product by $\langle \xi \otimes b, \eta \otimes c \rangle = n^* \varphi(\langle \xi, \eta \rangle) c$. the Hilbert B -module $\varphi_*(E)$ is then the Hausdorff completion of $E \otimes_A B$ for the family of seminorms obtained by composing the above inner product with the C^*-seminorms in $S(B)$. note that if $\psi: B \to C$ is another homomorphism of pro-C^*-algebras, then $\psi_*\big(\varphi_*(E)\big)$ is canonically isomorphic to $(\psi \circ \varphi)_*(E)$.

If E and F is a Hilbert A-modules, then we denote by $L(E, F)$ the space of all continuous adjointable A-module homomorphisms from E to F. we write $L(E)$ for the $*$-algebra $L(E, E)$. with $\varphi: A \to B$ as above, define $\varphi_*: L(E, F) \to L\big(\varphi_*(E), \varphi_*(F)\big)$ by $\varphi_*(t)(\xi \otimes b) = t\xi \otimes b$ we topologize $L(E, F)$ via the seminorms $\|t\|_P = \|(x_P)_*(t)\|$ as p turns through $S(A)$, where $x_p: A \to A_p$ is the quotient map. For $\xi \in F$ and $\eta \in E$, we define the rank one module homomorphism $\theta_{\xi,\eta} \in L(E, F)$ by $\theta_{\xi,\eta}(\lambda) = \xi \langle \eta, \lambda \rangle$ for $\lambda \in E$. then the space of compact module homomorphisms $K(E, F)$ is defined to be the closed linear span of $\{\theta_{\xi,\eta}: \xi \in F, \eta \in E\}$ in $L(E, F)$.we write $K(E)$ for the $*$-algebra $K(E, F)$.

The first three part of the following Theorem contain the statements needed to ensure that this definition makes sense. The other three statements are also analogs of standard results in the C^*-algebra case.

Lemma (1.1.44)[1]: Let $\varphi: A \to B$ be a homomorphism of pro-C^*-algebras, and let E be a pre-Hilbert A-module (except that we do not require that $\langle \xi, \xi \rangle = 0$ imply $\xi = 0$). Then $\{\xi \in E: \varphi(\langle \xi, \xi \rangle) = 0\}$ is a submodule of if B is a C^*-algebra, then the function $\xi \to \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$ is a seminorm on E.

Proof. We first observe that it's enough to prove the first statement in the case of C^*-algebra. Indeed, with $x_q: B \to B_q$ being the quotient map for $q \in S(B)$, we have

$$\{\xi \in E: \varphi(\langle \xi, \xi \rangle) = 0\} = \bigcup_{q \in S(B)} \{\xi \in E: x_q \circ \varphi(\langle \xi, \xi \rangle) = 0\},$$

And the union is increasing. Next, replacing B by $\varphi(A)$, we can assume that $B = A_{Pa}$ where $P(a) = \|\varphi(a)\|$.

Let $E_0 = E \cdot \ker(\varphi)$. the linear span of all products ξa for $\xi \in E$ and $a \in \ker(\varphi)$. then E/E_0 is a B-module with $(\xi + E_0)b = \xi a + E_0$ where $\varphi(a) = b$ and has a B-valued pre-inner product given by $\langle \xi + E_0, \eta + E_0 \rangle = \varphi(\langle \xi, \eta \rangle)$. It now follows from the C^*-algebra case that $\xi + E_0 \mapsto \|\langle \xi + E_0, \xi + E_0 \rangle\|^{1/2}$ is a seminorm on E/E_0, whence $\xi \mapsto \|\varphi(\langle \xi, \xi \rangle)\|^{1/2}$ is a seminorm in E in particular $\{\xi \in E: \varphi(\langle \xi, \xi \rangle) = 0\}$ is a vector subspace of E, which is readily seen to be a sub-module.

If A is a pro-C^*-algebra, $p \in S(A)$, and E is a pre-Hilbert A-module, then we write E_P for the Hilbert A_P-module obtained by completing the pre-Hilbert A_P-module $E/\{\xi \in E: P(\langle \xi, \xi \rangle) = 0\}$ as in the proof of the above Lemma. Note that the result of the Lemma ensures that this makes sense. Also not that, with $x_P: A \to A_P$ being the quotient map, we have $(x_P)_*(E) \cong E_P$, via the map $\xi \otimes b \mapsto \xi a$, where $x_P(a) = b$, and bars denote images in E_P of elements of E. In particular $(x_P)_*(E)$ is a Hilbert A_P-module. Similarly, for $P \geq q$ and $\pi_{P,q}: A_P \to A_q$ we have a canonical isomorphism $E_q \cong (\pi_{P,q})_*(E_P)$.

For the purpose of the next proposition, observe that if $\varphi: A \to B$ is a homomorphism of C^*-algebras and E is a Hilbert A-module, then there is a norm-reducing homomorphism σ from E to $\varphi_*(E)$ over φ, given by $\sigma(\xi) = \lim_{\lambda} \xi \otimes e_\lambda$ where $\{e_\lambda\}$ is an approximate identity for B. (note that this net is Cauchy, and its limit is does not depend on which approximate identity is chosen. In this case, $\varphi_*(E)$ is already known to be a Hilbert B-module by [35], Theorem (1.1.58).)

Proposition (1.1.45)[1]: Let $A = \lim_{\rightarrow} A_d$, with maps $\pi_{d,e}: A_d \to A_e$ and $x_d: A \to A_d$. if the x_d are all surjective, then each Hilbert A-module E is the inverse limit $\lim_{\rightarrow}(x_d)_*(E)$ of a system of A_d-modules. Conversely (without assuming surjectivty of the x_d), given Hilbert A_d-modules E_0 and a coherent family of isomorphisms $E_0 \cong (\pi_{d,e})_*(E)$, the inverse limit $E = \lim_{\rightarrow} E_d$ is a Hilbert A-module such that $(x_d)_*(E)$ is canonically identified with a closed submodule of E_d.

Proof. We do the second part first. The isomorphisms $E_0 \cong (\pi_{d,e})_*(E_0)$ yield coherent module maps $\sigma_{d,e}: E_d \to E_e$ over $\pi_{d,e}$ satisfying $\langle \sigma_{d,e}(\xi), \sigma_{d,e}(\eta) \rangle = \pi_{d,e}(\langle \xi, \eta \rangle)$, so it's clear how to make $\lim_{\rightarrow} E_d$ into a pre-Hilbert $\lim_{\rightarrow} A_d$-module completeness and the statement about $(x_d)_*(E)$ are immediate.

For the first part its enough to prove that $E \cong \lim_{\overrightarrow{P \in S(A)}} E_P$. There is an obvious isometry (in the sense of the A-valued, so is the image of E in $\lim_{\rightarrow} E_P$. Since E is complete, we have $E \cong \lim_{\rightarrow} E_P$.

Lemma (1.1.46)[1]: Let A be a pro-C^*-algebra, let E be a Hilbert A-module, and let $p \in S(A)$. Then the map $E \to E_p$ is surjective.

Proof. We let $b(E)$ be the set of bounded elements of, where $\xi \in E$ is bounded if $\langle \xi, \xi \rangle$ is a bounded elements of A. then $b(E)$ is a complex vector space and a right $b(A)$-module because, when E is identified with $\lim_{\rightarrow} E_p$ we see that $b(E)$ corresponding to the set of bounded coherent sequences .the Cauchy-Schwarz inequality, applied to the Hilbert modules E_p over the C^*-algebra A_p , yields , for $\xi,\eta \in b(E)$, the inequality $\|\langle\xi,\eta\rangle\|_\infty^2 \leq \|\langle\xi,\xi\rangle\|_\infty \|\langle\eta,\eta\rangle\|_\infty$, so that the restriction to $b(E)$ of the A-valued inner product on E is a $b(A)$-valued inner product on $b(E)$.

The proof of completeness in [37] , satz (1.1.28) , also applied here (compare with Proposition (1.1.11)(i)), and shows that $b(E)$ is complete for the norm $\|\xi\|_\infty = \|\langle\xi,\eta\rangle\|_\infty^{1\backslash 2}$. Therefore $b(E)$ is Hilbert $b(A)$-module.

Since $\varphi: b(A) \to A_P$ is a surjective map of C^*-algebras (Proposition (1.1.11) (v)), and since clearly $\varphi_*\big(b(E)\big) \cong E_P$, the Lemma will follow if we can show the following : whenever $\varphi: A \to B$ is a surjective map of a C^*algebras, and E is a Hilbert A-module, then the canonical map $\sigma: E \to \varphi_*(E)$ is a surjective . now in the case $\varphi_*(E)$ is a completion of E/E_0, where $E_0 = \{\xi \in E: \varphi(\langle\xi,\xi\rangle) = 0\}$, in its obvious pre-Hilbert B-module structure , as in proof of Lemma (1.1.44). so it is enough to show that E/E_0 is already complete , and this will follow if wwe can show that its norm $\|\xi + E_0\| = \|\varphi(\langle\xi,\xi\rangle)\|^{1/2}$ is just the quotient norm from E. (we know that E is complete .)Thus , we have to show that , for $\xi \in E$, we have $\|\varphi(\langle\xi,\xi\rangle)\|^{1/2} = \inf\limits_{\eta\in E_0} \|\xi+\eta\|$.

For one direction, we observe that if $\xi \in E$ and $\eta \in E_0$, then

$$\|\xi+\eta\|^2 = \|(\langle\xi+\eta,\xi+\eta\rangle)\| \geq \|\varphi(\langle\xi+\eta,\xi+\eta\rangle)\| = \|\varphi(\langle\xi,\xi\rangle)\|,$$

Where $\varphi(\langle\xi,\xi\rangle) = 0$ because $\varphi(\langle\eta,\eta\rangle) = 0$, by the Cauchy-Schwarz inequality in the form $\langle\xi,\eta\rangle^*\langle\xi,\eta\rangle \leq \|\langle\xi,\xi\rangle\|\langle\eta,\eta\rangle$ ([35], Proposition (1.1.23).) for the other direction , let $\xi \in E$ and choose an approximate identity $\{e_\lambda\}$ for $\ker(\varphi)$. then $\xi e_\lambda \in E_0$ for all λ, and we have

$$\begin{aligned}\lim_\lambda \|\xi - \xi e_\lambda\|^2 &= \lim_\lambda \|(1-e_\lambda)\langle\xi,\xi\rangle(1-e_\lambda)\| \\ &= \lim_\lambda \left\|\left((1-e_\lambda)\langle\xi,\xi\rangle^{\frac{1}{2}}\right)\right\|^2 = \lim_{x\subset\ker(\varphi)} \left\|\langle\xi,\xi\rangle^{\frac{1}{2}} + x\right\|^2 = \|\varphi(\langle\xi,\xi\rangle)\|,\end{aligned}$$

Where the second last equality is [29]. This shows that $\inf\limits_{\eta\in E_0}\|\xi+\eta\| \leq \|\varphi(\langle\xi,\xi\rangle)\|^{1/2}$, as needed.

Lemma (1.1.47)[1]: Let $\varphi: A \to B$ be a homomorphism from a pro-C^*-algebra to a C^*-algebra. Then for each Hilbert A-module$E$, the module $\varphi_*(E)$ is a Hilbert B-module , and φ_* defines a map from $L(E,F)$ to $L\big(\varphi_*(E),\varphi_*(F)\big)$ which sends $K(E,F)$ to $K\big(\varphi_*(E),\varphi_*(F)\big)$.

We note that the existence of the map from $L(E.F)$ to $L\big(\varphi_*(E),\varphi_*(F)\big)$ is exactly what is needed to define the topology on $L(E,F)$ under which $K(E,F)$ is the closure of the finite rank module homomorphism.

Proof. let $p \in S(A)$ be $p(a) = \|\varphi(a)\|$, and let $\psi: A_P \to B$ be the obvious map of C^*-algebras . then $\varphi_*(E) = \psi_*(E_P)$, which is Hilbert module by [35], Theorem (1.1.58). Now let $t \in L(E,F)$. choose an approximate identity $\{e_\lambda\}$ for $\ker(\varphi)$, and observe that , for $\xi \in E$ with $\varphi(\langle\xi,\xi\rangle) = 0$, we have

$$\lim_\lambda \langle\xi - \xi e_\lambda, \xi - \xi e_\lambda\rangle = \lim_\lambda (\langle\xi,\xi\rangle - \langle\xi,\xi\rangle e_\lambda) - \lim_\lambda e_\lambda(\langle\xi,\xi\rangle - \langle\xi,\xi\rangle e_\lambda) = 0,$$

Since $P(e_\lambda) \leq 1$ for all $P \in S(A)$. so $\xi e_\lambda \to \xi$. therefore

$$\langle t\xi,t\xi\rangle = \lim_\lambda \langle t\xi, t(\xi e_\lambda)\rangle = \lim_\lambda \langle t\xi,t\xi\rangle e_\lambda \in \ker(\varphi).$$

That is , $\langle\xi,\xi\rangle = 0$ implies $\langle t\xi,t\xi\rangle = 0$. so we obtain a map from $E/\{\xi \in E: \varphi(\langle\xi,\xi\rangle) = 0\}$ to $F/\{\xi \in F: \varphi(\langle\xi,\xi\rangle) = 0\}$ which is easily seen to be adjointable and a

B-module homomorphism . by the previous Lemma, this map is actually an adgointable module homomorphism from E_P to F_P, and hence an element t_P of $L(E_P, F_P)$. (the map t_P automatically continuous, by Lemma 2 of [19].) applying ψ_* and using the relations $\psi_*(E_P) = \psi_*(E)$ and $\psi_*(t_P) = \psi_*(t)$,we see that $\psi_*(t) \in L(\varphi_*(E), \varphi_*(F))$ is in fact well defined . obviously φ_* is homomorphism.

It remains to verify that φ_* sends $K(E, F)$ to $K(\varphi_*(E), \varphi_*(F))$ since φ_* is continuous ($t \mapsto t_P$ is continuous by definition , and φ_* is continuous because it comes from a map of C^*-algebra), its enough to show that $\varphi_*(0_{\xi,\eta})$ is a compact module homomorphism determined by the images of ξ and η if F_P and E_P repectively. Therefore $\varphi_*(0_{\xi,\eta}) = \psi_*(0_{\xi,\eta}) \in K(\psi_*(E), \psi_*(F))$ by [20].

Proposition (1.1.48)[1]: That an element of $L(E, F)$ defines a coherent sequence of elements of $L(E_P, F_P)$ follows from the previous Lemma, and similarly for $K(E, F)$ and $K(E_P, F_P)$. the converse for $L(E, F)$ is easily shown by using Proposition (1.1.45) to write $E = \lim\limits_{\rightarrow} E_P$ and $F = \lim\limits_{\rightarrow} F_P$. that the resulting map is homomorphism is essentially the definition of the topology on $L(E, F)$.

Now let $\{K_P\}$ be a coherent sequence of elements of $K(E_P, F_P)$. we have to show that the corresponding operator $k \in L(E, F)$ is actually in $K(E, F)$. for $p \in S(A)$ and $\varepsilon > 0$ choose $\bar{\xi}_1, \dots, \bar{\xi}_n \in F_n$ and $\bar{\eta}_1, \dots, \bar{\eta}_n \in E_P$ such that $\left\| \sum \acute{\theta}_{\xi_{i,\bar{\eta}_i}} - k_P \right\| < \varepsilon$. using Lemma (1.1.46), choose $\xi_1, \dots, \xi_n \in F$ and $\eta_1, \dots, \eta_n \in E$ whose images in F_P and E_P are $\bar{\xi}_1, \dots, \bar{\xi}_n$ and $\bar{\eta}_1, \dots, \bar{\eta}_n$. then set $l_{P,\varepsilon} = \sum \acute{\theta}_{\xi_{i,\eta_i}} \in K(E, F)$. we have $l_{P,\varepsilon} \to k$ as $(P, \varepsilon) \to \infty$(that is ,as p increases and $\varepsilon \to 0$), so $k \in K(E, F)$ as desired .

We are now able to prove Theorem (1.1.49).

Theorem (1.1.49)[1]: (i) the function $\| \quad \|_p$ of the previous definition are seminorms .
(ii) the pre-inner product defined on $E \otimes_A B$ satsfies all of the properties of an inner product except that $\langle \eta, \eta \rangle$ may be zero for nonzero $\eta \in E \otimes_A B$.
(iii) the map $\varphi_*: L(E, F) \to L(\varphi_*(E), \varphi_*(F))$ is well defined .
(iv) $\varphi_*(K(E, F)) \subset K(\varphi_*(E), \varphi_*(F))$.
(v) $L(E)$ and $K(E)$ are $pro - C^* - algebras.$
(vi) $L(E) \cong M(K(E))$ Canonically.

Proof. (i) this is Lemma (1.1.44).

(ii) for $\varphi: A \to B$ and $q \in S(B)$, let $\varphi_q$be the obvious homomorphism from A to B. then , for Hilbert A-module E, we have $\varphi_*(E) \cong \lim\limits_{\rightarrow}(\varphi_q)_*(E)$. the modules $(\varphi_q)_*(E)$ are Hilbert B_q-modules by Lemma (1.1.47), and the inverse limit is a Hilbert B-module by Proposition (1.1.45) the statement now follows.

(iii) this follows from Lemma (1.1.47), Proposition (1.1.48), and the expression of $\varphi_*(E)$ as $\lim(\varphi_q)_*(E)$ in the proof of part (ii).

(iv) this follows in the same way as (iii) .

(v) this is immediate from Proposition (1.1.48).

(vi) it follows from the argument used in the proof of Proposition (1.1.48) that the map $K(E) \to K(E_P)$ has dense range. By Corollary (1.1.12), it must be surjective. It now follows from Proposition (1.1.48) and Theorem (1.1.39) that $M(K(E)) \cong \lim M(K(E_P))$. since $M(K(E_P)) = L(E_P)$ by ([19], Theorem 1), we obtain $M(K(E)) = L(E)$ by another application of Proposition (1.1.48).

Example (1.1.50)[1]: Let $A = C(Z)^+$, which is just $\prod_{n=1}^{\infty} C$, and let $E = \prod_{n=1}^{\infty} C^n$. we make E into a Hilbert A-module via $(\xi a)_n = \xi_n a_n$ and $\langle \xi, \eta \rangle_n = \langle \xi_n, \eta_n \rangle$, where the right hand side is the usual C valued inner product on C^R. let A_R be the product of the first n factors of A , and let $E_R$be the product of the first n factors of E. then $A_n = A_{P_n}$. where $P_n(a) = \sup\{\|a_k\| : k \le n\}$. and $E_n = E_{P_n}$. so $A = \lim A_n, E = \lim E_n$, and $L(E) = \lim L(E_n) = \lim K(E_n) = K(E)$ using Proposition (1.1.48) however , E is not finitely generated as an A-module .

We will prove a stabilization Theorem for countably generated Hilbert modules over σ-C^*-algebras. The proof uses induction over the directed set, and we do not know if the result is true over general pro-C^*-algebras.

We restrict ourselves to the σ-C^*-algebras of Arveson. Which are the inverse limits of C^*-algebras whose topology is determined by countably many C^*-seminorms. equivalently. They are invert limits of countable inverse system of C^*-algebras. We do this because, in certain ways, the category of σ-C^*-algebras is much more manageable than the category of pro-C^*-algebras. In particular , we have no useful condition for the inverse limits of exact sequence of C^*-algebras to be exact , or for the maps $\lim\limits_{\rightarrow} A_P \to A_P$ to be surjective , we have also been unable to show that the quotient of a pro-C^*-algebra by a closed ideal is again a pro-C^*-algebra. (the issue here is completeness . it is known that in general the quotient of a complete topological vector space need not be complete → see [22],23.5 or 32.6) .however , we do have the corresponding results for σ-C^*-algebras. our proof use induction over the directed sets of our inverse system .

We discuses homomorphism, ideals, and of σ-C^*-algebras. we then give the σ-C^*-algebra versions of the important results from the previous in those cases in which they differ , and prove two additional results related to the earlier for which we need to begin with σ-C^*-algebras. We will assume that the countable directed set is always Z^+. this can always arranged since any countable directed set has a cofinal subset isomorphic to Z^+ (or else has a largest element),and limits are unchanged when the directed set is replaced by a cofinal subset. We will also always assume that the maps $A_{n+1} \to A_n$ are all surjective ; this can always be arranged by replacing each A_n by the intersection of the images of A_m for $m \ge n$. note that an inverse system indexed by Z^+ is determined by the maps $A_{n+1} \to A_n$, and that they can be arbitrary .

Finally , we assume that all ideals are closed two-sided .(it's shown in Theorem (1.1.22) of [18] that a closed two-sided ideal in an arbitrary pro-C^*-algebra is necessarily selfadjoint .)

Lemma (1.1.51)[1]: Let $A = \lim\limits_{\rightarrow} A_n$ be a σ-C^*-algebra (with all maps $A_{n+1} \to A_n$ surjective). Then $A \to A_n$ is a surjective.

Proof. We assume $n = 1$.(the proof is same for all n). Given $a_1 \in A_1$,construct inductivity a sequence $\{a_n\}$ defines an element of A whose image in A_1 is a_1.

Theorem (1.1.52)[1]: Let A be a σ-C^*-algebra, let B be a pro-C^*-algebra, and let $\varphi: A \to B$ be a $*$-homomorphism. Then φ is automatically continuous.

Proof. It is enough to prove that for $P \in S(B)$ the maps $A \to B_P$, determined by φ, are continuous. Thus we reduce to the case in which B is a C^*-algebra. Taking unitizations, we may assume that A, B and φ are unital. Now represent B faithfully on a Hilbert space and use Lemma (1.1.28) of [9].

However that a homomorphism of σ-C^*-algebras need not have closed range (consider the inclusion of $b(A)$ in A for any σ-C^*-algebra A , for instance $C(R)$, for which $b(A) \neq A$).
A sequence

$$0 \to I \overset{\alpha}{\to} A \overset{\beta}{\to} B \to 0 \qquad (1)$$

Is exact if and only if its an inverse limit (with surjective maps) of exact sequences of C^*-algebras.

For the if parts. The algebraic statements follow from Proposition 10.2 of [6], and the topological statements are easily verified. For the "only if" part of (i), write $B = \lim\limits_{\to} B_n$ with maps $\lambda_n \cong A_{q_n}$, and this hence a closed subalgebra of B_n, we clearly have φ equal to the inverse limit of the inclusions of A_n in B_n.

Now we do (ii) using (i) write α as the inverse limit of maps $\alpha_n: I_n \to A_n$. then I_n is an ideal in A_n and the sequence (1) is easily seen to be algebraically the inverse limit of the sequences

$$0 \to I_n \to A_n \to A_n/I_n \to 0.$$

To show that the identification is also topological, use Theorem (1.1.52).

For general inverse system, we know of no good criterion for the surjectivity of the last map in the inverse limit of a system of exact sequences. In the particular, if A is a general inverse limit of σ-C^*-algebra and I is an ideal in A, we have an obvious map from A/I to $\underline{\lim}_{P \in S(A)} A_P/I_P$, but we do not know whether its surjective in general.

The first part of the following Corollary has already been observed in [18] and [41].

Corollary (1.1.53)[1]: Let A be a σ-C^*-algebra and let I be an ideal in A. Then A/I is a σ-C^*-algebra, and every homomorphism $\varphi: A \to B$ of σ-C^*-algebra such that $\varphi|I = 0$ factors through A/I.

Proof. It essentially follows from the proof of the previous Proposition that with $A = \lim\limits_{\to} A_P$ and I_P being the image of I in A_P we have $A/I \cong \lim\limits_{\to} A_P/I_P$. the last statement follows from the definition of the quotient of topological vectors spaces. The categorical role played by A/I is presumably played in the category of pro-C^*-algebras by the closure of the image of A in $\lim\limits_{\to} A_P/I_P$.

Corollary (1.1.54)[1]: for the sequence of σ-C^*-algebra and $*$-homomorphism

$$0 \to I \overset{\alpha}{\to} A \overset{\beta}{\to} B \to 0$$

To be exact, it is sufficient that it be algebraically exact.

Proof. Use previous Corollary (once) and Theorem (1.1.52) (several times).

Proposition (1.1.55)[1]: let A be a σ-C^*-algebra , and let I and J be ideals in A. then $I \times J$ is a (closed) ideal in A.

Proof. Write $A = \lim\limits_{\to} A_n$ and let I_n and J_n be the images of I and J in A_n then we have $I = \lim\limits_{\to} I_n, J = \lim\limits_{\to} J_n$ and $I + J = \lim\limits_{\to}(I_n + J_n)$ (for the last statement ,one needs the fact that $I_{n+1} \cap J_{n+1} \to I_n \cap J_n$ is a surjective.) since $I_n + J_n$ is closed ([29], 1.5.8), so is $I + J$ the remaining properties are obvious.

We now identify the commutative σ-C^*-algebra. We will say that a topological space X is countably compactly generated if there is accountable family $\{K_n\}$ of compact subset of X such that a set $C \subset X$ is closed if and only if $C \cap K_n$ is closed for all n. Obviously we may require that $K_1 \subset K_2 \subset \cdots$. Thus, X is countably compactly generated if and only if its countable direct limit of compact spaces. (this is not the same as being σ-compact and compactly generated, as we will see in example (1.1.57).)

Proposition (1.1.56)[1]: The category of commutative unital σ - C^* -algebra is a contrvariantly equivalent to the category of countably compactly generated Hausdorff spaces.

Proof. We must prove two things: that a countably compactly generated Hausdorff space is completely Hausdorff, and that every σ - C^* -algebra is isomorphic to $C(X)$ for some countably compactly generated Hausdorff space.

For the first part, it is sufficient to show that if X is a topological space with a distinguished family F of compact subset which have countable cofinal subset , then X is countably compactly generated and F is equal to the set of all compact sets of X. Let $\{K_n : n \in Z^+\}$ be an increasing countable cofinal subset of F . Its immediate that $\{K_n : n \in Z^+\}$ determines the topology on X. If there is compact set $L \subset X$with $L \notin F,$ then for each n we can choose $x_n \in L \backslash K_n$. The set $T = \{x_n : n \in Z^+\}$ is closed because $T \cap K_n$ is finite for all n; similarly $T \backslash \{x_n\}$ is closed for each fixed n. Therefore T is a closed infinite discrete subset of the compact set, a contradiction.

We now give an example of something that looks like σ-C^*-algebra but is not.

Example (1.1.57)[1]: (note that $C(Q)$ is a pro-C^*-algebra, because metric spaces are compactly generated by [43], 1.4.3.) to prove this suppose that $C(Q)$ is a σ-C^*-algebra. By the previous proposition, we then have $C(Q) \cong C(X),$ where X is a countably compactly generated space. Both Q and X are σ-compact, hence Lindelof, hence realcompact by [17], Theorem 8.2. therefore [17], Theorem 10.6 implies that Q and X are homeomorphic. So it's enough to prove that $Q,$ in spite of being both countable and compactly generated, is not countably compactly generated.

The following argument was suggested by Bob Edwards. Let $K_1 \subset K_2 \subset \cdots$ be compact subsets of Q whose union is Q. each K_n is nowhere dense, so that there is $x_n \in Q \backslash K_n$ with $0 < x_n < 1/n$. then $x_n \to 0$ in Q, but $\{x_n\}$ does not converge in the direct limit topology on $\lim\limits_{\rightarrow} K_n$.(the only possible limit would be 0, which is not in $\{x_n\}$. but $\{x_n\}$ is closed since $K_m \cap \{x_n\}$ is finite for all m). Thus $C(Q)$ is not a σ-C^*-algebra. In fact, it cannot be σ-C^*-algebra for any topology on $C(Q)$.

We next specialize some of the results to σ-C^*-algebras.

Proposition (1.1.58)[1]: (i) the tensor product of two σ-C^*-algebra. In fact $\left(\lim\limits_{\rightarrow} A_n\right) \otimes \left(\lim\limits_{\rightarrow} B_n\right) \cong (A_n \otimes B_n)$.

(ii) A countable inverse limit of σ-C^*-algebras is a σ-C^*-algebra.

(iii) the multiplier algebra of σ-C^*-algebra is a σ-C^*-algebra is a σ-C^*-algebra. In fact, $M\left(\lim\limits_{\rightarrow} A_n\right) \cong \lim\limits_{\rightarrow} M(A_n)$. (recall that $A_{n+1} \to A_n$ is assumed surjective. However, $M(A_{n+1}) \to M(A_n)$ neet not be surjective.)

(iv) if A is a σ-C^*-algebra and E is a Hilbert A-module, then $K(E)$ and $L(E)$ are σ-C^*-algebras.

The proofs are trivial and are omitted. An uncountable inverse limit of σ-C^*-algebras obviously need not be a σ-C^*-algebra. And, as we now show, even a countable limit of σ-C^*-algebras need not be a σ-C^*-algebra.

Example (1.1.59)[1]: Let A be any σ-C^*-algebra which is not C^*-algebra, and write $A = \lim\limits_{\rightarrow} A_n$ with maps $x_n : A \to A_n$ and seminorms given by $p_n(a) = \|x_n(a)\|$ we can clearly arrange to have $P_n < p_m$ for $n < m$. let B_n be the direct sum of m copies of A, and define $\varphi_m: B_m \to B_{m+1}$ by $\varphi_m(a_1, \dots, a_m, 0)$. From the discussion, we see that $\lim\limits_{\to} B_m$ can be identified with the set B of all elements $a \in \prod_{m=1}^{\infty} A$ such that, for every function $s: Z^+ \to Z^+$, we have $\lim\limits_{m\to\infty} P_{s(m)}(a_m) = 0$. the topology on B is given by the C^*-seminorms $q_s(a) = \sup\{p_{s(m)}(a_m): m \in Z^+\}$. To show that B is not a σ-C^*-algebra, its enough to show that there is no countable cofinal subset of the set of the seminorms q_s. Notice that $q_s \le q$, if and only if $s \le t$. So suppose we had a cofinal subset $\{s_k\}$ of the set of all functions. Thus $\lim\limits_{\to} B_a$ is not σ-C^*-algebra.

By proving two results for σ-C^*-algebra for which we have been unable to prove analogous results for general pro-C^*-algebra. Note that the multiplier algebra of a σ-C^*-algebra is again a σ-C^*-algebra. We also point out that, by the Corollary to Theorem 14 of [45]. Multipliers (double centralizers) of a σ-C^*-algebra are automatically continuous.

Theorem (1.1.60)[1]: Let $\varphi: A \to B$ be a surjective homomorphism of σ-C^*-algebras, and assume that A has a countable approximate identity. Then the map $M(A) \to M(B)$ is surjective.

Proof. If A and B are C^*-algebras, this is Theorem 10 of [45]. In general ase, let $I = \ker(\varphi)$. using Proposition 5.3 (ii), write the exact sequence

$$0 \to I \to A \overset{Q}{\to} B \to 0$$

C as the inverse limit of exact sequences of C^*-algebras

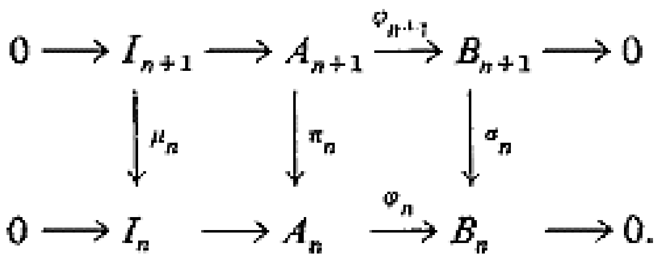
$$0 \to I_n \to A_n \overset{Q_n}{\to} B \to 0,$$

With, of course, all the maps in the inverse system being surjective. Let $\mu_n: I_{n+1} \to I_n, \pi_n: A_{n+1} \to A_n$, and $\sigma_n: B_{n+1} \to B_n$ be the maps of the inverse systems. Let J_n be the kernel of the obvious map $\overline{\varphi_n}: M(A_n) \to M(B_n)$. Since A has a countable approximate identity, so does each A_n and each B_n. Therefore $\overline{\varphi_n}$ is surjective, as are the maps $\overline{\pi_n}: M(A_{n+1}) \to M(A_n)$ and $\overline{\sigma_n}: M(B_{n+1}) \to M(B_n)$. We thus have an inverse system of exact sequences

$$0 \to J_n \to M(A_n) \overset{\overline{\varphi_n}}{\to} M(B_n) \to 0 \qquad (2).$$

In which the maps of the systems $\{M(A_n)\}$ and $\{M(B_n)\}$ are all surjective. Let $\bar{\mu}$ be the map from J_{n+1} to J_n.

Consider the commutative diagram with exact rows and surjective vertical maps:



Set $Q = \{(a, b) \in A_n \otimes B_{n+1}: \varphi_n(a) = \sigma_n(b)\}$. Then there is homomorphism

$$\psi: A_{n+1} \to Q$$

Given $\psi(a) = \left(\pi_n(a), \varphi_{n+1}(a)\right)$. A diagram shows that ψ is surjective. Therefore $\bar{\psi}: M(A_{n+1}) \to M(Q)$ is surjective (since A_{n+1} has a countable approximate identity).

The projection from Q to A_n and B_{n+1} are also surjective (since π_n and φ_{n+1} are), and its then easy to show that

$$M(Q) = \{(a, b) \in M(A_n) \oplus M(B_{n+1}): \overline{\varphi_n}(a) = \overline{\sigma_n}(b)\}.$$

In the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_{n+1} & \longrightarrow & M(A_{n+1}) & \xrightarrow{\bar{\varphi}_{n+1}} & M(B_{n+1}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \dot{\mu}_n} & & \Big\downarrow{\scriptstyle \ddot{\pi}_n} & & \Big\downarrow{\scriptstyle \dot{\sigma}_n} & & \\ 0 & \longrightarrow & J_n & \longrightarrow & M(A_n) & \xrightarrow{\ddot{\psi}_n} & M(B_n) & \longrightarrow & 0 \end{array}$$

With exact rows and in which $\overline{\pi_n}$ and $\overline{\sigma_n}$ are surjective, the surjectivity of

$$\bar{\psi}: M(A_{n+1}) \to M(Q)$$

Now implies that $\overline{\mu_n}$ is surjective. Therefore we can use Proposition (1.1.58)(ii) to take inverse limits in (2). In particular, $\lim\limits_{\to} M(A_n) \to \lim\limits_{\to} M(B_n)$ is surjective. By Theorem (1.1.39)(i), this is the same as saying that $M(A) \to M(B)$ is surjective.

For application of this Theorem, it should be pointed out that any separable σ-C^*-algebra A has a countable approximate identity: if $\{e_\lambda\}$ is an approximate identity for $b(A)$ and $\{a_k\}$ is countable dense subset of A, choose an increasing subsequence $\{x_n\}$ of $\{a_k\}$ such that, with $\{p_n\}$ be a cofinal sequence in $S(A)$. We have $p_n(x_n a_k - a_k) + p_n(a_k x_n - a_k) < 1/n$ for $1 \le k \le n$. Note that the separability of A is equivalent to A being the countable inverse limit of separable C^*-algebra.

However, $b(A)$ can fail to be separable when A is separable: consider $A = C(R)$.

Our final result is the stabilization Theorem promised.

Theorem (1.1.61)[1]: Let A be a σ-C^*-algebra with countable approximate identity, and let E be a countably generated (in the topological sense) Hilbert A-module. Then $E \oplus l^2(A) \cong l^2(A)$.

Proof. Write $A = \lim\limits_{\to} A_n$ (with surjective maps $\pi_n: A_{n+1} \to A_n$), and correspondingly write $E = \lim\limits_{\to} E_n$ with $(\pi_n)_*(E_{n+1}) = E_n$. Then each A_n has a countable approximate identity, and each E_n is countably generated.

We will construct by induction on n, isomorphisms $u_n: E_n \oplus l^3(A_n)^n \to l^2(A_n)^n$ such that $(\pi_n)_*(u_{n+1}) = u_n \oplus 1$ as the maps from $E_n \oplus l^2(A_n)^{n+1}$ to $l^2(A_n)^{n+1}$. We obtain u_1 from the stabilization Theorem for Hilbert modules over C^*-algebras, ([19], Theorem 2).

Given u_n, construct u_{n+1} as follows. First, use the stabilization Theorem to choose an isomorphism $l^2(A_n)^{n+1} \to E_{n+1} \oplus l^2(A_{n+1})^n$. Then $u_n(\pi_n)_*(t)$ is a unitary element of $L(l^2(A_n)^n)$, which we identify with which we identify with $M(K(H) \otimes A_n)$, where $H = l^2(C)$.

Since $H \cong \bigoplus_{k=1}^{\infty} H^n$, we see from (Proposition 2.2 of [28]) that $u_n(\pi_n)_*(v) \oplus 1$ is in the connected component of the identity in the unitary group of $M(K(H^{n+1}) \otimes A_n)$. Since $K(H^{n+1}) \otimes A_{n+1}$ has a countable approximate identity, the map

$$M(K(H^{n+1}) \otimes A_{n+1}) \to M(K(H^{n+1}) \otimes A_n)$$

is surjective by Theorem 10 of [45]. By (Proposition 4.8 of [41]), there is a therefore an invertible element w of $M(K(H^{n+1}) \otimes A_{n+1})$ whose image in $M(K(H^{n+1}) \otimes A_n)$ is $u_n(\pi_n)_*(v) \oplus 1$. Replacing w by $w(w^*w)^{-1/2}$.

We may assume that w is unitary. Now regard w as an element of $L(l^2(A_{n+1})^{n+1})$ and set $u_{n+1} = w(v \oplus 1)^{-1}$. Then $(\pi_n)_*(u_{n+1}) = u_n \oplus 1$, as desired.

We now let x_n be the direct sum of u_n and the identity on $\bigoplus_{k=1+1}^{\infty} l^2(A_n)$. Writing $l^2(A_n)^\infty$ for the direct sum $\bigoplus_{k=1}^{\infty} l^2(A_n)$, we see that $\{x_n\}$ is a coherent sequence of isomorphisms in $L(E_n \oplus l^2(A_n)^\infty, l^2(A_n)^\infty)$. Therefore $\{x_n\}$ defines an isomorphism $x: El^2(A_n)^\infty \to l^2(A_n)^\infty$. Since $l^2(A_n)^\infty \cong l^2(A_n)$, this completes proof.

Section (1.2): Hilber C^*-Modules

There are a number of results in the theory of C^*-algebras and the unitary Representation theory of groups concerned with various kinds of dilations. A unified Approach to such problems can be taken via the concept of a Kolmogorov decomposition for appositive definite kernel [51].The idea to write came from a reading of E. C. Lance's [52], where a dilation theorem for completely positive maps of Hilbert C^*-modules is derived by means of a certain tensor product construction. It seemed possible that the scalar theory of positive definite kernels would generalize to the Hilbert C^*-module context and that this could then be used to derive a more "natural" proof of the dilation theorem (as given in the scalar case in [51]). The purpose, therefore, to present a generalized theory of positive definite kernels in the Hilbert C^*-module context. Further justification for such a theory is provided by other applications we give below, where we use it to represent Hilbert C^*-modules as concrete spaces of operators and also to construct the exterior tensor product of Hilbert C^*-modules. An advantage of our construction of the latter is that we do not need to invoke the stabilization theorem of Kasparov, as is done in the standard construction [52].

It turns out that much of the scalar theory of positive definite kernels goes over to the context of Hilbert C^*-modules straightforwardly, although one has to be rather careful at certain points concerning the existence of adjoints for the linear maps under consideration. There are some important differences from the scalar case nevertheless –for instance, the proof of Theorem (1.2.3) below differs from its scalar analogue because a certain relevant Banach space may not admit a predual. It would be possible to shorten by omitting those parts of proofs that are parallel to the scalar case. However, it seemed preferable to give a self-contained account, partly because the scalar theory of positive definite kernels and Kolmogorov decompositions appears not to be as well known as it deserves to be and partly because such a full account illustrates clearly the elegance of the proofs and shows how easy and natural some Hilbert C^*-module results are if one uses the approach adopted here.

We begin by recalling the definition of positive definiteness.

If S is a non-empty set, a map k from $S \times S$ to a C^*-algebra A is said to be a positive definite kernel if, for every positive integer n and for all $n_1, \ldots, n_n \in S$, the matrix $(k(n_i, n_j))$ in $M_n(A)$ is positive.

It follows immediately from the definition that $k(s,s) \geq 0$ and that $k(s,s)^* = k(s,t)$, for all $s, t \in S$.

It follows that a kernel $k: S \times S \to A$ is positive definite if and only if for all $s_1, \ldots, s_n \in S$ and $a_1, \ldots, a_n \in A$, the sum $\sum_{ij=1}^{n} a_i^* \, k(s_i, s_j) a_j$ is positive in A.

We shall use these observations below.

Example (1.2.1)[50]: Let A and B be C^*-algebras. A linear map $p: A \to B$ is said to be completely positive if, for every positive integer n, the inflation $M_n(A) \to M_n(B)$, $(a_{ij}) \to (pa_{ij})$, is positive. Equivalently, p is completely positive if and only if the kernel $k: A \times A \to B$, $(a_1, a_2) \to p(a_1^* a_2)$, is positive definite. The equivalence follows easily from the fact that a positive matrix (a_{ij}) of $M_n(A)$ is a sum of matrices of the form $(a_i^* a_j)$, where $a_1, \ldots, a_n \in A$.

As we shall be studying positive definite kernels and completely positive maps in the context of Hilbert C^*-module theory, we recall now some basic terminology, notation and results of that theory. ([52] for details and examples. However, we mention in passing that the importance of Hilbert C-modules arises out of their applications to Morita equivalence, KK -theory and C^*-algebraic quantum group theory.)

(i) Let A be a C^*-algebra and E a linear space that is right A -module. A pair consisting of E and a map $\langle.,.\rangle$ from $E \times E$ to A is called an inner-product A -module if the map is linear in the second variable, conjugate-linear in the first, and satisfies the following conditions for all $x, y \in E$ and all $a \in A$:

(ii) $\langle x, ya\rangle = \langle x, y\rangle$;

(iii) $\langle x, y\rangle^* = \langle y, x\rangle$;

(iv) $\langle x, x\rangle \geq 0$ and if $\langle x, x\rangle = 0$, then $x = 0$.

If $\langle.,.\rangle$ satisfies all these requirements except possibly for the second part of Condition (3), it is called a semi-inner product on E. A version of the Cauchy-Schwarz inequality for semi-inner products holds, namely,

$$\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle \qquad (x, y \in E). \tag{3}$$

A Hilbert C^*-module over A, or Hilbert A-module, is an inner product A-module for which the associated norm, $x \to \|\langle x, x\rangle\|^{1/2}$, is complete.

If E, F are Hilbert A-modules, a map $V: E \to F$ is adjointable if there exists a map $W: F \to E$ such that $\langle Vx, y\rangle = \langle x, Wy\rangle$ for all $x \in E$ and $y \in F$. Automatically, V is then bounded and A-linear, that is, it is linear and $V(xa) = V(x)a$ for all $x \in E$ and $a \in A$. Moreover, W is unique and is denoted by V^*. The Banach space of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$ and $\mathcal{L}(E)$ denotes the C^*-algebra $\mathcal{L}(E, E)$.

A map $U: E \to F$ is a unitary if it is adjointable and $U^*U = 1$ and $UU^* = 1$. In this case U is isometric, surjective and A-linear. Conversely, if U has these properties, it is a unitary [52]. If a unitary mapping from E onto F exists, then E and F are said to be unitarily equivalent.

Before turning now to the theory of positive definite kernels, we need a few more items of notation that will be used frequently in the sequel.

We write $B(H, K)$ for the Banach space of all bounded linear operators from H to K, where H and K are Banach spaces, and we write $B(H)$ for the algebra $B(H, H)$.

If $(x, y) \to xy$ is a bilinear map on the product $H \times K$ with values in a Banach space L, and if S and T are subsets of H and K respectively, we denote by ST the closed linear span in L of all products xy, where $x \in S$ and $y \in T$.

We denote by [52] the closed linear span of S.

Let A be a C^*-algebra. If V is an arbitrary map from a non-empty set S to $\mathcal{L}(E, E_V)$, where E and E_V are Hilbert A-modules, then the kernel k, defined by setting $k(s, t) = V(s)^*V(t)$, is positive definite (k has values in $\mathcal{L}(E)$). The map V will be called a Kolmogorov decomposition for k. If the (scalar) linear span of the set $\cup_{s\in S} V(s)E$ is dense in E_V, then V will be said to be minimal.

Every positive definite kernel k with values in $\mathcal{L}(E)$ has an essentially unique minimal Kolmogorov decomposition. This is the content of the following result. The proof is modeled on the scalar (Hilbert space) case, see [51]. However, some care about adjointability of maps is required at certain points and a somewhat different approach is taken to demonstrating the properties of the inner product of the space constructed.

Theorem (1.2.2)[50]: Let E be a Hilbert C^*-module over a C^*-algebra A. Let S be a non-empty set and k a positive definite map from $S \times S$ to $\mathcal{L}(E)$. Then there exists a minimal

Kolmogorov decomposition for k. Moreover, if $V: S \to \mathcal{L}(E, E_V)$ and $W: S \to \mathcal{L}(E, E_W)$ are any two such minimal Kolmogorov decompositions, then there exists a unique unitary $\mathrm{U}: E_V \to E_W)$ such that $UV(s) = W(s)$, for all $s \in S$.

Proof. If $f: S \to E$ has finite support, define $kf: S \to E$ by setting $kf(s) = \sum_{t \in S} k(s,t)f(t)$ and denote by E_V^0 the set of all these maps kf. When endowed with the pointwise-defined operations, E_V^0 is a right module over A. Moreover, we may endow E_V^0 with a semi-inner product by setting

$$\langle kf, kg \rangle = \sum_{s,t \in S} \langle k(s,t)f(t), g(s) \rangle$$

(Positivity is given by positive definiteness of k.) In fact, we actually have an inner product. For, by the Cauchy-Schwarz inequality (5), if $\langle kf, kf \rangle = 0$, then $\langle kf, kg \rangle = \sum_{s \in S} \langle kf(s), g(s) \rangle = 0$, for any map $g: S \to E$ of finite support. If $x \in E$ and $t \in S$, define the map x, from S to E by setting $x_t(s) = 0$ if $x \neq t$ and by setting $x_t(t) = x$. Then with $g = x_t$, we get $\langle kf(t), x \rangle = \sum_{s \in S} \langle kf(s), x(s) \rangle = 0$. Hence, $kf(t) = 0$, for all $t \in S$, so $kf = 0$.

Thus, E_V^0 is an inner product A-module. We complete it to get a Hilbert A-module that we denote by E_V.

If $s \in S$, define $V(s): E \to E_V$ by setting $V(s)x = k(x_s)$. We show that $V(s) \in \mathcal{L}(E, E_V)$, that is, $V(s)$ is adjointable: Obviously, $V(s)$ is A-linear. Also, it is bounded, since $\|V(s)x\|^2 = \|\langle k(x_s), k(x_s) \rangle\| = \|\langle k(s,s)x, x \rangle\| \leq \|k(s,s)\| \|x\|^2$, and therefore, $\|V(s)\| \|k(s,s)\|^{1/2}$. Define $T: E_V^0 \to E$ by setting $T(kf)x = (kf)(s)$. Direct computation shows that

$$\langle x, T(kf) \rangle = \langle V(s)x, kf \rangle \tag{4}$$

And therefore,

$$\|T(kf)\| = \sup_{\|x\| \leq 1} \|\langle x, T(kf) \rangle\| = \sup_{\|x\| \leq 1} \|\langle V(s)x, kf \rangle\| \leq \|V(s)\| \|k(f)\|.$$

Hence, $\|T\| \leq \|V(s)\|$. Now extend T to a bounded linear operator from E_V to E. It follows from Equation (4) that $\langle x, T(g) \rangle = \langle V(s)x, g \rangle$ for all $x \in E$ and $g \in E_V$. Hence, $V(s)$ is adjointable, with adjoint $V(s)^* = T$. Moreover, if $s, t \in S$ and $x, y \in E$, then $\langle V(s)^* V(s)x, y \rangle = \langle k(x_t), k(y_t) \rangle = \langle k(s,t)x, y \rangle$, so $V(s)^* V(t) = k(s,t)$. Hence, the map, $V: S \to V(s)$, is a Kolmogorov decomposition for k.

If f is a map from S to E of finite support, then it can be written as a sum $f = f_1 + \cdots + f_n$, where $f_i = (x^i)_{s_i}$, for some vectors $x^i \in E$ and elements $s_i \in S$. Since $k(f_i) = V(s_i)x^i$ and $\sum_{i=1}^n k(f_i)$, the linear span of the set $\cup_s V(s)E$ contains $k(f)$.

Hence, $E_V = [\cup_s V(s)E]$ and V is minimal.

Suppose now that $V: S \to \mathcal{L}(E, E_V)$ and $W: S \to \mathcal{L}(E, E_W)$ are any two minimal Kolmogorov decompositions for k. If $s_1 + \cdots + s_n$ belong to S and $x_1 + \cdots + x_n$ belong to E, then

$$\left\| \sum_{i=1}^n V(s_i)x_i \right\|^2 = \left\| \left\langle \sum_{i=1}^n V(s_i)x_i, \sum_{j=1}^n V(s_j)x_j \right\rangle \right\| = \left\| \sum_{ij=1}^n \langle k(s_j, s_i)x_i, x_j \rangle \right\|$$

$$= \left\| \sum_{i=1}^n W(s_i)x_i \right\|^2.$$

Hence, there is a well-defined isometry from a dense linear subspace of $E_V$to E_W that maps $V(s)x$ to $W(s)x$. We extend this to get an isometry U from E_V to E_W. We may define similarly an isometry U' from E_W to E_V mapping $W(s)x$ to $V(s)x$. Clearly, U' and U are inverse to each other. Since $\langle UV(s)x, W(t)y\rangle = \langle W(s)x, W(t)y\rangle = \langle k(t,s)x, y\rangle = \langle V(s)x, V(t)y\rangle = \langle V(s)x, U'W(t)y\rangle$, we have $\langle Uf, g\rangle = \langle f, U'g\rangle$ for all $f \in E_V$ and $g \in E_w$. Hence, U is adjointable with $U^* = U' = U^{-1}$. Thus, U is a unitary. Also, $UV(s) = W(s)$, for all $S \in s$.

As observed earlier (in Example (1.2.1)), a completely positive map determines a positive definite kernel. We use this now to derive a dilation theorem, part of whose proof is parallel to the derivation of Theorem 2.13 in [51]. First, we need some definitions.

If E and F are Hilbert A-modules, the strict topology on $\mathcal{L}(E,F)$ is the one given by the seminorms

$$V \to \|V_X\| \quad (x \in E). \qquad V \to \|V^*y\| \quad (y \in F).$$

The closed 0-centred ball of $\mathcal{L}(E,F)$) of any finite radius is complete relative to the strict topology.

If A and B are C^*-algebras, and E is a Hilbert A-module, a completely positive map $\rho: B \to \mathcal{L}(E)$ is said to be strict [2,p.49] if, for some approximate unit (e_i) of B, the net $(\rho(e_i))$ satisfies the Cauchy condition for the strict topology in $\mathcal{L}(E)$.(If B is unital, ρ is automatically strict.)

Theorem (1.2.3)[50]: Let A and B be C^*-algebras, let E be a Hilbert A-module and let $\rho: B \to \mathcal{L}(E)$ be a strict completely positive map. Then there exists a Hilbert A -module E_π, a $*$ -homomorphism $\pi: B \to \mathcal{L}(E_\pi)$ and an element $W \in \mathcal{L}(E, E_\pi)$ such that $\rho(b) = W^*\pi(b)W$, for all $b \in B$. Moreover, $[\pi(B)WE] = E_\pi$.

Proof. Since the kernel $k: (b_1, b_2) \to \rho(b_1^* b_2)$ is positive definite, it has a minimal Kolmogorov decomposition $V: B \to \mathcal{L}(E, E_V)$, by Theorem (1.2.2). Moreover, V is linear. For, if $b_1, b_2, c \in B$ and $\lambda \in C$, then

$$\begin{aligned} V(b_1 + \lambda b_2)^* V(c) &= \rho((b_1 + \lambda b_2)^* c) = \rho(b_1^* c) + \bar{\lambda} p(b_2^* c) \\ &= V(b_1)^* V(c) + \bar{\lambda} V(b_2)^* V(c) = (V(b_1) + \lambda V(b_2))^* V(c); \end{aligned}$$

hence, since $[\cup_{c\in B} V(c)E] = E_V$, we have $V(b_1 + \lambda b_2) = V(b_1) + \lambda V(b_2)$.

If u is a unitary element of $\tilde{B} = B + C1$, the unitization of B, and $b, c \in B$, then $V(ub)^* V(uc) = \rho(b^* u^* uc) = \rho(b^* c) = V(b)^* V(c)$, so the map, $c \to V(uc)$, is a minimal Kolmogorov decomposition for k. Hence, there exists a unitary $\pi(u) \in \mathcal{L}(E_V)$ such that $\pi(u)V(c) = V(uc)$ $(c \in B)$. If b is a linear combination of unitaries of $\tilde{B}$, say $b = \sum_{i=1}^n \lambda_i u_i$, then $(\sum_{i=1}^n \lambda_i \pi(u_i))V(c) = V((\sum_{i=1}^n \lambda_i u_i)c) = V(bc)$. Using this and again using the fact that $E_V = [\cup_{c\in B} V(c)E]$, it follows that we may define $\pi(b) = \sum_{i=1}^n \lambda_i \pi(u_i)$, independent of the decomposition of b into a linear combination of unitaries. Thus, $\pi(b)V(c) = V(bc)$, and it follows easily that $\pi: B \to \mathcal{L}(E_V), b \to \pi(b)$, is a $*$-homomorphism. Set $E_\pi = E_V$.

Now let (e_i) be an approximate unit of B for which $(\rho(e_i))$ is a Cauchy net relative to the strict topology of $\mathcal{L}(E)$. We show that $(V(e_i))$ is a Cauchy net for the strict topology of $\mathcal{L}(E, E_\pi)$: First, observe that it is bounded. For, if $b \in B$, then

$$\|V(b)\|^2 = \|V(b)^* V(b)\| = \|\rho(b^* b)\| \le \|\rho\| \|b\|^2. \tag{5}$$

Since $V(e_i)^* V(b) = \rho(e_i b)$ and since $e_i b \to b$ in norm, the set $(V(e_i)^* V(b)x)$ is convergent in E for all $x \in E$. Hence, $(V(e_i)^* y)$ is convergent for all y in the linear span of $\cup_b V(b)E$. Using boundedness of the net $(V(e_i)^*)$ and density of the linear span of

$\cup_b V(b)E$ in E_π, it follows that $(V(e_i)^* y)$ is convergent for all $y \in E_\pi$. Now let $x \in E$ and suppose that, $e_j \le e_i$. Then

$$\left\|V(e_i)x - V(e_j)x\right\|^2 = \left\|\langle x, (V(e_i) - V(e_j))^* \left(V(e_i) - V(e_j)\right) x\rangle\right\|$$
$$= \left\|\langle x, \rho\left((e_i - e_j)^2\right)x\rangle\right\| \le \left\|\langle x, \rho\left(e_i - e_j\right)x\rangle\right\|$$

It follows that $(V(e_i)x)$ is a Cauchy net in E_π since $(\rho(e_i))$ forms a Cauchy net for the strict topology. Hence, $(V(e_i))$ is a Cauchy net for the strict topology in the closed 0-centred ball of radius $\|\rho\|^{1/2}$ in $\mathcal{L}(E, E_\pi)$ and therefore, by completeness, it is convergent in that topology to some element $W \in \mathcal{L}(E, E_\pi)$.

If $b \in B$ and $x \in E$, then $\pi(b)Wx = \lim \pi(b)V(e_i)x = \lim V(be_i)x = V(b)x$, since V is continuous. Therefore, $\pi(b)W = V(b)$. Since $[\cup_b V(b)E] = E_\pi$, it follows that $[\pi(B)WE] = E_\pi$. Finally, for any element $x \in E$, we have $W^*\pi(b)Wx = W^*V(b)x = \lim V(e_i)^*V(b)x = \lim \rho(e_i b)x = \rho(b)x$, so $W^*\pi(b)W = \rho(b)$.

It is an important fundamental result that every C^*-algebra has a faithful representation as a concrete algebra of operators. We are now going to show that an analogous result holds for Hilbert C^*-modules. First, however, we must make an appropriate definition.

Let H and K be Hilbert spaces and let A be a concrete C^*-algebra of operators acting on H. Let E be a closed linear subspace of $B(H, K)$ and suppose that the following two conditions are satisfied:

(a) If $x \in E$ and a $a \in A$, then $xa \in E$;

(b) If $x, y \in E$, then $x^* y \in A$.

Endowed with the multiplication $(x, a) \to xa$ (the product is just operator composition), E becomes a right, A-module. Setting $\langle x, y\rangle = x^* y$ makes E into a Hilbert A-module. (The induced norm is the operator norm.)

We call E a concrete Hilbert C^*-module.

The following result states that all Hilbert C^*-modules can be represented as concrete ones. A classical (Hilbert-space) Kolmogorov decomposition enters into the proof.

Theorem (1.2.4)[50]: Let A be a C^*-algebra and let E be a Hilbert A-module. Then there exists a faithful representation π of A on a Hilbert space H and an isometric, linear isomorphism U from E onto a concrete Hilbert $\pi(A)$-module F of operators from H to a Hilbert space K such that

$$\langle (U(x), U(y)\rangle = \pi(\langle x, y\rangle) \ and \ U(xa) = U(x)\pi(a)$$

For all $x, y \in E$ and $a \in A$.

Proof. Let (H, π) be any faithful representation of A. Then the kernel, $k: E \times E \to B(H)$, $(x, y) \to \pi(\langle x, y\rangle)$, is positive definite. To see this, suppose $x_1, \dots, x_n \in E$. Then, by Remark (1.2.1), the matrix $(\langle x_i, x_j\rangle) \in M_n(A)$ is positive, since, if $a_1, \dots, a_n \in A$, then $\sum_{i,j=1}^n a_j^* \langle a_j, a_i\rangle a_i = \langle \sum_{j=1}^n x_j a_j, \sum_{i=1}^n x_i a_i\rangle \ge 0$. Hence, the matrix $(\pi\langle x_i, x_j\rangle)$ is positive in $M_n(B(H))$.

Since k is positive definite, it admits a (classical) Kolmogorov decomposition $U: E \to B(H, K)$, where K is some Hilbert space. Using the fact that $U(x)^* U(y) = \pi(\langle x, y\rangle)$ for all $x, y \in E$, one easily verifies that U is linear and isometric and that $U(xa) = U(x)\pi(a)$ for all $x \in E$ and $a \in A$. Setting $F = U(E)$, it follows that F is a closed linear subspace of $B(H, K)$ for which $F\pi(A) \subseteq F$ and $F^* F \subseteq \pi(A)$. Hence, F is a concrete Hilbert $\pi(A)$-module. This proves the theorem.

We give an application of this representation to the construction of the exterior tensor product of two Hilbert C^*-modules. As remarked by Lance [2,p.34], the usual construction is hard: it uses the Kasparov stabilisation theorem [2,p.62]. However, our construction is quite straightforward, using the preceding theorem.

We write $E \otimes_{alg} F$ for the algebraic tensor product of two linear spaces and $H \otimes K$ for the Hilbert space tensor product of two Hilbert spaces.

Theorem (1.2.5)[50]: Suppose that B and C are C^*-algebras and that E and F are Hilbert C^*-modules over B and C, respectively. Suppose also that A is the minimal C^*-tensor product of B and C. Then there exists a Hilbert C^*-module G over A containing $E \otimes_{aig} F$ as a dense linear subspace such that for all $x, x' \in E$ and $y, y' \in F$, we have $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \otimes \langle y, y' \rangle$ and for all $b \in B$ and $c \in C$, we have $(x \otimes y)(b \otimes c) = xb \otimes yc$. Moreover, G is unique up to unitary equivalence: If G and G' are two Hilbert A-modules satisfying these conditions, then there is a unique unitary U from G onto G' which is the identity map when restricted to $E \otimes_{aig} F$.

Proof. The uniqueness of G is almost obvious. We show only its existence. Using Theorem (1.2.4), it is easily seen that we may suppose that B, C are concrete C^*-algebras acting on Hilbert spaces H and K, respectively and that E and F are concrete Hilbert C^*-modules; thus, they are closed linear subspaces of $B(H, H')$ and $B(K, K')$, respectively, for some Hilbert spaces H' and K'. Also, we regard A as a concrete C^*-algebra acting on the Hilbert space tensor product $H \otimes K$.

We can identify $E \otimes_{aig} F$ as a linear subspace of $B(H \otimes K, H' \otimes K')$ by identifying the elementary tensor $x \otimes y$ with the operator that maps $\eta \otimes \xi$ onto $x(\eta) \otimes y(\xi)$, where $\eta \in H$ and $\xi \in K$. We now define G to be the closure in $B(H \otimes K, H' \otimes K')$ of $E \otimes_{aig} F$. We have $GA \subseteq G$, since $(x \otimes y)(b \otimes c) = xb \otimes yc$ for all $x \in E, y \in F$, $b \in B$ and $c \in C$. Also, $G^*G \subseteq A$, since $(x_1 \otimes y_1)^*(x_2 \otimes y_2) = x_1^* x_2 \otimes y_1^* y_2$ for all $x_1, x_2 \in E$ and $y_1, y_2 \in F$. Hence, G is a concrete Hilbert A-module satisfying the condition s of the theorem.

The module G is the exterior tensor product of E and F.

Section (1.3): Means of Krein Spaces:

The Hilbert space $\mathcal{H}$ associated to a positive definite kernel K is an abstract version of the L^2 space associated to a positive measure and the Kolmogorov decomposition of K gives a useful expansion of the elements of $\mathcal{H}$ in terms of a geometrical model of a stochastic process with covariance kernel K. Therefore, it is quite natural to seek similar constructions for an arbitrary kernel. While the decomposition into a real and an imaginary part can be realized without difficulties, the study of Hermitian kernels is no longer straightforward. This was shown in the work of L. Schwartz [78], where a characterization of the Hermitian kernels admitting a Jordan decomposition was obtained in terms of a boundedness condition that we call the Schwartz condition (the statement (1) of Theorem (1.3.4)). A key difficulty of the theory was identified in [78] in the lack of uniqueness of the associated reproducing kernel spaces.

It was shown in [58] that the Schwartz condition is also equivalent to the existence of a Kolmogorov decomposition, while the uniqueness of the Kolmogorov decomposition was characterized in spectral terms (Theorem (1.3.4) and, respectively, Theorem (1.3.5)). The purpose is to continue these investigations by considering Hermitian kernels with additional symmetries given by the action of a semigroup. The main result gives a characterization of those Hermitian kernels that produce a representation of the action by bounded operators on a certain Krein space. It turns out that such a result has many applications and we discuss GNS representations on inner product spaces.

We review the concept of induced Krein space and we show its key role in the construction of Kolmogorov decompositions as described in [58]. A new result is added here in connection with a lifting property for induced Krein spaces that is related to an important inequality of M. G. Krein. Theorem (1.3.2) gives an example of an induced Krein space without the lifting property, adding one more pathology to the study of Hermitian kernels. Incidentally, this result answers negatively a question raised in [62].

We show the applicability of our results to questions concerning GNS representations of $*$-algebras on Krein spaces. The whole issue is motivated by the lack of positivity in some models in local quantum field theories. We relate these questions to properties of Kolmogorov decompositions so that we can characterize the existence (Theorem (1.3.7)) and the uniqueness (Theorem (1.3.8)) of the GNS data. This is also a motivation for considering the general case of semigroups with involution. For example, Theorem (1.3.19) characterizes the boundedness of the GNS data.

We consider the action of a semigroup on a Hermitian kernel and Theorem (1.3.10) gives the conditions that insure the representation of this action as a semigroup of bounded operators on a Krein space. We also address the uniqueness property of such representations. While the case of the trivial semigroup with one element is settled in [58] (Theorem (1.3.5)) and Theorem (1.3.13) gives another partial answer, the general case remains open. The proof used for the trivial semigroup cannot be easily extended precisely because Theorem (1.3.2) is true. We analyze the case when the projective representation given by Theorem (1.3.10) is fundamentally reducible or, equivalently, it is similar to a projective Hilbert space representation, a question closely related to other similarity problems and uniformly bounded representations.

We briefly review the concept of a Kolmogorov decomposition for Hermitian kernels. The natural framework to deal with these kernels is that of Krein spaces. We recall first some definitions and a few items of notation. An indefinite inner product space $(\mathcal{H}, [\cdot,\cdot])$ is called Krein space provided that there exists a positive inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ turning $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ into a Hilbert space such that $[\xi,\eta] = \langle J\xi,\eta\rangle,\ \xi,\eta \in \mathcal{H}$, for some symmetry $J(J^* = J^{-1} = J)$ on $\mathcal{H}$. Such a symmetry J is called a fundamental symmetry. The norm $\|\xi\|^2 = \langle\xi,\xi\rangle$ is called a unitary norm. The underlying Hilbert space topology of $\mathcal{K}$ is called the strong topology and does not depend on the choice of the fundamental symmetry.

For two Krein spaces $\mathcal{H}$ and $\mathcal{K}$ we denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the set of linear bounded operators from $\mathcal{H}$ to $\mathcal{K}$. For $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ we denote by $T^\sharp$ the adjoint of T with respect to $[\cdot,\cdot]$. We say that $A \in \mathcal{L}(\mathcal{H})$ is a selfadjoint operator if $A^\sharp = A$. A possibly unbounded operator V between two Krein spaces is called isometric if $[V\xi, V\eta] = [\xi,\eta]$ for all ξ,η in the domain of V. Also, we say that the operator $U \in \mathcal{L}(\mathcal{H})$ is unitary if $UU^\sharp = U^\sharp U = I$, where I denotes the identity operator on $\mathcal{H}$. The notation T^* is used for the adjoint of T with respect to the positive inner product $\langle\cdot,\cdot\rangle$.

Krein spaces induced by selfadjoint operators. Many difficulties in dealing with operators on Krein spaces are caused by the lack of a well-behaved factorization theory. The concept of induced space turned out to be quite useful in this direction. Thus, for a selfadjoint operator A in $\mathcal{L}(\mathcal{H})$ we define a new inner product $[\cdot,\cdot]_A$ on $\mathcal{H}$ by the formula

$$[\xi,\eta]_A = [A\xi,\eta], \qquad \xi,\eta \in \mathcal{H}, \tag{6}$$

and a pair $(\mathcal{K},\Pi)$ consisting of a Krein space $\mathcal{K}$ and a bounded operator $\Pi \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is called a Krein space induced by A provided that Π has dense range and the relation

$$[\Pi\xi, \Pi\eta]_{\mathcal{K}} = [\xi, \eta]_A \quad (7)$$
holds for all $\xi, \eta \in \mathcal{H}$, where $[\cdot,\cdot]_{\mathcal{K}}$ denotes the indefinite inner product on $\mathcal{K}$. One well-known example is obtained in the following way.

Example (1.3.1)[53]: Let J be a fundamental symmetry on $\mathcal{H}$ and let $\langle\cdot,\cdot\rangle_J$ be the associated positive inner product turning $\mathcal{H}$ into a Hilbert space. Then JA is a selfadjoint operator on this Hilbert space and let $\mathcal{H}_-$ and $\mathcal{H}_+$ be the spectral subspaces of JA corresponding to $(-\infty, 0)$ and, respectively, $(0, \infty)$. We obtain the decomposition
$$\mathcal{H} = \mathcal{H}_- \oplus kerA \oplus \mathcal{H}_+.$$
Note that $(\mathcal{H}_-, -[\cdot,\cdot]_A)$ and $(\mathcal{H}_+ ,\ [\cdot,\cdot]_A)$ are positive inner product spaces and hence they can be completed to the Hilbert spaces $\mathcal{K}_-$ and, respectively, $\mathcal{K}_+$. Let $\mathcal{K}_A$ be the Hilbert direct sum of $\mathcal{K}_-$ and $\mathcal{K}_+$ and denote by $\langle\cdot,\cdot\rangle\mathcal{K}_A$ the positive inner product on $\mathcal{K}_A$. Define
$$J_A(k_- \oplus k_+) = -k_- \oplus k_+$$
for $\mathcal{K}_- \in \mathcal{K}_-$ and $\mathcal{K}_+ \in \mathcal{K}_+$. We can easily check that J_A is a symmetry on $\mathcal{K}_A$ and then the inner product
$$[k,\ k']\mathcal{K}_A = \langle J_A \mathrm{k}, k'\rangle\mathcal{K}_A$$
turns $\mathcal{K}_A$ into a Krein space. The map Π_A: $\mathcal{H} \to \mathcal{K}_A$ is defined by the formula
$$\Pi_A\xi = \left[P_{\mathcal{H}_-}\ \xi\right] \oplus [P_{\mathcal{H}_+}\xi],$$
Where $\xi\ \in \mathcal{H}$, $P_{\mathcal{H}_\pm}$ denotes the orthogonal projection of the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_J)$ onto the subspace $\mathcal{H}_\pm$, and $[P_{\mathcal{H}_\pm}\ \xi]$ denotes the class of $P_{\mathcal{H}_\pm}\ \xi$ in $\mathcal{K}_\pm$. Then one checks that $(\mathcal{K}_A, \Pi_A)$ is a Krein space induced by A. In addition, if $JA = S_{JA}|JA|$ is the polar decomposition of JA, then we note that
$$J_A\Pi_A\ = \Pi_A S_{JA}. \quad (8)$$
This example proved to be very useful since it is accompanied by a good property concerning the lifting of operators, as shown by a classical result of M.G. Krein, [69]. The result was rediscovered by W.J. Reid [77], P.D. Lax [70], and J. Dieudonné [59]. The indefinite version presented below was proved in [60] by using a 2 × 2 matrix construction that reduces the proof to the positive definite case.

Theorem (1.3.1)[53]: Let A and B be bounded selfadjoint operators on the Kre˘ın spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Assume that the operators $T_1\ \in\ \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $T_2\ \in\ \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ satisfy the relation $T_2^{\#}A = BT_1$. Then there exist (unique) operators $\tilde{T}_1\ \in\ \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ and $\tilde{T}_2\ \in\ \mathcal{L}(\mathcal{K}_{\boldsymbol{B}}, \mathcal{K}_A)$ such that $\tilde{T}_1\Pi_A = \Pi_{\boldsymbol{B}}T_1, \tilde{T}_2\Pi_{\boldsymbol{B}} = \Pi_A T_2$ and $[\tilde{T}_1 f, g]_{\mathcal{K}_B} = [f, \tilde{T}_2 g]_{\mathcal{K}_A}$ for all $f \in \mathcal{K}_A, g \in \mathcal{K}_B$.

Theorem (1.3.1) will be used in an essential way in the proof of the main result and it is also related to the uniqueness property of a Kolmogorov decomposition for invariant Hermitian kernels. For these reasons we discuss one more question related to this result, namely whether this lifting property holds for other induced Krein spaces. More precisely, two Krein spaces $(\mathcal{K}_i\ , \Pi_i),\ i\ =\ 1, 2$, induced by the same selfadjoint operator $A\ \in\ \mathcal{L}(\mathcal{H})$ are *unitarily equivalent* if there exists a unitary operator U in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\Pi_1 = \Pi_2$. Theorem (1.3.7) in [58] shows that there exist selfadjoint operators with the property that not all of their induced Krein spaces are unitarily equivalent.

Let $(\mathcal{K}, \Pi)$ be a Krein space induced by A. We say that $(\mathcal{K}, \Pi)$ has the *lifting property* if for any pair of operators $T, S\ \in\ \mathcal{L}(\mathcal{H})$ satisfying the relation $AT = SA$ there exist unique operators $\tilde{T}, \tilde{S} \in \mathcal{L}(\mathcal{K})$ such that $\tilde{T}\Pi = \Pi T\ , \tilde{S}\Pi = \Pi S$. From Theorem (1.3.1) it follows that the induced Krein space $(\mathcal{K}_A\ , \Pi_A)$ constructed in Example (1.3.1) has the lifting property, as do all the others which are unitarily equivalent to it. However, as the following result shows, this is not true for all induced Krein spaces of A.

Theorem (3.1.2)[53]: There exists a selfadjoint operator that has an induced Krein space without the lifting property.

Proof. Let $\mathcal{H}_0$ be an infinite dimensional Hilbert space and A_0 is a bounded selfadjoint operator in $\mathcal{H}_0$ such that $0 \le A_0 \le I$, $\ker A_0 = 0$, and the spectrum of A_0 accumulates to 0, equivalently, its range is not closed. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ as well as the bounded selfadjoint operator

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & -A_0 \end{bmatrix}. \tag{9}$$

Let $\mathcal{K}$ be the Hilbert space $\mathcal{H}$ with the indefinite inner product [. , .] defined by the symmetry

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Consider the operator $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K})$,

$$\begin{bmatrix} I & -(I-A_0)^{1/2} \\ (I-A_0)^{1/2} & -I \end{bmatrix}. \tag{10}$$

It is a straightforward calculation to see that $\Pi^* J \Pi = A$ and, by performing a Frobenius–Schur factorization, it follows that Π has dense range. Thus, $(\mathcal{K}, \Pi)$ is a Krein space induced by A and we show that it does not have the lifting property.

Let T be an operator in $\mathcal{L}(\mathcal{H})$ such that, with respect to its 2×2 block-matrix representation, all its entries T_{ij}, $i, j = 1, 2$, commute with A_0. Define the operator $S = JTJ$ and note that $AT = SA$.

Let us assume that there exists a bounded operator $\tilde{T} \in \mathrm{L}(\mathcal{K})$ such that $\|T\| = \tilde{T}$. Then, there exists the constant $C = \|\tilde{T}\|_{\mathcal{K}} < \infty$ such that

$$\|\Pi T \xi\| \le C \|\Pi \xi\|, \quad \xi \in \mathcal{H},$$

or, equivalently, that

$$T^* H T \le C^2 H, \tag{11}$$

Where

$$H = \begin{bmatrix} 2 - A_0 & -2(I-A_0)^{1/2} \\ -2(I-A_0)^{1/2} & 2 - A_0 \end{bmatrix}.$$

Taking into account that A_0 commutes with all the other operator entries involved in (11), it follows that the inequality (11) is equivalent to

$$T^* \begin{bmatrix} I & -\Delta \\ -\Delta & I \end{bmatrix} T \le C^2 \begin{bmatrix} I & -\Delta \\ -\Delta & I \end{bmatrix}, \tag{12}$$

where we denoted

$$\Delta = 2(I-A_0)^{1/2}(2-A_0)^{-1}.$$

Note that, by continuous functional calculus, Δ is an operator in $\mathcal{H}$ such that $0 \le \Delta \le I$ and its spectrum accumulates to 1.

The use of the Frobenius–Schur factorization

$$\begin{bmatrix} I & -\Delta \\ -\Delta & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Delta & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I - \Delta^2 \end{bmatrix} \begin{bmatrix} I & -\Delta \\ 0 & I \end{bmatrix}. \tag{13}$$

suggests to take

$$T = \begin{bmatrix} I & -\Delta \\ 0 & I \end{bmatrix},$$

and this choice is consistent with our assumption that all its entries commute with A_0.

Since T is bounded invertible, from (12) we get

$$\begin{bmatrix} I & -\Delta \\ -\Delta & I \end{bmatrix} \leq C^2 \begin{bmatrix} I & 0 \\ 0 & I-\Delta^2 \end{bmatrix}.$$

Looking at the lower right corners of the matrices in the previous inequality we get $I \leq C^2(I-\Delta^2)$ which yields a contradiction since the spectrum of the operator $I-\Delta^2$ accumulates to 0.

We show that the answer to this question is negative. Indeed, an operator D as above produces the induced Krein space $(\mathcal{D}, D^{\#})$ for A. Let A be the operator defined by (9). Let us take

$$T = \begin{bmatrix} I & I \\ -I & I \end{bmatrix}.$$

One checks that $AT = T^*A$. Define $X = T^*$, then XA is selfadjoint. If $Y \in \mathcal{L}(\mathcal{D})$ exists such that $XD = DY$, then $Y^*D^* = D^*T$ and a similar reasoning as in the proof of Theorem (1.3.2) shows that from (12) and (14) we get $2(\Delta^3 + \Delta^2 - \Delta + I) \leq C^2(I - \Delta^2)$, which is impossible since the spectrum of the operator from the left side is bounded away from 0.

One might ask whether another additional assumption on the operator T that is frequently used in applications, namely that T is A-isometric, could enforce the lifting property. To see that this is not the case, let us take

$$T = \frac{2}{\sqrt{3}} \begin{bmatrix} I & -\frac{1}{2}I \\ \frac{1}{2}I & -I \end{bmatrix}.$$

It is easy to prove that $T^*AT = A$, that is, T is A-isometric. Noting that T is boundedly invertible, this corresponds to $S = T^{*-1}$. As before, from (12) and (13) we get $\frac{3}{4}(-\Delta^3 + \frac{15}{4}\Delta^2 - 3\Delta + \frac{5}{4}I) \leq C^2(I - \Delta^2)$. But this is again contradictory since the spectrum of the operator from the left side is bounded away from 0.

Kolmogorov decompositions of Hermitian kernels. We can use the concept of induced space in order to describe the Kolmogorov decomposition of a Hermitian kernel. Let X be an arbitrary set. A mapping K defined on $X \times X$ with values in $\mathcal{L}(\mathcal{H})$, where $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is a Krein space, is called a Hermitian kernel on X if $K(x,y) = K(y,x)^{\#}$ for all $x, y \in X$.

Let $\mathcal{F}_0(X,\mathcal{H})$ denote the vector space of $\mathcal{H}$-valued functions on X having finite support. We associate to K an inner product on $\mathcal{F}_0(X,\mathcal{H})$ by the formula:

$$[f,g]_K = \sum_{x,y \in X} [K(x,y)f(y), g(x)]_{\mathcal{H}} \tag{14}$$

for $f, g \in \mathcal{F}_0(X,\mathcal{H})$. We say that the Hermitian kernel $L: X \times X \to \mathcal{L}(\mathcal{H})$ is positive definite if the inner product $[\cdot,\cdot]_L$ associated to L by the formula (14) is positive. On the set of Hermitian kernels on X with values in $\mathcal{L}(\mathcal{H})$ we also have a natural partial order defined as follows: if A, B are Hermitian kernels, then $A \leq B$ means $[f,f]_A \leq [f,f]_B$ for all $f \in \mathcal{F}_0(X,\mathcal{H})$. Following L. Schwartz [78], we say that two positive definite kernels A and B are disjoint if for any positive definite kernel P such that $P \leq A$ and $P \leq B$ it follows that $P = 0$. A Kolmogorov decomposition of the Hermitian kernel K is a pair $(V; \mathcal{K})$, where $\mathcal{K}$ is a Krein space and $V = \{V(x)\}_{x \in X}$ is a family of bounded operators $V(x) \in \mathcal{L}(\mathcal{H},\mathcal{K})$ such that $K(x,y) = V(x)^{\#}V(y)$ for all $x, y \in X$, and the closure of $\bigvee_{x \in X} V(x)\mathcal{H}$ is $\mathcal{K}$ ([68],[75],[63]). Note that here and $\vee$ stands for the linear manifold generated by some set, without taking any closure.

The next result, obtained in [58], settles the question concerning the existence of a Kolmogorov decomposition for a given Hermitian kernel.

Theorem (1.3.4)[53]: Let $K: X \times X \to \mathcal{L}(\mathcal{H})$ be a Hermitian kernel. The following assertions are equivalent:

(i) There exists a positive definite kernel $L: X \times X \to \mathcal{L}(\mathcal{H})$ such that $-L \leq K \leq L$.

(ii) K has a Kolmogorov decomposition.

The condition $-L \leq K \leq L$ of the previous result appeared in the work of L. Schwartz [78] concerning the structure of Hermitian kernels. We will call it the Schwartz condition. It is proved in [78] that this condition is also equivalent to the Jordan decomposition of K, which means that the kernel K is a difference of two disjoint positive definite kernels. It is convenient for our purpose to review the construction of the Kolmogorov decomposition. We assume that there exists a positive definite kernel $L: X \times X \to \mathcal{L}(\mathcal{H})$ such that $-L \leq K \leq L$. Let $\mathcal{H}_L$ be the Hilbert space obtained by the completion of the quotient space $\mathcal{F}_0(X,\mathcal{H})/\mathcal{N}_L$ with respect to $[\cdot,\cdot]_L$, where $\mathcal{N}_L = \{f \in \mathcal{F}_0(X,\mathcal{H}) | [f,f]_L = 0\}$ is the isotropic subspace of the inner product space $(\mathcal{F}_0(X,\mathcal{H}), [.,.]_L)$. Since $-L \leq K \leq L$ is equivalent to

$$|[f,g]_K| \leq [f,f]_L^{1/2} [g,g]_L^{1/2} \tag{15}$$

for all $\{f, g \in \mathcal{F}_0(X,\mathcal{H})$ (see Proposition 38, [78]), it follows that $\mathcal{N}_L$ is a subset of the isotropic subspace $\mathcal{N}_K$ of the inner product space $(\mathcal{F}_0(X,\mathcal{H}), [.,.]_K)$. Therefore, $[.,.]_K$ uniquely induces an inner product on $\mathcal{H}_L$, still denoted by $[.,.]_K$, such that (15) holds for $f, g \in \mathcal{H}_L$. By the Riesz representation theorem we obtain a selfadjoint contractive operator $A_L \in \mathcal{L}(\mathcal{H}_L)$, referred to as the *Gram*, or metric operator of K with respect to L, such that

$$H_L[f,g]_K = [A_L f, g]_L \tag{16}$$

for all $f, g \in \mathcal{H}_L$. Let $(\mathcal{K}_{AL}, \Pi_{A_L})$ be the Kre˘ın space induced by A_L given by Example (1.3.1). For $\xi \in H$ and $x \in X$, we define the element $\xi_x \in (\mathcal{F}_0(X,\mathcal{H})$ by the formula:

$$\xi_x(y) = \begin{cases} \xi, & y = x \\ 0, & y \neq x. \end{cases} \tag{17}$$

Then we define

$$V(x)\xi = \Pi_{A_L}[\xi_x], \tag{18}$$

where $[\xi_x]$ denotes the class of ξ_x in $\mathcal{H}_L$ and it can be verified that $(V; \mathcal{K}_{AL})$ is a Kolmogorov decomposition of the kernel K.

We finally review the uniqueness property of the Kolmogorov decomposition. Two Kolmogorov decompositions $(V_1, \mathcal{K}_1)$ and $(V_1, \mathcal{K}_1)$ of the same Hermitian kernel K are unitarily equivalent if there exists a unitary operator $\Phi \in \mathcal{L}((\mathcal{K}_1, \mathcal{K}_2))$ such that for all $x \in X$ we have $V_2(x) = \Phi V_1(x)$. The following result was obtained in [58].

Theorem (1.3.5)[53]: Let K be a Hermitian kernel which has Kolmogorov decompositions. The following assertions are equivalent:

(i) All Kolmogorov decompositions of K are unitarily equivalent.

(ii) For each positive definite kernel L such that $-L \leq K \leq L$, there exists $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A_L)$ or $(-\epsilon, 0) \subset \rho(A_L)$, where A_L is the Gram operator of K with respect to L.

We give some motivation for the study of Hermitian kernels invariant under the action on a semigroup with involution. Thus, we first discuss the GNS representation for unital $*$-algebras from the point of view of Hermitian kernels, showing that considering only actions on groups is not sufficient. We make connections with some constructions of interest in quantum field theories such as those summarized in [79].

Another important issue is that we should consider projectively invariant Hermitian kernels. This is emphasized, for example, by the Fock representation of the canonical commutation relations obtained from an action of the rigid motions of a Hilbert space on the exponential vectors of a Fock space, since it is natural to consider a similar construction for other groups, the like the Poincaré group, involving an indefinite inner product. Various models involving Fock spaces associated to indefinite inner products were studied in [72], [79]. Here we emphasize that the Kolmogorov decomposition gives a simple construction of the Weyl exponentials (the related topic of the representations of the Heisenberg algebra in Krein spaces is taken up in [71]).

Representations of $*$-algebras associated to Hermitian forms. Let $\mathcal{A}$ be a $*$- algebra with identity 1 and let Z be a linear Hermitian functional on $\mathcal{A}$ with mass $1(Z(1) = 1)$. Then $\mathcal{A}$ is a unital multiplicative semigroup with involution acting on itself by

$$\phi(a, x) = xa^* \tag{19}$$

for $a, x \in \mathcal{A}$. We define

$$K_Z(x, y) = Z(xy^*) \tag{20}$$

for $x, y \in \mathcal{A}$. Then K_Z is a Hermitian kernel on $\mathcal{A}$ with scalar values and satisfies the symmetry relation

$$K_Z\big(x, \phi(a, y)\big) = Z(xay^*) = K_Z(\phi(a^*, x), y) \tag{21}$$

for $a, x, y \in \mathcal{A}$. In order to describe the GNS construction for Z we will use the concept of unbounded representations of $\mathcal{A}$. Thus, a mapping π of $\mathcal{A}$ into the set of closable operators defined on a common dense domain $\mathcal{D}(\pi)$ of a Banach space $\mathcal{K}$ is called a closable representation if it is linear, $\mathcal{D}(\pi)$ is invariant under all operators $\pi(a), a \in \mathcal{A}$, and $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in \mathcal{A}$. If, in addition, $\mathcal{K}$ is a Kre˘ın space and, for all $a \in \mathcal{A}$, the domain of $\pi(a)^\#$ contains $\mathcal{D}(\pi)$ and

$$\pi(a)^\# \mathcal{D}(\pi) = \pi(a^*), \tag{22}$$

then π is called a Hermitian closable representation on the Kre˘ın space $\mathcal{K}$ (or, a J-representation, as introduced in [73], see also [66]).

The GNS data $(\pi, \mathcal{K}, \Omega)$ associated to Z consists of a Hermitian closable representation of $\mathcal{A}$ on the Krein space $\mathcal{K}$ and a vector $\Omega, \in \mathcal{D}(\pi)$ such that

$$Z(a) = [\pi(a)\Omega, \Omega]_{\mathcal{K}} \tag{23}$$

for all $a \in \mathcal{A}$ and $\bigvee_{a\in\mathcal{A}} \pi(a)\Omega = \mathcal{D}(\pi)$. It was known that not every Hermitian functional Z admits GNS data. Characterizations of those Z that do admit GNS data appeared such as [72], [54], [66]. We first show that the GNS data associated to a Hermitian form can be equivalently described in terms of Kolmogorov decompositions of the kernel K_Z. The proof is straightforward and can be omitted.

Proposition (1.3.6)[53]: Let $\mathcal{A}$ be a unital $*$-algebra, let Z be a linear Hermitian functional on $\mathcal{A}$ with $Z(1) = 1$, and consider the kernel K_Z associated to Z by (20). For every GNS data $(\pi, \mathcal{K}, \Omega)$ of Z define

$$V(a)\lambda = \pi(a^*)\lambda\Omega, \quad a\in\mathcal{A}, \ \lambda\in\mathbb{C}. \tag{24}$$

Then $(V, \mathcal{K})$ is a Kolmogorov decomposition of the Hermitian kernel K_Z and (24) establishes a bijective correspondence between the set of all GNS data of Z and the set of all Kolmogorov decompositions of K_Z.

In particular, Z admits GNS data if and only if the Hermitian kernel K_Z has Kolmogorov decompositions.

As a consequence, Proposition (1.3.6) reduces the characterization of those Hermitian functionals that admit GNS data to Theorem (1.3.4) A different characterization was obtained in Theorem 2 in [66].

Theorem (1.3.7)[53]: Let $\mathcal{A}$ be a unital $*$-algebra and let Z be a linear Hermitian functional on $\mathcal{A}$ with $Z(1) = 1$. Then Z admits GNS data if and only if there exists a positive definite scalar kernel L on $\mathcal{A}$ such that

$$|Z(\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j x_i x_j^*)| \leq \sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j L(x_i, x_j), \quad n \in \mathbb{N}, \{\lambda_i\}_{i=1}^n \subset \mathbb{C}, \{x_i\}_{i=1}^n \subset \mathcal{A}. \qquad (25)$$

Proof. Note that (25) is equivalent to $-L \leq K_Z \leq L$ and then apply Proposition (1.3.6) and Theorem (1.3.4).

We now discuss the uniqueness property of the GNS data, an issue previously addressed in [66], but not completely solved. Two GNS data $(\pi_1, \mathcal{K}_1, \Omega_1)$ and $(\pi_2, \mathcal{K}_2, \Omega_2)$ are unitarily equivalent if there exists a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Phi\mathcal{D}(\pi_1) = \mathcal{D}(\pi_2), \pi_2(a)\Phi = \Phi\pi_1(a)$ for all $a \in \mathcal{A}$, and $\Phi\Omega_1 = \Omega_2$.

Theorem (1.3.8)[53]: Let $\mathcal{A}$ be a unital $*$-algebra and let Z be a linear Hermitian functional on $\mathcal{A}$ with $Z(1) = 1$, admitting GNS data. The following assertions are equivalent:

(i) All GNS data of Z are unitarily equivalent.

(ii) For each positive definite kernel L on $\mathcal{A}$ such that $-L \leq K_Z \leq L$, there exists $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A_L)$ or $(-\epsilon, 0) \subset \rho(A_L)$, where A_L is the Gram operator of K_Z with respect to L.

Proof. Let $(V_i, \mathcal{K}_i)$, $i = 1,2$, be two Kolmogorov decompositions of K_Z that are unitarily equivalent, that is, there exists a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $V_2(x) = \Phi V_1(x)$. Let $(\pi_i, \mathcal{K}_i, \Omega_i)$, $i = 1,2$, be the corresponding GNS data for Z as in Proposition (1.3.6). Then,

$$\mathcal{D}(\pi_2) = \bigvee_{x \in \mathcal{A}} V_2(x)\mathbb{C} = \bigvee_{x \in \mathcal{A}} \Phi V_1(x)\mathbb{C} = \Phi(\bigvee_{x \in \mathcal{A}} V_1(x)\mathbb{C}) = \Phi\mathcal{D}(\pi_1).$$

Also, for $a \in cA$ and $\lambda \in \mathbb{C}$,

$$\pi_2(a)V_1(x)\lambda = \pi_2(a)V_2(x)\lambda = V_2(xa^*)\lambda = \Phi V_1(xa^*)\lambda = \Phi\pi_1(a)V_1(x)\lambda,$$

which implies that $\pi_2(a)\Phi = \Phi\pi_1(a)$. Finally,

$$\Phi\Omega_1 = \Phi V_1(1)1 = V_2(1)1 = \Omega_2,$$

therefore $(\pi_1, \mathcal{K}_1, \Omega_1)$ and $(\pi_2, \mathcal{K}_2, \Omega_2)$ are unitarily equivalent GNS data for Z.

Conversely, let $(\pi_i, \mathcal{K}_i, \Omega_i)$, $i = 1,2$, be two unitarily equivalent GNS data for Z and let $(V_i, \mathcal{K}_i)$, $i = 1,2$, be the Kolmogorov decompositions of K_Z associated to these GNS data by Proposition (1.3.6). Therefore, there exists a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Phi\mathcal{D}(\pi_1) = \mathcal{D}(\pi_2), \pi_2(a)\Phi = \Phi\pi_1(a)$ for all $a \in \mathcal{A}$, and $\Phi\Omega_1 = \Omega_2$.. It follows that

$$V_2(x)\lambda = \pi_2(a^*)\lambda\Omega_2 = \pi_2(a^*)\lambda\Phi\Omega_1 = \pi_2(a^*)\Phi\lambda\Omega_1 = \Phi\pi_1(a^*)\lambda\Omega_1 = \Phi V_1(x)\lambda,$$

which shows that $(V_1, \mathcal{K}_1)$ and $(V_2, \mathcal{K}_2)$ are unitarily equivalent Kolmogorov decompositions of the kernel K_Z. Now, an application of Theorem (1.3.5) concludes the proof.

An example: We exponentials. Let $(\mathcal{H}, [\cdot,\cdot])$ be a Kreˇın space and consider $\mathcal{P}$ the group of its rigid motions. This is the semidirect product of the additive group $\mathcal{H}$ and the group of the bounded unitary operators on $\mathcal{H}$. The group law is given by

$$(\xi, U)(\xi', U') = (\xi + U\xi', UU')$$

and an action of $\mathcal{P}$ on $\mathcal{H}$ can be defined by the formula

$$\phi((\xi, U), \xi') = \xi + U\xi'.$$

In particular, the normal subgroup $\mathcal{H}$ of $\mathcal{P}$ acts on $\mathcal{H}$ by translations. For simplicity, we restrict here to this action by translations. The Hermitian kernel associated to this construction is defined by the formula:

$$K(\xi,\eta) = \exp(\frac{i\Im[\eta,\xi]}{2})\exp(-\frac{[\xi-\eta,\xi-\eta]}{4}), \tag{26}$$
for $\xi,\eta \in \mathcal{H}$. The additive group $\mathcal{H}$ acts on itself by the translations $\phi(\xi,\eta) = \xi + \eta$ and we notice that
$$K(\phi(\xi,\eta),\phi(\xi,\eta')) = \overline{\alpha(\xi,\eta)}\alpha(\xi,\eta')K(\eta,\eta') \quad (25\ )$$
for all $\xi,\eta,\eta' \in \mathcal{H}$, where
$$\alpha(\xi,\eta) = \exp(-\frac{i\Im[\xi,\eta]}{2})$$
and then
$$\sigma(\xi,\eta) = \alpha(\xi,\eta+\eta')^{-1}\alpha(\eta,\eta')^{-1}\alpha(\xi+\eta,\eta') = \exp(\frac{i\Im[\xi,\eta]}{2}).$$
In the terminology to be introduced, it is readily verified that α is a ϕ-multiplier and hence that σ has the 2-cocycle property. Then (27) means that the (scalar) Hermitian kernel K is projectively ϕ-invariant.

We can obtain a Kolmogorov decomposition of the kernel K by adapting the Fock space construction from the positive definite case, similar to the Kolmogorov decomposition that gives the Bose-Fock space (see [63] or [75]).

Proposition (1.3.9)[53]: The kernel K defined by (26) has a Kolmogorov decomposition $(V,\mathcal{K})$ with the property that the operators defined by the formula
$$\alpha(\xi,\eta)W(\xi)V(\eta) = V(\xi+\eta) \tag{28}$$
are defined on the common dense domain $\bigvee_{\xi\in\mathcal{H}} V(\xi)\mathbb{C}$ in $\mathcal{K}$ and satisfy the canonical commutation relations
$$W(\xi)W(\eta) = \sigma(\xi,\eta)W(\xi+\eta). \tag{29}$$

We study properties of the Kolmogorov decompositions of Hermitian kernels with additional symmetries. Let S be a unital semigroup and ϕ an action of S on the set X, this means that $\phi: S \times X \to X, \phi(a,\phi(b,x)) = \phi(ab,x)$ for all $a,b \in S, x \in X$, and $\phi(e,x) = x$, where e denotes the unit element of S. We are interested in those kernels K on X assumed to satisfy a certain invariance property with respect to the action ϕ because this leads to the construction of a representation of S on the space of a Kolmogorov decomposition of K. This kind of construction is well-known for a positive definite kernel (it just extends the construction of the regular representation, see for instance, [75]), but for the Krĕn space setting the question concerning the boundedness of the representation operators is more delicate. We deal with this matter in a more detailed way.

Let α be a ϕ-*multiplier*, that is, a complex-valued function on $S \times X$ such that $\alpha(a,x) \neq 0$ and subject to the following relation:
$$\alpha(ab,x)\overline{\alpha(ab,y)} = \alpha(a,\phi(b,x))\overline{\alpha(a,\phi(b,y))}\alpha(b,x)\overline{\alpha(b,y)} \tag{30}$$
for all $x,y \in X$. This implies that
$$\sigma(a,b) = \alpha(a,\phi(b,x))^{-1}\alpha(b,x)^{-1}\alpha(ab,x)$$
does not depend on x; moreover, $|\sigma(a,b)| = 1$, and σ has the 2-cocycle property:
$$\sigma(a,b)\sigma(ab,c) = \sigma(a,bc)\sigma(b,c) \tag{31}$$
for all $a,b,c \in S$ (see [75] in Lemma 2.2).

For each $a \in S$ we define a projective shift $\psi_a: \mathcal{F}_0(X,\mathcal{H}) \to \mathcal{F}_0(X,\mathcal{H})$ by
$$(\psi_a(f))(x) = \alpha(a,x)^{-1}f(\phi(a,x)),\ \ f \in \mathcal{F}_0(X,\mathcal{H}), x \in X. \tag{32}$$
In terms of the atoms of the vector space $\mathcal{F}_0(X,\mathcal{H})$, ψ_a acts as follows
$$\psi_a^0(\xi_x)\alpha(a,x)^{-1}\xi_{\phi(a,x)} = \left((\alpha(a,x)^{-1}\xi)_{\phi(a,x)}\right), \tag{33}$$

where ξ_x is defined as in (17). This can be used as an alternate definition of ψ_a since each element h of $\mathcal{F}_0(X,\mathcal{H})$ can be uniquely written as a finite sum $h=\sum_{k=1}^{n}\xi_{x_k}^{k}$ for vectors $\xi^1,\ldots,\xi^k \in \mathcal{H}$ and distinct elements $x_1,x_2,\ldots,x_n$ in X and then the projective shift ψ_a^0 is the extension by linearity to a linear map ψ_a, from $\mathcal{F}_0(X,\mathcal{H})$ into $\mathcal{F}_0(X,\mathcal{H})$,

$$\psi_a=\left(\sum_{k=1}^{n}\xi_{x_k}^{k}\right)=\sum_{k=1}^{n}\psi_a^0(\xi_{x_k}^{k}).$$

We say that a positive definite kernel L is projectively ϕ-bounded provided that for all $a\in S$, ψ_a is bounded with respect to the seminorm $[\cdot,\cdot]_L^{1/2}$ induced by L on $\mathcal{F}_0(X,\mathcal{H})$. We denote by $\mathcal{B}_\phi^+(X,\mathcal{H})$ the set of positive definite projectively ϕ-bounded kernels on X with values in $\mathcal{L}(\mathcal{H})$.

In addition, from now on we assume that S is a unital semigroup with involution, that is, there exists a mapping $\mathfrak{J}: S\to S$ such that $\mathfrak{J}(\mathfrak{J}(a))=a\ and\ \mathfrak{J}(ab)=\mathfrak{J}(b)\mathfrak{J}(a)$ for all $a,b\in S$. The connection between the involution $\mathfrak{J}$ and the ϕ-multiplier α is given by the assumption

$$\alpha(a\mathfrak{J}(a),x)=1,\qquad a\in S, x\in X. \tag{34}$$

Finally, with the notation and the assumptions as before, we say that the Hermitian kernel K on X is projectively ϕ -invariant if

$$K\big(x,\phi(a,y)\big)=\overline{\alpha\big(a,\phi(J(a),x)\big)}\alpha(a,y)K(\phi(\mathcal{J}(a),x),y) \tag{35}$$

for all $x,y\in X$ and $a\in S$. In order to keep the terminology simple, the function α and the involution $\mathcal{J}$ will be made each time precise, if not clear from the context. If $\alpha(a,x)=1$ for all $a\in S$ and $x\in X$ then the Hermitian kernel K satisfying (35) is called simply ϕ-ivariant.

Theorem (1.3.10)[53]: Let ϕ be an action of the unital semigroup S with involution $\mathfrak{J}$ satisfying (34) on the set X and let K be an $\mathcal{L}(\mathcal{H})$ -valued projectively ϕ-invariant Hermitian kernel on X.

The following assertions are equivalent:

(i) There exists $L\in \mathrm{B}_\phi^+(X,\mathcal{H})$ such that $-L\le K\le L$.

(ii) K has a Kolmogorov decomposition $(V;\mathcal{K})$ with the property that there exists a projective representation U of S on $\mathcal{K}$(that is, $U(a)U(b)=\sigma(a,b)U(ab)$ for all $a,b\in S$) such that

$$V\big(\phi(a,x)\big)=\alpha(a,x)U(a)V(x) \tag{36}$$

for all $x\in X, a\in S$. In addition, $\overline{\sigma(\mathcal{J}(a),a)}U(\mathcal{J}(a))=U(a)$ for all $a\in S$.

(iii) $K=K_1-K_2$ for two positive definite kernels such that $K_1+K_2\in \mathcal{B}_\phi^+(X,\mathcal{H})$.

(iv) $K=K_+-K_-$ for two disjoint positive definite kernels such that $K_+-K_-\in \mathcal{B}_\phi^+(X,\mathcal{H})$.

Proof. (i)⇒(ii). Let $\mathcal{H}_L$ be the Hilbert space obtained by the completion of the quotient space $\mathcal{F}_0(X,\mathcal{H})/\mathcal{N}_L$ with respect to $[\cdot,\cdot]_L$, where $\mathcal{N}_L=\{f\in\mathcal{F}_0(X,\mathcal{H})\,|\,[f,f]_L=0\}$ is the isotropic subspace of the inner product space $\mathcal{F}_0(X,\mathcal{H}),[\cdot,\cdot]_L)$. Let A_L be the Gram operator of K with respect to L and let $(V;\mathcal{K}_{A_L})$ be the Kolmogorov decomposition of the kernel K described. Since L is ϕ-bounded, it follows that each ψ_a extends to a bounded operator $F(a)$ on $\mathcal{H}_L$.We notice that

$$\begin{aligned}\big[\psi_a(\xi_x),\eta_y\big]_K&=\big[(\alpha(a,x)^{-1}\xi)_{\phi(a,x)},\eta_y\big]_K=\alpha(a,x)^{-1}\big[K\big(y,\phi(a,x)\big)\xi,\eta\big]_{\mathcal{H}}\\&=\overline{\alpha(a,\phi(\mathcal{J}(a),y))}[K(\phi(\mathcal{J}(a),y),x)\xi,\eta]_{\mathcal{H}}\\&=\overline{\alpha(a,\phi(\mathcal{J}(a),y))\alpha(\mathcal{J}(a),y)}[\xi_x,\psi_{\mathcal{J}(a)}(\eta_y)]_K\end{aligned}$$

From the definition of σ we have that for $y \in X$,

$$\sigma(a,\mathcal{J}(a)) = \alpha(a,\phi(\mathcal{J}(a),y))^{-1}\alpha(\mathcal{J}(a),y)^{-1}\alpha(a\mathcal{J}(a),y).$$

By our assumption (3.5), $\alpha(a\mathcal{J}(a),y) = 1$, so that

$$\sigma(a,\mathcal{J}(a)) = \alpha(a,\phi(\mathcal{J}(a),y))^{-1}\alpha(\mathcal{J}(a),y)^{-1}.$$

Since $|\sigma(a,\mathcal{J}(a))| = 1$, we deduce that

$$[\psi_a(\xi_x),\eta_y]_K = \sigma(a,\mathcal{J}(a))[\xi_x,\psi_{\mathcal{J}(a)}(\eta_y)]_K.$$

This relation can be extended by linearity to

$$[\psi_a(f),g]_K = \sigma(a,\mathcal{J}(a))[f,\psi_{\mathcal{J}(a)}(g)]_K$$

for all $f,g \in \mathcal{F}_0(X,\mathcal{H})$.We deduce that

$$[A_L\psi_a(f),g]_L = \sigma(a,\mathcal{J}(a))[A_Lf,\psi_{\mathcal{J}(a)}(g)]_L,$$

which implies that

$$A_LF(a) = \sigma(a,\mathcal{J}(a))F(\mathcal{J}(a))^*A_L. \tag{37}$$

Theorem (1.3.1) implies that there exists a unique operator $U(a) \in \mathcal{L}(\mathcal{K}_{A_L})$ such that

$$U(a)\Pi_{A_L} = \Pi_{A_L}F(a).$$

Moreover, for $h \in \mathcal{H}_L$,

$$U(a)U(b)\Pi_{A_L}h = U(a)\Pi_{A_L}F(b)h = \Pi_{A_L}F(a)F(b)h.$$

We also notice that

$$\psi_a\psi_b(\xi_x) = \psi_a\big(\alpha(b,x)\xi_{\phi(b,x)}\big) = \alpha(b,x)^{-1}\alpha(a,\phi(b,x))^{-1}\xi_{\phi(a,\phi(b,x))}$$
$$= \sigma(a,b)\alpha(ab,x)^{-1}\xi_{\phi(ab,x)} = \sigma(a,b)\psi_{ab}(\xi_x).$$

We deduce that $F(a)F(b) = \sigma(a,b)F(ab)$ and this relation implies that

$$U(a)U(b)\Pi_{A_L}h = \sigma(a,b)U(ab)\Pi_{A_L}h.$$

Since the set $\{\Pi_{A_L}h | h \in \mathcal{H}_L\}$ is dense in $\mathcal{K}_{A_L}$, we deduce that U is a projective representation of S on $\mathcal{K}_{A_L}$.

For $\xi \in \mathcal{H}$ we have

$$V(\phi(a,x))\xi = \Pi_{A_L}[\xi_{\phi(a,x)}]$$

and

$$U(a)V(x)\xi = U(a)\Pi_{A_L}[\xi_x] = \Pi_{A_L}F(a)[\xi_x].$$

Since $\psi(\xi_x) = \alpha(a,x)^{-1}\xi_{\phi(a,x)}$, we deduce that

$$F(a)[\xi_x] = \alpha(a,x)^{-1}[\xi_{\phi(a,x)}],$$

so that (36) holds.

Finally, the relation (37) implies that

$$[U(a)\Pi_{A_L}f,\Pi_{A_L}g]_{\mathcal{K}_{A_L}} = [\Pi_{A_L}F(a)f,\Pi_{A_L}g]_{\mathcal{K}_{A_L}} = [A_LF(a)f,g]_L$$
$$= \sigma(a,\mathfrak{I}(a))[F(\mathfrak{I}(a))^*A_Lf,g]_L = \sigma(a,(a))[f,F((a))g]_L$$
$$= \sigma(a,\mathfrak{I}(a))[\Pi_{A_L}f,\Pi_{A_L}F(\mathfrak{I}(a))g]_{\mathcal{K}_{A_L}}$$
$$= \sigma(a,\mathfrak{I}(a))[\Pi_{A_L}f,U(\mathfrak{I}\mathfrak{I}(a))\Pi_{A_L}g]_{\mathcal{K}_{A_L}}$$

for all $f,g \in \mathcal{H}_L$, which implies that $\overline{\sigma(a,\mathfrak{I}(a))}U(\mathfrak{I}(a)) = U(a)^\#$.We now notice that the relation (35) implies that $\sigma(a,\mathfrak{I}(a)) = \sigma(\mathfrak{I}(a),a)$, which concludes the proof of the relation $\overline{\sigma(a,\mathfrak{I}(a))}U(\mathfrak{I}(a)) = U(a)^\#$ for all $a \in S$.

(2) ⇒ (6). Let J be a fundamental symmetry on $\mathcal{K}$. Then J is a selfadjoint operator with respect to the positive definite inner product $\langle h,g\rangle_J = [Jh,g]_{\mathcal{K}}$. Let $J = J_+ - J_-$ be the Jordan decomposition of J and define the Hermitian kernels

$$K_\pm(x,y) = \pm V(x)^\# J \pm V(y)\,,\ L(x,y) = V(x)^\# JV(y), \quad x,y \in X.$$

From $J_+ + J_- = I$ and $\pm J_\pm = J_\pm J J_\pm$ we get $K(x,y) = K_+(x,y) - K_-(x,y)$ and $L(x,y) = K_+(x,y) + K_-(x,y)$. To prove that K_+ and K_- are positive definite kernels let $h \in \mathcal{F}_0(X,\mathcal{H})$. Then

$$\sum_{x,y\in X} [K_\pm(x,y)h(y), h(x)]_{\mathcal{H}} = \sum_{x,y\in X} [\pm V(x)^\sharp J_\pm V(y)h(y), h(x)]_{\mathcal{H}}$$

$$= \sum_{x,y\in X} [\pm J_\pm V(y)h(y), V(x)h(x)]_{\mathcal{K}} = \sum_{x,y\in X} [J_\pm J J_\pm V(y)h(y), V(x)h(x)]_{\mathcal{K}}$$

$$= \sum_{x,y\in X} \langle J_\pm V(y)h(y),\ J_\pm V(x)h(x)\rangle_J = \| \sum_{x\in X} J_\pm V(x)h(x) \|_J^2 \geq 0.$$

It remains to show that L is ϕ-bounded. If $h \in \mathcal{F}_0(X,\mathcal{H})$, then $h = \sum_{k=1}^n \xi_{x_k}^k$ for some $n \in \mathbb{N}$, vectors $\xi^1, \dots, \xi^n \in \mathcal{H}$ and distinct elements $x_1, x_2, \dots, x_n$ in X. Then

$$[\psi_a(h), \psi_a(h)]_L = \sum_{j,k=1}^n [\psi_a, (\xi_{x_j}^j)\psi_a(\xi_{x_k}^k)]_L$$

$$= \sum_{j,k=1}^n \alpha(a,x_j)^{-1}\overline{\alpha(a,x_k)^{-1}}[\xi_{\phi(a,x_j)}^j, \xi_{\phi(a,x_k)}^k]_L$$

$$= \sum_{j,k=1}^n \alpha(a,x_j)^{-1}\overline{\alpha(a,x_k)^{-1}}[L(\phi(a,x_k),\phi(a,x_j))\xi^j\xi^k]_{\mathcal{H}}$$

$$= \sum_{j,k=1}^n \alpha(a,x_j)^{-1}\overline{\alpha(a,x_k)^{-1}}\langle V\phi(a,x_j))\xi^j, V(\phi(a,x_k))\xi^k\rangle_J$$

$$= \sum_{j,k=1}^n \langle U(a)V(x_j)\xi^j, U(a)\,V(x_k))\xi^k\rangle_J = \| U(a)\sum_{k=1}^n V(x_k))\xi^k \|_J^2$$

$$\leq \| U(a) \|_J^2 \| \sum_{k=1}^n V(x_k))\xi^k \|_J^2 = \| U(a) \|_J^2 \sum_{j,k=1}^n \langle V(x_j)\xi^j, V(x_k))\xi^k\rangle_J =$$

$$\| U(a) \|_J^2 \sum_{j,k=1}^n [\xi_{x_j}^j, \xi_{x_k}^k]_L = \| U(a) \|_J^2\, [h,h]_L$$

so that L is ϕ-bounded.

We also deduce that $(V, (\mathcal{K}, \langle\cdot,\cdot\rangle_J))$ is the Kolmogorov decomposition of the positive definite kernel L and $J_\pm$V,($J_\pm\mathcal{K}, \langle\cdot,\cdot\rangle_J))$ is the Kolmogorov decomposition of $K_\pm$. Since $J_+J_- = 0$ we deduce that $J_+\mathcal{K} \cap J_-\mathcal{K} = \{0\}$ and, by Proposition 16, in [78] we deduce that K_+ and K_- are disjoint kernels.

Since (vi)⇒(iii) and (iii)⇒(i) are obvious implications, the proof is complete.

A Kolmogorov decomposition $(V,\mathcal{K})$ of the Hermitian kernel K for which there exists a projective representation U such that (36) holds is called a projectively invariant Kolmogorov decomposition. Also, a projective representation U satisfying the additional property $U(a)^\sharp = \overline{\sigma(\Im(a),a)}U(\Im(a))$ for all $a \in S$, is called symmetric projective representation.

A natural question that can be raised in connection with the previous result is whether $B_\phi^+(X,\mathcal{H})$ is a sufficiently rich class of kernels

Proposition (1.3.11)[53]: Assume that S is a group and $\Im(a) = a^{-1}$, $a \in S$. If K is a projectively ϕ-invariant Hermitian kernel on X then, for any $a \in S$ the operator ψ_a is isometric with respect to the inner product $[.,.]_K$. In particular, any projectively ϕ-invariant positive definite kernel on X belongs to $B_\phi^+(X,\mathcal{H})$.

Proof. Indeed, in this case (34) becomes $\alpha(e,x) = 1$ for all $x \in X$, where e is the unit of the group S. Also, if K is a Hermitian kernel then it is projectively ϕ-invariant if and only if

$$K(\phi(a,x),\phi(a,y)) = \overline{\alpha(a,x)}\alpha(a,y)K(x,y), \qquad x,y \in X, a \in S.$$

Let $\xi,\eta \in \mathcal{H}$ be arbitrary. Then

$$[\psi_a(\xi_x),\psi_a(\eta_y)]_K = \alpha(a,x)^{-1}\overline{\alpha(a,y)}^{-1}[\xi_{\phi(a,x)},\eta_{\phi(a,y)}]_K$$
$$= \alpha(a,x)^{-1}\overline{\alpha(a,y)}^{-1}[K(\phi(a,y),\phi(a,x))\xi,\eta]_{\mathrm{H}} = [K(y,x)\xi,\eta]_{\mathrm{H}}$$
$$= [\xi_x,\eta_y]_K,$$

and hence ψ_a is $[.,.]_K$ isometric.

Remark (1.3.12)[53]: (i) Theorem (1.3.10) is known when $\mathcal{H}$ is a Hilbert space and the kernel K is positive definite and satisfies

$$K(\phi(a,x),\phi(a,y)) = \overline{\alpha(a,x)}\alpha(a,y)K(x,y), \qquad a \in S, x,y \in X. \tag{38}$$

(see, [75]). In that case the proof is easily obtained by defining directly

$$U(a)V(x)\xi = \alpha(a,x)^{-1}V(\phi(a,x))\xi \tag{39}$$

for $\xi \in \mathcal{H}$ and verify that $U(a)$ satisfies all the required properties (we note that no involution is considered in this case). We have to emphasize that this direct approach does not work in the Hermitian case since the formula (39) does not necessarily give a bounded operator. In order to overcome this difficulty we have to replace the symmetry condition in (38) by the symmetry condition in (35) and then use Theorem (1.3.1). This was the main point in the proof of Theorem (1.3.10).

(ii) The positive definite version of Theorem (1.3.10) has many applications, some of them mentioned for instance in [63], [64], and [75]. Such a typical application gives a Naimark dilation for Toeplitz kernels. Thus, if $X = S, \phi(a,x) = ax$, and $\alpha(a,x) = 1$ for all $a,x \in S$, then (38) becomes the well-known Toeplitz condition

$$K(ab,ac) = K(b,c)$$

for all $a,b,c \in S$. If K is a positive definite kernel on S satisfying the Toeplitz condition and $(e,e) = I$, where e is the unit of S, then $\{U(a)\}_{a\in S}$ defined by (39) is a semigroup of isometries on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that

$$K(a,b) = P_{\mathcal{H}}U(a)^*U(b)|\mathcal{H},$$

for all $a,b \in S$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

(iii) The next example explores the fact that for positive definite kernels the representation $\{U(a)\}_{a\in S}$ given by (39) is unique up to unitary equivalence. Thus, consider the action of a group G on the Hilbert space $\mathcal{H}$ such that $\langle\phi(g,\xi),\phi(g,\eta)\rangle = \langle\xi,\eta\rangle$ for all $g \in G$ and $\xi,\eta \in \mathcal{H}$. We consider the kernel $K(\xi,\eta)=\eta,\xi$ on $\mathcal{H}$ and notice that K is positive definite. Its Kolmogorov decomposition is given by $V(\xi)\colon \mathbb{C} \to \mathcal{H}$, $(\xi)\lambda = \lambda\xi$, $\lambda \in \mathbb{C}, \xi \in \mathcal{H}$. If we use the positive definite version of Theorem (1.3.10), we deduce that there exists a Kolmogorov decomposition V' of K and a representation U' of G such that $V'(\phi(g,\xi)) = U'(g)V'(\xi)$ for all $g \in G$ and $\xi \in \mathcal{H}$. From the uniqueness of V up to unitary equivalence, it follows that there exists a unitary operator Φ such that $V(\phi(g,\xi)) = \Phi U'(g)\Phi^*V(\xi)$, or $\phi(g,\xi) = U(g)\xi$, with $\Phi U'(g)\Phi^*$. Therefore we obtained the well-known result that ϕ acts by linear unitary operators.

The last example was intended to emphasize the importance of the uniqueness up to unitary equivalence of the projectively invariant Kolmogorov decompositions. This issue turns out to be rather delicate in the Hermitian case. Theorem (1.3.5) settles this question only in the case of the trivial semigroup S with one element. It is easily seen that the spectral condition in Theorem (1.3.5) is also sufficient for the uniqueness of a projectively invariant Kolmogorov decomposition. However, Theorem (1.3.2) shows that the proof in [58] of Theorem (1.3.5) cannot be easily adapted to the case of an arbitrary semigroup S.

Given a Hermitian kernel K, the rank rank(K) is, by definition, the supremum of rank (K_Δ) taken over all finite subsets $\Delta\subset X$, where (K_Δ) is the restricted kernel $(K(x,y))_{x,y\in\Delta}$. By definition rank(K) is either a positive integer or the symbol ∞.A Hermitian kernel K has κ negative squares if the inner product space $(\mathcal{F}_0(X,\mathcal{H})[.,.]_K)$ has negative signature κ, that is, κ is the maximal dimension of all its negative subspaces. It is easy to see that this is equivalent to $K=K_+-K_-$, where $K_\pm$ are disjoint positive definite kernels such that rank$(K_-)=\kappa$, see e.g. [78]. This allows us to define $\kappa^-(K)=\kappa$, the number of negative squares of the kernel K. In particular, Hermitian kernels with a finite number of negative squares always have Kolmogorov decompositions and for any Kolmogorov decomposition $(V;\mathrm{K})$ of K we have $\kappa^-(\mathrm{K})=\kappa^-(K)<\infty$, hence K is a Pontryagin space with negative signature κ.

In Pontryagin spaces the strong topology is intrinsically characterized in terms of the indefinite inner product, e.g. see [65]. Therefore, by using Proposition (1.3.11) and Shmul'yan's Theorem (e.g. see Theorem 2.10 in [62]) we get:

Theorem (1.3.13)[53]: Let ϕ be an action of the group S on the set X and let K be an $\mathcal{L}(\mathcal{H})$-valued projectively ϕ-invariant Hermitian kernel on X with a finite number of negative squares. Then K has a projectively invariant Kolmogorov decomposition on a Pontryagin space that is unique up to unitary equivalence.

The symmetric projective representation U of S obtained in Theorem (1.3.10) acts on a Krein space. It would be of special interest to decide whether U is at least similar to a symmetric projective representation on a Hilbert space, a property related to the well-known similarity problem for group representations, see [76].

The above mentioned problem is also closely related to the characterization of those ϕ-invariant Hermitian kernels K with the property that the representation $K=K_+-K_-$ holds for two positive definite ϕ-invariant kernels.

We give an answer to these two questions in terms of fundamental reducibility. We say that the projective representation U of S on the Krein space K is fundamentally reducible if there exists a fundamental symmetry J on K such that $U(a)J=JU(a)$ for all $a\in S$. This condition is readily equivalent to the condition $U(a)^\# = U(a)^*$ for all $a\in S$, and further, equivalent to the diagonal representation of $U(a)$ with respect to a fundamental decomposition of the Krein space K.

Proposition (1.3.14)[53]: Let S be a semigroup with involution $\mathfrak{I}$ and σ satisfies the 2-cocycle property (31) on S. Let U be a symmetric projective representation of S on the Krein space K. Then the following assertions are equivalent:

(i) U is similar to a symmetric projective representation T on a Hilbert space.

(ii) U is fundamentally reducible.

Proof. (i)⇒(ii). Let $\Phi\in\mathcal{L}(\mathrm{K},\mathcal{G})$ be the similarity such that $T(a)\Phi=\Phi U(a)$ for $a\in S$.We first notice that Φ is also an involutory similarity (with the terminology from [67]), that is

$$T(a)^*=\Phi U(a)^\#\Phi^{-1}, \qquad a\in S. \tag{40}$$

Then, we consider on $\mathcal{K}$ the positive inner product $\langle \xi, \eta \rangle_\Phi = \langle \Phi\xi, \Phi\eta \rangle$, $\xi, \eta \in \mathcal{K}$. Since Φ is boundedly invertible, there exists a selfadjoint and boundedly invertible operator $G \in \mathcal{L}(\mathcal{K})$ such that $[\xi, \eta] = \langle G\xi, \eta \rangle_\Phi, \xi, \eta \in \mathcal{K}$. Therefore, for arbitrary $a \in S$ and $\xi, \eta \in \mathcal{K}$ we have

$$\begin{aligned}\langle U(a)\xi, \eta \rangle_\Phi &= \langle \Phi U(a)\xi, \Phi\eta \rangle = \langle \mathrm{T(a)}\Phi\xi, \Phi\eta \rangle \\ &= \langle \Phi\xi, \mathrm{T(a)}^*\Phi\eta \rangle = \langle \Phi\xi, \Phi U(a)^\#\eta \rangle \\ &= \langle \xi, U(a)^\#\eta \rangle_\Phi = [\mathrm{G}^{-1}\xi, U(a)^\#\eta] \\ &= [U(a)G^{-1}\xi, \eta] = \langle GU(a)G^{-1}\xi, \eta \rangle_\Phi\end{aligned}$$

Thus, $GU(a) = U(a)G$ and letting $J = sgn(G)$ it follows that J is a fundamental symmetry on the Krein space $\mathcal{K}$ such that $JU(a) = U(a)J$.

(ii)⇒(i). If J is a fundamental symmetry on the Krein space $\mathcal{K}$ such that $JU(a) = U(a)J$, for all $a \in S$, then U is a symmetric projective representation with respect to the Hilbert space $(\mathcal{K}\langle .,. \rangle_J)$.

With the notation as in Proposition (1.3.14), if σ has the 2-cocycle property (31) and $|\sigma(a,b)| = 1$ for all $a, b \in S$, then it follows that

$$U(a)^\# U(a) = U(\mathfrak{I}(a)a), \qquad a \in S \tag{41}$$

Thus, in certain applications where U consists of (Krein space) isometric operators, it is interesting to know whether U is similar to a symmetric projective representation of isometric operators on a Hilbert space. Clearly, a necessary condition is that for some (equivalently for all) unitary norm $\|\cdot\|$ on $\mathcal{K}$ there exists $C > 0$ such that

$$\frac{1}{C}\|\xi\| \le \|U(a)\xi\| \le C\|\xi\|, \quad a \in S, \xi \in \mathcal{K}. \tag{42}$$

As expected, the converse implication is related to the assumption of amenability of the semigroup S. More precisely, following closely the idea in the proof of Théorème 6 in [61], we get:

Theorem (1.3.15)[53]: Let S be an amenable semigroup, σ has the 2-cocycle property (31), $|\sigma(a,b)| = 1$ for all $a, b \in S$, and let U be a projective representation (without any assumption of symmetry) of S on a Hilbert space $\mathcal{K}$, such that (42) holds for some constant $C > 0$. Then U is similar to a projective representation T of S on a Hilbert space $\mathcal{G}$ such that $T(a)$ are isometric for all $a \in S$.

We come now to the problem of characterizing those Hermitian invariant kernels that can be represented as a difference of two positive invariant kernels.

Theorem (1.3.16)[53]: Let ϕ be an action of the unital semigroup S with involution $\mathfrak{I}$ satisfying (34) on the set X and let K be an $\mathcal{L}(\mathcal{H})$-valued ϕ-invariant Hermitian kernel on X. The following assertions are equivalent:

(i) There exists $L \in B_\phi^+(X,\mathcal{H})$ such that $-L \le K \le L$ and L is ϕ-invariant.

(ii) K has a projectively invariant Kolmogorov decomposition $(V;\mathcal{K})$ such that the associated projective representation is fundamentally reducible.

(iii) $K = K_+ - K_-$ for two disjoint positive definite kernels such that $K_+ + K_- \in B_\phi^+(X,\mathcal{H})$ and both $K_\pm$ are ϕ-invariant.

Proof. (i)⇒(ii). We use the same notation as in the proof of Theorem (1.3.10). Thus, $\mathcal{H}_L$ is the Hilbert space obtained by the completion of the quotient space $(\mathcal{F}_0(X,\mathcal{H}))/\mathcal{N}_L$ with respect to $[\cdot,\cdot]_L$, where $\mathcal{N}_L$ is the isotropic subspace of the inner product space $(\mathcal{F}_0(X,\mathcal{H}), [\cdot,\cdot]_L)$. Let A_L be the Gram operator of K with respect to L and let $(V;\mathcal{K}_{A_L})$ be the projectively invariant Kolmogorov decomposition of the kernel K described in the proof of (i)⇒(ii) in Theorem (1.3.10). Since L is ϕ-bounded, it follows that each ψ_a extends to a bounded operator $F(a)$ on $\mathcal{H}_L$. Since L is ϕ-invariant, we deduce that

$$[\psi_a(f), g]_L = \sigma(a, \Im(a))[f, \psi_{\Im(a)}(g)]_L$$
for all $f, g \in \mathcal{F}_0(X, \mathcal{H})$, which implies that
$$F(a) = \sigma(a, \Im(a))F\,(\Im(a))^*.$$
This relation and (37) imply that
$$A_L F(a) = F(a)A_L$$
for all $a \in S$. Let $A_L = S_{A_L}|A_L$ be the polar decomposition of A_L and let J_{A_L} be the symmetry introduced in Example (1.3.1). Using (8), we deduce that
$$\begin{aligned} U(a)J_{A_L}\Pi_{A_L} &= U(a)\Pi_{A_L}S_{A_L} \\ &= \Pi_{A_L}F(a)S_{A_L} \\ &= \Pi_{A_L}\, S_{A_L}F(a) \\ &= J_{A_L}\Pi_{A_L}F(a) \\ &= J_{A_L}U(a)\Pi_{A_L}, \end{aligned}$$
therefore the representation U is fundamentally reducible.

(ii)⇒(iii). We consider the elements involved in the proof of (2)⇒(6) in Theorem (1.3.10) for a fundamental symmetry J on K for which $U(a)J = JU(a), a \in S$. Therefore $U(a)J_\pm = J_\pm U(a)$ for all $a \in S$, and then
$$\begin{aligned} K_\pm(x, \phi(a, y)) &= \pm V(x)^\# J_\pm V\big(\phi(a, y)\big) \\ &= \pm\alpha(a, y)V(x)^\# J_\pm U(a)V(y) \\ &= \pm\alpha(a, y)V(x)^\# J_\pm U(a)V(y) \\ &= \pm\alpha(a, y)\overline{\sigma(\,\Im(a), a)}V(x)^\# U(\Im(a))^\# J_\pm V(y) \\ &= \pm\overline{\alpha(a, \phi\,(\Im(a), x))}\alpha(a, y)V(\phi(\Im(a)x))^\# J_\pm V(y) \\ &= \overline{\alpha(a, \phi\,(\Im(a), x))}\alpha(a, y)K_\pm(\phi(\Im(a)x), y) \end{aligned}$$
(iii)⇒(i). Just set $L(x, y) = K_+(x, y) + K_-(x, y)$.

In case S is a group with the involution $\Im(a) = a^{-1}$, then some of the assumptions in the previous results simplify to a certain extent. In this case, as a consequence of (41), the symmetric projective representation U associated to a ϕ-invariant Kolmogorov decomposition consists of unitary operators.

Theorem (1.3.17)[53]: Let S be a group and σ a 2-cocycle on S with $|\sigma(a, b)| = 1$ for all $a, b \in S$. Let U be a unitary projective representation of S on the Krein space K. Then the following assertions are equivalent:

(i) U is similar to a unitary projective representation T on a Hilbert space, that is, $T: S \to \mathcal{L}(\mathcal{G})$, $\mathcal{G}$ a Hilbert space, $T(a)T(b) = \sigma(a, b)T(ab)$ and $T(a)^* = \sigma(a^{-1}, a)T(a^{-1})$ for all $a \in S$.

(ii) U is fundamentally reducible.

Moreover, if U satisfies one (hence both) of the assumptions (i) and (ii) then U is uniformly bounded, that is,
$$\sup_{a \in S}\|U(a)\| < \infty. \tag{43}$$
If, in addition, S is amenable, then (43) is equivalent to (any of) the conditions (i) and (ii).

Proof. This follows from Proposition (1.3.14) and Theorem (1.3.15).

Theorem (1.3.18)[53]: Let ϕ be an action of the group S on the set X and let K be an $\mathcal{L}(\mathcal{H})$-valued ϕ-invariant Hermitian kernel on X.

The following assertions are equivalent:

(i) There exists a ϕ-invariant positive definite L on X such that $-L \le K \le L$.

(ii) K has a projectively invariant Kolmogorov decomposition $(V;\mathcal{K})$ such that the associated symmetric projective representation is similar to a symmetric projective representation on a Hilbert space.

(iii) $K = K_+ - K_-$ for two disjoint positive definite ϕ-invariant kernels.

Proof. This follows from Proposition (1.3.11) and Theorem (1.3.16).

Another consequence of the Kolmogorov decomposition approach is the possibility of obtaining a characterization of those Hermitian functionals Z that admit bounded GNS data, that is, the representation π is made of bounded operators.

Theorem (1.3.19)[53]: Let $\mathcal{A}$ be a unital $*$-algebra and let Z be a linear Hermitian functional on $\mathcal{A}$ with $Z(1) = 1$. Then Z admits bounded GNS data if and only if there exists a positive definite scalar kernel L on $\mathcal{A}$ having the property (25) and such that for every $a \in \mathcal{A}$ there exists $C_a > 0$ with the property that

$$\sum_{i.j=1}^{n} \lambda_i \bar{\lambda}_j L(x_i a^*, x_j a^*) \leq C_a \sum_{i.j=1}^{n} \lambda_i \bar{\lambda}_j L(x_i, x_j), n \in \mathbb{N}, \ \{\lambda_i\}_{i=1}^n \subset \mathbb{C}, \ \{x_i\}_{i=1}^n \subset \mathcal{A}.$$

Proof. This is a consequence of Theorem (1.3.10) and Proposition (1.3.6).

We conclude with a discussion of the Jordan decomposition of a linear Hermitian functional on a $*$-algebra $\mathcal{A}$, that is, the possibility of writing the Hermitian functional as the difference of two positive functionals. Let us first note that a functional $F: \mathcal{A} \to \mathbb{C}$ is positive, that is, $F(a^*a) \geq 0$ for all $a \in \mathcal{A}$, if and only if the kernel K_F associated to F by the formula (20) is positive definite. Also, if F is a positive functional on $\mathcal{A}$, then K_F is ϕ-bounded, with the action ϕ defined as in (19), if and only if for any $a \in \mathcal{A}$ there exists $C_a > 0$ such that

$$F(xa^*ax^* \leq C_a F(xx^*), \quad x \in \mathcal{A}. \tag{44}$$

For simplicity, we call the positive functional F ϕ-bounded if K_F is ϕ-bounded. Let F_1, F_2 be two positive functionals on the $*$-algebra $\mathcal{A}$. Then $F_1 \leq F_2$, by definition, if $F_2 - F_1$ is a positive functional. It is easy to see that $F_1 \leq F_2$ if and only if $K_{F_1} \leq K_{F_2}$. The functionals F_1 and F_2 are called disjoint if their associated kernels K_{F_1} and K_{F_2} are disjoint.

Theorem (1.3.20)[53]: Let $\mathcal{A}$ be a unital $*$-algebra, let Z be a linear Hermitian functional on $\mathcal{A}$ with $Z(1) = 1$, and let ϕ be the action given by (19). The following assertions are equivalent:

(i) There exists a linear positive ϕ-bounded functional Z_0 on $\mathcal{A}$ such that $-Z_0 \leq Z \leq Z_0$.

(ii) Z admits bounded GNS data $(\pi, \mathcal{K}, \Omega,)$ such that the representation π is similar with a $*$-representation on a Hilbert space.

(iii) $Z = Z_+ - Z_-$ for two disjoint linear positive definite functionals on $\mathcal{A}$ with the property that $(Z_+ + Z_-)$ is ϕ-bounded.

Proof. The implications (i)⇒(ii)⇒(iii) are direct consequences of Theorem (1.3.16) and Proposition (1.3.6). For (iii)⇒(i) we use the proof of Theorem (1.3.10) in order to deduce that there exists $L \in B_\phi^+(\mathcal{A}, \mathbb{C})$ such that $-L \leq K_Z \leq L$. Then Theorem (1.3.16) shows that $L(x, \phi(a, y)) = L(\phi(a^*, x), y)$ for all $x, y, a \in \mathcal{A}$. Also, in this case, L is linear in the first variable (hence, antilinear in the second variable). If we define

$$Z_0(x) = L(x, 1)$$

for $x \in \mathcal{A}$, then Z_0 is a linear functional on $\mathcal{A}$ and

$$K_{Z_0}(x, y) = Z_0(xy^*) = L(xy^*, 1) = L(x, y).$$

Now all the required properties of Z_0 follow from the corresponding properties of L.

Chapter 2
C^*-Algebras

We deal with some questions concerning the dual spaces of noncommutative C*-algebras, especially the group C*-algebras of certain groups. We also find a number of other conditions which for separable C^*-algebras are equivalent to being type I.

Section (2.1): The Dual Spaces

The idea of the structure space (or dual space) $\hat{A}$ of an associative algebra A was introduced by Jacobson in [88]. The space $\hat{A}$ consists of all kernels of irreducible representations of A, with the hull-kernel topology: An ideal I in $\hat{A}$ is in the closure of a subset B of $\hat{A}$ if I contains the intersection of the ideals in B. For unrestricted infinite-dimensional A, the dual space need not be Hausdorff or even T_1; and in many situations it is not very useful. However, Gelfand and others have shown that for commutative Banach algebras the dual space is a powerful tool. For noncommutative Banach algebras, too, the study of the dual space has been found fruitful. Kaplansky [92] has analyzed the dual spaces of C^*-algebras whose irreducible $*$-representations all consist of completely continuous operators. The importance of this study is emphasized by the fact that the group algebras of connected semi-simple Lie groups having faithful matrix representations all fall into this category (see [87]).

We deal with some questions concerning the dual spaces of noncommutative C^*-algebras, especially the group C^*-algebras of certain groups.

This is a theorem specifically about C^*-algebras. It states that, if $\mathcal{S}$ is a family of $*$-representations of a C^*-algebra A, and T is a $*$-representation of A which vanishes for those elements for which all S in $\mathcal{S}$ vanish, then positive functionals associated with T are weakly $*$-approximated by sums of positive functionals associated with $\mathcal{S}$. In another form, it states a one-to-one correspondence between closed two-sided ideals of a C^*-algebra and certain subsets of the positive cone of its conjugate space. In the latter form, the theorem was communicated to this author by R. Prosser, who also suggested the short proof of Theorem (2.1.1) given here. An interesting corollary of this theorem is the following: If G is a locally compact group, the hull-kernel topology of the dual space of its group C^*-algebra is equivalent to the topology which Godement defined in [85] for the space $\hat{G}$ of irreducible unitary representations of G, using functions of positive type. Let us refer to this simply as the topology of $\hat{G}$.

The equivalence theorem leads naturally to the ideas of weak containment and weak equivalence. Theorem (2.1.15) shows that every set of representations of a C^*-algebra A is weakly equivalent to a unique closed set of irreducible representations (compare the definition of the spectrum of a positive functional, in [85]). Theorem (2.1.17) relates weak equivalence to the construction of continuous direct integrals of representations. Theorem (2.1.20) is a digression, and relates the topology of $\hat{A}$ to the condition that a discrete direct sum of completely continuous representations be completely continuous. It should be noted that the elements of $\hat{A}$ are the, (topologically) irreducible $*$-representations of A, rather than the kernels of these.

From Kaplansky's observation [92] that the Hausdorff property of the dual space $\hat{A}$ of a C^*-algebra A is related to the continuity of the real-valued functions $T \to \|T_x\|$ $(T \in \hat{A}, x$ fixed). We ask what is the relation between the topology of $\hat{A}$ and the functions $T \to \|T_x\|$ for an arbitrary C^*-algebra A. The answer is Theorem (2.1.24). We also ask how the topology of $\hat{A}$ is related to the functions $T \to$ Trace (T_x) (supposing that Trace (T_x) exists for many x and T). Theorems (2.1.29) and (2.1.36) are partial answers to this question. We generalize Kaplansky's result [92] that a C^*-algebra A all of whose irreducible representations have the same finite dimension has a Hausdorff dual space. In fact, we show that if $\{T^i\}$ is a net of elements of $\hat{A}$, all of which are of dimension $\leqq n$, and if $T^i \to S^m$ $(m = 1, \dots, r)$, where $S^1, \dots, S^r$ are distinct elements of $\hat{A}$, then $\sum_{m=1}^{r} \dim S^m \leqq n$.

To calculate explicitly the topologies of the duals of the $n \times n$ complex unimodular groups G (all of whose irreducible representations together with their characters, are listed in [84]). The result is Theorem (2.1.49). The topologies are not Hausdorff, though their deviations from this property are rather weak (see Corollaries (2.1.50) and (2.1.56) of Theorem (2.1.49)). To illustrate, we recall that in the 2×2 case the elements of $\hat{G}$ fall into three classes: (i) the principal series of representations $T^{m,r}$ (m an integer, r real), (ii) the supplementary series of representations $T^s (0 < s < 1)$, and (iii) the identity representation I. Now the topology of $\hat{G}$ is the natural topology of the parameters with one exception: as $s \to 1-$, T^s converges both to I and to $T^{2,0}$. This failure of the Hausdorff property stems from the behavior of the characters. If $\Upsilon^{m,r}, \Upsilon^s, \Upsilon^I$ are the characters of $T^{m,r}$, T^s, and I respectively, it arises from the fact that

$$\lim_{s \to 1-,} \Upsilon^s = \Upsilon^I + \Upsilon^{2,0}.$$

A further fact about $\hat{G}$, true for all n, is that each principal series is closed in $\hat{G}$. This has the interesting consequence (Theorem (2.1.59) that the regular representation of G weakly contains the representations of the principal nondegenerate series, and no others.

Theorem (2.1.1)[80]: Let A be any norm-closed self-adjoint algebra of operators on a Hilbert space H. Then any continuous positive linear functional ϕ on A can be approximated in the weak* topology ($i.e.$, pointwise on A) by natural positive functionals on A, that is, positive functionals ψ of the form

$$\psi(a) = \sum_{i=1}^{k} (ax_i, x_i) \qquad (x_i \in H).$$

In fact, the approximating functionals ψ may be assumed to have norm equal to or less than $\|\phi\|$.

Proof. Let N he the family of all natural positive functionals ψ on A lor which $\|\psi\| \leqq 1$, considered as a subset of the conjugate space of the real Banach space B of all Hermitian elements of A. Then N is a convex set containing 0. We verify that the polar set $N_\pi = \{a \in B | \psi(a) \geqq -1 \text{for } \psi \text{ in } N\}$ (see [82]) consists of those a in B whose negative part a_- satisfies $||a_-|| \leqq 1$; and hence that the "bipolar" $(N_\pi)^\pi = \{\psi \in B^* | \psi(a) \geqq -1 \text{ for a in } N_\pi\}$ consists of all positive functionals ψ with $\|\psi\| \leqq 1$. Applying the theorem on "bipolars," [82], which says that $(N_\pi)^\pi$ is the weak* closure of N, we conclude that every positive functional ϕ with $\|\phi\| \leqq 1$ is a weak* limit of natural positive functionals ψ with $\|\psi\| \leqq 1$.

If A is a C^*-algebra, a $*$-representation T ol A is a homomorphism of A into the bounded operators on some Hilbert space $H = H(T)$, involution in A going into the adjoint operation. A positive functional ϕ on A is associated with a $*$-representation T if there is an x in $H(T)$ for which $\phi(a) = (T_a x, x) (a \in A)$; ϕ is associated with a family S of $*$-representations if it is associated with some S in S.

Theorem (2.1.2)[80]: Let A be any C^*-algebra, T a $*$-representation of A, and S a family of $*$-representations of A. The following four conditions are equivalent:

(i) The kernel J of T contains the intersection I of the kernels of the representations in S;

(ii) Every positive functional on A associated with T is a weak* limit of finite linear combinations of positive functionals associated with S;

(iii) Every positive functional on A associated with T is a weak* limit of finite sums of positive functionals associated with S;

(iv) Every positive functional ϕ on A associated with T is a weak* limit of finite sums ψ of positive functionals associated with S for which $\|\psi\| \leqq \|\phi\|$.

Proof. It is trivial that (iv) → (iii) → (ii). Assume (ii), and let a belong to $I - J$. Then there is a positive functional ϕ associated with T for which $\phi(a) \neq 0$. But, by (ii), ϕ is a weak* limit of linear combinations ψ of functionals associated with S; for such $\psi, \psi(a) = 0$. Hence $\phi(a) = 0$; and we have a contradiction. Thus $I \subset J$; and we have shown that (ii) implies (i).

Now assume (i), and let $S^0 = \sum_{s\in S} \oplus S$ Since $J \supset I = \text{Kernel } (S^0)$, a positive functional ϕ associated with T vanishes on I, and hence induces a continuous positive functional ϕ' on .A/I, which may be identified with the range $S^0(A)$ of S^0. Since the latter is norm-closed, apply Theorem (2.1.1) and approximate ϕ' eakly* by sums ψ' of natural positive functionals on $S^0(A)$ for which $\|\psi'\| \leqq \|\phi'\|$.. Thus, passing back to A, we approximate ϕ weakly* by sums ψ of positive functionals associated with S^0, for which $\|\psi\| \leqq \|\phi\|$. But each positive functional associated with S^0 is itself a norm-limit of sums of positive functionals associated with $\mathcal{S}$. The last two statements combine to give (iv). The proof of the Theorem is now complete.

This theorem will be referred to as the equivalence theorem. If $\mathcal{S}$ and T are such as to satisfy conditions (i)-(iv) of the theorem, we shall say that T is weakly contained in $\mathcal{S}$.

The equivalence of (i) and (ii) is evidently valid for any Banach $*$-algebra. However, the equivalence of (i) and (iii) or (iv) depends on the special properties of C^*-algebras; it fails, for example, for the L_1 group algebras of certain groups (see remark following Theorem (2.1.59)).

In case T is a cyclic representation, with cyclic vector ξ (that is, the $T_a\xi$, $a \in A$, are dense in $H(T)$), each condition of the equivalence theorem is equivalent to:

(ii′) Merely the positive functional $\phi_0(a) = (T_a\xi, \xi)$ is a weak* limit of finite linear combinations of positive functionals associated with $\mathcal{S}$.

Indeed: Assume (ii′); and let $\eta \in H(T), \phi(a) = (T_a\eta, \eta)$. Choose y in A so that $\|T_y\xi - \eta\|$ is small. Then ϕ', defined by $\phi'(a) = (T_aT_y\xi, T_y\xi) = \phi_0(y^*ay)$, approximates ϕ in the norm, hence weakly*. Pick a net $\{\psi_i\}$ of linear combinations of positive functionals associated with $\mathcal{S}$ so that $\psi_i \to \phi_0$ weakly*. If $\psi_i'(a) = \psi_i(y^*ay)$, then $\{\psi_i'\}$ is again a net of linear combinations of positive functionals associated with $\mathcal{S}$, and converges weakly* to ϕ', i.e., to a functional approximating ϕ weakly*. Thus, by the arbitrariness of ϕ, (ii) holds.

Let A he a C^*-algebra without unit, and A_1 the C^*-algebra obtained by adjoining a unit 1 to A (see [81], or [97]).

A $*$-representation T of A is nowhere trivial if $\xi = 0$ whenever $T_a\xi = 0$ for all a in A, or, equivalently, if the linear span of the $T_a\xi(a \in A, \xi \in H(T))$ is dense in $H(T)$. To each $*$-representation T of A, let T^1 be the $*$-representation of A_1 coinciding with T on A, and for which $T^1(1)$ is the identity operator in $H(T)$. If $\mathcal{S}$ is a family of $*$-representations of A, $\mathcal{S}^1$ will mean $\{T^1 | T \in \mathcal{S}\}$.

If I is a closed two-sided ideal of A, let $I^1 = \{a + \lambda .1 | a \in A, \lambda$ complex; $ay + \lambda y \in I$ for all y in A$\}$.

Lemma (2.1.3)[80]: I^1 is a closed two-sided ideal of A_1 with $A \cap I^1 = I$. In fact, if T is a nowhere trivial $*$-representation of A with kernel I, then I^1 is the kernel of I^1.

Proof. There exists a nowhere trivial $*$-representation T of A with kernel I. We have $T^1_{a+\lambda .1} = 0$ if and only if $0 = T^1_{a+\lambda .1} T_y \xi = T_{ay+\lambda y}\xi$ for all y in A and ξ in $H(T)$, i.e., if and only if $ay + \lambda y \in I$ lor all y in A. Thus $I^1 = \text{Kernel}(T^1)$.

Lemma (2.1.4)[80]: If $\mathcal{J}$ is a family of closed two-sided ideals of A, and J is a closed two-sided ideal of A, then $J \supset \bigcap_{I\in\mathcal{J}} I$ if and only if $J^1 \supset \bigcap_{I\in\mathcal{J}} I^1$.

Proof. If the second condition holds, intersect it with A to get the first (using Lemma (2.1.3)). Let the first condition hold; and suppose $a + \lambda .1 \in \bigcap_{I\in\mathcal{J}} I^1$. Then $ay + \lambda y \in I$ for all I in $\mathcal{J}$ and y in A; so that by the first condition $a + \lambda .1 \in J^1$.

Combining Lemmas (2.1.3) and (2.1.4), one obtains:

Lemma (2.1.5)[80]: If T is a nowhere trivial $*$-representation of A, and $\mathcal{S}$ is a family of nowhere trivial $*$-representations of A, then T is weakly contained in $\mathcal{S}$ if and only if T^1 is weakly contained in $\mathcal{S}^1$.

Let G be a locally compact topological group with unit element e. Its group algebra $L_1(G)$ with respect to left-invariant Haar measure is a Banach $*$-algebra, and there is a natural one-to-one correspondence between the unitary equivalence classes of unitary representations of G and those of the nowhere trivial $*$-representations of $L_1(G)$ (see [94]). In this correspondence irreducible representations of G correspond to irreducible representations of $L_1(G)$, and vice versa.

Now introduce into $L_1(G)$ a new norm $\| \ \|_c$ defined by

$$\|x\|_c = \sup_T \|T_x\|,$$

where T runs over all $*$-representations of $L_1(G)$. (This is the minimal regular norm; see [95], or [97], p. 235.) The completion of $L_1(G)$ under$\| \ \|_c$ is a C^*-algebra called $C^*(G)$, the group C^*-algebra of G. The correspondence between representations of G and of $L_1(G)$ carries over into an exactly similar correspondence between unitary representations of G and $*$-representations of $C^*(G)$, irreducible representations of one corresponding to irreducible representations of the other.

If T is a unitary representation of G, and $\mathcal{S}$ is a family of unitary representations of G, we say that T is weakly contained in $\mathcal{S}$ if this is the case when T and $\mathcal{S}$ are considered as representations of $C^*(G)$.

We shall now show that, in the case of groups, the weak containment relation can be defined in terms of the uniform convergence on compacta of functions of positive type. The essential argument for this is given in [85].

We observe first that the continuous positive functionals on $C^*(G)$ and on $L_1(G)$ are essentially the same.

Lemma (2.1.6)[80]: The restriction map is a one-to-one norm-preserving map of the set of all continuous positive functionals on $C^*(G)$ onto the set of all continuous positive functionals on $L_1(G)$.

Proof. It follows almost immediately from the definition of $C^*(G)$ that the restriction map is one-to-one and onto (see [95] or [97]). We need only prove that it preserves norm.

If x is a non-negative function in $L_1(G)$, and I is the one-dimensional identity representation of G, we have $\|x\|_c \geqq \|I_x\| = \int G^{x(g)dg} = \|x\|_{L_1(G)} \geqq \|x\|_c$. Hence

$$\|x\|_{L_1(G)} = \|x\|_c \qquad \text{for } x \in L_1(G),\ \ x \geqq 0 \tag{1}$$

Now let $\{U_i\}$ he a net of compact neighborhoods of e converging to e; and let x_i, be a continuous non-negative function on G, vanishing outside U_i, with $\int G^{x_i(g)dg} = 1$. By (1), $\|x_i\|_{L_1(G)} = \|x_i\|_c = 1$; hence $\{x_i\}$ is an approximate identity satisfying $\|x_i\| = 1$ in both $L_1(G)$ and $C^*(G)$. If/is a continuous positive linear functional on $C^*(G)$, and /' is its restriction to $L_1(G)$, we have (see [97], p. 172)

$$\|f'\| = \sup_i f'(x_i^* * x_i) = \sup_i f(x_i^* * x_i)$$
$$= \|f\|$$

Thus the restriction mapping preserves the norm.

By Lemma (2.1.6), the norm of a continuous positive functional f is the same Whether f acts on $L_1(G)$ or on $C^*(G)$. If $\{f_i\}$ is a net of such functionals, with uniformly bounded norm $\|f_i\|$, then weak* convergence of $\{f_i\}$ to f means the same with respect to $L_1(G)$ as it does with respect to $C^*(G)$.

If T is a unitary representation of G, and $0 \neq \xi \in H(T)$, the function F on G defined by $F(g) = (T_g\xi, \xi)$ is a function of positive type associated with T. If $\mathcal{S}$ is a family of unitary representations of G, a function of positive type is associated with $\mathcal{S}$ if it is associated with some T in $\mathcal{S}$. Functions F of positive type are extensively investigated in [85]. They are bounded and continuous, with $F(e) = \sup_{g\in G}|F(g)|$. Considered as elements of $L_\infty(G)$, or the dual of $L_1(G)$, they are precisely the positive continuous linear functionals on $L_1(G)$.

A family Φ of functions of positive type on G will be said to be closed invariant if:

(i) Φ is closed in the topology of uniform convergence on compacta ;

(ii) if $\phi \in \Phi$, n is a positive integer, $r_1, \dots, r_n$ are complex numbers, $h_1, \dots, h_n$ are elements of G, and ψ is defined on G by

$$\psi(g) = \sum_{i,j=1}^{n} \bar{r}_j r_i \phi(h_j^{-1} g h_i),$$

then $\psi \in \Phi$.

Now by an argument based on Gelfand's lemma on the weak* convergence of functionals, and similar to that used for the proof of Lemma C, [85], we derive the following:

Lemma (2.1.7)[80]: If Φ is a closed invariant family of functions of positive type, and if $\{\phi_i\}$ is a net of elements of Φ such that:

(i) $\|\phi_i\|$ is bounded uniformly in i;

(ii) $\phi_i \to \phi$ weakly* (as elements of $(L_1(G))^*$), then $\phi \in \Phi$.

Theorem (2.1.8)[80]: If G is a locally compact group, T is a unitary representation of G, and $\mathcal{S}$ is a family of unitary representations of G, then T is weakly contained in $\mathcal{S}$ if and only if every function of positive type on G associated with T can be approximated uniformly on compact sets by sums of functions of positive type associated with $\mathcal{S}$.

Proof. The "if" part of the theorem follows easily from Lemma (2.1.6) and the equivalence theorem. To prove the converse, suppose that T is weakly contained in $\mathcal{S}$; and let F be a function of positive type associated with T, corresponding to the positive functional ϕ on $C^*(G)$. By the equivalence theorem

$$\phi_i \to \phi \text{ weakly}^*, \quad (2)$$

where each ϕ_i is a sum of positive functionals on $C^*(G)$ associated with $\mathcal{S}$, and the $\|\phi_i\|$ are uniformly bounded in i. Let F_i be the function of positive type corresonding to ϕ_i.

We define Φ to be the set of all uniform-on-compacta limits of sums of functions of positive type associated with $\mathcal{S}$. It is easy to verify that Φ is closed invariant.-Now $F_i \in \Phi$, and, by (2) and Lemma (2.1.6), $F_i \to F$ weakly*(in$(L_1(G))^*$).

Also, by Lemma (2.1.6), the $\|F_i\|$ are uniformly bounded in i. Applying Lemma (2.1.7), we conclude that $F \in \Phi$, which completes the proof of the theorem.

Corollary(2.1.9) [80]:If $G, T, \mathcal{S}$ are as in the theorem, T is weakly contained in $\mathcal{S}$, and H is a closed subgroup of G, then the restriction of T to H is weakly contained in the family of all restrictions to H of members of $\mathcal{S}$.

The relation of weak containment, applied to irreducible representations, gives the closure operation in the dual space.

For any C^*-algebra, the dual space $\hat{A}$ will be the set of all unitary equivalence classes of irreducible $*$-representations of A. It $\mathcal{S} \subset \hat{A}$, the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$ will be defined as the set of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$, i.e., for which $\bigcap_{S\in\mathcal{S}}$ Kernel$(S) \subset$ Kernel(T).

This definition of closure in a set of ideals is essentially given in [98].

Our $\hat{A}$ differs from the Jacobson structure space (see [88]) in two minor respects: First, its elements are representations, not ideals (note that two different irreducible representations might have the same kernel). Secondly, the representations in $\hat{A}$ are required to be only topologically, not algebraically, irreducible. Kadison in [89] has shown that all irreducible $*$-representations of a C^*-algebra are algebraically irreducible; so that the importance of the second difference is much reduced. However, in proving that the closure defined above generates a topology, we need not use Kadison's rather abstruse result to make Jacobson's classical proof directly applicable; a slight modification of the latter will suffice.

Lemma (2.1.10) [80]: The above closure operation in $\hat{A}$ generates a topology.

The topology defined by this closure is called the hull-kernel topology of $\hat{A}$. Unless the contrary is stated, we assume $\hat{A}$ equipped with this topology.

If G is a locally compact group, the dual space $\hat{G}$ will be the set of all unitary equivalence classes of irreducible unitary representations of G, equipped with the hull-kernel topology, i.e., the topology of $(C^*(G))\hat{}$ transferred to G by the natural correspondence between G and $(C^*(G))\hat{}$.

Let A be any C^*-algebra, T any element of $\hat{A}$ belonging to the closure of a subset $\mathcal{S}$ of $\hat{A}$. By the equivalence theorem, each positive functional ϕ associated with T is a weak* limit of sums of positive functionals associated with $\mathcal{S}$. Since, however, we are dealing now with irreducible representations, it is now possible to make a stronger statement: Each such c6 is a weak* limit of positive functionals associated with $\mathcal{S}$; sums are unnecessary. We next prove this.

First let the C^*-algebra A have a unit 1. By P we denote the set of all normed (i.e., $\phi(1) = 1$) positive functionals on A, and by N the set of all indecomposable positive functionals. Let Q he a weakly* closed subset of P, and L the weak* closure of the set of all convex linear combinations of elements of Q.

Lemma (2.1.11) [80]: Every extreme point of L lies in Q.

Proof. Let $C(Q)$ he the space of all continuous complex functions on the compact Hausdorff space Q, and $M(Q)$ the set of all positive Baire measures on Q of total mass 1. Each element μ of $M(Q)$ corresponds naturally to an element of $(C(Q))^*$; and the weak* topology of $(C(Q))^*$ transferred to $M(Q)$ will be called the weak* topology of $M(Q)$. Evidently $M(Q)$ is weakly* compact.

To each μ in $M(Q)$ and x in A, let

$$\phi_\mu(x) = \int_Q \phi(x) d\mu\phi.$$

Evidently $\phi_\mu \in P$, and the map $\mu \to \phi_\mu$ is continuous in the weak* topologies of $M(Q)$ and P. So its range $\phi(M(Q))$ is compact, hence weakly* closed in P. On the other hand, $\phi(M(Q))$ contains all convex linear combinations of elements of Q, and the latter are dense in $\phi(M(Q))$. It follows that

$$L = \phi\big(M(Q)\big). \qquad (3)$$

Now let ψ be an extreme point of L. By (3) $\psi = \phi_\mu, \mu \in M(Q)$. The lemma will be proved if we show that μ is a point mass, i.e., that its closed hull contains only one point.

Let f_0 be a point in the closed hull of μ; and assume that the closed hull contains another point distinct from f_0. Then, for all sufficiently small open Baire neighborhoods U of f_0,

$$0 < \mu(U) < 1. \qquad (4)$$

Fix such a U. For each Baire set R let

$$\mu_1(R) = \frac{\mu(R \cap U)}{\mu(U)}, \qquad \mu_2(R) = \frac{\mu(R - U)}{\mu(Q - U)}.$$

Then the μ_i, belong to $M(Q)$ and, if $\psi_i = \phi_{\mu i}$, we have

$$\psi = \mu(U)\psi_1 + \big(1 - \mu(U)\big)\psi_2, \qquad (5)$$

and by (3)

$$\mu_i \in L. \qquad (6)$$

Since ψ is an extreme point of L, (4), (5), and (6) give

$$\psi_1 = \psi. \qquad (7)$$

But

$$\psi_1(x) = \int \phi(x) d\mu_1\phi = \frac{1}{\mu(U)} \int_U \phi(x) d\mu\phi \to f_0(x)$$

as U closes down on f_0. This combined with (7) shows that $\psi = f_0$ for each f_0 in the closed hull of μ. Hence there can only be one point in the closed hull of μ.

Theorem (2.1.12)[80]: Let A be an arbitrary C^*-algebra, $\mathcal{S}$ a subset of $\hat{A}$, and T an element of $\hat{A}$. Then the following three conditions are equivalent:

(i) $T \in \bar{\mathcal{S}}$;

(ii) some nonzero positive functional associated with T is a weak* limit of finite linear combinations of positive functionals associated with $\mathcal{S}$;

(iii) every nonzero positive functional ϕ associated with T is the weak* limit of some net $\{\psi_i\}$ of positive functionals associated with S such that $\|\psi_i\| \leqq \|\phi\|$.

Proof. It is trivial that (iii) implies (ii); the equivalence theorem gives that (ii) implies (i). To prove that (i) implies (iii), we assume that $T \in \bar{\mathcal{S}}$.

Suppose first that A has a unit 1. Let ϕ be a normed positive functional associated with T, Q the weak* closure of the set of all normed positive functionals associated with $\mathcal{S}$, and L the weak* closed convex hull of Q. The equivalence theorem assures us that $\phi \in L$. But now ϕ is indecomposable, hence is certainly an extreme point of L. By Lemma (2.1.11), $\phi \in Q$ This proves (iii) in case A has a unit.

If A has no unit, adjoin one to get A_1. Define T^1, $\mathcal{S}^1$. By Lemma (2.1.5), $T^1 \in (\mathcal{S}^1)^-$ If ϕ is a positive functional of norm 1 associated with T, extend it to a normed positive functional ϕ' on A_1 associated with T^1. So by the last paragraph ϕ' is the weak* limit of a net $\{\phi_i'\}$, where each ϕ_i' is a normed positive functional associated with $\mathcal{S}^1$. Restricting ϕ_i' to ϕ_i, on A, we have

$$\|\phi_i\| \leqq 1, \quad \phi = \lim \phi_i \text{ (in the weak}^* \text{ topology).}$$

The following theorem for groups bears the same relation to Theorem (2.1.12) as Theorem (2.1.8) does to the equivalence theorem. Its proof is omitted, since it is obtained by applying Lemma (2.1.7) to Theorem (2.1.12) in the same way that Lemma (2.1.7) was applied to the equivalence theorem in the proof of Theorem (2.1.8).

Theorem (2.1.13)[80]: Let G be a locally compact group, $\mathcal{S}$ a subset of $\hat{G}$, and T an element of $\hat{G}$. The following three conditions are equivalent:

(i) $T \in \bar{\mathcal{S}}$;

(ii) some function of positive type associated with T is a uniform-on-compacta limit of sums of functions of positive type associated with $\mathcal{S}$;

(iii) every function of positive type associated with T is a uniform-on-compacta limit of functions of positive type associated with $\mathcal{S}$.

It might be conjectured, by analogy with Theorem (2.1.12), that Theorem (2.1.13) would remain true on replacing "sums of" by "finite linear combinations of " in condition (ii). This however is not so. A counter-example is provided by Theorem (2.1.59), together with the observation that any bounded continuous complex function on G is a uniform-on-compacta limit of finite linear combinations of functions of positive type associated with the regular representation of G.

As we have already remarked, the essential difference between Theorems (2.1.2) and (2.1.12) (also between Theorems (2.1.8) and (2.1.13)) is the replacement of convergence of sums of positive functionals by convergence of single positive functionals, in the case that the representations are irreducible. There are other cases in which this replacement is possible. For example, Takenouchi has pointed out [99] that a representation V of a locally compact group G is weakly contained in the regular representation R if each function of positive type associated with V is a uniform-oncompacta limit of functions of positive type associated with the regular representation.

Let A be a C^*-algebra without unit, and .A_1 the C^*-algebra obtained by adjoining a unit 1 to A. For each T in $\hat{A}$, T^1 is in $\hat{A}_1$. Besides the T^1, there is only one other element of $\hat{A}_1$, namely, the onedimensional representation τ sending $a + \lambda.\,1$ into λ ($a \in A$). Thus $\hat{A}$ may be identified (as a set) with $\hat{A}_1 - \{\tau\}$. Lemma (2.1.5) now gives:

Lemma (2.1.14)[80]: The topology of $\hat{A}$ is that of $\hat{A}_1$ relativized to $\hat{A}_1 - \{\tau\}$.

Observe that τ belongs to the closure of $\hat{A}_1 - \{\tau\}$. Otherwise A_1 would contain an element $+\lambda\,.\,l$ not in A, belonging to the kernels of all $T^1 (T \in \hat{A})$. This means that $T_a + \lambda I = 0$ for all T in $\hat{A}$ (I is the identity operator in $H(T)$). But $\lambda \neq 0$; hence $T_{(-a/\lambda)} = I$ for all T in $\hat{A}$. It follows that $-a/\lambda$ is a unit element of A.

Let $\mathcal{S}$ and $\mathcal{G}$ be any two families of $*$-representations of a C^*-algebra A. If each S in $\mathcal{S}$ is weakly contained in $\mathcal{G}$. we say that $\mathcal{S}$ is weakly contained in $\mathcal{G}$. .If $\mathcal{S}$ and g are each weakly contained in the other, they are weakly equivalent.

The following remarks are trivial: (i) If $\mathcal{S} \subset \mathcal{G}, \mathcal{S}$ is weakly contained in $\mathcal{G}$; (ii) the relation of weak containment is reflexive and transitive; (iii) the relation of weak equivalence is an equivalence relation; (iv) if $\mathcal{S} \subset \hat{A},\ \mathcal{G} \subset \hat{A}$, then $\mathcal{S}$ is weakly contained in $\mathcal{G}$ if and only if $\mathcal{S} \subset \bar{\mathcal{G}}$; $\mathcal{S}$ is weakly equivalent to $\mathcal{G}$ if and only if $\bar{\mathcal{S}} = \bar{\mathcal{G}}$; (v) any $*$-representation T of A is weakly equivalent to any direct sum of copies of T.

Theorem (2.1.15)[80]: If $\mathcal{S}$ is any family of $*$-representations of a C^*-algebra A, there exists a (unique) closed subset $\mathcal{G}$ of $\hat{A}$ which is weakly equivalent to $\mathcal{S}$. It consists of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$.

Proof. (i) First assume A has a unit 1; and define P as the set of all positive linear functionals ϕ on A with $\phi(1) = 1$; and Q as the smallest convex weakly* closed subset of P containing all positive functionals associated with $\mathcal{S}$. I claim that an extreme point of Q is an extreme point of P (compare [85], Proposition 6, p. 40).

Let ϕ be an extreme point of Q; and assume

$$\phi = r\psi_1 + (1-r)\psi_2 \qquad (0 < r < 1, \psi_i \in P).$$

If ϕ is associated with a representation T, $\phi(a) = (T_a\xi, \xi)$ (ξ cyclic in $H(T)$), then a well-known majorization theorem [97] supplies us with an η in $H(T)$ for which $\psi_1(a) = (T_a\eta, \eta)$. Since ξ is cyclic, there is a sequence $\{y_n\}$ of elements of A with $\|T_{y_n}\xi\| = 1$, $T_{y_n}\xi \to \eta$. Put $\phi_n(a) = \phi({y_r}^* a y_n)$; then $\phi_n \in P$, and $\phi_n \to \psi_1$ weakly*. Since $\phi \in Q$, also $\phi_n \in Q$; and hence $\psi_1 \in Q$. Similarly $\psi_2 \in Q$. Since ϕ is an extreme point of Q, this gives $\psi_1 = \psi_2 = \phi$. Hence ϕ is an extreme point of P; and the claim is justified.

Now let $\mathcal{G}$ be the (closed) set of all T in $\hat{A}$ which are weakly contained in $\mathcal{S}$. It suffices to show that $\mathcal{S}$ is weakly contained in $\mathcal{G}$. If ϕ is in P and associated with $\mathcal{S}$, and if E is the set of all extreme points of Q, then by the Krein-Milman theorem ϕ is a weak* limit of convex linear combinations of elements of E. On the other hand, elements of E are extreme points of P, hence are associated with representations in $\mathcal{G}$. Thus $\mathcal{S}$ is weakly contained in $\mathcal{G}$.

(ii) If A has no unit, adjoin a unit 1 to get A_1. Defining $\mathcal{S}^1$, we obtain by (A) a closed subset $\mathcal{G}^1$ of $\hat{A}_1$ which is weakly equivalent to $\mathcal{S}^1$. Let $\mathcal{G}$ be the subset of $\hat{A}$ corresponding to $\mathcal{G}^1 - \{\tau\}$ (see Lemma (2.1.14)). $\mathcal{G}$ is obviously weakly contained in $\mathcal{S}$. That $\mathcal{S}$ is weakly contained in $\mathcal{G}$ follows from the two facts that $\mathcal{S}^1$ is weakly contained in $\mathcal{G}^1$ and that τ vanishes on A.

We will call the $\mathcal{G}$ of Theorem (2.1.15) the spectrum of $\mathcal{S}$. This definition is a generalization of [85], Definition 2, p. 43.

(See, [86]; and [97]) have studied direct integrals of $*$-representations of C^*-algebras. If this notion is defined topologically, rather than purely measure-theoretically, one can conclude the weak equivalence of the direct integral representation with the set of component representations.

For details concerning direct integrals, see [86].

Fix a locally compact Hausdorff space T; with each t in T associate a Hilbert space H_t. A vector field will be a function ξ on T such that $\xi(t) \in H_t$ for each t. An operator field will be a function B on T such that, for each t, $B(t)$ is a bounded linear operator on H_t.

A continuity basis is a family F of vector fields such that: (i) if $\xi, \eta \in F$ and r, s are complex, then $r\xi + s\eta \in F$; (ii) if $\xi \in F$, $\|\xi(t)\|$ is continuous on T; (iii) for each t_0 in T, $\{\xi(t_0) | \xi \in F$ is dense in H_{t_0}.

We fix a continuity basis F. A vector field ξ is continuous if for each t_0 in T, and each $\epsilon > 0$, there is a neighborhood U of t_0 and an η in F such that $\|\xi(t) - \eta(t)\| < \epsilon$ for all t in U. An operator field B is continuous if, for all continuous vector fields ξ, the map $t \to B(t)\xi(t)$ is a continuous vector field.

Now let m be a fixed regular Borel measure on T whose closed hull is T; and denote by H' the inner product space of all continuous vector fields ξ for which

$$\int_T \|\xi(t)\|^2 dmt < \infty\,,$$

equipped with the inner product

$$(\xi,\eta) = \int_T (\xi(t),\eta(t))dmt\,.$$

Following [97], the Hilbert space H obtained by completing H' will be denoted by

$$H = \int_T H_t(dmt)^{1/2}\,,$$

the direct integral of the H_t.

A bounded linear operator b on H is a direct integral operator if there is a continuous operator field B such that, for $\xi \in H'$, $b\xi \in H'$ and $(b\xi)(t) = B(t)\xi(t)$.

We describe this by writing $b = \int \oplus B(t)$. The operator field B is uniquely determined by b.

Lemma (2.1.16)[80]: (i) If $b = \int \oplus B(t)$, then $\|b\| = sup_t\|B(t)\|$.

(ii) If $b = \int \oplus B(t)$, $c = \int \oplus C(t)$, then

$$rb + sc = \int \oplus (rB(t) + sC(t)),$$

$$bc = \int \oplus B(t)C(t).$$

(iii) If $b = \int \oplus B(t)$, $b^* = \int \oplus C(t)$, then $C(t) = B(t)^*$

(Note that b might be a direct integral operator, without b^* being one.)

Now suppose A is a C^*-algebra, and S a $*$-representation of A in $H=\int_T H_t(dmt)^{1/2}$, such that $S_x = \int \oplus S_x^{(t)}$ is a direct integral operator for each x in A. It follows from Lemma (2.1.16) that, for each t in T, the map $S^{(t)}: x \to S_x^{(t)}$ is a $*$-representation of A in H_t. We then say that S is a direct integral of the $S^{(t)}$, and write $S = \int \oplus S^{(t)}$

Theorem (2.1.17)[80]: If $S = \int \oplus S^{(t)}$ is a direct integral representation of a C^*- algebra A, then S is weakly equivalent to $\{S^{(t)} | t \in T\}$.

Proof. (A) Let $t_0 \in T$. We shall show that $S^{(t_0)}$ is weakly contained in S. Letting $\xi \in F$, it is sufficient to approximate

$$\phi(x) = \left(S_x^{(t_0)}\xi(t_0), \xi(t_0)\right) \tag{8}$$

in the weak* topology by positive functionals associated with S. For each compact neighborhood U of t_0, f_U will be a continuous non-negative function on T vanishing outside U, for which

$$\int_T (f_U(t))^2 dmt = 1. \tag{9}$$

Putting $\xi_U(t) = f_U(t)\xi(t)$, $\phi_U(x) = (S_x\xi_U, \xi_U)$, we get (note $\xi_U \in H'$):

$$\begin{aligned}\phi_U(x) &= \int_T \left(S_x^{(t_0)}\xi_U(t), \xi_U(t)\right) dmt \\ &= \int_U \left(f_U(t)\right)^2 (S_x^{(t_0)}\xi(t), \xi(t))dmt.\end{aligned} \tag{10}$$

Now ϕ_U is a positive functional associated with S. By (8), (9) and (10), $\lim_{U\to t_0} \phi_U(x) = \phi(x)$ for $x \in A$.

(B) To prove that S is weakly contained in the set of all $S^{(t)}$, pick $\xi \in H'$ with compact support D, and put

$$\phi(x) = (S_x\,\xi,\xi) \\ = \int_D \phi^t(x)dmt, \tag{11}$$

where $\phi^t(x) = (S_x^{(t)}\xi(t),\xi(t))\cdot$ It is enough to show that ϕ is weakly* approximated by finite sums of the ϕ^t. But this is evident from the integral (11).

It is of interest to inquire which locally compact groups G have the property—which we shall refer to as property (R)—that their regular representation weakly contains all irreducible representations. Godement [85], p.77 has observed that, if the regular representation of G weakly contains merely the one-dimensional identity representation of G, then it has property (R). It is well known that this is the case for compact groups and locally compact Abelian groups. But it is not true for all locally compact groups. Takenouchi has shown in [99] that, if G is a locally compact group whose factor group modulo the connected component of the identity is compact, then property (R) holds if and only if G is of type (C) in the sense of Iwasawa.

In Theorem (2.1.40) we shall determine exactly which irreducible representations of the $n\times n$ complex unimodular group are weakly contained in the regular representation.

The following remark is mildly interesting:

Lemma (2.1.18)[80]: Let L be the left-regular representation of a locally compact group $G(L_y(x) = y * x$ for $x \in L_2(G)$, $y \in L_1(G)$. Then G has property (R) if and only if $\|y\|_c = \|L_y\|$ for all y in $L_1(G)$.

Proof. Property (R) holds if and only if the kernel of L on $C^*(G)$ is $\{o\}$ i.e., L is an isometry on $C^*(G)$. For this it suffices to know that L is an isometry on $L_1(G)$ with respect to $\|\ \ \|_c$.

A $*$-representation P of a C^*-algebra A is completely continuous if T_a is completely continuous for all a in A. Fix a C^*-algebra A.

Lemma (2.1.19)[80]: If T is a completely continuous element of $\hat{A}$, then $\{T\}$ is closed.

Proof. If I = Kernel (T), A/I is the algebra of all completely continuous operators on $H(T)$. If $S \in \{\bar{T}\}$, Kernel $(S) \supset I$, so that S induces an irreducible representation of A/I. But the latter is well-known to have no irreducible representation other than the identity map. Hence $S = T$.

Theorem (2.1.20)[80]: Let $T^i \in \hat{A}$ for each i in an index set N. Form the direct sum $T = \sum_{i\in N} \oplus\ T^i$. Then T is completely continuous if and only if the following three conditions hold:

(i) Each T^i is completely continuous;

(ii) For each i, there are only finitely many distinct j in N with $T^i \cong T^j$;

(iii) The set $\mathcal{G} = \{T^i | i \in N\}$ has no limit points in $\hat{A}$.

Proof. (A) Assume T completely continuous. Obviously (i) and (ii) hold. To show that $\mathcal{G}$ has no limit points, it is enough to show that it is closed in $\hat{A}$. Indeed, if this has been done, it will follow, since every subrepresentation of T is completely continuous, that every subset of $\mathcal{G}$ is closed in $\hat{A}$. But a set all of whose subsets are closed has no limit points.

Let I be the intersection of the kernels of the T^i; then $I =$ Kernel (T). Pick an irreducible representation S of A whose kernel contains I It suffices to show $S \in \mathcal{G}$

We will consider T as a faithful representation of $B = A/I$, and S as an irreducible representation of B. Now B is (via T) a C^*-algebra of completely continuous operators. By the structure theorem for such (see [90]), B is isomorphic to the C^∞-sum

$$B \cong \sum_{r \in W} \oplus \ B^{(r)},$$

where each $B^{(r)}$ is the algebra of all completely continuous operators on some Hilbert space. To each s in W corresponds an irreducible representation S^s of B:

$$S^s : \sum_{r \in W} \oplus \, a_r \to a_s \, ;$$

and all irreducible representations of B are of this form. In particular $S \cong$ some S^s. On the other hand, each $T^i (i \in N)$ gives rise to a representation of B, and these distinguish points of B. It follows that all S^s occur among the T^i In particular, S occurs among the T^i, i.e., $S \in \mathcal{G}$

(B) Now assume (i), (ii), and (iii); let I be the closed two-sided ideal of all x in A lor which T_x is completely continuous. It suffices to show that $I = A$.

Assume then that $I \neq A$. Then A has an irreducible representation S whose kernel contains I Now S is not a limit point of $\mathcal{G}$, so it is not in the closure of $\mathcal{G}' = \mathcal{G} - \{S\}$. Pick an x_0 belonging to the kernels of all T^i in $\mathcal{G}'$, but not to the kernel of S.

Now either $S \in \mathcal{G}$ or $S \notin \mathcal{G}$ If $S \notin \mathcal{G}$, then $\mathcal{G}' = \mathcal{G}$, so that $x_0 \in$ Kernel (T), i.e., $T_{x_0} = 0$. If $S \in \mathcal{G}$, then T_{x_0} is 0 on the subspaces of $H(T)$ corresponding to all $T^i \cong S$, while, on the subspace of $H(T)$ corresponding to the $T^i \cong S$, T_{x_0} is completely continuous by (i) and (ii). Thus in either case T_{x_0} is completely continuous; and $x_0 \in I \subset$ Kernel (S). This contradicts $x_0 \notin$ Kernel (S).

A will be an arbitrary fixed C^*-algebra.

For each x in A, $T \to \|T_x\|$ is a numerical function on $\hat{A}$. In general, this function is not continuous. It is, however, lower semi-continuous, as we now show.

Lemma (2.1.21)[80]: If $\mathcal{S} \subset \hat{A}$ and $T \in \hat{A}$, then $T \in \bar{\mathcal{S}}$ if and only if for each x in A,

$$\|T_x\| \leqq \sup_{s \in \mathcal{S}} \|S_x\| . \tag{12}$$

Proof. If $T \notin \hat{A}$, by the definition of closure there is an x in A for which $\sup_{s \in \mathcal{S}} \|S_x\| = 0$, $\|T_x\| > 0$. So (1) fails.

Assume $T \in \bar{\mathcal{S}}$. By Lemma (2.1.14) it is sufficient to assume that A has a unit 1. Again, since $\|T_{x*x}\| = \|T_x\|^2$, we may assume without loss of generality that x is positive.

Choose ξ in $H(T)$ so that $\|\xi\| = 1$ and

$$(T_x \xi, \xi) \geqq \|T_x\| - \frac{\epsilon}{2} . \tag{13}$$

By Theorem (2.1.12), there exist $S^1, \dots, S^n$ in $\mathcal{S}$, $\xi_i \in H(S^i)$ $(i = 1, \dots, n)$, and non-negative $\lambda_1, \dots, \lambda_n$, such that $\|\xi\| = 1$, $\sum_{i=1}^{n} \lambda_1 = 1$, and

$$\left| \left\{ \sum_{i=1}^{n} \lambda_1 (S_x^i \xi_i, \xi_i) \right\} - (T_x \xi, \xi) \right| < \frac{\epsilon}{2} . \tag{14}$$

But $(S_x^i \xi_i, \xi_i) \leqq \|S_x^i\|$, so that

$$\sum_{i=1}^{n} \lambda_1 (S_x^i \xi_i, \xi_i) \begin{matrix} n \\ max \\ i = 1 \end{matrix} \|S_x^i\| \leqq \sup_{s \in \mathcal{S}} \|S_x\|. \tag{15}$$

Combining (3) and (4), we get

$$(T_x \xi, \xi) \leqq \sup_{s \in \mathcal{S}} \|S_x\| + \frac{\epsilon}{2}. \tag{16}$$

Now (1) follows from (2), (5), and the arbitrariness of ϵ.

From this we immediately obtain:

Lemma (2.1.22)[80]: For each x in A, the function $T \to \|T_x\|$ is lower semi-continuous, i.e., if $T^i \to T$ in $\hat{A}$, then $\liminf_i \|T_x^i\| \geqq \|T_x\|$.

Lemma (2.1.23)[80]: For every net $\{T^i\}$ of elements of $\hat{A}$, and every element x of A,

$$\sup_{s \in \mathcal{G}} \|S_x\| \geqq \limsup_i \|T_x^i\|,$$

where $\mathcal{G}$ is the set of all cluster points of $\{T^i\}$.

Proof. For each index $J, \mathcal{S}_j$ will be the closure in $\hat{A}$ of $\{T^i | i > j\}$. The $\mathcal{S}_j$ form a decreasing net of closed sets, and

$$\mathcal{G} = \bigcap_j \mathcal{S}_j. \tag{17}$$

Let $I_j = \bigcap_{s \in \mathcal{S}_j} \text{Kernel}(S)$, $K = \bigcap_{s \in \mathcal{G}} \text{Kernel}(S)$. I claim that

$$\text{K} = \left(\bigcup_j I_j \right)^{-}. \tag{18}$$

Indeed, let $x \notin \left(\bigcup_j I_j \right)^{-}$. Since $\left(\bigcup_j I_j \right)^{-}$ is a closed two-sided ideal, there is a T in $\hat{A}$ such that $T_x \neq 0$, and

$$\left(\bigcup_j I_j \right)^{-} \subset \text{Kernel}(T). \tag{19}$$

Since each $\mathcal{S}_j$ is closed, (19) implies that T belongs to all $\mathcal{S}_j$, i.e., $T \in \mathcal{G}$. Therefore $T_x \neq 0$ gives $x \notin K$ Thus $K \subset \left(\bigcup_j I_j \right)^{-}$. The opposite inclusion is obvious. This proves (18).

Now $\{I_j\}$ is an increasing net of closed two-sided ideals of A. Denote as usual by x/I_j the element of the C^*-algebra A/I_j corresponding to x. Applying to (18) an elementary argument valid in all Banach spaces, we have for all x in A

$$\lim_j \|x/I_j\| = \|x/K\|. \tag{20}$$

Now, since $\|x\| = \sup_{T \in \hat{A}} \|T_x\|$ in any C^*-algebra (see p. 411 of [90]),

$$\|x/I_j\| = \sup_{s \in \mathcal{S}_j} \|S_x\|, \tag{21}$$

$$\|x/K\| = \sup_{s \in \mathcal{G}} \|S_x\|. \tag{22}$$

Combining (20), (21), and (22), we get for all x,

$$\lim_j \sup_{s \in \mathcal{S}_j} \|S_x\| = \sup_{s \in \mathcal{G}} \|S_x\|. \tag{23}$$

But, by the definition of $\mathcal{S}_j$,

$$\lim_j \sup_{s \in \mathcal{S}_j} \|S_x\| \geqq \lim_j \sup_{i > j} \|T_x^i\| = \limsup_i \|T_x^i\|.$$

This and (23) complete the proof.

Theorem (2.1.24)[80]: Let A be an arbitrary C^*-algebra, $\{T^i\}$ a net of elements of $\hat{A}$, and $\mathcal{S}$ a closed subset of $\hat{A}$. The following two conditions are equivalent:
(i) For all x in A, $\lim_i \|T_x^i\| = \sup_{s\in\mathcal{S}} \|S_x\|$.
(ii) For all subnets $\{T'^i\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have

$$T'^i \to S \text{ if and only if } s \in \mathcal{S}.$$

Proof. (A) Assume (ii). Then $\mathcal{S}$ is the set of all cluster points of $\{T^i\}$. By Lemma (2.1.23),

$$\lim_i \sup \|T_x^i\| \leqq \sup_{s\in\mathcal{S}} \|S_x\|. \tag{24}$$

On the other hand, if $s \in \mathcal{S}$, we have $T^i \to S$, so that by Lemma (2.1.22)

$$\|S_x\| \leqq \lim_i \inf \|T_x^i\|.$$

Combining this with (24) we get (i).

(B) Assume (i). Let $S^0 \notin \mathcal{S}$; then there exists x in A such that $S_x^0 \neq 0$, $S_x = 0$ for all S in $\mathcal{S}$. By (i), $\lim_i \|T_x^i\| = 0$. If at the same time S^0 is a cluster point of $\{T^i\}$, we pick a subnet { V} of {T{} for which T"-*S0. By Lemma (2.1.22)

$$\|S_x^0\| \leqq \lim_j \inf \|T_x'^j\| = 0.$$

This contradicts $S_x^0 \neq 0$. Thus all cluster points of $\{T^i\}$ are in $\mathcal{S}$.

Let $S^0 \in \mathcal{S}$. I claim that $T^i \to S^0$. Indeed, if this were not so, there would be a subnet $\{T'^i\}$ of $\{T^i\}$ and a neighborhood U of S^0 such that all T'^i are outside U. Hence there would be an x such that $S_x^0 \neq 0$, $T_x'^j = 0$ for all j Hence by(i)

$$\sup_{s\in\mathcal{S}} \|S_x\| = \lim_i \|T_x^i\| = \lim_j \|T_x'^j\| = 0,$$

which contradicts $S_x^0 \neq 0$. Thus every element of $\mathcal{S}$ is a limit of $\{T^i\}$.

We have proved that $\mathcal{S}$ coincides with the set of all limits, and also with the set of all cluster points, of $\{T^i\}$. But this is exactly (ii). The proof is complete.

It may be worth mentioning the status of Lemmas (2.1.22) and (2.1.23) and Theorem (2.1.24) for an arbitrary Banach $*$-algebra A. For such an A, one defines the topological space $\hat{A}$, as the set of all irreducible $*$-representations with the hull-kernel topology. Now Lemma (2.1.23) is still valid in the general case, but Lemma (2.1.22) is in general false; consider, for example, the Banach $*$-algebra of complex functions continuous on the closed unit disc and analytic in its interior. As for Theorem (2.1.24), it holds whenever Lemma (2.1.22) does.

Corollary (2.1.25)[80]: If A is a C^*-algebra, and $\{T^i\}$ a net of elements of $\hat{A}$, the following are equivalent:

(i) $\lim_i \|T_x^i\| = 0$ for all x in A;

(ii) no subnet of $\{T^i\}$ converges to any limit;

(iii) A has no unit, $(T^i)^1 \to \tau$ in $\hat{A}_1$, and no subnet of $\{(T^i)^1\}$ converges to any other limit.

Proof. Theorem (2.1.24), with $\mathcal{S}$ taken as the void set, shows that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from Lemmas (2.1.11) and (2.1.14).

Theorem (2.1.24) has as a simple corollary a connection between the Hausdorff property and the continuity of the functions $T \to \|T_x\|$.

Let the ideal structure space X of A be defined as the set of all kernels of irreducible $*$-representations, topologized with the hull-kernel topology. X is obtained by identifying elements of $\hat{A}$ with the same kernel. Since, for $T \in \hat{A}$, $\|T_x\|$ depends only on the kernel of T, we may define $N_x(I)$ as $\|T_x\|$ whenever $T \in \hat{A}$, Kernel $(T) = I$

Corollary (2.1.26)[80]: (see Theorem 4.1 of [92]). X is Hausdorff if and only if each N_x is continuous on X.

We fix a C^*-algebra A, with dual $\hat{A}$. The dimension of a representation S of A will be called dim S, and the trace of an operator B $\operatorname{Tr}(B)$.

Lemma (2.1.27)[80]: If $\{T^i\}$ is a net of elements of $\hat{A}$, n is an integer, and dim $T^i \leqq n$ for each i, then $\{T^i\}$ can converge to no more than a finite number of distinct limits, and, for each such limit S, dim $S \leqq n$.

Proof. (A) Let $T^i \to S$. To prove dim $S \leqq n$, we give a well-known argument using polynomial identities (see [91]). Let

$$B = A \Big/ \bigcap_i \text{Kernel } (T^i).$$

Since the elements of B are separated by the T^i, B must satisfy the standard polynomial identity for the $n \times n$ matrix algebra. Hence all its irreducible representations are of dimension $\leqq n$. But, since S is in the closure of the $\{T^i\}$, its kernel contains $\bigcap_i$ Kernel (T^i); hence it induces an irreducible representation of B. Combining these facts, we get dim $S \leqq n$.

(B) Pick such a positive integer p that there do not exist as many as p linear operators $A_1, \dots, A_p$ on an n-dimensional Hilbert space for which $\|A_k\| \leqq 1$ $(k = 1, \dots, p)$, $\|A_k - A_j\| \geqq 1$ $(k, j = 1, \dots, p;\ k \neq j)$. We complete the proof by contradicting the assumption that $\{T^i\}$ converges to p distinct limits $S^1, \dots, S^p$.

By (A) and Lemma (2.1.19), each one-point set $\{S^k\}$ is closed in $\hat{A}$. Thus, for $k = 1, \dots, p$, there is a Hermitian element x_k of A such that

$$\left\|A^j_{x_k}\right\| = \delta_{k_j}. \tag{25}$$

Let F be the following real-valued continuous function on the reals:

$$F(t) = \begin{cases} 1 \text{ if } & t \geqq 1, \\ t \text{ if } & -1 \leqq t \leqq 1, \\ -1 \text{ if } & t \leqq -1. \end{cases}$$

Applying F to x_k (see [92]), we have by (25)

$$\|F(x_i)\| \leqq 1, \quad S^j_{F(x_k)} = F\left(S^j_{x_k}\right) = S^j_{x_k}.$$

Replacing the x_k by the $F(x_k)$, we may assume $\|x_k\| \leqq 1$. Therefore

$$\left\|T^i_{x_k}\right\| \leqq 1, \quad \text{ for all } i, \text{and all } k = 1, \dots, p. \tag{26}$$

Fix $k \neq j$. Since S^k belongs to the closure of all subnets of $\{T^i\}$, Lemma (2.1.21) gives

$$1 = \left\|S^k_{x_{k-x_j}}\right\| \leqq \liminf \left\|T^i_{x_{k-x_j}}\right\|.$$

Hence there is an i_0 such that $\left\|T^i_{x_{k-x_j}}\right\| \geqq 1/2$ for all $i \succ i_0$. We may therefore pick an i such that

$$\left\|T^i_{x_k} - T^i_{x_j}\right\| \geqq 1/2 \quad \text{ for all } k, j = 1, \dots, p;\ k \neq j.$$

This combined with (26) contradicts the definition of p.

Lemma (2.1.28)[80]: Let $\{T^i\}$ be a net of n-dimensional representations in $\hat{A}$, and suppose $S^1, S^2, \dots, S^r$ are distinct elements of $\hat{A}$ such that

(i) for all subnets $\{T'^i\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have $T'^i \to S$ if and only if $S =$ some S^k;

(ii) for each x in A, $\mathrm{Tr}(T_x^i)$ approaches some limit $\sigma(x)$.

Then there exist positive integers $M_1, \dots, M_r$ such that

$$\sum_{k=1}^{r} M_k \dim S^k \leqq n, \tag{27}$$

$$\sigma(x) = \sum_{k=1}^{r} M_k Tr(S_x^k) \qquad \text{for all } x \text{ in } A. \tag{28}$$

Proof. By Lemma (2.1.27), r is finite, and $\dim S^k \leqq n$. By (i) and Theorem (2.1.24),

$$\lim\|T_x^i\| = \max_{k=1}\|S_x^k\| \qquad (x \in A) \tag{29}$$

Denote $\dim S^k$ by d_k.

(A) Let $P_{k,j} (j = 1, \dots, d_k)$ be orthogonal one-dimensional projections in $H(S^k)$. We shall first show that there are positive elements x_{kj} in $A (k = 1, \dots, r;\ j = 1, \dots, d_k)$ for which

(a) $S_{x_{kj}}^q = \delta_{qk} P_{kj}$;

(b) There exists i_0 such that, lor $i > i_0$, the $T_{x_{kj}}^i$ $(k = 1, \dots, r;\ j = 1, \dots, d_k)$ are orthogonal nonzero projections in $H(T^i)$, with $\dim T_{x_{kj}}^i$. independent of i and j.

To prove this, we select $\sum_{k=1}^{r} d_x$ distinct positive integers

$$\omega_{kj} (k = 1, \dots, r;\ j = 1, \dots, d_k),$$

and put $B_k = \sum_{j=1}^{d_x} \omega_{kj} P_{kj}$. Choose a positive element z of A such that

$$S_z^k = B_k \quad \text{for each } k. \tag{30}$$

Pick $1/4 \geqq \epsilon > 0$. I claim that i_0 can be found to satisfy the following, which we will call property (P):

For $i > i_0$, T_z^i has at least one eigenvalue in each interval $[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon]$, and no eigenvalue lying outside $[0, \epsilon]$ and also outside all $[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon]$.

Indeed: Fix k, j. Yet F be a continuous non-negative function on the reals which is 0 outside $[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon]$, and 1 at ω_{kj}. By (30)

$$S_{F(x)}^q = F(S_z^q) = F(B_q) = \delta_{qk} P_{kj}.$$

Hence $S_{F(z)}^q \neq 0$. So, by (29), $\lim_i \|T_{F(z)}^i\|$ exists and is not 0. Therefore we may choose i_0 so that, for $i > i_0$, $\|T_{F(z)}^i\| > 0$, i.e. $F(T_z^i) \neq 0$. But the latter implies that T_z^i has an eigenvalue in $[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon]$.

Thus i_0 can be chosen to satisfy the first half of property (P).

Now pick a non-negative continuous function G on the reals which is 0 at 0 and at each $\omega_{kj} (k = 1, \dots, r;\ j = 1, \dots, d_k)$, and is 1 at all points which lie outside all the intervals $[-\epsilon, \epsilon]$ and

$$[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon] (k = 1, \dots, r;\ j = 1, \dots, d_k).$$

Then

$$S_{G(x)}^q = G(S_z^q) = G(B_q) = 0,$$

so that by (29)

$$\lim_i \|T_{G(z)}^i\| = 0.$$

Now choose i_0 so that not only does the first half of property (P) hold, but also, for all $i > i_0$, $\|T^i_{G(z)}\| = G(T^i_z) < 1$, i.e., T^i_z has no eigenvalues at places where G is 1. Then i_0 satisfies property (P). Fix this i_0.

Now, for $(k = 1, \dots, r;\ j = 1, \dots, d_k)$, select a non-negative continuous function K_{kj} on the reals which is 1 on $[\omega_{kj} - \epsilon, \omega_{kj} + \epsilon]$ and 0 outside $[\omega_{kj} - 2\epsilon, \omega_{kj} + 2\epsilon]$. If $x_{kj} = K_{kj}(z)$, we have by (30)

$$\begin{aligned} S^q_{x_{kj}} &= K_{kj}(S^q_z) = K_{kj}(B_q) \\ &= \delta_{qk} P_{kj}, \end{aligned}$$

which is (a). If $i > i_0$, then by property (P), $T^i_{x_{kj}} = K_{kj}(T^i_z)$ is a nonzero projection on $H(T^i)$, and all the $T^i_{x_{kj}}(k = 1, \dots, r;\ j = 1, \dots, d_k)$ are orthogonal.

Let d^i_{kj} be the dimension of the range of $T^i_{x_{kj}}$. Now $d^i_{kj} = \mathrm{Tr}(T^i_{x_{kj}})$,w hich by hypothesis approaches $\sigma(x_{kj})$. But a convergent set of integers is eventually constant. Denoting the eventually constant value of d^i_{kj} by m_{kj}, I claim that m_{kj} depends only on k.

Indeed, fix k, and let j_1 and j_2 he two of the integers $1, \dots, d_k$. Select a partial isometry C on $H(S^k)$ so that $CC^* = P_{k,j_1}, CC^* = P_{k,j_2}$; and let $u \in A$, $S^q_u = \delta_{qk} C$. Then

$$S^q_{(uu^* - x_{k,j_1})} = 0 \qquad \text{for all } q.$$

Hence by (29)

$$\left\| T^i_{(uu^* - x_{k,j_1})} \right\| \underset{i}{\to} 0;$$

so that

$$Tr(T^i_{uu^*}) - \mathrm{Tr}(T^i_{x_{k,j_1}}) \underset{i}{\to} 0.$$

Similarly

$$Tr(T^i_{u^*u}) - \mathrm{Tr}(T^i_{x_{k,j_2}}) \underset{i}{\to} 0.$$

But $\mathrm{Tr}(T^i_{uu^*}) = \mathrm{Tr}(T^i_{u^*u})$, so that

$$\mathrm{Tr}\left(T^i_{x_{k,j_1}}\right) - \mathrm{Tr}(T^i_{x_{k,j_2}}) \underset{i}{\to} 0.$$

But this means that $m_{k,j_1} = m_{k,j_2}$, which proves that m_{kj} depends only on k. Write m_k for m_{kj}.

(B) I claim that the m_k thus defined have the properties (27) and (28).

Indeed, for large enough i, the $T^i_{x_{kj}}$. are orthogonal projections in $H(T^i)$ of dimension m_k. Hence

$$n \geqq \sum_{k,j} m_k = \sum_{k=1}^{r} m_k d_k,$$

which is (27).

Consider now any Hermitian element x of A such that

$$P_{kj} S^k_x = S^k_x P_{kj} \qquad (k = 1, \dots, r;\ j = 1, \dots, d_k). \tag{31}$$

Then for suitable real λ_{kj}, we have $S^k_x = \sum_j \lambda_{kj} P_{kj}$. Now look at $u = x - \sum_{k,j} \lambda_{kj} x_{kj}$. We have

$$S^q_u = \sum_j \lambda_{qj} P_{qj} - \sum_{k,j} \lambda_{kj} S^q_{x_{kj}} = 0$$

by (a). Hence by (29),

$$\lim_i \operatorname{Tr}\left(T_u^i\right) = 0.$$

Therefore,

$$0 = \lim_i \left\{ \operatorname{Tr}(T_x^i) - \sum_{k,j} \lambda_{kj} \operatorname{Tr}(T_{x_{kj}}^i) \right\}$$
$$= \lim_i \left\{ \operatorname{Tr}(T_x^i) - \sum_{k=1}^{r} m_k \sum_{j=1}^{d_k} \lambda_{kj} \right\}$$
$$= \lim_i \left\{ \operatorname{Tr}(T_x^i) - \sum_{k=1}^{r} m_k \operatorname{Tr}(S_x^k) \right\},$$

from which follows (28).

Now, for any preassigned Hermitian x in A, there is a set of P_{kj} for which (31) holds. Thus (28) will be established if we show that the m_k are independent of the P_{kj} with which we start. For this, choose a Hermitian x so that S_x^k is the identity operator on $H(S^k)$, while $S_x^q = 0$ for $q \neq k$. This x satisfies (31) for any set of P_{kj}. Hence using this x in (28), we have

$$\sigma(x) = m_k d_k.$$

Thus m_k is independent of the choice of the P_{kj}. We have therefore shown that (28) holds for all Hermitian elements, and hence for all elements. The proof is complete.

If A has a unit 1, then equality holds in (27). Indeed, substituting $x = 1$ in (28), we have $\sigma(1) = n$, $\operatorname{Tr}\left(S_1^k\right) = \dim S^k$.

Theorem (2.1.29)[80]: Let $\{T^i\}$ be a net of n-dimensional representations in $\hat{A}$ (n finite), and let $S^1, \dots, S^r$ be distinct elements of $\hat{A}$ such that for all subnets $\{T'^i\}$ of $\{T^i\}$, and all S in $\hat{A}$, we have $T'^i \to S$ if and only if $S =$ some S^k. Then there exists a subnet $\{T'^i\}$ of $\{T^i\}$, and positive integers $m_1, \dots, m_r$ such that

$$\sum_{k=1}^{r} m_k \dim S^k \leqq n, \tag{32}$$

and

$$\lim \operatorname{Tr}\,(T_x'^j) = \sum_{k=1}^{r} m_k \operatorname{Tr}(S_x^k) \tag{33}$$

for all x in A.

Proof. For fixed x in A, $\left|\operatorname{Tr}(T_x^i)\right| \leqq n\|x\|$ for all i. Hence, picking a universal subnet (see [93]) $\{T'^j\}$ of $\{T^i\}$, we find that $\operatorname{Tr}(T_x'^j)$ converges to some limit for each x in A. Now $\{T'^j\}$ satisfies the hypotheses of Lemma (2.1.28); and the conclusion of that lemma gives the theorem.

The following example shows that in Theorem (2.1.29) the subnet $\{T'^j\}$ is unavoidable; in general, (32) and (33) cannot be satisfied with the original $\{T^i\}$.

Let A be the C^*-algebra of all sequences $x = (x^{(1)}, x^{(2)}, \dots)$ of 2×2 complex matrices satisfying: (i) $\lim_{i\to\infty} x_{12}^{(i)} = \lim_{i\to\infty} x_{21}^{(i)} = 0$; (ii) $\lim_{i\to\infty} x_{11}^{(i)} = \lim_{i\to\infty} x_{22}^{(2i+1)}$ exists; call it $\sigma(x)$; (iii) $\lim_{i\to\infty} x_{22}^{(2i)} = 0$. Note that σ is a onedimensional representation of A. If $T^{(n)}$ is the irreducible representation sending x into $x^{(n)}$, we have $\lim_{n\to\infty} \left\| T_x^{(n)} \right\| = |\sigma(x)|$ for $x \in A$; so that, by Theorem (2.1.24), the hypotheses of Theorem (2.1.29) are satisfied with $\{S^1, \dots, S^r\} = \{\sigma\}$. On the other hand, $\mathrm{Tr} T_x^{(n)} \to \sigma(x)$ and $2\sigma(x)$ as $n \to \infty$ through even and odd values respectively.

Corollary (2.1.30)[80]: Let $\{T^i\}$ be a net of elements of $\hat{A}$, all of dimension equal to or less than the integer n; and let $S^1, \dots, S^r$ be distinct elements of $\hat{A}$ such that $T^i \to_i S^k$ for each k. Then

$$\sum_{k=1}^{r} \dim S^k \leqq n.$$

Proof. Pick a universal subnet $\{T'^j\}$ of $\{T^i\}$. Then $\dim T'^j$ is eventually equal to some $m \leqq n$; $\mathrm{Tr}(T_x'^j)$ approaches a limit for each x in A; and $\{S^1, \dots, S^r\}$ can be enlarged to a set $\{S^1, \dots, S^i\}$ (finite by Lemma (2.1.27)) for which (i) of Lemma (2.1.28) holds. Then the hypotheses of Lemma (2.1.28) hold for $\{T'^j\}$; and the conclusion of that lemma gives

$$\sum_{k=1}^{r} \dim S^k \leqq \sum_{k=1}^{i} dim\, S^k$$
$$\leqq \sum_{k=1}^{i} m_k \dim S^k \leqq n.$$

Corollary (2.1.31)[80]: see Theorem 4.2 of [92]. If all irreducible representations of A are of the same finite dimension n, $\hat{A}$ is a Hausdorff space.

Proof. By Corollary (2.1.30), no net of elements of $\hat{A}$ can converge to more than one limit.

Corollary (2.1.32)[80]: If $\{T^i\}$ is a net of n-dimensional elements of $\hat{A}$ (n finite), and $S^1, \dots, S^i$ are distinct elements of $\hat{A}$ for which (i) $T^i \to_i S^k$ for all $k = 1, \dots, r$, and (ii) $\sum_{k=1}^{r} \dim S^k = n$; then, for all x in A,

$$\lim_i \mathrm{Tr}\,(T_x^i) = \sum_{k=1}^{r} \mathrm{Tr}\,(S_x^k).$$

Proof. If the conclusion fails, there is an x in A, and a subnet $\{T'^j\}$ of $\{T^i\}$, such that $\mathrm{Tr}(T_x'^j)$ eventually lies outside some neighborhood U of $\sum_{k=1}^{r} \mathrm{Tr}\,(S_x^k)$. By (ii) and Corollary (2.1.30), no subnet of $\{T^i\}$, hence no subnet of $\{T'^j\}$, can converge to any element of $\hat{A}$ distinct from the S^k. Hence the hypotheses of Theorem (2.1.29) hold, and there are a subnet $\{T''^P\}$ of $\{T'^j\}$, and positive integers $m_1, \dots, m_r$, such that

$$\sum_{k=1}^{r} m_k \dim S^k \leqq n, \tag{34}$$

and

$$\lim_P \mathrm{Tr}\,(T''^P) = \sum_{k=1}^{r} m_k \mathrm{Tr}\,(S_x^k). \tag{35}$$

Now (ii) and (34) give $m_k = 1$. But then (35) contradicts the definition of $\{T'^j\}$.

The contain a partial converse of Theorem (2.1.29) (Corollary (2.1.38) of Theorem (2.1.36)).

Let H be an arbitrary Hilbert space, with bounded linear operators A, B, C. Denote the range of A by rng A, and the dimension of rng A by dim rng A. We will easily verify the following lemma:

Lemma (2.1.33)[80]: Suppose dim rng $A \leqq n$, dim rng $B \leqq m$. Then dim rng $A^* \leqq n$, dim rng $(A+B) \leqq n+m$, dim rng $(AC) \leqq n$, and dim rng $(CA) \leqq n$. There is a projection P such that dim rng $P \leqq 2n$, and $PAP = A$.

Now let A be a C^*-algebra, and $\mathcal{G}$ a fixed family of $*$-representations of A. An element x of A is boundedly represented in $\mathcal{G}$ if there is an integer n such that dim rng $T_x \leqq n$ for all T in $\mathcal{G}$.

Lemma (2.1.34)[80]: The elements of A which are boundedly represented in $\mathcal{G}$ form a self-adjoint two-sided ideal of A (not necessarily closed).

Proof. By Lemma (2.1.33).

Lemma (2.1.35)[80]: Let x be a positive element of A which is boundedly represented in $\mathcal{G}$. If $S \in \mathcal{G}$, and $\{T^i\}$ is a net of representations in $\mathcal{G}$, such that, for all $m = 1, 2, \dots$, we have

$$\lim_i \operatorname{Tr}\left(T^i_{x^m}\right) = \operatorname{Tr}(S_{x^m}), \tag{36}$$

then

$$\lim_i \left\|T^i_x\right\| = \|S_x\|.$$

Proof. By Lemma (2.1.33) and bounded representedness, choose an integer n, an n-dimensional projection P^i in $H(T^i)$, and an n-dimensional projection Q in $H(S)$, so that

$$T^i_x = P^i T^i_x P^i, \qquad S_x = Q S_x Q.$$

Since $x \geqq 0$, $T^i_x \geqq 0$; let $\lambda^i_1, \dots, \lambda^i_n$ be the eigenvalues of T^i_x in rng P^i. Similarly, let $\mu_1, \dots, \mu_n$ be the eigenvalues of S_x in rng Q. Then

$$\operatorname{Tr}(T^i_{x^m}) = \sum_{k=1}^{n} (\lambda^i_k)^m,$$

$$\operatorname{Tr}(S_{x^m}) = \sum_{k=1}^{n} {\mu_k}^m.$$

Taking mth roots of (36), we have for each m

$$\lim_i \left[\sum_{k=1}^{n} (\lambda^i_k)^m\right]^{1/m} = \left[\sum_{k=1}^{n} {\mu_k}^m\right]^{1/m} \tag{37}$$

But

$$\left\|T^i_x\right\| = \max_k \lambda^i_k \leqq \left[\sum_{k=1}^{n} (\lambda^i_k)^m\right]^{1/m} \leqq n^{1/m} \leqq \left\|T^i_x\right\|, \tag{38}$$

$$\|S_x\| = \max_k \mu_k \leqq \left[\sum_{k=1}^{n} {\mu_k}^m\right]^{1/m} \leqq n^{1/m} \leqq \|S_x\|. \tag{39}$$

Combining (37), (38), and (39), and letting m become arbitrarily large, we obtain the conclusion of the lemma.

Theorem (2.1.36)[80]: Let $\mathcal{G}$ be a family of $*$-representations of A, S an element of $\mathcal{G}$, and $\{T^i\}$ a net of elements of $\mathcal{G}$. Suppose A contains a dense self-adjoint subalgebra B such that (i) every x in B is boundedly represented in $\mathcal{G}$; (ii) for every x in B, $\lim_i \operatorname{Tr}\left(T^i_x\right) = \operatorname{Tr}(S_x)$. Then, for every x in A,

$$\lim_i \|T_x^i\| = \|S_x\|. \tag{40}$$

Proof. For positive elements x of B, (40) follows from Lemma (2.1.35). This, together with $\|T_{x^*x}^i\| = \|T_x^i\|^2$ and the same for S, implies. (40) for all x in B. By continuity and denseness, (40) holds for all x in A.

Corollary (2.1.37)[80]: Let $\mathcal{G}$, B, $\{T^i\}$ be as in the theorem. If $\lim_i \mathrm{Tr}(T_x^i) = 0$ for all x in B, then $\lim_i \mathrm{Tr}(T_x^i) = 0$ for all x in A.

Proof. Take S to be the zero representation in the theorem.

Corollary (2.1.38)[80]: Assume that A has a dense self-adjoint subalgebra B such that every x in B is boundedly represented in $\hat{A}$. Let $\{T^i\}$ be a net of elements of $\hat{A}$, and $S^1, \dots, S^r$ a finite sequence of elements of $\hat{A}$ (not necessarily distinct), such that for all x in B,

$$\lim_i \mathrm{Tr}\,(T_x^i) = \sum_{k=1}^{r} \mathrm{Tr}\,(S_x^k). \tag{41}$$

Then, for all S in $\hat{A}$, $T^i \to_i S$ if and only if $S =$some S^k.

Proof. Let $\mathcal{G}$ be the set of all direct sums of at most r elements of $\hat{A}$. Then B is boundedly represented in $\mathcal{G}$. If $S^0 = S^1 \oplus \dots \oplus S^r$, then S^0 and T^i all belong to $\mathcal{G}$, and, by (41), we may apply the theorem to conclude, for x in A,

$$\lim_i \|T_x^i\| = \|S_x^0\| = \max_{k=1}^{r} \|S_x^k\|.$$

Now Theorem (2.1.24) gives the required conclusion.

A semi-simple connected Lie group with a faithful continuous matrix representation. Let U be a maximal compact subgroup of G. Fix Haar measure dg in G and du in U (du being normalized so that U has measure 1). Then, for x in $L_1(G)$ and y in $L_1(U)$, there are natural convolutions $x * y$ and $y * x$, both lying in $L_1(G)$. This is a special case of the definition of the convolution of finite measures on G (see [85]).

Denote by E the family of all minimal central projections in$L_1(U)$ (see [93]).

The convolution on $L_1(G) \times L_1(U)$ can be extended to a convolution on $C^*(G) \times L_1(U)$. Indeed, if $x \in L_1(G), y \in L_1(U)$, and T is a unitary representation of G,

$$\|T_{y*x}\| = \|T_y T_x\| \leqq \|T_y\| \, \|T_x\| \leqq \|y\|_{L_1(U)} \|x\|_{C^*(G)}.$$

Hence $\|y * x\|_{C^*(G)} \leqq \|y\|_{L_1(U)} \|x\|_{C^*(G)}$; and similarly for $x * y$. It follows that $x * y$ can be defined on $C^*(G) \times L_1(U)$ so as to be jointly continuous in both variables; and similarly for $y * x$. The equations $T_{x*y} = T_x T_y, T_{y*x} = T_y T_x$ are preserved under this extension.

Since finite linear combinations of elements $e * y$ ($e \in E,\ y \in L_1(U)$) are dense in $L_1(U)$, we easily obtain:

Lemma (2.1.39)[80]: Finite linear combinations of the $e * x * f$ ($e, f \in E,\ x \in C^*(G)$) are dense in $C^*(G)$.

Now it is proved in [87] that, for each e in E, the subalgebra $e * L_1(G) * e$ satisfies a standard polynomial identity. Since this subalgebra is dense in $e * C^*(G) * e$, we have:

Lemma (2.1.40)[80]: For each e in E, $e * C^*(G) * e$ satisfies a standard polynomial identity. Consequently there is an integer n such that every irreducible representation of $e * C^*(G) * e$ is of dimension equal to or less than n.

Let T be an irreducible representation of G, or, equivalently, of $C^*(G)$. For e in E, T_{e*x*e} leaves the range of T_e invariant and annihilates its orthogonal subspace.

Lemma (2.1.41)[80]: T restricted to $e * C^*(G) * e$ is irreducible on the range of T_e.

This is proved as [87].

Now let B' be the set of all finite linear combinations of elements $e * x * f$ $(e, f \in E, x \in C^*(G))$.

Lemma (2.1.42)[80]: B' is a dense self-adjoint subalgebra of $C^*(G)$. Every element of B' is boundedly represented in $\hat{G}$ $(= C^*(G))^\wedge$.

Proof. By Lemmas (2.1.39), (2.1.40), and (2.1.41).

This lemma of course connects with Theorem (2.1.36) and its corollaries.

We shall be considering the $n \times n$ complex unimodular group $G = SL(n)$, the group of all complex $n \times n$ matrices of determinant 1 (n being fixed). The irreducible unitary representations of G belonging to the various principal and supplementary series, as well as their characters, are described in [84]; and are proved in [96] to exhaust all of $\hat{G}$. We find the topology of $\hat{G}$. The principal tool for this will be Corollary (2.1.38) of Theorem (2.1.36), applied to the characters of the representations.

It is shown in [84] that the elements of $\hat{G}$ may be specified by parameters as follows:

By a (proper) set of parameters we shall mean a triple ν, μ, ρ, where, for some $r = 1, 2, \ldots, n$, $\nu = \nu_1, \nu_2, \ldots, \nu_r$, $\mu = \mu_1, \mu_2, \ldots, \mu_r$, and $\rho = \rho_1, \rho_2, \ldots, \rho_r$ are three r-termed sequences satisfying:

(i) The ν_i are positive integers satisfying $\sum_{i=1}^{r} \nu_i = n$;

(ii) The μ_i are integers;

(iii) The ρ_i are complex;

(iv) For some permutation p of $\{1, \ldots, r\}$, and some non-negative integer τ such that $0 \leqq 2\tau \leqq r$, we have: (a) $\nu_{p(i)}$ tor $i = 1, \ldots, 2\tau$; (b) for $q = 1, \ldots, \tau, \mu_{p(2q-1)} = \mu_{p(2q)}, \rho_{p(2q-1)} = \bar{\rho}_{p(2q)}, 0 < \operatorname{Im} \rho_{p(2q)} < 1$; (c) for $i = 2\tau + 1, \ldots, r, \rho_{p(i)}$ is real.

In case ν, μ, ρ satisfy (i), (ii), (iii) and (iv'), where (iv') is obtained from (iv) by replacing in (b) "$0 < \operatorname{Im} \rho_{p(2q)} < 1$" by "$0 < \operatorname{Im} \rho_{p(2q)} \leqq 1$," we shall speak of ν, μ, ρ as an extended set of parameters.

Fix an extended set of parameters ν, μ, ρ. The sth block of integers (with respect to ν) will be the set $\{\nu_1 + \cdots + \nu_{s-1} + 1, \cdots, \nu_1 + \cdots + \nu_s\}$. If $g \in G$, let g^*_{st} he the submatrix of g consisting of the rows of the sth block and the columns of the tth block. Define K to be the subgroup of G consisting of those k for which $k^*_{st} = 0$ whenever $s > t$ (triangular block matrices). Finally, denote by X the complex homomorphism of K:

$$X(k) = \prod_{j=1}^{r} |A_j|^{\mu_j + i\rho_j} A_j^{-\mu_j}, \tag{42}$$

where $A_j = \det(k^*_{jj})$. Obviously, K and X depend on ν, μ, ρ.

If ν, μ, ρ is a proper set of parameters, the homomorphism X of K induces an irreducible unitary representation $T = T^{\nu,\mu,\rho}$ of G (see [84]); these exhaust all of $\hat{G}$ (see [96]).

There are equivalences among the $T^{\nu,\mu,\rho}$, as follows:

Lemma (2.1.43)[80]: $T^{\nu,\mu,\rho} \cong T^{\nu',\mu',\rho'}$ if and only if (a) the length r of the sequences ν, μ, ρ is equal to the length of ν', μ', ρ', and (b) there exist a permutation p of $\{1, \cdots, r\}$, an integer m, and a real s, such that, for $i = 1, \cdots, r$, we have

$$\nu'_i = \nu_{p(i)} \qquad \mu'_i = \mu_{p(i)} \qquad \rho'_i = \rho_{p(i)} + s.$$

The representations $T^{\nu,\mu,\rho}$ are classified into series as follows, in accordance with the values of $r, \nu_1, \cdots, \nu_r$, and τ:

In view of Lemma (2.1.43), we assume, without loss of generality, that $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_r$. 11 this is so, the $(\nu_1, \cdots, \nu_r; \tau)$ series will be the set of all $T^{\nu,\mu,\rho}$ having the given ν and the given value of τ. Each T in G lies in one and only one such series. If $\tau = 0$, the series is principal; otherwise it is supplementary, lf $r = n$ and all $\nu_i = 1$, the series is nondegenerate; otherwise it is degenerate.

For example, if $n = 2$, the elements of $\hat{G}$ are classified as follows:

(i) The principal nondegenerate series : $\tau = 0$, $r = 2$, $\nu_1 = \nu_2 = 1$. By Lemma (2.1.43), the representations of this series are determined by $m = \mu_1 - \mu_2$ and $r = \rho_1 - \rho_2$. We write $T^{\nu,\mu,\rho} = T^{m,r}$ in this case where m and r run over the integers and the reals respectively. One has $T^{m,r} = T^{m',r'}$ if and only if either $m = m'$, $r = r'$ or $m = -m'$, $r = -r'$.

(ii) The supplementary nondegenerate series: $\tau = 1$, $r = 2$, $\nu_1 = \nu_2 = 1$. By Lemma (2.1.43), we may take $\mu_1 = \mu_2 = 0$, $\rho_1 = -is$, $\rho_2 = is$, where $0 < s < 1$. Writing T^s for $T^{\nu,\mu,\rho}$, we have $T^s \cong T^{s'}$ if and only if $s = s'$.

(iii) The principal degenerate series: $\tau = 0$, $r = 2$, $\nu_1 = 2$. This contains only the identity representation I.

Let Q_r be the family of all extended sets of parameters with sequences of length r. As a subset of $3r$-dimensional complex space, Q_r acquires a natural topology. The set $Q = \bigcup_{r=1}^{n} Q_r$of all extended sets of parameters has a natural topology as the direct sum of the Q_r; with this topology we call it the extended parameter space. The family P of all proper sets of parameters is a dense subspace of Q, called the (proper) parameter space.

Define an equivalence relation $\sim$ in Q by requiring that $(\nu, \mu, \rho) \sim (\nu', \mu', \rho')$ if and only if conditions (a) and (b) hold in Lemma (2.1.43). The set $\tilde{Q}$ of equivalence classes of $\sim$ inherits a natural quotient topology from Q, which is clearly locally compact and Hausdorff. The set $\tilde{P}$ of those equivalence classes which are contained in P is a dense subset of $\tilde{Q}$. The topology of $\tilde{Q}$ relativized to $\tilde{P}$ will be called the natural topology of $\tilde{P}$. By Lemma (2.1.43) $\tilde{P}$ is in one-to-one correspondence with $\hat{G}$. The topology of $\hat{G}$ transferred to $\tilde{P}$ will be the hull-kernel topology of $\tilde{P}$. We may sometimes fail to distinguish corresponding elements of $\tilde{P}$ and $\hat{G}$.

It is shown in [84] that each T in $\hat{G}$ is characterized by a complex function $\gamma = \gamma^T$ on G, called the character of T, and given by

$$\gamma(\delta) = \frac{\sum_s' X\left(\delta^{(s)}\right) \prod_{i=1}^{r} \left|D\left(\delta_i^{(s)}\right)\right| \left|\det \delta_i^{(s)}\right|^{\nu_i}}{|D(\delta)|}. \tag{43}$$

The notation in (43) will first be explained. Let ν, μ, ρ he a set of parameters for T, the sequences being of length r.

If δ is a diagonal $n \times n$ matrix, and s is a permutation of $\{1, \cdots, n\}$, then $\delta^{(s)}$ will be the diagonal matrix $(\delta^{(s)})_{ii} = \delta_{s(i),s(i)}$. If $i = 1, \cdots, r$, then δ_i will mean δ_{ii}^*; and $\delta_i^{(s)}$ means $(\delta^{(s)})_i$. Denote by Z the group of permutations of $\{1, \cdots, n\}$ which leave setwise invariant each ith block of integers with respect to $\nu (i = 1, \cdots, r)$. The symbol $\sum_s'$ means summation over a set of permutations s of $\{1, \cdots, n\}$ which contains exactly one permutation from each left coset sZ. The X is the complex homomorphism defined in (1). $D(\delta)$ is the discriminant of the characteristic equation of δ, i.e., for a diagonal $n \times n$ matrix δ, $D(\delta) = \sum_{1 \leqq i < j \leqq n} (\delta_{ii} - \delta_{jj})^2$ (if $(n = 1,\ D(\delta) = 1)$.

Clearly $\gamma(\delta^{(s)}) = \gamma(\delta)$ for each permutation s of $\{1, \cdots, n\}$. If g is a matrix in G whose eigenvalues are all distinct, we may define $\gamma(\delta)$ without ambiguity by setting $\gamma(\mathrm{g}) =$

$\gamma(\delta)$, where δ is any diagonal matrix whose diagonal elements are the eigenvalues of g. The γ so defined (on almost all of G) we call γ^T or $\gamma^{\nu,\mu,\rho}$.

Lemma (2.1.44)[80]: If x is of the form $y * z$, where y, z are continuous complex functions with compact support on G, then for each proper set of parameters ν, μ, ρ,

$$\mathrm{Tr}\left(T_x^{\nu,\mu,\rho}\right) = \int_G x(\mathrm{g})\,\gamma^{\nu,\mu,\rho}(\mathrm{g})d\mathrm{g}.$$

For this lemma, see [84].

Now (42) and (43) will be used to define $\gamma = \gamma^{\nu,\mu,\rho}$ as a function on G even when ν, μ, ρ is only an extended set of parameters. A routine verification gives:

Lemma (2.1.45)[80]: If (ν, μ, ρ) and (ν', μ', ρ') are in Q, and $(\nu, \mu, \rho) \sim (\nu', \mu', \rho')$, then $\gamma^{\nu,\mu,\rho} = \gamma^{\nu',\mu',\rho'}$.

In view of this lemma, γ^q may be considered as defined for the equivalence classes q belonging to $\hat{Q}$. Indeed, if $(\nu, \mu, \rho) \in q \in \hat{Q}$, $\gamma^q = \gamma^{\nu,\mu,\rho}$.

Lemma (2.1.46)[80]: If $q \in \hat{Q}$, $\{q^i\}$ is a net of elements of $\hat{Q}$, and $q^i \to q$ in the natural topology of $\hat{Q}$, then, for each bounded function x on G with compact support,

$$\int x(\mathrm{g})\,\gamma^{q^i}(\mathrm{g})d\mathrm{g} \underset{i}{\to} \int x(g)\,\gamma^q(\mathrm{g})d\mathrm{g}. \tag{44}$$

Proof. Since $\hat{Q}$ has a countable basis of open sets, it is enough to suppose that $\{q^i\}$ is a sequence.

As g in G ranges over the compact support of x, its eigenvalues also range over a compact set. Hence $\gamma^q(\mathrm{g})$ and the $\gamma^{q^i}(\mathrm{g})$ are all majorized by a constant times $1/|D(\delta)|$; and the latter is summable over any compact subset of G. Further, it is clear that $\gamma^{q^i}(\mathrm{g}) \to \gamma^q(\mathrm{g})$ lor almost all g. Equation (44) now follows from the Lebesgue dominated-convergence theorem.

Lemma (2.1.47)[80]: $\{q^i\}$ is a net of elements of $\hat{Q}$ converging to the point at infinity of $\hat{Q}$, then, for each bounded function x on G with compact support,

$$\int x(\mathrm{g})\,\gamma^{q^i}(\mathrm{g})d\mathrm{g} \underset{i}{\to} 0. \tag{45}$$

Proof. Δ being the diagonal subgroup of G, it is clearly possible to decompose Haar measure dg thus:

$$\int_G f(\mathrm{g})\,d\mathrm{g} = \int_\Delta d\mathrm{g} \int_G f(\mathrm{g}^{-1}\delta\mathrm{g})\,d_{\mu_\delta \mathrm{g}}\,, \tag{46}$$

where $d\delta$ is Haar measure on Δ, and ps is some measure on G depending on δ.

Let q^i contain the extended set of parameters (ν^i, μ^i, ρ^i) ; and let X^i be the homomorphism (42) corresponding to this set. In proving (45) it is clearly sufficient to suppose the ν^i are all the same. Substituting the definition of γ^{q^i} into (46), we find that $\int x(\mathrm{g})\,\gamma^{q^i}(\mathrm{g})d\mathrm{g}$ is the sum of a finite number of terms (the number being independent of i), each of which is of the form

$$\int_\Delta X^i(\delta)L(\delta)d\delta, \tag{47}$$

L being summable on Δ and independent of i.

Now $X^i(\delta) = X_1^i(\delta)X_2^i(\delta)$,where $X_1^i(\delta) = (X^i(\delta))/(|X^i(\delta)|), X_2^i(\delta) = |X^i(\delta)|$. But X_1^i is a character belonging to the dual group $\hat{\Delta}$ of the commutative group Δ. As $q^i \to \infty$ in $\hat{Q}$, it is easy to see that $X_1^i \to \infty$ in $\hat{\Delta}$. If X_2^i were independent of i, the desired conclusion that (47) approaches 0 in i would follow from the well-known theorem that the Fourier transform of a summable function L on $\widehat{\Delta}$ is 0 at ∞ (in $\widehat{\Delta}$). On the other hand, though X_2^i does depend on i, we have $|\mathrm{Im}\rho_k^i| \leqq 1$; and a uniformity argument shows that the desired conclusion follows in this case also.

Let q be an element of $\hat{Q} - \hat{P}$. Then the function γ^q is not the character of any representation. We shall show that γ^q is the sum of a finite number of characters of representations. It will turn out that the non-Hausdorff character of $\hat{G}$ arises from this sum formula.

Suppose that ν, μ, ρ is an extended, but not a proper, set of parameters; in fact, suppose that $\nu_1 = \nu_2 = 1$, $\mu_1 = \mu_2 = \mu$, $\rho_1 = \sigma - i$, $\rho_2 = \sigma + i$. Z; and S will denote a cross-section of left Z cosets, i.e., a set of permutations of $\{1, \cdots, n\}$ containing exactly one element from each left coset sZ. Let δ be a diagonal matrix in G; $\delta_j^{(s)}$ means the same.

By (43), $\gamma^{\nu,\mu,\rho} = \gamma$ is given by

$$|D(\delta)|\,\gamma(\delta) = \sum_{s\in S} \prod_{j=1}^{r} \left\{ |\det \delta_j^{(s)}|^{\mu_j = i\rho_j - \nu_j} (\det \delta_j^{(s)})^{-\mu_j} |det(\delta_j^{(s)})| \right\}.$$

For each $u, v = 1, \cdots, n$, $u \neq v$, we define $S(u,v)$ to be the set of all s in S for which $s(1) = u, s(2) = v$. Noting that $S = \bigcup_{u \neq v} S(u,v)$, we transform the last equation to

$$\begin{aligned} |D(\delta)|\,\gamma(\delta) = &\sum_{u,v=1;u\neq v}^{n} [\,|\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu}|\delta_{vv}|^{\mu+i\sigma-2}\delta_{vv}^{-\mu}] \\ &\times \sum_{s\in S(u,v)} \prod_{j=3}^{r} [\,|\det \delta_j^{(s)}|^{\mu_j = i\rho_j - \nu_j} (\det \delta_j^{(s)})^{-\mu_j} |\,D(\delta_j^{(s)})|] \\ = &\sum_{u<v} [\{|\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu}|\delta_{vv}|^{\mu+i\sigma-2}\delta_{vv}^{-\mu} + |\delta_{vv}|^{\mu+i\sigma}\delta_{vv}^{-\mu}|\delta_{uu}|^{\mu+i\sigma-2}\delta_{uu}^{-\mu}\} \\ &\times \sum_{s\in S(u,v)} \prod_{j=3}^{r} |\det \delta_j^{(s)}|^{\mu_j = i\rho_j - \nu_j} (\det \delta_j^{(s)})^{-\mu_j} |\,D(\delta_j^{(s)})|]. \end{aligned} \tag{48}$$

Now let $\nu' = (2, \nu_2, \cdots, \nu_r)$, $\mu' = (\mu, \mu_3, \cdots, \mu_r)$, , $\rho' = (\sigma, \rho_3, \cdots, \rho_r)$,. If Z' is the group of permutations of $\{1, \cdots, n\}$ leaving setwise invariant the blocks with respect to ν', then $S' = \bigcup_{u,v=1;u<v}^{n} S(u,v)$ is a cross-section of left Z' cosets. Therefore, abbreviating $\gamma^{\nu',\mu',\rho'}$ to γ', we have

$$\begin{aligned} |D(\delta)|\,\gamma'^{(\delta)} = &\sum_{u<v} \{|\delta_{uu}\delta_{vv}|^{\mu+i\sigma-2}(\delta_{uu}\delta_{vv})^{-\mu_j}|\delta_{uu} - \delta_{vv}|^2 \\ &\times \sum_{s\in S(u,v)} \prod_{j=3}^{r} |det\, \delta_j^{(s)}|^{\mu_j + i\rho_j - \nu_j} (det\, \delta_j^{(s)})^{-\mu_j} |\,D(\delta_j^{(s)})| \}. \end{aligned} \tag{49}$$

Note that for any complex x, y,

$$\frac{1}{|x|^2} + \frac{1}{|y|^2} - \frac{|x-y|^2}{|xy|^2} = \frac{y}{x|y|^2} + \frac{x}{y|x|^2}.$$

Equations (48) and (49) thus give

$$|D(\delta)|\left(\gamma(\delta)-\gamma'(\delta)\right)=\sum_{u\neq v}\Big\{|\delta_{uu}|^{\mu+i\sigma}\delta_{uu}^{-\mu-1}|\delta_{vv}|^{\mu+i\sigma-2}\,\delta_{vv}^{-\mu+1}$$
$$\times\sum_{s\in S(u,v)}\prod_{j=3}^{r}\,|\det\delta_j^{(s)}\,|^{\mu_j+i\rho_j-\nu_j}(\det\delta_j^{(s)})^{-\mu_j}|\,D(\delta_j^{(s)})|\Big\}$$
$$=|D(\delta)|\gamma''(\delta),$$
where $\gamma''=\gamma^{\nu'',\mu'',\rho''}$, and $\nu"=\nu$,
$$\mu''=(\mu+1,\mu-1,\mu_3,\cdots,\mu_r),\qquad \rho''=(\sigma,\sigma,\rho_3,\cdots,\rho_r).$$
We thus obtain
$$\gamma^{\nu,\mu,\rho}=\gamma^{\nu',\mu',\rho'}+\gamma^{\nu'',\mu'',\rho''}. \tag{50}$$
For convenience we describe an extended set of parameters ν,μ,ρ by a $3\times r$ matrix
$$A=\begin{bmatrix}\nu_1 & \nu_2 & \cdots & \nu_r\\ \mu_1 & \mu_2 & \cdots & \mu_r\\ \rho_1 & \rho_2 & \cdots & \rho_r\end{bmatrix}, \tag{51}$$
and write γ^A for $\gamma^{\nu,\mu,\rho}$. Now, iterating (50), we obtain the following lemma:

Lemma (2.1.48)[80]: If
$$A=\begin{bmatrix}1 & 1 & 1 & 1 & \cdots & 1 & 1 & \nu_{2\tau+1} & \cdots & \nu_r\\ \mu_1 & \mu_1 & \mu_2 & \mu_2 & \cdots & \mu_r & \mu_r & \mu_{2\tau+1} & \cdots & \mu_r\\ \sigma_1-i & \sigma_1+i & \sigma_2-i & \sigma_2+i & \cdots & \sigma_\tau-i & \sigma_\tau+i & \rho_{2\tau+1} & \cdots & \rho_r\end{bmatrix}, \tag{52}$$
we have $\gamma^A=\sum_{M_1,\cdots,M_\tau}\gamma^{(M_1\cdots M_\tau R)}$, where
$$R=\begin{bmatrix}\nu_{2\tau+1} & \cdots & \nu_r\\ \mu_{2\tau+1} & \cdots & \mu_r\\ \rho_{2\tau+1} & \cdots & \rho_r\end{bmatrix} \tag{53}$$
and each $M_j(j=1,\cdots,\tau)$ runs over the two possibilities
$$\begin{bmatrix}2\\ M_j\\ \sigma_j\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1 & 1\\ M_j+1 & M_j-1\\ \sigma_j & \sigma_j\end{bmatrix}. \tag{54}$$
For illustration, let us apply Lemma (2.1.48) to the 2×2 case. Here $\tilde{Q}-\tilde{P}$ contains exactly one element q, described by the matrix
$$\begin{bmatrix}1 & 1\\ 0 & 0\\ -i & i\end{bmatrix};$$
q is the limit, in the natural topology, of the representations T^s (of the supplementary series) as $s\to 1$. Since
$$\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}$$
corresponds to the identity representation I, and
$$\begin{bmatrix}1 & 1\\ 1 & -1\\ 0 & 0\end{bmatrix}$$
to the representation $T^{2,0}$ of the principal series, Lemmas (2.1.46) and (2.1.48) give
$$\lim_{s\to 1-}\gamma^{T^s}=\gamma^q=\gamma^I+\gamma^{T^{2,0}}. \tag{55}$$
We are now ready to describe completely the hullkernel topology of $\tilde{P}(\cong\hat{G})$.

If A is the matrix of ν,μ,ρ (see (51)), let $[A]$ he the point of $\tilde{Q}$ to which ν,μ,ρ belongs.

Theorem (2.1.49)[80]: With each point q of $\tilde{Q}$, we associate one or more points of $\tilde{P}$ as follows:

(i) If $q \in \tilde{P}$, with q is associated just q itself;

(ii) If $q \in \tilde{Q} - \tilde{P}$, $q = [A]$, where A is the matrix (52), with $|\operatorname{Im} \rho_j| < 1$ for $j > 2\tau$, then with q are associated precisely the

$$[M_1 \cdots M_\tau R],$$

where R is as in (53), and each M_j runs over the two alternatives (54).

Now if $A \subset \tilde{P}$, the hull-kernel closure of A consists exactly of those p in $\tilde{P}$ which are associated with some q in the natural closure of A (with respect to $\tilde{Q}$).

Proof. Recalling the definition of E, we define B as the set of all finite linear combinations of elements $e * x * y * f$, where $e, f \in E$, and x and y are continuous functions with compact support on G. Since the $x * y$ are dense in $C^*(G)$, B is dense in B'; hence, by Lemma (2.1.42), B is a dense self-adjoint subalgebra of $C^*(G)$, all of whose elements are boundedly represented in $\hat{G}$. By Lemma (2.1.44),

$$\operatorname{Tr}\left(T_x^{\nu,\mu,\rho}\right) = \int_G x\,(\mathrm{g})\gamma^{\nu,\mu,\rho}(\mathrm{g})d\mathrm{g} \qquad \text{for } x \in B. \tag{56}$$

Let $A \subset \tilde{P}$ If q belongs to the natural closure of A (in $\tilde{Q}$), there is a sequence $\{q^n\}$ of elements of A with $q^n \to q$. By (56) and Lemma (2.1.46),

$$\lim_n \operatorname{Tr}(T_x^{q^n}) = \int x\,(\mathrm{g})\gamma^q(\mathrm{g})d\mathrm{g}$$

for $x \in B$. By Lemma (2.10.48) and the definition of associated elements, this implies that for $x \in B$,

$$\lim_n \operatorname{Tr}\left(T_x^{q^n}\right) = \operatorname{Tr}\left(\sum_{q'} \oplus T_x^{q'}\right), \tag{57}$$

where q' runs over the elements of $\tilde{P}$ which are associated with q. This combined with Corollary (2.1.38) of Theorem (2.1.36) shows that every q' associated with q belongs to the hull-kernel closure of A.

Conversely, let p in. $\tilde{P}$ belong to the hull-kernel closure of A. Select a net $\{p^i\}$ of elements of A converging hull-kernelwise to p. Now no subnet $\{p'^i\}$ of $\{p^i\}$ converges (in the natural topology) to the point at infinity of $\tilde{Q}$. For, if it did, Lemma (2.1.47) and Corollary (2.1.38) of Theorem (2.1.36) would tell us that $\{p'^i\}$ converged hull-kernelwise to no limit at all; which is impossible.

Thus all natural cluster points of $\{p^i\}$ are in the finite part of $\tilde{Q}$. Let q be such a cluster point; and $\{p'^i\}$ a subnet of $\{p^i\}$ converging naturally to q. Again by Lemmas (2.1.46) and (2.1.48), and Corollary (2.1.38) of Theorem (2.1.36), $\{p'^i\}$ can converge in the hull-kernel topology to no p' except those associated with q; and the same is true of $\{p^i\}$. We have shown that every point p in the hullkernel closure of A is associated with a point in the natural closure of A. This completes the proof.

For illustration, consider the 2×2 case. We see that $\tilde{Q}$ with the natural topology is homeomorphic to the subset W of the plane consisting of:

(i) all (m, r), where m is a positive integer and r is real;

(ii) all $(0, r)$ where $r \geqq 0$;

(iii) all $(-s, 0)$, where $0 < s \leqq 1$;

(iv) an isolated point, say $(-2, 0)$.

Here the (m, r) or (i) or (ii) corresponds to the representation $T^{m,r}$ of the principal series; $(-s, 0)$ corresponds to the representation T^s of the supplementary series for $0 < s < 1$; $(-2, 0)$ corresponds to the identity representation I; and $(-1, 0)$ corresponds to the one and only point of $\tilde{Q} - \tilde{P}$. $(-1, 0)$ is associated with $(-2, 0)$ and $(2, 0)$.

Corollary (2.1.50)[80]: In the 2×2 case, transfer the hull-kernel topology of $\hat{G}$ to the subset W of the plane by means of the above correspondence. If $A \subset W - \{(-1,0)\}$ the hull-kernel closure of A is equal to

(a) the natural closure $\bar{A}$ of A unless $\bar{A}$ contains $(-1,0)$;

(b) $(\bar{A} - \{(-1,0)\}) \cup \{(-2, 0), (2, 0)\}$ if $\bar{A}$ contains $(-1,0)$.

From Corollary (2.1.50) we see that $\hat{G}$ is not Hausdorff. In fact, if $s \to 1-$, T^s approaches both I and $T^{2,0}$. The same is true for the group G with general n. However, the deviation from the Hausdorff property is rather weak, as is shown by the next corollary:

Corollary (2.1.51)[80]: If G is the $n \times n$ complex unimodular group, no net of elements of $\hat{G}$ converges to more than $2^{[n/2]}$ distinct limits.

Proof. By an argument exactly similar to the second half of the proof of Theorem (2.1.49), for each net of elements of $\hat{G}$ there is a q in $\tilde{Q}$ such that every limit of the net is associated with q. But, by the definition of associated elements, the largest number of elements in $\tilde{P}$ associated with any one q in $\tilde{Q}$ is $2^{[n/2]}$.

Recall that each extended set of parameters ν, μ, ρ, hence each q in $\tilde{Q}$, is associated with a certain value of τ, called its τ-value, namely, half the number of nonreal terms in the sequence ρ. Let $\tilde{Q}_\tau$ be the set of q in Q having τ –value τ. The following fact is immediate:

Lemma (2.1.52)[80]: An element q' associated with a q in $\tilde{Q}_\tau$ has τ -value $\tau' < \tau$.

Corollary (2.1.53)[80]: The topology of $\hat{G}$ relativized to the set A_τ of those representations having fixed τ -valued τ, is Hausdorff.

Proof. Consider A_τ, as a subset of $\tilde{P}$. If $B \subset A_\tau$, denote by $\bar{B}$ and $\bar{B}^H$ the closure of B in the natural and hull-kernel topologies respectively. It follows from Theorem (2.1.49) and Lemma (2.1.52) that

$$\bar{B} \cap A_\tau = \bar{B}^H \cap A_\tau .$$

Hence the hull-kernel and the natural topologies coincide when relativized to A_τ. Thus the former is Hausdorff on A_τ.

Corollary (2.1.54)[80]: The topology of $\hat{G}$, relativized lo the union of all the principal series, is Hausdorff.

Proof. Put $\tau = 0$ in Corollary (2.1.53).

Corollary (2.1.55)[80]: For each fixed τ_0, the set $\mathcal{G}(\tau_0)$ of all T in $\hat{G}$ with τ -values equal to or less than to is closed in $\hat{G}$.

Proof. By Lemma (2.1.52).

Note that $r = [n/2]$ is the largest permissible value of τ. Corollaries (2.1.53) and (2.1.55) show that the sequence

$$\mathcal{G}(0), \mathcal{G}(1), \cdots, \mathcal{G}(r) = \hat{G}$$

is an ascending sequence of closed subsets of $\hat{G}$ such that each $\mathcal{G}(i) - \mathcal{G}(i-1)$ is Hausdorff in the relativized topology of $\hat{G}$. An easy argument (see [92]) now shows:

Corollary (2.1.56)[80]: There is a finite increasing sequence

$$I_0 = \{0\}, \qquad I_1, I_2, \cdots, I_r = C^*(G)$$

of closed two-sided ideals of $C^*(G)$ such that each $I_i/I_{i-1} (i = 1, \cdots, r)$ has a Hausdorff structure space.

Corollary (2.1.57)[80]: A subset A of $\hat{G}$ which (considered as a subset of $\tilde{Q}$) is closed in the natural topology is also closed in $\hat{G}$, and is Hausdorff as a subspace of $\hat{G}$.

Proof. The closure of A in $\hat{G}$ follows immediately from Theorem (2.1.49). If $B \subset A$, the hull-kernel closure of B is equal to the natural closure. Thus the hull-kernel and natural topologies relative to A coincide, and the former is Hausdorff.

Corollary (2.1.58)[80]: Each principal series is a closed subset of $\hat{G}$.

Proof. By Corollary (2.1.57).

It is well known (see [84]) that the regular representation L of G is a direct integral of representations of the principal non-degenerate series. In fact, if $\mathcal{G}$ is the locally compact Hausdorff space consisting of the principal nondegenerate series (see Corollary (2.1.58)), then

$$L = \int_{\mathcal{G}} \oplus \ (\aleph_0 \cdot T)$$

with respect to a measure on $\mathcal{G}$ whose closed hull is $\mathcal{G}$. By Theorem (2.1.17), L is weakly equivalent to the set of all $\aleph_0 \cdot T$, $T \in \mathcal{G}$; hence (see remark preceding Theorem (2.1.15)) L is weakly equivalent to $\mathcal{G}$. It follows that the spectrum of L (see Theorem (2.1.15)) is the closure of $\mathcal{G}$ in G, i.e., $\mathcal{G}$ itself.

Theorem (2.1.59)[80]: The spectrum of the regular representation of G is precisely the principal nondegenerate series.

Thus G is an example of a locally compact group whose regular representation does not weakly contain all irreducible representations.

Theorem (2.1.59) also shows that the implication (i) → (iii) in Theorem (2.1.2) fails for general Banach $*$-algebras. Indeed, let A be obtained by adjoining a unit element to the group algebra $L_1(G)$, G being as usual the $n \times n$ unimodular group. Let T be an element of $\hat{G}$ not belonging to the principal nondegenerate series; and consider it as acting on A. If a positive functional ϕ on A associated with T is a weak* limit of sums ψ_i of positive functionals on A associated with the regular representation L, an easy argument shows that $\psi_i \to \phi$ weakly* even after ψ_i and ϕ are extended to $C^*(G)$; and hence that the regular representation weakly contains T. But this is untrue by Theorem (2.1.59). Therefore condition (iii) of Theorem (2.1.2) fails, when A, T are as defined above, and $\mathcal{S}$ consists of L only. On the other hand, L is well known to be faithful on A; so that (i) holds. Thus the implication (i) → (iii) fails in this situation.

One naturally asks what is the relationship between the topology of $\hat{G}$ discussed and the Borel structure on $\hat{G}$ defined by Mackey in [94].

Corollaries (2.1.51) and (2.1.56) of Theorem (2.1.49) have shown that the departure in $\hat{G}$ from the Hausdorff property is fairly weak, when G is a complex unimodular group. Presumably the same result is true for arbitrary connected semisimple Lie groups with faithful matrix representations.

Form the detailed study of the structure of the group C^*-algebras of semi-simple groups, or, more generally, of C^*-algebras A whose irreducible representations are all completely continuous. Perhaps, as suggested by Kaplansky in [94], the cases where the structure space is Hausdorff form the appropriate building-blocks for the general case. If so, it would appear that further progress must take two directions: (a) the analysis of A, in case $\hat{A}$ is Hausdorff, in terms of fibre bundles with $\hat{A}$ as base space; (b) the extension problem—how a construct A when I and A/I are known (I being a closed two-sided ideal of A). We made some headway in problem (a), in the case that all irreducible representations are of the same finite dimension.

Section (2.2): Type I

In [92], Kaplansky studied the class of those C^*-algebras $\mathfrak{A}$ such that every irreducible representation of $\mathfrak{A}$ maps $\mathfrak{A}$ into the completely continuous operators (CCR algebras). He proved that such an algebra $\mathfrak{A}$ has a composition series $\{\mathfrak{J}_\alpha\}$ (an increasing family $\{\mathfrak{J}_\alpha\}$ of ideals indexed by the set of ordinals less than or equal to some ordinal γ, such that $\mathfrak{J}_0 = 0$, $\mathfrak{J}_\gamma = \mathfrak{A}$ and if α is a limit ordinal then $\bigcup_{\beta<\alpha} \mathfrak{J}_\beta$ is dense in $\mathfrak{J}_\alpha$) such that the Jacobson structure space X_α of $\mathfrak{J}_{\alpha+1}/\mathfrak{J}_\alpha$ is Hausdorff. Kaplansky proved that X_α is locally compact and that $\mathfrak{J}_{\alpha+1}/\mathfrak{J}_\alpha$ is closed under multiplication by continuous functions on X_α. CCR algebras are not the most general algebras with such a composition series, and Kaplansky called a C^*-algebra GCR if it has a composition series $\{\mathfrak{J}_\alpha\}$ such that each $\mathfrak{J}_{\alpha+1}/\mathfrak{J}_\alpha$ is CCR, or equivalently a composition series $\{\mathfrak{J}_\alpha\}$ such that each $\mathfrak{J}_{\alpha+1}/\mathfrak{J}_\alpha$ is CCR and has a Hausdorff structure space.

If φ is a representation of a C^*-algebra $\mathfrak{A}$, then φ is type I (resp. II, III) if the weak closure of $\varphi(\mathfrak{A})$ is of type I (resp. II, III) in the sense of Murray and von Neumann. If all representations of $\mathfrak{A}$ are type I then we say a is type I. The C^*-group algebras of many locally compact groups are known to be type I.

We show that if $\mathfrak{A}$ is a separable C^*-algebra then $\mathfrak{A}$ is type I if and only if $\mathfrak{A}$ is GCR and $\mathfrak{A}$ is type I if and only if $\mathfrak{A}$ has a smooth dual (Theorems (2.2.7) and (2.2.8)). We also find a number of other conditions which for separable C^*-algebras are equivalent to being type I (Theorems (2.2.7) and (2.2.8)). We devoted to the proof of Theorems (2.2.7) and (2.2.8). We show that the structure space of a GCR algebra $\mathfrak{A}$ is T_1 if and only if $\mathfrak{A}$ is CCR. We derive necessary and sufficientc onditions for a C^*-algebra $\mathfrak{A}$ to have the property: every w^*-limit point of the pure states of $\mathfrak{A}$ is proportional to a pure state of $\mathfrak{A}$ (Theorem (2.2.16). It is necessary but not sufficientt hat such an $\mathfrak{A}$ be CCR with a Hausdorff structure space. We apply the technique of the proof of Theorem (2.2.7) to show that there are representations of the commutation relations of quantum field theory which generate factors of type II_∞. (resp. III). A result of this nature has been announced by Garding and Wightman [118] but the proof has not been published. We find an analogue for C^*-algebras of a theorem of Bishop and DeLeeuw characterizing the Choquet boundary of a uniformly closed separating subalgebra of $C_c(X)$ [101]. This is a simple application of Lemma (2.2.2).

If a C^*-algebra $\mathfrak{A}$ has no non-zero GCR ideals (ideals which as sub- algebras of $\mathfrak{A}$ are GCR algebras) then we call $\mathfrak{A}$ an NGCR algebra. Kaplan- sky proved [92] that any C^*-algebra $\mathfrak{A}$ contains a largest GCR ideal $\mathcal{A}$ and W/9 is an NGCR algebra. In Theorem (2.2.7) we find a number of properties which for separable C^*-algebras are equivalent to being NGCR. A major step in the proof of this equivalence and, in fact, in the proof of Theorems (2.2.7) and (2.2.8) is Lemma (2.2.4). Lemma (2.2.4) states that an NGCR algebra contains an "approximately ascending" sequence of "approximate matrix algebras" of order $2, 4, \cdots, 2^n, \cdots$. It follows that NGCR algebras behave in many ways like von Neumann algebras of type II and III (von Neumann algebras of type II and III are NGCR, see [106]). The infinite tensor product $\mathfrak{A} = \prod_{i=0}^{\infty} \oplus \mathfrak{A}_i$ (owhere $\mathfrak{A}_0$ is an arbitrary C^*-algebra, and $\mathfrak{A}_i$ is a 2×2 matrix algebra for $i \neq 0$ seems to come closer to approximating the structure of an arbitrary NGCR algebra.

The hypothesis of the lemma should be strengthened to include the assumption: If $f(A) \geqq 0$ for all f in P then $A \geqq 0$. Under the augmented hypothesis, $\mathfrak{A}$ is order isomorphic (in the natural fashion) to a linear space of functions on P and so the proof cited in [106] proves this weakened Lemma (2.2.7). It should not be difficultt o verify that the extra hypothesis is satisfied each place that Lemma (2.2.7) is used in [106].

A C^*-algebra $\mathfrak{A}$ is a uniformly closed self-adjoint algebra of operators on a (complex) Hilbert space $\mathfrak{H}$. Since we wish our results to be applicable to group representations, we do not assume that $\mathfrak{A}$ has a unit, but if $\mathfrak{A}$ does have a unit we suppose that it is the identity operation I on $\mathfrak{H}$, and in any case we suppose that the closed linear span of Ax for A in $\mathfrak{A}$ and x in $\mathfrak{H}$ is $\mathfrak{H}$. A state f of $\mathfrak{A}$ is a positive linear functional on $\mathfrak{A}$ such that $\sup\{f(A): A \in \mathfrak{A}, 0 \leqq A \leqq I\} = 1$. If $I \in \mathfrak{A}$ this is equivalent to $f(I) = 1$. If $I \in \mathfrak{A}$, the set of states of $\mathfrak{A}$ is convex and w^*-compact. The extreme points of the set of states of $\mathfrak{A}$ are called pure states. Each state f of $\mathfrak{A}$ gives rise to a representation φ_f of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}_f$ such that $f = \omega_x \varphi_f$ (ω_x is defined below) and $[\varphi_f(\mathfrak{A})X] = \mathfrak{H}_f$ for some x in $\mathfrak{H}_f$, the representation φ_f is uniquely determined (up to unitary equivalence) by these properties and, furthermore, f is pure if and only if φ_f is irreducible, see [115]. We denote by $\mathcal{S}$ the set of states of $\mathfrak{A}$ with the relative w^*-topology, by $\mathcal{P}_0$ the set of pure states of $\mathfrak{A}$ and by $\mathcal{P}$ the relative w^*-closure of $\mathcal{P}_0$ in $\mathcal{S}$. If $I \in \mathfrak{A}$ and if $\mathfrak{M}$ is a self-adjoint linear subspace of $\mathfrak{A}$ containing I then states of $\mathfrak{M}$ (positive functionals f such that $f(I) = 1$) have extensions to states of $\mathfrak{A}$, and it follows from the Krein-Milman theorem that pure states of $\mathfrak{M}$ (extreme points of the set of states of $\mathfrak{M}$) have extensions to pure states of $\mathfrak{A}$. If $x \in \mathfrak{H}$ then ω_x is the linear functional $(\cdot\, x, x)$ defined on the algebra of bounded operators on $\mathfrak{H}$). (If $f = \{(x, f(x)): x \in \text{domain } f\}$ is a function we also let $f(\cdot)$ denote f). If $\|x\| = 1$ then $\omega_x|\mathfrak{A}$ is a state. If E is a (self-adjoint) projection on H, we also denote by E the set $\{x: x \in \mathfrak{H}, x = Ex\}$. If $F \subset \mathfrak{H}$ then $[F]$ is the closed linear span of F (or the orthogonal projection on this subspace of $\mathfrak{H}$). If x and y are in $\mathfrak{H}$ then $|(x, y)| = \|x\|\|y\|$ if and only if x and y are proportional.

If A is self-adjoint and in $\mathfrak{A}$, if f is a continuous complex valued function of a real variable and if either $f(0) = 0$ or $I \in \mathfrak{A}$ then $f(A)$ is in $\mathfrak{A}$ W. If f_n, is a sequence of polynomials converging to f uniformly on the spectrum of A and such that $f_n(0) = 0$ if $f(0) = 0$ then $f(A)$ is by Definition $\lim_n f_n(A)$. Thus if A is a self-adjoint operator on $\mathfrak{H}$ and if $Ax = \lambda x$ for some complex λ and some vector x in $\mathfrak{H}$ then $f(A)x = \lim_n f_n(A)x = \lim_n f_n(\lambda)x = f(\lambda)x$. Another elementary fact we shall need is that if A is a positive matrix then determinant $A \geqq 0$.

The structure space X of an arbitrary algebra $\mathfrak{A}$ is the set of primitive ideals of $\mathfrak{A}$, and the closed sets in X are the sets K such that if $\mathcal{Y} \supset \bigcap_{x \in K} x$ then $\mathcal{Y} \in K$, see [88]. We let $\mathfrak{A}(x) = \mathfrak{A}/x$,we let ψ_x be the canonical map, A$\psi_x: \mathfrak{A} \to \mathfrak{A}(x)$. By [92] we can suppose $\mathfrak{A}(x)$ acts faithfully and irreducibly on a Hilbert space $\mathfrak{H}_x$. If $A \in \mathfrak{A}$, we let $A(x) = \psi_x(A)$.

Lemma (2.2.1)[100]: Let A be a C^*-algebra with a unit I, let φ be an irreducible representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$, let $x_1, \cdots, x_k$ be unit vectors in $\mathfrak{H}$, let $\mathcal{K}$ be the w^*-closedc onvexh ull of the set $\{\omega_x \varphi: x \text{ is a unit vector in } [\{x_1, \cdots, x_k\}]\}$ of states of $\mathfrak{A}$. Then

(∗) $\mathcal{K} = \{\lambda: \lambda \text{ is a state of } \mathfrak{A} \text{ and if } B \in \mathfrak{A}, \text{ if } 0 \leqq B \leqq I, \text{ if } \varphi(B)x_j = x_j, \text{ for all j, then } \lambda(B) = 1\}$.

Let $\mathcal{K}_1$ be the right member of (∗). Then $\mathcal{K}_1$ is w^*-compact and convex and since $\omega_x \varphi \in \mathcal{K}_1$ if x is a unit vector in $K = [\{x_1, \cdots, x_k\}]$, $\mathcal{K} \subset \mathcal{K}_1$. Let λ be an arbitrary extreme point of $\mathcal{K}_1$. We will show that $\lambda \in \mathcal{K}$ and by the Krein-Milman theorem this will imply that $\mathcal{K}_1 \subset \mathcal{K}$ and the proof will be complete.

Suppose $\lambda = (\lambda_1 + \lambda_2)/2$ where λ_i is a state of $\mathfrak{A}$. If B is in $\mathfrak{A}$ and $0 \leqq B \leqq I$ then $\lambda_1(B),\ \lambda_2(B) \leqq 1$. If further $\varphi(B)x_j = x_j$, for all j then $1 = \lambda(B) = (\lambda_1(B) + \lambda_2(B))/2$. Thus $\lambda_1(B),\ \lambda_2(B) = 1$ and $\lambda_1, \lambda_2 \in \mathcal{K}_1$. Since λ is an extreme point of $\mathcal{K}_1$, $\lambda_1 = \lambda = \lambda_2$ and so λ is a pure state.

If $\mathcal{Y}$ is a unit vector in $\mathfrak{H}$ but not in K then $\mathcal{Y} = x + z$ where $x \in K, z \perp K$ and $\|x\| < 1$. There is a self-adjoint B in $\mathfrak{A}$ with $\varphi(B)x_j = x_j$, for all j and $\varphi(B)z = 0$ [89]. Let f be the function defined by $f(x) = 0, x, 1$, as $x \leqq 0$, $x \in [0,1]$ or $x \geqq 1$ respectively. Then

$$\varphi\big(f(B)\big)z = f\big(\varphi(B)\big)z = f(0)z = 0$$
$$\varphi\big(f(B)\big)x_j = f\big(\varphi(B)\big)x_j = f(1)x_j = x_j$$

for all j and since $0 \leqq f(x) \leqq 1$, $0 \leqq f(B) \leqq I$ and it follows that $\omega_{\mathcal{Y}}\varphi \notin \mathcal{K}_1$ and $\lambda \neq \omega_{\mathcal{Y}}\varphi$. If $\lambda \neq \omega_z\varphi$ for any z in $\mathfrak{H}$ then φ_λ and φ are not unitarily equivalent. Let w be in $\mathfrak{H}_\lambda$. so that $\lambda = \omega_w\varphi$. By [107] there is a self-adjoint C in $\mathfrak{A}$ with $\varphi_\lambda(C)w = 0$ and $\varphi(C)x_j = x_j$ for all j. As above $\varphi(f(C))x_j = x_j$, and $\varphi_\lambda(f(C))w = 0$ and $\lambda(f(C)) = 0$. This is a contradiction and so $\lambda = \omega_x\varphi$ for some x in K, $\lambda \in \mathcal{K}$ and the proof is complete.

Lemma (2.2.2)[100]: Using the notation of Lemma (2.2.1), if $\mathcal{V}$V is a relatively w^*-open neighborhood of $\mathcal{K}$ in $\mathcal{S}$ then there is a positive number δ and a B in $\mathfrak{A}$ such that $0 \leqq B \leqq I$, $\varphi(B)x_j = x_j$ for all j and

$$\{\lambda : \lambda \in \mathcal{S}, \lambda(B) \geqq 1 - \delta\} \subset \mathcal{V}\text{V} .$$

The sets $\{\lambda : \lambda \in \mathcal{S}, \lambda(B) \geqq 1 - \delta\} \cap (\mathcal{S} \sim \mathcal{V}\text{V})$, where $0 \leqq B \leqq I$, $\varphi(B)x_j = x_j$, for all j and $\delta > 0$, are closed and by Lemma (2.2.1) this family of sets has an empty intersection. Since $\mathcal{S}$ is compact, there are a finite number of these sets with an empty intersection, and so there are $B_1, \cdots, B_n$ in $\mathfrak{A}$ and there is a positive δ_0, such that $0 \leqq B_i \leqq I$, $\varphi(B_i)x_j = x_j$, and

$$\{\lambda : \lambda \in \mathcal{S}, \lambda(B_i) \geqq 1 - \delta_0, i = 1, \cdots, n\} \subset \mathcal{V}\text{V} .$$

Let $B = (1/n)\sum_{i=1}^n B_i$. Then $0 \leqq B \leqq I$ and $\varphi(B)x_j = x_j$. If $\lambda \in \mathcal{S}$ and $\lambda(B) \geqq 1 - (\delta_0/n)$, then $\lambda(\sum_{i=1}^n B_i) \geqq n - \delta_0$, $\lambda(B_i) \geqq 1 - \delta_0$ for $i = 1, \cdots, n$, and so $\lambda \in \mathcal{V}$V . Let $\delta = \delta_0/n$ and the proof is complete.

Lemma (2.2.3)[100]: Let ε be a positive number, let n be a positive integer. There is a positive number $\delta(\varepsilon, n) = \delta$ such that if $A_1, \cdots, A_n$ are operators on a Hilbert space $\mathfrak{H}$, if $0 \leqq A_i \leqq I$, if x is a vector in the unit sphere of $\mathfrak{H}$ and if

$$(A_n \cdots A_1 x, x) > 1 - \delta ,$$

then $(A_i x, x) > 1 - \varepsilon$ and $\|A_i x - x\| < \varepsilon$ for $i = 1, \cdots, n$.

Let $\delta(\varepsilon, n) = \min\{1, 2^{-1}\varepsilon^2$ and let $\delta(\varepsilon, n) = \delta(\varepsilon, n-1)^2/4$. If $0 \leqq A_i \leqq I$ and $(A_i x, x) > 1 - 2^{-1}\varepsilon^2$ then

$$\|A_i x - x\|^2 = (A_1 x, A_1 x) - 2(A_1 x, x) + (x, x) < 1 - 2 + \varepsilon^2 + 1 = \varepsilon^2$$

and so the lemma is true in the case $n = 1$. Suppose inductively that $n \geqq 2$ and that the lemma is true for $n - 1$. If $(A_n \cdots A_1 x, x) > 1 - \delta(\varepsilon, n)$ then

$$1 - \delta(\varepsilon, n) < (A_n^{1/2} A_{n-1} \cdots A_1 x, A_n^{1/2} x) \leqq (A_n x, x).$$

By the Definition of $\delta(\varepsilon, n)$, $\|A_n x - x\| < \delta(\varepsilon, n-1)/2$ and so

$$(A_{n-1} \cdots A_1 x, x) \geqq (A_n \cdots A_1 x, x) - \|A_n x - x\| > 1 - \delta(\varepsilon, n-1).$$

By the induction hypothesis, $(A_i x, x) > 1 - \varepsilon$ and $\|A_i x - x\| < \varepsilon$ for $i = 2, \cdots, n$ and the proof is complete.

The next lemma may be regarded as an analogue for C^*-algebras of the theorem of Murray and von Neumann which states that every factor of type II_1 contains a factor of type II_1, which is hyperfinite. (A factor of type II_1 is hyperfinite if it is generated by an increasing sequence of factors of type $\mathrm{I}_2, \mathrm{I}_4, \cdots, \mathrm{I}_{2^n}, \cdots$.) If $\mathfrak{A}$ is a C^*-algebra with no non-zero GCR ideals then we choose a subalgebra of $\mathfrak{A}$ which is generated by operators in $\mathfrak{A}$ which are strong approximations to a sequence of matrix units of order $2^n, n = 1, 2, \cdots$ (i.e., a sequence of sets $\{E_{ij}^{(n)}: i, j = 1, \cdots, 2^n\}$, $n = 1, 2, \cdots$ of non-zero operators, in general not in $\mathfrak{A}$, such that $E_{ij}^{(n)} E_{st}^{(n)} = \delta_s^j E_{it}^{(n)}$, $E_{ij}^{(n)} = E_{ji}^{(n)*}$ but in general $\sum_i E_{ij}^{(n)} \neq I$). The operators which generate the subalgebra will be denoted by $V(a_1, \cdots, a_n)$, where $a_i = 0$ or 1 and $n = 1, 2, \cdots$, and $V(a_1, \cdots, a_n)$ will be a strong approximation to $E_{j1}^{(n)}$ for some j in $1, \cdots, 2^n$. This construction is fundamental to the proofs of Theorems (2.2.7) and (2.2.8).

Lemma (2.2.4)[100]: Let $\mathfrak{A}$ be a C^*-algebra with unit I and with no non-zero GCR ideals, let $S_0, S_1, S_2, \cdots$ be a sequence of self-adjoint elements in $\mathfrak{A}$, and suppose S_0, is positive and II $\|S_0\| = 1$. There are non-zero operators $V(a_1, \cdots, a_n)$ in the unit sphere of $\mathfrak{A}$, $a_i = 0$ or $1, n = 1, 2, \cdots$ and there is a self-adjoint $2^n \times 2^n$ matrix $(\alpha_{(a_1,\cdots,a_n)(b_1,\cdots,b_n)})$ such that if we let

$$T_n = \sum_{\substack{(a_1,\cdots,a_n)\\(b_1,\cdots,b_n)}} \alpha_{(a_1,\cdots,a_n)(b_1,\cdots,b_n)} V(a_1, \cdots, a_n)\, V(b_1, \cdots, b_n)^*$$

$$E(n) = \sum_{(a_1,\cdots,a_n)} V(a_1, \cdots, a_n)\, V((a_1, \cdots, a_n)^*$$

then the following properties are satisfied:

if $k \geqq 2, \|E(k)(S_{k-1} - T_{k-1})E(k)\| < 1/k$; (58)

$S_0 + (1/4)I \geqq V(0)V(0)^*$; (59)

if $j \leqq k$, if $a_1, \cdots, a_j \neq b_1, \cdots, b_j, V(a_1, \cdots, a_j)^*\, V(b_1, \cdots, b_k) = 0$; (60)

if $k \geqq 2, V(a_1, \cdots, a_k) = V(a_1, \cdots, a_k) = V(a_1, \cdots, a_{k-1})V(0, \cdots, 0, a_k)$; (61)

if $j < k, V(a_1, \cdots, a_j)^* V(a_1, \cdots, a_j)V(0, \cdots, 0, a_k) = V(0, \cdots, 0, a_k)$; (62)

$V(0, \cdots, 0) \geqq 0$. (63)

Let $B(0) = S_0^{1/2}$, let $V(\varnothing) = V(a_1, \cdots, a_0) = I$. If n is a non-negative Integer we suppose inductively that non-zero operators $B(n)$ and $V(a_1, \cdots, a_j)$ in the unit sphere of $\mathfrak{A}$ are defined if $0 \leqq j \leqq n$ and the self-adjoint matrix $(\alpha_{(a_1,\cdots,a_j),\ (b_1,\cdots,b_j)})$, is defined if $0 < j < n$, and if $0 < j \leqq k \leqq n$ then (58), $\cdots$, (63) are satisfied, and $B(n) \geqq 0$, $\|B(n)\| = 1$ and

$$V(a_1, \cdots, a_n)^*\, V(a_1, \cdots, a_n)B(n) = B(n). \quad (64)$$

If $n = 0$ the inductive assumption is true.

Let μ be a pure state of $\mathfrak{A}$ with $\mu(B(n)) = 1$. (There is a pure state μ_0 of the C^*-algebra generated by $B(n)$ with $\mu_0(B(n)) = 1$, namely evaluation at a point in the spectrum of $B(n)$, and μ is an extension of μ_0 to a pure state of $\mathfrak{A}$.) Suppose $n \neq 0$ and let

$$\alpha_{(a_1,\cdots,a_n),\ (b_1,\cdots,b_n)} = \mu(V(a_1, \cdots, a_n)^*\, S_n V(a_1, \cdots, a_n)) .$$

Let $\mu = \omega_x \varphi_\mu$ where x is a vector in $\mathfrak{H}_\mu$, let $\mathcal{K}$ be the closed convex hull of the set of states $\omega_{\mathcal{Y}} \varphi_\mu$ where $\mathcal{Y}$ is a unit vector in the linear span K of the $\varphi_\mu(V(a_1, \cdots, a_n))x$. If $z_1 = \varphi_\mu(V(a_1, \cdots, a_n))x$ and $z_2 = \varphi_\mu(V(b_1, \cdots, b_n))x$ then, using (60),

$$(\varphi_\mu(T_n)z_2, z_1) = \alpha_{(a_1,\cdots,a_n),\ (b_1,\cdots,b_n)}(\varphi_\mu(V(a_1, \cdots, a_n)V(b_1, \cdots, b_n)^*)z_2, z_1).$$

However

$$\varphi_\mu(V(a_1,\cdots,a_n)^* V(a_1,\cdots,a_n)) \mid \text{range } \varphi_\mu(B(n)) = \text{identity}$$

by (64) and since $(\varphi_\mu(B(n))x,x) = 1 = \|\varphi_\mu(B(n))x\| \, \|x\|$, x is equal to $\varphi_\mu(B(n))x$, and so $x \epsilon$ range $\varphi_\mu(B(n))$ and $\varphi_\mu(V(a_1,\cdots,a_n)^* V(a_1,\cdots,a_n))x = x$. Thus

$$(\varphi_\mu(T_n)z_2,z_1) = \alpha_{(a_1,\cdots,a_n),\ (b_1,\cdots,b_n)}$$
$$\times (\varphi_\mu(V(b_1,\cdots,b_n)^* V(b_1,\cdots,b_n))x, \varphi_\mu(V(a_1,\cdots,a_n)^* V(a_1,\cdots,a_n))x$$
$$= \alpha_{(a_1,\cdots,a_n),\ (b_1,\cdots,b_n)} = (\varphi_\mu(S_n)z_2,z_1).$$

This implies that $K\varphi_\mu(S_n)\mathrm{K} = K\varphi_\mu(T_n)K$ and so $\omega_{\mathcal{Y}}(\varphi_\mu(S_n - T_n)) = 0$ for $\mathcal{Y}$ in K. Let

$\mathsf{W} = \{\lambda: \lambda \in S,\ |\lambda(S_n - T_n)| < 1/(n+1)\}$.

Then W is an open subseto f $\mathcal{S}$ and by what we have just proved, $\mathcal{K} \subset \mathsf{W}$. By Lemma (2.2.2), there is a positive γ and a B in $\mathfrak{A}$ such that $0 \leqq B \leqq I$, $\varphi_\mu(B)\varphi_\mu V(a_1,\cdots,a_n)x = \varphi_\mu V(a_1,\cdots,a_n)x$ for all $a_1,\cdots,a_n$, and

$$\{\lambda: \lambda \in S, \lambda(\mathrm{B}) > 1-\gamma\} \subset \mathsf{W}.$$

We no longer suppose $n \neq 0$. For each ε in (0,1), let f_ε be the function defined by: $f_\varepsilon((-\infty, 1-\varepsilon]) = 0, f_\varepsilon([1-(\varepsilon/2),+\infty)) = 1, f_\varepsilon$ is linear on $[1-\varepsilon, 1-(\varepsilon/2)]$. If $n = 0$, let $A = B(0)$, if $n \neq 0$, let

$$A = B(n)\prod_{a_1,\cdots,a_n} V(a_1,\cdots,a_n)^* \, BV(a_1,\cdots,a_n).$$

(We have not specified the order of the factors in the product on the right, however any order will do.) Let $B_\varepsilon = f_\varepsilon(AA^*)$, let $\sigma = \min\{1/16,\ \delta(\gamma/2^{2n+1}, 2^{n+1}+2)/5\}$ where $\delta(\gamma/2^{2n+1}, 2^{n+1}+2)$ is defined by Lemma (2.2.3).

It was proved in [106] that, if C is a non-zeroe lement of $\mathfrak{A}$ then there is an irreducible representation φ of $\mathfrak{A}$ such that $\varphi(C)$ is not completelyc ontinuous. In fact it is easy to see that the set of all C in $\mathfrak{A}$ such that $\varphi(C)$ is completely continuous for each irreducible representation φ of $\mathfrak{A}$ is a CCR ideal in a and so is zero. We have already proved that $\varphi_\mu(B(n))x = x$. Thus $\varphi_\mu(A)x = x = \varphi_\mu(A^*)x$, $\varphi_\mu(B_\sigma)x = x$ and so $B_\sigma \neq 0$. Let φ be an irreducible representation of $\mathfrak{A}$ such that $\varphi(B_\sigma)$ is not completely continuous. Then the range of $\varphi(B_\sigma)$ is infinite dimensional, and we can choose orthogonal unit vectors $\mathcal{Y}$ and z in the range of $\varphi(B_\sigma)$. There is a self-adjoint C_0 in $\mathfrak{A}$ such that $\varphi(C_0)\mathcal{Y} = \mathcal{Y}$, $\varphi(C_0)z = 0$ [89]. Let f be the function defined by: $f(x) = 0, x, 1$ as $x \leqq 0, \epsilon[0,1], \geqq 1$ respectively, let $C = f(C_0)$. Then $0 \leqq C \leqq I$ and $\varphi(C)\mathcal{Y} = \mathcal{Y}$, $\varphi(C)z = 0$. Let $D_0 = B_{2\sigma}CB_{2\sigma}$, let $D_1 = B_{4\sigma} - D_0$. let U be a unitary operator in $\mathfrak{A}$ such that $\varphi(U)\mathcal{Y} = z$ [107]. Let $V = f_{\sigma'}(D_1)Uf_{\sigma'}(D_0)$, let k be the function defined by $k(x) = (f_{1/2}(x)x^{-1})^{1/2}$ for non-zero x, $k(0) = 0$, let $V(0_n, 1) = Vk(V^* V)$, $V(0_{n+1}+1) = (f_{1/2}(V^* V))^{1/2}$, and if $a_1,\cdots,a_n \neq 0_n$, let

$$V(a_1,\cdots,a_{n+1}) = V(a_1,\cdots,a_n)V(0_n,a_{n+1}). \qquad (65)$$

(Here and throughout, we use the symbol 0_j ,t o indicate the family $0,\cdots,0$ of j zeros.) Let $B(n+1) = f_{1/4}(V^* V)$.

Since $0 \leqq f_{1/2}, V(0_{n+1})$ satisfies (63). Also $B(n+1)$ and $V(a_1,\cdots,a_{n+1})$ are in the unit sphere of $\mathfrak{A}$. It follows from (64) that

$$V(a_1,\cdots,a_n)^* V(a_1,\cdots,a_n)AA^* = AA^*$$

and, since $f_{8\sigma}$, is a limit of polynomials without constant terms,

$$V(a_1,\cdots,a_n)^* V(a_1,\cdots,a_n)B_{8\sigma} = B_{8\sigma}.$$

Since $f_{8\sigma}(x)f_{4\sigma}(x) = f_{4\sigma}(x)$ and $f_{8\sigma}(x)f_{2\sigma}(x) = f_{2\sigma}(x)$, $f_{8\sigma}D_i = D_i$. Since f_{σ}. is a limit of polynomials without constant terms, $B_{8\sigma}V = V$, $B_{8\sigma}V^* V = V^* V$ and $B_{8\sigma}f_{1/2}(V^* V) = f_{1/2}(V^* V)$. Thus

$$\begin{aligned} &V(a_1, \cdots, a_n)^* V(a_1, \cdots, a_n) V(0_n, a_{n+1}) \\ &\qquad = V(a_1, \cdots, a_n)^* = V(a_1, \cdots, a_n) B_{8\sigma} V(0_n, a_{n+1}) \\ &\qquad = B_{8\sigma} V(0_n, a_{n+1}) = V(0_n, a_{n+1}) \end{aligned}$$

which proves (62) if $j = n$, $k = n + 1$. If $j < n$, then

$$\begin{aligned} &V(a_1, \cdots, a_j)^* V\big(a_1, \cdots, a_j\big) V(0_n, a_{n+1}) \\ &\qquad = V(a_1, \cdots, a_j)^* V(a_1, \cdots, a_j) V(0_n)^* \, V(0_n) V(0_n, a_{n+1}) \\ &\qquad = V(0_n)^* \, V(0_n) V(0_n, a_{n+1}) = V(0_n, a_{n+1}) \end{aligned}$$

Which proves (62). In particular, $V(0_n)^* \, V(0_n) V(0_n, a_{n+1}) = V(0_n, a_{n+1})$, and so

$$\begin{aligned} &(V(0_n)^* \, V(0_n))^{1/2} \mid \text{range}\, V(0_n, a_{n+1}) \\ &\qquad = V(0_n)^* \, V(0_n) \mid \text{range}\, V(0_n, a_{n+1}) = I \mid range\, V(0_n, a_{n+1}) \,, \\ &V(0_n) V(0_n, a_{n+1}) = (V(0_n)^* \, V(0_n))^{1/2} \, V(0_n, a_{n+1}) = V(0_n, a_{n+1}) \,, \end{aligned}$$

using the fact that $0 \leqq V(0_n)$ by (63). This together with (65) shows that (61) is satisfied for $k = n + 1$. Furthermore

$$V(a_1, \cdots, a_{n+1})^* V(a_1, \cdots, a_{n+1}) B(n+1) = V(0_n, a_{n+1})^* \, V(0_n, a_{n+1}) B(n+1)$$

and since

$$\begin{aligned} V(0_n, 1)^* \, V(0_n, 1) &= k(V^* \, V) V^* \, V k(V^* \, V) = f_{1/2}(V^* \, V) \\ V(0_{n+1})^* \, V(0_{n+1}) &= f_{1/2}(V^* \, V) \,, \end{aligned}$$

we have

$$\begin{aligned} V(a_1, \cdots, a_{n+1})^* V(a_1, \cdots, a_{n+1}) B(n+1) &= f_{1/2}(V^* \, V) \, f_{1/4}(V^* \, V) \\ &= f_{1/4}(V^* \, V) = B(n+1) \,, \end{aligned}$$

and (64) is true for $n + 1$.

Since $\mathcal{Y}$ and $\mathcal{z}$ are in the range of $\varphi(B_\sigma)$, and since $B_{4\sigma}B_\sigma = B_{2\sigma}B_\sigma = B_\sigma$, $\varphi(B_{2\sigma})\mathcal{Y} = \mathcal{Y}$ and $\varphi(B_{4\sigma})\mathcal{z} = \varphi(B_{2\sigma})\mathcal{z} = \mathcal{z}$. Thus $\varphi(D_0)\mathcal{Y} = \mathcal{Y}$, $\varphi(D_0)\mathcal{z} = 0$ and $\varphi(D_1)\mathcal{z} = \mathcal{z}$, and since $f_\sigma(1) = 1$, $\varphi(f_\sigma(D_0))\mathcal{Y} = \mathcal{Y}$, $\varphi(f_\sigma(D_1))\mathcal{z} = \mathcal{z}$. Since U is unitary, $\varphi(U^*)\mathcal{z} = \varphi(U^{-1})\mathcal{z} = \mathcal{Y}$, and so $\varphi(V^* \, V)\mathcal{Y} = \varphi(V^*)\mathcal{z} = \mathcal{Y}$ and $\varphi(B(n+1))\mathcal{Y} = \mathcal{Y}$. Thus I $\|B(n+1)\| = 1$; also $B(n+1) \geqq 0$.

We show (60) is satisfied for $k = n + 1$. If $0 < j < k$ then (60) follows from (61) and the validity of (60) for $k = n$. If $j = k$ but $a_1, \cdots, a_n \neq b_1, \cdots, b_n$, it follows from (61) and the validity of (60) for $k = n$ that (60) is satisfied. Suppose $j = k$ and $a_1, \cdots, a_n = b_1, \cdots, b_n$ but $a_{n+1} \neq b_{n+1}$. Since $f_{2\sigma}(D_i)f_\sigma(D_i) = f_\sigma(D_i)$, we have

$$\begin{aligned} V(b_1, \cdots, b_{n+1})^* V(a_1, \cdots, a_{n+1}) &= V(0_n, b_{n+1})^* \, V(0_n, a_{n+1}) \\ &= V(0_n, b_{n+1})^* \, f_{2\sigma}\big(D_{b_{n+1}}\big) f_{2\sigma}\big(D_{a_{n+1}}\big) V(0_n, a_{n+1}). \end{aligned}$$

If ρ is a homomorphism of the (commutative) C^*-algebra generated by D_0 and $B_{4\sigma}$, and if $\rho(D_0) \neq 0$ then $\rho(B_{4\sigma}) = 1$ and $\rho(D_1) = 1 - \rho(D_0)$. Thus

$$\rho\big(f_{2\sigma}(D_0) f_{2\sigma}(D_1)\big) = f_{2\sigma}(\rho(D_0)) f_{2\sigma}(\rho(D_1)) = 0$$

since $2\sigma < 1/2$ and so $f_{2\sigma}(D_0) f_{2\sigma}(D_1) = 0$ and

$$V(b_1, \cdots, b_{n+1})^* V(a_1, \cdots, a_{n+1}) = 0 \,.$$

In this case (60) is satisfied for $k = n + 1$, and thus (60) is satisfied for $k = n + 1$ for all cases.

We show (58) is true for $k = n + 1$. If $n = 0$, this is trivial; we suppose $n \geqq 1$. We suppose that $\mathfrak{A}$ is acting on a Hilbert space $\mathfrak{H}$. Let $w = V(a_1, \cdots, a_{n+1})v$ be a unit vector, for some vector v in $\mathfrak{H}$. By (61) and (62),

$$V(a_1,\cdots,a_n)^*w = V(a_1,\cdots,a_n)^*\, V(a_1,\cdots,a_n)V(0_n,a_{n+1})v \qquad (66)$$
and since $x \geqq f_{8\sigma}(x) - 4\sigma$ for all positive numbers x,
$$(A^*A\, V(a_1,\cdots,a_n)^*w, V(a_1,\cdots,a_n)^*w) \geqq (B_{8\sigma}V(0_n,a_{n+1})v, V(0_n,a_{n+1})v - 4\sigma$$
$$= (V(0_n,a_{n+1})v, V(0_n,a_{n+1})v - 4\sigma$$
$$\geqq \|V(a_1,\cdots,a_n)(V(0_n,a_{n+1})v\|^2 - 4\sigma = \|w\|^2 - 4\sigma$$
$$= 1 - 4\sigma > 1 - \delta(\gamma/2^{2n+1}, 2^{n+1}+2)$$
By Lemma (2.2.3),
$(V(a_1,\cdots,a_n)^*BV(a_1,\cdots,a_n)V(a_1,\cdots,a_n)^*w, V(a_1,\cdots,a_n)^*w) > 1 - \delta(\gamma/2^{2n+1})$. and by (66), $V(a_1,\cdots,a_n)V(a_1,\cdots,a_n)^*w = w$, so $(Bw,w) > 1 - \delta(\gamma/2^{2n+1})$. If u is a unit vector in the range of $V(b_1,\cdots,b_{n+1})$ then also $(Bu,u) > 1 - \delta(\gamma/2^{2n+1})$. If $a_1,\cdots,a_{n+1} \neq b_1,\cdots,b_{n+1}$ then $u \perp w$ by (60) and since $[\{u,w\}]B[\{u,w\}] \leqq I$,
$$|(Bu,w)|^2 \leqq (1-(Bu,u))(1-(Bw,w)) < (\gamma/2^{2n+1})^2 .$$
If $s = E(n+1)t$ is a unit vector, then by the Definition of $E(n+1)$, $s = \sum_{a_1,\cdots,a_n} s(a_1,\cdots,a_{n+1})$, where we have let
$$s(a_1,\cdots,a_{n+1}) = V(a_1,\cdots,a_{n+1})V(a_1,\cdots,a_{n+1})^*t .$$
By (60), $\{s(a_1,\cdots,a_{n+1})\}$i s an orthogonal family, and so
$$(Bs,s) \geqq \sum_{a_1,\cdots,a_n} (Bs(a_1,\cdots,a_{n+1}), s(a_1,\cdots,a_{n+1}))$$
$$-\sum_{a_1,\cdots,a_{n+1}\neq b_1,\cdots,b_{n+1}} |Bs(a_1,\cdots,a_{n+1}), s(b_1,\cdots,b_{n+1})|$$
$$> \sum_{a_1,\cdots,a_n} (1-(\gamma/2))\|s(a_1,\cdots,a_{n+1})\|^2$$
$$-\sum_{a_1,\cdots,a_{n+1}\neq b_1,\cdots,b_{n+1}} (\gamma/2^{2n+1})\|s(a_1,\cdots,a_{n+1})\|\|s(b_1,\cdots,b_{n+1})\|$$
$$\geqq 1-(\gamma/2)-(\gamma/2) = 1-\gamma .$$
Thus $\omega_s|\mathfrak{A} \in \mathcal{W}$ and so $|((S_n - T_n)s,s)| < 1/(n+1)$. Let r be a vector in the unit sphere of $\mathfrak{H}$. Then $|(E(n+1)(S_n-T_n)E(n\ +1)r,r)| < 1/(n+1)$, since $E(n\ +1)r$ is in the range of $E(n\ +1)$ and has norm at most 1. Since. $(E(n+1)(S_n-T_n)E(n\ +1)$ is self-adjoint, $\|(E(n+1)(S_n-T_n)E(n\ +1)\ )\|| < 1/(n+1)$ and the proof of (58) is complete.

To prove (59) is true, we can suppose $n = 0$. Then
$$V(O)V(O)^* = f_{1/2}(V^*V) \leqq B_{8\sigma} \leqq 4\sigma I + AA^*$$
$$\leqq (1/4)I + B(O)B(O)^* = (1/4)I + S_0,$$
and the proof of Lemma (2.2.4) is complete.

We remark that the hypothesis, no non-zero GCR ideals, enters the proof only one point: the choice of a representation φ such that dim range $\varphi(B_\sigma) \geqq 2$. Instead of approximating S_n at the nths tep, we could have approximated $S_1,\ldots,S_n$.

Lemma (2.2.5)[100]: Let $\mathfrak{A}$ and the V's be as in Lemma (2.2.4). Let φ be a representa- tion of a which does not annihilate $E(n), n\ =\ 1,2,\cdots$. Let $\mathfrak{M}(n)$ be the linear span of
$$\{V(a_1,\cdots,a_n)V(b_1,\cdots,b_n)^*: a_i, b_i = 0 \ or\ 1.$$
Then $\varphi(\mathfrak{M}(n))$ [range $\varphi(E(n+1))$] is a $2^n \times 2^n$ matrix algebra with matrix units
$$\varphi(V(a_1,\cdots,a_n)V(b_1,\cdots,b_n)^*)|[\text{range } \varphi(E(n+1))],$$
and if $n > 1$,
$$\varphi(V(a_1,\cdots,a_{n-1})V(b_1,\cdots,b_{n-1})^*)|[\text{range } \varphi(E(n+1))]$$
$$= \varphi(V(a_1,\cdots,a_{n-1},0)V(b_1,\cdots,b_{n-1},0)^*$$
$$+(V(a_1,\cdots,a_{n-1},1)V(b_1,\cdots,b_{n-1},1)^*|[\text{range } \varphi(E(n+1))].$$

Also $\varphi(\mathfrak{M}(n))$ leaves [range $\varphi(E(n+1))$] invariant and the sequence $\{[\text{range } \varphi(E(n))]\}$ is monotone decreasing.

Let $x = \varphi(V(a_1, \cdots, a_{n+1}))\mathcal{Y}$ for some $\mathcal{Y}$ in the Hilbert space $\mathfrak{H}$ upon which $\varphi(\mathfrak{A})$ acts. By (60), (61) and (62),

$$\begin{aligned} &\varphi(V(b_1, \cdots, b_n)V(c_1, \cdots, c_n)^*)x \\ &\quad = \varphi(V(b_1, \cdots, b_n)V(c_1, \cdots, c_n)^*)V(a_1, \cdots, a_n)V(0_n, a_{n+1}))\mathcal{Y} \\ &\quad = \delta_{a_1,\cdots,a_n}^{c_1,\cdots,c_n}\varphi(V(b_1, \cdots, b_n, a_{n+1}))\,\mathcal{Y} \in [range\ \varphi(E(n+1))] \end{aligned} \tag{67}$$

and so $\varphi(\mathfrak{M}(n))$ leaves [range $\varphi(E(n+1))$] invariant. By (67),

$$\begin{aligned} &\varphi(V(b_1, \cdots, b_n)V(c_1, \cdots, c_n)^*)\,V(d_1, \cdots, d_n)V(e_1, \cdots, e_n)^*)x \\ &\quad = \delta_{a_1,\cdots,a_n}^{e_1,\cdots,e_n}\varphi(V(b_1, \cdots, b_n, a_{n+1}))\,\mathcal{Y} \\ &\quad = \delta_{d_1,\cdots,d_n}^{c_1,\cdots,c_n}\varphi(V(b_1, \cdots, b_n)V(e_1, \cdots, e_n)^*)x. \end{aligned}$$

Thus

$$\begin{aligned} &(\varphi(V(b_1, \cdots, b_n)V(c_1, \cdots, c_n)^*)|[\text{range } \varphi(E(n+1))]) \\ &\quad \times (\varphi(V(d_1, \cdots, d_n)V(e_1, \cdots, e_n)^*)|[range\ \varphi(E(n+1))]) \\ &= \delta_{d_1,\cdots,d_n}^{c_1,\cdots,c_n}(\varphi(V(b_1, \cdots, b_n)V(e_1, \cdots, e_n)^*)\,|[range\ \varphi(E(n+1))]) \end{aligned}$$

It follows from(10) that

$$\varphi(V(a_1, \dots, a_{n-1})V(b_1, \dots, b_n)^*)|[range\ \varphi(E(n+1))] \neq 0.$$

and so the proof of the first statement is complete.

By a calculation similar to (67),

$$\varphi(V(b_1, \dots, b_{n-1})V(c_1, \dots, c_{n-1})^*)x = \delta_{a_1,\dots,a_{n-1}}^{c_1,\dots,c_{n-1}}\varphi\big(V(b_1, \dots, b_{n-1}, a_n, a_{n+1})\big)y$$

However

$$\begin{aligned} &\varphi(V(b_1, \dots, b_{n-1}, 0)V(c_1, \dots, c_{n-1}, 0)^* + V(b_1, \dots, b_{n-1}, 1)V(c_1, \dots, c_{n-1}, 1)^*)x \\ &\quad = \delta_{a_1,\dots,a_n}^{c_1,\dots,c_{n-1},0}\varphi\big(V(b_1, \dots, b_{n-1}, 0, a_{n+1})\big)y \\ &\quad + \delta_{a_1,\dots,a_n}^{c_1,\dots,c_{n-1},1}\varphi\big(V(b_1, \dots, b_{n-1}, 1, a_{n+1})\big)y \\ &\quad = \delta_{a_1,\dots,a_{n-1}}^{c_1,\dots,c_{n-1}}\varphi\big(V(b_1, \dots, b_{n-1}, a_n, a_{n+1})\big)y \end{aligned}$$

and the second statement is proved.

It follows from (4) that [range $\varphi\big(E(n)\big)$] is monotone decreasing as $n \to \infty$

Lemma (2.2.6)[100]: Using the notation of Lemma (2.2.4), let f be a state of $\mathfrak{A}$ such that $f\big(E(n)\big) = 1$. Then $f\big(E(n)AE(n)\big) = f(A)$ for all A in $\mathfrak{A}$. If $f(V(a_1, \dots, a_n)V(a_1, \dots, a_n)^*) = 1$ then

$$f = f(V(a_1, \dots, a_n)V(a_1, \dots, a_n) \cdot V(a_1, \dots, a_n)V(a_1, , \dots, a_n)^*).$$

Let x be a vector in $\mathfrak{H}_f$ such that $f = w_x\varphi_f$.If $f(B) = 1$ for a B in $\mathfrak{A}$ of norm 1 then

$$\big(\varphi_f(B)x, x\big) - 1 = \big\|\varphi_f(B)x\big\|\|x\|$$

and so $\varphi_f(B)x$ is proportional to x, and thus is equal to x. Hence $f(B \cdot B) = f$, and this proves Lemma (2.2.6).

Theorem (2.2.7)[100]: Let $\mathfrak{A}$ be a separable C*-algebra. Then

(a) the following are equivalent:

(a1) $\mathfrak{A}$ is GCR,

(a2) $\mathfrak{A}$ is type I,

(a3) $\mathfrak{A}$ has no representations of type II,

(a4) $\mathfrak{A}$ has no representations of type III,

(a5) every irreducible image of $\mathfrak{A}$ contains the completely continuous operators.

(a6) If φ_1 and φ_2 are any two irreducible representations of $\mathfrak{H}$ such that kernel $\varphi_1 = kernel\ \varphi_2$ then $\varphi_{1\mathfrak{H}}^{\mathfrak{A}}$ is unitarily equivalent to φ_2.

(b) The following are equivalent:

(b1) $\mathfrak{A}$ has no non-zero GCR ideals,

(b2) $\mathfrak{A}$ has a faithful type II representation,

(b3) $\mathfrak{A}$ has a faithful type III representation,

(b4) there is a family $\{\varphi_\alpha\}$ of irreducible representations of a such that $\mathfrak{A}\sum_\alpha\bigoplus\varphi_\alpha$ is faithful and $\varphi_\alpha(\mathfrak{A})$ does not contain the completely continuous operators.

(b5) There are families $\{\varphi_\alpha: \alpha \in A\}$ and $\{\psi_\alpha: \alpha \in A\}$ of irreducible representations of $\mathfrak{A}$ such that $\sum_\alpha\bigoplus\varphi_\alpha$ and $\sum_\alpha\bigoplus\psi_\alpha$ are faithful and $kernel\ \varphi_\alpha = kernel\ \psi_\alpha$ but ψ_α is not unitarily equivalent to ψ_α

(c) The implications (al)⇒(a2) ⇒(a3), (a2) ⇒ (a4) ⇒ (al) ⇒ (a5) ⇒ (a6), (b2) ⇒ (b1), (b3) ⇒ (b1), (b5) ⇒(b4) ⇒(b1) are valid for non-separable C*-algebras.

It is no loss of generality to suppose $\mathfrak{A}$ has a unit I. The implications (a2)⇒(a3) and (a2) ⇒(a4) are evident while (b2) ⇒(b1) and (b3) ⇒(b1) follow from the Corollary of Lemma (2.2.1)2 of [106]. The last two implications can also be deduced from I. Kaplansky, Group algebras in the large, Tohoku Math. J., 3 (1951), 249-255; we are indebted to J. Dixmier for calling this work of Kaplansky to our attention. Furthermore [106] and its corollary are essentially the same as Lemma (2.2.3) and Theorem (2.2.17) of I. Kaplansky, op. cit. The implication (al) ⇒ (a2) is Theorem (2.2.17) of I. Kaplansky, op. cit.

(al) ⇒ (a5) and (b4) ⇒(b1): Suppose $\mathfrak{A}$ is GCR and φ is an irreducible representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{S}$. Then $\varphi(\mathfrak{A})$ is a GCR algebra [92] and so contains a non-zero CCR ideal $\mathfrak{J}$. If E is a nonzero subspace of $\mathfrak{S}$ invariant under $\mathfrak{J}$ then $[\mathfrak{J}E]$ is invariant under $\varphi(\mathfrak{A})$, and so $[\mathfrak{J}E] = 0$ or $\mathfrak{S}$. However $[\mathfrak{J}E] \supset [\mathfrak{J}\mathfrak{A}E] = [\mathfrak{J}\mathfrak{S}] \neq 0$, so $[\mathfrak{J}E] = \mathfrak{S}$ Since $[\mathfrak{J}E] \subset E, E = \mathfrak{S}$ and we have proved that $\mathfrak{J}$ acts irreducibly on $\mathfrak{S}$. By the Definition of CCR algebras, $\mathfrak{J}$ consists of completely continuous operators, and by irreducibility $\mathfrak{J}$ is all completely continuous operators (see [89]). Thus $\varphi(\mathfrak{A})$ contains the completely continuous operators, and (al) ⇒(a5). Assume (b4) and let $\mathfrak{K}$ be a GCR ideal in $\mathfrak{A}$. If $\mathfrak{K}$ is not zero then $\varphi_\alpha(\mathfrak{K})$ is not zero for some α. As above, was $\varphi_\alpha|\mathfrak{K}$ is an irreducible representation of $\mathfrak{K}$ and by the implication (al) ⇒(a5), $\varphi_\alpha(\mathfrak{K})$ contains the completely continuous operators. However this contradicts the assumption, (b4), so $\mathfrak{K} = 0$ and (b1) holds.

(a5) ⇒ (a6): We prove the stronger statement: If $\mathfrak{A}$ is a C*algebra, if φ_1 and φ_2 are irreducible representations of $\mathfrak{A}$, if $\varphi_1(\mathfrak{A})$ contains the completely continuous operators and if $kernel\ \varphi_1 = kernel\ \varphi_2$ then φ_1 is unitarily equivalent to φ_2.

Let $\pi: \mathfrak{A} \to \mathfrak{A}/kernel\ \varphi_1$, Then $\varphi_1\pi^{-1}$ and $\varphi_2\pi^{-1}$ are faithful irreducible representations of $\pi(\mathfrak{A})$, unitarily equivalent if and only if φ_1 and φ_2 are unitarily equivalent. Thus we can suppose φ_1 and φ_2 are faithful. Then $\mathfrak{A}$ contains an ideal $\mathfrak{J}$ isomorphic to the completely continuous operators, and so $\mathfrak{J}$ has a unique (up to unitary equivalence) irreducible representation, and since $\varphi_1(\mathfrak{J})$ and $\varphi_2(\mathfrak{J})$ act irreducibly, $\varphi_1|\mathfrak{J}$ is unitarily equivalent to $\varphi_2|\mathfrak{J}$. Let U be a unitary operator which implements this equivalence, let x be in the representation space $\mathfrak{S}$ of φ_2. If $A, B \in \mathfrak{A}$, and $B \in \mathfrak{A}$ then

$$\varphi_2(A)(\varphi_2(B)x) = \varphi_2(AB)x = U^*\varphi_1(AB)Ux = U^*\varphi_1(A)UU^*\varphi_1(B)Ux$$
$$= U^*\varphi_p(A)U(\varphi_2(B)x).$$

Since $\varphi_2(\mathfrak{J})\mathfrak{S}$ is dense in $\mathfrak{S}, \varphi_2(A) = U^*\varphi_1(A)U, \varphi_1$ is unitarily equivalent to φ_2, and the statement is proved.

(b5) ⇒(b4). Given the families $\{\varphi_\alpha: \alpha \in A\}$ and $\{\psi_\alpha: \alpha \in A\}$ as in (b5), $\varphi_\alpha(\mathfrak{A})$ does not contain the completely continuous operators, since if it $did\ \varphi_\alpha$ and ψ_α would be unitarily equivalent, by the preceding paragraph.

We have not yet used the assumption that $\mathfrak{A}$ is separable and so the implications already proved are valid for an arbitrary C*-algebra. Our proof of each of the remaining implications will use the assumption of separability.

(b1) ⇒(b5): Suppose (b1) is true and in Lemma (2.2.4) choose $S_1, S_2, \dots$ to be a sequence of self-adjoint elements of $\mathfrak{A}$ which is dense in the selfadjoint elements. Let S_0 be an arbitrary positive element of $\mathfrak{A}$ of norm one. Let $\mathfrak{M}(n)$ be defined as in Lemma (2.2.5) and let $\mathfrak{M}(0) = I$, let $\mathfrak{M}$ be the closed linear subspace of $\mathfrak{A}$ generated by the $\mathfrak{M}(n)$'s. Let $d_1, d_2, \dots$ be a sequence of zeros and ones, let $D_r = d_1, \dots, d_r$. Define a linear functional $f(\{d_i\})(\cdot) = f(\cdot)$ on $\mathfrak{M}$ by

$$f(I) = 1, \qquad f(V(a_1, \dots, a_n)V(b_1, \dots, b_n)^*) = \delta^{d_1,\dots,d_n}_{a_1,\dots,a_n}\delta^{d_1,\dots,d_n}_{b_1,\dots,b_n}. \tag{68}$$

In Lemma (2.2.5), take φ to be the identity representation. Then $f|$ (the span of $U^k_{n=0}\mathfrak{M}(n)$) is a state (in fact a vector state) for $k = 1, 2, \dots$, and so f is a state. Suppose $f|\mathfrak{M}(n) = 2^{-1}g|\mathfrak{M}(n) + 2^{-1}h|\mathfrak{M}(n)$ where g and h are states of $\mathfrak{M}$. Since $0 \leqq V(D_n)V(D_n)^* \leqq I, g(V(D_n)V(D_n)^*) = 1$. If $a_1, \dots, a_n \neq D_n$ or $b_1, \dots, b_n \neq D_n$,

$$g(V(a_1, \dots, a_n)V(b_1, \dots, b_n)^*)$$
$$= g(V(D_n)V(D_n)^*V(a_1, \dots, a_n)V(b_1, \dots, b_n)^*V(D_n)V(D_n)^*) = 0$$

by Lemma (2.2.6) and (3), and so $g|\mathfrak{M}(n) = f|\mathfrak{M}(n)$. Thus f is a pure state of $\mathfrak{M}$. Let $e_{\{de_i\}} = \varphi$ be an extension of f to a pure state of $\mathfrak{M}$, let $\varphi_e = \varphi_{\{d_i\}} = \varphi$ be the representation defined by e on a Hilbert space $\mathfrak{H}_{\{d_i\}} = \mathfrak{M}$, let $e = \omega_x\varphi$ for some vector $x = x(\{d_i\})$ in $\mathfrak{H}$ such that $[\varphi(\mathfrak{A})x] = \mathfrak{H}$.We remark that $\varphi(S_0) \neq 0$. In fact $\left(\varphi\left(E(n)\right)x, x\right) = e\left(E(n)\right) = f\left(E(n)\right) = 1$ so $\varphi(E(n)) \neq 0$. By Lemma (2.2.5), $\|\varphi(V(0)V(0)^*)\| = 1$ and by (2), $\varphi(S_0) \neq 0$.

For later use we observe: The representation space of φ is denumerably infinite dimensional. In fact $\mathfrak{H}$ is not finite dimensional since matrix units of arbitrarily high order act on $\mathfrak{H}$ (Lemma (2.2.5)) and $\mathfrak{H}$ is separable since $\mathfrak{H} = [\varphi(\mathfrak{A})x]$ and $\mathfrak{A}$ is separable.

Let $\mathfrak{J}$ be the set of elements A of $\mathfrak{A}$ such that for each B and C in $\mathfrak{A}$, $\lim_{n\to\infty}\|E(n)BACE(n)\| = 0$. We assert that $\mathfrak{J}$ is the kernel of φ. Let A be in $\mathfrak{J}$, let B and C be in $\mathfrak{A}$ Then

$$|\varphi(A)\varphi(C)x, \varphi(B)x| = \left|\omega_x\left(\varphi(B^*AC)\right)\right| = |\omega_x\left(\varphi\left(E(n)B^*ACE(n)\right)\right)$$
$$\leqq \|E(n)B^*ACE(n)\|$$

and since $[\varphi(\mathfrak{A})x] = \mathfrak{H}, \varphi(A) = 0$. Thus $\mathfrak{J} \subset \mathit{kernel}\,\varphi$. Let A be in $\mathfrak{A}$ but not in a, we show that $A \notin \mathit{kernel}\,p$. Let B and C be elements of A such that $\lim_{n\to\infty}\|E(n)BACE(n)\| = b > 0$. If we show that $BAC \notin \mathit{kernel}\,\varphi$ then it follows that $A \notin \mathit{kernel}\,\varphi$. Thus to prove $A \notin \mathit{kernel}\,\varphi$, it suffices to consider the case where $\lim_{n\to\infty}\|E(n)AE(n)\| = b > 0$. One of $\|E(n)(A + A^*)E(n)\|, \|E(n)(A - A^*)E(n)\|$, has a non-zero limit and the corresponding one of $A + A^*, A - A^*$ is not in $\mathfrak{J}$. Kernel φ is self-adjoint, and so if one of $A + A^*, A - A^* \notin \mathit{kernel}\,\varphi$ then $A \notin \mathit{kernel}\,\varphi$, and we can suppose A is self-adjoint. There is a k such that $\|E(k)(A - T_{k-1})E(k)\| < b/2$, where T_{k-1} is in $\mathfrak{M}(k-1)$ and is defined in Lemma (2.2.4), and so $\|E(k)T_{k-1}E(k)\| > b/2$, and $\|T_{k-1}\,|[\mathit{range}\,E(k)]\| > b/2$. Applying Lemma (2.2.5) first to the identity representation of $\mathfrak{M}(k-1)$ and then to the representation φ, we see that the map

$$\mathfrak{M}(k-1)|[\mathit{range}\,E(k)] - \varphi\left(\mathfrak{M}(k-1)\right)\left[\mathit{range}\,\varphi\left(E(k)\right)\right]$$

defined by φ is an isomorphism (of one matrix algebra onto another), and in particular is norm preserving. Thus

$$\|\varphi(T_{k-1})|[range\, p(E(k))]\| > b/2 .$$

Hence $\|\varphi(E(k))\varphi(T_{k-1})\varphi(E(k))\| > b/2$ and if $\varphi(A) = 0$ then

$$\varphi(E(k))\varphi(A - T_{k-1})\varphi(E(k)) > b/2,$$

which is a contradiction, and so $\varphi(A) \neq 0$ and $\mathfrak{J} = kernel\, \varphi$. This implies that kernel $\varphi_{\{d_i\}} = kernel\, \varphi_{\{c_j\}}$ for all sequences d_1, d_2, and $c_1, c_2, \ldots$ of zeros and ones.

We assert: If $s_1, s_2, \ldots (resp. t_1, t_2, \ldots)$ is a sequence of zeros and ones, then $\varphi_{\{s_i\}}$ is unitarily equivalent to $\varphi_{\{t_i\}}$ only if $s_i = t_i$ for all but a finite number of i. We will need this statement later and we will prove the converse later, that $\varphi_{\{s_i\}}$ is unitarily equivalent to $\varphi_{\{t_i\}}$ if $s_i = t_i$ for all but a finite number of i. Let $\varphi_s = \varphi_{\{s_i\}}, \varphi_i = \varphi_{\{t_i\}}, e_s = e(\{S_i\}), e_t = e(\{t_i\})$, and suppose that φ_s is unitarily equivalent to φ_t, and recall that $\varphi_s(resp. \varphi_t)$ is the representation defined by $e_s(resp. e_i)$. It follows from [89] (as in the proof of Corollary 8 of [107]) that there is a B in $\mathfrak{A}$ such that $e_s(A) = e_t(B^*AB)$ for all $\mathfrak{A}$ in X, and by Lemma (2.2.6),

$$e_s = e_t(B^* \cdot B) = e_t(E(r)B^*E(r) \cdot E(r)BE(r))$$

for $r = 1, 2, \ldots$ There are r_1 and r_2 and T_{r_1} and T_{r_2} in $\mathfrak{M}(r_1)$ and $\mathfrak{M}(r_2)$ respectively such

$$\|E(p)(2^{-1}(B + B^*) - T_{r_1})E(p)\|$$
$$= \|E(p)E(r_1 + 1)(2^{-1}(B + B^*) - T_{r_1})E(r + 1)E(p)\| < 1/8$$

and

$$\|E(p)(2^{-1}(B - B^*) - iT_{r_2})E(p)\| < 1/8$$

for $p > \max r_1, r_2$. Let $C = T_{r_1} + iT_{r_2}$ let $p > \max r_l, r_2$. Then $\varphi_t(C)x(\{t_j\}) = \varphi_t(E(p)CE(p))x(\{t_i\})$ by Lemma (2.2.5), since $x(\{t_i\}) \in [range\, \varphi_t(E(p + 1))]$, and so

$$\|e_s - e_t(C^* \cdot C)\| = e_t(E(p)B^*E(p) \cdot E(p)BE(p)) - e_t(E(p)C^*E(p) \cdot E(p)CE(p))$$
$$\leqq 2\|E(p)(B - C)E(p)\| < 1/2.$$

Let

$$E(p, v) = \sum a_1, \ldots, a_{p-1}\ V(a_1, \ldots, a_{p-1}, v)V(a_1, \ldots, a_{p-1}, v)^*,$$

$v = 0$ or 1. Then $E(p, t_p)$ commutes with C, and if $t_p \neq s_p$,

$$1 = e_s\left(E(p, s_p)\right) \leqq 1/2 + e_t(C^*E(p, s_p)C) - 1/2 + e_t\left(E(p, t_p)C^*E(p, s_p)CE(p, t_p)\right)$$
$$= 1/2$$

by Lemma (2.2.6) (since $e_t\left(E(p, t_p)\right) = 1$), for $p > \max r_1, r_2$. This is a contradiction and if $p > \max r_1, r_2$ then $t_p = s_p$ which proves our assertion.

Let φ_{s_0} be the representation of $\mathfrak{A}$ defined by the choice $s_i = 0$, let ψ_{s_0} be the representation of $\mathfrak{A}$ defined by the choice $t_i = 1$. Then φ_{S_0} and ψ_{s_0} are not unitarily equivalent but have the same kernel. The family $\{\varphi_{s_0}\}(resp. \{\psi_{s_0}\})$, as S_0 ranges over the positive norm one elements of $\mathfrak{A}$, satisfies (b5) and the proof that (b1)⇒(b5) is complete.

We remark that we have shown that there are not just two families $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$, there is a continuum of families, $\{\{\varphi_{\alpha x}\}_{\alpha \in A} : x \in \tilde{X}\}$ such that $\varphi_{\alpha x}$ is not unitarily equivalent to $\varphi_{\alpha y}$ for $x \neq y$, but $kernel\, \varphi_{\alpha x} = kernel\, \varphi_{\alpha y}$ and $\sum_\alpha \varphi_{\alpha x}$ is faithful. ($\tilde{X}$ will be defined in the proof of Theorem (2.2.8).)

(a6)⇒(al): Suppose (a6) is satisfied and let $\mathfrak{K}$ be the maximum GCR ideal in $\mathfrak{A}$ [92]. If $\mathfrak{K} \neq \mathfrak{A}$ then $\mathfrak{A}/\mathfrak{K}$ (and so $\mathfrak{A}$) has two unitarily inequivalent representations with the same kernel, by (b1)⇒(b5). This contradicts (a6) and so $\mathfrak{K} = \mathfrak{A}$ and $\mathfrak{A}$ is GCR.

(b1)⇒(b2); (b1)⇒(b3): Let SO be an arbitrary positive element of $\mathfrak{A}$ of norm 1, let $S_1, S_2, \ldots$ be a sequence of self-adjoint operators which is dense in the self-adjoint operators in $\mathfrak{A}$. Let p be in $(0, 1/2)$, let $q = 1 - p$. Let g be the functional on $\mathfrak{M}$ ($\mathfrak{M}$ defined as before) defined by

$$g(I) = 1, \qquad g(V(a_1, \ldots, a_n)V(b_1, \ldots, b_n)^*) - \delta_{a_1,\ldots,a_n}^{b_1,\ldots,b_n} q^{\sum_{i=1}^n a_i} p^{n - \sum_{i=1}^n a_i}.$$

In Lemma (2.2.5), take φ to be the identity representation. Then it is evident that $g|$ (the span of $\cup_{n=1}^n \mathfrak{M}(n)$) is a state (in fact a vector state) for $k = 1,2, \ldots$. This g is a state of $\mathfrak{M}$. Let h be any extension of g to a state of $\mathfrak{A}$.

We assert that the weak closure $\varphi_h(\mathfrak{A})^-$ of $\varphi_h(\mathfrak{A})$ is a von Neumann algebra of type II if $p = 1/2$, and of type III if $p \neq 1/2$. Let $\varphi = \varphi_h$. Since $h(V(0_2)V(0_2)^*) \neq 0, \varphi(V(0_2)V(0_2)^*) \neq 0$. This and (5) imply $\|\varphi(V(0))\| = 1$, and (2) implies that φ does not annihilate S_0. Since S_0 was an arbitrary positive element of $\mathfrak{A}$ of norm one, this assertion will show that (b2) and (b3) are satisfied. Let

$$F(n) = [range\ \varphi(E(n))].$$

The sequence $\{F(n)\}$ is monotone decreasing (Lemma (2.2.5)), and so $F = \inf F(n)$ exists and is in $\varphi(\mathfrak{A})^-$. Let $h = \omega_x\varphi$ for some x in $\mathfrak{H}_h$ such that $\mathfrak{H}_h = [\varphi(\mathfrak{A})x]$. Since $\varphi(E(n)) \leqq F(n) \leqq 1$ and since $\omega_x(\varphi_h E(n)) = 1, \omega_x F(n) = 1$. Thus $\omega_x(F) = 1$ and $x \in F$. By Lemma (2.2.5), $\varphi(\mathfrak{M}(n))$ leaves $F(r)$ invariant for all $r \geqq n + 1$, and so $\varphi(\mathfrak{M}(n))$ leaves F invariant. However $\varphi(\mathfrak{M}(n))F = \varphi(\mathfrak{M}(n))F(n + 1)F$ is a homomorphic image of the $2^n \times 2^n$ matrix algebra $\varphi(\mathfrak{M}(n))F(n + 1)$ and since $x \in F$, the image is not zero and thus is a $2^n \times 2^n$ matrix algebra. Let A be a self-adjoint element of $\mathfrak{A}$. For each n there is a $j \geqq 2n$ such that $\|A - S_j\| < 1/2n$ and a T_j in $\mathfrak{M}(j)$ such that $\|E(j + 1)(S_j - T_j)E(j + 1)\| < \frac{1}{2n+1}$. Thus

$$\|F_\varphi(A - T_j)F\| \leqq \varphi\left(E(j + 1)(A - T_j)\right)E(j + 1)1/n$$

and we have proved that $\varphi(\mathfrak{M})F$ is uniformly dense in $F\varphi(\mathfrak{A})F$.

Suppose we show $\varphi(\mathfrak{M})F^-$, the weak closure of $\varphi(\mathfrak{M})F$, is type II if $p = 1/2$, and type III if $p \neq 1/2$. (In the case $p = 1/2$, this is well known and could be deduced from the published literature. However, we include a proof . Our proof gives rise to a new demonstration of the existence of factors of type II_i. In the case $p \neq 1/2$, this is known, but not published.) Then $F\varphi(\mathfrak{A})F^-$ is type II (resp. III) and the commutant $F\varphi(\mathfrak{A})F'$ of $F\varphi(\mathfrak{A})F^-$ (acting on $F\mathfrak{H}$) is type II (resp. III) [102]. However $[\varphi(\mathfrak{A})x] = \mathfrak{H}$ and a fortiori $[\varphi(\mathfrak{A})F] = \mathfrak{H}$, so F is separating for $\varphi(\mathfrak{A})'$. That is, the map $A' \to A'F$ is an isomorphism from $\varphi(\mathfrak{A})'$ onto $\varphi(\mathfrak{A})'F$. Since $\varphi(\mathfrak{A})'F = (F\varphi(\mathfrak{A})F)'$ [102], $\varphi(\mathfrak{A})'$ is of type II (resp. III) and $\varphi(\mathfrak{A})$ is of type II (resp. III) [102] and the proof of (bi) => (b2), (b3) will be complete.
First we suppose $p = 1/2$. Then $\omega_x|\varphi(\mathfrak{M}(n))F$ is the trace (normalized by trace $(I) = 1$) for $(\varphi(\mathfrak{M}(n))F$. Thus for A_n and B_n in $(\varphi(\mathfrak{M}(n))F, \omega_x(A_nB_n) = \omega_x(B_nA_n)$. Let A and B be in $\varphi(\mathfrak{M})F^-$ and suppose $\{A_n\}$ and $\{B_n\}$ are norm bounded sequences converging strongly to A and B respectively, and suppose A_n and B_n are in $\varphi(\mathfrak{M}(n))F$. Then $\{A_nB_n\}$ and $\{B_nA_n\}$ converge strongly to AB and BA respectively so $\omega_x(AB) = \omega_x(BA)$. Thus ω_x is a trace for $\varphi(\mathfrak{M})F^-$. We assert $\varphi(\mathfrak{M})F^-$ is a factor. In fact let R be a projection in the center of $\varphi(\mathfrak{M})F^-$. Then $\omega_{Rx}|\varphi(\mathfrak{M}(n))F$, and $\omega_{(1-R)x}|\varphi(\mathfrak{M}(n))F$ are both proportional to traces and so are proportional to each other. Thus $\omega_{Rx}|\varphi(\mathfrak{M})F$ and $\omega_{(1-R)x}|\varphi(\mathfrak{M})F$ are proportional and by weak continuity, $|\omega_{Rx}|\varphi(\mathfrak{M})F^-$ and $\omega_{(I-R)x}|\varphi(\mathfrak{M})F^-$ are

proportional. In particular $\omega_{Rx}(R)$ and $\omega_{(I-R)x}(R)$ are proportional. If $Rx \neq 0$, $0\omega_{Rx}(R) = \omega_{(I-R)x}(R)$,$0\omega_{Rx} = \omega_{(I-R)x}$, and $(I - R)x = 0$. Thus one of $Rx, (I - R)x$ is zero, one of $F\varphi(\mathfrak{A})FRx = RF\varphi(\mathfrak{A})Fx, F\varphi(\mathfrak{A})F(I - R)x = (I - R)F\varphi(\mathfrak{A})Fx$ Is zero, and so one of $R, I - R$ is zero. Thus $\varphi(\mathfrak{M})F^-$ is a factor as asserted, and it is finite since x is a trace vector. It is not of type Ir for any $r < \infty$ since it is infinite dimensional as a linear space. Hence $\varphi(\mathfrak{M})F^-$ is of type II, and the proof (b1)⇒(b2) is complete.

We suppose $p \neq 1/2$ and we construct an isomorphism θ of $\varphi(\mathfrak{M})F^-$ with a factor of type III constructed by Pukanszky [114] by means of von Neumann's construction [113]. We introduce the notation of [114]. Let X_0 be the measure space $\{0, 1\}$, let S_0 be the set of subsets of $\{0, 1\}$, let μ_0 be the measure on X_0 defined by $\mu_0(\{0\}) = p, \mu_0(\{1\}) = q$. Let $\{X_n, S_n, \mu_n\} = \{X_0, S_n, \mu_0\}$ for $n = 1, 2, ...$, let $\{X, S', \mu'\} = \{X_{i=1}^{\infty} X_i, X_{i=1}^{\infty} S_i, X_{i=1}^{\infty} A\}$ and let $\{X, S, \mu\}$ be the measure space formed by the completion of μ'. If x is in X, x is identified with the sequence (x_n), where $x_n = 0$ or 1. If $y = (y_n)$ is in X, we define $x + y$ to be the sequence $\left(x_n + n_y\right)$ reduced mod 2. Then X is a group, and $\Delta = \{(x_n): x_n \neq 0$ for at most a finite number of n$\}$ is a countable subgroup of X generated by the elements $\gamma_k = (\gamma_k)_n)$ where $(\gamma_K)_n = \delta_k^n$. For γ in Δ we define a mapping of X onto itself by $x\gamma = x + \gamma$. The transformation defined by an a in Δ maps measurable sets onto measurable sets and sets of μ-measure zero onto sets of μ-measure zero [114]. Thus the Radon-Nikodym derivative $d\mu_a/d\mu(X)$ of the translated measure with respect to the original measure exists. Let $\mathfrak{H}$ be the Hilbert space of functions $F(\gamma, x)(\gamma \in \Delta, x \in X)$ for which

$$\sum_{\gamma \in \Delta} \int_X |F(\gamma, x)|^2 d\mu < +\infty$$

with inner product

$$(F, G) = \sum_{\gamma \in \Delta} \int_X F(\gamma, x)\overline{G(\gamma, x)} d\mu,$$

for F and G in $\mathfrak{H}$. The ring of operators M generated by U_a and $L_{\rho(x)}$ $(a\ in\ \Delta, \rho(x)a$ bounded measurable function on X) is a factor of type III, where

$$(U_a F)(\gamma, x) = \left(\frac{d\mu_a}{d\mu}(x)\right)^{\frac{1}{2}} F(\gamma + a, xa)$$

$$\left(L_{\rho(x)}F\right)(\gamma, x) = \rho(x)F(\gamma, x).$$

We observe that M is also generated by the operators U_a and $L_{\rho(a_1,\ldots,a_k)(x)}$ (a in Δ, a_i in $X_i, k = 1, 2, ...$) where $\rho(a_1, \ldots, a_k)(x)$ is the characteristic function of the set $\{(x_n) : = x_i = a_i, i = 1, \ldots, k\}$. In fact, let C be the algebra of linear combinations of the functions $\rho(a_1, \ldots, a_k)(x)$. Then the strong closure L_σ^- of L_σ is a subalgebra of $L_{L_\infty(x)}$ which is closed under monotone limits and thus it contains L_ρ, where ρ is the characteristic function of an arbitrary measurable set, and so $L_\sigma^- = L_{L_\infty(x)}$.

Let

$$W(a_1, \ldots, a_n; b_1, \ldots, b_n) = \varphi(V(a_1, \ldots, a_n)V(b_1, \ldots, b_n)^*)F$$
$$\theta\left(L_{\rho(a_1,\ldots,a_n)}\right) = W(a_1, \ldots, a_n; a_1, \ldots, a_n)$$
$$\theta\left(U_{\gamma_n}\right) = \sum\_(a_1, \ldots, a_{n-1}\left(W(a_1, \ldots, a_{n-1}, 0; a_1, \ldots, a_{n-1}, 1)\right.$$
$$\left. + W(a_1, \ldots, a_{n-1}, 1; a_1, \ldots, a_{n-1}, 0)\right)$$

Then θ extends uniquely to an isomorphism θ of the C^*-algebra Ngenerated by U_Δ and L_σ onto $\varphi(\mathfrak{M})F$. Let $G(\gamma, x) = 1$ if $\gamma = e$, the identity of Δ, $G(\gamma, x) = 0$ otherwise. Then $G \in \mathfrak{H}$,

$$\omega_\theta | N = \omega_x \theta$$

and $[NG] = \mathfrak{H}, [\theta(N)x] = [\varphi(\mathfrak{M})Fx] = F$. Thus there is a unitary transformation V from F to $\mathfrak{H}$ such that $\theta(\cdot) = V \cdot V^*$. Then $V \cdot V^*$ is an isomorphism of the weak closure Mof N onto the weak closure $\varphi(\mathfrak{M})F^-$ of $\varphi(\mathfrak{M})F$. Thus $\varphi(\mathfrak{M})F^-$ is type III and the proof of (bl)>(b3) is complete.

We remark that if φ is as above then $\varphi(\mathfrak{A})$ is a factor.

(a3)⟹(al); (a4)⟹(al): Let $\mathfrak{K}$ be the largest GCR ideal in $\mathfrak{A}$. If $\mathfrak{A} \neq \mathfrak{K}$ then $\mathfrak{A}/\mathfrak{K}$ satisfies (bl) [92] and so $\mathfrak{A}/\mathfrak{K}$ (and hence $\mathfrak{A}$) has representations of type II and representations of type III. This contradicts (a3) and (a4) so $\mathfrak{A} = \mathfrak{K}$ and $\mathfrak{A}$ is GCR. This completes the proof of Theorem (2.2.7).

It would be interesting to know which portions of Theorem (2.2.7) remain true when $\mathfrak{A}$ is not separable. It follows from Theorem (2.2.7) and [92] that there is a "type decomposition" for separable C*-algebras which is somewhat analogous to the decomposition of a von Neumann algebra into a direct sum of von Neumann algebras of types I, II and III. In fact, the maximum GCR ideal $\mathfrak{K}$ of a separable C*-algebra $\mathfrak{A}$ is the type I portion of $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{K}$ might be called continuous or type II and III. However these latter terms are somewhat misleading (in this terminology the algebra of continuous funcions on a compact space is not a continuous C*-algebra) and it is not known whether they are appropriate for non-separable C*- algebras. We will not use these terms and we will call a C*-algebra with no non-zero GCR ideals an NGCR algebra. Of course $\mathfrak{A}$ need not be isomorphic to a direct sum of $\mathfrak{K}$ and $\mathfrak{A}/\mathfrak{K}$. This decomposition is fairly reasonable with respect to the "global behavior" (i.e., faithful representations) of $\mathfrak{A}$, but it may behave poorly with respect to arbitrary representations (or arbitrary irreducible representations). For example a might be an NGCR algebra, but have an ideal $\mathfrak{J}$ such that $\mathfrak{A}/\mathfrak{J}$ is GCR. To see this, let $\mathfrak{A}_i, i = 1, 2, \ldots$, be an NGCR algebra acting on a Hilbert space $\mathfrak{K}_i$. Let $\mathfrak{J}$ be the set $A_1, A_2, \ldots$ of seqences of operators, A_i in $\mathfrak{A}_i$, such that $\lim_{i\to\infty} \|A_i\| = 0$. $\mathfrak{J}$ S is a C*-algebra (acting on $\sum \oplus \mathfrak{H}_i$) and the C*-algebra $\mathfrak{A}$ generated by $\mathfrak{J}$ and I, the identity operator on $\sum_i \oplus \mathfrak{H}_i$ has no non-zero GCR ideals. However $\mathfrak{J}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}$ is GCR, in fact it is the complex numbers.

There are C*-algebras $\mathfrak{A}$ which are not GCR but which have a family $\{\varphi_\alpha\}$ of irreducible representations such that $\sum_\alpha \varphi_\alpha$ is faithful and $\varphi_\alpha(\mathfrak{A})$ contains the completely continuous operators. In fact take $\mathfrak{A}$ to be the direct sum $\sum_{i=1}^\infty \oplus \mathfrak{M}_i$, where $\mathfrak{M}_i$ is an $i \times i$ matrix algebra [92]. Of course $\mathfrak{A}$ is not separable; for an example of a separable C*-algebra, take $\mathfrak{A}$ to be the C*-algebra generated by the completely continuous operators and a separable NGCR algebra $\mathfrak{B}$ acting on $\mathfrak{H}$, for some Hilbert space $\mathfrak{H}$.

One might ask to what extent an arbitrary C*-algebra $\mathfrak{A}$ could be studied by means of representations φ of $\mathfrak{A}$ such that the weak closure $\varphi(\mathfrak{A})^-$ of $\varphi(\mathfrak{A})$ is a factor of type I or II and $\varphi(\mathfrak{A})$ contains the trace class operators in $\varphi(\mathfrak{A})^-$. There are (non-separable) C*-algebras which have no such representations, for example the algebra of all bounded operators on separable Hilbert space reduced modulo the completely continuous operators. See also the C*-algebras of &6.

The implication (a2)⇒(a5) suggests that possibly every locally compact separable type I group has sufficiently many "characters". By a character of a locally compact group, we mean a complex or infinite valued functional on the L_1 group algebra $\mathfrak{L}$ of the form $f \to trace(\varphi(\mathfrak{L}))$ where $f \in \mathfrak{L}$ and φ is a * representation of $\mathfrak{L}$ such that the weak closure of $\varphi(\mathfrak{L})$ is a factor, and where $0 \neq trace(\varphi(f)) < \infty$ for some f in $\mathfrak{L}$. This is a modification of Godement's Definition [107], the essential change being that we do not require the character to be finite on a dense subset of $\mathfrak{L}$. This modification seems to be necessary in order to deal with non-unimodular groups. Indeed it is quite likely that the "$ax + b$" group has characters in the above sense for which the set {If: trace $(\varphi(f)) < \infty$} is not dense in $\mathfrak{L}$. With regard to this, cf. [111].

The implications (b2)⟺(b3) show that a separable C*-algebra or locally compact group has representations of type II if and only if it has representations of type III. In particular Mautner's five-dimensional connected Lie group (described, for example, in [110]) has type III representations.

If $\mathfrak{A}$ is a simple C*-algebra, $\mathfrak{A}$ is either the completely continuous operators on a Hilbert space or $\mathfrak{A}$ is NGCR. In fact by [92], $\mathfrak{A}$ is GCR or NGCR and the statement follows from (a1)⇒(a5). If $\mathfrak{A}$ is a simple C*-algebra with a unit then $\mathfrak{A}$ is either a full $n \times n$ matrix algebra or NGCR.

We show Mackey's conjecture [110] (cf. [94]) that a separable locally compact group is type I if and only if it has a smooth dual. We make the following Definitions (cf. [94]). For each $n = 1, 2, \ldots, \infty$, let $\mathfrak{H}_n$ be a fixed Hilbert space of dimension n. ($\mathfrak{H}_\infty$ is separable.) Let $\mathfrak{G}$ be a separable locally compact group, let $\mathfrak{A}$ be a separable C*-algebra, let $\mathfrak{L}$ be the L_1 group algebra of $\mathfrak{G}$. Let $\mathfrak{G}_n^c(resp. \mathfrak{L}_n^c, \mathfrak{A}_n^c)$ be the set of all (unitary or *, respectively) representations of $\mathfrak{G}(resp. \mathfrak{L}, \mathfrak{A})$ on $\mathfrak{H}_n$. (For L to be a representation of $\mathfrak{L}$, we require that the closed linear span of $L_{\mathfrak{L}}\mathfrak{G}_n$ be $\mathfrak{G}_n$, and similarly for $\mathfrak{A}$. Mackey does not impose this restriction on representations and so the present meaning of $\mathfrak{L}_n^c$, etc., differs from that of [94].) Let $\mathfrak{G}_n^c$ have the smallest possible Borel structure (i.e., σ-field of subsets of $\mathfrak{G}_n^c$) such that for each x in $\mathfrak{G}$, and φ and ψ in $\varphi\mathfrak{H}_n$ the complex valued function $f(L) = (L_x\varphi, \psi)(L \in \mathfrak{G}_n^c)$ defined on $\mathfrak{G}_n^c$ is a Borel function. Let $\mathfrak{L}_n^c$ and $\mathfrak{A}_n^c$ have analogous Borel structures. Let $\mathfrak{G}^c = U_n\mathfrak{G}_n^c$, let the Borel sets of $\mathfrak{G}^c$ be those which meet each $\mathfrak{G}_n^c$ in a Borel set. Let $\widehat{\mathfrak{G}}^c$ be the subset of $\widehat{\mathfrak{G}}$ consisting of irreducible representations; let $\widehat{\mathfrak{G}}$ be the set of unitary equivalence classes of representations in $\widehat{\mathfrak{G}}^c$. Let $\widehat{\mathfrak{G}}^c$ have a Borel structure as a subspace of $\mathfrak{G}^c$. (The Borel subsets of $\widehat{\mathfrak{G}}^c$ are defined to be the intersections of $\widehat{\mathfrak{G}}^c$ with the Borel subsets of Tic.) Let $\widehat{\mathfrak{G}}$ have a Borel structure as a quotient of $\mathfrak{G}^c$. (The Borel subsets of $\widehat{\mathfrak{G}}^c$ are the sets E such that the sets $\{x: x \in \tilde{x} \in E$ for some $\tilde{x}$ in E$\}$ are Borel subsets of $\widehat{\mathfrak{G}}^c$.) In an analogous fashion define $\mathfrak{L}^c, \widehat{\mathfrak{L}}^c, \widehat{\mathfrak{L}}, \mathfrak{A}^c, \widehat{\mathfrak{A}}^c, \mathfrak{A}$. Then $\backslash fraktur$ G (resp. $\mathfrak{L}, \mathfrak{A}$) has a smooth dual if there is a countable family of Borel subsets of $\widehat{\mathfrak{G}}(resp. \widehat{\mathfrak{L}}, \widehat{\mathfrak{A}})$ which separate points of $\widehat{\mathfrak{G}}(resp. \widehat{\mathfrak{L}}, \widehat{\mathfrak{A}})$. A Borel space is standard if it is isomorphic to the Borel space of a Borel subset of a complete separable metric space. We say $\mathfrak{G}(resp. \mathfrak{A})$ has a metrically smooth dual if $\widehat{\mathfrak{G}}(resp. \widehat{\mathfrak{A}})$ is metrically countably separated, that is, for each σ-finite measure μ on $\mathfrak{G}(resp. \mathfrak{A})$, there is a Borel set N contained in $\mathfrak{G}(resp. \mathfrak{A})$ such that $\mu(N) = 0$ and $\widehat{\mathfrak{G}} \sim N(resp. \widehat{\mathfrak{A}} \sim N)$ is countably separated (that is, there is a countable family of Borel subsets of $\mathfrak{G} \sim N$ or $\widehat{\mathfrak{A}} \sim N$ which separates points of $\widehat{\mathfrak{G}} \sim N$ or $\widehat{\mathfrak{A}} \sim N$ respectively). We can state the following theorem, the portions (a1)⟺(a2), (a1)⟺(a3) and (g1)⟺(g2), (g1)⟺(g3) of which, were conjectured by Mackey.

Theorem (2.2.8)[100]: Let $\mathfrak{G}$ be a separable locally compact group, let $\mathfrak{A}$ be a, separable C*-algebra. Then the statements $(al), \ldots, (a4)$ are equivalent and the statements $(g1), \ldots, (g4)$ are equivalent.

(a1) (resp. (g1)) $\mathfrak{A}$ (resp. $\mathfrak{G}$) is type I.

(a2) (resp. (g2)) $\mathfrak{A}$ (resp. $\mathfrak{G}$) has a metrically smooth dual.

(a3) (resp. (g3)) $\mathfrak{A}$ (resp. $\mathfrak{G}$) has a smooth dual.

(a4) (resp. (g4)) $\widehat{\mathfrak{A}}$ (resp. $\widehat{\mathfrak{G}}$) is a standard Borel space.

Suppose for the time being that we have proved the equivalence of $(a1), \ldots, (a4)$ for all separable C*-algebras. Let $\mathfrak{A}$ be the completion of the L_1 group algebra $\mathfrak{L}$ of $\mathfrak{G}$ in the norm

$$\|A\|_c = \sup\{\|\varphi(A)\|: \varphi \; is \; a * -representation \; of \; \mathfrak{L}\}.$$

If A is self-adjoint then $\|A\|_c \leqq \|A\|_1$, where $\|A\|_1$ is the norm of A as an element of $\mathfrak{L}$. Thus $\|A\|_c$ is finite in all cases, and furthermore $\mathfrak{A}$ is a C*-algebra. Mackey has proved [94] that $\mathfrak{A}^c$ is a standard Borel space. Every *-representation of $\mathfrak{L}$ can be uniquely extended to a *-representation of $\mathfrak{A}$ and so the map $L \to L|\mathfrak{L}$ is one-one from $\mathfrak{A}^c$ onto $\mathfrak{L}^c$. It is obviously a Borel map and by [94] or by direct calculation, it is a Borel isomorphism. This map is an isomorphism with respect to the properties of irreducibility and unitary equivalence, and so $\widehat{\mathfrak{A}}$ is Borel isomorphic to $\widehat{\mathfrak{L}}$. By [94], $\widehat{\mathfrak{L}}$ is Borel isomorphic to $\widehat{\mathfrak{G}}$. (What we call $\widehat{\mathfrak{L}}$ is called $\mathcal{A}_{\mathfrak{G}}^{p}$ in [94].) It is well known that $\mathfrak{G}$ is type I if and only if $\mathfrak{L}$ is type I and it is trivial to see that $\mathfrak{G}$ is type I if and only if $\mathfrak{L}$ is type I (since the weak closure of $L(\mathfrak{A})$ is the weak closure of $L(\mathfrak{L})$ for any representation L of $\mathfrak{A}$). Thus the equivalence of (g1),...,(g4) follows from the equivalence of (al),…,(a4).

It is obvious that (a4)⟹(a3)⟹(a2). We prove (a2)⟹(a1). Suppose $\mathfrak{A}$ is not type I. We must show that $\widehat{\mathfrak{A}}$ is not metrically countably separated and to do this, it is sufficient to find a subset K of $\widehat{\mathfrak{A}}$ (not necessarily a Borel subset) such that K as a subspace of $\widehat{\mathfrak{A}}$ is not metrically countably separated. In fact suppose there is a σ-finite measure μ on K such that for any Borel subset N of K of μ-measure zero, $K \sim N$ is not countably separated. Define

$$\tilde{\mu}(E) = \mu(E \cap K)$$

for E a Borel subset of $\widehat{\mathfrak{A}}$. Then $\tilde{\mu}$ is a σ-finite measure on $\widehat{\mathfrak{A}}$. Let N be a Borel subset of $\widehat{\mathfrak{A}}$ such that $\tilde{\mu}(N) = 0$, let $E_1, E_2, \ldots$ be Borel subsets of $\mathfrak{A}$. Then $\mu(N \cap K) = 0$ and so the sets $E_1 \cap K, E_2 \cap K, \ldots$ do not separate $K \sim N$ and this implies that the sets $E_1, E_2, \ldots$ do not separate $\widehat{\mathfrak{A}} \sim N$, and that $\widehat{\mathfrak{A}}$ is not metrically countably separated.

Since $\mathfrak{A}$ is not type I, Theorem (2.2.7) implies that the maximum GCR ideal $\mathfrak{K}$ of $\mathfrak{A}$ is not equal to $\mathfrak{A}$. Let K^c be the set of representations in $\widehat{\mathfrak{A}}^c$ which annihilate $\mathfrak{K}$, let K be the set of unitary equivalence classes of $\widehat{\mathfrak{A}}^c$ contained in K^c. Let $\pi: A \to \mathfrak{A}/\mathfrak{K}$. The map

$$\pi^* : (\mathfrak{A}/\mathfrak{K})^{\wedge c} \to K^c$$

defined by: $\pi^*(L) = L \circ \pi$ for L in $(\mathfrak{A}/\mathfrak{K})^{\wedge c}$ is one-one and onto and is a Borel isomorphism. Since K^c contains each unitary equivalence class it meets, π^* defines a one-one map from $(\mathfrak{A}/\mathfrak{K})^{\wedge}$onto K, and furthermore this map is a Borel isomorphism, if we give K the quotient Borel structure $\mathfrak{B}_q$ derived from K^c. If we can show that $\mathfrak{A}/\mathfrak{K}$ does not have a metrically smooth dual then K with the Borel structure $\mathfrak{B}_q$ is not metrically countably separated. However $\mathfrak{B}_q$ contains the Borel structure $\mathfrak{B}_s$ on K which makes K a subspace of $\widehat{\mathfrak{A}}$. In fact if $\tilde{E} \subset \widehat{\mathfrak{A}}$, let E be the set of elements of elements of $\tilde{E}$. If $\tilde{E} \subset K$ then $\tilde{E} \in \mathfrak{B}_s$, if and only if there is a Borel set $\tilde{F}$ contained in $\widehat{\mathfrak{A}}$ such that $\tilde{F} \cap K = \tilde{E}$, or equivalently $F \cap K^c = E$, while $E \in \mathfrak{B}_q$ if and only if there is a Borel set D contained in $\widehat{\mathfrak{A}}^c$ such that $D \cap K^c = E$. (Actually $\mathfrak{B}_s = \mathfrak{B}_q$ since K^c is a Borel set, but we do not need this.) Thus K with the Borel structure $\mathfrak{B}_s$ is not metrically countably separated. By the previous paragraph, this implies that $\mathfrak{A}$ does not have a metrically smooth dual. Thus it is sufficient to consider the case $\mathfrak{K} = 0$.

We suppose $\mathfrak{A}$ is NGCR and we use the notation of the proof of (bl)⇒(b5) of Theorem (2.2.7). In particular we suppose $S_1, S_2, \ldots$ and $V(a_1, .., a_n), n = 1, \ldots$ chosen as in (bi)⇒(b5). Choose a unit vector ψ in $\mathfrak{H}_\infty$. and let $K^c = \{L: L \in \widehat{\mathfrak{A}}^c \cap \mathfrak{A}^c_\infty, (L_{E(k)}\psi, \psi) = 1 \; and \; (L_{E(k,1)}\psi, \psi) = 0 \; or \; 1 \; for \; all \; k\}$ where $E(k)$ is defined in Lemma (2.2.4), $E(k, 1)$ is defined in (b1)⇒(b5). Then K^c is a Borel subset of $\widehat{\mathfrak{A}}^c$. Let X be the measure space defined in the proof of Theorem (2.2.7), (b1)⇒(b3) and let Δ be the group of measurability preserving transformations defined there. If $L \in K^c$, we define $\theta(L)$ to be the sequence $\{(L_{E(k,1)}\psi, \psi): k = 1, 2, \ldots\}$ in X. Then θ maps K^c onto X. To see this, let $(d_1, d_2, \ldots)$ be an element of X and let $\varphi_{\{d_i\}} = \varphi, \mathfrak{H}_{\{d_i\}} = \mathfrak{H}$ and $x(\{d_i\}) = x$ be as in (b1)⇒(b5). Let U be a unitary transformation from $\mathfrak{H}$ onto $\mathfrak{H}_\infty$ such that $Ux = \psi$. (Recall that $\mathfrak{H}$ and $\mathfrak{H}_\infty$ have the same dimension.) Define L by the equation $L_A = U\varphi(A)U^{-1}$ for A in $\mathfrak{A}$. Then $L \in \widehat{\mathfrak{A}}^c \cap \mathfrak{A}^c_\infty$, and

$$(L_{E(k)}\psi, \psi) = (U\varphi(E(k))x, Ux) = 1$$
$$(L_{E(k-1)}\psi, \psi) = (U\varphi(E(k, 1)x, Ux) = (\varphi(E(k, 1)x, x) = d_k$$

so $L \in K^c$ and $\theta(L) = (d_1, d_2, \ldots)$, and θ is onto.

We show that the Borel structure on X is the same as the quotient Borel structure on X derived from θ and K^c, where K^c has a Borel structure as a subset of $\widehat{\mathfrak{A}}^c$. Let $F(a_1, \ldots, a_k)$ be the cylinder $\{(x_1, x_2, \ldots): x_1 = a_1, \ldots, x_k = a_k\}$ in X. Then

$$\theta^{-1}(F(a_1, \ldots, a_k)) = \{L: L \in K^c, (L_{E(j,1)}\psi, \psi) = a_j, j = 1, \ldots, k\}$$

is a Borel subset of K^c. Thus $F(a_1, \ldots, a_n)$ is a Borel set in the quotient Borel structure and the quotient Borel structure contains the original Borel structure of X. Thus the quotient Borel structure is countably separated and since K^c is standard, the quotient structure is analytic [94]. (A countably generated Borel space which is the image under a Borel map of a standard Borel space is called analytic.) The identity map of X onto itself is a Borel map from the quotient Borel structure to the original Borel structure, and so the two Borel structures coincide, as asserted ([109], [94]).

Let L^1 and L^2 be in K^c. We assert that L^1 is unitarily equivalent to L^2 if and only if $\theta(L^1) = \theta(L^2) + \delta$ for some δ in Δ. First suppose L^1 is unitarily equivalent to L^2, and let $\theta(L^1) = (s_1, s_2, \ldots)$, let $\theta(L^2) = (t_1, t_2, \ldots)$. Let $\varphi_1 = \varphi_{\{s_j\}}$ let $\varphi_2 = \varphi_{\{t_j\}}$ let $x_1 = x_{\{s_j\}}$, let $x_2 = x_{\{t_j\}}$. We assert L^i is unitarily equivalent to φ_i. Let V be the transformation from $\mathfrak{H}_\infty$. to $\mathfrak{H}_i$, the representation space of φ_i, defined by

$$V(L^i_A\psi) = \varphi_i(A)x_i.$$

Then

$$\|V(L^i_A\psi)\|^2 = (\varphi_i(A^*A)x_i, x_i) = (L^i_{A*A}\psi, \psi) = \|L^i_A\psi\|^2$$

since $\omega_{x_i}\varphi_i|\mathfrak{M} = \omega_\psi L^i|\mathfrak{M}$ (a consequence of Lemma (2.2.6) and the Definition of φ_i) and so $\omega_{x_i}\varphi_i = \omega_\psi L^i$ (a consequence of Lemma (2.2.6), formula (1) and the choice of $S_1, S_2, \ldots$). Thus V is well defined, isometric, and admits a unitary extension. This extension implements the desired equivalence and it follows that φ_1, is unitarily equivalent to φ_2, and as we saw in the proof of (b1)⇒(b5), $s_i = t_i$ for all but finitely many i. Thus $\theta(L^1) = \theta(L^2) + \delta$ for some δ in Δ.

Conversely suppose $\theta(L^1) = \theta(L^2) + \delta$. Since $\delta = \gamma_{k_1} + \cdots + \gamma_{k_s}$ for some $k_1, \ldots, k_s$, where γ_k is defined as in (b1)⇒(b3), it is sufficient to consider the case $\delta = \gamma_k$. Let

$$U(k) = \sum_{a_1,\ldots,a_{k-1}} (V(a_1, \ldots, a_{k-1}, 0)V(a_1, \ldots, a_{k-1}, 1)^* + V(a_1, \ldots, a_{k-1}, 1)V(a_1, \ldots, a_{k-1}, 0)^*)$$

let W be a unitary operator on $\mathfrak{H}_\infty$. such that $W\psi = L^2_{U(k)}\psi$; W exists since $U(k)^*U(k)E(k+1) = E(k+1)$ (Lemma (2.2.5)) and so

$$\left\|L^2_{U(k)}\psi\right\|^2 = \left\|L^2_{U(k)}L^2_{E(k+1)}\psi\right\|^2 = (L^2_{E(k+1)^2}\psi, \psi) = 1.$$

(We use the fact that $L^2_{E(k+1)}\psi = \psi$; see the proof of Lemma (2.2.6).) Let $L^3 = W^*L^2W$. Then for any positive integer r,

$$\left(L^3_{E(r)}\psi, \psi\right) = \left(L^2_{E(r)}L^2_{U(k)}\psi, L^2_{U(k)}\psi\right) = 1$$

since $U(k)E(r)U(k)E(s) = E(r)E(s) = E(s)$ if $s > \max r, k$ and $L^2_{E(S)}\psi = \psi$;

$$(L^3_{E(r,1)}\psi, \psi) = \left(L^2_{U(k)E(r,1)U(k)}\psi, \psi\right) \begin{cases} \left(L^2_{E(r,1)}\psi, \psi\right) & if\ r \neq k \\ \left(L^2_{E(r,0)}\psi, \psi\right) & if\ r = k \end{cases}$$

since $U(k)E(r,1)U(k)E(s) = E(r,1)E(s)$ if $r \neq k, = E(r,0)E(s)$ if $r = k$, and if $s > max\, r, k$. Thus $L^3 \in K^c$ and $\theta(L^3) = \theta(L^2) + \gamma_k = \theta(L^1)$. Let $\theta(L^3) = (S_1, s_2, ..)$. We saw in the preceding paragraph that L^3 and L^1 are each unitarily equivalent to the representation $\varphi_{\{s_i\}}$ of $\mathfrak{A}$ constructed in the proof of (b1)⇒(b5). Thus L^1 and L^3 are unitarily equivalent and L^1 and L^2 are unitarily equivalent.

Let K be the set of unitary equivalence classes of elements of K^c. That is, if $\tilde{x} \in K$ then for some x in $K^c, \tilde{x} = \{y: y$ is unitarily equivalent to x and $y \in K^c\}$. Then $\tilde{x} \notin \widehat{\mathfrak{A}}$; we let $\rho(\tilde{x})$ be the unitary equivalence class in $\widehat{\mathfrak{A}}$ containing $\tilde{x}$. Let $\mathfrak{B}_q$ be the quotient Borel structure on K derived from K^c and let $\mathfrak{B}_s$, be the Borel structure on K which makes ρ a Borel isomorphism of K with $\rho(K)$, where $\rho(K)$ has a Borel structure as a subspace of $\widehat{\mathfrak{A}}$. If $\tilde{E} \subset K$, let E be the set of elements of elements of $\tilde{E}$. Then $\tilde{E} \in \mathfrak{B}_s$ if and only if there is a Borel subset $\tilde{F}$ of $\widehat{\mathfrak{A}}$ such that $\rho(\tilde{E}) = p(K) \cap \tilde{F}$ or equivalently $E = K^c \cap F$, while $\tilde{E} \in \mathfrak{B}_q$ if and only if there is a Borel subset D of $\widehat{\mathfrak{A}}^c$ such that $E = K^c \cap D$. Thus $\mathfrak{B}_q \supset \mathfrak{B}_s$.

Let $\tilde{\theta}$ be the one-one map defined by θ from K onto the set $\tilde{X}$ of Δ-equivalence classes of X, let $\tilde{X}$ have the quotient Borel structure derived from X. We show that $\tilde{\theta}$ is a Borel isomorphism with respect to the Borel structure $\mathfrak{B}_q$ on K. Let $\tilde{E}$ be a subset of $\tilde{X}$. Then $\tilde{E}$ is a Borel set if and only if the set E of elements of elements of $\tilde{E}$ is a Borel set and this is a Borel set if and only if $\theta^{-1}(E)$ is a Borel set. However $\theta^{-1}(E)$ contains each unitary equivalence class in K^c that it meets, and so $\theta^{-1}(E)$ is a Borel set if and only if the set $\theta^{-1}(E)^\sim$ of unitary equivalence classes of elements of $\theta^{-1}(E)$ is in $\mathfrak{B}_q$. Since $\theta^{-1}(E)^\sim = \tilde{\theta}^{-1}(\tilde{E}), \tilde{\theta}$ is a Borel isomorphism.

X is a compact group, Δ is a dense subgroup and $\tilde{X} = X/\Delta$. It follows from [94] that $\tilde{X}$ is not metrically countably separated and so K with the Borel structure $\mathfrak{B}_q, K$ with the Borel structure $\mathfrak{B}_q, \rho(K)$ and $\widehat{\mathfrak{A}}$ are not metrically countably separated, and so $\mathfrak{A}$ does not have a metrically smooth dual. The proof of (a2)⇒(a1) is complete.

We prove (al)⇒(a4). Suppose that $\mathfrak{A}$ is a separable type I C*-algebra. By Theorem (2.2.7), $\mathfrak{A}$ is GCR and by [92], $\mathfrak{A}$ has a composition series $\{\mathfrak{K}_\alpha\}$ such that each $\mathfrak{K}_{\alpha+1}/\mathfrak{K}_\alpha$ is

CCR with a Hausdorff structure space. We assert that there are at most a countable number of terms in the composition series. Let $B(1), B(2), \ldots$ be a countable dense subset of $\mathfrak{A}$. For each index α, choose an x_α in the structure space X_α of $\mathfrak{K}_{\alpha+1}/\mathfrak{K}_\alpha$ and an A_α in $\mathfrak{K}_{\alpha+1}$ with $\|\mathfrak{A}_\alpha(x_\alpha)\| = 1$. (We identify X_α with the subset $\{x: \mathfrak{K}_\alpha(x) = 0, \mathfrak{K}_{\alpha+1}(x) \neq 0\}$ of the structure space X of $\mathfrak{A}$. The topology on X_α as a subspace of X is the same as the topology on X_α as the structure space of $\mathfrak{K}_{\alpha+1}/\mathfrak{K}_\alpha$; cf. [92].) Then

$$\|A_\alpha - A_\beta\| \geqq \|(A_\alpha - A_\beta)(x_\alpha)\| = \|A_\alpha(x_\alpha)\| = 1$$

for $\alpha > \beta$. Thus for any given i there is at most one a(i) for which $\|A_{\alpha(i)} - B(i)\| < 1/2$, and so $i - \alpha(i)$ is a function. Since $B(1), B(2), \ldots$ is dense, this function is onto the set of all indexing ordinals except the largest, and this proves the assertion.

For each α choose a sequence $B(\alpha,1), B(\alpha,2), \ldots$ dense in $\mathfrak{K}_\alpha$. The functions $x \to \|B(\alpha+1, i)(x)\|$ are continuous on X_α. [92] and separate points of X_α, and so X_α has a countable base for open sets. (In the above and in what follows, topological statements concerning subsets of X_α will be regarded as referring to the topological space X_α which has the relative topology from X.)

Let K be a compact subset of X_α, let $\widehat{\mathfrak{A}}^c(K)$ be the set of L in $\widehat{\mathfrak{A}}^c$ such that kernel $L \in K$. We show that $\widehat{\mathfrak{A}}^c(K)$ is a Borel set. Since X_α has a countable base for open sets, there is a sequence $\{U(j)\}$ of open neighborhoods of K in X_α such that $\cap_j U_j = K$. There is no difficulty in finding a C in $\mathfrak{K}_{\alpha+1}$, such that $\|C(x)\| \geqq 1$ for x in K and since $\mathfrak{K}_{\alpha+1}$ is closed under multiplication by continuous functions on X_α [92], we can find a $C(j)$ in $\mathfrak{K}_{\alpha+1}$ such that $\|C(j)(x)\| \geqq 1$ for x in K and $\|C(j)(x)\| = 0$ for x not in $U(j)$. Choose an orthonormal basis $\{\varphi_s: s = 1, 2, \ldots, n\}$ for $\mathfrak{H}_n$ Then

$$\widehat{\mathfrak{A}}^c(K) \cap \mathfrak{A}_n^c = \widehat{\mathfrak{A}}^c \cap \{L: L \in \mathfrak{A}_n^c, (L_{B(\alpha,i)}\varphi_s, \varphi_i) = 0 \text{ for all } i, s, t,\}$$
$$\cap \{L: L \in \mathfrak{A}_n^c, \text{for each j}, \exists s \text{ and t such that } (L_{o(j)}\varphi_s, \varphi_i) \neq 0\}$$

and so $\widehat{\mathfrak{A}}^c(K)$ is a Borel set. Since $\widehat{\mathfrak{A}}^c(K)$ contains each unitary equivalence class it meets, the set $\widehat{\mathfrak{A}}(K)$ of unitary equivalence classes in $\widehat{\mathfrak{A}}^c(K)$ is a Borel subset of $\widehat{\mathfrak{A}}$.

Let $\tilde{L}$ be in $\widehat{\mathfrak{A}}$, let $\pi(\tilde{L})$ be the kernel of a representative L of $\tilde{L}$. By 11l, Theorem (2.2.7)] or [92], L is algebraically irreducible and so $\pi(\tilde{L}) \in X$. By [92] and the separability of $\mathfrak{A}, \pi(\widehat{\mathfrak{A}}) = X$ and by Theorem (2.2.7), π is $1-1$. If K is a compact subset of X_α then $\pi^{-1}(K) = \widehat{\mathfrak{A}}(K)$ and it follows from the preceding paragraph that $\pi|X_\alpha$ is a Borel map. Since X_α is the intersection of the closed set $\{x: \mathfrak{K}_\alpha(x) = 0\}$ and the open set $\{x: \mathfrak{K}_{\alpha+1}(x) \neq 0\}, X_\alpha$ is a Borel subset of X. Also $\pi^{-1}(X_\alpha)$ is a Borel subset of $\widehat{\mathfrak{A}}$ and so π is a Borel map. We show that X is a standard Borel space. In fact X_α is a Hausdorff locally compact space [92] and an open (and hence Borel) subset of its one-point compactification $X_\alpha \cup \{\infty\}$. If we define $\|B(\alpha,i)(\infty)\| = 0$ then the functions $x \to \|B(\alpha,i)(x)\|$ on $X_\alpha \cup \{\infty\}$ are continuous [92] and so $X_\alpha \cup \{\infty\}$ is a separable metrizable space, and since $X_\alpha \cup \infty$ is compact, it is complete. Thus the Borel space of $X_\alpha \cup \{\infty\}$ is standard and the Borel space of X_α is standard. Since X_α is a Borel subset of X, if $E \subset X$ then E is a Borel set if and only if $E \cap X_\alpha$ is a Borel set for each a, and by [94], X is standard. Thus W is countably separated and so is analytic [94]. Hence π is a Borel isomorphism [109], [94], $\widehat{\mathfrak{A}}$ is standard and the proof of Theorem (2.2.8) is complete.

We remark that (al)⇒(a3) and the isomorphism π of $\widehat{\mathfrak{A}}$ onto X derived in the proof of the theorem could also be deduced from Theorem (2.2.7), (al)⟹(a6) and [105]. In [94],[108], Mackey has a theory of direct integral decomposition of multiplicity free representations of separable locally compact type I groups or algebras with metrically smooth duals. It follows that the hypothesis "with metrically smooth dual" can be deleted, and the theory is applicable to a wide range of groups, for example all real algebraic Lie groups [103].

If $\mathfrak{A}$ is a separable GCR algebra then we have already proved that Mackey's dual, $\mathfrak{A}$, is Borel isomorphic to the Borel space generated by the structure space X of $\mathfrak{A}$. Even in the non-separable case, the structure space seems to be the natural dual to a GCR algebra. The next theorem identifies X as a topological quotient of the set $\mathcal{B}_0$ of pure states of $\mathfrak{A}$, it is essentially a reformulation [104].

Theorem (2.2.9)[100]: Let $\mathfrak{A}$ be a C*-algebra, let X be the structure space of $\mathfrak{A}$, let $\mathcal{B}_0$ be the set of pure states of $\mathfrak{A}$ with the relative w*-topology. If $f \in \mathcal{B}_0$ let $\pi(f) = kernel\ \varphi_f$. Then π is a continuous open map of $\mathcal{B}_0$ onto X.

That π maps into X follows from [89], that π is onto was proved in [92]. Let K be a closed subset of X, let $\mathfrak{L}$ be the intersection of the ideals in K, let $\mathfrak{L}^{\perp}$ be the subset of $\mathcal{B}_0$ which annihilates $\mathfrak{L}$. Let f be in $\mathcal{B}_0$ and let $f = \omega_x\varphi_f$ for some x in $\mathfrak{H}_f$ (and $[\varphi_f(\mathfrak{A})x] \neq \mathfrak{H}_f$). If $f \in \mathfrak{L}^{\perp}$ then $0 = f(\mathfrak{L}) = \left(\varphi_f(\mathfrak{L})x, x\right) = \left(\varphi_f(\mathfrak{L})\varphi_f\,(A)x, \varphi_f\,(B)x\right)$ for all A and B in $\mathfrak{A}$ and so kernel $\varphi_f \supset \mathfrak{L}, f \in \pi^{-1}(K)$. Conversely if $f \in \pi^{-1}(K)$ then $\varphi_f\,(\mathfrak{L}) = 0, f(\mathfrak{L}) = \omega_x\varphi_f\,(\mathfrak{L}) = 0$, and $f \in \mathfrak{L}^{\perp}$. Thus $\pi^{-1}(K) = \mathfrak{L}^{\perp}, \pi^{-1}(K)$ is closed, and π is continuous. We interrupt the proof to prove a lemma.

Lemma (2.2.10)[100]: Let $\mathfrak{A}$ be a C*-algebra, let $\mathfrak{A}_1$ be the C*-algebra generated by $\mathfrak{A}$ and I, let $\mathcal{U}$ be the group of unitary operators in $\mathfrak{A}_1$, let Δ be a relatively w*-closed subset of $\mathcal{B}_0$ and suppose Δ is $\mathcal{U}$-invariant (i.e., $f \in \Delta$ implies $f(U^* \cdot U) \in \Delta$ for U in $\mathcal{U}$. Then $\Delta^{\perp}$ is an ideal, $\Delta^{\perp}$=$\cap_{f\in\Delta}\ kernel\ \varphi_f$ and $\Delta^{\perp\perp} \cap \mathcal{B}_0 = \Delta$.

We remark that $\mathfrak{A}$ is an ideal in $\mathfrak{A}_1$ and so $A \to U^*AU$ is an automorphism of $\mathfrak{A}$, for U in $\mathcal{U}$, and so $f(U^* \cdot U) \in \mathcal{B}_0$ if $f \in \mathcal{B}_0$. Let $\varphi\ = \sum\oplus \left\{\varphi_f : f \in \Delta\right\}$. Let A be an element of $\mathfrak{A}$. If $\varphi\,(A)\ = 0$ then $\varphi_f\,(A)\ = 0$ for allf in $\Delta, f(A) = 0$ for all f in Δ and $A \in \Delta^{\perp}$. If $\varphi\,(A) \neq 0$ then $\varphi_f\,(A) \neq 0$ for some f in Δ and $(\varphi_f\,(A)y, y) \neq 0$ for some y in $\mathfrak{H}_f$. Let $f = \omega_x\varphi_f$ for some x in $\mathfrak{H}_f$, let $\bar{\varphi}_f$ be the extension of φ_f to $\mathfrak{A}$, defined by $\bar{\varphi}_f(I) =$ identity. Let U be in $\mathcal{U}$ such that $\bar{\varphi}_f(U)x = y$ [107]. Then $f(U^*AU) = \left(\varphi_f\,(A)\bar{\varphi}_f(U)x, \bar{\varphi}_f(U)x\right) \neq 0$ and since $f(U^* \cdot U) \in \Delta, A \notin \Delta^{\perp}$. Thus $\Delta^{\perp}= kernel\ \varphi, \Delta^{\perp}$ is an ideal and $\Delta^{\perp}$=$\cap_{f\in\Delta}\ kernel\ \varphi_f$. If $\varphi(A)$ is a non-zero element of $\varphi(\mathfrak{A})$ then we have shown $f(\varphi^{-1}\varphi(A)) \neq 0$ for some f in Δ. If $f(A) \geqq 0$ for all f in Δ then $\omega_x\varphi_f(A) \geqq 0$ for all y in $\mathfrak{H}_f$ and f in $\Delta, \varphi_f\,(A) \geqq 0$ for all f in Δ and $\varphi(A) \geqq 0$. Thus the mapping which sends $\varphi(A)$ into the function $\left\{\left(f\varphi^{-1}, f(A)\right): f \in \Delta\right\}$ is an order isomorphism of $\varphi(\mathfrak{A})$ onto a linear space of functions on $\Delta\varphi^{-1}$. It follows from the discussion of [108] that the w*-closure of $\{f\varphi^{-1}: f \in \Delta\}$ contains the pure states of $\varphi(A)$ and so the w*-closure of Δ contains the pure states of $\mathfrak{A}$ which annihilate kernel φ [116]. Since Δ is relatively closed in $\mathcal{B}_0, \Delta\supset \Delta^{\perp\perp} \cap \mathcal{B}_0$. The reverse inclusion is evident and the proof is complete.

Let Vbe an open subset of $\mathcal{B}_0$. Let $\mathcal{U}(V) = \{f(U^* \cdot U): U \in \mathcal{U}, f \in V\}$, let $K = \pi\left(\mathcal{B}_0 \sim \mathcal{U}(V)\right)$. Then $\mathcal{U}(V)$ is open and $\mathcal{B}_0 \sim \mathcal{U}(V)$ is closed and $\mathcal{U}$-invariant. Let $\mathfrak{L} =\cap \{x: x \in K\}$. Then $\mathfrak{L} = \left(\mathcal{B}_0 \sim \mathcal{U}(V)\right)^{\perp}$ and if $g \in \mathcal{U}(V)$, then $g(\mathfrak{L}) \neq 0, \varphi_g(\mathfrak{L}) \neq 0$ and $\pi(g)$ is not in the closure of K. Thus

$$X \sim K = X \sim \pi\left(\mathcal{B}_0 \sim \mathcal{U}(V)\right) \subset \pi\left(\mathcal{U}(V)\right) \subset X \sim K \subset X \sim K.$$

Hence K is closed and $\pi\left(\mathcal{U}(V)\right) = X \sim K$ is open. Since $\pi(V) = \pi\left(\mathcal{U}(V)\right), \pi$ is open and the proof is complete.

We remark that if $\mathfrak{A}$ is GCR then

$$\pi^{-1}(x) = \{\omega_\xi \psi_x : \xi \text{ is a unit vector in } \mathfrak{H}_x\}$$

If $f = \omega_\xi \psi_x$ for some such ξ then φ_f is unitarily equivalent to ψ_x, $kernel\, \varphi_f = kernel\, \psi_x = x$ and $f \in \pi^{-1}(x)$. If $f \in \pi^{-1}(x)$ then kernel $\varphi_f = x$ and by Theorem (2.2.7), $\varphi_f = U^{-1}\psi_x U$, where U is some unitary transformation from $\mathfrak{H}_f$ to $\mathfrak{H}_x$. Since $f = (\varphi_f(\cdot)\gamma, \gamma)$ for some unit vector γ in $\mathfrak{H}_f$,

$$f = (\psi_x(\cdot)U\gamma, U\gamma) = \omega_{U\gamma}\psi_x$$

and the remark is proved.

We show that a GCR algebra has a T_1 structure space if and only if it is CCR (cf. [104]). We denote by $\mathfrak{G}(\mathfrak{H})$ the algebra of completely continuous operators on a Hilbert space $\mathfrak{H}$.

Theorem (2.2.11)[100]: Let $\mathfrak{A}$ be a GCR algebra with structure space X. If $x \in X$ then {x} is closed if and only if $\mathfrak{A}(x)$ is the set of all completely continuous operators.

By Theorem (2.2.7), $\mathfrak{A}(x)$ contains $\mathfrak{G}(\mathfrak{H}_x)$. However $\mathfrak{A}(x) = \mathfrak{G}(\mathfrak{H}_x)$ if and only if all irreducible representations of $\mathfrak{A}(x)$ are faithful and this is true if and only if all primitive ideals y which contain x are equal to x; that is, if and only if {x} is closed.

Theorem (2.2.11) would not be true if we deleted the hypothesis: $\mathfrak{A}$ is GCR.

If $\mathfrak{A}$ is commutative or if $\mathfrak{A}$ is $\mathfrak{G}(\mathfrak{H})$ then it is known that each point in the w*-closure of the set of pure states of $\mathfrak{A}$ is proportional to a pure state of $\mathfrak{A}$ (and if $I \in \mathfrak{A}$, is a pure state). Theorem (2.2.17) gives necessary and sufficient conditions for $\mathfrak{A}$ to have this property. First we need to extend [106] to C*-algebras without units.

Lemma (2.2.12)[100]: Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space $\mathfrak{H}$, and suppose $I \in (resp. \notin)\mathfrak{A}$. Then the w*-closure of the vector states of $\mathfrak{A}$ is the set $a\omega_x|\mathfrak{A} + bg$ where $a, b \in [0,1], a + b = (resp. \leqq 1)$, x is a unit vector and g is a state of $\mathfrak{A}$ which annihilates the completely continuous operators in $\mathfrak{A}$.

Let $\mathfrak{B}$ be the set of $\lambda I + A$ for λ a complex number and A in $\mathfrak{A}$. Then $\mathfrak{B}$ is a C*-algebra and if $\mathfrak{B} \neq \mathfrak{A}$ the positive linear functionals of $\mathfrak{A}$ of norm not greater than one have unique extensions to states of $\mathfrak{B}$. Furthermore a net of such functionals w*-converges if and only if the net of extensions to states of $\mathfrak{B}$ w*-converges. Thus w*-closure of the vector states of $\mathfrak{A}$ is the set of restrictions to $\mathfrak{A}$ of elements of the w*-closure of the vector states of $\mathfrak{B}$. Thus Lemma (2.2.12) follows from [106]. (See the first paragraph of the proof of Theorem (2.2.8) of [106] for the fact that the f of Theorem (2.2.8) of [106] is a vector state. Observe that $\omega_x = \omega_{x_1} + \omega_{x_2}$ where $x_1 = [\mathfrak{A} \cap \mathfrak{G}(\mathfrak{H})x]x$ and $x_2 = x - x_1$.)

The next two lemmas are concerned with the continuous extension of matrix units in $\mathfrak{A}(y)$ to matrix units in $\mathfrak{A}(x)$ for x near y.

Lemma (2.2.13)[100]: Let $\mathfrak{A}$ be a CCR algebra with a Hausdorff structure space X. Let y be in X, let N be a neighborhood of y and let E and F be in $\mathfrak{A}$ and suppose that E(x) and F(x) are projections for all x in N. Suppose further that there is a V in $\mathfrak{A}$ such that V(y) is a partial isometry from E(y) to F(y). Then there is a neighborhood M of y contained in N and a W in $\mathfrak{A}$ such that if $x \in M$ then W(x) is a partial isometry from E(x) to F(x) and W(y)=V(y).

Let M be the subset of N consisting of those x in N for which $\|V(x)\|^2 < 2, \|F(x) - FVEV^*F(x)\| < 1/4$ and $\|E(x) - EV^*FVE(x)\| < 1/4$. By [92] Mis a neighborhood of y. Let $W = FVEk(EV^*FVE)$, where k is the function defined by: $k(x) = 0$ or $\left(\frac{1}{x}\right)^{\frac{1}{2}}$ as $x \leqq 1/4$ or $x \geqq 1/2$ and k is linear on $[1/4, 1/2]$. Let x be in M and let γ be a homomorphism of the (commutative) C*-algebra generated by E(x) and $EV^*FVE(x)$.

Then $\gamma(EV^*FVE(x))$ is in $[-1/4, 1/4]$ or $[3/4, 1+(1/4)]$, and the first possibility occurs if and only if $\gamma(E(x)) = 0$, and so $\gamma\big(W^*W(x)\big) = 0$ or 1 and the first possibility occurs if and only if $\gamma\big(E(x)\big) = 0$. Thus $\gamma(E(x)) = \gamma\big(W^*W(x)\big)$ and $E(x) = W^*W(x)$. It follows that $W(x)$ is a partial isometry, $WW^*(x)$ is a projection, and since $FW(x) = W(x), WW^*(x) \leqq F(x)$. However

$$\|F(x) - WW^*(x)\| < \|F(x) - FVEV^*F(x)\| + \|FVEV^*F(x) - WW^*(x)\|$$
$$< 1/4 + \|V(x)\|^2 \left\|E(x) - \big(k(EV^*FVE)\big)^2\right\| \leqq 1/4 + 2(1/3) < 1$$

since if γ is as above, $\left|\gamma\big(E(x)\big) - \gamma\big(k(EV^*FVE)\big)^2\right| \leqq 1/3$. Thus $WW^*(x) = F(x)$. Since $W(y) = V(y)$, the proof is complete.

Definition (2.2.14)[100]: A point y in X is called singular if there is an E in $\mathfrak{A}$ with E(x) a projection for all x in some neighborhood N of y, with E(y) one dimensional, and such that for each neighborhood M of y contained in N, there is an x in M such that dim $E(x) > 1$. If y is not singular, y is called -regular.

We remark that there is always an E in $\mathfrak{A}$ and a neighborhood N of y with E(x) a projection for all x in N and E(y) one dimensional (see Lemma (2.2.15)). If dim $E(x) = 1$ for x sufficiently near y then by Lemma (2.2.13), dim $F(x) = 1$ for x sufficiently near y, where F is any element of $\mathfrak{A}$ such that F(x) is a projection for x near y and dim $F(y) = 1$, and so y is regular.

Lemma (2.2.15)[100]: Let $\mathfrak{A}$ be a CCR algebra with a Hausdorff structure space X, let y be in X, let N be a neighborhood of y and let A and F be in $\mathfrak{A}$. Suppose that F(x) is a projection for x in N, A(y) is a non-zero projection and $A(y)F(y) = 0$. Then there is a neighborhood M of y contained in N and a B in $\mathfrak{A}$ such that B(x) is a non-zero projection and $B(x)F(x) = 0$ for x in M and $B(y) = A(y)$. Furthermore if there is an E in $\mathfrak{A}$ with E(x) a projection greater than F(x) for x in N and if $E(y) \geqq A(y)$ then M and B can be chosen to satisfy the above conditions and also to satisfy: $E(x) \geqq B(x)$ for all x in M.

Replacing A by $(A + A^*)/2$ if necessary, we can suppose A is self-adjoint. Let $C = A$ if E is not given, let $C = E^*AE$ if E is given as above. Let $D = C - F^*C - CF + F^*CF$. Then D is self-adjoint, $D(y) = A(y), EDE(x) = D(x)$ if E is given and $x \in N$, and $DF(x) = 0$ if $x \in N$. Let γ be the function defined by: $\gamma(\left(-\infty, -\frac{1}{2}\right] \cup \{1/2\}\, U\, [3/2, +\infty)) = 1, \gamma(0) = \gamma(1) = 0$ and γ is linear on $[-1/2, 0], [0, 1/2], [1/2, 1], [1, 3/2]$. Let M be the set of x in N for which $\|\gamma(D)(x)\| < 1/4, \|D(x)\| > 3/4$. Since $\gamma(D)(y) = 0$, M is a neighborhood of y. Let δ be the function defined by: $\delta((-\infty, 1/4]) = 0, \delta([3/4, +\infty)) = 1, \delta$ is linear on $[1/4, 3/4]$. If $x \in M$ then $\sigma(D(x))$, the spectrum of (D)(x) is contained in $\left[-\frac{1}{4}, \frac{1}{4}\right] \cup [3/4, 5/4]$ but not in $[-1/4, 1/4]$ and so $\delta(D)(x)$ is a non-zero projection. Since $\delta(D)$ is a limit of polynomials in D without constant terms, $\delta(D)F(x) = 0$ and if E is given, $E\delta(D)E(x) = \delta(D)(x)$, for x in M. We let $B = \delta(D)$ and this completes the proof.

The next result will not be used in the sequel, however it clarifies the concept of regularity.

Theorem (2.2.16)[100]: Let $\mathfrak{A}$ be a CCR algebra with a Hausdorff structure space X. The set of regular elements is open and dense in X.

It follows from the remark following the Definition that the set of regular elements is open. Let N be a non-empty open subset of X. We must show that N contains a regular element. Let y be in N and let A be in $\mathfrak{A}$ so that A(y) is a non-zero projection. By Lemma

(2.2.15) there is a B in $\mathfrak{A}$ and a neighborhood M of y contained in N such that B(x) is a nonzero projection, for x in M. Since X is locally compact [92] we can suppose that M is compact and the closure of its interior. Let $M_n = \{x: x \in M, 1 \leqq Dim\, B(x) \leqq n\}$. There is a polynomial identity satisfied by all matrices of order n but not by all matrices of order $n + 1$ [91]. Thus M_n is closed. Since $M = \cup_n M_n$, the Baire category theorem implies that for some n, M_n, has a non-void interior as a subset of M, M having the relative topology as a subset of X. That is, there is an open set U in X and $0\emptyset \neq U \cap M \subset M_n$. Then $U \cap$ Interior M is not empty since M is the closure of its interior. Thus M_n has a non-void interior as a subset of X. Let m be the smallest integer for which M_m has a non-void interior, let P be the interior of M_m, let z be in P but not in M_{m-1}. Then $B(z) = \sum_{i=1}^{m} F_i$, where F_i is an orthogonal family of onedimensional projections. By Lemma (2.2.15) and by induction we can choose $B_1, \dots, B_m$ in $\mathfrak{A}$ and an open neighborhood R of z such that $\{B_1(x), \dots, B_m(x)\}$ is an orthogonal family of non-zero projections in $B(x)$ for x in R and $B_i(z) = F_i$. Thus if $x \in R$

$$m \geqq dim\, B(x) \geqq \sum_{i=1}^{m} dim\, B_i(x) \geqq m\,.$$

There is equality throughout this equation, and so $dim\, B_i(x) = 1$ for x in R. By the remark following the Definition of regularity, z is regular, so N contains a regular element, namely z, and the proof is complete.

It follows that a GCR algebra has a composition series $\{I_\alpha\}$ such that the structure space of $I_{\alpha+1}/I_\alpha$ is Hausdorff and has no singular points.

Theorem (2.2.17)[100]: Let $\mathfrak{A}$ be a C*-algebra. The elements of the w*-closure of the pure states of $\mathfrak{A}$ are proportional to pure states if and only if $\mathfrak{A}$ is CCR with a Hausdorff structure space X and at every singular x in X, $\mathfrak{A}(x)$ is 1-dimensional.

Suppose $\mathfrak{A}$ is CCR with a Hausdorff structure space and suppose $\mathfrak{A}(x)$ is 1-dimensional for x singular in X. Let $f(\beta)$ be a net of pure states of $\mathfrak{A}$ converging to a non-zero linear functional f. Then af is a state for some non-negative a. Let $x(\beta) = kernel\, \varphi_{f(\beta)}$. Then $f(\beta) = g_\beta \psi_{x(\beta)}$, where g_β is a pure state of $A\big(x(\beta)\big)$ [116]. The sets $\{x: \|A(x)\| \geqq r\}$ for A in $\mathfrak{A}$ and r positive are compact [92] and so if $x(\beta)$ is not eventually in some compact set in X then $f = 0$, a contradiction. Thus $x(\beta)$ is eventually in some compact subset of X, and by passing to a subnet we can suppose $x(\beta)$ converges to some x in X. If $A \in x$ then $\big\|A\big(x(\beta)\big)\big\|$ is small for β sufficiently large and $f(\beta)(A)$ is small. Thus $f(A) = 0$, $f(x) = 0$, and $af = g\psi_x$ where g is a state of $\mathfrak{A}(x)$. If x is singular then $\mathfrak{A}(x)$ is 1-demensional and so af is pure. Suppose x is regular. Since $\mathfrak{A}(x) = \mathfrak{G}(\mathfrak{H}_x)$, it follows from [102] that $g = \sum_i b_i \omega_{\xi_i}$. where $b_i \geqq 0$ and $\{\xi_i\}$ is an orthonormal family in $\mathfrak{H}_x$. If $j \neq k$, there is a neighborhood U of x and E, F, W in $\mathfrak{A}$ such that if $y \in U$ then E(y) and F(y) are orthogonal one-dimensional projections and W(y) is a partial isometry from E(y) to F(y), and $E(x) = [\zeta_k], F(x) = [\zeta_k]$. Let $g_\beta = (\cdot\, \gamma_\beta, \gamma_\beta)$ for some γ_β in $\mathfrak{H}_{x(\beta)}$. For large $\beta, F(x(\beta))\gamma_\beta$ is proportional to $W\big(x_\beta\big)E\big(x_\beta\big)\gamma_\beta$, and

$$\begin{aligned} 0 = |f(W)|^2 &= \lim_\beta |f(\beta)(W)|^2 = \lim_\beta \big|W\big(x(\beta)\big)E\big(x(\beta)\big)\gamma_\beta, F\big(x(\beta)\big)\gamma_\beta\big)\big|^2 \\ &= \lim_\beta \big\|W\big(x(\beta)\big)E\big(x(\beta)\big)\gamma_\beta\big\|^2 \big\|F\big(x(\beta)\big)\gamma_\beta\big\|^2 \\ &= \lim_\beta \big\|E\big(x(\beta)\big)\gamma_\beta\big\|^2 \big\|F\big(x(\beta)\big)\gamma_\beta\big\|^2 = \lim_\beta f(\beta)(E) f(\beta)(F) = b_j b_k. \end{aligned}$$

Thus there is at most one j with $b_j > 0$, and af is pure. We have proved that the elements of the w*-closure of the pure states are proportional to pure states.

Conversely suppose that the elements of the closure of the pure states of $\mathfrak{A}$ are proportional to pure states. Then the same is true for all homomorphic images of $\mathfrak{A}$. By Lemma (2.2.12) and [106], all irreducible images of $\mathfrak{A}$ consist of completely continuous operators and $\mathfrak{A}$ is CCR. Suppose X is not Hausdorff and let $z(\beta)$ be a net in X, let x and y be distinct elements of X such that $z(\beta) \to x, z(\beta) \to y$. Let ξ be a unit vector in $\mathfrak{H}_x$ let ζ be a unit vector in $\mathfrak{H}_y$. If U is an open set in $\mathcal{B}_0$ containing $\omega_\xi \psi_x$ then it follows from Theorem (2.2.9) and the remark following Theorem (2.2.9) that for sufficiently large β there is a $\xi(\beta)$ in $\mathfrak{H}_{z(\beta)}$ such that $\omega_{\xi(\beta)}\psi_{z(\beta)} \in U$. Thus by passing to a subnet (which we denote z_β) of $z(\beta)$, we can choose unit vectors ξ_β and ζ_β in $\mathfrak{H}_{z_\beta}$ such that $\left(\psi_{z_\beta}(\cdot)\xi_\beta, \xi_\beta\right) \to \omega_\xi \psi_x$ and $\left(\psi_{z_\beta}(\cdot)\zeta_\beta, \zeta_\beta\right) \to \omega_\zeta \psi_y$.

We assert that there is an A in $\mathfrak{A}$ such that $0 \leqq A \leqq I, A(y) = 0$ and $A(x)\xi = \xi$. By [89] there is a self-adjoint A_0 in $\mathfrak{A}$ such that $A_0(x)\xi = \xi$. Letf be the function defined by: $f((-\infty, 1/2]) = 0, f([1, +\infty)) = 1$ and f is linear on $[1/2, 1]$. Then $A_1 = f(A_0)$ is self-adjoint, $A_1(x)\xi = \xi$ and $A_1(y)\xi = f\left(A_0(y)\right)$ has a finite dimensional range. By [107] there is a self-adjoint A_2 in such that $A_2(x)\xi = \xi$ and $A_2(y)$ range $A_1(y) = 0$. Let $A = f(A_2A_1A_2)$; then A has the the desired properties.

Let B be in $\mathfrak{A}$. We show that $\lim_\beta\left(B(z_\beta)\zeta_\beta, \xi_\beta\right) = 0$. Let $B(z_\beta)\zeta_\beta = a_\beta\xi_\beta + b_\beta\tau_\beta$ where a_β and b_β are complex numbers and τ_β is a unit vector in $\mathfrak{H}_{z_\beta}$ orthogonal to ξ_{z_β}. Then

$$\begin{aligned} 0 = (B^*AB(y)\zeta, \zeta) &= \lim_\beta\left(B^*AB(z_\beta)\zeta_\beta, \zeta_\beta\right) = \lim_\beta\left(A(z_\beta)\left(a_\beta\xi_\beta + b_\beta\tau_\beta\right), a_\beta\xi_\beta + b_\beta\tau_\beta\right) \\ &\geqq \lim_\beta\left(\left|a_\beta\right|^2\left(A(z_\beta)\xi_\beta, \xi_\beta\right) - 2\left|a_\beta\right|\left|b_\beta\right|\left|A(z_\beta)\xi_\beta, \tau_\beta\right|\right). \end{aligned}$$

Since $A \leqq I$,

$$\begin{aligned} \lim_\beta 2\left|a_\beta\right|\left|b_\beta\right|\left(A_\beta(z_\beta)\xi_\beta, \tau_\beta\right) &\leqq< \lim_\beta 2\left|a_\beta\right|\left|b_\beta\right|\left|1 - \left(A(z_\beta)\xi_\beta, \xi_\beta\right)\right|^{\frac{1}{2}} \\ &\leqq \lim_\beta 2\|B\|^2\left|1 - \left(A(z_\beta)\xi_\beta, \xi_\beta\right)\right|^{\frac{1}{2}} = 2\|B\|^2|1 - (A(x)\xi, \xi)|^{\frac{1}{2}} = 0 \end{aligned}$$

and so

$$0 > \lim_\beta\left|a_\beta\right|^2\left(A(z_\beta)\xi_\beta, \xi_\beta\right) = \lim_\beta\left|a_\beta\right|^2(A(x)\xi, \xi) = \lim_\beta\left|a_\beta\right|^2$$

Thus $\lim_\beta\left|a_\beta\right|^2 = 0$ and

$$\lim_\beta\left(B(z_\beta)\zeta_\beta, \xi_\beta\right) = \lim_\beta a_\beta\left(\xi_\beta, \xi_\beta\right) + b_\beta\left(\tau_\beta, \xi_\beta\right) = 0$$

as asserted.

Let $c_\beta = \left\|\zeta_\beta + \xi_\beta\right\|$, let $v_\beta = (\zeta_\beta + \xi_\beta)/c_\beta$. By Lemma (2.2.3) $\lim_\beta\left\|A(z_\beta)\xi_\beta - \xi_\beta\right\| = 0$ and so

$$\begin{aligned} \lim_\beta\left|\left(\xi_\beta, \zeta_\beta\right)\right| = \lim_\beta\left|\left(A(z_\beta)\xi_\beta, \zeta_\beta\right)\right| = \lim_\beta\left|\left(A(z_\beta)\zeta_\beta, \xi_\beta\right)\right| &\leqq \lim_\beta\left|A^*A(z_\beta)\zeta_\beta, \zeta_\beta\right|^{\frac{1}{2}} \\ &= (A^*A(y)\zeta, \zeta) = 0. \end{aligned}$$

Thus $\lim_\beta c_\beta = \sqrt{2}$ and

$$\lim_\beta \big(B(z_\beta)v_\beta, v_\beta\big) = \lim_\beta \left(c_\beta^{-2}\big(B(z_\beta)\xi_\beta, \xi_\beta\big) + c_\beta^{-2}\big(B(z_\beta)\zeta_\beta, \zeta_\beta\big)\right)$$
$$= \left(\frac{1}{2}\right)(B(x)\xi, \xi) + (1|2)(B(y)\zeta, \zeta),$$

and

$$\lim_\beta \left(\psi_{z_\beta}(\cdot)v_\beta, v_\beta\right) = (\omega_\xi \psi_x + \omega_\zeta \psi_y)/2. \tag{69}$$

The right member of (69) is a limit of pure states of $\mathfrak{A}$ but is not proportional to a pure state. This is a contradiction and so X is Hausdorff.

Let x be a singular point in X and suppose $\mathfrak{A}(x)$ is not 1-dimensional. Then we can choose orthogonal unit vectors ξ and ζ in $\mathfrak{H}(x)$, and by Lemma (2.2.15), there is a neighborhood N_1 of x and E and Fin $\mathfrak{A}$ such that $E(x) = [\xi], F(x) = [\zeta]$ and if $y \in N$, then E(y) and F(y) are orthogonal projections. By Lemma (2.2.13) there is a neighborhood N of x contained in N_1 and a W in $\mathfrak{A}$ such that if $y \in N$ then W(y) is a partial isometry from E(y) to F(y). Since x is singular, we can choose a net y_β in N such that $y_\beta \to x$ and $dim\, E\big(y_\beta\big) = dim\, F(y_\beta) \geqq 2$, and so we can choose orthogonal unit vectors $\xi_{1\beta}$ and $\xi_{2\beta}$ in $E(y_\beta)$. Let $\zeta_{1\beta} = W\big(y_\beta\big)\xi_{1\beta}$ let $\zeta_{2\beta} = W\big(y_\beta\big)\xi_{2\beta}$.

Let A be in $\mathfrak{A}$. We assert that $\lim_\beta\big(A\big(y_\beta\big)\xi_{1\beta}, \zeta_{2\beta}\big) = 0$. Since $E(x)$ is one-dimensional, $FAE(x) = \lambda W(x)$ for some complex number λ, and by the continuity of the norm,

$$\lim_\beta \big\|(\lambda W - FAE)\big(y_\beta\big)\big\| = 0.$$

Thus

$$\lim_\beta\big(A\big(y_\beta\big)\xi_{1\beta}, \zeta_{2\beta}\big) = \lim_\beta\left\{\big(FAE\big(y_\beta\big)\xi_{1\beta}, \zeta 2\beta\ \big) + \left((\lambda W - FAE)\big(y_\beta\big)\xi_{1\beta}, \zeta_{2\beta}\right)\right\}$$
$$= \lim_\beta\big(\lambda X W\big(y_\beta\big)\xi_{1\beta}, \zeta_{2\beta}\big) = \lim_\beta\big(\lambda W\big(y_\beta\big)\xi_{1\beta}, W\big(y_\beta\big)\xi_{2\beta}\big) = \lim_\beta\big(\lambda\xi_{1\beta}, \xi_{2\beta}\big)$$
$$= 0$$

as asserted. Let $v_\beta = (\xi_{1\beta} + \zeta_{2\beta})/\sqrt{2}$. Since E(x) and F(x) are one-dimensional, $EAE(x) = \lambda_{11}(x), FAF(x) = \lambda_{22}F(x)$ for some complex numbers λ_{11} and λ_{22} and so

$$\lim_\beta\big\|(\lambda_{11}E - EAE)\big(y_\beta\big)\big\| = 0 = \lim_\beta\big\|(\lambda_{22}F - FAF)\big(y_\beta\big)\big\|.$$

Thus

$$\lim_\beta\big(A\big(y_\beta\big)v_\beta, v_\beta\big)$$
$$= 2^{-1}\lim_\beta\{\big(A\big(y_\beta\big)\xi_{1\beta}, \xi_{1\beta}\big) + \big(A\big(y_\beta\big)\zeta_{2\beta}, \zeta_{2\beta}\big) + \big(A\big(y_\beta\big)\zeta_{2\beta}, \xi_{1\beta}\big)$$
$$+ \big(A\big(y_\beta\big)\xi_{1\beta}, \zeta_{2\beta}\big)\}$$
$$= 2^{-1}\lim_\beta\{\big(\lambda_{11}E\big(y_\beta\big)\xi_{1\beta}, \xi_{1\beta}\big) + \big(\lambda_{22}E\big(y_\beta\big)\zeta_{2\beta}, \zeta_{2\beta}\big)\}$$
$$- 2^{-1}\{(A(x)\xi, \xi) + (A(x)\zeta, \zeta)\}$$

and so

$$\lim_\beta \left(\psi_{v_\beta}(\cdot)v_\beta, v_\beta\right) = (\omega_\xi \psi_x + \omega_\zeta \psi_x)/2\,. \tag{70}$$

The right member of (70) is a limit of pure states of $\mathfrak{A}$ but is not proportional to a pure state of $\mathfrak{A}$, and this is a contradiction. Thus $\mathfrak{A}(x)$ is onedimensional and the proof is complete.

The technique might be useful in determining the closure $\mathcal{B}^-$ of the set $\mathcal{B}$ of elementary positive definite normalized functions f on a locally compact separable type I group G. (By normalized, we mean $f(e) = 1$ where e is the identity of G.) Theorem (2.2.17) suggests the question: are there any locally compact groups G other than direct products of compact and abelian groups for which $\mathcal{B}^- \subset \cup_{1 \geqq \lambda \geqq 0} \{\lambda f : f \in \mathcal{B}\} = [0\ 1] \times \mathcal{B}$? The topology in which the closure $\mathcal{B}^-$ of $\mathcal{B}$ is to be taken is the w*-topology in the dual to L_1.

We show that certain C*-algebras associated with representations of the commutation relations with an infinite number of degrees of freedom are simple, NGCR, and have representations the weak closures of the images of which are factors of type II_∞ (resp. III). The existence of type II and type III representations for one of the C*-algebras we consider has been announced by GArding and Wightman in [118] but the proof has not been published.

We present the terminology of [117]. A single particle structure $\sum$ is defined as a system $(\mathfrak{H}, \mathfrak{H}', B)$ where $\mathfrak{H}$ and $\mathfrak{H}'$ are real linear spaces and B is a real non-singular bilinear form on $(\mathfrak{H}, \mathfrak{H}')$. A canonical system over $\sum$ is defined as a pair of linear maps $p(\cdot)$ and $q(\cdot)$ from $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, to respective commutative families of (unbounded) selfadjoint operators on a complex Hilbert space (called the representation space) such that

$$e^{ip(x)}e^{iq(x')} = e^{iB(x,x')}e^{iq(x')}e^{ip(x)} \tag{71}$$

for arbitrary x in $\mathfrak{H}$ and x' in $\mathfrak{H}'$. Linearity is with respect to the strong operations on the unbounded linear operators on the representation space. Since we wish to deal only with the case of an infinite number of degrees of freedom, we assume that $\mathfrak{H}$ (and so $\mathfrak{H}'$) is infinite dimensional.

A bounded linear operator T (on the representation space) is said to depend on submanifolds $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{M}''$ of $\mathfrak{H}'$ in case T is in the weak closure of the algebra generated by $e^{ip(x)}$ and $e^{iq(x')}$ as x and x' range over $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The collection of all bounded linear operators dependent on a pair of fixed manifolds $\mathfrak{M}$ and $\mathfrak{M}'$ forms a weakly closed von Neumann algebra $\mathfrak{A}(\mathfrak{M}, \mathfrak{M}')$ while the union over all finite-dimensional $\mathfrak{M}$ and $\mathfrak{M}'$ forms an algebra whose uniform closure $\mathfrak{A}$ is called the representation algebra of field observables. It is proved in [117] that the algebra $\mathfrak{A}$ is determined (up to isomorphism) by $\sum$.

Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are n-dimensional and that $B|\mathfrak{M} \times \mathfrak{M}'$ is nondegenerate. Let a base $\{e_1, \dots, e_n\}(resp. \{e_1', \dots, e_n'\})$ be chosen in $\mathfrak{M}$ (resp. $\mathfrak{M}'$) so that if $x = \sum_i a_i e_i$ $x' = \sum_i a_i' e_i'$ then $B(x, x') = \sum_i a_i a_i'$. It follows from [112] (at least in the case where the representation space is separable, which is all we need here) that there is an isomorphism θ from $\mathfrak{A}(\mathfrak{M}, \mathfrak{M}')$ onto the bounded operators on $L^2(E^n)$, such that

$$\begin{cases} \left(\theta\left(\exp\left(ip(a_j e_j)\right)\right) f\right)(\xi_1, \dots, \xi_n) = f(\xi_1, \dots, \xi_{j-1}, \xi_j + a_j, \xi_{j+1}, \dots, \xi_n) \\ \left(\theta\left(\exp\left(iq(a_j e_j)\right)\right) f\right)(\xi_1, \dots, \xi_n) = \exp(i a_j \xi_j) f(\xi_1, \dots, \xi_j) \end{cases} \tag{72}$$

for f in $L^2(E^n)$. (E^n is euclidean n-space.)

Theorem (2.2.18)[100]: $\mathfrak{A}$ is simple and NGCR. If φ is a representation of $\mathfrak{A}$ and if a trace is defined on the weak closure $\varphi(\mathfrak{A})^-$ of $\varphi(\mathfrak{A})$ then the only operator in $\varphi(\mathfrak{A})$ with finite trace is zero. Thus $\mathfrak{A}$ has no representations of type $I_n (n < \infty)$ or of type II_1.

Let φ be a representation of $\mathfrak{A}$, let $\mathfrak{M}_0$ and $\mathfrak{M}_0'$ be finite dimensional subspaces of $\mathfrak{H}$ and $\mathfrak{H}'$ respectively and let $\mathfrak{M}$ (resp. $\mathfrak{M}'$) be a finite dimensional subspace of $\mathfrak{H}$ (resp. $\mathfrak{H}'$) such that $\mathfrak{M}_0 \subsetneqq \mathfrak{M}$ (resp. $\mathfrak{M}_0' \subsetneqq \mathfrak{M}'$) and $B|\mathfrak{M} \times \mathfrak{M}'$ is non-degenerate. Then $\varphi\theta^{-1}$ is a representation of the bounded operators on $L^2(E^n)$, when $n = \dim \mathfrak{M}$, and kernel $\varphi\theta^{-1}$ is zero or the completely continuous operators. No non-zero element of $\theta\left(\mathfrak{A}(\mathfrak{M}_0, \mathfrak{M}_0')\right)$ is completely continuous and so $\varphi\theta^{-1}|\theta\left(\mathfrak{A}(\mathfrak{M}_0, \mathfrak{M}_0')\right)$ and $\varphi|\mathfrak{A}(\mathfrak{M}_0, \mathfrak{N}_0')$ are faithful. Since

$\mathfrak{M}_0$ and $\mathfrak{M}'_0$ were arbitrary finite dimensional subspaces, φ is faithful and $\mathfrak{A}$ is simple. Since $\mathfrak{A}$ is not an $n \times n$ matrix algebra, $\mathfrak{A}$ is NGCR.

Let t be a trace on $\varphi(\mathfrak{A})^-$. Since I is an infinite projection in $\mathfrak{A}(\mathfrak{M}, \mathfrak{M}')$, $\varphi(I)$ is an infinite projection in $\varphi(\mathfrak{A})^-$, and so $t\big(\varphi(I)\big) = \infty$. The set of operators A in $\mathfrak{A}$ such that $\big|t\big(\varphi(A)\big)\big| < \infty$ is a proper two sided ideal in $\mathfrak{A}$ and so is zero.

$\mathfrak{A}$ is never separable and so we cannot use Theorem (2.2.7) to conclude that $\mathfrak{A}$ has type II and type III representations. By making further restrictions on $\mathfrak{H}$ and $\mathfrak{H}'$, however, we can make use of the techniques of Theorem (2.2.7). Let $\mathfrak{K}$ be a separable Hilbert space, let $e_1, e_2, \ldots$ be an orthonormal base for $\mathfrak{K}$. Let $\mathfrak{H}_0$ be the real linear span of $e_1, e_2, \ldots$, let $\mathfrak{H}_1$ be the set of elements $\sum a_i e_i$ where $\{a_i\}$ is a square summable sequence of real numbers. Let $B_1(x, x') = i(x, x')$ for x in $i\mathfrak{H}_1$ and x' in $i\mathfrak{H}_1$. We suppose that $\mathfrak{H}$ is dense in $\mathfrak{H}_1$ and $\mathfrak{H}'$ is dense in $i\mathfrak{H}_i$ and that B is the restriction to $\mathfrak{H} \times \mathfrak{H}'$ of B_1.

Theorem (2.2.19)[100]: If $\mathfrak{H}, \mathfrak{H}'$ and B are as above then there is a representation φ of $\mathfrak{A}$ such that the weak closure $\varphi(\mathfrak{A})$ of $\varphi(\mathfrak{A})$ is a factor of type II_∞ (resp. type III) and this representation φ arises from a canonical system over $\sum$.

Let $\mathfrak{M}_n$ be the subspace of $\mathfrak{H}_0$ spanned by $e_1, \ldots, e_n$, let $\mathfrak{M}'_n = i\mathfrak{M}_n$. Let $S_n = \mathfrak{A}(\mathfrak{M}_n, \mathfrak{M}'_n)$. Let $\mathfrak{N}_n$ be the subspace of $\mathfrak{H}_0$ spanned by e_n, let $\mathfrak{N}'_n = i\mathfrak{N}_n$, let $T_n = \mathfrak{A}(\mathfrak{N}_n, \mathfrak{N}'_n)$ let $\mathfrak{A}_0$ be the C*-algebra generated by the $S_n, n = 1,2, \ldots$ Let θ_n (resp. ψ_n) be the isomorphism θ of S_n (resp. T_n) given by (72) in the case $\mathfrak{M}_n = \mathfrak{M}, \mathfrak{M}'_n = \mathfrak{M}'$ (resp. $\mathfrak{N}_n = \mathfrak{M}, \mathfrak{N}'_n = \mathfrak{M}'$).

Let $g_0(x) = \sin x$ if $x \in [-\pi, \pi], g_0(x) = 0$ otherwise, let $g_1(x) = \sin x$ if $x \in [-3\pi, \pi] \cup [\pi, 3\pi], g_1(x) = 0$ otherwise, let $f_j = g_j/\|g_j\|$, where $\|g_j\|$ is the norm of g_j as an element of $L^2(E^1)$. We assert that there is a positive number b such that

$$\left|1 - \left(e^{ax} f_j(x), f_j(x)\right)\right| < a^2 b \tag{73}$$

$$\left|1 - \left(f_j(x + a), f_j(x)\right)\right| < a^2 b \tag{74}$$

for all real a. In fact (73) follows from

$$\int_{-c}^{c} \left(e^{iax} - 1\right) \sin^2(x)\, dx = \sum_{n=1}^{\infty} \left(\frac{(ia)^n}{n!}\right) \int_{-c}^{c} x^n \sin^2(x)\, dx$$

and from $\int_{-c}^{c} x \sin^2(x)\, dx = 0$ for c real while (74) follows from

$$\int_{d}^{2\pi+d} \sin(x + a) \sin(x)\, dx = \int_{d}^{2\pi+d} \sin^2(x) \cos(a)\, dx + \int_{d}^{2\pi+d} \sin(x) \cos(x) \sin(a)\, dx$$

$$= \cos(a) \int_{d}^{2\pi+d} \sin^2(x)\, dx$$

for d real. Let F_{ii} be the projection on f_i, let F_{ij} be the partial isometry from F_{jj} to F_{ii} which takes f_j onto f_i for $i \neq j$. Let $E_{ij}^{(n)} = \psi_n^{-1}\big(F_{ij}\big) \in T_n$, let

$$E(i_1, \ldots, i_n; j_1, \ldots, j_n) = E_{i_1 j_1}^{(1)} \cdots E_{i_n j_n}^{(n)}.$$

Since T_s and T_i commute for $s \neq t$ it is easy to see that

$$\{E(i_1, \ldots, i_n; j_1, \ldots, j_n) \colon i_k, j_k = 0, 1\}$$

is a family of $2^n \times 2^n$ matrix units in S_n and that $E(i_1, \ldots, i_n; i_1, \ldots, i_n)$ is a minimal projection in S_n. Let

$$E(n) = \sum_{i_1, \ldots, i_n} E(i_1, \ldots, i_n; i_1, \ldots, i_n)$$

Let $M(n)$ be the linear span of the $E(i_1, \dots, i_n; j_1, \dots, j_n)$. Since $E(i_1, \dots, i_n; i_1, \dots, i_n)$ is a minimal projection, $M(n) = E(n)S_nE(n)$. Let s be in (0, 1/2], let $t = 1 - s$, let

$$\lambda_0\big(E(i_1, \dots, i_n; j_1, \dots, j_n)\big) = \delta_{i_1,\dots,i_n}^{j_1,\dots,j_n} g^{\sum_k i_k} t^{n-\sum_k i_k}.$$

As in the proof of (b1)⇒(b2), (b3) of Theorem (2.2.7), there is a state λ of $\mathfrak{A}_0$ which is an extension of λ_0. Let $E = \inf_n \varphi_\lambda\big(E(n)\big)$. The sequence $\varphi_\lambda\big(E(n)\big)$ is monotone decreasing and so E exists and is in $\varphi_\lambda(\mathfrak{A}_0)^-$, the weak closure of $\varphi_\lambda(\mathfrak{A}_0)$. By an argument as in Lemma (2.2.5), $\varphi_\lambda\big(M(n)\big)$ leaves $\varphi_\lambda\big(E(n+r)\big)$ invariant for r a positive integer, and so $\varphi_\lambda(M(n))$ leaves E invariant. Thus $A \to \varphi_\lambda(A)E$ is a representation of M(n). Let M be the linear span of the M(n). Then $\varphi_\lambda(M)E$ is a*-algebra which is generated by an ascending sequence $\varphi_\lambda(M(n))E$ of $2^n \times 2^n$ matrix algebras, and it follows as in the proof of (b1)⇒(b2) and (b1)⟹(b3) of Theorem (2.2.7) that $\varphi_\lambda(M)E^-$ is a factor of type II_1 if $s = 1/2$ and a factor of type III if $s \neq 1/2$. Since

$$E\varphi_\lambda(S_n)E = E\varphi_\lambda\big(E(n)S_nE(n)\big)E = \varphi_\lambda(M(n))E$$

and since the union of the $S_n{}'s$ is dense in $\mathfrak{A}_0, \varphi_\lambda(M)E$ is uniformly dense in $E\,\varphi_\lambda(\mathfrak{A}_0)E$. As in the proof of (b1)⇒(b2), (b3) of Theorem (2.2.7), this implies that $\varphi_\lambda(\mathfrak{A}_0)^-$ is type II if $s = 1/2$ and type III if $s \neq 1/2$ and the same argument shows that $\varphi_\lambda(\mathfrak{A}_0)^-$ is a factor. If $s = 1/2$ then by Theorem (2.2.18), $\varphi_\lambda(\mathfrak{A}_0)^-$ is a factor of type II_∞.

Let B_k be in T_k, let $F_k = E_{00}^{(k)} + E_{11}^{(k)}$. Then

$$F_kB_kF_k = \sum_{ij} b_{ij}^{(k)} E_{ij}^{(k)}$$

for some complex numbers $b_{ij}^{(k)}$. As in Lemma (2.2.6), $\lambda = \lambda(F_k \cdot F_k)$ and since F_k commutes with T_j for $j \neq k$,

$$\begin{aligned}\lambda(B_1, \dots, B_n) &= \lambda(F_1B_1F_1 \dots F_nB_nF_n)\\ &= \sum_n \left\{b_{i_1i_1}^{(1)} \dots, b_{i_ni_n}^{(n)} g^{\sum_k i_k} t^{n-\sum_k i_k} : i_1 = 0 \text{ or } 1, \dots, i_n = 0 \text{ or } 1\right\}\\ &= \prod_{k=1} \left(b_{00}^{(k)} t + b_{11}^{(k)} s\right) = \lambda(F_1B_1F_1) \dots (F_nB_nF_n) = \lambda(B_1) \dots \lambda(B_n).\end{aligned}$$

Let z be a vector in $\mathfrak{H}_\lambda$ such that $\lambda = \omega_s\varphi_\lambda$ and $\mathfrak{H}_\lambda = [\varphi_\lambda(\mathfrak{A}_0)z]$. Let $x = \sum_k a_ke_k$ be in $\mathfrak{H}_0$, let $r = p$ or q, let $j = 1$ or i respectively. Let $R(a)$ be a translation by a (resp. multiplication by e^{iax}) acting on $L^2(E^1)$. Then

$$1 - \left|\big(\varphi_\lambda\big(\exp\big(ir(jx)\big)\big)z, z\big)\right|^2 = 1 - \prod_k \left|\lambda\big(\exp\big(ir(ja_ke_k)\big)\big)\right|^2$$

$$1 - \prod_k |t(R(a_k)f_0, f_0) + s(R(a_k)f_1, f_1)|^2 \leqq \left|1 - \prod_k (1 + a_k^2 b)^2\right|$$

$$= \left|1 - \exp\left(\sum_k 2\, log(1 + a_k^2 b)\right)\right| \leqq \left|1 - \exp\left(\sum_k 2a_k^2 b\right)\right| = |1 - \exp(b2\|x\|)|$$

Let $U_x = \varphi_\lambda\big(\exp\big(ir(jx)\big)\big)$, let $U_zz = az + \beta z'$ where α and β are complex numbers and z' is a unit vector in $\mathfrak{H}_\lambda$ orthogonal to z. Then

$$\|U_zz - z\|^2 = |1 - \alpha|^2 + |\beta|^2 < 2(1 - |\alpha^2|) \leqq 2|1 - \exp(b2\|x\|)|$$

If $A \in S_n$ and if $x = u + v$ where $u \in \mathfrak{M}_n$and $v \perp \mathfrak{M}_n$ then

$$\begin{cases} \|U_x Az - Az\| \leqq \|U_u U_v Az - U_u Az\| + \|U_u Az - Az\| \\ \qquad = \|U_u A U_v z - U_u Az\| + \|U_u Az - Az\| \\ \qquad \leqq 2\|A\||1-\exp(b2\|x\|)|^{\frac{1}{2}} \\ \qquad + \|\varphi_\lambda(\exp(ir(ju)))Az - Az\| \end{cases} \tag{75}$$

The map $\pi: B \to \varphi_\lambda(B)|[\varphi_\lambda(S_n)z]$ for B in S_n is normal [102] since $\lambda|S_n$ is normal and so the restriction of π to bounded sets is strongly continuous [102]. It follows from this and (75) that the map

$$x \to \varphi_\lambda(\exp(ir(jx))) \tag{76}$$

is continuous from the norm topology on $\mathfrak{H}_0$ (resp. i$\mathfrak{H}_0$) to the strong operator topology on $\varphi_\lambda(\mathfrak{A}_0)$. It is easy to see that the map (76) can be extended to a strongly continuous unitary representation

$$x \to \bar{\varphi}_\lambda(\exp(ir(jx)))$$

of the normed space $\mathfrak{H}_1$ (resp. $i\mathfrak{H}_1$). Thus there is a canonical system $\bar{p}$ and $\bar{q}$ over $\sum_i = (\mathfrak{H}_1, i\mathfrak{H}_1, B_1)$ such that

$$\bar{\varphi}_\lambda(\exp(ip(x))) = \exp(\bar{p}(x))$$
$$\bar{\varphi}_\lambda(\exp(iq(x))) = \exp\bar{q}(ix)$$

and so $\bar{p}|\mathfrak{H}, \bar{q}|\mathfrak{H}'$ is a canonical system over $\sum = (\mathfrak{H}, \mathfrak{H}', B)$. It follows from [112] (as in [23)]) that there is an isomorphism φ of $\mathfrak{A}$ onto the representation algebra of field observables defined by $\bar{p}|\mathfrak{H}, \bar{q}|\mathfrak{H}'$ which extends $\bar{\varphi}_\lambda|\exp(ip(\mathfrak{H})) \cup \exp(iq(\mathfrak{H}'))$ and such that the restriction of φ to operators depending on finite dimensional submanifolds $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{H}'$ of $\mathfrak{H}'$ is ultrastrongly continuous. The weak closure of $\varphi(\mathfrak{A})$ is the von Neumann algebra generated by $\bar{\varphi}_\lambda(\exp(ip(x)))$ and $\bar{\varphi}_\lambda(\exp(ip(x')))$ for x in $\mathfrak{H}_0$, x', in $\mathfrak{H}'$. This is the von Neumann algebra generated by $\varphi_\lambda(\exp(ip(x)))$ and $\varphi_\lambda(\exp(ip(ix)))$ for x in $\mathfrak{H}_0$ and this is $\varphi_\lambda(\mathfrak{A}_0)^-$, a factor of type II, if $s = 1/2$ and of type III if $s \neq 1/2$. The proof is complete.

The case considered by Garding and Wightman was the case $\mathfrak{A}_0 = \mathfrak{A}$. We remark that the Hilbert space $\mathfrak{H}_\lambda$ is separable. In fact for any $n, [\varphi_\lambda(S_n)z]$ is separable and $\mathfrak{H}_\lambda = [\cup_{n=1}^{\infty} \varphi_\lambda(S_n)z]$. In view of Theorem (2.2.18), it seems likely that type II representations will not play a distinguished role in the study of $\mathfrak{A}$.

We show that a slight variant of a theorem ([101]) of Bishop and de Leeuw characterizing the Choquet boundary of a uniformly closed subalgebra of $C_c(X)$ which contains constants and separates points is true for certain linear subspaces of $C_c(X)$, the linear subspaces being those which correspond to C*-algebras. ($C_c(X)$ is the algebra of continuous complex valued functions on the compact Hausdorff space X.).

Let $\mathfrak{A}$ be a C*-algebra with a unit I. Let A be in $\mathfrak{A}$ and let $\theta(A)$ be the function $\{(f, f(A)): f \in S\}$. Then θ is a*-linear map of $\mathfrak{A}$ onto a subspace of $C_c(S), \theta$ is norm decreasing and the restriction of θ to the set $\mathfrak{A}_*$ of self-adjoint elements in $\mathfrak{A}$ is isometric. Although $\mathfrak{A}$ is an algebra, $\theta(\mathfrak{A})$ is not a subalgebra of $C_c(S)$ unless $\mathfrak{A}$ is the complex numbers. The states of $\mathfrak{A}$ are precisely the linear functionals on $\mathfrak{A}$ such that $f|\mathfrak{A}_*$ is real and $f(I) = \|f|\mathfrak{A}_*\| = 1$ (for example see [107]). It follows from [101] that $\mathcal{B}_0$ is the Choquet boundary of $\theta(\mathfrak{A})$. (See [101] for a Definition of Choquet boundary.) Observe that our condition I below is a modification of condition Iof [101].

Theorem (2.2.20)[100]: Let $\mathfrak{A}$ be a C*-algebra with a unit I and let f be a state of $\mathfrak{A}$. Then the followings are equivalent:

(i) For each open neighborhood U of f in S there is a positive δ and an A in $\mathfrak{A}$ such that $0 \leqq A \leqq I, f(A) = 1$ and $g(A) < 1 - \delta$ for all g in $S \sim U$.

(ii) For each closed $G_\delta S$ containing f, there is an A in $\mathfrak{A}$ such that $\|A\| = |f(A)|$ and $\{g: g \in S, |g(A)| = \|A\|\} \subset S$.

(iii) f is a pure state.

(iii)⇒(i): Suppose III and let $f = \omega_x \varphi_f$ for some x in $\mathfrak{H}_f$, let U be an open neighborhood of f. Since f is pure, φ_f is irreducible, and by Lemma (2.2.2) we can choose a positive δ and an A in $\mathfrak{A}$ such that $0 \leqq A \leqq I, \varphi_f(A)x = x$ and

$$\{g: g \in S, g(A) \geqq 1 - \delta\} \subset U.$$

Since $f(A) = \left(\varphi_f(A)x, x\right) = 1$, I is satisfied.

I⇒II: Suppose I and let S be a closed G_s containing f, let $\{U_n\}$ be a decreasing sequence of open sets with $S = \cap U_n$. For each U_n choose an A_n and a δ_n to satisfy I and let $A = \sum_{n=1}^{\infty} A_n/2^n$. Then $\|A\| \leqq 1$ and since $f(A) = 1, 1 = \|A\| = f(A)$. If $g \in S \sim S$ then $g(A_n) < 1 - \delta_n$ for some n and so $|g(A)| = g(A) < 1$ and II is satisfied.

(ii)⟹(iii): Suppose II and suppose $f = 2^{-1}(g_1 + g_2)$ where g_1 and g_2 are in S. Let S be a closed G_s containing f, let A be chosen to satisfy II. Then

$$\|A\| = |f(A)| < 2^{-1}(|g_1(A)| + |g_2(A)|) \leqq \|A\|$$

and so $|g_1(A)| = \|A\| = |g_2(A)|$. By II, $g_i \in S$. If x and y are distinct points of S then there are disjoint Gs's S_x and S_y containing x and y respectively. This implies $g_i = f$, so f is pure and the proof of Theorem (2.2.20) is complete.

Let $aj)_k$ be condition $aj)$ of Theorem k. Then J. Dixmier has proved $(a1)_1 \Leftrightarrow (a5)_1 \Leftrightarrow (a6)_1$ [119] and he has written me that he has proved $(a6)_1 \Leftrightarrow (a3)_2 \Leftrightarrow (a4)_2$ [120]. The proof of $(a6)_1 \Leftrightarrow (a5)_1 \Leftrightarrow (a1)_1$ in [119] is different from the proof. Lemma (2.2.13) is a consequence of [121]. Lemma (2.2.15) and Theorem (2.2.16) are closely related in [121] respectively. See [119], [120],[121].

Section (2.3): Topological Representation

For A be a C^*-algebra, Ω the structure space of A, i.e. the space of all primitive ideals in A with hull-kernel topology. At every point P of Ω we associate a primitive C^*-algebra A/P (which we denote by $A(P)$) and we may associate for any element $a \in A$ the function $a(P)$ whose value at P is the homomorphic image of a in $A(P)$. Then the most difficult parts of the noncommutative structure theory of C^*-algebras are the restrictions such as to destroy the main feature of the commutative case the Gelfand representation of A by the continuous function $a(P)$ on Ω Even if Ω is a Hausdorff space, it has long been observed hopeless to discuss the continuity of the function $a(P)$ since Kaplansky [92] proposed a method to study the structure of general C^*- algebras and instead of these discussions the continuity of the function $\|\alpha(P)\|$ was studied. Unfortunately this property does not give directly the suitable topological representation of algebras.

On the other hand, in [125], in the case that A satisfies the condition that any irreducible representation of A is n-dimensional (such a C^*-algebra is called n-dimensionally homogeneous) we have defined a topology in the set $\mathcal{B} = \cup_{P \in \Omega} A(P)$ and represented A as the algebra of all $\mathcal{B}$ -valued functions $a(P)$ on Ω with $a(P) \in A(P)$ which is continuous in this topology (we call these functions the (continuous) cross-sections of $\mathcal{B}$).

Now the above treatment offers a non-commutative model of the classical Gelfand representation theorem in the case that the structure space Ω is a Hausdorff space. Is it always possible to define a natural topology in the set $\mathcal{B} = \cup_{P \in \Omega} A(P)$ so that A is represented as the algebra of all continuous cross-sections of $\mathcal{B}$ vanishing at infinity ? we give a positive answer for this question and to analyse the algebras by their topological representations.

We devoted to define a suitable topology in $\mathcal{B}$ in somewhat general situations and to discuss the general structure theory of algebras of cross-sections. Some fundamental results corresponding to the algebras of continuous functions are proved here, including the Stone-Weierstrass theorem and as a direct consequence of their results we can settle the problems remained unsolved in Kaplansky [92]. We treat the above mentioned problem stating our result in rather general form so that it may be applicable to the case where Ω is not a Hausdorff space. The result (Theorem (2.3.13)) is the following one if there exists an appropriate decomposition of Ω (called a continuous decomposition), then we get a locally compact Hausdorff space X at each point of which a suitable C^*-algebra $A(x)$ is given and, setting $\mathcal{B} = \bigcup_{x \in X} A(x)$, A is represented as the algebra of all cross-sections of $\mathcal{B}$, continuous in a suitable topology in $\mathcal{B}$ and vanishing at infinity on X. The case where Ω is a Hausdorff space is the one where every classes in the decomposition reduce to one point.

In [90], Kaplansky denned a class of C^*-algebras, central C^*-algebras, to which commutative methods are applicable to some extent. The structure spaces of these are always Hausdorff spaces. However, the above result shows that there are no distinctions between the centrality and the Hausdorff property of the structure spaces of C^*-algebras and we get, as a direct consequence of our representation theorem, the following: If the center of a C^*-algebra A is not contained in any primitive ideal in A then A is central if and only if the structure space of A is a Hausdorff space.

We show the case where there exists always the nontrivial (or rather finest) continuous decomposition. Theorem (2.3.16). is an another interpretation of the decomposition considered in Glimm [105] and we prove later more sharpened results for this decomposition than those of [105].

Let X be a Hausdorff topological space at each point x of which a Banach algebra $A(x)$ is given. All $A(x)'s$ are considered to be different each other. Put $\mathcal{B} = \bigcup_{x \in X} A(x)$. We suppose that, for each element $b \in \mathcal{B}$, there exists uniquely a point $x \in X$ such as $b \in A(x)$. The projection mapping π from $\mathcal{B}$ to X is defined by $\pi(b) = x$ and $A(x)$ is called the fibre over the point $x \in X$. A function *a(x)* on X is called a cross-section of $\mathcal{B}$ if $a(x)$ £ $A(x)$ for each $x \in X$.

Let $f(x)$ be a complex-valued function on X and $a(x)$ a cross-section of $\mathcal{B}$. We denote by $f \cdot a$ the cross-section of $\mathcal{B}$ defined by $f \cdot a(x) = f(x)a(x)$.

Let A be a family of cross-sections of $\mathcal{B}$. A is said to be closed under multiplication by $f(x)$ if $f \cdot a \in A$ for every $a \in A$.

We consider an arbitrary fixed family F of cross-sections $a(x)$ of $\mathcal{B}$ satisfying the following condition:

(i) $\|a(x)\|$ is continuous and bounded on X,

(i i) at each point $x \in X$, $F(x)$ fills out the algebra $A(x)$,

(iii) F forms an algebra under pointwise operations.

Then we get the following

Theorem (2.3.1)[122]: The family F defines a Hausdorff topology $\mathcal{T}_F$ in $\mathcal{B}$ and the algebra of all bounded $\mathcal{T}_F$-continuous cross-sections of $\mathcal{B}$ becomes a Banach algebra, which is closed under multiplication by $C(X)$, the algebra of all bounded complex-valued continuous functions on X.

Proof. Take an arbitrary element $b_0 \in \mathcal{B}$, an element $a \in F$ with $a(x_0) = b_0$, and a neighborhood U of $x_0 = \pi(b_0)$. Put $\mathcal{U}(b_0, U, \varepsilon, a(x)) = \bigcup_{x \in U}\{ b \in \mathcal{B} \mid b \in a(x)$ and $\|b - a(x)\| < \varepsilon\} = \{b \in \mathcal{B} \mid \pi(b) = x \in U$ and $\|b - a(x)\| < \varepsilon\}$,
where ε is an arbitrary positive number. Then a straight-forward calculation shows that the family $\{ \mathcal{U}(b_0, U, \varepsilon, a(x)) \mid b_0 \in \mathcal{B}\}$ forms a neighborhood system of $\mathcal{B}$ and defines a topology $\mathcal{T}_F$ in $\mathcal{B}$.

Besides, one sees that $\mathcal{T}_F$ is a Hausdorff topology and the relative topology of $\mathcal{T}_F$ in $A(x)$ coincides with the original norm topology of $A(x)$.

Let $\tilde{C}_F(X, \mathcal{B})$ be the set of all bounded cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_F$-topology. We notice that the function $\|a(x)\|$ is a continuous function on X for each $a \in \tilde{C}_F(X, \mathcal{B})$. In fact, let an arbitrary positive number ε and a point $x_0 \in X$ be given. Take an element $a_0 \in F$ with $a_0(x_0) = a(x_0)$. Since each of the functions of F is norm continuous, we can find a neighborhood U of x_0 such as

$$|\ \|a_0(x)\| - \|a_0(x_0)\|\ | < \varepsilon/2 \text{ for every } x \in U.$$

On the other hand, the continuity of $a(x)$ in $\mathcal{T}_F$ implies that there exists a neighborhood V of x_0 such as

$$a(x) \in \mathcal{U}(a(x_0), U, \frac{\varepsilon}{2}, a_0(x)) \text{ for every } x \in V.$$

Hence we have

$$\begin{aligned} |\ \|a(x)\| - \|a(x_0)\|\ | &\leqq |\ \|a(x)\| - \|a_0(x)\|\ | + |\ \|a_0(x)\| - \|a_0(x_0)\|\ | \\ &\leqq \|a(x) - a_0(x)\| + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

at eaoh point $x \in V$.

Now, it is not difficult to see that $\tilde{C}_F(X, \mathcal{B})$ is closed under pointwise addition, multiplication and scalar multiplication. Define the norm $\|a\| = \sup\|a(x)\|$ for $a \in \tilde{C}_F(X, \mathcal{B})$, then $\tilde{C}_F(X, \mathcal{B})$ becomes a Banach algebra. The one non-trivial point here is the completeness of $\tilde{C}_F(X, \mathcal{B})$. Let $\{a_n\}$ be a Cauchy sequence in $\tilde{C}_F(X, \mathcal{B})$. One easily verifies that the sequences $\{a_n(x) \mid x \in X\}$ are uniformly Cauchy sequences and, as $A(x)'s$ are complete, $\{a_n(x) \mid x \in X\}$ define a cross-section $a(x) = \lim_{n\to\infty} a_n(x)$. Clearly $a(x)$ is a bounded cross-section of $\mathcal{B}$. We assert that this is continuous in $\mathcal{T}_F$. Let x_0 be an arbitrary point of X and $\mathcal{U}(a(x_0), U_0, \varepsilon, a'(x))$ a neighborhood of $a(x_0)$ There exists a number n_0 such that $\|a(x) - a_n(x)\| < \varepsilon/3$ for every $n \geqq n_0$ and $x \in X$.

Let $a'' \in F$ be an element with $a''(x_0) = a_{n_0}(x_0)$. Since

$$\|a'(x_0) - a''(x_0)\| = \left\|a(x_0) - a_{n_0}(x_0)\right\| < \varepsilon/3,$$

there exists a neighborhood U_1 of x_0 such as

$$\|a'(x) - a''(x)\| < \varepsilon/3. \text{ for every } x \in U_1.$$

Moreover $a''(x_0) = a_{n_0}(x_0)$ and a'', $a_{n_0} \in \tilde{C}_F(X, \mathcal{B})$ imply that we can find a neighborhood U_2 *of* x_0 such as $\left\|a''(x) - a_{n_0}(x)\right\| < \varepsilon/3$ for every $x \in U_2$. Then at each point x in the neighborhood U of x_0 which is contained in all of U_0 , U_1 and U_2, we have

$$\begin{aligned} \|a(x) - a'(x)\| &\leqq \left\|a(x) - a_{n_0}(x)\right\| + \left\|a_{n_0}(x) - a''(x)\right\| + \|a''(x) - a'(x)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

That is, $a(x) \in \mathcal{U}(a(x_0), U_0, \varepsilon, a'(x))$. Thus the first half part of the theorem is proved.

Now let $f(x)$ be an arbitrary bounded complex-valued continuous function on X and take a cross-section $a \in \tilde{C}_F(X, \mathcal{B})$. It is clear that $f \cdot a$ is a bounded cross-section of $\mathcal{B}$. Let x_0 be a point of X and consider a neighborhood $\mathcal{U}(f(x_0)a(x_0), U_0, \varepsilon, x_0(x))$ of

$f(x_0)a(x_0)$. Take an element $a_1 \in F$ with $a(x_0) = a_1(x_0)$. Since $a(x)$ is continuous in $\mathcal{T}_F$ we can find a neighborhood U_1 of x_0 such that

$$\|a(x) - a_1(x)\| < \varepsilon/3m \text{ for every } x \in U_1,$$

where $m = \sup_{x \in X} |f(x)|$. On the other hand, the continuity of $f(x)$ implies that there exists a neighborhood U_2 of x such as

$$\|f(x) - f(x_0)\| < \varepsilon/3\|a_1\| \text{ for every } x \in U_2.$$

Finally, as $f(x_0)a_1 \in F$ and $f(x_0)a_1(x_0) = f(x_0)a(x_0) = a_0(x_0)$ there exists a neighborhood U_3 of x_0 at each point x of which

$$\|f(x_0)a_1(x) - a_0(x)\| < \varepsilon/3.$$

Therefore, at each point x of the neighborhood U of x_0 which is contained in all of the above neighborhoods, we have

$$\begin{aligned} \|f(x) - a_1(x)\| &\leqq \|f(x)a(x) - f(x)a_1(x)\| \\ &\quad + \|f(x)a_1(x) - f(x_0)a_1(x)\| + \|f(x_0)a_1(x) - a_0(x_0)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Hence $f \cdot a(x)$ is a bounded continuous cross-section of $\mathcal{B}$. That is, $f \cdot a \in \tilde{C}_F(X, \mathcal{B})$. This completes the proof.

Now we assume, for the rest of the discussions, that X is a locally compact Hausdorff space and $A(x)'s$ are C^*-algebras. We consider a fixed family F of cross-sections $a(x)$ of $\mathcal{B}$ satisfying the following conditions

(a) $\|a(x)\|$ is continuous on X and vanishes at infinity,

(b) at each point $x \in X$, $F(x)$ fills out the algebra $A(x)$,

(c) F forms a self-adjoint algebra under pointwise operations.

Denote by $C_F(X, \mathcal{B})$ the algebra of all cross-sections of $\mathcal{B}$, continuous in $\mathcal{T}_F$-topology and vanishing at infinity of X. (Here we mean a cross-section $a(x)$ vanishing at infinity if the function $\|a(x)\|$ vanishes at infinity). We notice that the proof of Theorem (2.3.1). can be applicable to the algebra $C_F(X, \mathcal{B})$ and we see that $C_F(X, \mathcal{B})$ is a C^*-algebra. Moreover for any cross-section $a(x)$ in $C_F(X, \mathcal{B})$ and any bounded complex-valued continuons function $f(x)$, the cross-section $f \cdot a(x)$ is $\mathcal{T}_F$-continuous and vanishes at infinity. It follows that $C_F(X, \mathcal{B})$ is closed under multiplication by $C(X)$, the algebra of all bounded complex-valued continuous function on X.

If X is compact and all $A(x)'s$ are isomorphic to a fixed C^*-algebra A and F is a family of so-called constant cross-sections, then $C_F(X, \mathcal{B})$ is isomorphic to the usual A-valued continuous function algebra $C(X, A)$. Moreover it is not difficult to see that in this case the space $\mathcal{B}$ with $\mathcal{T}_F$-topology is homeomorphic with the product space $X \times A$. But generally the situation is not so simple as we shall see from the discussions in section 3 and Tomiyama-Takesaki [125].

The next theorem shows that the cross-section algebra $C_F(X, \mathcal{B})$ satisfies the condition corresponding to the regularity in commutative function algebras.

Theorem (2.3.2)[122]: For any closed set G in X, any point $x_0 \notin G$ and an arbitrary element b in $A(x_0)$, $C_F(X, \mathcal{B})$ contains a cross-section $a(x)$ such that $a(x_0) = b$ and $a(x) = 0$ for every $x \in G$.

Proof. Let $a'(x)$ be an element of $C_F(X, \mathcal{B})$ with $a'(x_0) = b$ and $f(x)$ a bounded complex-valued continuous function on X with $f(x_0) = 1$ and $f(G) = 0$. Then $a = f \cdot a' \in C_F(X, \mathcal{B})$ satisfies the property.

Lemma (2.3.3)[122]: Let P be a primitive ideal in $C_F(X, \mathcal{B})$. Then there exists uniquely a point x_0 in X and a primitive ideal $P(x_0)$ in $A(x_0)$ such that

$$P = \{a \in C_F(X, \mathcal{B}) | a(x_0) \in P(x_0)\}.$$

Proof. Let X_0 be the one-point compactification of X. Adding new fibre $A(x_\infty) = 0$ at the exceptional point x_∞, $C_F(X, \mathcal{B})$ may be considered to be the algebra of all cross-sections of $\mathcal{B} = \mathcal{B} \cup A(x_\infty)$ continuous in $\mathcal{T}_F$-topology. Hence, by Lemma 3.2 in Kaplansky [92], we see that the algebra $C_F(X, \mathcal{B})$ on X_0 satisfies all the conditions (a) to (d) in [92]. Thus, coming back to the algebra $C_F(X, \mathcal{B})$ on X one easily see that we can freely use Theorem 3.1 in [92] on X.

By Theorem 3.1 in [92] we have

$$P = \{a \in C_F(X, \mathcal{B}) | a(x) \in P(x) \text{ for every } x \in X\}$$

where $P(x)$ means the closed ideal in $A(x)$ consisting of all $a(x)'s$ for $a \in P$. Suppose that there exist different points x_1, x_2 such that $P(x_1)$ and $P(x_2)$ are proper closed ideals in $A(x_1)$ and $A(x_2)$ respectively. Let $U(x_1)$ and $U(x_2)$ be disjoint neighborhoods of x_1 and x_2, and put

$$P_1 = \{a \in C_F(X, \mathcal{B}) | a(x) \in P(x) \text{ for } x \in U(x_2)^c\},$$
$$P_2 = \{a \in C_F(X, \mathcal{B}) | a(x) \in P(x)\ for\ x \in U(x_1)^c\},$$

where $U(x_1)^c$ and $U(x_2)^c$ mean the complements of $U(x_1)$ and $U(x_1)$. P_1 and P_2 are proper closed ideals in $C_F(X, \mathcal{B})$ and since $U(x_1)^c \cup U(x_2)^c = X$ we have $P_1 \cap P_2 = P$. On the other hand, by Theorem (2.3.2). $C_F(X, \mathcal{B})$ contains a crosssection $a(x)$ such that $a(x_2) \notin P(x_2)$ and $a(x) = 0$ for $x \in U(x_2)^c$. Hence we get $P_1 \supsetneqq P$ and similarly $P_1 \supsetneqq P$, which is a contradiction. Therefore there exists only one point $x_0 \in X$ where $A(x_0)$ is a proper ideal $A(x_0)$. We have

$$P = \{a \in C_F(X, \mathcal{B}) | a(x_0) \in P(x_0)\}.$$

It is not difficult to see that the ideal $A(x_0)$ is a primitive ideal in $A(x_0)$. This completes the proof.

Now let Ω be the structure space of $C_F(X, \mathcal{B})$, i.e. the space of all primitive ideals in $C_F(X, \mathcal{B})$ with hull-kernel topology.

$$I_x = \{a \in C_F(X, \mathcal{B}) | a(x) = 0\}.$$

Clearly I_x is a closed ideal in $C_F(X, \mathcal{B})$. We denote by $h(I_x)$ the hull of I_x in Ω, that is, $h(I_x) = \{P \in \Omega | P \supset I_x\}$..

The following lemma is almost clear.

Lemma (2.3.4)[122]: $h(I_x)$ is homeomorphic with the structure space of $A(x)$.

Then we get the structure theorem for Ω.

Theorem (2.3.5)[122]: $\Omega = \bigcup_{x \in X} h(I_x)$ is a decomposition of Ω into closed sets $h(I_x)$ and the space X is homeomorphic with the quotient space of this decomposition. In particular, if all $A(x)'s$ are simple C^*-algebras, X is homeomorphic with Ω, hence in this case Ω is a Hausdorff space.

Proof. By Lemma (2.3.3) we see that $\bigcup_{x \in X} h(I_x)$ is a decomposition of Ω Let O be an open set in X and put $\tilde{O} = \bigcup_{x \in O} h(I_x)$. We show that $\tilde{O}$ is an open set in Ω Let P be a primitive ideal in $C_F(X, \mathcal{B})$ such as $P \supset k(\tilde{O}^c)$, where $k(\tilde{O}^c)$ means the kernel of the complement of O in Ω. P belongs to some $h(I_{x_0})$ by Lemma (2.3.3). Suppose that $P \in \tilde{O}$, then $x_0 \in O$. By Theorem (2.3.2), there exists a cross-section $a(x)$ in $C_F(X, \mathcal{B})$ satisfying the condition that $a(x_0) \notin P(x_0)$ and $a(x) = 0$ for each $x \in \tilde{O}^c$, then $a \in k(\tilde{O}^c)$ and $a \notin P$. This is a contradiction. Hence $P \in \tilde{O}^c$ and $\tilde{O}$ is an open set in Ω.

Conversely let $\tilde{O} = \bigcup_{x \in O} h(I_x)$ be an open set in Ω and x_0 be an arbitrary point of the closure of O^c, the complement of O in X. We must show that $x_0 \in O^c$. Suppose on the contrary that $x_0 \in O$, then for an ideal $P \in h(I_{x_0})$ we can find a cross-section $a \in C_F(X, \mathcal{B})$

such as $a \in k(\tilde{O}^c)$ and $a \notin P$ because P does not belong to the closed set $\tilde{O}^c$. Since $\tilde{O}^c = \bigcup_{x \in O} h(I_x)$, this means that $a(x) = 0$ for every $x \in O^c$ and $a(x_0) \neq 0$. However this contradicts the continuity of $a(x)$. Thus $x_0 \in O^c$ and O is an open set in X.

Since there is one-to-one correspondence between X and the quotient space of the decomposition $\Omega = \bigcup_{x \in O} h(I_x)$, we have shown that this correspondence is bicontinuous.

In order to prove the non-commutative Stone-Weierstrass theorem for cross-section algebras, we need the following theorem which is a direct consequence of Glimm's strengthened non-commutative Stone-Weierstrass theorem of pure state type (cf. Glimm [105]).

Theorem (2.3.6)[122]: Let A be a C^*-algebea and B a C^*-sub algebra of A. Suppose that B separates the w^*-closure of the pure states of A. Then $A = B$ if both A and B have a unit or A has no unit. If A has a unit and B has not, A coincides with the algebra generated by B and a unit.

Proof. Let A_1 be a C^*-algebra obtained by adjoining a unit to A, then the algebra B_1 obtained also by adjoining a unit to B is naturally considered to be a C^*sub-algebra of A_1. Let φ be an element of the w^*-closure of the pure states of A_1 and $\{\varphi_a\}$ a net of pure states of A_1 converging weakly to φ. If φ is a non-zero functional on A, we may suppose that all $\varphi_a{}'s$ are non-zero functionals on A and, since A is a closed ideal in A_1 this implies that all $\varphi_a{}'s$ are pure states of A by an argument in the proof of Theorem 2 in Tomiyama-Takesaki [125]. Hence $\varphi|A$, the restriction of φ to A, belongs to the pure states of A, too. On the other hand, it is clear that the w^*-closure of the pure states of A contains zero-functional if A has no unit (cf. Glimm [100]). Now let φ and ψ be different elements of the w^*-closure of the pure states of A_1. Then we have $\varphi \neq \psi$ on A. Since $\varphi|A$ and $\psi|A$ belong to the w^*-closure of the pure states of A as mentioned above, we can find an element $a \in B$ such as $\varphi(a) \neq \psi(a)$. Hence B_1 separates the w^*-closure of the pure states of A_1 and we get $A_1 = B_1$ by Glimm [105]. Therefore we can deduce the conclusion in each case stated in the theorem.

It is not difficult to see that the last case in Theorem (2.3.6). really arises even if A is a CCR algebra. This case corresponds to the case in usual Stone-Weierstrass theorem that B coincides with the algebra of all continuous functions vanishing at a single point. Thus the non-commutative Stone-Weierstrass theorem of CCR algebras stated in Kaplansky [92] is generally insufficient if we do not restrict the case to a certain limit. Using Theorem (2.3.6) we can prove the following non-commutative Stone-Weierstrass theorem for the cross-section algebra $C_F(X, \mathcal{B})$.

Theorem (2.3.7)[122]: Let C be a self-adjoint subalgebra of $\mathrm{C_F(X}, \mathcal{B})$ where $\mathcal{B} = \bigcup_{x \in X} A(X)$. Suppose that for any distinct points $x, y \in X, C$ contains cross-sections taking arbitrary pairs of values in $A(x)$, $A(y)$ at x, y. Then C is dense in $\mathrm{C_F(X}, \mathcal{B})$.

Proof. Let ψ be an element of the w^*-closure of the pure states of $C_F(X, \mathcal{B})$ and $\{\varphi_\alpha\}$ a net of pure states converging weakly to φ. Put $P_\alpha = \{a \in C_F(X, \mathcal{B}) | \varphi_\alpha(b^*ac) = 0$ for every $b, c \in C_F(X, \mathcal{B})$. Then it is known that $P_\alpha \in \Omega$ for each α. Suppose that $\{P_\alpha\}$ is not eventually in any compact set of Ω. Denote by $a(P)$ the homomorphic image of $a \in C_F(X, \mathcal{B})$ in $C_F(X, \mathcal{B})/P$ for an ideal P. Since the sets $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ for ε positive are compact (cf. [92]), one easily verifies that $\varphi = 0$. Hence if $\varphi \neq 0$, $\{P_\alpha\}$ must be eventually in some compact set in Ω and in this case we may suppose, without loss of generality, that P converges to some point P_0 in Ω.

Now let φ and ψ be different elements of the w^*-closure of the pure states of $C_F(X, \mathcal{B})$ and $\{\varphi_\alpha\}$, $\{\psi_\alpha\}$ nets of pure states converging to φ and ψ respectively. Put

$$P_\alpha = \{a \in C_F(X, \mathcal{B}) | \varphi_\alpha(b^*ac) = 0 \text{ for every } b, c \in C_F(X, \mathcal{B})\}$$

and

$$Q_\beta = \{a \in C_F(X,\mathcal{B}) | \psi_\beta(b^*ac) = 0 \text{ for every } b,c \in C_F(X,\mathcal{B})\}.$$

We assume at first that both φ and ψ are non-zero functional on $C_F(X,\mathcal{B})$. Then we may suppose that $\{P_\alpha\}$ and $\{Q_\beta\}$ converge to some points P_0 and Q_0 in Ω. By Lemma (2.3.3) for each primitive ideal P_α there exists a point $x_\alpha \in X$ and a primitive ideal $P(x_\alpha)$ in $A(x_\alpha)$ such that

$$P_\alpha = \{a \in C_F(X,\mathcal{B}) |\ a(x_\alpha) \in P(x_\alpha)\ \}.$$

Similarly Q_β may be written as

$$Q_\beta = \{a \in C_F(X,\mathcal{B}) |\ a(y_\beta) \in Q(y_\beta)\}$$

for some point $y_\beta \in X$ and primitive ideal $Q(y_\beta)$ in $A(y_\beta)$.

Let

$$P_0 = \{a \in C_F(X,\mathcal{B}) |\ a(x_0) \in P(x_0)\ \}$$

And

$$Q_0 = \{a \in C_F(X,\mathcal{B}) |\ a(y_0) \in Q(y_0)\}$$

where $P(x_0)$ and $Q(y_0)$ mean the primitive ideals in $A(x_0)$ and $A(y_0)$ respectively. Then, by Theorem (2.3.5), x_α converges to x_0 and y_β to y_0. Takea cross-section $a \in C_F(X,\mathcal{B})$ with $a(x_0) = 0$, then $\|a(x_\alpha)\|$ converges to $\|a(x_\alpha)\| = 0$ and as

$$|\varphi_a(a)| \leqq \|a(P_\alpha)\| = \|a(x_\alpha)P(x_\alpha)\| \leqq \|a(x_\alpha)\|$$

we get $\varphi(a) = 0$. Similarly $\psi(a) = 0$ for any cross-section $a \in C_F(X,\mathcal{B})$ with $a(y_0) = 0$. Here we have two cases in question.

(i) the case $x_0 = y_0$. Let a be an element of $C_F(X,\mathcal{B})$ such as $\varphi(a) \neq \psi(a)$. We can find an element a' in C with $a(x_0) = a'(x_0)$. Then $a(x_0) - a'(x_0) = a(y_0) - a'(y_0) = 0$ and

$$\varphi(a') = \varphi(a) \neq \psi(a) = \psi(a').$$

(ii) the case $x_0 \neq y_0$. Let

$$P'_0 = \{a \in C_F(X,\mathcal{B}) | \varphi(b^*ac) = 0 \text{ for every } b,c \in C_F(X,\mathcal{B})\}$$

and

$$Q'_0 = \{a \in C_F(X,\mathcal{B}) | \psi(b^*ac) = 0 \text{ for every } b,c \in C_F(X,\mathcal{B})\}.$$

P'_0 and Q'_0 are not contained in each other, for P'_0 contains the ideal $\{a \in C_F(X,\mathcal{B}) | a(x_0) = 0\}$ and Q'_0 the ideal $\{a \in C_F(X,\mathcal{B}) | a(y_0) = 0\}$. Hence there exists an element $a \in C_F(X,\mathcal{B})$ such as $a \in P'_0$ and $a \notin Q'_0$, so that we get some elements b, c in $C_F(X,\mathcal{B})$ such as $\varphi(b^*ac) = 0$ and $\psi(b^*ac) = 0$. Take an element $a' \in C$ with $a'(x_0) = b^*ac(x_0)$ and $a'(y_0) = b^*ac(y_0)$. We have,

$$\varphi(a') = \varphi(b^*ac) = 0, \text{ and } \psi(a') = \psi(b^*ac) \neq 0.$$

On the other hand, if one of φ and ψ is zero, say φ, then ψ determines a point $x_0 \in X$ and $\psi(a) = 0$ whenever $a(x_0) = 0$. Hence one verifies easily that the restriction of ψ to C is a nonzero functional, too.

Now let $\tilde{C}$ be the closure of C in $C_F(X,\mathcal{B})$. We must show that $\tilde{C} = C_F(X,\mathcal{B})$. Clearly $\tilde{C}$ is a C^*-subalgebra of $C_F(X,\mathcal{B})$ and the above discussion shows that $\tilde{C}$ separates the w^*-closure of the pure states of $C_F(X,\mathcal{B})$. Hence if $C_F(X,\mathcal{B})$ has no unit we get directly $\tilde{C} = C_F(X,\mathcal{B})$ by Theorem (2.3.6). In the case that $C_F(X,\mathcal{B})$ has a unit, it is sufficient to show that C has a unit, too. Otherwise, C is a maximal ideal in $C_F(X,\mathcal{B})$ whose quotient algebra is one-dimensional but this is a contradiction as it is easily seen from [92] and the condition for C Therefore in any case $\tilde{C} = C_F(X,\mathcal{B})$. This completes the proof.

Theorem (2.3.7). offers the affirmative answer to the question in Kaplansky [92], that is, Theorem 3.3 and 3. 4 in [92] can be proved without any restriction on the fibre $A(x)$. Both Corollary (2.3.8) and (2.3.9) are readily deduced from Theorem (2.3.7).

Corollary (2.3.8)[122]: Let X be a locally compact Hausdorff space at each point of which a C^*-algebra $A(x)$ is given. Let A be a C^*-algebra of crosssections $a(x)$ of $\mathcal{B} = (\bigcup_{x\in X} A(x))$ satisfying the postulate that $\|a(x)\|$ is continuous and vanishing at infinity. Suppose further that for any distinct points $x, y \in X$, A contains functions taking arbitrary pairs of values in $A(x)$, $A(y)$ at x, y. Then A is closed under multiplication by $C(X)$, the algebra of all bounded continuous functions on X.
Corollary (2.3.9)[122]: Let X be a locally compact Hausdorff space, D a C^*-algebra and A the C^*-algebra of all continuous functions vanishing at infinity from X to D. Let B be a C^*-subalgebra of A, which contains functions taking arbitrary prescribed pairs of values in D at every distinct points $x, y \in X$. Then $A = B$.

Let C be a self-adjoint subalgebra of $C_F(X, \mathcal{B})$. As in the case of commutative function algebras the weakest topology in X for which each $a(x) \in C$ is norm continuous (that is, the function $\|a(x)\|$ is continuous) is called the C-topology in X.
Theorem (2.3.10)[122]: If C is a self-adjoint subalgebra of $C_F(X, \mathcal{B})$ which contains cross-sections taking arbitrary pairs of values in $A(x)$, $A(y)$ at any distinct points x, y in X, then the given topology in X is equivalent to the C-topology.
Proof. Since the function $\|a(x)\|$ is continuous in the original topology in X for any cross-section $a(x) \in C$, it is clear that the original topology is stronger than the C-topology. Hence any closed set in C -topology is closed in the original topology, too. Conversely, let G be a closed set X in the original topology. We assert that
$$G = \{x \in X | I_x \supset \bigcap_{y\in G} I_y\}.$$
In fact, it is clear that $G \subseteqq \{x \in X | I_x \supset \bigcap_{y\in G} I_y\}$. Take a point x_0 in the right member.If x_0 does not belong to G,then we can find a cross-section $a(x)$ in $C_F(X, \mathcal{B})$ such that $a(x) = 0$ on G and $a(x_0) \neq 0$, a contradiction. Let x_0 be a point in the closure of G in the C-topology and take a cross-section a $a \in \bigcap_{x\in G} I_x$. Clearly $a(x) = 0$ for every $x \in G$. Since C is dense in $C_F(X, \mathcal{B})$ by Theorem (2.3.6), all cross-sections in $C_F(X, \mathcal{B})$ are norm continuous in the C-topology. Therefore $a(x_0) = 0$, hence $I_{x_0} \supset \bigcap_{x\in G} I_x$. We have $x_0 \in G$ and G is closed in the C-topology. This completes the proof.
Theorem (2.3.11)[122]: Let G be closed set in X. Then any $\mathcal{T}_F$– continuous cross-section $a(x)$ defined on G and vanishing at infinity can always be extended to the whole space X.
Proof. Let
$$I = \{a \in C_F(X, \mathcal{B}) | a(x) = 0 \text{for } x \in G\}$$
and C_0 the algebra of all $\mathcal{T}_F$–-continuous cross-section on G vanishing at infinity. Consider the factor algebra $C_F(X, \mathcal{B})/I$, then the mapping $[a] \to a(x)|G$ is the natural embedding of $C_F(X, \mathcal{B})/I$ into C_0 where $[a]$ means the class to which $a(x)$ belongs and $a(x)|G$ the restriction of $a(x)$ to G. By Theorem (2.3.6) this embedding is onto. Hence any $\mathcal{T}_F$–-continuous cross-section on G vanishing at infinity is the restriction of an element in $C_F(X, \mathcal{B})$.

Let A be a C^*-algebra and Ω the structure of A, that is, the space of all primitive ideals in A with hull-kernel topology. We denote by $a(P)$ the homomorphic image of $a \in A$ in the quotient algebra A/P by an ideal P in A. Let $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ be a decomposition into closed sets of Ω and put $x_\alpha = k(\Omega_\alpha)$ (kernel Ω_x). Then there is a one-to-one correspondence between the set of ideals $X = \{x_\alpha | \alpha \in \Gamma\}$ and the quotient space of Ω with respect to this decomposition, so that we can consider on X the quotient topology of this decomposition.

Definition (2.3.12) : Let $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ be a Hausdorff decomposition of Ω and put $X = \{x_\alpha |\ \alpha \in \Gamma\}$ where $x_\alpha = k(\Omega_\alpha)$. We call X the decomposition space of Ω. If we have

$$\tilde{S} = \{x \in X | x \supset \bigcap_{y\in S} y\}$$

for any subset S in X where $\tilde{S}$ means' the closure of S in the quotient topology, this decomposition is called a continuous decomposition of Ω.

With this definition we get the following topological representation theorem of C^*-algebras.

Theorem (2.3.13)[122]: Let A be a C^*-algebra and $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ a continuous decomposition of the structure space Ω of A. Then the decomposition space $X = \{x_\alpha | \alpha \in \Gamma\}$ with quotient topology is a locally compact Hausdorff space on which each element $a \in A$ is represented as the cross-section $a(x)$ satisfying the postulate that $\|a(x)\|$ is continuous on X and vanishing at infinity. Put $\mathcal{B} = \bigcup_{x\in X} A\,(x)$. Then A is represented as $C_A(X, \mathcal{B})$ the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of X.

Proof. From the definition of X, X is a Hausdorff space. Let a be an element of A and ε a positive number. Put $K = \{x \in X | \|a(x)\| \geqq \varepsilon\}$. Then K is an image of the set $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ in Ω by the quotient map, for it is clear that the latter is mapped into K and moreover for any point $x \in X$ there exists a primitive ideal P which contains x and $\|a(P)\| = \|a(x)\|$ (cf. Kaplansky [92]). Since the set $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ is compact by [92], K is a compact subset of X. Hence K is closed in X because X is a Hausdorff space. Therefore the function $\|a(x)\|$ is upper semi-continuous in X.

In order to prove the lower semi-continuity of $\|a(x)\|$ we must show that the sets $\{x \in X | \|a(x)\| \geqq \varepsilon\}$ for ε positive are closed in X. Because of the identity $\|a^*a\| = \|a\|^2$, we need consider only the case where a is self-adjoint. Suppose that x_0 is in the closure of the set $S = \{x \in X | \|a(x)\| \leqq \varepsilon\}$ and $\|a(x_0)\| = \rho > \varepsilon$. Let $\gamma(x)$ be a real-valued continuous function defined as follows: $\gamma((-\infty, \varepsilon]) = 0$, $\gamma([\rho, +\infty]) = 1$ and $\gamma(x)$ is linear on $[\varepsilon, \rho]$. Then $\gamma(a)(x) = \gamma(a(x)) = 0$ for every $x \in S$ hence $\gamma(a) \in k(S)$, the kernel of S and $\gamma(a)(x_0) \neq 0$, that is, $\gamma(a) \notin x_0$ However this contradicts the definition of a continuous decomposition. Hence $x_0 \in S$.

Therefore, $\|a(x)\|$ is a continuous function on X and X is a locally compact space.

Now put $\mathcal{B} = \bigcup_{x\in X} A\,(x)$, then the above argument shows that we can associate with any $a \in A$ the cross-section $a(x)$ of $\mathcal{B}$ such as $\|a(x)\|$ is continuous and vanishing at infinity. Moreover one easily see that $\|a\| = \sup_{x\in X} \|a(x)\|$. Hence we may identify A with the represented algebra of cross-sections of $\mathcal{B}$. Consider the topology $\mathcal{T}_A$ in $\mathcal{B}$ and let $C_F(X, \mathcal{B})$ be the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of X. We assert that A contains cross-sections taking arbitrary pairs of values in $A(x)$, $A(y)$ at distinct points $x, y \in X$. In fact, consider the ideal

$$x + y = \{a + b | a \in x, b \in y\}.$$

Then $x + y$ is dense in A, for otherwise there exists a primitive ideal P in A containing $x + y$, that is, $P \in h(x) \cap h(y)$. However, since the decomposition $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ is a Hausdorff decomposition, each class Ω_α is closed in Ω and $\Omega_\alpha = hk(\Omega_\alpha)$ which implies that $h(x) \cap h(y) = \phi$ whenever $x \neq y$, a contradiction. Thus . $x + y$ is dense in A and by Lemma 8.1 in [90] we have $A = x + y$. Let $a_1(x), a_2(y)$ be an arbitrary pair of values in $A(x), A(y)$ at distinct points $x, y \in X$. We can find an element $a_1' \in x$ and an element $a_2' \in y$ such that $a_1 - a_2 = a_1' - a_2'$

$$a_0 = a_1 - a_1' = a_2 - a_2' .$$

Then clearly $a_0(x) = a_1(x)$ and $a_0(y) = a_2(y)$.

Therefore, by Theorem (2.3.7), the represented algebra A coincides with $C_F(X, \mathcal{B})$. This completes the proof.

As a direct consequence of this theorem we get the following representation theorem of C^*-algebras whose structure spaces are Hausdorff.

Corollary (2.3.14)[122]: Let A be a C^*-algebra and Ω the structure space of A. Suppose that Ω is a Hausdorff space and put $\mathcal{B} = \bigcup_{P\in\Omega} A(P)$. Then A is represented as $C_A(\Omega, \mathcal{B})$, the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of Ω.

Though the above defined topology is slightly different from the bundle space topology defined in Tomiyama-Takesaki [125] in the case that A is an n-homogeneous C^*-algebra, one may easily see that they are equivalent. Therefore Corollary (2.3.14) is a natural generalization of Theorem 5 in [125].

Now the above result shows that the commutative method is always applicable to the class of C^*-algebras whose structure spaces are Hausdorff. Hence there is no reason to distinguish the central C^*-algebras from the C^*-algebras whose structure spaces are Hausdorff spaces and we get naturally the following

Corollary (2.3.15)[122]: Let A be a C^*-algebra and Ω the structure space of A. Suppose that any $P \in \Omega$ does not contain the center Z of A. Then A is central if and only if Ω is a Hausdorff space.

Proof. It is sufficient to prove the "if" part of this corollary. Suppose that Ω is a Hausdorff space. Let P and Q be different primitive ideals in A and take an element z in Z such as $z(F) \neq 0$. Let f be a bounded complex-valued continuous function on Ω such as $f(P) = 1$ and $f(Q) = 0$, then by Corollary (2.3.14) we have $f \cdot z \in A$. Since $f \cdot z(P) = z(P) \neq 0$ and $f \cdot z(Q) = 0$, one sees that $f \cdot z \notin P \cap Z$ and $f \cdot z \in Q \cap Z$. Thus $P \cap Z \neq Q \cap Z$, hence A is a central C^*-algebra.

We show that there exists always the finest continuous decomposition in the structure space of a W^*-algebra A. As we see below, this is an another interpretation of the decomposition considered by Glimm [105]. We shall make clear the situation of Glimm's theorems by [105] on the pure state spaces of W^*-algebras and give more sharpened results for them.

Let A be a C^*-algebra and Ω the structure space of A. A decomposition $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ is called finer than the decomposition $\Omega = \bigcup_{\lambda\in\Lambda} \Omega'_\lambda$ if each Ω_α is contained in some class Ω'_λ.

Thus, setting $\mathcal{B} = \bigcup_{x\in X} A(x)$, A is represented as $C_A(\Omega, \mathcal{B})$, the algebra of all bounded $\mathcal{T}_A$-continuous cross-sections of $\mathcal{B}$. Notice that in this case a continuous function f on X is considered to be a continuous function on Ω_0, hence an element in Z and $f \cdot a (a \in A)$ coincides with the usual product of the central element f and a in A.

Theorem (2.3.16)[122]: Let A be a W^*-algebra, Ω the structure space of A and Ω_0 the structure space of the center Z of A, Then $\Omega = \bigcup_{\xi\in\Omega_0} h(\xi)$ is the finest continuous decomposition of Ω whose decomposition space X with quotient topology is homeomorphic with Ω_0.

Proof. Since the map $: P \in \Omega \to P \cap Z \in \Omega_0$ is a continuous map from Ω to Ω_0, it is not difficult to see that the decomposition $\Omega = \bigcup_{\xi\in\Omega_0} h(\xi)$ is a Hausdorff decomposition. Let $\tilde{O} = \bigcup_{\xi\in O} h(\xi)$ be an open set in Ω. We assert that O is an open set of Ω_0, so let ξ_0 be a point of O and P_0 a primitive ideal in $h(\xi_0)$. Since $\tilde{O}^c$, the complement of $\tilde{O}$, is closed in Ω

we can find an element $a \in A$ such as and $a(P_0) \neq 0$ for every $P \in \tilde{O}^c$. Let X be the decomposition space of $\Omega = \bigcup_{\xi \in \Omega_0} h(\xi)$, that is, $X = \{x(\xi) = kh(\xi) | \xi \in \Omega_0\}$, then one easily see that $a(x(\xi_0)) \neq 0$ and $a(x(\xi)) = 0$ for every $\xi \in \tilde{O}^c$, the compplement of O in Ω_0. Hence, by Lemma 10 in Glimm [105], there exists a neighborhood U of ξ_0 contained in O and this implies that O is an open set in Ω_0,

Now it is clear that there is a one-to-one correspondence between X and Ω_0, and the above discussion shows that this correspondence is bicontinuous where the set X is endowed with the quotient topology with respect to the decomposition $\Omega = \bigcup_{\xi \in \Omega_0} h(\xi)$.

Next, let S be an arbitrary subset of X and $\bar{S}$ the closure of S in X. Put

$$\tilde{S} = \{x \in X | x \supset k(S)\}.$$

Then, by the definition of quotient topology, it is not difficult to see that $\tilde{S} \subset \bar{S}$. Conversely suppose that a is in $k(S)$, the kernel of S. Then $a(x) = 0$ on S and by [105] $a(x) = 0$ on $\bar{S}$, hence $a \in x$ for every $x \in \bar{S}$. That is, $\bar{S} \subset \tilde{S}$ and we get, $\bar{S} = \tilde{S}$. Therefore the above decomposition is a continuous one.

We shall show that the above decomposition is the finest continuous decomposition of Ω. Suppose on the contrary that there exists a continuous decomposition $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ exactly finer than the decomposition $\Omega = \bigcup_{\xi \in \Omega_0} h(\xi)$. Then we get at least two distinct class Ω_α and Ω_β in some class $h(\xi)$. Let $x = k(\Omega_\alpha)$ and $y = k(\Omega_\beta)$. As $x \not\supseteq Z$, there exists an element $z \in Z$ such as $z(x) \neq 0$ hence taking a bounded continuous function f on the decomposition space of the decomposition $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ such as $f(x) = 1$ and $f(y) = 0$ we have, by Theorem (2.3.13), $fz \in Z$ and $fz \notin Z$, $fz \in y$. This is a contradiction.

By the pure state space of a C^*-algebra A with unit, we mean the w^*-closure of the pure states of A and denote it by $\mathfrak{B}(A)$. $\mathfrak{S}(A)$ means the state space of A.

We keep the above notations in Theorem (2.3.16).

Next lemma concerns with the first half part of Theorem 4 in Glimm [105].

Lemma (2.3.17)[122]: If $A(x)$ has a non-zero GCR ideal, then $A(x)$ is a primitive algebra and contains a minimal projection.

Proof. Let I_x be a non-zero GCR ideal in $A(x)$, then I_x has no ideal divisors of zero because $A(x)$ has no ideal divisors of zero (cf. [105]). Hence, by Kaplansky [92], I_x is primitive and there exists a primitive ideal P_x in $A(x)$ such as $P_x \cap I_x = \{0\}$, which implies $P_x = \{0\}$, Therefore $A(x)$ is a primitive algebra. On the other hand, I_x contains a minimal projection and, as I_x is an ideal in $A(x)$, this is also a minimal projection of $A(x)$.

Lemma (2.3.18)[122]: Every projection in $A(x)$ is the image of some projection in A.

Proof. Let e_{x_0} be a projection in $A(x_0)$. By the proof of Lemma 12 in Glimm [105], we can find an element $a \in A$ and a neighborhood U of x_0 such that $a(x)$ is a non-zero projection in $A(x)$ for every $a \in U$ and $a(x_0) = e_{x_0}$ Moreover as X is homeomorphic with Ω_0 which is known to be a totally disconnected space, there exists an open and closed neighborhood V of x_0 contained in U. let f be the characteristic function of V and put $e = f \cdot a$, then it is not difficult to see that e is a projection of A and $e(x_0) = e_{x_0}$.

Now we get

Theorem (2.3.19)[122]: Lei A *be a* W^*-algebra. Then the following statements are equivalent:

(i) A is of continuous type, that is, A has no type I portion,

(ii) A has no non-zero GCR ideal,

(iii) $A(x)$ has no non-zero GCR ideal for every $x \in X$,

(iv) $\mathfrak{P}(A(x)) = \mathfrak{S}(A(x))$ for every $x \in X$,
(v) $\mathfrak{P}(A) = \{{}^t\psi_x(\varphi) | \varphi \in \mathfrak{S}(A(x)), x \in X\}$, where ψ_x means the canonical map from A to $A(x)$.

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) were established in Glimm [105] but we prove here all implications for the completeness.

Proof. (i) $\Longrightarrow$ (iii). Suppose that there exists a point $x \in X$ such that $A(x)$ has a non-zero GCR ideal. Then, by Lemma (2.3.17). $A(x)$ contains a minimal projection e_x, which is the image of a projection e in A. Since A is of continuous type it is well known that e is the sum of two equivalent orthogonal projections e_1, e_2 in A. Hence, $e_x = e(x) = e_1(x) + e_2(x)$ and both of $e_1(x)$ and $e_2(x)$ are non-zero projections in $A(x)$. This contradicts the minimality of e_x. Therefore every $A(x)'s$ have no non-zero GCR ideals.

(iii) $\Longrightarrow$ (iv). Since $A(x)$ has no ideal divisors of zero, (iii) implies (iv) by [125]. The implication (iv) $\Longrightarrow$ (v) is clear.

(v) $\Longrightarrow$ (i). Suppose that A has a non-zero type I portion Az where z is a central projection of A. By [105], we have

$\mathfrak{P}(Az) = \{{}^t\tilde{\psi}_x(\varphi) | \varphi \in \mathfrak{P}(A(x))$ for $x \in X$ with $z(x) \neq 0$

and $\mathfrak{P}(A(x)) \neq \mathfrak{S}(A(x))$ for all such $x's$ where $\tilde{\psi}_x$ means the restriction of ψ_x to Az. Take a functional $\varphi \in \mathfrak{S}(A(x))$ and $\varphi \notin \mathfrak{P}(A(x))$ for some point $x \in X$ with $z(x) \neq 0$. Then ${}^t\psi_x \in \mathfrak{P}(A)$ by the assumption, hence ${}^t\tilde{\psi}_x \in \mathfrak{P}(Az)$, a contradiction. Therefore A has no type I portion.

The implication (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is clear.

It is perhaps worth to notice that though we can not generally conclude that the weak closure of a C^*-algebra having no non-zero GCR ideal is of continuous type, it is true in the case of a W^*-algebra.

Theorem (2.3.20)[122]: Let A be a W^*-algebra. Then the following statements are equivalent
(i) A is of type I,
(ii) $A(x)$ has a non-zero GCR ideal for everx $x \in X$.

Proof. The implication (i) $\Longrightarrow$ (ii) is due to [105]. Roughly speaking, the discussion is as follows : the canonical image of an abelian projection in A by ψ_x is a minimal projection in $A(x)$ or zero and as A has stufficiently many abelian projections this means that each of $A(x)'s$ has a minimal projection, hence a non-zero GCR ideal.

The converse is clear from Theorem (2.3.19).

Combining (iv) of Theorem (2.3.19) and Theorem 4 in [105] we can easily show that the pure state space of a W^*-algebra is determined completely by the pure state spaces of its component algebras.

Theorem (2.3.21)[122]: Let A be a W^*-algebra. Then

$$\mathfrak{P}(A) = \{{}^t\psi_x(\varphi) | \varphi \in \mathfrak{P}(A(x)) \text{ for } x \in X\}.$$

Chapter 3
Complex and Skew Symmetric Operators

We show that results explain why the compressed shift and Volterra integration operator are complex symmetric. We attempt to describe all complex symmetric partial isometries, obtaining the sharpest possible statement given only the data (dim ker T, dim ker T^*). We obtain a canonical decomposition of complex symmetric operators. This result decomposes general complex symmetric operators into direct sums of three kinds of elementary ones. We classify up to approximate unitary equivalence those skew symmetric operators $T \in \mathcal{B}(\mathcal{H})$ satisfying $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. This is used to characterize when a unilateral weighted shift with nonzero weights is approximately unitarily equivalent to a skew symmetric operator.

Section (3.1): Some New Classes

$\mathcal{H}$ will denote a separable complex Hilbert space and all operators considered will be bounded. We first require a few preliminary definitions:

Definition (3.1.1)[127]: A conjugation is a conjugate-linear operator $C: \mathcal{H} \to \mathcal{H}$ that is both involutive ($C^2 = I$) and isometric. We say that a bounded linear operator $T \in B(\mathcal{H})$ is C-symmetric if $T = CT^*C$ and complex symmetric if there exists a conjugation C with respect to which T is C-symmetric.

It is not hard to see that T is a complex symmetric operator if and only if T is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [136] or [134]).

The class of complex symmetric operators includes all normal operators, operators defined by Hankel matrices, compressed Toeplitz operators (including finite Toeplitz matrices and the compressed shift), and the Volterra integration operator. See [134], [135] (or [136] for a more expository pace). Other concerning complex symmetric operators include [130], [138].

We exhibit several additional classes of complex symmetric operators. In particular, we establish that

(i) All binormal operators are complex symmetric (Theorem (3.1.3)) and that n-normal operators that are not complex symmetric exist for each $n \geq 3$ (Example (3.1.5)).

(ii) Operators that are algebraic of degree two are complex symmetric (Theorem (3.1.9)). This includes all idempotents and all operators that are nilpotent of order 2.

(iii) Large classes of rank-one perturbations of normal operators are complex symmetric (Theorem (3.1.12)). On abstract grounds, this explains why the compressed shift operator (Example (3.1.13)) and Volterra integration operator (Example (3.1.16)) are complex symmetric.

(iv) We attempt to describe all complex symmetric partial isometries, obtaining the sharpest possible statement (Theorem (3.1.19)) given only the data (dim ker T, dim ker T^*).

Definition (3.1.2)[127]: An operator $T \in B(\mathcal{H})$ is called binormal if T is unitarily equivalent to an operator of the form

$$\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \tag{1}$$

where the entries N_{ij} are commuting normal operators. We say that T is n-normal if T is unitarily equivalent an $n \times n$ operator matrix whose entries are commuting normal operators.

Needless to say, each $n \times n$ scalar matrix trivially defines an n-normal operator on $\mathbb{C}^n$. For further information concerning binormal and n-normal operators, see [139], [142].

Theorem (3.1.3)[127]: If $T \in B(\mathcal{H})$ is a binormal operator, then T is a complex symmetric operator. This result is sharp in the sense that if $n \geq 3$, then there exists an n-normal operator that is not a complex symmetric operator.

Proof. We focus our attention on the first statement, since the second will follow from the construction of explicit examples (see Example (3.1.5)). Given an operator of the form (1), the Spectral Theorem asserts that we may assume that each N_{ij} is a multiplication operator $M_{u_{ij}}$ on a Lebesgue space $L^2(\mu)$ where μ is a Borel measure on $\mathbb{C}$ with compact support Δ and that the corresponding symbols u_{ij} belong to $L^\infty(\mu)$. To simplify our notation, we will henceforth identify multiplication operators M_u with their symbols u.

Without loss of generality, we may further restrict our attention to operators on $L^2(\mu)^{(2)}$ (the two-fold inflation of $L^2(\mu)$) of the form

$$T = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}, \tag{2}$$

since any binormal operator is unitarily equivalent to an operator of form (2) [142].

We denote by E the subset of Δ upon which $u_{11} = u_{22}$:

$E = \{z \in \Delta: u_{11}(z) = u_{11}(z) \ \mu\text{-a.e.}\}$.

Letting χ_E denote the characteristic function of E, we note that the subspace $\mathcal{E}_1 = \chi_E L^2(\mu)$ and its orthogonal complement $\mathcal{E}_1 = \chi_{\Delta \setminus E} L^2(\mu)$ are both reducing subspaces for $M_z: L^2(\mu) \to L^2(\mu)$, the operator of multiplication by the independent variable. In particular, their inflations $\mathcal{E}_1^{(2)}$ and $\mathcal{E}_2^{(2)}$ are both reducing subspaces for T, and we see that

$$T = T|_{\mathcal{E}_1^{(2)}} \oplus T|_{\mathcal{E}_2^{(2)}}.$$

Since the direct sum of complex symmetric operators is complex symmetric, we need only consider the following two special cases:

(i) $u_{11} = u_{22}$ μ-a.e.,

(ii) $u_{11} \neq u_{22}$ μ-a.e..

Case (i): Suppose that $u_{11} = u_{22}$ μ-a.e. In this case, we may write (2) as

$$\begin{pmatrix} u & v \\ 0 & u \end{pmatrix},$$

where $u, v \in L^\infty(\mu)$. One can immediately verify that T is C-symmetric with respect to the conjugation $C(f_1, f_2) = (\overline{f_2}, \overline{f_1})$ on $L^2(\mu)^{(2)}$.

Case (ii). Suppose that $u_{11} \neq u_{22}$ μ-a.e. In this case, T has the form

$$\begin{pmatrix} u_1 & v \\ 0 & u_2 \end{pmatrix}, \tag{3}$$

where $u_1 \neq u_2$ μ-a.e. Let F denote the subset of Δ upon which v vanishes, and observe that $T = T|_{\mathcal{F}_1^{(2)}} \oplus T|_{\mathcal{F}_2^{(2)}}$, where $\mathcal{F}_1 = \chi_F L^2(\mu)$ and $\mathcal{F}_1 = \chi_{\Delta \setminus F} L^2(\mu)$. Since v vanishes on F, it follows from (3) that $T|_{\mathcal{F}_1^{(2)}}$ is normal and hence complex symmetric. On the other hand, $T|_{\mathcal{F}_2^{(2)}}$ is an operator of the form (3), where v is μ-a.e. nonvanishing. Without loss of generality, we may therefore assume that v does not vanish on a set of positive μ-measure. Since $u_1 - u_2$ and v are nonvanishing μ-a.e., we may define a unimodular function γ by the formula

$$\gamma = \frac{v}{|v|} \cdot \frac{|u_1 - u_2|}{(u_1 - u_2)}. \tag{4}$$

Letting

$$a(z) = \frac{\gamma|u_1 - u_2|}{\sqrt{|u_1 - u_2|^2 + |v|^2}}, \qquad b(z) = \frac{|v|}{\sqrt{|u_1 - u_2|^2 + |v|^2}}, \tag{5}$$

we note that the operator

$$U = \begin{pmatrix} a & b \\ b & -\bar{a} \end{pmatrix}$$

on $L^2(\mu)^{(2)}$ is unitary since b is real and $|a|^2 + |b|^2 = 1$ μ-a.e.

Let $Jf = \bar{f}$ denote the canonical conjugation on $L^2(\mu)$ and let $K = L^{(2)}$ denote its two-fold inflation:

$$K = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

Clearly K is a conjugation on $L^2(\mu)^{(2)}$, and a short computation shows that $U^* = KUK$ (i.e. U is a K-symmetric operator).

We now claim that $C = UK$ is a conjugation on $L^2(\mu)^{(2)}$. Since C is obviously conjugate-linear and isometric, we need only verify that $C^2 = I$:

$$C^2 = (UK)(UK) = U(KUK) = UU^* = I.$$

Thus C is a conjugation on $L^2(\mu)^{(2)}$, as claimed.

We conclude the proof by showing that T is C-symmetric. We will do this by directly verifying that $CT^* = TC$. First note that

$$\begin{aligned} TC &= TUK \\ &= \begin{pmatrix} u_1 & v \\ 0 & u_2 \end{pmatrix}\begin{pmatrix} a & b \\ b & -\bar{a} \end{pmatrix} \mathrm{K} \qquad (6) \\ &= \begin{pmatrix} au_1 + bv & bu_1 - \bar{a}v \\ bu_2 & -\bar{a}u_2 \end{pmatrix} K. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} CT^* &= UKT^* \\ &= \begin{pmatrix} a & b \\ b & -\bar{a} \end{pmatrix} \mathrm{K} \begin{pmatrix} \overline{u_1} & 0 \\ \bar{v} & \overline{u_2} \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & -\bar{a} \end{pmatrix}\begin{pmatrix} u_1 & 0 \\ v & u_2 \end{pmatrix} K \\ &= \begin{pmatrix} au_1 + bv & bu_2 \\ bu_1 - \bar{a}v & -\bar{a}u_2 \end{pmatrix} K. \qquad (7) \end{aligned}$$

To verify the equality of (6) and (7), we need only show that $bu_2 = bu_1 - \bar{a}v$. However, the preceding equation follows directly from (4) and (5).

One might regard Theorem (3.1.3) as a generalization of the following well-known result (alternate proofs of which can be found in [130], [134], or [145]):

Corollary (3.1.4)[127]: Every linear operator on $\mathbb{C}^2$ is complex symmetric. In other words, every 2×2 matrix is unitarily equivalent to a symmetric matrix with complex entries.

In order to verify the second claim of Theorem (3.1.3), we must exhibit examples of n-normal operators ($n \geq 3$) that are not complex symmetric. The following example does just this.

Example (3.1.5)[127]: We first claim that the operator $T: \mathbb{C}^3 \to \mathbb{C}^3$ defined by the matrix

$$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

(with respect to the standard basis) is a complex symmetric operator if and only if $ab = 0$ or $|a| = |b|$. There are several possible cases to investigate: (i) $a = 0$ or $b = 0$, (ii)

$|a| = |b| \neq 0$, (iii) $a \neq 0$, $b \neq 0$, and $|a| \neq |b|$. In particular, the final case yields 3-normal operators that are not complex symmetric.
Case (i). If $a = 0$ or $b = 0$, then T is the direct sum of complex symmetric operators by Corollary (3.1.4).
Case (ii). If $|a| = |b| \neq 0$, then a short computation shows that (8) is unitarily equivalent to a constant multiple of a 3×3 nilpotent Jordan matrix. It follows from [134] or [136] that T is a complex symmetric operator.
Case (iii). Let $a \neq 0$, $b \neq 0$, and $|a| \neq |b|$, and suppose toward a contradiction that $T = CT^*C$ for some conjugation C. Let e_1, e_2, e_3 denote the standard basis for $\mathbb{C}^3$ and observe that e_1 and e_3 span the eigenspaces of T and T^*, respectively, corresponding to the eigenvalue zero. Since $T^i x = 0$ if and only if $(T^*)^i(Cx) = 0$, we see that

$$Ce_1 = \alpha_1 e_3, \qquad Ce_2 = \alpha_2 e_2, \qquad Ce_3 = \alpha_3 e_1,$$

where $\alpha_1, \alpha_2, \alpha_3$ are certain unimodular constants. The desired contradiction will arise from computing $\|Te_2\|$ in two different ways. On one hand, we have

$$\|Te_2\| = \|T^*Ce_2\| = \|T^*(\alpha_2 e_2)\| = \|T^*e_2\| = \left\|0, 0, \bar{b}\right\| = |b|.$$

On the other hand, we also have $\|Te_2\| = \|a, 0, 0\| = |a|$. However, this contradicts the fact that $|a| \neq |b|$. Therefore T is not a complex symmetric operator.

If $n > 3$, then we can use the preceding ideas to construct examples of n-normal operators that are not complex symmetric. Specifically, let $T: \mathbb{C}^3 \to \mathbb{C}^3$ be defined as in (8), with $ab \neq 0$ and $|a| \neq |b|$ as in Case (iii). The operator $T \oplus I$ on $\mathbb{C}^n$, where I denotes the identity operator on $\mathbb{C}^{n-3}$, is trivially n-normal. The same argument used in Case (iii) reveals that $T \oplus I$ is not complex symmetric.

We remark that matrices of the form (8) arose in a related unitary equivalence problem. Consideration of Jordan canonical forms reveals that each $n \times n$ matrix is similar to its transpose. On the other hand, the matrix

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not unitarily equivalent to its transpose [141].
Corollary (3.1.6)[127]: If N is a normal operator having spectral multiplicity ≤ 2 and if T is an operator commuting with N, then T is a complex symmetric operator.
Proof. If N is a normal operator having spectral multiplicity ≤ 2, then we may write $N = N_1 \oplus N_2^{(2)}$, where N_1 and N_2 are mutually singular $*$-cyclic normal operators [129]. Moreover, we also have $T = T_1 \oplus T_2$, where T_1 commutes with N_1 and T_2 commutes with $N_2^{(2)}$ (see the discussion following [129] or [138]). From this we immediately see that T_1 is normal and T_2 is binormal [129]. It then follows from Theorem (3.1.3) that T is a complex symmetric operator.

In the preceding, observe that if N has spectral multiplicity two, then the conjugation corresponding to the operator T depends on T (as well as N).

Our next corollary asserts that any square root (normal or otherwise) of a normal operator is itself a complex symmetric operator:
Corollary (3.1.7)[127]: If T^2 is normal, then T is a complex symmetric operator.
Proof. This follows immediately from Theorem (3.1.3) and the fact that T must be of the form

$$T = A \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix},$$

where A and B are normal and C is a positive operator that commutes with B [143].

Definition (3.1.8)[127]: An operator $T \in B(\mathcal{H})$ is algebraic if $p(T) = 0$ for some polynomial $p(z)$. The degree of an algebraic operator is defined to be the degree of the polynomial $p(z)$ of least degree for which $p(T) = 0$.

Although the following theorem is essentially a corollary of Theorem (3.1.3), we choose to state it as a theorem since it will have several useful corollaries of its own.

Theorem (3.1.9)[127]: If $T \in B(\mathcal{H})$ is algebraic of degree ≤ 2, then T is a complex symmetric operator. This result is sharp in the sense that for each finite $n \geq 3$ and for each $\mathcal{H}$ satisfying $\dim \mathcal{H} \geq n$, there exists an algebraic operator on $\mathcal{H}$ of degree n that is not a complex symmetric operator.

Proof. The first statement follows from Theorem (3.1.3) and an old lemma of Brown Lem. 7.1 in [128] that asserts that if T is algebraic of degree ≤ 2, then T is binormal. Suppose now that $3 \leq n \leq \dim \mathcal{H}$ and consider the operator $T \oplus D$, where T has a matrix of the form (8) with $ab \neq 0$ and $|a| \neq |b|$ and D is a diagonal operator chosen so that $T \oplus D$ is algebraic of degree n. An argument similar to that used in Case (iii) of Example (3.1.5) shows that this operator is not complex symmetric.

Two particular classes of operators stand out for special consideration:

Corollary (3.1.10)[127]: Let $T \in B(\mathcal{H})$. If T is idempotent (i.e. $T^2 = T$) or nilpotent of order 2 (i.e. $T^2 = 0$), then T is a complex symmetric operator.

A direct proof of the second portion of Corollary (3.1.10), involving the explicit construction of the associated conjugation, can be found in [132]. Yet another basic class of operators that happen to be complex symmetric are the rank-one operators:

Corollary (3.1.11)[127]: If $T \in B(\mathcal{H})$ and $\operatorname{rank}(T) = 1$, then T is a complex symmetric operator.

Proof. Any rank-one operator T is of the form $Tf = \langle f, v \rangle u$ for certain vectors u, v (this operator is frequently denoted $u \otimes v$). Since $T^2 - \langle u, v \rangle T = 0$, it follows from Theorem (3.1.9) that T is a complex symmetric operator.

It is important to note that although every operator on $\mathbb{C}^2$ is a complex symmetric operator (Corollary (3.1.4)), there are certainly operators having rank two that are not complex symmetric operators (Example (3.1.5)).

In light of Corollary (3.1.11) and the fact that all normal operators are complex symmetric (see [136] or [134]), it is natural to attempt to identify those rank-one perturbations of normal operators that are also complex symmetric.

Theorem (3.1.12)[127]: If $N \in B(\mathcal{H})$ is a normal operator, U is a unitary operator in $W^*(N)$ (the von Neumann algebra generated by N), $a \in C$, and $v \in \mathcal{H}$, then the operator $T = N + a(Uv \otimes v)$ is a complex symmetric operator.

Proof. We may without loss of generality assume that N is a $*$-cyclic normal operator with cyclic vector v. Otherwise let $\mathcal{H}_1$ denote the reducing subspace of N generated by v and let $\mathcal{H}_2 = \mathcal{H}_1^\perp$. Now write $N = N_1 \oplus N_2$ relative to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. It follows that N_1 is $*$-cyclic and, since $\mathcal{H}_1$ reduces U, we have $T = N_1 + a(U_1 v \otimes v)) \oplus N_2$, where $U_1 = |_{\mathcal{H}_1}$ belongs to $W^*(N_1)$.

By the Spectral Theorem, we may further presume that $N = M_z$ (the operator of multiplication by the independent variable on a Lebesgue space $L^2(\mu)$), that v is the constant function 1, and that $U = M_\theta$ (the operator of multiplication by some unimodular function θ in $L^\infty(\mu)$). At this point, a straightforward computation shows that $Cf = \theta \bar{f}$ is a conjugation on $L^2(\mu)$ with respect to which both M_z and $\theta \otimes 1$ are C-symmetric.

On an abstract level, the preceding theorem indicates that compressed shifts are complex symmetric operators. In other words, starting from the fact that the Aleksandrov-Clark unitary operators are complex symmetric, we can directly derive the fact that the compressed shift is also complex symmetric. In essence, this is the reverse of the path undertaken in [136] (which we may consult for further details concerning the following example).

Example (3.1.13)[127]: Let φ denote a nonconstant inner function and let H^2 denote the Hardy space on the unit disk $\mathbb{D}$. For each λ in the open unit disk $\mathbb{D}$, we define the unit vectors

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}, \tag{9}$$

$$k_\lambda(z) = \sqrt{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}} \cdot \frac{1-\overline{\varphi(\lambda)}\varphi(z)}{1-\bar{\lambda}z}, \tag{10}$$

$$q_\lambda(z) = \sqrt{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}} \cdot \frac{\varphi(z)-\varphi(\lambda)}{z-\lambda}. \tag{11}$$

In particular, the function k_λ is a normalized reproducing kernel for the so-called model space $H^2 \ominus \varphi H^2$.

For each unimodular constant α, we define the generalized Aleksandrov-Clark operator by setting

$$U_\lambda f = \begin{cases} b_\lambda f, & f \perp q_\lambda, \\ \alpha k_\lambda, & f = q_\lambda. \end{cases}$$

Each U_λ is C-symmetric with respect to the conjugation (defined in terms of boundary functions) $[Cf](z) = \overline{fz}\varphi$ on $H^2 \ominus \varphi H^2$. Moreover, we also note that $q_\lambda = Ck_\lambda$ for each λ.

By Theorem (3.1.12), it follows that the operator

$$S_\lambda = U_\lambda - (\alpha + \varphi(\lambda))(k_\lambda \otimes q_\lambda) \tag{12}$$

is complex symmetric since it is of the form $U_\lambda + a(U_\lambda v \otimes v)$, where a is a complex constant and $v = q_\lambda$. More specifically, tracing through the proof of Theorem (3.1.12), we expect that S_λ will be C-symmetric with respect to the C described above.

The significance of this example lies in the fact that, for the choice

$$\alpha = -\varphi(\lambda)/|\varphi(\lambda)|,$$

the operator (12) turns out to be

$$S_\lambda f = P_\varphi(b_\lambda f), \tag{13}$$

the compression of the operator $M_{b_\lambda}: H^2 \to H^2$ to the subspace $H^2 \ominus \varphi H^2$. Here P_φ denotes the orthogonal projection from H^2 onto $H^2 \ominus \varphi H^2$. The operator $S_0 f = P_\varphi(zf)$ is commonly known as the compressed shift or Jordan model operator corresponding to φ. In summary, purely operator-theoretic considerations guarantee that the operators S_λ are complex symmetric. See [136] and [144] for more information.

In fact, the preceding example can be greatly generalized (without any reference to function theory whatsoever). Given a contraction $T \in B(\mathcal{H})$, there is a unique decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_u$, where $\mathcal{H}_0$ and $\mathcal{H}_u$ are both T-invariant, $T|\mathcal{H}_u$ is unitary, and $T|\mathcal{H}_0$ is completely nonunitary (i.e., $T|\mathcal{H}_0$ is not unitary when restricted to any of its invariant subspaces). The operator $D_T = (I - T^*T)^{1/2}$ is called the defect operator of T, and the defect spaces of T are defined to be the subspaces $\mathcal{D}_T = \overline{D_T\mathcal{H}}$ and $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$. The defect indices of T are the numbers $\partial_T = \dim \mathcal{D}_T$ and $\partial_{T^*} = \dim \mathcal{D}_{T^*}$. We say that $T \in C_{0\cdot}$ if $T^n \to 0$ (SOT) and that $T \in C_{\cdot 0}$ if $T^* \in C_{0\cdot}$. Finally, we also define $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

It turns out that any Hilbert space contraction with defect indices $\partial_T = \partial_{T^*} = 1$ is complex symmetric. Although this is known (see Cor. 3.2 in [130] for a general proof) and easy to prove if $T \in C_{00}$ (see Thm. 5.1 in [136], Prop. 3 in [134], and Lem. 2.1 in [144]), we are able to establish this result in the abstract-without the use of characteristic functions and complex analysis.

Corollary (3.1.14)[127]: If $T \in B(\mathcal{H})$ is a contraction such that $\partial_T = \partial_{T^*} = 1$, then T is a complex symmetric operator.

Proof. Since $\partial_T = 1$, it follows that $I - T^*T = u \otimes u$ for some nonzero vector u. If x is any vector orthogonal to u, then we have

$$\|x\|^2 - \|Tx\|^2 \langle (I - T^*T)x, x\rangle = |\langle u, x\rangle|^2 = 0.$$

Thus T is isometric on a subspace of $\mathcal{H}$ having codimension one. Similarly, we see that $I - TT^*$ is also of rank one, whence $I - TT^* = v \otimes v$ for some nonzero vector v. Putting this together, we find that $T = T|_{u^\perp} + c(u \otimes v)$ for some constant c. In particular, there exists a unitary U such that $T = U + c'(u \otimes v)$ is a rank-one perturbation of U. Since T is of the form $T = U + a(Uv \otimes v)$ where U is unitary, it follows from Theorem (3.1.12) that T is a complex symmetric operator.

Following Theorem (3.1.12) in another direction, we obtain the following:

Corollary (3.1.15)[127]: Let $T = A + iB$ denote the Cartesian decomposition of $T \in B(\mathcal{H})$ (i.e. $A = A^*$ and $B = B^*$). If $\operatorname{rank}(A) = 1$ or $\operatorname{rank}(B) = 1$, then T is a complex symmetric operator.

Proof. If A has rank one, then $A = a(v \otimes v)$ for some $a \in \mathbb{R}$ and $v \in \mathcal{H}$. Apply Theorem (3.1.12) with $N = iB$ and $U = I$.

The preceding corollary easily furnishes many examples of nonnormal complex symmetric operators. Indeed, if A is an arbitrary selfadjoint operator and B is a rank-one selfadjoint operator that does not commute with A, then $T = A + iB$ is a non-normal complex symmetric operator. Despite the apparent simplicity of such a recipe, nontrivial examples abound. Consider the following example:

Example (3.1.16)[127]: It is well known that the Volterra integration operator

$$[V f](x) = \int_0^x f(y)dy$$

on $L^2[0,1]$ is a rank-one selfadjoint perturbation of a skew-selfadjoint operator (see [131] or [140] for further details). Indeed, a short computation shows that the selfadjoint component of V is

$$A = \tfrac{1}{2}(V + V^*) = \int_0^1 f(y)dy = \tfrac{1}{2}(1 \otimes 1),$$

where the 1 above denotes the constant function. By Corollary (3.1.15), we conclude that V is a complex symmetric operator. In fact, $V = CV^*C$, where C denotes the conjugation $[Cf](x) = \overline{f(1-x)}$ on $L^2[0,1]$ (see [135] and [136]).

Setting $U = I$ in Theorem (3.1.12) provides a generalization of Corollary (3.1.15):

Corollary (3.1.17)[127]: If $N \in B(\mathcal{H})$ is a normal operator, P is a rank-one orthogonal projection, and $a \in \mathbb{C}$, then $T = N + aP$ is a complex symmetric operator.

It is important to note that not every rank-one perturbation of a normal operator will be complex symmetric (unless $\dim \mathcal{H} = 2$; see Corollary (3.1.4)). In fact, even a rank-one perturbation of an orthogonal projection may fail to be complex symmetric:

Example (3.1.18)[127]: We claim that the operator $T: \mathbb{C}^3 \to \mathbb{C}^3$ defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(with respect to the standard basis) is not a complex symmetric operator. First observe that the eigenspaces of T (and hence of T^*) for the eigenvalues 0 and 1 are both one-dimensional. The eigenspaces of T corresponding to the eigenvalues 0 and 1 are spanned by the unit vectors $v_0 = (1,0,0)$ and $v_1 = (0,1,0)$, respectively. The eigenspaces of T^* corresponding to the eigenvalues 0 and 1 are spanned by the unit vectors $w_0 = (1,0,0)$ and $w_1 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, respectively. If C is a conjugation such that $T = CT^*C$, then $0 = |\langle v_0, v_1\rangle| = |\langle Cv_1, Cv_0\rangle| = |\langle w_1, w_0\rangle| = \frac{1}{\sqrt{2}}$, which is absurd.

We attempt to classify those partial isometries that are complex symmetric. This question is related to the preceding material in the sense that if $\varphi(0) = 0$ in Example (3.1.13), then the corresponding compressed shift operator is a complex symmetric partial isometry.

Given only the dimensions of the kernels of a partial isometry and its adjoint, the following theorem is as definitive as possible:

Theorem (3.1.19)[127]: Let $T \in B(\mathcal{H})$ be a partial isometry.

(i) If $\dim \ker T = \dim \ker T^* \le 1$, then T is a complex symmetric operator.

(ii) If $\dim \ker T \neq \dim \ker T^*$, then T is not a complex symmetric operator.

(iii) If $2 \le \dim \ker T = \dim \ker T^* \le \infty$, then either possibility can (and does) occur.

Proof. (i) If $\dim \ker T = \dim \ker T^* = 0$, then $T^*T = TT^* = I$, whence T is unitary and hence complex symmetric. Suppose that T is a partial isometry satisfying $\dim \ker T = \dim \ker T^* = 1$ and that $\ker T$ and $\ker T^*$ are spanned by the unit vectors v and w, respectively. Since the operator $N = T + w \otimes v$ is unitary, it follows that $T = N - Nv \otimes v$ is a complex symmetric operator by Theorem (3.1.12). For a different proof, see [130].

(ii) We show the contrapositive. If T is C-symmetric, then it is easy to see that $Tx = 0$ if and only if $T^*(Cx) = 0$. Therefore C furnishes an isometric, conjugate-linear bijection between $\ker T$ and $\ker T^*$, whence $\dim \ker T = \dim \ker T^*$.

(iii) This portion of the theorem follows upon consideration of several examples. It is trivial to produce complex symmetric partial isometries with $\dim \ker T = \dim \ker T^* = n$ for any n. In fact, $T = I \oplus 0$, where 0 is the zero operator on an n-dimensional Hilbert space, is such an example. On the other hand, finding partial isometries that are not complex symmetric when $2 \le n \le \infty$ is more involved.

For the remainder of this proof, we choose not to distinguish between matrices and the operators they induce (with respect to the standard basis). We must first study a certain auxiliary matrix that will be used in our construction. Specifically, we intend to prove that

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix}$$

is not a complex symmetric operator. This will follow from a careful study of the eigenstructures of A and A^*. First, note that the eigenvalues of A are

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = -\frac{1}{4} + i\frac{\sqrt{3}}{4}, \qquad \lambda_3 = -\frac{1}{4} - i\frac{\sqrt{3}}{4},$$

and that these are also the eigenvalues of A^*. A straightforward computation shows that corresponding unit eigenvectors of A are

$$v_1 = \frac{1}{\sqrt{6}}(1,1,2),$$
$$v_2 = \frac{1}{2\sqrt{6}}(-1+i\sqrt{3},-1-i\sqrt{3},4),$$
$$v_3 = \frac{1}{2\sqrt{6}}(1+i\sqrt{3},1-i\sqrt{3},-4).$$

Since A has three distinct eigenvalues, it follows that v_1, v_2, v_3 must be sent to unimodular scalar multiples of the corresponding unit eigenvectors

$$w_1 = \frac{1}{3}(2,2,1),$$
$$w_2 = \frac{1}{3}(-1-i\sqrt{3},-1+i\sqrt{3},1),$$
$$w_3 = \frac{1}{3}(-1+i\sqrt{3},-1-i\sqrt{3},1)$$

of A^*. Now observe that

$$|\langle v_1,v_2\rangle| = |\langle v_2,v_3\rangle| = |\langle v_3,v_1\rangle| = \frac{1}{2},$$

whereas

$$|\langle w_1,w_2\rangle| = |\langle w_2,w_3\rangle| = |\langle w_3,w_1\rangle| = \frac{1}{3}.$$

The same argument used in Example (3.1.18) now reveals that A cannot be a complex symmetric operator.

We are now ready to construct our desired partial isometry. Noting that

$$A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{16} \end{pmatrix},$$

we see that if $2 \le n \le \infty$, then the $n \times 3$ matrix

$$B = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{15}}{4} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies $A^*A = B^*B = I$ (the 3×3 identity matrix). The $(n+3)\times(n+3)$ matrix

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

is a partial isometry since T^*T is the orthogonal projection

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Since it is clear from the construction of T that $\dim\ker T = \dim\ker T^* = n$, we need only prove that T is not a complex symmetric operator.

Suppose toward a contradiction that T is C-symmetric. By , Thm. 2 & Cor. 1 in [135], we may write $T = CJP$, where J is an auxiliary conjugation that commutes with P. Since $JP = PJ$ we find that

$$J(PT)J = J(PCJP)J = PJCP = T^*P = (PT)^*,$$

whence PT is J-symmetric. However,

$$PT = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

and the same argument that showed that A was not a complex symmetric operator also shows that PT is not a complex symmetric operator. This contradiction shows that our partial isometry T is not a complex symmetric operator, as desired.

We remark that in the final paragraph of the proof, we could have appealed to the fact that the Aluthge transform of a complex symmetric operator is also complex symmetric [132].

We can prove that every partial isometry on a three-dimensional Hilbert space is complex symmetric:

Corollary (3.1.20)[127]: If $\dim \mathcal{H} = 3$, then every partial isometry $T \in B(\mathcal{H})$ is complex symmetric.

Proof. Suppose that $\dim \mathcal{H} = 3$ and that T is a partial isometry on $\mathcal{H}$. There are four cases to discuss:

(i) If $\dim \ker T = 0$, then T is unitary and thus complex symmetric. Indeed, the Spectral Theorem asserts that T has a diagonal matrix representation with respect to some orthonormal basis of $\mathcal{H}$.

(ii) If $\dim \ker T = 1$, then T is complex symmetric by (i) of Theorem (2.1.19). The condition $\dim \ker T = \dim \ker T^*$ holds trivially since $\mathcal{H}$ is finite-dimensional.

(iii) If $\dim \ker T = 2$, then $\operatorname{rank}(T) = 1$. By Corollary (2.1.11), it follows that T is complex symmetric.

(iv) If $\dim \ker T = 3$, then $T = 0$ and the result is trivial.

Based on the construction used in the proof of Theorem (3.1.19), it is clear that many partial isometries that are not complex symmetric exist if the dimension of the underlying Hilbert space is ≥ 5. On the other hand, we were for a considerable time unable to determine whether all partial isometries on a four-dimensional Hilbert space are complex symmetric (they are). In this setting, the method of Corollary (3.1.20) suffices to resolve all but the case $\dim \ker T = 2$.

Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has been produced by J. Tener [145]. See [137] for the resolution of this problem.

Section (3.2): A Canonical Decomposition

In linear algebra, there is a lot of work on the theory of symmetric matrices (that is, $T = T^t$), which have many motivations in function theory, matrix analysis and other mathematical disciplines. They have many applications even in engineering disciplines. In [135], Garcia and Putinar initiated the study of complex symmetric operators which can be viewed as a generalization of symmetric matrices in the setting of Hilbert space.

We denote by $\mathcal{H}$ a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle \cdot,\cdot \rangle$. We let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$.

Definition (3.2.1)[146]: Let C be a map on $\mathcal{H}$. C is called an antiunitary operator if C is conjugate-linear, invertible and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$; if, in addition, $C^{-1} = C$, then C is called a conjugation on $\mathcal{H}$.

Definition (3.2.2)[146]: An operator $T \in B(\mathcal{H})$ is called a complex symmetric operator (CSO) if there is a conjugation C on $\mathcal{H}$ so that $CTC = T^*$. We let $S(\mathcal{H})$ denote the set of all $CSOs$ on $\mathcal{H}$.

Note that an operator $T \in B(\mathcal{H})$ is complex symmetric if and only if T admits a symmetric matrix representation relative to some orthonormal basis of $\mathcal{H}$ ([135]). Some important results concerning CSOs have been obtained (see [131], [151], [158], [136], [138],

[128], [138], [170], [171], [172], [173]). In particular, CSOs are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason's [145] and has led to a rapid progress in related areas [149], [151], [152], [159], [160], [168], [169]. See [135] for more about the history of CSOs and its connections to other subjects.

A fundamental question about CSOs is how to develop a model theory [156], [159]. A natural thought is to decompose CSOs into "simple blocks" and then represent them in concrete terms. Some known results suggest that truncated Toeplitz operators may play the role of "simple blocks" ([151], [157], [159], [160]).

In [161], Garcia and Tener gave a canonical decomposition of matrices which are unitarily equivalent to complex symmetric matrices. To formulate their result in an operator theoretic form, we first introduce a notion.

Definition (3.2.3)[146]: Let $T \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$is called a transpose of T, if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

The notion "transpose" for operators is in fact a generalization of that for matrices. In general, an operator has more than one transpose. As indicated in [164], any two transposes of an operator are unitarily equivalent. We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$. Here and in what follows, the notation $\cong$ denotes unitary equivalence.

Theorem (3.2.4)[146]: ([161], Proposition 3.6). If T is a CSO on a finite dimensional Hilbert space, then T is unitarily equivalent to a direct sum of

(i) irreducible CSOs, and

(ii) operators of the form $A \oplus A^t$, where A is irreducible and not a CSO.

Thereafter Garcia [155] asked what is the infinite-dimensional analogue of the preceding result. In this aspect, Guo Ji and Zhu [164] recently obtained a decomposition theorem for essentially normal CSOs.

Theorem (3.2.5)[146]: ([164], Theorem 2.8). Let $T \in B(\mathcal{H})$be essentially normal. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;

(ii) $T \in S(H)$;

(iii) T is unitarily equivalent to a direct sum of normal operators, irreducible CSOs and operators with form of $A \oplus A^t$, where A is irreducible and not a CSO.

Let $T \in B(\mathcal{H})$and $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. If $\mathcal{M}$ reduces T and $T|_{\mathcal{M}}$ is irreducible, then $\mathcal{M}$ is called a minimal reducing subspace of T. An operator is said to be completely reducible if it does not admit any minimal reducing subspace [154]. Note that a normal operator is completely reducible if and only if it has no eigenvalues.

We give a decomposition theorem for UET operators (Theorem (3.2.1)), which generalizes two recent results ([161], Theorem 1.2 and [164], Theorem 6.1). Recall that an operator $T \in B(\mathcal{H})$is said to be UET if $T \cong T^t$ ([164], Definition 1.8). By definitions, each CSO is UET. However the converse does not hold (see Example (3.2.21)). As one can see in [164], UET operatorsare closely related to CSOs and its norm closure problem. In particular, a bilateral weighted shift T with nonzero weights is complex symmetric if and only if it is UET. On the other hand, each C^*-algebra generated by a UET operator possesses at least one anti-automorphism on it.

The notion of UET operators also has its motivations in linear algebra. In his problem book ([142], Proposition 159), Halmos asked when a matrix is unitarily equivalent to its transpose (UET). There are matrices that are not UET (see [163]). Recently, Garcia and Tener ([161], Theorem 12) gave a canonical decomposition for UET matrices. Also we note that UET matrices are closely related to the work on linear preservers ([150], [162], [165], [166], [167]).

Lemma (3.2.6)[146]: ([164], Lemma 3.6). If $T \in B(\mathcal{H})$, then $T \oplus T^t$ is complex symmetric.

Lemma (3.2.7)[146]: Let $T \in B(\mathcal{H})$ and $A = CT^*C^{-1}$, where C is an antiunitary operator on $\mathcal{H}$. If $\mathcal{M}$ is a reducing subspace of T, then $C(\mathcal{M})$ is a reducing subspace of A and $A|_{C(\mathcal{M})} \cong (T|_{\mathcal{M}})^t$ In particular, $T|_{\mathcal{M}}$ is irreducible if and only $A|_{C(\mathcal{M})}$is irreducible.

Proof. Denote $\mathcal{N} = C(\mathcal{M})$. It is easy to check that $\mathcal{N}$ is a reducing subspace of A. For $x \in \mathcal{M}$, define $Dx = Cx$. Thus $D: \mathcal{M} \to \mathcal{N}$ is an antiunitary operator. Since $AC = CT^*$, we obtain $(A|_{\mathcal{N}})(C|_{\mathcal{M}}) = (C|_{\mathcal{M}})(T^*|_{\mathcal{M}})$, that is, $(A|_{\mathcal{N}})D = (T^*|_{\mathcal{M}})$. Arbitrarily choose a conjugation E on $\mathcal{M}$. Then we have

$$A|_{\mathcal{N}} = D(T^*|_{\mathcal{M}}) = D^{-1} = (DE)[E(T^*|_{\mathcal{M}})E](ED^{-1}) = (DE)[E(T^*|_{\mathcal{M}})^*E](ED^{-1}).$$

Noting that $DE: \mathcal{M} \to \mathcal{N}$ is unitary, it follows that $A|_{\mathcal{N}} \cong (T|_{\mathcal{M}})^t$. The assertion about minimal reducing subspaces follows readily.

For $A \in B(\mathcal{H})$), we let $C^*(A)$ denote the C^*-algebra generated by A and the identity.

Proposition (3.2.8)[146]: Let $T \in B(\mathcal{H})$ and $T = T_0 \oplus (\oplus_{i\in\Lambda} T_i^{(n_i)})$, where T_0 is completely reducible, T_i is irreducible and $1 \le n_i \le \infty$ for $i \in \Lambda$; moreover, $T_i \ncong T_i$ whenever $i, j \in \Lambda$ and $i \ne j$. Then each reducing subspace $\mathcal{M}$ of T has the form of $\mathcal{M}_0 \oplus (\oplus_{i\in\Lambda} \mathcal{M}_i)$, where $\mathcal{M}_0$ is a reducing subspace of T_0 and $\mathcal{M}_i$ is a reducing subspace of $T_i^{(n_i)}$ for $i \in \Lambda$.

Proof. Without loss of generality we assume that $\Lambda = \mathbb{N}$ and $T_i \in B(\mathcal{H}_i)$ for $i \in \{0\} \cup \mathbb{N}$. Thus $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\ge 1} \mathcal{H}_i^{(n_i)})$. It suffices to prove that for any two $i, j \in \{0\} \cup \mathbb{N}$ with $i \ne j$ there exists no nonzero X such that $T_i X = X T_j$ and $T_i^* X = X T_j^*$. If this holds, then each projection $P \in B(\mathcal{H})$ commuting with T can be written as $P = P_0 \oplus (\oplus_{i\ge 1} P_i)$, where $p_0 \in B(\mathcal{H}_0)$ and $P_i \in B(\mathcal{H}_i^{(n_i)})$for $i \ge 1$. So the desired result follows readily.

For a proof by contradiction, we assume $i, j \in \{0\} \cup \mathbb{N}$, $i \ne j$ and $X \in B(\mathcal{H}_i, \mathcal{H}_j)$isnonzero satisfying, $T_i X = X T_j$, and $T_i^* X = p(T_i^*, T_i)X = Xp(T_j^*, T_j) X T_j^*$. It implies that

$$p(T_i^*, T_i)X = Xp(T_j^*, T_j) \tag{14}$$

for any polynomial $p(\cdot,\cdot)$ in two free variables.

Set $A = T_i \oplus T_j$. Denote by ρ the identity representation of $C^*(A)$ on $\mathcal{H}_i \oplus \mathcal{H}_j$. Thus $\rho_1 := \rho|\mathcal{H}_i$ and $\rho_2 := \rho|\mathcal{H}_i$ are two sub-representations of ρ. By (14), it follows that $\rho_1(\Upsilon)X = X\rho_2(\Upsilon)$ for any $\Upsilon \in C^*(A)$. So, by Proposition 2.1.4 of [147], there exist a nonzero sub-representation $\sigma_1$of $\rho_1$and a nonzero sub-representation σ_2 of $\rho_2$such that $\sigma_1 \cong \sigma_2$.

Case 1. $i, j \ge 1$. Note that T_i, T_j are both irreducible. Then $\sigma_1 = \rho_1$and $\sigma_2 = \rho_2$. Thus there exists unitary $V: \mathcal{H}_j \to \mathcal{H}_i$such that $\rho_1(\Upsilon)V = V\rho_2(\Upsilon), \forall \Upsilon \in C^*(A)$. In particular, we have

$T_i V = \rho_1(A) V = V\rho_2(A) = VT_j$,

contradicting the fact that $T_i \ncong T_j$ since $i \ne j$.

Case 2. $i = 0$ or $j = 0$. Without loss of generality, we assume that $i = 0$ and hence $j \ge 1$. Since T_j is irreducible, we have $\sigma_2 = \rho_2$. Thus there is a unitary V from $\mathcal{H}_i$ onto a reducing subspace $\mathcal{N}$ of T_0 such that $(\Upsilon|_{\mathcal{N}})V = \sigma_1(\Upsilon)V = VP_2(\gamma), \forall \Upsilon \in C^*(A)$. In particular,

$$(T_0|_{\mathcal{N}})V = (A|_{\mathcal{N}})V = \sigma_1(A)V = V\sigma_2(A) = VT_j.$$

Then $T_0|_{\mathcal{N}} \cong T_j$ is irreducible. This contradicts the fact that T_0 is completely reducible. Thus we conclude the proof.

Corollary (3.2.9)[146]: Let $T \in B(\mathcal{H})$ and $T = T_0 \oplus (\oplus_{i\in\Lambda} T_i^{(n_i)})$, where T_0 is completely reducible, T_i is irreducible and $1 \le n_i \le \infty$ for $i \in \Lambda$; moreover, $T_i \ncong T_j$ whenever $i, j \in \Lambda$ and $i \neq j$. Let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. ThenMis a minimal reducing subspace of T if and only if there exists $i \in \Lambda$such that $\mathcal{M} \in \mathcal{H}_i^{(n_i)}$ and $\mathcal{M}$ is a minimal reducing subspace of $T_i^{(n_i)}$.

Corollary (3.2.10)[146]: Let $T \in B(\mathcal{H})$ and *and* $T = T_0 \oplus (\oplus_{i\geq 1} T_i^{(n_i)})$, where T_0 is completely reducible, T_i is irreducible and $1 \le n_i \le \infty$ for $i \in \Lambda$; moreover, $T_i \ncong T_j$ whenever $i, j \in \Lambda$ and $i \neq j$. If $T \in \mathcal{S}(\mathcal{H})$, then T_0 and $\oplus_{i\in\Lambda}(T_i^{(n_i)})$ are both complex symmetric.

Proof. Since $T \in \mathcal{S}(\mathcal{H})$, there is a conjugation C on $\mathcal{H}$ such that $CTC = T^*$. By Lemma (3.2.7), C maps one minimal reducing subspace of T to another. It follows from Corollary (3.2.9) that $C(\oplus_{i\in\Lambda}(\mathcal{H}_i^{(n_i)}) = \oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$. Since C is a conjugation and $C^2 = I$, one can see that $C(\oplus_{i\in\Lambda}(\mathcal{H}_i^{(n_i)}) = \oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$. and hence $C(\mathcal{H}_0) = \mathcal{H}_0$.

Set $C_1 = C|_{\mathcal{H}_0}$ and $C_2 = C|_{\mathcal{H}_0^{\perp}}$. Then C_1, C_2 are conjugations and $C = C_1 \in C_2$. It follows from $T = CT^*C$ that $T_0 = C_{\,1} T_0^{\,*} C_{\,1}$and

$$(\oplus_{\,i\geq\Lambda} T_i^{(n_i)})^* = C_2(\oplus_{\,i\geq\Lambda} T_i^{(n_i)})\, C_2.$$

This completes the proof.

If $A \in B(\mathcal{H})$is irreducible, then the commutant algebra $\{A, A^*\}$ of $\{A, A^*\}$ equals $\mathbb{C}I$; whence the following result is clear. We also referred to Proposition 7.4 of [164] for a proof.

Lemma (3. 2.11)[146]: Let $T = A^{(n)}$, where $A \in B(\mathcal{H})$ is irreducible and $1 \le n \le \infty$.If $\mathcal{M}$is a nonzero reducing subspace of T, then the following are equivalent:

(i) $T|_{\mathcal{M}} \cong A$;

(ii) $T|_{\mathcal{M}}$ is irreducible;

(iii) there exist complex numbers $\{\alpha_i\}_{i=1}^n$ *with* $0 < \sum_{i=1}^n |\alpha_i|^2 < \infty$ such that

$$\mathcal{M} = \{\oplus_{i=1}^n \alpha_i : \xi : \xi \in \mathcal{H}\}.$$

Proposition (3.2.12)[146]: Let $T = A^{(n)}$, where $A \in B(\mathcal{H})$is irreducible and $1 \le n \le \infty$. Then T is complex symmetric if and only if exactly one of the following holds:

(i) A is complex symmetric;

(ii) $n \in \{2i : i \in \mathbb{N} \cup \{\infty\}$, A is UET and not complex symmetric.

Proof. Here we need only deal with the case that $n = \infty$. The proof for $n < \infty$ is contained in Proposition 7.8 of [164].

"⇐" By the hypothesis, we have $A \cong A^t$. Thus

$$T = (A)^{(\infty)} = (A \oplus A)^{(\infty)} \cong (A \oplus A^t)^{(\infty)}.$$

It follows from Lemma(3. 2.6) that T is a CSO.

"⇒" Now we assume that T is a CSO and A is not a CSO. It suffices to prove that A is UET. Since T is a CSO, there is a conjugation C on $\mathcal{H}^{(\infty)}$ such that $CTC = T^*$. For convenience, we write

$$T = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_1 = \cdots = \mathcal{H}$.

Denote $\mathcal{N} = C(\mathcal{H}_1)$. Since $A = T|_{\mathcal{H}_1}$ is irreducible and $T = CT^*C$, it follows from Lemma(3.2.7) that $\mathcal{N}$ is a minimal reducing subspace of T and $T|_{\mathcal{N}} \cong (T|_{\mathcal{H}_1})^t = A^t$; whence $T|_{\mathcal{N}}$ is irreducible. By Lemma (3.2.11), this implies $T|_{\mathcal{N}} \cong A$. Therefore we obtain $A \cong A^t$, which completes the proof.

Lemma (3.2.13)[146]: ([154], Proposition 2.4). If $A \in B(\mathcal{H})$, then T admits the decomposition $T = T_0 \oplus (\oplus_{i\in\Lambda} T_i)$, where $T_0 \in B(\mathcal{H}_0)$ is completely reducible and $T_i \in B(\mathcal{H}_i)$ is irreducible for all $i \in \Lambda$.

Theorem (3.2.14)[146]: (Main theorem). Let $T \in B(\mathcal{H})$. Then T is a CSO if and only ifit is unitarily equivalent to a direct sum of (some of the summands may be absent)

(i) completely reducible CSOs,

(ii) irreducible CSOs, and

(iii) operators of the form $A \oplus A^t$, where A is irreducible and not a CSO.

Proof. The sufficiency is obvious. It suffices to prove the necessity.

"⇒" By Lemma (3.2.13) and Corollary (3.2.10), we may directly assume that $T = \oplus_{i=1}^{\infty} T_i^{(n_i)}$, where $T_i \in B(\mathcal{H}_i)$ is irreducible and $1 \le n_i \le \infty$; for $i \ge 1$ moreover, $T_i \ncong T_j$ whenever $i \neq j$. Thus $\mathcal{H} = \oplus_{i\ge1} \mathcal{H}_i^{(n_i)}$. To be convenient, for each $i \ge 1$, we write

$$T_i^{(n_i)} = \begin{bmatrix} T_i & & & \\ & T_i & & \\ & & \ddots & \\ & & & T_i \end{bmatrix} \begin{matrix} \mathcal{H}_{i,1} \\ \mathcal{H}_{i,2} \\ \vdots \\ \mathcal{H}_{i,n_i} \end{matrix},$$

where $\mathcal{H}_{i,1} = \mathcal{H}_{i,2} = \cdots = \mathcal{H}_{i,n_i} = \mathcal{H}_i$.

Since T is a CSO, there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. For each $i \ge 1$, denote $\mathcal{M}_i = C(\mathcal{H}_{i,1})$. Since $T_i = T|_{\mathcal{H}_i}$ is irreducible, it follows from Lemma (3.2.7) that $T|_{\mathcal{M}_i}$ is irreducible and $T|_{\mathcal{M}_i} \cong (T_i)^t$. By Corollary(3.2.9), there exists unique $\tau_i \in \mathbb{N}$ such that $\mathcal{M}_i \subset \mathcal{H}_{i,1}^{(n_{\tau_i})}$. This defines a map τ on $\mathbb{N}$. It follows from Lemma(3. 2.11) that $T|_{\mathcal{M}_i} = T_{\mathcal{T}_i}^{(n_{\tau_i})}|_{\mathcal{M}_i} \cong T_{\tau_i}$. Then we obtain

$$T_{\tau_i} \cong T_i^t \ \forall i \ge 1. \tag{15}$$

Claim 1: For each $i \ge 1$, if $\mathcal{N} \subset \mathcal{H}_i^{(n_i)}$ is a minimal reducing subspace of T, then $C(\mathcal{N}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$.

Since $T|\mathcal{N}$ is irreducible, it follows from Lemmas (3.2.7) and 2.6 that $T|_{C(\mathcal{N})}$ is irreducible and

$$T|_{C(\mathcal{N})} \cong (T|_{(\mathcal{N})})^t = (T_i)^t. \tag{16}$$

Moreover there exists $j \in \Lambda$ such that $C(\mathcal{N}) \subset \mathcal{H}_j^{(n_j)}$. Thus $T|_{C(\mathcal{N})} \cong T_j$. In view of (15) and (16), we obtain $T_j \cong (T_i)^t \cong T_{\tau_i}$. By the hypothesis, it follows that $j = \tau_i$. This proves the claim.

Claim 2: $C(\mathcal{H}_i^{n(\tau_i)}) = \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$ for any $i \ge 1$.

Fix an $i \ge 1$and denote $j = \tau_i$. In view of (15), we have $T_{\tau_i} \cong T_i^t$ and $T_{\tau_i} \cong T_j^t$. It follows that $T_{\tau_i} \cong T_i$. By the hypothesis on the decomposition T $= \oplus_{i\ge1} T_i^{(n_i)}$, one can deduce that $i = \tau_i$. It follows immediately from Claim 1 that

$C(\mathcal{H}_i^{(n_i)}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$ and $C(\mathcal{H}_{\tau_i}^{(n_{\tau_i})}) \subset \mathcal{H}_i^{(n_i)}$.

Since $C^{-1} = C$, we have

$$C(\mathcal{H}_i^{(n_i)}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})} \subset C(\mathcal{H}_i^{(n_i)}),$$

That is, $C(\mathcal{H}_i{}^{(n_i)}) = \mathcal{H}_{\tau_i}{}^{(n_{\tau_i})}$. This proves Claim 2.

By the above argument, the map $\mathcal{T} : i \longmapsto \mathcal{T}_i$ is invertible and $\tau^{-1} = \tau$. Thus τ induces the following partition of $\mathbb{N}$

$$\{\{i, \tau_i\}: i \geq 1\},$$

denoted by $\{\Lambda_r: r \in \Gamma\}$. Then $\bigcup_{r\in\Gamma} \Lambda_r = \mathbb{N}$ and $1 \leq \text{card}\, \Lambda_r \leq 2$ for all $r \in \Gamma$. Thus T can be written as

$$T = \oplus_{r\in\Gamma}(\oplus_{i\in\Lambda_r} T_i{}^{(n_i)})$$

with respect to the decomposition

$$\mathcal{H} = \oplus_{r\in\Gamma}(\oplus_{i\in\Lambda_r} \mathcal{H}_i{}^{(n_i)}).$$

Noting that $c(\oplus_{i\in\Lambda_r} \mathcal{H}_i{}^{(n_i)}) = \oplus_{i\in\Lambda_r} \mathcal{H}_i{}^{(n_i)}$ for all $r \in \Gamma$, it follows that $\oplus_{i\in\Lambda_r} T_i{}^{(n_i)}$ is complex symmetric for all $r \in \Gamma$. So, in order to complete the proof, it suffices to prove that $\oplus_{i\in\Lambda_r} T_i{}^{(n_i)}$ admits the desired decomposition for all $r \in \Gamma$.

Claim 3: $n_i = n_{\tau_i}$ for all $i \geq 1$.

Now fix an $i \geq 1$. For each $1 \leq j \leq n_i$, denote $\mathcal{N}_j = C(\mathcal{H}_{i,j})$. Then, by Claim 2, $\oplus_{j=1}^{n_i} \mathcal{N}_j = C(\mathcal{H}_i^{(n_i)}) = \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$. Hence

$$T_{\tau_i}^{(n_{\tau_i})} = T|_{\mathcal{H}_{\tau_i}^{(n_{\tau_i})}} = \oplus_{j=1}^{n_i}(T|_{\mathcal{N}_j}) \cong T_{\tau_i}^{(n_i)}.$$

Since T_{τ_i} is irreducible, by comparing commutant algebras, one can see that $n_i = n_{\tau_i}$. This proves Claim 3.

Now we can conclude our proof. Fix an $r \in \Gamma$.

If $\text{card}\, \Lambda_r = 1$ and $k \in \Lambda_r$, then $\oplus_{i\in\Lambda_r} T_i{}^{(n_i)} = T_k{}^{(n_k)}$. Since $T_k{}^{(n_k)}$ is a CSO, by Proposition (3.2.12), it admits the desired decomposition.

If $\text{card}\, \Lambda_r = 2$ and $k \in \Lambda_r$, then $k \neq \tau_k$ and $\oplus_{i\in\Lambda_r} T_i{}^{(n_i)} = T_k{}^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})}$. In view of Claim 3 and (15), we have

$$T_k^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})} = \left(T_k \oplus T_{\tau_k}\right)^{(n_k)} \cong (T_k \oplus T_k{}^t)^{(n_k)}.$$

We claim that T_k is not complex symmetric. In fact, if not, then $T_k \cong T_k{}^t$. In view of (15), we have $T_k \cong T_{\tau_k}$. This contradicts the fact that $T_k \ncong T_{\tau_k}$ since $k \neq \tau_k$. Therefore $\oplus_{i\in\Lambda_r} T_i{}^{(n_i)}$ admits the desired decomposition. This completes the proof.

The main result is the following theorem which gives a canonical decomposition of UET operators.

Lemma (3.2.15)[146]: ([164], Lemma 3.8). An operator $T \in B(\mathcal{H})$is UET if and only if there exists an antiunitary operator D on $\mathcal{H}$ such that $TD = DT^*$.

Lemma (3.2.16)[146]: Let $T \in B(\mathcal{H})$ and $T = T_0 \oplus (\oplus_{i\in\Lambda} T_i{}^{(n_i)}g)$, where T_0 is completely reducible, T_i is irreducible and $1 \leq n_i \leq \infty$ for $i \in \Lambda$; moreover, $T_i \ncong T_j$ whenever $i, j \in \Lambda$ and $i \neq j$. If T is UET, then T_0 and $\oplus_{i\in\Lambda} T_i{}^{(n_i)}$ are both UET.

Proof. Since T is UET, it follows from Lemma (3.2.15) that there is an antiunitary operator D on $\mathcal{H}$ such that $TD = DT^*$. By Lemma (3.2.7), D maps each minimal reducing subspace of T to another. In view of Corollary (3.2.9), we obtain $D(\oplus_{i\in\Lambda} \mathcal{H}_i{}^{(n_i)} \subset \oplus_{i\in\Lambda} \mathcal{H}_i{}^{(n_i)}$.

Denote $\mathcal{N} = D(\mathcal{H}_0)$. Then, by Lemma (3.2.7), $\mathcal{N}$ reduces T and

$$T|_{\mathcal{N}} \cong (T|_{\mathcal{H}_0})^t\ (T_0)^t,$$

which implies that $T|_{\mathcal{N}}$ is completely reducible. By Proposition (3.2.8), this implies that $\mathcal{N} \subset \mathcal{H}_0$. Whence we obtain $D(\oplus_{i\in\Lambda} \mathcal{H}_i{}^{(n_i)}) = \oplus_{i\in\Lambda} \mathcal{H}_i{}^{(n_i)}$ and $D(\mathcal{H}_0) = \mathcal{H}_0$.

Set $D_1 = D|_{\mathcal{H}_0}$ and $D_2 = D|_{\mathcal{H}_0^{\perp}}$. Then D_1, D_2 are antiunitary operators and $D = D_1 \oplus D_2$. It follows from $T = DT^*D^{-1}$ that $T_0 = D_1 T_0^* D_1^{-1}$ and

$$\oplus_{i\in\Lambda} T_i^{(n_i)} = D_2(\oplus_{i\in\Lambda} T_i^{(n_i)})^* D_2^{-1}.$$

This completes the proof.

Lemma (3.2.17)[146]: Let $T = A^{(n)}$, where $T \in B(\mathcal{H})$is irreducible and $1 \le n \le \infty$ Then T is UET if and only if A is UET.

Proof. The sufficiency is obvious. We need only prove the necessity. "⇒" Assume that T is UET. Thus there exists an antiunitary operator D on $\mathcal{H}^{(n)}$such that $T = DT^*D^{-1}$. For convenience, we write

$$T = \begin{bmatrix} A & * & * & * \\ * & A & * & * \\ * & * & \ddots & * \\ * & * & * & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_n \end{matrix},$$

where $\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$.

Denote $\mathcal{N} = D(\mathcal{H}_1)$.. Since $A = T|_{\mathcal{H}_0}$ is irreducible and $T = DT^*D^{-1}$, it follows from Lemma (3.2.7) that $\mathcal{N}$ is a minimal reducing subspace of T and $T|_{\mathcal{N}} \cong (T|_{\mathcal{H}_1})^t = A^t$; whence $T|_{\mathcal{N}}$ is irreducible. By Lemma (3.2.11), this implies $T|_{\mathcal{N}} \cong A$. Therefore we obtain $A \cong A^t$, which completes the proof.

Theorem (3.2.18)[146]: Let $T \in B(\mathcal{H})$. Then T is UET if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent)

(i) completely reducible UET operators,

(ii) irreducible UET operators, and

(iii) operators of the form $A \oplus A^t$, where A is irreducible and not UET.

Proof. The sufficiency is obvious. It suffices to prove the necessity.

"⇒" By Lemma (3.2.16), we may directly assume that $T = \oplus_{i=1}^{\infty} T_i^{(n_i)}$, where $T_i \in B(\mathcal{H}_i)$ is irreducible and $1 \le n_i \le \infty$ for $i \ge 1$; moreover, $T_i \ncong T_j$ whenever $i \ne j$. Thus $\mathcal{H} = \oplus_{i\ge1}^{\infty} \mathcal{H}_i^{(n_i)}$. To be convenient, for each $i \ge 1$, we write

$$T_i^{(n_i)} = \begin{bmatrix} T_i & & & \\ & T_i & & \\ & & \ddots & \\ & & & T_i \end{bmatrix} \begin{matrix} \mathcal{H}_{i,1} \\ \mathcal{H}_{i,2} \\ \vdots \\ \mathcal{H}_{i,n_i} \end{matrix},$$

where $\mathcal{H}_{i,1} = \mathcal{H}_{i,2} = \cdots = \mathcal{H}_{i,n_i} = \mathcal{H}_i$.

Since T is UET, there exists an antiunitary operator D on $\mathcal{H}$ such that $T = DT^*D^{-1}$. For each $i \ge 1$, denote $\mathcal{M}_i = D(\mathcal{H}_{i,1})$. Since $T_i = T|_{\mathcal{H}_i}$ is irreducible, it follows from Lemma`(3.2.7) that $T|_{\mathcal{M}_i}$ is irreducible and $T|_{\mathcal{M}_i} \cong (T_i)^t$. By Corollary(3.2.9), there exists unique $\tau_i \in \Lambda$ such that $\mathcal{M}_i \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$. This defines a map τ on $\mathbb{N}$. It follows from Lemma (3.2.11) that $T|_{\mathcal{M}_i} = T_{\tau_i}^{(n_{\tau_i})}|_{\mathcal{M}_i} \cong T_{\tau_i}$.Then we obtain,for all $i \ge 1$,

$$T_{\tau_i} \cong T_i^{\,t}. \tag{17}$$

Claim 1: For each $i \ge 1$, if $\mathcal{N} \subset \mathcal{H}_i^{(n_i)}$ is a minimal reducing subspace of T,then $D(\mathcal{N}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$.

Since $T|_{\mathcal{N}}$ is irreducible, it follows from Lemmas(3.2.7) and(3.2.11) that $T|_{D(\mathcal{N})}$ is irreducible and

$$T|_{D(\mathcal{N})} \cong (T|_{\mathcal{N}})^t = (T_i)^t. \tag{18}$$

Moreover there exists $j \in \Lambda$such that $D(\mathcal{N}) \subset \mathcal{H}_{\tau_j}^{(n_{\tau_j})}$. Thus $T|_{D(\mathcal{N})} \cong T_j$. In view of (17) and (18), we obtain $T_i \cong (T_i)^t \cong T_{\tau_i}$. By the hypothesis, it follows that $j = \tau_i$. This proves the claim.

Claim 2: τ is an invertible map on $\mathbb{N}$ and $\tau^{-1} = \tau$.

Fix an $i \geq 1$ and denote $j = \tau_i$. In view of (17), we have $T_{\tau_i} \cong T_i{}^t$ and $T_{\tau_j} \cong T_j{}^t$. It follows that $T_{\tau_i} \cong T_i$. By the hypothesis on the decomposition $T = \oplus_{i\geq 1} T_i{}^{(n_i)}$, one can deduce that $i = \tau_i$. This implies that the map $\tau: k \mapsto \tau_k$ is invertible and $\tau^{-1} = \tau$. This proves Claim 2.

Thus τ induces the partition $\{\{i, \tau_i\}: i \geq 1\}$ of $\mathbb{N}$, denoted by $\{\Lambda_r: r \in \Gamma\}$. Then $\bigcup_{r \in \Gamma} \Lambda_r = \mathbb{N}$ and $1 \leq \text{card } \Lambda_r \leq 2$ for all $r \in \Gamma$. Thus T can be written as

$$T = \oplus_{r \in \Gamma} (\oplus_{i \in \Lambda_r} T_i{}^{(n_i)})$$

With respectto the decomposition $\mathcal{H} = \oplus_{r \in \Gamma} (\oplus_{i \in \Lambda_r} \mathcal{H}_i{}^{(n_i)})$.

On the other hand, it follows from Claim 1 that

$$D(\mathcal{H}_i{}^{(n_i)}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})} \; and \; D(\mathcal{H}_{\tau_i}^{(n_{\tau_i})}) \subset \mathcal{H}_i{}^{(n_i)} .$$

Thus $\mathcal{H}_r := \oplus_{i \in \Lambda_r} \mathcal{H}_i{}^{(n_i)}$ reduces D for each $r \in \Gamma$. For each $r \in \Gamma$, denote $D_r = D|_{\mathcal{H}_r}$. Then D_r is an antiunitary operator on $\mathcal{H}_r$, $r \in \Gamma$, and $D = \oplus_{r \in \Gamma} D_r$. It follows immediately that $D(\mathcal{H}_i{}^{(n_i)}) = \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$ for each $i \geq 1$; moreover, $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)}$ is UET for each $r \in \Gamma$.

In order to complete the proof, it suffices to prove that $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)}$ admits the desired decomposition for each $r \in \Gamma$.

Claim 3: $n_i = n_{\tau_i}$ for all $i \geq 1$.

Now fix an $i \geq 1$. For each $1 \leq j \leq n_i$, denote $\mathcal{N}_i = D(\mathcal{H}_{i,j})$. Then $\oplus_{j=1}^{n_i} \mathcal{N}_i = D(\mathcal{H}_i{}^{(n_i)}) = \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$. Hence

$$\mathcal{H}_{\tau_i}^{(n_{\tau_i})} = T|_{\mathcal{H}_{\tau_i}^{(n_{\tau_i})}} = \oplus_{j=1}^{n_i} (T|_{\mathcal{N}_i}) \cong T_{\tau_i}^{(n_i)}.$$

Since T_{τ_i} is irreducible, by comparing commutant algebras, one can see that $n_i = n_{\tau_i}$. This proves Claim 3.

Now we can conclude our proof. Fix an $r \in \Gamma$.

If card $\Lambda_r = 1$ and $k \in \Lambda_r$, then $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)} = T_i{}^{(n_k)}$. Since $T_k{}^{(n_k)}$ is UET, by Lemma (3.2.17), T_k is UET. Thus $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)}$ admits the desired decomposition.

If card $\Lambda_r = 2$ and $k \in \Lambda_r$, then $k \neq \mathcal{T}_k$ and $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)} = T_k{}^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})}$. In view of Claim 3 and (17), we have

$$T_k{}^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})} = (T_k \oplus T_{\mathcal{T}_k})^{(n_k)} \cong (T_k \oplus T_k{}^t)^{(n_k)}.$$

We claim that T_k is not UET. In fact, if not, then $T_k \cong T_k{}^t$. In view of (17), we have $T_k \cong T_k{}^t$. This contradicts the fact that $T_k \ncong T_{\mathcal{T}_k}$ since $k \neq \mathcal{T}_k$. Therefore $\oplus_{i \in \Lambda_r} T_i{}^{(n_i)}$ admits the desired decomposition. This completes the proof.

We shall give several examples of completely reducible CSOs, irreducible CSOs and irreducible UET operators which are not complex symmetric.

Example (3.2.19)[146]: Let $\mathcal{H} = L^2([0,1], \text{dm})$ and A be the "multiplication by z" operator on $\mathcal{H}$. Then A is self-adjoint and completely reducible. Set

$$T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Then $T^2 = 0$ and it is obvious that $T|_{\mathcal{M}} = 0$ for any nonzero reducing subspace $\mathcal{M}$of T. By Theorem 2 of [128], any nilpotent operator of order 2 is complex symmetric. It follows that $T|_{\mathcal{M}}$ is complex symmetric for any nonzero reducing subspace $\mathcal{M}$of T.

Now we shall prove that T is completely reducible. For convenience, we write

$$T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Let $P \in B(\mathcal{H}^{(15)})$ be an orthogonal projection commuting with T. Assume that

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Since $\ker T = \mathcal{H}_1$, $\ker T^* = \mathcal{H}_2$ are hyperinvariant subspace of T, we obtain $P(\mathcal{H}_1) \subset \mathcal{H}_1$ and $P(\mathcal{H}_2) \subset \mathcal{H}_2$. It follows that $P_{1,2} = P_{2,1} = 0$.

Note that

$$|T| = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}, \qquad |T^*| = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}. \tag{19}$$

Since $P|T| = |T|P$ and $P|T^*| = |T^*|P$, it follows from (19) that $P_{i,i}A = AP_{i,i}$, $i = 1, 2$. On the other hand, since $PT = TP$, we obtain $P_{1,1}A = AP_{2,2}$. Hence $AP_{1,1} = AP_{2,2}$. Furthermore we obtain $P_{1,1} = P_{2,2}$. Thus we have proved that each orthogonal projection P commuting with T has the form of $Q^{(15)}$, where Q is an orthogonal projection on $\mathcal{H}$ commuting with A. Since A is completely reducible, we deduce that T is completely reducible.

The above argument also shows that T can not be written as the direct sum of operators with form $R \oplus R^t$, where R is not a CSO.

Example (3.2.20)[146]: We shall construct a completely reducible CSO T which admitsa nonzero reducing subspace M such that $T|_{\mathcal{M}}$ is not complex symmetric.

Denote $\mathcal{H} = L^2([0,1], \mathrm{dm})$. Let A be the “multiplication by z” operator on $\mathcal{H}$ and set

$$T_1 = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 2I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}, \quad T_2 = \begin{bmatrix} 0 & 2I & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}$ and I is the identity operator on $\mathcal{H}$. Define $T = T_1 \oplus T_2$.

Since $\|A\| = 1 < 2 = \|2I\|$, it follows from Proposition 5.4 of [171] that neither T_1 nor T_2 is complex symmetric. Since A is self-adjoint, there exists a conjugation C on $\mathcal{H}$ such that $CAC = A$. Define

$$D = \begin{bmatrix} 0 & 0 & C \\ 0 & C & 0 \\ C & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}.$$

Then one can verify that D is a conjugation on $\mathcal{H}^{(16)}$ and a matricial calculation shows that $DT_1^*D = T_2$. Thus T_2. is a transpose of T_1., and hence $T = T_1 \oplus T_2$ is a CSO.

Now it remains to prove that T is completely reducible. Using a similar argument as in Example (3.2.19), one can prove that T_1 and T_2 are both completely reducible. Thus it suffices to prove that there exists no nonzero $X \in B(\mathcal{H}^{(16)})$ such that $T_1X = XT_2$ and $T_1^*X = XT_2^*$. In fact, if this holds, then each projection $P \in B(\mathcal{H}^{(19)})$commuting with T can be written as $P = P_1 \oplus P_2$, where P_1, $P_2 \in B(\mathcal{H}^{(16)})$, $P_1T_1 = T_1P_1$ and $P_2T_2 = T_2P_2$. Since T_1, T_2 are both completely reducible, it follows that T is completely reducible.

Now fix an $X \in B(\mathcal{H}^{(16)})$which satisfies $T_1X = XT_2$ and $T_1^*X = XT_2^*$. Assume that

$$X = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2. \\ \mathcal{H}_3 \end{matrix}$$

By the hypothesis, we have $|T_1|X = X|T_2|$ and $|T_1^*|X = X|T_2^*|$. A direct calculation shows that $X = 0$. So T is completely reducible.

Example (3.2.21)[146]: Let $S \in B(\mathcal{H})$ be the unilateral shift defined by $Se_i = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an ONB of $\mathcal{H}$. Define $F \in B(\mathcal{H})$ as

$$Fe_1 = -e_2, \quad Fe_2 = e_1; \quad Fe_i = 0, \quad \forall i \geq 3.$$

Set

$$T_1 = \begin{bmatrix} S^* & I \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}, \quad T_2 = \begin{bmatrix} S^* & F \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and I is the identity operator on $\mathcal{H}$. Then it is easy to verify that T_1, T_2 are both irreducible. We shall show that T_1 is complex symmetric, T_2 is UET and not complex symmetric.

For $x \in \mathcal{H}$ with $x = \sum_i \alpha_i e_i$, define $Cx = \sum_i \bar{\alpha}_i e_i$. Then C is a conjugation on $\mathcal{H}$. It is easy to check that CSC $= S$ and $CS^*C = S^*$. Also the map C defined as

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

is a conjugation and one can verify that $DT_1D = T_1^*$. Hence T_1 is an irreducible CSO. Define a unitary operator on $\mathcal{H}^{(15)}$as

$$U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Then one can verify that

$$DT_2^*D = UT_2U^*.$$

This shows that T_2 is UET. It remains to show that T_2 is not a CSO.

For a proof by contradiction, we assume that E is a conjugation on $\mathcal{H}^{(15)}$ satisfying $ET_2E = T_2$. Then $E(T_2^n)E = (T_2^n)^*$ for any $n \geq 1$. Thus $E(\ker T_2^n) = \ker(T_2^n)^*$ for any $n \geq 1$. Note that

$$\ker T_2^n = \vee\left\{\begin{pmatrix} e_i \\ 0 \end{pmatrix} : i = 1,2,\cdots,n\right\} \text{ and } \ker(T_2^n)^* = \vee\left\{\begin{pmatrix} 0 \\ e_i \end{pmatrix} : i = 1,2,\cdots,n\right\}.$$

Since E is a conjugation, there exist complex numbers $\{\lambda_i\}$ with $|\lambda_i| = 1$ such That

$$E\begin{pmatrix} e_i \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} 0 \\ e_i \end{pmatrix}, \quad \forall i \geq 1.$$

Now compute to see that

$$ET_2^*\begin{pmatrix} e_1 \\ 0 \end{pmatrix} = E\begin{pmatrix} e_2 \\ e_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 e_2 \\ \lambda_2 e_2 \end{pmatrix} \text{ and } T_2E\begin{pmatrix} e_1 \\ 0 \end{pmatrix} = T_2\begin{pmatrix} 0 \\ \lambda_1 e_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 e_2 \\ \lambda_1 \ e_2 \end{pmatrix};$$

noting that $ET_2^* = T_2E$, this is absurd. This shows that T_2 is not complex symmetric. Then, by Proposition (3.2.12), $T_2^{(k)}$ is complex symmetric if and only if k is even or $k = \infty$

Now we shall give an irreducible Toeplitz operator which is complex symmetric.

Example (3.2.22)[146]: Let $\varphi(z) = z^2 + z - \bar{z} + \bar{z}^2$ and T_φ be the Toeplitz operator on the Hilbert space H^2 of the unit disk $\mathbb{D}$ induced by φ. Denote $e_n(z) = z^2$ for $n \geq 0$. Thus $\{e_n\}_{n=0}^{\infty}$ is an ONB of H^2. For $x \in H^2$ with $x = \sum_{k=0}^{\infty} \alpha_k e_k$, define

$$Cx = \sum_{k=0}^{\infty} \bar{\alpha}_k (-1)^k e_k .$$

Thus C is a conjugation on H^2. Note that

$$CT_\varphi e_0 = C(z^2+z) = z^2 - z = T_\varphi^* e_0 = T_\varphi^* C e_0 \text{ and}$$
$$CT_\varphi e_1 = C(z^3 + z^2 - 1) = -z^3 + z^2 - 1 = -T_\varphi^* e_1 = T_\varphi^* C e_1.$$

For $n \geq 2$, we have

$$\begin{aligned} CT_\varphi e_n &= C(z^{n+2}+z^{n+1}-z^{n-1}+z^{n-2}) \\ &= (-1)^n(z^{n+2}+z^{n+1}-z^{n-1}+z^{n-2}) = (-1)^n T_\varphi^* e_n = T_\varphi^* C e_n. \end{aligned}$$

Hence T_φ is a CSO. We shall prove that T_φ is an irreducible operator. In fact, we shall prove that $C^*(T_\varphi)$contains all compact operators on H^2.

A direct computation shows that the commutator $[T_\varphi^*, T_\varphi]$ equals $2(e_0 \otimes e_1 + e_1 \otimes e_0)$. By the Riesz-Dunford functional calculus, one can see that $e_0 \otimes e_0 + e_1 \otimes e_1 \in C^*(T_\varphi)$. $Denote$ $P = I - e_0 \otimes e_0 - e_1 \otimes e_1$. Again compute to see that
$[T_\varphi^*, T_\varphi] T_\varphi (I - P) = 2(e_0 \otimes e_0 - e_1 \otimes e_1)$.

Using functional calculus again, one can see $e_i \otimes e_i \in C^*(T_\varphi)$, $i = 0, 1$. Since $PT_\varphi|_{\operatorname{ran} P} \cong T_\varphi$, it follows that

$$\left[PT_\varphi^* P, PT_\varphi P\right] = 2(e_2 \otimes e_3 + e_3 \otimes e_2).$$

Thus $e_i \otimes e_i \in C^*(T_\varphi)$, $i = 2, 3$. Using a similar argument, one can check that $e_i \otimes e_i \in C^*(T_\varphi)$ for any $i \geq 0$. It readily follows that $C^*(T_\varphi)$ contains all compact operators on H^2. Hence T_φ is irreducible.

Example (3.2.22) motivates us to study the following problem.

Problem (3.2.23)[146]: Characterize complex symmetric Toeplitz operators on the Hilbert space H^2 of the unit disk.

Lemma (3.2.24)[146]: Let $\varphi(z) = \sum_{k=-n}^{n} \alpha_k z^k$. If T_φ is UET, then

$$\sum_{k=1}^{n} k(|\alpha_k|^2 - |\alpha_{-k}|^2) = 0$$

Proof. For convenience we write $T = T_\varphi$. Since T is UET, there exists an an-tiunitary operator D on H^2 such that $DTD^{-1} = T^*$. So we obtain $D[T^*, T]D^{-1} = -[T^*, T]$. It is trivial to see that

$$\operatorname{ran}[T^*, T] \subset \vee\{1, z, \ldots, z^{n-1}\}.$$

Thus the trace $\operatorname{tr}[T^*, T]$ of $[T^*, T]$ exists and

$$-\operatorname{tr}[T^*, T] = \operatorname{tr} D[T^*, T]D^{-1} = \operatorname{tr}[T^*, T]^* = \operatorname{tr}[T^*, T],$$

which implies that $\operatorname{tr}[T^*, T] = 0$. Since $\{z^n : n \geq 0\}$ is an ONB of H^2, it follows that

$$\begin{aligned} \operatorname{tr}[T^*, T] &= \sum_{k=0}^{n} \langle T^*Tz^k, z^k\rangle - \sum_{k=0}^{n} \langle TT^*z^k, z^k\rangle \\ &= \sum_{k=0}^{n} \left\|Tz^k\right\|^2 - \sum_{k=0}^{n} \left\|T^*z^k\right\| = \sum_{k=1}^{n} k(|\alpha_k|^2 - |\alpha_{-k}|^2). \end{aligned}$$

This completes the proof.

Proposition (3.2.25)[146]: Let $\varphi(z) = z^n + \alpha z^{-m}$ and T_φ be the Toeplitz operator on the Hilbert space H^2 of the unit disk $\mathbb{D}$ induced by φ, where $\alpha \in \mathbb{C}$, $n \geq m \geq 0$ and $n > 0$. Then the following are equivalent:

(i) T_φ is a CSO;

(ii) T_φ is UET;

(iii) $m = n$ and $|\alpha| = 1$;

(iv) T_φ is normal.

Proof. The implications (iv) ⇒ (i) ⇒ (ii) are obvious.

(ii) ⇒ (iii) By Lemma (3.2.24), it suffices to prove that $m = n$. For a proof by cpontradiction, we assume that $m < n$. Using Lemma (3.2.24) again, one can see $|\alpha| = \sqrt{n/m} > 1$. For convenience we write $T = T_\varphi$. Since T is UET, there exists an an-tiunitary operator D on H^2 such that $\mathrm{DTD}^{-1} = T^*$. Denote $A = [T^*, T]$. It follows that $\mathrm{DTD}^{-1} = -A$. Compute to see that

$$A = (1 - |\alpha|^2) \sum_{k=0}^{m-1} e_k \otimes e_k + \sum_{k=m}^{n-1} e_k \otimes e_k. \tag{20}$$

Here $e_k(z) = z^k, k \geq 0$. Thus $\{e_k\}$ is an ONB of H^2.

Since $\mathrm{DTD}^{-1} = -A$, one can see that

$$n - m = \dim \ker(A - I) = \dim \ker(A + I).$$

It implies that -1 is an eigenvalue of A of multiplicity $n - m$. In view of (20), we obtain $1 - |\alpha|^2 = -1$ and $m = n - m$, that is, $m = n/2$ and $|\alpha| = \sqrt{2}$.

Note that A is self-adjoint. Since $\mathrm{DTD}^{-1} = -A$, it follows that

$$D \frac{A^2 - A}{2} \mathrm{D}^{-1} = \frac{A^2 + A}{2},$$

that is, $\mathrm{DQ_1D}^{-1} = Q_2$, where $\mathrm{Q}_1 = \sum_{k=0}^{m-1} e_k \otimes e_k$ and $Q_2 = \sum_{k=m}^{n-1} e_k \otimes e_k$. Thus $\mathrm{DQ_1D}^{-1} = T^* Q_2$ and hence

$$\|TQ_1\| = \|T^* Q_2\|. \tag{21}$$

Noting that $\mathrm{T} = \sum_{k=0}^{\infty} e_{n+k} \otimes e_k + \sum_{k=0}^{\infty} \alpha e_k \otimes e_{m+k}$, one can deduce that $\|TQ_1\| = 1$ and $\|T^* Q_2\| = |\alpha|$. By (21), this implies $|\alpha| = 1$, a contradiction.

(iii) ⇒ (iv) Since $m = n$ and $|\alpha| = 1$, it is easy to see that $\varphi(\partial\mathbb{D})$ lies in a straight line. According to a classical result of Brown and Halmos [148], this implies that T_φ is normal.

Section (3.3): Approximate Unitary Equivalence

Many classical results in matrix theory deal with complex symmetric matrices (that is, $= T^t$) and skew symmetric matrices (that is, $T = -T^t$), which appear naturally in a variety of applications such as complex analysis, functional analysis, and even quantum mechanics. Garcia and Putinar [135],[136] initiated the study of complex symmetric operators, which is an infinite dimensional analogue of complex symmetric matrices. The class is surprisingly large and encompasses many important special operators such as normal operators, truncated Toeplitz operators, Hankel operators, and many integral operators. Many important results concerning complex symmetric operators have been obtained (see [131],[158],[159]–[164],[171],[172]).

Recently, there has been growing interest in skew symmetric operators (see [187],[188], [197],[198]), which are closely related to the study of complex symmetric operators.

We denote by $\mathcal{H}$ a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We let $K(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$.

Definition (3.3.1)[174]: A map C on $\mathcal{H}$ is called an antiunitary operator if C is conjugatelinear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. If, in addition, $C^{-1} = C$, then C is called a conjugation.

Definition (3.3.2)[174]: [197] An operator $T \in B(\mathcal{H})$ is said to be skew symmetric if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$. T is said to be complex symmetric if $CTC = T^*$ for some conjugation C on $\mathcal{H}$.

For convenience, we write SSO to denote the set of all skew symmetric operators on $\mathcal{H}$.

Using Lemma 1 in [135], one can see that $T \in B(\mathcal{H})$ is skew symmetric if and only if there exists an orthonormal basis (ONB for short) $\{e_n\}$ of $\mathcal{H}$ such that $\langle Te_n, e_m\rangle = -\langle Te_m, e_n\rangle$ for all m, n; that is, T admits a skew symmetric matrix representation with respect to $\{e_n\}$. Thus skew symmetric operators can be viewed as an infinite dimensional analogue of skew symmetric matrices. The most obvious examples of skew symmetric operators on finite dimensional spaces are those Jordan blocks with odd orders (see Example 1.7 in [188]).

The following lemma, which contains some elementary facts about skew symmetric operators, firstly appeared in [188].

Lemma (3.3.3)[174]: Let C be a conjugation on $\mathcal{H}$. Denote $S_C(\mathcal{H}) = \{X \in B(\mathcal{H}): CXC = -X^*\}$. Then

(i) if $A, B \in B(\mathcal{H})$, $CAC = A^*$ and $CBC = B^*$, then $[A, B] := AB - BA \in S_C(\mathcal{H})$;

(ii) if $T \in S_C(\mathcal{H})$, then $CT^{2n}C = (T^{2n})^*$ for all $n \in \mathbb{N}$;

(iii) the class $S_C(\mathcal{H})$ is norm-closed and forms a Lie algebra under the commutator bracket $[\cdot,\cdot]$;

(iv) if $T \in S_C(\mathcal{H})$, then $\sigma(T) = -\sigma(T)$.

By Lemma (3.3.3) (i), one can use complex symmetric operators to construct new skew symmetric operators. In particular, if $T \in B(\mathcal{H})$ is complex symmetric, then $T^*T - TT^*$ is always skew symmetric. By Proposition 3 in [135], all truncated Toeplitz operators are complex symmetric with respect to the same conjugation. Then it follows from Lemma (3.3.3) (i) that any commutator of two truncated Toeplitz operators is skew symmetric.

There are several motivations for the study of skew symmetric operators.

For one thing, skew symmetric operators have been extensively studied for many years in the finite dimensional setting [181], and have many applications in pure mathematics, applied mathematics and even in engineering disciplines. In particular, real skew symmetric matrices are very important in applications such as function theory [185],[186], the solution of linear quadratic optimal control problems, robust control problems, model reduction, crack following in anisotropic materials and others (see [175],[189],[190],[193]). Many important results related to canonical forms for symmetric matrices or skew symmetric matrices are obtained [181],[185]. In particular, Hua [186] proved that each skew symmetric matrix A can be written as $A = UBU^t$,where U is a unitary matrix and

$$B = \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{bmatrix} \oplus 0 \oplus \cdots \oplus 0.$$

Thus it is natural to study skew symmetric operators in infinite dimensional case and their applications.

The second motivation for the study of skew symmetric operators lies in its connections to complex symmetric operators. From Lemma (3.3.7) (i) and (ii), one may see this point. It is often difficult to determine whether a given operator is complex symmetric. By Lemma (3.3.7) (i), if T is complex symmetric, then $T^*T - TT^*$ is skew symmetric. In view of the description of skew symmetric normal operators Theorem 1.10 in [188], this provides another approach to describing complex symmetric operators. In [147], one can see such an application to Toeplitz operators. On the other hand, each operator T on $\mathcal{H}$ can be written as the sum of a complex symmetric operator and a skew symmetric operator. In fact, arbitrarily choose a conjugation C on $\mathcal{H}$ and set $A = \frac{1}{2}(T + CT^*C)$, $B = \frac{1}{2}(T - CT^*C)$. Then $CAC = A^*$, $CBC = -B^*$ and $T = A + B$. This reflects some universality of complex symmetric operators and skew symmetric operators.

Another motivation for this study lies in the connection between skew symmetric operators and anti-automorphisms of singly generated C^*-algebras. Recall that an anti-automorphism of a C^*-algebra $\mathcal{A}$ is a vector space isomorphism $\varphi: \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. An anti-automorphism or an automorphism ρ is said to be involutory if $\rho^{-1} = \rho$. Involutory anti-automorphisms play an important role in the study of the real structure of C^*-algebras [176],[194],[195]. It is not necessary that each C^*-algebra possesses an involutory anti-automorphism on it [177],[178],[191]. We explore some C^*-algebra information contained in the notion of skew symmetry. Certain connections between skew symmetric operators and anti-automorphisms of singly generated C^*-algebras are established. We shall prove that if $T \in SSO$, then $C^*(T)$ admits an involutory anti-automorphism on it (see Corollary (3.3.8)).

Recently, some interesting results concerning skew symmetric operators have been obtained [187],[188]. In particular, it is proved in Theorem 1.10 in [188] that a normal operator A is skew symmetric if and only if $A|_{(ker\ A)^\perp} \cong N \oplus (-N)$ for some normal operator N. Here $\cong$ denotes unitary equivalence. This result classifies skew symmetric normal operators up to unitary equivalence. However, there is no denying that it is difficult to classify general skew symmetric operators. Here we consider another important equivalence relation, that is, approximate unitary equivalence. Recall that two operators A, $B \in B(\mathcal{H})$ are approximately unitarily equivalent if there exists a sequence $\{U_n\}_{n=1}^\infty$ of unitary operators such that $U_n A - B U_n \to 0$ as $n \to \infty$ (see p. 57 in [180]), or equivalently, the closures of the unitary equivalence classes of A and B coincide. The notion of approximate unitary equivalence ignores some geometry of operators, and concentrates more on algebraic aspects.

In [182], Hadwin introduced operator-valued generalizations of spectrum, essential spectrum and eigenvalue to characterize approximate unitary equivalence of operators. It is proved in [183] that two operators are approximately unitarily equivalent if and only if they have the same operator-valued spectrum. The chief tool in this study is Voiculescu's theorem [196] concerning representations of separable C^*-algebras. In fact, the representation theory of C^*-algebras provides many useful tools to deal with those problems related to the approximate unitary equivalence of operators.

We explore algebraic information contained in the skew symmetry, and to classify certain skew symmetric operators up to approximate unitary equivalence. It is proved that any skew symmetric operator $T \in B(\mathcal{H})$ satisfying $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ is approximately unitarily equivalent to an operator of the form $A \oplus (-A^t)$, where A^t denotes a transpose of A (see Definition (3.3.4)).We shall begin by investigating the approximation of skew symmetric operators, and some results from the representation theory of C^*-algebras play an important role.

For $A \in B(\mathcal{H})$, we let $C^*(A)$ denote the C^*-algebra generated by A and the identity operator on $\mathcal{H}$. The notation $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence. As usual, given two representations ρ_1 and ρ_2 of a C^*-algebra, we also write $\rho_1 \cong \rho_2 (\rho_1 \cong_a \rho_2)$ to denote that ρ_1 and ρ_2 are unitarily equivalent (approximately unitarily equivalent, respectively). To state our main theorem, we have

Definition (3.3.4)[174]: Let $T \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$ is called a transpose of , if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

If $T \in B(\mathcal{H})$ is normal, then, by [135], T is complex symmetric. Thus there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. It follows that T is a transpose of itself. Likewise, one can see that each complex symmetric operator is a transpose of itself. In general, an operator has more than one transpose. Here is an example.

Example (3.3.5)[174]: Let $S \in B(\mathcal{H})$ be the unilateral shift defined as

$$Se_n = e_{n+1}, \quad \forall n \geq 1,$$

where $\{e_n\}_{n=1}^{\infty}$ is an OPN of $\mathcal{H}$. For $x \in \mathcal{H}$ with $x = \sum_{n=1}^{\infty} \alpha_n e_n$, define

$$Cx = \sum_{n=1}^{\infty} \overline{\alpha_n} e_n, \quad Dx = \sum_{n=1}^{\infty} \overline{\alpha_n} (-1)^n e_n.$$

Then one can check that C, D are two conjugations on $\mathcal{H}$. Set $A = CS^*C$ and $B = DS^*D$. Then A, B are transposes of S. Note that

$$Ae_2 = CS^*Ce_2 = CS^*e_2 = Ce_1 = e_1$$

and

$$Be_2 = DS^*De_2 = DS^*e_2 = De_1 = -e_1.$$

This shows that $A \neq B$.

As indicated in [164], any two transposes of an operator are unitarily equivalent.We often write T^t to denote a transpose of . In general, there is no ambiguity especially when we write $T \cong T^t$ or $T \cong_\alpha T$.

The proof of our main result depends heavily on connections between skew symmetric operators and anti-automorphisms of singly generated C^*-algebras. Now let us show some algebraic information contained in the skew symmetry.

Let $T \in B(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$ satisfying $CTC = -T^*$. So $CT^*C = -T$. If $p(x, y)$ is a polynomial in two free variables, then it is easy to verify that $\tilde{p}(-T, -T^*) = Cp(T^*, T)C$, where $\tilde{p}(x, y)$ is obtained from $p(x, y)$ by conjugating each coefficient. Since C is isometric, it follows that

$$\|p(T^*, T)\| = \|\tilde{p}(-T, -T^*)\|.$$

This motivates the following definition.

Definition (3.3.6)[174]: An operator $T \in B(\mathcal{H})$ is called $\mathcal{Z}$-normal if it satisfies

$$\|p(T^*, T)\| = \|\tilde{p}(-T, -T^*)\|$$

for any polynomial $p(x, y)$ in two free variables. Here $\tilde{p}(x, y)$ is obtained from $p(x, y)$ by conjugating each coefficient.

By the discussion before Definition (3.3.6), each skew symmetric operator is $\mathcal{Z}$-normal. It is easy to see that each norm limit of $\mathcal{Z}$-normal operators is still $\mathcal{Z}$-normal. So each norm limit of skew symmetric operators is $\mathcal{Z}$-normal.

We remark that this definition is essentially inspired by an observation of Garcia, Lutz and Timotin Question 1 in [157] about complex symmetric operators. The notion of skew symmetry depends on the existence of certain conjugations. While the $\mathcal{Z}$-normality is defined in terms of a norm equality. As we shall see later, this notion implies a C^*-algebra approach to skew symmetric operators.

Let $\mathcal{E}$ be a subset of $B(\mathcal{H})$. We denote by $\mathcal{E}$ the norm closure of $\mathcal{E}$. The compact closure $\overline{\mathcal{E}}^c$ of $\mathcal{E}$ is defined to be the set of all operators $A \in B(\mathcal{H})$ satisfying: for any $\varepsilon > 0$, there exists $K \in B(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A + K \in \mathcal{E}$. It is clear that $\mathcal{E} \subset \overline{\mathcal{E}}^c \subset \overline{\mathcal{E}}$ and $\overline{\mathcal{E}}^c \subset [\mathcal{E} + \mathcal{K}(\mathcal{H})]$.

Lemma (3.3.7)[174]: An operator $T \in B(\mathcal{H})$ is $\mathcal{Z}$-normal if and only if there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = -T$.

Proof. "⟹". Assume that T is $\mathcal{Z}$-normal. Then the map

$$\rho: C^*(T) \longrightarrow C^*(T)$$
$$p(T^*, T) \longmapsto \tilde{p}(-T, -T^*)$$

is isometric and densely defined. Hence ρ can be extended to a map on $C^*(T)$, which is also denoted by ρ. One can check that ρ is a conjugate automorphism of $C^*(T)$; that is, $\rho: C^*(T) \longrightarrow C^*(T)$ is an invertible conjugate-linear map, $\rho(X^*) = \rho(X)^*$ and $\rho(XY) = \rho(X)\rho(Y)$ for $X, Y \in C^*(T)$. So, if we define $\varphi(X) = \rho(X)^*$ for $X \in C^*(T)$, then φ is an anti-automorphism of $C^*(T)$ and $\varphi(T) = -T$.

"⟸". Let φ be an anti-automorphism of $C^*(T)$ satisfying $\varphi(T) = -T$. Then $\varphi(T^*) = -T^*$ and, given a polynomial $p(\cdot,\cdot)$ in two free variables, one can see

$$\varphi(p(T^*, T)) = \tilde{p}(-T, -T^*).$$

Since each anti-automorphism of $C^*(T)$ is isometric, it follows that

$$\|p(T^*, T)\| = \|\tilde{p}(-T, -T^*)\|.$$

So T is $\mathcal{Z}$-normal.

Corollary (3.3.8)[174]: If $T \in \overline{SSO}$, then $C^*(T)$ admits an involutory anti-automorphism on it.

Proof. Since $T \in \overline{SSO}$, by the discussion right after Definition (3.3.6), it follows that T is $\mathcal{Z}$-normal. By Lemma (3.3.7), there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = -T$. Then $\varphi^2(T) = \varphi(-T) = -\varphi(T) = T$. Furthermore, $\varphi^2(T^*) = \varphi^2(T)^* = T^*$. Since φ is an anti-automorphism of $C^*(T)$, it follows that φ^2 is an automorphism. Thus $\varphi^2(X) = X$ for all $X \in C^*(T)$, that is, φ is involutory.

Corollary (3.3.9)[174]: If $T \in B(\mathcal{H})$ is $\mathcal{Z}$-normal, then $\sigma(T) = -\sigma(T)$.

Proof. Since T is $\mathcal{Z}$-normal, by Lemma (3.3.7), there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = -T$. Since φ is linear, $\varphi(I) = I$ and maps invertible elements to invertible ones, one can see that $\sigma(\varphi(X)) = \sigma(X)$. Thus

$$\sigma(T) = \sigma(\varphi(T)) = \sigma(-T) = -\sigma(T).$$

Lemma (3.3.10)[174]: Let $T \in B(\mathcal{H})$. Then $T \in \overline{SSO}$ if and only if there exist conjugations $\{C_n\}_{n=1}^{\infty}$ on $\mathcal{H}$ such that $C_n T C_n + T^* \to 0$.

Proof. "⟹". Since $T \in \overline{SSO}$, there exist skew symmetric operators $\{T_n\}$ on $\mathcal{H}$ such that $T_n \to T$. Thus there exist conjugations $\{C_n\}$ on $\mathcal{H}$ such that $C_n T_n C_n = -T_n^*$ for all $n \geq 1$. It follows readily that $C_n T C_n \to -T^*$.

"⟸". Assume that $\{C_n\}$ are conjugations on $\mathcal{H}$ such that $C_n T C_n + T^* \to 0$ as $n \to \infty$. For each n, set

$$T_n = \frac{T - C_n T^* C_n}{2}.$$

Then one can check that

$$C_n T_n C_n = \frac{C_n T C_n - T^*}{2} = \frac{(C_n T^* C_n - T)^*}{2} = -T_n^*,$$

that is, $T_n \in SSO$ for $n \geq 1$. Note that $T_n \to T$. This completes the proof.

Corollary (3.3.11)[174]: If $T \in \overline{SSO}$, then $T \cong_a (-T^t)$.

Proof. Since $T \in \overline{SSO}$, by Lemma (3.3.10), there exist conjugations $\{C_n\}_{n=1}^{\infty}$ on $\mathcal{H}$ such that $C_n T C_n \to -T^*$. Arbitrarily choose a conjugation C on $\mathcal{H}$. Then $C C_n T C_n C \to -C T^* C$.

For $n \geq 1$, set $U_n = CC_n$. Then each U_n is linear, invertible and isometric. So each U_n is unitary and $U_n^{-1} = CC_n$. It follows that $T \cong_a (-CT^*C)$.

Lemma (3.3.12)[174]: If $A \in B(\mathcal{H})$ and $T = A \oplus (-A^t)$, then T is skew symmetric.

Proof. Since A^t is a transpose of A, by the definition, there exists a conjugation C on $\mathcal{H}$ such that $A^t = CA^*C$. Define

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then it is easy to see that D is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ and

$$DTD = (-A^*) \oplus CAC = -(A \oplus (-CA^*C)) = -T^*.$$

It follows that T is skew symmetric.

Lemma (3.3.13)[174]: Let $T \in B(\mathcal{H})$. If $T \cong_a (-T^t)$, then T is $\mathcal{Z}$-normal.

Proof. Since T^t is a transpose of , there exists a conjugation C on $\mathcal{H}$ such that $T^t = CT^*C$. Since $T \cong_a (-T^t)$, there exist unitary operators $\{U_n\}$ such that $U_n^*TU_n \to -CT^*C$. Thus $CU_n^*TU_nC \to -T^*$. Set $D_n = U_nC$ for $n \geq 1$. Then D_n is an antiunitary operator and $D_n^{-1} = CU_n^*$ for $n \geq 1$. If $p(\cdot,\cdot)$ is a polynomial in two free variables, then one can check that $\tilde{p}(-T, -T^*) = \lim_n D_n^{-1} p(T^*, T)D_n$. Since each D_n is isometric, it follows that

$$\|p(T^*, T)\| = \|\tilde{p}(-T, -T^*)\|.$$

Hence T is $\mathcal{Z}$-normal.

Let $T \in B(\mathcal{H})$. If dim ran $T < \infty$, the rank of T is rank T = dim ran T; otherwise, we define rank $T = \infty$.

Lemma (3.3.14)[174]: [180] Let $\mathcal{A}$ be a separable C^*-algebra, and let ρ_1 and ρ_2 be non-degenerate representations of $\mathcal{A}$ on separable Hilbert spaces. Then the following are equivalent:

(i) $\rho_1 \cong_a \rho_2$,

(ii) rank $\rho_1(X)$ = rank $\rho_2(X)$ for all $X \in \mathcal{A}$.

Theorem (3.3.15)[174]: (Main theorem) *Let* $T \in B(\mathcal{H})$ *and* assume that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then the following are equivalent:

(i) $T \in \overline{SSO}$;

(ii) $T \in \overline{SSO}^c$;

(iii) $\exists A \in SSO$ such that $T \cong_\alpha A$;

(iv) $\exists A \in B(\mathcal{H})$ such that $T \cong_\alpha A \oplus (-A^t)$;

(v) $T \cong_\alpha T \oplus (-T^t)$;

(vi) $T \cong_\alpha (-T^t)$;

(vii) T is $\mathcal{Z}$-normal;

(viii) There exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = -T$.

Proof. The implications "(v)⇒(iv)⇒(iii)" follow from Lemma (3.3.12). "(iii)⇒(ii)". By [184]), $T \cong_\alpha A$ implies that for any $\varepsilon > 0$ there exists $K \in B(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \cong A$. Hence $T + K \in SSO$. This implies that $T \in \overline{SSO}^c$.

The implication "(ii)⇒(i)" is trivial. The implications "(i)⇒(vi)" and "(vi)⇒(vii)" follow from Corollary (3.3.11) and Lemma (3.3.13) respectively. The equivalence "(vii) ⟺ (viii)" follows from Lemma (3.3.7). Now it remains to prove "(viii)⇒(v)". "(viii)⇒(v)".Assume that φ is an anti-automorphism of $C^*(T)$ such that $\varphi(T) = -T$. Arbitrarily choose a conjugation C on $\mathcal{H}$ and define

$$\rho: C^*(T) \longrightarrow B(\mathcal{H})$$
$$X \longmapsto C\varphi(X)^*C.$$

Let $X, Y \in C^*(T)$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}\rho(\alpha X+Y) &= C\varphi(\alpha X+Y)^*C = C(\alpha\varphi(X)+\varphi(Y))^*C\\ &= C(\bar{\alpha}\varphi(X)^*+\varphi(Y)^*)C = \alpha C\varphi(X)^*C + C\varphi(Y)^*C\\ &= \alpha\rho(X)+\rho(Y),\\ \rho(XY) &= C\varphi(XY)^*C = C(\varphi(Y)\varphi(X))^*C\\ &= C(\varphi(X)^*\varphi(Y)^*)C = (C\varphi(X)^*C)(C\varphi(Y)^*C)\\ &= \rho(X)\rho(Y),\end{aligned}$$

and $\rho(X^*) = C\varphi(X^*)^*C = (C\varphi(X)^*C)^* = \rho(X)^*$. Moreover, $\rho(I = I$ and $\rho(T) = -CT^*C$. It follows that ρ is a non-degenerate representation of $C^*(T)$ on $\mathcal{H}$.

Claim rank $\rho(X)$ = rank X for all $X \in C^*(T)$.

Let $X \in C^*(T)$ be fixed. Since φ is an anti-automorphism, it follows that $\|X\| = \|\varphi(X)\|$. Noting that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we obtain rank $\varphi(X)$ = rank X. Since

rank $\varphi(X)$ = rank $\varphi(X)^*$ = rank $C\varphi(X)^*C$ = rank $\rho(X)$,

one can obtain rank $\rho(X)$ = rank X. This proves the claim.

In view of Lemma (3.3.14), the above claim implies that $\rho \cong_\alpha$ id, where id is the identity representation of $C^*(T)$. Then we can choose unitary operators $\{U_n\}$ such that $\lim_n U_n^* X U_n = \rho(X)$. In particular,

$$\lim_n U_n^* T U_n = \rho(T) = -CT^*C,$$

that is, $T \cong_\alpha (-CT^*C)$. Since $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, it follows from Proposition 42.9 in [179] that $T \cong_\alpha T \oplus T$. Furthermore, we obtain $T \cong_\alpha T \oplus (-CT^*C)$. This completes the proof.

Given $T \in B(\mathcal{H})$ and a cardinal n, $1 \le n \le \aleph_0$, we let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$ and let $T^{(n)}$ denote the direct sum of n copies of T , acting on $\mathcal{H}^{(n)}$ (see Definition 6.3 in [179]). For convenience, $\mathcal{H}(\aleph_0)$ and $T(\aleph_0)$ are denoted by $\mathcal{H}^{(\infty)}$ and $T^{(\infty)}$. It is clear that an operator T is $\mathcal{Z}$-normal if and only if $T^{(\infty)}$ is $\mathcal{Z}$-normal. So the following corollary is immediate from Theorem (3.3.15).

Corollary (3.3.16)[174]: If $T \in B(\mathcal{H})$, then $T^{(\infty)}$ *is* approximately unitarily equivalent to a skew symmetric operator if and only if T is $\mathcal{Z}$-normal.

We shall give an example to show that the condition $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ in Theorem (3.3.15) is necessary. We first make some preparation.

Given $A, B \in B(\mathcal{H})$, we denote $[A, B] = AB - BA$.

Let $T \in B(\mathcal{H})$. Denote $M = \bigcap_{m,n\ge 1} \ker [T^{*m}, T^n]$. Then M and $M^\perp$ both reduce . In fact, $T|_M$ is normal and $T|_{M^\perp}$ is abnormal [184]. Recall that an operator A is said to be abnormal if A has no nonzero reducing subspace $\mathcal{N}$ such that $A|_\mathcal{N}$ is normal. We call $T|_M$ the normal part of T and $T|_{M^\perp}$ the abnormal part of , denoted by T_{nor} and T_{abnor} respectively.

Proposition (3.3.17)[174]: Let $T \in B(\mathcal{H})$. Then T is skew symmetric if and only if T_{abnor} and T_{nor} are both skew symmetric.

Proof. The sufficiency is clear. We need only prove the necessity.

Denote $M = \bigcap_{m,n\ge 1} \ker [T^{*m}, T^n]$. Assume that C is a conjugation on $\mathcal{H}$ and $CTC = -T^*$. Choose an $x \in M$. Then, for any $m, n \ge 1$, we have

$$\begin{aligned}[T^{*m}, T^n]Cx &= (T^{*m}T^n - T^nT^{*m})Cx\\ &= (-1)^{m+n}C(T^mT^{*n} - T^{*n}T^m)x\\ &= -1)^{m+n+1}C[T^{*n}, T^m]x = 0.\end{aligned}$$

Hence $C(M) \subset M$. Noting that C is a conjugation, we deduce that $C(M) = M$ and $C(M^\perp) = M^\perp$. Thus $D_1 = C|_M$ and $D_1 = C|_{M^\perp}$ are two conjugations. It follows from

$CTC = -T^*$ that $D_1 = T_{nor}D_1 = -T_{nor}^*$ and $D_2 = T_{abnor}D_2 = -T_{abnor}^*$. This completes the proof.

Proposition (3.3.18)[174]: Let $S \in B(\mathcal{H})$ be the unilateral shift defined by $Se_i = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an ONP of $\mathcal{H}$. Define $F \in B(\mathcal{H})$ as

$$Fe_1 = -e_2, \quad Fe_2 = e_1; \qquad Fe_i = 0, \qquad \forall i \geq 3.$$

Set

$$T = \begin{bmatrix} S^* & F \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Then

(i) T is irreducible, abnormal and $C^*(T)$ contains all compact operators on $\mathcal{H}^{(2)}$,

(ii) $T \cong (-T^t)$ and T is not approximately unitarily equivalent to any skew symmetric operator.

Proof. (i) Assume that P is an orthogonal projection on $\mathcal{H}^{(2)}$ such that $PT = TP$. In order to prove that T is irreducible, it suffices to prove that $P = 0$ or P is the identity operator on $\mathcal{H}^{(2)}$.

For any $n \geq 1$, we have $PT^n = T^nP$, and hence $P(ker\, T^n) \subset \ker T^n$. Note that

$$\ker T^n = \bigvee \left\{ \begin{pmatrix} e_i \\ 0 \end{pmatrix}; i = 1, 2, \ldots, n \right\}$$

and $\vee_{n\geq 1} \ker T^n = \mathcal{H}_1$, where $\vee$ denotes the closed linear span. We deduce that $P(\mathcal{H}_1) \subset \mathcal{H}_1$. Then P can be written as

$$P = \begin{bmatrix} P_1 & P_{1,2} \\ 0 & P_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Since P is self-adjoint, we deduce that $P_{1,2} = 0$. Hence each P_i is an orthogonal projection and $P_iS = SP_i$. Noting that $K(\mathcal{H}) \subset C^*(S)$, S is irreducible. Thus $P_i = 0$ or , where I is the identity operator on $\mathcal{H}$. Since $PT = TP$, we have $P_1F = FP_2$. It follows that either $P_1 = P_2 = I$ or $P_1 = P_2 = 0$. This shows that T is irreducible. Since dim $\mathcal{H} = \infty$, it follows immediately that T is abnormal.

Note that F is of finite rank and S is essentially normal. Then T is essentially normal. Since T is abnormal, we deduce that $T^*T - TT^* \in C^*(T)$ is a nonzero compact operator. Noting that T is irreducible, we conclude that $C^*(T)$ contains all compact operators on $\mathcal{H}^{(2)}$.

(ii) For $x \in \mathcal{H}$ with $x = \sum_i \alpha_i e_i$, define $Cx = \sum_i \bar{\alpha}_i (-1)^i e_i$. Then C is a conjugation on $\mathcal{H}$. Compute to see that

$$CF^*Ce_1 = -CF^*e_1 = -Ce_2 = -e_2 = Fe_1,$$
$$CF^*Ce_2 = CF^*e_2 = -Ce_1 = e_1 = Fe_2$$

and $CF^*Ce_i = (-1)^i\, CF^*e_i = 0 = F^*e_i$ for $i \geq 3$. So we obtain $CF^*C = F$. For each $i \geq 1$, we have

$$CSCe_i = (-1)^i CSe_i = (-1)^i Ce_{i+1}.$$
$$= (-1)^{2i+1} e_{i+1} = -e_{i+1} = -Se_i.$$

So $CSC = -S$. Likewise, one can check that $CS^*C = -S^*$.

It is obvious that the map D defined as

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

is a conjugation on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Furthermore, we have

$$-DT^*D = -\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{bmatrix} S & C \\ F^* & S^* \end{bmatrix} \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} CS^*C & CF^*C \\ 0 & CSC \end{bmatrix} = \begin{bmatrix} S^* & -F \\ 0 & S \end{bmatrix}.$$

Define a unitary operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ as

$$U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Then one can verify that

$$UTU^* = \begin{bmatrix} S^* & -F \\ 0 & S \end{bmatrix} = -DT^*D;$$

whence we obtain $T \cong (-T^t)$.Nowit remains to prove that T is not approximately unitarily equivalent to any skew symmetric operator.

For a proof by contradiction, we assume that A is a skew symmetric operator on $\mathcal{H}^{(2)}$ such that $T \cong_a A$. Since F is of finite rank and S is essentially normal, one can see that T, A are both essentially normal. By (i) and Proposition 4.27 in [184], we have $T = T_{abnor} \cong A_{abnor}$. Since A is skew symmetric, it follows from Proposition (3.3.17) that A_{abnor} is skew symmetric. So T is skew symmetric. Then there is a conjugation E on $\mathcal{H}^{(2)}$ satisfying $ETE = -T^*$. Then $E(T^n)E = (-1)^n(T^n)^*$ for any $n \geq 1$. Thus $E(\ker T^n) = \ker(T^n)^*$ for any $n \geq 1$. Note that

$$\ker T^n = \bigvee \left\{ \begin{pmatrix} e_i \\ 0 \end{pmatrix}; i = 1, 2, \dots, n \right\}$$

and

$$\ker (T^n)^* = \bigvee \left\{ \begin{pmatrix} 0 \\ e_i \end{pmatrix}; i = 1, 2, \dots, n \right\}.$$

Since E is a conjugation, there exist complex numbers $\{\lambda_i\}$ with $|\lambda_i| = 1$ such that

$$E \begin{pmatrix} e_i \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} 0 \\ e_i \end{pmatrix}, \quad \forall i \geq 1.$$

Now compute to see that

$$ET^* \begin{pmatrix} e_1 \\ 0 \end{pmatrix} E \begin{pmatrix} e_2 \\ e_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 e_2 \\ \lambda_2 e_2 \end{pmatrix}$$

and

$$-TE \begin{pmatrix} e_1 \\ 0 \end{pmatrix} - T \begin{pmatrix} 0 \\ \lambda_1 e_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 e_2 \\ \lambda_1 e_2 \end{pmatrix};$$

noting that $ET^* = -TE$, this is absurd. Thus we conclude the proof.

Recall that a (forward) unilateral weighted shift T on $\mathcal{H}$ with weight sequence $\{w_n\}_{n\geq 1}$ is the operator defined by $Te_n = w_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n\geq 1}$ is an ONP of $\mathcal{H}$.

we shall characterize when a unilateral weighted shift with nonzero weights is approximately unitarily equivalent to a skew symmetric operator. By Corollary 1 in [192], we need only deal with unilateral weighted shifts with positive weights. This provides nontrivial examples of $\mathcal{Z}$-normal operators.

Theorem (3.3.19)[174]: Let $T \in B(\mathcal{H})$ be a unilateral weighted shift with positive weights. Then the following are equivalent:

(i) $T \in \overline{SSO}$;

(ii) $T \in \overline{SSO}^c$;

(iii) $\exists A \in SSO$ such that $T \cong_a A$;

(iv) $\exists A \in B(H)$ such that $T \cong_a A \oplus (-A^t)$;

(v) $T \cong_a T \oplus (-T^t)$;

(vi) $T \cong_a T^*$;

(vii) T is $\mathcal{Z}$-normal.

Proof. The proofs of the implications "(v)⇒(iv)⇒(iii)⇒(ii)⇒(i)" follow the same lines as that of Theorem (3.3.15).

"(vi)⇒(v)". Since $T \cong_a T^*$, there exist unitary operators $\{U_n\}$ such that $\lim_n U_n^* UTU_n = T^*$. This induces an automorphism φ of $C^*(T)$ such that $\varphi(T) = T^*$.

We claim that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In fact, if not, then $C^*(T)$ contains some nonzero compact operators. Since T is irreducible, we have $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. Then it follows from Corollary 5.41 in [153] that φ is unitarily implemented. Thus $T \cong T^*$. Since $\dim\ker T = \dim\ker T^*$, this is a contradiction. Thus we have proved that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. By Proposition 42.9 in [179], we have $T \cong_a T \oplus T$. By the hypothesis, it follows that $T \cong_a T \oplus T^*$. Since T is a weighted shift, it follows that $T^* \cong (-T^*)$. Hence we obtain $T \cong_a T \oplus (-T^*)$.

By Lemma (3.3.12), it remains to check that T^* is a transpose of T. Since T is a unilateral weighted shift with positive weights, we may assume that $Te_i = w_i e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an ONP of $\mathcal{H}$. For $x \in \mathcal{H}$ with $x = \sum_{i\geq 1} \alpha_i e_i$, define $Cx = \sum_{i\geq 1} \overline{\alpha_i} e_i$. One can check that C is a conjugation on $\mathcal{H}$ satisfying $CT^*C = T^*$. Hence T^* is a transpose of T.

"(i)⇒(vi)". Since $T \in \overline{SSO}$, it follows from Corollary (3.3.11) that $T \cong_a (-T^*)$. It can be seen from the proof of "(vi)⇒(v)" that T^* is a transpose of T and $T^* \cong (-T^*)$. Thus we obtain $T \cong_a T^*$.

"(vi)⇒(vii)". Assume that $T \cong_a T^*$. Since T^* is a transpose of T and $T^* \cong (-T^*)$, it follows from Lemma (3.3.13) that T is $\mathcal{Z}$-normal.

"(vii)_⇒(vi)". Still, T^* is a transpose of T and $T^* \cong (-T^*)$. By Theorem (3.3.15), it suffices to prove that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Since T is $\mathcal{Z}$-normal, by Lemma (3.3.7), there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = -T$. Arbitrarily choose a conjugation C on $\mathcal{H}$ and define

$$\rho: C^*(T) \longrightarrow B(\mathcal{H})$$
$$X \longmapsto C\varphi(X)^*C.$$

Then, as one can see from the proof of "(viii)⇒(v)" in Theorem (3.3.15), the map ρ is a faithful representation of $C^*(T)$ on $\mathcal{H}$. Note that $\rho(T) = -CT^*C$ is irreducible. Hence ρ is irreducible.

Note that T is irreducible. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. It follows from Corollary 5.41 in [153] that ρ is unitarily implemented. It follows that $T \cong (-CT^*C)$. Since $\dim\ker T \neq \dim\ker T^* = \dim\ker(-CT^*C)$, this is a contradiction. Thus we conclude the proof.

Example (3.3.20)[174]: The Kakutani shift [125] is a unilateral weighted shift with weight sequence $\{w_n\}_{n=1}^{\infty}$, where

$$w_n = \frac{1}{\gcd\{n, 2^n\}}, n \geq 1.$$

Here $\gcd\{i, j\}$ denotes the greatest common divisor of i and . Now we shall check that W is an approximately Kakutani shift.

Given $\varepsilon > 0$ and $n \geq 1$, we set $m = [\frac{1}{\varepsilon}] + 1 + n$ and $N = 2^m$, where $[\frac{1}{\varepsilon}]$ is the integer part of $\frac{1}{\varepsilon}$. Thus $n < m < N$, $\frac{1}{N} < \varepsilon$ and

$$w_N = \frac{1}{\gcd\{N, 2^N\}} = \frac{1}{2^m} < \varepsilon.$$

For each $1 \leq i \leq n$, assume that $i = 2^k \cdot j$, where $k \geq 0$ and $j \geq 1$ is odd. Then $2^k < N = 2^m$ and $N - i = 2^m - 2^k \cdot j = 2^k(2^{m-k} - j)$. Since $2^{m-k} - j$ is odd, it follows that

$$w_i = \frac{1}{\gcd\{i, 2^i\}} = \frac{1}{2^k} = \frac{1}{\gcd\{N - i, 2^{N-i}\}} = w_{N-i}.$$

So W is approximately Kakutani.

Chapter 4
Strict Completely Positive Maps and Dilations of Some VH-Spaces with Representations of ∗-Semigroups

We describe the structure of the continuous strict completely positive linear maps between locally C^*-algebras. We obtain a general dilation theorem in both Kolmogorov and reproducing kernel space representations, that unifies many dilation results, in particular B. Sz.-Nagy's and Stinesprings' dilation type theorems. We point out the reproducing kernel fabric of dilation theory and we show that the general theorem unifies many dilation results at the non-topological level.

Section (4.1): Locally C^*-Algebras and Representations on Hilbert Modules

Hilbert modules over C^*-algebras generalize, in a certain sense, the notion of Hilbert space by allowing the inner product to take values in a C^*-algebra.

The notion of a Hilbert module over a unital, commutative C^*-algebra appeared by Kaplansky [201]. He used Hilbert C^*-modules to show that derivations of type I AW^*-algebras are inner. Hilbert modules over an arbitrary C^*-algebra were first considered independently by Paschke [203] and Rieffel [35]. In [203], Paschke showed that most of the basis properties of Hilbert modules over a commutative C^*-algebra are valid for Hilbert modules over an arbitrary C^*-algebra. Rieffel, in [35], used Hilbert C^*-modules for the study of induced representations of C^*-algebras. The next important step in the development of the theory of Hilbert C^*-modules was made by Kasparov [19]. Hilbert C^*-modules are also used as the technical basis for the C^*-algebraic theory of quantum groups.

Locally C^*-algebras were first systematically studied by Inoue [18], and they were also studied by Phillips [1] (under the name of pro-C^*-algebras), M. Fragoulopoulou and other people. A locally C^*-algebra is a complete topological involutive algebra whose topology is determined by a directed family of C^*-seminorms.

Hilbert modules over a locally C^*-algebra were first considered by Phillips [1]. He showed that many properties of Hilbert C^*-modules are valid for Hilbert modules over a locally C^*-algebra, such as a stabilization theorem for countably generated Hilbert modules over a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. The proofs are not always straightforward.

We obtain a version of KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction of Hilbert modules over locally C^*- algebras.

The continuous completely positive linear maps between locally C^*-algebras are investigated. In [200], Bhatt and Karia showed that a continuous unital map ρ between the locally C^*-algebras A and B is completely positive if and only if, by restriction, it defines a completely positive linear map between C^*-algebras $b(A)$ and $b(B)$ consisting of all bounded elements of A and B. We show that this result is valid for a continuous linear map $\rho: A \to B$ with the property that $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$ for some approximate unit $\{e_i\}_{i\in I}$ of A (Proposition (4.1.2)). Also we show that a continuous linear map $\rho: A \to B$ with the property that $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$ for some approximate unit $\{e_i\}_{i\in I}$ of A can be extended to a continuous completely positive linear map $\rho^+: A_1 \to B_1$, where A_1 (respectively B_1) is the unitization of A (respectively B) (Proposition (4.1.4)).

We show that all the continuous strict completely positive linear maps from a locally C^*-algebra A into $L_B(E)$, the locally C^*-algebra of all adjointable B-module morphisms on a Hilbert B-module E, have the form $a \to V^*\phi(a)V$, $a \in A$, where ϕ is a nondegenerate continuous $*$-representation of A on a Hilbert B-module F and $V \in L_B(E; F)$ (Theorem (4.1.9)).

Let A be a locally C^*-algebra and let $S(A)$ be the set of all continuous C^*- seminorms on A. The set of all bounded elements of A, $b(A) = \{a \in A; \|a\|_\infty = \sup\{p(a); p \in S(A)\} < \infty$ with the C^*-norm $\|\cdot\|_\infty$ is a C^*-algebra that is dense in A Proposition 1.11 in [1].

An approximate unit of A is an increasing net $\{e_i\}_{i \in I}$ of positive elements of A such that $p(e_i) \leq 1$ for all $i \in I$ and for all $p \in S(A)$, $p(ae_i - a) \to 0$, and $p(e_i a - a) \to 0$ for all $p \in S(A)$ and for all $a \in A$. Any locally C^*-algebra has an approximate unit Proposition 3.11 in [1].

For $p \in S(A)$, $A_p = A/\ker(p)$ is a C^*-algebra in the norm induced by p. The canonical map from A onto A_p is denoted by π_p and $\pi_p(a) = a_p$, $a \in A$. Thus, for all $p; q \in S(A)$, $p \geq q$, there is a canonical morphism π_{pq} from A_p onto A_q such that $\pi_{pq}(a_p) = a_q$, $a_p \in A_p$. Then $\{A_p, \pi_{pq}: A_p \to A_q, p \geq q, p, q \in S(A)\}$ is an inverse system of C^*-algebras, and the locally C^*-algebras A and

$$\lim_{\overleftarrow{p}} A_p$$

are isomorphic.

A multiplier on A is a pair (l, r), where $l: A \to A$ and $r: A \to A$ are morphisms of left, respectively right, A-modules such that $al(b) = r(a)b$ for all $a, b \in A$. The set $M(A)$ of all multipliers on A with the topology determined by the family of C^*- seminorms $\{\|\cdot\|_p\}_{p \in S(A)}$, where $\|(l; r)\|_p = \sup\{p(l(a)), a \in A, p(a) \leq 1\}$, $(l; r) \in M(A)$, is a locally C^*-algebra from Theorem 3.14 in [1].

We denote by $M_n(A)$ the set of all $n \times n$ matrices over A. $M_n(A)$ with the usual algebraic operations and the topology obtained by reading it as a direct sum of n^2 copies of A is a locally C^*-algebra, and moreover it can be identified with

$$\lim_{\overleftarrow{p}} M_n(A_p).$$

Thus the topology on $M_n(A)$ is determined by the family of C^*-seminorms $\{p^{(n)}\}_{p \in S(A)}$, where $p^{(n)}([a_{ij}]) = \left\|[\pi_p(a_{ij})]\right\|_{M_n(A_p)}$, $[a_{ij}] \in M_n(A)$.

Definition (4.1.1)[199]: A pre-Hilbert A-module is a complex vector space E that is also a right A-module, compatibly with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle: E \times E \to A$, which is $\mathbb{C}$-linear and A-linear in its second variable and satisfies the following relations:

(i) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$.

(ii) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$.

(iii) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\|\cdot\|_p\}_{p \in S(A)}$, where $\|\cdot\|_p = \sqrt{P(\langle \xi, \xi \rangle)}$, $\xi \in E$ [7, Definition 4.1].

Let E be a Hilbert A-module. For $p \in S(A)$, $\mathcal{E}_p = \{\xi \in E; P(\langle \xi, \xi \rangle) = 0\}$ is a closed submodule of E, and $E_p = E/\mathcal{E}_p$ is a Hilbert A_p-module with $(\xi + \mathcal{E}_p)a_p = \xi a + \mathcal{E}_p$ and $\langle \xi + \mathcal{E}_p, \eta + \mathcal{E}_p \rangle = \pi_p(\langle \xi, \xi \rangle)$ see Lemma 4.5 in [1]. The canonical map from E onto E_p is denoted by σ_p, and $\sigma_p(\xi) = \xi_p$, $\xi \in E$. Thus, for all $p, q \in S(A)$, $p \geq q$, there is a canonical morphism of vector spaces σ_{pq} from E_p onto E_p such that $\sigma_{pq}(\xi_p) = \xi_q$, $\xi_p \in E_p$. Then

$\{E_p, A_p, \sigma_p: E_p \to E_q, p \geq q,\ p, q \in S(A)\}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}(\xi_p a_p) = \sigma_{pq}(\xi_p)\, \pi_{pq}(a_p)$, $\xi_p \in E_p$, $a_p \in A_p$, $\langle \sigma_{pq}(\xi_p), \sigma_{pq}(\eta_p) \rangle = \pi_{pq}(\langle \xi_p, \eta_p \rangle)$, $\xi_p, \eta_p \in E_p$, $\sigma_{pp}(\xi_p) = \xi_p, \xi_p \in E_p$, and $\sigma_{qr} \circ \sigma_{pq} = \sigma_{pr}$ if $p \geq q \geq r$ and

$$\lim_{\overleftarrow{p}} E_p$$

is a Hilbert A-module with $(\xi_p)_{p \in S(A)}(a_p)_{p \in S(A)} = (\xi_p a_p)_{p \in S(A)}$ and

$$\langle (\xi_p)_{p \in S(A)}, (\eta_p)_{p \in S(A)} \rangle_{\lim\limits_{\leftarrow} E_p} = (\langle \xi_p, \eta_p \rangle_{E_p})_{p \in S(A)}.$$

Moreover, the Hilbert A-modules E and

$$\lim_{\overleftarrow{p}} E_p$$

are isomorphic in the sense that there is an isomorphism of right A-modules

$$\phi: E \to \lim_{\overleftarrow{p}} E_p$$

such that

$$\langle \phi(\xi), \phi(\eta) \rangle_{\lim\limits_{\leftarrow} E_p} = (\langle \xi, \eta \rangle_E)$$

for every $\xi, \eta \in E$ see Proposition 4.4 in [1].

Let E and F be Hilbert A-modules. We say that an A-module morphism $T: E \to F$ is adjointable if there is an A-module morphism $T^*: F \to E$ such that $\langle T\xi, \eta \rangle_F = \langle \xi, T^*\eta \rangle_E$ for every $\xi \in E$ and $\eta \in F$. Any adjointable A-module morphism is continuous. We denote by $L_A(E; F)$ the set of all adjointable. A-module morphisms from E into F. We write $L_A(E)$ for $L_A(E; E)$.

We consider on $L_A(E; F)$ the topology de_ned by the family of seminorms $\{\tilde{P}\}_{p \in S(A)}$, where $\tilde{P}(T) = \left\| (\pi_p)_*(T) \right\|_{L_{A_p}(E_p; E_p)}, T \in L_A(E; F)$, and $(\pi_p)_*(T)(\xi + \mathcal{E}_p) = T\xi + \mathcal{F}_p, \xi \in E$. Thus topologized, $L_A(E)$ is a locally C^*-algebra. Moreover $\{L_{A_p}(E_p; E_p), (\pi_{pq})_* : L_{A_p}(E_p; E_p) \to L_{A_q}(E_q; E_q),\ p \geq q,\ p, q \in S(A)\}$, where $(\pi_{pq})_*(T_p)(\sigma_p(\xi)) = \chi_{pq}(T_p(\sigma_p(\xi)))$, $T_p \in L_{A_p}(E_p; E_p)$, $\xi \in E$, and $\chi_{pq}, p, q \in S(A), p \geq q$, are the connecting maps of the inverse system $\{F_p\}_{p \in S(A)}$, is an inverse system of Banach spaces, and $L_A(E; F)$ is isomorphic to

$$\lim_{\overleftarrow{p}} L_{A_p}(E_p; E_p)$$

see Proposition 4.7 in [1].

The strict topology on $L_A(E)$ is defined by the family of seminorms $\{\|\cdot\|_{p,\xi}\}_{(p,\xi) \in S(A) \times E}$, where $\|T\|_{p,\xi} = \|T\xi\|_p + \|T^*\xi\|_p$, $T \in L_A(E)$.

Given a locally C^*-algebra A, A is a Hilbert A-module with $\langle a, b \rangle = a^*b,\ a, b \in A$, and the locally C^*-algebras $M(A)$ and $L_A(E)$ are isomorphic from Theorem 4.2 in [1].

Let A and B be C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert$B$-module, and let $\varphi: A \to L_B(F)$ be a completely positive linear map. The map $\langle \cdot, \cdot \rangle : (E \otimes_{\mathrm{alg}} F) \times (E \otimes_{\mathrm{alg}} F) \to B$ defined by

$$\left\langle \sum_{i=1}^{n} \xi_i \otimes \eta_i, \sum_{j=1}^{m} \xi_j \otimes \gamma_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \eta_i, \varphi(\langle \xi_i, \xi_j \rangle) \gamma_j \rangle$$

is an inner product on $E \otimes_{alg} F$ that satisfies the conditions of Definition (4.1.1) except for condition (iii). We denote by $E \otimes_\varphi F$ the Hilbert B-module obtained by completing the pre-

Hilbert B-module $E\otimes_{alg}F/\mathcal{N}_\varphi$, where $\mathcal{N}_\varphi = \{\xi \in E\otimes_{alg}F;\ \langle\xi,\xi\rangle = 0\}$, and by $\xi\otimes_\varphi\eta$ the element $\xi\otimes\eta + \mathcal{N}_\varphi$ (see, for example, [202]).

Let A and B be two locally C^*-algebras. We say that a linear map ρ from A into B is completely positive if, for all positive integers n, the linear maps $\rho^{(n)} : M_n(A) \to M_n(B)$ defined by $\rho^{(n)}([a_{ij}]) = [\rho(a_{ij})],\ [a_{ij}] \in M_n(A)$, are positive.

Proposition (4.1.2)[199]: Let $\rho: A \to B$ be a continuous linear map between locally C^*-algebras. Then the following statements are equivalent:

(i) ρ is completely positive, and, for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$.

(ii)$\rho(b(A)) \subseteq b(B)$ and $\rho|_{b(A)}: b(A) \to b(B)$ is completely positive.

Proof. (i) ⇒ (ii): Let f$\{e_i\}_{i\in I}$ be an approximate unit of A such that $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is an $M > 0$ such that $\{\rho(e_i)\}_\infty \le M$ for all $i \in I$.

To show that $\rho(b(A)) \subseteq b(B)$, it is sufficient to prove that $\rho(b(A)^+) \subseteq b(B)^+$, since an arbitrary element of $b(A)$ can be written as a linear combination of positive elements in $b(A)$.

Let $i_0 \in I$ and $a \in b(A)^+$. Since $e_{i_0}ae_{i_0} \le \|a\|_\infty e_{i_0}$,

$$q(p(e_{i_0}ae_{i_0})) \le \|a\|_\infty q(\rho(e_{i_0})) \le M\|a\|_\infty$$

for every $a \in S(B)$. Therefore $\{p(e_iae_i)\}_{i\in I}$ is a bounded net in $b(B)$, and since ρ is continuous and $\{p(e_iae_i - a) \to 0$ for every $p \in S(B)$, $\rho(a) \in b(B)$. Clearly $\rho|_{b(A)}: b(A) \to b(B)$ is completely positive.

(ii) ⇒ (i): Let $\{e_i\}_{i\in I}$ be an approximate unit of $b(A)$. Then $\{e_i\}_{i\in I}$ is an approximate unit of A and $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$ since $\|\rho(e_i)\|_\infty \le \left\|\rho|_{b(A)}\right\|$ for all $i \in I$.

Let n be a positive integer. Since $M_n(b(A)) = b(M_n(A))$ see Lemma 2.1 in [200], we have

$$\rho^{(n)}(b(M_n(A))) = \rho^{(n)}(M_n(b(A))) \subseteq M_n(b(B)) = b(M_n(B)).$$

Now, using the facts that $b(M_n(A))^+$ is dense in $M_n(A)^+$ and $\rho^{(n)}$ is continuous, we deduce that $\rho^{(n)}$ is positive. Hence ρ is completely positive.

Corollary (4.1.3)[199]: Let $\rho: A \to B$ be a continuous completely positive linear map between locally C^*-algebras such that, for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is an $M > 0$ such that

$$\rho^{(n)}([a_{jk}]^*)\rho^{(n)}([a_{ij}]) \le M\rho^{(n)}([a_{jk}]^*[a_{ij}])$$

for every $[a_{ij}] \in M_n(A)$, and consequently $[\rho(a_j^*)\rho(a_k) \le M[\rho(a_j^*a_k)$ for every $a_1, \dots, a_n \in A$.

Proposition (4.1.4)[199]: Let $\rho: A \to B$ be a continuous completely positive linear map between locally C^*-algebras such that, for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is a continuous completely positive linear map ρ^+ from A_1 into B_1 such that $\rho^+|_A = \rho$, where A_1 (respectively B_1) is the unitization of A (respectively B).

Proof. According to Proposition (4.1.2), $\rho|_{b(A)} : b(A) \to b(B)$ is a completely positive liner map between C^*-algebras. Let $q \in S(B)$. The continuity of ρ implies that there is a $K_q > 0$ and $p_q \in S(B)$ such that $q(\rho(a)) \le K_q p_q(a)$ for all $a \in A$. Hence there is a continuous linear map $\rho_q: A_{\rho_q} \to B_q$ such that $\rho_q \circ \pi_{\rho_q} = \pi_q \circ \rho$. Clearly, $\rho_q^{(n)} \circ \pi_{\rho_q}^{(n)} = \pi_q^{(n)} \circ \rho^{(n)}$ for all positive integers n, and so ρ_q is a completely positive linear map between C^*-algebras. Since $\|\rho_q\| \le \left\|\rho|_{b(A)}\right\|$, the map $\tilde{\rho}_q: (A_{\rho_q})_1 \to (B_q)_1$ defined by $\tilde{\rho}_q(a,\lambda) =$

$\rho_q(a) + \lambda\|\rho|_{b(A)}\|$ is a completely positive linear map between C^*-algebras. Then the map $\rho_q^+ : (A)_1 \to (B)_1$ defined by $\rho_q^+ = \tilde{\rho}_q \circ \pi_{p_q}^+$, where $\pi_{p_q}^+$ is the canonical map from A_1 into $(A_{\rho_q})_1$, is a continuous completely positive linear map from A_1 into $(B_q)_1$.

It is easy to verify that $\pi_{qr}^+ \circ \rho_q^+ = \rho_r^+$ for all $q, r \in S(B)$, $q \geq r$, whereπ_{qr}^+, $q, r \in S(B)$, $q \geq r$, are the connecting maps of the inverse system $\{(B_q)_1\}_{q\in S(B)}$. This implies that there is a continuous linear map ρ^+ from A_1 into B_1 such that $\pi_q^+ \circ \rho^+ = \rho_q^+$ for all $q \in S(B)$, where π_q^+ is the canonical map from $(B)_1$ into $(B_q)_1$. Evidentlyρ^+ is completely positive and $\rho^+|_A = \rho$.

Definition (4.1.5)[199]: Let A and B be locally C^*-algebras, and let E be a Hilbert B-module. A continuous $*$-representation of A on E is a continuous $*$-morphism ϕ from A into $L_B(E)$. We say that the continuous $*$-representation ϕ is nondegenerate if $\phi(A)E$ is dense in E.

Proposition (4.1.6)[199]: Let A and B be locally C^*-algebras, let E be a Hilbert B-module, and let ϕ be a continuous $*$-representation of A on E. Then the following statements are equivalent:

(i) ϕ is nondegenerate.

(ii) There is a unique unital continuous $*$-morphism $\bar{\phi}$ from $M(A)$ into $L_B(E)$ such that

(a) $\bar{\phi}|_A = \phi$;

(b) $\bar{\phi}|_X$ is strictly continuous whenever X is a bounded selfadjoint subset of $M(A)$.

(iii) For some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(E)}$, the identity map on E.

Proof. (i) $\Rightarrow$ (ii): Let $c \in M(A)$. We consider the map $\bar{\phi}(c)$ from $\phi(A)E$ into E defined by

$$\bar{\phi}(c)\left(\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right) = \sum_{j=1}^{n} \phi\,(ca_j)\xi_j.$$

Let $\{e_i\}_{i\in I}$ be an approximate unit of A, $\sum_{j=1}^{n} \phi\,(a_j)\xi_j \in \phi(A)E$, and $q \in S(B)$. Then

$$\left\|\bar{\phi}(c)\left(\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right)\right\|_q = \lim_i \left\|\sum_{j=1}^{n} \phi\,(ce_ia_j)\xi_j\right\|_q$$
$$\leq \lim_i \|\phi(ce_i)\|_q \left\|\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right\|_q$$
$$\leq p_q(c)\left\|\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right\|_q$$

for some $p_q \in S(A)$. Hence $\bar{\phi}(c)$ can be extended on E by continuity. It is easy to verify that $\bar{\phi}(c) \in L_B(E)$ and the map $\bar{\phi}$ from $M(A)$ into $L_B(E)$ is a unital continuous $*$-morphism. Evidently $\bar{\phi}|_A = \phi$.

Let X be a bounded selfadjoint subset of $M(A)$. If $\{x_i\}_{i\in I}$ is a net in X that converges strictly to $x, x \in X$, $\xi \in E$ and $q \in S(B)$, then, since $\phi(A)E$ is dense in E, for every $\epsilon > 0$, there is $\sum_{j=1}^{n} \phi\,(a_j)\xi_j \in \phi(A)E$ such that

$$\left\| \xi - \sum_{j=1}^{n} \phi\,(a_j)\xi_j \right\|_q \leq \epsilon.$$

This implies that

$$\left\|\bar{\phi}(x_i) - \bar{\phi}(x)\xi\right\|_q \leq \left\|\bar{\phi}(x_i - x)\left(\xi - \sum_{j=1}^{n} \phi\,(a_j)\xi_j\right)\right\|_q + \sum_{j=1}^{n} \left\|\phi(x_i a_j - x a_j)\xi_j\right\|_q$$

$$\leq p_q(x_i - x)\left\|\xi - \sum_{j=1}^{n} \phi\,(a_j)\xi_j\right\|_q + \sum_{j=1}^{n} p_q(x_i a_j - x a_j)\left\|\xi_j\right\|_q$$

$$\leq 2M_{p_q}\epsilon + \sup_{j=\overline{1,n}}\left\{\left\|\xi_j\right\|_q\right\}\sum_{j=1}^{n} p_q\,(x_i a_j - x a_j)$$

and

$$\left\|\bar{\phi}(x_1^*) - \bar{\phi}(x^*)\xi\right\|_q \leq 2M_{p_q}\epsilon + \sup_{j=\overline{1,n}}\left\{\left\|\xi_j\right\|_q\right\}\sum_{j=1}^{n} p_q\,(x_i^* a_j - x^* a_j),$$

where $M_{p_q} = \sup\{\, p_q(y); y \in X\}$. Hence $\{\bar{\phi}(x_i)\}_{i\in I}$ converges strictly to $\bar{\phi}(x)$.

To show that $\bar{\phi}$ is unique, let $\tilde{\phi}$ be another unital continuous $*$-morphism from $M(A)$ into $L_B(E)$ such that $\tilde{\phi}|_A = \phi$. Then, for each $c \in M(A)$ and $\sum_{j=1}^{n} \phi\,(a_j)\xi_j \in \phi(A)E$,

$$\bar{\phi}(c)\left(\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right) = \sum_{j=1}^{n} \phi\,(ca_j)\xi_j$$

$$= \sum_{j=1}^{n} \tilde{\phi}\,(ca_j)\xi_j = \tilde{\phi}(c)\left(\sum_{j=1}^{n} \phi\,(a_j)\xi_j\right),$$

whence$\bar{\phi}(c) = \tilde{\phi}(c)$, since $\phi(A)E$ is dense in E.

(ii) ⇒ (iii): Let $\{e_i\}_{i\in I}$ be an approximate unit of A. Then $\{e_i\}_{i\in I} \cup \{1_{M(A)}\}$ is a bounded selfadjoint subset of $M(A)$, and since $\{e_i\}_{i\in I}$ converges strictly to $1_{M(A)}$, $\{\phi(e_i)\}_{i\in I}$ converges strictly to $\bar{\phi}(1_{M(A)}) = 1_{L_B(E)}$.

(iii) ⇒ (i): Let $\{e_i\}_{i\in I}$ be an approximate unit of A such that $\{\phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(E)}$, and let $\xi \in E$. Then $\{\phi(e_i)\xi\}_{i\in I}$ converges to ξ. Hence $\phi(A)E$ is dense in E.

Definition (4.1.7)[199]: Let A and B be locally C^*-algebras, and let E be a Hilbert B-module. We say that a continuous completely positive linear map $\rho : A \to L_B(E)$ is strict if, for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\xi\}_{i\in I}$ is strictly Cauchy in $L_B(E)$.

Proposition (4.1.8)[199]: Let A and B be locally C^*-algebras, let E and F be Hilbert B-modules, let $\phi : A \to L_B(E)$ be a nondegenerate continuous $*$-representation of A on F, and let V be an element in $L_B(E; F)$. Then the map $\rho : A \to L_B(E)$ de_ned by

$$\rho(a) = V^*\phi(a)V, \qquad a \in A,$$

is a continuous strict completely positive linear map.

Theorem (4.1.9)[199]: Let A and B be two locally C^*-algebras, let E be a Hilbert B-module, and let $\rho : A \to L_B(E)$ be a continuous strict completely positive linear map.

(i) Then there is a Hilbert B-module E_ρ, a continuous $*$-representation of A on E_ρ, $\phi_\rho : A \to L_B(E_\rho)$, and an element V_ρ in $L_B(E; E_\rho)$ such that

(a) $\rho(a) = V_\rho^*\phi_\rho(a)V_\rho$ for every $a \in A$;

(b) $\phi_\rho(A)V_\rho E$ is dense in E_ρ.

(ii) If F is a Hilbert B-module, $\phi : A \to L_B(F)$ is a continuous $*$-representation of A on F, and W is an element in $L_B(E;\ F)$ such that

(a) $\rho(a) = W^*\ \phi(a)W$ for every $a \in A$;

(b) $\phi(a)WE$ is dense in F;

then there is a unitary operator U in $L_B(E_\rho; F)$ such that

$\phi(a) = U\phi_\rho(a)U^*$, for every $a \in A$,

and $W = UV_\rho$.

The triple $(E_\rho, \phi_\rho, V_\rho)$ constructed in Theorem (4.1.9) is called the KSGNS construction associated with the continuous strict completely positive linear map ρ.

Proof. First we suppose that B is a C^*-algebra.

(i) The continuity of ρ implies that there is a $p \in S(A)$ and $M > 0$ such that $\|\rho(a)\| \leq Mp(a)$ for all $a \in A$ and so there is a linear map $\rho_p : A_p \to L_B(E)$ such that $\rho_p \circ \pi_p = \rho$. Clearly ρ_p is a strict completely positive linear map between C^*-algebras. Let $(E_\rho, \phi_{\rho_p}, V_\rho)$ be the ordinary KSGNS construction associated with ρ_p see Theorem 5.6 in [202]. Moreover, we know the following:

(i) $E_p = A_p \otimes_{\rho_p} E$ (up to unitary equivalence).

(ii) $\phi_{\rho_p}(a_p)(c_p \otimes_{\rho_p} \xi) = a_p c_p \otimes_{\rho_p} \xi$, $a_p c_p \in A_p$, $\xi \in E$.

(iii) $V_\rho \xi = \lim_i (e_i \otimes_{\rho_p} \xi)$, where $\{e_i\}_{i\in I}$ is an approximate unit of A_p.

We consider the map $\phi_p : A_p \to L_B(E_\rho)$ defined by $\phi_\rho(a) = (\phi_{\rho_p} \circ \pi_p)(a)$, $a \in A$. It is easy to see that ϕ_ρ is a continuous $*$-representation of A on E_ρ, $\rho(a) = V_\rho^*\phi_\rho(a)\, V_\rho$ for all $a \in A$, and $\phi_\rho(A)\, V_\rho E$ is dense in E_ρ.

(ii) The continuity of ϕ implies that there is an $r \in S(A)$ and $K > 0$ such that $\|\phi(a)\| \leq Kr(a)$ for all $a \in A$, and since $S(A)$ is directed, we can suppose that $r \geq p$. Hence there is a map $\phi_r : A_r \to L_B(F)$ such that $\phi_r \circ \pi_r = \phi$. Evidently$\phi_r$ is a $*$-representation of A_r on F, $(\rho_p \circ \pi_{rp})(a_r) = W^*\phi_r(a_r)W$ for all $a_r \in A_r$, and $\phi_r(A_r)WE$ is dense in F.

On the other hand, $(E_\rho, \phi_{\rho_p} \circ \pi_{rp}, V_\rho)$ is the ordinary KSGNS construction associated with the strict completely positive linear map $\rho_p \circ \pi_{rp}$, since $(\rho_p \circ \pi_{rp})(a_r) = V_\rho^*(\phi_{\rho_p} \circ \pi_{rp})(a_r)V_\rho$ for all $a_r \in A_r$, and $(\phi_{\rho_p} \circ \pi_{rp})(A_r)V_\rho E = \phi_{\rho_p}(A_\rho)V_\rho E$ is dense in E_ρ.

Then, according to the ordinary KSGNS construction, there is a unitary operator U in $L_B(E_\rho, F)$ such that $UV_\rho = W$ and $\phi_r(a_r) = U(\phi_{\rho_p} \circ \pi_{rp})(a_r)U^*$ for all $a_r \in A_r$. Therefore we have found a unitary operator U in $L_B(E_\rho, F)$ such that $UV_\rho = W$ and $\phi(a) =$

$U\phi_\rho(a)U^*$ for all $a \in A$, and thus, in the particular case when B is a C^*-algebra, the theorem is proved.

Now we suppose that B is an arbitrary locally C^*-algebra.

(i) Let $q \in S(A)$. Since ρ is continuous, there is a $p_q \in S(A)$ and $M_q > 0$ such that $\tilde{q}(\rho(a)) \leq M_q p_q(a)$ for all $a \in A$, and so there is a linear map $\rho_q : A_{p_q} \to L_{B_q}(E_q)$ such that $\rho_q \circ \pi_{p_q} = (\pi_q)_* \circ \rho$. Evidently $\rho_q \circ \pi_{p_q}$ is a continuous strict completely positive linear map from A into $L_{B_q}(E_q)$. Let $(E_{\rho_q}, \phi_{\rho_q}, V_{\rho_q})$ be the KSGNS construction associated with $\rho_q \circ \pi_{p_q}$ according to the first half of this proof.

Let $q, r \in S(B)$, $q \geq r$. We may suppose that $p_q \geq p_r$, since $\tilde{r}(\rho(a)) \leq \tilde{q}(\rho(a)) \leq M_q p_q(a)$ for all $a \in A$. We consider the linear map $\tilde{\psi}_{q_r} : A_{p_q} \otimes_{alg} E_q \to A_{p_r} \otimes_{alg} E_r$ defined by

$$\tilde{\psi}_{q_r}\left(a_{p_q p_r} \otimes \xi_q\right) = \pi_{q_q}(a_{p_q}) \otimes \sigma_{q_r}(\xi_q).$$

For $a_{p_q}, c_{p_q} \in A_{p_q}$ and $\xi_q, \eta_q \in E_q$, we have

$$\begin{aligned} \rho_r(\pi_{p_q p_r}(a_{p_q}^* c_{p_q}))\, \sigma_{q_r}(\eta_q) &= (\pi_r)_*(\rho(a^* c))\sigma_{q_r}(\eta_q) \\ &= \sigma_{q_r}((\pi_q)_*(\rho(a^* c))\eta_q) \\ &= \sigma_{q_r}(\rho_q(a_{p_q}^* c_{p_q})\eta_q) \end{aligned}$$

and so

$$\langle \tilde{\psi}_{q_r}\left(a_{p_q} \otimes \xi_q\right), \tilde{\psi}_{q_r}\left(c_{p_q} \otimes \eta_q\right)\rangle = \pi_{q_r}\left(\langle a_{p_q} \otimes \xi_q, c_{p_q} \otimes \eta_q\rangle\right).$$

Therefore $\tilde{\psi}_{q_r}$ defines a linear map from $A_{p_q} \otimes_{alg} E_q / \mathcal{N}_{\rho_q}$ into $A_{p_r} \otimes_{alg} E_r / \mathcal{N}_{\rho_r}$ that may be extended to a linear map $\psi_{q_r} : E_{\rho_q} \to E_{\rho_r}$. It is easy to see that $\{E_{\rho_q}, B_q, \psi_{q_r} : E_{\rho_q} \to E_{\rho_r}, q \geq r, q, r \in S(B)\}$ is an inverse system of Hilbert C^*-modules. We denote by E_ρ the Hilbert B-module

$$\lim_{\leftarrow q} E_{\rho_q}.$$

We want to show that $L_B(E, E_\rho)$ (respectively $L_B(E_\rho)$) is isomorphic to

$$\lim_{\leftarrow q} L_{B_q}(E_q, E_{\rho_q})$$

(respectively

$$\lim_{\leftarrow q} L_{B_q}(E_{\rho_q})).$$

For this, according to Proposition 4.4, Proposition 4.7 in [1], it is sufficient to show that, for all $q, r \in S(B)$, $q \geq r$, the Hilbert B_r-modules $E_{\rho_q} \otimes_{\pi_{q_r}} B_r$ and E_{ρ_r} are isomorphic. Since

$$\begin{aligned} \langle \xi_{\rho_q} \otimes B_r, \xi_{\rho_q} \otimes B_r\rangle &= b_r^* \pi_{q_r}(\langle \xi_{\rho_q}, \xi_{\rho_q}\rangle) b_r \\ &= \langle \psi_{q_r}(\xi_{\rho_q}) b_r, \psi_{q_r}(\xi_{\rho_q}) b_r\rangle \end{aligned}$$

for all $\xi_{\rho_q} \in E_{\rho_q}$ and for all $b_r \in B_r$, we can consider the linear map $\widetilde{U} : E_{\rho_q} \otimes_{\pi_{q_r}} B_r \to E_{\rho_r}$ defined by

$$\widetilde{U}\left(\xi_{\rho_q} \otimes_{\pi_{2r}} b_r\right) = \psi_{q_r}(\xi_{\rho_q}) b_r.$$

Evidently $\widetilde{U}$ is an isometric B_r-linear map, and since $\psi_{q_r}(E_{\rho_q})$ is dense in E_{ρ_r}, $\widetilde{U}$ is unitary see Theorem 3.5 in [202]. Therefore the Hilbert B_r-modules $E_{\rho_q} \otimes_{\pi_{q_r}} B_r$ and E_{ρ_r} are isomorphic.

It is easy to verify that $\psi_{q_r} \circ V_{\rho_q} = V_{\rho_r} \circ \sigma_{q_r}$ for all $q, r \in S(B)$, $q \geq r$, and consequently

$$(V_{\rho_q})_{q\in S(B)} \in \lim_{\overleftarrow{q}} L_{B_q}(E_q, E_{\rho_q}).$$

For $a \in A$, it is easy to see that $\psi_{q_r} \circ \phi_{\rho_q}(a) = \phi_{\rho_r}(a) \circ \psi_{q_r}$ for all $q, r \in S(B)$, $q \geq r$, and consequently

$$(\phi_{\rho_q}(a))_{q\in S(B)} \in \lim_{\overleftarrow{q}} L_{B_q}(E_{\rho_q}).$$

Define

$$\phi_\rho : A \to \lim_{\overleftarrow{q}} L_{B_q}(E_{\rho_q}).$$

by

$$\phi_\rho(a) = (\phi_{\rho_q}(a))_{q\in S(B)}, \qquad a \in A.$$

Since, for each $q \in S(B)$, ϕ_{ρ_q} is a continuous $*$-representation of A on E_{ρ_q}, ϕ_ρ is a continuous $*$-representation of A on E_ρ. Clearly $\rho(a) = V_\rho^* \phi_\rho(a) V_\rho$ for all $a \in A$, where $V_\rho = (V_{\rho_q})_{q\in S(B)}$. As we know that, for each $q \in S(B)$, $\phi_{\rho_q}(A)V_{\rho_q}E_q$ is dense in E_{ρ_q}, we have

$$\begin{aligned}\overline{\phi_\rho(A)V_\rho E} &= \lim_{\overleftarrow{q}} \overline{(\pi_q)_*(\phi_\rho(A)V_\rho)E_q} \\ &= \lim_{\overleftarrow{q}} \overline{\phi_{\rho_q}(A)V_{\rho_q}E_q} = \lim_{\overleftarrow{q}} E_{\rho_q} = E_\rho,\end{aligned}$$

and so $\phi_\rho(A)V_\rho E$ is dense in E_ρ.

(ii) Let $q \in S(B)$. We have the following:

(i) $(\pi_q)_* \circ \rho$ is a continuous strict completely positive linear map from A into $L_{B_q}(E_q)$.

(ii) $(\pi_q)_* \circ \phi$ is a continuous $*$-representation of A of F_q such that

$$((\pi_q)_* \circ \rho)(a) = W_q^*((\pi_q)_* \circ \phi)(a)W_q, \qquad a \in A,$$

and

$((\pi_q)_* \circ \phi)(A)W_qE_q$ is dense in F_q.

Then, according to the first part of the proof, there is a unitary operator U_q in $L_{B_q}(E_{\rho_q}, E_q)$ such that $U_qV_{\rho_q} = W_q$ and $((\pi_q)_* \circ \phi)(a)U_q = U_q\phi_{\rho_q}(a)$ for all $a \in A$.

To show that $(U_q)_{q\in S(B)}$ is an element in $L_B(E_\rho, F)$, it is su_cient to show that $\chi_{q_r} \circ U_q = U_r \circ \psi_{q_r}$ for all $q, r \in S(B)$, $q \geq r$, where χ_{q_r}, $q, r \in S(B)$, $q \geq r$, are the connecting maps of the inverse system $(F_q)_{q\in S(B)}$. Let $q, r \in S(B)$, $q \geq r$. Since $\phi_{\rho_q}(A)V_{\rho_q}E_q$ is dense in E_{ρ_q} and

$$(\chi_{q_r} \circ U_q)(\phi_{\rho_q}(a)V_{\rho_q}\xi_q) = (U_r \circ \psi_{q_r})(\phi_{\rho_q}(a)V_{\rho_q}\xi_q)$$

for all $a \in A$ and for all $\xi_q \in E_q$, we have $\chi_{q_r} \circ U_q = U_r \circ \psi_{q_r}$. Therefore $(U_q)_{q\in S(B)} \in L_B(E_\rho, F)$.

Let $U = (U_q)_{q\in S(B)}$. Then evidently U is a unitary operator in $L_B(E_\rho, F)$, $UV_\rho = W$ and $\phi(a) = U\phi_\rho(a)U^*$ for all $a \in A$.

If $\rho : A \to L_B(E)$ is a continuous completely positive linear map that is not strict but $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(L_B(E))$ for some approximate unit $\{e_i\}_{i\in I}$ of A, then we can find a continuous $*$-representation of A on a Hilbert B-module E_ρ, $\phi_\rho : A \to L_B(E_\rho)$, and an element V_ρ in $L_B(E, E_\rho)$ such that $\rho(a) = V_\rho^* \phi_\rho(a) V_\rho$ for every $a \in A$. In this case, the $*$-representation ϕ_ρ is not nondegenerate.

Corollary (4.1.10)[199]: Let A and B be two locally C^*-algebras, let E be a Hilbert B-module, and let $\rho : A \to L_B(E)$ be a continuous completely positive linear map such that, for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(L_B(E))$. Then there is a Hilbert B-module E_ρ, a continuous $*$-representation of A on E_ρ, $\phi_\rho : A \to L_B(E_\rho)$, and an element V_ρ in $L_B(E, E_\rho)$ such that

$\rho(a) = V_\rho^* \phi_\rho(a) V_\rho, \quad$ for every $a \in A$.

Proof. According to Proposition (4.1.4), there is a continuous completely positive linear map ρ^+ from A_1 into $L_B(E)$ such that $\rho^+|_A = \rho$. Then, according to Theorem (4.1.9), there is a Hilbert B-module E_ρ, a continuous $*$- representation of A_1 on E_ρ, $\phi_{\rho^+} : A_1 \to L_B(E_\rho)$, and an element V_ρ in $L_B(E, E_\rho)$ such that

$\rho^+(a) = V_\rho^* \phi_{\rho^+}(a) V_\rho, \quad$ for every $a \in A_1$.

Let $\phi_\rho = \phi_{\rho^+}|_A$. Then ϕ_ρ is a continuous $*$-representation of A on E_ρ and $\rho(a) = V_\rho^* \phi_\rho(a) V_\rho$ for every $a \in A$.

Corollary (4.1.11)[199]: Let A and B be two locally C^*-algebras, let E be a Hilbert B-module, and let $\rho : A \to L_B(E)$. Then the following statements are equivalent:

(i) ρ is a continuous strict completely positive linear map.

(ii) There is a unique continuous completely positive linear map $\bar{\rho} : M(A) \to L_B(E)$ such that

(a) $\bar{\rho}|_A = \rho$;

(b) $\bar{\rho}|_X$ is strictly continuous whenever X is a bounded selfadjoint subset of $M(A)$.

Proof. (i) ⇒ (ii): Let $(E_\rho, \phi_\rho, V_\rho)$ be the KSGNS construction associated with ρ. Since ϕ_ρ is nondegenerate, there is a unique continuous $*$-representation $\overline{\phi_\rho} : M(A) \to L_B(E_\rho)$ such that $\overline{\phi_\rho}|_A = \phi_\rho$ and $\overline{\phi_\rho}|_X$ is strictly continuous whenever X is a bounded selfadjoint subset of $M(A)$. Evidently the map $\bar{\rho} : M(A) \to L_B(E)$ defined by $\bar{\rho}(a) = V_\rho^* \overline{\phi_\rho}(a) V_\rho$ is a continuous completely positive linear map that satisfies conditions (a) and (b).

To show the uniqueness $\bar{\rho}$, let $\tilde{\rho} : M(A) \to L_B(E)$ be another continuous completely positive linear map that satisfies conditions (a) and (b). Let $\{e_i\}_{i\in I}$ be an approximate unit of A, and let a be a selfadjoint element in $M(A)$. Then, since $\{e_i a e_i\}_{i\in I}$ is a bounded selfadjoint net in A and it converges strictly to a, $\bar{\rho}(a) = \tilde{\rho}(a)$. Therefore $\bar{\rho} = \tilde{\rho}$.

(ii) ⇒ (i): From condition (a), it follows that ρ is a continuous completely positive linear map. If $\{e_i\}_{i\in I}$ is an approximate unit of A, then, from condition (b), $\{\rho(e_i)\}_{i\in I}$ converges strictly to $\bar{\rho}(1_{M(A)})$.

Section (4.2): Operator Valued Invariant Kernels

The dilation theorem of Sz-Nagy [225], which generalizes a dilation theorem for groups of Naimark [221], says that any operator valued positive semidefinite map on a $*$-semigroup can be dilated to a $*$-representation of the $*$-semigroup on a “larger” Hilbert space. A generalization to VH-spaces (Vector Hilbert spaces) operator valued maps, motivated by questions in multivariable stochastic processes, was obtained by Loynes [214]. A slightly stronger version of this generalization was obtained in [212].

The Stinespring's Theorem [223], which generalizes another dilation theorem, for semispectral measures, of Naimark [220], says that, for the case of a Hilbert space $\mathcal{H}$ and a C^*-algebra $\mathcal{A}$, any positive semidefinite map $\varphi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ can be dilated to a $*$-homomorphism π of $\mathcal{A}$ on $\mathcal{B}(\mathcal{K})$, for Some “larger” Hilbert space $\mathcal{K}$. A result of

Szafraniec [224] says that Stinespring's Theorem is logically equivalent with the Sz-Nagy Dilation Theorem. An enhanced version of Stinespring's Theorem, e.g. see [207], states that Stinespring's Theorem is also true when $\mathcal{A}$ is a B^*-algebra. It was proven in [212] that a variant of this equivalence can be put in the framework of VH-spaces, that is, the theorem of Szafraniec extends to the setting of VH-spaces as well.

We show that these two dilation theorems, even at the level of generality of VH-spaces operator valued maps, can be unified under the same concept, those of positive semidefinite kernels that are invariant under the action of a $*$-semigroup. We prove in Theorem (4.2.10) that these kernels can be equivalently characterized by a "linearization" of the kernel, that is called a Kolmogorov decomposition, together with a $*$-representation of the $*$-semigroup onto a "larger" VH-space, and we also show that, even more, this is equivalent with a VH-space reproducing kernel onto which the $*$-representation holds

Positive semidefinite kernels have been first considered mainly with respect to reproducing kernel Hilbert spaces, see Aronszajn [205]. An equivalent description can be obtained by Kolmogorov decomposition, which is a linearization (or separation of variables) of the kernel, named this way after Kolmogorov's seminal [68]. For scalar valued kernels, this linearization was first obtained by J. Mercer back in 1909, cf. [218], in connection with the theory of integral equations as developed by D. Hilbert, while the reproducing kernel aspects have been systematically considered by Moore [219]. Basically, a Kolmogorov decomposition is a dilation phenomenon that is strongly related with many other problems in operator algebras and mathematical physics, see Parthasarathy and Schmidt [75], Evans and Lewis [63]. When the kernel presents a certain symmetry, that can be modeled, e.g. by an invariance with respect to an action of a $*$-semigroup (any group can be organized in a natural way as a $*$-semigroup with the involution defined by the inverse operation), this turns out to be a powerful method of producing representations of the underlying $*$-semigroup on the Hilbert space of dilation. A consequence is that this unifies both Sz-Nagy's type dilations and Stinespring's type dilations. These ideas have been used in investigating dilations for indefinite Hermitian kernels in [58],[53].

Motivated by questions in operator algebras and mathematical physics, some generalizations of Hilbert spaces to the case when the inner product takes vector values have been investigated: we mention here Hilbert modules, see [202],[217], notably a generalization to Hilbert C^*-module operator valued maps of the Stinespring's Theorem obtained by Kasparov [216], Hilbert modules over locally C^*-algebras, cf. Inoue [215] and Phillips [1], as well as a different type generalization, that was performed by R. M. Loynes, notably his generalization of the Sz-Nagy's dilation theorem as in [214], and followed by a study of operators on these spaces, as in [215]. The latter vector valued Hilbert spaces, that have been acronymed by VH-spaces, show many common features with Hilbert spaces but there are many anomalies as well, the most notable ones due to missing a Schwarz Inequality and an analog of the Riesz's Representation Theorem. Motivation for studying these VH-spaces and their linear operators originally came mainly from multi-variable stochastic processes, as explained in [216], see also [209] and the rich bibliography cited there for applications of this theory and for an update review of these applications.

It is worth noting that VH-spaces are so general that they contain Hilbert modules over either C^* or locally C^*-algebras. From this perspective, more recently, Murphy [50] considered Kolmogorov decompositions in connection with Hilbert C^*-modules, Gaspar and Gaspar studied reproducing kernel Hilbert $B(X)$-modules in [211] and reproducing kernel Hilbert modules over locally C^*-algebras in [210], while Heo [213] investigated reproducing kernel Hilbert C^*-modules and kernels associated with cocycles.

We start with a brief presentation of notation and basic facts on VH-spaces, their linear operators and the C^*-algebra of adjointable operators, to which we add an inequality related to tensor products of Hilbert spaces with VH-spaces, as a technical result needed later. Then we consider Hermitian kernels that take values in $\mathcal{B}^*(\mathcal{H})$, the C^*- algebra of adjointable operators on a VH-space $\mathcal{H}$ and investigate different levels of positivity and their consequences. There are two main results here: one is Theorem (4.2.5) that shows that Kolmogorov decompositions characterize positive semidefinte kernels and the second is Theorem (4.2.7) that adds the characterization by reproducing kernel VH-spaces. There are advantages and disadvantages for each one of these: Kolmogorov decomposition gives much more freedom in dealing with it, while its reproducing kernel counter-part has a "function space" look and enjoys uniqueness. In view of experience with applications of the operator valued kernels to moment problems, dilations theory, and multi-variable holomorphy as in [208], we think that having both of them available is an advantage on the flexibility side, which offers a choice depending on the particular problem that requires this model.

The main result is Theorem (4.2.10) that shows that, when the kernel is invariant under the action of a certain $*$-semigroup, then the Kolmogorov decomposition, as well as its underlying reproducing kernel VH-space, yields a $*$-representation of the $*$-semigroup on the VH-space of dilation, that can be viewed also on the underlying reproducing kernel VH-space. Then we show that the Loynes-Sz.-Nagy dilation type theorem, see Theorem (4.2.12) obtained in [212], is a particular case of Theorem (4.2.10) to which we add an equivalent characterization in terms of reproducing kernels. In addition, we transfer Kolmogorov decompositions to linearizations of positive semidefinite maps on $*$-semigroups. Finally, we show that the Stinespring's type theorem for VH-spaces operator valued completely positive maps on B^*-algebras obtained in [212], see Theorem (4.2.15), can be obtained from Theorem (4.2.10) as well.

We review most of the definitions and some theorems on VH-spaces and their operator theory, cf. Loynes [214]-[216]. A few facts are added in connection to taking tensor products, in order to obtain a technical result that can be considered as a surrogate of a multivariable Schwarz inequality.

A complex vector space Z is called admissible if:

(a1) Z is a complete locally convex space.

(a2) Z has an involution $*$, that is, a map $Z \ni z \mapsto z^* \in Z$ that is conjugate linear ($(\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^*$ for all $\alpha, \beta \in \mathbb{C}$ and all $x, y \in Z$) and involutive ($(z^*)^* = z$ for all $z \in Z$).

(a3) In Z there is a convex cone Z_+($\alpha x + \beta y \in Z_+$ for all numbers $\alpha, \beta \geq 0$ and all $x, y \in Z_+$), that is closed, strict ($Z_+ \cap -Z_+ = \{0\}$), and consisting of selfadjoint elements only ($z^* = z$ for all $z \in Z_+$). This cone is used to define a partial order in Z by: $z_1 \geq z_2$ if $z_1 - z_2 \in Z_+$.

(a4) The topology of Z is compatible with the partial ordering in the sense there exists a base of the topology, linearly generated by a family of neighbourhoods $\{N_j\}_{j\in\mathcal{J}}$ of the origin, such that all of them are convex and solid, that is, whenever $x \in N_j$ and $0 \leq \mathcal{Y} \leq x$ then $\mathcal{Y} \in N_j$.

It can be proven that axiom (a4) is equivalent with the following one:

(a4') There exists a collection of seminorms $\{p_j\}_{j\in\mathcal{J}}$ defining the topology of Z that are increasing, that is, $0 \leq x \leq \mathcal{Y}$ implies $p_j(x) \leq p_j(\mathcal{Y})$.
If, in addition, to the axioms (a1)−(a4), the space Z satisfies also the following:
(a5) With respect to the specified partial ordering, any bounded monotone sequence is convergent.

Then Z is called a strongly admissible space.

Given a complex linear space $\mathcal{E}$ and an admissible space Z, a Z-valued inner product or Z-gramian is, by definition, a mapping $\mathcal{E} \times \mathcal{E} \ni (x, \mathcal{Y}) \mapsto [x, \mathcal{Y}] \in Z$ subject to the following properties:
(ve1) $[x, x] \geq 0$ for all $x \in \mathcal{E}$, and $[x, x] = 0$ if and only if $x = 0$.
(ve2) $[x, \mathcal{Y}] = [\mathcal{Y}, x]^*$ for all $x, \mathcal{Y} \in \mathcal{E}$.
(ve3) $[ax_1 + bx_2, \mathcal{Y}] = a[x_1, \mathcal{Y}] + b[x_2, \mathcal{Y}]$ for all $a, b \in \mathbb{C}$ and all $x_1, x_2 \in \mathcal{E}$.

A complex linear space $\mathcal{E}$ onto which a Z-valued inner product $[\cdot,\cdot]$ is specified, for a certain admissible space Z, is called a VE-space (Vector Euclidean space).

In any VE-space $\mathcal{E}$ over an admissible space Z the familiar polarization formula

$$4[x, \mathcal{Y}] = \sum_{k=0}^{3} \mathrm{i}^k \left[(x + \mathrm{i}^k \mathcal{Y}, x + \mathrm{i}^k \mathcal{Y}\right], x, \mathcal{Y} \in \mathcal{E}, \tag{1}$$

holds, which shows that the Z-valued inner product is perfectly defined by the Z-valued quadratic form $\mathcal{E} \ni x \mapsto [x, x]$.

Any VE-space $\mathcal{E}$ can be made in a natural way into a Hausdorff separated locally convex space by considering the weakest locally convex topology on $\mathcal{E}$ that makes the mapping $\mathcal{E} \ni h \mapsto [h, h] \in Z$ continuous, more precisely, letting $\{N_j\}_{j\in\mathcal{J}}$ be the collection of convex and solid neighbourhoods of the origin in Z as in axiom (a4), the collection of sets

$$U_j = \{x \in \mathcal{E} | \, [x, x] \in N_j\}, j \in \mathcal{J}, \tag{2}$$

is a topological base of neighbourhoods of the origin of $\mathcal{E}$ that linearly generates the weakest locally convex topology on $\mathcal{E}$ that makes the mapping $\mathcal{E} \ni h \mapsto [h, h] \in Z$ continuous, cf. Theorem 1 in [214]. In terms of seminorms, this topology can be defined in the following way: let $\{p_j\}_{j\in\mathcal{J}}$ be a family of increasing seminorms defining the topology of Z and let

$$q_j(h) = p_j([h, h])^{1/2}, h \in \mathcal{E}, j \in \mathcal{J}. \tag{3}$$

Then the specified topology of E is fully determined by the family of seminorms $\{q_j\}_{j\in\mathcal{J}}$.

If $\mathcal{E}$ is complete with respect to this locally convex topology then it is called a VH-space (Vector Hilbert space). In case the admissible space Z is strongly admissible, a VH-space is called an LVH-space (Limit Vector Hilbert space), cf. [215], or Loynes space. LVH-spaces are more suitable for spectral representations of their unitary or selfadjoint operators, but we do not use them.

The concept of VE-spaces isomorphism is also naturally defined: this is just a linear bijection $U: \mathcal{E} \to \mathcal{F}$, for two VE-spaces over the same admissible space Z, such that, $[U_x, U_y] = [x, \mathcal{Y}]$ for all $x, y \in \mathcal{E}$. Any VE-space $\mathcal{E}$ can be embedded as a dense subspace of a VH-space $\mathcal{H}$, uniquely determined up to an isomorphism, cf. Theorem 2 in [214].

In general VH-spaces, an analog of the Schwarz Inequality does not hold. However, some of its consequences can be proven using slightly different techniques. One such consequence is the following lemma.

Lemma (4.2.1)[204]: (Loynes [214]) Let Z be an admissible space, $\mathcal{E}$ a complex vector space and $[\cdot,\cdot]: \mathcal{E} \times \mathcal{E} \to Z$ a positive semidefinite sesqui-linear map, that is, $[\cdot,\cdot]$ is linear in the first variable and conjugate linear in the second variable, and $[x, x] \geq 0$ for all $x \in \mathcal{E}$. If $f \in \mathcal{E}$ is such that $[f, f] = 0$, then $[f, f'] = [f', f] = 0$ for all $f' \in \mathcal{E}$.

The collection $\mathcal{L}(\mathcal{E}, \mathcal{F})$ of all linear and continuous operators between VE-spaces $\mathcal{E}$ and $\mathcal{F}$ is naturally organized as a complex vector space. In particular, the set $\mathcal{L}(\mathcal{E})$ of all linear and continuous operators $T: \mathcal{E} \to \mathcal{E}$ is naturally organized as a complex algebra. Given two VH-spaces $\mathcal{H}$ and $\mathcal{K}$, a linear operator $A: \mathcal{H} \to \mathcal{K}$ is called bounded if there exists a constant $k \geq 0$ such that

$$[Ax, Ax] \leq k[x, x], x \in \mathcal{H}. \tag{4}$$

Taking into account the definition of the underlying topology of a VH-space, any linear and bounded operator T is continuous but the converse is not true, in general. We denote the special class of bounded operators by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. For a bounded operator $A \in B(\mathcal{H}, \mathcal{K})$ we define its operator norm $\|A\|$ by the square root of the least k satisfying (4), that is,

$$\|A\| = \inf\{\sqrt{k} \mid [Ax, Ax] \leq k[x, x], \text{for all } x \in \mathcal{H}\}. \tag{5}$$

It is easy to see that the infimum is actually a minimum and hence, that we have

$$[Ax, Ax] \leq \|A\|^2[x, x], \qquad x \in \mathcal{H}. \tag{6}$$

If $\mathcal{H}$ is a VH-space then $B(H) = B(H, H)$ is a Banach algebra with respect to the usual algebraic operations and the operator norm, cf. Theorem 1 in [215].

Given two VH-spaces $\mathcal{H}$ and $\mathcal{K}$, an operator $A \in B(\mathcal{H}, \mathcal{K})$ is called adjointable if there exists a bounded operator $A^*: \mathcal{K} \to \mathcal{H}$ such that for all $x \in \mathcal{H}$ and $\mathcal{Y} \in \mathcal{K}$

$$[Ax, \mathcal{Y}] = [x, A^*\mathcal{Y}]. \tag{7}$$

We denote by $B^*(\mathcal{H}, \mathcal{K})$ the collection of all adjointable elements in $B(\mathcal{H}, \mathcal{K})$. We emphasize the fact that, in a general VH-space setting, not all bounded operators are adjointable. This is mostly due to the lack of an analog of the Riesz Representation Theorem.

The definitions of selfadjoint, unitary, and normal operators are the same as in the Hilbert space case. It is clear that A is selfadjoint if and only if $[Ax, \mathcal{Y}] = [x, A\mathcal{Y}]$ for all $x, \mathcal{Y} \in \mathcal{H}$, and also, by the polarization formula (1), this is equivalent to

$$[Ax, x] = [Ax, x]^*, \qquad x \in \mathcal{H}. \tag{8}$$

A bounded operator A in $\mathcal{H}$ is called positive if $[Ax, x] \geq 0$ for all $x \in \mathcal{H}$. From (8) it follows that a positive operator is necessarily selfadjoint. A contraction is a linear operator T such that $[Tx, Tx] \leq [x, x]$ for all $x \in \mathcal{H}$. By Theorem 2 in [215], the involution $*$ is isometric, that is, $T^* = T$. If $A \in B^*(\mathcal{H})$ is selfadjoint, then we have

$$-\|A\|[x, x] \leq [Ax, x] \leq \|A\|[x, x]. \tag{9}$$

The importance of the previous inequality, cf. Theorem 3 in [215], is that, sometimes, it may be used instead of the Schwarz Inequality which, in general, does not hold for a VH-space. Moreover, assume that A is a linear operator in $\mathcal{H}$ and that for some real numbers m, M we have

$$m[x, x] \leq [Ax, x] \leq M[x, x], \qquad x \in \mathcal{H}.$$

Then $A \in B^*(\mathcal{H})$ and $A = A^*$. If, in addition, m is the maximum and M is the minimum with these properties, then $|A| = \max\{|m|, |M|\}$.

It is now clear that $B^*(\mathcal{H})$ is a Banach $*$-algebra with isometric involution. According to Theorem 4 in [215], for any VH-space $\mathcal{H}$ the algebra $B^*(\mathcal{H})$ is a C^*-algebra, more precisely, we have $\|A^*A\| = \|A\|^2$ for all $A \in B^*(\mathcal{H})$.

On the other hand, the natural cone of positive elements in a C^*-algebra $\mathcal{A}$ is $\mathcal{A}^+ = \{a^*a \mid a \in \mathcal{A}\}$. According to Theorem 5 in [215], given $\mathcal{H}$ a VH-space and $A \in B^*(\mathcal{H})$, then A is positive (that is, $[Ax, x] \geq 0$ for all $x \in \mathcal{H}$) if and only if $A = B^*B$ for some $B \in B^*(\mathcal{H})$. So, the two notions coincide.

A subspace $\mathcal{M}$ of a VH-space $\mathcal{H}$ is orthocomplemented or accessible if every element $x \in \mathcal{H}$ can be written as $x = \mathcal{Y} + z$ where $\mathcal{Y}$ is in $\mathcal{M}$ and z is such that $[z, m] = 0$ for all $m \in \mathcal{M}$, that is, z is in the orthogonal companion $\mathcal{M}^{\perp}$ of $\mathcal{M}$. Observe that if such a decomposition exists it is unique and hence the orthogonal projection $P_{\mathcal{M}}$ onto $\mathcal{M}$ can be defined by $P_{\mathcal{M}} x = \mathcal{Y}$. Any orthogonal projection P is selfadjoint and idempotent, in particular we have $[Px, \mathcal{Y}] = [Px, P\mathcal{Y}]$ for all $x, y \in \mathcal{H}$, hence P is positive and contractive. Conversely, any selfadjoint idempotent operator is an orthogonal projection onto its range subspace. Any orthocomplemented subspace is closed.

Let Z be an admissible space and $\mathcal{H}_k$, $k = 1, \ldots, n$ be VH-spaces with Z-gramian $[\cdot,\cdot]_k$, respectively. On the algebraic direct sum $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ one can consider a two variable Z-valued map

$$[x, \mathcal{Y}] = \sum_{k=1}^{n} [x_k, \mathcal{Y}_k]_k, \qquad x = (x_1, \ldots, x_n), \;\; \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_n), \tag{10}$$

and it is easy to see that it is a Z-valued inner product on $\mathcal{H}$. Letting $\{p_j\}_j$ denote a collection of increasing seminorms on Z that define its underlying topology, we consider the family of seminorms on $\mathcal{K}$

$$\mathcal{K} \ni x \mapsto \left(p_j \sum_{k=1}^{n} [x_k, \mathcal{Y}_k]_k \right)^{1/2}, \qquad x = (x_1, \ldots, x_n),$$

and it is easy to see that the locally convex topology of $\mathcal{K}$ defined by this family of seminorms is complete, hence $\mathcal{K}$ is a VH-space.

If we have $\mathcal{H}_{\mathcal{K}} = \mathcal{H}$ for all $k = 1, \ldots, n$ then we let $\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (the sum has exactly n terms). An alternate characterization of the VH-space $\mathcal{H}^n$ can be obtained as a tensor product. More precisely, let $\mathbb{C}^n$ denote the canonical n-dimensional complex vector space and consider the algebraic tensor product $\mathbb{C}^n \otimes \mathcal{H}$, on which a Z- valued two variable map can be defined by

$$[\sum_{k=1}^{l} e_k \otimes x_k, \sum_{j=1}^{m} f_j \otimes \mathcal{Y}_j] = \sum_{k=1}^{l} \sum_{k=1}^{n} \langle e_k, f_j \rangle [x_k, \mathcal{Y}_j], \tag{11}$$

for $x_k, \mathcal{Y}_j \in \mathbb{C}^n$ and $e_k, f_j \in \mathcal{H}$, $k = 1, \ldots, l$ and $j = 1, \ldots, m$. By Proposition 2.4 in [212], given $\mathcal{H}$ a VH-space and $n \in \mathbb{N}$, the vector space $\mathbb{C}^n \otimes \mathcal{H}$, endowed with the Z-valued map $[\cdot,\cdot]$ defined by (11), is a VH-space, canonically isomorphic with the VH-space $\mathcal{H}^n$.

Let M_n denote the C^*-algebra of all $n \times n$ matrices with complex entries. There is a canonical identification of M_n with the C^*-algebra $B(\mathbb{C}^n)$ given by the action on the canonical orthonormal basis of $\mathbb{C}^n$. We consider $M_n(B^*(\mathcal{H}))$ as the collection of all $n \times n$ matrices with entries in $B^*(\mathcal{H})$ that has a natural structure of $*$-algebra: for instance, letting $A = [A_{i,j}]_{i,j=1}^{n}$ we have $A^* = [A_{j,i}^*]_{i,j=1}^{n}$, addition is entry-wise, while multiplication is matrix-wise. Since $B^*(\mathcal{H})$ is a C^*-algebra, $M_n(B^*(\mathcal{H}))$ is a C^*-algebra, e.g. see [222], in a natural fashion. There is a canonical identification of $M_n(B^*(\mathcal{H}))$ with the C^*-algebra $B^*(\mathcal{H}^n)$ which provides a $*$-isomorphism of C^*-algebras, more precisely, any $A = [A_{i,j}]_{i,j=1}^{n}$ in $M_n(B^*(\mathcal{H}))$ is identified with the operator A in $\mathcal{H}^n$ defined by left matrix multiplication with column vectors of size n with entries in $\mathcal{H}$.

Consider now the vector space $M_n \otimes B^*(\mathcal{H})$. There is a natural structure of $*$-algebra on $M_n \otimes B^*(\mathcal{H})$: for elementary tensors $A \otimes T$ and $B \otimes S$, we have

$$(A \otimes T)(B \otimes S) = (AB \otimes TS), \qquad (A \otimes T)^* = A^* \otimes T^*.$$

Moreover, an identification of the $*$-algebra $M_n \otimes B^*(\mathcal{H})$ with the C^*-algebra $M_n(B^*(\mathcal{H}))$ is obtained in the following way: for an elementary tensor $A \otimes T$ the corresponding element in $M_n(B^*(\mathcal{H}))$ is $[a_{i,j}T]_{i,j=1}^n$. This provides a natural C^*-algebra structure on $M_n \otimes B^*(\mathcal{H})$ with respect to which this identification becomes a $*$-isomorphism.

On the other hand, since the C^*-algebras M_n and $B(\mathbb{C}^n)$ are identified canonically, we actually have a canonical identification of the C^*-algebras $M_n \otimes B^*(\mathcal{H})$ with the C^*-algebra $B^*(\mathbb{C}^n \otimes \mathcal{H})$: an arbitrary elementary tensor $A \otimes T$ in $M_n \otimes B^*(\mathcal{H})$ is identified with the operator on the VH-space $\mathbb{C}^n \otimes \mathcal{H}$ by

$$(A \otimes T)(x \otimes h), \qquad x \in \mathbb{C}^n,\ \ h \in \mathcal{H}, \tag{12}$$

and then extended by linearity. We are particularly interested in positive elementary tensors: if $A \in M_n^+$ and $T \in B^*(\mathcal{H})^+$, then $A \otimes T \in (M_n \otimes B^*(\mathcal{H}))^+$, more precisely, if $A = B^*B$ for some $B \in M_n$ and similarly, $T = S^*S$ for some $S \in B^*(\mathcal{H})$, hence, $A \otimes T = B^*B \otimes S^*S = (B^* \otimes S^*)(B \otimes S) = (B \otimes S)^*(B \otimes S)$. The following inequality is a surrogate of a Schwarz inequality and will be needed later.

Lemma (4.2.2)[204]: Let T be a positive operator in the VH-space $\mathcal{H}$. Then, for all $h_1, h_2, \ldots, h_n \in \mathcal{H}$ we have

$$0 \leq \sum_{j,k=1}^{n} [Th_j, h_k]_{\mathcal{H}} \leq \|T\| \sum_{j,k=1}^{n} [h_j, h_k]_{\mathcal{H}}. \tag{13}$$

Proof. We consider the VH-space $\mathcal{H}^n = \mathbb{C}^n \otimes \mathcal{H}$ and then the C^*-algebra

$$M_n \otimes B^*(\mathcal{H}) \simeq M_n \otimes B^*(\mathcal{H}) \simeq B^*(\mathcal{H}^n).$$

We consider $E \in M_n$ the $n \times n$ matrix with all entries equal to 1 and note that it is positive. Since T is positive it follows that $\|T\|I - T \geq 0$ and hence

$$E \otimes (\|T\|I) - E \otimes T = E \otimes (\|T\|I - T) \geq 0,$$

as an element in the C^*-algebra $M_n \otimes B^*(\mathcal{H})$ as before, equivalently,

$$E \otimes T \leq E \otimes (\|T\|I) = \|T\|(E \otimes I),$$

which, when evaluated at the vector $h_1, h_2, \ldots, h_n \in \mathcal{H}^n \simeq \mathbb{C}^n \otimes \mathcal{H}$, provides the inequality (13).

Let X be a nonempty set and let $\mathcal{H}$ be a VH-space over the admissible space Z. A map $K : X \times X \to B(\mathcal{H})$ is called a kernel on X and valued in $B(\mathcal{H})$. In case the kernel K has values in $B^*(\mathcal{H})$, an adjoint kernel $K^* : X \times X \to B^*(\mathcal{H})$ can be associated by $K^*(x, \mathcal{Y}) = K(\mathcal{Y}, x)^*$ for all $x, \mathcal{Y} \in X$. The kernel K is called Hermitian if $K^* = K$.

Let $\mathcal{F} = \mathcal{F}(X; \mathcal{H})$ denote the complex vector space of all functions $f : X \to \mathcal{H}$, and $\mathcal{G} = \mathcal{G}(X; \mathcal{H})$ its subspace of those functions having finite support. A pairing $[\cdot,\cdot]_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to Z$ can be defined by

$$[g, h]_{\mathcal{G}} = \sum_{\mathcal{Y} \in X} [g(\mathcal{Y}), h(\mathcal{Y})]_{\mathcal{H}}, \qquad g, h \in \mathcal{G}. \tag{14}$$

This pairing is clearly a Z-gramain on $\mathcal{G}$, hence $(\mathcal{G};\ [\cdot,\cdot]_{\mathcal{G}})$ is a VE-space. Let us observe that the sum in (14) makes sense even when only one of the functions g or h has finite support, the other can be arbitrary in $\mathcal{F}$. Thus, another pairing $[\cdot,\cdot]_K$ can be defined on $\mathcal{G}$ by

$$[g, h]_{\mathcal{G}} = \sum_{x, \mathcal{Y} \in X} [K(\mathcal{Y}, x)g(x), h(\mathcal{Y})]_{\mathcal{H}}, \qquad g, h \in \mathcal{G}. \tag{15}$$

In general, the pairing $[\cdot,\cdot]_K$ is linear in the first variable and conjugate linear in the second variable. If, in addition, $K = K^*$ then the pairing $[\cdot,\cdot]_K$ is Hermitian as well, that is,

$$[g, h]_K = [h, g]_K^*, \qquad g, h \in \mathcal{G}.$$

A convolution operator $K : \mathcal{G} \to \mathcal{F}$ can be associated to the kernel K by

$$(Kg)(\mathcal{Y}) = \sum_{x,\mathcal{Y} \in X} K(\mathcal{Y}, x) g(x), \qquad g \in \mathcal{G}. \tag{16}$$

and it is easy to see that K is a linear operator. There is a natural relation between the pairing $[\cdot,\cdot]_K$ and the convolution operator K given by

$$[g, h]_K = [Kg, h]_{\mathcal{G}}, \qquad g, h \in \mathcal{G}.$$

If K is adjointable, and letting K^* denote the convolution operator of the adjoint kernel K^*, we have

$$[g, h]_K = [Kg, h]_{\mathcal{G}} = [g, K^* h]_{\mathcal{G}} = [K^* h, g]_{\mathcal{G}}^* = [h, g]_{K^*}^*, \quad g, h \in \mathcal{G},$$

and hence, the pairing $[\cdot,\cdot]_K$ is Hermitian if and only the kernel K is Hermitian.

Given $n \in \mathbb{N}$, the kernel K is called n-positive if for any $x_1, x_2, \ldots, x_n \in X$ and any $h_1, h_2, \ldots, h_n \in \mathcal{H}$ we have

$$\sum_{j,k=1}^{n} [K(x_k, x_j) h_j, h_k]_{\mathcal{H}} \geq 0. \tag{17}$$

The kernel K is called positive semidefinite (or of positive type) if it is n-positive for all natural numbers n.

Lemma (4.2.3)[204]: Assume that the kernel $K : X \times X \to B^*(\mathcal{H})$ is 2-positive. Then:

(a) K is Hermitian.

(b) If, for some $x \in X$, we have $K(x, x) = 0$, then $K(x, y) = 0$ for all $y \in X$.

Proof. (a) Since K is 2-positive it is 1-positive, hence $K(x, x) \geq 0$ for all $x \in X$. On the other hand, writing down the 2-positivity condition, for any $x, y \in X$ and any $g, h \in \mathcal{H}$ we have

$$[K(x, y) g, h]_{\mathcal{H}} + [K(y, x) h, g]_{\mathcal{H}} + [K(x, x) h, h]_{\mathcal{H}} + [K(y, y) g, g]_{\mathcal{H}} \geq 0, \tag{18}$$

hence the sum of the first two terms is in the real span of the cone Z_+, in particular, it is selfadjoint. Thus,

$$[K(x, y) g, h]_{\mathcal{H}} + [K(y, x) h, g]_{\mathcal{H}} = [h, K(x, y) g]_{\mathcal{H}} + [g, K(y, x) h]_{\mathcal{H}},$$

equivalently,

$$[(K(x, y) - K(y, x)^*) g, h]_{\mathcal{H}} + [(K(y, x) - K(x, y)^*) h, g]_{\mathcal{H}} = 0.$$

Letting $h = \mathrm{i}(K(x, y) - K(y, x)^*) g$ it follows

$$2\mathrm{i}[(K(x, y) - K(y, x)^*) g, (K(x, y) - K(y, x)^*) g]_{\mathcal{H}} = 0,$$

hence $K(x, y) = K(y, x)^*$ for all $x, y \in X$, that is, K is Hermitian.

(b) Let $K(x, x) = 0$ and consider (18) for arbitrary $y \in X$ and $g, h \in \mathcal{H}$. Then

$$[K(x, y) g, h]_{\mathcal{H}} + [K(y, x) h, g]_{\mathcal{H}} \geq -[K(y, y) g, g]_{\mathcal{H}}. \tag{19}$$

We claim that

$$[K(x, y) g, h]_{\mathcal{H}} + [K(y, x) h, g]_{\mathcal{H}} = 0. \tag{20}$$

Indeed, taking into account the K is 1-positive, we have two choices only: if $[K(y, y) g, g]_{\mathcal{H}} = 0$, from(19) it follows that $[K(x, y) g, h]_{\mathcal{H}} + [K(y, x) h, g]_{\mathcal{H}} \geq 0$ and then, replacing g by $-g$ we obtain the opposite inequality, hence (20) holds. The second possible choice is $[K(y, y) g, g]_{\mathcal{H}} > 0$ when, observing that the rightmost term in (19) does not depend on h, we can replace h by th, for $t \in \mathbb{R}$. But then, from (19) it follows that the only possibility is that (20) should hold, since the opposite leads to a contradiction. Thus, (20) is proven.

To finish the proof, in (20) we replace g by $\mathrm{i}g$ and get $[K(x, y) g, h]_{\mathcal{H}} - [K(y, x) h, g]_{\mathcal{H}} = 0$ which, in combination with (20) implies $[K(x, y) g, h]_{\mathcal{H}} = 0$ for all $g, h \in \mathcal{H}$, hence $K(x, y) = 0$.

The following result is a surrogate of a Schwarz inequality for kernels and it will have a technical role .

Proposition (4.2.4)[204]: Assume that the kernel K is $2n$-positive for some natural number n. Then, for any $x, y_1, y_2, \ldots, y_n \in X$ and any $g_1, g_2, \ldots, g_n \in \mathcal{H}$, the following inequality holds

$$\sum_{j,k=1}^{n} [K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}} \leq \|K(x,x)\| \sum_{j,k=1}^{n} [K(y_k, y_j)g_j, g_k]_{\mathcal{H}}. \tag{21}$$

Proof. Since K is $2n$-positive, it follows that for any $x_1, \ldots, x_{2n} \in X$ and $h_1, \ldots, h_{2n} \in \mathcal{H}$ we have

$$\sum_{j,k=1}^{2n} [K(x_k, x_j)h_j, h_k]_{\mathcal{H}} \geq 0. \tag{22}$$

For each $k = 1, \ldots, n$ we make the following choice

$$x_k = x,\ x_{n+k} = y_k, \qquad h_k = -K(x, y_k)g_k, \qquad h_{n+k} = \|K(x,x)\|g_k,$$

in (22) and get

$$0 \leq \sum_{j,k=1}^{n} [K(x,x)K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}}$$
$$-2\|K(x,x)\| \sum_{j,k=1}^{n} [K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}}$$
$$+\|K(x,x)\|^2 \sum_{j,k=1}^{n} [K(y_k, y_j)g_j, g_k]_{\mathcal{H}}. \tag{23}$$

Taking into account that, by Lemma (4.2.2), when applied for $T = K(x,x) \geq 0$ and $h_j = K(x, y_j)g_j$, have

$$\sum_{j,k=1}^{n} [K(x,x)K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}} \leq \|K(x,x)\|$$
$$\times \sum_{j,k=1}^{n} [K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}},$$

which, when used in (23), yields

$$\|K(x,x)\| \sum_{j,k=1}^{n} [K(x, y_j)g_j, K(x, y_k)g_k]_{\mathcal{H}} \leq \|K(x,x)\|^2 \sum_{j,k=1}^{n} [K(y_k, y_j)g_j, g_k]_{\mathcal{H}},$$

which, by Lemma (4.2.3).(2), implies (21).

Given a $B^*(\mathcal{H})$-valued kernel K on a nonempty set X, for some VH-space $\mathcal{H}$ on an admissible space Z, a Kolmogorov decomposition of K is, by definition, a pair $(V\,;\mathcal{K})$, subject to the following conditions:

(kd1) K is a VH-space over the same admissible space Z.

(kd2) $V : X \to B^*(\mathcal{H}, \mathcal{K})$ satisfies $K(y,x) = V^*(y)V(x)$ for all $x, y \in X$.

If, in addition, the Kolmogorov decomposition satisfies the following condition

(kd3) Lin $V(X)\mathcal{H}$ is dense in $\mathcal{K}$.

then it is called minimal.

Two Kolmogorov decompositions $(V\,;\mathcal{K})$ and $(V'\,;\mathcal{K}')$ of the same kernel K are called unitary equivalent if there exists a unitary operator $U:\mathcal{K}\to\mathcal{K}'$ such that $UV(x)=V'(x)$ for all $x\in X$.

Theorem (4.2.5)[204]: Given a $B^*(\mathcal{H})$-valued kernel K, for some VH-space $\mathcal{H}$ on an admissible space Z, on a nonempty set X, the following assertions are equivalent:

(i) K is positive semidefinite.

(ii) K admits a Kolmogorov decomposition $(V\,;\mathcal{K})$.

In addition, the Kolmogorov decomposition $(V\,;\mathcal{K})$ can always be chosen minimal and, in this case, it is unique up to unitary equivalence.

Proof. (i) ⇒ (ii). Assuming that K is positive semidefinite, by Lemma (4.2.2).(a) it follows that K is Hermitian, that is, $K(x,y)^*=K(y,x)$ for all $x,y\in X$. We consider the convolution operator K defined at (16) and let $\mathcal{F}_0=\mathcal{F}_0(X;\mathcal{H})$ be its range, more precisely,

$$\begin{aligned}\mathcal{F}_0&=\{f\in\mathcal{F}\mid f=Kg \text{ for some } g\in\mathcal{G}\} \qquad (24)\\ &=\{f\in\mathcal{F}\mid f(y)=\sum_{x\in X}K(y,x)\,g(x)\ for\ some\ g\in\mathcal{G}\ and\ all\ y\in X\}.\end{aligned}$$

A pairing $[\cdot,\cdot]_{\mathcal{F}_0}:X\times X\to Z$ can be defined by

$$\begin{aligned}[e,f]_{\mathcal{F}_0}=[g,h]_K=[Kg,h]_{\mathcal{G}}&=\sum_{y\in X}[e(y),h(y)]_{\mathcal{H}}\\ &=\sum_{x,y\in X}[K(y,x)g(x),h(y)]_{\mathcal{H}}, \qquad (25)\end{aligned}$$

where $f=Kh$ and $e=Kg$ for some $g,h\in\mathcal{G}$, that is, g and h are finitely supported $\mathcal{H}$-valued functions on X. We observe that

$$\begin{aligned}[e,f]_{\mathcal{F}_0}&=\sum_{y\in X}[e(y),h(y)]_{\mathcal{H}}=\sum_{x,y\in X}[K(y,x)g(x),h(y)]_{\mathcal{H}}\\ &=\sum_{x,y\in X}[g(x),K(y,x)h(y)]_{\mathcal{H}}=\sum_{x\in X}[g(x),f(x)]_{\mathcal{H}},\end{aligned}$$

which shows that the definition in (25) is correct (that is, independent of g and h such that $e=Kg$ and $f=Kh$).

We claim that $[\cdot,\cdot]_{\mathcal{F}_0}$ is a Z-valued inner product, that is, it satisfies all the requirements (ve1)–(ve3). The only fact that needs a proof is $[f,f]_{\mathcal{F}_0}=0$ implies $f=0$. To see this we use Lemma (4.2.1) and first get that $[f,f']_{\mathcal{F}_0}=0$ for all $f'\in\mathcal{F}_0$. For each $x\in X$ and each $h\in\mathcal{H}$ let $\delta_x h\in\mathcal{G}$ denote the function

$$(\delta_x h)(y)=\begin{cases}h, & \text{if } y=x,\\ 0, & \text{if } y\neq x.\end{cases} \qquad (26)$$

(A correct notation would be $\delta_x h=\delta_x\otimes h$, when identifying $\mathcal{G}(X;\mathcal{H})$ with $\mathcal{G}(X;\mathbb{C})\otimes\mathcal{H}$, but we resisted the temptation of using it in order to keep the notation simpler.) Letting $f=K\delta_x h$ we have

$$0=[f,f']_{\mathcal{F}_0}=[f,K\delta_x h]_{\mathcal{G}}=\sum_{y\in X}[f(y),(\delta_x h)(y)]_{\mathcal{H}}=[f(x),h]_{\mathcal{H}},$$

hence, since $h\in\mathcal{H}$ and $x\in X$ are arbitrary, it follows that $f=0$.

Thus, $(\mathcal{F}_0;\ [\cdot,\cdot]_{\mathcal{F}_0})$ is a VE-space that can be completed to a VH-space that we denote by K, that contains $\mathcal{F}_0$ as a dense linear manifold. For each $x\in X$ we define $V(x):\mathcal{H}\to\mathcal{F}_0$ by

$$V(x)h=K\delta_x h,\qquad h\in\mathcal{H}. \qquad (27)$$

Actually, there is an even more explicit way of expressing $V(x)$, namely,

$$(V(x)h)(y) = (K\delta_x h)(y) = \sum_{z\in X} K(y,z)(\delta_x h)(z)$$
$$= K(y,x)h, \qquad y \in X. \tag{28}$$

We first show that $V(x)$ is a bounded operator from the VH-space $\mathcal{H}$ to the VE-space $\mathcal{F}_0$. Indeed,

$$[V(x)h, V(x)h]_{\mathcal{F}_0} = [K\delta_x h, K\delta_x h]_{\mathcal{F}_0} = \sum_{y,z\in X} K(y,z)(\delta_x h)(z)\,, (\delta_x h)(y)]_{\mathcal{H}}$$
$$= [K(x,x)h, h]_{\mathcal{H}} \leq \|K(x,x)\|[h,h]_{\mathcal{H}}.$$

Thus, $V(x)$ is bounded and hence can be uniquely extended by continuity to an operator $V(x) \in B(\mathcal{H},\mathcal{K})$.

We now show that $V(x)$ is adjointable for all $x \in X$. To see this, let us fix $x \in X$ and take $h \in \mathcal{H}$ and $f \in \mathcal{F}_0$ arbitrary. Then,

$$[V(x)h, f]_{\mathcal{F}_0} = \sum_{y,z\in X} [(\delta_x h)(y), f(y)]_{\mathcal{H}} = [h, f(x)]_{\mathcal{H}}, \tag{29}$$

which shows that, if $V(x)$ is adjointable then its adjoint, when restricted to $\mathcal{F}_0$ should be $\mathcal{F}_0 \ni f \longmapsto f(x) = W(x)f \in \mathcal{H}$. We prove that $W(x)$ is bounded as a linear operator from the VE-space $\mathcal{F}_0$ to the VH-space $\mathcal{H}$. To this end, let $f \in \mathcal{F}_0$ be arbitrary, hence $f = Kg$ for some finitely supported g. Then

$$[W(x)f, W(x)]_{\mathcal{H}} = [f(x), f(x)]_{\mathcal{H}} = [\sum_{z\in X} K(y,z)(\delta_x h)(z)\,, \sum_{y\in X} K(x,y)\, g(y)]_{\mathcal{H}}$$
$$= \sum_{z,y\in X} [K(x,z)g(z), K(x,y)g(y)]_{\mathcal{H}}$$

and, by Proposition (4.2.4), we get

$$\leq \|K(x,x)\| \sum_{z,y\in X} [K(y,z)g(z), g(y)]_{\mathcal{H}} = \|K(x,x)\|[f,f]_{\mathcal{F}_0}.$$

This proves that $W(x)$ is bounded and hence can be extended uniquely, by continuity, to an operator $V(x) \in B(\mathcal{H},\mathcal{K})$. By (29) it follows that $V(x)$ is adjointable and $W(x) = V(x)^*$ for all $x \in X$, more precisely,

$$V(x)^* f = f(x), \qquad f \in \mathcal{F}_0. \tag{30}$$

On the other hand, for any $x, y \in X$, by (30) and (28), we have

$$V(y)^* V(x)h = (V\,(x)h)(y) = K(y,x)h, \qquad h \in \mathcal{H},$$

hence $(V\,;\mathcal{K})$ is a Kolmogorov decomposition of K. We prove that it is minimal as well. To see this, note that for any $g \in \mathcal{G}$, with the notation as in (26), we have

$$g = \sum_{x\in supp(g)} \delta_x g(x),$$

hence, by (27), the linear span of $V(X)\mathcal{H}$ equals $\mathcal{F}_0$ which is dense in $\mathcal{K}$.

The uniqueness of the minimal Kolmogorov decomposition $(V\,;\mathcal{K})$ just constructed follows in the usual way: if $(V'\,;\mathcal{K}')$ is another minimal Kolmogorov decomposition of K, for arbitrary $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and arbitrary $g_1, \ldots, m_m, h_1, \ldots, h_n \in \mathcal{H}$, we have

$$[\sum_{j=1}^{m} V(x_j)\, g_j, \sum_{k=1}^{m} V(y_k)\, h_k]_{\mathcal{K}} = \sum_{j=1}^{m} \sum_{k=1}^{n} V(y_k)^* V(x_j)\, g_j, h_k]_{\mathcal{H}}$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n} [K(y_k, x_j)V(x_j)\, g_j, h_k]_{\mathcal{H}}$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{n} V'(y_k)^* V'(x_j)\, g_j, h_k]_{\mathcal{H}}$$

$$= [\sum_{j=1}^{m} V'(x_j)\, g_j, \sum_{k=1}^{m} V'(y_k)\, h_k]_{\mathcal{K}'},$$

hence U: Lin $V(X)\mathcal{H} \to$ Lin $V'(X)\mathcal{H}$ defined by

$$\sum_{j=1}^{m} V(x_j)\, g_j \mapsto \sum_{j=1}^{m} V'(x_j)\, g_j \tag{31}$$

is a linear operator, correctly defined, isometric, densely defined, and with dense range. Thus, U extends uniquely to a unitary operator $U \in B^*(\mathcal{K}, \mathcal{K}')$ and $UV(x) = V'(x)$ for all $x \in X$, by construction.

(ii) ⇒ (i). This is proven exactly as in the classical case:

$$\sum_{j,k=1}^{n} [K(x_k, x_j)h_j, h_k]_{\mathcal{H}} = \sum_{j,k=1}^{n} [V(x_k)^* V(x_j)h_j, h_k]_{\mathcal{H}}$$

$$= [\sum_{j=1}^{n} V(x_j)\, h_j, \sum_{j=1}^{n} V(x_j)\, h_j]_{\mathcal{H}} \geq 0,$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $h_1, \ldots, h_n \in \mathcal{H}$.

Let $\mathcal{H}$ be a VH-space over the admissible space Z, and let X be a nonempty set. We consider the complex vector space $\mathcal{F}(X;\mathcal{H})$ of all functions $f : X \to \mathcal{H}$. A VH-space $\mathcal{R}$, over the same admissible space Z, is called an $\mathcal{H}$-reproducing kernel VH-space on X if there exists a Hermitian kernel $K : X \times X \to B^*(\mathcal{H})$ such that the following axioms are satisfied:

(rk1) $\mathcal{R}$ is a subspace of $\mathcal{F}(X;\mathcal{H})$, with all algebraic operations.

(rk2) For all $x \in X$ and all $h \in \mathcal{H}$, the $\mathcal{H}$-valued function $K_x h = K(\cdot, x)h \in \mathcal{R}$.

(rk3) For all $f \in \mathcal{R}$ we have $[f(x), h]_{\mathcal{H}} = [f, Kxh]_{\mathcal{R}}$, for all $x \in X$ and $h \in \mathcal{H}$. In addition, as a consequence of (rk3), the following minimality property holds as well:

(rk4) Lin$\{K_x h \mid x \in X, h \in \mathcal{H}\}$ is dense in $\mathcal{R}$.

Proposition (4.2.6)[204]: Assume that $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VH-space on X with kernel K.

(i) K is positive semidefinite and uniquely determined by $\mathcal{R}$.

(ii) $\mathcal{R}$ is uniquely determined by K.

Proof. (i) Using the reproducing axiom (rk3) it follows

$$\sum_{j,k=1}^{n} [K(x_k, x_j)h_j, h_k]_{\mathcal{H}} = \sum_{j,k=1}^{n} [K_{x_j} h_j, K_{x_k} h_k]_{\mathcal{H}}$$

$$= [\sum_{j=1}^{n} K_{x_j}\, h_j, \sum_{j=1}^{n} K_{x_j}\, h_j]_{\mathcal{H}} \geq 0$$

hence K is positive semidefinite.

On the other hand, by (rk3) it follows that all the functions $K_x\, h, x \in X, h \in \mathcal{H}$ are uniquely determined by $(\mathcal{R};\ [\cdot,\cdot]_{\mathcal{R}})$, hence all the operators $K(y,x) = K_x(y), x, y \in X$, are uniquely determined.

(ii) Let $\mathcal{R}'$ be another $\mathcal{H}$-reproducing kernel VH-space on X with kernel K. By axiom (rk2) and (rk4), $\mathcal{R}_0 = \operatorname{Lin}\{K_x h \mid x \in X, h \in \mathcal{H}\}$ is a linear space that lies and is dense in both of $\mathcal{R}$ and $\mathcal{R}'$. By axiom (rk3), the Z-valued inner products $[\cdot,\cdot]_{\mathcal{R}}$ and $[\cdot,\cdot]_{\mathcal{R}'}$ coincide on $\mathcal{R}_0$ and then it is easy to see that, due to the special way in which the topologies on $\mathcal{R}$ and $\mathcal{R}'$ are defined (see (ii) and (iii)) and the density of $\mathcal{R}_0$, we actually have $\mathcal{R} = \mathcal{R}'$ as VH-spaces.

Consequently, given $\mathcal{R}$ an $\mathcal{H}$-reproducing kernel VH-space on X we can talk about the $\mathcal{H}$-reproducing kernel K corresponding to $\mathcal{R}$.

The following theorem adds one more equivalent characterization of VH-spaces operator valued positive semidefinite kernels in terms of reproducing kernel VH-spaces. Our point of view is to obtain this equivalent statement through Kolmogorov decompositions.

Theorem (4.2.7)[204]: Let $\mathcal{H}$ be a VH-space over the admissible space Z,X a nonempty set, and $K : X \times X \to B^*(\mathcal{H})$ a Hermitian kernel. The following assertions are equivalent:

(i) K is positive semidefinite.

(ii) K has a Kolmogorov decomposition.

(iii) K is the $\mathcal{H}$-reproducing kernel on X of a VH-space $\mathcal{R}$.

Proof. The equivalence (i) ⇔ (ii) was proven in Theorem (4.2.5). Even though we already have the implication (iii) ⇒ (i) by Proposition (4.2.6), we prefer to prove the equivalence of assertions (ii) and (iii) independently of this, in order to show explicitly both ways of the connection between Kolmogorov decompositions and reproducing kernel VH-spaces.

(ii) ⇒ (iii). Let $(\mathcal{K};\ V)$ be a Kolmogorov decomposition of K. As shown by Theorem (4.2.5), without loss of generality we can assume it to be minimal as well. Define

$$\mathcal{R} = \{V(\cdot)^* f \mid f \in \mathcal{K}\}, \tag{32}$$

that is, $\mathcal{R}$ consists of all functions $X \ni x \longmapsto V(x)^* f \in \mathcal{H}$, in particular $\mathcal{R} \subseteq \mathcal{F}(X;\mathcal{H})$, and we endow $\mathcal{R}$ with the algebraic operations inherited from the complex vector space $\mathcal{F}(X;\mathcal{H})$.

We want to show that the correspondence

$$\mathcal{K} \ni f \longmapsto Uf = V(\cdot)^* f \in \mathcal{R} \tag{33}$$

is bijective. By the definition of $\mathcal{R}$, this correspondence is surjective. In order to verify that it is injective as well, let $f, g \in \mathcal{K}$ be such that $V^*(\cdot)f = V^*(\cdot)g$. Then, for all $x \in X$ and all $h \in \mathcal{H}$ we have

$$[V(x)^* f, h]_{\mathcal{H}} = [V(x)^* g, h]_{\mathcal{H}},$$

equivalently,

$$[f - g, V(x)h]_{\mathcal{K}} = 0, \in X, h \in \mathcal{H}.$$

By the minimality of the Kolmogorov decomposition $(\mathcal{K};\ V)$ it follows that $g = f$. Thus, U is a bijection.

Clearly, the bijective map U defined at (33) is linear, hence a linear isomorphism of complex vector spaces $\mathcal{K} \to \mathcal{R}$. On R we introduce a Z-valued pairing

$$[Uf, Ug]_{\mathcal{R}} = [V(\cdot)^* f, V(\cdot)^* g]_{\mathcal{R}} = [f, g]_{\mathcal{K}}, f, g \in \mathcal{K}. \tag{34}$$

Since $(\mathcal{K};\ [\cdot,\cdot]_{\mathcal{K}})$ is a VH-space over Z, it follows that $(\mathcal{R};\ [\cdot,\cdot]_{\mathcal{R}})$ is a VH-space over Z: just note that, by (34) we transported the Z-gramian from $\mathcal{K}$ to $\mathcal{R}$ or, in other words, we have defined on $\mathcal{R}$ the Z-gramian that makes the linear isomorphism U a unitary operator between the VH-spaces $\mathcal{K}$ and $\mathcal{R}$.

We show that $(\mathcal{R};\ [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VH-space with corresponding reproducing kernel $\mathcal{K}$. By definition, $\mathcal{R} \subseteq \mathcal{F}(X;\mathcal{H})$. On the other hand, since

$$K_x(y)h = K(y,x)h = V(y)^*V(x)h, \quad \text{for all } x,y \in X \text{ and all } h \in \mathcal{H},$$

taking into account that $V(x)h \in \mathcal{K}$, by (32) it follows that $K_x \in \mathcal{R}$ for all $x \in X$. Further, for all $f \in \mathcal{R}$, $x \in X$, and $h \in \mathcal{H}$, we have

$$\begin{aligned}[f, K_x h]_{\mathcal{R}} &= [V(\cdot)^* g, K_x h]_{\mathcal{R}} = [V(\cdot)^* g, V(\cdot)^* V(x)h]_{\mathcal{R}} \\ &= [g, V(x)h]_{\mathcal{K}} = [V(x)^* g, h]_{\mathcal{H}} = [f,h]_{\mathcal{H}},\end{aligned}$$

where $g \in \mathcal{K}$ is the unique vector such that $V(x)^* g = f$, which shows that $\mathcal{R}$ satisfies the reproducing axiom as well. Finally, taking into account the minimality of the Kolmogorov decomposition $(\mathcal{K};\ V)$ and the definition (32), it follows that $\overline{\mathrm{Lin}\{K_x \mid x \in X\}} = \mathcal{R}$. Thus, we finish proving that $(\mathcal{R};\ [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VH-space with reproducing kernel K.

(iii) ⇒ (ii). Assume that $(\mathcal{R};\ [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VH-space on X, with reproducing kernel K. We let $\mathcal{K} = \mathcal{R}$ and define

$$V(x)h = K_x h, \qquad x \in X,\ \ h \in \mathcal{H}. \tag{35}$$

Note that $V(x)\colon \mathcal{H} \to \mathcal{K}$ is linear for all $x \in X$.

We have to show that $V(x) \in B^*(\mathcal{H},\mathcal{K})$ for all $x \in X$. To see this, first note that, by the reproducing property, for all $h \in \mathcal{H}$ we have

$$[V(x)h, V(x)h]_{\mathcal{K}} = [K_x h, K_x h]_{\mathcal{R}} = [K(x,x)h, h]_{\mathcal{H}} \le \|K(x,x)\| [h,h]_{\mathcal{H}},$$

hence $V(x)$ is bounded for all $x \in X$. On the other hand,

$$[f, V(x)h]_{\mathcal{K}} = [f, K_x h]_{\mathcal{R}} = [f(x), h]_{\mathcal{H}}, \qquad x \in X,\ \ h \in \mathcal{H}. \tag{36}$$

Let us then, for fixed $x \in X$, consider the linear operator $W(x)\colon \mathcal{R} = \mathcal{K} \to \mathcal{H}$ defined by $W(x)f = f(x)$ for all $f \in \mathcal{R} = \mathcal{K}$. In order to show that $W(x)$ is bounded, by the minimality property (rk4) it follows that it is sufficient to consider only functions $f \in \mathrm{Lin}\{K_x h \mid x \in X,\ \ h \in \mathcal{H}\}$. Thus, if $f = K_{x_1}h_1 + \cdots + K_{x_n}h_n$ it follows that

$$\begin{aligned}[W(x)f, W(x)f]_{\mathcal{H}} &= [f(x), f(x)]_{\mathcal{H}} = [\sum_{j=1}^{n} K(x,x_j)\, h_j\, , \sum_{k=1}^{n} K(x,x_k)\, h_k]_{\mathcal{H}} \\ &\le \|K(x,x)\| \sum_{j,k=1}^{n} [K(x_k,x_j)h_j, h_k]_{\mathcal{H}},\end{aligned}$$

where the inequality follows by Proposition (4.2.4). Since, by the reproducing axiom, we have

$$\begin{aligned}[f,f]_{\mathcal{R}} &= \sum_{j,k=1}^{n} [K_{x_j}h_j, K_{x_k}h_k]_{\mathcal{R}} \\ &= \sum_{j,k=1}^{n} [K_{x_j}(x_k)h_j, h_k]_{\mathcal{H}} = \sum_{j,k=1}^{n} [K(x_k,x_j)h_j, h_k]_{\mathcal{H}}\end{aligned}$$

it follows that, for all $f \in \mathrm{Lin}\{K_x h \mid x \in X,\ \ h \in \mathcal{H}\}$, we have

$$[W(x)f, W(x)f]_{\mathcal{H}} \le \|K(x,x)\| [f,f]_{\mathcal{R}}. \tag{37}$$

Thus, $W(x)$ is bounded on a dense linear manifold of $\mathcal{R}$ and hence it extends by continuity to an operator $W(x) \in B(\mathcal{H},\mathcal{K})$. From (35) we conclude that $V(x)$ is adjointable and $V(x)^* = W(x)$ for all $x \in X$.

Finally, by the reproducing axiom, for all $x,y \in X$ and all $h,g \in \mathcal{H}$ we have

$$[V(y)^*V(x)h, g]_{\mathcal{H}} = [V(x)h, V(y)g]_{\mathcal{R}} = [K_x h, Kyg]_{\mathcal{R}} = [K(y,x)h, g]_{\mathcal{H}},$$

hence $V(y)^*V(x) = K(y,x)$ for all $x, y \in X$. Thus, $(\mathcal{K};\, V)$ is a Kolmogorov decomposition of K (actually, a minimal one).

Given $K: X \times X \to B^*(\mathcal{H})$ a positive semidefinite kernel, as a consequence of Theorem (4.2.7) and statement (b) in Proposition (4.2.6), we can denote, without any ambiguity, by $\mathcal{R}_K$ the unique $\mathcal{H}$-reproducing kernel VH-space on X associated to K.

Remark (4.2.8)[204]: There is another by-product of the proofs of Theorem (4.2.5) and that of Theorem (4.2.7) namely, that, up to the abstract completion, the construction in Theorem (4.2.5) is essentially an $\mathcal{H}$-reproducing kernel one. More precisely, with the notation as in the proof of the implication (i) ⇒ (ii) of Theorem (4.2.5), we first note that, for arbitrary $f \in \mathcal{F}(X;\mathcal{H})$, $f = Kg$ with $g \in \mathcal{G}(X;\mathcal{H})$, we have

$$f = \sum_{x\in X} K(y,x)g(x) = \sum_{x\in X} K_x(y)g(x), \tag{38}$$

hence $\mathcal{F}_0(X;\mathcal{H}) = \mathrm{Lin}\{K_x h \mid x \in X,\ \ h \in \mathcal{H}$. Then, for arbitrary $f \in \mathcal{F}_0$ we have

$$[f, K_x h]_{\mathcal{K}} = [f, K_x h]_{\mathcal{F}_0} = [f, K\delta_x h]_{\mathcal{F}_0} = \sum_{y\in X} [f(y), (\delta_x h)(y)]_{\mathcal{H}}$$

$$= [f(x), h]_{\mathcal{H}} = [f, K_x h]_{\mathcal{R}(K)}, \qquad x \in X,\ \ h \in \mathcal{H},$$

hence $[\cdot,\cdot]_{\mathcal{K}} = [\cdot,\cdot]_{\mathcal{R}(K)}$ on $\mathcal{F}_0(X;\mathcal{H}) = \mathrm{Lin}\{K_x h \mid x \in X,\ \ h \in \mathcal{H}$, that is dense in both $\mathcal{K}$ and $\mathcal{R}(K)$. Therefore, we can take $\mathcal{K} = \mathcal{R}(K)$ as the completion of $\mathcal{F}_0(X;\mathcal{H})$ to a VH-space, with the advantage that it consists entirely of $\mathcal{H}$-valued functions on X and hence, it is very "concrete".

This fact can be put in the following way as well: the completion of the VE-space $\mathcal{F}_0(X;\mathcal{H})$ can be performed within $\mathcal{F}(X;\mathcal{H})$, and this is exactly the $\mathcal{H}$-reproducing kernel VH-space $\mathcal{H}(K)$. In order to prove this statement there are, at least, two different paths. One way is that we just mentioned, going through the Kolmogorov decomposition $(\mathcal{K};\, V)$ obtained in the proof of Theorem (4.2.5). Alternatively, there is a more direct way that we can briefly outline: if (f_j) is a net in $\mathcal{F}_0(X;\mathcal{H})$ that is Cauchy with respect to the locally convex topology of the VE-space $\mathcal{F}_0(X;\mathcal{H})$, one can prove that for all $x \in X$ the net $(f_j(x))$ is Cauchy within the VH-space $\mathcal{H}$, which is complete, hence let $f(x) \in \mathcal{H}$ be its limit. In this way, we obtained $f \in \mathcal{F}(X;\mathcal{H})$ and it remains to prove that $f \in \mathcal{H}(K)$ and the net (f_j) converges to f in $\mathcal{H}(K)$.

Let X be a nonempty set, a (multiplicative) semigroup Γ, and an action of Γ on X, denoted by $\xi \cdot x$, for all $\xi \in \Gamma$ and all $x \in X$. By definition, we have

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x \text{ for all } \alpha, \beta \in \Gamma \text{ and all } x \in X. \tag{39}$$

Alternatively, this means that we have a semigroup morphism $\Gamma \ni \xi \longmapsto \xi\, \cdot \in G(X)$, where $G(X)$ denotes the semigroup, with respect to composition, of all maps $X \to X$. In case the semigroups Γ has a unit ϵ, the action is called unital if $\epsilon \cdot x = x$ for all $x \in X$, equivalently, $\epsilon\, \cdot = \mathrm{Id}_X$.

We assume further that Γ is a $*$-semigroup, that is, there is an involution $*$ on Γ; this means that $(\xi\eta)^* = \eta^*\xi^*$ and $(\xi^*)^* = \xi$ for all $\xi, \eta \in \Gamma$. Note that, in case Γ has a unit ϵ then $\epsilon^* = \epsilon$.

Given a VH-space H we are interested in those Hermitian kernels $K: X \times X \to B^*(\mathcal{H})$ that are invariant under the action of Γ on X, that is,

$$K(y, \xi \cdot x) = K(\xi^* \cdot y, x) \text{ for all } x, y \in X \text{ and all } \xi \in \Gamma. \tag{40}$$

A triple $(\mathcal{K};\, \pi;\, V\,)$ is called an invariant Kolmogorov decomposition, of the kernel K and the action of Γ on X, if:

(ikd1) $(\mathcal{K};\ V)$ is a Kolmogorov decomposition of the kernel K.
(ikd2) $\pi: \Gamma \to B * (K)$ is a $*$-representation, that is, a multiplicative $*$-morphism.
(ikd3) V and π are related by the formula: $V(\xi \cdot x) = \pi(\xi)V(x)$, for all $x \in X$, $\xi \in \Gamma$.

In order to explain this definition, let $(\mathcal{K};\ \pi;\ V\ )$ be an invariant Kolmogorov decomposition of the kernel K. Since $(\mathcal{K};\ V)$ is a Kolmogorov decomposition and taking into account the axiom (ikd3), for all $x, y \in X$ and all $\xi \in \Gamma$, we have

$$\begin{aligned} K(y, \xi \cdot x) &= V(y)^* V(\xi \cdot x) = V(y)^*\, \pi(\xi) V(x) \\ &= (\pi(\xi^*) V\ (y))^* V(x) = K(\xi^* \cdot y, x), \end{aligned} \tag{41}$$

hence K is invariant under the action of Γ on X.

If, in addition to the axioms (idk1)–(idk3), the triple $(\mathcal{K};\ \pi;\ V\ )$ has also the property (ikd4) Lin $V(X)\mathcal{H}$ is dense in $\mathcal{K}$,
that is, the Kolmogorov decomposition $(\mathcal{K};\ V)$ is minimal, then $(\mathcal{K};\ \pi;\ V\ )$ is called a minimal invariant Kolmogorov decomposition of K and the action of Γ on X.

The next proposition shows that minimal invariant Kolmogorov decompositions have a built-in linearity property.

Proposition (4.2.9)[204]: Assume that, given an VH-operator valued kernel K, invariant under the action of the $*$-semigroup Γ on X, for some fixed $\alpha, \beta, \gamma \in \Gamma$ we have $K(y, \alpha \cdot x) + K(y, \beta \cdot x) = K(y, \gamma \cdot x)$ for all $x, y \in X$. Then for any minimal invariant Kolmogorov decomposition $(\mathcal{K};\ \pi;\ V\ )$ of K, the representation satisfies $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$.

Proof. For any $x, y \in X$ and any $h, k \in \mathcal{H}$ we have

$$\begin{aligned} [(\pi(\alpha) + \pi(\beta))V(x)h, V(y)k]_{\mathcal{H}} &= [V^*(y)\pi(\alpha)V(x)h + V^*(y)\pi(\beta)V(x)h, k]_{\mathcal{H}} \\ &= [K(y, \alpha \cdot x)h\ +\ K(y, \beta \cdot y), k]_{\mathcal{H}} \\ &= [K(y.\gamma \cdot x)h, k]_{\mathcal{H}} \\ &= [V(y)^* \pi(\gamma)V(x)h, k]_{\mathcal{H}} \\ &= [\pi(\gamma)V(x)h, V(y)k]_{\mathcal{H}}, \end{aligned}$$

hence, since $V(X)\mathcal{H}$ is total in $\mathcal{H}$, it follows that $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$.

Two invariant Kolmogorov decompositions $(\mathcal{K};\ \pi;\ V\ )$ and $(\mathcal{K}';\ \pi';\ V'\ )$, of the same Hermitian kernel K, are called unitary equivalent if there exists a unitary operator U : $\mathcal{K} \to \mathcal{K}'$ such that $U\pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$, and $UV(x) = V'(x)$ for all $x \in X$. Let us note that, in case both of these invariant Kolmogorov decompositions are minimal, then this is equivalent with the requirement that the Kolmogorov decompositions $(\mathcal{K};\ V)$ and $(\mathcal{K}';\ V')$ are unitary equivalent.

Theorem (4.2.10)[204]: Let Γ be a unital $*$-semigroup that acts on the nonempty set X, and let $K: X \times X \to B^*(\mathcal{H})$ be a kernel, for some VH-space $\mathcal{H}$ over an admissible space Z. The following assertions are equivalent:

(i) K satisfies the following conditions:
(a) K is positive semidefinite.
(b) K is invariant under the action of Γ on X, that is, (40) holds.
(c) For any $\alpha \in \Gamma$ there exists $c(\alpha) \geq 0$ such that

$$\sum_{j,k=1}^{n} [K(\alpha \cdot x_k, \alpha \cdot x_j)h_j, h_k]_{\mathcal{H}} \leq c(\alpha)^2 \sum_{j,k=1}^{n} [K(x_k, x_j)h_j, h_k]_{\mathcal{H}}, \tag{42}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $h_1, \ldots, h_n \in \mathcal{H}$.

(ii) K has an invariant Kolmogorov decomposition $(\mathcal{K};\ \pi;\ V)$.

(iii) K admits an $\mathcal{H}$-reproducing kernel VH-space $\mathcal{R}$ and there exists a $*$-representation $\rho\colon \Gamma \to B^*(\mathcal{R})$ such that $\rho(\xi)K_x h = K_{\xi\cdot x}h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

Moreover, in case any of the assertions (i), (ii), or (iii) holds, then a minimal invariant Kolmogorov decomposition can be constructed, any minimal invariant Kolmogorov decomposition is unique up to unitary equivalence, and the pair $(\mathcal{R};\ \rho)$ as in assertion (iii) is uniquely determined by K as well.

Proof. (i) ⇒ (ii). We consider the notation and the minimal Kolmogorov decomposition $(\mathcal{K};\ V)$ constructed as in the proof of the implication (i) ⇒ (ii) of Theorem (4.2.5). For each $\xi \in \Gamma$ we let $\pi(\xi)\colon \mathcal{F} \to \mathcal{F}$ be defined by

$$(\pi(\xi)f)(y) = f(\xi^* \cdot y), \qquad y \in X,\ \ \xi \in \Gamma. \tag{43}$$

We claim that $\pi(\xi)$ leaves $\mathcal{F}_0$ invariant. To see this, let $f \in \mathcal{F}_0$, that is, $f = Kg$ for some $g \in \mathcal{G}$ or, even more explicitly, by (24),

$$f(y) = \sum_{x\in X} K(y,x)g(x)\,, \qquad y \in X. \tag{44}$$

Then,

$$f(\xi^* \cdot y) = \sum_{x\in X} K(\xi^* \cdot y, x)g(x) \tag{45}$$
$$= \sum_{x\in X} K(y, \xi \cdot x)g(x) = \sum_{z\in X} K(y,z)g_\xi(z),$$

where,

$$g\xi(z) = \begin{cases} 0, & \text{if } \xi \cdot x = z \text{ has no solution } x \in X, \\ \sum_{\xi\cdot x = z} g(x)\,, & \text{otherwise.} \end{cases} \tag{46}$$

Since clearly $g_\xi \in \mathcal{G}$, that is, g_ξ has finite support, it follows that $\pi(\xi)$ leaves $\mathcal{F}_0$ invariant. In the following we denote by the same symbol $\pi(\xi)$ the map $\pi(\xi)\colon \mathcal{F}_0 \to \mathcal{F}_0$.

In the following we prove that π is a representation of the semigroup Γ on the complex vector space $\mathcal{F}_0$, that is,

$$\pi(\alpha\beta)f = \pi(\alpha)\pi(\beta)f, \qquad \alpha,\beta \in \Gamma, f \in \mathcal{F}_0. \tag{47}$$

To see this, let $f \in \mathcal{F}_0$ be fixed and denote $h = \pi(\beta)f$, that is, $h(y) = f(\beta^* \cdot y)$ for all $y \in X$. Then $\pi(\alpha)\pi(\beta)f = \pi(\alpha)h$, that is, $(\pi(\alpha)h)(y) = h(\alpha^* \cdot\ y) = h(\beta^*\alpha^* \cdot y) = h((\alpha\beta)^* \cdot y) = (\pi(\alpha\beta))(y)$, for all $y \in X$, which proves (47) Next we show that π is actually a $*$-representation, that is,

$$[\pi(\xi)f, f']_{\mathcal{F}_0} = [f, \pi(\xi^*)f']_{\mathcal{F}_0}, \qquad f, f' \in \mathcal{F}_0. \tag{48}$$

To see this, let $f = Kg$ and $f' = Kg'$ for some $g, g' \in \mathcal{G}$. Then, recalling (25) and (45),

$$[\pi(\xi)f, f']_{\mathcal{F}_0} = \sum_{y\in X} [f(\xi^* y), g'(y)]_{\mathcal{H}} = \sum_{x,y\in X} [K(\xi^* \cdot y, x)g(x), g'(y)]_{\mathcal{H}}$$
$$= \sum_{x,y\in X} [K(y, \xi \cdot x)g(x), g'(h)]_{\mathcal{H}} = \sum_{x,y\in X} [g(x), K(\xi \cdot x, y)g'(y)]_{\mathcal{H}}$$
$$= \sum_{y\in X} [g(x), f'(\xi \cdot x)]_{\mathcal{H}} = [f, \pi(\xi^*)f']_{\mathcal{H}},$$

and hence the formula (48) is proven.

Considering $\mathcal{F}_0$ as a VE-space, we prove now that $\pi(\xi)$ is bounded for all $\xi \in \Gamma$. Indeed, let $f = Kg$ for some $g \in \mathcal{G}$. Using the definition of $\pi(\xi)$ and the boundedness condition (c), we have

$$
\begin{aligned}
[\pi(\xi)f,\pi(\xi)f]_{\mathcal{F}_0} &= [\pi(\xi^*)\pi(\xi)f,f]_{\mathcal{F}_0} = [\pi(\xi^*\xi)f,f]_{\mathcal{F}_0} \\
&= \sum_{x,y\in X} [K(\xi^*\xi\cdot y,x)g(x),g(y)]_{\mathcal{H}} \\
&= \sum_{x,y\in X} [K(\xi\cdot y,\xi\cdot x)g(x),g(y)]_{\mathcal{H}} \\
&\le c(\xi)^2 \sum_{x,y\in X} [K(y,x)g(x),g(y)]_{\mathcal{H}} \\
&= c(\xi)^2[f,f]_{\mathcal{F}_0},
\end{aligned}
$$

and hence the boundedness of $\pi(\xi)$ is proven. This implies that $\pi(\xi)$ can be uniquely extended by continuity to an operator $\pi(\xi)\in B(\mathcal{H})$. In addition, since $\pi(\xi^*)$ also extends by continuity to an operator $\pi(\xi^*)\in B(\mathcal{H})$ and taking into account (48), it follows that $\pi(\xi)$ is adjointable and $\pi(\xi^*)=\pi(\xi)^*$. We conclude that π is a $*$-representation of Γ in $B^*(\mathcal{H})$, that is, the axiom (ikd2) holds.

In order to show that the axiom (ikd3) holds as well, we use (28). Thus, for all $\xi\in\Gamma$, $x,y\in X$, $h\in\mathcal{H}$, and taking into account that K is invariant under the action of Γ on X, we have

$$
\begin{aligned}
(V(\xi\cdot x)h)(y) &= K(y,\xi\cdot x)h = K(\xi^*\cdot y,x)h \\
&= (V(x)h)(\xi^*\cdot y) = (\pi(\xi)V(x)h)(y),
\end{aligned}
\tag{49}
$$

which proves (ikd3). Thus, $(\mathcal{K};\ \pi;\ V)$, here constructed, is an invariant Kolmogorov decomposition of the Hermitian kernel K. Note that $(\mathcal{K};\ \pi;\ V)$ is minimal, that is, the axiom (ikd4) holds, since the Kolmogorov decomposition $(\mathcal{K};\ V)$ is minimal, by the proof of Theorem (4.2.5).

In order to prove the uniqueness of the minimal invariant Kolmogorov decomposition, let $(\mathcal{K}';\ \pi';\ V')$ be another minimal invariant Kolmogorov decomposition of K. We consider the unitary operator $U:\mathcal{K}\to\mathcal{K}'$ defined as in (31) and we already know that $UV(x)=V'(x)$ for all $x\in X$. Since, for any $\xi\in\Gamma$, $x\in X$, and $h\in\mathcal{H}$, we have

$$
U\pi(\xi)V(x)h = UV\,(\xi\cdot x)h = V'(\xi\cdot x)h = \pi'(\xi)V'(x)h = \pi'(\xi)UV\,(x)h,
$$

and taking into account the minimality, it follows that $U\pi(\xi)=\pi'(\xi)U$ for all $\xi\in\Gamma$.

(ii) ⇒ (i). Let $(\mathcal{K};\ \pi;\ V)$ be an invariant Kolmogorov decomposition of the kernel K. We already know from the proof of Theorem (4.2.5) that K is positive semidefinite and it was shown in (41) that K is invariant under the action of Γ on X. In order to show that the boundedness condition (c) holds as well, let $\alpha\in\Gamma$, $n\in\mathrm{N}$, $x_1,\ldots,x_1\in X$, and $h_1,\ldots,h_1\in\mathcal{H}$ be arbitrary. Then

$$
\begin{aligned}
\sum_{j,k=1}^{n}[K(\alpha\cdot x_k,\alpha\cdot x_j)h_j,h_k]_{\mathcal{H}} &= \sum_{j,k=1}^{n}[V(x_k)*\pi(\alpha *)\pi(\alpha)V(x_j)h_j,h_k]_{\mathcal{H}} \\
&= \sum_{j,k=1}^{n}[\pi(\alpha)V(x_j)h_j,\pi(\alpha)V(x_k)h_k]_{\mathcal{H}} \\
&\le \|\pi(\alpha)\|^2\sum_{j,k=1}^{n}[V(x_j)h_j,V(x_k)h_k]_{\mathcal{H}} \\
&= \|\pi(\alpha)\|^2\sum_{j,k=1}^{n}[K(x_k,x_j)h_j,h_k]_{\mathcal{H}},
\end{aligned}
$$

and hence (c) holds with $c(\alpha) = \pi(\alpha) \geq 0$.

(ii) ⇒ (iii). Let $(\mathcal{K};\ \pi;\ V)$ be an invariant Kolmogorov decomposition of the kernel K and the action of Γ on X. Without loss of generality, we can assume that it is minimal. We use the notation and the facts established during the proof of the implication (ii) ⇒ (iii) in Theorem (4.2.7). Then, for all $x, y \in X$ and $h \in \mathcal{H}$ we have

$$K_{\xi \cdot x}(y)h = K(y, \xi \cdot x)h = V(y)^* V(\xi \cdot x) = V(y)^* \pi(\xi) V(x)h,$$

hence, letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U : \mathcal{K} \to \mathcal{R}$ is the unitary operator defined as in (33), we obtain a $*$-representation of Γ on the VH-space $\mathcal{R}$ such that $K_{\xi \cdot x} = \rho(\xi) K_x$ for all $\xi \in \Gamma$ and $x \in X$.

(iii) ⇒ (ii). Let $(\mathcal{R};\ \rho)$, where $\mathcal{R} = \mathcal{R}(K)$ is the $\mathcal{H}$-reproducing kernel VH-space of K and $\rho : \Gamma \to B^*(\mathcal{R})$ is a $*$-representation such that $K_{\xi \cdot x} = \rho(\xi) K_x$ for all $\xi \in \Gamma$ and $x \in X$. As in the proof of the implication (iii) ⇒ (ii) in Theorem (4.2.7), we show that$(\mathcal{R};\ V)$, where V is defined as in (35), is a minimal Kolmogorov decomposition of K. Letting $\pi = \rho$, it is then easy to see that $(\mathcal{R};\ \pi;\ V)$ is an invariant Kolmogorov decomposition of the kernel K and the action of Γ on X.

We show that Theorem (4.2.10) contains both the Loynes generalization of the Sz.-Nagy's Dilation Theorem and the VH-space operator valued generalization of Stinespring's Dilation Theorem.

Recall that a $*$-semigroup is a (multiplicative) semigroup Γ on which there exists an involution, denoted by $*$, that is, $\Gamma \ni \gamma \longmapsto \gamma^* \in \Gamma$ having the properties: $(\beta\gamma)^* = \gamma^*\beta^*$ and $(\gamma^*)^* = \gamma$, for all $\beta, \gamma \in \Gamma$. If Γ has a unit ϵ then $\epsilon^* = \epsilon$. In case Γ is a group and we use the multiplicative notation, we can take $\gamma^* = \gamma^{-1}$, but other choices are also possible.

Let $\mathcal{H}$ be a VH-space and consider a family $T = \{T\xi\}_{\xi \in \Gamma}$ of operators in $B^*(\mathcal{H})$ indexed by a $*$-semigroup Γ. However, taking into account the framework of this article, it is preferable to think T as a function on Γ and valued in $B^*(\mathcal{H})$. Given n an arbitrary natural number, we call T n-positive if for any $\eta_1, \ldots, \eta_n \in \Gamma$ and any $h_1, \ldots, h_n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^{n} [T_{\eta_i^* \eta_j} h_j, h_i] \geq 0. \tag{50}$$

It is easy to see that, if T is n-positive then it is k-positive for all natural numbers $k \leq n$. T is called positive semidefinite if it is n-positive for all natural numbers n.

To any map $T : \Gamma \to B^*(\mathcal{H})$ we associate a kernel $K : \Gamma \times \Gamma \to B^*(\mathcal{H})$ by

$$K(\xi, \eta) = T_{\eta^* \xi}, \qquad \xi, \eta \in \Gamma. \tag{51}$$

Then, the kernel K is n-positive, in the sense of (17), if and only if T is n-positive, in the sense of (50). Consequently, K is positive semidefinite if and only if T is positive semidefinite. Having in mind the Kolmogorov decompositions of Hermitian kernels, we introduce the following definition. A pair $(\mathcal{K}; U)$ is called a linearization of T if:

(a) $\mathcal{K}$ is a VH-space on Z.

(b) $U : \Gamma \to B^*(\mathcal{K}, \mathcal{H})$.

(c) $T_{\eta^* \xi} = U(\xi) U^*(\eta)$ for all $\xi, \eta \in \Gamma$.

Proposition (4.2.11)[204]: Given $\mathcal{H}$ a VH-space on the admissible space Z, a unital $*$-semigroup Γ, and a map $T : \Gamma \to B^*(\mathcal{H})$, the following assertions are equivalent:

(i) T is positive semidefinite.

(ii) T admits a linearization $(\mathcal{K}; U)$.

(iii) T yields a reproducing kernel VH-space $\mathcal{R}$ over Z, that is:

(a) $\mathcal{R}$ consists of functions $f : \Gamma \to \mathcal{H}$ only.

(b) $T_{\xi}.h \in R$, that is, the map $\Gamma \ni \eta \longmapsto T_{\xi\eta}h \in \mathcal{H}$ belongs to $\mathcal{R}$, for all $\xi \in \Gamma$ and all $h \in \mathcal{H}$.

(c) $[f(\xi), h]_{\mathcal{H}} = [f, T_{\xi^*}.h]_{\mathcal{R}}$ for all $\xi \in \Gamma$, $h \in \mathcal{H}$, and $f \in \mathcal{R}$.

Proof. The equivalence of (i) and (ii) is a consequence of the observations from before applying Theorem (4.2.5) to the kernel $K(\xi,\eta) = T_{\eta^*\xi}$, for $\xi,\eta \in \Gamma$, and letting $U(\xi) = V^*(\xi)$, for all $\xi \in \Gamma$.

In order to prove the equivalence of (ii) and (iii), we apply Theorem (4.2.7) to the kernel $K(\xi,\eta) = T_{\eta^*\xi}$, for $\xi,\eta \in \Gamma$, and observing that $K\xi h = T_{\xi^*}.h$, where, $T_{\xi^*}.h$ denotes the map $\Gamma \ni \eta \longmapsto T_{\xi\eta}h \in \mathcal{H}$, for all $\xi \in \Gamma$ all $h \in \mathcal{H}$.

The kernel $K(\xi,\eta) = T_{\eta^*\xi}$ has an additional property, namely that it is invariant under the action of Γ on itself by left multiplication: $\xi \cdot \eta = \xi\eta$. With the definition as in (40), this is proven as follows: for all $\xi,\eta,\gamma \in \Gamma$, we have

$$K(\eta, \xi \cdot \gamma) = T_{(\xi\cdot\gamma)^*\eta} = T_{\gamma^*(\xi^*\cdot\eta)} = K(\xi^* \cdot \eta, \gamma). \tag{52}$$

Thus, we can consider T having in mind the invariant Kolmogorov decomposition and its reproducing kernel counter-part, as in Theorem (4.2.10), in order to obtain the following:

Theorem (4.2.12)[204]: Let Γ be a $*$-semigroup with unit ϵ and $T = \{T\xi\}_{\xi\in\Gamma} \subseteq B^*(\mathcal{H})$, for some VH-space $\mathcal{H}$. The following assertions are equivalent:

(i) (a)T satisfies the following conditions:

(b) T is positive semidefinite as a function on Γ, in the sense that for any finitely supportedfamily $\{g\xi\}_{\xi\in\Gamma}$ in $\mathcal{H}$ we have

$$\sum_{\xi,\eta\in\Gamma} [T_{\xi^*\eta} g_\eta, g_\xi] \geq 0.$$

(c) For any $\alpha \in \Gamma$ there exists a nonnegative number $c(\alpha)$ such that for any finitely supported family $g = \{g\xi\}_{\xi\in\Gamma}$ in $\mathcal{H}$ we have

$$\sum_{\xi,\eta\in\Gamma} [T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi] \leq c(\alpha)^2 \sum_{\xi,\eta\in\Gamma} [T_{\xi^*\eta} g_\eta, g_\xi]. \tag{53}$$

(ii) There exists a VH-space $\mathcal{K}$, a $*$-representation $D = \{D_\xi\}_{\xi\in\Gamma}$ of Γ in $B^*(\mathcal{K})$, and an operator $V \in B^*(\mathcal{H},\mathcal{K})$, such that

$$T_\xi = V^* D_\xi V, \qquad \xi \in \Gamma. \tag{54}$$

Moreover, under the assumption of Theorem (4.2.12), the VH-space $\mathcal{K}$ can be obtained minimal in the sense that it is spanned by elements of the form $D_\xi Vf$, where $f \in \mathcal{H}$ and $\xi \in \Gamma$ and, in this case, the triple $(\mathcal{K}; D;\ V)$ is uniquely determined up to an isomorphism of VH-spaces that intertwines the $*$-representations and keeps the corresponding operators V.

In addition, in case $T_\epsilon = I, \mathcal{H}$ can be isometrically embedded as an orthocomplemented subspace into $\mathcal{K}$ and, letting $P_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$, we have

$$T_\xi = P_{\mathcal{H}} D_\xi|_{\mathcal{H}}, \qquad \xi \in \Gamma. \tag{55}$$

As a consequence of the reproducing kernel version for invariant kernels, see Theorem (42.10).(iii), and the reproducing kernel version of the linearization as in Proposition (4.2.11), we get the following:

Corollary (4.2.13)[204]:. With the notation as in Theorem (4.2.12), each of the assertions (i) and (ii) is equivalent with (iii) T admits a reproducing kernel VH-space $\mathcal{R}$ over Z, in the sense of the properties (a)–(c) in Proposition (4.2.11)(iii), and a $*$-representation $\rho : \Gamma \to B^*(\mathcal{R})$, such that

(d) $\rho : \Gamma \to B^*(\mathcal{R})$ is a $*$-representation such that $\rho(\xi)T_\eta.h = T_{\eta\xi^*}.h$, in the sense that map $\Gamma \ni \gamma \longmapsto \rho(\xi)T_{\eta\gamma}h \in \mathcal{H}$ coincides with the map $\Gamma \ni \gamma \longmapsto T_{\eta\xi^*\gamma}h \in \mathcal{H}$, for all $\xi, \eta \in \Gamma$ and all $h \in \mathcal{H}$.

Also, the pair $(\mathcal{R};\ \rho)$ is uniquely determined by T, with properties (a)–(d).

Let $\mathcal{A}$ be a complex $*$-algebra with unit 1. Recall that the involution $*$ is supposed to be conjugate linear, $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$, $(a^*)^* = a$ for all $a \in \mathcal{A}$, and that $1^* = 1$. In particular, $\mathcal{A}$ has an underlying structure of a unital multiplicative $*$-semigroup. Also, by definition, an element $a \in \mathcal{A}$ is positive if $a = x^*x$ for some $x \in \mathcal{A}$. This definition, for general $*$-algebras, may not mean too much, but it is the right definition in the case of C^*-algebras.

For an arbitrary VH-space $\mathcal{H}$ over the admissible space Z, let $\varphi : \mathcal{A} \to B^*(\mathcal{H})$ be a linear map. A kernel $K\colon \mathcal{A} \times \mathcal{A} \to B^*(\mathcal{H})$ can be defined by

$$K(b, a) = \phi(a^*b), a, b \in \mathcal{A}. \tag{56}$$

With this observation, the types of positivity for kernels, we have transcriptions to this setting: φ is n-positive, for some natural number n, if for any $a_1, \dots, a_n \in \mathcal{A}$ and $h_1, \dots, h_n \in \mathcal{H}$ we have

$$\sum_{i,j=1}^{n} [\varphi(a_i^*\, a_j)h_j, h_i]_{\mathcal{H}} \geq 0,$$

and, respectively, φ is positive semidefinite if it is n-positive for all $n \in \mathbb{N}$.

There is another notion of positivity that has been considered, following the original terminology of Stinespring [223]. Given $\mathcal{A}$ a $*$-algebra, a linear map $\varphi : \mathcal{A} \to B^*(\mathcal{H})$ is called positive if $\varphi(a^*a) \geq 0$ for any $a \in \mathcal{A}$. Given $n \in \mathbb{N}$, there is a natural identification of $*$-algebras of $M_n(A)$, the algebra of all $n \times n$ matrices with entries in $\mathcal{A}$, with $M_n \otimes \mathcal{A}$, organized as a $*$-algebra similarly in a natural way (e.g. see [222]). A linear map $\varphi_n : M_n(A) \to M_n(B^*(\mathcal{H}))$ is naturally associated to φ by

$$\varphi_n([a_{i,j}\,]_{i,j=1}^{n}) = [\varphi[a_{i,j}\,]_{i,j=1}^{n}, [a_{i,j}\,]_{i,j=1}^{n} \in M_n(\mathcal{A}). \tag{57}$$

The importance of this construction, and its consequences in terms of positivity, relies on its "quantization" interpretation, which basically comes from the interpretation of the tensor product of two Hilbert spaces as the aggregate of two quantum systems.

Taking into account that any positive element $A = [a_{i,j}\,]_{i,j=1}^{n} \in M_n(\mathcal{A})$ can be decomposed

$$A = A_1^*A_1 + \cdots + A_n^*A_n,$$

where A_k is the $n \times n$ matrix having its k-th row with entries $a_{k,j}, j = 1, \dots, n$, and all the other entries null, we get the following fact, essentially proven in [223]:

Proposition (4.2.14)[204]: Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{H}$ a VH-space, and a linear map $\varphi : \mathcal{A} \to B^*(\mathcal{H})$.

(i) For arbitrary $n \in \mathbb{N}$, φ is n-positive if and only if φ_n is positive.

(ii) φ is positive semidefinite if and only if φ_n is positive for all $n \in \mathbb{N}$.

φ is called completely positive if φ_n is positive for all $n \in \mathbb{N}$, hence, Proposition (4.2.14) says that complete positivity is the same with positive semidefiniteness, in this setting.

We make now the observation that the kernel K defined as in (56) is invariant under the action of $\mathcal{A}$ on itself by left multiplication,

$$K(b, c \cdot a) = \varphi((ca)^* b) = \varphi(a^* c^* b) = K(c^* \cdot b, a), \qquad a, b, c \in \mathcal{A}. \tag{58}$$

Thus, dilations of completely positive maps φ fall under Theorem (4.2.10) for dilations of positive semidefinite kernels that are invariant under actions of $*$-semigroup. However, there is an important difference due to the fact that the rich structure of the B^*-algebra yields the boundedness condition (42) for free, in this special setting; we briefly recall the argument in [212]. Let $a \in \mathcal{A}$ and finitely supported $\{h_b\}_{b \in \mathcal{A}}$ in $\mathcal{H}$. Since φ is positive semidefinite, for any $y \in \mathcal{A}$ we have

$$\sum_{b,c \in \mathcal{A}} [\varphi(c^* y^* y b) h_b, h_c]_{\mathcal{H}} \geq 0. \tag{59}$$

Without loss of generality we can assume that $a < 1$ and let $x = a^* a$, hence $\|x\| \leq \|a\|^2 < 1$. Following an idea in [207] and using an exercise in [206] at page 125, we consider the power series of the analytic complex function $(1 - \lambda)^{1/2}$ that converges in the open unit disc

$$(1 - \lambda)^{1/2} = 1 - \sum_{n \geq 1} c_n \lambda^n$$

and let

$$y = 1 - \sum_{n \geq 1} c_n x^n \in \mathcal{A}. \tag{60}$$

It is easy to see that $y = y^*$, since $x = x^*$, and that $1 - a^* a = 1 - x = y^2$, hence, from (59) it follows

$$\sum_{b,c \in \mathcal{A}} [\varphi(c^* a^* a b) h_b, h_c]_{\mathcal{H}} \leq \sum_{b,c \in \mathcal{A}} [\varphi(c^* b) h_b, h_c]_{\mathcal{H}}, \tag{61}$$

which proves that, there exists a nonnegative number $c(a)$ such that

$$\sum_{b,c \in \mathcal{A}} [\varphi(c^* a^* a b) h_b, h_c]_{\mathcal{H}} \leq c(a)^2 \sum_{b,c \in \mathcal{A}} [\varphi(c^* b) h_b, h_c]_{\mathcal{H}}, \tag{62}$$

The following theorem summarizes the consideration from above in the form of a Stinespring type theorem that falls under Theorem (4.2.10). In [212] this theorem has been proven in two other different ways.

Theorem (4.2.15)[204]: Let $\mathcal{A}$ be a unital B^*-algebra, $\mathcal{H}$ a VH-space over an admissible space Z, and let $\varphi : \mathcal{A} \to B^*(\mathcal{H})$ be a linear map. Then φ is a completely positive if and only if there exists $\mathcal{K}$ a VH-space over the same admissible space Z, an operator $V \in B^*(\mathcal{H}, \mathcal{K})$ and a $*$-representation $\rho: \mathcal{A} \to B^*(\mathcal{K})$ such that

$$\varphi(a) = V^* \rho(a) V, \text{ for all } a \in \mathcal{A}. \tag{63}$$

Moreover, the VH-space $\mathcal{K}$ can be obtained minimal in the sense that $\mathcal{K} = \overline{\operatorname{Lin}\{\varphi(\mathcal{A})\mathcal{H}\}}$ and, in this case, the triple $(\rho;\ V;\mathcal{K})$ is unique, modulo a unitary operator of VH-spaces that intertwines the $*$-representations and keeps the operators V.

In addition, if φ is unital, $\mathcal{H}$ can be isometrically embedded as an orthocomplemented subspace of $\mathcal{K}$ and, letting $P_{\mathcal{H}}$ denote the orthogonal projection onto $\mathcal{H}$, we have

$$\varphi(a) = P_{\mathcal{H}} \rho(a)|_{\mathcal{H}}, \text{ for all } a \in \mathcal{A}. \tag{64}$$

Finally, we point out that, in Theorem (4.2.15), a reproducing kernel representation can always be obtained, similarly as in Corollary (4.2.13).

Corollary (4.2.16)[204]: Let $\mathcal{A}$ be a unital B^*-algebra, $\mathcal{H}$ a VH-space over an admissible space Z, and $\varphi : \mathcal{A} \to B^*(\mathcal{H})$ be a completely positive linear map. Then, there exists a pair $(\mathcal{R};\ \pi)$ subject to, and uniquely determined by, the following properties:

(i) $\mathcal{R}$ consists of functions $f : \mathcal{A} \to \mathcal{H}$ only.
(ii) $\varphi(a\,\cdot)h \in \mathcal{R}$, that is, the function $\mathcal{A} \ni b \longmapsto \varphi(ab)h \in \mathcal{H}$ belongs to $\mathcal{R}$, for all $a \in \mathcal{A}$ and all $h \in \mathcal{H}$.
(iii) $[f(a), h]_{\mathcal{H}} = [f, \varphi(a^*\,\cdot)h]_{\mathcal{R}}$ for all $f \in \mathcal{R}$, $a \in \mathcal{A}$, and $h \in \mathcal{H}$.
(iv) $\pi : \mathcal{A} \to B^*(\mathcal{R})$ is a $*$-algebra representation such that $\pi(a)\varphi(b\,\cdot)h = \varphi(ba^*\,\cdot)h$ for all $a, b \in \mathcal{A}$ and all $h \in \mathcal{H}$.

Section (4.3): Invariant Kernels with Values Adjointable Operators

Starting with the celebrated Naimark's dilation theorems in [220] and [221], a powerful dilation theory for operator valued maps was obtained through results of B.Sz.-Nagy [225], W. F.Stinespring [223], and their generalisations to VH-spaces (Vector Hilbert spaces) by R. M.Loynes [214], or to Hilbert C^*-modules by G. G.Kasparov [216]. Taking into account the diversity of dilation theorems for operator valued maps, there is a natural question, whether one can unify all, or the most, of these dilation theorems, under one theorem. An attempt to approach this question was made in [204] by using the notion of VH-space over an admissible space, introduced by R. M.Loynes [214],[215]. As a second step in this enterprise, an investigation at the "ground level", that is, a non-topological approach, makes perfectsense. In addition, an impetus to pursue this way was given to us by the recent investigation on closely related problems, e.g. non-topological theory for operator spaces and operator systems, cf. [230],[227],[229],[228].

We present a general non-topological approach to dilation theory based on positive semidefinite kernels that are invariant under actions of $*$-semigroups and with values adjointable operators on VE-spaces (Vector Euclidean spaces) over or-dered $*$-spaces. More precisely, we show that at the level of conjunction of order with $*$-spaces or $*$-algebras and operator valued maps, one can obtain a reasonable dilation theory that contains the fabric of most of the more or less topological versions of classi-cal dilation theorems. In addition, we integrate into non-topological dilation theory, on equal foot, the reproducing kernel technique and show that almost each dilation theorem is equivalent to a realisation as a reproducing kernel space with additional properties. Our approach is based on ideas already present under different topological versions of dilation theorems in [75],[63],[214],[58],[53],[50],[209]–[211],[232],[213],[204].

We fix some terminology and facts on ordered $*$-spaces, ordered $*$-algebras, VE-spaces over ordered $*$-spaces, and VE-modules over ordered $*$-algebras. On these basic objects, one can build the ordered $*$-algebras of adjointable operators on VE-spaces or VE-modules. We provide many examples that illustrate the richness of this theory, even at the non-topological level.

We consider the main object of investigation which refers to positive semidefinite kernels with values adjointable operators on VE-spaces. We make a preparation by showing that, although analogs of Schwarz Inequality is missing at this level of generality, some basic results can be obtained by different techniques. In order to achieve a sufficient generality that allowsto recover known dilation theorems for both $*$-semigroups (B. Sz.-Nagy) and $*$-algebras (Stinespring), in view of [204], we consider positive semidefinite kernels that are invariant under actions of $*$-semigroups and that have values adjointable operators on VE-spaces. In Lemma (4.3.8), we show that, for a 2-positive kernel, if boundedness in the sense of Loynes is assumed for all the operators on the diagonal, then the entire kernel is made up by bounded operators. In this way we explain how the investigation is situated with respect to that in [204]. We briefly show the connection between linearisations and reproducing kernel spaces at this level of generality. It is this stage when we are able to state and prove the main result, Theorem (4.3.14) that, basically, shows that this kind of kernels produce $*$-representations on "dilated" VE-spaces that linearise the kernel or, equivalently, on reproducing kernel VE-spaces that can be explicitly described.

We show how non-topological versions of most of the known dilation theorems [225],[223],[214],[216],[199] can be obtained. On the other hand, in order to unify the known dilation theorems in topological versions, one needs certain topological structures on ordered $*$-spaces and VE-spaces, that lead closely to the VH-spaces over admissible spaces, as considered in [214]. This way was followed, to a certain extent, in [204] but, in order to obtain a sufficiently large generality allowing to cover most of the known topological dilation theory, one needs more flexibility by moving beyond bounded operators. We briefly review most of the definitions and some basic facts on VE-spaces over ordered $*$-spaces, inspired by cf. R. M. Loynes, [214]–[216]. We slightly modify some definitions in order to match the requirements of this investigation, notably by separating the non-topological from the topological cases and by adhering to the convention, that is very popular in Hilbert C^*-modules, to let gramians be linear in the second variable and conjugate linear in the first variable, for consistency.

A complex vector space Z is called ordered $*$-space, cf.[230], if:
(a1) Z has an involution$*$, that is, a map $Z \ni z \mapsto z^* \in Z$ that is conjugate linear ($(sx + ty)^* = \bar{s}x^* + \bar{t}y^*$ for all s, $t \in \mathbb{C}$ and all $x, y \in Z$) and involutive ($(z^*)^* = z$ for all $z \in Z$).
(a2) In Z there is a cone Z^+($sx + ty \in Z^+$ for all numbers $s, t \geq 0$ and all $x, y \in Z^+$), that is strict ($Z^+ \cap -Z^+ = \{0\}$), and consisting of self-adjoint elementsonly ($z^* = z$ for all $z \in Z^+$). This cone is used to define a partial orderon the real vector space of all selfadjoint elements in Z: $z_1 \geq z_2$ if $z_1 - z_2 \in Z^+$.

Recall that a $*$-algebra $\mathcal{A}$ is a complex algebra onto which there is defined an involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$, that is, $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(ab)^* = b^*a^*$, and $((a^*)^* = a$, for all $a, b \in \mathcal{A}$ and all $\lambda, \mu \in \mathbb{C}$.

An ordered $*$-algebra $\mathcal{A}$ is a $*$-algebra such that it is an ordered $*$-space, more precisely, it has the following property.
(osa1)There exists a strict cone $\mathcal{A}^+$ in Asuch that for any $a \in \mathcal{A}^+$ we have $a = a^*$. Clearly, any ordered $*$-algebra is an ordered $*$-space. In particular, given $a \in \mathcal{A}$, we denote $a \geq 0$ if $a \in \mathcal{A}^+$ and, for $a = a^* \in \mathcal{A}$ and $b = b^* \in \mathcal{A}$, we denote $a \geq b$ if $a - b \geq 0$.
Remark (4.3.1)[226]: In analogy with the case of C^*-algebras, given a $*$-algebra $\mathcal{A}$, one defines an element $a \in \mathcal{A}$ to be $*$-positiveif $a = \sum_{k=1}^{n} a_k^* a_k$ for some natural number nand some elements $a_1, \ldots, a_n \in \mathcal{A}$. The collection of all $*$-positive elements in a $*$-algebra is a cone, but it may fail to be strict and hence, associated is only a quasi-order, e.g. see [227] for a recent investigation. Thus, our definition of an ordered $*$-algebra specifies a strict cone $\mathcal{A}^+$ and, in general, it does not refer to the cone of $*$-positive elements as defined above, except special cases as, for example, pre C^*-algebras or pre locally C^*-algebras.
Examples (4.3.2)[226]: (i) Any C^*-algebra, e.g. see [206], Ais an ordered $*$-algebra and any $*$-subspace $\mathcal{S}$ of a C^*-algebra $\mathcal{A}$, with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations (addition, multiplication with scalars, and involution) inherited from $\mathcal{A}$, is a $*$-space.

(ii) Any pre-C^*-algebra is an ordered $*$-algebra. Any $*$-subspace $\mathcal{S}$ of a pre-C^*-algebra $\mathcal{A}$ is an ordered $*$-space, with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$.

(iii) Any locally C^*-algebra, see [215],[1], is an ordered $*$-algebra. In particular, any $*$-subspace $\mathcal{S}$ of a locally C^*-algebra $\mathcal{A}$, with the cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$, is an ordered $*$-space.

(iv) Any locally pre-C^*-algebra is an ordered $*$-algebra. Any $*$-subspace $\mathcal{S}$ of a locally pre-C^*-algebra is an ordered $*$-space, with $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$.

(v) Let V be a complex vector space and let V' be its conjugate dual space. On the vector space $\mathcal{L}(V, V')$ of all linear operators $T: V \to V'$, a natural notion of positive operator can be defined: T is positiveif $(Tv)(v) \geq 0$ for all $v \in V$. Let $\mathcal{L}^+(V, V')$be the collection of all positive operators and note that it is a strict cone. The involution $*$ in $\mathcal{L}(V, V')$ is defined in the following way: for any $T \in \mathcal{L}(V, V')$, $T^* = T'|V$, that is, the restriction to $V$of the dual operator $T': V'' \to V'$. With respect to the cone $\mathcal{L}^+(V, V')$ and the involution $*$ just defined, $\mathcal{L}(V, V')$ becomes an ordered $*$-space. See A.Weron [233], as well as D.Gaşpar and P. Gaşpar [209].

(vi) Let X be a nonempty set and denote by $\mathcal{K}(X)$ the collection of all complex valued kernels on X, that is, $\mathcal{K}(X) = \{k \mid k: X \times X \to \mathbb{C}\}$, considered as a complex vector space with the operations of addition and multiplication of scalars defined elementwise. An involution $*$ can be defined on $\mathcal{K}(X)$ as follows: $k^*(x, y) = \overline{k(y, x)}$, for all $x, y \in X$ and all $k \in \mathcal{K}(X)$. The cone $\mathcal{K}(X)^+$ consists on all positive semidefinite kernels, that is, those kernels $k \in \mathcal{K}(X)$ with the property that, for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$, the complex matrix $[k(x_i, x_j)]_{i,j=1}^n$ is positive semidefinite.

On $\mathcal{K}(X)$ a multiplication can be defined elementwise: if k, $l \in \mathcal{K}(X)$ then $(kl)(x, y) = k(x, y)l(x, y)$ for all $x, y \in X$. With respect to this multiplication and the other operations described before, $\mathcal{K}(X)$ is an ordered $*$-algebra.

Using the notion of Schur product, e.g. see [222], it can be proven that the ordered $*$-algebra $\mathcal{K}(X)$ has the following property: if $k, l \in \mathcal{K}(X)$ are positive semidefinite kernels, then kl is positive semidefinite. However, this is a case that illustrates Remark (4.3.1): it is not true, in general, that kernels of type k^*k are positive semidefinite.

(vii) Let $\mathcal{A}$ and $\mathcal{B}$ be two ordered $*$-spaces. In addition, we assume that the specified strict cone $\mathcal{A}^*$ linearly generates $\mathcal{A}$. On $\mathcal{L}(\mathcal{A}, \mathcal{B})$, the vector space of all linear maps $\varphi: \mathcal{A} \to \mathcal{B}$, we define an involution: $\varphi^*(a) = \varphi(a^*)^*$, for all $a \in \mathcal{A}$. Alinear map $\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{B})$is called positive if $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}^+$. It is easy to see that $\mathcal{L}(\mathcal{A}, \mathcal{B})^+$, the collection of all positive maps from $\mathcal{L}(\mathcal{A}, \mathcal{B})$, is a cone, and that it is strict because $\mathcal{A}^+$ linearly generates $\mathcal{A}$. In addition, any $\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{B})^+$ is selfadjoint, again due to the fact that $\mathcal{A}^+$ linearly generates $\mathcal{A}$. Consequently, $\mathcal{L}(\mathcal{A}, \mathcal{B})$ has a natural structure of ordered $*$-space.

(viii) Let $\{Z_\alpha\}_{\alpha \in A}$ be a family of ordered $*$-spaces such that, for each $\alpha \in A$, Z_α^+ is the specified strict cone of positive elements in Z_α. On the product space $Z = \prod_{\alpha \in A} Z_\alpha$ let $Z^+ = \prod_{\alpha \in A} Z_\alpha^+$ and observe that Z^+ is a strict cone. Letting the involution $*$ on Z be defined elementwise, it follows that Z^+ consists on selfadjoint elements only. In this way, Z is an ordered $*$-space.

(ix) Let Z be an ordered $*$-space with the specified strict cone Z^+. A subspace J of Z is called an order idealif it is selfadjoint, that is, $J = J^* = \{z^* \mid z \in J\}$, and solid, that is, for any $z \in J \cap Z^+$ and any $y \in Z^+$ such that $y \leq z$ it follows $y \in J$. Then, on the quotient vector space Z/J, an involution $*$ can be defined by: $(z + J)^* = z^* + J$, for $z \in Z$. Also, letting $(Z/J)^+ = \{z + J \mid z \in Z^+\}$, it follows that $(Z/J)^+$ is a strict cone in Z/J consisting of selfadjoint elements only and, hence, Z/J is an ordered $*$-space. See [230].

Given a complex linear space $\mathcal{E}$ and an ordered $*$-space space Z, a Z-gramian, also called a Z-valued inner product, is a mapping $\mathcal{E} \times \mathcal{E} \ni (x,y) \longmapsto [x,y] \in Z$ subject to the following properties:

(ve1) $[x,x] \geq 0$ for all $x \in \mathcal{E}$, and $[x,x] = 0$ if and only if $x = 0$.

(ve2) $[x,y] = [y,x]^*$ for all $x,y \in \mathcal{E}$.

(ve3) $[x, \alpha y_1 + \beta y_2] = \alpha[x,y_1] + \beta[x,y_2]$ for all $\alpha, \beta \in \mathbb{C}$ and all $x_1, x_2 \in \mathcal{E}$.

A complex linear space $\mathcal{E}$ onto which a Z-gramian $[\cdot,\cdot]$ is specified, for a certain ordered $*$-space Z, is called a VE-space(Vector Euclidean space) over Z, cf. [214].

Remark (4.3.3)[226]: In any VE -space Eover an ordered $*$-space Z, the familiar polarisation formula

$$4[x,y] = \sum_{k=0}^{3} i^k [(x + i^k y, x + i^k y] \, \mathrm{i}^k, \qquad x,y \in \mathcal{E}, \tag{65}$$

holds, which shows that the Z-valued inner product is perfectly defined by the Z-valued quadratic map $\mathcal{E} \ni x \longmapsto [x,x] \in Z$.

Actually, the formula (65) is more general: given a pairing $[\cdot,\cdot]: \mathcal{E} \times \mathcal{E} \to Z$, where $\mathcal{E}$ is some vector space and Z is a $*$-space, and assuming that $[\cdot,\cdot]$ satisfies only the axioms (ve2) and (ve3), then (65) is still valid.

The concept of VE-spaces isomorphismis also naturally defined: this is just a linear bijection $U: \mathcal{E} \to \mathcal{F}$, for two VE-spaces over the same ordered $*$-space Z, which is isometric, that is, $[Ux, Uy]_{\mathcal{F}} = [x,y]_{\mathcal{E}}$ for all $x,y \in \mathcal{E}$.

In general VE-spaces, an analog of the Schwarz Inequality may not hold but some of its consequences can be proven using slightly different techniques. One such method is provided by the following lemma.

Lemma (4.3.4)[226]: (See Loynes [214].) Let Z be an ordered $*$-space, $\mathcal{E}$ a complex vector space and $[\cdot,\cdot]: \mathcal{E} \times \mathcal{E} \to Z$ a positive semidefinite sesquilinear map, that is, $[\cdot,\cdot]$ is linear in the second variable, conjugate linear in the first variable, and $[x,x] \geq 0$ for all $x \in \mathcal{E}$. If $f \in \mathcal{E}$ is such that $[f,f] = 0$, then $[f,f'] = [f',f] = 0$ for all $f' \in \mathcal{E}$.

Given two VE-spaces $\mathcal{E}$ and $\mathcal{F}$, over the same ordered $*$-space Z, one can consider the vector space $\mathcal{L}(\mathcal{E},\mathcal{F})$ of all linear operators $T: \mathcal{E} \to \mathcal{F}$. Alinear operator $T \in \mathcal{L}(\mathcal{E},\mathcal{F})$ is called adjointable if there exists $T^* \in \mathcal{L}(\mathcal{E},\mathcal{F})$ such that

$$[Te,f]_{\mathcal{F}} = [e, T^*f]_{\mathcal{E}}, \qquad e \in \mathcal{E}, f \in \mathcal{F}. \tag{66}$$

The operator T^*, if it exists, is uniquely determined by T and called its adjoint. Since an analog of the Riesz Representation Theorem for VE-spaces may not exist, in general, there may be not so many adjointable operators. Denote by $\mathcal{L}^*(\mathcal{E},\mathcal{F})$ the vector space of all adjointable operators from $\mathcal{L}(\mathcal{E},\mathcal{F})$. Note that $\mathcal{L}^*(\mathcal{E}) = \mathcal{L}^*(\mathcal{E},\mathcal{E})$ is a $*$-algebra with respect to the involution $*$ determined by the operation of taking the adjoint.

An operator $A \in \mathcal{L}(\mathcal{E})$ is called selfadjoint if $[Ae,f] = [e,Af]$, for all $e,f \in \mathcal{E}$. Clearly, any selfadjoint operator A is adjointable and $A = A^*$. By the polarisation formula (65), A is selfadjoint if and only if $[Ae,e] = [e,Ae], e \in \mathcal{E}$. An operator $A \in \mathcal{L}(\mathcal{E})$ is positive if

$$[Ae,e] \geq 0, \qquad e \in \mathcal{E}. \tag{67}$$

Since the cone Z^+ consists of selfadjoint elements only, any positive operator is selfadjoint and hence adjointable. On the other hand, note that any VE-space isomorphism is adjointable and hence, it makes sense to call it unitary.

Examples (4.3.5)[226]: (i) If $\mathcal{E}$ is some VE-space over an ordered $*$-space Z, then $\mathcal{L}^*(\mathcal{E})$ is an ordered $*$-algebra, where the cone of positive elements is defined by (67). Note that this cone is strict. In connection with Remark (4.3.1), note that any operator $A \in \mathcal{L}^*(\mathcal{E})$ that can be represented $A = \sum_{j=1}^{N} A_j^* A_j$ is positive, but the converse, in general, is not true.

(ii) Let $\{\mathcal{E}_\alpha\}_{\alpha \in A}$ be a family of VE-spaces such that, for each $\alpha \in A$, $\mathcal{E}_\alpha$ is a VE-space over the ordered $*$-space Z_α. Consider the ordered $*$-space $Z = \prod_{\alpha \in A} Z_\alpha$ as in Example (4.3.2). Consider the vector space $\mathcal{E} = \prod_{\alpha \in A} \mathcal{E}_\alpha$ on which we define

$$[(e_\alpha)_{\alpha \in A}, (f_\alpha)_{\alpha \in A}] = ([e_\alpha, f_\alpha])_{\alpha \in A} \in Z, (e_\alpha)_{\alpha \in A}, (f_\alpha)_{\alpha \in A} \in \mathcal{E}.$$

Then $\mathcal{E}$ is a VE-space over Z.

(iii) Let $\mathcal{H}$ be a pre-Hilbert space having an orthonormal basis and $\mathcal{E}$ a VE-space over the ordered $*$-space Z. On the algebraic tensor product $\mathcal{H} \otimes \mathcal{E}$ define a gramian by

$$[h \otimes e, l \otimes f]_{\mathcal{H} \otimes \mathcal{E}} = \langle h, l \rangle_{\mathcal{H}} [e, f]_{\mathcal{E}} \in Z, \qquad h, l \in \mathcal{H}, \;\; e, f \in \mathcal{E},$$

and then extend it to $\mathcal{H} \otimes \mathcal{E}$ by linearity. By a standard but rather long argument, e.g. similar to p.6 in [202], it can be proven that, in this way, $\mathcal{H} \otimes \mathcal{E}$ becomes a VE-space over Z as well.

Remark (4.3.6)**[226]:** Given a finite collection of VE-spaces $\mathcal{E}_1, \ldots, \mathcal{E}_N$, over the same ordered $*$-space Z, one can define naturally the VE-space $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_N$ over Z where, for $e_j, f_j \in \mathcal{E}_j, j = 1, \ldots, N$ we define

$$[e_1 \oplus \cdots \oplus e_N, f_1 \oplus \cdots \oplus f_N] = Nj = \sum_{j=1}^{N} [e_j, f_j].$$

We use the notation $\mathcal{E}^N$ for $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_N$ when $\mathcal{E} = \mathcal{E}_j$ for all $j = 1, \ldots, N$. Then observe that $\mathcal{L}^*(\mathcal{E}^N)$ can be naturally identified with $M_N(\mathcal{E})$, the space of all $N \times N$ matrices with entries in $\mathcal{L}^*(\mathcal{E})$. This identification provides a natural structure of ordered $*$-algebra of $\mathcal{L}^*(\mathcal{E}^N)$ over Z, with an even richer structure, see Remarks (4.3.20).

An operator $A \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, for two VE-spaces over the same ordered $*$-space Z, is called boundedif, for some $\mu \geq 0$,

$$[Ah, Ah]_{\mathcal{F}} \leq \mu [h, h]_{\mathcal{E}}, \qquad h \in \mathcal{E}. \tag{68}$$

We denote the class of bounded operators by $\mathcal{B}(\mathcal{E}, \mathcal{F})$. For a bounded operator $A \in \mathcal{B}(\mathcal{E}, \mathcal{F})$, its operator normis denoted by $\|A\|$ and it is defined by square root of the least $\mu \geq 0$ satisfying (68), that is,

$$\|A\| = \inf\{\sqrt{\mu} \mid \mu \geq 0, [Ah, Ah] \leq \mu [h, h], \text{for all } h \in \mathcal{H}\}. \tag{69}$$

It is easy to see that the infimum is actually a minimum and hence, that we have

$$[Ah, Ah] \leq \|A\|^2 [h, h], \qquad x \in \mathcal{H}. \tag{70}$$

$\mathcal{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E}, \mathcal{F})$ is a normed algebra with respect to the usual algebraic operations and the operator norm, cf. Theorem1 in [215].

Let $\mathcal{B}^*(\mathcal{E}, \mathcal{F})$ denote the collection of all bounded and adjointable linear operators $A: \mathcal{E} \to \mathcal{F}$. Acontractionis a linear operator $T: \mathcal{E} \to \mathcal{F}$ such that $[Tx, Tx] \leq [x, x]$ for all $x \in \mathcal{H}$. By Theorem2 in [215], if $T \in \mathcal{B}^*(\mathcal{E}, \mathcal{F})$ is a contraction then T^* is a contraction as well, hence, for all $T \in \mathcal{B}^*(\mathcal{E}, \mathcal{F})$ we have $\|T^*\| = \|T\|$.

If $A \in \mathcal{B}^*(\mathcal{E})$ is selfadjoint, then, by Theorem3 in [215],

$$-\|A\| [h, h] \leq [Ah, h] \leq \|A\| [h, h], \qquad h \in \mathcal{E}. \tag{71}$$

Moreover, if A is a linear operator in $\mathcal{E}$ and, for some real numbers m, M, we have

$$m[h, h] \leq [Ah, h] \leq M[h, h], \qquad h \in \mathcal{E}, \tag{72}$$

then $A \in \mathcal{B}^*(\mathcal{E})$ and $A = A^*$. If, in addition, mis the minimum and M is the maximum with these properties, then $\|A\| = \min\{|m|, |M|\}$, see Theorem3 in [215].

According to Theorem 4 in [215], the algebra $\mathcal{B}^*(\mathcal{E})$ of bounded and adjointable operators on $\mathcal{E}$ is a pre C^*-algebra and we have $\|AA^*\| = \|A\|^2$ for all $A \in \mathcal{B}^*(\mathcal{E})$.

A VE-module Eover an ordered $*$-algebra $\mathcal{A}$ is an ordered $*$-space over $\mathcal{A}$, that is, (ve1)–(ve3) hold, subject to the following additional properties

(vem1) $\mathcal{E}$ is a right module over $\mathcal{A}$, compatible with the multiplication with scalars: $\lambda(ea) = (\lambda e)a = e(\lambda a)$ for all $\lambda \in \mathbb{C}, e \in \mathcal{E}$, and $a \in \mathcal{A}$.

(vem2) $[e, fa + gb]_{\mathcal{E}} = [e,f]_{\mathcal{E}} a + [e,g]_{\mathcal{E}} b$ for all $e,f,g \in \mathcal{E}$ and all $a, b \in \mathcal{A}$.

Given an ordered $*$-algebra $\mathcal{A}$ and two VE-modules $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{A}$, an operator $T \in \mathcal{L}(\mathcal{E},\mathcal{F})$ is called a module map if

$$T(ea) = T(e)a, \qquad e \in \mathcal{E}, a \in \mathcal{A}.$$

Any operator $T \in \mathcal{L}^*(\mathcal{E},\mathcal{F})$ is a module map: given arbitrary $e \in \mathcal{E}$, $f \in \mathcal{F}$ and $a \in \mathcal{A}$ we have

$$[T(ea), f]_{\mathcal{F}} = [ea, T^*(f)]_{\mathcal{E}} = a^*[e, T^*(f)]_{\mathcal{E}} = a^*[T(e), f]_{\mathcal{F}} = [T(e)a, f]_{\mathcal{F}},$$

hence T is a module map. See [202],[217],[231], for the more special case of Hilbert modules over C^*-algebras.

Examples (4.3.7)[226]: Let $\mathcal{E}$ and $\mathcal{F}$ be two VE-spaces over the same ordered $*$-space Z.

(i) The vector space $\mathcal{L}^*(\mathcal{E},\mathcal{F})$ has a natural structure of VE-module over the ordered $*$-algebra $\mathcal{L}^*(\mathcal{E})$, see Example (4.3.5), more precisely,

$$[T,S] = T^*S, \qquad T,S \in \mathcal{L}^*(\mathcal{E},\mathcal{F}). \tag{73}$$

(ii) Let $\mathcal{S}$ be a $*$-subspace of $\mathcal{L}^*(\mathcal{E},\mathcal{F})$ and define a gramian $[\cdot,\cdot]$ on $\mathcal{S}$ by (73). Let $\mathcal{Z}$be the $*$-subspace of $\mathcal{L}^*(\mathcal{E})$ generated by all operators T^*S, where $T,S \in \mathcal{S}$. $\mathcal{Z}$ has a natural structure of ordered $*$-space, where positivity of $T \in \mathcal{S}$ is in the sense of (67). Thus, $\mathcal{S}$ is a VE-space over $\mathcal{Z}$ that, in general, is not a VE-module.

We present the main dilation theorem for kernels. We start with some preliminary results.

Let X be a nonempty set and let $\mathcal{H}$ be a VE-space over the ordered $*$-space Z. A map $\mathbf{k}: X \times X \to \mathcal{L}(\mathcal{H})$ is called a kernel on X and valued in $\mathcal{L}(\mathcal{H})$. In case the kernel $\mathbf{k}$ has all its values in $\mathcal{L}^*(\mathcal{H})$, an adjoint kernel $\mathbf{k}^*: X \times X \to \mathcal{L}^*(\mathcal{H})$ can be associated by $\mathbf{k}^*(x,y) = \mathbf{k}(y,x)^*$ for all $x, y \in X$. The kernel $\mathbf{k}$ is called Hermitian if $\mathbf{k}^* = \mathbf{k}$.

Let $\mathcal{F} = \mathcal{F}(X;\, \mathcal{H})$ denote the complex vector space of all functions $f: X \to \mathcal{H}$ and let $\mathcal{F}_0 = \mathcal{F}_0(X;\, \mathcal{H})$ be its subspace of those functions having finite support. A pairing $[\cdot,\cdot]_{\mathcal{F}_0}: \mathcal{F}_0 \times \mathcal{F}_0 \to Z$ can be defined by

$$[g,h]_{\mathcal{F}_0} = \sum_{y \in X} [g(y), h(y)]_{\mathcal{H}}\,, \qquad g,h \in \mathcal{F}_0. \tag{74}$$

This pairing is clearly a Z-gramianon $\mathcal{F}_0$, hence $(\mathcal{F}_0;\, [\cdot,\cdot]_{\mathcal{F}_0})$ is a VE-space.

Let us observe that the sum in (74) makes sense even when only one of the functions g or h has finite support, the other can be arbitrary in $\mathcal{F}$. Thus, another pairing $[\cdot,\cdot]_{\mathbf{k}}$ can be defined on $\mathcal{F}_0$ by

$$[g,h]_{\mathbf{k}} = \sum_{x,y \in X} [\mathbf{k}(y,x)g(x), h(y)]_{\mathcal{H}}\,, \qquad g,h \in \mathcal{F}_0. \tag{75}$$

In general, the pairing $[\cdot,\cdot]_{\mathbf{k}}$ is linear in the first variable and conjugate linear in the second variable. If, in addition, $\mathbf{k} = \mathbf{k}^*$ then the pairing $[\cdot,\cdot]_{\mathbf{k}}$ is Hermitian as well, that is,

$$[g,h]_{\mathbf{k}} = [h,g]_{\mathbf{k}}^*, \qquad g,h \in \mathcal{F}_0.$$

A convolution operator $K: \mathcal{F}_0 \to \mathcal{F}$ can be associated to the kernel $\mathbf{k}$ by

$$(Kg)(y) = \sum_{x\in X} \mathbf{k}(y,x)g(x)\,, \qquad g \in \mathcal{F}_0, \tag{76}$$
and it is easy to see that K is a linear operator. There is a natural relation between the pairing $[\cdot,\cdot]_{\mathbf{k}}$ and the convolution operator K given by

$$[g,h]_{\mathbf{k}} = [Kg,h]_{\mathcal{F}_0}, \qquad g,h \in \mathcal{F}_0.$$

Given $n \in \mathbb{N}$, the kernel $\mathbf{k}$ is called n-positiveif for any $x_1, x_2, \dots, x_n \in X$ and any $h_1, h_2, \dots, h_n \in \mathcal{H}$ we have

$$\sum_{i,j=1} [\mathbf{k}(x_i,x_j)h_j,h_i]_{\mathcal{H}} \geq 0. \tag{77}$$

The kernel $\mathbf{k}$ is called positive semidefinite (or of positive type) if it is n-positive for all natural numbersn. The proof of the following lemma is the same as the proof of Lemma 3.1 from [204].

The third assertion in the next result makes the connection with the kernels made up of bounded operators only as in [204].

Lemma (4.3.8)[226]: Assume that the kernel $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ is 2-positive. Then:

(i) $\mathbf{k}$ is Hermitian.

(ii) If, for some $x \in X$, we have $\mathbf{k}(x,x) = 0$, then $\mathbf{k}(x,y) = 0$ for all $y \in X$.

(iii) Assume that, for $x, y \in X$ the operators $\mathbf{k}(x,x)$ and $\mathbf{k}(y,y)$ are bounded. Then $\mathbf{k}(x,y)$ and $\mathbf{k}(y,x) = \mathbf{k}(x,y)^*$ are bounded and

$$\|\mathbf{k}(x,y)\|^2 \leq \|\mathbf{k}(x,x)\|\|\mathbf{k}(y,y)\|. \tag{78}$$

In particular, if $\mathbf{k}(x,x) \in \mathcal{B}^*(\mathcal{E})$ for all $x \in X$, then $\mathbf{k}(y,x) \in \mathcal{B}^*(\mathcal{E})$ for all $x, y \in X$.

Proof. The proof of (i) and (ii) is the same as the proof of Lemma 3.1 from [204].

(iii) Assume that both operators $\mathbf{k}(x,x)$ and $\mathbf{k}(y,y)$ are bounded, hence $\mathbf{k}(x,x)$, $\mathbf{k}(y,y) \in \mathcal{B}^*(\mathcal{E})$. If $\mathbf{k}(y,y) = 0$ then, by (ii), $\mathbf{k}(x,y) = 0$ and $\mathbf{k}(y,x) = \mathbf{k}(x,y)^* = 0$, hence bounded, and the inequality (78) holds trivially.

Assume that $\mathbf{k}(y,y) \neq 0$, hence $\|\mathbf{k}(y,y)\| > 0$. Since $\mathbf{k}$ is 2-positive, for any $h, g \in \mathcal{H}$ we have

$$[\boldsymbol{k}(x,x)h,h] + [\boldsymbol{k}(x,y)g,h] + [\boldsymbol{k}(y,x)h,g] + [\boldsymbol{k}(y,y)g,g] \geq 0. \tag{79}$$

We let $g = -\mathbf{k}(\boldsymbol{x},\boldsymbol{y})^* h/\|\mathbf{k}(y,y)\|$ in (79), take into account (72) and get

$$\begin{aligned}\frac{2}{\|\mathbf{k}(y,y)\|}[\mathbf{k}(y,x)h,\mathbf{k}(y,x)h] &\leq [\mathbf{k}(x,x)h,h] + \frac{1}{\|\mathbf{k}(y,y)\|^2}[\mathbf{k}(y,y)\mathbf{k}(y,x)h,\mathbf{k}(y,x)h] \\ &\leq [\mathbf{k}(x,x)h,h] + \frac{\|\mathbf{k}(y,y)\|}{\|\mathbf{k}(y,y)\|^2}[\mathbf{k}(y,x)h,\mathbf{k}(y,x)h] \\ &= [\mathbf{k}(x,x)h,h] + \frac{1}{\|\mathbf{k}(y,y)\|}[\mathbf{k}(y,x)h,\mathbf{k}(y,x)h],\end{aligned}$$

hence

$$[\mathbf{k}(y,x)h,\mathbf{k}(y,x)h] \leq \|\mathbf{k}(y,y)\|[\mathbf{k}(x,x)h,h] \leq \|\mathbf{k}(x,x)\|\|\mathbf{k}(y,y)\|[h,h],$$

which proves that $\mathbf{k}(y,x)$ is a bounded operator and the inequality (78).

Example (4.3.9)[226]: This example is a generalisation of Example (4.3.2).(vi). Let X be a nonempty set, $\mathcal{E}$ be a VE-space over the ordered $*$-space Z. Let $\mathcal{K}(X;\,\mathcal{E})$ be the vector space of all kernels $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{E})$, and let $\mathcal{K}(X;\,\mathcal{E})^+$ be the set of all positive semidefinite kernels. Then $\mathcal{K}(X;\,\mathcal{E})^+$ is a cone and, by Lemma (4.3.8), it consists only of selfadjoint elements. If $\mathbf{k} \in (\mathcal{K}(X;\,\mathcal{E})^+ \cap -\mathcal{K}(X;\,\mathcal{E})^+)$, we obtain $[\mathbf{k}(x,x)h,h]_{\mathcal{E}} = 0$ for all $x \in X$ and $h \in \mathcal{E}$ by strictness of the cone of Z. Since $\mathbf{k}(x,x)$ is a positive operator, hence selfadjoint, by means of the analog of the polarisation formula (65), see the second part of Remark (4.3.3), it follows that $\mathbf{k}(x,x)=0$ for any $x\in X$. Then, by Lemma (4.3.8) again, $\mathbf{k}(x,y)=0$ for all $x,y\in X$, i.e. $\mathbf{k}=0$. Therefore $\mathcal{K}(X;\ \mathcal{E})$ is an ordered $*$-space with cone $\mathcal{K}(X;\ \mathcal{E})^+$. Amultiplication can be defined on $\mathcal{K}(X;\ \mathcal{E})$: for $\mathbf{k},\mathbf{l}\in\mathcal{K}(X;\ \mathcal{E})$ we let $(\mathbf{kl})(x,y)=\mathbf{k}(x,y)\mathbf{l}(x,y)$ for all $x,y\in X$. With respect to this multiplication, $\mathcal{K}(X;\ \mathcal{E})$ is an ordered $*$-algebra.

Given an $\mathcal{L}^*(\mathcal{H})$-valued kernel $\mathbf{k}$ on a nonempty set X, for some VE-space $\mathcal{H}$ on an ordered $*$-space Z, a VE-space linearisationor, equivalently, a VE-space Kolmogorov decomposition of $\mathbf{k}$ is, by definition, a pair $(\mathcal{K};\ V)$, subject to the following conditions:

(kd1) $\mathcal{K}$ is a VE-space over the same ordered $*$-space Z.

(kd2) $V:X\to\mathcal{L}^*(\mathcal{H},\mathcal{K})$ satisfies $\mathbf{k}(x,y)=V(x)^*V(y)$ for all $x,y\in X$.

The VE-space linearisation $(\mathcal{K};\ V)$ is called minimal if

(kd3) $\operatorname{Lin} V(X)\mathcal{H}=\mathcal{K}$.

Two VE-space linearisations $(\mathcal{K};\ V)$ and $(V';\ \mathcal{K}')$ of the same kernel $\mathbf{k}$ are called unitary equivalent if there exists a unitary operator $U:\mathcal{K}\to\mathcal{K}'$ such that $UV(x)=V'(x)$ for all $x\in X$.

The uniqueness of a minimal VE-space linearisation $(\mathcal{K};V)$ of a positive semidefinite kernel $\mathbf{k}$, modulo unitary equivalence, follows in the usual way: if $(\mathcal{K}';\ V')$ is another min-imal VE-space linearisation of $\mathbf{k}$, for arbitrary $x_1,\ldots,x_m,y_1,\ldots,y_n\in X$ and arbitrary $h_1,\ldots,h_m,g_1,\ldots,g_n\in\mathcal{H}$, we have

$$\begin{aligned}[\sum_{j=1}^{m}V(x_j)h_j,\sum_{i=1}^{n}V(y_i)g_i]_{\mathcal{K}} &= \sum_{j=1}^{m}\sum_{i=1}^{n}[V(x_j)h_j,V(y_i)g_i]_{\mathcal{K}}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}[\mathbf{k}V(y_i,x_j)h_j,g_i]_{\mathcal{K}}\\ &= \sum_{j=1}^{m}\sum_{i=1}^{n}[V'(x_j)h_j,V'(y_i)g_i]_{\mathcal{K}'}\\ &= [\sum_{j=1}^{m}V'(x_j)h_j,\sum_{i=1}^{n}V'(y_i)g_i]_{\mathcal{E}'},\end{aligned}$$

hence $U:\operatorname{Lin}V(X)\to\operatorname{Lin}V'(X)$ defined by

$$\sum_{j=1}^{m}V(x_j)h_j\mapsto\sum_{i=1}^{n}V'(y_i)g_i \tag{80}$$

is a correctly everywhere defined linear operator, isometric and onto. Thus, U is a VE-space isomorphism $U:\mathcal{K}\to\mathcal{K}'$ and $UV(x)=V'(x)$ for all $x\in X$, by construction.

Let $\mathcal{H}$ be a VE-space over the ordered $*$-space Z, and let X be a nonempty set. A VE-space $\mathcal{R}$, over the same ordered $*$-space Z, is called an $\mathcal{H}$-reproducing kernel VE-space on X if there exists a Hermitian kernel $\mathbf{k}:X\times X\to\mathcal{L}^*(\mathcal{H})$ such that the following axioms are satisfied:

(rk1) $\mathcal{R}$ is a subspace of $\mathcal{F}(X;\ \mathcal{H})$, with all algebraic operations.

(rk2) $\mathcal{F}$ or all $x\in X$ and all $h\in\mathcal{H}$, the $\mathcal{H}$-valued function $\mathbf{k}_x h=\mathbf{k}(\cdot,x)h\in\mathcal{R}$.

(rk3) $\mathcal{F}$ or all $f\in\mathcal{R}$ we have $[f(x),h]_{\mathcal{H}}=[f,\mathbf{k}_x xh]_{\mathcal{R}}$, for all $x\in X$ and $h\in\mathcal{H}$.

As a consequence of (rk2), $\operatorname{Lin}\{\mathbf{k}_x h\mid x\in X,h\in\mathcal{H}\}\subseteq\mathcal{R}$. The reproducing kernel VE-space $\mathcal{R}$ is called minimal if the following property holds as well:

(rk4) $\operatorname{Lin}\{\mathbf{k}_x h\mid x\in X,h\in\mathcal{H}\}=\mathcal{R}$.

Observe that if $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VE-space on X, with kernel $\mathbf{k}$, then $\mathbf{k}$ is positive semidefinite and uniquely determined by $\mathcal{R}$ hence we can talk about the $\mathcal{H}$-reproducing kernel $\mathbf{k}$ corresponding to $\mathcal{R}$. On the other hand, a minimal reproducing kernel VE-space $\mathcal{R}$ is uniquely determined by its reproducing kernel $\mathbf{k}$.

The classical reproducing kernel Hilbert spaces, e.g. see [205], are characterised, within the Hilbert function spaces, by the continuity of the evaluation functionals. In the following, we generalise this by showing that, in the absence of an analogue of the Riesz Representation Theorem, it is the adjointability which makes the difference. Letting $\mathcal{H}$ be a VE-space over an ordered $*$-space Z, for X a nonempty set, an evaluation operator $E_x : \mathcal{F}(X;\ \mathcal{H}) \to \mathcal{H}$ can be defined for each $x \in X$ by letting $E_x f = f(x)$ for all $f \in \mathcal{F}(X;\ \mathcal{H})$. Clearly, E_x is linear.

Proposition (4.3.10)[226]: Let X be a nonempty set, $\mathcal{H}$ a VE-space over an ordered $*$-space Z, and let $\mathcal{R} \subseteq \mathcal{F}(X;\ \mathcal{H})$, with all algebraic operations, be a VE-space over Z. Then $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VE-space if and only if, for all $x \in X$, the restriction of the evaluation operator E_x to $\mathcal{R}$ is adjointable as a linear operator $\mathcal{R} \to \mathcal{H}$.

Proof. Assume first that $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VE-space on X and let $\mathbf{k}$ be its reproducing kernel. For any $h \in \mathcal{H}$ and any $f \in \mathcal{R}$

$$[E_x f, h]_{\mathcal{H}} = [f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}}. \tag{81}$$

Since $\mathbf{k}_x \in \mathcal{L}(\mathcal{H}, \mathcal{R})$, it follows that E_x is adjointable and, in addition, $E_x^* = \mathbf{k}_x$, for all $x \in X$.

Conversely, assume that, for all $x \in X$, the evaluation operator $E_x \in \mathcal{L}^*(\mathcal{R}, \mathcal{H})$. Equation (81) shows that, in order to show that $\mathcal{R}$ is a reproducing kernel VE-space, we should define the kernel $\mathbf{k}$ in the following way:

$$\mathbf{k}(y, x)h = (E_x^* h)(y), \qquad x, y \in X,\ \ h \in \mathcal{H}. \tag{82}$$

It is clear that $k(y, x) : \mathcal{H} \to \mathcal{H}$ is a linear operator and observe that $\mathbf{k}_x h = E_x^* h$ for all $x \in X$ and all $h \in \mathcal{H}$. The reproducing property (rk3) holds:

$$[f(x), h]_{\mathcal{H}} = [E_x f, h]_{\mathcal{H}} = [f, E_x^* h]_{\mathcal{R}} = [f, \mathbf{k}_x h]_{\mathcal{R}}, \qquad f \in R, h \in H, x \in X.$$

The axioms (rk1) and (rk2) are clearly satisfied, so it only remains to prove that $\mathbf{k}$ is a Hermitian kernel. To see this, fix $x, y \in X$ and $h, l \in \mathcal{H}$. Then

$$\begin{aligned}[\mathbf{k}(y, x)h, l]_{\mathcal{H}} &= [(\mathbf{k}_x h)(y), l]_{\mathcal{H}} = [\mathbf{k}_x h, \mathbf{k}_y l]_{\mathcal{R}} \\ &= [\mathbf{k}_y l, \mathbf{k}_x h]_{\mathcal{R}}^* = [\mathbf{k}(x, y)l, h]_{\mathcal{R}}^* = [h, \mathbf{k}(x, y)l]_{\mathcal{R}}.\end{aligned}$$

Therefore, $\mathbf{k}(y, x)$ is adjointable and $\mathbf{k}(y, x)^* = \mathbf{k}(x, y)$, hence $\mathbf{k}$ is a Hermitian kernel. We have proven that $\mathbf{k}$ is the reproducing kernel of $\mathcal{R}$.

There is a very close connection between VE-space linearisations and reproducing kernel VE-spaces.

Proposition (4.3.11)[226]: Let X be a nonempty set, $\mathcal{H}$ a VE-space over an ordered $*$-space Z, and let $\mathbf{k} : X \times X \to \mathcal{L}^*(\mathcal{H})$ be a Hermitian kernel.

(i) Any $\mathcal{H}$-reproducing kernel VE-space Rwith kernel $\mathbf{k}$ is a VE-space linearisation $(\mathcal{R};\ V)$ of $\mathbf{k}$, with $V(x) = \mathbf{k}_x$ for all $x \in X$.

(ii) For any minimal VE-space linearisation $(\mathcal{K};\ V)$ of $\mathbf{k}$, letting

$$\mathcal{R} = \{V(\cdot)^* f \mid f \in \mathcal{K}\}, \tag{83}$$

we obtain the minimal $\mathcal{H}$-reproducing kernel VE-space with reproducing kernel $\mathbf{k}$.

Proof. (ii)⇒(i). Let $(\mathcal{K};\ \pi;\ V)$be a minimal VE-space linearisation of the kernel $\mathbf{k}$ on X. Let Rbe the set of all functions $X \ni x \mapsto V(x)^* f \in \mathcal{H}$, in particular $\mathcal{R} \subseteq \mathcal{F}(X;\ \mathcal{H})$, and we endow Rwith the algebraic operations inherited from the complex vector space $\mathcal{F}(X;\ \mathcal{H})$. The correspondence

$$K \ni f \mapsto Uf = V(\cdot)^* f \in \mathcal{R} \tag{84}$$
is bijective. By the definition of $\mathcal{R}$, this correspondence is surjective. In order to verify that it is injective as well, let $f, g \in \mathcal{K}$ be such that $V^*(\cdot)f = V^*(\cdot)g$. Then, for all $x \in X$ and all $h \in \mathcal{H}$ we have
$$[V(x)^* f, h]H = [V(x)^* g, h]_{\mathcal{H}},$$
equivalently,
$$[f - g, V(x)h]_{\mathcal{K}} = 0, \qquad x \in X,\ \ h \in \mathcal{H}.$$
By the minimality of the VE-space linearisation $(\mathcal{K};\ V)$ it follows that $g = f$. Thus, U is a bijection.

Clearly, the bijective map U defined at (84) is linear, hence a linear isomorphism of complex vector spaces $\mathcal{K} \to \mathcal{R}$. On $\mathcal{R}$ we introduce a Z-valued pairing
$$[Uf, Ug]\ = [V(\cdot)^* f, V(\cdot)^* g]_{\mathcal{R}} = [f, g]_{\mathcal{K}}, \qquad f, g \in \mathcal{K}. \tag{85}$$
Then $(R;\ [\cdot,\cdot]_{\mathcal{R}})$ is a VE-space over Z since, by (85), we transported the Z-gramian from $\mathcal{K}$ to $\mathcal{R}$ or, in other words, we have defined on $\mathcal{R}$ the Z-gramian that makes the linear isomorphism U a unitary operator between the VE-spaces $\mathcal{K}$ and $\mathcal{R}$.

We show that $(R;\ [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VE-space with corresponding reproducing kernel $\mathbf{k}$. By definition, $\mathcal{R} \subseteq \mathcal{F}(X;\ \mathcal{H})$. On the other hand, since
$\mathbf{k}_x(y)h = \mathbf{k}(y, x)h = V(y)^* V(x)h,$ for all $x, y \in X$ and all $h \in \mathcal{H}$,
taking into account that $V(x)h \in \mathcal{K}$, by (83) it follows that $\mathbf{k}_x \in \mathcal{R}$ for all $x \in X$. Further, for all $f \in \mathcal{R}$, $x \in X$, and $h \in \mathcal{H}$, we have
$$\begin{aligned}[f, \mathbf{k}_x h]_{\mathcal{R}} &= [V(\cdot)^* g, \mathbf{k}_x h]_{\mathcal{R}} = [V(\cdot)^* g, V(\cdot)^* V(x)h]_{\mathcal{R}} \\ &= [g, V(x)h]_{\mathcal{K}} = [V(x)^* g, h]_{\mathcal{H}} = [f(x), h]_{\mathcal{H}},\end{aligned}$$
where $g \in \mathcal{K}$ is the unique vector such that $V(x)^* g = f(x)$, which shows that Rsatisfies the reproducing axiom as well.

(i) ⇒ (ii). Assume that $(R;\ [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VE-space on X, with reproducing kernel $\mathbf{k}$. We let $\mathcal{K} = \mathcal{R}$ and define
$$V(x)h = \mathbf{k}_x h, \qquad x \in X, h \in \mathcal{H}. \tag{86}$$
Note that $V(x){:}\, \mathcal{H} \to \mathcal{K}$ is linear for all $x \in X$.

We show that $V(x) \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ for all $x \in X$. To see this, first note that, by the reproducing property,
$$[f, V(x)h]_{\mathcal{K}} = [f, \mathbf{k}_x h]_{\mathcal{R}} = [f(x), h]_{\mathcal{H}}, \qquad x \in X, h \in \mathcal{H}. \tag{87}$$
Let us then, for fixed $x \in X$, consider the linear operator $W(x){:}\, \mathcal{R} = \mathcal{K} \to \mathcal{H}$ defined by $W(x)f = f(x)$ for all $f \in \mathcal{R} = \mathcal{K}$. From (87) we conclude that $V(x)$ is adjointable and $V(x)^* = W(x)$ for all $x \in X$.

Finally, by the reproducing axiom, for all $x, y \in X$ and all $h, g \in \mathcal{H}$ we have
$$[V(y)^* V(x)h, g]_{\mathcal{H}} = [V(x)h, V(y)g]_{\mathcal{R}} = [\mathbf{k}_x h, \mathbf{k}_y g]_{\mathcal{R}} = [\mathbf{k}(y, x)h, g]_{\mathcal{H}},$$
hence $V(y)^* V(x) = \mathbf{k}(y, x)$ for all $x, y \in X$. Thus, $(\mathcal{K};\ V)$ is a VE-space linearisation of $\mathbf{k}$ (actually, a minimal one).

Let a (multiplicative) semigroup Γ act on X, denoted by $\xi \cdot x$, for all $\xi \in \Gamma$ and all $x \in X$. By definition, we have
$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x \text{ for all } \alpha, \beta \in \Gamma \text{ and all } x \in X. \tag{88}$$
Equivalently, this means that we have a semigroup morphism $\Gamma \ni \xi \mapsto \xi \cdot \in G(X)$, where $G(X)$ denotes the semigroup, with respect to composition, of all maps $X \to X$. In case the semigroup Γ has a unit ϵ, the action is called unital if $\epsilon \cdot x = x$ for all $x \in X$, equivalently, $\epsilon \cdot = \mathrm{Id}_X$.

We assume further that Γ is a $*$-semigroup, that is, there is an involution $*$ on Γ; this means that $(\xi\eta)^* = \eta^*\xi^*$ and $(\xi^*)^* = \xi$ for all $\xi, \eta \in \Gamma$. Note that, in case Γ has a unit ϵ then $\epsilon^* = \epsilon$.

Given a VE-space $\mathcal{H}$ we are interested in those Hermitian kernels $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ that are invariantunder the action of Γ on X, that is,

$\mathbf{k}(y, \xi \cdot x) = \mathbf{k}(\xi *\cdot y, x)$ for all $x, y \in X$ and all $\xi \in \Gamma$. (89)

A triple $(\mathcal{K}; \pi; V)$ is called an invariant VE-space linearisationof the kernel $\mathbf{k}$ and the action of Γ on X, shortly a Γ-invariant VE-space linearization of $\mathbf{k}$, if:

(ikd1) $(\mathcal{K}; V)$ is a VE-space linearisation of the kernel $\mathbf{k}$.

(ikd2) $\pi: \Gamma \to \mathcal{L}^*(\mathcal{H})$ is a $*$-representation, that is, amultiplicative $*$-morphism.

(ikd3) V and π are related by the formula: $V(\xi \cdot x) = \pi(\xi)V(x)$, for all $x \in X$, $\xi \in \Gamma$.

Remarks (4.3.12)[226]: (i) Let $(\mathcal{K}; \pi; V)$ be a Γ-invariant VE-space linearisation of the kernel $\mathbf{k}$. Since $(\mathcal{K}; V)$ is a VE-space linearisation and taking into account the axiom (ikd3), we have

$$\begin{aligned}\mathbf{k}(y, \xi \cdot x) &= V(y)^*V(\xi \cdot x) = V(y)^*\pi(\xi)V(x)\\ &= (\pi(\xi^*)V(y))^*V(x) = \mathbf{k}(\xi^* \cdot y, x), \qquad x, y \in X,\ \ \xi \in \Gamma, \qquad (90)\end{aligned}$$

hence $\mathbf{k}$ is invariant under the action of Γon X.

(ii) Observe that, if the action of Γon X is unital then, for a Γ-invariant VE-space linearisation $(\mathcal{K}; \pi; V)$, the two conditions $\mathbf{k}(x, y) = V(x)^*V(y)$, $x, y \in X$, and $V(\xi \cdot x) = \pi(\xi)V(x)$, $\xi \in \Gamma$ and $x \in X$, can be equivalently combined into two slightly different conditions, namely, π unital and $\mathbf{k}(x, \xi \cdot y) = V(x)^*\pi(\xi)V(y)$, $\xi \in \Gamma$ and $x, y \in X$.

If, in addition to the axioms (ikd1)–(ikd3), the triple (K; π; V)has the property

(ikd4) Lin $V(X)\mathcal{H} = \mathcal{K}$,

that is, the VE-space linearisation $(\mathcal{K}; V)$ is minimal, then $(\mathcal{K}; \pi; V)$ is called a minimal Γ-invariant VE-space linearization of $\mathbf{k}$ and the action of Γ on X.

Minimal invariant VE-space linearisations have a built-in linearity property; the proof is the same with that of Proposition 4.1 in [204].

Proposition (4.3.13)[226]: Assume that, given a VE-space adjointable operator valued kernel $\mathbf{k}$, invariant under the action of the $*$-semigroup Γ on X, for some fixed $\alpha, \beta, \gamma \in \Gamma$ we have $\mathbf{k}(y, \alpha \cdot x) + \mathbf{k}(y, \beta \cdot x) = \mathbf{k}(y, \gamma \cdot x)$ for all $x, y \in X$. Then, for any minimal invariant VE-space linearisation $(\mathcal{K}; \pi; V)$ of $\mathbf{k}$, the representation satisfies $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$.

Two Γ-invariant VE-space linearisations $(\mathcal{K}; \pi; V)$ and $(\mathcal{K}'; \pi'; V')$, of the same Hermitian kernel $\mathbf{k}$, are called unitary equivalentif there exists a unitary operator $U: \mathcal{K} \to \mathcal{K}'$ such that $U\pi(\xi)\ \pi'(\xi)U$ for all $\xi \in \Gamma$, and $UV(x) = V(x)$ for all $x \in X$. Let us note that, in case both of these invariant VE-space linearisations are minimal, then this is equivalent with the requirement that the VE-space linearisations $(\mathcal{K}; V)$ and $(\mathcal{K}'; V')$ are unitary equivalent.

The main result is the following theorem. It is stated in terms of both linearisations and reproducing kernels and the proof points out essentially a reproducing kernel and operator range construction.

Theorem (4.3.14)[226]: Let Γ be a $*$-semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) $\mathbf{k}$ is positive semidefinite, in the sense of (77), and invariant under the action of Γ on X, that is, (89) holds.

(ii) $\mathbf{k}$ has a Γ-invariant VE-space linearisation $(\mathcal{K}; \pi; V)$.

(iii) $\mathbf{k}$ admits an $\mathcal{H}$-reproducing kernel VE-space $\mathcal{R}$ and there exists a $*$-representation $\rho\colon \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi\cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (i),(ii), or (iii) holds, then a minimal Γ-invariant VE-space linearisation can be constructed, any minimal Γ-invariant VE-space linearisation is unique up to unitary equivalence, a pair $(\mathcal{R};\, \rho)$ as in assertion (iii) with $\mathcal{R}$ minimal can be always obtained and, in this case, it is uniquely determined by $\mathbf{k}$ as well.

Proof. (i) ⇒ (ii). Assuming that $\mathbf{k}$ is positive semidefinite, by Lemma (4.3.8).(i) it follows that $\mathbf{k}$ is Hermitian, that is, $\mathbf{k}(x,y)^* = \mathbf{k}(y,x)$ for all $x, y \in X$. We consider the convolution operator K defined at (76) and let $\mathcal{G} = \mathcal{G}(X;\, \mathcal{H})$ be its range, more precisely,

$$\begin{aligned} \mathcal{G} &= \{f \in \mathcal{F} \mid f = Kg \text{ for some } g \in \mathcal{F}_0\} \\ &= \{f \in \mathcal{F} \mid f(y) = \sum_{x\in X} \mathbf{k}(y,x)g(x) \text{ for some } g \in \mathcal{F}_0 \text{ and all } x \in X\}. \end{aligned} \tag{91}$$

A pairing $[\cdot,\cdot]_{\mathcal{G}}\colon \mathcal{G}\times\mathcal{G} \to Z$ can be defined by

$$\begin{aligned} [e,f]_{\mathcal{G}} = [Kg,h]_{\mathcal{F}_0} &= \sum_{y\in X} [e(y),h(y)]_{\mathcal{H}} \\ &= \sum_{x,y\in X} [\mathbf{k}(y,x)g(x),h(y)]_{\mathcal{H}}, \end{aligned} \tag{92}$$

where $f = Kh$ and $e = Kg$ for some $g, h \in \mathcal{F}_0$. We observe that

$$\begin{aligned} [e,f]_{\mathcal{G}} &= \sum_{y\in X} [e(y),h(y)]_{\mathcal{H}} = \sum_{x,y\in X} [\mathbf{k}(y,x)g(x),h(y)]_{\mathcal{H}} \\ &= \sum_{x,y\in X} [g(x),\mathbf{k}(y,x)h(y)]_{\mathcal{H}} = \sum_{x\in X} [g(x),f(x)]_{\mathcal{H}}, \end{aligned}$$

which shows that the definition in (92) is correct, that is, independent of g and h such that $e = Kg$ and $f = Kh$.

We claim that $[\cdot,\cdot]_{\mathcal{G}}$ is a Z-valued gramian, that is, it satisfies all the requirements (ve1)–(ve3). The only fact that needs a proof is $[f,f]_{\mathcal{G}} = 0$ implies $f = 0$ and this follows by Lemma (4.3.4).

Thus, $(\mathcal{G};\, [\cdot,\cdot]_{\mathcal{G}})$ is a VE-space that we denote by $\mathcal{K}$. For each $x \in X$ we define $V(x)\colon \mathcal{H} \to \mathcal{G}$ by

$$V(x)h = Kh_x, \qquad h \in \mathcal{H}, \tag{93}$$

where $h_x = \delta_x h \in \mathcal{F}_0$ is the function that takes the value h at x and is null elsewhere. Equivalently,

$$(V(x)h)(y) = (Kh_x)(y) = \sum_{z\in X} \mathbf{k}(y,z)(h_x)(z) = \mathbf{k}(y,x)h, \quad y \in X. \tag{94}$$

Note that $V(x)$ is an operator from the VE-space $\mathcal{H}$ to the VE-space $\mathcal{G} = \mathcal{K}$ and it remains to show that $V(x)$ is adjointable for all $x \in X$. To see this, let us fix $x \in X$ and take $h \in \mathcal{H}$ and $f \in \mathcal{G}$ arbitrary. Then,

$$[V(x)h,f]_{\mathcal{G}} = \sum_{y\in X} [(h_x)(y),f(y)]_{\mathcal{H}} = [h,f(x)]_{\mathcal{H}}, \tag{95}$$

which shows that $V(x)$ is adjointable and that its adjoint $V(x)^*$ is the operator $\mathcal{G} \ni f \mapsto f(x) \in \mathcal{H}$ of evaluation at x.

On the other hand, for any $x, y \in X$, by (94), we have

$$V(y)^*V(x)h = (V(x)h)(y) = \mathbf{k}(y,x)h, \qquad h \in \mathcal{H},$$

hence $(V;\ \mathcal{K})$ is a VE-space linearisation of $\mathbf{k}$. We prove that it is minimal as well. To see this, note that a typical element in the linear span of $V(X)\mathcal{H}$ is, for arbitrary $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $h_1, \dots, h_n \in \mathcal{H}$,

$$\sum_{j=1}^{n} V(x_j)h_j = \sum_{j=1}^{n} Kh_{j,x_j}$$
$$= \sum_{j=1}^{n} \sum_{y \in X} \mathbf{k}(\cdot, y) h_{j,x_j}(y) = \sum_{j=1}^{n} \mathbf{k}(\cdot, x_j) h_j\,,$$

and then take into account that $\mathcal{G}$ is the range of the convolution operator K defined at (76). The uniqueness of the minimal VE-space linearisation $(V;\ \mathcal{K})$ just constructed follows as in (80).

For each $\xi \in \Gamma$ we let $\pi(\xi)\colon \mathcal{F} \to \mathcal{F}$ be defined by

$$(\pi(\xi)f)(y) = f(\xi^* \cdot y), \qquad y \in X, \xi \in \Gamma. \tag{96}$$

We prove that $\pi(\xi)$ leaves $\mathcal{G}$ invariant. To see this, let $f \in \mathcal{G}$, that is, $f = Kg$ for some $g \in \mathcal{F}_0$ or, even more explicitly, by (91),

$$f(y) = \sum_{x \in X} \mathbf{k}(y,x) g(x)\,, \qquad y \in X. \tag{97}$$

Then,

$$f(\xi^* \cdot y) = \sum_{x \in X} \mathbf{k}(\xi^* \cdot y, x) g(x)$$
$$= \sum_{x \in X} \mathbf{k}(y, \xi \cdot x) g(x) = \sum_{z \in X} \mathbf{k}(y,z) g\xi(z), \tag{98}$$

where,

$$g^{\xi}(z) = \begin{cases} 0, & \text{if } \xi \cdot x = z \text{ has no solution } x \in \operatorname{supp} g, \\ \displaystyle\sum_{\xi \cdot x = z} g(x), & \text{otherwise.} \end{cases} \tag{99}$$

Since $g^{\xi} \in \mathcal{F}_0$, it follows that $\pi(\xi)$ leaves $\mathcal{G}$ invariant. In the following we denote by the same symbol $\pi(\xi)$ the map $\pi(\xi)\colon \mathcal{G} \to \mathcal{G}$.

We prove that π is a representation of the semigroup Γ on the complex vector space $\mathcal{G}$, that is,

$$\pi(\alpha\beta)f = \pi(\alpha)\pi(\beta)f, \qquad \alpha, \beta \in \Gamma, f \in \mathcal{G}. \tag{100}$$

To see this, let $f \in \mathcal{G}$ be fixed and denote $h = \pi(\beta)f$, that is, $h(y) = f(\beta^* \cdot y)$ for all $y \in X$. Then $\pi(\alpha)\pi(\beta)f = \pi(\alpha)h$, that is, $(\pi(\alpha)h)(y) = h(a^* \cdot y) = h(\beta^* a^* \cdot y) = h((\alpha\beta)^* \cdot y) = (\pi(\alpha\beta))(y)$, for all $y \in X$, which proves (100).

We show that π is actually a $*$-representation, that is,

$$[\pi(\xi)f, f']_{\mathcal{G}} = [f, \pi(\xi^*)f']_{\mathcal{G}}, \qquad f, f' \in \mathcal{G}. \tag{101}$$

To see this, let $f = Kg$ and $f' = Kg'$ for some $g, g' \in \mathcal{F}_0$. Then, recalling (92) and (98),

$$[\pi(\xi)f, f']_{\mathcal{G}} = \sum_{y \in X} [f(\xi^* y), g'(y)]_{\mathcal{H}} = \sum_{x,y \in X} [\mathbf{k}(\xi^* \cdot y, x) g(x), g'(y)]_{\mathcal{H}}$$
$$= \sum_{x,y \in X} [\mathbf{k}(y, \xi \cdot x) g(x), g'(y)]_{\mathcal{H}} = \sum_{x,y \in X} [g(x), \mathbf{k}(\xi \cdot x, y) g'(y)]_{\mathcal{H}}$$

$$= \sum_{x \in X} [g(x), f'(\xi \cdot x)]_{\mathcal{H}} = [f, \pi(\xi^*) f']_{\mathcal{H}},$$

and hence the formula (101) is proven.

In order to show that the axiom (ikd3) holds as well, we use (94). Thus, for all $\xi \in \Gamma$, $x, y \in X$, $h \in \mathcal{H}$, and taking into account that $\mathbf{k}$ is invariant under the action of Γ on X, we have

$$(V(\xi \cdot x)h)(y) = \mathbf{k}(y, \xi \cdot x)h = \mathbf{k}(\xi^* \cdot y, x)h$$
$$= (V(x)h)(\xi^* \cdot y) = (\pi(\xi)V(x)h)(y), \tag{102}$$

which proves (ikd3). Thus, $(\mathcal{K};\ \pi;\ V)$, here constructed, is a Γ-invariant VE-space linearisation of the Hermitian kernel $\mathbf{k}$. Note that $\mathcal{K};\ \pi;\ V)$ is minimal, that is, the axiom (ikd4) holds, since the VE-space linearisation $(\mathcal{K};\ V)$ is minimal.

Let $\mathcal{K}';\ \pi';\ V')$ be another minimal invariant VE-space linearisation of K. We consider the unitary operator $U: \mathcal{K} \to \mathcal{K}'$ defined as in (80) and we already know that $UV(x) = V'(x)$ for all $x \in X$. Since, for any $\xi \in \Gamma$, $x \in X$, and $h \in \mathcal{H}$, we have

$$U\pi(\xi)V(x)h = UV(\xi \cdot x)h = V'(\xi \cdot x)h = \pi'(\xi)V'(x)h = \pi'(\xi)UV(x)h,$$

and taking into account the minimality, it follows that $U\pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$.

(ii) ⇒ (i). Let $\mathcal{K};\ \pi;\ V)$ be a Γ-invariant VE-space linearisation of $\mathbf{k}$. Then

$$\sum_{j,i=1}^{n} [\mathbf{k}(x_i, x_j)h_j, h_i]_{\mathcal{H}} = \sum_{j,i=1}^{n} [V(x_i)^* V(x_j)h_j, h_i]_{\mathcal{H}}$$
$$= [\sum_{j=1}^{n} V(x_j)h_j, \sum_{j=1}^{n} V(x_j)h_j]_{\mathcal{H}} \geq 0,$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $h_1, \ldots, h_n \in \mathcal{H}$, hence $\mathbf{k}$ is positive semidefinite. It was shown in (90) that $\mathbf{k}$ is invariant under the action of Γ on X.

(ii) ⇒ (iii). This follows from Proposition (4.3.11) with the following observation: with notation as in the proof of that proposition, for all $x, y \in X$ and $h \in \mathcal{H}$ we have

$$\mathbf{k}_{\xi \cdot x}(y)h = \mathbf{k}(y, \xi \cdot x)h = V\ (y)^* V(\xi \cdot x)h = V(y)^* \pi(\xi)V(x)h,$$

hence, letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U: \mathcal{K} \to \mathcal{R}$ is the unitary operator defined as in (84), we obtain a $*$-representation of Γ on the VH-space $\mathcal{R}$ such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$.

(iii) ⇒ (ii). Let $\rho: \Gamma \to \mathcal{L}^*(\mathcal{R})$ is a $*$-representation such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. Again, we use Proposition (4.3.11). Letting $\pi = \rho$, it is then easy to see that $(\mathcal{R};\ \pi;\ V)$ is a Γ-invariant VE-space linearisation of the kernel $\mathbf{k}$.

Remarks (4.3.15)[226]: (i) Given $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ a positive semidefinite kernel, as a consequence of Theorem (4.3.14) we can denote, without any ambiguity, by $\mathcal{R}_{\mathbf{k}}$ the unique minimal $\mathcal{H}$-reproducing kernel VE-space on X associated to $\mathbf{k}$.

(ii) The construction in the proof of (i) ⇒ (ii) in Theorem (4.3.14) is essentially a minimal $\mathcal{H}$-reproducing kernel VE-space one. More precisely, we first note that, for arbitrary $f \in \mathcal{F}(X;\ \mathcal{H})$, $f = Kg$ with $g \in \mathcal{F}_0(X;\ \mathcal{H})$, we have

$$f = \sum_{x \in X} \mathbf{k}(y, x)g(x) = \sum_{x \in X} \mathbf{k}_x(y)g(x), \tag{103}$$

hence $\mathcal{G}(X;\ \mathcal{H}) = \mathrm{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\}$. Then, for arbitrary $f \in \mathcal{G}$ we have

$$[f, \mathbf{k}_x h]_{\mathcal{K}} = [f, \mathbf{k}_x h]_{\mathcal{G}} = [f, Kh_x]_{\mathcal{G}} = \sum_{y \in X} [f(y), (h_x)(y)]_{\mathcal{H}}$$

$$= [f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}(K)}, \qquad x \in X, \;\; h \in \mathcal{H},$$

hence $[\cdot,\cdot]_{\mathcal{K}} = [\cdot,\cdot]_{\mathcal{R}(K)}$ on $\mathcal{G}(X;\, \mathcal{H}) = \mathrm{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\}$, that coincides with both $\mathcal{K}$ and $\mathcal{R}(K)$. Therefore, we can take $\mathcal{K} = \mathcal{R}(K) = \mathcal{G}(X;\, \mathcal{H})$ to be a VE-space, with the advantage that it consists entirely of $\mathcal{H}$-valued functions on X.

This idea was used in [204] as well and the source of inspiration is [225].

We obtain, as consequences of the main result, different versions of known dilation theorems in non-topological versions.

Given a VE-space Hover an ordered $*$-space Z and a $*$-semigroup Γ, a map $\varphi\colon \Gamma \to \mathcal{L}^*(\mathcal{H})$ is called positive semidefinite or of positive type if, for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \Gamma$, and $h_1, \ldots, h_n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^{n} [\phi(\xi_i^* \, \xi_j) h_j, h_i]_{\mathcal{H}} \geq 0. \tag{104}$$

Given a map $\varphi\colon \Gamma \to \mathcal{L}^*(\mathcal{H})$ we consider the kernel $\mathbf{k} : \Gamma \times \Gamma \to \mathcal{L}^*(\mathcal{H})$ defined by

$$\mathbf{k}(\alpha, \beta) = \phi(\alpha^* \beta), \qquad \alpha, \beta \in \Gamma, \tag{105}$$

and observe that φ is positive semidefinite, in the sense of (104), if and only if $\mathbf{k}$ is positive semidefinite, in the sense of (77).

On the other hand, considering the action of Γ on itself by left multiplication, the kernel $\mathbf{k}$, as defined at (105), is Γ-invariant, in the sense of (89). Indeed,

$$\mathbf{k}(\xi, \alpha \cdot \zeta) = \varphi(\xi^* \alpha \zeta) = \varphi((\alpha^* \xi)^* \zeta) = \mathbf{k}(\alpha^* \cdot \xi, \zeta), \qquad \alpha, \xi, \zeta \in \Gamma.$$

Therefore, the following corollary is a direct consequence of Theorem (4.3.14).

Corollary (4.3.16)[226]: Let $\varphi\colon \Gamma \to \mathcal{L}^*(\mathcal{H})$ be a map, for some $*$-semigroup Γ and some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) The map φ is positive semidefinite.

(ii) There exists a VE-space $\mathcal{K}$ over Z, a map $V\colon \Gamma \to \mathcal{L}^*(\mathcal{H}, \mathcal{K})$, and a $*$-representation $\pi\colon \Gamma \to \mathcal{L}^*(\mathcal{K})$, such that:

(i) $\varphi(\xi^* \zeta) = V(\xi)^* V(\zeta)$ for all $\xi, \zeta \in \Gamma$.

(ii) $V(\xi\zeta) = \pi(\xi) V(\zeta)$ for all $\xi, \zeta \in \Gamma$.

In addition, if this happens, then the triple $(\mathcal{K};\, \pi;\, V)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $V(\Gamma)\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

(iii) There exist an $\mathcal{H}$-reproducing kernel VE-space $\mathcal{R}$ on Γ and a $*$-representation $\rho\colon \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that:

(a) $\mathcal{R}$ has the reproducing kernel $\Gamma \times \Gamma \ni (\xi, \zeta) \mapsto \varphi(\xi^* \zeta) \in \mathcal{L}^*(\mathcal{H})$.

(b) $\rho(\alpha) \varphi(\cdot\, \xi) h = \varphi(\cdot\, \alpha \xi) h$ for all $\alpha, \xi \in \Gamma$ and $h \in \mathcal{H}$.

In addition, the reproducing kernel VE-space $\mathcal{R}$ as in (3) can be always constructed minimal and in this case it is uniquely determined by φ.

As can be observed from condition (2).(a) in Corollary (4.3.16), we do not have a representation of φ on the whole $*$-semigroup Γ but only on its $*$-subsemigroup $\{\xi^* \zeta \mid \xi, \zeta \in \Gamma\}$, which may be strictly smaller than Γ. This situation can be remedied, for example, in case the $*$-semigroup Γ has a unit, when the previous corollary takes a form similar with B. Sz.-Nagy Theorem, cf. [225].

Corollary (4.3.17)[226]: Assume that the $*$-semigroup Γ has a unit ϵ. Let $\varphi : \Gamma \to \mathcal{L}^*(\mathcal{H})$ be a map, for some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) The map φ is positive semidefinite.

(ii) There exist a VE-space $\mathcal{K}$ over Z, a linear operator $W \in \mathcal{L}^*(\mathcal{H},\mathcal{K})$, and a unital $*$-representation $\pi: \Gamma \to \mathcal{L}^*(\mathcal{K})$, such that:

$$\varphi(\alpha) = W^*\pi(\alpha)W, \qquad \alpha \in \Gamma. \tag{106}$$

In addition, if this happens, then the triple $(\mathcal{K};\ \pi;\ V)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $\pi(\Gamma)W\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

Given a $*$-algebra $\mathcal{A}$, a linear map $\varphi: \Gamma \to \mathcal{L}^*(\mathcal{H})$, for some VE-space Hover an ordered $*$-space Z, is called positive semidefiniteif for all $a_1, \ldots, a_n \in \mathcal{A}$, and $h_1, \ldots, h_n \in \mathcal{H}$ we have

$$\sum_{i,j=1}^{n} [\varphi(a_i^* a_j)h_j, h_i]_{\mathcal{H}} \geq 0, \tag{107}$$

where the inequality is understood in Z with respect to the given cone Z^+ and the underlying partial order. Observe that for a Hermitian linear map $\varphi: \Gamma \to \mathcal{L}^*(\mathcal{H})$ one can define a Hermitian kernel $\boldsymbol{k}_\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{L}^*(\mathcal{H})$ by letting

$$\boldsymbol{k}_\varphi(a,b) = \varphi(a^*b), \qquad a,b \in \mathcal{A}.$$

Also, observe that the $*$-algebra Acan be viewed as a multiplicative $*$-semigroup and, letting Aact on itself by multiplication, the kernel $\boldsymbol{k}_\varphi$ is invariant under this action. With this notation, another consequence of Theorem (4.3.14) is the following:

Corollary (4.3.18)[226]: Let $\varphi : \mathcal{A} \to \mathcal{L}^*(\mathcal{H})$ be a linear map, for some $*$-algebra Aand some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) The map φ is positive semidefinite.

(ii) There exist a VE-space $\mathcal{K}$ over the ordered $*$-space Z, a linear map $V : \mathcal{A} \to \mathcal{L}^*(\mathcal{H},\mathcal{K})$, and a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}^*(\mathcal{K})$, such that:

(i) $\varphi(a^*b) = V(a)^*V(b)$ for all $a, b \in \mathcal{A}$.

(ii)$V(ab) = \pi(a)V(b)$ for all $a, b \in \mathcal{A}$.

In addition, if this happens, then the triple $(\mathcal{K};\ \pi;\ V)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $V(\mathcal{A})\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

(iii) There exist an $\mathcal{H}$-reproducing kernel VE-space $\mathcal{R}$ on $\mathcal{A}$ and a $*$-representation $\rho: \mathcal{A} \to \mathcal{L}^*(\mathcal{R})$ such that:

(a) $\mathcal{R}$ has the reproducing kernel $\mathcal{A} \times \mathcal{A} \ni (a,b) \mapsto \varphi(a^*b) \in \mathcal{L}^*(\mathcal{H})$.

(b) $\rho(a)\varphi(\cdot\, b)h = \varphi(\cdot\, ab)h$ for all $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$.

In addition, the reproducing kernel VE-space $\mathcal{R}$ as in (iii) can be always constructed minimal and in this case it is uniquely determined by φ.

In case the $*$-algebra has a unit, the previous corollary yields a Stinespring type Representation Theorem, cf. [223], or its generalisations [204]. More precisely, letting edenote the unit of the $*$-algebra $\mathcal{A}$ and with the notation as in Corollary (4.3.18).(ii), letting $W = V(e)$, we have

Corollary (4.3.19)[226]: Let $\mathcal{A}$ be a unital $*$-algebra $\mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{L}^*(\mathcal{H})$ a linear map, for some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) φ is positive semidefinite.

(ii) There exist $\mathcal{K}$ a VE-space over the same ordered $*$-space Z, a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}^*(\mathcal{K})$, and $W \in \mathcal{L}^*(\mathcal{H},\mathcal{K})$ such that

$$\varphi(a) = W^*\pi(a)W, \qquad a \in \mathcal{A}. \tag{108}$$

In addition, if this happens, then the triple $(\mathcal{K};\ \pi;\ W)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $\pi(\mathcal{A})W\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

Remarks (4.3.20)[226]: (i) In dilation theory, one encounters also the notion of completely positive, e.g. see [222]. In our setting, we can consider a linear map $\varphi : \mathcal{V} \to \mathcal{L}^*(\mathcal{E})$, where $\mathcal{V}$ is a $*$-space and $\mathcal{E}$ is some VE-space over an ordered $*$-space Z. For each none can consider the $*$-space $M_n(\mathcal{V})$ of all $n \times n$ matrices with entries in $\mathcal{V}$. Then the n-th amplification map $\varphi_n: M_n(\mathcal{V}) \to M_n(\mathcal{L}^*(\mathcal{E})) = \mathcal{L}^*(\mathcal{E}^n)$ is defined by

$$\varphi_n([a_{i,j}]_{i,j=1}^n) = [\varphi_n(a_{i,j})]_{i,j=1}^n, \qquad [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{V}). \tag{109}$$

Basically, φ would be called completely positive if φ_n is "positive" for all n, where "positive" should mean that, whenever $[a_{i,j}]_{i,j=1}^n$ is "positive" in $M_n(\mathcal{V})$ then $\varphi_n([a_{i,j}]_{i,j=1}^n)$ is positive in $M_n(\mathcal{L}^*(\mathcal{E}))$. Since positivity in $M_n(\mathcal{L}^*(\mathcal{E}))$ is perfectly defined, see Remark (4.3.6), the only problem is to define positivity in $M_n(\mathcal{V})$. One of the possible approaches, e.g. see [228], is to assume $\mathcal{V}$ be a matrix quasi ordered $*$-space, that is, there exists $\{C_n\}_{n\geq 1}$ a matrix quasi ordering of $\mathcal{V}$, in the following sense

(mo1) For each $n \in \mathbb{N}$, C_n is a cone on $M_n(\mathcal{V})$.

(mo2)For each $m, n \in \mathbb{N}$ and each $m \times n$ matrix with complex entries, we have $T^* C_m T \subseteq C_n$, where multiplication is the usual matrix multiplication.

In the special case when (mo1) is changed such that for each $n \in \mathbb{N}$, the cone C_n is strict, one has the concept of matrix ordering and, respectively, of matrix ordered $*$-space, e.g. see [229]. For example, $\mathcal{L}^*(\mathcal{E})$ has a natural structure of matrix ordered $*$-algebra, see Remark (4.3.6). Observe that, in the latter case, each $M_n(\mathcal{V})$ is an ordered $*$-space hence, in this case, the concept of completely positive map $\varphi : \mathcal{V} \to \mathcal{L}^*(\mathcal{E})$ makes perfectly sense.

In the former case, that of matrix quasi ordered $*$-space $\mathcal{V}$, the concept of completely positive map φ makes sense as well.

(ii) Assuming that instead of $\mathcal{V}$ we have a $*$-algebra $\mathcal{A}$ and that the concept of a completely positive map $\varphi : \mathcal{A} \to \mathcal{L}^*(\mathcal{E})$ is defined, a natural question is what is the relation of this concept with that of positive semidefinite map φ. By inspection, it can be observed that, in order to relate the two concepts, the matrix (quasi) ordering on $\mathcal{A}$ should be related with that of $*$-positivity, see Remark (4.3.1). More precisely, observe first that $*$-positivity provides in a natural way a matrix quasi ordering of $\mathcal{A}$. Then, one can prove that if φ is completely positive, with definition as in item (i) and with respect to the $*$-positivity, then φ is positive semidefinite, with definition as in (107). The converse is even more problematic, depending on whether any $*$-positive matrix $[a_{i,j}]_{i,j=1}^n$ can be represented as a sum of matrices a^*a, where ais a special matrix with only one non-null row. This special situation happens for C^*-algebras [223], or even for locally C^*-algebras [215], but it may fail even for pre C^*-algebras, in general.

Given an ordered $*$-algebra $\mathcal{A}$ and a VE-module $\mathcal{E}$ over $\mathcal{A}$, an $\mathcal{E}$-reproducing kernel VE-module over $\mathcal{A}$ is just an $\mathcal{E}$-reproducing kernel VE-space over $\mathcal{A}$, with definition, which is also a VE-module over $\mathcal{A}$.

Proposition (4.3.21)[226]: Let Γ be a $*$-semigroup that acts on the nonempty set X and let $\mathbf{k} : X \times X \to \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-module $\mathcal{H}$ over an ordered $*$-algebra $\mathcal{A}$. The following assertions are equivalent:

(i) $\mathbf{k}$ is positive semidefinite, in the sense of (77), and invariant under the action of Γ on X, that is, (89) holds.

(ii) $\mathbf{k}$ has a Γ-invariant VE-module (over $\mathcal{A}$) linearisation $(\mathcal{K};\ \pi;\ V)$.

(iii) $\mathbf{k}$ a dmits an $\mathcal{H}$-reproducing kernel VE-module $\mathcal{R}$ and there exists a $*$-representation $\rho: \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (i), (ii), or (iii) holds, then a minimal Γ-invariant VE-module linearisation can be constructed, any minimal Γ-invariant VE-module linearisation is unique up to unitary equivalence, a pair $(\mathcal{R}; \rho)$ as in assertion (iii) with $\mathcal{R}$ minimal can be always obtained and, in this case, it is uniquely determined by $\mathbf{k}$ as well.

Proof. We use the notation as in the proof of Theorem (4.3.14). We actually prove only the implication (i) ⇒ (ii) since, as observed in Remark (4.3.15), that construction provides a Γ-invariant reproducing kernel VE-space linearisation, while the other implications are not much different.

(i) ⇒ (ii). We first observe that, since $\mathcal{H}$ is a module over $\mathcal{A}$, the space $\mathcal{F}(X; \mathcal{H})$ has a natural structure of right module over $\mathcal{A}$, more precisely, for any $f \in \mathcal{F}(X; \mathcal{H})$ and $a \in \mathcal{A}$

$$(fa)(x) = f(x)a, x \in X.$$

In particular, the space $\mathcal{F}_0(X; \mathcal{H})$ is a submodule of $\mathcal{F}(X; \mathcal{H})$. On the other hand, by assumption, for each $x, y \in X$, $\mathbf{k}(x, y) \in \mathcal{L}^*(\mathcal{H})$, hence $\mathbf{k}(x, y)$ is a module map. These imply that the convolution operator $K: \mathcal{F}_0(X; \mathcal{H}) \to \mathcal{F}(X; \mathcal{H})$ defined as in (76) is a module map. Indeed, for any $f \in \mathcal{F}_0(X; \mathcal{H})$, $a \in \mathcal{A}$, and $y \in X$,

$$((Kf)a)(x) = \sum_{x \in X} \mathbf{k}(y, x) f(x) a = K(fa)(x).$$

Then, the space $\mathcal{G}(X; \mathcal{H})$ which, with the definition as in (91), is the range of the convolution operator K, is a module over $\mathcal{A}$ as well.

We show that, when endowed with the $\mathcal{A}$ valued gramian $[\cdot,\cdot]_{\mathcal{G}}$ defined as in (92), we have

$$[e, fa]_{\mathcal{G}} = [e, f]_{\mathcal{G}}\, a, \quad e, f \in \mathcal{G}(X; \mathcal{H}), \qquad a \in \mathcal{A}. \tag{110}$$

To see this, let $e = Kg$ and $f = Kh$ for some $g, h \in \mathcal{F}_0(X; \mathcal{H})$. Then,

$$[e, fa]_{\mathcal{G}} = [Kg, ha]_{\mathcal{F}_0} = \sum_{y \in X} [e(y), h(y)a]_{\mathcal{H}}$$

$$= \sum_{y \in X} [e(y), h(y)]_{\mathcal{H}}\, a = [Kg, h]_{\mathcal{F}_0} a = [e, f]_{\mathcal{G}}\, a.$$

From (110) and the proof of the implication (i) ⇒ (ii) in Theorem (4.3.14), it follows that $\mathcal{K} = \mathcal{G}(X; \mathcal{H})$ is a VE-module over the ordered $*$-algebra $\mathcal{A}$ and hence, the triple $(\mathcal{K}; \pi; V)$ is a minimal Γ-invariant VE-module linearisation of $\mathbf{k}$.

Corollary (4.3.22)[226]: Let $\varphi : \mathcal{B} \to \mathcal{L}^*(\mathcal{H})$ be a linear map, for some $*$-algebra Band some VE-module Hover an ordered $*$-algebra $\mathcal{A}$. The following assertions are equivalent:

(i) The map φ is positive semidefinite.

(ii) There exist a VE-module $\mathcal{K}$ over the ordered $*$-algebra $\mathcal{A}$, a linear map $V: \mathcal{B} \to \mathcal{L}^*(\mathcal{H}, \mathcal{K})$, and a $*$-representation $\pi: \mathcal{B} \to \mathcal{L}^*(\mathcal{K})$, such that:

(i) $\varphi(a^* b) = V(a)^* V(b)$ for all $a, b \in \mathcal{B}$.

(ii) $V(ab) = \pi(a) V(b)$ for all $a, b \in \mathcal{B}$.

In addition, if this happens, then the triple $(\mathcal{K}; \pi; V)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $V(\mathcal{B})\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

(iii) There exist an $\mathcal{H}$-reproducing kernel VE-module $\mathcal{R}$ on $\mathcal{A}$ and a $*$-representation $\rho: \mathcal{B} \to \mathcal{L}^*(\mathcal{R})$ such that:

(a) $\mathcal{R}$ has the reproducing kernel $\mathcal{B} \times \mathcal{B} \ni (a, b) \mapsto \varphi(a^*b) \in \mathcal{L}^*(\mathcal{H})$.

(b) $\rho(a)\varphi(\cdot\, b)h = \varphi(\cdot\, ab)h$ for all $a, b \in \mathcal{B}$ and $h \in \mathcal{H}$.

In addition, the reproducing kernel VE-module $\mathcal{R}$ as in (3) can be always constructed minimal and in this case it is uniquely determined by φ.

In case the $*$-algebra $\mathcal{B}$ is unital, Corollary (4.3.22) takes a form that reveals the fact that it is actually a non-topological version of Kasparov's Theorem [216] and its generalization [199].

Corollary (4.3.23)[226]: Let $\mathcal{B}$ be a unital $*$-algebra and $\varphi : \mathcal{B} \to \mathcal{L}^*(\mathcal{H})$ a linear map, for some VE-module Hover an ordered $*$-algebra $\mathcal{A}$. The following assertions are equivalent:

(i) φ is positive semidefinite.

(ii) There exist a VE-module $\mathcal{K}$ over $\mathcal{A}$, a $*$-representation $\pi: \mathcal{B} \to \mathcal{L}^*(\mathcal{K})$, and $W \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ such that

$$\varphi(b) = W^*\pi(b)W, \qquad b \in \mathcal{B}. \tag{111}$$

In addition, if this happens, then the triple $(\mathcal{K};\ \pi;\ W)$ can always be chosen minimal, in the sense that $\mathcal{K}$ is the linear span of the set $\pi(\mathcal{A})W\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

Corollary (4.3.24)[295]: Assume that the kernel $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ is 2-positive. Then:

(i) $\mathbf{k}$ is Hermitian.

(ii) If, for some $x \in X$, we have $\mathbf{k}(x, x) = 0$, then $\mathbf{k}(x, x + \epsilon) = 0$ for all $(x + \epsilon) \in X$.

(iii) Assume that, for $x, (x + \epsilon) \in X$ the operators $\mathbf{k}(x, x)$ and $\mathbf{k}(x + \epsilon, x + \epsilon)$ are bounded. Then $\mathbf{k}(x, x + \epsilon)$ and $\mathbf{k}(x + \epsilon, x) = \mathbf{k}(x, x + \epsilon)^*$ are bounded and

$$\|\mathbf{k}(x, x + \epsilon)\|^2 \le \|\mathbf{k}(x, x)\|\|\mathbf{k}(x + \epsilon, x + \epsilon)\|. \tag{112}$$

In particular, if $\mathbf{k}(x, x) \in \mathcal{B}^*(\mathcal{E})$ for all $x \in X$, then $\mathbf{k}(x + \epsilon, x) \in \mathcal{B}^*(\mathcal{E})$ for all $x, (x + \epsilon) \in X$.

Proof. The proof of (i) and (ii) is the same as the proof of Lemma 3.1 from [204].

(iii) Assume that both operators $\mathbf{k}(x, x)$ and $\mathbf{k}(x + \epsilon, x + \epsilon)$ are bounded, hence $\mathbf{k}(x, x)$, $\mathbf{k}(x + \epsilon, x + \epsilon) \in \mathcal{B}^*(\mathcal{E})$. If $\mathbf{k}(x + \epsilon, x + \epsilon) = 0$ then, by (ii), $\mathbf{k}(x, x + \epsilon) = 0$ and $\mathbf{k}(x + \epsilon, x) = \mathbf{k}(x, x + \epsilon)^* = 0$, hence bounded, and the inequality (112) holds trivially.

Assume that $\mathbf{k}(x + \epsilon, x + \epsilon) \neq 0$, hence $\|\mathbf{k}(x + \epsilon, x + \epsilon)\| > 0$. Since $\mathbf{k}$ is 2-positive, for any $h_n, g_n \in \mathcal{H}$ we have

$$[\boldsymbol{k}(x, x)h_n, h_n] + [\boldsymbol{k}(x, x + \epsilon)g_n, h_n] + [\boldsymbol{k}(x + \epsilon, x)h_n, g_n] + [\boldsymbol{k}(x + \epsilon, x + \epsilon)g_n, g_n] \ge 0. \tag{113}$$

We let $g_n = -\mathbf{k}(\boldsymbol{x}, x + \epsilon)^* h_n / \|\mathbf{k}(x + \epsilon, x + \epsilon)\|$ in (113), take into account (72) and get

$$\frac{2}{\|\mathbf{k}(x + \epsilon, x + \epsilon)\|}[\mathbf{k}(x + \epsilon, x)h_n, \mathbf{k}(x + \epsilon, x)h_n]$$
$$\le [\mathbf{k}(x, x)h_n, h_n] + \frac{1}{\|\mathbf{k}(x + \epsilon, x + \epsilon)\|^2}[\mathbf{k}(x + \epsilon, x + \epsilon)\mathbf{k}(x + \epsilon, x)h_n, \mathbf{k}(x + \epsilon, x)h_n]$$
$$\le [\mathbf{k}(x, x)h_n, h_n] + \frac{\|\mathbf{k}(x + \epsilon, x + \epsilon)\|}{\|\mathbf{k}(x + \epsilon, x + \epsilon)\|^2}[\mathbf{k}(x + \epsilon, x)h_n, \mathbf{k}(x + \epsilon, x)h_n]$$

$$= [\mathbf{k}(x,x)h_n, h_n] + \frac{1}{\|\mathbf{k}(x+\epsilon, x+\epsilon)\|}[\mathbf{k}(x+\epsilon, x)h_n, \mathbf{k}(x+\epsilon, x)h_n],$$

hence

$$[\mathbf{k}(x+\epsilon, x)h_n, \mathbf{k}(x+\epsilon, x)h_n] \le \|\mathbf{k}(x+\epsilon, x+\epsilon)\|[\mathbf{k}(x,x)h_n, h_n]$$
$$\le \|\mathbf{k}(x,x)\|\|\mathbf{k}(x+\epsilon, x+\epsilon)\|[h_n, h_n],$$

which proves that $\mathbf{k}(x+\epsilon, x)$ is a bounded operator and the inequality (112).

Corollary (4.3.25)[295]: [226] Let X be a nonempty set, $\mathcal{H}$ a VE-space over an ordered $*$-space Z, and let $\mathcal{R} \subseteq \mathcal{F}(X;\ \mathcal{H})$, with all algebraic operations, be a VE-space over Z. Then $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VE-space if and only if, for all $x \in X$, the restriction of the evaluation operator E_x to $\mathcal{R}$ is adjointable as a linear operator $\mathcal{R} \to \mathcal{H}$.

Proof. Assume first that $\mathcal{R}$ is an $\mathcal{H}$-reproducing kernel VE-space on X and let $\mathbf{k}$ be its reproducing kernel. For any $h_n \in \mathcal{H}$ and any $f_n \in \mathcal{R}$

$$[E_x f_n, h_n]_{\mathcal{H}} = [f_n(x), h_n]_{\mathcal{H}} = [f_n, \mathbf{k}_x h_n]_{\mathcal{R}}. \tag{114}$$

Since $\mathbf{k}_x \in \mathcal{L}(\mathcal{H}, \mathcal{R})$, it follows that E_x is adjointable and, in addition, $E_x^* = \mathbf{k}_x$, for all $x \in X$.

Conversely, assume that, for all $x \in X$, the evaluation operator $E_x \in \mathcal{L}^*(\mathcal{R}, \mathcal{H})$. Equation (114) shows that, in order to show that $\mathcal{R}$ is a reproducing kernel VE-space, we should define the kernel $\mathbf{k}$ in the following way:

$$\mathbf{k}(x+\epsilon, x)h_n = (E_x^* h_n)(x+\epsilon), \qquad x, (x+\epsilon) \in X,\ \ h_n \in \mathcal{H}. \tag{115}$$

It is clear that $k(x+\epsilon, x)\colon \mathcal{H} \to \mathcal{H}$ is a linear operator and observe that $\mathbf{k}_x h_n = E_x^* h_n$ for all $x \in X$ and all $h_n \in \mathcal{H}$. The reproducing property (rk3) holds:

$$[f_n(x), h_n]_{\mathcal{H}} = [E_x f_n, h_n]_{\mathcal{H}} = [f_n, E_x^* h_n]_{\mathcal{R}} = [f_n, \mathbf{k}_x h_n]_{\mathcal{R}}, \qquad f_n \in R, h_n \in H, x \in X.$$

The axioms (rk1) and (rk2) are clearly satisfied, so it only remains to prove that $\mathbf{k}$ is a Hermitian kernel. To see this, fix $x, (x+\epsilon) \in X$ and $h_n, l_n \in \mathcal{H}$. Then

$$[\mathbf{k}(x+\epsilon, x)h_n, l_n]_{\mathcal{H}} = [(\mathbf{k}_x h_n)(x+\epsilon), l_n]_{\mathcal{H}} = [\mathbf{k}_x h_n, \mathbf{k}_{x+\epsilon} l_n]_{\mathcal{R}}$$
$$= [\mathbf{k}_{x+\epsilon} l_n, \mathbf{k}_x h_n]_{\mathcal{R}}^* = [\mathbf{k}(x, x+\epsilon) l_n, h_n]_{\mathcal{R}}^* = [h_n, \mathbf{k}(x, x+\epsilon) l_n]_{\mathcal{R}}.$$

Therefore, $\mathbf{k}(x+\epsilon, x)$ is adjointable and $\mathbf{k}(x+\epsilon, x)^* = \mathbf{k}(x, x+\epsilon)$, hence $\mathbf{k}$ is a Hermitian kernel. We have proven that $\mathbf{k}$ is the reproducing kernel of $\mathcal{R}$.

There is a very close connection between VE-space linearisations and reproducing kernel VE-spaces.

Corollary (4.3.26)[295]: [226] Let X be a nonempty set, $\mathcal{H}$ a VE-space over an ordered $*$-space Z, and let $\mathbf{k}\colon X \times X \to \mathcal{L}^*(\mathcal{H})$ be a Hermitian kernel.

(i) Any $\mathcal{H}$-reproducing kernel VE-space Rwith kernel $\mathbf{k}$ is a VE-space linearisation $(\mathcal{R};\ V)$ of $\mathbf{k}$, with $V(x) = \mathbf{k}_x$ for all $x \in X$.

(ii) For any minimal VE-space linearisation $(\mathcal{K};\ V)$ of $\mathbf{k}$, letting

$$\mathcal{R} = \{V(\cdot)^* f_n \mid f_n \in \mathcal{K}\}, \tag{116}$$

we obtain the minimal $\mathcal{H}$-reproducing kernel VE-space with reproducing kernel $\mathbf{k}$.

Proof. (ii)⇒(i). Let $(\mathcal{K};\ \pi;\ V)$be a minimal VE-space linearisation of the kernel $\mathbf{k}$ on X. Let Rbe the set of all functions $X \ni x \mapsto V(x)^* f_n \in \mathcal{H}$, in particular $\mathcal{R} \subseteq \mathcal{F}(X;\ \mathcal{H})$, and we endow Rwith the algebraic operations inherited from the complex vector space $\mathcal{F}(X;\ \mathcal{H})$.

The correspondence

$$K \ni f_n \mapsto Uf_n = V(\cdot)^* f_n \in \mathcal{R} \tag{117}$$

is bijective. By the definition of $\mathcal{R}$, this correspondence is surjective. In order to verify that it is injective as well, let $f_n, g_n \in \mathcal{K}$ be such that $V^*(\cdot)f_n = V^*(\cdot)g_n$. Then, for all $x \in X$ and all $h_n \in \mathcal{H}$ we have

$$[V(x)^* f_n, h_n]H = [V(x)^* g_n, h_n]_{\mathcal{H}},$$

equivalently,

$$[f_n - g_n, V(x)h_n]_{\mathcal{K}} = 0, \qquad x \in X, \;\; h_n \in \mathcal{H}.$$

By the minimality of the VE-space linearisation $(\mathcal{K};\, V)$ it follows that $g_n = f_n$. Thus, U is a bijection.

Clearly, the bijective map U defined at (117) is linear, hence a linear isomorphism of complex vector spaces $\mathcal{K} \to \mathcal{R}$. On $\mathcal{R}$ we introduce a Z-valued pairing

$$[Uf_n, Ug_n] = [V(\cdot)^* f_n, V(\cdot)^* g_n]_{\mathcal{R}} = [f_n, g_n]_{\mathcal{K}}, \qquad f_n, g_n \in \mathcal{K}. \tag{118}$$

Then $(R;\, [\cdot,\cdot]_{\mathcal{R}})$ is a VE-space over Z since, by (118), we transported the Z-gramian from $\mathcal{K}$ to $\mathcal{R}$ or, in other words, we have defined on $\mathcal{R}$ the Z-gramian that makes the linear isomorphism U a unitary operator between the VE-spaces $\mathcal{K}$ and $\mathcal{R}$.

We show that $(R;\, [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VE-space with corresponding reproducing kernel $\mathbf{k}$. By definition, $\mathcal{R} \subseteq \mathcal{F}(X;\, \mathcal{H})$. On the other hand, since

$\mathbf{k}_x(x+\epsilon)h_n = \mathbf{k}(x+\epsilon, x)h_n = V(x+\epsilon)^* V(x)h_n$, for all $x, (x+\epsilon) \in X$ and all $h_n \in \mathcal{H}$,

taking into account that $V(x)h_n \in \mathcal{K}$, by (116) it follows that $\mathbf{k}_x \in \mathcal{R}$ for all $x \in X$. Further, for all $f_n \in \mathcal{R}$, $x \in X$, and $h_n \in \mathcal{H}$, we have

$$\begin{aligned}[f_n, \mathbf{k}_x h_n]_{\mathcal{R}} &= [V(\cdot)^* g_n, \mathbf{k}_x h_n]_{\mathcal{R}} = [V(\cdot)^* g_n, V(\cdot)^* V(x)h_n]_{\mathcal{R}} \\ &= [g_n, V(x)h_n]_{\mathcal{K}} = [V(x)^* g_n, h_n]_{\mathcal{H}} = [f_n(x), h_n]_{\mathcal{H}},\end{aligned}$$

where $g_n \in \mathcal{K}$ is the unique vector such that $V(x)^* g_n = f_n(x)$, which shows that Rsatisfies the reproducing axiom as well.

(i) ⇒ (ii). Assume that $(R;\, [\cdot,\cdot]_{\mathcal{R}})$ is an $\mathcal{H}$-reproducing kernel VE-space on X, with reproducing kernel $\mathbf{k}$. We let $\mathcal{K} = \mathcal{R}$ and define

$$V(x)h_n = \mathbf{k}_x h_n, \qquad x \in X, h_n \in \mathcal{H}. \tag{119}$$

Note that $V(x)\colon \mathcal{H} \to \mathcal{K}$ is linear for all $x \in X$.

We show that $V(x) \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ for all $x \in X$. To see this, first note that, by the reproducing property,

$$[f_n, V(x)h_n]_{\mathcal{K}} = [f_n, \mathbf{k}_x h_n]_{\mathcal{R}} = [f_n(x), h_n]_{\mathcal{H}}, \qquad x \in X, h_n \in \mathcal{H}. \tag{120}$$

Let us then, for fixed $x \in X$, consider the linear operator $W(x)\colon \mathcal{R} = \mathcal{K} \to \mathcal{H}$ defined by $W(x)f_n = f_n(x)$ for all $f_n \in \mathcal{R} = \mathcal{K}$. From (120) we conclude that $V(x)$ is adjointable and $V(x)^* = W(x)$ for all $x \in X$.

Finally, by the reproducing axiom, for all $x, (x+\epsilon) \in X$ and all $h_n, g_n \in \mathcal{H}$ we have

$$\begin{aligned}[V(x+\epsilon)^* V(x)h_n, g_n]_{\mathcal{H}} &= [V(x)h_n, V(x+\epsilon)g_n]_{\mathcal{R}} = [\mathbf{k}_x h_n, \mathbf{k}_{x+\epsilon} g_n]_{\mathcal{R}} \\ &= [\mathbf{k}(x+\epsilon, x)h_n, g_n]_{\mathcal{H}},\end{aligned}$$

hence $V(x+\epsilon)^* V(x) = \mathbf{k}(x+\epsilon, x)$ for all $x, (x+\epsilon) \in X$. Thus, $(\mathcal{K};\, V)$ is a VE-space linearisation of $\mathbf{k}$ (actually, a minimal one).

Corollary (4.3.27)[295]: Let Γ be a $*$-semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \to \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-space Hover an ordered $*$-space Z. The following assertions are equivalent:

(i) $\mathbf{k}$ is positive semidefinite, in the sense of (77), and invariant under the action of Γ on X, that is, (89) holds.

(ii) $\mathbf{k}$ has a Γ-invariant VE-space linearisation $(\mathcal{K};\, \pi;\, V)$.

(iii) $\mathbf{k}$ admits an $\mathcal{H}$-reproducing kernel VE-space $\mathcal{R}$ and there exists a $*$-representation $\rho: \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h_n = \mathbf{k}_{\xi \cdot x} h_n$ for all $\xi \in \Gamma$, $x \in X$, $h_n \in \mathcal{H}$.

In addition, in case any of the assertions (i), (ii), or (iii) holds, then a minimal Γ-invariant VE-space linearisation can be constructed, any minimal Γ-invariant VE-space linearisation is unique up to unitary equivalence, a pair $(\mathcal{R};\, \rho)$ as in assertion (iii) with $\mathcal{R}$ minimal can be always obtained and, in this case, it is uniquely determined by $\mathbf{k}$ as well.

Proof. (i) ⇒ (ii). Assuming that $\mathbf{k}$ is positive semidefinite, by Lemma (4.3.8).(i) it follows that $\mathbf{k}$ is Hermitian, that is, $\mathbf{k}(x, x+\epsilon)^* = \mathbf{k}(x+\epsilon, x)$ for all $x, (x+\epsilon) \in X$. We consider the convolution operator K defined at (76) and let $\mathcal{G} = \mathcal{G}(X;\, \mathcal{H})$ be its range, more precisely,

$$\mathcal{G} = \{f_n \in \mathcal{F} \mid f_n = K g_n \text{ for some } g_n \in \mathcal{F}_0\}$$
$$= \{f_n \in \mathcal{F} \mid f_n(x+\epsilon) = \sum_{x \in X} \mathbf{k}(x+\epsilon, x) g_n(x) \text{ for some } g_n \in \mathcal{F}_0 \text{ and all } x \in X\}. \quad (121)$$

A pairing $[\cdot,\cdot]_{\mathcal{G}}: \mathcal{G} \times \mathcal{G} \to Z$ can be defined by

$$[e_n, f_n]_{\mathcal{G}} = [K g_n, h_n]_{\mathcal{F}_0} = \sum_{(x+\epsilon) \in X} [e_n(x+\epsilon), h_n(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x,(x+\epsilon) \in X} [\mathbf{k}(x+\epsilon, x) g_n(x), h_n(x+\epsilon)]_{\mathcal{H}}, \quad (122)$$

where $f_n = K h_n$ and $e_n = K g_n$ for some $g_n, h_n \in \mathcal{F}_0$. We observe that

$$[e_n, f_n]_{\mathcal{G}} = \sum_{(x+\epsilon) \in X} [e_n(x+\epsilon), h_n(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x,(x+\epsilon) \in X} [\mathbf{k}(x+\epsilon, x) g_n(x), h_n(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x,(x+\epsilon) \in X} [g_n(x), \mathbf{k}(x+\epsilon, x) h_n(x+\epsilon)]_{\mathcal{H}} = \sum_{x \in X} [g_n(x), f_n(x)]_{\mathcal{H}},$$

which shows that the definition in (122) is correct, that is, independent of g_n and h_n such that $e_n = K g_n$ and $f_n = K h_n$.

We claim that $[\cdot,\cdot]_{\mathcal{G}}$ is a Z-valued gramian, that is, it satisfies all the requirements (ve1)–(ve3). The only fact that needs a proof is $[f_n, f_n]_{\mathcal{G}} = 0$ implies $f_n = 0$ and this follows by Lemma (4.3.4).

Thus, $(\mathcal{G};\ [\cdot,\cdot]_{\mathcal{G}})$ is a VE-space that we denote by $\mathcal{K}$. For each $x \in X$ we define $V(x)\colon \mathcal{H} \to \mathcal{G}$ by

$$V(x)h_n = K(h_n)_x, \qquad h_n \in \mathcal{H}, \tag{123}$$

where $(h_n)_x = \delta_x h_n \in \mathcal{F}_0$ is the function that takes the value h_n at x and is null elsewhere. Equivalently,

$$\begin{aligned}(V(x)h_n)(x+\epsilon) &= (K(h_n)_x)(x+\epsilon)\\ &= \sum_{(x+2\epsilon)\in X} \mathbf{k}(x+\epsilon, x+2\epsilon)((h_n)_x)(x+2\epsilon) = \mathbf{k}(x+\epsilon, x)h_n\,,\\ &\quad (x+\epsilon) \in X.\end{aligned} \tag{124}$$

Note that $V(x)$ is an operator from the VE-space $\mathcal{H}$ to the VE-space $\mathcal{G} = \mathcal{K}$ and it remains to show that $V(x)$ is adjointable for all $x \in X$. To see this, let us fix $x \in X$ and take $h_n \in \mathcal{H}$ and $f_n \in \mathcal{G}$ arbitrary. Then,

$$[V(x)h_n, f_n]_{\mathcal{G}} = \sum_{(x+\epsilon)\in X} [((h_n)_x)(x+\epsilon), f_n(x+\epsilon)]_{\mathcal{H}} = [h_n, f_n(x)]_{\mathcal{H}}, \tag{125}$$

which shows that $V(x)$ is adjointable and that its adjoint $V(x)^*$ is the operator $\mathcal{G} \ni f_n \mapsto f_n(x) \in \mathcal{H}$ of evaluation at x.

On the other hand, for any $x, (x+\epsilon) \in X$, by (124), we have

$$V(x+\epsilon)^* V(x)h_n = (V(x)h_n)(x+\epsilon) = \mathbf{k}(x+\epsilon, x)h_n, \qquad h_n \in \mathcal{H},$$

hence $(V;\ \mathcal{K})$ is a VE-space linearisation of $\mathbf{k}$. We prove that it is minimal as well. To see this, note that a typical element in the linear span of $V(X)\mathcal{H}$ is, for arbitrary $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $(h_n)_1, \ldots, (h_n)_n \in \mathcal{H}$,

$$\begin{aligned}\sum_{j=1}^{n} V(x_j)(h_n)_j &= \sum_{j=1}^{n} K(h_n)_{j,x_j}\\ &= \sum_{j=1}^{n} \sum_{(x+\epsilon)\in X} \mathbf{k}(\cdot, x+\epsilon)(h_n)_{j,x_j}(x+\epsilon) = \sum_{j=1}^{n} \mathbf{k}(\cdot, x_j)(h_n)_j\,,\end{aligned}$$

and then take into account that $\mathcal{G}$ is the range of the convolution operator K defined at (76). The uniqueness of the minimal VE-space linearisation $(V;\ \mathcal{K})$ just constructed follows as in (80).

For each $\xi \in \Gamma$ we let $\pi(\xi)\colon \mathcal{F} \to \mathcal{F}$ be defined by

$$(\pi(\xi)f_n)(x+\epsilon) = f_n(\xi^* \cdot x + \epsilon), \qquad (x+\epsilon) \in X, \xi \in \Gamma. \tag{126}$$

We prove that $\pi(\xi)$ leaves $\mathcal{G}$ invariant. To see this, let $f_n \in \mathcal{G}$, that is, $f_n = Kg_n$ for some $g_n \in \mathcal{F}_0$ or, even more explicitly, by (121),

$$f_n(x+\epsilon) = \sum_{x\in X} \mathbf{k}(x+\epsilon, x)g_n(x)\,, \qquad (x+\epsilon) \in X. \tag{127}$$

Then,

$$f_n(\xi^* \cdot (x+\epsilon)) = \sum_{x\in X} \mathbf{k}(\xi^* \cdot (x+\epsilon), x)g_n(x)$$

$$= \sum_{x \in X} \mathbf{k}((x+\epsilon), \xi \cdot x) g_n(x) = \sum_{(x+2\epsilon) \in X} \mathbf{k}(x+\epsilon, x+2\epsilon) g_n \xi(x+2\epsilon), \qquad (128)$$

where,

$$g_n^{\xi}(x+2\epsilon) = \begin{cases} 0, & \text{if } \xi \cdot x = x+2\epsilon \text{ has no solution } x \in \operatorname{supp} g_n, \\ \displaystyle\sum_{\xi \cdot x = x+2\epsilon} g_n(x), & \text{otherwise.} \end{cases} \qquad (129)$$

Since $g_n^{\xi} \in \mathcal{F}_0$, it follows that $\pi(\xi)$ leaves $\mathcal{G}$ invariant. In the following we denote by the same symbol $\pi(\xi)$ the map $\pi(\xi): \mathcal{G} \to \mathcal{G}$.

We prove that π is a representation of the semigroup Γ on the complex vector space $\mathcal{G}$, that is,

$$\pi(\alpha\beta) f_n = \pi(\alpha)\pi(\beta) f_n, \qquad \alpha, \beta \in \Gamma, f_n \in \mathcal{G}. \qquad (130)$$

To see this, let $f_n \in \mathcal{G}$ be fixed and denote $h_n = \pi(\beta) f_n$, that is, $h_n(x+\epsilon) = f_n(\beta^* \cdot (x+\epsilon))$ for all $(x+\epsilon) \in X$. Then $\pi(\alpha)\pi(\beta) f_n = \pi(\alpha) h_n$, that is, $(\pi(\alpha) h_n)(x+\epsilon) = h_n(a^* \cdot (x+\epsilon)) = h_n(\beta^* a^* \cdot (x+\epsilon)) = h_n((\alpha\beta)^* \cdot (x+\epsilon)) = (\pi(\alpha\beta))(x+\epsilon)$, for all $(x+\epsilon) \in X$, which proves (130)

We show that π is actually a $*$-representation, that is,

$$[\pi(\xi) f_n, f_n']_{\mathcal{G}} = [f_n, \pi(\xi^*) f_n']_{\mathcal{G}}, \qquad f_n, f_n' \in \mathcal{G}. \qquad (131)$$

To see this, let $f_n = K g_n$ and $f_n' = K g_n'$ for some $g_n, g_n' \in \mathcal{F}_0$. Then, recalling (122) and (128),

$$[\pi(\xi) f_n, f_n']_{\mathcal{G}} = \sum_{(x+\epsilon) \in X} [f_n(\xi^*(x+\epsilon)), g_n'(x+\epsilon)]_{\mathcal{H}} \sum_{x, (x+\epsilon) \in X} [\mathbf{k}(\xi^* \cdot (x+\epsilon), x) g_n(x), g_n'(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x, (x+\epsilon) \in X} [\mathbf{k}(x+\epsilon, \xi \cdot x) g_n(x), g_n'(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x, (x+\epsilon) \in X} [g_n(x), \mathbf{k}(\xi \cdot x, x+\epsilon) g_n'(x+\epsilon)]_{\mathcal{H}}$$
$$= \sum_{x \in X} [g_n(x), f_n'(\xi \cdot x)]_{\mathcal{H}} = [f_n, \pi(\xi^*) f_n']_{\mathcal{H}},$$

and hence the formula (131) is proven.

In order to show that the axiom (ikd3) holds as well, we use (124). Thus, for all $\xi \in \Gamma$, $x, (x+\epsilon) \in X$, $h_n \in \mathcal{H}$, and taking into account that $\mathbf{k}$ is invariant under the action of Γ on X, we have

$$(V(\xi \cdot x) h_n)(x+\epsilon) = \mathbf{k}(x+\epsilon, \xi \cdot x) h_n = \mathbf{k}(\xi^* \cdot (x+\epsilon), x) h_n$$
$$= (V(x) h_n)(\xi^* \cdot (x+\epsilon)) = (\pi(\xi) V(x) h_n)(x+\epsilon), \qquad (132)$$

which proves (ikd3). Thus, $(\mathcal{K}; \pi; V)$, here constructed, is a Γ-invariant VE-space linearisation of the Hermitian kernel $\mathbf{k}$. Note that $\mathcal{K}; \pi; V)$ is minimal, that is, the axiom (ikd4) holds, since the VE-space linearisation $(\mathcal{K}; V)$ is minimal.

Let $\mathcal{K}'$; π'; V') be another minimal invariant VE-space linearisation of K. We consider the unitary operator $U: \mathcal{K} \to \mathcal{K}'$ defined as in (80) and we already know that $UV(x) = V'(x)$ for all $x \in X$. Since, for any $\xi \in \Gamma$, $x \in X$, and $h_n \in \mathcal{H}$, we have

$$U\pi(\xi)V(x)h_n = UV(\xi \cdot x)h_n = V'(\xi \cdot x)h_n = \pi'(\xi)V'(x)h_n = \pi'(\xi)UV(x)h_n,$$

and taking into account the minimality, it follows that $U\pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$.

(ii) ⇒ (i). Let $\mathcal{K}$; π; V) be a Γ-invariant VE-space linearisation of $\mathbf{k}$. Then

$$\sum_{j,i=1}^{n} [\mathbf{k}(x_i, x_j)(h_n)_j, (h_n)_i]_{\mathcal{H}} = \sum_{j,i=1}^{n} [V(x_i)^* V(x_j)(h_n)_j, (h_n)_i]_{\mathcal{H}}$$

$$= [\sum_{j=1}^{n} V(x_j)(h_n)_j, \sum_{j=1}^{n} V(x_j)(h_n)_j]_{\mathcal{H}} \geq 0,$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $(h_n)_1, \ldots, (h_n)_n \in \mathcal{H}$, hence $\mathbf{k}$ is positive semidefinite. It was shown in (90) that $\mathbf{k}$ is invariant under the action of Γ on X.

(ii) ⇒ (iii). This follows from Corollary (4.3.26) with the following observation: with notation as in the proof of that proposition, for all $x, (x+\epsilon) \in X$ and $h_n \in \mathcal{H}$ we have

$$\mathbf{k}_{\xi \cdot x}(x+\epsilon)h_n = \mathbf{k}(x+\epsilon, \xi \cdot x)h_n = V\,(x+\epsilon)^* V(\xi \cdot x)h_n = V(x+\epsilon)^* \pi(\xi)V(x)h_n,$$

hence, letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U: \mathcal{K} \to \mathcal{R}$ is the unitary operator defined as in (117), we obtain a $*$-representation of Γ on the VH-space $\mathcal{R}$ such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$.

(iii) ⇒ (ii). Let $\rho: \Gamma \to \mathcal{L}^*(\mathcal{R})$ is a $*$-representation such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. Again, we use Corollary (4.3.26). Letting $\pi = \rho$, it is then easy to see that $(\mathcal{R}; \pi; V)$ is a Γ-invariant VE-space linearisation of the kernel $\mathbf{k}$.

Corollary (4.3.28)[295]: [226] Let Γ be a $*$-semigroup that acts on the nonempty set X and let $\mathbf{k} : X \times X \to \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-module $\mathcal{H}$ over an ordered $*$-algebra $\mathcal{A}$. The following assertions are equivalent:

(i) $\mathbf{k}$ is positive semidefinite, in the sense of (77), and invariant under the action of Γ on X, that is, (89) holds.

(ii) $\mathbf{k}$ has a Γ-invariant VE-module (over $\mathcal{A}$) linearisation $(\mathcal{K}; \pi; V)$.

(iii) $\mathbf{k}$ a dmits an $\mathcal{H}$-reproducing kernel VE-module $\mathcal{R}$ and there exists a $*$-representation $\rho: \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h_n = \mathbf{k}_{\xi \cdot x} h_n$ for all $\xi \in \Gamma$, $x \in X$, $h_n \in \mathcal{H}$.

In addition, in case any of the assertions (i), (ii), or (iii) holds, then a minimal Γ-invariant VE-module linearisation can be constructed, any minimal Γ-invariant VE-module linearisation is unique up to unitary equivalence, a pair $(\mathcal{R}; \rho)$ as in assertion (iii) with $\mathcal{R}$ minimal can be always obtained and, in this case, it is uniquely determined by $\mathbf{k}$ as well.

Proof. We use the notation as in the proof of Corollary (4.3.27). We actually prove only the implication (i) ⇒ (ii) since, as observed in Remark (4.3.15), that construction provides a Γ-invariant reproducing kernel VE-space linearisation, while the other implications are not much different.

(i) ⇒ (ii). We first observe that, since $\mathcal{H}$ is a module over $\mathcal{A}$, the space $\mathcal{F}(X;\ \mathcal{H})$ has a natural structure of right module over $\mathcal{A}$, more precisely, for any $f_n \in \mathcal{F}(X;\ \mathcal{H})$ and $a \in \mathcal{A}$

$$(f_n a)(x) = f_n(x)a, x \in X.$$

In particular, the space $\mathcal{F}_0(X;\ \mathcal{H})$ is a submodule of $\mathcal{F}(X;\ \mathcal{H})$. On the other hand, by assumption, for each $x, (x+\epsilon) \in X, \mathbf{k}(x, x+\epsilon) \in \mathcal{L}^*(\mathcal{H})$, hence $\mathbf{k}(x, x+\epsilon)$ is a module map. These imply that the convolution operator $K: \mathcal{F}_0(X;\ \mathcal{H}) \to \mathcal{F}(X;\ \mathcal{H})$ defined as in (76) is a module map. Indeed, for any $f_n \in \mathcal{F}_0(X;\ \mathcal{H})$, $a \in \mathcal{A}$, and $(x+\epsilon) \in X$,

$$((Kf_n)a)(x) = \sum_{x \in X} \mathbf{k}(x+\epsilon, x) f_n(x) a = K(f_n a)(x).$$

Then, the space $\mathcal{G}(X;\ \mathcal{H})$ which, with the definition as in (121), is the range of the convolution operator K, is a module over $\mathcal{A}$ as well.

We show that, when endowed with the $\mathcal{A}$ valued gramian $[\cdot,\cdot]_{\mathcal{G}}$ defined as in (122), we have

$$[e_n, f_n a]_{\mathcal{G}} = [e_n, f_n]_{\mathcal{G}}\, a, \quad e_n, f_n \in \mathcal{G}(X; \mathcal{H}), \qquad a \in \mathcal{A}. \tag{133}$$

To see this, let $e_n = Kg_n$ and $f_n = Kh_n$ for some $g_n, h_n \in \mathcal{F}_0(X;\ \mathcal{H})$. Then,

$$[e_n, f_n a]_{\mathcal{G}} = [Kg_n, h_n a]_{\mathcal{F}_0} = \sum_{(x+\epsilon) \in X} [e_n(x+\epsilon), h_n(x+\epsilon)a]_{\mathcal{H}}$$

$$= \sum_{(x+\epsilon) \in X} [e_n(x+\epsilon), h_n(x+\epsilon)]_{\mathcal{H}}\, a = [Kg_n, h_n]_{\mathcal{F}_0} a = [e_n, f_n]_{\mathcal{G}}\, a.$$

From (133) and the proof of the implication (i) ⇒ (ii) in Corollary (4.3.27), it follows that $\mathcal{K} = \mathcal{G}(X;\ \mathcal{H})$ is a VE-module over the ordered $*$-algebra $\mathcal{A}$ and hence, the triple $(\mathcal{K};\ \pi;\ V)$ is a minimal Γ-invariant VE-module linearisation of $\mathbf{k}$.

Chapter 5
C^*-Algebras with Application of Jacobi's Representation Theorem and Seminormed

We show that the results are more natural when the C*-algebra is singly generated. For singly generated C*-aIgebras with unbounded representation dimension, we reduce the problem to the case when the generator is an infinite direct sum of irreducible finite scalar matrices, and we have partial results in this case. We obtain a representation of any linear functional L : A → R which is continuous with respect to any such ρ or τ and non-negative on S as integration with respect to a unique Radon measure on the space of all real valued Ralgebra homomorphisms on A, and we characterize the support of the measure obtained in this way. We study lifting of positive measures from (X, Σ) to the Gelfand spectrum of $M_b(X, \Sigma)$ and observe an unexpected shift in the support of measures. In the case that Σ is the Borel algebra of a topology, we study the relation of the underlying topology of X and the topology of the Gelfand spectrum of $M_b(X, \Sigma)$.

Section (5.1): Hausdorff Spectrum

For a C^*-algebra A let $\hat{A}$ be the spectrum of A; that is, $\hat{A}$ is the set of unitary equivalence classes of nonzero irreducible representations of A equipped with the hull-kernel topology see paragraph 3 in [239]. We attempt to characterize those C^*-algebras with identity that have Hausdorff spectrum. For A a bounded linear operator on a Hubert space let $C^*(A)$ be the C^*-algebrag enerated by A and the identity. We say that A has Hausdorff spectrum if $C^*(A)^\wedge$ is Hausdorff. We started our research in this direction as a result of John Ernest's question of characterizing the operators A with Hausdorff spectrum [241]. Although we state many of our results for arbitrary separable C^*-algebras with identity, most of our results have more natural interpretations in the case of singly generated C^*-algebras.

It follows from J. Glimm's theorem see p. 582 in [100] that if A is a separable C^*-algebra, then $\hat{A}$ is T_0 if and only if A is GCR (or postliminal), and $\hat{A}$ is T_1 if and only if A is CCR (or liminal). I. Kaplansky see Theorem 4.2 in [90] proved that if A is a C^*-algebra such that all irreducible representations of A are of the same finite dimension, then the primitive ideal space of A is Hausdorff in the hull-kernel topology. In this case, the primitive ideal space is homeomorphic to $\hat{A}$, so that $\hat{A}$ is also Hausdorff. J. M. G. Fell proved a theorem see CoroUary 1, p. 388 in [80] which has Kaplansky's result as a corollary.

We recall one characterization of Hausdorff spectrum that is in the literature [12], [122]. If A is a C^*-algebraw ith identity, and center C, then A is called central if for all primitive ideals I and J of A, $I \cap C = J \cap C$ implies that $I = J$. It follows from [12], [122] that if A is a separable C^*-algebra with identity, then $\hat{A}$ is Hausdorff if and only if A is central and GCR. However we know of no natural way to compute the center of a singly generated C^*-algebra $C^*(A)$ in terms of the operator A, so we do not regard this necessary and sufficient condition as a satisfactory answer to the problem. We will make no use of this condition.

We say that a C^*-algebra A has bounded representation dimension if there is an integer N such that every irreducible representation of A acts on a Hubert space of dimension less than or equal to N. We prove decomposition theorems, (5.1.2), (5.1.6), and (5.1.10), which give necessary and sufficient conditions for C^*-algebras with identity and bounded representation dimension to have Hausdorff spectrum. We show that, to characterize which operators A have Hausdorff spectrum, it suffices to consider only operators A which are (possibly infinite) direct sums of irreducible finite complex matrices.

We are not able to give a complete characterization in the case of unbounded representation dimension, but we do give some partial results. We conclude with some remarks concerning the lifting of matrix units. We show that matrix units cannot necessarily be lifted from the Calkin algebra.

We recall the hull-kernel topology on $\hat{A}$. For $S \subset \hat{A}$, the closure of S is the set $\{\pi \in \hat{A}: \text{Ker}\,\pi \supseteq \bigcap\{\text{Ker}\,p: p \in S\}\}$. The open sets of $\hat{A}$ are all of the form $\{\pi \in \hat{A}: \pi|J \neq 0$ where J is a closed two-sided ideal in A. If $\{A_i\}$ is a dense subset of A, then the sets $Z_i = \{\pi \in \hat{A}: \|\pi(A_i)\| > 1\}$ form a base for the topology of $\hat{A}$. The tools used are standard and are mostly contained in [239]. We will make frequent and extensive use of [239], which was mostly taken from Fell's [80].

We will use script letters $A, B, \ldots$ for C^*-algebras and Latin letters $A, B, \ldots$ for operators on a Hubert space. We will denote the algebra of all bounded operators on a Hubert space $\mathcal{H}$ by $B(\mathcal{H})$ and the ideal of all compact operators by $\mathcal{K}(\mathcal{H})$. We will use $\mathcal{H}_\pi$ to denote the Hubert space associated with some representation $\pi: A \longrightarrow B(\mathcal{H}_\pi)$. We denote $\{A \in A: \pi(A) = 0\}$ by Ker π.

We state our results both for C^*-algebras A and singly generated C^*-algebras $C^*(A)$. However, we state our results only for algebras or operators.

Let A be a separable C^*-algebra with identity I and assume that $\hat{A}$ is Hausdorff. If ρ and θ are irreducible representations of A and Ker $\rho =$ Ker θ, then p is in the closure of the singleton set $\{\theta\}$ in $\hat{A}$. Thus ρ and θ must be unitarily equivalent since $\hat{A}$ is Hausdorff. Hence, by [239], A is GCR and $\rho(A)$ must contain $\mathcal{K}(\mathcal{H}_\rho)$. If $\mathcal{K}(\mathcal{H}_\rho)$ were properly contained in $\rho(A)$, then there would exist an irreducible representation π of A with Ker p properly contained in Ker θ. This would contradict $\hat{A}$ being Hausdorff, hence we must have $\rho(A) = \mathcal{K}(\mathcal{H}_\rho)$, and $\mathcal{K}(\mathcal{H}_\rho)$ must have an identity. Thus $\mathcal{H}_\rho$ must be finite dimensional. We have thus proved the following theorem.

Theorem (5.1.1)[234]: If A is a separable C^*-algebra with identity such that A is Hausdorff, then every irreducible representation of A must be finite dimensional.

We note that Theorem (5.1.1) is not true if we drop the hypothesis of A containing an identity. For if $\mathcal{H}$ is a separable infinite dimensional Hubert space, then $\mathcal{K}(\mathcal{H})$ is a separable C^*-algebra whose spectrum consists of a single point, and thus is Hausdorff, but has no representation of finite dimension. Theorem (5.1.1.) is basic to much of what follows. The problem of characterizing when $\hat{A}$ is Hausdorff for algebras without identity seems more difficult, and we will usually assume A has an identity.

As in [245] we call a C^*-algebra A n-normal if for all $A_1, A_2, \ldots, A_{2n}$ in A we have

$$\sum \text{sgn}\,(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0 \qquad (*)$$

where the summation is taken over all permutations of $2n$ objects. An operator A is called n-normal if $C^*(A)$ is n -normal. We note that the identity $(*)$ is satisfied by the full $n \times n$ matrix algebra over a commutative C^*-algebra [245]. A representation π of a C^*-algebra A is called n-normal if the C^*-algebra $\pi(A)$ is n-normal. An operator A is called pure n-normal if A is n-normal but no direct summand of A is k-normal for any $k < n$. A representation is called pure n-normal if it is n-normal but no subrepresentation is k-normal for any $k < n$.

Theorem (5.1.2)[234]: Let A be a separable C^*-algebra such that every irreducible representation of A is finite dimensional. If $\pi: A \to B(\mathcal{H}_\pi)$ is a nondegenerate representation of A on a separable Hubert space, then $\pi \cong \sum_{k \in I} \oplus \pi_k$ where each π_k is pure

k-normal. If π is an n-normal representation then each $k \in I$ is less than or equal to n, and one need not assume that A or $\mathcal{H}_\pi$ are separable.

Proof. Since A is GCR we may apply Theorem 8.6.6 of [239] to obtain measures $\mu_1, \mu_2, \dots, \mu_\infty$ on $\hat{A}$, pairwise disjoint, such that

$$\pi \cong \int^{\oplus} \rho\, d\mu_1(\rho) \oplus 2 \int^{\oplus} \rho\, d\mu_2(\rho) \oplus \dots \oplus \aleph_0 \int^{\oplus} \rho\, d\mu_\infty(\rho). \tag{1}$$

If $\hat{A}_n = \{\theta \in \hat{A}: dim(\mathcal{H}_\theta) < n)$, then $\hat{A} = \cup_{1\le n} \hat{A}_n$ and each $\hat{A}_n$ is closed in A [239]. We may thus write, for each i and n,

$$\int^{\oplus} \rho\, d\mu_i(\rho) \int_{\hat{A}_n}^{\oplus} \rho\, d\mu_i(\rho) \oplus \int_{X_n}^{\oplus} \rho\, d\mu_i(\rho). \tag{2}$$

where X_n is the complement of $\hat{A}_n$ in $\hat{A}$. Now for some n and / we must have that $\mu_i(\hat{A}_n) \neq 0$, in which case the first representation in the right-hand side of equation (2) is nondegenerate and n-normal. By applying the decomposition of (2) to (1) and rearranging, we have for some n that $\pi \cong \theta \oplus \theta'$, where θ is an n-normal nondegenerate representation. We have thus shown that if n is any nondegenerate representation of A on a separable Hubert space, then some subrepresentation of π is n-normal for some n. Now let n_0 be the smallest integer such that π has a subrepresentation that is n_0-normal. By Zorn's lemma there is a maximal family $F_{n_0} = \{\mathcal{H}_\alpha^{(n_0)}\}$ of orthogonal subspaces of $\mathcal{H}_\pi$, each reducing π, such that each $\pi|\mathcal{H}_\alpha^{(n_0)}$ is n_0-normal. Let $\mathcal{H}^{(n_0)} = \sum_\alpha \oplus \mathcal{H}_\alpha^{(n_0)}$. Now pick a maximal family $F_{n_0+1} = \{\mathcal{H}_\beta^{(n_0+1)}\} \supset F_{n_0}$ of orthogonal subspaces of $\mathcal{H}_\pi$, each reducing π, such that each $\pi|\mathcal{H}_\beta^{(n_0+1)}$ is (n_0+1)-normal. Let $\mathcal{H}^{(n_0+1)} = \sum_\beta \oplus \mathcal{H}_\beta^{(n_0+1)}$. Continue to choose maximal families $F_{n+1} = \{\mathcal{H}_\gamma^{(n+1)}\} \supseteq F_n$ of orthogonal subspaces of $\mathcal{H}_\pi$ each reducing π, such that $\pi|\mathcal{H}_\gamma^{(n+1)}$ is (n_0+1)- normal. Again let $\mathcal{H}^{(n+1)} = \sum_\gamma \oplus \mathcal{H}_\gamma^{(n+1)}$. Then $\mathcal{H}^{(n_0)} \subseteq \mathcal{H}^{(n_0+1)} \subseteq \dots$,and each $\pi|\mathcal{H}^{(n)}$ is n-normal. Now let $\mathcal{H}_{n_0} = \mathcal{H}^{(n)}$ and $\mathcal{H}_{n+1} = \mathcal{H}^{(n+1)} \ominus \mathcal{H}^{(n)}$ for $n \ge n_0$. Let $I = \{k \in Z^+ : \mathcal{H}_k \neq \{0\}\}$, and let $\mathcal{H}_\infty = \sum_{k\in I} \oplus \mathcal{H}_k$ fife. Since $\pi|\mathcal{H} \ominus \mathcal{H}_\infty$, is a subrepresentation of π which by maximality contains no k-normal subrepresentation for any k, we obtain that $\mathcal{H} = \mathcal{H}_\infty$. Furthermore, for $k \in I$, $\pi_k = \pi|\mathcal{H}_k$ is clearly k-normal, and must be pure k-normal by the maximality of F_{k-1}. Hence $\pi = \sum_{k\in I} \oplus \pi_k$ with each π_k pure k-normal. If π is a nondegenerate n-normal representation of any C^*-algebra A then the first part of the proof is superfluous since any subrepresentation of π is n-normal. In this case $\mathcal{H} = \mathcal{H}^{(n)}$ and we must have that $k \in I$ implies $k \le n$.

Corollary (5.1.3)[234]: If A is a bounded operator on a separable Hubert space such that every irreducible representation of $C^*(A)$ is finite dimensional, then $A \cong \sum_{k\in I} \oplus A_k$ where each A_k is pure k-normal.

Proof. Apply Theorem (5.1.2) to the identity representation of $C^*(A)$ and let $A_i = \pi_i(A)$.

Let A be a C^*-algebra and let $\theta: A \to B(\mathcal{H})$ be a representation. Then the map $\hat{\theta}: \theta(A)^\wedge \to \hat{A}$ defined by $\hat{\theta}(\rho) = \rho \circ \theta$ is one-to-one and continuous. Thus $\theta(A)^\wedge$ must be Hausdorff if $\hat{A}$ is Hausdorff. Hence if A is a separable C^*-algebra with identity and Hausdorff spectrum, then the C^*-algebras $\pi_i(A)$ in Theorem (5.1.2.) also have Hausdorff spectrum. Likewise, if A has Hausdorff spectrum, then the A_i in Corollary (5.1.3.) also have Hausdorff spectrum. We need the following two lemmas which will be used several times.

Lemma (5.1.4)[234]: Let $\theta: A \to B(\mathcal{H})$ be a representation of a C^*-algebra A. If $\theta(A)^\wedge$ is Hausdorff, then $\hat{\theta}(\theta(A)^\wedge)$ is closed in $A^\wedge$ and is a Hausdorff space in the relative topology from $A^\wedge$.

Proof. This follows immediately from [239].

Lemma (5.1.5)[234]: Let $\pi: A \to B(\mathcal{H})$ be a representation of a C^*-algebra A such that $\pi = \sum_{i \in I} \oplus \pi_i$, each π_i a representation of A. Then each π_i gives rise to a representation of $\pi(A)$ into $\pi_i(A)$, which we call P_i, defined by $P_i(\pi(A)) = \pi_i(A)$. Then $\bigcup_{i \in I} \hat{P}_i(\pi_i(A)^\wedge)$ dense in $\pi(A)^\wedge$.

Proof. Let $U = \{\theta \in \pi(A)^\wedge : \theta|J \neq 0\}$ be any nonempty open set in $\pi(A)^\wedge$ where J is a nonzero closed ideal in $\pi(A)$. Let $0 \neq \pi(A) \in J$ and choose i with $\pi_i(A) \neq 0$. Then for $p_i \in \pi_i(A)^\wedge$ with $p_i(\pi_i(A)) \neq 0$ we have that $\hat{P}_i(p_i) \in U$. Hence $\bigcup_{i \in I} \hat{P}_i(\pi_i(A)^\wedge)$ is densei n $\pi(A)^\wedge$.

The following theorem is now immediate from Lemmas (5.1.4) and (5.1.5).

Theorem (5.1.6)[234]: Let π be a nondegenerate representation of a separable C^*-algebra A with identity such that $\pi = \sum_{i=1}^{n} \oplus \pi_i$, n a finite integer, and each π_i a nondegenerate representation of A. Then $\pi(A)$ has Hausdorff spectrum if and only if each $\pi_i(A)$ has Hausdorff spectrum. If every irreducible representation of A is finite dimensional, then we do not need to assume that A is separable.

Proof. Since we have representations $P_i: \pi(A) \to \pi_i(A)$, each $\pi_i(A)$ has Hausdorff spectrum if $\pi(A)$ has Hausdorff spectrum. On the other hand if each $\pi_i(A)$ has Hausdorff spectrum, then by Lemmas (5.1.4) and (5.1.5) we have that $\pi(A)^\wedge = \bigcup_{i=1}^{n} \hat{P}_i(\pi_i(A)^\wedge)$ and each $\hat{P}_i(\pi_i(A)^\wedge)$ is closed in $\pi(A)^\wedge$ and is a Hausdorff space in the relative topology from $\pi(A)^\wedge$. It is then clear that every net in $\pi(A)^\wedge$ has a unique limit point.

Corollary (5.1.7)[234]: If A is a bounded operator on a Hubert space and A is a finite direct sum of operators A_i, $A = \sum_{i=1}^{n} \oplus A_i$, then A has Hausdorff spectrum if and only if each A_i has Hausdorff spectrum.

We will give an example to show that if $\pi = \sum_{i=1}^{\infty} \oplus \pi_i$, then each $\pi_i(A)^\wedge$ can be Hausdorff without $\pi(A)^\wedge$ being Hausdorff; thus the hypothesis of only a finite number of direct summands in Theorem (5.1.6) and Corollary (5.1.7) is necessary. Theorem (5.1.6) implies the following (probably known) corollary.

Corollary (5.1.8)[234]: Every finite dimensional C^*-algebra has Hausdorff spectrum.

Proof. First, we include a proof that a finite dimensional C^*-algebra A has only a finite number of unitarily inequivalent irreducible representations. Let $p_1, p_2, \dots, p_n$ be unitarily inequivalent irreducible representations of A and let $p = \sum_{i=1}^{n} \oplus p_i$, $p: A \to B(\mathcal{H}_p)$. By Kadison's transitivity theorem [239], p is a cyclic representation. Hence $n \leq \dim \mathcal{H}_p \leq \dim A$. Now let $p_1, p_2, \dots, p_k$ be all the irreducible representations of A up to unitary equivalence, and let $p = \sum_{i=1}^{k} \oplus p_i$. Then A is isomorphic to $p(A)$. Since each $p_i(A)$ is a full finite matrix algebra, each $p_i(A)^\wedge$ is a single point and thus Hausdorff. Theorem (5.1.6) then implies that $p(A)^\wedge$ and hence A is Hausdorff.

If A has bounded representation dimension N,t hen A is N-normal. For A a C^*-algebra with identity and bounded representation dimension, let π be a faithful nondegenerate representation of A. Then by Theorem (5.1.2) we can write π as a finite direct sum of pure k-normalr epresentation. Theorem (5.1.6) then implies that in order to characterize when a C^*-algebra with identity and bounded representation dimension has

Hausdorff spectrum, one need only characterize when $\pi(A)$ has Hausdorff spectrum for π a pure k-normal representation of A. This is done in Theorem (5.1.10).

Let $\pi: A \to B(\mathcal{H})$ be an n-normal representation. Then, by [245], $\pi(A)'' \cong \sum_{k=1}^{n} \oplus M_k(C_k)$, where $M_k(C_k)$ is the algebra of $k \times k$ matrices with entries from the abelian W^*-algebra C_k. Let I_i be the element in $\sum_{k=1}^{n} \oplus M_k(C_k)$ with the identity in the ith coordinate and zeros elsewhere. Then $\pi(A) \to I_i \pi(A) I_i$. is an i-normal subrepresentation of π. Thus if π is a pure n-normal representation then $\pi(A)'' \cong M_n(C_n)$. In this case, let D be the C^*-subalgebra of C_n generated by the matrix entries from elements of $\pi(A)$. Let $X(\pi)$ be the maximal ideal space of D. For $p \in X(\pi)$, let $\hat{p}$ be defined on $M_n(D)$ by $\hat{p}((D_{ij})) = (p(D_{ij}))$. Then $\hat{p}$ is a representation of $M_n(D)$ into $B(C^n)$. As Proposition 2 in [236] every irreducible representation of $\pi(A)$ is of the form $\pi(A) \to \hat{p}(\pi(A))|M$ for some $p \in X(\pi)$ and for some subspace M reducing for $\hat{p}(\pi(A))$.

Lemma (5.1.9)[234]: Let $\pi: A \to B(\mathcal{H})$ be a pure n-normal representation of a C^*-algebra A with identity. Then $Y \equiv \{\omega \in X(\pi): \widehat{\omega}|\pi(A)$ is irreducible) is dense in $X(\pi)$.

Proof. Let D and $X(\pi)$ be defined as above. Then $D \cong C(X(\pi))$ the set of all continuous complex-valued functions on $X(\pi)$. Suppose that $\theta: C(X(\pi)) \to B(\mathcal{H}')$ is a faithful representation of $C(X(\pi))$ on a Hilbert space $\mathcal{H}'$, and let E be the regular spectral measure associated with θ by the general spectral theorem. That is, $\theta(f) = \int f\, dE$ for all $f \in C(X(\pi))$. Let $\hat{\theta}$ be the associated representation of $M_n(C(X(\pi)))$ on the direct sum of n-copies of $\mathcal{H}'$, defined by $\hat{\theta}((f_{ij})) = (\theta(f_{ij}))$ Then π is unitarily equivalent to the representation $A \to \hat{\theta}(\pi(A))$, where we identify $\pi(A)$ as an element of $M_n(C(X(\pi)))$. Now let S be the complement of $\bar{Y}$ in $X(\pi)$. Define the representation $\sigma: C(X(\pi)) \to B(E(S)\mathcal{H}')$ by $\sigma(f) = \theta(f)|E(S)\mathcal{H}'$, and let $\hat{\sigma}$ be the associated representation of $M_n(C(X(\pi)))$ on the direct sum of n-copies of $E(S)\mathcal{H}'$. Now, for $\omega \in S$, $\widehat{\omega}(\pi(A))$ is $(n-1)$-normal, and a computation then shows that $\hat{\sigma}(\pi(A))$ is also $(n-1)$-normal. But the representation $A \longmapsto \hat{\sigma}(\pi(A))$ is a subrepresentation of the representation $A \longmapsto \hat{\theta}(\pi(A))$. Since π, and hence $A \longmapsto \hat{\theta}(\pi(A))$, is pure n-normal we must have that $E(S) = 0$. Hence support $(F) \subset \bar{Y}$. But 8 is faithful, so if $f \in C(X(\pi))$ satisfies $f(\text{support } E) = 0$ then $f = 0$. Hence $\bar{Y} = X(\pi)$ and the lemma is proved.

Theorem (5.1.10)[234]: Let $\pi: A \to B(\mathcal{H})$ be a pure n-normal representation of a C^*-algebra A with identity. Then $\pi(A)$ has Hausdorff spectrum if and only if, for every $p \in X(\pi)$, $\hat{p}|\pi(A)$ is a direct sum of unitarily equivalent irreducible representations of $\pi(A)$.

Proof. First assume that, for $p \in X(\pi)$, $\hat{p}|\pi(A)$ is a direct sum of unitarily equivalent irreducible representations of $\pi(A)$. Let π_α be a net in $\pi(A)^\wedge$ onverging to both π_1 and π_2 in $\pi(A)^\wedge$. Again applying Proposition 2 in [236] we obtain that for each a there is a $\omega_\alpha \in X(\pi)$ and a reducing subspace M_α for $\widehat{\omega}_\alpha(\pi(A))$ such that $\pi_\alpha(\pi(A)) = \widehat{\omega}_\alpha(\pi(A))|M_\alpha$. Since $X(\pi)$ is compact we can assume by passing to a subset that ω_α converges to $\omega_\alpha \in X(\pi)$. Now consider a neighborhood of π_1 of the form $U = [\theta \in \pi(A)^\wedge: \|\theta(\pi(A))\| > 1\}$, where A is a fixed but arbitrary element of A. Let $\epsilon > 0$ be such that $\|\pi_1(\pi(A))\| > 1 + \epsilon$. Then there is an α_0 such that $\|\pi_\alpha(\pi(A))\| > 1 + \epsilon$ for all $\alpha \geq \alpha_0$. Since ω_0 converges to ω_0, we have that $\widehat{\omega}_\alpha(\pi(A))$ converges to $\widehat{\omega}_0(\pi(A))$ in norm, for all $A \in \mathcal{A}$. But, by our assumption, $\widehat{\omega}_0|\pi(A) \cong \sum_{i=1}^{k} \oplus \pi_0$ for some irreducible representation π_0 of $\pi(A)$, and $\widehat{\omega}_\alpha|\pi(A) \cong \sum_{i=1}^{k} \oplus \pi_\alpha$ (this fact follows from our assumption and [239]). Then $\|\widehat{\omega}_0(\pi(A))\| = \|\pi_0(\pi(A))\|$ and $\|\widehat{\omega}_\alpha(\pi(A))\| = \|\pi_\alpha(\pi(A))\|$ for all α. Hence $\|\pi_\alpha(\pi(A))\|$ converges to $\|\pi_0(\pi(A))\|$ and we obtain that $\|\pi_0(\pi(A))\| \geq 1 + \epsilon$. Hence

$\pi_0 \in \mathcal{U}$ and $\{\pi_0\}$ is in the closure of the singleton set $\{\pi_1\}$. But since $\pi(A)$ is a CCR algebra and CCR algebras have T_1 spectrum see 4.1.10 and 4.1.11 in [239], this implies that $\pi_0 \cong \pi_1$. Likewise $\pi_0 \cong \pi_1$ and $\pi(A)^\wedge$ is Hausdorff.

Now assume that there is a $\omega_0 \in X(\pi)$ such that $\widehat{\omega}_0|\pi(A) \cong \pi_1 \oplus \pi_2 \oplus \pi'$ where π_1 and π_2 are unitarily inequivalent irreducible representations of $\pi(A)$. Now by Lemma (5.1.9) there exists a net $\omega_0 \in X(\pi)$ such that ω_0 converges to ω_0 and $\widehat{\omega}|\pi(A)$ irreducible. Let $U = [\theta \in \pi(A)^\wedge : \|\theta(\pi(A))\| > 1\}$ be an open set containing π_1 Since $\|\widehat{\omega}_\alpha(\pi(A))\|$ converges to $\|\widehat{\omega}_0(\pi(A))\|$ which is greater than or equal to $\|\pi_1(\pi(A))\|$, we have that $\widehat{\omega}_\alpha|\pi(A) \in U$ for all $\alpha \geq \alpha_0$ for some α_0. Hence $\widehat{\omega}_\alpha|\pi(A)$ converges to π_1 , and likewise to π_2. Thus $\pi(A)^\wedge$ is not Hausdorff.

Theorem (5.1.2), Theorem (5.1.6), and Theorem (5.1.10) together give concrete necessary and sufficient conditions for a C^*-algebra with identity and bounded representation dimension to have Hausdorff spectrum. This result includes Kaplansky's result from Theorem 4.2 in [90] in the case when the algebra has an identity.

With the aid of Theorem (5.1.10) we now give some simple examples. Let M_t be defined on $L^2(0,1)$ by $(M_t f)(x) = xf(x)$ for all $f \in L^2(0,1)$. For $\alpha, \beta \in C$, let $A_{\alpha,\beta}$ ß be the operator matrix

$$\begin{bmatrix} \alpha I & M_t \\ 0 & \beta I \end{bmatrix}$$

defined on $L^2(0,1) \oplus L^2(0,1)$. It is easily seen that $A_{\alpha,\beta}$ is always pure 2-normal. Theorem (5.1.10) implies that $A_{\alpha,\beta}$ has Hausdorff spectrum if and only if $\alpha = \beta$. We remark that this result also follows easily from the results in [236]. Thus the operator $A_{1,1}$ is an example of a nonnormal operator with Hausdorff spectrum which has irreducible representations of dimensions one and two, while the operator $A_{1,-1}$ is such an example with non-Hausdorff spectrum. Thus $C^*(A_{1,-1})$ is a C^*-algebra whose spectrum is non-Hausdorff, but which is a C^*-subalgebra of $M_2(C[0,1])$ whose spectrum is Hausdorff. Thus the property of having Hausdorff spectrum is not inherited by subalgebras, as is the property of separable C^*-algebras having T_0 or T_1 spectrum see 4.2.4 and 4.3.5 in [239].

We deal only with singly generated C^*-algebras. Let A be a bounded operator on a separable Hubert space and assume that every irreducible representation of $C^*(A)$ is finite dimensional. Then by Corollary (5.1.3) we can write $A = \sum_{k \in I} \oplus A_k$ where each A_k is pure k-normal. Theorem (5.1.11) will show that in order to determine when $C^*(A)$ has Hausdorff spectrum, it suffices to solve the case when each A_k is actually an irreducible finite complex matrix.

Theorem (5.1.11)[234]: Let A be a bounded operator on a Hubert space and assume that A is the direct sum of operators A_k, $A = \sum_{k=1}^{\infty} \oplus A_k$. Then A has Hausdorff spectrum if and only if each A_k has Hausdorff spectrum and for all choices $\theta_k \in C^*(A_k)^\wedge$ the operator $\sum_{k=1}^{\infty} \oplus \theta_k(A_k)$ has Hausdorff spectrum.

Proof. Assume that A has Hausdorff spectrum. Then for every k the operator A_k has Hausdorff spectrum since the mapping $A \to A_k$ is a representation of $C^*(A)$ (recall that such a representation induces a continuous one-to-one mapping from $C^*(A_k)^\wedge$ to $C^*(A_k)^\wedge$). Likewise for all choices $\theta_k \in C^*(A_k)^\wedge$ the mapping $A \to \sum_{k=1}^{\infty} \oplus \theta_k(A_k)$ is a representation of $C^*(A)$ so that $\sum_{k=1}^{\infty} \oplus \theta_k(A_k)$ has Hausdorff spectrum.

Now assume that each A_i has Hausdorff spectrum and that $\sum_{k=1}^{\infty} \oplus \theta_k(A_k)$ has Hausdorff spectrum for all choices $\theta_k \in C^*(A_k)^\wedge$. Furthermore suppose there exist $p_0, p_1 \in C^*(A)^\wedge$ which cannot be separated by open sets. Let $\{U_i\}$ and $\{V_j\}$ be countable bases for the open sets containing p_0 and p_1 respectively. Since p_0 and p_1 cannot be separated by open sets we must have that $U_i \cap V_j \neq \phi$ for all j. By Lemma (5.1.5) for each j we can choose an element $\varphi_j \in U_j \cap V_j$with $\varphi_j \in \cup_{k=1} \hat{P}_k\,(C^*(A_k)^\wedge)$ where $P_k(A) = A_k$. Then the sequence φ_j. Converges to both p_0 and p_1. Since each A_k has Hausdorff spectrum, Lemma (5.1.4) implies that each $\hat{P}_k(C^*(A_k)^\wedge)$ is closed in $C^*(A)^\wedge$ and is a Hausdorff space in the relative topology from $C^*(A)^\wedge$. Hence only a finite number of the $\{\varphi_j\}$ belong to any one $\hat{P}_k(C^*(A_k)^\wedge)$, and by passing to a subsequence we may assume that if $i \neq j$ then φ_j and φ_j, belong to different $\hat{P}_k(C^*(A_k)^\wedge)$. Thus for every j there exists k_j such that $\varphi_j \in \hat{P}_k(C^*(A_k)^\wedge)$ and $i \neq j$ implies $k_i \neq k_j$. Now since $\varphi_j \in \hat{P}_{k_j}(C^*(A_{k_j})^\wedge)$ there is a $\varphi_{k_j} \in C^*(A_{k_j})^\wedge$ such that $\varphi_j(A) = \theta_{k_j}(P_{k_j}(A)) = \theta_{k_j}(A_{k_j})$. Now let 0 be a representation of $C^*(A)$ defined by $\theta(A) = \sum_{j=1}^{\infty} \oplus\, \theta_{k_j}(A_{k_j})$. Since, by Corollary (5.1.7), a direct summand of an operator with Hausdorff spectrum has Hausdorff spectrum, $\theta(A)$ has Hausdorff spectrum. Define representations $\tilde{\varphi}_{k_j}, \tilde{p}_0, \tilde{p}_1$ on $C^*(\theta(A))$ by

$$\tilde{\varphi}_{k_j}(p(\theta(A), \theta(A)^*)) = \theta_{k_j}(p(A_{k_j}, A_{k_j}^*))$$

and

$$\tilde{p}_i(p(\theta(A), \theta(A)^*)) = p_i(p(A, A^*)),$$

for p a polynomial in two noncommuting variables. It is clear that θ_{k_j}. Extends to an irreducible representation of $C^*(\theta(A))$. Since φ_j converges to both p_0 and p_1 in $C^*(A)^\wedge$, lower semicontinuity of the norm 3.3.2 in [238] gives

$$\|p_i(p(A, A^*))\| \leq \lim\inf \|\varphi_j(p(A, A^*))\|$$
$$\leq \sup \left\|\theta_{k_j}(p(A, A^*))\right\| = \|\theta(p(A, A^*))\|.$$

So that $\tilde{p}_i$ also extend to irreducible representations of $C^*(\theta(A))$. Also the same computation shows that $\tilde{\theta}_{k_j}$. converges to both $\tilde{p}_0$ and $\tilde{p}_1$ in $C^*(A)^\wedge$. Hence $\tilde{p}_0 \cong \tilde{p}_1$, which implies $p_0 \cong p_1$. Contradiction. Hence $C^*(A)^\wedge$ must be Hausdorff.

As previously mentioned, if A is a bounded operator on a separable Hubert space such that every irreducible representation of $C^*(A)$ is finite dimensional, then by means of Corollary (5.1.3), Theorem (5.1.6), Theorem (5.1.10), and Theorem (5.1.11) one could decide if $C^*(A)^\wedge$ had Hausdorff spectrum if one could settle the question for operators of the form $\sum_{k \in I} \oplus A_k$ with each A_k an irreducible $k \times k$ complex matrix. At the present time we are unable to resolve this question, but we do present some partial results.

Let B be any operator on a Hubert space of dimension N,a nd let I_n denote the identity operator on a Hubert space of dimension n. Since the set of irreducible operators on any separable Hubert space is dense see p. 920 in [243], for every n there is an operator K_n with $\|K_n\| < 1/n$ such that $A_n = B \otimes I_n + K_n$ is an irreducible operator on the Hubert space of dimension N . Let $A = \sum_{n=2}^{\infty} \oplus A_n$. Then the following theorem completely describes $C^*(A)^\wedge$ and its topology in this case.

Theorem (5.1.12)[234]: let A be as above and let $B_1, B_2, \ldots, B_k$ be all the unitarily inequivalent irreducible direct summands of B. Then for every $1 \leq i \leq k$ there is an irreducible representation π_i of $C^*(A)$ determined by $\pi_i(A) = B_i$; and $C^*(A)^\wedge = \{\pi_i : 1 \leq i \leq k\} \cup \{P_n : 2 \leq n\}$, where as before $P_n(A) = A_n$. The topology is determined by the fact that singleton sets are closed and the sequence $\{P_n\}_{n=2}^{\infty}$ onverges to each π_i Thus A has Hausdorff spectrum if and only if B is a direct sum of unitarily equivalent irreducible matrices.

Proof. For any polynomial p in two noncommuting variables we have that

$$\|p(B,B^*)\| = \|p(B\otimes I_n, B^*\otimes I_n)\| = \|p(A_n,A_n^*) + K_n'\|$$

where $\|K_n'\|$ onverges to zero as n tends to infinity. Hence

$$\|p(B,B^*)\| \leq \lim\sup(\|p(A_n,A_n^*)\| + \|K_n'\|)$$
$$\leq \sup(\|p(A_n,A_n^*)\| = \|p(A,A^*)\|.$$

Hence there is a representation π of $C^*(A)$ determined by $\pi(A) = B$, and then there are irreducible representations π_i of $C^*(A)$ determined by $\pi_i(A) = B_i$. Now, for any polynomial p and for all i and n,

$$\|\pi_i(p(A,A^*))\| = \|p(B_i,B_i^*)\| \leq \|p(B,B^*)\|$$
$$= \|p(A_n,A_n^*) + K_n'\| \leq \|p(A_n,A_n^*)\| + \|K_n'\|$$
$$= \|P_n(p(A,A^*))\| + \|K_n'\|.$$

Hence if $\|\pi_i(p(A,A^*))\| > 1$ then there is an n_0 such that $\|P_n(p(A,A^*))\| > 1$ for all $n \geq n_0$. Since the-algebra of polynomials in A and A^* is dense in $C^*(A)$ we obtain that the sequence $\{P_n\}_{n=2}^{\infty}$ converges to each π_i.

We now show that the $\{\pi_i\}$ and $\{P_n\}$ are the only irreducible representations of $C^*(A)$ up to unitary equivalence. Let $\theta: C^*(A) \to B(\mathcal{H}_\theta)$ be an irreducible representation of $C^*(A)$. By [239], we can "extend" θ to an irreducible representation $\theta': \sum_{n=2}^{\infty} \oplus C^*(A_n) \to B(\mathcal{H}_{\theta'})$ where $\mathcal{H}_\theta$ is a subspace of $\mathcal{H}_{\theta'}$. reducing $\theta'(A)$ and $\theta'(A)|\mathcal{H}_\theta = \theta(A)$. Here $\sum_{n=2}^{\infty} \oplus C^*(A_n)$ is the C^*-algebra of all bounded sequences with entries from the $C^*(A_n)$. Let I_j be the operator in $\sum_{n=2}^{\infty} \oplus C^*(A_n)$ with I in the jth coordinate and zeros elsewhere. Since θ' is irreducible and since $\theta'(I_j)$ is a projection, $\theta'(I_j)$ is either zero or I and $\theta'(I_j) = I$ for at most one j. Suppose $\theta'(I_j) = I$ Then $\theta'(AI_j) = \theta'(A)$, and we have an irreducible representation of $C^*(A_j)$, determined by $A_j \to \theta'(AI_j)|\mathcal{H}_\theta = \theta'(A)|\mathcal{H}_\theta = \theta(A)$. But since A_j is a finite irreducible matrix this implies that $\theta(A) \cong A_j$ and $\theta \cong P_j$. Now suppose $\theta'(I_j) = 0$ for all j . Then $\theta'(\sum_{n=2}^{\infty} \oplus K_n) = 0$ so that $\theta'(\sum_{n=2}^{\infty} \oplus (B\otimes I_n)) = \theta'(\sum_{n=2}^{\infty} \oplus A_n) = \theta'(A)$. Hence one has an irreducible representation of $C^*(B)$ determined by $B \to \theta'(\sum_{n=1}^{\infty} \oplus (B\otimes I_n))|\mathcal{H}_\theta = \theta'(A)|\mathcal{H}_\theta = \theta(A)$. Hence $\theta(A) \cong B_i$ for some i and thus $\theta \cong \pi_i$ some i. Thus $C^*(A)^\wedge = \{\pi_i: 1 \leq i \leq k\} \cup \{P_n: 2 \leq n\}$ and all the points are distinct; notice that dim $(\pi_i) \leq N <$ dim(P_n) for all $1 \leq i \leq k$; and $2 \leq n$. Since all irreducible representations of $C^*(A)$ are finite dimensional $C^*(A)$ is CCR and hence $C^*(A)^\wedge$ is T_1 [239], so that singleton sets are closed. The topology is then completely determined by the fact that the sequence $\{P_n\}$ converges to each π_i For then each P_n is open and for each / the sets $\{P_n: m \leq n) \cup \{\pi_i\}$ form a base for the open sets containing π_i. It follows that A has Hausdorff spectrum if and only if $k = 1$, that is, B is the direct sum of unitarily equivalent irreducible matrices.

Theorem (5.1.12) shows that Theorem (5.1.6) does not extend to the case of an infinite number of direct summands. Also, Theorem (5.1.12) shows that there exist operators A with Hausdorff spectrum such that $C^*(A)$ does not have bounded representation dimension.

Let A be an operator with the structure of the operator A in Theorem (5.1.12); that is, $A = (\sum_{n=2}^{\infty} \oplus A_n)$ with $A_n = B\otimes I_n + K_n, A_n$, irreducible, $\|K_n\| \to 0$, and B a finite matrix. Since the set $\{P_n: m \leq n)$ is discrete in $C^*(A)^\wedge$, an application of the Dauns-Hofmann theorem see Remark 7 in [240] implies that $I_j \in C^*(A)$ for all j. Hence $C^*(A)$ contains the C^*-algebra $\sum' = \sum' \oplus C^*(A_n)$ of all sequences in $\sum \oplus C^*(A_n)$ which converge to zero in norm. This observation motivates our next considerations.

Suppose $A = \sum \oplus A_n$ with each A_n an irreducible finite matrix and that $\sum' \subseteq C^*(A_n)$. Furthermore, assume that the sequence $\{P_n\}_{n=2}^{\infty}$ converges to a unique irreducible representation π with $\pi(A) = B$ (thus $C^*(A)^{\wedge}$ consists of a discrete sequence with a single limit point and is hence Hausdorff, thus π must be finite dimensional). As a small step toward completing the characterization of operators with Hausdorff spectrum, we will show that there is an n_0 such that, for all $n \geq n_0$, $A_n \cong B \otimes I_n + K_n$ with $\|K_n\| \to 0$, and I is the identity on an appropriate space (note that included in this result is the fact that the dimension of the space that π acts on must divide the dimension of the space that P_n acts on for all $n \geq n_0$). Thus we will obtain a concrete characterization of those operators $A = \sum_{n=1}^{\infty} \oplus A_n$, A_n an irreducible finite matrix, such that $C^*(A)^{\wedge}$ is a countable set with a single limit point. Since $C^*(A)^{\wedge} = \{P_n : 2 \leq n) \cup \{\pi\}$, the C^*-algebra $C^*(A)/\sum'$ is isomorphic to $B(\mathcal{H}_\pi)$, and we consider π as the quotient map of $C^*(A)$ onto $C^*(A)/\sum'$, and also as the quotient map of $\sum \oplus C^*(A_n)$ onto $(\sum \oplus C^*(A_n))/\sum'$.

If the dimension of π is one, then $\pi(A)$ is a scalar, say $\pi(A) = \lambda$. But then $\pi(A - \lambda I) = 0$ so $A - \lambda I \in \sum'$ and A has the desired structure. The case when π has higher dimension is somewhat harder. We need to show that matrix units in $C^*(A)/\sum'$ can be lifted to "almost matrix units" in $C^*(A)$. We first need some lemmas. Lemma (5.1.13) is known and was shown to us by the late David Topping, so we include a proof.

Lemma (5.1.13)[234]: Let E and F be projections in $B(\mathcal{H})$. If $\|EF\| < 1$, then $E(\mathcal{H}) \cap F(\mathcal{H}) = \{0\}$ and $E(\mathcal{H}) + F(\mathcal{H})$ is closed. Furthermore, if P is defined by $Pz = 0$ for $z \in (E(\mathcal{H}) + F(\mathcal{H}))^{\perp}$ and $P(x + y) = x$ for $x \in E(\mathcal{H})$, $y \in F(\mathcal{H})$, then P is bounded and $\|P\| \leq (1 - \|EF\|)^{-1/2}$.

Proof. Actually, a more detailed analysis than we will do shows that $\|P\| = (1 - \|EF\|^2)^{-1/2}$. However, the estimate in the lemma is all that we will need. Clearly $E(\mathcal{H}) \cap F(\mathcal{H}) = \{0\}$. If $Ex = x$ and $Fy = y$, then $|(x, y)| \leq \|EF\|\|x\|\|y\|$, so that

$$\begin{aligned}\|x + y\|^2 &\geq \|x\|^2 - 2\|EF\|\|x\|\|y\| + \|y\|^2 \\ &= (1 - \|EF\|)(\|x\|^2 + \|y\|^2) + \|EF\|(\|x\| - \|y\|)^2 \\ &\geq (1 - \|EF\|)\|x\|^2.\end{aligned}$$

It then follows that $E(\mathcal{H}) + F(\mathcal{H})$ is closed and $\|P\| \leq (1 - \|EF\|)^{-1/2}$.

Lemma (5.1.14)[234]: Let E and F be projections in $B(\mathcal{H})$ and assume that $\|EF\| < 1$. then

$$\|(E \vee F) - E - F\| \leq 2\|EF\|(1 - \|EF\|)^{-1/2}.$$

Here $E \vee F$ is the supremum of the projections E and F.

Proof. The right-hand side of the inequality can be improved to $(2\|EF\|).(1 - \|EF\|)^{-1/2}$,butthe stated estimate is all that we need. Let P and Q in $B(\mathcal{H})$ be defined by $Pz = 0 = Q(z)$ if $z \in (E(\mathcal{H}) + F(\mathcal{H}))^{\perp}$ and, for $Ex = x$, $Fy = y$, let $P(x + y) = x$, $Q(x + y) = y$. Then for $z = x + y$ with $Ex = x$, $Fy = y$ we obtain that

$$\begin{aligned}\|(E \vee F - E - F)z\| &= \|x + y - x - Ey - Fx - y\| \\ &= \|Ey + Fx\| = \|EQz + FPz\| \\ &\leq \|EFQz + FEPz\| \leq \|EF\|(\|Q\| + \|P\|)\|z\| \\ &\leq 2\|EF\|(1 - \|EF\|)^{-1/2}\|z\|,\end{aligned}$$

where the last inequality is from Lemma (5.1.13). Hence Lemma (5.1.14) follows.

We now prove that one can lift a finite family of orthogonal projections from $C^*(A)/\sum'$ back to $C^*(A)$. It is known that orthogonal projections can be lifted out of the Calkin algebra Lemma 3.4 in [248], but we want to get our projections in the C^*-algebra $C^*(A)$.

Lemma (5.1.15)[234]: Suppose that $A = \sum_{n=2}^{\infty} \oplus A_n$, with each A_n an irreducible finite matrix and $\sum' \subseteq C^*(A)$. If $\{e_i: 1 \le i \le N\}$is a finite family of orthogonal projections in $C^*(A)/\sum'$, then there is a family of orthogonal projections $\{E_i: 1 \le i \le N\}$ in $C^*(A)$ with $\pi(E_i) = e_i$, for all i, where $\pi: C^*(A) \to C^*(A)/\sum'$ is the quotient map.

Proof. Let $B_i \in C^*(A)$ be such that $\pi(B_i) = e_i$. We may assume that $B_i = B_i^*$. Then $B_i^2 - B_i \in \sum'$ for all i, so e_{ij}^*. Hence, by changing each B_i in only finitely many coordinates, we may assume that $(B_i^2 - B_i) < 1/100$ for all i. Then there exist α, β, $0 < \alpha < \beta < 1$, such that $\text{sp}(B_i) \cap (\alpha, \beta) = \phi$, and a function f continuous on $\text{sp}(B_i)$ with $f(x) = 0$ for $x \le \alpha$ and $f(x) = 1$ for $\beta \le x$. Then $f(B_i)$ is a projection in $C^*(A)$ and $\pi(f(B_i)) = f(e_i) = e_i$. Hence we can assume that all our original B_i are projections in $C^*(A)$. Now, for $i \ne j$, $\pi(B_i B_j) = 0$, so $P_n(B_i B_j) \to 0$ as $n \to \infty$. By changing each B_i in only finitely many coordinates we may assume that $\|B_i B_j\| < 1\ 1$ for all i, j.

Let $E_1 \equiv B_1$. Now, by Lemma (5.1.14),

$$\|P_n(B_1) \vee P_n(B_2) - P_n(B_1) - P_n(B_2)\| \le 2\|P_n(B_1)P_n(B_2)\|(1 - \|P_n(B_1)P_n(B_2)\|)^{-1/2}.$$

Hence the element of $\sum \oplus C^*(A_n)$ given by

$$(P_n(B_1) \vee P_n(B_2)) = (P_n(B_1) \vee P_n(B_2) - P_n(B_1) - P_n(B_2)) + B_1 + B_2$$

is in $C^*(A)$. Let $E_2 \equiv (P_n(B_1) \vee P_n(B_2)) - B_1$. Then $E_2 \in C^*(A)$, $E_1 E_2 = 0$ and $\pi(E_2) = \pi(B_2) = e_2$. Likewise, if we let

$$E_3 \equiv (P_n(E_1) \vee P_n(E_2) \vee P_n(B_3)) - E_1 - E_2\,,$$

then $E_3 \in C^*(A)$, $\pi(B_3) = e_3$ and $\{E_1, E_2, E_3\}$ are an orthogonal family. The proof is completed by an induction argument which we omit.

The next theorem is the key to proving the structure theorem that we promised in the remarks before Lemma (5.1.13). The method used in the proof of Theorem (5.1.16) is closely related to the proofs of Lemmas 1.9 and 1.10 in [242]. The fact that matrix units in the Calkin algebra lift to "almost matrix units" (namely, $E_{ij}^* = E_{ji}$, $E_{ij}E_{ki} = \delta_{jk}E_{ii}$, and $\sum E_{ii}$ is a projection of finite codimension) was stated in the preliminary version of [235]. A more general theorem has been proved by F. J. Thayer [Liftings in the category of C^*-algebras, Thesis, Harvard Univ., 1972], and the result for the Calkin algebra has been proved in [246]. But we cannot use this fact, since we again need to insure that the matrix units lift to $C^*(A)$.

Theorem (5.1.16)[234]: Suppose $A = \sum_{n=1}^{\infty} \oplus A_n$, with each A_n an irreducible finite matrix and $C^*(A) \supset \sum'$. Further suppose that the sequence $\{P_n: 1 \le n\}$ converges to a unique irreducible representation π with $N = \dim(\mathcal{H}_\pi) \ge 2$. Let $\{e_{ij}: 1 \le i, j \le N\}$ be elements $C^*(A)/\sum'$ such that $e_{ij}^* = e_{ji}$, $e_{ij}e_{ik} = \delta_{ji}e_{ik}$, $\sum_{i=1}^{N} e_{ii} = I$.

Then there exist $E_{ij} \in C^*(A)$ such that $E_{ij}^* = E_{ji}$, $E_{ij}E_{ik} = \delta_{ji}E_{ik}$, $\pi(E_{ij}) = e_{ij}$, and $P_n(\sum_1^N E_i) = I_n$ for all n greater than some n_0.

Proof. Let $B_{ij} \in C^*(A)$ be such that $\pi(B_{ii}) = e_{ij}$. By Lemma (5.1.15) we may assume that the $\{B_{ii}: 1 \le i \le N\}$ are orthogonal projections. Now for every $i \ne 1$ we have $\pi(B_{11} - B_{11}B_{i1}^* B_{i1} B_{11}) = 0$.

Hence $P_n(B_{11} - B_{11}B_{i1}^* B_{i1} B_{11}) \to 0$ as $n \to \infty$. Thus by making B_{11} zero in the first few coordinates we can assume that

$\|B_{11} - B_{11}B_{i1}^* B_{i1} B_{11}\| < 1$ for all i.

Now if C is the abelian C^*-algebra (without I) generated by B_{11} and $B_{11}B_{i1}^* B_{i1} B_{11}$ then C has B_{11} as identity and $B_{11}B_{i1}^* B_{i1} B_{11}$ is positive and invertible in C. Hence if $X_{1i} \in C \subset$

$C^*(A)$ is the positive square root of the inverse of $B_{11}B_{i1}^*B_{i1}B_{11}$, then $B_{11} = X_{1i}^2 B_{11}B_{i1}^*B_{i1}B_{11}$.W e note that

$$e_{11} = \pi(X_{1i})^2 e_{11}e_{1i}e_{i1}e_{11} = \pi(X_{1i})^2 e_{11} = \pi(X_{1i})^2,$$

and hence $\pi(X_{1i})^2 = e_{11}$. Now let $W_{1i} = X_{1i}B_{11}B_{i1}^*$. Then $W_{1i}W_{i1}^* = X_{1i}B_{11}B_{i1}^*B_{11}X_{1i} = B_{11}$, also $\pi(W_{1i}) = e_{11}e_{11}e_{1i} = e_{1i}$ and $\pi(W_{1i}^*W_{1i}) = e_{ii}$. Hence $P_n(W_{1i}^*W_{1i} - B_{ii}) \to 0$ as $n \to \infty$, and there is an m_i such that

$\|P_n(W_{1i}^*W_{1i} - B_{ii})\| < 1$ for all $n \geq m_i$.

Now for $m \geq m_0 = \sup\{m_i : 2 \leq i \leq n\}$, let

$$C_{im} = P_m(I - B_{ii} - W_{1i}^*W_{1i} + B_{ii}W_{1i}^*W_{1i} + W_{1i}^*W_{1i}B_{ii}).$$

Now by [238] each C_{im} is positive and invertible and if we let

$$S_{im} = C_{im}^{-1/2} P_m(W_{1i}^*W_{1i} + B_{ii} - I)$$

then S_{im} is a selfadjoint unitary and

$$S_{im}P_m(W_{1i}^*W_{1i})S_{im} = P_m(B_{ii}).$$

Let $S_i \in \sum \oplus C^*(A_n)$ be defined by

$$P_m(S_i) = I \quad \text{if } m < m_0$$

and

$$P_m(S_i) = S_{im} \quad \text{if } m \geq m_0.$$

Then, for each i, $2 \leq i \leq N$, S_i is a selfadjoint unitary. Now

$$\pi(I - B_{ii} - W_{1i}^*W_{1i} + B_{ii}W_{1i}^*W_{1i} + W_{1i}^*W_{1i}B_{ii}) = I - e_{ii} - e_{ii} + 2e_{ii} = I,$$

so $\|C_{im} - I_m\| \to 0$ as $m \to \infty$, and $\left\|C_{im}^{-1/2} - I_m\right\| \to 0$ as $m \to \infty$. Also $\pi(W_{1i}W_{1i} + B_{ii} - I) = 2e_{ii} - I$. Hence $\|S_{im} - P_m(2B_{ii} - I)\| \to 0$ as $m \to \infty$. Thus $(S_i - (2B_{ii} - I)) \in \sum' \subseteq C^*(A)$ and hence $S_i \in C^*(A)$ and $\pi(S_i) = 2e_{ii} - I$. Now change all the $B_{ii}, B_{11}, B_{i1}, W_{1i}, X_{1i}$ to be zero in the first $m_0 - 1$ coordinates. Then all our previous equations still are true. (We do not go back and redo the proof, we just alter the operators we have.) Also

$$S_i(W_{1i}^*W_{1i})S_i = B_{ii}.$$

Let $U_{i1} = S_iW_{1i}^* \in C^*(A)$. Then $U_{i1}^*U_{i1} = B_{11}$, $U_{i1}^*U_{i1} = B_{ii}$, and $\pi(U_{i1}) = e_{i1}$. Now let $E_{ii} = B_{ii}$ for all i and let $E_{ij} = U_{i1}U_{i1}^*$ if $i \neq j$, where we let $U_{11} = B_{11}$. It then follows that $E_{ij}E_{ik} = \delta_{ji}E_{ik}$, $\pi(E_{ij}) = e_{ij}$, $E_{ij}^* = E_{ji}$. Also $\pi(\sum_1^N E_{ii}) = I$, so, $\|P_n(I - \sum_1^N E_i)\| < 1$ for all n greater than some n_0. But $P_n(I - \sum_1^N E_i)$ is a projection, hence $P_n(\sum_1^N E_i) = I_n$ for all n greater than some n_0.

We have stated and proved Theorem (5.1.16) in the form given for notational convenience, but the same proof proves a more general statement: Suppose $A = \sum_{n=1}^{\infty} \oplus A_n$ with each A_n an irreducible finite matrix and $C^*(A) \supset \sum'$. Further suppose that the sequence $\{P_n : 1 \leq n\}$ converges to a finite number of unitarily in equivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_m$, each of which is finite dimensional. Then $C^*(A)/\sum'$ is isomorphic to $B(\mathcal{H}_{\pi_1}) \oplus B(\mathcal{H}_{\pi_2}) \oplus \ldots \oplus B(\mathcal{H}_{\pi_m})$. If $\{e_{ij}^{(k)} : 1 \leq i, j \leq \dim \mathcal{H}_{\pi_k}\}$ is a set of matrix units for $B(\mathcal{H}_{\pi_k})$, $1 \leq k \leq m$, then there are elements $E_{ij}^{(k)} \in C^*(A)$ such that $(E_{ij}^{(k)})^* = E_{ji}^{(k)}, E_{ij}^{(k)}E_{it}^{(k)} = \delta_{ji}E_{it}^{(k)}, \pi(E_{ij}^{(k)}) = e_{ij}^{(k)}$, and $P_n(\sum_1^N E_{ii}^{(k)}) = I_n$ for all n greater than some n_0, where the summation is taken over all i and k.

We also remark that by using C. Olsen's theorem [244] J. Calkin's original method of lifting partial isometries out of the Calkin algebra can be altered to give a more elegant proof of Theorem (5.1.16) in the case $N = 2$.

Corollary (5.1.17)[234]: Let A be as in Theorem (5.1.16). Suppose that relative to the matrix units (e_{ij}), $\pi(A)$ has the matrix $(\beta_{ij}) \cong B$. Then there is an n_0 such that, for all $n \geq n_0$, $A_n \cong B \otimes I + K_n$ with $\|K_n\| \to 0$.

Proof. Let $D = \sum \beta_{ij} E_{ij}$. where the E_{ij} are as in Theorem (5.1.16). Then $\pi(D) = \pi(A)$, so $A - D \in \sum'$ and $A = D + K$ with $K \in \sum'$. But, for $n \geq n_0$, the E_{ij} are matrix units for $C^*(A_n)$. Hence $D_n \cong I \otimes B \cong B \otimes I$ for $n \geq n_0$ and $\|K_n\| \to 0$.

By using the remarks just after the proof of Theorem (5.1.16), a structure theorem for A similar to Corollary (5.1.17) could be stated and proved in the case that the sequence $\{ P_n : 1 \leq n \}$ converges to a finite number of unitarily inequivalent irreducible representations $\pi_1, \pi_2, \dots, \pi_m$, each of which is finite dimensional. In this case we would have that for large n: $A_n \cong \sum_{i=1}^{m} \oplus (\pi_i(A) \otimes I_i) + K_n$, where each I_i. is the identity on an appropriate space.

Also, if we do not assume that theP_n are discrete in $C^*(A)^\wedge$ and assume instead that the sequence $\{P_n : 1 \leq n\}$ converges to P_i for some i, then by considering $A' = \sum' \oplus A_n$, where now the prime denotes the fact that i is not included in the summation, we still have that the conclusion of Corollary (5.1.17) is valid, except that the n_0 must be chosen greater than i. If $A = \sum \oplus A_n$, each A_n a finite irreducible matrix and $C^*(A)^\wedge$ Hausdorff with a finite number of cluster points for $\{P_n : 1 \leq n \}$, then by partitioning $\{P_n : 1 \leq n \}$ into a finite number of disjoint subsequences, we immediately have a structure theorem similar to Corollary (5.1.17).

In order to prove a structure theorem for arbitrary operators with Hausdorff spectrum, the only case that remains is the case when $\{P_n : 1 \leq n \}$ has infinitely many cluster points. This case presents many complications, and we have no results in this case.

In Theorem (5.1.16) we showed that $n \times n$ matrix units can be lifted to "almost matrix units". We conclude by showing that (even 2×2) matrix units in $C^*(A)/\sum'$ or in $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ cannot in general be lifted to matrix units in $C^*(A)$ or in $B(\mathcal{H})$. Although this result is known (see the preliminary version of [235]) for $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ we include a proof and comments for completeness.

Lemma (5.1.18)[234]: Let U_+ denote the unilateral shift of multiplicity one on $\mathcal{H}$. Then the operator $T = \begin{pmatrix} 0 & U_+ \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ is not unitarily equivalent to a compact perturbation of $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$.

Proof. Suppose that $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ is a unitary operator implementing the unitary equivalence between

$$\begin{pmatrix} 0 & U_+ \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} K_1 & I + K_2 \\ K_3 & K_4 \end{pmatrix}$$

where $K_1, K_2, K_3, K_4 \in \mathcal{K}(\mathcal{H})$. Straightforward calculations yield that $Z = U_+^*(XK_1 + YK_3)$, $K_1 = X^*U_+Z$, $I + K_2 = X^*U_+W$, $K_3 = Y^*U_+Z$, $K_1 = Y^*U_+W$, and $XX^* + YY^* = X^*X + Z^*Z = Y^*Y + W^*W = WW^* + ZZ^* = I$. Hence $Z \in \mathcal{K}(\mathcal{H})$ and $Y^*U_+ = Y^*U_+WW^* + Y^*U_+ZZ^* = K_4W^* + K_3Z^*$. It then follows that $Y \in \mathcal{K}(\mathcal{H})$. Since $Z, Y \in \mathcal{K}(\mathcal{H})$, we have that $I - XX^*, I - X^*X, I - WW^*$ and $I - W^*W \in \mathcal{K}(\mathcal{H})$. By Theorem 3.1 [235], X and W are compact perturbations of isometries or co-isometries of finite defect. In particular, they are Fredholm operators. Since the index of the product of Fredholm operators is the sum of the indices we obtain that

$$0 = \operatorname{ind}(I + K_2) = \operatorname{ind}(X^*) + \operatorname{ind}(U_+) + \operatorname{ind}(W) = -\operatorname{ind}(X) - 1 + \operatorname{ind}(W).$$

But we also have that

$$0=\operatorname{ind}\begin{pmatrix}X & Y\\ Z & W\end{pmatrix}=\operatorname{ind}\begin{pmatrix}X & 0\\ 0 & W\end{pmatrix}=\operatorname{ind}(X)+\operatorname{ind}(W).$$

Adding these two equations yields $1 = 2\operatorname{ind}(W)$, which is a contradiction. One easily verifies that the operator T in Lemma (5.1.18) is unitarily equivalent to the operator

$$0\oplus\sum_{n=1}^{\infty}\oplus\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\quad\text{on}\quad \mathrm{C}\oplus\sum_{n=1}^{\infty}\oplus\mathrm{C}^2.$$

We remark that the conclusion of Lemma (5.1.18) holds if and only if the multiplicity of the shift U_+ is odd. The above-mentioned description of T and Lemma (5.1.18) establish that there exist operators S (for example, $S=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$ and one dimensional operators F (for example $F=0$) such that $S\oplus F$ is not unitarily equivalent to a compact perturbation of S.

Proposition (5.1.19)[234]: Matrix units in $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ do not necessarily lift to matrix units in $B(\mathcal{H})$. In fact, there exist operators A as in Theorem (5.1.16) such that matrix units in $C^*(A)/\sum'$ do not lift to matrix units in$C^*(A)$.

Proof. Let $B=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $A_1=0$ on C, and for $n\geq 2$ let $A_n=B\otimes I_n+K_n$ be an irreducible operator on C^{2n} with $\|K_n\|<1/n$. Then, $A=\sum_{n=1}^{\infty}\oplus A_n$ is as in Theorem (5.1.16). Now A is a compact $\sum'$-perturbation of the operator $0\oplus\sum_{n=1}^{\infty}\oplus(B\otimes I_n)$, which in turn is unitarily equivalent to the operator T of Lemma (5.1.18). Hence $\pi(A)$ generates a set of 2×2 matrix units in $C^*(A)/\sum'$ and hence in $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. However, if A were a compact perturbation of a partial isometry V whose initial space V^*V and final space VV^* sum to I, then, since with respect to this decomposition V is of the form $\begin{pmatrix}0 & U\\ 0 & 0\end{pmatrix}$ where U is a unitary operator from V^*V onto VV^* which is in turn unitarily equivalent to $\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$ we would obtain a contradiction from Lemma (5.1.18).

Section (5.2): Locally Multiplicatively Convex Topological ℝ-Algebras

It was known to Hilbert [258] that a nonnegative real multivariable polynomial $f=\sum_\alpha f_\alpha \underline{X}^\alpha\in\mathbb{R}[\underline{X}]:=\mathbb{R}[X_1,\dots,X_n]$ is not necessarily a sum of squares of polynomials. However, every such polynomial can be approximated by elements of the cone $\sum\mathbb{R}[\underline{X}]^2$:= sums of squares of polynomials, with respect to the topology induced by the norm $\|\cdot\|_1$ (given by $\left\|\sum_\alpha f_\alpha \underline{X}^\alpha\right\|_1:=\sum_\alpha|f_\alpha|$). In fact, every polynomial $f\in\mathbb{R}[\underline{X}]$, nonnegative on $[-1;\,1]^n$ is in the $\|\cdot\|_1$-closure of $\sum\mathbb{R}[\underline{X}]^2$ see Theorem 9.1 in [251]. Moreover, it is known that for every $f\in\operatorname{Pos}([-1;\,1]^n)$:= the cone of nonnegative polynomials on $[-1;\,1]^n$, and $\epsilon>0$, there exists $N>0$ depending on n,ϵ, $\deg f$ and the size of coefficients of f such that for every integer $r\geq N$, the polynomial $f_{\epsilon,r}:=f+\epsilon(1+\sum_{i=1}^n X_i^{2r})\in\sum\mathbb{R}[\underline{X}]^2$. This gives an effective way of approximating f by sums of squares in $\|\cdot\|_1$ see Theorem 3.9 in [260]. The closure of $\sum\mathbb{R}[\underline{X}]^2$ with respect to the family of weighted $\|\cdot\|_p$-norms has been studied in [253]. Note that an easy application of Stone-Weierstrass Theorem shows that the same result holds for the coarser norm $\|f\|_\infty:=\sup_{x\in[-1;\,1]^n}|f(x)|$; i.e., $\overline{\sum\mathbb{R}[\underline{X}]^2}^{\|\cdot\|_\infty}=\operatorname{Pos}([-1;\,1]^n)$, but in practice, finding $\|f\|_\infty$ is a computationally difficult optimization problem, whereas $\|f\|_1$ is easy to compute. Therefore to gain more computational exibility it is interesting to study such closures with respect to various norms on $\mathbb{R}[\underline{X}]$.

The general set-up we consider is the following. Let C be a cone in $\mathbb{R}[\underline{X}]$, τ a locally convex topology on $\mathbb{R}[\underline{X}]$ and $K \subseteq \mathbb{R}^n$ be a closed set. Consider the condition:

$$\bar{C}^{\tau} \supseteq \mathrm{Pos}(\mathrm{K}), \qquad (3)$$

(where as above, $\mathrm{Pos}(K)$ denotes the set of polynomials nonnegative on K). An application of Hahn-Banach Separation Theorem together with Haviland's Theorem (see Theorem (5.2.2)) shows that (3) holds if and only if for every τ-continuous linear functional L with $L(C) \subseteq [0, \infty)$, there exists a Borel measure μ on K such that

$$\forall f \in \mathbb{R}[\underline{X}] \qquad L(f) = \int_K f \, d\mu. \qquad (4)$$

We study closure results of type (3) and their corresponding representation results of type (4) for any locally multiplicatively convex (unital, commutative) topological $\mathbb{R}$-algebra.

We introduce recall Jacobi's Theorem and a generalized version of Haviland's Theorem, results which play a crucial role.

We consider the case of a submultiplicative seminorm ρ on an $\mathbb{R}$- algebra A. In Theorem (5.2.8) we prove that for any integer $d \geq 1$ and any $\sum A^{2d}$- module S of A, $\bar{S}^{\rho}$ consists of all elements of A with nonnegative image under every ρ-continuous $\mathbb{R}$-algebra homomorphism $\alpha: A \to \mathbb{R}$ such that $\alpha(S) \subseteq [0, \infty)$. This generalizes Theorem 5.3 in [254] on the closure of $\sum A^{2d}$ with respect to a submultiplicative norm. The application of Theorem (5.2.8) to the representation of linear functionals by measures is explained in Corollary (5.2.9).

We explain how Theorem (5.2.8) apply in the case of a (unital, commutative) $*$-algebra equipped with a submultiplicative $*$-seminorm. Corollary (5.2.11) generalizes results on $*$-semigroup algebras in Theorem 4.2.5 [252] and Theorem 4.3 and Corollary 4.4 [255].

We explain how Theorem (5.2.8) extends to the class of locally multiplicatively convex topologies. Such topologies are induced by families of submultiplicative seminorms. Theorem (5.2.15) can viewed as a strengthening (in the commutative case) of the result in Lemma 6.1 and Proposition 6.2 in [263] about enveloping algebras of Lie algebras.

Throughout A denotes a unitary commutative $\mathbb{R}$-algebra. The set of all unitary $\mathbb{R}$-algebra homomorphisms from A to $\mathbb{R}$ will be denoted by $\mathcal{X}(A)$. Note that $\mathcal{X}(A)$ as a subset of $\mathbb{R}^A$ carries a natural topology, where $\mathbb{R}^A$ is endowed with the product topology. This topology coincides with the weakest topology on $\mathcal{X}(A)$ which makes all the evaluation maps $\hat{a}: \mathcal{X}(A) \to \mathbb{R}$, defined by $\hat{a}(\alpha) = \alpha(a)$ continuous in [262].

For an integer $d \geq 1$, $\sum A^{2d}$ denotes the set of all finite sums of $2d$ powers of elements of A. A $\sum A^{2d}$-module of A is a subset S of A such that $1 \in S$, $S + S \subseteq S$ and $a^{2d} \cdot S \subseteq S$ for each $a \in A$. We say S is archimedean if for each $a \in A$ there exists an integer $n \geq 1$ such that $n + a \in S$. For any subset S of A, the non-negativity set of S, denoted by $\mathcal{K}_S$, is defined by

$$\mathcal{K}_S := \{\alpha \in \mathcal{X}(A): \hat{a}(\alpha) \geq 0 \text{ for all } a \in S\}.$$

Also, for $K \subseteq \mathcal{X}(A)$, we define $\mathrm{Pos}(K)$ by

$$\mathrm{Pos}(K) := \{a \in A: \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in S\}.$$

Theorem (5.2.1)[249]: (Jacobi). Suppose S is an archimedean $\sum A^{2d}$-module of A for some integer $d \geq 1$. Then for each $a \in A$,

$$\hat{a} > 0 \text{ on } \mathcal{K}_S \Rightarrow a \in S.$$

Proof. See Theorem 4 [259].

Recall that a Radon measure on a Hausdorff topological space X is a measure on the σ-algebra of Borel sets of X that is locally finite and inner regular. Locally finite means that every point has a neighbourhood of finite measure. Inner regular means each Borel set can be approximated from within using a compact set. We will use the following version of Haviland's Theorem to get representations of linear functionals on A.

Theorem (5.2.2)[249]: Suppose A is an $\mathbb{R}$-algebra, X is a Hausdorff space, and $\hat{}: A \to C(X)$is an $\mathbb{R}$-algebra homomorphism such that for some $p \in A$, $\hat{p} \geq 0$ on X, the set $X_i = \hat{p}^{-1}([0; i])$ is compact for each $i = 1, 2 \cdots$. Then for every linear functional $L : A \to \mathbb{R}$ satisfying

$$L(\{a \in A: \hat{a} \geq 0 \text{ on } X\}) \subseteq [0, \infty),$$

there exists a Radon measure μ on X such that $\forall a \in A \quad L(a) = \int_X \hat{a}\, d\mu$.

Here, $C(X)$ denotes the ring of all continuous real valued functions on X. A proof of Theorem (5.2.2) can be found in Theorem 3.1 in [261] or Theorem 3.2.2 in [262] (also see [256], [257] for the original version). Note that the hypothesis of Theorem (5.2.2) implies in particular that X is locally compact (so μ is actually a Borel measure).

Definition (5.2.3)[249]: A seminorm ρ on A is a map $\rho: A \to [0, \infty)$ such that

(i) for $x \in A$ and $r \in \mathbb{R}$, $\rho(rx) = |r|\rho(x)$, and

(ii) for all $x, y \in A$, $\rho(x + y) \leq \rho(x) + \rho(y)$.

Moreover, ρ is called a submultiplicative seminorm if in addition:

(iii) for all $x, y \in A$, $\rho(xy) \leq \rho(x)\rho(y)$.

The algebra A together with a submultiplicative seminorm ρ on A is called a seminormed algebra and is denoted by the symbolism (A, ρ). We denote the set of all ρ-continuous $\mathbb{R}$-algebra homomorphisms from A to $\mathbb{R}$ by $\mathfrak{sp}(\rho)$, which we refer to as the Gelfand spectrum of (A, ρ). The topology on $\mathfrak{sp}(\rho)$ is the topology induced as a subspace of $\mathcal{X}(A)$.

Lemma (5.2.4)[249]: For any submultiplicative seminorm ρ on A,

$$\mathfrak{sp}(\rho) = \{\alpha \in \mathcal{X}(A): |\alpha(x)| \leq \rho(x) \text{ for all } x \in A\}.$$

Proof. Suppose $\alpha \in \mathcal{X}(A)$ and there exists $x \in A$ such that $|\alpha(x)| > \delta > \rho(x)$. Set $y = \frac{x}{\delta}$ where $\delta \in \mathbb{R}$ is such that $|\alpha(x)| > \rho(x)$. Then $\rho(y) < 1$ and $|\alpha(y)| > 1$ so, as $n \to \infty$, $\rho(y^n) \to 0$ and $|\alpha(y^n)| \to \infty$. This proves ($\subseteq$). The other inclusion is clear.

Corollary (5.2.5)[249]: For any submultiplicative seminorm ρ on A, $\mathfrak{sp}(\rho)$ is compact.

Proof. The map $\alpha \mapsto (\hat{a}(\alpha))_{a \in A}$ identifies $\mathfrak{sp}(\rho)$ with a closed subset of the compact space $\prod_{a \in A}[-\rho(a), \rho(a)]$.

Lemma (5.2.6)[249]: For any unital Banach $\mathbb{R}$-algebra (B, φ), any $a \in A$ and $r \in \mathbb{R}$ such that $r > \varphi(a)$, and any integer $k \geq 1$, there exists $p \in B$ such that $p^k = r + a$.

Proof. This is well-known. The standard power series expansion

$$(r + x)^{1/k} = r^{1/k}(1 + \tfrac{x}{r})^{1/k} = r^{1/k} \sum_{i=0}^{\infty} \frac{\frac{1}{k}\left(\frac{1}{k} - 1\right) \ldots \left(\frac{1}{k} - i\right)}{i!} (\tfrac{x}{r})^i$$

converges absolutely for $|x| < r$. This implies that

$$p := r^{1/k} \sum_{i=0}^{\infty} \frac{\frac{1}{k}\left(\frac{1}{k} - 1\right) \ldots \left(\frac{1}{k} - i\right)}{i!} (\tfrac{a}{r})^i$$

is a well-defined element of B and $p^k = r + a$.

Corollary (5.2.7)[249]: For any unital Banach $\mathbb{R}$-algebra (B,φ) and any linear functional $L:B\to\mathbb{R}$, if $L(b^{2d})\ge 0$ for all $b\in B$ for some $d\ge 1$ then L is φ-continuous. In particular, each $\alpha\in\mathcal{X}(B)$ is φ-continuous.

Proof. By Lemma (5.2.6), $\frac{1}{n}+\varphi(a)\pm a=\frac{1}{n}+\varphi(\pm a)+(\pm a)\in B^{2d}$ for all $a\in B$ and all $n\ge 1$. Applying L this yields $|L(a)|\le(\frac{1}{n}+\varphi(a))L(1)$ for all $a\in B$ and all $n\ge 1$ so $|L(a)|\le\varphi(a)L(1)$ for all $a\in B$.

Theorem (5.2.8)[249]: Let ρ be a submultiplicative seminorm on A and let S be a $\sum A^{2d}$-module of A. Then $\bar{S}^{\rho}=\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$. In particular, $\overline{\sum A^{2d}}^{\rho}=\operatorname{Pos}(\mathfrak{sp}(\rho))$.

Proof. Since each $\alpha\in\mathcal{K}_S\cap\mathfrak{sp}(\rho)$ is continuous and

$$\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))=\bigcap_{\alpha\in\mathcal{K}_S\cap\mathfrak{sp}(\rho)}\alpha^{-1}([0,\infty),$$

$\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$ is ρ-closed. Since $S\subseteq\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$ this implies $\bar{S}^{\rho}\subseteq\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$. For the reverse inclusion we have to show that if $b\in\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$ then $b\in\bar{S}^{\rho}$. Let $\tilde{S}$ denote the closure of the image of S in $(\tilde{A},\tilde{\rho})$. Then $\tilde{S}$ is a $\sum\tilde{A}^{2d}$-module of $\tilde{A}$. By Lemma (5.2.6), $\frac{1}{n}+\tilde{\rho}(a)+a\in\tilde{A}^{2d}\subseteq\tilde{S}$ for all $a\in\tilde{A}$ and all $n\ge 1$, so $\tilde{\rho}(a)+a\in\tilde{S}$ for all $a\in\tilde{A}$. This implies that $\tilde{S}$ is archimedean. By Corollary (5.2.7) every element of $\mathcal{K}_{\tilde{S}}$ restricts to an element of $\mathcal{K}_S\cap\mathfrak{sp}(\rho)^1$ so, by our hypothesis on b, $\alpha(\tilde{\rho})=\alpha|_A(b)\ge 0$ for all $\alpha\in\mathcal{K}_{\tilde{S}}$, where $\tilde{b}$ denotes the image of b in $\tilde{A}$. Then $\alpha(\tilde{b}+\frac{1}{n})>0$ for all $\alpha\in\mathcal{K}_{\tilde{S}}$ so, by Jacobi's Theorem (5.2.1), $\tilde{b}+\frac{1}{n}\in\tilde{S}$ for all $n\ge 1$. Then $\tilde{b}\in\tilde{S}$, so $b\in\bar{S}^{\rho}$.

Corollary (5.2.9)[249]: Let ρ be a submultiplicative seminorm on A, S a $\sum A^{2d}$-module of A. If $L:B\to\mathbb{R}$ is a ρ-continuous linear functional such that $L(s)\ge 0$ for all $s\in S$ then there exists a unique Radon measure μ on $\mathcal{X}(A)$ such that

$$\forall a\in A\quad L(a)=\int\hat{a}\,d\mu.$$

Moreover, $supp(\mu)\subseteq\mathcal{K}_S\cap\mathfrak{sp}(\rho)$.

Proof. By our hypothesis and Theorem (5.2.8) L is non-negative on $\operatorname{Pos}(\mathcal{K}_S\cap\mathfrak{sp}(\rho))$. Applying Theorem (5.2.2), with $X:=\mathcal{K}_S\cap\mathfrak{sp}(\rho)$ and $\hat{}:A\to C(X)$ the map defined by $a\mapsto\hat{a}|_x$, yields a Radon measure μ'. on X such that $L(a)=\int_X\hat{a}\,d\mu'$. for all $a\in A$. Observe that X is compact, by Corollary (5.2.5), so we can take $p=1$. The Radon measure μ on $\mathcal{X}(A)$ that we are looking for is just the extension of μ'. to $\mathcal{X}(A)$, i.e., $\mu(E):=\mu'(E\cap X)$ for all Borel sets E in $\mathcal{X}(A)$. Uniqueness of μ is a consequence of the following easy result.

Lemma (5.2.9)[249]: Suppose μ is a Radon measure on $\mathcal{X}(A)$ having compact support. Then μ is determinate, i.e., if ν is any Radon measure on $\mathcal{X}(A)$ satisfying $\int\hat{a}\,d\nu=\int\hat{a}\,d\mu$ for all $a\in A$ then $\nu=\mu$.

Proof. Set $Y=supp(\mu)$. Suppose first that $\operatorname{supp}(\nu)\nsubseteq Y$. Then there exists a compact set $Z\subseteq\mathcal{X}(A)\backslash Y$ with $\nu(Z)>0$. Choose $\epsilon>0$ so that $\epsilon<\frac{\nu(Z)}{\mu(Y)+\nu(Z)}$. Since Y,Z are compact and disjoint, the Stone-Weierstrass Theorem implies there exists $a\in A$ such that $|\hat{a}(\alpha)|\le\epsilon$ for all $\alpha\in Y$ and $|\hat{a}(\alpha)-1|\le\epsilon$ for all $\alpha\in Z$. Replacing a by a^2 if necessary, we can suppose $\hat{a}\ge 0$ on $\mathcal{X}(A)$. Then $\int\hat{a}\,d\mu\le\epsilon\mu(Y)$, but $\int\hat{a}\,d\nu\ge\int_Z\hat{a}\,d\nu\ge(1-\epsilon)\nu(Z)$, which is a contradiction. It follows that $\operatorname{supp}(\nu)\subseteq Y$, so μ,ν both have support in the same compact set . Then, using the Stone-Weierstrass Theorem again, $\int\varphi\,d\nu=\int\varphi\,d\mu$, for all $\varphi\in C(Y)$ so $\mu=\nu$ by the Riesz Representation Theorem.

We consider a $*$-algebra R equipped with a submultiplicative $*$-seminorm φ, i.e., R is a (unital, commutative) $\mathbb{C}$-algebra equipped with an involution $*: R \to R$ satisfying

$$(\lambda a)^* = \bar{\lambda} a^*;\ (a+b)^* = a^* + b^*;\ (ab)^* = a^* b^* \text{ and } a^{**} = a$$

for all $\lambda \in \mathbb{C}$ and all $a;\ b \in R$, and $\varphi: R \to [0;\infty)$ satis_es

$$\varphi(\lambda a) = |\lambda|\varphi(a); \varphi(a+b) \le \varphi(a) + \varphi(b); \varphi(ab) \le \varphi(a)\varphi(b) \text{ and } \varphi(a^*) = \varphi(a)$$

for all $\lambda \in \mathbb{C}$ and all $a;\ b \in R$.

We denote by $\mathcal{X}(R)$ set of all $*$-algebra homomorphisms $\alpha: R \to \mathbb{C}$ equipped with its natural topology as a subspace of the product space $\mathbb{C}^R$ and by $\mathfrak{sp}(\varphi)$ the subspace of $\mathcal{X}(R)$ consisting of all φ-continuous $*$-algebra homomorphisms $\alpha: R \to \mathbb{C}$. The symmetric part of R is

$$H(R) := \{a \in R: a^* = a\}.$$

Since $R = H(R) \oplus iH(R)$, one sees that $\mathcal{X}(R)$ and $\mathfrak{sp}(\varphi)$ are naturally identified via restriction with $\mathcal{X}(H(R))$ and $\mathfrak{sp}(\varphi|_{H(R)})$, respectively, and φ continuous $*$-linear functionals $L: R \to \mathbb{C}$ are naturally identified via restriction with $\varphi|_{H(R)}$-continuous $\mathbb{R}$-linear functionals $L: H(R) \to \mathbb{R}$.

Applying Theorem (5.2.8) and Corollary (5.2.9) to the symmetric part of (R, φ) yields the following result.

Corollary (5.2.11)[249]: Let R be a $*$-algebra equipped with a submultiplicative $*$-seminorm φ, S a $\sum H(R)^{2d}$-module of $H(R)$. Then $\bar{S}^{\varphi} = \operatorname{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\varphi))$. If $L: R \to \mathbb{C}$ is any φ -continuous $*$-linear functional such that $L(s) \ge 0$ for all $s \in S$ then there exists a unique Radon measure on $\mathcal{X}(R)$ such that $L(a) = \int \hat{a}\, d\mu$ for all $a \in R$. Moreover, $supp(\mu) \subseteq (\mathcal{K}_S \cap \mathfrak{sp}(\varphi))$.

Corollary (5.2.11) applies, in particular, to any $*$-semigroup algebra $\mathbb{C}[W]$ equipped with a $*$-seminorm $\|\cdot\|_{\phi}$ arising from an absolute value ϕ on the $*$-semigroup W, i.e., $\|\sum \lambda_\omega \omega\|_\phi := \sum_\omega |\lambda_\omega| \phi(\omega)$. In this way Corollary (5.2.11) extends Theorem 4.2.5 in [252] and Theorem 4.3 and Corollary 4.4 in [255].

Let A be an $\mathbb{R}$-algebra. A subset U of A is called a multiplicative set (an m-set for short) if $U \cdot U \subseteq U$. A locally convex vector space topology on A is said to be locally multiplicatively convex (lmc for short) if there exists a system of neighbourhoods for 0 consisting of m-sets. It is immediate from the definition that multiplication is continuous in any lmc-topology. We recall the following result.

Theorem (5.2.12)[249]: A locally convex vector space topology τ on A is lmc if and only if τ is generated by a family of submultiplicative seminorms on A.

Proof. See 4.3-2 [250].

A family $\mathcal{F}$ of submultiplicative seminorms of A is said to be saturated if, for any $\rho_1, \rho_2 \in \mathcal{F}$, the seminorm ρ of A defined by

$$\rho(x) := \max\{\rho_1(x);\ \rho_2(x)\} \text{ for all } x \in A$$

belongs to $\mathcal{F}$. For an lmc topology τ on A one can always assume that the family $\mathcal{F}$ of submultiplicative seminorms generating τ is saturated. In this situation the topology τ is the inductive limit topology, i.e., the balls $B_r^{\rho}(0) := \{a \in A: \rho(a) < r\}, \rho \in \mathcal{F}, r > 0$ form a system of τ-neighbourhoods of zero. This is clear.

We record the following more-or-less obvious result:

Lemma (5.2.13)[249]: Suppose τ is an lmc topology on A generated by a saturated family $\mathcal{F}$ of submultiplicative seminorms of A and $L: A \to \mathbb{R}$ τ-continuous linear functional. Then there exists $\rho \in \mathcal{F}$ such that L is ρ-continuous.

Proof. The set $\{a \in A: |SL(a)| < 1\}$ is an open neighbourhood of 0 in A so there exists $\rho \in \mathcal{F}$ and $r > 0$ such that$B_r^\rho(0) \subseteq \{a \in A: |SL(a)| < 1\}$. Then $B_{r\epsilon}^\rho(0) = \epsilon B_r^\rho(0)$ so

$$L(B_{r\epsilon}^\rho(0)) = L(\epsilon B_r^\rho(0)) = \epsilon L(B_r^\rho(0)) \subseteq \epsilon(-1,1) = (-\epsilon,\epsilon)$$

for all $\epsilon > 0$, i.e., L is ρ-continuous.

We denote the Gelfand spectrum of $(A;\tau)$, i.e., the set of all τ -continuous $\alpha \in \mathcal{X}(A)$, by $\mathfrak{sp}(\tau)$ for short.

Corollary (5.2.14)[249]: Suppose τ is an lmc topology on A generated by a saturated family $\mathcal{F}$ of submultiplicative seminorms of A. Then $\mathfrak{sp}(\tau) = \bigcup_{\rho\in\mathcal{F}} \mathfrak{sp}(\rho)$.

The main result extends to general lmc topologies, as follows:

Theorem (5.2.15)[249]: Let τ be an lmc topology on A and let S be any $\sum A^{2d}$-module of A. Then $\bar{S}^\tau = \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\tau))$. In particular, $\overline{\sum A^{2d}}^\tau = Pos(\mathfrak{sp}(\tau))$.

Proof. Let $\mathcal{F}$ be a saturated family of submultiplicative seminorms generating τ. Then $\bar{S}^\tau = \bigcap_{\rho\in\mathcal{F}} \bar{S}^\rho = \bigcap_{\rho\in\mathcal{F}} Pos(\mathcal{K}_S \cap \mathfrak{sp}(\rho)) = Pos(\mathcal{K}_S \cap \mathfrak{sp}(\tau))$. _

In view of Lemma (5.2.13), Corollary (5.2.9) also extends to general lmc topologies in an obvious way. The unique Radon measure corresponding to a τ-continuous linear functional $L: A \to \mathbb{R}$ such that $L(s) \geq 0$ for all $s \in S$ has support contained in the compact set $\mathcal{K}_S \cap \mathfrak{sp}(\rho)$ for some $\rho \in \mathcal{F}$.

The finest lmc topology on A is the lmc topology generated by the family of all submultiplicative seminorms of A. Theorem (5.2.15) can thought of as a strengthening (in the commutative case) of the result of Lemma 6.1 and Proposition 6.2 in [263] about enveloping algebras for $\mathbb{R}$-algebras. Note also the following:

Corollary (5.2.16)[249]: Let η be the finest lmc topology on A. Then, for any $\sum A^{2d}$-module

S of A, $\bar{S}^\eta = \text{Pos}(\mathcal{K}_S)$. In particular, $\overline{\sum A^{2d}}^\eta = Pos(\mathcal{X}(A))$

Proof. Apply Theorem (5.2.15) with $\tau = \eta$, using the fact that $\mathfrak{sp}(\eta) = \mathcal{X}(A)$.

Section (5.3): $*$-Subalgebras of $\ell^\infty(X)$

It is common to look at rings and algebras as families of functions over a nonempty set with values in a suitable ring or field. This is especially helpful if one wants to study the ideal structure of a ring or algebra which naturally involves topological notions, mainly compactness.

We summarize some observations about topological algebras in an abstract manner. One motivation comes from [266] which attempts to represent positive linear functionals on a given commutative unital algebra as an integral with respect to a positive measure on the space of characters of the algebra. This is done by realizing the algebra as a subalgebra of continuous functions over the character space.

We always assume that A is an involutive commutative algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a seminorm ρ. We provide a brief overview of the theory of seminormed algebras and their Gelfand spectrum. Then, we assume that A can be embedded into $(\mathbb{F}^X,\rho)$ for a nonemty set X where ρ is a submultiplicative seminorm on a subalgebra of $\mathbb{F}^X$ that contains the image of A. This induces a seminormed structure on A as well. Theorem (5.3.4) gives a necessary and sufficient condition for X to be dense in the Gelfand spectrum of A, that is, when the topology induced by the seminorm is equivalent to the topology induced by the sup-norm defined in (6).

Motivated by [266], where positive linear functionals on an algebra are presented as integrals with respect to constructibly Radon measures, we consider a measurable structure Σ on X and study the spectrum of the algebra of bounded measurable functions on (X,Σ), denoted by $M_b(X,\Sigma)$. We prove that positive measures on X lift to positive measures over the spectrum of $M_b(X,\Sigma)$, but this lifting shifts the support of the original measure out of X modulo at most a countable subset of X (Propositions (5.3.10) and (5.3.11)). At the end we choose Σ to be the Borel algebra of a topology τ on X and observe some connections between τ and the spectrum of $M_b(X,\Sigma)$ (Proposition (5.3.13) and Theorem (5.3.14)).

Let (X,τ) be a topological space. We denote the set of all τ-continuous $\mathbb{F}$-valued functions on X by $C(X,\tau)$ or $C(X)$ if there is no risk of confusion. We use $C_b(X)$ (or $C_b(X,\tau)$) to denote the set of all $f \in C(X)$ which are bounded on X. If (X,τ) is locally compact, $C_0(X)$ denotes the set of all $f \in C_b(X)$ which are vanishing at infinity.

Let $P(X)$ be the power set of X. The σ-algebra of sets induced on X by a set $\Lambda \subseteq P(X)$ is denoted by $\sigma(\Lambda)$. In particular if τ is a topology on X, then $\sigma(\tau)$ is the σ-algebra of all Borel subsets of (X,τ) denoted by B_τ.

The set theory which is applied is ZFC. Throughout all algebras are assumed to be involutive (also called $*$-algebra) and commutative over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$ as specified). Subsequently, all $\mathbb{F}$-valued $*$-algebra homomorphisms are also supposed to be $\mathbb{F}$-module maps.

Definition (5.3.1)[264]: Let A be a commutative $*$-algebra. A function $\rho: A \to [0,\infty]$ is called a quasi-norm on A if

(i) $\forall a \in A \quad \rho(a^*) = \rho(a)$,

(ii) $\forall a,b \in A \quad \rho(a+b) \leq \rho(a) + \rho(b)$ (subadditive),

(iii) $\forall r \in \mathbb{F} \ \ \forall a \in A \quad \rho(ra) = |r|\rho(a)$.

ρ is called submultiplicative if

(iv) $\forall a,b \in A, \quad \rho(ab) \leq \rho(a)\rho(b)$ where the product of ∞ and 0 is ∞.

A quasi-norm ρ on A is called a seminorm if $\rho(a) < \infty$ for every $a \in A$.

Let A be a commutative $*$-algebra and let ρ be a quasi-norm on A. The set of all elements of A with a finite quasi-norm ρ is denoted by $B_\rho(A)$, i.e.,

$$B_\rho(A) = \{a \in A : \rho(a) < \infty\}.$$

If ρ is a submultiplicative quasi-norm, it is clear that $B_\rho(A)$ is a $*$-subalgebra of A and the restriction of ρ to $B_\rho(A)$ is a seminorm. A $*$-algebra A with a seminorm ρ forms a seminormed algebra if ρ is submultiplicative. For a seminormed algebra (A,ρ), the set of all non-zero $*$-algebra homomorphisms $\alpha: A \to \mathbb{F}$ is denoted by $\mathcal{X}(A)$. The set $\mathfrak{sp}_\rho(A)$ of all ρ-continuous $*$-algebra homomorphisms belonging to $\mathcal{X}(A)$ is called the Gelfand spectrum of (A,ρ). Every element $a \in A$ induces a map $\hat{a}: \mathcal{X}(A) \to \mathbb{F}$ defined by $\hat{a}(\alpha) := \alpha(a)$ for each $\alpha \in \mathcal{X}(A)$. Next, we have a characterization of all ρ-continuous $\mathbb{F}$-valued $*$-algebra homomorphisms. The following lemma was proved as Lemma 3.2 in [267].

Lemma (5.3.2)[264]: *Let* (A,ρ) be a commutative seminormed $*$-algebra and $\alpha \in \mathcal{X}(A)$. Then $\alpha \in \mathfrak{sp}_\rho(A)$ if and only if $|\alpha(a)| \leq \rho(a)$, for all $a \in A$.

The Gelfand spectrum $\mathfrak{sp}_\rho(A)$ (as well as $\mathcal{X}(A)$) naturally carries a Hausdorff topology as a subspace of $\mathbb{F}^A$ with the product topology. For a real number $r > 0$, let $Dr := \{c \in \mathbb{F}: |c| \leq r\}$. According to Lemma (5.3.2), $\mathfrak{sp}_\rho(A) \subseteq \prod_{a\in A} D_{\rho(a)}$. One simple approximation argument implies that every element in the closure of $\mathfrak{sp}_\rho(A)$ is a $*$-algebra homomorphism. But it also belongs to $\prod_{a\in A} D_{\rho(a)}$. Therefore, the closure of $\mathfrak{sp}_\rho(A)$ is a subset of $\mathfrak{sp}_\rho(A) \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is the constant linear functional zero on A. From now on, we use $\mathfrak{sp}_\rho(A)$ to denote it as a topological subspace of $\prod_{a\in A} D_{\rho(a)}$. Note that, for each $a \in$

A, $\hat{a}$ is an element in $C(\mathcal{X}(A))$ and subsequently, its restriction to $\mathfrak{sp}_\rho(A)$ belongs to $C(\mathfrak{sp}_\rho(A))$.

Note that the difference between the following corollary and Corollary 3.3 in [267] is due to the fact that we exclude zero in the definition of $\mathcal{X}(A)$.

Corollary (5.3.3)[264]: Let (A, ρ) be a commutative seminormed $*$-algebra. If A is unital then $\mathfrak{sp}_\rho(A)$ is compact. If $\mathfrak{sp}_\rho(A)$ is compact then there exists an element $a_0 \in A$ such that $|\alpha(a_0)| \geq 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$.

Proof. If A is unital, one may use the identity element, $\mathbf{1}$, (for which we have $\alpha(\mathbf{1}) = 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$) to show that $\mathbf{0}$ does not belong to the closure of $\mathfrak{sp}_\rho(A)$. Therefore, $\mathfrak{sp}_\rho(A)$ is indeed a closed set in $\prod_{a\in A} D_{\rho(a)}$, and subsequently, $\mathfrak{sp}_\rho(A)$ is compact.

Now suppose that $\mathfrak{sp}_\rho(A)$ is compact. Therefore, $\mathfrak{sp}_\rho(A)$ is a closed subset of $\prod_{a\in A} D_{\rho(a)}$, not containing $\mathbf{0}$. So, there exist a finite set $\{a_1, \dots, a_m\}$ and $\epsilon > 0$ such that for each α there is an i with $|\alpha(a_i)| \geq \epsilon^{1/2}$. Now set $a := a_1^* a_1, \dots, a_m^* a_m$. Then, this particular element a, satisfies $|\alpha(a)| \geq \epsilon$ for each $\alpha \in \mathfrak{sp}_\rho(A)$. Let $k = \inf\{|\alpha(a)| : \alpha \in \mathfrak{sp}_\rho(A)\} \geq \epsilon$ and $a_0 := a/k$. The claim follows for a_0.

Every $*$-algebra homomorphism $\phi: A \to B$ induces a mapping $\phi_*: \mathcal{X}(B) \to \mathcal{X}(A) \cup \{\mathbf{0}\}$ defined by $\phi_*(\beta) = \beta \circ \phi$ for each $\beta \in \mathcal{X}(B)$. Suppose that B is equipped with a seminorm ρ. The homomorphism ϕ induces a seminorm ρ_ϕ on A defined by $\rho_\phi(a) = \rho(\phi(a))$. If ρ is submultiplicative, then so is ρ_ϕ. The map ϕ as a homomorphism between seminormed $*$-algebras (A, ρ_ϕ) and (B, ρ) is continuous. Therefore ϕ_* maps $\mathfrak{sp}_\rho(B)$ continuously into $\mathfrak{sp}_{\rho_\phi}(A)$.

Here we are mainly interested in the case where B is a $*$-subalgebra of $\mathbb{F}^X$ for a non-empty set X where $\mathbb{F}^X$ is the space of all $\mathbb{F}$-valued functions on X furnished with pointwise multiplication and the canonical $\mathbb{F}$-conjugate involution. This generally enables us to realize $\mathfrak{sp}(A)$ relative to X as follows.

Let ρ be a submultiplicative quasi-norm on $\mathbb{F}^X$ with $\rho(\mathbf{1}) \geq 1$ where $\mathbf{1}$ denotes the constant function which takes 1 all over the X. There is a natural map $e: X \to \mathcal{X}(\mathbb{F}^X)$ which, to every $x \in X$, assigns the evaluation map $e_x: \mathbb{F}^X \to \mathbb{F}$, defined by $e_x(f) := f(x)$. It is clear that $e_x \in \mathcal{X}(\mathbb{F}^X)$. We denote the set of all ρ-continuous evaluations by X_ρ. Note that by Lemma (5.3.2), for every $x \in X$, $e_x \in X_\rho$ if and only if $e_x \in \mathfrak{sp}_\rho(B_\rho(\mathbb{F}^X))$. In symbols:

$$X_\rho = \{e_x : x \in X, e_x \in \mathfrak{sp}_\rho(B\rho(B_\rho(\mathbb{F}^X)))\}. \quad (5)$$

Let $\iota: A \to B_\rho(\mathbb{F}^X)$ be a $*$-algebra homomorphism. We use ι_* to denote the induced map $\iota_*|_X : X \to \mathfrak{sp}_\rho(A)$.

Theorem (5.3.4)[264]: Let A be a commutative $*$-algebra and $\iota: A \to B_\rho(\mathbb{F}^X)$ be a $*$-algebra homomorphism, where ρ is a submultiplicative quasi-norm on $\mathbb{F}^X$ with $\rho(\mathbf{1}) \geq 1$. Define $\rho_\iota := \rho \circ \iota$ on A. Then $\iota_*(X_\rho)$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ if and only if there exists $D > 0$ such that

$$\rho_\iota(a) \leq D \cdot \sup_{x \in X_\rho} |e_x(\iota a)|,$$

for all $a \in A$.

Proof. Note that by Lemma (5.3.2), for each $a \in A$,

$$\sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\hat{a}(\beta)| \leq \rho_\iota(a).$$

($\Rightarrow$) Since $\iota_*(X_{\rho_\iota})$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ we have

$$\sup_{x\in X_\rho} | e_x(\iota a)| = \sup_{\beta\in \mathfrak{sp}_{\rho_\iota}(A)} | \beta(a)|,$$
and it suffices to take $D = 1$.

($\Leftarrow$) In contrary, suppose that $\alpha \in \mathfrak{sp}_{\rho_\iota}(A)\backslash\overline{\iota_*(X_\rho)}$. There exists $f \in C(\mathfrak{sp}_{\rho_\iota}(A))$ such that $f(\alpha) = 1$ and $f|_{\iota_*(X_\rho)} = 0$, by Urysohn's lemma. Since $\hat{A}$ separates points of $\mathfrak{sp}_{\rho_\iota}(A)$, $\hat{a} \in C(\mathfrak{sp}_{\rho_\iota}(A))$, and $\mathfrak{sp}_{\rho_\iota}(A)$ is compact, by Stone-Weierstrass theorem, it is dense in $C(\mathfrak{sp}_{\rho_\iota}(A))$. Therefore, for $\epsilon > 0$, there is $a_\epsilon \in A$ with $\|f - \hat{a}_\epsilon\| < \epsilon$. Take an $\epsilon > 0$ such that $\frac{1-\epsilon}{\epsilon} > D$.
Then $|f(\alpha) - \alpha(a_\epsilon)| = |1 - \alpha(a_\epsilon)| < \epsilon$ or $1 - \epsilon < |\alpha(a_\epsilon)| < 1 + \epsilon$. Also $|f(\iota_* e_x) - e_x(\iota a_\epsilon)| = |0 - \iota a_\epsilon(x)| < \epsilon$ for all $x \in X_\rho$. Now
$$\sup_{\beta\in \mathfrak{sp}_{\rho_\iota}(A)} | \beta(a_\epsilon)| \le \rho_\iota(a_\epsilon) \le D \sup_{x\in X_\rho} | e_x(\iota a)| \le D\epsilon < 1 - \epsilon,$$
and hence $|\alpha(a\epsilon)| < 1 - \epsilon$, a contradiction which completes the proof.

The immediate implication of Theorem (5.3.4) is that if one is to realise a unital commutative algebra as a subalgebra of $(\mathbb{F}^X, \rho)$ the natural choice for ρ is the sup-norm over X which is defined by
$$\|f\|_X = \sup_{x\in X} | f(x)|. \quad (6)$$
We denote $B_{\|\cdot\|_X}(\mathbb{F}^X)$ by $\ell^\infty(X)$. According to Theorem (5.3.4) the image of X under the map $x \mapsto e_x$ is dense in $\mathfrak{sp}_{\|\cdot\|_X}(\ell^\infty(X))$ and also for $\iota: A \to \ell^\infty(X)$, we have $\overline{\iota_*(X_{\|\cdot\|_X})}^{\|\cdot\|_{X_\iota}} = \mathfrak{sp}_{\|\cdot\|_{X_\iota}}(A)$.

It is well known that if (X, τ) is a completely regular Hausdorff space, then $\mathfrak{sp}_{\|\cdot\|_{X_\iota}}(C_b(X))$ is the Stone-Cech compactification of (X, τ). Moreover, every Hausdorff compactification of (X, τ) is homeomorphic to the spectrum of a unital subalgebra of $C_b(X)$. We study the algebra of bounded measurable functions for a measurable structure on X.

Let Σ be a σ-algebra of subsets of X. Let $M_b(X, \Sigma)$ be the $*$-algebra of all bounded Σ-measurable $\mathbb{F}$-valued functions on (X, Σ). Suppose that $M_b(X, \Sigma)$ separates the points of X. Hence, there is an injection from X onto a dense subset of $\mathfrak{sp}_{\|\cdot\|_X}(M_b(X, \Sigma))$. Although we are assuming that $M_b(X, \Sigma)$ separates points of X, this does not imply that Σ contains singletons as we see in the following example.

Example (5.3.5)[264]: Recall that a topological space (X, τ) is called a T_0 space if for each pair x, y of distinct points of X, either $x \notin \overline{\{y\}}^\tau$ or $y \notin \overline{\{x\}}^\tau$. Then characteristic functions of open sets clearly separate points of X. Let $\omega 1$ be the first uncountable ordinal and $X = \omega_1 + 1$. The family of sets $R_a := \{x \in X: x > a\}$ $(a \in X)$ forms a basis for a topology on $Y = X\backslash\{0\}$. This topology is evidently T_0 and satisfies the first axiom of countability at every point except ω_1. Although $\{\omega_1\} = \bigcap_{\omega_1>a} R_a$, every countable intersection of sets R_a for $a < \omega_1$ contains ordinals smaller than ω_1. Thus $\{\omega_1\}$ does not belong to the σ-algebra Σ_r generated by $\{R_a: a \in X\}$, while $M_b(Y, \Sigma_r)$ separates points of Y. Note that the topology of Y in this case is not first countable. Singletons always belong to the σ-algebra of Borel subsets of first countable spaces.

We denote $\mathfrak{sp}_{\|\cdot\|_X}(M_b(X, \Sigma))$ by $\xi_\Sigma X$ which is a compact Hausdorff space. Since $M_b(X, \Sigma)$ separates the points of X, there is an injection $\psi: X \to \xi_\Sigma X$ such that $\psi(X)$ is a dense subset of $\xi_\Sigma X$. Further, for every bounded Σ-measurable function f on X, the function $f \circ \psi^{-1}$ is continuously extendible over $\xi_\Sigma X$. Also, $\xi_\Sigma X$ is unique (up to a homeomorphism) with this property in the sense that for every other compact Hausdorff space, say γX, with

X as a dense subset, so that the elements of $M_b(X,\Sigma)$ are continuously extendible to γX, there is a continuous map $\iota: \gamma X \to \xi_\Sigma X$ agreeing on the images of X in $\xi_\Sigma X$ and γX. For $E \in \Sigma$, let χ_E be the characteristic function of E, defined on X. Denoting its continuous extension to $\xi_\Sigma X$ with $\tilde{\chi}_E$ we have:

$$(\tilde{\chi}_E)^2 = (\chi_E^2)^\sim = \tilde{\chi}_E;$$

thus it ranges over $\{0,1\}$, which implies that $\tilde{\chi}_E$ itself must be the characteristic function of a set, say $\hat{E}$ in $\xi_\Sigma X$.

Lemma (5.3.6)[264]: Let $E \in \Sigma$. Then $\bar{E} = \tilde{E}$ where $\bar{E}$ is the closure of E in $\xi_\Sigma X$.

Proof. It is clear that $\tilde{E} = \tilde{\chi}_E^{-1}(\{1\})$ is closed and $E \subseteq \tilde{E}$. Thus $\bar{E} \subseteq \tilde{E}$. If $z \notin \bar{E}$, then for an open neighbourhood U of z we have $U \cap E = \emptyset$. Therefore there is a function $f \in M_b(X,\Sigma)$ and an open interval I in $\mathbb{R}$ such that $z \in \tilde{f}^{-1}(I) \subseteq U$. Let $F = f^{-1}(I) \in \Sigma$, then $E \cap F = \emptyset$, so $\chi_E \cdot \chi_F = 0$ and $\tilde{\chi}_E \cdot \tilde{\chi}_F = 0$. Since $\tilde{\chi}_F(z) = 1$ the later equation implies $\tilde{\chi}_E(z) = 0$. This contradicts the assumption $z \in \tilde{E}$, therefore $\tilde{E} = \bar{E}$.

Using the above lemma, we investigate some properties of X as a subspace of $\xi_\Sigma X$.

Corollary (5.3.7)[264]: Let Σ be a σ-algebra of subsets of X.

(i) $\bar{E}$ is a clopen subset of $\xi_\Sigma X$ for every $E \in \Sigma$;

(ii) $\tilde{\Sigma} := \{\bar{E}: E \in \Sigma\}$ forms a basis for the topology of $\xi_\Sigma X$;

(iii) $\tilde{\Sigma}$ is the set of all clopen subsets of $\xi_\Sigma X$.

In addition, if Σ contains all singletons, then

(iv) X is an open dense subspace of $\xi_\Sigma X$ whose subspace topology is discrete;

(v) For a subset $Y \subset X, \bar{Y} = Y$ if and only if Y is finite.

(vi) For $x \in X$ and $E \in \Sigma, x \in \tilde{E}$ if and only if $x \in E$.

Proof. (i) Since $\bar{E} = \tilde{E} = \tilde{\chi}_E^{-1}(\{1\}) = \tilde{\chi}_E^{-1}(\frac{1}{2},\infty)$ and $\tilde{\chi}_E$ is continuous, we conclude that $\tilde{E}$ is clopen.

(ii) The family $\{\tilde{E}: E = f^{-1}([0,1])$ for $f \in M_b(X,\Sigma)\}$ forms a basis for the closed subsets of $\xi_\Sigma X$. Note that $E = f^{-1}([0,1]) \in \Sigma$ and $\tilde{E}: E = \tilde{f}^{-1}([0,1])$ which is clopen by (i) and the conclusion follows.

(iii) By (i) and (ii), $\xi_\Sigma X$ is totally disconnected. Suppose that $Y \subseteq \xi_\Sigma X$ is clopen. Since $\xi_\Sigma X$ is compact, so is Y. By (ii), as an open set, $Y = \bigcup_{i\in I} \tilde{E}_i$ for a family $\{E_i\}_{i\in I} \subset \Sigma$. Therefore, $Y = \tilde{E}_{i_1} \cup \cdots \cup \tilde{E}_{i_n}$ for some $i_1,\ldots,i_n \in I$, which belongs to $\tilde{\Sigma}$.

(iv) By (i), the closure of every element of Σ is open in $\xi_\Sigma X$. Since the topology of $\xi_\Sigma X$ is Hausdorff and Σ contains all singletons, singletons are closed. Therefore $\{x\}$ is clopen for every $x \in X$ and hence X is open in $\xi_\Sigma X$. Moreover, by Theorem (5.3.4), X is dense in $\xi_\Sigma X$.

(v) If Y is finite, then since the topology of $\xi_\Sigma X$ is Hausdorff, it is also closed. Let Y be an arbitrary subset of X. The set $Y \subseteq \xi_\Sigma X$ is compact. If $= \bar{Y}$, then $\{\{x\}: x \in Y\}$ is an open cover of Y which will not have a finite subcover, if Y is infinite.

(vi) Clearly if $x \in E$ then $x \in \tilde{E}$. Conversly, suppose that $x \in \tilde{E} \setminus E$. Then $E \subseteq \tilde{E} \setminus \{x\}$. The superset is closed since $\{x\}$ and $\tilde{E}$ are both clopen in $\xi_\Sigma X$ by (v) and (i) respectively. Thus $\bar{E} = \tilde{E} \subseteq \tilde{E} \setminus \{x\}$, a contradiction.

A topological space is called extremely disconnected if the closure of every open set is open. In the following we study this property for $\xi_\Sigma X$. For the relation between Boolean algebras and extremely disconnected spaces see [269] or [270]. Commutative algebras with extremely disconnected Gelfand spectra are forming the commutative class of AW*-algebras where $\mathbb{F} = \mathbb{C}$.

An algebra of sets is said to be *complete* if it is closed under arbitrary union and hence intersection

Proposition (5.3.8)[264]: Let Σ be a σ-algbera on X including all singletones. Then $\xi_\Sigma X$ is extremely disconnected if and only if Σ is complete.

Proof. Suppose that $\xi_\Sigma X$ is extremely disconnected and let $\Delta \subseteq \Sigma$. Then $U = \bigcup_{Y\in\Delta}\tilde{Y}$ is open and hence $\overline{U}$ is also open, thus by Corollary (5.3.7)(iii), it belongs to $\tilde{\Sigma}$, say $U = \tilde{E}$ for some $E \in \Sigma$. We show that $E = \bigcup_{Y\in\Delta} Y$. To do so, first suppose that $\exists x \in (\bigcup_{Y\in\Delta} Y)\setminus E$. Clearly $\in \overline{U} = \tilde{E}$. This violates Corollary (5.3.7)(vi). Conversely, if $\exists x \in E \setminus \bigcup_{Y\in\Delta} Y$, then $\bigcup_{Y\in\Delta} Y \subseteq F$ for $F = E\setminus\{x\} \in \Sigma$. Therefore, $U \subseteq \tilde{F}$. Also, by Corollary (5.3.7)(vi), $\tilde{F} \subset \tilde{E}\setminus\{x\}$. On the other hand, since $\tilde{F}$ is clopen,

$$\tilde{E} = \overline{U} \subseteq \tilde{F} \subseteq \tilde{E}\setminus\{x\} \subsetneq \tilde{E},$$

a contradiction and hence the claim is proved.

Now, suppose that Σ is complete and let U be an open set in $\xi_\Sigma X$. Take $\Delta \subset \Sigma$ such that $U = \bigcup_{F\in\Delta}\tilde{F}$. Since Σ is complete, $E = \bigcup_{F\in\Delta}\tilde{F} \in \Sigma$ and $\overline{U} \subseteq \overline{E} = \tilde{E}$. If $\tilde{E}\setminus\overline{U}$, which is open, is not empty, then it contains a nonempty clopen $Y \in \tilde{\Sigma}$. Now $V = \tilde{E}\setminus Y$ is a clopen set such that

$$E \subseteq U \subseteq V = \overline{V} \subsetneq \tilde{E} = \overline{E}$$

which is a contradiction. Thus $\overline{U} = \tilde{E}$ is clopen and hence $\xi_\Sigma X$ is extremely disconnected. Let $\theta: A \to \ell^\infty(X)$ be an algebra homomorphism and τ be a topology on X. Then one can show that the induced map $\theta_*|_X: (X,\tau) \to \mathfrak{sp}_{\|\cdot\|_{X\theta}}(A)$ is continuous if and only if $\theta A \subseteq C_b(X,\tau)$. The following proposition is an analogue of this result for $M_b(X,\Sigma)$ and Σ-measurability.

Proposition (5.3.9)[264]: Suppose that Σ is a σ-algebra on $X \neq \emptyset$ such that every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. Let $\iota: A \to \ell^\infty(X)$ be an algebra homomorphism. Then the induced map $\iota_*|_X: (X,\Sigma) \to \mathfrak{sp}_{\|\cdot\|_{X\iota}}(A)$ is Σ-measurable if and only if $\iota A \subseteq M_b(X,\Sigma)$.

Proof. By assumption, every Borel subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. A basic open set of $\mathfrak{sp}_{\|\cdot\|_{X\iota}}(A)$ is of the form $\hat{a}^{-1}(O)$ where $O \subseteq \mathbb{F}$ is open and $a \in A$. Looking at the inverse image of $\hat{a}^{-1}(O)$ under ι_*, we have

$$\iota_*|_X^{-1}\hat{a}^{-1}(O) = \iota\hat{a}^{-1}(O) \cap X \tag{7}$$

($\Rightarrow$) Suppose that ι_* is Σ-measurable. If in contrary $\iota a \notin C_b(X,\tau)$ for some $a \in A$, then there exists a set $O \subseteq \mathbb{F}$, such that $\iota\hat{a}^{-1}(O) \cap X$ is not Σ-measurable and hence by (7), $\iota_*|_X$ cannot be Σ-measurable which is a contradiction.

($\Leftarrow$) If each ιa is Σ-measurable, then $\iota\hat{a}^{-1}(O)$ is Σ-measurable for any open $O \subseteq \mathbb{F}$ and again by (7), $\iota_*|_X$ is Σ-measurable.

It is not known to the authors if the assumption "every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$" in Proposition 3.6 is essential or not. One can show that this assumption rules out some examples including $X = \mathbb{N}$ and $\Sigma = P(\mathbb{N})$, the power set of X.

Starting with a measurable structure (X,Σ) such that $M_b(X,\Sigma)$ separates the points of X. We identified X as an open dense subset of a totally disconnected compact space $\xi_\Sigma X$ where every bounded Σ- measurable function on X extends continuously to $\xi_\Sigma X$. This naturally leads one to ask about the relation between measures on (X,Σ) and $\xi_\Sigma X$.

Proposition (5.3.10)[264]: Let μ be a finite positive measure on (X,Σ). Then μ extends to a Borel measure ${}^*\mu$ on $\xi_\Sigma X$, satisfying

$$\forall E \in \Sigma \quad {}^*\mu(\tilde{E}) = \mu(E).$$

Proof. Define a linear functional $L: C(\xi_\Sigma X) \to \mathbb{F}$ by

$$L(f) = \int_X f|X \, d\mu, \qquad \forall f \in C(\xi_\Sigma X).$$

Clearly L is positive and hence by Riesz representation theorem, there exists a Borel measure ${}^*\mu$ on $\xi_\Sigma X$ such that

$$L(f) = \int_{\xi_\Sigma X} f \, d^*\mu, \qquad \forall f \in C(\xi_\Sigma X).$$

Note that for every $E \in \Sigma$, ${}^*\mu(\tilde{E}) = \int \tilde{\chi}_E \, d^*\mu = L(\tilde{\chi}_E) = \int \chi_E \, d\mu = \mu(E)$.

Although the measure ${}^*\mu$ obtained in Proposition (5.3.10) seems to be mainly supported on X, but in fact, the size of $X \cap supp({}^*\mu)$ is rather small as it is pointed out in the following proposition.

Proposition (5.3.11)[264]: Let μ be a finite Borel measure on $\xi_\Sigma X$ and Σ contains all singletons. Then $X \cap supp(\mu)$ is at most countable.

Proof. By definition, a point $x \in \xi_\Sigma X$ belongs to $supp(\mu)$ if and only if for every neighbourhood U of x, $\mu(U) > 0$. Every singleton $\{z\}$ for $z \in X$ is open in$\xi_\Sigma X$, thus for every $z \in X \cap supp(\mu)$, $\mu(\{z\}) > 0$. Since $\mu(\xi_\Sigma X) < \infty$, $X \cap supp(\mu)$ cannot be uncountable.

Corollary (5.3.12)[264]: Let μ be a finite positive measure on (X, Σ) where Σ contains all singletons. If $\mu(\{x\}) = 0$, for some $x \in X$, then $x \notin supp({}^*\mu)$.

Proof. Since $\{x\} \in \Sigma$ and ${}^*\mu(\{x\}) = 0$, $\chi_x \in M_b(X, \Sigma)$ and $\int_X \chi_x \, d\mu = 0$. Thus ${}^*\mu(\{x\}) = \int_{\xi_\Sigma X} \tilde{\chi}_x \, d^*\mu = 0$. But $\{x\}$ is open and hence $x \notin supp({}^*\mu)$.

Let (X, τ) be a T_1 topological space. Since the topology τ is T_1, singletons are Borel and hence $M_b(X, B_\tau)$ separates points of X. Clearly the inclusion $\iota: C_b(X, \tau) \to M_b(X, B_\tau)$ is continuous and hence $\iota_*: \xi_{B_\tau} X \to \mathfrak{sp}_{\|\cdot\|_X}(C_b(X, \tau))$ is onto. Consequently, if τ is completely regular, then βX is a continuous image of $\xi_{B_\tau X}$ where βX is the Stone- Čech compactification of X (look at 6.5 in [249]). If $B_\tau = \tau$ then ξ_{B_τ} and β are identical and ι_* is injective. It is natural to ask if there is any relation between topological structures of (X, τ) and $\xi_{B_\tau} X$.

Let $x \in X$ and $\mathcal{N}_\tau(x)$ be the family of open neighbourhoods of x in τ and $\widetilde{\mathcal{N}}_\tau(x) = \{\widetilde{U}: U \in \mathcal{N}_\tau(x)\}$. Define the halo of x in $\xi_{B_\tau} X$ as

$$h(x) := \bigcap \widetilde{\mathcal{N}}_\tau(x).$$

The set $h(x)$ is compact and contains all points of $\xi_{B_\tau} X$ that cannot be distinguished from x via the image of τ. If τ is Hausdorff, then for each $y \in X$ such that $x \neq y$, there are open sets $U_x, U_y \in \tau$ with $U_x \cap U_y = \emptyset$. Thus $\widetilde{U}_y \cap \widetilde{U}_y = \emptyset$, and therefore $h(x) \cap h(y) = \emptyset$.

Proposition (5.3.13)[264]: If τ is Hausdorff, then $h(x)$ is open if and only if $\{x\}$ is open in (X, τ).

Proof. If $\{x\}$ is open, then $\{x\} \in \mathcal{N}_\tau(x)$. Since $\{\tilde{x}\} = \{x\}$, clearly, $x \in h(x) \subseteq \{x\}$. Conversely, if $h(x)$ is open, then it is clopen and hence, by Corollary (5.3.7)(iii), $h(x) = \tilde{E}$ for some $E \in B_\tau$. If $E \neq \{x\}$, then E contains another point $y \in X$, $y \neq x$. Thus $y \in h(x)$ which implies that $h(x) \cap h(y) \neq \emptyset$, contradicting the above argument before the proposition.

Proposition (5.3.13) can be read as $h(x) = \{x\}$ if and only if $\{x\}$ is open in (X, τ). The following shows how the compactness of a Borel subset of (X, τ) is reflected in $\xi_{B_\tau} X$.

Theorem (5.3.14)[264]: Let $Y \subseteq (X, \tau)$ be a Borel subspace. Then Y is compact if and only if $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$.

Proof. (⇒) Suppose that Y is compact and let $z \in \xi_{B_\tau}X \setminus \bigcup_{y\in Y} h(y)$. We show $z \notin \tilde{Y}$. Since $z \notin \bigcup_{y\in Y} h(y)$, for each $y \in Y$, there exists $O_y \in \mathcal{N}_\tau(y)$ such that $z \notin \tilde{O}_y$. Now $\{O_y: y \in Y\}$ is an open cover of the compact set Y. Let $\{O_{y_1}, \ldots, O_{y_k}\}$ be such that $Y \subseteq \bigcup_{i=1}^{k} O_y$, then $\tilde{Y} \subseteq \bigcup_{i=1}^{k} \tilde{O}_y$ which proves $z \notin \tilde{Y}$, and hence $\tilde{Y} \subseteq \bigcup_{y\in Y} h(y)$.

(⇐) Suppose that $\tilde{Y} \subseteq \bigcup_{y\in Y} h(y)$, but Y is not compact. Then there exists an open cover $\{O_i\}_{i\in I}$ of Y with no finite subcover. So, for every finite subset $\{i_1, \ldots, i_n\}$ of I,

$$Y \cap \left(\bigcap_{k=1}^{n} O_{i_k}^c\right) \neq \emptyset.$$

Since Y is Borel, $\tilde{Y}$ is compact and hence $\{\tilde{O}_i^c\}_{i\in I}$ forms a basis for an ultrafilter $\mathcal{F}$ in $\xi_{B_\tau}X$. Clearly $\tilde{Y} \in \mathcal{F}$ and hence $z = \lim \mathcal{F} \in \tilde{Y}$ (for more detail on filters see [271]). For every $y \in Y$, there exists $i \in I$ such that $O_i \in \mathcal{N}_\tau(y)$ and hence $z \notin O_i$. Thus $z \notin h(y) \subseteq \tilde{O}_i$. This proves

$$z \in \tilde{Y} \setminus \bigcup_{y\in Y} h(y),$$

as desired.

It is worth mentioning that the results resemble significant similarities between properties of $\xi_{B_\tau}X$ and nonstandard extensions of (X, τ). We can consider $\xi_{B_\tau}X$ as a nonstandard model of (X, τ) and characterize halos as analogue to monads and so on. In this scope, Theorem (5.3.14) is the analogue of Robinson’s theorem [268] on nonstandard extensions of compact spaces.

Chapter 6
Complex Symmetric Operators and Complex

We give a general solution to the norm closure problem for complex symmetric operators. As an application, we provide a concrete description of partial isometries which are norm limits of complex symmetric operators. We give concrete characterizations for weighted shifts with nonzero weights to be norm limits of complex symmetric operators. We show a conjecture of Garcia and Poore. On the other hand, it is proved that an essentially normal operator is a norm limit of complex symmetric operators if and only if it is complex symmetric. We obtain a canonical decomposition for essentially normal operators which are complex symmetric. Also it is completely determined when $C^*(T)$ is $*$-isomorphic to a C^*-algebra singly generated by complex symmetric operators. These both depend only on the singular part of T.

Section (6.1): Approximation of Complex Symmetric Operators

We a continuation of a recent by Guo et al. [164], which provides a C^*-algebra approach to complex symmetric operators. We shall develop further some C^*-algebra techniques to solve in a general sense the norm closure problem for complex symmetric operators. Our approach employs some classical results from the representation theory of C^*-algebras.

We let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. We always denote by $\mathcal{B}(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we let $C^*(A)$ denote the C^*-algebra generated by A and the identity operator.

Definition (6.1.1)[272]: A map C on $\mathcal{H}$ is called a conjugation if C is conjugate-linear, $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

Definition (6.1.2)[272]: An operator $T \in \mathcal{B}(\mathcal{H})$ is called a complex symmetric operator (CSO) if there is a conjugation C on $\mathcal{H}$ such that $CTC = T^*$. We denote by $S(\mathcal{H})$ the set of all CSOs on $\mathcal{H}$.

Note that an operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if there exists an orthonormal basis (onb, for short) $\{e_n\}$ of $\mathcal{H}$ such that $\langle T e_i, e_j\rangle = \langle T e_j, e_i\rangle$ for all i, j, that is, T admits a symmetric matrix representation with respect to $\{e_n\}$ (see [16, Lemma 1]). Thus the notion of CSO can be viewed as a generalization of symmetric matrix of Hilbert space.

The general study of CSOs was initiated by Garcia and Putinar [135],[136], and has many motivations in function theory, matrix analysis and other areas. Many significant results concerning CSOs have been obtained (see [131],[138],[158],[276],[139],[171],[172]). It is worth mentioning that CSOs are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason's seminal [145] and has led to rapid progress in function-theoretic operator theory [149],[152],[151],[159], [160],[168],[169]. See [135],[136] for more about CSOs and its connections to other subjects.

We will concentrate on the following norm closure problem.

Problem (6.1.3)[272]: Characterize the norm closure of the set $S(\mathcal{H})$.

There are several motivations for us to study Problem (6.1.3). Firstly, although much attention has been paid to CSOs, the internal structure of CSOs is still not well understood. In particular, Garcia posed many concrete questions concerning CSOs (see [18–20,25]). A basic problem is to give a characterization, in "simple terms", of when an operator is complex symmetric. In a real sense such a characterization is very far from existing even in finite dimensional spaces. So people naturally restrict attention to special classes of operators. In this aspect, partial isometries, weighted shifts and some other operators are studied [131],[138],[171],[172]. Another alternative is to consider the approximation of CSOs, that is, to characterize which operators are the norm limits of CSOs. Maybe the answer is relatively easy to state. This may help to achieve a meaningful classification. In fact, Problem (6.1.3) has inspired many interesting results [158],[276],[164],[147],[171]–[172]. One of the main results gives a classification of CSOs up to approximate unitary equivalence. Recall that two operators $A, B \in \mathcal{B}(\mathcal{H})$ are approximately unitarily equivalent if there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators such that $U_n A - B U_n \to 0$ as $n \to \infty$ (see p. 57 [180]).

The second motivation lies in connections between CSOs and antiautomorphisms of singly generated C^*-algebras. Recall that an anti-automorphism of a C^*-algebra A is a vector space isomorphism $\varphi: \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. Anti-automorphisms play an important role in the study of the real structure of C^*-algebras [176],[277],[191],[194],[195]. It is not necessary that each C^*-algebra possesses an anti-automorphism on it [178],[177]. So a basic problem is to determine when a C^*-algebra possesses an anti-automorphism on it.

Definition (6.1.4)[272]: We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is g-normal if it satisfies

$$\|p(T^*, T)\| = \|\tilde{p}(T, T^*)\|$$

for any polynomial $p(z_1, z_2)$ in two free variables. Here $\tilde{p}(z_1, z_2)$ is obtained from $p(z_1, z_2)$ by conjugating each coefficient.

The notion of g-normal operator was suggested in [157]. It is proved in [164] that (1) an operator $T \in \mathcal{B}(\mathcal{H})$ is g-normal if and only if there exists an anti-automorphism φ of $C^*(T)$ such that $(T) = T$, and (2) each operator in $\overline{S(\mathcal{H})}$ is g-normal. Moreover each C^*-algebra generated by an operator in $\overline{S(\mathcal{H})}$ possesses a real structure. This suggests a C^*-algebra approach to CSOs and its norm closure problem.

Another motivation of our study stems from some recent interest in the study of $S(\mathcal{H})$ itself as a subset of $\mathcal{B}(\mathcal{H})$. In [133], Garcia showed that the set $S(\mathcal{H})$ is invariant under the Aluthge transform, an important transformation which originally arose in the study of hyponormal operators. In [138], Garcia and Wogen showed that $S(\mathcal{H})$ is not closed in the strong operator topology (sot). In [158], Garcia and Poore proved that the sot closure of $S(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$.

The norm closure problem for CSOs was posed and first studied by Garcia and Wogen [138]. In particular, they asked whether or not the set $S(\mathcal{H})$ is norm closed. Zhu et al. [173] answered this question negatively by proving that the Kakutani shift is not complex symmetric but belongs to $\overline{S(\mathcal{H})}$. The proof there depends on a construction of finite-dimensional truncated weighted shifts. Almost immediately, using the unilateral shift and its adjoint, Garcia and Poore [276] constructed another completely different operator in $S(\mathcal{H}) \backslash S(\mathcal{H})$.

Generalizing the Kakutani shift, Garcia and Poore [158] constructed some special weighted shifts, the so-called approximately Kakutani shifts. A unilateral weighted shift $T \in \mathcal{B}(\mathcal{H})$ with positive weights $\{\alpha_k\}_{k=1}^{\infty}$ is said to be approximately Kakutani if for each $n \geq 1$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $0 < \alpha_N < \varepsilon$ and

$$1 \leq k \leq n \;\Rightarrow\; |\alpha_k - \alpha_{N-k}| < \varepsilon.$$

It was proved that approximately Kakutani shifts are norm limits of CSOs ([158], Theorem 10). Moreover they conjectured the converse also holds (see [158], Conjecture 1). Guo et al. provided a C^*-algebra approach to the norm closure problem for CSOs, and gave a positive answer to the conjecture (see [164], Theorem 2.4). In fact, more results were obtained there.

As observed in many significant results in operator theory, there is a subtle interplay between compact perturbation and approximation. In fact, in a large number of interesting cases, the norm closure of a subset E of B($\mathcal{H}$) is contained in the set of all compact perturbations of operators in $\mathcal{E}$. For example, an operator T is a norm limit of triangular operators if and only if T is a compact perturbation of triangular operators if and only if there exist triangular operators $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to T$ and $T_n - T$ is compact for each $n \geq 1$ (see [184], Theorem 6.4). This motivates the following definition.

Definition (6.1.5)[272]: Let $\mathcal{E}$ be a subset of $\mathcal{B}(\mathcal{H})$. The compact closure of $\mathcal{E}$, denoted by $\overline{\mathcal{E}}^c$, is defined to be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ satisfying: for any $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A + K \in \mathcal{E}$.

It is clear that $\mathcal{E} \subset \overline{\mathcal{E}}^c \subset \overline{\mathcal{E}}$ and $\overline{\mathcal{E}}^c \subset [\mathcal{E} + \mathcal{K}(\mathcal{H})]$. Thus $\mathcal{E}^c$ can be viewed as the set of all small compact perturbations of operators in $\mathcal{E}$.

Definition (6.1.6)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a transpose of T, if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

The notion "transpose" for operators is in fact a generalization of that for matrices. In general, an operator has more than one transpose (see [174], Example 2.2). However, as indicated in [164], any two transposes of an operator are unitarily equivalent. We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$ or $\cong_a T^t$. Here and in what follows, the notation $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence.

Guo, Ji and Zhu obtained the following theorem which characterizes irreducible unilateral weighted shifts in $\overline{S(\mathcal{H})}$.

Theorem (6.1.7)[272]: ([164], Theorem 2.4)Let $T \in \mathcal{B}(\mathcal{H})$ be a unilateral weighted shift with positive weights. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;
(ii) $T \in \overline{S(\mathcal{H})}^c$;
(iii) $\exists A \in S(\mathcal{H})$ such that $A \cong_a T$;
(iv) $T \cong_a T^*$;
(v) T is g-normal;
(vi) T is approximately Kakutani.

Furthermore, Guo, Ji and Zhu gave a description of those operators T in $\overline{S(\mathcal{H})}$ satisfying $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Theorem (6.1.8)[272]: ([164], Theorem 2.1) Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$
(ii) $T \in \overline{S(\mathcal{H})}^c$;
(iii) $\exists A \in S(\mathcal{H})$ such that $A \cong_a T$;
(iv) T is g-normal;
(v) $T \cong_a T^t$.

All these results mentioned above suggest that the structure of the set $\overline{S(\mathcal{H})}$ may admit some special form, and it needs and deserves much more study. On the other hand, these results suggest a C^*-algebra approach to CSOs. By virtue of an intensive analysis of compact operators in singly generated C^*-algebras, we employ the representation theory of C^*-algebras to give a complete description of operators in $\overline{S(\mathcal{H})}$.

One of the main results is the following theorem which extends Theorem (6.1.8) and gives a general solution to the norm problem for CSOs.

As an application of Theorem (6.1.35), we shall give a concrete description of partial isometries which are norm limits of CSOs. In [138], Garcia and Wogen proved that a partial isometry T is complex symmetric if and only if the compression of T to its initial space is complex symmetric. We shall prove the following theorem, which can be viewed as an analogue of their result in the setting of approximation.

We shall make some preparation and give some auxiliary results mainly concerning the representations of C^*-algebras and compact operators in singly generated C^*-algebras. The proof of Theorem (6.1.35) shall be provided; also, some corollaries of Theorem (6.1.35) shall be stated. The concluding is devoted to the proof of Theorem (6.1.41).

Given $A \in \mathcal{B}(\mathcal{H})$, we let $\sigma(A)$ and $\sigma_e(A)$ denote the spectrum and the essential spectrum of A respectively. Denote by ker A and ran A the kernel of A and the range of A respectively. As usual, given two representations ρ_1 and ρ_2 of a C^*-algebra, we write $\rho_1 \cong \rho_2 (\rho_1 \cong_a \rho_2)$ to denote that ρ_1 and ρ_2 are unitarily equivalent (approximately unitarily equivalent, respectively).

Definition (6.1.9)[272]: Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. We write $A \trianglelefteq B$ if there is a *-homomorphism ρ of $C^*(B)$ into $C^*(A)$ such that $\rho(B) = A$; if, in addition, ρ annihilates $C^*(B) \cap \mathcal{K}(\mathcal{H}_2)$, then we write $A \lhd B$.

It is easy to see that $A \trianglelefteq B$ if and only if $\|p(A^*, A)\| \leq \|p(B^*, B)\|$ for any polynomial $p(z_1, z_2)$ in two free variables z_1, z_2.

Lemma (6.1.10)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ and $T = A \oplus B$, where $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Assume that $A \lhd B$. Then

$$C^*(T) \cap \mathcal{K}(\mathcal{H}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} : K \in C^*(B) \cap \mathcal{K}(\mathcal{H}_2) \right\};$$

in particular, if $C^*(T)$ contains an operator of the form $Y_1 \oplus Y_2$, where $Y_1 \in \mathcal{B}(\mathcal{H}_1)$ and $Y_2 \in \mathcal{K}(\mathcal{H}_2)$, then $Y_1 = 0$.

Proof. Assume that $A \trianglelefteq B$. Then, by definition, there is a*-homomorphism ρ of $C^*(B)$ into $C^*(A)$ such that $\rho(B) = A$. For a polynomial $p(\cdot,\cdot)$ in two free variables, note that

$$p(T^*, T) = \begin{bmatrix} p(A^*, A) & 0 \\ 0 & p(B^*, B) \end{bmatrix} = \begin{bmatrix} \rho\big(p(B^*, B)\big) & 0 \\ 0 & p(B^*, B) \end{bmatrix}.$$

It follows immediately that

$$C^*(T) = \left\{ \begin{bmatrix} \rho(X) & 0 \\ 0 & X \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} : X \in C^{*(B)} \right\}.$$

In view of this, the result follows readily.

The following lemma is clear.

Lemma (6.1.11)[272]: Let A, B and C be three Hilbert space operators satisfying $C \lhd B$. Then

(i) $A \trianglelefteq B$ if and only if $A \trianglelefteq C \oplus B$,

(ii) $A \lhd B$ if and only if $A \lhd C \oplus B$.

Lemma (6.1.12)[272]: ([153], Corollary 5.41) If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains K($\mathcal{H}$) and ρ is an irreducible representation of A on some Hilbert space $\mathcal{H}_\rho$ such that $\rho|_{K(\mathcal{H})}$ is not zero, then there exists unitary $U: \mathcal{H} \to \mathcal{H}_\rho$ such that $\rho(X) = UXU^*$ for $X \in \mathcal{A}$.

Lemma (6.1.13)[272]: ([180], Corollary II.5.5) Suppose ρ is a non-degenerate representation of a separable C^*-subalgebra $\mathcal{A}$ of B($\mathcal{H}$) into B($\mathcal{H}_\rho$) such that $\rho(\mathcal{A} \cap \mathcal{K}(\mathcal{H})) = \{0\}$. Then $id \ \cong_a \ id \oplus \rho$, where id is the identity representation of $\mathcal{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a nonempty clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma\,(T)\backslash\sigma] \cap \overline{\Omega} \ = \emptyset$. We let $E(\sigma;\, T)$ denote the Riesz idempotent of T corresponding to σ (see page 2 in [184]), that is,

$$E(\sigma;\, T) = \frac{1}{2\pi i}\int_{\Gamma}(\lambda - T)^{-1} d\lambda,$$

Where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. If T is self-adjoint, then it is obvious that $E(\sigma;\, T)$ is an orthogonal projection.

Lemma (6.1.14)[272]: Let $T = A \oplus C \oplus B$, where $A \in \mathcal{B}(\mathcal{H}_1), B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_3)$.

(i) If $A \ntrianglelefteq B$, then $C^*(T)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & Z \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$ and each omitted entry is 0.

(ii)If $A \trianglelefteq B$ but $A \ntriangleleft B$, then $C^*(T)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

here $X \neq 0$ and $K \in \mathcal{K}(\mathcal{H}_2)$.

Proof. (i) If $A \ntrianglelefteq B$, then, by definitions, there exists a polynomial $p(\cdot,\cdot)$ in two free variables such that $\|p(A^*, A)\| > |p(B^*, B)|$. Denote $D = |p(A^*, A)|, E = |p(C^*, C)|$ and $F = |p(B^*, B)|$. Then it follows that

$$|p(T^*, T)| = \begin{bmatrix} D & & \\ & E & \\ & & F \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_2 \end{matrix}$$

is a positive operator in $C^*(T)$ with $\|D\| > \|F\|$. Set $\delta = \frac{\|D\| + \|F\|}{2}$ and define

$$h(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ t - \delta, & t > \delta. \end{cases}$$

Thus h is continuous on $[0, +\infty)$ and one can see that $h(D) \neq 0$ and $h(F) = 0$. Set $X = h(D), Y = h(E)$ and $Z = h(|p(T^*, T)|)$. Then $Z = X \ \oplus Y \ \oplus 0 \in C^*(T)$ and $X = 0$.

(ii) Since $A \trianglelefteq B$, there is a $*$-homomorphism of $C^*(B)$ onto $C^*(A)$ such that $\rho(B) = A$. It is easy to see that $\rho(p(B^*, B)) = p(A^*, A)$ for any polynomial $p(\cdot,\cdot)$ in two free variables. By the hypothesis, there exists $D \in C^*(B) \cap \mathcal{K}(\mathcal{H}_2)$ such that $\rho(D) \neq 0$. So $D \neq 0$. Obviously, we can choose polynomials $\{p_n(\cdot,\cdot)\}$ in two free variables such that $p_n(B^*, B) \to D$. Thus

$$p_n(A^*, A) = \rho(p_n(B^*, B)) \to \rho(D).$$

Hence $|p_n(B^*, B)| \to |D|$ and $|p_n(A^*, A)| \to \rho(|D|)$. Note that $|\rho(D)| = \rho(|D|)$.

Since $|D|$ is compact, there exists $\delta < \frac{\|\rho(|D|)\|}{2}$ such that $\delta \notin \sigma\,(|D|)$. Noting that $\|\rho(|D|)\| \le \|D\|$, we have $\sigma(|D|) = \sigma_1 \cup \sigma_2$, where $\sigma_1 \subset (-1, \delta)$ and $\emptyset \neq \sigma_2 \subset (\delta, \|D\| + 1)$. Moreover, the Riesz idempotent of $|D|$ corresponding to σ_2, denoted by $E(\sigma_2;\ |D|)$, is of finite rank and $E(\sigma_2;\ |D|) \neq 0$.

By the upper semi-continuity of spectra in approximation, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\sigma(|p_n(B^*,B)|) = \sigma_1' \cup \sigma_2'$ with $\sigma_1' \subset (-1,\delta)$ and $\sigma_2' \subset (\delta, \|D\|+1)$; moreover, rank $E(\sigma_2';\ |p_n(B^*,B)|) = rank\ E(\sigma_2;\ |D|) < \infty$ (see [184], Corollary 1.6). Also it can be required that $\|p_n(A^*,A)\| > \delta$ for any $n > N$.

Define

$$h(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ t-\delta, & t > \delta. \end{cases}$$

Then h is nonnegative and continuous on $[0,+\infty)$. Now fix an $n > N$. Set $X = h(|p_n(A^*,A)|), Y = h(|p_n(C^*,C)|)$ and $K = h(|p_n(B^*,B)|)$. It is evident that $X \neq 0$, $K \in \mathcal{K}(\mathcal{H}_2)$ and $X \oplus Y \oplus K \in C^*(T)$. This completes the proof.

For convenience, we write $0_{\mathcal{H}}$ to denote the subalgebra $\{0\}$ of $\mathcal{B}(\mathcal{H})$.

Lemma (6.1.15)[272]: Let $T = A \oplus C \oplus B$, where $A \in \mathcal{B}(\mathcal{H}_1), B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_3)$. If $\mathcal{K}(\mathcal{H}_1) \oplus 0_{\mathcal{H}_3} \subset C^*(A \oplus C)$ and $C^*(T)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$, then $\mathcal{B}(\mathcal{H}_1) \oplus 0_{\mathcal{H}_3} \oplus\ 0_{\mathcal{H}_2} \subset C^*(T)$.

Proof. Arbitrarily choose two unit vectors $e, f \in \mathcal{H}_1$. It suffices to prove that $C^*(T)$ contains the operator

$$\begin{bmatrix} f \otimes e & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3. \\ \mathcal{H}_2 \end{matrix}$$

For convenience we denote $D = A \oplus C$. Since $X \neq 0$, there exist nonzero vectors $e_0, f_0 \in \mathcal{H}_1$ such that $Xe_0 = f_0$. On the other hand, noting that $\mathcal{K}(\mathcal{H}_1) \oplus 0_{\mathcal{H}_3} \subset C^*(D)$, we can choose polynomials $\{p_n(\cdot,\cdot)\}$ and $\{q_n(\cdot,\cdot)\}$ in two free variables such that

$$p_n(D^*,D) \to \begin{bmatrix} f \otimes \frac{f_0}{\|f_0\|^2} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \end{matrix}$$

and

$$q_n(D^*,D) \to \begin{bmatrix} e_0 \otimes e & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \end{matrix}.$$

It follows that

$$p_n(T^*,T) Z q_n(T^*,T) \to \begin{bmatrix} f \otimes e & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3,, \\ \mathcal{H}_2 \end{matrix}$$

which completes the proof.

Lemma (6.1.16)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. Assume that $= \oplus_{i=1}^n T_i$, where $T_i \in \mathcal{B}(\mathcal{H}_i)$ and $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ for each $1 \le i \le n$ and $T_1 \ntrianglelefteq T_j$ whenever $j \neq 1$. If there exists $K = \oplus_{i=1}^n K_i \in C^*(T) \cap \mathcal{K}(\mathcal{H})$ with $K_1 \neq 0$, then

$$\mathrm{K}(\mathcal{H}_1)\ \oplus 0_{\mathcal{H}_2} \oplus \cdots \oplus 0_{\mathcal{H}_n} \subset C^*(T).$$

Proof. We shall proceed by induction on n. When $n = 1$, the result is clear.

Now assume that the result is true when $n \le k$. We shall prove that the result holds when $n = k+1$.

Since $T_1 \ntrianglelefteq T_n$, by Lemma (6.1.14)(i), $C^*(T)$ contains an operator $X = (\bigoplus_{i=1}^{n-1} X_i) \oplus 0$, where $X_i \in \mathcal{B}(\mathcal{H}_i)$ for $1 \le i \le n-1$ and $X_1 \ne 0$. Since $\|X_1\| \cdot \|K_1\| \ne 0$, there exist nonzero vectors $e_i, f_i (i = 1, 2)$ such that $X_1 e_1 = e_2$ and $K_1\, f_1 = f_2$. Noting that $f_1 \otimes e_2 \in C^*(T_1) \cap \mathcal{K}(\mathcal{H}_1)$, there exists a sequence $\{p_n(\cdot,\cdot)\}_{n=1}^{\infty}$ of polynomials in two free variables such that $p_n(T_1^*, T_1) \to f_1 \otimes e_2$; hence we have $K_1 p_n(T_1^*, T_1) X_1 \to K_1(\, f_1 \otimes e_2) X_1 \ne 0$. Then some $n_0 \in \mathbb{N}$ exists such that $K_1\, p_{n_0}(T_1^*, T_1) X_1 \ne 0$. Set $C_i = K_i p_{n_0}(T_i^*, T_i) X_i$ for each $1 \le i \le n-1$. Then $C := K\, p_{n_0}\,(T^*, T) X = \left(\bigoplus_{i=1}^{n-1} C_i\right) \oplus 0$ is a compact operator in $C^*(T)$ with $C_1 \ne 0$. In particular, $\bigoplus_{i=1}^{n-1} C_i$ is a compact operator in $C^*\left(\bigoplus_{i=1}^{n-1} T_i\right)$ with $C_1 \ne 0$. By the induction hypothesis, we have

$$\mathcal{K}(\mathcal{H}_1) \oplus\ 0_{\mathcal{H}_2} \oplus \cdots \oplus\ 0_{\mathcal{H}_{n-1}} \subset C^*\left(\bigoplus_{i=1}^{n-1} T_i\right).$$

Since we have proved that $C^*(T)$ contains $C_1 \oplus \left(\bigoplus_{i=2}^{n-1} C_i\right) \oplus 0$ with $C_1 \ne 0$, the desired result follows immediately from Lemma (6.1.15).

Given a set Γ, we write card for the cardinality of Γ. For $T \in \mathcal{B}(\mathcal{H})$ and a cardinal n with $1 \le n \le \aleph_0$, we let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$ and let $T^{(n)}$ denote the direct sum of n copies of T , acting on $\mathcal{H}^{(n)}$ (see [179] , Definition 6.3). For convenience, $\mathcal{H}^{(\aleph_0)}$ and $T^{(\aleph_0)}$ are denoted by $\mathcal{H}^{(\infty)}$ and $T^{(\infty)}$.

Lemma (6.1.17)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ and $T = \bigoplus_{i\in\Lambda} T_i^{(n_i)}$, where $T_i \in \mathcal{B}(\mathcal{H}_i)$ for each $i \in \Lambda$. If there exists nonzero $K \in C^*(T) \cap \mathcal{K}(\mathcal{H})$ with $K = \bigoplus_{i\in\Lambda} K_i^{(n_i)}$, then $C^*(T)$ contains nonzero $C \in C^*(T) \cap \mathcal{K}(\mathcal{H})$ with the form $C = \bigoplus_{i\in\Lambda} C_i^{(n_i)}$ satisfying card $\{i \in \Lambda: C_i \ne 0\} < \infty$.

Proof. Without loss of generality, we may directly assume that K is positive. Set $\delta = \frac{\|K\|}{2}$ and define

$$h(t) = \begin{cases} 0, & 0 \le t < \delta, \\ t - \delta, & \delta \le t \le \|K\|. \end{cases}$$

Then h is a nonnegative, continuous function on $[0, \|K\|]$. Set $C = h(K)$ and $C_i = h(K_i)$ for each $i \in \Lambda$. Then $= \bigoplus_{i\in\Lambda} C_i^{(n_i)}$. It remains to show that C satisfies all requirements.

Noting that K is compact and $h(0) = 0$, we have $C \in C^*(T) \cap \mathcal{K}(\mathcal{H})$. Since K is compact and $K \ne 0$, it immediately follows that $0 < card\{i \in \Lambda: K_i > \delta\} < \infty$. For each i, note that $C_i = h(K_i) \ne 0$ if and only if $\|K_i\| > \delta$. Thus one can deduce that $\{i \in \Lambda: C_i \ne 0\}$ is finite. This completes the proof.

Corollary (6.1.18)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ with $= \bigoplus_{i\in\Lambda} T_i$, where $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for each $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \ne j$. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \ne \{0\}$, then there exists $i_0 \in \Lambda$ such that

$$\mathcal{K}(\mathcal{H}_{i_0}) \oplus 0_{H_{i_0}^{\perp}} \subset C^*(T).$$

Proof. By the hypothesis, we can choose a nonzero $K \in C^*(T) \cap \mathcal{K}(\mathcal{H})$. Since $T = \bigoplus_{i\in\Lambda} T_i$, K can be written as $= \bigoplus_{i\in\Lambda} K_i$, where $K_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Lambda$. By Lemma (6.1.17), we may assume that $\Lambda_0 := \{i \in :\ K_i \ne 0\}$ is a finite set. Now fix an $i_0 \in 0$. Set $A = T_{i_0}, C = \bigoplus_{i\in\Lambda} 0\backslash\{i_0\} T_i$ and $= \bigoplus_{i\in\Lambda\backslash \Lambda_0} T_i$. Then $T = A \oplus C \oplus B$. Denote $\mathcal{H}_A = \mathcal{H}_{i_0}, \mathcal{H}_C = \bigoplus_{i\in\Lambda_0\backslash\{i_0\}} \mathcal{H}_i$ and $\mathcal{H}_B = \bigoplus_{i\in\backslash\Lambda_0} \mathcal{H}_i$.

Claim: $T_{i_0} \ntrianglelefteq T_i$ for any $i \in \Lambda_0$ with $i \ne i_0$.

In fact, if not, then there exists $j \in \Lambda_0$ with $j \ne \Lambda_{i_0}$ such that $T_{i_0} \ntrianglelefteq T_j$. So there exists a *-homomorphism ρ of $C^*(T_j)$ onto $C^*(T_{i_0})$ such that $(T_j) = T_{i_0}$. Then ρ is an irreducible representation of $C^*(T_j)$. Noting that $K_j \in C^*(T_j)$ and T_j is irreducible, we have $\mathcal{K}(\mathcal{H}_j) \subset C^*(T_j)$. On the other hand, since $K_{i_0} \oplus K_j \in C^*(T_{i_0} \oplus T_j)$, one can see that $K_{i_0} = \rho(K_j)$. It follows that $\mathcal{K}(\mathcal{H}_j) \not\subseteq ker\ \rho$. Then, by Lemma (6.1.12), ρ is unitarily implemented, which implies that $T_{i_0} \cong T_j$, a contradiction. This proves the claim.

Set $S = \oplus_{i\in \Lambda_0} T_i$. Then $S = A \oplus C$ and, by the hypothesis, $\oplus_{i\in\Lambda_0} K_i$ is a compact operator in $C^*(S)$ with $K_1 \neq 0$. Since $T_{i_0} \not\trianglelefteq T_i$ for any $i \in \Lambda_0$ with $i \neq i_0$, it follows from Lemma (6.1.16) that

$$\mathcal{K}(\mathcal{H}_{i_0}) \oplus 0_{\mathcal{H}_C} \subset C^*(S).$$

Note that K is an operator in $C^*(T)$ which can be written as $K = X \oplus Y \oplus 0$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_C \oplus \mathcal{H}_B$, where $X \neq 0$. Hence the desired result follows immediately from Lemma (6.1.15).

Corollary (6.1.19)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ with $T = \oplus_{i\in\Lambda} T_i^{(n_i)}$, where $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for each $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then there exists $i \in \Lambda$ such that

$$\left\{\begin{bmatrix} X^{(n_i)} & \\ & 0\end{bmatrix}\begin{matrix}\mathcal{H}^{(n_i)} \\ \mathcal{H} \ominus \mathcal{H}_i^{(n_i)}\end{matrix} : X \in \mathcal{K}(\mathcal{H}_i)\right\} \subset C^*(T) \cap \mathcal{K}(\mathcal{H}).$$

Lemma (6.1.20)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ with $T = A \oplus B$, where $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Assume that $\{C_n\}_{n=1}^{\infty}$ is a sequence of conjugations on $\mathcal{H}$ such that $C_nTC_n \to T^*$. If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$ and $(I-P)C_nPx \to 0$ for any $x \in \mathcal{H}$, then A is g-normal.

Proof. For each $n \geq 1$, we may assume that

$$C_n = \begin{bmatrix} C_{1,1}^n & C_{1,2}^n \\ C_{2,1}^n & C_{2,2}^n\end{bmatrix}\begin{matrix}\mathcal{H}_1 \\ \mathcal{H}_2\end{matrix}.$$

By the hypothesis, we have $C_{2,1}^n x \to 0$ for any $x \in \mathcal{H}_1$.

Now fix a polynomial $p(\cdot,\cdot)$ in two free variables. Assume that

$$p(T^*,T) = \begin{bmatrix} X & 0 \\ 0 & Y\end{bmatrix}\begin{matrix}\mathcal{H}_1 \\ \mathcal{H}_2\end{matrix} \quad \text{and} \quad \tilde{p}(T,T^*) = \begin{bmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y}\end{bmatrix}\begin{matrix}\mathcal{H}_1 \\ \mathcal{H}_2\end{matrix}.$$

Thus $X = p(A^*,A)$ and $\tilde{X} = \tilde{p}(A,A^*)$. For each n, a matrix multiplication shows that

$$C_np(T^*,T)C_n = \begin{bmatrix} C_{1,1}^nXC_{1,1}^n + C_{1,2}^nYC_{2,1}^n & * \\ * & *\end{bmatrix}\begin{matrix}\mathcal{H}_1 \\ \mathcal{H}_2\end{matrix}.$$

Since $C_np(T^*,T)C_n \to \tilde{p}(T,T^*)$, it follows that

$$C_{1,1}^nXC_{1,1}^n + C_{1,2}^nY\,C_{2,1}^n \to \tilde{X} \quad as\ \ n \to \infty.$$

For $x \in \mathcal{H}_1$, noting that $\|C_{1,2}^nYC_{2,1}^nx\| \leq \|Y\,C_{2,1}^nx\| \to 0$, we have $C_{1,1}^nXC_{1,1}^nx \to \tilde{X}x$. Thus

$$\|\tilde{X}x\| = \lim_n \|C_{1,1}^nXC_{1,1}^nx\| \leq \limsup_n \|XC_{1,1}^nx\| \leq \|X\| \cdot \|x\|.$$

Thus we deduce that $\|\tilde{X}\| \leq \|X\|$, that is, $\|\tilde{p}(A,A^*)\| \leq \|p(A^*,A)\|$. By symmetry, we obtain $\|\tilde{p}(A,A^*)\| = \|p(A^*,A)\|$, which implies that A is g-normal.

Lemma (6.1.21)[272]: ([154], Proposition 2.4) If $T \in \mathcal{B}(\mathcal{H})$, then T admits the decomposition $= T_0 \oplus (\oplus_{i\in\Gamma} T_i)$, where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible and $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for all $i \in \Gamma$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called completely reducible if T has no nonzero minimal reducing subspace. Following Arveson [147], we let $\sum_{i\in\Lambda} \mathcal{A}_i$ denote the direct sum of a family $\{\mathcal{A}_i\}_{(i\in\Lambda)}$ of C^*-algebras. Given a C^*-algebra $\mathcal{A}$ of operators and n with $1 \le n \le \aleph_0$, we denote by $\mathcal{A}^{(n)}$ the C^*-algebra $\{A^{(n)}: A \in \mathcal{A}\}$.

Lemma (6.1.22)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then T is unitarily equivalent to an operator $A = T_0 \oplus \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right)$, where $T_0 \in \mathcal{B}(\mathcal{H}_0)$, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ for $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$; moreover, $C^*(A) \cap \mathcal{K}(\mathcal{H}) = 0_{\mathcal{H}_0} \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$, where $\widehat{H} = \mathcal{H}_0 \oplus \left(\oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}\right)$.

Proof. The proof is omitted since it is a minor modification of [164].

The following result is a consequence of Voiculescu's theorem [196]. See [179] for a proof.

Lemma (6.1.23)[272]: Each operator in $\mathcal{B}(\mathcal{H})$ is approximately unitarily equivalent to a direct sum of irreducible operators.

Corollary (6.1.24)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ and $T \in \overline{S(\mathcal{H})}$, then T is approximately unitarily equivalent to a direct sum of operators of the form $\oplus A^t$, where A is irreducible.

Proof. Since $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, it follows from Proposition 42.9 in [179] that $T \cong_a T \oplus T$. By Theorem2, $T \in \overline{S(\mathcal{H})}$ implies that $\cong_a T^t$. Hence we obtain $T \cong_a T \oplus T^t$.

On the other hand, by Lemma (6.1.23), there exists a family $\{A_i\}_{i\in\Lambda}$ of irreducible operators such that $\cong_a \oplus_{\in\Lambda} A_i$. Therefore we obtain

$$T \cong_a \oplus_{i\in\Lambda} (A_i \oplus A_i^t),$$

which completes the proof.

We first give some auxiliary results.

Lemma (6.1.25)[272]: Let $P \in \mathcal{B}(\mathcal{H})$ be a finite-rank projection and $\{C_n\}_{n=1}^{\infty}$ be a sequence of conjugations on $\mathcal{H}$ so that $\{C_n P C_n\}_{n=1}^{\infty}$ converges to an operator $Q \in \mathcal{B}(\mathcal{H})$. Then Q is a projection on $\mathcal{H}$ with rank $P = rank\ Q$ and there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that for each $x \in ran\ P$ the sequence $\{C_{n_j} x\}_{j=1}^{\infty}$ converges to a vector in ran Q.

Proof. Since P is a finite-rank projection, we may assume that rank $P = m$ and $= \sum_{i=1}^{m} e_i \otimes e_i$, where $\{e_i\}_{i=1}^{m}$ is an orthonormal subset of $\mathcal{H}$. First, it is evident that Q is a projection. By the hypothesis, $\lim_n C_n P C_n = Q$; by the lower semi-continuity of the rank in approximation (see [184], Proposition 1.12), it follows that

$$rank\ P = \liminf_n rank\ C_n P C_n \ge rank\ Q.$$

On the other hand, since $C_n^2 = I$ for any n, we have $\lim_n C_n Q C_n = P$; by the lower semi-continuity of the rank in approximation again, we have rank $Q \ge rank\ P$. So we obtain rank $Q = rank\ P$.

Now fix a k with $1 \le k \le m$. By the hypothesis,

$$\|(I - Q) C_n e_k\| = \|C_n P_{e_k} - Q C_n e_k\| \to 0 \quad as \quad n \to \infty. \qquad (1)$$

It follows that $\|Q C_n e_k\| \to 1$. Since $dim\ ran\ Q < \infty$ and $\sup_n \|Q C_n e_k\| \le 1$, there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\left\{Q C_{n_j} e_k\right\}_{j=1}^{\infty}$ converges to a unit vector x_k in ran Q. In view of (1), it follows that $C_{n_j} P e_k = C_{n_j} e_k \to x_k$ as $j \to \infty$.

In view of the above discussion, applying the diagonal process, we can find a subsequence, denoted by $\{n_j\}_{j=1}^{\infty}$ again, such that for each $1 \le k \le m$ we have $C_{n_j} e_k \to x_k$ as $j \to \infty$. Since $\{e_k\}_{k=1}^{m}$ is an onb of $ran\ P$, it can be seen that $\{C_j^n x\}_{j=1}^{\infty}$ converges to a vector in ran Q for each $x \in ran\ P$.

Definition (6.1.26)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. Denote by $\mathcal{H}_e$ the closed linear span of the following set

$$\{K\,x : K \in C^*(T) \cap \mathcal{K}(\mathcal{H})\ and\ x \in \mathcal{H}\}$$

and set $\mathcal{H}_r = \mathcal{H} \ominus \mathcal{H}_e$. It is easy to see that $\mathcal{H}_e$ and $\mathcal{H}_r$ both reduce T. Denote $T_e = T|_{\mathcal{H}_e}$ and $T_r = T|_{\mathcal{H}_r}$.

The proof of the following lemma follows readily from Lemma (6.1.22) and is omitted.

Lemma (6.1.27)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. Then $C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e)$ is non-degenerate and

$$C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{\begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{matrix} \mathcal{H}_r \\ \mathcal{H}_e \end{matrix} : K \in C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e).$$

Theorem (6.1.28)[272]: If $T \in \overline{S(\mathcal{H})}$, then T_e is a CSO.

Proof. By Lemma (6.1.22), we may directly assume that

$$T = T_0 \oplus \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right),$$

where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ and each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Lambda$. Moreover, we assume that

$$C^*(T) \cap \mathcal{K}(\mathcal{H}) = 0_{\mathcal{H}_0} \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}. \qquad (2)$$

It is obvious that $n_i < \infty$ for all $i \in \Lambda$. Note that $T_e = \oplus_{i\in\Lambda} T_i^{(n_i)}$ and $\mathcal{H}_e = \oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$. Since $T \in \overline{S(\mathcal{H})}$, it follows from Lemma 6 in [158] that there is a sequence $\{C_n\}_{n=1}^{\infty}$ of conjugations on $\mathcal{H}$ such that $C_n T C_n \to T^*$. Then it is easy to check that $C_n p(T^*, T) C_n \to \tilde{p}(T, T^*)$ for each polynomial $p(\cdot,\cdot)$ in two free variables. So $\lim_n C_n\, X C_n$ exists for each $X \in C^*(T)$. Define $\varphi(X) = \lim_n C_n X^* C_n$ for $X \in C^*(T)$. Then φ is an anti-automorphism of $C^*(T)$ and $\varphi^{-1} = \varphi$.

In view of (2), we can choose a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank projections in $C^*(T) \cap \mathcal{K}(\mathcal{H})$ with $P_m\, P_l = 0$ whenever $m \ne l$ such that $\oplus_{n=1}^{\infty} ran\ P_n = \mathcal{H}_e$. For each k, denote $Q_k = \varphi(P_k)$, that is, $Q_k = \lim_n C_n P_k C_n$. By Lemma (6.1.25), each Q_k is a projection in $C^*(T)$ with rank $Q_k = rank\ P_k$, and there is a subsequence $\{n_j(k)\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that for each $x \in ran\ P_k$ the sequence $\left\{C_{n_{j(k)}} x\right\}_{j=1}^{\infty}$ converges to a vector in ran Q_k. Applying the diagonal process, we can choose a subsequence $\{n_j\}$ of $\mathbb{N}$ such that for each $x \in \cup_{k=1}^{\infty} ran\ P_k$ the sequence $\left\{C_{n_j} x\right\}_{j=1}^{\infty}$ converges to a vector in $\cup_{k=1}^{\infty} ran\ Q_k$.

Noting that $Q_k \in C^*(T) \cap \mathcal{K}(\mathcal{H})$ and ran $Q_k \subset \mathcal{H}_e$, we have found a subsequence $\{n_j\}$ of $\mathbb{N}$ such that $\lim_j C_{n_j}\, x \in \mathcal{H}_e$ for each $\in \cup_{k=1}^{\infty} ran\ P_k$. Since each C_{n_j} is isometric, one can easily see that $\lim_j C_{n_j}\, x$ exists for each $x \in \oplus_{k=1}^{\infty} ran\ P_k = \mathcal{H}_e$ and $\lim_j C_{n_j}\, x \in \mathcal{H}_e$.

For $x \in \mathcal{H}_e$, define $Ex = \lim_j C_{n_j} x$. Then, by the discussion above, the map $E: \mathcal{H}_e \to \mathcal{H}_e$ is well defined. Since each C_{n_j} is a conjugation, it is obvious that E is isometric and conjugate-linear. We claim that E is indeed a conjugation on $\mathcal{H}_e$. For fixed $x \in \mathcal{H}_e$, it suffices to check that $E^2x = x$. In fact,

$$\|E^2x - x\| = \lim_j \left\|C_{n_j}Ex - x\right\| = \lim_j \left\|C_{n_j}\left(Ex - C_{n_j}x\right)\right\| = \lim_j \left\|Ex - C_{n_j}x\right\| = 0.$$

Thus E is a conjugation.

Now it remains to check that $E\,T_e = T_e^*E$. Given a vector $y \in \mathcal{H}_e$, we have

$$ET_ey = \lim_j C_{n_j}T_ey = \lim_j C_{n_j}Ty = \lim_j T^*C_{n_j}y = T^*\left(\lim_j C_{n_j}y\right) = T^*Ey = T_e^*Ey.$$

This shows that $E\,T_e = T_e^*E$. Therefore T_e is a CSO.

Corollary (6.1.29)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. If $T = T_e$, then $T \in \overline{S(\mathcal{H})}$ if and only if $T \in S(\mathcal{H})$.

Now we can give a short proof of the following result which was first proved in Theorem 2.8 [164].

Corollary (6.1.30)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then $T \in \overline{S(\mathcal{H})}$ if and only if $T \in S(\mathcal{H})$.

Proof. It suffices to prove the necessity. Note that $T = T_r \oplus T_e$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_e$. Since $T \in \overline{S(\mathcal{H})}$, it follows from Theorem (6.1.28) that T_e is a CSO. Note that $T^*T - T\,T^* = (T_r^*T_r - T_rT_r^*) \oplus (T_e^*T_e - T_eT_e^*)$ is compact. Thus $ran(T^*T - TT^*) \subset \mathcal{H}_e$, which implies that $T_r^*T_r - T_rT_r^* = 0$. Then T_r is normal; furthermore, $T = T_r \oplus T_e$ is a CSO, see Theorem 4 in [158]

Corollary (6.1.31)[272]: Let $R_{T,n}$ be a Foguel operator of order n, where $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. Then $R_{T,n}$ is a norm limit of CSOs if and only if $R_{T,n}$ is a CSO.

Proof. By Corollary (6.1.29), it suffices to prove that $\left(R_{T,n}\right)_e = R_{T,n}$, that is, $\mathcal{H}^{(2)} = \left(\mathcal{H}^{(2)}\right)_e$.

For convenience, we write

$$R_{T,n} = \begin{bmatrix} (S^*)^n & T \\ 0 & S^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Fix an $m \in \mathbb{N}$. Denote $A = R_{T,n}^m$. Since $R_{T,n}$ is a Fredholm operator, it follows that A and A^*A are both Fredholm and $ker\ A = ker\ A^*A$. Since $dim\ ker\ A^*A < \infty$, one can see that $P_{\ker A} = P\_(\ker A^*A \in C^*(A^*A) \subset C^*\left(R_{T,n}\right)$. Then $P_{\ker A} \in \left[C^*\left(R_{T,n}\right) \cap \mathcal{K}\left(\mathcal{H}^{(2)}\right)\right]$. Noting that $\vee_{m\geq 1} \ker R_{T,n}^m = \mathcal{H}_1$, we obtain $\mathcal{H}_1 \subset \left(\mathcal{H}^{(2)}\right)_e$.

Applying the above argument to $R_{T,n}^*$, one can prove that $\mathcal{H}_2 \subset \mathcal{H}_e^{(2)}$. Thus $\mathcal{H}^{(2)} = \left(\mathcal{H}^{(2)}\right)_e$. This completes the proof.

Lemma (6.1.32)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. If T can be written as a direct sum of some irreducible operators, then there exists no nonzero reducing subspace M of T such that $T|_M$ is completely reducible.

Proof. Since T can be written as a direct sum of some irreducible operators, it follows from Theorem 3.1 in [275] that the commutant algebra $\{T, T^*\}'$ is *-isomorphic to $\sum_{i\in\Lambda} M_{n_i}(\mathbb{C})$, where $1 \leq n_i \leq \infty$ for each $i \in \Lambda$. Thus each nonzero projection $P \in \{T, T^*\}'$ admits a nonzero minimal subprojection. Then each nonzero reducing reducing subspace M of T contains a nonzero minimal reducing subspace of T . This completes the proof.

Proposition (6.1.33)[272]: If $T \in S(\mathcal{H})$ and $T = T_e$, then T can be written as a direct sum of irreducible CSOs and operators of the form $A \oplus A^t$, where A is irreducible and not a CSO.

Proof. By Theorem 1.6 in [147], T can be written as a direct sum of completely reducible CSOs, irreducible CSOs and operators of the form $\oplus A^t$, where A is irreducible and not a CSO. In view of Lemma (6.1.32), it suffices to prove that T is a direct sum of irreducible operators. Since $T = T_e$, by Lemma (6.1.22), T can be written as a direct sum of irreducible operators. This completes the proof.

Lemma (6.1.34)[272]: (see page 793 [274]) If $T \in \mathcal{B}(\mathcal{H})$, then $T \oplus T^t$ is complex symmetric.

Theorem (6.1.35)[272]: Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;

(ii) $T \in \overline{S(\mathcal{H})^c}$;

(iii) $\exists R \in S(\mathcal{H})$ such that $T \cong_a R$;

(iv) T is approximately unitarily equivalent to an operator which can be written as a direct sum of irreducible CSOs and operators of the form $\oplus A^t$, where A is irreducible and not a CSO.

Proof. The implication "(iv)⇒(iii)" follows from Lemma (6.1.34). By definitions, the implications "(iii)⇒(ii) ⇒(i)" are obvious. It suffices to prove "(i) ⇒(iv)".

"(i)⇒(iv)" $T \in \overline{S(\mathcal{H})}$ implies that any operator approximately unitarily equivalent to T lies in $\overline{S(\mathcal{H})}$. Thus, in view of Lemma (6.1.23), we may directly assume that T is a direct sum of irreducible operators.

Note that $T = T_r \oplus T_e$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_e$. If T_e or T_r is absent, then, by Corollary (6.1.24) and Proposition (6.1.33), the result is clear. So we may assume that neither T_e nor T_r is absent. By Theorem (6.1.28), T_e is a CSO.

By Lemma (6.1.21), we may also assume that $T_r = T_0 \oplus \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right)$, where T_0 is completely reducible, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$. Since T is a direct sum of irreducible operators, it follows from Lemma (6.1.32) that T_0 is absent. Thus $\mathcal{H}_r = \oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$. Hence we have

$$T = T_r \oplus T_e = \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right) \oplus T_e$$

with respect to the decomposition $\mathcal{H} = (\oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)} \oplus \mathcal{H}_e$.

Denote $\Lambda_2 = \{i \in \Lambda: T_i \lhd T_e\}$ and $\Lambda_1 = \Lambda\backslash\Lambda_2$. Set

$$A = \bigoplus_{i\in\Lambda_1} T_i^{(n_i)} \quad and \quad B = \bigoplus_{i\in\Lambda_2} T_i^{(n_i)}.$$

Denote $\mathcal{H}_A = \oplus_{i\in\Lambda_1} \mathcal{H}_i^{(n_i)}$ and $\mathcal{H}_B = \oplus_{i\in\Lambda_2} \mathcal{H}_i^{(n_i)}$. Then $A \in \mathcal{B}(\mathcal{H}_A)$ and $B \in \mathcal{B}(\mathcal{H}_B)$. Moreover $T_r = A \oplus B$ and $T = A \oplus B \oplus T_e$.

We give the rest of the proof by proving three claims.

Claim 1: $B \oplus T_e \cong_a T_e$.

Since $T_j \lhd T_e$ for all $j \in \Lambda_2$ and $B = \oplus_{j\in\Lambda_2} T_j^{(n_j)}$, it follows that $B \lhd T_e$ and there exists a unital *-homomorphism ρ of $C^*(T_e)$ into $C^*(B)$ such that $\rho(T_e) = B$ and ρ annihilates $C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e)$. Then, by Lemma (6.1.13), we obtain $\cong_a id \oplus \rho$, where id is the identity representation of $C^*(T_e)$. It follows that $T_e \cong_a T_e \oplus B$.

Claim 2: $C^*(A) \cap \mathcal{K}(\mathcal{H}_A) = \{0\}$.

For a proof by contradiction, we assume that $C^*(A) \cap \mathcal{K}(\mathcal{H}_A) \neq \{0\}$. In view of Corollary (6.1.19), this implies that there exists $j \in \Lambda_1$ such that

$$\left\{ \begin{bmatrix} X^{(n_j)} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)} \end{matrix} : X \in \mathcal{K}(\mathcal{H}_j) \right\} \subset C^*(A).$$

Since $T_j \ntrianglelefteq T_e$, by Lemma (6.1.11), we have $T_j \ntriangleleft B \oplus T_e$. Now there are two possible cases.

Case 1: $T_j \ntrianglelefteq B \oplus T_e$.

In this case, it follows from Lemma (6.1.14)(i) that there exists $Z \in C^*(T)$ with

$$Z = \begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

and $X \neq 0$. By Lemma (6.1.15), it follows that $\mathcal{K}(\mathcal{H}_j)^{(n_j)} \oplus 0 \oplus 0 \subset C^*(T) \cap \mathcal{K}(\mathcal{H})$ and $\mathcal{H}_j^{(n_j)} \subset \mathcal{H}_e$, which is absurd.

Case 2: $T_j \trianglelefteq B \oplus T_e$ and $T_j \ntriangleleft B \oplus T_e$.

In this case, it follows from Lemma (6.1.14)(ii) that there exists $Z \in C^*(T)$ with

$$Z = \begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)}, \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

where $X \neq 0$ and K is compact acting on $\mathcal{H}_B \oplus \mathcal{H}_e$. Since $B \triangleleft T_e$, it follows from Lemma (6.1.10) that K has the form

$$K = \begin{bmatrix} 0 & \\ & \bar{K} \end{bmatrix} \begin{matrix} \mathcal{H}_B \\ \mathcal{H}_e \end{matrix},$$

where $\bar{K} \in C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e)$. Then, by Lemma (6.1.27), $C^*(T)$ contains the element

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & \bar{K} \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ H_B \\ \mathcal{H}_e \end{matrix} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)}. \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

Therefore we deduce that $C^*(T)$ contains

$$\begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & 0 \end{bmatrix}.$$

Using a same argument as in Case 1, one can prove that $\mathcal{K}\left(\mathcal{H}_j^{(n_j)}\right) \oplus 0 \oplus 0 \in C^*(T) \cap \mathcal{K}(\mathcal{H})$, a contradiction. This proves Claim 2.

Claim 3: A is g-normal. Denote $D = A \oplus T_e$ and $\mathcal{H}_D = \mathcal{H}_A \oplus \mathcal{H}_e$. Then $D \in \mathcal{B}(\mathcal{H}_D)$. By Claim 2, $C^*(A) \cap \mathcal{K}(\mathcal{H}_A) = \{0\}$. One can see that $D_e = T_e$. Denote by P_e the orthogonal projection of $\mathcal{H}_D$ onto $\mathcal{H}_e$. By Claim 1, we have $T = A \oplus B \oplus T_e \cong_a A \oplus T_e = D$. So $T \in \mathcal{S}(\mathcal{H})$ implies that D a norm limit of CSOs. Hence we can choose a sequence $\{C_n\}$ of conjugations on $\mathcal{H}_D$ such that $C_n D C_n \to D^*$. For $R \in C^*(D)$, define $\varphi(R) = \lim_n C_n R^* C_n$. Then φ is an anti-automorphism of $C^*(D)$ and $\varphi^{-1} = \varphi$. By Lemma (6.1.25), $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(D)$. So $\varphi\left(C^*(D) \cap \mathcal{K}(\mathcal{H}_D)\right) = C^*(D) \cap \mathcal{K}(\mathcal{H}_D)$. Since $\mathcal{H}_A =$

$\bigoplus_{i\in\Lambda_1}\mathcal{H}_i^{(n_i)}$, by Lemma 9, it suffices to prove for each $i\in\Lambda_1$ and each $x\in\mathcal{H}_i^{(n_i)}$ that the sequence $\{P_eC_nx\}$ converges to 0.

Now fix an $i_0\in\Lambda_1$. For convenience we may directly assume that $n_{i_0}=1$. The proof for $n_{i_0}>1$ follows easily. Arbitrarily choose a vector $f\in\mathcal{H}_{i_0}$. It suffices to prove that $P_eC_nf\to 0$.

Since $T_{i_0}\not\lhd T_e$, using a similar argument as in the proof of Claim 2, one can check that $C^*(D)$ contains an operator Z of the form

$$Z=\begin{bmatrix}X & & \\ & Y & \\ & & 0\end{bmatrix}\begin{matrix}\mathcal{H}_{i_0}\\ \mathcal{H}_A\ominus\mathcal{H}_{i_0}\\ \mathcal{H}_e\end{matrix}$$

where $X\neq 0$. Then there exist nonzero vectors $g_0,f_0\in\mathcal{H}_{i_0}$ such that $Xg_0=f_0$.

Note that T_{i_0} is irreducible. By the well-known Double Commutant Theorem, we can find a sequence $\{p_k(\cdot,\cdot)\}$ of polynomials in two free variables such that

$$p_k\left(T_{i_0}^*,T_{i_0}\right)\xrightarrow{sot}\frac{f\otimes f_0}{\|f_0\|^2}\quad as\quad k\to\infty. \tag{3}$$

Set $\tilde{Z}=\varphi(Z)$. For any $K\in C^*(D)\cap\mathcal{K}(\mathcal{H}_D)$, since $K(I-P_e)=0$, it follows that $KZ=0$ and $\tilde{Z}\varphi(K)=\varphi(K\,Z)=0$. Noting that $\varphi\big(C^*(D)\cap\mathcal{K}(\mathcal{H}_D)\big)=C^*(D)\cap\mathcal{K}(\mathcal{H}_D)$, we obtain $\tilde{Z}P_e=0$. Thus $\tilde{Z}$ admits the following matrix representation

$$Z=\begin{bmatrix}\tilde{X} & & \\ & \tilde{Y} & \\ & & 0\end{bmatrix}\begin{matrix}\mathcal{H}_{i_0}\\ \mathcal{H}_A\ominus\mathcal{H}_{i_0}\\ \mathcal{H}_e\end{matrix}$$

For any k, we have

$$\lim_n C_np_k(D^*,D)ZC_n=\lim_n(C_np_k(D^*,D)C_n)\cdot(C_n\,ZC_n)=\widetilde{p_k}\,(D,D^*)\varphi(Z^*)$$
$$=\widetilde{p_k}(D,D^*)\varphi(Z)^*=\widetilde{p_k}(D,D^*)(Z)^*.$$

For each k, noting that $P_e\widetilde{p_k}(D,D^*)\big(\tilde{Z}\big)^*=0$, we obtain

$$\|P_eC_np_k(D^*,D)Z\|=\|P_eC_n\,p_k(D^*,D)ZC_n\|\to 0$$

as $n\to\infty$. It follows that

$$\limsup_n P_eC_nf=\limsup_n\|P_eC_nf-P_eC_np_k(D^*,D)Zg_0\|\leq\ \|f-p_k(D^*,D)f_0\|$$
$$=\left\|f-p_k\left(T_{i_0}^*,T_{i_0}\right)f_0\right\|$$

for any $k\geq 1$. In view of (3), one can deduce that $\|P_eC_nf\|\to 0$. This proves Claim 3.

In view of Claims 2 and 3, it follows from Theorem (6.1.8) that A is a norm limit of CSOs. By Corollary (6.1.24), A is approximately unitarily equivalent to an operator which can be written as a direct sum of irreducible CSOs and operators of the form $\oplus\,R^t$, where R is irreducible and not a CSO.

Note that T_e is a CSO and $C^*(T_e)\cap\mathcal{K}(\mathcal{H}_e)$ is non-degenerate. In view of Proposition (6.1.33), T_e can be written as a direct sum of irreducible CSOs and operators of the form $R\oplus R^t$, where R is irreducible and not a CSO. Noting that

$$T=A\oplus B\oplus T_e\cong_a A\oplus T_e,$$

we conclude the proof.

We devoted to the proof of Theorem (6.1.41). We first give some auxiliary results.

Proposition (6.1.36)[272]: Let $T\in\mathcal{B}(\mathcal{H})$ have the form

$$T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$
relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A \in \mathcal{B}(\mathcal{H}_1)$ satisfies $C^*(A) \cap \mathcal{K}(\mathcal{H}_1) = \{0\}$ and $\overline{ran\, A + ran\, A^*} = \mathcal{H}_1$. Then $T \in \overline{S(\mathcal{H})}$ if and only if A is a norm limit of CSOs.

Proof. It is obvious that we need only prove the necessity. Since $C^*(A) \cap \mathcal{K}(\mathcal{H}_1) = \{0\}$, by Theorem (6.1.8), it suffices to prove that A is g-normal.

Since $T \in \overline{S(\mathcal{H})}$, we can choose conjugations $\{C_n\}_{n=1}^{\infty}$ on $\mathcal{H}$ such that $C_nTC_n \to T^*$. For each $n \geq 1$, assume that
$$C_n = \begin{bmatrix} C_{1.1}^n & C_{1.2}^n \\ C_{2.1}^n & C_{2.2}^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$
By Lemma (6.1.32), we need only verify for each $x \in \mathcal{H}_1$ that $C_{2,1}^n x \to 0$.
A direct matrix calculation shows that
$$C_nT - T^*C_n = \begin{bmatrix} * & * \\ C_{2.1}^n A & * \end{bmatrix} \quad \text{and} \quad C_nT^* - TC_n = \begin{bmatrix} * & * \\ C_{2.1}^n A^* & * \end{bmatrix}.$$
Since $C_nT - T^*C_n \to 0$ and $C^nT^* - TC_n \to 0$, we deduce that $C_{2,1}^n Ax \to 0$ and $C_{2,1}^n A^*x \to 0$ for each $x \in \mathcal{H}_1$. Noting that $\overline{ran\, A + ran\, A^*} = \mathcal{H}_1$ and $\sup_n \|C_{2,1}^n\| \leq 1$, one can see that $C_{2,1}^n y \to 0$ for each $y \in \mathcal{H}_1$. This completes the proof.

Lemma (6.1.37)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ have the form
$$T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$
relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A \in \mathcal{B}(\mathcal{H}_1)$ and
$$\overline{ran\, A + ran\, A^*} = \mathcal{H}_1.$$
If $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then $C^*(A) \cap \mathcal{K}(\mathcal{H}_1) = \{0\}$.

Proof. For a proof by contradiction, we assume that $C^*(A) \cap \mathcal{K}(\mathcal{H}_1) = \{0\}$. Then, by Lemma (6.1.22), we may directly assume that
$$A = \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_{1,1}^{(n)} \\ \mathcal{H}_{2,2} \end{matrix},$$
where $\mathcal{H}_1 = \mathcal{H}_{1,1}^{(n)} \oplus H_{1,2}, A_i \in \mathcal{B}(H_{1,i}) (i = 1, 2),\ n \in \mathbb{N}$ and
$$\mathrm{K}(\mathcal{H}_{1,1})^{(n)} \oplus 0_{H_{1,2}} \subset C^*(A).$$
For convenience, we may directly assume that $n = 1$.

Case 1: $\|p(A_1^*, A_1)\| \leq |p(0,0)|$ for any polynomial $p(\cdot,\cdot)$ in two free variables. In this case, it follows readily that $\|A_1^*\, A_1\| \leq 0$. So $A_1 = 0$, contradicting the fact that $ran\, A + ran\, A^* = \mathcal{H}_1$.

Case 2: There exists a polynomial $p(\cdot,\cdot)$ in two free variables such that
$$p(A_1^*, A_1) > |p(0,0)|.$$
In this case, by Lemma (6.1.14)(i), $C^*(T)$ contains an operator of the form
$$\begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_{1,1} \\ \mathcal{H}_{1,2} \\ \mathcal{H}_2 \end{matrix},$$
where $X \neq 0$. Note that $\mathcal{K}(\mathcal{H}_{1,1}) \oplus 0_{\mathcal{H}_{1,2}} \subset C^*(A)$. Then it follows from Lemma (6.1.15) that $\mathcal{K}(\mathcal{H}_{1,1}) \oplus 0_{\mathcal{H}_{1,2}} \oplus 0_{\mathcal{H}_2} \subset C^*(T)$, a contradiction. Therefore we conclude the proof.

Lemma (6.1.38)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ have the form

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \end{matrix},$$

where $\mathcal{H}_A \oplus \mathcal{H}_B = \mathcal{H}, A \in \mathcal{B}(\mathcal{H}_A)$ and $B \in \mathcal{B}(\mathcal{H}_B)$. Assume that $C^*(B) \cap \mathcal{K}(\mathcal{H}_B)$ is non-degenerate. Denote $M = \overline{ran\, B + ran\, B^*}$. If $A \lhd B$, then $A \lhd (B|_M)$.

Proof. It is obvious that M reduces B. Denote $B_1 = B|_M$. Then T can be written as

$$T = \begin{bmatrix} A & & \\ & 0 & \\ & & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M \\ M \end{matrix}.$$

It suffices to prove $A \lhd B_1$.

Since $C^*(B) \cap \mathcal{K}(\mathcal{H}_B)$ is non-degenerate, we can see that $dim\, \mathcal{H}_B \ominus M < \infty$.

We claim that $A \lhd B_1$. In fact, if not, then there exists $p(\cdot,\cdot)$ such that $\|p(A^*, A)\| > \|p(B_1^*, B_1)\|$. Since

$$\|p(A^*, A)\| \le \|p(B^*, B)\| = \max\{|p(0,0)|, \|p(B_1^*, B_1)\|\},$$

we obtain $\|p(B_1^*, B_1)\| < \|p(A^*, A)\| \le |p(0,0)|$. Let δ be a positive number satisfying $\|p(B_1^*, B_1)\| < \delta < \|p(A^*, A)\|$. Define

$$f(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ t - \delta, & t > \delta. \end{cases}$$

Then $f(|p(T^*, T)|) \in C^*(T)$ has the form of

$$\begin{bmatrix} A & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M \\ M \end{matrix},$$

where $X \neq 0$ and $Y \neq 0$. Noting that $dim\, \mathcal{H}_B \ominus M < \infty$, Y is a nonzero compact operator and hence $Y \oplus 0 \in C^*(B) \cap \mathcal{K}(\mathcal{H}_B)$. Since $A \lhd B$, it follows from Lemma (6.1.10) that $X = 0$, a contradiction. Thus we have proved that $A \lhd B_1$.

It remains to prove $A \lhd B_1$. In fact, if not, then, by Lemma (6.1.14), $C^*(T)$ contains an operator of the form

$$\begin{bmatrix} X_1 & & \\ & Y_1 & \\ & & K_1 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M \\ M \end{matrix},$$

where $X_1 \neq 0$ and $K_1 \in \mathcal{K}(M)$. Since $dim\, \mathcal{H}_B \ominus M < \infty, Y_1 \oplus K_1 \in C^*(B)$ is compact on $\mathcal{H}_B$. Noting that $A \lhd B$, it follows from Lemma (6.1.10) that $X_1 = 0$, a contradiction. This completes the proof.

The following result extends (see Lemma 1 in [138]) in the sense of approximation

Theorem (6.1.39)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ have the form

$$T = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $R \in \mathcal{B}(\mathcal{H}_1)$. Then $T \in \overline{S(\mathcal{H})}$ if and only if R is a norm limit of CSOs.

Proof. It is obvious that we need only prove the necessity. Assume that $T \in \overline{S(\mathcal{H})}$. By Lemma (6.1.23), we may assume that R is a direct sum of irreducible operators. Then, under this hypothesis, T is also a direct sum of irreducible operators. Without loss of generality, we may also assume that $\overline{ran\, R + ran\, R^*} = \mathcal{H}_1$.

It can be seen from the proof of "(i)⇒(iv)" in Theorem (6.1.35) that T admits the following matrix representation

$$T = \begin{bmatrix} A & & \\ & B & \\ & & T_e \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \\ \mathcal{H}_e \end{matrix},$$

where $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \oplus \mathcal{H}_e$ and

(a) A is a norm limit of CSOs and $C^*(A) \cap \mathcal{K}(\mathcal{H}_A) = \{0\}$,

(b) T_e is a CSO, $T_e \cong_a B \oplus T_e, B \lhd T_e$ and $C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e)$ is non-degenerate,

(c) if M_1, M_2, M_3 are nonzero minimal reducing subspaces of A, B and T_e respectively, then any two of $A|_{M_1}$, $B|_{M_2}$ and $T_e|_{M_3}$ are not unitarily univalent.

By condition (c), one can deduce that exactly one of the following three holds: $\mathcal{H}_2 \subset \mathcal{H}_A, \mathcal{H}_2 \subset \mathcal{H}_B$ and $\mathcal{H}_2 \subset \mathcal{H}_e$. So the rest of the proof is divided into three cases.

Case 1: $\mathcal{H}_2 \subset \mathcal{H}_A$. In this case, we can write

$$A = \begin{bmatrix} 0 & \\ & A_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_A \ominus \mathcal{H}_2 \end{matrix}.$$

Since $\overline{ran\ R + ran\ R^*} = \mathcal{H}_1$, one can see $\overline{ran\ A_1 + ran\ A_1^*} = \mathcal{H}_A \mathcal{H}_2$. By Lemma (6.1.37), $C^*(A_1)$ contains no nonzero compact operator. Since A is a norm limit of CSOs, it follows from Proposition (6.1.36) that A_1 is a norm limit of CSOs. Note that $R = A_1 \oplus B \oplus T_e \cong_a A_1 \oplus T_e$. Thus we deduce that R is a norm limit of CSOs.

Case 2: $\mathcal{H}_2 \subset \mathcal{H}_B$. In this case, we can write

$$B = \begin{bmatrix} 0 & \\ & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_B \ominus \mathcal{H}_2 \end{matrix}.$$

Since $B \lhd T_e$, one can see $B_1 \lhd T_e$. Then, by Lemma (6.1.13), we obtain $B_1 \oplus T_e \cong_a T_e$. Noting that $R = A \oplus B_1 \oplus T_e \cong_a A \oplus T_e$, we can deduce that R is a norm limit of CSOs.

Case 3: $\mathcal{H}_2 \subset \mathcal{H}_e$. In this case, we can write

$$T_e = \begin{bmatrix} 0 & \\ & T_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_e \ominus \mathcal{H}_2 \end{matrix}.$$

Since T_e is a CSO, it follows from (see Lemma 1 in [138] that T_1 is a CSO. Noting that $\overline{ran\ R + ran\ R^*} = \mathcal{H}_1$, one can see $\overline{ran\ T_1 + ran\ T_1^*} = \mathcal{H}_e \ominus \mathcal{H}_2$.

Since $C^*(T_e) \cap \mathcal{K}(\mathcal{H}_e)$ is non-degenerate and $B \lhd T_e$, it follows from Lemma (6.1.38) that $B \lhd T_1$. By Lemma (6.1.13), we obtain $B \oplus T_1 \cong_a T_1$. Thus $R = A \oplus B \oplus T_1 \cong_a A \oplus T_1$. Noting that A is a limit of CSOs and T_1 is a CSO, we deduce that R is a norm limit of CSOs. Thus we conclude the proof.

Lemma (6.1.40)[272]: Let T, R be two partial isometries on $\mathcal{H}$ and $T \cong_a R$. Denote by A_1 the compression of T to $(ker\ T)^\perp$, and by A_2 the compression of R to $(ker\ R)^\perp$. Then $A_1 \cong_a A_2$.

Proof. We first assume that

$$A = \begin{bmatrix} A_1 & 0 \\ B_1 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \end{matrix}. \quad \text{And} \quad R = \begin{bmatrix} A_2 & 0 \\ B_2 & 0 \end{bmatrix} \begin{matrix} (\ker R)^\perp \\ \ker R \end{matrix}.$$

Since T, R are two partial isometries, it follows that $A_1^*A_1 + B_1^*B_1 = I_1$ and $A_2^*A_2 + B_2^*B_2 = I_2$, where I_1 is the identity operator on $(ker\ T)^\perp$ and I_2 is the identity operator on $(ker\ R)^\perp$. Noting that $T \cong_a R$, we can choose unitary operators $\{U_n\}_{n=1}^\infty$ on $\mathcal{H}$ such that $TU_n - U_nR \to 0$ as $n \to \infty$. Without loss of generality, we assume that

$$U_n = \begin{bmatrix} U_{1.1}^n & U_{1.2}^n \\ U_{2.1}^n & U_{2.2}^n \end{bmatrix},$$

where

$$U_{1,1}^n \in B((\ker R)^\perp, (\ker T)^\perp), \qquad U_{1,2}^n \in \mathcal{B}(ker\ R, (ker\ T)^\perp)$$

and

$$U_{2,1}^n \in \mathcal{B}((ker\ R)^\perp, ker\ T), \qquad U_{2,2}^n \in B(ker\ R, ker\ T).$$

A matrix computation shows that

$$TU_n - U_n\, R = \begin{bmatrix} * & A_1 U_{1.2}^n \\ * & B_1 U_{1.2}^n \end{bmatrix}.$$

Thus $A_1 U_{1,2}^n \to 0$ and $B_1 U_{1,2}^n \to 0$ as $n \to \infty$. Noting that

$$U_{1,2}^n = (B_1^* B_1 + A_1^*\, A_1) U_{1,2}^n = B_1^* B_1 U_{1,2}^n + A_1^* A_1 U_{1,2}^n,$$

we obtain $U_{1,2}^n \to 0$ as $n \to \infty$.

Note that

$$RU_n^* - U_n^* T = \begin{bmatrix} * & A_1 (U_{1.2}^n)^* \\ * & B_1 (U_{1.2}^n)^* \end{bmatrix} \to 0$$

as $n \to \infty$. Using a similar argument as above, one can prove that $U_{2,1}^n = \left(U_{2,1}^n\right)^* \to 0$ as $n \to \infty$. Since $U_n^* U_n = I = U_n U_n^*$, we have

$$\left(U_{1,1}^n\right)^* U_{1,1}^n + \left(U_{2,1}^n\right)^* U_{2,1}^n = I_2 \text{ and } U_{1,1}^n \left(U_{1,1}^n\right)^* + U_{1,2}^n \left(U_{1,2}^n\right)^* = I_1.$$

It follows readily that $U_{1,1}^n$ is invertible for n large enough and $\left(U_{1,1}^n\right)^* U_{1,1}^n \to I_2$. Hence $|U_{1,1}^n| \to I_2$ as $n \to \infty$.

For each $n \geq 1$, assume that $U_{1,1}^n = V_{1,1}^n \left|U_{1,1}^n\right|$ is the polar decomposition of $U_{1,1}^n$, where $V_{1,1}^n \colon (ker\ R)^\perp \to (ker\ T)^\perp$ is a partial isometry. Then, by the discussion above, $V_{1,1}^n$ is invertible and hence unitary for n large enough. Moreover, since $|U_{1,1}^n| \to I_2$, we deduce that $\left\|V_{1,1}^n - U_{1,1}^n\right\| \to 0$ as $n \to \infty$.

Since $T\, U_n - U_n R \to 0$, a direct calculation shows that

$$A_1 U_{1,1}^n - U_{1,1}^n A_2 - U_{1,2}^n B_2 \to 0.$$

Noting that $V_{1,1}^n - U_{1,1}^n \to 0$ and $U_{1,2}^n \to 0$ as $n \to \infty$, we deduce that $A_1 V_{1,1}^n - V_{1,1}^n A_2 \to 0$, that is, $A_1 \cong_a A_2$.

Theorem (6.1.41)[272]: Let $T \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Denote by A the compression of T to its initial space. Then $T \in \overline{S(\mathcal{H})}$ if and only if A is a norm limit of CSOs.

Proof. "⇒" Since $T \in \overline{S(\mathcal{H})}$, it follows from Theorem (6.1.35) that there exists $F \in S(\mathcal{H})$ such that $T \cong_a F$. It is easy to check that F is also a partial isometry. Denote by A_1 the compression of F to $(ker\ F)^\perp$. By Theorem 2 [138], $F \in S(\mathcal{H})$ implies that A_1 is a CSO. Noting that $T \cong_a F$, it follows from Lemma (6.1.40) that $A \cong_a A_1$. Thus A is a norm limit of CSOs.

"⇐" Since A is a norm limit of CSOs, there exists a CSO A_1 on $(ker\ T)^\perp$ such that $A \cong_a A_1$. We assume that

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \end{matrix}.$$

Since T is a partial isometry, it follows that $A^* A + B^* B = I_1$ and hence $|B| = \sqrt{I_1 - A^* A}$, where I_1 is the identity operator on $(ker\ T)^\perp$. The rest of the proof is divided into three cases.

Case 1: $dim\ ker\ B = dim\ ker\ B^*$. Assume that $B = V|B|$ is the polar decomposition of B, where $V \colon (ker\ T)^\perp \to ker\ T$ is a partial isometry. Since $dim\ ker\ B = dim\ ker\ B^*$, we have $dim\ ker\ V = dim\ ker\ V^*$. Then V can be extend to a unitary operator : $(ker\ T)^\perp \to ker\ T$. Then $U^* V|B| = |B|$. Define $W \colon (ker\ T)^\perp \oplus (ker\ T)^\perp \to \mathcal{H}$ as $W \colon (x, y)\ \to x + Uy$. Thus W is a unitary operator. A direct matrical calculation shows that

$$W^* T W = \begin{bmatrix} A & 0 \\ |B| & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ \sqrt{I_1 - A^* A} & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ (\ker T)^\perp \end{matrix}.$$

Noting that $A\ \cong_a A_1$, it is easy to check that

$$W^*TW \cong_a \begin{bmatrix} A_1 & 0 \\ \sqrt{I_1 - A_1^*A_1} & 0 \end{bmatrix} \triangleq L.$$

It is clear that

$$L^*L = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ (\ker T)^\perp \end{matrix}.$$

Thus L is a partial isometry and the compression of L to its initial space is A_1. Since A_1 is a CSO, it follows from Theorem 2 [138] that L is complex symmetric, and hence $T \in \overline{S(\mathcal{H})}$.

Case 2: $dim\ ker\ B < dim\ ker\ B^*$.

Since $ker\ B^* = ker\ T\ \overline{ran\ B}$, there exists a subspace M of ker T such that ran $B \subset M$ and $dim\ M \ominus \overline{ran\ B} = dim\ ker\ B$. Then T admits the following matrix representation

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \end{matrix} = \begin{bmatrix} A & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ M \\ (\ker T) \ominus M \end{matrix},$$

where $B_1 \in \mathcal{B}((ker\ T)^\perp, M)$. Note that $ker\ B = ker\ B_1$ and $ran\ B = ran\ B_1$. We have

$$dim\ ker\ B_1 = dim\ ker\ B = dim\ M \ominus ran\ B = dim\ M \ominus ran\ B_1 = dim\ ker\ B_1^*.$$

Set

$$F = \begin{bmatrix} A & 0 \\ B_1 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \end{matrix}$$

One can check that F is a partial isometry and $(ker\ F)^\perp = (ker\ T)^\perp$.

Noting that $A \cong_a A_1$ and $ker\ B_1 = dim\ ker\ B_1^*$, it can be seen from the argument in Case 1 that F is a norm limit of CSOs. Then it follows readily that $T \in \overline{S(\mathcal{H})}$.

Case 3: $dim\ ker\ B > dim\ ker\ B^*$.

In this case, we can choose a Hilbert space M such that $dim\ M + dim\ ker\ B^* = dim\ ker\ B$. Set

$$R = T = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ M \end{matrix} = \begin{bmatrix} A & 0 & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \\ M \end{matrix} \triangleq \begin{bmatrix} A & 0 \\ B_2 & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \oplus M \end{matrix}.$$

Noting that $ker\ B_2 = ker\ B, ran\ B_2 = ran\ B$ and $ker\ B_2^* = ker\ B^* \oplus M$, we obtain

$$dim\ ker\ B_2^* = dim\ ker\ B^* + dim\ M = dim\ ker\ B = dim\ ker\ B_2.$$

Obviously, R is still a partial isometry and A is the compression of R to its initial space. Since $A \cong_a A_1$ and $im\ ker\ B_2 = dim\ ker\ B_2^*$, it can be seen from the proof in Case 1 that R is a norm limit of CSOs. By Theorem (6.1.39), it follows that $T \in \overline{S(\mathcal{H})}$. This completes the proof.

Let $T \in \overline{B(\mathcal{H})}$. Assume that $T = U|T|$ be the polar decomposition of T. Recall that the Aluthge transform of T is defined to be the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (see [273]).

Corollary (6.1.42)[272]: Let $T \in B(\mathcal{H})$ be a partial isometry. Then $T \in \overline{S(\mathcal{H})}$ if and only if the Aluthge transform of T is a norm limit of CSOs.

Proof. We first assume that

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^\perp \\ \ker T \end{matrix}.$$

Then

$$T^*T = \begin{bmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where I_1 is the unit operator on ker T . Since T is a partial isometry, the Aluthge transform of T is

$$|T|^{\frac{1}{2}}T|T|^{\frac{1}{2}} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T \end{matrix}.$$

Section (6.2): A C^*-Algebra Approach

We always denote by $\mathcal{H}$ a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$, and let π denote the canonical quotient map of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H})$, we let $C^*(A)$ denote the C^*-algebra generated by A and the identity operator on $\mathcal{H}$. We let $\sigma(A)$ and $\sigma_e(A)$ denote the spectrum and the essential spectrum of A respectively.

We give a brief introduction to complex symmetric operators and their norm closure problem.

Definition (6.2.1)[164]: A map C on $\mathcal{H}$ is called an anti-unitary operator if C is conjugate linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. If, in addition, $C^{-1} = C$, then C is called a conjugation on $\mathcal{H}$.

Definition (6.2.2)[164]: An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be complex symmetric if there is a conjugation C on $\mathcal{H}$ such that $CTC = T^*$.

The study of complex symmetric operators was initiated by Garcia and Putinar [135], [136] and has recently received much attention. Many significant results concerning the internal structure of complex symmetric operators have been obtained (see [131],[158], [276],[138]–[139],[170]–[173]). Complex symmetric operators have many motivations in function theory, matrix analysis and other areas; in particular, complex symmetric operators are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason's seminal [145] and has led to rapid progress in related areas [149], [151], [152], [159], [160],[168],[169]. See [135],[159] for more about the history of this topic and its connections to other subjects.

Following Garcia and Poore [158], we denote by CSO the set of all complex symmetric operators on $\mathcal{H}$. People have recently paid much attention to the structure of the set CSO. Among other things, people consider the closures of CSO in several important topologies, including the weak operator topology (wot), the strong operator topology (sot) and the norm topology. Garcia and Poore [158] recently proved that CSO is dense in $\mathcal{B}(\mathcal{H})$ with respect to both wot and sot. As for the norm topology, things become very complicated. In the following, we let $\overline{CSO}$ denote the norm closure of CSO. Although CSO encompasses many important special operators, the set $\overline{CSO}$ is indeed nowhere dense in $\mathcal{B}(\mathcal{H})$. In fact, one can easily verify that each operator in CSO is biquasitriangular. Recall that an operator A is said to be biquasitriangular if there exists no $\lambda \in \mathbb{C}$ such that $A - \lambda$ is semi-Fredholm and $ind(A - \lambda) \neq 0$ (see [184]). Then, using an approximation result Theorem (6.2.62)7 in [184], one can see that $\overline{CSO}$ is nowhere dense in $\mathcal{B}(\mathcal{H})$.

In [138], Garcia and Wogen posed the norm closure problem for complex symmetric operators, which asked whether or not the set CSO is norm closed. Zhu, Li and Ji [173] gave a negative answer to the norm closure problem by proving that the Kakutani shift is not complex symmetric but belongs to $\overline{CSO}$. Almost immediately, using the unilateral shift and its adjoint, Garcia and Poore [276] constructed a completely different counterexample.

Generalizing the Kakutani shift, Garcia and Poore [158] constructed some special weighted shifts in $\overline{CSO}\backslash CSO$ which they called approximately Kakutani weighted shifts. A unilateral weighted shift $T \in \mathcal{B}(\mathcal{H})$ with nonzero weights $\{\alpha_k\}_{k=1}^{\infty}$ is said to be approximately Kakutani if for each $n \geq 1$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$0 < |\alpha_N| < \varepsilon$$

and

$$1 \le k \le n \Rightarrow -\varepsilon < |\alpha_k| - |\alpha_{N-k}| < \varepsilon.$$

Garcia and Poore raised the following conjecture.

Conjecture (6.2.3)[164]: ([158], Conjecture 1). Every irreducible unilateral weighted shift in $\overline{CSO}$ is approximately Kakutani.

In general, if $T \in \overline{CSO}$, then it follows that T is a "small perturbation" of operators in CSO; however, we find that in many cases T is in fact a "small compact perturbation" of operators in CSO. To be precise, we first give a definition.

Given a subset $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$, we denote by $\overline{\mathcal{E}}^c$ the set of all operators $A \in \mathcal{B}(\mathcal{H})$ satisfying: for any $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A + K \in \mathcal{E}$. We call $\overline{\mathcal{E}}^c$ the compact closure of $\mathcal{E}$. It is clear that $\mathcal{E} \subset \overline{\mathcal{E}}^c \subset \mathcal{E}$ and $\overline{\mathcal{E}}^c \subset [\mathcal{E} + \mathcal{K}(\mathcal{H})]$.

If we let W denote the Kakutani shift, then a minor modification of the proof of Theorem 0.5 [173] shows that $\in \overline{CSO}^c$. Garcia and Poore proved that a compact operator belongs to $\overline{CSO}$ if and only if it is complex symmetric Theorem 4 in [158]. These two results motivate the following question.

Question (6.2.4)[164]: Does $\overline{CSO}$ coincide with $\overline{CSO}^c$?

For some special classes of operators, including completely reducible operators, essentially normal operators, hyponormal operators and many weighted shifts, we give a positive answer to Question (6.2.4). All these results mentioned above suggest that the structure of the set $\overline{CSO}$ may admit some special form, and it needs and deserves much more study.

We provide a C^*-algebra approach to the norm closure problem for complex symmetric operators, which exhibits an interplay between complex symmetric operators and operator algebras. In fact, certain connections between complex symmetric operators and anti-automorphisms of singly generated C^*-algebras are established. Several new notions are introduced to characterize $\overline{CSO}$. Our main results apply to several special classes of operators, including completely reducible operators, irreducible operators, weighted shifts and essentially normal operators. We give a positive answer to Garcia and Poore's conjecture. These results generalize and update some recent results on complex symmetric operators [158], [161], [172], [173].

The proofs of the main results depend heavily on connections between complex symmetric operators and anti-automorphisms of singly generated C^*-algebras. Now let us show some C^*- algebra information contained in the notion of complex symmetry.

Let $T \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$ satisfying $CTC = T^*$. If $p(x,y)$ is a polynomial in two free variables, then it is easy to verify that $p(T,T^*) = Cp(T^*,T)C$, where $\tilde{p}(x,y)$ is obtained from $p(x,y)$ by conjugating each coefficient. Since C is isometric, it follows that

$$\|p(T^*,T)\| = \|\tilde{p}(T,T^*)\|. \tag{4}$$

This was first observed by Garcia, Lutz and Timotin see Question 1 in[157]. They asked whether the converse holds; that is, if $T \in \mathcal{B}(\mathcal{H})$ satisfies (4) for every polynomial $p(\cdot,\cdot)$ in two free variables, does it follow that T is complex symmetric?

Definition (6.2.5)[164]: For convenience, we say that an operator T is g-normal if it satisfies

$$\|p(T^*,T)\| = \|p(T,T^*)\|$$

for any polynomial $p(\cdot,\cdot)$ in two free variables. So, by the above discussion, each complex symmetric operator is g-normal. In particular, each normal operator is g-normal.

It is easy to see that each norm limit of g-normal operators is still g-normal. So each operator in $\overline{CSO}$ is g-normal. By Theorem 0.5 in [173], the Kakutani shift W satisfies $W \in \overline{CSO} \backslash CSO$. This shows that

$$CSO \subsetneq \overline{CSO} \subset \{g - \text{normal operators on } \mathcal{H}\},$$

which gives a negative answer to the question of Garcia, Lutz and Timotin.

In view of the above discussion, the following is perhaps an appropriate revision of the question of Garcia, Lutz and Timotin.

Question (6.2.6)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is g-normal, does it follow that $T \in \overline{CSO}$?

As we shall see later, although the answer to Question (6.2.6) is in general negative, g-normality of operators is closely related to complex symmetry. In fact, we shall prove that if $T \in \mathcal{B}(\mathcal{H})$ and $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then $T \in \overline{CSO}$ if and only if T is g-normal (Theorem (6.2.37)). In particular, it follows that if $A \in \mathcal{B}(\mathcal{H})$, then $A^{(\infty)}$ is a norm limit of complex symmetric operators if and only if A is g-normal.

We depend heavily on the observation that the g-normality of an operator T implies the existence of anti-automorphisms on $C^*(T)$. Recall that an anti-automorphism of a C^*-algebra A is a vector space isomorphism $\phi: \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. Anti-automorphisms play an important role in the study of the real structure of C^*-algebras [176], [194]–[278]. It is not necessary that each C^*-algebra possesses an anti-automorphism on it. Connes [177], [178] constructed von Neumann factors of type II_1 or type III which are not antiisomorphic to themselves. Jones [279] constructed another example of a type II_1 factor which is anti-isomorphic to itself but not by an involutory anti-automorphism. An anti-automorphism or an automorphism ρ is said to be involutory if $\rho^{-1} = \rho$. See [282], [283], [284],[285] for more results on anti-automorphisms of C^*-algebras.

The following lemma shows that there must exist involutory anti-automorphisms on the C^*-algebra generated by a g-normal operator. Hence each C^*-algebra generated by an operator in $\overline{CSO}$ possesses a real structure. This makes it possible for us to use C^*-algebra methods to study complex symmetric operators and their norm closure problem.

Lemma (6.2.7)[164]: An operator $T \in \mathcal{B}(\mathcal{H})$ is g-normal if and only if there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = T$.

Proof. "⇒". Assume that T is g-normal. Then the map

$$\rho: C^*(T) \to C^*(T),$$
$$p(T^*, T) \to \tilde{p}(T, T^*)$$

is isometric and densely defined. Hence ρ can be extended to a map on $C^*(T)$, which is also denoted by ρ. One can check that ρ is a conjugate automorphism of $C^*(T)$; that is, $\rho: C^*(T) \to C^*(T)$ is an invertible conjugate-linear map, $\rho(X^*) = \rho(X)^*$ and $\rho(XY) = \rho(X)\rho(Y)$ for $X, Y \in C^*(T)$. So, if we define $\varphi(X) = \rho(X)^*$ for $X \in C^*(T)$, then φ is an anti-automorphism of $C^*(T)$ and $\varphi(T) = T$.

"⇐". Let φ be an anti-automorphism of $C^*(T)$ satisfying $\varphi(T) = T$. Then $\varphi(T^*) = T^*$ and, given a polynomial $p(\cdot,\cdot)$ in two free variables, one can see

$$\varphi\big(p(T^*, T)\big) = \tilde{p}(T, T^*)^*.$$

Since each anti-automorphism of $C^*(T)$ is isometric, it follows that $\|p(T^*, T)\| = \|\tilde{p}(T, T^*)\|$. So T is g-normal.

One of our main results characterizes when an essentially normal operator is g-normal; in fact, a decomposition theorem is given (Theorem (6.2.58)). Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be essentially normal if $A^*A - AA^* \in \mathcal{K}(\mathcal{H})$. For an irreducible operator T, we shall show that T is g-normal if and only if T is an AUET operator.

Besides g-normal operators, two other important classes of operators, namely UET operators and AUET operators, are closely related to our results. To give the definitions, we need to define transposes of Hilbert space operators.

Definition (6.2.8)[164]: Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a transpose of T if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

The notion "transpose" for operators is in fact a generalization of that for matrices. Let $T \in \mathcal{B}(\mathcal{H})$. Assume that C is a conjugation on $\mathcal{H}$. Then there exists an orthonormal basis (ONb, for short) $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see Lemma 1 in [135]). Thus T has a matrix representation $[a_{i,j}]$ with respect to $\{e_n\}$, where $a_{i,j} = \langle Te_j, e_i\rangle$. Set $A = CT^*C$. Note that

$$\langle Ae_i, e_j\rangle = \langle CT^*Ce_i, e_j\rangle = \langle CT^*e_i, e_j\rangle = \langle Ce_j, T^*e_i\rangle = \langle e_j, T^*e_i\rangle = \langle Te_j, e_i\rangle.$$

Thus the matrix representation of A with respect to $\{e_n\}$ is just the transpose of the matrix $[a_{i,j}]$. So, given an operator T, a transpose of T is obtained from T by transposing the matrix representation of T with respect to some on b.

By the above discussion, an operator may have more than one transpose. In fact, any two transposes of an operator are unitarily equivalent. Assume that A, B, $T \in \mathcal{B}(\mathcal{H})$ and A, B are two transposes of T. Then there are two conjugations C and D on $\mathcal{H}$ such that $A = CT^*C$ and $B = DT^*D$. Set $U = CD$. Then it is easy to see that $U \in \mathcal{B}(\mathcal{H})$ is unitary and $AU = (CT^*C)(CD) = CT^*D = (CD)(DT^*D) = UB$; that is, A, B are unitarily equivalent.

We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$ or $\cong_a T^t$. Here and in what follows, the notation $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence. As usual, given two representations ρ_1 and ρ_2 of a C^*-algebra, we also write $\rho_1 \cong \rho_2$ ($\rho_1 \cong_a \rho_2$) to denote that ρ_1 and ρ_2 are unitarily equivalent (approximately unitarily equivalent, respectively).

Definition (6.2.9)[164]: An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be UET if $T \cong T^t$, and T is said to be AUET if $T \cong_a T^t$.

By definitions, each complex symmetric operator is UET. But the converse does not hold (see Example (6.2.29)).

The notion of UET operators has its motivations in linear algebra. In his problem book see Proposition 159 in [142], Halmos asked when a matrix is unitarily equivalent to its transpose (UET). There are matrices that are not UET (see [161]). Recently, Garcia and Tener [161] gave a canonical decomposition for UET matrices. As an application, they gave a canonical decomposition for complex symmetric operators on finite dimensional Hilbert spaces.

We give a characterization for an essentially normal operator to be UET (Proposition (6.2.57)); in particular, one of our main results gives a canonical decomposition for essentially normal operators which are UET (Theorem (6.2.62)). Also we give a canonical decomposition for essentially normal operators which are complex symmetric (Theorem (6.2.71)). The notion of AUET operators is useful for us to characterize $\overline{CSO}$. In fact, when $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we shall prove that $T \in \overline{CSO}$ if and only if T is AUET (Theorem (6.2.37)).

The first main result focuses on those operators $T \in \mathcal{B}(\mathcal{H})$ satisfying $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

It is clear that an operator T is g-normal if and only if $T^{(\infty)}$ is g-normal. So the following corollary is immediate from Theorem (6.2.37).

Corollary (6.2.10)[164]: If $T \in \mathcal{B}(\mathcal{H})$, then $T^{(\infty)}$ is a norm limit of complex symmetric operators if and only if T is g-normal.

As an application of Theorem (6.2.37), we shall characterize when a weighted shift with nonzero weights belongs to $\overline{CSO}$. Recall that a (forward) weighted shift T on $\mathcal{H}$ with weight sequence $\{w_n\}$ is the operator defined by $T\,e_n = w_n e_{n+1}$ for all n, where $\{e_n\}$ is an onb of $\mathcal{H}$. If the index n runs over positive integers, then T is called a unilateral weighted shift, while if n runs over integers, then T is called a bilateral weighted shift. According to a result of Shields [192], each weighted shift is unitarily equivalent to a weighted shift with nonnegative weights. So we need only deal with weighted shifts with positive weights.

Let T be a bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. For each $n \geq 1$, the n-spectrum of T (denoted by $\Sigma_n(T)$) is defined to be the closure (in the usual topology on $\mathbb{R}^n$) of the set

$$\{(w_{i+1}, w_{i+2}, \cdots, w_{i+n}) : i \in \mathbb{Z}\}.$$

This notion was first introduced to estimate the distance between unitary orbits of invertible bilateral weighted shifts [280].

Given a subset G of $\mathbb{R}^n$, we denote

$$G^t = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n : (\alpha_n, \alpha_{n-1}, \dots, \alpha_1) \in G\}.$$

For a weighted shift T with positive weights, although the equality $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ generally does not hold, we still have the following three theorems which completely characterize weighted shifts with positive weights in $\overline{CSO}$. In particular, Theorem (6.2.39) answers Conjecture (6.2.3) in the positive.

Theorem (6.2.11)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. If T is reducible or $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then the following are equivalent:

(i) $T \in \overline{CSO}$;

(ii) $T \in CSO$;

(iii) $T \cong T^*$;

(iv) T is g-normal;

(v) $\exists k \in \mathbb{Z}$ such that $w_i = w_{k-i}$ for all $i \in \mathbb{Z}$.

Example (6.2.12)[164]: Let G be the set of all rational numbers in $(0,1]$. Since G is denumerable, one can construct a bilateral weighted shift T with positive weights such that $\Sigma_n(T) = [0,1]^n$ for all $n \geq 1$. Thus $\Sigma_n(T)^t = \Sigma_n(T)$ for all $n \geq 1$. By Theorem (6.2.51), it follows that $T \in \overline{CSO}$.

In general, g-normality, UET property and complex symmetry are quite different. To see the difference, we characterize when an essentially normal operator is g-normal or UET. The following theorem gives a canonical decomposition for essentially normal operators in $\overline{CSO}$.

A fundamental question about complex symmetric operators is how to develop a model theory [159]. A natural thought is to decompose complex symmetric operators into "simple blocks" and then represent them in concrete terms. Some known results suggest that truncated Toeplitz operators may play the role of "simple blocks" [159]. Let $T \in \mathcal{B}(\mathcal{H})$ be complex symmetric and M be a nontrivial reducing subspace of T. It is known that each normal operator is complex symmetric [135]. If T is normal, then $T|_M$ must be complex symmetric; if T is not normal, it is possible that $T|_M$ is not complex symmetric (see [161]). This motivates the following definition:

Definition (6.2.13)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be complex symmetric. T is said to be completely complex symmetric if T is reducible and $T|_M$ is complex symmetric for any nontrivial reducing subspace M of T; T is called a minimal complex symmetric operator if there exists no nontrivial reducing subspace M of T such that $T|_M$ is complex symmetric.

Thus each normal operator on Hilbert spaces of dimension greater than 1 is completely complex symmetric. Note that each operator on a Hilbert space of dimension 1 is normal, irreducible and hence a minimal complex symmetric operator. Thus each normal operator is either completely complex symmetric or a minimal complex symmetric operator. On the other hand, if A is irreducible and not complex symmetric, we shall prove later that $A \oplus A^t$ is a minimal complex symmetric operator (Proposition (6.2.68)). So Theorem (6.2.71) shows that if an essentially normal operator T is complex symmetric, then T can be written as a direct sum of completely complex symmetric operators and minimal complex symmetric operators. We shall show some completely complex symmetric operators which are nonnormal (Proposition (6.2.50)).

Given $A, B \in \mathcal{B}(\mathcal{H})$, we denote $[A, B] = AB - BA$.

Let $T \in \mathcal{B}(\mathcal{H})$. Denote $M = \bigcap_{m,n\geq 1} ker[T^{*m}, T^n]$. Then M and $M^\perp$ both reduce T. In fact, $T|_M$ is normal and $T_{M^\perp}$ is abnormal (see page 116 in [184]). Recall that an operator A is said to be abnormal if A has no nonzero reducing subspace $\mathcal{N}$ such that $A|_{\mathcal{N}}$ is normal. We call $T|_M$ the normal part of T and $T|_{M^\perp}$ the abnormal part of T, denoted by T_{nor} and T_{abnor} respectively.

Lemma (6.2.14) ([135], page 1295). Each normal operator is complex symmetric.

Lemma (6.2.15)[164]: An operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if T_{abnor} is complex symmetric.

Proof. By Lemma (6.2.14), the sufficiency is clear. We need only prove the necessity. Denote $M = \cap_{m,n\geq 1}\ ker[T_m^*, T^n]$. Assume that C is a conjugation on $\mathcal{H}$ and $CT^*C = T$. Thus, for any $m, n \geq 1$, we have $C[T^{*m}, T^n]C = -[T^{*n}, T^m]$. Hence we deduce that $C(ker[T^{*n}, T^m]) = ker[T^{*m}, T^n]$. Since $m, n \geq 1$ are arbitrary, we obtain $C(M) = M$. Noting that C is a conjugation, we deduce that $C(M^\perp) = M^\perp$ and $D = C|_{M^\perp}$ is a conjugation. It follows from $CT^*C = T$ that $DT_{abnor}^*D = T_{abnor}$. This completes the proof.

Proposition (6.2.16)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then the following are equivalent:

(i) $T \in \overline{CSO}$;
(ii) T is g-normal;
(iii) T is normal;
(iv) $T \in CSO$.

Proof. Note that each normal operator is complex symmetric. By definition, the implications "(iii) ⇒(iv) ⇒(i) ⇒(ii)" are obvious.

"(ii)⇒(iii)". Since T is g-normal, by Lemma (6.2.7), the map φ defined as

$$\varphi: C^*(T) \to C^*(T), p(T^*, T) \mapsto p(T, T^*)^*$$

is an anti-automorphism of $C^*(T)$ and $\varphi(T) = T$. Thus the map φ preserves the *-operation and preserves the spectra of operators. Hence an operator $X \in C^*(T)$ is positive if and only if $\varphi(X)$ is positive. Set $A = [T^*, T]$. Since T is hyponormal, it follows that $A \geq 0$; furthermore, $-A = \varphi(A) \geq 0$. So $A = 0$ and T is normal.

Lemma (6.2.17)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $T = A^{(m)} \oplus B^{(n)}$, where $1 \leq m, n \leq \infty$. Then T is g-normal if and only if $A \oplus B$ is g-normal.

Proof. Set $R = A \oplus B$. Let $p(\cdot,\cdot)$ be a polynomial in two free variables. It is easy to check that

$$\|p(T^*,T)\| = \max\{\|p(A^*,A)\|, \|p(B^*,B)\|\} = \|p(R^*,R)\|$$

and

$$\|\tilde{p}(T,T^*)\| = \max\{\|\tilde{p}(A,A^*)\|, \|\tilde{p}(B,B^*)\|\} = \|\tilde{p}(R,R^*)\|.$$

It immediately follows that T is g-normal if and only if R is g-normal.

Corollary (6.2.18)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $T = \bigoplus_{i\in\Lambda} T_i$. Then the following hold:

(i) if each T_i is g-normal, then T is g-normal;

(ii) T is g-normal if and only if $\bigoplus_{i\in\Lambda} T_i^{(n_i)}$ is g-normal for some sequence $\{n_i\}$ with $1 \le n_i \le \infty$ $(i \in \Lambda)$ if and only if $\bigoplus_{i\in\Lambda} T_i^{(n_i)}$ is g-normal for any sequence $\{n_i\}$ with $1 \le n_i \le \infty$ $(i \in \Lambda)$.

Lemma (6.2.19)[164]: If $T \in \mathcal{B}(\mathcal{H})$, then $T \oplus T^t$ is complex symmetric.

Proof. Assume that $T^t = CT^*C$, where C is a conjugation on $\mathcal{H}$. Define

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}.$$

Then it is easy to see that D is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ and

$$D(T \oplus CT^*C)D = T^* \oplus CTC = (T \oplus CT^*C)^*,$$

which implies that $T \oplus T^t$ is complex symmetric.

Example (6.2.20)[164]: Let $A, B \in \mathcal{B}(\mathbb{C}^3)$ and assume that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to some onb of $\mathbb{C}^3$. It is obvious that $\cong A^t$. Then, by Lemma (6.2.19), $A \oplus B$ is complex symmetric and hence g-normal. By Corollary (6.2.18), $A^{(m)} \oplus B^{(n)}$ is g-normal for all $1 \le m, n \le \infty$. However, neither A nor B is g-normal. In fact, if we set $p(x,y) = x^2 y$, then

$$p(A^*,A) = A^{*^2}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \ \ and \ \ \tilde{p}(A,A^*) = A^2A^* = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have $\|p(A^*,A)\| \ne \|\tilde{p}(A,A^*)\|$. Similarly, one can check that $\|p(B,B^*)\| \ne \|\tilde{p}(B^*,B)\|$. So neither A nor B is g-normal.

Lemma (6.2.21)[164]: An operator $T \in \mathcal{B}(\mathcal{H})$ is UET if and only if there exists an antiunitary operator D on $\mathcal{H}$ such that $DT = T^*D$.

Proof. "⇒". If T is UET, then there exist unitary $U \in \mathcal{B}(\mathcal{H})$ and a conjugation C on $\mathcal{H}$ such that $U^*T\,U = CT^*C$. Set $D = CU^*$. Then D is an anti-unitary operator on $\mathcal{H}$ and $DT = T^{*D}$.

"⇐". Arbitrarily choose a conjugation C on $\mathcal{H}$. If D is an anti-unitary operator on $\mathcal{H}$ such that $DT = T^*D$, then $(CD)T = (CT^*C)(CD)$. Set $U = CD$. Then $U \in \mathcal{B}(\mathcal{H})$ is unitary. It follows that $T \cong CT^*C$.

Lemma (6.2.22) ([180], Theorem II.5.8). Let $\mathcal{A}$ be a separable C^*-algebra, and let ρ_1 and ρ_2 be nondegenerate representations of $\mathcal{A}$ on separable Hilbert spaces. Then the following are equivalent:

(i) $\rho_1 \cong_a \rho_2$,

(ii) $rank\ \rho_1(X) = rank\ \rho_2(X)$ for all $X \in \mathcal{A}$.

Proposition (6.2.23)[164]: Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:

(i) T is AUET.

(ii) There exists a sequence $\{D_n\}$ of anti-unitary operators on $\mathcal{B}(\mathcal{H})$ such that $\lim_n \|D_n T - T^* D_n\| = 0$.

(iii) There exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = T$ and $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(T)$.

Proof. "(i)⇒(ii)". Since T is AUET, we can find a sequence $\{U_n\}$ of unitary operators and a conjugation C on $\mathcal{H}$ such that $U_n^* T U_n \to CT^*C$. Hence $D_n := CU_n^*$ is an anti-unitary operator on $\mathcal{H}$ for $n \geq 1$ and $D_n^{-1} = U_n C$. One can check that $\lim_n D_n T - T^* D_n = 0$.

"(ii) ⇒(iii)". By the hypothesis, we have $\lim_n D_n T^* D_n^{-1} = T$ and $\lim_n D_n T D_n^{-1} = T^*$. Then, given a polynomial $p(\cdot,\cdot)$ in two free variables, it can be verified that

$$\tilde{p}(T, T^*) = \lim_n D_n p(T^*, T) D_n^{-1}. \quad (5)$$

Since each D_n is isometric, T is g-normal. By Lemma (6.2.7), the map φ defined by

$$\varphi: C^*(T) \to C^*(T),$$
$$p(T^*, T) \to \tilde{p}(T, T^*)^*$$

is an anti-automorphism of $C^*(T)$. Moreover, φ is involutory and $\varphi(T) = T$. In view of (5), we deduce that $\varphi(X) = \lim_n D_n X^* D_n^{-1}$ for $X \in C^*(T)$. By the lower semi-continuity of the rank in approximation (see [184]), it follows that

$$rank\ X = \liminf_n rank\ D_n X^* D_n^{-1} \geq rank\ \varphi(X).$$

Moreover, we have $rank\ \varphi(X) \geq rank\ \varphi^2(X)$. Since φ is involutory, it follows that $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(T)$.

"(iii)⇒(i)". Let C be a conjugation on $\mathcal{H}$. For each $X \in C^*(T)$, define $\rho(X) = C\varphi(X)^*C$. It is easily seen that ρ is a faithful representation of $C^*(T)$ on $\mathcal{H}$ and

$$rank\ \rho(X) = rank\ \varphi(X) = rank\ X = rank\ Id(X)$$

for all $X \in C^*(T)$, where $Id(\cdot)$ is the identity representation on $\mathcal{H}$. Noting that $Id(\cdot)$ and ρ are both nondegenerate, it follows from Lemma (6.2.22) that $\rho \cong_a Id$. Furthermore we obtain $T \cong_a CT^*C$, that is, T is AUET.

By Lemma (6.2.7), the following corollary is immediate from Proposition (6.2.23).

Corollary (6.2.24)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is AUET, then T is g-normal.

Lemma (6.2.25)[164]: If $T \in \overline{CSO}$, then T is AUET.

Proof. Since $T \in \overline{CSO}$, there exists a sequence $\{T_n\}$ of complex symmetric operators such that $T_n \to T$. For each n, since $T_n \in CSO$, we can choose a conjugation C_n on $\mathcal{H}$ such that $C_n T_n C_n = T_n^*$. One can verify that $C_n T\ C_n \to T^*$. In view of Proposition (6.2.23), this implies that T is AUET.

Example (6.2.26)[164]: The Kakutani shift is a unilateral weighted shift with weight sequence $\{w_n\}_{n=1}^{\infty}$, where

$$w_n = \frac{1}{gcd\{n, 2^n\}}, n \geq 1.$$

Here $gcd\{i, j\}$ denotes the greatest common divisor of i and j.

Denote by W the Kakutani shift. By ([173], Theorem 0.5), W is a norm limit of complex symmetric operators and hence it is AUET. However, note that

$$dim\ ker\ W = 0 < 1 = dim\ ker\ W^* = dim\ ker\ DW^*D^{-1}$$

for any anti-unitary operator D. Thus, by Lemma (6.2.21), W is not UET. This example shows that

$$\overline{CSO} \nsubseteq \{UET \text{ operators}\}\{AUET \text{ operators}\}.$$

Theorem (6.2.27)[164]: Let $S \in \mathcal{B}(\mathcal{H})$ be the unilateral shift defined by $S_{e_i} = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an onb of $\mathcal{H}$. Assume that $T \in \mathcal{B}(\mathcal{H})$ and define

$$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{\cdot} \end{matrix}$$

Then

(i) R_T is complex symmetric if and only if $\langle T\, e_i, e_j \rangle = \langle T\, e_j\,, e_i \rangle$ for all $i, j \geq 1$;

(ii)R_T is UET if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda \langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle$ for all $i, j \geq 1$.

Proof. (i) "⇐". For each $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_\iota} e_i$. Then C is a conjugation on $\mathcal{H}$. One can verify that $CS = SC$ and $CS^* = S^*C$. Since $\langle Te_i, e_j \rangle = T\langle e_j, e_i \rangle$ for all i, j, we also have $CTC = T^*$. Define

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{\cdot} \end{matrix}$$

Then D is a conjugation on $H \oplus H$ and one can check that $R_T\, D = R_T^*$. "⇒". Assume that C is a conjugation on $H \oplus H$ and $CRT\, C = R_T^*$. For convenience, we write

$$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = H$. Note that $CR_T^n C = (R_T^*)^n$ for all $n \geq 1$. So $C(ker\, R_T^n) = ker(R_T^*)^n$ for all $n \geq 1$. It follows that $C(\bigvee_n ker\, R_T^n\,) = \bigvee_n ker(R_T^*)^n$, that is, $C(\mathcal{H}_1) = \mathcal{H}_2$. Since $C^{-1} = C$, we have $C(\mathcal{H}_2) = \mathcal{H}_1$. Hence C admits the following matrix representation:

$$C = \begin{bmatrix} 0 & E \\ D & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Also one can see that D is an anti-unitary operator on $\mathcal{H}$ and $E = D^{-1}$. Since $CR_TC = R_T^*$, a straightforward computation shows that $DS = SD, DS^* = S^*D$ and $DTD = T^*$.

For $n \geq 1, DS^* = S^*D$ implies that $D(S^*)^n = (S^*)^n D$. Since D is invertible and $ker(S^*)^n = \{e_i\colon 1 \leq i \leq n\}$, we deduce that $D(ker(S^*)^n) = ker(S^*)^n$. Since D is isometric, there exists a sequence $\{\lambda_i\}$ of complex numbers with $|\lambda_i| = 1$ such that $De_i = \lambda_i e_i$ for all i.

Now fix an $i \geq 1$. Hence

$$\lambda_{i+1} e_{i+1} = De_{i+1} = DSe_i = SDe_i = \lambda_i Se_i = \lambda_i e_{i+1}.$$

So we have $\lambda_i = \lambda_{i+1}$. Thus the sequence $\{\lambda_i\}$ is constant.

On the other hand, for given $i, j \geq 1$, one can verify that

$$\langle T^* e_i, e_j \rangle = \langle DTDe_i, e_j \rangle = \langle D^{-1} e_j\,, TDe_i \rangle = T^* D^{-1} e_j, De_i = \langle T^* e_j\,, e_i \rangle,$$

that is, $\langle T\, e_i, e_j \rangle = \langle T\, e_j, e_i \rangle$. This completes the proof.

(ii) For each $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_\iota} e_i$. This defines a conjugation on $\mathcal{H}$. It is easy to verify that $CS = SC$ and $CS^* = S^*C$. Define

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{\cdot} \end{matrix}$$

Thus D is a conjugation on $\mathcal{H} \oplus \mathcal{H}$.

"⇐". For $i, j \geq 1$, we have

$$\langle CT^* Ce_i, e_j \rangle = \langle Ce_j, T^* Ce_i \rangle = \langle e_j, T^* e_i \rangle = \langle Te_j, e_i \rangle = \lambda \langle T\, e_i, e_j \rangle.$$

It follows that $CT^*C = \lambda T$.

Define

$$U = \begin{bmatrix} \lambda_I & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}' \end{matrix}$$

where I is the identity operator on $\mathcal{H}$. So $U \in \mathcal{B}(\mathcal{H}^{(2)})$ is unitary and
$DR_TD = \begin{bmatrix} CSC & 0 \\ CTC & CS^*C \end{bmatrix} = R_T = \begin{bmatrix} S & 0 \\ \bar{\lambda}T^* & S^* \end{bmatrix} = R_T = \begin{bmatrix} S^* & \lambda T \\ 0 & S \end{bmatrix}^* = (URTU^*)^* = UR_T^*U^*$;
that is, $DR_TD = UR_T^*U^*$. Hence R_T is UET.
"⇒". Since D is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, the operator

$$A := DR_T^*D = \begin{bmatrix} CS^*C & CT^*C \\ 0 & CSC \end{bmatrix} = \begin{bmatrix} S^* & CT^*C \\ 0 & S \end{bmatrix}$$

is a transpose of R_T. For convenience, we write

$$A = \begin{bmatrix} S^* & CT^*C \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \quad \text{and} \quad R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Since R_T is UET, we can choose a unitary operator U on $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $UR_T = AU$. Thus for each $n \geq 1$ we have $UR_T^n = A^nU$ and $U(ker\, R_T^n) = ker\, A^n$. Furthermore we have $U(\bigvee_n ker\, R_T^n) = \bigvee_n ker\, A^n$. Since $\bigvee_n ker\, R_T^n = \bigvee_n ker\, A^n = \mathcal{H}_1$, it follows that $U(\mathcal{H}_1) = \mathcal{H}_1$.

On the other hand, since $R_T^*U^* = U^*A^*$, using a similar argument as above, we can prove that $U^*(\bigvee_n ker(A^*)^n) = \bigvee_n ker(R_T^*)^n$. Noting that $\bigvee_n ker(R_T^*)^n = \bigvee_n ker(A^*)^n = \mathcal{H}_2$, we obtain $U^*(\mathcal{H}_2) = \mathcal{H}_2$, that is, $U(\mathcal{H}_2) = \mathcal{H}_2$. Then U admits the matrix representation

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Since $UR_T = AU$, it follows that $U_1S^* = S^*U_1, U_2S = SU_2$ and $U_1T = CT^*CU_2$. Noting that S is irreducible and each U_i is unitary, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_1| = |\lambda_2| = 1$ such that $U_1 = \lambda_1 I$ and $U_2 = \lambda_2 I$. Thus we conclude that $CT^*C = \lambda T$, where $\lambda = \lambda_1/\lambda_2$. One can check that $\lambda\langle Te_i, e_j\rangle = T\langle e_j, e_i\rangle$ for all $i, j \geq 1$. This completes the proof.

Remark (6.2.28)[164]: (i) In Theorem (6.2.27), when S is replaced by any unilateral weighted shift with positive weights, the results remain true.

(ii) In Theorem (6.2.27), if $T \neq 0$, then R_T is irreducible. This shows that there exist irreducible operators which are UET but not complex symmetric.

Example (6.2.29)[164]: Let A be the unilateral weighted shift defined as $Ae_i = \frac{1}{i}e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an onb of $\mathcal{H}$. Define a finite-rank operator F on $\mathcal{H}$ as

$$F\,e_1 = -e_2, Fe_2 = e_1 \quad \text{and} \quad F\,e_i = 0 \quad \text{for} \quad i \geq 3.$$

Then the operator K on $\mathcal{H} \oplus \mathcal{H}$ given by

$$K = R_T = \begin{bmatrix} A^* & F \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}$$

is compact. By Theorem (6.2.27) and Remark (6.2.28) (i), K is UET but not complex symmetric. Then, by Theorem 4 in [158], K is not a norm limit of complex symmetric operators. This combined with Lemma (6.2.25) shows that

$$\text{CSO} \subsetneq \{\text{UET operators}\} \nsubseteq \overline{\text{CSO}} \subsetneq \{\text{AUET operators}\}.$$

Now, using K, we shall construct an AUET operator which is neither UET nor a norm limit of complex symmetric operators.

Let $W \in \mathcal{B}(\mathcal{H})$ be the Kakutani shift defined in Example (6.2.26). Define

$$T = K \oplus (2I + W),$$

where I is the identity operator on $\mathcal{H}$. Thus $K^t \oplus (2I + W^t)$ is a transpose of T.

We claim that T is not UET. In fact, if not, then $K \oplus (2I + W) \cong K^t \oplus (2I + W^t)$. Note that $\sigma(2I + W) = \sigma(2I + W^t), \sigma(K) = \sigma(K^t)$ and $\sigma(2I + W) \cap \sigma(K) = \emptyset$. By

Rosenblum's Theorem [184], it follows that $K \cong K^t$ and $2I + W \cong 2I + W^t$; in particular, $W \cong W^t$. By Example (6.2.26), W is AUET but not UET, a contradiction.

In view of Theorem (6.2.27) and Remark (6.2.28), K is UET and hence AUET. Thus T is AUET.

Now it remains to prove that T is not a norm limit of complex symmetric operators. For a proof by contradiction, we assume that T is a norm limit of complex symmetric operators. Then there exists a sequence $\{C_n\}$ of conjugations on $\mathcal{H}^{(3)}$ such that $C_n T^* C_n - T \to 0$ as n tends to ∞. For each n, we assume that

$$C_n = \begin{bmatrix} C_{1.1}^n & C_{1.2}^n & C_{1.3}^n \\ C_{2.1}^n & C_{2.2}^n & C_{2.3}^n \\ C_{3.1}^n & C_{3.2}^n & C^{3.3} \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}. \\ \mathcal{H} \end{matrix}$$

Since $C_n T^* C_n - T \to 0$, we have

$$K^* \begin{bmatrix} C_{1,3}^n \\ C_{2,3}^n \end{bmatrix} - \begin{bmatrix} C_{1,3}^n \\ C_{2,3}^n \end{bmatrix} (2I + W) \to 0.$$

Arbitrarily choose a conjugation E on $\mathcal{H}^{(2)}$. Then

$$(EK^*E)\left(E\begin{bmatrix} C_{1,3}^n \\ C_{2,3}^n \end{bmatrix}\right) - \left(E\begin{bmatrix} C_{1,3}^n \\ C_{2,3}^n \end{bmatrix}\right)(2I + W) \to 0.$$

Since $\sigma(EK^*E) = \sigma(K)$ and $\sigma(K) \cap \sigma(2I + W) = \emptyset$, using Rosenblum's Theorem again, one can see that

$$\left\| E \begin{bmatrix} C_{1,3}^n \\ C_{2,3}^n \end{bmatrix} \right\| \to 0.$$

So $\|C_{1,3}^n\| + \|C_{2,3}^n\| \to 0$. Similarly one can prove that $\|C_{3,1}^n\| + \|C_{3,2}^n\| \to 0$. Thus

$$\|C_{1,3}^n\| + \|C_{2,3}^n\| + C_{3,1}^n + C_{3,2}^n \to 0.$$

For each $n \geq 1$, denote

$$D_n = \begin{bmatrix} C_{1,1}^n & C_{1.2}^n \\ C_{2.1}^n & C_{2.2}^n \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{\cdot} \end{matrix}$$

Then one can deduce that $D_n K^* D_n \to K$ and $\{D_n^2\}$ converges to the identity operator on $H^{(2)}$. So D_n is conjugate-linear and invertible provided that n is large enough.

Since K is compact and $\overline{ranK + ranK^*} = \mathcal{H}^{(2)}$, using a similar argument as in the proof of Theorem 4 in [158], one can prove that there exists a subsequence $\{n_j\}$ of ℕ such that $\{D_{n_j}\}$ converges to a conjugation D on $\mathcal{H}^{(2)}$. Noting that $D_{n_j} K^* D_{n_j} - K \to 0$, we obtain $DK^*D = K$, contradicting the fact that K is not complex symmetric. Thus we have proved that T is an AUET operator; however, T is neither UET nor a norm limit of complex symmetric operators. This shows that

$$[\overline{\text{CSO}} \cup \{\text{UET operators}\} \subsetneq \{\text{AUET operators}\}.$$

Example (6.2.30)[164]: Let S be the unilateral shift defined by $Se_i = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an onb of $\mathcal{H}$. Define $T = S^{(2)} \oplus S^*$. By Theorem (6.2.27) (i), $S \oplus S^*$ is complex symmetric and hence g-normal. By Corollary (6.2.18), it follows that T is g-normal. Note that each AUET operator is biquasitriangular and T is not biquasitriangular. We deduce that T is not AUET. This example combined with Corollary (6.2.24) implies that

$$\{\text{AUET operators}\} \subsetneq \{\text{g} - \text{normal operators}\}.$$

Proposition (6.2.31)[164]: Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying K($\mathcal{H}$) ⊂ $\mathcal{A}$. If φ is an anti-automorphism of $\mathcal{A}$, then there exists an anti-unitary operator D on $\mathcal{H}$ such that

$$\varphi(X) = DX^*D^{-1}, \forall X \in \mathcal{A}.$$

Proof. Arbitrarily choose a conjugation C on $\mathcal{H}$ and define $\rho(X) = C\varphi(X)^*C$ for $X \in C^*(T)$. Then it is easy to see that ρ is a faithful representation of $\mathcal{A}$ on $\mathcal{H}$. Since K($\mathcal{H}$) ⊂ $\mathcal{A}$, we have

$$\mathcal{A}(\mathcal{H}) = C\mathcal{K}(\mathcal{H})C \subset C\mathcal{A}C = \rho(\mathcal{A}).$$

It follows that ρ is irreducible. Then, by Corollary (6.2.54)1 in [153], there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\rho(X) = U^*XU$ for $X \in \mathcal{A}$. Then $\varphi(X) = C\rho(X)^*C = CU^*X^*UC$. Set $D = CU^*$. Then D is an anti-unitary operator on $\mathcal{H}$ and $\varphi(X) = DX^*D^{-1}$ for $X \in \mathcal{A}$.

Corollary (6.2.32)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that K($\mathcal{H}$) ⊂ $C^*(T)$. Then T is g-normal if and only if T is UET.

Proof. The sufficiency follows from Corollary (6.2.24). It suffices to prove the necessity. "⇒". Since T is g-normal, by Lemma (6.2.7), there is an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = T$. Since K($\mathcal{H}$) ⊂ $C^*(T)$, by Proposition (6.2.31), there exists an anti-unitary operator D on $\mathcal{H}$ such that

$$\varphi(X) = DX^*D^{-1}, \forall X \in C^*(T).$$

In particular, $T = \varphi(T) = DT^*D^{-1}$. By Lemma (6.2.21), it follows that T is UET.

Proposition (6.2.33)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. If ρ is an anti-automorphism of $C^*(T)$, then there exists a sequence $\{D_n\}$ of anti-unitary operators on $\mathcal{H}$ such that

$$\rho(X) = \lim_n D_n X^* D_n^{-1} \ , \forall X \in C^*(T).$$

Proof. Arbitrarily choose a conjugation C on $\mathcal{H}$ and define $\varphi(X) = C\rho(X)^*C$ for $X \in C^*(T)$. Then φ is a faithful, nondegenerate representation of $C^*(T)$ on $\mathcal{H}$. Noting that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we have

$$rank\ \varphi(X) = rank\ X = rank\ Id(X)$$

for all $X \in C^*(T)$, where $Id(\cdot)$ is the identity representation of $C^*(T)$. By Lemma (6.2.22), we have $\varphi \cong_a Id$. Hence there exists a sequence $\{U_n\}$ of unitary operators on $\mathcal{H}$ such that $\varphi(X) = \lim_n U_n^* X U_n$ for $X \in C^*(T)$. Thus we have

$$\rho(X) = C\varphi(X)^*C = \lim_n (CU_n^*)X^*(U_nC)$$

for $X \in C^*(T)$.

For each n, set $D_n = CU_n^*$. Then $\{D_n\}$ satisfies all requirements.

Corollary (6.2.34)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be irreducible. If ρ is an anti-automorphism of $C^*(T)$, then there exists a sequence $\{D_n\}$ of anti-unitary operators on $\mathcal{H}$ such that

$$\rho(X) = \lim_n D_n X^* D_n^{-1} \ , \forall X \in C^*(T).$$

Proof. Since T is irreducible, we have either K($\mathcal{H}$) ⊂ $C^*(T)$ *or* $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In view of Propositions (6.2.31) and (6.2.33), one can see the conclusion.

Corollary (6.2.35)[164]: If $T \in \mathcal{B}(\mathcal{H})$ and $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then T is g-normal if and only if $\cong_a T^t$.

Proof. By Corollary (6.2.24), we need only prove the necessity.

Since T is g-normal, by Lemma (6.2.7), there is an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = T$. By Proposition (6.2.33), there exists a sequence $\{D_n\}$ of antiunitary operators on $\mathcal{H}$ such that

$$\varphi(X) = \lim_n D_n X^* D_n^{-1}, X \in C^*(T).$$

In particular, we have $= \varphi(T) = \lim_n D_n T^* D_n^{-1}$. In view of Proposition (6.2.23), it follows that $\cong_a T^t$. This completes the proof.

Theorem (6.2.36)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is irreducible, then T is g-normal if and only if $T \cong_a T^t$.

Proof. The sufficiency follows from Corollary (6.2.24). It remains to prove the necessity. Since T is irreducible, we have either $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ or $\mathrm{K}(\mathcal{H}) \subset C^*(T)$. By Corollaries (6.2.32) and (6.2.35), it follows in either case that $T \cong_a T^t$.

Theorem (6.2.37)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then the following are equivalent:

(i) $T \in \overline{CSO}$;

(ii) $T \in \overline{CSO}^c$;

(iii) $\exists A \in CSO$ such that $A \cong_a T$;

(iv) $T \cong_a T^t$;

(v) T is g-normal.

Proof. By the discussion, "(i)⇒ (v)" is clear. The equivalence of (iv) and (v) follows from Corollary (6.2.35).

"(iii) ⇒ (ii)". By Proposition 4.21 in [184], $A \cong_a T$ implies that for any $\varepsilon > 0$ there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \cong A$. Hence $T + K \in CSO$. This implies that $T \in \overline{CSO}^c$.

The implication "(ii) ⇒ (i)" is trivial. Now it remains to prove "(iv)⇒ (iii)".

"(iv)⇒ (iii)". For $X \in C^*(T)$, define $\rho_1(X) = X$ and $\rho_2(X) = X^{(\infty)}$. Then ρ_1 and ρ_2 are two nondegenerate faithful representations of $C^*(T)$. Since $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we deduce that rank $\rho_1(X) = rank\ \rho_2(X)$ for all $X \in C^*(T)$. Then, by Lemma (6.2.22), $\rho_1 \cong_a \rho_2$. Hence $= \rho_1(T) \cong_a \rho_2(T) = T^{(\infty)}$. Noting that $T \cong_a T^t$, we have

$$T \cong_a T^{(\infty)} \cong (T \oplus T)^{(\infty)} \cong_a (T \oplus T^t)^{(\infty)}.$$

By Lemma (6.2.19), $(T \oplus T^t)^{(\infty)}$ is complex symmetric. Arbitrarily choose an operator A on $\mathcal{H}$ satisfying $\cong (T \oplus T^t)^{(\infty)}$. Hence $A \in CSO$ and $T \cong_a A$. This completes the proof.

Let $\{A_i : 1 \le i \le n\}$ be a commuting family of normal operators on $\mathcal{H}$. Denote by $C^*(A_1, A_2, \dots, A_n)$ the C^*-algebra generated by $A_1, A_2, \dots, A_n$ and the identity I. The joint spectrum of the n-tuple $(A_1, A_2, \dots, A_n)$ is defined as the set of n-tuples of scalars $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that the ideal of $C^*(A_1, A_2, \dots, A_n)$ generated by $A_1 - \lambda_1, A_2 - \lambda_2, A_n - \lambda_n$ is different from $C^*(A_1, A_2, \dots, A_n)$ (see [278]). We let $\sigma(A_1, A_2, \dots, A_n)$ denote the joint spectrum of $(A_1, A_2, \dots, A_n)$.

Proposition (6.2.38)[164]: Let $W \in \mathcal{B}(\mathcal{H})$ be a unilateral weighted shift with positive weights $\{d_i\}_{i=0}^{\infty}$. If $W \cong_a W^*$, then there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ of $\mathbb{N}$ such that $d_{1+n_i} \to 0$, and, for each $k \ge 0$, we have $d_{n_i - k} \to d_k$.

Proof. Since $W \cong_a W^*$, we can choose a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators on $\mathcal{H}$ such that $U_n^* W U_n \to W^*$. As consequences, we have for each $k \ge 1$ that $U_n^* |W^k| U_n \to |(W^k)^*|$ and $U_n^* |(W^k)^*| U_n \to |W^k|$.

Denote $A_0 = |W^*|$ and $A_k = |W^k|$ for $k \ge 1$. Denote $B_0 = |W|$ and $B_k = |(W^k)^*|$ for $k \ge 1$. So, for each $k \ge 0$, one can see that

$$U_n^* A_k U_n \to B_k. \tag{6}$$

Without loss of generality, we may assume that $We_i = d_i e_{i+1}$ for all $i \ge 0$, where $\{e_i\}_{i=0}^{\infty}$ is an onb of $\mathcal{H}$. Note that all A_k's and B_k's are diagonal operators with respect to $\{e_i\}$. For each $k \ge 0$, we assume that

$$A_k = diag\left\{a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \dots\right\} and\ B_k = diag\left\{b_0^{(k)}, b_1^{(k)}, b_2^{(k)}, \dots\right\} \tag{7}$$

with respect to $\{e_i\}$.

Let $k \geq 1$ be fixed. Note that $(A_0, A_1, \dots, A_k)$ and $(B_0, B_1, \dots, B_k)$ are two $(k+1)$-tuples of commuting diagonal operators. In view of (6), there is an isomorphism ρ from $C^*(A_0, A_1, \cdots, A_k)$ onto $C^*(B_0, B_1, \dots, B_k)$ such that $\rho(A_i) = B_i$ for $0 \leq i \leq k$. Thus we have

$$\sigma(A_0, A_1, \dots, A_k) = \sigma(B_0, B_1, \dots, B_k). \tag{8}$$

For $i \geq 0$, define $\omega_i(X) = x_i$ if $X \in C^*(A_0, A_1, \dots, A_k)$ and

$$X = diag\{x_0, x_1, x_2, \dots\}$$

with respect to $\{e_i\}$. Then each ω_i is a multiplicative linear functional on the C^*- algebra $C^*(A_0, A_1, \dots, A_k)$. Moreover, $\{\omega_i : i \geq 0\}$ is dense in the maximal ideal space of $C^*(A_0, A_1, \dots, A_k)$. Then, by [278], it follows from (7) that

$$\sigma(A_0, A_1, \dots, A_k) = \left\{\left(a_i^{(0)}, a_i^{(1)}, \dots, a_i^{(k)}\right) : i \geq 0\right\}^-, \tag{9}$$

where the closure is taken in the usual topology on $\mathbb{R}^n$. Similarly we have

$$\sigma(B_0, B_1, \dots, B_k) = \left\{\left(b_i^{(0)}, b_i^{(1)}, \dots, b_i^{(k)}\right) : i \geq 0\right\}^-. \tag{10}$$

We choose the desired subsequence $\{n_k\}$ of $\mathbb{N}$ as follows.

Step 1. The choice of n_1.

Note that $B_1 = A_0 = diag\{0, d_0, d_1, d_2, \dots\}$ and $B_0 = A_1 = diag\{d_0, d_1, d_2, \dots\}$ with respect to $\{e_i\}$. In view of (9) and (10), $(0, d_0) \in \sigma(A_0, A_1)$ and $\sigma(B_0, B_1)$ is the closure of

$$\{(d_0, 0)\} \cup \{(d_{i+1}, d_i) : i \geq 0\}.$$

By (8), there exists $i > n_1$ such that $d_{i+1} + |d_i - d_0| < \frac{1}{2}$. Denote $n_1 = i$.

Step 2. The choice of n_2.

Note that $A_2 = diag\{d_0d_1, d_1d_2, d_2d_3, \dots\}$ and $B_2 = diag\{0, 0, d_0d_1, d_1d_2, \dots\}$ with respect to $\{e_i\}$. In view of (9) and (10), $(0, d_0, d_0d_1) \in \sigma(A_0, A_1, A_2)$ and $\sigma(B_0, B_1, B_2)$ is the closure of

$$\{(d_0, 0, 0), (d_1, d_0, 0)\} \cup \{(d_{i+1}, d_i, d_{i-1}d_i) : i \geq 1\}.$$

By (8), there exists $i > n_1$ such that

$$\max\{d_{i+1}, |d_i - d_0|, |d_{i-1}d_i - d_0d_1|\} < \frac{d_0}{24(1 + \|W\|)(1 + d_0)};$$

furthermore, we have

$$d_{i+1} + |d_i - d_0| + |d_{i-1} - d_1| < \frac{1}{24} + \frac{1}{24} + \frac{|d_0d_1 - d_0d_{i-1}|}{d_0}$$
$$\leq \frac{1}{12} + \frac{|d_0d_1 - d_{i-1}d_i| + |d_{i-1}d_i - d_0d_{i-1}|}{d_0}$$
$$< \frac{1}{12} + \frac{1}{12} + \frac{|d_{i-1}d_i - d_0d_{i-1}|}{d_0} \leq \frac{1}{6} + \frac{\|W\| \cdot |d_i - d_0|}{d_0} < \frac{1}{6} + \frac{1}{12} = \frac{1}{2^2}.$$

Denote $n_2 = i$.

Step 3. The choice of n_3.

Note that $A_3 = diag\{d_0d_1d_2, d_1d_2d_3, d_2d_3d_4, \dots\}$ and $B_3 = diag\{0, 0, 0, d_0d_1d_2, d_1d_2d_3, d_2d_3d_4, \dots\}$ with respect to $\{e_i\}$. In view of (9) and (10), $(0, d_0, d_0d_1, d_0d_1d_2) \in \sigma(A_0, \dots, A_3)$ and $\sigma(B_0, \dots, B_3)$ is the closure of

$$\{(d_0, 0, 0, 0), (d_1, d_0, 0, 0), (d_2, d_1, d_0d_1, 0)\} \cup \{(d_{i+1}, d_i, d_{i-1}d_i, d_{i-2}d_{i-1}d_i) : i \geq 2\}.$$

By (8), there exists $i > n_2$ such that

$$\max\{d_{i+1}, |d_i - d_0|, |d_{i-1}d_i - d_0d_1|, |d_{i-2}d_{i-1}d_i - d_0d_1d_2|\}$$

is small enough to guarantee $d_{i+1} + |d_i - d_0| + |d_{i-1} - d_1| + |d_{i-2} - d_2| < \frac{1}{2^3}$. Denote $n_3 = i$.

Using a similar argument as above, one can choose a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that

$$d_{n_k+1} + |d_{n_k} - d_0| + |d_{n_k-1} - d_1| + \cdots + |d_{n_k-k+1} - d_{k-1}| < 1/2^k$$

for each $k \geq 1$. Thus $\{n_k\}$ is the desired subsequence of $\mathbb{N}$.

Theorem (6.2.39)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a unilateral weighted shift with positive weights. Then the following are equivalent:

(i) $T \in \overline{CSO}$;

(ii) $T \in \overline{CSO^c}$;

(iii) $\exists A \in CSO$ such that $A \cong_a T$;

(iv) $T \cong_a T^*$;

(v) T is g-normal;

(vi) T is approximately Kakutani.

Proof. By definitions, "(iii)⇒(ii)⇒(i)" are obvious. Without loss of generality, we may directly assume that $T e_i = w_i e_{i+1}$ for all $i \geq 1$, where $\{e_i\}_{i\in\mathbb{N}}$ is an onb of $\mathcal{H}$ and $w_i > 0$ for all i. Thus we can define a conjugation C on $\mathcal{H}$ satisfying $Ce_i = e_i$ for all i. Noting that $CT^*C = T^*$, we deduce that T^* is a transpose of T. Then the implication "(i)⇒(iv)" follows from Lemma (6.2.25). Since T is irreducible, "(iv)⇔(v)" follows from Theorem (6.2.36). "(iv)⇒(iii)". Since $T \cong_a T^*$, we claim that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In fact, if not, then $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. Note that $T \cong_a T^*$ induces an automorphism ρ of $C^*(T)$ satisfying $\rho(T) = T^*$. Since $\mathcal{K}(\mathcal{H}) \subset C^*(T)$, it follows from [153] that ρ is unitarily implemented. Thus $T \cong T^*$, a contradiction. So $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. $T \cong_a T^*$ implies that T is AUET. By Theorem (6.2.37), there exists $A \in CSO$ such that $A \cong_a T$.

The implications "(vi)⇒(i)" and "(iv)⇒(vi)" follow from [158] and Proposition (6.2.38) respectively. This completes the proof.

Corollary (6.2.40)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is an irreducible unilateral weighted shift and $T \in \overline{CSO}$, then $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

We deal with bilateral weighted shifts with positive weights.

Proposition (6.2.41)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights. If T is reducible, then T is invertible and completely reducible; moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Proof. Assume that $\{w_i\}_{i\in\mathbb{Z}}$ is the weight sequence of T. Since T is reducible, by [141], $\{w_i\}$ is periodic. Thus $\inf_i w_i > 0$ and T is invertible. By [154], if T is completely reducible, then $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. So it suffices to prove that T is completely reducible.

Now we may assume that $\{w_i\}$ is of period n. When $n = 1, \{w_i\}$ is constant; in this case, T is normal without eigenvalues and hence completely reducible. In the rest of this proof we deal with the case that $n > 1$.

Without loss of generality, we assume that $T e_i = w_i e_{i+1}$ for $i \in \mathbb{Z}$, where $\{e_i\}_{i\in\mathbb{Z}}$ is an onb of $\mathcal{H}$. Let U be the bilateral shift on $\mathcal{H}$ defined by $Ue_i = e_{i+1}$ for $i \in \mathbb{Z}$. Set

$$A = \begin{bmatrix} 0 & & & & \\ w_1 I & 0 & & & w_n U \\ & w_2 I & \ddots & & \\ & & \ddots & 0 & \\ & & & w_{(n-1)} I & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \vdots \\ \mathcal{H} \\ \mathcal{H} \end{matrix},$$

where I is the identity operator on $\mathcal{H}$ and all omitted entries are zero. Then $A \in \mathcal{B}\left(\mathcal{H}^{(n)}\right)$ is invertible. Since $w_i = w_{i+n}$ for all $i \in \mathbb{Z}$, it is easy to see that $A \cong T$. So we need only prove that A is completely reducible.

Let $P \in \mathcal{B}\left(\mathcal{H}^{(n)}\right)$ be a nonzero projection which commutes with A. Assume that P admits the following matrix representation:

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,n} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,1} & P_{n,2} & \cdots & P_{n,n} \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \vdots \\ \mathcal{H} \end{matrix}.$$

Since $P\,A = AP$, a straightforward matrical calculation shows that there exist positive numbers $\{\lambda_{i,j}\}_{1 \le i,j \le n}$ satisfying

$$\frac{P_{l,l+k}}{\lambda_{l,l+k}} = \frac{P_{n-k,n}}{\lambda_{n-k,n}} = \frac{P_{n-k+s,s}U}{\lambda_{n-k+s,s}} \qquad (11)$$

whenever $1 \le k \le n-1, 1 \le l \le n-k$ and $1 \le s \le k$; in particular, we have $P_{1,1} = P_{2,2} = \cdots = P_{n,n}$.

On the other hand, since P is self-adjoint, $P\,A = AP$ implies that $P|A| = |A|P$. Noting that

$$|A| = \begin{bmatrix} w_1 I & & & \\ & w_2 I & & \\ & & \ddots & \\ & & & w_n I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \vdots \\ \mathcal{H} \end{matrix},$$

it follows from $P|A| = |A|P$ that

$$\text{“}1 \le i,j \le n, w_i \ne w_j\text{”} \Rightarrow \text{“}P_{i,j} = 0\text{”}. \qquad (12)$$

Now let $1 \le k \le n-1$ be fixed. We claim that there exist $1 \le i,j \le n$ with $i + k = j$ or $i + n - k = j$ such that $w_i \ne w_j$. In fact, since $\{w_m\}$ is of period n, there must exist $l \in \mathbb{Z}$ such that $w_{l+k} \ne w_l$; in addition, we may directly assume that $1 \le l \le n$. If $l + k \le n$, then set $i = l$ and $j = l + k$; if $l + k > n$, set $i = l + k - n$ and $j = l$. In either case, one can verify that $1 \le i,j \le n$ with $i + k = j$ or $i + n - k = j$; moreover, $w_i \ne w_j$. In view of (12), it follows that $P_{i,j} = 0$. Furthermore, by (11), we have either $P_{1,1+k} = 0$ or $P_{1,1+n-k} = 0$. We claim that the latter also implies $P_{1,1+k} = 0$. In fact, since P is self-adjoint, the latter implies $P_{n-k+1,1} = 0$; using (11), we have $P_{1,1+k} = 0$. Thus we have proved that $P_{1,1+k} = 0$. Since $1 \le k \le n-1$ is arbitrary, by (11), we have $P_{i,j} = 0$ for any i, j with $1 \le i < j \le n$. Noting that P is self-adjoint, it follows that $P = \bigoplus_{i=1}^{n} P_{i,i}$.

Since we have proved $P_{1,1} = P_{2,2} = \cdots = P_{n,n}, P$ can be written as $P = P_{1,1}^{(n)}$. Then it follows that $P_{1,1}$ is a nonzero projection on $\mathcal{H}$ commuting with U. Since U is completely reducible, we can choose a nonzero proper subprojection Q of $P_{1,1}$ such that $QU = UQ$. Thus $Q^{(n)}$ is a nonzero proper subprojection of P commuting with A. Hence we conclude that A is completely reducible.

Lemma (6.2.42)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $T\,e_i = w_i e_{i+1}$ for $i \in \mathbb{Z}$, where $w_i > 0$ for all i and $\{e_i\}_{i \in \mathbb{Z}}$ is an onb of $\mathcal{H}$. Assume that $V \in \mathcal{B}(\mathcal{H})$ is unitary and $T\,V = VT^*$. Then

(i) $V\,e_k \in \{e_j : w_j = w_{k-1}\}$ for all $k \in \mathbb{Z}$;

(ii)if $k, l \in \mathbb{Z}$ and $V\,e_k, e_l \ne 0$, then $\langle V\,e_{k-j}, e_{l+j}\rangle \ne 0$ for all $j \in \mathbb{Z}$.

Proof. Since $T\,V = V\,T^*$, one can see that $|T|V = V|T^*|$.

(i) It is easy to check that $|T|e_j = w_j e_j$ and $|T^*|e_j = w_{j-1}e_j$ for all $j \in \mathbb{Z}$. Then, given $k \in \mathbb{Z}$, we have

$$w_{k-1}Ve_k = V(w_{k-1}e_k) = V|T^*|e_k = |T|Ve_k;$$

that is, $Ve_k \in \ker(|T| - w_{k-1}) = \bigvee\{e_j : w_j = w_{k-1}\}$.

(ii) For any $s, t \in \mathbb{Z}$, we have

$$w_{s-1}\langle Ve_{s-1}, e_{t+1}\rangle = \langle V(w_{s-1}e_{s-1}), e_{t+1}\rangle = \langle VT^*e_s, e_{t+1}\rangle = \langle TVe_s, e_{t+1}\rangle$$
$$= \langle Ve_s, T^*e_{t+1}\rangle = \langle Ve_s, w_t e_t\rangle = \langle w_t Ve_s, e_t\rangle.$$

Thus $\langle Ve_{s-1}, e_{t+1}\rangle = 0$ if and only if $\langle Ve_s, e_t\rangle \neq 0$. Using an obvious inductive argument, one can see the conclusion.

Lemma (6.2.43)[164]: ([172], Theorem (6.2.41)). Let $\{e_i\}_{i\in\mathbb{Z}}$ be an onb of $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ with $Te_i = w_i e_{i+1}$ for $i \in \mathbb{Z}$. If $w_i \neq 0$ for all $i \in \mathbb{Z}$, then T is complex symmetric if and only if there exists $k \in \mathbb{Z}$ such that $|w_{k-j}| = |w_j|$ for all $j \in \mathbb{Z}$.

Theorem (6.2.44)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then the following are equivalent:

(i) $T \in \overline{CSO}$;
(ii) $T \in CSO$;
(iii) $T \cong T^*$;
(iv) $T \cong_a T^*$;
(v) T is g-normal;
(vi) $\exists k \in \mathbb{Z}$ such that $w_i = w_{k-i}$ for all $i \in \mathbb{Z}$.

Proof. Without loss of generality, we may directly assume that $Te_i = w_i e_{i+1}$ for all i, where $\{e_i\}_{i\in\mathbb{Z}}$ is an onb of $\mathcal{H}$. Thus we can define a conjugation C on $\mathcal{H}$ satisfying $Ce_i = e_i$ for all i. Noting that $CT^*C = T^*$, we deduce that T^* is a transpose of T. Then, by Corollary (6.2.24) and Lemma (6.2.25), the implications "(i)⇒(iv)⇒(v)" are obvious.

By Proposition (6.2.41), it follows from $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$ that T is irreducible and $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. So the equivalence between (iii) and (v) is given by Corollary (6.2.32). The equivalence between (ii) and (vi) is given by Lemma (6.2.43). The implication "(ii)⇒(i)" is trivial. Now it remains to prove "(iii)⇒(vi)". "(iii)⇒(vi)". Since $T \cong T^*$, we can choose a unitary operator V on $\mathcal{H}$ such that $T\,V = V\,T^*$. Thus there exists $k \in \mathbb{Z}$ such that $\langle Ve_1, e_k\rangle \neq 0$. Then, by Lemma (6.2.42) (ii), $\langle Ve_{j+1}, e_{k-j}\rangle \neq 0$ for all $j \in \mathbb{Z}$. By Lemma (6.2.42) (i), it follows that $w_j = w_{k-j}$ for all $j \in \mathbb{Z}$. This completes the proof.

Lemma (6.2.45)[164]: ([280], Prop. 2.2.14 and Thm. (6.2.14).7). Let $A, B \in \mathcal{B}(\mathcal{H})$ be two bilateral weighted shifts with positive weights. If A, B are both invertible, then $A \cong_a B$ if and only if $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$.

Lemma (6.2.46)[164]: ([281], Lemma (6.2.39).1). Let $A \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $A = U|A|$. If ρ is any representation of $C^*(A)$ on $\mathcal{H}_\rho$ for which $\ker\rho(A) = \ker\rho(A)^* = \{0\}$, then ρ has an extension to a representation of $C^*(A, U)$ on the same Hilbert space $\mathcal{H}_\rho$.

Theorem (6.2.47)[164]: Let $A, B \in \mathcal{B}(\mathcal{H})$ be two bilateral weighted shifts with positive weights. Then $A \cong_a B$ if and only if $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$.

Proof. Without loss of generality, we assume that $\{e_i\}_{i\in\mathbb{Z}}$ is an onb of $\mathcal{H}$ and

$$Ae_i = w_i e_{i+1}, Be_i = v_i e_{i+1},$$

where $w_i, v_i > 0$ for all $i \in \mathbb{Z}$.

"⇐". Arbitrarily choose an $\varepsilon > 0$. Set

$$a_i = \begin{cases} w_i, & if\ w_i \geq \varepsilon, \\ \varepsilon, & if\ w_i < \varepsilon, \end{cases} \quad \text{and} \quad b_i = \begin{cases} v_i, & if\ v_i \geq \varepsilon, \\ \varepsilon, & if\ v_i < \varepsilon. \end{cases}$$

Define $A_\varepsilon e_i = a_i e_{i+1}$ and $B_\varepsilon e_i = b_i e_{i+1}$ for $i \in \mathbb{Z}$. Then A_ε, B_ε are two invertible bilateral weighted shifts and $\max\{\|A_\varepsilon - A\|, \|B_\varepsilon - B\|\} \leq \varepsilon$.

Since $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$, it is not difficult to see that $\Sigma_n(A_\varepsilon) = \Sigma_n(B_\varepsilon)$ for all $n \geq 1$. So, by Lemma (6.2.45), there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\|U^* A_\varepsilon U - B_\varepsilon\| < \varepsilon$. Hence

$$\|U^* A U - B\| \leq \|U^*(A - A_\varepsilon)U\| + \|U^* A_\varepsilon U - B_\varepsilon\| + \|B_\varepsilon - B\| < 3\varepsilon.$$

Since ε was arbitrarily chosen, we deduce that $A \cong_a B$.

"⇒". If A is reducible, then, by Proposition (6.2.41), A is invertible. Since $A \cong_a B$, it follows that B is also invertible. In view of Lemma (6.2.45), we obtain $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$. When B is reducible, the proof is similar. So, in the following, we may assume that A and B are both irreducible.

Denote by U the bilateral shift on $\mathcal{H}$ defined by $Ue_i = e_{i+1}$ for all $i \in \mathbb{Z}$. Thus $A = U|A|$ and $B = U|B|$ are respectively the polar decomposition of A and the polar decomposition of B.

Since $A \cong_a B$, there exists an isomorphism $\rho: C^*(A) \to C^*(B)$ so that $\rho(A) = B$. Thus ρ is a faithful representation of $C^*(A)$ on $\mathcal{H}$. Note that $\ker B = \{0\} = \ker B^*$. Then, by Lemma (6.2.46), ρ has an extension to a representation ρ' of $C^*(A, U)$ on $\mathcal{H}$. Noting that

$$B = \rho'(A) = \rho'(U)\rho'(|A|) = \rho'(U)\rho(|A|) = \rho'(U)|B|,$$

one can deduce that $\rho'(U) = U$. Thus, for any polynomial $p(\cdot,\cdot,\cdot)$ in three free variables, we have $\|p(B, B^*, U)\| \leq \|p(A, A^*, U)\|$. By the symmetry, we can also prove that $\|p(A, A^*, U)\| \leq \|p(B, B^*, U)\|$. Hence $\rho': C^*(A, U) \to C^*(B, U)$ is an isomorphism.

Define $\tilde{A} = U(|A| + I)$ and $\tilde{B} = U(|B| + I)$. It is easy to see that

$$C^*(A, U) = C^*(\tilde{A}), C^*(B, U) = C^*(\tilde{B})$$

and $\rho'(\tilde{A}) = \tilde{B}$. Since A and B are both irreducible, it follows from [141] that $\tilde{A}$ and $\tilde{B}$ are both irreducible. Thus we have either $\mathcal{K}(\mathcal{H}) \subset C^*(\tilde{A})$ or $C^*(\tilde{A}) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Case 1. $\mathcal{K}(\mathcal{H}) \subset C^*(\tilde{A})$. In this case, since ρ' is a faithful representation of $C^*(\tilde{A})$, by Corollary (6.2.54)1 in [153], it must be unitarily implemented; that is, there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\rho'(X) = U^* X U$ for all $X \in C^*(\tilde{A})$. In particular, $\tilde{B} = U^* \tilde{A} U$. Note that $\tilde{A}, \tilde{B}$ are both invertible bilateral weighted shifts with positive weights. Thus, by Lemma (6.2.45), we have $\Sigma_n(\tilde{A}) = \Sigma_n(\tilde{B})$ for all $n \geq 1$. So we conclude that $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$.

Case 2. $C^*(\tilde{A}) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In this case, since $\tilde{B}$ is irreducible, we claim that $C^*(\tilde{B}) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In fact, if not, then $\mathcal{K}(\mathcal{H}) \subset C^*(\tilde{B})$ and, using a similar argument as in Case 1, one can prove that $\tilde{A} \cong \tilde{B}$. Hence $\mathcal{K}(\mathcal{H}) \subset C^*(\tilde{A})$, a contradiction. By the claim, we have $rank\ X = rank\ \rho'(X)$ for all $X \in C^*(\tilde{A})$. By Lemma (6.2.22), it follows that $\rho' \cong_a Id$, where $Id(\cdot)$ is the identity representation of $C^*(\tilde{A})$. So we obtain $A \cong_a B$. By Lemma (6.2.45), it implies that $\Sigma_n(\tilde{A}) = \Sigma_n(\tilde{B})$ for all $n \geq 1$; that is, $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$. This completes the proof.

Corollary (6.2.48)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. Then $T \cong_a T^*$ if and only if $\Sigma_n(T)^t = \Sigma_n(T)$ for all $n \geq 1$.

Proof. Note that T^* is unitarily equivalent to a bilateral weighted shift A with weights $\{v_i\}_{i\in\mathbb{Z}}$, where $v_i = w_{-i}$ for all $i \in \mathbb{Z}$. It is easy to see that $\Sigma_n(A) = \Sigma_n(T)^t$ for all $n \geq 1$. Then, by Theorem (6.2.47), we have

$$T \cong_a T^* \Rightarrow T \cong_a A \Leftrightarrow \Sigma_n(T) = \Sigma_n(T)^t, \forall n \geq 1.$$

Theorem (6.2.49)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. If T is reducible, then the following are equivalent:

(i) $T \in \overline{CSO}$;

(ii) $T \in CSO$;

(iii) T is g-normal;

(iv) $T \cong_a T^*$;

(v) $T \cong T^*$;

(vi) $\exists k \in \mathbb{Z}$ such that $w_i = w_{k-i}$ for all $i \in \mathbb{Z}$.

Proof. We first note that T^* is also a transpose of T.

The implication "(vi)⇒(ii)" follows from Lemma (6.2.43). By definition, the implications "(ii)⇒(v)⇒(iv)" are obvious.

Since T is reducible, by Problem 159 in [141], $\{w_i\}$ is periodic. By Proposition (6.2.41), T is invertible and $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Thus, by Theorem (6.2.37), (i), (iii) and (iv) are equivalent. Now it remains to prove "(iv)⇒(vi)".

"(iv)⇒(vi)". Note that T^* is unitarily equivalent to a bilateral weighted shift A with weights $\{v_i\}_{i\in\mathbb{Z}}$, where $v_i = w_{-i}$ for all $i \in \mathbb{Z}$. Then we have $T \cong_a A$. It follows from Theorem (6.2.47) that $\Sigma_i(T) = \Sigma_i(A)$ for all $i \geq 1$.

We may assume that $\{w_i\}$ is of period n. Since $\Sigma_n(T) = \Sigma_n(A) = \Sigma_n(T)^t$, there exists $i \in \mathbb{Z}$ such that

$$(w_1, w_2, \dots, w_n) = (w_{i+n}, w_{i+n-1}, \dots, w_{i+1}).$$

Case 1. n divides i. Noting that $\{w_i\}_{i\in\mathbb{Z}}$ is of period n, it follows that

$$(w_1, w_2, \dots, w_n) = (w_n, w_{n-1}, \dots, w_1).$$

So $w_i = w_{n+1-i}$ for all $i \in \mathbb{Z}$.

Case 2. n does not divide i. Thus there exists $1 \leq m < n$ such that

$$(w_1, w_2, \dots, w_n) = (w_m, w_{m-1}, \dots, w_1, w_n, w_{n-1}, \dots, w_{m+1}).$$

Noting that $\{w_i\}_{i\in\mathbb{Z}}$ is of period n, we deduce that $w_i = w_{m+1-i}$ for all $i \in \mathbb{Z}$. This completes the proof.

Proposition (6.2.50)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a reducible bilateral weighted shift with positive weights $\{w_i\}_{i\in\mathbb{Z}}$. If $T \in CSO$, then T is completely complex symmetric.

Proof. Since T is reducible, by Problem 159 in [141], $\{w_i\}$ is periodic and we may assume that $\{w_i\}$ is of period n. If $n = 1$, then T is normal without eigenvalues and hence completely complex symmetric. It suffices to give the proof in the case that $n > 1$.

Let U be the bilateral shift on $\mathcal{H}$ defined by $Ue_i = e_{i+1}$ for $i \in \mathbb{Z}$. Set

$$A = \begin{bmatrix} 0 & & & & w_n U \\ w_1 I & 0 & & & \\ & w_2 I & \ddots & & \\ & & \ddots & 0 & \\ & & & w_{n-1} I & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \vdots \\ \mathcal{H} \\ \mathcal{H} \end{matrix},$$

where I is the identity operator on $\mathcal{H}$ and all omitted entries are zero. Then $A \in \mathcal{B}\left(\mathcal{H}^{(n)}\right)$. Since $w_i = w_{i+n}$ for all $i \in \mathbb{Z}$, it is easy to see that $A \cong T$. So we need only prove that A is completely complex symmetric.

Arbitrarily choose a nontrivial reducing subspace M of A. It suffices to prove that $A|_M$ is complex symmetric. Let P be the projection of $\mathcal{H}^{(n)}$ onto M. Then $PA = AP$. By the proof of Proposition (6.2.41), P can be written as $P = P_0^{(n)}$, where P_0 is a projection on $\mathcal{H}$ commuting with U.

Since U is unitary and hence complex symmetric, there exists a conjugation D on $\mathcal{H}$ such that $DUD = U^*$. Then, for each polynomial $p(\cdot,\cdot)$ in two free variables, we have $Dp(U^*, U)D = p(U^*, U)^*$. Note that there exists a sequence $\{p_n\}$ of polynomials in two free variables such that $\{p_n(U^*, U)\}$ converges to P_0 in the weak operator topology (see page 282, Thm. (6.2.70) in [130]). It follows that $DP_0D = P_0$.

Since T is complex symmetric, by the proof of "(iv)⇒(vi)" in Theorem (6.2.49), we have either (a)

$$(w_1, w_2, \dots, w_n) = (w_k, w_{k-1}, \dots, w_1, w_n, w_{n-1}, \dots, w_{k+1})$$

for some $1 \le k < n$, or (b) $(w_1, w_2, \dots, w_n) = (w_n, w_{n-1}, \dots, w_1)$. Thus the rest of the proof is divided into two cases.

In case (a), we set

$$C = \left[\begin{array}{cccc|cccc} & & & UD & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & \\ & UD & & & & & & \\ UD & & & & & & & \\ \hline & & & & & & & D \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & D & & \\ & & & & D & & & \end{array}\right] \begin{array}{l} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_k \\ \mathcal{H}_{k+1} \\ \mathcal{H}_{k+2} \\ \mathcal{H}_{k+3} \\ \vdots \\ \mathcal{H}_n \end{array},$$

where $\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$ and all omitted entries are zero. In case (b), set

$$C = \left[\begin{array}{c|cccc} UD & & & & \\ \hline & & & & D \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & D & & \\ & D & & & \end{array}\right] \begin{array}{l} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_{n-1} \\ \mathcal{H}_n \end{array}$$

In either case, one can verify that C is a conjugation on $\mathcal{H}^{(n)}$ and $CAC = A^*$. Since $DP_0 = P_0D$ and $P_0U = UP_0$, a direct calculation shows that $CP = P\,C$, that is, M reduces C. Set $C = C|_M$. Then $\tilde{C}$ is a conjugation and it follows from $CAC = A^*$ that $\tilde{C}(A|_M)\tilde{C} = (A|_M)^*$. So $A|_M$ is complex symmetric. This completes the proof.

Theorem (6.2.51)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights. Then the following are equivalent:

(i) $T \in \overline{CSO}$;
(ii) $T \in \overline{CSO^c}$;
(iii) $\exists A \in CSO$ such that $A \cong_a T$;
(iv) $T \cong_a T^*$;
(v) T is g-normal;
(vi) $\Sigma_n(T)^t = \Sigma_n(T)$ for all $n \ge 1$.

Proof. If $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then, by Theorem (6.2.37), (i)-(v) are equivalent. By Corollary (6.2.48), $T \cong_a T^*$ if and only if $\Sigma_n(T) = \Sigma_n(T)^t$ for all $n \ge 1$. So, in this case, (i)-(vi) are all equivalent.

If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \ne \{0\}$, then, by Proposition (6.2.41), T is irreducible and $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. In view of Theorem (6.2.44), one can see the following implications:

(i)⇔(iv)⇔(v) ⇒ (iii) ⇒ (ii) ⇒ (i); that is, (i)-(v) are all equivalent. On the other hand, it follows from Corollary (6.2.48) that (iv) and (vi) are equivalent. This completes the proof.

The main theorem is the following theorem which characterizes when an essentially normal operator is g-normal. We obtain a canonical decomposition for such operators.

Corollary (6.2.52)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then T is g-normal if and only if T_{abnor} is g-normal.

Proof. Since each normal operator is g-normal, the sufficiency is evident.

"⇒". Note that if R is an irreducible operator, then R is either normal or abnormal. If T is essentially normal and g-normal, then, by Theorem (6.2.58), Tabnor is unitarily equivalent to a direct sum of irreducible UET operators and operators with a form of $A^{(m)} \oplus (A^t)^{(n)}$, where A is irreducible but not UET. In view of Theorem (6.2.58), T_{abnor} is still g-normal.

Corollary (6.2.53)[164]: If $T \in \mathcal{K}(\mathcal{H})$ is essentially normal, then T is g-normal if and only if $T^{(\infty)}$ is complex symmetric.

Proof. The sufficiency follows from Corollary (6.2.18).

"⇒". If $R \in \mathcal{B}(\mathcal{H})$ is UET, then it is easy to see that

$$R^{(\infty)} \cong (R \oplus R)^{(\infty)} \cong (R \oplus R^t)^{(\infty)}.$$

If $= A^{(m)} \oplus (A^t)^{(n)}$, then $Q^{(\infty)} = A^{(\infty)} \oplus (A^t)^{(\infty)} = (A \oplus A^t)^{(\infty)}$.

By the above discussion and Theorem (6.2.58), if T is g-normal, then $T^{(\infty)}$ is unitarily equivalent to a direct sum of normal operators and operators with a form of $B \oplus B^t$. By Lemmas (6.2.14) and (6.2.19), $T^{(\infty)}$ is complex symmetric.

To give the proof of Theorem (6.2.62), we need to make some preparations. Let $\{\mathcal{A}_i\}_{i\in\Lambda}$ be a family of C^*-algebras. Following Arveson [147], we let $\sum_{i\in\Lambda}\mathcal{A}_i$ denote the direct sum of $\{\mathcal{A}_i\}_{i\in\Lambda}$.

Lemma (6.2.54)[164]: ([180], Theorem I.10.8). Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{K}(\mathcal{H})$. Then there are Hilbert spaces $\mathcal{H}_0, \mathcal{H}_i$ for $i \in \Lambda$ and nonnegative integers n_i so that

$$\mathcal{H} \cong \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}\right) \quad and \quad \mathcal{A} \cong 0 \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Corollary (6.2.55)[164]: Let $T \in \mathcal{B}(\mathcal{H})$. If T is essentially normal, then $T \cong N \oplus \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right)$, where N is normal, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \ncong T_j$ whenever $i \neq j$; moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) \cong 0 \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$.

Proof. Denote $A = C^*(T) \cap \mathcal{K}(\mathcal{H})$. Then $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{K}(\mathcal{H})$. By Lemma (6.2.54), there are Hilbert spaces H_0, H_i for $i \in \Lambda$ and nonnegative integers n_i so that

$$\mathcal{H} \cong \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}\right) \quad and \quad \mathcal{A} \cong 0 \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Then there exists an operator A acting on $\widetilde{\mathcal{H}} \cong \mathcal{H}_0 \oplus \left(\oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}\right)$ so that $T \cong A$ and

$$C^*(A) \cap \mathcal{K}(\widetilde{\mathcal{H}}) = 0 \oplus \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}. \tag{13}$$

Since $C^*(A) \cap \mathcal{K}(\widetilde{\mathcal{H}})$ is an ideal of $C^*(A)$, one can see that $A = T_0 \oplus_{i\in\Lambda} T_i^{(n_i)}$ with respect to the decomposition $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus (\oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$, where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ and $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Lambda$; moreover, $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ for each $i \in \Lambda$. So each T_i is irreducible. On the other hand, it is evident that $T_i \ncong T_j$ whenever $i, j \in \Lambda$ and $i \neq j$.

Since $A \cong T$ is essentially normal, $A^*A - AA^*$ is compact. By (13), it follows that $T_0^*T_0 - T_0T_0^* = 0$, that is, T_0 is normal. Denote $N = T_0$. This completes the proof.

Lemma (6.2.56)[164]: ([184], Proposition 4.27). Let $S, T \in \mathcal{B}(\mathcal{H})$ and assume that T is essentially normal. Then $T \cong_a S$ if and only if $S_{abnor} \cong T_{abnor}, \sigma_e(S) = \sigma_e(T)$ and $dim \ker(\lambda - S) = dim \ker(\lambda - T)$ for all $\lambda \in [\sigma(S) \cup \sigma(T)] \backslash \sigma_e(T)$.

Proposition (6.2.57)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then the following are equivalent:

(i) T is UET;

(ii) T is AUET;

(iii) there exists an anti-automorphism φ of $C^*(T)$ such that $\varphi(T) = T$ and $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(T)$.

Proof. The implication "(i)⇒(ii)" is clear, and the equivalence of (ii) and (iii) follows from Proposition (6.2.23).

"(ii)⇒(i)". Let $T = N \oplus A$, where $N \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_2)$ are the normal part and the abnormal part of T respectively. Arbitrarily choose a conjugation C_1 on $\mathcal{H}_1$ and a conjugation C_2 on H_2. Then $S := (C_1 N^* C_1) \oplus (C_2 A^* C_2)$ is a transpose of T. Since T is AUET, we have $T \cong_a S$.

Note that $S_{nor} = C_1 N^* C_1$ and $S_{abnor} = C_2 A^* C_2$. Since N is normal, it follows that $N \cong C_1 N^* C_1$, that is, $T_{nor} \cong S_{nor}$. Since T is essentially normal and $T \cong_a S$, it follows from Lemma (6.2.56) that $T_{abnor} \cong S_{abnor}$. Thus $T \cong S$ and T is UET.

Theorem (6.2.58)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then T is g-normal if and only if it is unitarily equivalent to a direct sum of the following three kinds of gnormal operators (some of the summands may be absent):

(i) normal operators;

(ii) irreducible UET operators;

(iii) operators with form of $A^{(m)} \oplus (A^t)^{(n)}$, where A is irreducible, not UET and $m, n \in \mathbb{N}$.

Proof. By Corollary (6.2.18), Lemma (6.2.19) and Corollary (6.2.24), the sufficiency is obvious. We need only prove the necessity.

By Corollary (6.2.55), we may directly assume that $T = N \oplus \left(\oplus_{i \in \Lambda} T_i^{(n_i)}\right)$, where $N \in \mathcal{B}(\mathcal{H}_0)$ is normal, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \ncong T_j$ whenever $i \neq j$. Moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$. Since T is essentially normal, it is obvious that $1 \leq ni < \infty$ for all i.

Noting that T is g-normal, it follows from Corollary (6.2.18) that $S := N \oplus (\oplus_{i \in \Lambda} T_i)$ is also g-normal. Denote $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus (\oplus_{i \in \Lambda} \mathcal{H}_i)$. Thus $S \in \mathcal{B}(\mathcal{H})$ and $C^*(S) \cap \mathcal{K}(\mathcal{H}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)$. Moreover, by Lemma (6.2.7), the map ρ defined by

$$\rho: C^*(S) \to C^*(S),$$
$$p(S^*, S) \longmapsto p(S, S^*)^*$$

is an anti-automorphism of $C^*(S)$. Note that $\rho^{-1} = \rho$.

Claim 1. $rank\ X = rank\ \rho(X)$ for all $X \in C^*(S)$.

Let $p(\cdot,\cdot)$ be a polynomial in two free variables. Noting that S is essentially normal, it can be verified that $p(S^*, S) = \tilde{p}(S, S^*)^* + K$ for some $K \in \mathcal{K}(\widetilde{\mathcal{H}})$. So $\pi\big(p(S^*, S)\big) = \pi\left(\rho\big(p(S^*, S)\big)\right)$, where π is the canonical quotient map of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\widetilde{\mathcal{H}})/\mathcal{K}(\widetilde{\mathcal{H}})$. Furthermore, one can deduce that $\pi(X) = \pi(\rho(X))$ for all $X \in C^*(S)$. It follows that an operator $X \in C^*(S)$ is compact if and only if $\rho(X)$ is compact.

We first note that if P is a rank-one projection in $C^*(S)$, then $\rho(P)$ is a minimal projection of $C^*(S)$ and $\rho(P)$ is compact. Since $C^*(S) \cap \mathcal{K}(\mathcal{H}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)$, one can see that $\rho(P)$ is of rank one. It shows that ρ maps each rank-one projection in $C^*(S)$ to another rank-one projection. On the other hand, if Y is a finite-rank positive operator in

$C^*(S)$, then Y can be written as $Y = \sum_{i=1}^{m} \lambda_i P_i$, where $\lambda_i > 0, \{P_j\}_{j=1}^{n}$ are pairwise orthogonal projections in $C^*(S)$ and $rank\ P_i = 1$ for all i. Noting that ρ is an anti-automorphism, we have

$$rank\ \rho(Y\ ) = rank \sum_{i=1}^{m} \lambda_i \rho(P_i) = \sum_{i=1}^{m} rank\ \rho(P_i) = \sum_{i=1}^{m} rank\ P_i = rank\ Y.$$

Now fix an operator $X \in C^*(S)$. We shall prove that rank $\rho(X) = rank\ X$. For a proof by contradiction, we assume that $rank\ \rho(X) \neq rank\ X$. Noting that $\rho^{-1} = \rho$, without loss of generality, we may directly assume that rank $X < rank\ \rho(X)$. Thus $rank\ X < \infty$. Denote $Z = \rho(X)$. Since ρ is an anti-automorphism of $C^*(S)$, we have $\rho(X^*) = Z^*$ and $\rho(X^*X) = ZZ^*$. Using functional calculus, we obtain $\rho(|X|) = |Z^*|$. Since $|X|$ is positive and $rank\ |X| = rank\ X < \infty$, by the discussion in the last paragraph, we have

$$rank\ X = rank\ |X| = rank\ |Z^*| = rank\ Z,$$

a contradiction. This proves Claim 1.

Claim 2. There exists an anti-unitary operator D on $\widetilde{\mathcal{H}}$ such that $\rho(X) = DX^*D^{-1}$ for $X \in C^*(S)$. In particular, $S = DS^*D^{-1}$ and $S^* = DSD^{-1}$.

Since ρ is an anti-automorphism of $C^*(S)$ and $\rho(S) = S$, in view of Proposition (6.2.57), Claim 1 implies that S is UET. By Lemma (6.2.21), there is an anti-unitary operator D on $\widetilde{\mathcal{H}}$ such that $DS = S^*D$. Thus, given a polynomial $p(\cdot,\cdot)$ in two free variables, it is easy to see that

$$\tilde{p}(S, S^*) = Dp(S^*, S)D^{-1}.$$

It follows that $\rho(X) = DX^*D^{-1}$ for $X \in C^*(S)$. This proves Claim 2.

For convenience we denote $\mathcal{A} = C^*(S)$. Note that $D\mathcal{A}D^{-1} = \mathcal{A}$ and $D^{-1}\mathcal{A}D = A$.

Claim 3. For each $i \in \Lambda$, there exists a unique $\tau_i \in \Lambda$ such that $D(\mathcal{H}_i) = \mathcal{H}_{\tau_i}$ and $D(\mathcal{H}_{\tau_i}) = \mathcal{H}_i$.

Now fix an $i \in \Lambda$. Arbitrarily choose a unit vector $e_i \in \mathcal{H}_i$ and set $P_i = e_i \otimes e_i$. Then $P_i \in C^*(S)$. Denote $Q_i = \rho(P_i)$ and $f_i = De_i$. For $x \in \widetilde{\mathcal{H}}$, we have

$$Q_i x = DP_iD^{-1}x = D(\langle D^{-1}x, e_i\rangle e_i) = \langle e_i, D^{-1}x\rangle f_i = \langle x, De_i\rangle f_i = (f_i \otimes f_i)(x).$$

So we obtain $Q_i = f_i \otimes f_i$. Note that $\mathcal{A} \cap \mathcal{K}(\widetilde{\mathcal{H}}) = 0 \oplus \sum_{j\in\Lambda} \mathcal{K}(\mathcal{H}_j)$. Since $Q_i \in \mathcal{A}$ is of rank one, there exists $\tau_i \in \Lambda$ such that $Q_i \in \mathcal{K}(\mathcal{H}_{\tau_i})$. So $f_i \in \mathcal{H}_{\tau_i}$. Thus

$$\mathcal{H}_{\tau_i} = [\mathcal{A}f_i] = [\mathcal{A}(De_i)] = D[(D^{-1}\mathcal{A}D)e_i] = D[\mathcal{A}e_i] = D(\mathcal{H}_i).$$

Noting that $\rho = \rho^{-1}$, we have $DQ_iD^{-1} = P_i$ and $Df_i = \alpha_i e_i$ for some $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$. So

$$\mathcal{H}_i = [\mathcal{A}e_i] = [\mathcal{A}(Df_i)] = D[(D^{-1}\mathcal{A}D)f_i] = D[\mathcal{A}f_i] = D(\mathcal{H}_{\tau_i}).$$

This proves Claim 3.

Claim 4. The map $\tau: i \mapsto \tau_i$ is bijective on Λ and $\tau^{-1} = \tau$.

Let $i, j \in \Lambda$ with $i \neq j$. If $\tau_i = \tau_j$, then $\mathcal{H}_i = D(\mathcal{H}_{\tau_i}) = D\left(\mathcal{H}_{\tau_j}\right) = \mathcal{H}_j$, a contradiction. Given a $j \in \Lambda$, since $\mathcal{H}_j = D\left(\mathcal{H}_{\tau_j}\right) = \mathcal{H}_{\tau(\tau_j)}$, we have $j = \tau(\tau_j) = \tau^2(j)$. This means that τ is bijective and $\tau^{-1} = \tau$.

By Claim 4, τ induces a partition $\Lambda = \cup_{r\in\Gamma} \Lambda_r$, where each Λ_r can be written as $\Lambda_r = \{j, \tau_j\}$ for some $j \in \Lambda$. So $S = N \oplus \left[\oplus_{r\in\Gamma} \left(\oplus_{j\in\Lambda_r} T_j\right)\right]$. Thus

$$T = N \oplus \left(\bigoplus_{r \in \Gamma} \left(\bigoplus_{j \in \Lambda_r} T_j^{(n_j)} \right) \right). \quad (14)$$

Claim 5. $T_j \cong T_{\tau_j}^t$ for each $j \in \Lambda$.

Let $j \in \Lambda$ be fixed. Denote $D_j = D|_{\mathcal{H}_j}$. Then $D_j: \mathcal{H}_j \to \mathcal{H}_{\tau_j}$ is an antiunitary operator. Since $DS = S^*D$, we have $(D|_{\mathcal{H}_j})(S|_{\mathcal{H}_j}) = (S^*|_{\mathcal{H}_{\tau_j}})(D|_{\mathcal{H}_j})$, that is, $D_jT_j = T_{\tau_j}^*D_j$. Arbitrarily choose a conjugation E on $\mathcal{H}_{\tau_j}$. Thus $(ED_j)T_j = \left(ET_{\tau_j}^*E\right)\left(ED_j\right)$. Noting that ED_j is unitary, we obtain $T_j \cong T_{\tau_j}^t$.

Claim 6. If $i \in \Lambda$ and $i \neq \tau i$, then T_i is not UET.

In fact, if not, then $T_i^t \cong T_i$. By Claim 5, $T_i \cong T_{\tau_i}^t$. So we have $T_i \cong T_i^\tau$, contradicting the hypothesis that $T_l \ncong T_s$ whenever $l \ncong s$.

Now we can conclude the proof. Let $r \in \Gamma$ be fixed.

If card $\Lambda_r = 1$ and $k \in \Lambda_r$, then $k = \tau_k$. By Claim 5, $T_k \cong T_{\tau_k}^t = T_k^t$. Hence T_k is an irreducible UET operator and $\bigoplus_{i \in \Lambda_r} T_i^{(n_i)} = T_k^{(n_k)}$.

If card $\Lambda_r = 2$ and $k \in \Lambda_r$, then $k \neq \tau_k$. By Claim 6, T_k is not UET. So $T_k \oplus T_{\tau_k} \cong T_k \oplus T_k^t$. Hence $j \in \Lambda_r T_j^{(n_j)}$ is unitarily equivalent to an operator with a form of $A^{(m)} \oplus (A^t)^{(n)}$, where A is irreducible, not UET and $m, n \geq 1$.

In view of (14), we complete the proof.

The main result is the following theorem which gives a canonical decomposition for essentially normal operators which are UET.

To give the proof of Theorem (6.2.62), we need to make some preparations. The proof of the following result is immediate

Lemma (6.2.59)[164]: Let $T \in \mathcal{B}(\mathcal{H})$. Then T is UET if and only if Tabnor is UET.

Proposition (6.2.60)[164]: Let S be the unilateral shift on $\mathcal{H}$ defined by

$$Se_i = e_{i+1}, \forall i \geq 1,$$

where $\{e_i\}_{i=1}^\infty$ is an onb of $\mathcal{H}$. Set $T = S^{(m)} \oplus (S^*)^{(n)}$, where $1 \leq m, n \leq \infty$. Then the following are equivalent:

(i) T is complex symmetric;

(ii) $T \cong T^*$;

(iii) $T \cong_a T^*$;

(iv) T is g-normal and $rank\, p(T^*, T) = rank\, \tilde{p}(T, T^*)$ for any polynomial $p(\cdot,\cdot)$;

(v) $m = n$.

Proof. We first note that T^* is a transpose of T. By definition, the implications "(i) ⇒ (ii) ⇒ (iii)" are obvious.

"(iii)⇒(iv)". Since T is AUET, by Proposition (6.2.23), there exists an antiautomorphism φ of $C^*(T)$ such that $\varphi(T) = T$ and $rank\, \varphi(X) = rank\, X$ for all $X \in C^*(T)$. For each polynomial $p(\cdot,\cdot)$ in two free variables, one can see that $\varphi\big(p(T^*, T)\big) = \tilde{p}(T, T^*)^*$. Thus the conclusion follows.

"(iv)⇒(v)". Set $p(x, y) = 1 - xy$. One can check that $rank\, p(T^*, T) = n$ and $rank\, \tilde{p}(T, T^*) = m$. By the hypothesis, we have $m = n$.

"(v)⇒(i)". Note that S^* is a transpose of S. Then, by Lemma (6.2.19), $S \oplus S^*$ is complex symmetric. Since $m = n$, one can see that $T = (S \oplus S^*)^{(n)}$ and T is complex symmetric.

Question (6.2.61)[164]: Let $T \in \mathcal{B}(\mathcal{H})$. If T is g-normal and $rank\, p(T^*,T) = rank\, \tilde{p}(T,T^*)$ for any polynomial $p(\cdot,\cdot)$ in two free variables, then does it follow that T is AUET?

Theorem (6.2.62)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then the following are equivalent:

(i) $T \cong T^t$;

(ii) $T \cong_a T^t$;

(iii) T is unitarily equivalent to a direct sum of (some of the summands may be absent): normal operators, irreducible UET operators and operators with the form of $\oplus A^t$, where A is irreducible and not UET.

Proof. By Proposition (6.2.57), the implication "(i)⇔(ii)" is obvious. The implication "(iii)⇒(i)" follows from Lemma (6.2.19).

"(i)⇒(iii)". Assume that T is UET. In view of Lemma (6.2.59), it follows that T_{abnor} is UET. Thus we may directly assume that T is abnormal. By Corollary (6.2.55), we may also assume that $T = \oplus_{i\in\Lambda} T_i^{(n_i)}$, where each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \ncong T_j$ whenever $i \neq j$. Moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$.

Since T is g-normal, by Lemma (6.2.7), the map φ defined by

$$\varphi: C^*(T) \to C^*(T),$$
$$p(T^*,T) \longmapsto \tilde{p}(T,T^*)^*$$

is an anti-automorphism of $C^*(T)$. Since $\cong T^t$, there is an anti-unitary operator D on $\mathcal{H}$ such that $DT = T^*D$. It follows that $\varphi(X) = DX^*D^{-1}$ for $X \in C^*(T)$. So $rank\, \varphi(X) = rank\, X$ for all $X \in C^*(T)$.

Noting that T is g-normal, it follows from Corollary (6.2.18) that $S := \oplus_{i\in\Lambda} T_i$ is also g-normal. Thus the map ρ defined by

$$\rho: C^*(S) \to C^*(S),$$
$$p(S^*,S) \longmapsto p(S,S^*)^*$$

is an anti-automorphism of $C^*(S)$. Denote $\widetilde{\mathcal{H}} = \oplus_{i\in\Lambda} \mathcal{H}_i$. Then $S \in \mathcal{B}(\widetilde{\mathcal{H}})$ and $C^*(S) \cap \mathcal{K}(\widetilde{\mathcal{H}}) = \sum_{i\in\Lambda} \mathcal{K}(\mathcal{H}_i)$.

Let $X, Y \in C^*(S)$. Assume that $X = \oplus_{i\in\Lambda} X_i$ and $Y = \oplus_{i\in\Lambda} Y_i$ with respect to the decomposition $\widetilde{\mathcal{H}} = \oplus_{i\in\Lambda} \mathcal{H}_i$. By definition, if $\rho(X) = Y$, then $\bar{X}$, $\bar{Y} \in C^*(T)$ and $\varphi(\bar{X}) = \bar{Y}$, where $\bar{X} = \oplus_{i\in\Lambda} X_i^{(n_i)}$ and $\bar{Y} = \oplus_{i\in\Lambda} Y_i^{(n_i)}$.

In the following, we are going to establish some facts about $C^*(S)$ and ρ. Since the proof follows the same lines as the proof of Theorem (6.2.58), we omit it.

(a) S is UET and there is an anti-unitary operator E on $\overline{\mathcal{H}}$ such that $\rho(X) = EX^*E^{-1}$ for $X \in C^*(S)$.

(b) For each $i \in \Lambda$, choose a unit vector $e_i \in \mathcal{H}_i$ and denote $P_i = e_i \otimes e_i$. Then $P_i \in C^*(S)$. Denote $Q_i = \rho(P_i)$ and $f_i = Ee_i$ for $i \in \Lambda$. One can verify that $Q_i = f_i \otimes f_i$.

(c) For each $i \in \Lambda$, there exists a unique $\tau_i \in \Lambda$ such that $\mathcal{H}_i = E(\mathcal{H}_{\tau_i})$ and $\mathcal{H}_{\tau_i} = E(\mathcal{H}_i)$. Thus $f_i \in \mathcal{H}_{\tau_i}$ and $Q_i \in \mathcal{K}(\mathcal{H}_{\tau_i})$.

(d) The map $\tau: i \to \tau_i$ is bijective on Λ and $\tau^{-1} = \tau$.

(e) $T_i \cong T_{\tau_i}^t$ for each $i \in \Lambda$.

(f) If $i \in \Lambda$ and $i \neq \tau_i$, then T_i is not UET.

For each $i \in \Lambda$, note that $P_i, Q_i \in C^*(S)$ and $\rho(P_i) = Q_i$. Then, by definition, we have $P_i^{(n_i)}, Q_i^{(n_{\tau_i})} \in C^*(T)$ and $(P_i^{(n_i)}) = Q_i^{(n_{\tau_i})}$. Since $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(T)$, one can deduce that $n_i = n_{\tau_i}$.
By statement (d), τ induces a partition $\Lambda = \cup_{r\in\Gamma} \Lambda_r$, where each Λ_r can be written as $\Lambda_r = \{j, \tau_j\}$ for some $j \in \Lambda$. So S is the direct sum of $\bigoplus_{i\in\Lambda_r} T_i, r \in \Gamma$. Thus

$$T = \bigoplus_{r\in\Gamma}\left(\bigoplus_{i\in\Lambda_r} T_i^{(n_i)}\right). \tag{15}$$

Now we can conclude the proof. Let $r \in \Gamma$ be fixed.
If card $\Lambda_r = 1$ and $k \in \Lambda_r$, then $k = \tau_k$. By statement (e), $T_k \cong T_{\tau_k}^t = T_k^t$. Hence T_k is an irreducible UET operator and $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} = T_k^{(n_k)}$.

If card $\Lambda_r = 2$ and $k \in \Lambda_r$, then $k \neq \tau_k$. Hence $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} = T_k^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})}$. Since we have proved that $n_k = n_{\tau_k}$, it follows from statement (e) that $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} \cong (T_k \oplus T_k^t)^{(n_k)}$. By statement (f), $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)}$ is unitarily equivalent to a direct sum of operators with a form of $\oplus A^t$, where A is irreducible and not UET.

In view of (15), we complete the proof.

The mainly devoted to the proof of Theorem (6.2.71).

Lemma (6.2.63)[164]: ([158], Theorem 4). Let $T \in \mathcal{K}(\mathcal{H})$ and $\{C_n\}_{n=1}^\infty$ be a sequence of conjugations on $\mathcal{H}$ satisfying $C_nT^*C_n - T \to 0$ as $n \to \infty$. If P is the projection of $\mathcal{H}$ onto $\overline{ranT + ranT^*}$, then there exists a subsequence $\{n_j\}_{j=1}^\infty$ of $\mathbb{N}$ such that $\left\{P\ C_{n_j}|_{ranP}\right\}_{j=1}^\infty$ converges to a conjugation on ranP.
Using a similar argument as in the proof of Lemma (6.2.63), one can prove the following result.

Corollary (6.2.64)[164]: Let $T \in \mathcal{K}(\mathcal{H})$ and $\{C_n\}_{n=1}^\infty$ be a sequence of conjugations on $\mathcal{H}$ satisfying $C_nT^*C_n - T \to 0$ as $n \to \infty$. If P is the projection of $\mathcal{H}$ onto $\overline{ranT + ranT^*}$, then there exists a subsequence $\{n_j\}_{j=1}^\infty$ of $\mathbb{N}$ such that $\left\{PC_{n_j}|_{ranP}\right\}_{j=1}^\infty$ converges to a conjugation on ranP.

Theorem (6.2.65)[164]: If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T \in \overline{CSO}$ if and only if $T \in CSO$.

Proof. We need only prove the necessity. Since $T \in \overline{CSO}$, it follows from the proof of Lemma (6.2.25) that there exists a sequence $\{C_n\}$ of conjugations on $\mathcal{H}$ such that $C_nT^*C_n \to T$ as n tends to ∞.
Now let $m, l \geq 1$ be fixed. Let $P_{m,l}$ be the projection of $\mathcal{H}$ onto the subspace spanned by $ran[T^{*m}, T^l]$ and $ran[T^{*m}, T^l]^*$. It is easy to verify that

$$C_n[T^{*m}, T^l]C_n + [T^{*m}, T^l]^* \to 0 \quad \text{as} \quad n \to \infty.$$

Since T is essentially normal, it follows that $[T^{*m}, T^l] \in \mathcal{K}(\mathcal{H})$. Then, by Corollary (6.2.64), $\{C_n\}$ has a subsequence $\left\{C_{n_j}\right\}$ such that $\left\{P_{m,l}C_{n_j}|_{ranP_{m,l}}\right\}$ converges to a conjugation on $ranP_{m,l}$. Hence, for any $x \in ranP_{m,l}, \left\{C_{n_j}x\right\}_{j\geq 1}$ converges to a vector in $ranP_{m,l}$.

Since $\left\{ranP_{m,l}: m, l \geq 1\right\}$ is at most denumerable, applying the diagonal process we can find a subsequence $\left\{n_j\right\}$ of $\mathbb{N}$ satisfying: for any $m, l \geq 1$,

$$x \in ranP_{m,l} \Rightarrow \{C_{n_j}x\} \text{ converges to a vector in } ranP_{m,l}.$$

Denote by M_0 the subset of $\mathcal{H}$ consisting of all finite linear combinations of vectors in $\cup_{m,l\geq 1} ranP_{m,l}$. Then for each $x \in M_0$ the sequence $\{C_{n_j}x\}$ converges to a vector in M_0.

Denote $M = \overline{M_0}$. Then M is in fact the subspace of $\mathcal{H}$ spanned by all $ranP_{m,l}(m, l \geq 1)$; moreover, $M, M^{\perp}$ both reduce $T, A := T|_M$ is abnormal and $N := T|_{M^{\perp}}$ is normal. In order to complete the proof, we need only prove that A is complex symmetric. We give the rest of the proof by proving the following three claims.

Claim 1. For each $x \in M, \{C_{n_j}x\}$ converges to a vector in M.

Note that $\{C_{n_j}x\}$ converges to a vector in M_0 for each $x \in M_0$. Define $Ex = \lim_j C_{n_j}x$ for $x \in M_0$. Thus E is a conjugate-linear map on M_0. Noting that $\|Ex\| = \|x\|$ for each $x \in M_0$ and M_0 is a dense subset of M, E can be extended to an isometric map on M, denoted by C_M. So $C_M x = \lim_j C_{n_j}x \in M$ for each $x \in M$.

Claim 2. C_M is a conjugation on M.

It is obvious that C_M is conjugate-linear and isometric. By the polarization identity, it implies $\langle C_M x, C_M y\rangle = \langle y, x\rangle$ for all $x, y \in M$. So it suffices to prove that $C_M^2 x = x$ for all $x \in M$. Now fix an $x \in M$. Since x and $C_M x$ both belong to M, given $\varepsilon > 0$, there exists j_0 such that $\left\|C_{n_{j_0}}x - C_M x\right\| + \left\|C_{n_{j_0}}C_M x - C_M^2 x\right\| < \varepsilon$. Then

$$\begin{aligned}\|C_M^2 x - x\| &\leq \left\|C_M^2 x - C_{n_{j_0}}C_M x\right\| + \left\|C_{n_{j_0}}C_M x - C_{n_{j_0}}^2 x\right\| \\ &= \left\|C_M^2 x - C_{n_{j_0}}C_M x\right\| + \left\|C_M x - C_{n_{j_0}}x\right\| < \varepsilon.\end{aligned}$$

Since ε is arbitrary, we deduce that $C_M^2 x = x$. Thus CM is a conjugation on M.

Claim 3. $C_M A C_M = A^*$.

Fix an $x \in M$. Since $\lim_j C_{n_j}x = C_M x \in M$, we have

$$A^* x = T^* x = \lim_j C_{n_j}TC_{n_j}x = \lim_j C_{n_j}TC_M x = \lim_j C_{n_j}AC_M x = C_M A C_M x.$$

It follows that $A^* = C_M A C_M$, that is, A is complex symmetric.

Proposition (6.2.66)[164]: Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathcal{H})$ is irreducible. If M is a nonzero reducing subspace of T, then the following are equivalent:

(i) $T|_M \cong A$;

(ii) $T|_M$ is irreducible;

(iii) there exists a nonzero $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that $M = \{\bigoplus_{i=1}^n \alpha_i\xi : \xi \in \mathcal{H}\}$.

Proof. The implication "(i)⇒(ii)" is obvious.

"(ii)⇒(iii)". Denote by P the projection of $\mathcal{H}^{(n)}$ onto M. Then $PT = TP$ and $PT^* = T^*P$. We may assume that P admits the following matrix representation:

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,n} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,1} & P_{n,2} & \cdots & P_{n,n}\end{bmatrix} \begin{matrix}\mathcal{H} \\ \mathcal{H} \\ \vdots \\ \mathcal{H}\end{matrix},$$

It follows that $AP_{i,j} = P_{i,j}A$ and $AP_{i,j}^* = P_{i,j}^*A$ for all i, j. Let i, j be fixed. Thus

$$AP_{i,j}P_{i,j}^* = P_{i,j}P_{i,j}^*A \quad \text{and} \quad AP_{i,j}^*P_{i,j} = P_{i,j}^*P_{i,j}A.$$

Noting that $P_{i,j}P_{i,j}^*$ is positive and A is irreducible, $\sigma(P_{i,j}P_{i,j}^*)$ is a singleton set. Similarly $\sigma(P_{i,j}^*P_{i,j})$ is also a singleton set. So there exist unitary $U \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that

$P_{i,j} = \lambda U$. If $\lambda \neq 0$, then we obtain $AU = UA$; by the irreducibility of A, we obtain $U = e^{i\theta}I$ for some $\theta \in \mathbb{R}$, where I is the identity operator on $\mathcal{H}$. Thus we conclude that there exists $\lambda_{i,j} \in \mathbb{C}$ such that $P_{i,j} = \lambda_{i,j}I$. Set $R = [\lambda_{i,j}]_{1\le i,j\le n}$. Then R is a nonnegative-definite matrix, $R^2 = R$ and $P = R \otimes I$.

We claim that $rank\ R = 1$. In fact, if not, then we can choose an $n \times n$ nonnegative-definite matrix R_1 such that $R_1^2 = R_1, rank\ R_1 = 1$ and $R_1 \le R$. Then $ranR_1 \otimes I$ is a nonzero reducing subspace of T and $ranR_1 \otimes I \subsetneq ranP$, contradicting the fact that $T|_{ranP}$ is irreducible. So we have rank $R = 1$. Then there exists $1 \le j_0 \le n$ such that each column of R is a scalar multiple of the j_0-th column. For $1 \le i \le n$, set $\alpha_i = \lambda_{i,j_0}$. Then one can see that

$$M = ranP = \left\{\bigoplus_{i=1}^{n} \alpha_i x : x \in \mathcal{H}\right\}.$$

"(iii)⇒(i)". Set $\delta = \sqrt{\Sigma_{i=1}^{n}|\alpha_i|^2}$. For $\xi \in \mathcal{H}$, define $U\xi = \oplus_{i=1}^{n} \frac{\alpha_i}{\delta}\xi$. Then one can see that $U{:}\mathcal{H} \to M$ is a unitary operator. For each $\xi \in \mathcal{H}$, we have

$$UA\xi = \bigoplus_{i=1}^{n} \frac{\alpha_i}{\delta} A\xi = (T|_M)\left(\bigoplus_{i=1}^{n} \frac{\alpha_i}{\delta}\xi\right) = (T|_M)U\xi.$$

This implies that $T|_M \cong A$.

Corollary (6.2.67)[164]: Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathcal{H})$ is irreducible. If M is a nonzero reducing subspace of T, then there exists $1 \le m \le n$ such that $T|_M \cong A^{(m)}$.

Proof. Let P be the projection of $\mathcal{H}^{(n)}$ onto M. It can be seen from the proof of "(ii)⇒(iii)" in Proposition (6.2.66) that $P = R \otimes I$, where I is the identity operator on $\mathcal{H}$ and R is an $n \times n$ nonnegative-definite matrix satisfying $R^2 = R$.

Denote $m = rank\ R$. So $1 \le m \le n$. Then there exist nonnegative-definite matrices $R_1,\cdots,R_m$ with rank $R_i = 1$ and $R_i^2 = R_i z$ for all $1 \le i \le m$ such that $R = \sum_{i=1}^{m} R_i$ and $R_iR_j = 0$ whenever $i \ne j$. For each $1 \le i \le m$, set $P_i = R_i \otimes I$. Then $P_iP_j = 0$ whenever $i \neq j, P_iT = T\ P_i$ and $ranP_i$ reduces T for each i. Hence $M = \oplus_{i=1}^{m} ranP_i$. Let i be fixed. Since $P_i = R_i \otimes I$, there exists a nonzero $(\alpha_1, \dots, \alpha_n) \in C_n$ such that

$$ranP_i = \left\{\bigoplus_{j=1}^{n} \alpha_j \xi : \xi \in \mathcal{H}\right\}.$$

By Proposition (6.2.66), $T|_{ranP_i} \cong A$. Thus we conclude that $T|_M \cong A^{(m)}$.

Proposition (6.2.68)[164]: Let $T = A \oplus A^t$, where $A \in \mathcal{B}(\mathcal{H})$ is irreducible. If A is not complex symmetric, then T is a minimal complex symmetric operator.

Proof. It is obvious that T is complex symmetric. Assume that M is a nontrivial reducing subspace of T. Denote by P the projection of $\mathcal{H}^{(2)}$ onto M. We shall prove that $T|_M$ is not complex symmetric.

Case 1. A is UET. In this case, we have $T \cong A^{(2)}$. We may directly assume that $T = A^{(2)}$. Since A is irreducible and P is a projection commuting with T, using a similar argument as in the proof of "(ii)⇒(iii)" in Proposition (6.2.66), one can prove that $P = R \otimes I$, where I is the identity operator on $\mathcal{H}$ and R is a 2×2 nonnegative-definite matrix satisfying $R^2 = R$.

Since $ranP \neq \{0\}$ and $ranP \neq \mathcal{H}^{(2)}$, we have $rank\ R = 1$. Then there exists a nonzero $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ such that

$$M = \left\{ \bigoplus_{i=1}^{2} \alpha_i \xi : \xi \in \mathcal{H} \right\}.$$

By Proposition (6.2.66), $T|_M \cong A$. Since A is not complex symmetric, this completes the proof in Case 1.

Case 2. A is not UET. In this case, we have $\ncong A^t$. For convenience we write

$$T = \begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Then, by Theorem (6.2.37) in [275], T has only four reducing subspaces. Then $\{0\}, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2$ are all reducing subspaces of T. Since M is nontrivial, we have either $M = \mathcal{H}_1$ or $M = \mathcal{H}_2$. Hence $T|_M = A$ or $T|_M = A^t$. Since A is not complex symmetric, it follows that $T|_M$ is not complex symmetric.

In view of [172], the following result immediately follows from Propositions (6.2.50) and (6.2.68).

Corollary (6.2.69)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be a weighted shift (unilateral or bilateral) and assume that $T \in CSO$. Then T is either completely complex symmetric or a direct sum of the following two kinds of minimal complex symmetric operators: irreducible complex symmetric operators and operators with a form of $\oplus A^t$, where A is irreducible and not complex symmetric.

Proposition (6.2.70)[164]: Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathcal{H})$ is irreducible. Then T is complex symmetric if and only if exactly one of the following holds:

(i) A is complex symmetric;

(ii)n is even, and A is UET and not complex symmetric.

Proof. "⇐". If A is complex symmetric, then the conclusion is evident. If A is UET and n is even, then

$$T = A^{(n)} = (A \oplus A)^{\left(\frac{n}{2}\right)} \cong (A \oplus A^t)^{\left(\frac{n}{2}\right)}.$$

By Lemma (6.2.19), T is complex symmetric.

"⇒". Now we assume that T is complex symmetric and A is not complex symmetric. We shall prove that A is UET and n is even. By the hypothesis, there is a conjugation C on $\mathcal{H}^{(n)}$ such that $CTC = T^*$. For convenience, we write

$$T = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_n \end{matrix},$$

where $\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$.

Denote $M = C(\mathcal{H}_1)$. Since $C^{-1} = C$, one can see that $\mathcal{H}_1 = C(M)$. So we have

$$T(M) = CT^*C(M) = CT^*(\mathcal{H}_1) \subset C(\mathcal{H}_1) = M$$

and

$$T^*(M) = CTC(M) = CT(\mathcal{H}_1) \subset C(\mathcal{H}_1) = M.$$

It follows that M is a nonzero reducing subspace of T. Denote $D = C|_M$. Then $D: M \to \mathcal{H}_1$ is an anti-unitary operator. Since $CT = T^*C$, we have $(C|_M)(T|_M) = \left(T^*|_{\mathcal{H}_1}\right)(C|_M)$, that is, $D(T|_M) = A^*D$. Arbitrarily choose a conjugation E on $\mathcal{H}_1$. Thus $T|_M = D^{-1}A^*D = (D^{-1}E)(EA^*E)(ED)$. Noting that ED is unitary and $(ED)^{-1} = D^{-1}E$, we obtain $T|_M \cong$

EA^*E. So $T|_M$ is irreducible and, by Proposition (6.2.66), we have $T|_M \cong A$; furthermore, we have $A \cong EA^*E$ (i.e., A is UET). Also there exists a nonzero $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that

$$M = \left\{ \bigoplus_{i=1}^{n} \alpha_i x : x \in \mathcal{H} \right\}.$$

Note that $C(\vee\{M, \mathcal{H}_1\}) = \vee\{M, \mathcal{H}_1\}$ and $C(\{M, \mathcal{H}_1\}^\perp) = \{M, \mathcal{H}_1\}^\perp$. Thus $\{M, \mathcal{H}_1\}$ is a common reducing subspace of T and C. So $T|_{\{M,\mathcal{H}_1\}}$ and $T|_{\{M,\mathcal{H}_1\}^\perp}$ are both complex symmetric. Also it can be seen that $\vee\{M, \mathcal{H}_1\} = \mathcal{H}_1 \oplus M_1$, where $M_1 = \{\oplus_{i=2}^{n} \alpha_i x : x \in \mathcal{H}\} \subset \oplus_{i=2}^{n} \mathcal{H}_i$.

We claim that $(\alpha_2, \dots, \alpha_n) \neq 0$. Otherwise, we have $M_1 = \{0\}$ and $\mathcal{H}_1 = M$. Thus

$$T|_{\vee\{M,\mathcal{H}_1\}} = T|_{\mathcal{H}_1} = A$$

is complex symmetric, contradicting the hypothesis that A is not complex symmetric. Thus $M_1 \neq \{0\}$. It is easy to see that M_1 is a nonzero reducing subspace of T and, by Proposition (6.2.66), we have $T|_{M_1} \cong A$. So $T|_{\vee\{M,\mathcal{H}_1\}} = A \oplus (T|_{M_1}) \cong A^{(2)}$.

By Corollary (6.2.67), there exists some positive integer m less than n such that $T|_{\{M,\mathcal{H}_1\}^\perp} \cong A^{(m)}$. So $T \cong A^{(2)} \oplus \left(T_{\{M,\mathcal{H}_1\}^\perp}\right) = A^{(2+m)}$. Since $T = A^{(n)}$ and A is irreducible, one can deduce that $m = n - 2$. This shows that $T|_{\{M,\mathcal{H}_1\}^\perp} \cong A^{(n-2)}$. Note that $T_{\{M,\mathcal{H}_1\}^\perp}$ is complex symmetric. If n is odd, then, using an inductive argument, it can eventually be proved that A is complex symmetric, contradicting the hypothesis. So n is even. Since we have proved that A is UET, this completes the proof.

Theorem (6.2.71)[164]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then the following are equivalent:

(i) $T \in \overline{CSO}$;

(ii) $T \in CSO$;

(iii) T is unitarily equivalent to a direct sum of (some of the summands may be absent): normal operators, irreducible complex symmetric operators and operators with form of $\oplus A^t$, where A is irreducible and not complex symmetric.

Proof. By Theorem (6.2.65), (i) and (ii) are equivalent. "(iii)⇒(ii)" follows from Lemmas (6.2.14) and (6.2.19). It suffices to prove "(ii)⇒(iii)".

By Lemma (6.2.15), we may directly assume that T is abnormal. Then, by Corollary (6.2.55), we may also assume that $= \oplus_{i \in \Lambda} T_i^{(n_i)}$, where each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \ncong T_j$ whenever $i \neq j$. Moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$.

Since T is g-normal, by Lemma (6.2.7), the map defined by

$$\varphi: C^*(T) \to C^*(T),$$
$$p(T^*, T) \to p(T, T^*)^*$$

is an anti-automorphism of $C^*(T)$. Since T is complex symmetric, there is a conjugation D on $\mathcal{H}$ such that $DT = T^*D$. It follows that $\varphi(X) = DX^*D$ for $X \in C^*(T)$. So $rank\ X = rank\ \varphi(X)$ for all $X \in C^*(T)$.

Noting that T is g-normal, it follows from Corollary (6.2.18) that $S := \oplus_{i \in \Lambda} T_i$ is also g-normal. Thus the map ρ defined by

$$\rho: C^*(S) \to C^*(S),$$
$$p(S^*, S) \to p(S, S^*)^*$$

is an anti-automorphism of $C^*(S)$. Denote $\mathcal{H} = \oplus_{i \in \Lambda} \mathcal{H}_i$. Then $S \in \mathcal{B}(\widetilde{\mathcal{H}})$ and $C^*(S) \cap \mathcal{K}(\widetilde{\mathcal{H}}) = \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)$.

Let $X, Y \in C^*(S)$. Assume that $X = \bigoplus_{i \in \Lambda} X_i$ and $Y = \bigoplus_{i \in \Lambda} Y_i$ with respect to the decomposition $\widetilde{\mathcal{H}} = \bigoplus_{i \in \Lambda} \mathcal{H}_i$. By definition, if $\rho(X) = Y$, then $\bar{X}, \bar{Y} \in C^*(T)$ and $\varphi(\bar{X}) = \bar{Y}$, where $\bar{X} = \bigoplus_{i \in \Lambda} X_i^{(n_i)}$ and $\bar{Y} = \bigoplus_{i \in \Lambda} Y_i^{(n_i)}$.

Denote $\mathcal{A} = C^*(S)$. Just as we have proved in the proof of Theorem (6.2.58), the following statements hold:

(a) S is UET and hence there is an anti-unitary operator E on $\widetilde{\mathcal{H}}$ such that $\rho(X) = EX^*E^{-1}$ for $X \in \mathcal{A}$. Then $E\mathcal{A}E^{-1} = \mathcal{A}$ and $E^{-1}\mathcal{A}E = \mathcal{A}$.

(b) For each $i \in \Lambda$, there exists a unique $\tau_i \in \Lambda$ such that $\mathcal{H}_i = E(\mathcal{H}_{\tau_i})$ and $\mathcal{H}_{\tau_i} = E(\mathcal{H}_i)$.

(c) The map $\tau: i \longmapsto \tau_i$ is bijective on Λ and $\tau^{-1} = \tau$.

(d) $T_i \cong T_{\tau_i}^t$ for each $i \in \Lambda$.

(e) If $i \in \Lambda$ and $i \neq \tau_i$, then T_i is not UET.

By the above statements, we have the following claim.

Claim 1. If $i \in \Lambda$ and $x \in \mathcal{H}_i$ with $\|x\| = 1$, then $\rho(x \otimes x) \in \mathcal{K}(\mathcal{H}_{\tau_i})$.

Fix an $i \in \Lambda$ and a unit vector $x \in \mathcal{H}_i$. Denote $y = Ex$. In view of (b), we obtain $y \in \mathcal{H}_{\tau_i}$. Note that $\rho(X) = EX^*E^{-1}$ for all $X \in \mathcal{A}$. Thus, for each $z \in \widetilde{\mathcal{H}}$, we have

$$\rho(x \otimes x)(z) = E(x \otimes x)E^{-1}(z) = E(\langle E^{-1}z, x\rangle x) = \langle x, E^{-1}z\rangle Ex = \langle z, Ex\rangle y$$
$$= (y \otimes y)(z).$$

It follows that $\rho(x \otimes x) = y \otimes y \in \mathcal{K}(\mathcal{H}_{\tau_i})$. This proves Claim 1.

Claim 2. $n_i = n_{\tau_i}$ and $D\left(\mathcal{H}_i^{(n_i)}\right) = \mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)}$ for all $i \in \Lambda$.

Let $i \in \Lambda$ be fixed. Arbitrarily choose a rank-one projection $P \in \mathcal{K}(\mathcal{H}_i)$. Then, by Claim 1, $Q := \rho(P)$ is a rank-one projection in $\mathcal{K}(\mathcal{H}_{\tau_i})$. Noting that $P^{(n_i)}, Q^{\left(n_{\tau_i}\right)} \in C^*(T)$, we have

$$DP^{(n_i)}D = \varphi\left(P^{(n_i)}\right) = Q^{\left(n_{\tau_i}\right)} \in \mathcal{K}\left(\mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)}\right).$$

Thus $ank\, P^{(n_i)} = rank\, Q^{\left(n_{\tau_i}\right)}$. So $n_i = n_{\tau_i}$ and $D\left(ranP^{(n_i)}\right) \subset ranQ^{\left(n_{\tau_i}\right)} \subset \mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)}$. Since P was arbitrarily chosen in $\mathcal{H}(\mathcal{H}_i)$, we deduce that $D\left(\mathcal{H}_i^{(n_i)}\right) \subset \mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)}$. Since $\tau^2(i) = i$, by the symmetry, we have $D(\mathcal{H}^{\left(n_{\tau_i}\right)}{}_{\tau_i} \subset \mathcal{H}_i^{(n_i)}$. Noting that $D^{-1} = D$, we obtain $\mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)} \subset D(\mathcal{H}_i^{(n_i)})$. It follows that

$$D\left(\mathcal{H}_i^{(n_i)}\right) = \mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)} \text{ and } D\left(\mathcal{H}_{\tau_i}^{\left(n_{\tau_i}\right)}\right) = \mathcal{H}_i^{(n_i)}.$$

By statement (c), τ induces a partition $\Lambda = \cup_{r \in \Gamma} \Lambda_r$, where each Λ_r can be written as $\Lambda_r = \{j, \tau_j\}$ for some $j \in \Lambda$. Note that S is the direct sum of $\bigoplus_{i \in \Lambda_r} T_i, r \in \Gamma$. Then T can be written as

$$T = \bigoplus_{r \in \Gamma} \left(\bigoplus_{i \in \Lambda_r} T_i^{(n_i)}\right).$$

For each $r \in \Gamma$, set

$$M_r = \bigoplus_{i \in \Lambda_r} \mathcal{H}_i^{(n_i)}.$$

Then $\mathcal{H} = \bigoplus_{r\in\Gamma} M_r$ and each M_r is a common reducing subspace of D and T; in particular, $T|_{M_r} = \bigoplus_{i\in\Lambda_r} T_i^{(n_i)}$. So $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)}$ is complex symmetric for all $r \in \Gamma$. We shall prove that $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)}$ admits the desired decomposition for every $r \in \Gamma$.

Now let $r \in \Gamma$ be fixed.

If card $\Lambda_r = 1$ and $k \in \Lambda_r$, then $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} = T_k^{(n_k)}$ and, by statement (d), $T_k \cong T_k^t$. So T_k is an irreducible UET operator.

If card $\Lambda_r = 2$ and $k \in \Lambda_r$, then $k \neq \tau_k$. Hence $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} = T_k^{(n_k)} \oplus T_{\tau_k}^{(n_{\tau_k})}$. Since we have proved that $n_k = n_{\tau_k}$, it follows from statement (d) that $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)} \cong (T_k \oplus T_k^t)^{(n_k)}$. Then, by statement (e), we have proved that $\bigoplus_{i\in\Lambda_r} T_i^{(n_i)}$ is unitarily equivalent to a direct sum of operators with a form of $A \oplus A^t$, where A is irreducible, not UET and hence not complex symmetric. This completes the proof.

Section (6.3): Generators of C^*-Algebras

$\mathcal{H}(\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{K}\, etc.)$ will always denote a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we let $C^*(T)$ denote the C^*-algebra generated by T and the identity I. If A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathscr{A} = C^*(T)$ for some $T \in \mathcal{B}(\mathcal{H})$, then T is called a generator of $\mathscr{A}$.

We are interested in C^*-algebras which are singly generated by complex symmetric operators.

Definition (6.3.1)[286]: A map C on $\mathcal{H}$ is called a conjugation if C is conjugate-linear, $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

Definition (6.3.2)[286]: An operator $T \in \mathcal{B}(\mathcal{H})$ is called a complex symmetric operator (CSO for short) if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$.

CSOs can be viewed as a generalization of symmetric matrices to the case of operators on Hilbert spaces. The general study of CSOs was initiated by Garcia, Putinar and Wogen in [138],[136],[138],[128]. CSOs have many motivations in function theory, matrix analysis and other areas. In particular, CSOs are closely related to the study of truncated Toeplitz operators [159], which was initiated in Sarason's seminal [145]. Some interesting results concerning CSOs have been obtained (see [131],[158],[139]–[147],[292],[272],[172]).

Since CSOs have certain nice structural properties, it is natural to explore the algebraic aspects of the theory of CSOs. Recently certain connections between CSOs and C^*-algebras generated by them are established, and a C^*-algebraic approach has been developed to answer a number of open questions concerning CSOs (see [164],[147],[272]).

The present is a continuation of [286], where many von Neumann algebras and C^*-algebras prove to have a single complex symmetric generator. We shall concentrate on those C^*-algebras singly generated by essentially normal operators, which have been the subject of much interest since the seminal [235] by Brown, Douglas and Fillmore.

First we are interested in the following question.

Question (6.3.3)[286]: When does an essentially normal operator T have $C^*(T)$ generated by a complex symmetric operator?

There exist operators T lying outside the class of CSOs such that $C^*(T)$ admits a complex symmetric generator (see Examples (6.3.11) and (6.3.42)). Hence the above question is natural and worth answering.

We give a complete answer to Question (6.3.3) (see Theorem (6.3.27)). We give a decomposition of such operators. Our result shows that whether or not $C^*(T)$ has a complex symmetric generator depends heavily on the spectral picture of the restrictions of T to its minimal reducing subspaces. The proof of our result depends on some approximation results, which are developed using tools from BDF theory, Voiculescu's theorem and noncommutative approximation theory of operators [184].

Two $*$-isomorphic C^*-algebras have the same algebraic properties. The following question arises naturally.

Question (6.3.4)[286]: When is $C^*(T)$ $*$-isomorphic to a C^*-algebra singly generated by CSOs?

When T is essentially normal, we give an answer to the above question (see Theorem (6.3.54)). In order to answer Question (6.3.4), we need to introduce an algebraic analogue of CSOs.

Given a polynomial $p(z_1, z_2)$ in two free variables z_1, z_2, we let $\tilde{p}(z_1, z_2)$ denote the polynomial obtained from $p(z_1, z_2)$ by conjugating each coefficient.

Definition (6.3.5)[286]: An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be g-normal if it satisfies $\|p(A^*, A)\| = \|\tilde{p}(A, A^*)\|$ for any polynomial $p(\cdot,\cdot)$ in two free variables.

The above concept was inspired by Garica, Lutz and Timotin [157], and posed by Guo, Ji and the author [164]. It was proved that an operator A is g-normal if and only if there is an anti-automorphism φ of $C^*(A)$ such that $\varphi(A) = A$ (see Lem. 1.7 in [164]). G-normal operators, containing all CSOs, play an important role in solving the norm closure problem for CSOs (see [164],[272]). Obviously, g-normal elements in a C^*-algebra can be defined in the same manner as in Definition (6.3.5).

We shall show that an operator is g-normal if and only if it is algebraically equivalent to a CSO (see Theorem(6.3.44)). Thus the notion of g-normal operator is a suitable algebraic analogue of CSOs. Recall that two operators A, B are algebraically equivalent (write $A \approx B$) if there is a $*$-isomorphism of $C^*(A)$ onto $C^*(B)$ which carries A into B.

We shall solve Question (6.3.3) in the irreducible case. We shall prove some approximation results and give necessary and sufficient conditions for an essentially normal operator to have a complex symmetric generator for its C^*-algebra. We study the algebraical equivalence of certain special operators and give a complete answer to Question (6.3.4) in the essentially normal case (see Theorem (6.3.54)).

For convenience, we write $A \in (cs)$ to denote that $C^*(A)$ admits a complex symmetric generator.

We shall use the BDF Theorem to derive a necessary spectral condition for an essentially normal operator T to satisfy $T \in (cs)$ (see Lemma (6.3.7)), and then prove that the spectral condition is also sufficient when T is irreducible (see Theorem (6.3.9)).

In the following, unless otherwise stated, $\mathcal{H}$ is always assumed to be a complex separable infinite-dimensional Hilbert space. We let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $\sigma(T)$ the spectrum of T. Denote by ker T and ran T the kernel of T and the range of T respectively. T is called a semi-Fredholm operator, if ran T is closed and either dim ker T or dim ker T^* is finite; in this case, $ind\, T := dim\, ker\, T - dim\, ker\, T^*$ is called the index of T. In particular, if $-\infty < ind\, T < \infty$, then T is called a Fredholm operator. The Wolf spectrum of T and the essential spectrum of T are defined respectively as

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi} - \text{Fredholm}\}$$

and

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\}.$$

The spectral picture of an operator T, denoted by $\Lambda(T)$, consists of the Wolf spectrum and the values of the index function off the Wolf spectrum. So two operators A, B have the same spectral picture if and only if $\sigma_{lre}(A) = \sigma_{lre}(B)$ and $ind\ (A-\lambda) = ind\ (B-\lambda)$ for $\lambda \notin \sigma_{lre}(A)$.

Recall that an operator T is essentially normal if $T^*T - TT^*$ is compact. It is well known that $\sigma_{lre}(T) = \sigma_e(T)$ when T is essentially normal. The classical BDF Theorem classifies essentially normal operators up to unitary equivalence modulo compacts.

Theorem (6.3.6)[286]: ([235]). Let $A, B \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then there exists $K \in \mathscr{K}(\mathcal{H})$ such that $A \cong B + K$ if and only if $\Lambda(A) = \Lambda(B)$.

Here and in what follows,≅denotes unitary equivalence.

Following Berg and Davidson [288], we say that an operator T is almost normal if $T = N + K$ for some normal N and some compact K. Then almost normal operators are always essentially normal. By Theorem (6.3.6), an essentially normal operator A is almost normal if and only if ind $(A-\lambda) = 0$ for all $\lambda \notin \sigma_e(A)$. By the continuity of the index function, one can see that the class of almost normal operators on $\mathcal{H}$ is norm closed.

Lemma (6.3.7)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If $C^*(T)$ admits a complex symmetric generator, then T is almost normal.

Proof. Assume that $A \in \mathcal{B}(\mathcal{H})$ is complex symmetric and $C^*(T) = C^*(A)$. Then there is a conjugation C on $\mathcal{H}$ such that $CAC = A^*$. Then for each $\lambda \notin \sigma_{lre}(A)$ one can check that

$$\text{ind }(A-\lambda) = \text{ind } C(A-\lambda)C = \text{ind }(A-\lambda)^* = -\text{ind }(A-\lambda).$$

So $ind\ (A-\lambda) = 0$ for $\lambda \notin \sigma_{lre}(A)$. On the other hand, since T is essentially normal and $A \in C^*(T)$, it follows that A is essentially normal. By the BDF Theorem, A has the form "normal plus compact". Since $T \in C^*(A)$, T is also of the form "normal plus compact".

The proof of the preceding result depends on a key approximation result.

Proposition (6.3.8)[286]: Given a normal operator $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is an irreducible CSO.

Proof. By the Weyl–von Neumann Theorem, we may directly assume that T is a diagonal operator with respect to some onb $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$. Assume that $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of T satisfying $Te_n = \lambda_n e_n$ for $n \geq 1$. For each $n \geq 1$, denote $a_n = Re\lambda_n$ and $b_n = Im\lambda_n$. Up to a small compact perturbation, we may assume that $a_i \neq a_j$ for $i \neq j$. Set

$$A = \sum_{i=1}^{\infty} a_i e_i \otimes e_i, \qquad B = \sum_{i=1}^{\infty} b_i e_i \otimes e_i$$

Then $T = A + iB$. For $i, j \geq 1$, set $d_{i,j} = \frac{\varepsilon}{2^{i+j}}$. Define a compact operator $K_1 \in \mathscr{K}(\mathcal{H})$ by

$$K_1 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}.$$

It is obvious that $K_1 \in \mathscr{K}(\mathcal{H})$ is self-adjoint and $\|K_1\| < 2\left(\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{1+n}}\right) = \varepsilon$. Set $K = iK_1$. Then it remains to check that $T + K$ is an irreducible CSO.

Note that $T + K = A + iB_1$, where $B_1 = B + K_1$. Then A, B_1 are both self-adjoint. Assume that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection commuting with $T + K$. It follows that PA=AP and PB1=B1P. Since $A = \sum_{i=1}^{\infty} a_i e_i \otimes e_i$ and $a_i \neq a_j$ whenever $i \neq j$, it follows from $AP = PA$ that $P = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$, where $\mu_i = 0$ or $\mu_i = 1$ for each $i \geq 1$. On the other hand, for $i, j \geq 1$ with $i \neq j$, we have

$$\langle PB_1 e_j, e_i \rangle = \langle B_1 e_j, Pe_i \rangle = \langle B_1 e_j, \mu_i e_i \rangle = \mu_i \langle Be_j, e_i \rangle + \mu_i \langle K_1 e_j, e_i \rangle = \mu_i / d_{i,j} = \frac{\mu_i}{2^{i+j}}$$

and

$$\langle B_1 Pe_j, e_i \rangle = \langle Pe_j, B_1 e_i \rangle = \mu_j \langle e_j, B_1 e_i \rangle = \mu_j \langle Be_j, e_i \rangle + \mu_j \langle K_1 e_j, e_i \rangle = \mu_j d_{i,j} = \frac{\mu_j}{2^{i+j}}.$$

Since $PB_1 = B_1 P$, it follows that $\mu_i = \mu_j$. Then either $P = 0$ or P is the identity operator on $\mathcal{H}$, which implies that $T + K$ is irreducible.

Now it remains to show that $T + K$ is a CSO. In fact, if C is the conjugation on $\mathcal{H}$ defined by $Ce_i = e_i$ for $i \geq 1$, then one can check that $C(A+K)C = (A+K)^*$. Since each of the operators A, B, K_1 admits a complex symmetric matrix representation with respect to the same onb $\{e_n\}$, one can also see that $T + K = A + i(B + K_1)$ is complex symmetric. We remark that the proof of Proposition (6.3.8) is inspired by the proof of Lemma (6.3.31) (see [184] or [294]).

Theorem (6.3.9)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If T is irreducible, then $T \in (cs)$ if and only if T is almost normal.

Proof. The necessity follows from Lemma (6.3.7).

"⇐". Since T is almost normal, there exist a normal operator N and $K \in \mathscr{K}(\mathcal{H})$ so that $T = N + K$. By Proposition (6.3.8), we can find compact K_0 such that $R := N + K_0$ is an irreducible CSO. Since T, R are both irreducible and essentially normal, we have $\mathscr{K}(\mathcal{H}) \subset C^*(T) \cap C^*(R)$. It follows that $T - R = K - K_0 \in C^*(T) \cap C^*(R)$. Thus $C^*(T) = C^*(R)$. This completes the proof.

In general, the condition of irreducibility in Theorem (6.3.9) can not be canceled. That is, the spectral condition "$ind\ (T - \lambda) = 0, \forall \lambda \notin \sigma_e(T)$" is necessary and not sufficient for $T \in (cs)$. Before giving an example, we first introduce a useful result.

Recall that an operator A is said to be abnormal if A has no nonzero reducing subspace $\mathcal{M}$ such that $A|_{\mathcal{M}}$ is normal. If an irreducible operator is not normal, then it is abnormal. Each Hilbert space operator T admits the unique decomposition

$$T = T_{nor} \oplus T_{abnor},$$

where T_{nor} is normal and T_{abnor} is abnormal. The operators Tnor and Tabnor are called the normal part and the abnormal part of T respectively. See p. 116 in [184].

Lemma (6.3.10)[286]: ([164], Lem. (6.3.13)). An operator T is complex symmetric if and only if Tabnor is complex symmetric.

Example (6.3.11)[286]: Let $S \in \mathcal{B}(\mathcal{H}_1)$ be the unilateral shift of multiplicity one and $N \in \mathcal{B}(\mathcal{H}_2)$ be a normal operator with $\sigma(N) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$. Denote $T = N \oplus S$. Then T is essentially normal. Note that $\sigma_{lre}(N) = \sigma(N) \supset \sigma(S)$. Thus $\sigma(T) = \sigma_{lre}(T)$ and $ind\ (T - \lambda) = 0$ for $\lambda \notin \sigma_{lre}(T)$. It follows from Theorem (6.3.6) that T is almost normal.

Now we shall show that $C^*(T)$ does not have a complex symmetric generator. For a proof by contradiction, we assume that $A \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a complex symmetric generator of $C^*(T)$. Obviously, A can be written as $A = A_1 \oplus A_2$, where $A_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$. So $C^*(A_1) = C^*(N)$ and $C^*(A_2) = C^*(S)$. It follows immediately that A_1 is normal, A_2 is irreducible and not normal. So A_2 is abnormal. Hence $A_1 = A_{nor}$ and $A_2 = A_{abnor}$. Since A is complex symmetric, it follows from Lemma (6.3.10) that A_2 is complex symmetric. Thus $C^*(S)$ has a complex symmetric generator A_2. By Lemma (6.3.7), S is almost normal. This is a contradiction, since S is Fredholm and ind $S = -1$.

We shall characterize when an essentially normal operator has a complex symmetric generator for its C^*-algebra. To state our main result, we need several extra definitions.

Definition (6.3.12)[286]: ([164], Def. 1.8). Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a transpose of T if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

The concept "transpose" of an operator is in fact a generalization of that for matrices. By definition, an operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if T is a transpose of itself. In general, an operator has more than one transpose [174]. However, one can check that any two transposes of an operator are unitarily equivalent ([164]). We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $\cong T^t$. It is easy to check that $\sigma(T) = \sigma(T^t), \sigma_{lre}(T) = \sigma_{lre}(T^t)$ and $ind\ (T - \lambda) = -ind\ (T^t - \lambda)$ for $\lambda \notin \sigma_{lre}(T)$.

If $\mathcal{M}$ is a nonzero reducing subspace of $T \in \mathcal{B}(\mathcal{H})$ and $T|_{\mathcal{M}}$ is irreducible, then $\mathcal{M}$ is called a minimal reducing subspace (m.r.s. for short) of T. Given an essentially normal operator $T \in \mathcal{B}(\mathcal{H})$, define

$$\mathcal{H}_s = \vee\{\mathcal{M} \subset \mathcal{H}: \mathcal{M} \text{ is a m.r.s. of } T \text{ and } T|_{\mathcal{M}} \text{ is not almost normal}\},$$

where $\vee$ denotes closed linear span. It is obvious that $\mathcal{H}_s$ is either absent or a nonzero reducing subspace of T. Denote by T_s the restriction of T to H_s. We call Ts the singular part of T.

We say that two operators A, B are disjoint if there exist no nonzero reducing subspace $\mathcal{M}_1$ of A and nonzero reducing subspace M_2 of B such that $A|_{\mathcal{M}_1} \cong B|_{\mathcal{M}_2}$.

Definition (6.3.13)[286]: An essentially normal operator T is called type C, if $T = T_s$ and T is unitarily equivalent to an operator of the form $A \oplus B$, where (a) $A, B \in \mathcal{B}(\mathcal{H})$ are disjoint, (b) $C^*(A) \cap \mathcal{K}(\mathcal{H}) = C^*(B) \cap \mathcal{K}(\mathcal{H})$, and (c) there exists compact $K \in C^*(A)$ such that $A + K$ is a transpose of B and $C^*(A + K) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$.

One can check that if an essentially normal operator T is of type C, then T is almost normal. In fact, by the discussion right after Definition (6.3.12), we have

$$\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B) = \sigma_e(A)$$

and

$$ind\ (T - \lambda) = ind\ (A - \lambda) + ind\ (B - \lambda) = 0 \text{ for all } \lambda \notin \sigma_e(T).$$

By Theorem (6.3.6), T is almost normal.

By Theorem (6.3.27), whether or not an essentially normal operator T has a complex symmetric generator for its C^*-algebra depends only on the behavior of T_s.

We give a concrete description of the essentially normal operators of type C. We first make some preparation.

Let $\{\mathcal{A}_i\}_{i\in\Gamma}$ be a family of C^*-algebras. We denote by $\prod_{i\in\Gamma} \mathcal{A}_i$ the direct product of $\{\mathcal{A}_i\}_{i\in\Gamma}$, and by $\bigoplus_{i\in\Gamma} \mathcal{A}_i$ the direct sum of $\{\mathcal{A}_i\}_{i\in\Gamma}$.

Let $A \in \mathcal{B}(\mathcal{H})$. We let $W^*(A)$ denote the von Neumann algebra generated by A. By the von Neumann Double Commutant Theorem, we have $W^*(A) = C^*(A)''$. Here and in what follows, $\mathcal{A}'$ denotes the commutant algebra of $\mathcal{A}$.

See [275] for a proof of the following result.

Lemma (6.3.14)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $T = \bigoplus_{i\in\Gamma} T_i$, where $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_i \ncong T_j$ whenever $i \neq j$. Then

$$C^*(T)' = \prod_{i\in\Gamma} \mathbb{C}I_i, \qquad W^*(T) = \prod_{i\in\Gamma} \mathcal{B}(\mathcal{H}_i),$$

where I_i is the identity operator on $\mathcal{H}_i$ and $\mathbb{C}I_i = \{\lambda I_i: \lambda \in \mathbb{C}\}$ for $i \in \Gamma$.

For convenience, we let $0_{\mathcal{H}}$ denote the subalgebra $\{0\}$ of $\mathcal{B}(\mathcal{H})$. Given $e, f \in \mathcal{H}$, the operator $e \otimes f$ is defined as $(e \otimes f)(x) = \langle x, f\rangle e$ for $x \in \mathcal{H}$.

Corollary (6.3.15)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal and $T = N \oplus (\oplus_{i=1}^{\infty} T_i)$, where

(i) $N \in \mathcal{B}(\mathcal{H}_0)$ is normal,

(ii)$T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \geq 1$, and

(iii) $T_i \not\cong T_j$ whenever $i \neq j$.

Then $0_{\mathcal{H}_0} \oplus (\oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i) \subset C^*(T)$. Moreover, if N is absent, then

$$C^*(T) \cap \mathcal{K}(\mathcal{H}) = \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i).$$

Proof. For any fixed $i \geq 1$ and fixed $e, f \in \mathcal{H}_i$, it suffices to prove that $f \otimes e \in C^*(T)$. Set $K = T^*T - TT^*$. By the hypothesis, we may assume $K = 0 \oplus \left(\oplus_{j=1}^{\infty} K_j\right)$, where $K_j \in \mathcal{K}\left(\mathcal{H}_j\right)$ for $j \geq 1$. It is obvious that $K_j \neq 0$ for all $j \geq 1$ since T_j is not normal. There exist nonzero $e_1, f_1 \in \mathcal{H}_i$ such that $K_i e_1 = f_1$. We may assume that $\|f_1\| = 1$.

Set $=\oplus_{j=1}^{\infty} T_j$. Since each T_j is irreducible and $T_{j_1} \not\cong T_{j_2}$ for $j_1 \neq j_2$, it follows from Lemma (6.3.14) that each operator commuting with both A and A^* has the form $\oplus_{j=1}^{\infty} \lambda_j I_j$, where I_j is the identity operator on $\mathcal{H}_j$. Moreover, we have

$$W^*(A) = \prod_{j=1}^{\infty} \mathcal{B}\left(\mathcal{H}_j\right).$$

So $f \otimes e \in W^*(A)$ and, by the von Neumann Double Commutant Theorem, we have $\otimes f_1, e_1 \otimes e, f \otimes e \in \overline{C^*(A)}^{sot}$. Here sot denotes the strong operator topology. Using the Kaplansky Density Theorem ([180], Thm. I.7.3, Rem. I.7.4), we can choose polynomials $\{p_n(\cdot,\cdot)\}$ and $\{q_n(\cdot,\cdot)\}$ in two free variables so that

$$p_n(A^*, A) \xrightarrow{sot} f \otimes f_1, \qquad q_n(A^*, A) \xrightarrow{sot} e_1 \otimes e.$$

Since $\oplus_{j=1}^{\infty} K_j$ is compact, we obtain

$$p_n(A^*, A)\left(\oplus_{j=1}^{\infty} K_j\right) q_n(A^*, A) \xrightarrow{\|\cdot\|} f \otimes e.$$

Moreover, we obtain

$$p_n(T^*, T) K q_n(T^*, T) = \begin{bmatrix} 0 & 0 \\ 0 & p_n(A^*, A)\left(\oplus K_j\right) q_n(A^*, A) \end{bmatrix} \xrightarrow{\|\cdot\|} \begin{bmatrix} 0 & 0 \\ 0 & f \otimes e \end{bmatrix},$$

which completes the proof.

Recall that an operator is said to be completely reducible if it does not admit any minimal reducing subspace ([154]).

Lemma (6.3.16)[286]: If an essentially normal operator T is completely reducible, then T is normal.

Proof. Assume that $T \in \mathcal{B}(\mathcal{H})$. Since T is completely reducible, by [154], we have $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Noting that T is essentially normal, we obtain $T^*T - TT^* \in C^*(T) \cap \mathcal{K}(\mathcal{H})$. Thus $T^*T - TT^* = 0$.

If d is a cardinal number and $\mathcal{H}$ is a Hilbert space, let $\mathcal{H}^{(d)}$ denote the direct sum of $\mathcal{H}$ with itself d times. If $A \in \mathcal{B}(\mathcal{H}), A^{(d)}$ is the direct sum of A with itself d times.

Lemma (6.3.17)[286]: ([154]). Each operator $T \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to an operator of the form

$$T_0 \oplus (\oplus_{i \in \Gamma} T_i^{(n_i)},$$

where T_0 is completely reducible, each T_i is irreducible and $T_i \not\cong T_j$ for $i, j \in \Gamma$ with $i \neq j$.

Lemma (6.3.18)[286]: ([147]). Let $T \in \mathcal{B}(\mathcal{H})$ and $T = T_0 \oplus (\bigoplus_{i\in\Gamma} T_i^{(n_i)}$, where T_0 is completely reducible, T_i is irreducible and $1 \le n_i \le \infty$ for $i \in \Gamma$; moreover, $T_i \ncong T_j$ whenever $i, j \in \Gamma$ and $i \neq j$. Then each reducing subspace $\mathcal{M}$ of T has the form of $\mathcal{M}_0 \oplus (\bigoplus_{i\in\Gamma} \mathcal{M}_i)$, where $\mathcal{M}_0$ is a reducing subspace of T_0 and $\mathcal{M}_i$ is a reducing subspace of $T_i^{(n_i)}$ for $i \in \Gamma$.

Lemma (6.3.19)[286]: ([147]). Let $T = A^{(n)}$, where $A \in \mathcal{B}(\mathcal{H})$ is irreducible and $1 \le n \le \infty$. If $\mathcal{M}$ is a nonzero reducing subspace of T, then $T|_{\mathcal{M}} \cong A$ if and only if $T|_{\mathcal{M}}$ is irreducible.

Lemma (6.3.20)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then Tabnor is unitarily equivalent to an operator of the form

$$\bigoplus_{i\in\Gamma} T_i^{(n_i)},$$

where each T_i is irreducible, not normal and $T_i \ncong T_j$ for $i, j \in \Gamma$ with $i \neq j$. Moreover, T_s is the restriction of T_{abnor} to a reducing subspace and

$$T_s \cong \bigoplus_{i\in\Gamma_0} T_i^{(n_i)},$$

where $\Gamma_0 = \{i \in \Gamma : T_i \text{ is not almost normal}\}$.

Proof. By Lemma (6.3.17), T_{abnor} is unitarily equivalent to an operator of the form

$$T_0 \oplus \left(\bigoplus_{i\in\Gamma} T_i^{(n_i)}\right),$$

where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and $T_i \ncong T_j$ for $i, j \in \Gamma$ with $i \neq j$. Note that T_i is abnormal for $i \in \Gamma$. Since T_0 is completely reducible and essentially normal, it follows from Lemma (6.3.16) that T_0 is normal. Note that T_{abnor} is abnormal; so T_0 is absent. Then $T_{abnor} \cong \bigoplus_{i\in\Gamma} T_i^{(n_i)}$. For convenience we directly assume that $T_{abnor} = \bigoplus_{i\in\Gamma} T_i^{(n_i)}$. Thus

$$T = T_{nor} \oplus \left(\bigoplus_{i\in\Gamma} T_i^{(n_i)}\right).$$

By definition, it is obvious that $\bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)} \subset \mathcal{H}_s$. On the other hand, if $\mathcal{M}$ is a m.r.s. of T and $T|_{\mathcal{M}}$ is not almost normal, then, by Lemmas (6.3.18) and (6.3.19), there exists $i_0 \in \Gamma$ such that $\mathcal{M} \subset \mathcal{H}_{i_0}^{(n_{i_0})}$ and $T|_{\mathcal{M}} \cong T_{i_0}$. So T_{i_0} is not almost normal and $\mathcal{H}_{i_0}^{(n_{i_0})} \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Thus $\mathcal{M} \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Furthermore we obtain $\mathcal{H}_s \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Therefore $\mathcal{H}_s = \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$.

Corollary (6.3.21)[286]: If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T = T_s$ if and only if T is the direct sum of a family of essentially normal operators which are irreducible and not almost normal.

Proposition (6.3.22)[286]: An essentially normal operator T is of type C if and only if T is unitarily equivalent to an operator of the form

$$\bigoplus_{1\le i<v} (A_i \oplus B_i)^{(n_i)}, \qquad 1 \le n_i < \infty,$$

where (i) $v \in \mathbb{N}$ or $v = \infty, \{A_i, B_i : 1 \le i < v\}$ are irreducible and no two of them are unitarily equivalent, (ii) A_i is not almost normal and there exists compact K_i such that $A_i + K_i$ is a transpose of B_i for each i, and (iii) $\|K_i\| \to 0$ if $v = \infty$.

Proof. "⇐". Assume that $A_i, B_i \in \mathcal{B}(\mathcal{H}_i)$ for $1 \le i < v$. Denote $\mathcal{H} = \bigoplus_{1\le i<v} \mathcal{H}_i^{(n_i)}$ and

$$A = \bigoplus_{1\le i<v} A_i^{(n_i)}, \qquad B = \bigoplus_{1\le i<v} B_i^{(n_i)}.$$

Then $A, B \in \mathcal{B}(\mathcal{H})$ are essentially normal and $T \cong A \oplus B$. For convenience we directly assume that $T = A \oplus B$ and $v = \infty$.

Since $\{A_i, B_i: 1 \le i < v\}$ are irreducible, not normal and no two of them are unitarily equivalent, it follows from Corollary (6.3.15) that

$$C^*(A) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{1 \le i < v} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(B) \cap \mathcal{K}(\mathcal{H}). \tag{16}$$

Moreover, if $\mathcal{M}$ is a m.r.s. of T, then, by Lemmas (6.3.18) and (6.3.19), there exists unique i_0 with $1 \le i_0 < v$ such that exactly one of the following holds

$$T|_{\mathcal{M}} \cong A_{i_0}, \qquad T|_{\mathcal{M}} \cong B_{i_0}.$$

It follows that A, B are disjoint; moreover, $T|_{\mathcal{M}}$ is not almost normal. Thus, by Corollary (6.3.21), $T = T_s$.

By statement (ii), for each $1 \le i < v$, we can find a conjugation C_i on $\mathcal{H}_i$ so that $A_i + K_i = C_i B_i^* C_i$. Set

$$K = \bigoplus_{1 \le i < v} K_i^{(n_i)}, \qquad C = \bigoplus_{1 \le i < v} C_i^{(n_i)}.$$

Then C is a conjugation on $\mathcal{H}$ and, by (16), $K \in C^*(A) \cap \mathcal{K}(\mathcal{H})$, since $\|K_j\| \to 0$; moreover, $CB^*C = A + K$.

On the other hand, since $\{B_i\}$ are irreducible, not normal and no two of them are unitarily equivalent, so are $\{A_i + K_i\}$. It follows from Corollary (6.3.15) that

$$C^*(A + K) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{1 \le i < v} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(A) \cap \mathcal{K}(\mathcal{H}),$$

"⇒". Now assume that $T = T_s$ and $T = A \oplus B$, where $A, B \in \mathcal{B}(\mathcal{H})$ satisfy conditions (a), (b) and (c) in Definition (6.3.13). Since $T = T_s$, it follows that $A = A_s$. Then, by Corollary (6.3.21), we may assume that

$$A = \oplus_{i \in \Gamma} A_i^{(n_i)}, \qquad 1 \le n_i < \infty,$$

where each $A_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible, not almost normal and $A_i \ncong A_j$ whenever $i \ne j$. By Corollary (6.3.15), we have

$$C^*(A) \cap \mathcal{K}(\mathcal{H}) = \oplus_{i \in \Gamma} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Then K can be written as

$$K = \oplus_{i \in \Gamma} K_i^{(n_i)},$$

where $K_i \in \mathcal{K}(\mathcal{H}_i)$ for $i \in \Gamma$, and $\|K_i\| \to 0$ if Γ is infinite. Since $C^*(B) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$ is an ideal of $C^*(B)$, B can be written as

$$B = \oplus_{i \in \Gamma} E_i^{(n_i)};$$

moreover, this means that $\mathcal{K}(\mathcal{H}_i) \subset C^*(E_i)$, E_i is irreducible and $E_i \ncong E_j$ whenever $i \ne j$. Since A, B are disjoint, we deduce that no two of $\{A_i, E_i: i \in \Gamma\}$ are unitarily equivalent. Note that $A + K = \oplus_{i \in \Gamma} (A_i + K_i)^{(n_i)}$ and $C^*(A + K) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$. As we have done to B, we can also deduce that $\{A_i + K_i\}$ are irreducible and no two of them are unitarily equivalent.

By the hypothesis, $A + K$ is a transpose of B. Thus $\oplus_{i \in \Gamma} (A_i + K_i)^{(n_i)}$ and $\oplus_{i \in \Gamma} (E_i^t)^{(n_i)}$ are unitarily equivalent, and their m.r.s.'s correspond one to one. Then, by Lemmas (6.3.18) and (6.3.19), there exists a bijective map $\tau: \Gamma \to \Gamma$ such that $A_i + K_i \cong E_{\tau(i)}^t$ and $n_i = n_{\tau(i)}$ for all $i \in \Gamma$. For each $i \in \Gamma$, set $B_i = E_{\tau(i)}$. Then, up to unitary equivalence, $A_i + K_i$ is a transpose of B_i for each $i \in \Gamma$.

We first make some preparation.

Lemma (6.3.23)[286]: Let $H =\bigoplus_{i\in\Gamma} \mathcal{H}_i$ and $A \in \mathcal{B}(\mathcal{H})$ with $A =\bigoplus_{i\in\Gamma} A_i$, where $A_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$. If $B \in \mathcal{B}(\mathcal{H})$ and $C^*(A) = C^*(B)$, then there exist $B_i \in \mathcal{B}(\mathcal{H}_i), i \in \Gamma$, such that $B =\bigoplus_{i\in\Gamma} B_i$ and

(i) for any subset Γ_0 of $\Gamma, C^*\left(\bigoplus_{i\in\Gamma_0} A_i\right) = C^*\left(\bigoplus_{i\in\Gamma_0} B_i\right)$,

(ii) for each $i \in \Gamma$, the reducing subspaces of A_i coincide with that of B_i,

(iii) for each $i \in \Gamma, A_i$ is irreducible if and only if B_i is irreducible,

(iv) for any $i, j \in \Gamma, A_i \cong A_j$ if and only if $B_i \cong B_j$.

Proof. Since $C^*(A) = C^*(B)$, it is clear that B has the form $B =\bigoplus_{i\in\Gamma} B_i$, where $B_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$.

Statement (i) is also clear.

(ii) By (i), we have $C^*(A_i) = C^*(B_i)$. Thus $C^*(A_i) = C^*(B_i)$ and the assertion holds.

(iii) This follows immediately from (ii).

(iv) We directly assume $i \neq j$. By (i), we have $C^*(A_i \oplus A_j) = C^*\left(B_i \oplus B_j\right)$. If $A_i \cong A_j$, then there exists unitary operator $U: \mathcal{H}_j \to \mathcal{H}_i$ such that $A_j = U^*A_iU$. Then, for any polynomial $p(\cdot,\cdot)$ in two free variables, we have $p\left(A_j^*, A_j\right) = U^*p(A_i^*, A_i)U$. It follows immediately that each operator in $C^*\left(A_i \oplus A_j\right)$ has the form $X \oplus U^*XU$, where $X \in C^*(A_i)$. Sine $B_i \oplus B_j \in C^*\left(A_i \oplus A_j\right)$, we obtain $B_j = U^*B_iU$, that is, $B_i \cong B_j$. Thus $A_i \cong A_j$ implies $B_i \cong B_j$. Likewise, one can see the converse.

Lemma (6.3.24)[286]: Let $T, R \in \mathcal{B}(\mathcal{H})$ be essentially normal. If $C^*(T) = C^*(R)$, then

(i) T_s is absent if and only if R_s is absent, and

(ii) $C^*(T_s) = C^*(R_s)$.

Proof. In view of Lemma (6.3.20), we may assume that

$$T = T_{nor} \oplus \left(\bigoplus_{i\in\Gamma} T_i^{(n_i)}\right), \qquad 1 \leq n_i < \infty,$$

where $T_{nor} \in \mathcal{B}(\mathcal{H}_0), T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \in \Gamma$; moreover, $T_i \ncong T_j$ whenever $i \neq j$. Since $C^*(T) = C^*(R), R$ can be written as

$$R = R_0 \oplus \left(\bigoplus_{i\in\Gamma} R_i^{(n_i)}\right),$$

where $R_0 \in \mathcal{B}(\mathcal{H}_0)$ and $R_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Thus $C^*(R_0) = C^*(T_{nor})$ and $C^*(R_i) = C^*(T_i)$ for $i \in \Gamma$. Then R_0 is normal; moreover, by Lemma (6.3.23), each R_i is irreducible, not normal and $R_i \ncong R_j$ whenever $i \neq j$. For each $i \in \Gamma$, we note that R_i is almost normal if and only if T_i is almost normal.

Denote $\Gamma_0 = \{i \in \Gamma: T_i \text{ is not almost normal}\}$. Then $\Gamma_0 = \{i \in \Gamma: R_i \text{ is not almost normal}\}$. Thus, by Lemma (6.3.20),

$$T_s =\bigoplus_{i\in\Gamma_0} T_i^{(n_i)}, \qquad R_s =\bigoplus_{i\in\Gamma_0} R_i^{(n_i)}.$$

From $C^*(T) = C^*(R)$, we deduce that $C^*(T_s) = C^*(R_s)$. This completes the proof.

Lemma (6.3.25)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal and $T =\bigoplus_{i=1}^{\infty} A_i$, where $Ai \in \mathcal{B}(\mathcal{H}_i)$ for $i \geq 1$. Assume that $B_i \in \mathcal{B}(\mathcal{H}_i)$ is a transpose of A_i for $i \geq 1$. If $p(z_1, z_2)$ is a polynomial in two free variables, then there exists $\bigoplus_{i=1}^{\infty} K_i \in\bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$ such that $p(B_i^*, B_i) + K_i$ is a transpose of $p(A_i^*, A_i)$ for $i \geq 1$.

Proof. By the hypothesis, there exist conjugations $\{C_i\}_{i=1}^{\infty}$ such that $B_i = C_iA_i^*C_i, i \geq 1$. Set $E_i = A_i^*A_i - A_iA_i^*$ for $i \geq 1$. Since T is essentially normal, we have $T^*T - TT^* = \bigoplus_{i=1}^{\infty} E_i \in \mathcal{K}(\mathcal{H})$. So $E_i \in \mathcal{K}(\mathcal{H}_i)$ for $i \geq 1$ and $\|E_i\| \to 0$.

For convenience, we assume that $p(z_1, z_2) = z_1^2z_2z_1$. The proof in general case is similar. Compute to see that

$$C_i p(A_i^*, A_i)^* C_i = C_i A_i A_i^* A_i^2 C_i = B_i^* B_i (B_i^*)^2 = B_i^*(B_i B_i^*) B_i^* - B_i^*(B_i^* B_i) B_i^* + B_i^*(B_i^* B_i) B_i^*$$
$$= B_i^*(B_i B_i^* - B_i^* B_i) B_i^* + p(B_i^*, B_i) = B_i^*(C_i E_i C_i) B_i^* + p(B_i^*, B_i).$$
Set $K_i = B_i^*(C_i E_i C_i) B_i^*$. So K_i is compact and $p(B_i^*, B_i) + K_i$ is a transpose of $p(A_i^*, A_i)$; moreover, we have
$$\|K_i\| \le \|B_i\|^2 \cdot \|E_i\| = \|A_i\|^2 \cdot \|E_i\| \le \|T\|^2 \cdot \|E_i\| \to 0.$$
Hence $\bigoplus_{i=1}^{\infty} K_i \in \bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$. This completes the proof.

Proposition (6.3.26)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal and
$$T = \bigoplus_{j=1}^{\infty} (A_j \oplus B_j),$$
where $A_j, B_j \in \mathcal{B}(\mathcal{H}_j)$ and B_j is a transpose of A_j for $j \ge 1$. Then each operator $R \in C^*(T)$ can be written as $R = \bigoplus_{j=1}^{\infty} (F_j \oplus G_j)$, where $G_j \in \mathcal{B}(\mathcal{H}_j)$ is a compact perturbation of some transpose F_j^t of F_j and $\|G_j - F_j^t\| \to 0$.

Proof. Since B_j is a transpose of A_j, there exists a conjugation C_j such that $B_j = C_j A_j^* C_j$. Assume that $\{p_n\}_{n=1}^{\infty}$ are polynomials in two free variables and $p_n(T^*, T) \to R$. Note that $\bigoplus_{j=1}^{\infty} A_j$ is essentially normal. Then, by Lemma (6.3.25), for each $n \ge 1$, there exist compact operators $\{K_{j,n}\}_{j \ge 1}$ such that
$$p_n(B_j^*, B_j) + K_{j,n} = C_j p_n(A_j^*, A_j)^* C_j$$
and $\|K_{j,n}\| \to 0$ as $j \to \infty$. Then $\bigoplus_{j=1}^{\infty} K_{j,n}$ is compact for each $n \ge 1$.
Note that $p_n(T^*, T) \to R$ as $n \to \infty$ and
$$p_n(T^*, T) = \bigoplus_{j=1}^{\infty} \left(p_n(A_j^*, A_j) \oplus p_n(B_j^*, B_j)\right), \qquad n \ge 1.$$
Then $\bigoplus_{j=1}^{\infty} p_n(A_j^*, A_j)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} F_j$ and $\bigoplus_{j=1}^{\infty} p_n(B_j^*, B_j)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} G_j$ as $n \to \infty$. Then
$$\bigoplus_{j=1}^{\infty} C_j p_n(A_j^*, A_j)^* C_j \to \bigoplus_{j=1}^{\infty} C_j F_j^* C_j.$$
So, as $n \to \infty$, we have
$$\bigoplus_{j=1}^{\infty} K_{j,n} = \bigoplus_{j=1}^{\infty} \left(C_j p_n(A_j^*, A_j)^* C_j - p_n(B_j^*, B_j)\right) \to \bigoplus_{j=1}^{\infty} (C_j F_j^* C_j - G_j).$$
For each $n \ge 1$, note that $\bigoplus_{j=1}^{\infty} K_{j,n}$ is compact. Thus their norm limit $\bigoplus_{j=1}^{\infty} (C_j F_j^* C_j - G_j)$ is also compact. Hence $C_j F_j^* C_j - G_j$ is compact for each j and $\|C_j F_j^* C_j - G_j\| \to 0$ as $j \to \infty$. Note that $R = \lim_n p_n(T^*, T) = \bigoplus_{j=1}^{\infty} (F_j \oplus G_j)$. This completes the proof.

Now we are going to give the proof for the necessity of Theorem (6.3.27).

Theorem (6.3.27)[286]: If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T \in (cs)$ if and only if T_s is either absent or of type C.

Proof. Assume that $R \in \mathcal{B}(\mathcal{H})$ is complex symmetric and $C^*(T) = C^*(R)$. Also we assume that T_s is not absent. Then, by Lemma (6.3.24), R_s is not absent. Since T is essentially normal, so is R. By Lemma (6.3.10), Rabnor is complex symmetric. By ([164], Thm. 2.8), R_{abnor} is a direct sum of irreducible CSOs and operators with form of $\oplus Z^t$, where Z is irreducible and not complex symmetric. Note that each essentially normal CSO is almost normal. Then, up to unitary equivalence, we may assume that

$$R = N \oplus \left(\oplus_{i\in\Gamma_1} R_i^{(m_i)}\right) \oplus \left(\oplus_{j\in\Gamma_2} \left(A_j \oplus B_j\right)^{(n_j)}\right), \qquad (17)$$

where

(i) $N = R_{nor}$ is normal, $\{R_i, A_j, B_j : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) each R_i is almost normal and not normal;

(iii) A_j is not almost normal and B_j is a transpose of A_j for $j \in \Gamma_2$.

Note that each of $\{R_i, A_j, B_j : i \in \Gamma_1, j \in \Gamma_2\}$ is abnormal. Since R is essentially normal, it follows that $1 \le m_i, n_j < \infty$ for all i, j.

We assume that $N \in \mathscr{B}(\mathcal{H}_0), R_i \in \mathscr{B}(\mathcal{H}_{1,i})$ and $A_j, B_j \in \mathscr{B}(\mathcal{H}_{2,j})$ for $i \in \Gamma_1$ and $j \in \Gamma_2$. Hence

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\oplus_{i\in\Gamma_1} \mathcal{H}_{1,i}^{(m_i)}\right) \oplus (\oplus_{j\in\Gamma_2} \left(\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j}\right)^{(n_j)}. \qquad (18)$$

Since $C^*(T) = C^*(R)$, in view of Lemma (6.3.23), T can be written as

$$T = D \oplus \left(\oplus_{i\in\Gamma_1} E_i^{(m_i)}\right) \oplus \left(\oplus_{j\in\Gamma_2} \left(F_j \oplus G_j\right)^{(n_j)}\right) \qquad (19)$$

with respect to the decomposition (18); moreover, by statements (1)–(3), we have

(iv) D is normal, $\{E_i, F_j, G_j : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(v) each E_i is almost normal and not normal for $i \in \Gamma_1$;

(vi) F_j, G_j are essentially normal and not almost normal for $j \in \Gamma_2$.

By Lemma (6.3.20), we have $T_s = \oplus_{j\in\Gamma_2} \left(F_j \oplus G_j\right)^{(n_j)}$. On the other hand, note that

$$\oplus_{j\in\Gamma_2} \left(F_j \oplus G_j\right) \in C^*\left(\oplus_{j\in\Gamma_2} \left(A_j \oplus B_j\right)\right).$$

It follows from Proposition (6.3.26) that G_j is a compact perturbation of a transpose F_j^t of F_j for $j \in \Gamma_2$, and $\|G_j - F_j^t\| \to 0$ if Γ_2 is infinite. By Proposition (6.3.22), T_s is of type C. This proves the necessity.

To give the proof for the sufficiency of Theorem (6.3.27), we need to prove several approximation results.

Lemma (6.3.28)[286]: ([291]). Let $T \in \mathscr{B}(\mathcal{H})$ and suppose that $\emptyset \ne \Delta \subset \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where N is a diagonal normal operator of uniformly infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \bar{\Delta}, \sigma(T) = \sigma(A)$ and $\Lambda(T) = \Lambda(A)$.

Corollary (6.3.29)[286]: Let $T \in \mathscr{B}(\mathcal{H})$ and suppose that $\lambda \in \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} \lambda & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $A \in \mathscr{B}(\{e\}^\perp)$ satisfies $\sigma(T) = \sigma(A)$.

Proof. By Lemma (6.3.28), there exists $K \in \mathscr{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} \lambda I_1 & * \\ 0 & A_0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}, dim\, \mathcal{H}_1 = \infty, I_1$ is the identity operator on $\mathcal{H}_1$ and $A_0 \in \mathscr{B}(\mathcal{H}_2)$ satisfies $\sigma(A_0) = \sigma(T)$. Choose a unit vector $e \in \mathcal{H}_1$. Then $T + K$ can be written as

$$T+K=\begin{bmatrix}\lambda & 0 & E\\ 0 & \lambda I_2 & F\\ 0 & 0 & A_0\end{bmatrix}\begin{matrix}\mathbb{C}e\\ \mathcal{H}_1\ominus\mathbb{C}e,\\ \mathcal{H}_2\end{matrix}$$

where I_2 is the identity operator on $\mathcal{H}_1 \ominus \mathbb{C}e$. Set

$$A=\begin{bmatrix}\lambda I_2 & F\\ 0 & A_0\end{bmatrix}\begin{matrix}\mathcal{H}_1\ominus\mathbb{C}e\\ \mathcal{H}_2\end{matrix}.$$

Since $\lambda \in \sigma(T) = \sigma(A_0)$, it follows that $\sigma(A) = \sigma(T)$. Noting that

$$T+K=\begin{bmatrix}\lambda & *\\ 0 & A\end{bmatrix}\begin{matrix}\mathbb{C}e\\ \mathcal{H}\ominus\mathbb{C}e'\end{matrix}$$

we conclude the proof.

Given a subset Δ of $\mathbb{C}$, we write iso Δ for the set of all isolated points of Δ. For $\lambda \in \mathbb{C}$ and $\varepsilon > 0$, denote $B(\lambda, \varepsilon) = \{z \in \mathbb{C}: |z-\lambda| < \varepsilon\}$.

Lemma (6.3.30)[286]: Let $A, B \in \mathcal{B}(\mathcal{H})$. Assume that $\lambda \in iso\ \sigma(A)$ and $\lambda \notin \sigma(B)$. Then there exists $\delta > 0$ such that

$$\text{“}E, F \in \mathcal{B}(\mathcal{H}), E < \delta, F < \delta\text{”} \Rightarrow \text{“}\sigma(A+E) \neq \sigma(B+F)\text{”}.$$

Proof. Since $\lambda \in iso\ \sigma(A)$ and $\lambda \notin \sigma(B)$, there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon)^- \cap \sigma(A) = \{\lambda\}$ and $B(\lambda, \varepsilon)^- \cap \sigma(B) = \emptyset$. Then, by the upper semi-continuity of spectrum (see [184]), there exists $\delta > 0$ such that

(i) $B(\lambda, \varepsilon)^- \cap \sigma(A+E) \neq \emptyset$ for any $E \in \mathcal{B}(\mathcal{H})$ with $\|E\| < \delta$, and

(ii)$B(\lambda, \varepsilon)^- \cap \sigma(B+F) = \emptyset$ for any $F \in \mathcal{B}(\mathcal{H})$ with $\|F\| < \delta$.

Hence we conclude the proof.

In the preceding lemma, A, B can be operators acting on different separable Hilbert spaces.

Lemma (6.3.31)[286]: ([290]). Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is irreducible.

Lemma (6.3.32)[286]: Let $\{A_i\}_{i=1}^n$ be operators on separate Hilbert spaces with pairwise distinct spectra. Then, given $B \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A_{n+1} := B + K$ is irreducible, and $\{\sigma(A_i)\}_{i=1}^{n+1}$ are pairwise distinct.

Proof. Choose a point λ_0 in $\partial\sigma(B) \cap \sigma_{lre}(B)$. By Corollary (6.3.29), there exists compact K_0 with $\|K_0\| < \frac{\varepsilon}{2}$ such that

$$B+K_0=\begin{bmatrix}\lambda_0 & E\\ 0 & B_0\end{bmatrix}\begin{matrix}\mathbb{C}e\\ \{e\}^{\perp}\end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $\sigma(B_0) = \sigma(B)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ outside $\sigma(B)$ such that $\sup\limits_{1\le i\le n+1} |\lambda_i - \lambda_0| < \frac{\varepsilon}{4}$. For each $1 \le i \le n+1$, set

$$B_i=\begin{bmatrix}\lambda_i & E\\ 0 & B_0\end{bmatrix}\begin{matrix}\mathbb{C}e\\ \{e\}^{\perp}\end{matrix}$$

Then $\|B + K_0 - B_i\| < \frac{\varepsilon}{4}$, $\lambda_i \in iso\ \sigma(B_i)$ and $\lambda_j \notin \sigma(B_i)$ whenever $i \neq j$. By Lemma (6.3.31), there exist compact operators F_i with $\|F_i\| < \frac{\varepsilon}{4}$ such that each $B_i + F_i$ is irreducible; moreover, by Lemma (6.3.30), we may also assume that $\{\sigma(B_i + F_i)\}_{i=1}^{n+1}$ are pairwise distinct. Then there exists some $i_0, 1 \le i_0 \le n+1$, such that $\sigma(B_{i_0} + F_{i_0}) \neq \sigma(A_j)$ for $1 \le j \le n$. Set $K = F_{i_0} + B_{i_0} - B$ and $A_{n+1} = B + K$. Then $A_{n+1} = B_{i_0} + F_{i_0}$ is irreducible. Noting that $K = F_{i_0} + B_{i_0} - (B + K_0) + K_0$ is compact,

$$\|K\| \le \|F_{i_0}\| + \|B_{i_0} - (B+K_0)\| + \|K_0\| < \varepsilon$$

and $\{\sigma(A_i)\}_{i=1}^{n+1}$ are pairwise distinct, we complete the proof.

In view of Lemma (6.3.32), the following corollary is clear.

Corollary (6.3.33)[286]: Given a sequence $\{A_i\}_{i=1}^{\infty}$ of operators and $\varepsilon > 0$, there exist compact operators $\{K_i\}_{i=1}^{\infty}$ with

$$\sup_i K_i < \varepsilon \quad \text{and} \quad \lim_i \|K_i\| = 0$$

such that each $A_i + K_i$ is irreducible for $i \geq 1$ and $\{\sigma(A_i + K_i)\}_{i=1}^{\infty}$ are pairwise distinct.

Lemma (6.3.34)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then, given $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist irreducible CSOs $T_1, T_2, \dots, T_n \in \mathcal{B}(\mathcal{H})$ with pairwise distinct spectra such that $\|T_i - T\| \in \mathcal{B}(\mathcal{H})$ and $T_i - T < \varepsilon$ for all $1 \leq i \leq n$.

Proof. Choose a point λ in $\partial\sigma(T) \cap \sigma_{lre}(T)$. By the classical Weyl–von Neumann Theorem, there exists compact K with $\|K\| < \frac{\varepsilon}{2}$ such that

$$T + K = \begin{bmatrix} \lambda & 0 \\ 0 & N \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^{\perp} \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, N is normal and $\sigma(N) = \sigma(T)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1, \lambda_2, \dots, \lambda_n$ outside $\sigma(T)$ such that $\sup_{1\leq i\leq n} |\lambda_i - \lambda_0| < \frac{\varepsilon}{4}$. For each $1 \leq i \leq n$, set

$$A_i = \begin{bmatrix} \lambda_i & E \\ 0 & N \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^{\perp} \end{matrix}.$$

Then $\|T + K - A_i\| < \frac{\varepsilon}{4}, \lambda_i \in iso\,\sigma(A_i)$ and $\lambda_j \notin \sigma(A_i)$ whenever $i \neq j$. By Proposition (6.3.8), there exist compact operators F_i with $\|F_i\| < \frac{\varepsilon}{4}$ such that each $A_i + F_i$ is irreducible and complex symmetric; moreover, by Lemma (6.3.30), it can be required that $\{\sigma(A_i + F_i)\}_{i=1}^{n}$ are pairwise distinct. Set $T_i = A_i + F_i$ for $1 \leq i \leq n$. Then $\{T_i : 1 \leq i \leq n\}$ satisfy all requirements.

Corollary (6.3.35)[286]: Let $\{T_i\}_{i=1}^{\infty}$ be normal operators on separable Hilbert spaces. Then, given $\varepsilon > 0$, there exist compact operators $\{K_i\}_{i=1}^{\infty}$ with

$$\sup_i \|K_i\| < \varepsilon, \qquad \lim_i \|K_i\| = 0$$

such that

(i) $T_i + K_i$ is complex symmetric and irreducible for $i \geq 1$, and

(ii)$\sigma(T_i + K_i) \neq \sigma(T_j + K_j)$ whenever $i \neq j$.

Proof. For convenience, we assume that $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \geq 1$. We shall construct $\{K_i\}_{i=1}^{\infty}$ by induction. By Proposition (6.3.8), we can choose $K_1 \in \mathscr{K}(\mathcal{H}_1)$ with $\|K_1\| < \varepsilon$ such that $T_1 + K_1$ is irreducible and complex symmetric.

Now assume that we have chosen compact operators $K_i \in \mathscr{K}(\mathcal{H}_i), 1 \leq i \leq n$, satisfying that (a) $\|K_i\| < \varepsilon/i$ for $1 \leq i \leq n$, (b) $T_i + K_i$ is complex symmetric and irreducible for $1 \leq i \leq n$, and (c) $\sigma(T_i + K_i) \neq \sigma(T_j + K_j)$ whenever $1 \leq i \neq j \leq n$. We are going to choose $K_{n+1} \in \mathscr{K}(\mathcal{H}_{n+1})$ with $\|K_{n+1}\| < \varepsilon/(n+1)$ such that $T_{n+1} + K_{n+1}$ is irreducible and complex symmetric; moreover, $\sigma(T_i + K_i) \neq \sigma(T_{n+1} + K_{n+1})$ for $1 \leq i \leq n$.

By Lemma (6.3.34), we can find $F_1, F_2, \dots, F_{n+1} \in \mathscr{K}(\mathcal{H}_{n+1})$ with $F_i < \varepsilon/(n+1)$ such that $T_{n+1} + F_i$ is irreducible and complex symmetric for $1 \leq i \leq n+1$; moreover, $\sigma(T_{n+1} + F_i) = \sigma(T_{n+1} + F_j)$ whenever $i \neq j$. So some $i_0, 1 \leq i_0 \leq n+1$, exists such that $\sigma(T_{n+1} + F_{i_0}) = \sigma(T_j + K_j)$ for all $1 \leq j \leq n$. Set $K_{n+1} = F_{i_0}$. Then K_{n+1} satisfies all requirements. By induction, this completes the proof.

In [293], Huaxin Lin solved the problem that an approximate normal matrix is close to a normal matrix in the affirmative. As an application, he proved a conjecture of Berg [287], which implies the following result.

Lemma (6.3.36)[286]: ([293]). Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of almost normal operators. Assume that $\sup_n \|T_n\| < \infty$ and $\|T_n^* T_n - T_n T_n^*\| \to 0$ as $n \to \infty$. Then there exists a sequence $\{N_n\}_{n=1}^{\infty}$ of normal operators such that $T_n - N_n$ is compact for $n \geq 1$ and $\|T_n - N_n\| \to 0$.

By the hypothesis, Lemma (6.3.20) and Proposition (6.3.22), we may assume that

$$T = N \oplus \left(\oplus_{i \in \Gamma_1} T_i^{(n_i)}\right) \oplus \left(\oplus_{j \in \Gamma_2} \left(A_j \oplus B_j\right)^{(n_j)}\right),$$

where

(i) N is normal, $\{T_i, A_j, B_j : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) T_i is almost normal and not normal for $i \in \Gamma_1$;

(iii) A_j is not almost normal and there exists a compact operator K_j such that $B_j + K_j$ is a transpose of A_j for $j \in \Gamma_2$;

(iv) $1 \leq n_i, n_j < \infty$ for all $i \in \Gamma_1$ and $j \in \Gamma_2$, and $\|K_j\| \to 0$ if Γ_2 is infinite.

Assume that $N \in \mathcal{B}(\mathcal{H}_0), T_i \in \mathcal{B}(\mathcal{H}_{1,i})$ for $i \in \Gamma_1$ and $A_j, B_j \in \mathcal{B}(\mathcal{H}_{2,j})$ for $j \in \Gamma_2$.

For convenience, we may directly assume that Γ_1, Γ_2 are countable and $n_i = 1$ for all $i \in \Gamma_1 \cup \Gamma_2$. The proof for the general case is similar. Then

$$T = N \oplus (\oplus_{i=1}^{\infty} T_i) \oplus \left(\oplus_{j=1}^{\infty} \left(A_j \oplus B_j\right)\right)$$

and

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i=1}^{\infty} \mathcal{H}_{1,i}\right) \oplus \left(\bigoplus_{j=1}^{\infty} (\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j})\right). \qquad (20)$$

The rest of the proof is divided into three steps.

Step 1. Compact perturbations of the operators $\{T_i : i \geq 1\}$.

Since T is essentially normal, it follows that $T^*T - TT^* \in \mathscr{K}(\mathcal{H})$ and hence $\|T_i^* T_i - T_i T_i^*\| \in \mathscr{K}(\mathcal{H}_{1,i})$ and $\|T_i^* T_i - T_i T_i^*\| \to 0$. By Lemma (6.3.36), we can choose $D_{1,i} \in \mathscr{K}(\mathcal{H}_{1,i}), i \geq 1$, so that $\|D_{1,i}\| \to 0$ and $N_i := T_i + D_{1,i}$ is normal for all $i \geq 1$. By Corollary (6.3.35), there are compact operators $D_{2,i} \in \mathscr{K}(\mathcal{H}_{1,i}) (i \geq 1)$ with $\|D_{2,i}\| \to 0$ such that $S_i := N_i + D_{2,i}$ is irreducible, complex symmetric and $S_i \not\cong S_j$ whenever $i \neq j$.

Set $D_i = D_{1,i} + D_{2,i}$ for $i \geq 1$. Then $S_i = T_i + D_i, D_i \in \mathscr{K}(\mathcal{H}_{1,i})$ and $\|D_i\| \to 0$. From statement (ii), each T_i acts on a space of dimension ≥ 2. Thus S_i is almost normal and not normal.

Step 2. Compact perturbations of the operators $\{A_j, B_j : j \geq 1\}$.

For each $j \geq 1$, by the hypothesis, there exists a conjugation C_j on $\mathcal{H}_{2,j}$ such that $C_j A_j^* C_j = B_j + K_j$. Note that $\|K_j\| \to 0$.

Since each A_j is irreducible, it follows from Corollary (6.3.33) that we can find compact operators $\{E_j\}_{j=1}^{\infty}$ with $E_j \to 0$ such that $R_j := A_j + E_j$ is irreducible for all $j \geq 1$ and $\{\sigma(R_j)\}_{j=1}^{\infty}$ are pairwise distinct.

For each $j \geq 1$, set $G_j = K_j + C_j E_j^* C_j$. Then $G_j \in \mathscr{K}(\mathcal{H}_{2,j})$ and $\|G_j\| \to 0$. On the other hand, note that

$$C_j R_j^* C_j = C_j A_j^* C_j + C_j E_j^* C_j = B_j + K_j + C_j E_j^* C_j = B_j + G_j.$$

Step 3. Construction and verification.
Set

$$R = N \oplus \left(\bigoplus_{i=1}^{\infty} S_i\right) \oplus \left(\bigoplus_{j=1}^{\infty} (R_j \oplus C_j R_j^* C_j)\right).$$

By ([147], Thm. 1.6) or [164], R is complex symmetric. Define $K \in \mathcal{B}(\mathcal{H})$ with respect to the decomposition (20) as

$$K = 0 \oplus \left(\bigoplus_{i=1}^{\infty} D_i\right) \oplus \left(\bigoplus_{j=1}^{\infty} (E_j \oplus G_j)\right). \quad (21)$$

Then K is compact and one can check that $R = T + K$. Now it remains to prove $C^*(T) = C^*(R)$. Clearly, we need only prove $K \in C^*(T) \cap C^*(R)$.

In view of (21), it suffices to prove that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathscr{K}(\mathcal{H}_{1,i})\right) \oplus \left(\bigoplus_{j=1}^{\infty} \left(\mathscr{K}(\mathcal{H}_{2,j}) \oplus \mathscr{K}(\mathcal{H}_{2,j})\right)\right) \subset C^*(T) \cap C^*(R).$$

By statements (i)–(iii), it follows from Corollary (6.3.15) that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathscr{K}(\mathcal{H}_{1,i})\right) \oplus \left(\bigoplus_{j=1}^{\infty} \left(\mathscr{K}(\mathcal{H}_{2,j}) \oplus \mathscr{K}(\mathcal{H}_{2,j})\right)\right) \subset C^*(T).$$

Since $\{S_i, R_i, C_i R_i^* C_i : i \geq 1\}$ are irreducible and not normal, by Corollary (6.3.15), it suffices to prove that no two of them are unitarily equivalent.

Noting that $\sigma(C_i R_i^* C_i) = \sigma(R_i) \neq \sigma(R_j) = \sigma(C_j R_j^* C_j)$ whenever $i \neq j$, we deduce that $R_i \ncong R_j, R_i C_j R_j^* C_j$ and $C_i R_i^* C_i \ncong C_j R_j^* C_j$ whenever $i \neq j$. On the other hand, note that R_j is a compact perturbation of A_j and A_j is not almost normal for $j \geq 1$. Then, for each $j \geq 1$, we can choose $\lambda \in \mathbb{C}$ such that $R_j - \lambda$ is Fredholm and ind $(R_j - \lambda) \neq 0$. So

$$ind(R_j - \lambda) = -ind\,(R_j - \lambda)^* = -ind\, C_j (R_j - \lambda)^* C_j = -ind\,(C_j R_j^* C_j - \lambda),$$

which implies that $R_j \ncong C_j R_j^* C_j$.

By the preceding argument, $S_i \ncong S_j$ whenever $i \neq j$. Since each of $\{S_i : i \geq 1\}$ is almost normal, we have $S_i \ncong C_j R_j^* C_j$ and $S_i \ncong R_j$ for all $i, j \geq 1$. Hence we deduce that no two of $\{S_i, R_i, C_i R_i^* C_i : i \geq 1\}$ are unitarily equivalent. This completes the proof.

Corollary (6.3.37)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If the restriction of T to its every reducing subspace is almost normal, then $T \in (cs)$.

Corollary (6.3.38)[286]: Each compact operator has a complex symmetric generator for its C^*-algebra.

Proof. Assume that $T \in \mathcal{B}(\mathcal{H})$ is compact. Then the restrictions of T to its minimal reducing subspaces are all compact and hence almost normal. Hence the result follows readily from Corollary (6.3.37).

Corollary (6.3.39)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If T_s is not absent, then the following are equivalent:

(i) $T \in (cs)$.

(ii)$T_{abnor} \in (cs)$.

(iii)$T_s \in (cs)$.

Proof. Note that $(T_s)_s = T_s = (T_{abnor})_s$. Then the result follows readily from Theorem (6.3.27).

Corollary (6.3.40)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal and assume that $= N \oplus A^{(n)}$, where $1 \le n < \infty$, N is normal, A is abnormal and irreducible. Then $T \in (cs)$ if and only if A is almost normal.

Proof. If A is almost normal, then T_s is absent. By Theorem (6.3.27), we have $T \in (cs)$. If A is not almost normal, then $T_s = A^{(n)}$ is not almost normal. So T_s is not of type C. By Theorem (6.3.27), we have $T \notin (cs)$.

Using the above corollary, one can deduce immediately that the operator T in Example (6.3.11) does not have a complex symmetric generator for its C^*-algebra.

Corollary (6.3.41)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal and $T = A^{(m)} \oplus B^{(n)}$, where A, B are irreducible, not normal and $A \not\cong B$. Then $T \in (cs)$ if and only if exactly one of the following holds:

(i) both A and B are almost normal;

(ii)neither A nor B is almost normal, $m = n$ and $\Lambda(At) = \Lambda(B)$.

Proof. Since T is essentially normal, it follows immediately that $1 \le m, n < \infty$.

"⇐". If (i) holds, then T_s is absent. By Theorem (6.3.27), we have $T \in (cs)$. If (ii) holds, then $T = T_s$; moreover, by the BDF Theorem, $\Lambda(A^t) = \Lambda(B)$ implies that B is a compact perturbation of A^t. So, by Proposition (6.3.22), T is of type C. The conclusion follows immediately from Theorem (6.3.27).

"⇒". We assume that $T \in (cs)$ and (i) does not hold. It suffices to prove that (ii) holds. For convenience we assume that $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$.

We claim that neither A nor B is almost normal. For a proof by contradiction, without loss of generality, we assume that A is almost normal. Then, by the hypothesis, B is not almost normal. So $T_s = B^{(n)}$ is not almost normal. Then T_s is not of type C and $T \notin (cs)$, a contradiction. This proves the claim, which means that $T = T_s$.

Since $T \in (cs)$, it follows that T is of type C. Noting that $A \not\cong B$, by the definition, it follows that $m = n$ and there exists compact K such that $A + K$ is unitarily equivalent to a compact perturbation of B^t. So $\Lambda(A) = \Lambda(B^t)$ and, equivalently, $\Lambda(A^t) = \Lambda(B)$.

Here we give another example of essentially normal operator which lies outside the class of CSOs and has a complex symmetric generator for its C^*-algebra.

Example (6.3.42)[286]: Let $\{e_i\}_{i=1}^{\infty}$ be an onb of $\mathcal{H}$. Define $A, B \in \mathcal{B}(\mathcal{H})$ as

$$Ae_i = \begin{cases} \frac{e_2}{2}, & i = 1 \\ e_{i+1}, & i \ge 2. \end{cases}, \qquad Be_i = e_{i+1}, \qquad \forall i \ge 1.$$

It is easy to verify that A, B are both essentially normal and irreducible; moreover, A is a compact perturbation of B. Note that A, B are Fredholm operators and ind $A = -1 = ind\ B$. So neither A nor B is complex symmetric. Set $T = A \oplus B^*$. It is obvious that $T = T_s$.

Define a conjugation C on $\mathcal{H}$ as $C: \sum_{i=1}^{\infty} \alpha_i e_i \mapsto \sum_{i=1}^{\infty} \overline{\alpha}_i e_i$. It is easy to check that $CB^*C = B^*$, so B^* is a transpose of B and, equivalently, B is a transpose of B^*. Then A is a compact perturbation of a transpose of B^*. Then T is of type C. By Theorem (6.3.27), we have $T \in (cs)$. In view of [172] or Thm. 1.6 in [147], T is not complex symmetric.

By Example (6.3.11), a compact perturbation of CSOs need not have its C^*-algebra generated by a CSO. It is natural to ask if $T \in \mathcal{B}(\mathcal{H})$ and there exists compact $K \in C^*(T)$

such that $T + K$ is complex symmetric, then does it follow that $C^*(T)$ can be generated by a CSO? No. Here is a counterexample.

Example (6.3.43)[286]: Let S be the unilateral shift of multiplicity one on $\mathcal{H}$. By Lemma (6.3.31), there exists compact K on $\mathcal{H} \oplus \mathcal{H}$ such that $A := (S \oplus 2I) + K$ is irreducible.

Set $T = A \oplus S^*$. Note that A, S^* are irreducible, essentially normal and neither A nor S^* is almost normal. So $T_s = T$. Since $\sigma_e(A^t) = \sigma_e(A) \neq \sigma_e(S^*)$, we deduce that $A \not\cong S^*$ and A^t is not unitarily equivalent to a compact perturbation of S^*. So T is not of type C. By Theorem (6.3.27), $C^*(T)$ does not admit a complex symmetric generator.

We write $\mathcal{H}_1$ and $\mathcal{H}_2$ for the underlying subspace of A and S^* respectively. By Corollary (6.3.15), we have

$$\mathscr{K}(\mathcal{H}_1) \oplus \mathscr{K}(\mathcal{H}_2) \subset C^*(T).$$

So $K_0 = (-K) \oplus 0$ is a compact operator in $C^*(T)$, and $T + K_0 = S \oplus 2I \oplus S^*$. Since S^* is a transpose of S, it follows from [172] that $T + K_0$ is complex symmetric.

For convenience, we write $T \in (wcs)$ to denote that $C^*(T)$ is $*$-isomorphic to some C^*-algebra singly generated by CSOs. The main result is the following theorem.

Note that $A \approx B$ if and only if $\|p(A^*, A)\| = \|p(B^*, B)\|$ for all polynomials $p(z_1, z_2)$ in two free variables. It is obvious that g-normal operators are invariant under algebraical equivalence.

Two operators A, B are approximately unitarily equivalent (write $A \cong_a B$) if there is a sequence of unitary operators U_n such that $\lim_n U_n A U_n^* = B$. It is obvious that approximate unitary equivalence implies algebraical equivalence.

Theorem (6.3.44)[286]: For $T \in \mathscr{B}(\mathcal{H})$, the following are equivalent:

(i) there is a faithful representation ρ of $C^*(T)$ such that $\rho(T)$ is complex symmetric;

(ii) T is g-normal;

(iii) T is algebraically equivalent to a CSO.

Proof. "(i)⇒(ii)". Assume that ρ is a faithful representation of $C^*(T)$ on $\mathcal{H}_\rho$ with $A = \rho(T)$ being complex symmetric. Then, for any polynomial $p(z_1, z_2)$ in two free variables, we have $\rho\big(p(T^*, T)\big) = p(A^*, A)$ and $\rho\big(\tilde{p}(T, T^*)\big) = \tilde{p}(A, A^*)$. Since ρ is faithful, we have

$$\|p(T^*, T)\| = \|p(A^*, A)\|, \qquad \|\tilde{p}(T, T^*)\| = \|\tilde{p}(A, A^*)\|.$$

Since each CSO is g-normal, it follows that

$$\|p(T^*, T)\| = \|p(A^*, A)\| = \|\tilde{p}(A, A^*)\| = \|\tilde{p}(T, T^*)\|.$$

So T is g-normal.

"(ii) ⇒ (iii)". Denote $= T^{(\infty)}$. Then R is still g-normal and $R \approx T$; moreover, $C^*(R)$ contains no nonzero compact operator. By [164], R is approximately unitarily equivalent to some complex symmetric operator X. Then $T \approx X$.

"(iii) ⇒ (i)". By definition, the implication is obvious.

An operator $T \in \mathscr{B}(\mathcal{H})$ is said to be multiplicity-free if $T|_{\mathcal{M}} T|_{\mathcal{N}}$ for any distinct minimal reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of T.

Lemma (6.3.45)[286]: Each operator is algebraically equivalent to a multiplicity-free operator.

Proof. Let $T \in \mathscr{B}(\mathcal{H})$. By Lemma (6.3.17), we may assume that

$$T = T_0 \oplus \left(\oplus_{i \in \Gamma} T_i^{(n_i)}\right),$$

where T_0 is completely reducible, $T_i \in \mathscr{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \not\cong T_{i_2}$ whenever $i_1 \neq i_2$.

Set $R = T_0 \oplus (\bigoplus_{i\in\Gamma} T_i)$. Then it is obvious that $\|p(T^*,T)\| = \|p(R^*,R)\|$ for any polynomial $p(z_1,z_2)$ in two free variables. So $T \approx R$. It remains to prove that R is multiplicity-free.

By Lemma (6.3.18), $\{\mathcal{H}_i : i \in \Gamma\}$ are all minimal reducing subspaces of R. For $i_1, i_2 \in \Gamma$ with $i_1 \neq i_2$, we have $R|_{\mathcal{H}_{i_1}} = T_{i_1} \not\cong T_{i_2} = R|_{\mathcal{H}_{i_2}}$. This completes the proof.

Recall that two representations ρ_1 and ρ_2 of a separable C^*-algebra $\mathscr{A}$ are approximately unitarily equivalent (write $\rho_1 \cong_a \rho_2$) if there is a sequence of unitary operators U_n such that

$$\rho_1(A) = \lim_n U_n^* \rho_2(A) U_n \quad \text{for all} \quad A \in \mathscr{A}.$$

The following result can be viewed as a consequence of Voiculescu's Theorem [196].

Lemma (6.3.46)[286]: ([180], Thm. II.5.8). Let $\mathscr{A}$ be a separable C^*-algebra, and let ρ_1 and ρ_2 be non-degenerate representations of $\mathscr{A}$ on separable Hilbert spaces. Then the following are equivalent:

(i) $\rho_1 \cong_a \rho_2$;

(ii) $rank\ \rho_1(X) = rank\ \rho_2(X)$ for all $X \in \mathscr{A}$.

Lemma (6.3.47)[286]: ([153], Thm. 5.40). If φ is a $*$-homomorphism of $\mathscr{K}(\mathcal{H})$ into $\mathscr{B}(\mathcal{K})$, then there exists a unique direct sum of $\mathcal{K} = \mathcal{K}_0 \oplus (\bigoplus_{\alpha\in\Gamma} \mathcal{K}_\alpha)$ such that each $\mathcal{K}_\alpha$ reduces $\varphi(\mathscr{K}(\mathcal{H})), \varphi(T)|_{\mathcal{K}_0} = 0$ for $T \in \mathscr{K}(\mathcal{H})$, and there exists a unitary operator U_α from $\mathcal{H}$ onto $\mathcal{K}_\alpha$ for $\alpha \in \Gamma$ such that $\varphi(T)|_{\mathcal{K}_\alpha} = U_\alpha T U_\alpha^*$ for $T \in \mathscr{K}(\mathcal{H})$.

Theorem (6.3.48)[286]: Let $T, R \in \mathscr{B}(\mathcal{H})$ be multiplicity-free. Then $T \approx R$ if and only if $T \cong_a R$.

Proof. The sufficiency is obvious.

"⇒". We let $\varphi: C^*(T) \to C^*(R)$ denote the $*$-isomorphism carrying T into R. It suffices to prove that

$$rank\ X = rank\ \varphi(X), \qquad \forall X \in C^*(T) \cap \mathscr{K}(\mathcal{H}) \tag{22}$$

and

$$rank\ \varphi^{-1}(Y) = rank\ Y, \qquad \forall Y \in C^*(R) \cap \mathscr{K}(\mathcal{H}). \tag{23}$$

In fact, if these equalities hold, then rank $\varphi(X) = rank\ X$ for all $X \in C^*(T)$. By Lemma(6.3.46), this implies $\varphi \cong_a id$, where $id(\cdot)$ denotes the identity representation of $C^*(T)$. So $R = \varphi(T) \cong_a id(T) = T$.

Denote $\mathscr{A} = C^*(T) \cap \mathscr{K}(\mathcal{H})$. By Thm.I.10.8 in [180] we may assume that

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{H}_i^{(k_i)} \right), \ \mathscr{A} = 0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i\in\Gamma} \mathscr{K}(\mathcal{H}_i)^{(k_i)} \right),$$

where the dimensions of $\mathcal{H}_0$ and $\mathcal{H}_i (i \in \Gamma)$ may be finite or $\aleph_0$, and $1 \le k_i < \infty$ for $i \in \Gamma$. Since $\mathscr{A}$ is an ideal of $C^*(T)$, T can be written as

$$T = D_0 \oplus \left(\bigoplus_{i\in\Gamma} D_i^{(k_i)} \right),$$

where $D_0 \in \mathscr{B}(\mathcal{H}_0)$ and $D_i \in \mathscr{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Then $\mathscr{K}(\mathcal{H}_i) \subset C^*(D_i)$ for each $i \in \Gamma$. Hence each D_i is irreducible. Noting that T is multiplicity-free, we have $k_i = 1$ for all $i \in \Gamma$. Then each compact operator in $C^*(T)$ has the form $0 \oplus (\bigoplus_{i\in\Gamma} X_i)$, where $X_i \in \mathscr{K}(\mathcal{H}_i)$. For $i \in \Gamma$, denote by P_i the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i$.

Claim 1. For each $i \in \Gamma$, there exist unique subspace $\mathcal{K}_i$ of $\mathcal{H}$ and a unitary operator $U_i: \mathcal{K}_i \to \mathcal{H}_i$ such that

$$\varphi(P_i K P_i) = 0 \oplus U_i^* K U_i, \qquad \forall K \in \mathscr{K}(\mathcal{H}_i).$$

Now fix an $i \in \Gamma$. Define $\varphi_i: \mathscr{K}(\mathcal{H}_i) \to \mathscr{B}(\mathcal{H})$ as

$$\varphi_i(F) = \varphi(P_iFP_i), \qquad \forall F \in \mathscr{K}(\mathcal{H}_i).$$

Then φ_i is an isometric $*$-homomorphism. By Lemma(6.3.47), there exists a unique direct sum of $\mathcal{H} = \mathcal{K}_0 \oplus (\oplus_{\alpha\in\Upsilon} \mathcal{K}_\alpha)$ with respect to which

$$\varphi_i(K) = 0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* K U_\alpha\right), \qquad \forall K \in \mathscr{K}(\mathcal{H}_i),$$

where $U_\alpha: \mathcal{K}_\alpha \to \mathcal{H}_i$ is unitary for each $\alpha \in \Upsilon$. To prove Claim 1, it suffices to prove that card $\Upsilon = 1$. Here "card" denotes cardinality. For a proof by contradiction, we assume that card $\Upsilon > 1$.

Note that $\mathscr{I} := \{P_iKP_i: K \in \mathscr{K}(\mathcal{H}_i)\}$ is an ideal of $C^*(T)$ and φ is an $*$-isomorphism. Then $\varphi(\mathscr{I}) = \varphi_i(\mathscr{K}(\mathcal{H}_i))$ is an ideal of $C^*(R)$. One can directly check that R can be written as

$$R = X_0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} X_\alpha\right)$$

with respect to the decomposition $\mathcal{H} = \mathcal{K}_0 \oplus (\oplus_{\alpha\in\Upsilon} \mathcal{K}_\alpha)$. Then $\mathscr{K}(\mathcal{K}_\alpha) \subset C^*(X_\alpha)$ and X_α is irreducible for each $\alpha \in \Upsilon$.

Since card $\Upsilon > 1$, we can find distinct $\alpha_1, \alpha_2 \in \Upsilon$. Since $\varphi_i(\mathscr{K}(\mathcal{H}_i))$ is an ideal of $C^*(R)$, for any $F \in \mathscr{K}(\mathcal{H}_i)$, we have $\varphi_i(F)R \in \varphi_i(\mathscr{K}(\mathcal{H}_i))$. So there exists unique $G \in \mathscr{K}(\mathcal{H}_i)$ such that $\varphi_i(F)R = \varphi_i(G)$, that is,

$$0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* F U_\alpha X_\alpha\right) = 0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* G U_\alpha\right).$$

It follows that $U\wedge * \_\alpha_1 FU\alpha_1 X_{\alpha_1} = U_{\alpha_1}^* G U_{\alpha_1}$ and $U_{\alpha_2}^* F U_{\alpha_2} X_{\alpha_2} = U_{\alpha_2}^* G U_{\alpha_2}$. So

$$FU_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = FU_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*.$$

Since $F \in \mathscr{K}(\mathcal{H}_i)$ is arbitrary, one can see that $U_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = U_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*$. Then $X_{\alpha_1} \cong X_{\alpha_2}$, contradicting the fact that R is multiplicity-free. This proves Claim 1.

Claim 2. $\{\mathcal{K}_i: i \in \Gamma\}$ are pairwise orthogonal.

For i_1, i_2 with $i_1 \neq i_2$, if $K_1 \in \mathscr{K}(\mathcal{H}_{i_1})$ and $K_2 \in \mathscr{K}(\mathcal{H}_{i_2})$, then

$$\varphi(P_{i_1}K_1P_{i_1})\varphi(P_{i_2}K_2P_{i_2}) = \varphi(P_{i_1}K_1P_{i_1}P_{i_2}K_2P_{i_2}) = 0.$$

Since $K_1 \in \mathscr{K}(\mathcal{H}_{i_1})$ and $K_2 \in \mathscr{K}(\mathcal{H}_{i_2})$ are arbitrary, one can deduce that $\mathcal{K}_{i_1}$ is orthogonal to $\mathcal{K}_{i_2}$.

Now we can conclude the proof by verifying that (22) and (23) hold.

Let $K \in C^*(T) \cap \mathscr{K}(\mathcal{H})$ Then, by our hypothesis, K can be written as

$$K = 0 \oplus \left(\bigoplus_{i\in\Gamma} K_i\right),$$

where $K_i \in \mathscr{K}(\mathcal{H}_i)$. It is obvious that $\|K_i\| \to 0$ if Γ is infinite. By Claims 1 and 2, we have

$$\varphi(K) = \varphi\left(\sum_{i\in\Gamma} P_iK_iP_i\right) = \sum_{i\in\Gamma} \varphi(P_iK_iP_i) = 0 \oplus (\oplus_{i\in\Gamma} U_i^* K_i U_i).$$

It follows immediately that rank $\varphi(K) = \sum_{i\in\Gamma} rank\ K_i = rank\ K$. This proves (22). By the symmetry, one can also deduce that (23) holds.

Lemma (6.3.49)[286]: ([184]). Let $T, R \in \mathscr{B}(\mathcal{H})$ and assume that T is essentially normal. If $T \cong_a R$, then $T_{abnor} \cong R_{abnor}$.

Corollary (6.3.50)[286]: Let $T, R \in \mathcal{B}(\mathcal{H})$ be essentially normal. If $T \approx R$, then $T_{abnor} \approx R_{abnor}$ and $T_s \approx R_s$.

Proof. By Lemma (6.3.17), we may assume that

$$T = T_0 \oplus \left(\oplus_{i\in\Gamma} T_i^{(m_i)}\right),$$

where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible, $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_1 \neq i_2$. Likewise, we assume that

$$R = R_0 \oplus \left(\oplus_{j\in\Upsilon} R_j^{(n_j)}\right),$$

where $R_0 \in \mathcal{B}(\mathcal{K}_0)$ is completely reducible, $R_j \in \mathcal{B}(\mathcal{K}_j)$ is irreducible for $j \in \Upsilon$ and $R_{j_1} \ncong R_{j_2}$ whenever $j_1 \neq j_2$. Noting that T_0, R_0 are essentially normal, it follows from Lemma (6.3.16) that T_0, R_0 are normal.

Denote

$$\Gamma_1 = \{i \in \Gamma: T_i \text{ is not normal}\}, \qquad \Gamma_2 = \{i \in \Gamma: T_i \text{ is not almost normal}\}.$$

Then $\Gamma_2 \subset \Gamma_1$ and

$$T_{abnor} = \oplus_{i\in\Gamma_1} T_i^{(m_i)}, \qquad T_s = (T_{abnor})_s = \oplus_{i\in\Gamma_2} T_i^{(m_i)}.$$

Denote

$$\Upsilon_1 = \{j \in \Upsilon: R_j \text{ is not normal}\}, \qquad \Upsilon_2 = \{j \in \Upsilon: R_j \text{ is not almost normal}\}.$$

Then $\Upsilon_2 \subset \Upsilon_1$ and

$$R_{abnor} = \oplus_{j\in\Upsilon_1} R_j^{(n_j)}, \qquad R_s = (R_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j^{(n_j)}.$$

Set

$$A = T_0 \oplus (\oplus_{i\in\Gamma} T_i), \qquad B = R_0 \oplus \left(\oplus_{j\in\Upsilon} R_j\right).$$

From the proof of Lemma(6.3.45), one can see that A, B are both multiplicity-free, $T \approx A$ and $R \approx B$. Since $T \approx R$, we obtain $A \approx B$. By Theorem(6.3.48), we have $A \cong_a B$. Note that A, B are both essentially normal. In view of Lemma (6.3.49), it follows that $A_{abnor} \cong B_{abnor}$. Hence $(A_{abnor})_s \cong (B_{abnor})_s$.

Note that

$$A_{abnor} = \oplus_{i\in\Gamma_1} T_i, \qquad A_s = (A_{abnor})_s = \oplus_{i\in\Gamma_2} T_i,$$

and

$$B_{abnor} = \oplus_{j\in\Upsilon_1} R_j, \qquad B_s = (B_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j.$$

We obtain

$$\oplus_{i\in\Gamma_1} T_i \cong \oplus_{j\in\Upsilon_1} R_j, \qquad \oplus_{i\in\Gamma_2} T_i \cong \oplus_{j\in\Upsilon_2} R_j.$$

This implies that

$$\oplus_{i\in\Gamma_1} T_i^{(m_i)} \approx \oplus_{j\in\Upsilon_1} R_j^{(n_j)}, \qquad \oplus_{i\in\Gamma_2} T_i^{(m_i)} \approx \oplus_{j\in\Upsilon_2} R_j^{(n_j)}.$$

Thus we obtain $T_{abnor} \approx R_{abnor}$ and $T_s \approx R_s$.

Lemma (6.3.51)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be multiplicity-free. Then each generator of $C^*(T)$ is multiplicity-free.

Proof. By Lemma(6.3.45), we may assume that

$$T = T_0 \oplus (\oplus_{i\in\Gamma} T_i),$$

where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible, $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_1 \neq i_2$. Note that $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$.

Assume that $R \in \mathcal{B}(\mathcal{H})$ and $C^*(T) = C^*(R)$. Then R can be written as $R = R_0 \oplus (\oplus_{i\in\Gamma} R_i)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$. By Lemma (6.3.23),

R_0 is completely reducible and R_i is irreducible for $i \in \Gamma$; moreover, $R_i \ncong R_j$ for $i, j \in \Gamma$ with $i \neq j$. In view of the proof of Lemma(6.3.45), R is multiplicity-free.

An operator is said to be UET if $\cong T^t$. In view of the BDF Theorem, if an essentially normal operator T is UET, then T is almost normal.

Lemma (6.3.52)[286]: ([164], Thm. 5.1). Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then T is g-normal if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent)

(i) normal operators,

(ii) irreducible UET operators, and

(iii) operators with the form of $A^{(m)} \oplus (A^t)^{(n)}$, where A is irreducible, not UET and $1 \leq m, n < \infty$.

Lemma (6.3.53)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If T is multiplicity-free and g-normal, then $T \in (cs)$.

Proof. Since T is essentially normal and g-normal, by Lemma (6.3.52), we may assume that
$$T = N \oplus \left(\oplus_{i \in \Gamma} T_i^{(l_i)}\right) \oplus (\oplus_{j \in \Upsilon} A_j^{(m_j)} \oplus B_j^{(n_j)},$$
where $N = T_{nor}$ is normal, $\{T_i, A_j, B_j : i \in \Gamma, j \in \Upsilon\}$ are abnormal, irreducible and no two of them are unitarily equivalent; moreover, each T_i is UET and A_j is a transpose of B_j for $j \in \Upsilon$. So $\Lambda(A_j) = \Lambda(B_j^t)$ for $j \in \Upsilon$. It follows that A_j is almost normal if and only if B_j^t (or, equivalently, B_j) is almost normal. On the other hand, since T is multiplicity-free, we deduce that $l_i = m_j = n_j = 1$ for all $i \in \Gamma$ and all $j \in \Upsilon$.

Denote $\Upsilon_0 = \{j \in \Upsilon : A_j$ is not almost normal$\}$. Note that T_i is almost normal for $i \in \Gamma$. It follows that
$$T_s = \oplus_{j \in \Upsilon_0} (A_j \oplus B_j).$$
By Proposition (6.3.22), T_s is of type C. In view of Theorem (6.3.27), we have $T \in (cs)$.

Theorem (6.3.54)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then $T \in (wcs)$ if and only if there exists an essentially normal operator $R \in (cs)$ such that $T \approx R$.

Proof. The sufficiency is obvious.

"⇒". Assume that
$$T = T_0 \oplus \left(\oplus_{i \in \Gamma} T_i^{(n_i)}\right),$$
where T_0 is completely reducible, $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_i \neq i_2$. Set $A = T_0 \oplus (\oplus_{i \in \Gamma} T_i)$. Then $A \approx T$ is essentially normal and, by Lemma(6.3.45), A is multiplicity-free.

Assume that $S \in \mathcal{B}(\mathcal{K})$ is complex symmetric and $C^*(S)$ is $*$-isomorphic to $C^*(T)$. By Lemma(6.3.45), S is algebraically equivalent to some multiplicity-free operator B. By Theorem(6.3.44), B is g-normal.

Since $C^*(S)$ is $*$-isomorphic to $C^*(T), A \approx T$ and $B \approx S$, we can find a$*$-isomorphism $\varphi : C^*(A) \to C^*(B)$. Denote $R = \varphi(A)$. Then $A \approx R$ and $C^*(B) = C^*(R)$. Noting that B is multiplicity-free, it follows from Lemma (6.3.51) that R is also multiplicity-free. By Theorem(6.3.48), we obtain $A \cong_a R$. Since A is essentially normal, so is R. This combining $C^*(B) = C^*(R)$ implies that B is also essentially normal. Since B is multiplicity-free and g-normal, it follows from Lemma (6.3.53) that $C^*(B) = C^*(R)$ admits a complex symmetric generator, that is, $R \in (cs)$. Noting that $T \approx A$ and $A \cong_a R$, we obtain $T \approx R$.

Corollary (6.3.55)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If T_s is not absent, then the following are equivalent:

(i) $T \in (wcs)$;
(ii) $T_{abnor} \in (wcs)$;
(iii) $T_s \in (wcs)$;
(iv) T_s is algebraically equivalent to an essentially normal operator of type C.

Proof. "(i) ⇒ (ii)". By Theorem (6.3.54), $T \in (wcs)$ implies that there exists an essentially normal operator $A \in \mathcal{B}(\mathcal{H})$ such that $A \in (cs)$ and $T \approx A$. By Corollary (6.3.50), we have $T_{abnor} \approx A_{abnor}$, and it follows from Corollary (6.3.39) that $A_{abnor} \in (cs)$. Using Theorem (6.3.54), we obtain $T_{abnor} \in (wcs)$.

"(ii) ⇒ (iii)". By Theorem (6.3.54), $T_{abnor} \in (wcs)$ implies that there exists an essentially normal operator $A \in \mathcal{B}(\mathcal{H})$ such that $A \in (cs)$ and $T_{abnor} \approx A$. By Corollary (6.3.50), we have $T_s = (T_{abnor})_s \approx A_s$, and it follows from Corollary (6.3.39) that $A_s \in (cs)$. Using Theorem (6.3.54), we obtain $T_s \in (wcs)$.

"(iii) ⇒ (iv)". By Theorem (6.3.54), $T_s \in (wcs)$ implies that there exists an essentially normal operator $A \in \mathcal{B}(\mathcal{H})$ such that $A \in (cs)$ and $T_s \approx A$. Then, by Corollary (6.3.50), $T_s = (T_s)_s \approx A_s$. By Theorem (6.3.27), $A \in (cs)$ implies that A_s is of type C. This proves the implication "(iii) ⇒ (iv)".

"(iv) ⇒ (i)". Assume that $A \in \mathcal{B}(\mathcal{H})$ is an essentially normal operator of type C and $T_s \approx A$. Denote by B the restriction of T to $\mathcal{H} \ominus \mathcal{H}_s$. Then the restriction of B to its each nonzero reducing subspace is almost normal. It follows that $T = T_s \oplus B \approx A \oplus B$. Noting that $(A \oplus B)_s = A_s = A$ is of type C, by Theorem (6.3.27), we have $A \oplus B \in (cs)$. By Theorem (6.3.54), we conclude that $T \in (wcs)$.

Now we shall conclude by giving a concrete form of those essentially normal operators T with $T \in (wcs)$. We need an auxiliary result.

Lemma (6.3.56)[286]: ([289]). Let $A, B \in \mathcal{B}(\mathcal{H})$ be essentially normal. If A, B are abnormal, then $A \approx B$ if and only if $A^{(\infty)} \cong B^{(\infty)}$.

Corollary (6.3.57)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then $T \in (wcs)$ if and only if T_s is either absent or unitarily equivalent to an essentially normal operator of the form

$$\bigoplus_{1\le i<v}\left(A_i^{(m_i)} \oplus B_i^{(n_i)}\right), 1 \le m_i, n_i < \infty,$$

where $\{A_i, B_i : 1 \le i < v\}$ are essentially normal operators satisfying the conditions (i), (ii) and (iii) in Proposition (6.3.22).

Proof. Obviously, we need only consider the case that T_s is not absent. By Lemma(6.3.45) and Proposition (6.3.22), each essentially normal operator of type C is algebraically equivalent to a multiplicity-free operator of the form

$$R = \bigoplus_{1\le i<v}(A_i \oplus B_i) \qquad (24)$$

Where $\{A_i, B_i : 1 \le i < v\}$ satisfy the conditions (i), (ii) and (iii) in Proposition (6.3.22). Then, by Corollary (6.3.55), an essentially normal operator T satisfies $T \in (wcs)$ if and only if T_s is algebraically equivalent to an operator R of the form (24). Noting that both T_s and R are abnormal, in view of Lemma (6.3.56), the latter is equivalent to

$$T_s^{(\infty)} \cong \bigoplus_{1\le i<v}\left(A_i^{(\infty)} \oplus B_i^{(\infty)}\right), \qquad (25)$$

By Lemmas (6.3.18) and (6.3.19), the condition (25) holds if and only if there exist $m_i, n_i, 1 \le i < v$ such that

$$T_s \cong \bigoplus_{1 \le i < v} \left(A_i^{(m_i)} \oplus B_i^{(n_i)} \right)$$

For each i, note that both $A_i^* A_i - A_i A_i^*$ and $B_i^* B_i - B_i B_i^*$ are nonzero compact operators. Since T_s is essentially normal, if such m_i, n_i exist, then it is necessary that $m_i, n_i < \infty$ for each i.

Corollary (6.3.58)[286]: Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. If T is irreducible, then the following are equivalent:

(i) $T \in (cs)$;

(ii) $T \in (wcs)$;

(iii) T is almost normal.

Proof. The implication "(i) ⇒ (ii)" is trivial, and the equivalence "(i) ⇔ (iii)" follows from Theorem (6.3.9).

"(ii) ⇒ (iii)". If T is not almost normal, then $T = T_s$ and T_s is not absent. By Corollary (6.3.57), T_s is reducible, a contradiction. This ends the proof.

Corollary (6.3.59)[295]: Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If $C^*(T^2)$ admits a complex symmetric square generator, then T^2 is almost normal.

Proof. Assume that $A^2 \in \mathcal{B}(\mathcal{H})$ is complex symmetric and $C^*(T^2) = C^*(A^2)$. Then there is a conjugation C on $\mathcal{H}$ such that $CA^2C = (A^*)^2$. Then for each $\lambda^2 \notin \sigma_{lre}(A^2)$ one can check that

$$ind\,(A^2 - \lambda^2) = ind\,C(A^2 - \lambda^2)C = ind\,(A^2 - \lambda^2)^* = -ind\,(A^2 - \lambda^2).$$

So $ind\,(A^2 - \lambda^2) = 0$ for $\lambda^2 \notin \sigma_{lre}(A^2)$. On the other hand, since T^2 is essentially square normal and $A^2 \in C^*(T^2)$, it follows that A^2 is essentially square normal. By the BDF Theorem, A^2 has the form "normal plus compact". Since $T^2 \in C^*(A^2)$, T^2 is also of the form "normal plus compact".

Corollary (6.3.60)[295]: Given a square normal operator $T^2 \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K^2 \in \mathscr{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that $T^2 + K^2$ is an irreducible CSO.

Proof. By the Weyl–von Neumann Theorem, we may directly assume that T^2 is a diagonal operator with respect to some ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$. Assume that $\{\lambda_n^2\}_{n=1}^{\infty}$ are the eigenvalues of T^2 satisfying $T^2 e_n = \lambda_n^2 e_n$ for $n \ge 1$. For each $n \ge 1$, denote $a_n^2 = \mathrm{Re}\lambda_n^2$ and $b_n^2 = Im\lambda_n^2$. Up to a small compact perturbation, we may assume that $a_i^2 \ne a_j^2$ for $i \ne j$. Set

$$A^2 = \sum_{i=1}^{\infty} a_i^2 e_i \otimes e_i\,, \qquad B^2 = \sum_{i=1}^{\infty} b_i^2 e_i \otimes e_i$$

Then $T^2 = A^2 + iB^2$. For $i, j \ge 1$, set $d_{i,j} = \frac{\varepsilon}{2^{i+j}}$. Define a compact operator $K_1^2 \in \mathscr{K}(\mathcal{H})$ by

$$K_1^2 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}^2 \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}.$$

It is obvious that $K_1^2 \in \mathscr{K}(\mathcal{H})$ is self-adjoint and $\|K_1^2\| < 2\left(\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{1+n}}\right) = \varepsilon$. Set $K^2 = iK_1^2$. Then it remains to check that $T^2 + K^2$ is an irreducible CSO.

Note that $T^2 + K^2 = A^2 + iB_1^2$, where $B_1^2 = B^2 + K_1^2$. Then A^2, B_1^2 are both self-adjoint. Assume that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection commuting with $T^2 + K^2$. It follows that $PA^2 = A^2 P$ and $PB_1^2 = B_1^2 P$. Since $A^2 = \sum_{i=1}^{\infty} a_i^2 e_i \otimes e_i$ and $a_i^2 \ne a_j^2$

whenever $i \neq j$, it follows from $A^2P = PA^2$ that $P = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$, where $\mu_i = 0$ or $\mu_i = 1$ for each $i \geq 1$. On the other hand, for $i, j \geq 1$ with $i \neq j$, we have

$$\langle PB_1^2 e_j, e_i\rangle = \langle B_1^2 e_j, Pe_i\rangle = \langle B_1^2 e_j, \mu_i e_i\rangle = \mu_i\langle B^2 e_j, e_i\rangle + \mu_i\langle K_1^2 e_j, e_i\rangle = \mu_i/d_{i,j} = \frac{\mu_i}{2^{i+j}}$$

and

$$\langle B_1^2 P e_j, e_i\rangle = \langle Pe_j, B_1^2 e_i\rangle = \mu_j\langle e_j, B_1^2 e_i\rangle = \mu_j\langle B^2 e_j, e_i\rangle + \mu_j\langle K_1^2 e_j, e_i\rangle = \mu_j d_{i,j} = \frac{\mu_j}{2^{i+j}}.$$

Since $PB_1^2 = B_1^2 P$, it follows that $\mu_i = \mu_j$. Then either $P = 0$ or P is the identity operator on $\mathcal{H}$, which implies that $T^2 + K^2$ is irreducible.

Now it remains to show that $T^2 + K^2$ is a CSO. In fact, if C is the conjugation on $\mathcal{H}$ defined by $Ce_i = e_i$ for $i \geq 1$, then one can check that $C(A^2 + K^2)C = (A^2 + K^2)^*$. Since each of the operators A^2, B^2, K_1^2 admits a complex symmetric matrix representation with respect to the same onb $\{e_n\}$, one can also see that $T^2 + K^2 = A^2 + i(B^2 + K_1^2)$ is complex symmetric.

Corollary (6.3.61)[295]: Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If T^2 is irreducible, then $T^2 \in (cs)$ if and only if T^2 is almost normal.

The proof of the preceding result depends on a key approximation result (see [286]).

Proof. [286] The necessity follows from Corollary (6.3.59).

"$\Leftarrow$". Since T^2 is almost normal, there exist a normal operator N^2 and $K^2 \in \mathcal{K}(\mathcal{H})$ so that $T^2 = N^2 + K^2$. By Corollary (6.3.60), we can find compact K_0^2 such that $R^2 := N^2 + K_0^2$ is an irreducible CSO. Since T^2, R^2 are both irreducible and essentially square normal, we have $\mathcal{K}(\mathcal{H}) \subset C^*(T^2) \cap C^*(R^2)$. It follows that $T^2 - R^2 = K^2 - K_0^2 \in C^*(T^2) \cap C^*(R^2)$. Thus $C^*(T^2) = C^*(R^2)$. This completes the proof.

In general, the condition of irreducibility in Corollary (6.3.61) can not be canceled. That is, the spectral condition "$ind\ (T^2 - \lambda^2) = 0, \forall \lambda^2 \notin \sigma_e(T^2)$" is necessary and not sufficient for $T^2 \in (cs)$. Before giving an example, we first introduce a useful result.

Recall that an operator A^2 is said to be abnormal if A^2 has no nonzero reducing subspace $\mathcal{M}$ such that $A^2|_{\mathcal{M}}$ is normal. If an irreducible operator is not normal, then it is abnormal. Each Hilbert space operator T^2 admits the unique decomposition

$$T^2 = T_{nor}^2 \oplus T_{\text{abnor}}^2,$$

where T_{nor}^2 is normal and T_{abnor}^2 is abnormal. The operators T_{nor}^2 and T_{abnor}^2 are called the normal part and the abnormal part of T^2 respectively. See [21, p. 116] for more details.

Corollary (6.3.62)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal and $T^2 = N^2 \oplus \left(\bigoplus_{i=1}^{\infty} T_i^2\right)$, where

(i) $N^2 \in \mathcal{B}(\mathcal{H}_0)$ is normal,

(ii) $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \geq 1$, and

(iii) $T_i^2 \not\cong T_j^2$ whenever $i \neq j$.

Then $0_{\mathcal{H}_0} \oplus (\bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i) \subset C^*(T^2)$. Moreover, if N^2 is absent, then

$$C^*(T^2) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i).$$

Proof. For any fixed $i \geq 1$ and fixed $e, f \in \mathcal{H}_i$, it suffices to prove that $f \otimes e \in C^*(T^2)$. Set $K^2 = (T^*)^2 T^2 - T^2 (T^*)^2$. By the hypothesis, we may assume $K^2 = 0 \oplus \left(\bigoplus_{j=1}^{\infty} K_j^2\right)$, where $K_j^2 \in \mathcal{K}(\mathcal{H}_j)$ for $j \geq 1$. It is obvious that $K_j^2 \neq 0$ for all $j \geq 1$ since T_j^2 is not normal. There exist nonzero $e_1, f_1 \in \mathcal{H}_i$ such that $K_i^2 e_1 = f_1$. We may assume that $\|f_1\| = 1$.

Set $A^2 = \bigoplus_{j=1}^{\infty} T_j^2$. Since each T_j^2 is irreducible and $T_{j_1}^2 \ncong T_{j_2}^2$ for $j_1 \neq j_2$, it follows from Lemma (6.3.14) that each operator commuting with both A^2 and $(A^*)^2$ has the form $\bigoplus_{j=1}^{\infty} \lambda_j^2 I_j$, where I_j is the identity operator on $\mathcal{H}_j$. Moreover, we have

$$W^*(A^2) = \prod_{j=1}^{\infty} \mathcal{B}(\mathcal{H}_j).$$

So $f \otimes e \in W^*(A^2)$ and, by the von Neumann Double Commutant Theorem, we have $\otimes f_1, e_1 \otimes e, f \otimes e \in \overline{C^*(A^2)}^{sot}$. Here sot denotes the strong operator topology. Using the Kaplansky Density Theorem ([180], Thm. I.7.3, Rem. I.7.4]), we can choose polynomials $\{p_n(\cdot,\cdot)\}$ and $\{q_n(\cdot,\cdot)\}$ in two free variables so that

$$p_n((A^*)^2, A^2) \xrightarrow{sot} f \otimes f_1, \qquad q_n((A^*)^2, A^2) \xrightarrow{sot} e_1 \otimes e.$$

Since $\bigoplus_{j=1}^{\infty} K_j^2$ is compact, we obtain

$$p_n((A^*)^2, A^2)\left(\bigoplus_{j=1}^{\infty} K_j^2\right) q_n((A^*)^2, A^2) \xrightarrow{\|\cdot\|} f \otimes e.$$

Moreover, we obtain

$$p_n((T^*)^2, T^2) K^2 q_n((T^*)^2, T^2)$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & p_n((A^*)^2, A^2)\left(\bigoplus K_j^2\right) q_n((A^*)^2, A^2) \end{bmatrix} \xrightarrow{\|\cdot\|} \begin{bmatrix} 0 & 0 \\ 0 & f \otimes e \end{bmatrix},$$

which completes the proof.

Corollary (6.3.63)[295]: [286]. If an essentially square normal operator T^2 is completely reducible, then T^2 is square normal.

Proof. Assume that $T^2 \in \mathcal{B}(\mathcal{H})$. Since T^2 is completely reducible, by [9, Lem. 2.5], we have $C^*(T^2) \cap \mathscr{K}(\mathcal{H}) = \{0\}$. Noting that T^2 is essentially square normal, we obtain $(T^*)^2 T^2 - T^2 (T^*)^2 \in C^*(T^2) \cap \mathscr{K}(\mathcal{H})$. Thus $(T^*)^2 T^2 - T^2 (T^*)^2 = 0$.

If d is a cardinal number and $\mathcal{H}$ is a Hilbert space, let $\mathcal{H}^{(d)}$ denote the direct sum of $\mathcal{H}$ with itself d times. If $A^2 \in \mathcal{B}(\mathcal{H}), A^{2(d)}$ is the direct sum of A^2 with itself d times.

Corollary (6.3.64)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. Then T_{abnor}^2 is unitarily equivalent to an operator of the form

$$\bigoplus_{i \in \Gamma} T_i^{2(n_i)},$$

where each T_i^2 is irreducible, not normal and $T_i^2 \ncong T_j^2$ for $i, j \in \Gamma$ with $i \neq j$. Moreover, T_s^2 is the restriction of T_{abnor}^2 to a reducing subspace and

$$T_s^2 \cong \bigoplus_{i \in \Gamma_0} T_i^{2(n_i)},$$

where $\Gamma_0 = \{i \in \Gamma : T_i^2 \text{ is not almost normal}\}$.

Proof. By Lemma (6.3.17), T_{abnor}^2 is unitarily equivalent to an operator of the form

$$T_0^2 \oplus \left(\bigoplus_{i \in \Gamma} T_i^{2(n_i)}\right),$$

where $T_0^2 \in \mathcal{B}(\mathcal{H}_0)$ is completely r educible, each $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and $T_i^2 \ncong T_j^2$ for $i, j \in \Gamma$ with $i \neq j$. Note that T_i^2 is abnormal for $i \in \Gamma$. Since T_0^2 is completely reducible and essentially square normal, it follows from Corollary (6.3.63) that T_0^2 is normal. Note that T_{abnor}^2 is abnormal; so T_0^2 is absent. Then $T_{abnor}^2 \cong \bigoplus_{i \in \Gamma} T_i^{2(n_i)}$. For convenience we directly assume that $T_{abnor}^2 = \bigoplus_{i \in \Gamma} T_i^{2(n_i)}$. Thus

$$T^2 = T_{nor}^2 \oplus \left(\bigoplus_{i \in \Gamma} T_i^{2(n_i)}\right).$$

By definition, it is obvious that $\bigoplus_{i\in\Gamma_0}\mathcal{H}_i^{(n_i)}\subset\mathcal{H}_s$. On the other hand, if $\mathcal{M}$ is a m.r.s. of T^2 and $T^2|_{\mathcal{M}}$ is not almost normal, then, by Lemmas (6.3.18) and (6.3.19), there exists $i_0\in\Gamma$ such that $\mathcal{M}\subset\mathcal{H}_{i_0}^{(n_{i_0})}$ and $T^2|_{\mathcal{M}}\cong T_{i_0}^2$. So $T_{i_0}^2$ is not almost normal and $\mathcal{H}_{i_0}^{(n_{i_0})}\subset\bigoplus_{i\in\Gamma_0}\mathcal{H}_i^{(n_i)}$. Thus $\mathcal{M}\subset\bigoplus_{i\in\Gamma_0}\mathcal{H}_i^{(n_i)}$. Furthermore we obtain $\mathcal{H}_s\subset\bigoplus_{i\in\Gamma_0}\mathcal{H}_i^{(n_i)}$. Therefore $\mathcal{H}_s=\bigoplus_{i\in\Gamma_0}\mathcal{H}_i^{(n_i)}$.

Corollary (6.3.65)[295]: [286]. An essentially square normal operator T^2 is of type C if and only if T^2 is unitarily equivalent to an operator of the form

$$\bigoplus_{1\le i<v}\left(A_i^2\oplus B_i^2\right)^{(n_i)},\qquad 1\le n_i<\infty,$$

where (i) $v\in\mathbb{N}$ or $v=\infty$, $\{A_i^2,B_i^2:1\le i<v\}$ are irreducible and no two of them are unitarily equivalent, (ii) A_i^2 is not almost normal and there exists compact K_i^2 such that $A_i^2+K_i^2$ is a transpose of B_i^2 for each i, and (iii) $\|K_i^2\|\to 0$ if $v=\infty$.

Proof. "⇐". Assume that $A_i^2,B_i^2\in\mathcal{B}(\mathcal{H}_i)$ for $1\le i<v$. Denote $\mathcal{H}=\bigoplus_{1\le i<v}\mathcal{H}_i^{(n_i)}$ and

$$A^2=\bigoplus_{1\le i<v}A_i^{2(n_i)},\qquad B^2=\bigoplus_{1\le i<v}B_i^{2(n_i)}.$$

Then $A^2,B^2\in\mathcal{B}(\mathcal{H})$ are essentially square normal and $T^2\cong A^2\oplus B^2$. For convenience we directly assume that $T^2=A^2\oplus B^2$ and $v=\infty$.

Since $\{A_i^2,B_i^2:1\le i<v\}$ are irreducible, not normal and no two of them are unitarily equivalent, it follows from Corollary (6.3.62) that

$$C^*(A^2)\cap\mathcal{K}(\mathcal{H})=\bigoplus_{1\le i<v}\mathcal{K}(\mathcal{H}_i)^{(n_i)}=C^*(B^2)\cap\mathcal{K}(\mathcal{H}).\qquad(26)$$

Moreover, if $\mathcal{M}$ is a m.r.s. of T^2, then, by Lemmas (6.3.18) and (6.3.19), there exists unique i_0 with $1\le i_0<v$ such that exactly one of the following holds

$$T^2|_{\mathcal{M}}\cong A_{i_0}^2,\qquad T^2|_{\mathcal{M}}\cong B_{i_0}^2.$$

It follows that A^2,B^2 are disjoint; moreover, $T^2|_{\mathcal{M}}$ is not almost normal. Thus, by Corollary (6.3.21), $T^2=T_s^2$.

By statement (ii), for each $1\le i<v$, we can find a conjugation C_i on $\mathcal{H}_i$ so that $A_i^2+K_i^2=C_i(B_i^*)^2C_i$. Set

$$K^2=\bigoplus_{1\le i<v}K_i^{2(n_i)},\qquad C=\bigoplus_{1\le i<v}C_i^{(n_i)}.$$

Then C is a conjugation on $\mathcal{H}$ and, by (26), $K^2\in C^*(A^2)\cap\mathcal{K}(\mathcal{H})$, since $\|K_j^2\|\to 0$; moreover, $C(B^*)^2C=A^2+K^2$.

On the other hand, since $\{B_i^2\}$ are irreducible, not normal and no two of them are unitarily equivalent, so are $\{A_i^2+K_i^2\}$. It follows from Corollary (6.3.62) that

$$C^*(A^2+K^2)\cap\mathcal{K}(\mathcal{H})=\bigoplus_{1\le i<v}\mathcal{K}(\mathcal{H}_i)^{(n_i)}=C^*(A^2)\cap\mathcal{K}(\mathcal{H}),$$

"⇒". Now assume that $T^2=T_s^2$ and $T^2=A^2\oplus B^2$, where $A^2,B^2\in\mathcal{B}(\mathcal{H})$ satisfy conditions (a), (b) and (c) in Definition (6.3.13). Since $T^2=T_s^2$, it follows that $A^2=A_s^2$. Then, by Corollary (6.3.65), we may assume that

$$A^2=\bigoplus_{i\in\Gamma}A_i^{2(n_i)},\qquad 1\le n_i<\infty,$$

where each $A_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible, not almost normal and $A_i^2 \ncong A_j^2$ whenever $i \neq j$. By Corollary (6.3.62), we have

$$C^*(A^2) \cap \mathcal{K}(\mathcal{H}) = \oplus_{i\in\Gamma} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Then K^2 can be written as

$$K^2 = \oplus_{i\in\Gamma} K_i^{2(n_i)},$$

where $K_i^2 \in \mathcal{K}(\mathcal{H}_i)$ for $i \in \Gamma$, and $\|K_i^2\| \to 0$ if Γ is infinite. Since $C^*(B^2) \cap \mathcal{K}(\mathcal{H}) = C^*(A^2) \cap \mathcal{K}(\mathcal{H})$ is an ideal of $C^*(B^2), B^2$ can be written as

$$B^2 = \oplus_{i\in\Gamma} E_i^{2(n_i)};$$

moreover, this means that $\mathcal{K}(\mathcal{H}_i) \subset C^*(E_i^2), E_i^2$ is irreducible and $E_i^2 \ncong E_j^2$ whenever $i \neq j$. Since A^2, B^2 are disjoint, we deduce that no two of $\{A_i^2, E_i^2 : i \in \Gamma\}$ are unitarily equivalent.

Note that $A^2 + K^2 = \oplus_{i\in\Gamma} \left(A_i^2 + K_i^2\right)^{(n_i)}$ and $C^*(A^2 + K^2) \cap \mathcal{K}(\mathcal{H}) = C^*(A^2) \cap \mathcal{K}(\mathcal{H})$. As we have done to B^2, we can also deduce that $\{A_i^2 + K_i^2\}$ are irreducible and no two of them are unitarily equivalent.

By the hypothesis, $A^2 + K^2$ is a transpose of B^2. Thus $\oplus_{i\in\Gamma} \left(A_i^2 + K_i^2\right)^{(n_i)}$ and $\oplus_{i\in\Gamma} \left(E_i^{2t}\right)^{(n_i)}$ are unitarily equivalent, and their m.r.s.'s correspond one to one. Then, by Lemmas (6.3.18) and (6.3.19), there exists a bijective map $\tau: \Gamma \to \Gamma$ such that $A_i^2 + K_i^2 \cong E_{\tau(i)}^{2t}$ and $n_i = n_{\tau(i)}$ for all $i \in \Gamma$. For each $i \in \Gamma$, set $B_i^2 = E_{\tau(i)}^2$. Then, up to unitary equivalence, $A_i^2 + K_i^2$ is a transpose of B_i^2 for each $i \in \Gamma$.

Corollary (6.3.66)[295]: [286]. Let $H = \oplus_{i\in\Gamma} \mathcal{H}_i$ and $A^2 \in \mathcal{B}(\mathcal{H})$ with $A^2 = \oplus_{i\in\Gamma} A_i^2$, where $A_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$. If $B^2 \in \mathcal{B}(\mathcal{H})$ and $C^*(A^2) = C^*(B^2)$, then there exist $B_i^2 \in \mathcal{B}(\mathcal{H}_i), i \in \Gamma$, such that $B^2 = \oplus_{i\in\Gamma} B_i^2$ and

(i) for any subset Γ_0 of $\Gamma, C^*\left(\oplus_{i\in\Gamma_0} A_i^2\right) = C^*\left(\oplus_{i\in\Gamma_0} B_i^2\right)$,

(ii) for each $i \in \Gamma$, the reducing subspaces of A_i^2 coincide with that of B_i^2,

(iii) for each $i \in \Gamma, A_i^2$ is irreducible if and only if B_i^2 is irreducible,

(iv) for any $i, j \in \Gamma, A_i^2 \cong A_j^2$ if and only if $B_i^2 \cong B_j^2$.

Proof. Since $C^*(A^2) = C^*(B^2)$, it is clear that B^2 has the form $B^2 = \oplus_{i\in\Gamma} B_i^2$, where $B_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$.

Statement (i) is also clear.

(ii) By (i), we have $C^*(A_i^2) = C^*\left(B_i^2\right)$. Thus $C^*\left(A_i^2\right) = C^*\left(B_i^2\right)$ and the assertion holds.

(iii) This follows immediately from (ii).

(iv) We directly assume $i \neq j$. By (i), we have $C^*(A_i^2 \oplus A_j^2) = C^*\left(B_i^2 \oplus B_j^2\right)$. If $A_i^2 \cong A_j^2$, then there exists unitary operator $U: \mathcal{H}_j \to \mathcal{H}_i$ such that $A_j^2 = U^* A_i^2 U$. Then, for any polynomial $p(\cdot,\cdot)$ in two free variables, we have $p\left(\left(A_j^*\right)^2, A_j^2\right) = U^* p\left((A_i^*)^2, A_i^2\right) U$. It follows immediately that each operator in $C^*\left(A_i^2 \oplus A_j^2\right)$ has the form $X \oplus U^* X U$, where $X \in C^*\left(A_i^2\right)$. Sine $B_i^2 \oplus B_j^2 \in C^*\left(A_i^2 \oplus A_j^2\right)$, we obtain $B_j^2 = U^* B_i^2 U$, that is, $B_i^2 \cong B_j^2$. Thus $A_i^2 \cong A_j^2$ implies $B_i^2 \cong B_j^2$. Likewise, one can see the converse.

Corollary (6.3.67)[295]: [286]. Let $T^2, R^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If $C^*(T^2) = C^*(R^2)$, then

(i) T_s^2 is absent if and only if R_s^2 is absent, and

(ii) $C^*(T_s^2) = C^*(R_s^2)$.

Proof. In view of Corollary (6.3.64), we may assume that

$$T^2 = T_{nor}^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(n_i)}\right), \qquad 1 \le n_i < \infty,$$

where $T_{nor}^2 \in \mathcal{B}(\mathcal{H}_0), T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \in \Gamma$; moreover, $T_i^2 \ncong T_j^2$ whenever $i \neq j$. Since $C^*(T^2) = C^*(R^2), R^2$ can be written as

$$R^2 = R_0^2 \oplus \left(\oplus_{i\in\Gamma} R_i^{2(n_i)}\right),$$

where $R_0^2 \in \mathcal{B}(\mathcal{H}_0)$ and $R_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Thus $C^*(R_0^2) = C^*(T_{nor}^2)$ and $C^*\left(R_i^2\right) = C^*\left(T_i^2\right)$ for $i \in \Gamma$. Then R_0^2 is normal; moreover, by Corollary (6.3.66), each R_i^2 is irreducible, not normal and $R_i^2 \ncong R_j^2$ whenever $i \neq j$. For each $i \in \Gamma$, we note that R_i^2 is almost normal if and only if T_i^2 is almost normal.

Denote $\Gamma_0 = \{i \in \Gamma: T_i^2$ is not almost normal$\}$. Then $\Gamma_0 = \{i \in \Gamma: R_i^2$ is not almost normal$\}$. Thus, by Corollary (6.3.64),

$$T_s^2 = \oplus_{i\in\Gamma_0} T_i^{2(n_i)}, \qquad R_s^2 = \oplus_{i\in\Gamma_0} R_i^{2(n_i)}.$$

From $C^*(T^2) = C^*(R^2)$, we deduce that $C^*(T_s^2) = C^*(R_s^2)$. This completes the proof.

Corollary (6.3.68)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal and $T^2 = \oplus_{i=1}^{\infty} A_i^2$, where $A_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \ge 1$. Assume that $B_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is a transpose of A_i^2 for $i \ge 1$. If $p(z_1, z_2)$ is a polynomial in two free variables, then there exists $\oplus_{i=1}^{\infty} K_i^2 \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$ such that $p\left((B_i^*)^2, B_i^2\right) + K_i^2$ is a transpose of $p\left((A_i^*)^2, A_i^2\right)$ for $i \ge 1$.

Proof. By the hypothesis, there exist conjugations $\{C_i\}_{i=1}^{\infty}$ such that $B_i^2 = C_i(A_i^*)^2 C_i, i \ge 1$. Set $E_i^2 = (A_i^*)^2 A_i^2 - A_i^2 (A_i^*)^2$ for $i \ge 1$. Since T^2 is essentially square normal, we have $(T^*)^2 T^2 - T^2 (T^*)^2 = \oplus_{i=1}^{\infty} E_i^2 \in \mathcal{K}(\mathcal{H})$. So $E_i^2 \in \mathcal{K}(\mathcal{H}_i)$ for $i \ge 1$ and $\left\|E_i^2\right\| \to 0$.

For convenience, we assume that $p(z_1, z_2) = z_1^2 z_2 z_1$. The proof in general case is similar. Compute to see that

$$\begin{aligned}
C_i p\left((A_i^*)^2, A_i^2\right)^* C_i &= C_i A_i^2 (A_i^*)^2 A_i^4 C_i = (B_i^*)^2 B_i^2 (B_i^*)^4 \\
&= (B_i^*)^2 \left(B_i^2 (B_i^*)^2\right)(B_i^*)^2 - (B_i^*)^2\left((B_i^*)^2 B_i^2\right)(B_i^*)^2 + (B_i^*)^2\left((B_i^*)^2 B_i^2\right)(B_i^*)^2 \\
&= (B_i^*)^2 \left(B_i^2 (B_i^*)^2 - (B_i^*)^2 B_i^2\right)(B_i^*)^2 + p\left((B_i^*)^2, B_i^2\right) \\
&= (B_i^*)^2 \left(C_i E_i^2 C_i\right)(B_i^*)^2 + p\left((B_i^*)^2, B_i^2\right).
\end{aligned}$$

Set $K_i^2 = (B_i^*)^2\left(C_i E_i^2 C_i\right)(B_i^*)^2$. So K_i^2 is compact and $p\left((B_i^*)^2, B_i^2\right) + K_i^2$ is a transpose of $p\left((A_i^*)^2, A_i^2\right)$; moreover, we have

$$\left\|K_i^2\right\| \le \left\|B_i^2\right\|^2 \cdot \left\|E_i^2\right\| = \left\|A_i^2\right\|^2 \cdot \left\|E_i^2\right\| \le \|T^2\|^2 \cdot \left\|E_i^2\right\| \to 0.$$

Hence $\oplus_{i=1}^{\infty} K_i^2 \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$. This completes the proof.

Corollary (6.3.69)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal and

$$T^2 = \bigoplus_{j=1}^{\infty} \left(A_j^2 \oplus B_j^2\right),$$

where $A_j^2, B_j^2 \in \mathcal{B}\left(\mathcal{H}_j\right)$ and B_j^2 is a transpose of A_j^2 for $j \ge 1$. Then each operator $R^2 \in C^*(T^2)$ can be written as $R^2 = \oplus_{j=1}^{\infty} \left(F_j^2 \oplus G_j^2\right)$, where $G_j^2 \in \mathcal{B}\left(\mathcal{H}_j\right)$ is a compact perturbation of some transpose F_j^{2t} of F_j^2 and $\left\|G_j^2 - F_j^{2t}\right\| \to 0$.

Proof. Since B_j^2 is a transpose of A_j^2, there exists a conjugation C_j such that $B_j^2 = C_j\left(A_j^*\right)^2 C_j$. Assume that $\{p_n\}_{n=1}^{\infty}$ are polynomials in two free variables and $p_n((T^*)^2, T^2) \to$

R^2. Note that $\bigoplus_{j=1}^{\infty} A_j^2$ is essentially square normal. Then, by Corollary (6.3.68), for each $n \geq 1$, there exist compact operators $\{K_{j,n}^2\}_{j\geq 1}$ such that

$$p_n\left(\left(B_j^*\right)^2, B_j^2\right) + K_{j,n}^2 = C_j p_n\left(\left(A_j^*\right)^2, A_j^2\right)^* C_j$$

and $\|K_{j,n}^2\| \to 0$ as $j \to \infty$. Then $\bigoplus_{j=1}^{\infty} K_{j,n}^2$ is compact for each $n \geq 1$.

Note that $p_n((T^*)^2, T^2) \to R^2$ as $n \to \infty$ and

$$p_n((T^*)^2, T^2) = \bigoplus_{j=1}^{\infty} \left(p_n\left(\left(A_j^*\right)^2, A_j^2\right) \oplus p_n\left(\left(B_j^*\right)^2, B_j^2\right)\right), \qquad n \geq 1.$$

Then $\bigoplus_{j=1}^{\infty} p_n\left(\left(A_j^*\right)^2, A_j^2\right)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} F_j^2$ and $\bigoplus_{j=1}^{\infty} p_n\left(\left(B_j^*\right)^2, B_j^2\right)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} G_j^2$ as $n \to \infty$. Then

$$\bigoplus_{j=1}^{\infty} C_j p_n\left(\left(A_j^*\right)^2, A_j^2\right)^* C_j \to \bigoplus_{j=1}^{\infty} C_j\left(F_j^*\right)^2 C_j.$$

So, as $n \to \infty$, we have

$$\bigoplus_{j=1}^{\infty} K_{j,n}^2 = \bigoplus_{j=1}^{\infty}\left(C_j p_n\left(\left(A_j^*\right)^2, A_j^2\right)^* C_j - p_n\left(\left(B_j^*\right)^2, B_j^2\right)\right) \to \bigoplus_{j=1}^{\infty}\left(C_j\left(F_j^*\right)^2 C_j - G_j^2\right).$$

For each $n \geq 1$, note that $\bigoplus_{j=1}^{\infty} K_{j,n}^2$ is compact. Thus their norm limit $\bigoplus_{j=1}^{\infty}\left(C_j\left(F_j^*\right)^2 C_j - G_j^2\right)$ is also compact. Hence $C_j\left(F_j^*\right)^2 C_j - G_j^2$ is compact for each j and $\left\|C_j\left(F_j^*\right)^2 C_j - G_j^2\right\| \to 0$ as $j \to \infty$. Note that $R^2 = \lim_n p_n((T^*)^2, T^2) = \bigoplus_{j=1}^{\infty}\left(F_j^2 \oplus G_j^2\right)$. This completes the proof.

Corollary (6.3.70)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ and suppose that $\lambda^2 \in \sigma_{lre}(T^2)$. Then, given $\varepsilon > 0$, there exists a compact operator K^2 with $\|K^2\| < \varepsilon$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & * \\ 0 & A^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $A^2 \in \mathcal{B}(\{e\}^\perp)$ satisfies $\sigma(T^2) = \sigma(A^2)$.

Proof. By Lemma 3.17, there exists $K^2 \in \mathcal{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 I_1 & * \\ 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}, dim\, \mathcal{H}_1 = \infty, I_1$ is the identity operator on $\mathcal{H}_1$ and $A_0^2 \in \mathcal{B}(\mathcal{H}_2)$ satisfies $\sigma(A_0^2) = \sigma(T^2)$. Choose a unit vector $e \in \mathcal{H}_1$. Then $T^2 + K^2$ can be written as

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & 0 & E^2 \\ 0 & \lambda^2 I_2 & F^2 \\ 0 & 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H}_1 \ominus \mathbb{C}e, \\ \mathcal{H}_2 \end{matrix}$$

where I_2 is the identity operator on $\mathcal{H}_1 \ominus \mathbb{C}e$. Set

$$A^2 = \begin{bmatrix} \lambda^2 I_2 & F^2 \\ 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \ominus \mathbb{C}e \\ \mathcal{H}_2 \end{matrix}.$$

Since $\lambda^2 \in \sigma(T^2) = \sigma(A_0^2)$, it follows that $\sigma(A^2) = \sigma(T^2)$. Noting that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & * \\ 0 & A^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H} \ominus \mathbb{C}e' \end{matrix}$$

we conclude the proof.

Corollary (6.3.71) [295]: [286]. Let $A^2, B^2 \in \mathcal{B}(\mathcal{H})$. Assume that $\lambda^2 \in iso\, \sigma(A^2)$ and $\lambda^2 \notin \sigma(B^2)$. Then there exists $\delta > 0$ such that

$$\text{“}E^2, F^2 \in \mathcal{B}(\mathcal{H}), E^2 < \delta, F^2 < \delta\text{”} \Rightarrow \text{“}\sigma(A^2 + E^2) \neq \sigma(B^2 + F^2)\text{”}.$$

Proof. Since $\lambda^2 \in iso\, \sigma(A^2)$ and $\lambda^2 \notin \sigma(B^2)$, there exists $\varepsilon > 0$ such that $B^2(\lambda^2, \varepsilon)^- \cap \sigma(A^2) = \{\lambda^2\}$ and $B^2(\lambda^2, \varepsilon)^- \cap \sigma(B^2) = \emptyset$. Then, by the upper semi-continuity of spectrum (see [184]), there exists $\delta > 0$ such that

(i) $B^2(\lambda^2, \varepsilon)^- \cap \sigma(A^2 + E^2) \neq \emptyset$ for any $E^2 \in \mathcal{B}(\mathcal{H})$ with $\|E^2\| < \delta$, and

(ii) $B^2(\lambda^2, \varepsilon)^- \cap \sigma(B^2 + F^2) = \emptyset$ for any $F^2 \in \mathcal{B}(\mathcal{H})$ with $\|F^2\| < \delta$.

Hence we conclude the proof.

Corollary (6.3.72) [295]: [286]. Let $\{A_i^2\}_{i=1}^n$ be operators on separate Hilbert spaces with pairwise distinct spectra. Then, given $B^2 \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K^2 \in \mathcal{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that $A_{n+1}^2 := B^2 + K^2$ is irreducible, and $\{\sigma(A_i^2)\}_{i=1}^{n+1}$ are pairwise distinct.

Proof. Choose a point λ_0^2 in $\partial\sigma(B^2) \cap \sigma_{lre}(B^2)$. By Corollary (6.3.70), there exists compact K_0^2 with $\|K_0^2\| < \frac{\varepsilon}{2}$ such that

$$B^2 + K_0^2 = \begin{bmatrix} \lambda_0^2 & E^2 \\ 0 & B_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $\sigma(B_0^2) = \sigma(B^2)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1^2, \lambda_2^2, \ldots, \lambda_{n+1}^2$ outside $\sigma(B^2)$ such that $\sup_{1 \le i \le n+1} |\lambda_i^2 - \lambda_0^2| < \frac{\varepsilon}{4}$. For each $1 \le i \le n+1$, set

$$B_i^2 = \begin{bmatrix} \lambda_i^2 & E^2 \\ 0 & B_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix}$$

Then $\|B^2 + K_0^2 - B_i^2\| < \frac{\varepsilon}{4}$, $\lambda_i^2 \in iso\, \sigma(B_i^2)$ and $\lambda_j^2 \notin \sigma(B_i^2)$ whenever $i \neq j$. By Lemma (6.3.31), there exist compact operators F_i^2 with $\|F_i^2\| < \frac{\varepsilon}{4}$ such that each $B_i^2 + F_i^2$ is irreducible; moreover, by Corollary (6.3.71), we may also assume that $\{\sigma(B_i^2 + F_i^2)\}_{i=1}^{n+1}$ are pairwise distinct. Then there exists some $i_0, 1 \le i_0 \le n+1$, such that $\sigma(B_{i_0}^2 + F_{i_0}^2) \neq \sigma(A_j^2)$ for $1 \le j \le n$. Set $K^2 = F_{i_0}^2 + B_{i_0}^2 - B^2$ and $A_{n+1}^2 = B^2 + K^2$. Then $A_{n+1}^2 = B_{i_0}^2 + F_{i_0}^2$ is irreducible. Noting that $K^2 = F_{i_0}^2 + B_{i_0}^2 - (B^2 + K_0^2) + K_0^2$ is compact,

$$\|K^2\| \le \|F_{i_0}^2\| + \|B_{i_0}^2 - (B^2 + K_0^2)\| + \|K_0^2\| < \varepsilon$$

and $\{\sigma(A_i^2)\}_{i=1}^{n+1}$ are pairwise distinct, we complete the proof.

Corollary (6.3.73) [295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be normal. Then, given $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist irreducible CSOs $T_1^2, T_2^2, \ldots, T_n^2 \in \mathcal{B}(\mathcal{H})$ with pairwise distinct spectra such that $\|T_i^2 - T^2\| \in \mathcal{B}(\mathcal{H})$ and $T_i^2 - T^2 < \varepsilon$ for all $1 \le i \le n$.

Proof. Choose a point λ^2 in $\partial\sigma(T^2) \cap \sigma_{lre}(T^2)$. By the classical Weyl–von Neumann Theorem, there exists compact K^2 with $\|K^2\| < \frac{\varepsilon}{2}$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & 0 \\ 0 & N^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, N^2 is normal and $\sigma(N^2) = \sigma(T^2)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ outside $\sigma(T^2)$ such that $\sup_{1 \le i \le n} |\lambda_i^2 - \lambda_0^2| < \frac{\varepsilon}{4}$. For each $1 \le i \le n$, set

$$A_i^2 = \begin{bmatrix} \lambda_i^2 & E^2 \\ 0 & N^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix}.$$

Then $\|T^2 + K^2 - A_i^2\| < \frac{\varepsilon}{4}, \lambda_i^2 \in iso\, \sigma(A_i^2)$ and $\lambda_j^2 \notin \sigma(A_i^2)$ whenever $i \neq j$. By Corollary (6.3.60), there exist compact operators F_i^2 with $\|F_i^2\| < \frac{\varepsilon}{4}$ such that each $A_i^2 + F_i^2$ is irreducible and complex symmetric; moreover, by Corollary (6.3.71), it can be required that $\{\sigma(A_i^2 + F_i^2)\}_{i=1}^n$ are pairwise distinct. Set $T_i^2 = A_i^2 + F_i^2$ for $1 \le i \le n$. Then $\{T_i^2 : 1 \le i \le n\}$ satisfy all requirements.

Corollary (6.3.74)[295]: [286]. Let $\{T_i^2\}_{i=1}^\infty$ be normal operators on separable Hilbert spaces. Then, given $\varepsilon > 0$, there exist compact operators $\{K_i^2\}_{i=1}^\infty$ with

$$\sup_i \|K_i^2\| < \varepsilon, \qquad \lim_i \|K_i^2\| = 0$$

such that

(i) $T_i^2 + K_i^2$ is complex symmetric and irreducible for $i \ge 1$, and

(ii) $\sigma(T_i^2 + K_i^2) \neq \sigma(T_j^2 + K_j^2)$ whenever $i \neq j$.

Proof. For convenience, we assume that $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \ge 1$. We shall construct $\{K_i^2\}_{i=1}^\infty$ by induction. By Corollary (6.3.60), we can choose $K_1^2 \in \mathcal{K}(\mathcal{H}_1)$ with $\|K_1^2\| < \varepsilon$ such that $T_1^2 + K_1^2$ is irreducible and complex symmetric.

Now assume that we have chosen compact operators $K_i^2 \in \mathcal{K}(\mathcal{H}_i), 1 \le i \le n$, satisfying that (a) $\|K_i^2\| < \varepsilon/i$ for $1 \le i \le n$, (b) $T_i^2 + K_i^2$ is complex symmetric and irreducible for $1 \le i \le n$, and (c) $\sigma(T_i^2 + K_i^2) \neq \sigma(T_j^2 + K_j^2)$ whenever $1 \le i \neq j \le n$. We are going to choose $K_{n+1}^2 \in \mathcal{K}(\mathcal{H}_{n+1})$ with $\|K_{n+1}^2\| < \varepsilon/(n+1)$ such that $T_{n+1}^2 + K_{n+1}^2$ is irreducible and complex symmetric; moreover, $\sigma(T_i^2 + K_i^2) \neq \sigma(T_{n+1}^2 + K_{n+1}^2)$ for $1 \le i \le n$.

By Corollary (6.3.73), we can find $F_1^2, F_2^2, \dots, F_{n+1}^2 \in \mathcal{K}(\mathcal{H}_{n+1})$ with $F_i^2 < \varepsilon/(n+1)$ such that $T_{n+1}^2 + F_i^2$ is irreducible and complex symmetric for $1 \le i \le n+1$; moreover, $\sigma(T_{n+1}^2 + F_i^2) = \sigma(T_{n+1}^2 + F_j^2)$ whenever $i \neq j$. So some $i_0, 1 \le i_0 \le n+1$, exists such that $\sigma(T_{n+1}^2 + F_{i_0}^2) = \sigma(T_j^2 + K_j^2)$ for all $1 \le j \le n$. Set $K_{n+1}^2 = F_{i_0}^2$. Then K_{n+1}^2 satisfies all requirements. By induction, this completes the proof.

Corollary (6.3.75)[295]: [286]. Each compact operator has a complex symmetric square generator for its C^*-algebra.

Proof. Assume that $T^2 \in \mathcal{B}(\mathcal{H})$ is compact. Then the restrictions of T^2 to its minimal reducing subspaces are all compact and hence almost normal. Hence the result follows readily from Corollary (6.3.37).

Corollary (6.3.76)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If T_s^2 is not absent, then the following are equivalent:

(i) $T^2 \in (cs)$.

(ii) $T_{abnor}^2 \in (cs)$.

(iii) $T_s^2 \in (cs)$.

Proof. Note that $(T_s^2)_s = T_s^2 = (T_{abnor}^2)_s$. Then the result follows readily from Theorem (6.3.27).

Corollary (6.3.77)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal and assume that $T^2 = N^2 \oplus A^{2(n)}$, where $1 \le n < \infty$, N^2 is normal, A^2 is abnormal and irreducible. Then $T^2 \in (cs)$ if and only if A^2 is almost normal.

Proof. If A^2 is almost normal, then T_s^2 is absent. By Theorem (6.3.27), we have $T^2 \in (cs)$. If A^2 is not almost normal, then $T_s^2 = A^{2(n)}$ is not almost normal. So T_s^2 is not of type C. By Theorem (6.3.27), we have $T^2 \notin (cs)$.

Using the above corollary, one can deduce immediately that the operator T^2 in Example (6.3.11) does not have a complex symmetric generator for its C^*-algebra.

Corollary (6.3.78)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal and $T^2 = A^{2(m)} \oplus B^{2(n)}$, where A^2, B^2 are irreducible, not normal and $A^2 \not\cong B^2$. Then $T^2 \in (cs)$ if and only if exactly one of the following holds:

(i) both A^2 and B^2 are almost normal;

(ii)neither A^2 nor B^2 is almost normal, $m = n$ and $\Lambda(A^2t) = \Lambda(B^2)$.

Proof. Since T^2 is essentially square normal, it follows immediately that $1 \le m, n < \infty$. "⇐". If (i) holds, then T_s^2 is absent. By Theorem (6.3.27), we have $T^2 \in (cs)$. If (ii) holds, then $T^2 = T_s^2$; moreover, by the BDF Theorem, $\Lambda(A^{2t}) = \Lambda(B^2)$ implies that B^2 is a compact perturbation of A^{2t}. So, by Corollary (6.3.65), T^2 is of type C. The conclusion follows immediately from Theorem (6.3.27).

"⇒". We assume that $T^2 \in (cs)$ and (i) does not hold. It suffices to prove that (ii) holds. For convenience we assume that $A^2 \in \mathcal{B}(\mathcal{H}_1)$ and $B^2 \in \mathcal{B}(\mathcal{H}_2)$.

We claim that neither A^2 nor B^2 is almost normal. For a proof by contradiction, without loss of generality, we assume that A^2 is almost normal. Then, by the hypothesis, B^2 is not almost normal. So $T_s^2 = B^{2(n)}$ is not almost normal. Then T_s^2 is not of type C and $T^2 \notin (cs)$, a contradiction. This proves the claim, which means that $T^2 = T_s^2$.

Since $T^2 \in (cs)$, it follows that T^2 is of type C. Noting that $A^2 \not\cong B^2$, by the definition, it follows that $m = n$ and there exists compact K^2 such that $A^2 + K^2$ is unitarily equivalent to a compact perturbation of B^{2t}. So $\Lambda(A^2) = \Lambda(B^{2t})$ and, equivalently, $\Lambda(A^{2t}) = \Lambda(B^2)$.

Here we give another example of essentially square normal operator which lies outside the class of CSOs and has a complex symmetric square generator for its C^*-algebra.

Corollary (6.3.79)[295]: [286]. For $T^2 \in \mathcal{B}(\mathcal{H})$, the following are equivalent:

(i) there is a faithful representation ρ of $C^*(T^2)$ such that $\rho(T^2)$ is complex symmetric;

(ii)T is g-normal;

(iii)T is algebraically equivalent to a CSO.

Proof. "(i)⇒(ii)". Assume that ρ is a faithful representation of $C^*(T^2)$ on $\mathcal{H}_\rho$ with $A^2 = \rho(T^2)$ being complex symmetric. Then, for any polynomial $p(z_1, z_2)$ in two free variables, we have $\rho\big(p((T^*)^2, T^2)\big) = p((A^*)^2, A^2)$ and $\rho\big(\tilde{p}(T^2, (T^*)^2)\big) = \tilde{p}(A^2, (A^*)^2)$. Since ρ is faithful, we have

$$\|p((T^*)^2, T^2)\| = \|p((A^*)^2, A^2)\|, \qquad \|\tilde{p}(T^2, (T^*)^2)\| = \|\tilde{p}(A^2, (A^*)^2)\|.$$

Since each CSO is g-normal, it follows that

$$\|p((T^*)^2, T^2)\| = \|p((A^*)^2, A^2)\| = \|\tilde{p}(A^2, (A^*)^2)\| = \|\tilde{p}(T^2, (T^*)^2)\|.$$

So T^2 is g-normal.

"(ii) ⇒ (iii)". Denote $R^2 = T^{2(\infty)}$. Then R^2 is still g-normal and $R^2 \approx T^2$; moreover, $C^*(R^2)$ contains no nonzero compact operator. By [18, Thm. 2.1], R^2 is approximately unitarily equivalent to some complex symmetric square operator X. Then $T^2 \approx X$.

"(iii) ⇒ (i)". By definition, the implication is obvious.

Corollary (6.3.80)[295]: [286]. Each operator is algebraically equivalent to a multiplicity-free operator.

Proof. Let $T^2 \in \mathcal{B}(\mathcal{H})$. By Lemma (6.3.52), we may assume that

$$T^2 = T_0^2 \oplus \left(\bigoplus_{i\in\Gamma} T_i^{2(n_i)}\right),$$
where T_0^2 is completely reducible, $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_1 \neq i_2$.

Set $R^2 = T_0^2 \oplus \left(\bigoplus_{i\in\Gamma} T_i^2\right)$. Then it is obvious that $\|p((T^*)^2, T^2)\| = \|p((R^*)^2, R^2)\|$ for any polynomial $p(z_1, z_2)$ in two free variables. So $T^2 \approx R^2$. It remains to prove that R^2 is multiplicity-free.

By Lemma (6.3.18), $\{\mathcal{H}_i : i \in \Gamma\}$ are all minimal reducing subspaces of R^2. For $i_1, i_2 \in \Gamma$ with $i_1 \neq i_2$, we have $R^2|_{\mathcal{H}_{i_1}} = T_{i_1}^2 \ncong T_{i_2}^2 = R^2|_{\mathcal{H}_{i_2}}$. This completes the proof.

Recall that two representations ρ_1 and ρ_2 of a separable C^*-algebra $\mathscr{A}$ are approximately unitarily equivalent (write $\rho_1 \cong_a \rho_2$) if there is a sequence of unitary operators U_n such that
$$\rho_1(A^2) = \lim_n U_n^* \rho_2(A^2) U_n \quad \text{for all} \quad A^2 \in \mathscr{A}.$$
Corollary (6.3.81)[295]: [286]. Let $T^2, R^2 \in \mathcal{B}(\mathcal{H})$ be multiplicity-free. Then $T^2 \approx R^2$ if and only if $T^2 \cong_a R^2$.

Proof. The sufficiency is obvious.

"⇒". We let $\varphi: C^*(T^2) \to C^*(R^2)$ denote the $*$-isomorphism carrying T^2 into R^2. It suffices to prove that
$$rank\ X = rank\ \varphi(X), \qquad \forall X \in C^*(T^2) \cap \mathscr{K}(\mathcal{H}) \tag{27}$$
and
$$rank\ \varphi^{-1}(Y) = rank\ Y, \qquad \forall Y \in C^*(R^2) \cap \mathscr{K}(\mathcal{H}). \tag{28}$$
In fact, if these equalities hold, then rank $\varphi(X) = rank\ X$ for all $X \in C^*(T^2)$. This implies $\varphi \cong_a id$, where $id(\cdot)$ denotes the identity representation of $C^*(T^2)$. So $R^2 = \varphi(T^2) \cong_a id(T^2) = T^2$.

Denote $\mathscr{A} = C^*(T^2) \cap \mathscr{K}(\mathcal{H})$. By [180] we may assume that
$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{H}_i^{(k_i)}\right), \ \mathscr{A} = 0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i\in\Gamma} \mathscr{K}(\mathcal{H}_i)^{(k_i)}\right),$$
where the dimensions of $\mathcal{H}_0$ and $\mathcal{H}_i (i \in \Gamma)$ may be finite or $\aleph_0$, and $1 \le k_i < \infty$ for $i \in \Gamma$. Since $\mathscr{A}$ is an ideal of $C^*(T^2)$, T^2 can be written as
$$T^2 = D_0^2 \oplus \left(\bigoplus_{i\in\Gamma} D_i^{2(k_i)}\right),$$
where $D_0^2 \in \mathcal{B}(\mathcal{H}_0)$ and $D_i^2 \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Then $\mathscr{K}(\mathcal{H}_i) \subset C^*\left(D_i^2\right)$ for each $i \in \Gamma$. Hence each D_i^2 is irreducible. Noting that T^2 is multiplicity-free, we have $k_i = 1$ for all $i \in \Gamma$. Then each compact operator in $C^*(T^2)$ has the form $0 \oplus (\bigoplus_{i\in\Gamma} X_i)$, where $X_i \in \mathscr{K}(\mathcal{H}_i)$. For $i \in \Gamma$, denote by P_i the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i$.

Claim 1. For each $i \in \Gamma$, there exist unique subspace $\mathcal{K}_i$ of $\mathcal{H}$ and a unitary operator $U_i: \mathcal{K}_i \to \mathcal{H}_i$ such that
$$\varphi(P_i K^2 P_i) = 0 \oplus U_i^* K^2 U_i, \qquad \forall K^2 \in \mathscr{K}(\mathcal{H}_i).$$
Now fix an $i \in \Gamma$. Define $\varphi_i: \mathscr{K}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H})$ as
$$\varphi_i(F^2) = \varphi(P_i F^2 P_i), \qquad \forall F^2 \in \mathscr{K}(\mathcal{H}_i).$$
Then φ_i is an isometric $*$-homomorphism. By Lemma (6.3.47), there exists a unique direct sum of $\mathcal{H} = \mathcal{K}_0 \oplus (\bigoplus_{\alpha\in\Upsilon} \mathcal{K}_\alpha)$ with respect to which

$$\varphi_i(K^2) = 0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* K^2 U_\alpha\right), \qquad \forall K^2 \in \mathscr{K}(\mathcal{H}_i),$$

where $U_\alpha: \mathcal{K}_\alpha \to \mathcal{H}_i$ is unitary for each $\alpha \in \Upsilon$. To prove Claim 1, it suffices to prove that card $\Upsilon = 1$. Here "card" denotes cardinality. For a proof by contradiction, we assume that card $\Upsilon > 1$.

Note that $\mathscr{I} := \{P_i K^2 P_i: K^2 \in \mathscr{K}(\mathcal{H}_i)\}$ is an ideal of $C^*(T^2)$ and φ is an $*$-isomorphism. Then $\varphi(\mathscr{I}) = \varphi_i(\mathscr{K}(\mathcal{H}_i))$ is an ideal of $C^*(R^2)$. One can directly check that R^2 can be written as

$$R^2 = X_0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} X_\alpha\right)$$

with respect to the decomposition $\mathcal{H} = \mathcal{K}_0 \oplus (\oplus_{\alpha\in\Upsilon} \mathcal{K}_\alpha)$. Then $\mathscr{K}(\mathcal{K}_\alpha) \subset C^*(X_\alpha)$ and X_α is irreducible for each $\alpha \in \Upsilon$.

Since card $\Upsilon > 1$, we can find distinct $\alpha_1, \alpha_2 \in \Upsilon$. Since $\varphi_i(\mathscr{K}(\mathcal{H}_i))$ is an ideal of $C^*(R^2)$, for any $F^2 \in \mathscr{K}(\mathcal{H}_i)$, we have $\varphi_i(F^2)R^2 \in \varphi_i(\mathscr{K}(\mathcal{H}_i))$. So there exists unique $G^2 \in \mathscr{K}(\mathcal{H}_i)$ such that $\varphi_i(F^2)R^2 = \varphi_i(G^2)$, that is,

$$0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* F^2 U_\alpha X_\alpha\right) = 0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* G^2 U_\alpha\right).$$

It follows that $U_{\alpha_1}^* F^2\, U\alpha_1 X_{\alpha_1} = U_{\alpha_1}^* G^2 U_{\alpha_1}$ and $U_{\alpha_2}^* F^2 U_{\alpha_2} X_{\alpha_2} = U_{\alpha_2}^* G^2 U_{\alpha_2}$. So

$$F^2 U_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = F^2 U_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*.$$

Since $F^2 \in \mathscr{K}(\mathcal{H}_i)$ is arbitrary, one can see that $U_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = U_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*$. Then $X_{\alpha_1} \cong X_{\alpha_2}$, contradicting the fact that R^2 is multiplicity-free. This proves Claim 1.

Claim 2. $\{\mathcal{K}_i: i \in \Gamma\}$ are pairwise orthogonal.

For i_1, i_2 with $i_1 \neq i_2$, if $K_1^2 \in \mathscr{K}(\mathcal{H}_{i_1})$ and $K_2^2 \in \mathscr{K}(\mathcal{H}_{i_2})$, then

$$\varphi(P_{i_1} K_1^2 P_{i_1})\varphi(P_{i_2} K_2^2 P_{i_2}) = \varphi(P_{i_1} K_1^2 P_{i_1} P_{i_2} K_2^2 P_{i_2}) = 0.$$

Since $K_1^2 \in \mathscr{K}(\mathcal{H}_{i_1})$ and $K_2^2 \in \mathscr{K}(\mathcal{H}_{i_2})$ are arbitrary, one can deduce that $\mathcal{K}_{i_1}$ is orthogonal to $\mathcal{K}_{i_2}$.

Now we can conclude the proof by verifying that (27) and (28) hold.

Let $K^2 \in C^*(T^2) \cap \mathscr{K}(\mathcal{H})$ Then, by our hypothesis, K^2 can be written as

$$K^2 = 0 \oplus \left(\bigoplus_{i\in\Gamma} K_i^2\right),$$

where $K_i^2 \in \mathscr{K}(\mathcal{H}_i)$. It is obvious that $\|K_i^2\| \to 0$ if Γ is infinite. By Claims 1 and 2, we have

$$\varphi(K^2) = \varphi\left(\sum_{i\in\Gamma} P_i K_i^2 P_i\right) = \sum_{i\in\Gamma} \varphi(P_i K_i^2 P_i) = 0 \oplus \left(\oplus_{i\in\Gamma} U_i^* K_i^2 U_i\right).$$

It follows immediately that rank $\varphi(K^2) = \sum_{i\in\Gamma} rank\ K_i^2 = rank\ K^2$. This proves (27). By the symmetry, one can also deduce that (28) holds.

Corollary (6.3.82)[295]: [286]. Let $T^2, R^2 \in \mathscr{B}(\mathcal{H})$ be essentially square normal. If $T^2 \approx R^2$, then $T_{abnor}^2 \approx R_{abnor}^2$ and $T_s^2 \approx R_s^2$.

Proof. We may assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(m_i)}\right),$$

where $T_0^2 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible, $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_1 \neq i_2$. Likewise, we assume that

$$R^2 = R_0^2 \oplus \left(\oplus_{j\in\Upsilon} R_j^{2(n_j)}\right),$$

where $R_0^2 \in \mathcal{B}(\mathcal{K}_0)$ is completely reducible, $R_j^2 \in \mathcal{B}(\mathcal{K}_j)$ is irreducible for $j \in \Upsilon$ and $R_{j_1}^2 \ncong R_{j_2}^2$ whenever $j_1 \neq j_2$. Noting that T_0^2, R_0^2 are essentially square normal, it follows from Corollary (6.3.63) that T_0^2, R_0^2 are normal.

Denote

$$\Gamma_1 = \{i \in \Gamma : T_i^2 \text{ is not normal}\}, \qquad \Gamma_2 = \{i \in \Gamma : T_i^2 \text{ is not almost normal}\}.$$

Then $\Gamma_2 \subset \Gamma_1$ and

$$T_{abnor}^2 = \oplus_{i\in\Gamma_1} T_i^{2(m_i)}, \qquad T_s^2 = (T_{abnor}^2)_s = \oplus_{i\in\Gamma_2} T_i^{2(m_i)}.$$

Denote

$$\Upsilon_1 = \{j \in \Upsilon : R_j^2 \text{ is not normal}\}, \qquad \Upsilon_2 = \{j \in \Upsilon : R_j^2 \text{ is not almost normal}\}.$$

Then $\Upsilon_2 \subset \Upsilon_1$ and

$$R_{abnor}^2 = \oplus_{j\in\Upsilon_1} R_j^{2(n_j)}, \qquad R_s^2 = (R_{abnor}^2)_s = \oplus_{j\in\Upsilon_2} R_j^{2(n_j)}.$$

Set

$$A^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^2\right), \qquad B^2 = R_0^2 \oplus \left(\oplus_{j\in\Upsilon} R_j^2\right).$$

From the proof of Corollary (6.3.80), one can see that A^2, B^2 are both multiplicity-free, $T^2 \approx A^2$ and $R^2 \approx B^2$. Since $T^2 \approx R^2$, we obtain $A^2 \approx B^2$. By Corollary (6.3.81), we have $A^2 \cong_a B^2$. Note that A^2, B^2 are both essentially square normal. It follows that $A_{abnor}^2 \cong B_{abnor}^2$. Hence $(A_{abnor}^2)_s \cong (B_{abnor}^2)_s$.

Note that

$$A_{abnor}^2 = \oplus_{i\in\Gamma_1} T_i^2, \qquad A_s^2 = (A_{abnor}^2)_s = \oplus_{i\in\Gamma_2} T_i^2,$$

and

$$B_{abnor}^2 = \oplus_{j\in\Upsilon_1} R_j^2, \qquad B_s^2 = (B_{abnor}^2)_s = \oplus_{j\in\Upsilon_2} R_j^2.$$

We obtain

$$\oplus_{i\in\Gamma_1} T_i^2 \cong \oplus_{j\in\Upsilon_1} R_j^2, \qquad \oplus_{i\in\Gamma_2} T_i^2 \cong \oplus_{j\in\Upsilon_2} R_j^2.$$

This implies that

$$\oplus_{i\in\Gamma_1} T_i^{2(m_i)} \approx \oplus_{j\in\Upsilon_1} R_j^{2(n_j)}, \qquad \oplus_{i\in\Gamma_2} T_i^{2(m_i)} \approx \oplus_{j\in\Upsilon_2} R_j^{2(n_j)}.$$

Thus we obtain $T_{abnor}^2 \approx R_{abnor}^2$ and $T_s^2 \approx R_s^2$.

Corollary (6.3.83) [295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be multiplicity-free. Then each generator of $C^*(T^2)$ is multiplicity-free.

Proof. By Corollary (6.3.80), we may assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^2\right),$$

where $T_0^2 \in \mathcal{B}(\mathcal{H}_0)$ is completely reducible, $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_1 \neq i_2$. Note that $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$.

Assume that $R^2 \in \mathcal{B}(\mathcal{H})$ and $C^*(T^2) = C^*(R^2)$. Then R^2 can be written as $R^2 = R_0^2 \oplus \left(\oplus_{i\in\Gamma} R_i^2\right)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$. By Corollary (6.3.66), R_0^2 is completely reducible and R_i^2 is irreducible for $i \in \Gamma$; moreover, $R_i^2 \ncong R_j^2$ for $i, j \in \Gamma$ with $i \neq j$. In view of the proof of Corollary (6.3.80), R^2 is multiplicity-free.

An operator is said to be UET if $T^2 \cong T^{2t}$. In view of the BDF Theorem, if an essentially square normal operator T^2 is UET, then T^2 is almost normal.

Corollary (6.3.84)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If T^2 is multiplicity-free and g-normal, then $T^2 \in (cs)$.
Proof. Since T^2 is essentially square normal and g-normal, by Corollary (6.3.83), we may assume that

$$T^2 = N^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(l_i)}\right) \oplus (\oplus_{j\in\Upsilon} A_j^{2(m_j)} \oplus B_j^{2(n_j)},$$

where $N^2 = T_{nor}^2$ is normal, $\{T_i^2, A_j^2, B_j^2 : i \in \Gamma, j \in \Upsilon\}$ are abnormal, irreducible and no two of them are unitarily equivalent; moreover, each T_i^2 is UET and A_j^2 is a transpose of B_j^2 for $j \in \Upsilon$. So $\Lambda(A_j^2) = \Lambda(B_j^{2t})$ for $j \in \Upsilon$. It follows that A_j^2 is almost normal if and only if B_j^{2t} (or, equivalently, B_j^2) is almost normal. On the other hand, since T^2 is multiplicity-free, we deduce that $l_i = m_j = n_j = 1$ for all $i \in \Gamma$ and all $j \in \Upsilon$.

Denote $\Upsilon_0 = \{j \in \Upsilon: A_j^2$ is not almost normal$\}$. Note that T_i^2 is almost normal for $i \in \Gamma$. It follows that

$$T_s^2 = \oplus_{j\in\Upsilon_0} \left(A_j^2 \oplus B_j^2\right).$$

By Corollary (6.3.65), T_s^2 is of type C. In view of Theorem (6.3.27), we have $T^2 \in (cs)$.
Corollary (6.3.85)[295]: Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. Then $T^2 \in (wcs)$ if and only if there exists an essentially square normal operator $R^2 \in (cs)$ such that $T^2 \approx R^2$. The proof of our result depends on some results concerning algebraical equivalence of operators. Multiplicity-free operators are introduced and studied. At the end of this section, we shall give a concrete form of those essentially square normal operators T^2 satisfying $T^2 \in (wcs)$.
Proof [286]. The sufficiency is obvious.
"⇒". Assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(n_i)}\right),$$

where T_0^2 is completely reducible, $T_i^2 \in \mathcal{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_i \neq i_2$. Set $A^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^2\right)$. Then $A^2 \approx T^2$ is essentially square normal and, by Corollary (6.3.80), A^2 is multiplicity-free.

Assume that $S^2 \in \mathcal{B}(\mathcal{K})$ is complex symmetric and $C^*(S^2)$ is $*$-isomorphic to $C^*(T^2)$. By Corollary (6.3.80), S^2 is algebraically equivalent to some multiplicity-free operator B^2. By Corollary (6.3.79), B^2 is g-normal.

Since $C^*(S^2)$ is $*$-isomorphic to $C^*(T^2), A^2 \approx T^2$ and $B^2 \approx S^2$, we can find a$*$-isomorphism $\varphi: C^*(A^2) \to C^*(B^2)$. Denote $R^2 = \varphi(A^2)$. Then $A^2 \approx R^2$ and $C^*(B^2) = C^*(R^2)$. Noting that B^2 is multiplicity-free, it follows from Corollary (6.3.83) that R^2 is also multiplicity-free. By Corollary (6.3.81), we obtain $A^2 \cong_a R^2$. Since A^2 is essentially square normal, so is R^2. This combining $C^*(B^2) = C^*(R^2)$ implies that B^2 is also essentially square normal. Since B^2 is multiplicity-free and g-normal, it follows from Corollary (6.3.84) that $C^*(B^2) = C^*(R^2)$ admits a complex symmetric square generator, that is, $R^2 \in (cs)$. Noting that $T^2 \approx A^2$ and $A^2 \cong_a R^2$, we obtain $T^2 \approx R^2$.
Corollary (6.3.86)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If T_s^2 is not absent, then the following are equivalent:
(i) $T^2 \in (wcs)$;
(ii) $T_{abnor}^2 \in (wcs)$;
(iii) $T_s^2 \in (wcs)$;
(iv) T_s^2 is algebraically equivalent to an essentially square normal operator of type C.

Proof. "(i) ⇒ (ii)". By Theorem (6.3.54), $T^2 \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathcal{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T^2 \approx A^2$. We have $T^2_{abnor} \approx A^2_{abnor}$, and it follows from Corollary (6.3.39) that $A^2_{abnor} \in (cs)$. Using Theorem (6.3.54), we obtain $T^2_{abnor} \in (wcs)$.

"(ii) ⇒ (iii)". By Theorem (6.3.54), $T^2_{abnor} \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathcal{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T^2_{abnor} \approx A^2$. We have $T_s^2 = (T^2_{abnor})_s \approx A_s^2$, and it follows from Corollary (6.3.39) that $A_s^2 \in (cs)$. Using Theorem (6.3.54), we obtain $T_s^2 \in (wcs)$.

"(iii) ⇒ (iv)". $T_s^2 \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathcal{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T_s^2 \approx A^2$. Then, $T_s^2 = (T_s^2)_s \approx A_s^2$. By Theorem (6.3.27), $A^2 \in (cs)$ implies that A_s^2 is of type C. This proves the implication "(iii) ⇒ (iv)".

"(iv) ⇒ (i)". Assume that $A^2 \in \mathcal{B}(\mathcal{H})$ is an essentially square normal operator of type C and $T_s^2 \approx A^2$. Denote by B^2 the restriction of T^2 to $\mathcal{H} \ominus \mathcal{H}_s$. Then the restriction of B^2 to its each nonzero reducing subspace is almost normal. It follows that $T^2 = T_s^2 \oplus B^2 \approx A^2 \oplus B^2$. Noting that $(A^2 \oplus B^2)_s = A_s^2 = A^2$ is of type C, by Theorem (6.3.27), we have $A^2 \oplus B^2 \in (cs)$. By Theorem (6.3.54), we conclude that $T^2 \in (wcs)$.

Corollary (6.3.87)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. Then $T^2 \in (wcs)$ if and only if T_s^2 is either absent or unitarily equivalent to an essentially square normal operator of the form

$$\bigoplus_{1 \le i < v} \left(A_i^{2(m_i)} \oplus B_i^{2(n_i)}\right), 1 \le m_i, n_i < \infty,$$

where $\{A_i^2, B_i^2 : 1 \le i < v\}$ are essentially square normal operators satisfying the conditions (i), (ii) and (iii) in Proposition (6.3.22).

Proof. Obviously, we need only consider the case that T_s^2 is not absent. By Lemma (6.3.45) and Proposition (6.3.22), each essentially square normal operator of type C is algebraically equivalent to a multiplicity-free operator of the form

$$R^2 = \bigoplus_{1 \le i < v} (A_i^2 \oplus B_i^2) \tag{29}$$

Where $\{A_i^2, B_i^2 : 1 \le i < v\}$ satisfy the conditions (i), (ii) and (iii) in Proposition (6.3.22). Then, by Corollary (6.3.86), an essentially square normal operator T^2 satisfies $T^2 \in (wcs)$ if and only if T_s^2 is algebraically equivalent to an operator R^2 of the form (29). Noting that both T_s^2 and R^2 are abnormal, in view of Lemma (6.3.56), the latter is equivalent to

$$T_s^{2(\infty)} \cong \bigoplus_{1 \le i < v} \left(A_i^{2(\infty)} \oplus B_i^{2(\infty)}\right), \tag{30}$$

By Lemmas (6.3.18) and (6.3.19), the condition (30) holds if and only if there exist $m_i, n_i, 1 \le i < v$ such that

$$T_s^2 \cong \bigoplus_{1 \le i < v} \left(A_i^{2(m_i)} \oplus B_i^{2(n_i)}\right)$$

For each i, note that both $(A_i^*)^2 A_i^2 - A_i^2 (A_i^*)^2$ and $(B_i^*)^2 B_i^2 - B_i^2 (B_i^*)^2$ are nonzero compact operators. Since T_s^2 is essentially square normal, if such m_i, n_i exist, then it is necessary that $m_i, n_i < \infty$ for each i.

Corollary (6.3.88)[295]: [286]. Let $T^2 \in \mathcal{B}(\mathcal{H})$ be essentially square normal. If T^2 is irreducible, then the following are equivalent:

(iv) $T^2 \in (cs)$;

(v) $T^2 \in (wcs)$;

(vi) T^2 is almost square normal.

Proof. The implication "(i) ⇒ (ii)" is trivial, and the equivalence "(i) ⇔ (iii)" follows from Theorem (6.3.6).

"(ii) ⇒ (iii)". If T^2 is not almost square normal, then $T^2 = T_s^2$ and T_s^2 is not absent. By Corollary (6.3.87), T_s^2 is reducible, a contradiction. This ends the proof.

List of Symbols

Symbol		Page
Ker	: Kernel	3
$\oplus$	: Direct sum	4
sp	: spectrum	5
sup	: Supremum	5
ℓ^∞	: Hilbert space of sequences	6
I. m. c	: locally multiplicatiply convex	13
$\otimes$	: Tensor product	13
ℓ^2	: Hilbert space of square sammble	17
al$g$	: algebra space of sequences	31
L^2	: Hilbert space	31
GNS	: Gelfond Naimark and Segal	32
sgn	: sign	45
L_1	: Lebesgue space on the real line	52
L_∞	: Essential Lebesgue space	52
max	: maximum	60
inf	: infimum	61
tr	: trace	62
dim	: dimension	62
rng	: range	67
det	: determinant	70
Im	: Imaginary	70
CCR	: completely continuous operators algebras	77
min	: minimum	80
ℓ^2	: Hilbert space of sequences	119
μ-a. e	: measurable almost everywhere	120
H^2	: Hardy space	124
$\ominus$	: Direct difference	124
SOT	: strong operator topology	125
CSO	: complex symmetric operator	129
UET	: unitarity equivalent traispase	130
ONB	: orthonarmal basis	137
SSO	: skew symmetric operator	140
nor	: normal	146
abnor	: abnormal	146
ONP	: orthonarmal projection	146
$g$cd	: greatest common divisor	149
KSGNS	: Kasparor, Stinespring, Gelfand, Naimark, Segal	150
VH	: Vector Hilbert	150
LVH	: Limit Vector Hilbert	162
Ind	: induced	219
POS	: The cone of nonnegative polynomials	219
supp	: support	222

References

[1] N. Christopher Phillips, Inverse Limits of C*-algebras, J. Operator Theory 19(1988), 159-195.

[2] Akemann, C. A.; Pedersen, G. K.; Tomiyama, J., Multipliers Of C*-algebras, J. Funct. Anal

[3] Arens, R., A generalization Of normed rings. Pacific J.Math 2(1952);455-471.

[4] Arens, R., Dense inverse limit rings, Michigan Math. 5(1958),169-182.

[5] Arveson, W., The harmonic analysis of automorphism groups, Proc. Sympos. Pure Math.. vol 38 part1 pp.199-269. Math. Soc 1982.

[6] Atiyah, M. F., Macdonald, L. G., Introduction to commutative algebra, Addison-Wesley, Reading, Mass

[7] Atiyah, M. F.; Segal, G. B., Equivariant K-theory and completion, J. Differential Geom.,

[8] Blackadar, B., Shape theory for CO-algebras, Math. Scand., S

[9] Brooks, R. M., on representing re-algebras, pacific 0'. Math

[10] Brown, L. G., Ext Of certain free product CO-algebras, J. Operator Theory.,

[11] Busby, R. C. Double centralizers and extensions Of CO-algebras, Trans. *mer. Math. Soc.,

[12] Dugundji, J., Topology, Allyn and Bacon, Boston.

[13] Fragoutopoulot, M., Spaces of representations and enveloping l.m.c. "-algebras, Pacific Math.

[14] Fragoutopoulou, M., Representations Of tensor product J.m.c. *-algcbras, Manuscripta Math.,

[15] Fraooutdpoulou. M.. Kadison's transitivity for locally CO-algebras. J. Math. Anal. Appt..

[16] Is. Fragoulopocloc. M., topological o-algcbras. Abstracts Amer. Math. Soc

[17] Gillman, L,. M., Rings of eontinuoes functions. (Graduate TcĂts in Malhem,atics NO. 4 3). Springer.Vcrlag. New York, Heidelberg. Berlin. 1976.

[18] Inoue, A. Locally C-algebra, Mem. Fac. Sei. Kyushu Univ., 25(1971), 197-235.

[19] Kasparov, G. G., Hilcert C-module: theorems of Stinespring and Voiculescu, J. Operator Theory, 4(1980), 133-150.

[20] Kasparov, G. G., The Operator K-funtor and extensions of C*-algebras (Russian), Izv. Aka. Nauk SSSr, Ser. Mat., 44(1980), 571-636

[21] Kiley, J.L., General Topology, Van Nostrand, New York, 1955.

[22] kothe, G., Topologische lineara Raume. I, (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen No. 107), Springer- verlag, Berlin, Heidelberg, 1960.

[23] Lazar, A.J.; Taylor, D. C., Multipliers of Pedersen’s ideal, Mem. Amer. Math. See. 169 (1976).

[24] Mallios, A., heermitian K-theroy over topogical *-algebras, J. Math. Anal. Appl. 106(1985), 454-539.

[25] Mallios, A., Contionuous vector bundles on topological algebras, J. Math. Anal. App.113(1986), 245-254.

[26] Michael, E. A., S0-spaces, J.. Math. Mech., 15(1966), 983-1002.

[27] Michael,E. A., Locally multipliteatively-convex topogical algebras, Mem. Amer, Math. Soc., 11(1952).

[28] Mingo, J, A., K-theory and multipliers of stable C*-algebras, Trans. Amer. Math. Soc., 299(1987), 397-411.
[29] Pedfrsen, G. K.m C*-algebras and their automorphism groups, Academic Press London, New York, San Francisco, 1979.
[30] Phillips, N.C., Equivariant K-theoy and freeness of group action on C*-algebras, Lecture Notes in Mathematics, no. 1274, Springer –Verlag, Berlin, Heidelnerg, New York, London, Parirs, Tokyo, 1987.
[31] Phillips, N.C., Representable K-theory for σ C*-algebras, preprint..
[32] Phillips, N.C., The Atiyah-Segal Completion theorem for C*-algebras, Proc. Amer. Soc,. To appear.
[33] Phillips, N.C., A new approach to the multipliers of Pedersen's Ideal, preprint.
[34] Phillips, N.C., Inverse limits of C*-algebras and applications, preprint.
[35] Rieffel, M. A., Induced representations of C*-algebras, Adv. In math., 13(1974), 176-257.
[36] Rieffel, M. A., Morita equivalence for operator algebras, Proc. Sympos. Pure Math., vol. 38, part 1 pp. 285-298.
[37] Shmudgen, K. Uber LMC*-algebren, Math. Nachr., 68(1975), 167-182.
[38] Segal, G. B., Equivariant K-theoy, Publ. Math. Inst. Hautes Etudes Sci., 34(1968), 129-151.
[39] Stefnrod, N.E., A convenient Category of topological spaces, Michigan Math. J. 14(1967), 133-152.
[40] Takisaki, M., Theroy of operator algebras. I, Speinger-Verlag, New York, Heidelberg, Berlin, 1979.
[41] Tayior, J. L., Bananch algecras and topology, in Algebras in Analysis, pp. 118-186, Academic press, London, New York, San Francisco, 1975.
[42] Voiculescu, D., Dual algebraic Structures on operator algebras related to free products, J. Operator Theory, 17(1987), 85-98.
[43] walker, R. C., The Stone-Cech compactification, (Ergebnisse der Mathematik und ihrer Grenzgebiete No. 83), Springer-Verlag, New York, Heidelberg, Berlin, 1974.
[44] Whitehead, G. W., Elements of homotopy theory(Graduate Texts in Mathematics No.61). Springer-verlag, new York, Heidelberg, Berlin, 1978.
[45] Jonson, B. E., An introduction to the theory of centralizers, Proc. Londan Math, Soc.(3) 14(2964), 299-320.
[46] Pedersen, G. K., SAW*-algebras, and corona C*-algebras, contributions to noncommutative topology, J. Operator Theory, 15(1986), 15-32.
[47] Spanier, E., Quasitopologies, Duke Math. J., 30(1963), 1-14.
[48] Weinder, J., Topological invariants for generalized operator algebras, Ph. D. Thesis Heidel-berg, 1987.
[49] Wenjen, C., On scminormal *-algebras, Pacific J. Math,. 8(1985), 177-186.
[50] Gerard J. Murphy, Positive definite kernels and hilbert C*-modules, Proceedings of the Edinburgh Mathematical Society (1997) 40, 367-374.
[51] D. E. EVANS and J. T. LEWIS, Dilations of Irreversible Evolutions in Algebraic Quantum Theory (Comm. DIAS, Series A, no. 24, Dublin, 1977).
[52] E. C. LANCE, Hilbert C*-modules. (Cambridge University Press, Cambridge, 1995).
[53] T. Constantinescu, A. Gheondea, Representations of Hermitian Kernels by Means of Krein Spaces. II. Invariant Kernels, Commun. Math. Phys. 216, 409 – 430 (2001).

[54] Albeverio, S., Gottschalk, H.,Wu, J.-L.: Models of local relativistic quantum fields with indefinite metric (in all dimensions). Commun. Math. Phys. 184, 509–531 (1997)
[55] Antoine, J.-P., Ôta, S.: Unbounded GNS representations of a ∗-algebra in a Kreˇın space. Lett. Math. Phys. 18, 267–274 (1989)
[56] Araki, H.: Indecomposable representations with invariant inner product. A theory of the Gupta–Bleuler triplet. Commun. Math. Phys. 97, 149–159 (1985)
[57] Bonsall, F.F., Duncan, J.: Complete Normed Algebras, Berlin–Heidelberg–New York: Springer-Verlag, 1973
[58] Constantinescu, T., Gheondea, A.: Representations of Hermitian kernels by means of Kreˇın spaces. Publ. RIMS. Kyoto Univ. 33, 917–951 (1997)
[59] Dieudonné, J.: Quasi-Hermitian operators. In: Proceedings of International Symposium on Linear Spaces, Jerusalem 1961, pp. 115–122
[60] Dijksma, A., Langer, H., de Snoo, H.S.: Unitary colligations in Kreˇın spaces and their role in extension theory of isometries and symmetric linear relations in Hilbert spaces. In: Functional Analysis. II, Lecture Notes in Math. 1242, Berlin: Springer Verlag, 1987, pp. 1–42
[61] Dixmier, J.: Les moyennes invariante dans les semi-groupes et leurs applications. Acta. Sci. Math. Szeged 12, 213–227 (1950)
[62] Dritschel, M.A.,Rovnyak, J.: Operators on indefinite inner product spaces. In: Lectures on Operator Theory and its Applications, Fields Institute Monographs, Vol. 3, Providence, RI: Amer. Math. Soc., 1996
[63] Evans, D.E., Lewis, J.T.: Dilations of Irreducible Evolutions in Algebraic Quantum Theory. Dublin Institute for Advanced Studies, Dublin, 1977
[64] Evans, D.E., Kawahigashi,Y.: Quantum Symmetries on Operator Algebras. Oxford Science Publications, 1998
[65] Iokhvidov, I.S., Kreˇın, M.G., Langer, H.: Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric. Berlin: Akademie-Verlag, 1982
[66] Hofmann, G.: On GNS representations on inner product spaces I. The structure of the representation space. Commun. Math. Phys. 191, 299–323 (1998)
[67] Kissin, E., Shulman, V.: Representations of Krein Spaces and Derivations of C∗-Algebras. Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 89, Harlow: Longman, 1997
[68] Kolmogorov, A.N.: Stationary sequences in Hilbert space. Bull. Math. Univ. Moscow 2, 1–40 (1941)
[69] Kreˇın, M.G.: On linear completely continuous operators in functional spaces with two norms. [Ukrainian], Zbirnik Prak. Inst. Mat. Akad. Nauk USSR 9, 104–129 (1947)
[70] Lax, P.: Symmetrizable linear transformations. Comm. Pure Appl. Math. 7, 633–647 (1954)
[71] Mnatsakanova, M., Morchio, G.,Strocchi, F., Vernov, Yu.: Irreducible representations of the Heisenberg algebra in Kreˇın spaces. J. Math. Phys. 39, 2969–2982 (1998)
[72] Morchio, G., Pierotti, D., Strocchi, F.: Infrared and vacuum structure in two-dimensional local quantum field theory models. The massless scalar field. J. Math. Phys. 31, 1467–1477 (1990)

[73] Ôta, S.: Unbounded representation of a ∗-algebra on indefinite metric space. Ann. Inst. Henri Poincaré 48, 333–353(1988)
[74] Paulsen, V.I.: Completely bounded maps and dilations. Pitman Research Notes in Math. 146, New York: Longman, Wiley, 1986
[75] Parthasaraty, K.R., Schmidt, K.: Positive-Definite Kernels, Continous Tensor Products and Central Limit Theorems of Probability Theory. Lecture Notes in Mathematics, Vol. 272, Berlin: Springer-Verlag, 1972
[76] Pisier, G.: Similarity Problems and Completely Bounded Maps. Springer Lecture Notes 1618, Berlin– Heidelberg–New York: Springer, 1996.
[77] Reid, W.T.: Symmetrizable completely continuous linear transformations in Hilbert space Duke Math. J. 18, 41–56 (1951)
[78] Schwartz, L.: Sous espace Hilbertiens d'espaces vectoriel topologiques et noyaux associés (noyaux reproduisants). J. Analyse Math. 13, 115–256 (1964)
[79] Strocchi, F.: Selected Topics on the General Properties of Quantum Field Theory. Lecture Notes Phys. 51, Singapore: World Scientific, 1993
[80] J.M.G. FELL, The dual spaces of C*-algebras, Trans. Amer. Math. Soc. 94 (1960), 365 405.
[81] R. Arens, Representation of *-algcbras, Duke Math. J. vol. 14 (1947) pp. 269-282.
[82] M. M. Day, Normed linear spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1958.
[83] J. Dixmier, Les algebres d'opSrateurs dans Vespace Hilbertien, Gauthier-Villars, Paris, 1957.
[84] I. M. Gelfand and M. A. Naimark, Unitarnye Predstavleniya Klassicheskih Grupp, Trudy Mat. Inst. Steklov., 1950.
[85] R. Godement, Les fonctions de type positif et la theorie des groupes, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 1-84.
[86] Sur la thiorie 'des representations unitaires, Ann. of Math. vol. 53 (1951) pp. 68-124.
[87] A theory of spherical functions I, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 496-556.
[88] N. Jacobson, A topology for the set of primitive ideals in an arbitrary ring, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 333-338.
[89] R. V. Kadison, Irreducible operator algebras, Proc. Nat. Acad. Sci. U.S.A. vol. 43 (1957) pp. 273-276
[90] I. Kaplansky, Normed algebras, Duke Math. J. vol. 16 (1949) pp. 399-418.
[91] Groups with representations of bounded degree, Canad. J. Math. vol. 1 (1949) pp. 105-112.
[92] The structure of certain operator algebras, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 219-255.
[93] L. H. Loomis, An introduction to abstract harmonic analysis, New York, Van Nostrand, 1953.
[94] G. W. Mackey, Borel structure in groups and their duals, Trans. Amer. Math. Soc. vol. 85 (1957) pp. 134-165.
[95] M. A. Naimark, Rings with involution, Uspehi Mat. Nauk vol. 3 (1948) pp. 52-145; Amer. Math. Soc. Translations, no. 25.
[96] Description of all irreducible unitary representations of the classical groups, Dokl. Akad. Nauk SSSR vol. 84 (1952) pp. 883-886.
[97] Normirovannye Koltsa, Gos. Izdat. Tech.-Teor. Lit., Moskva, 1956.

[98] I. E. Segal, The group ring of a locally compact group, I, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 348-352.
[99] O. Takenouchi, Sur une classe de fonctions continues de type positif sur un groupe localement compact, Math. J. Okayama Univ. vol. 4 (1954) pp. 143-173.
[100] James Glimm, Type I C*-algebras, Annals of Math., 73(1961), 572-612.
[101] E. BISHOP and K. DELEEUW, The representation of linearfunctionals by measureson sets of extreme points, Ann. Inst. Fourier, Grenoble, 9 (1959), 305-331.
[102] J. DIxMIER, Les algebres d'operateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1957.
[103] Sur les representations unitaires des groupes de Lie algetbriques, Ann. Inst. Fourier, Grenoble, 7 (1957), 315-328.
[104] C*-algebras with smooth dual, Illinois J. Math., 4 (1960), 221-230.
[105] J. GLIMM, A Stone- Weierstrass theorem for C*-algebras, Ann. of Math., 72 (1960), 216- 244.
[106] R. KADISON, Unitary operators in C*-algebras, Pacific J. Math., 10 (1960), 547-556.
[107] R. GODEMENT, Theorie des characteres II, Ann. of Math., 59 (1954), 63-85.
[108] R. KADISON, Unitary invariants for representations of operator algebras, Ann. of Math., 66 (1957), 304-379.
[109] C. KURATOWSKI, Topologie I. Espaces metrisables, espaces complets, 2d ed. Warsaw, 1948.
[110] G. MACKEY, The theory of group representations, mimeographed notes, University of Chicago, 1955.
[111] E. NELSON and W. F. STINESPRING, Representation of elliptic operators in an enveloping algebra, Amer. J. Math., 81 (1959), 547-560.
[112] J. VON NEUMANN, Die Eindeutigkeit der Schrodingerschen Operatoren, Math. Ann., 104 (1931), 570-578.
[113] On rings of operators, III, Ann. of Math., 41 (1940), 94-161.
[114] L. PUKANSZKY, Some examples of factors, Publicationes Mathematicae, 4 (1956),135-156.
[115] I. SEGAL, Irreducible representations of operator algebras, Bull. Amer. Math. Soc., 53 (1947), 73-88.
[116] Two-sided ideals in operator algebras, Ann. of Math., 50 (1949), 856-865.
[117] Foundations of the theory of dynamical systems of infinitely many degrees of freedom, I, Det Konglige Danske Videnskabernes Selskab Matematisk-fysiske Meddelelser, 31 no. 12, 1-38.
[118] A. WIGHTMAN and S. SCHWEBER, Configuration space methods, Phys. Rev. 98 (1955), 812-837.
[119] J. DIxMIER, Sur les C*-algebres, Bull. Soc. Math. France, 88 (1960), 95-112.
[120] Sur les structures Borelienne du spectre d'une C*-algebre, Publications Mathematique Institut Hautes Etudes Scientifiques, 6, 297-303.
[121] J. M. G. FELL, The structure of algebras of operator fields, to appear.
[122] J. Tomiyama, Topological representation of C*-algebras, Tôhoku Math. J. (2) 14 (1962), 187-204. MR 26-619.
[123] Topological representation of algebras II, Trans. Amer. Math. Soc, 68(1950), 62-75.
[124] C. E. RlCKART, General theory of Banach algebras, Van Nostrand, New York, 1960.

[125] J. TOMIYAMA-M. TAKESAKI, Applications of fibre bundle to the certain class of C*-alge bras, Tóhoku Math. Journ., 13(1961), 498-522.
[126] A. B. WlLLCOX, Some structure theorems for a class of Banach algebras, Pacific Journ. Math., 6(1956), 177-192.
[127] S.R. GARCIA, W.R. WOGEN, Some new classes of complex symmetric operators, Trans. Amer. Math. Soc. 362(2010), 6065–6077.
[128] Brown, A., The unitary equivalence of binormal operators, Amer. J. Math. 76 (1954), 414– 434. MR0062355 (15:967e)
[129] Conway, J.B., A Course in Functional Analysis (second edition), Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990. MR1070713 (91e:46001)
[130] Chevrot, N., Fricain, E., Timotin, D., The characteristic function of a complex symmetric contraction, Proc. Amer. Math. Soc. 135 (2007), 2877–2886. MR2317964 (2008c:47025)
[131] Garcia, S.R., Approximate antilinear eigenvalue problems and related inequalities, Proc. Amer. Math. Soc. 136 (2008), no. 1, 171–179 . MR2350402 (2008k:47022)
[132] Garcia, S.R., Aluthge transforms of complex symmetric operators, Integral Equations Operator Theory 60 (2008), no. 3, 357–367. MR2392831 (2008m:47052)
[133] Garcia, S.R., Means of unitaries, conjugations, and the Friedrichs operator, J. Math. Anal. Appl. 335 (2007), 941–947. MR2345511 (2008i:47070)
[134] Garcia, S.R., Putinar, M., Complex symmetric operators and applications, Trans. Amer. Math. Soc. 358 (2006), 1285-1315. MR2187654 (2006j:47036)
[135] Garcia, S.R., Putinar, M., Complex symmetric operators and applications II, Trans. Amer. Math. Soc. 359 (2007), 3913-3931. MR2302518 (2008b:47005)
[136] Garcia, S.R., Conjugation and Clark Operators, Contemp. Math. 393 (2006), 67-112. MR2198373 (2007b:47073)
[137] Garcia, S.R., Wogen, W.R., Complex symmetric partial isometries, J. Funct. Anal. 257 (2009), no. 4, 1251–1260. MR2535469.
[138] Gilbreath, T.M., Wogen, W.R., Remarks on the structure of complex symmetric operators, Integral Equations Operator Theory 59 (2007), no. 4, 585–590. MR2370050
[139] Hadwin, D., Laurie, C., Reflexive binormal operators, J. Funct. Anal. 123 (1994), no. 1, 99–108. MR1279297 (96c:47049)
[140] Halmos, P.R., A Hilbert Space Problem Book (Second Edition), Springer-Verlag, New York, 1982. MR675952 (84e:47001)
[141] Halmos, P.R., A Linear Algebra Problem Book, The Dolciani Mathematical Expositions, 16, Mathematical Association of America, Washington, DC, 1995. MR1310775 (96e:15001)
[142] Radjavi, H., Rosenthal, P., Invariant Subspaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77, Springer-Verlag, New York-Heidelberg, 1973. MR0367682 (51:3924)
[143] Radjavi, H., Rosenthal, P., On roots of normal operators, J. Math. Anal. Appl. 34 (1971), 653–664. MR0278097 (43:3829)
[144] Sarason, D., Algebraic properties of truncated Toeplitz operators, Oper. Matrices 1 (2007), no. 4, 491–526. MR2363975 (2008i:47060)
[145] Tener, J.E., Unitary equivalence to a complex symmetric matrix: An algorithm, J. Math. Anal. Appl. 341 (2008), no. 1, 640–648. MR2394112 (2008m:15062)

[146] Kunyu Guo and Sen Zhu, A canonical decomposition of complex symmetric operators, J. OPERATOR THEORY (2014), 529–547, doi: 10.7900/jot.2013aug15.2007.
[147] W. ARVESON, An Invitation to C*-Algebras, Graduate Texts in Math., vol. 39, SpringerVerlag, New York 1976.
[148] A. BROWN, P.R. HALMOS, Algebraic properties of Toeplitz operators, J. Reine Angew. Math. 213(1963/1964), 89–102.
[149] I. CHALENDAR, E. FRICAIN, D. TIMOTIN, On an extremal problem of Garcia and Ross, Oper. Matrices 3(2009), 541–546.
[150] J.-T. CHAN, C.-K. LI, N.-S. SZE, Mappings preserving spectra of products of matrices, Proc. Amer. Math. Soc. 135(2007), 977–986 (electronic).
[151] J.A. CIMA, S.R. GARCIA, W.T. ROSS, W.R. WOGEN, Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity, Indiana Univ. Math. J. 59(2010), 595–620.
[152] J.A. CIMA, W.T. ROSS, W.R. WOGEN, Truncated Toeplitz operators on finite dimensional spaces, Oper. Matrices 2(2008), 357–369.
[153] R.G. DOUGLAS, Banach Algebra Techniques in Operator Theory, second ed., Graduate Texts in Math., vol. 179, Springer-Verlag, New York 1998.
[154] N.S. FELDMAN, Essentially subnormal operators, Proc. Amer. Math. Soc. 127(1999), 1171–1181.
[155] S.R. GARCIA, Complex symmetric operators, Tech. report, in ESF Exploratory Workshop on Mathematical Aspects of the Physics with Non-self-adjoint Operators, Czech Academy of Sciences, town 2010.
[156] S.R. GARCIA, Three questions about complex symmetric operators, Integral Equations Operator Theory 72(2012), 3–4.
[157] S.R. GARCIA, B. LUTZ, D. TIMOTIN, Two remarks about nilpotent operators of order two, Proc. Amer. Math. Soc. 142(2014), 1749–1756.
[158] S.R. GARCIA, D.E. POORE, On the norm closure of the complex symmetric operators: compact operators and weighted shifts, J. Funct. Anal. 264(2013), 691–712.
[159] S.R. GARCIA, W. ROSS, Recent progress on truncated Toeplitz operators, Fields Inst. Commun. 65(2013), 275–319.
[160] S.R. GARCIA, W. ROSS, W.R. WOGEN, $C*$-algebras generated by truncated toeplitz operators, in Concrete Operators, Spectral Theory, Operators in Harmonic analysis and Approximation, Oper. Theory Adv. Appl., vol. 236, Birkhäuser Verlag, Basel 2014, pp. 181–192,
[161] S.R. GARCIA, J.E. TENER, Unitary equivalence of a matrix to its transpose, J. Operator Theory 68(2012), 179–203.
[162] H.-L. GAU, C.-K. LI, $C*$-isomorphisms, Jordan isomorphisms, and numerical range preserving maps, Proc. Amer. Math. Soc. 135(2007), 2907–2914.
[163] A. GEORGE, K.D. IKRAMOV, Unitary similarity of matrices with quadratic minimal polynomials, Linear Algebra Appl. 349(2002), 11–16.
[164] K. GUO, Y. JI, S. ZHU, A $C*$-algebra approach to complex symmetric operators, Trans. Amer. Math. Soc., to appear.
[165] C.-K. LI, E. POON, A. RASTOGI, Maps preserving spectral radius, numerical radius, spectral norm, Electron. J. Linear Algebra 16(2007), 347–365 (electronic).

[166] C.-K. LI, L. RODMAN, Preservers of spectral radius, numerical radius, or spectral norm of the sum on nonnegative matrices, Linear Algebra Appl. 430(2009), 1739–1761.
[167] C.-K. LI, N.-S. SZE, Product of operators and numerical range preserving maps, Studia Math. 174(2006), 169–182.
[168] N.A. SEDLOCK, Properties of truncated Toeplitz operators, ProQuest LLC, Ann Arbor, MI, Ph.D. Dissertation, Washington Univ. in St. Louis, St. Louis 2010.
[169] N.A. SEDLOCK, Algebras of truncated Toeplitz operators, Oper. Matrices 5(2011), 309– 326.
[170] X. WANG, Z. GAO, A note on Aluthge transforms of complex symmetric operators and applications, Integral Equations Operator Theory 65(2009), 573–580.
[171] S. ZHU, C.G. LI, Complex symmetry of a dense class of operators, Integral Equations Operator Theory 73(2012), 255–272.
[172] S. ZHU, C.G. LI, Complex symmetric weighted shifts, Trans. Amer. Math. Soc. 365(2013), 511–530.
[173] S. ZHU, C.G. LI, Y.Q. JI, The class of complex symmetric operators is not norm closed, Proc. Amer. Math. Soc. 140(2012), 1705–1708.
[174] Sen Zhu, Approximate Unitary Equivalence to Skew Symmetric Operators, Complex Anal. Oper. Theory DOI 10.1007/s11785-014-0369-z.
[175] Benner, P., Byers, R., Mehrmann, V., Xu, H.G.: Numerical computation of deflating subspaces of skew-Hamiltonian/Hamiltonian pencils. SIAM J. Matrix Anal. Appl. 24(1), 165–190 (2002)
[176] Boersema, J.L.: The range of united K-theory. J. Funct. Anal. 235(2), 701–718 (2006)
[177] Connes, A.: A factor not anti-isomorphic to itself. Ann. Math. (2) 101, 536–554 (1975)
[178] Connes, A.: Sur la classification des facteurs de type II, C. R. Acad. Sci. Paris Sér. A-B 281(1), Aii, A13–A15 (1975)
[179] Conway, J.B.: A Course in Operator Theory, Graduate Studies in Mathematics, vol. 21. American Mathematical Society, Providence, RI (2000)
[180] Davidson, K.R.: C∗-Algebras by Example, Fields Institute Monographs, vol. 6. American Mathematical Society, Providence, RI (1996)
[181] Gantmacher, F.R.: The Theory of Matrices, vols. 1, 2 (trans: Hirsch, K.A.). Chelsea Publishing Co., New York (1959)
[182] Hadwin, D.: Closures of unitary equivalence classes. Ph.D. thesis, Indiana University (1975)
[183] Hadwin, D.: An operator-valued spectrum. Indiana Univ. Math. J. 26(2), 329–340 (1977)
[184] Herrero, D.A.: Approximation of Hilbert Space Operators. Vol. 1, 2nd edn, Pitman Research Notes in Mathematics Series, vol. 224. Longman Scientific and Technical, Harlow (1989)
[185] Hua, L.-K.: On the theory of automorphic functions of a matrix variable. II. The classification of hypercircles under the symplectic group. Am. J. Math. 66, 531–563 (1944)
[186] Hua, L.-K.: On the theory of automorphic functions of a matrix level. I. Geometrical basis. Am. J. Math. 66, 470–488 (1944)

[187] Li, C.G., Zhou, T.T.: Skew symmetry of a class of operators. Banach J. Math. Anal. 8(1), 279–294 (2014)
[188] Li, C.G., Zhu, S.: Skew symmetric normal operators. Proc. Am. Math. Soc. 141(8), 2755–2762 (2013)
[189] Mehrmann, V.,Watkins, D.: Polynomial eigenvalue problems with Hamiltonian structure. Electr. Trans. Numer. Anal. 13, 106–113 (2002)
[190] Mehrmann, V., Xu, H.: Numerical methods in control. J. Comput. Appl. Math. 123, 371–394 (2000)
[191] Phillips, N.C., Viola, M.G.: A simple separable exact C∗-algebra not anti-isomorphic to itself. Math. Ann. 355(2), 783–799 (2013)
[192] Shields, A.L.: Weighted shift operators and analytic function theory. Topics in operator theory. American Mathematical Society, Providence, RI, pp. 49–128. Mathematical Surveys, No. 13 (1974)
[193] Sorensen, D.: Passivity preserving model reduction via interpolation of spectral zeros. Syst. Control Lett. 54, 347–360 (2005)
[194] Stacey, P.J.: Real structure in unital separable simple C∗-algebras with tracial rank zero and with a unique tracial state, New York. J. Math. 12, 269–273 (2006). (electronic)
[195] Stacey, P.J.: Antisymmetries of the CAR algebra. Trans. Am. Math. Soc. 363(12), 6439–6452 (2011) (With an appendix by J. L. Boersema and N. C. Phillips)
[196] Voiculescu, D.: A non-commutative Weyl–von Neumann theorem. Rev. Roumaine Math. Pures Appl. 21(1), 97–113 (1976)
[197] Zagorodnyuk, S.M.: On a J -polar decomposition of a bounded operator and matrices of J -symmetric and J -skew-symmetric operators. Banach J. Math. Anal. 4(2), 11–36 (2010)
[198] Zagorodnyuk, S.M.: On the complex symmetric and skew-symmetric operators with a simple spectrum. Symmetry, Integr. Geom. Methods Appl. 7, 1–9 (2011)
[199] M. Joița, Maps between locally C∗-algebras, J. Lond. Math. Soc. 2 (66) (2002) 421–432.
[200] J. S. Bhatt and J. D. Karia, ‘Complete positivity, tensor products and C∗-nuclearity for inverse limits of C∗-algebras’, Proc. Indian Acad. Sci. Math. Sci. 101 (1991) 149–167.
[201] I. Kaplansky, ‘Modules over operator algebras’, Amer. J. Math. 75 (1953) 839–853.
(19).
[202] E. C. Lance, Hilbert C∗-modules. A toolkit for operator algebraists, London Mathematical Society Lecture Note Series 210 (Cambridge University Press, Cambridge, 1995).
[203] W. L. Paschke, ‘Inner product modules over B∗-algebras’, Trans. Amer. Math. Soc. 182 (1973) 443–468.
[204] A. Gheondea, Dilations of some VH-spaces operator valued kernels, Integral Equations Operator Theory 74 (2012) 451–479.
[205] Aronszajn, N.: Theory of reproducing kernels. Trans. Am. Math. Soc. 68, 337– 404 (1950)
[206] Arveson, W.B.: A Short Course on Spectral Theory. Graduate Texts in Mathematics, vol. 209. Springer, Berlin-Heidelberg-New York (2002)

[207] Arveson, W.B.: Dilation theory yesterday and today. In: A Glimpse at Hilbert Space Operators. pp. 99–123. Oper. Theory Adv. Appl., vol. 207. Birkh¨auser Verlag, Basel (2010)
[208] Constantinescu, T., Gheondea, A.: On L. Schwartz's boundedness condition for kernels. Positivity 10, 65–86 (2006)
[209] Ga¸spar, D., Ga¸spar, P.: An operatorial model for Hilbert B(X)-modules. Analele Univ. de Vest Timi¸soara. Ser. Mat.-Inform. 40, 15–29 (2002)
[210] Ga¸spar, D., Ga¸spar, P.: Reproducing kernel Hilbert B(X)-modules. An. Univ. Vest Timi¸soara Ser. Mat.-Inform. 43(2), 47–71 (2005)
[211] Ga¸spar, D., Ga¸spar, P.: Reproducing kernel Hilbert modules over locally C∗ algebras. An. Univ. Vest Timi¸soara Ser. Mat.-Inform. 45(1), 245–252 (2007)
[212] Gheondea, A., U˘gurcan, B.E.: On two equivalent dilation theorems in VH-spaces. Complex Anal. Operator Theory 6, 625–650 (2012)
[213] Heo, J.: Reproducing kernel Hilbert C∗-modules and kernels associated to cocycles. J. Math. Phys. 49, 103507 (2008)
[214] Loynes, R.M.: On generalized positive-definite function. Proc. London Math. Soc. III. Ser. 15, 373–384 (1965)
[215] Loynes, R.M.: Linear operators in V H-spaces. Trans. Am. Math. Soc. 116, 167– 180 (1965)
[216] Loynes R.M.: Some problems arising from spectral analysis. In: Symposium on Probability Methods in Analysis (Loutraki, 1966), pp. 197–207. Springer, Berlin (1967)
[217] Manuilov, V., Troitsky, E.: Hilbert C∗-Modules. American Mathematical Society, Providence (2005)
[218] Mercer, J.: Functions of positive and negative type and their connection with the theory of integral equations. Philos. Trans. Roy. Soc. Lond. Ser. A 209, 415– 446 (1909)
[219] Moore, E.H.: General Analysis. Mem. Amer. Philos. Soc., Part I, Philadelphia 1935; Part II, Philadelphia (1939)
[220] Na˘ımark, M.A.: On the representations of additive operator set functions. C.R. (Doklady) Acad. Sci. USSR 41, 359–361 (1943)
[221] Na˘ımark, M.A.: Positive definite operator functions on a commutative group. Bull. (Izv.) Acad. Sci. USSR 7, 237–244 (1943)
[222] Paulsen, V.R.: Completely Bounded Maps and Operator Algebras. Cambridge University Press, Cambridge (2002)
[223] Stinespring, W.F.: Positive functions on C∗-algebras. Proc. Am. Math. Soc. 6, 211–216 (1955)
[224] Szafraniec, F.H.: Dilations of linear and nonlinear maps. In: Functions, Series, Operators, vols. I, II (Budapest, 1980), pp. 1165–1169. Colloq. Math. Soc. Janos Bolyai, 35. North-Holland, Amsterdam (1983)
[225] Sz.-Nagy B.: Prolongement des transformations de l'espace de Hilbert qui sortent de cet espace. In: Appendice au livre "Le¸cons d'analyse fonctionnelle" par F. Riesz et B. B. Sz.-Nagy, pp. 439–573. Akademiai Kiado, Budapest (1955)
[226] Serdar Ay , Aurelian Gheondea, Representations of ∗-semigroups associated to invariant kernels with values adjointable operators, Linear Algebra and its Applications 486 (2015) 361–388.

[227] G.H. Esslamzadeh, F. Taleghani, Structure of quasi operator systems, Linear Algebra Appl. 438 (2013) 1327–1392.
[228] G.H. Esslamzadeh, L. Turowska, Arveson's extension theorem in ∗-algebras, arXiv:1311.5065.
[229] V.R. Paulsen, I.G. Todorov, M. Tomforde, Operator system structures on ordered spaces, Proc. Lond. Math. Soc. 102 (2011) 25–49.
[230] V.R. Paulsen, M. Tomforde, Vector spaces with an order unit, Indiana Univ. Math. J. 58 (2009) 1319–1359.
[231] M. Skeide, Hilbert Modules and Applications in Quantum Probability, Habilitationschrift, Cottbus, 2001, http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html.
[232] F.H. Szafraniec, Murphy's positive definite kernels and Hilbert C∗-modules reorganized, in: Banach Center Publications, vol. 89, Warsawa 2010, pp. 275–295.
[233] A. Weron, Prediction theory in Banach spaces, in: Proceedings of the Winter School of Probability, Karpacz, Springer Verlag, Berlin, 1975, pp. 207–228.
[234] John W. Bunce, James A. Deddens, C*-Algebras with Hausdorff Spectrum, TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY Volume 212, 1975.
[235] L. G. Brown, R. G. Douglas and P. A. Fillmore, Unitary equivalence modulo the compact operators, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 58-128.
[236] J. W. Bunce and 1. A. Deddens, Irreducible representations of the C*-algebra generated by an n-normal operator, Trans. Amer. Math. Soc. 171 (1972), 301-307. MR 46 #6051.
[237] J. W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hubert space, Ann. of Math. (2) 42 (1941), 839-873. MR 3, 208.
[238] C. Davis, Separation of two linear subspaces, Acta Sei. Math. Szeged 19 (1958), 172-187. MR 20 #5425.
[239] J. Dixmier, Les C*-algèbres et leurs représentations, Cahiers Scientifiques, fase. 29, Gauthier-Villars, Paris, 1964. MR 30 #1404.
[240] Ideal center of a C-algebra, Duke Math. J. 35 (1968), 375-382. MR 37 #5703.
[241] J. Ernest, A classification, decomposition and spectral multiplicity theory for bounded operators on a separable Hilbert space (to appear).
[242] J. G. Glimm, On a certain class of operator algebras, Trans. Amer. Math. Soc. 95 (1960), 318-340. MR 22 #2915.
[243] P. R. Halmos, Ten problems in Hilbert space. Bull. Amer. Math. Soc. 76 (1970), 887-933. MR 42 #5066.
[244] C. L. Olsen, A structure theorem for polynomially compact operators, Amer. J. Math. 93 (1971), 686-698.
[245] C. Pearcy, A complete set of unitary invariants for operators generating finite W*-algebras of type I, Pacific J. Math. 12 (1962), 1405-1416. MR 26 #6816.
[246] C. Pearcy and N Salinas, Finite-dimensional representations of C*-algebras and the reducing matricial spectra of an operator, Rev. Roumaine Math. Pures Appl. (to appear).
[247] N. B. Vasil'ev, C*-algebras with finite dimensional irreducible representations, Uspehi Mat. Nauk 21 (1966), no. 1 (127), 135-154 = Russian Math. Surveys 21 (1966), no. 1, 137-155. MR 34 #1871.

[248] F. B. Wright, A reduction for algebras of finite type, Ann. of Math. (2) 60 (1954), 560-570. MR 16, 375.
[249] Mehdi Ghasemi , Salma Kuhlmann , Murray Marshall, Application Of Jacobi's Representation Theorem To Locally Multiplicatively Convex Topological R-Algebras, arXiv:1209.2966v1 [math.FA]13 Sep 2012.
[250] E. Beckenstein, L. Narici, C. Suffel, Topological Algebras, North-Holland Math. Stud., 24,. Elsevier Sci. Publ., Amsterdam, (1977).
[251] C. Berg, J.P.R. Christensen, P. Ressel, Positive definite functions on abelian semigroups, Math. Ann. 223, 253–272, (1976).
[252] C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups, Theory of Positive Definite and Related Functions, Springer-Verlag, (1984).
[253] M. Ghasemi, S. Kuhlmann, E. Samei, Moment problem for continuous positive semidefinite linear functionals, arXiv:1010.2796v3 [math.AG].
[254] M. Ghasemi, S. Kuhlmann, Closure of the cone of sums of 2d-powers in real topological algebras, arXiv:1110.3016v3 [math.RA].
[255] M. Ghasemi, M. Marshall, S. Wagner, Closure results for the cone of sums of 2d-powers, arXiv:1109.0048v1 [math.AC].
[256] E. K. Haviland, On the momentum problem for distribution functions in more than one dimension, Amer. J. Math. 57, 562–572, (1935).
[257] E. K. Haviland, On the momentum problem for distribution functions in more than one dimension II, Amer. J. Math. 58, 164–168, (1936).
[258] D. Hilbert, Uber die Darstellung definiter Formen als Summe von Formenquadraten ¨ , Math. Ann. 32, 342–350 (1888).
[259] T. Jacobi, A representation theorem for certain partially ordered commutative rings, Math. Z. 237, 259–273, (2001).
[260] J. B. Lasserre, T. Netzer, SOS approximation of nonnegative polynomials via simple high degree perturbations, Math. Z. 256, 99–112, (2006).
[261] M. Marshall, Approximating positive polynomials using sums of squares, Canad. Math. Bull. 46 (3), 400–418, (2003).
[262] M. Marshall, Positive Polynomials and Sum of Squares, Mathematical Surveys and Monographs 146, (2007).
[263] K. Schm¨udgen, Positive cones in enveloping algebras, Reports on Mathematical Physics 14 (3), 385–404, (1978).
[264] Mahmood Alaghmandan, Mehdi Ghasemi, Seminormed ∗-Subalgebras Of l∞(X), S0022-247X(17)30501-2, Journal of Mathematical Analysis and Applications (2016).
[265] C-Y. Chou. Notes on the separability of C∗-algebras. Taiwanese J. Math., 16(2):555–559, 2012.
[266] M. Ghasemi, S. Kuhlmann, and M. Marshall. Moment problem in infinitely many variables. Israel J. Math., (to appear), 2015.
[267] L. Gillman and M. Jerison. Rings of continuous functions. Springer-Verlag, New YorkHeidelberg, 1976. Reprint of the 1960 edition, Graduate Texts in Mathematics, No. 43.
[268] A. E. Hurd and P. A. Loeb. An Introduction to Nonstandard Real Analysis. Academic Press Inc., Orlando, Florida 32887, 1985.

[269] P. T. Johnstone. Stone spaces, volume 3 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
[270] R. Sikorski. Boolean algebras. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25. Springer-Verlag New York Inc., New York, 1969.
[271] S. Willard. General topology. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.
[272] S. Zhu, Approximation of complex symmetric operators, Math. Ann. 364 (1–2) (2016) 373–399.
[273] Aluthge, A.: On p-hyponormal operators for 0 < p < 1. Integral Equ. Oper. Theory 13(3), 307–315 (1990)
[274] Danciger, J., Garcia, S.R., Putinar, M.: Variational principles for symmetric bilinear forms. Math. Nachr. 281(6), 786–802 (2008)
[275] Fang, J.S., Jiang, C.L., Wu, P.Y.: Direct sums of irreducible operators. Stud. Math. 155(1), 37–49 (2003)
[276] Garcia, S.R., Poore, D.E.: On the norm closure problem for complex symmetric operators. Proc. Am. Math. Soc. 141(2), 549–549 (2013)
[277] Størmer, E.: On anti-automorphisms of von Neumann algebras. Pac. J. Math. 21, 349–370 (1967)
[278] Lars H¨ormander, An introduction to complex analysis in several variables, 3rd ed., NorthHolland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639 (91a:32001)
[279] V. F. R. Jones, A II1 factor anti-isomorphic to itself but without involutory antiautomorphisms, Math. Scand. 46 (1980), no. 1, 103–117. MR585235 (82a:46075)
[280] Laurent Marcoux, On the distance between unitary orbits of weighted shifts, Trans. Amer. Math. Soc. 326 (1991), no. 2, 585–612, DOI 10.2307/2001775. MR1010887 (91k:47062)
[281] Donal P. O'Donovan, Weighted shifts and covariance algebras, Trans. Amer. Math. Soc. 208 (1975), 1–25. MR0385632 (52 #6492)
[282] N. Christopher Phillips, Continuous-trace C∗-algebras not isomorphic to their opposite algebras, Internat. J. Math. 12 (2001), no. 3, 263–275, DOI 10.1142/S0129167X01000642. MR1841515 (2002f:46110)
[283] N. Christopher Phillips, A simple separable C∗-algebra not isomorphic to its opposite algebra, Proc. Amer. Math. Soc. 132 (2004), no. 10, 2997–3005 (electronic), DOI 10.1090/S0002-9939- 04-07330-7. MR2063121 (2005b:46127)
[284] P. J. Stacey, Inductive limit decompositions of real structures in irrational rotation algebras, Indiana Univ. Math. J. 51 (2002), no. 6, 1511–1540, DOI 10.1512/iumj.2002.51.2042. MR1948458 (2004e:46077)
[285] P. J. Stacey, Real structure in purely infinite C∗-algebras, J. Operator Theory 49 (2003), no. 1, 77–84. MR1978322 (2004c:46111)
[286] Sen Zhu, Jiayin Zhao, Complex symmetric generators of C* -algebras, J. Math. Anal.
[287] I.D. Berg, Index theory for perturbations of direct sums of normal operators and weighted shifts, Canad. J. Math. 30 (6) (1978) 1152–1165.
[288] I.D. Berg, K.R. Davidson, Almost commuting matrices and a quantitative version of the Brown–Douglas–Fillmore theorem, Acta Math. 166 (1–2) (1991) 121–161.
[289] N.S. Feldman, The Self-Commutator of a Subnormal Operator, Thesis (Ph.D.), The University of Tennessee, 1997.

[290] P.R. Halmos, Irreducible operators, Michigan Math. J. 15 (1968) 215–223.
[291] C.L. Jiang, Z.Y. Wang, Structure of Hilbert Space Operators, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
[292] S. Jung, Y. Kim, E. Ko, J.E. Lee, Complex symmetric weighted composition operators on H2(D), J. Funct. Anal. 267 (2) (2014) 323–351.
[293] H. Lin, Almost commuting selfadjoint matrices and applications, in: Operator Algebras and Their Applications, Waterloo, ON, 1994/1995, in: Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 193–233.
[294] H. Radjavi, P. Rosenthal, The set of irreducible operators is dense, Proc. Amer. Math. Soc. 21 (1) (1969) 256.
[295] Shawgy Hussein and Aboubakr Khalifa, Representations of $*$-Semigroups Associated to Invariant Kernels and Seminormed $*$-Subalgebras with Symmetric Generators of C^*-Algebras, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2021).