Sudan University of Science and Technology
College of Graduate Studies

Quasiconformal Harmonic Mappings and Steinhaus Lattice-Point Problem with Partitions of Unity on Banach Spaces

الرواسم التوافقية شبه حافظة الزوايا ومسألة نقطة-شبكة شتاينهاوس مع تجزئيات الوحدة على فضاءات باناخ

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

The bi-Lipschitz type inqualities with the quasi – isometries of harmonic quasiconformal mappings and between smooth Jordan domains are studied. The quasiconformal maps with controlled Laplacian and coefficients estimates for harmonic υ – Bloch mappings and harmonic k- quasiconformal mappings with the curvature of the boundary are considered. We give the smooth functions, partitions of unity, lines and spaces and partitions of unity on certain Banach spaces. We show the dual locally uniformaly rotund and convex norms, the structure of WUR Banach spaces. Smooth norms and approximation in Banach spaces of type C(k). the three lattice – point problems of Steinhaus and Steinhaus tilling problem for Banach spaces are dealt with.

الخلاصة

قمنا بدراسة المتباينات نوع ثنائية – لبشيتز مع شبه الايزومتريس للرواسم شبه حافظات الزوايا التوافقية وبين مجالات جوردان الملساء. تم اعتبار الرواسم شبه حافظات الزوايا مع اللابلاسيان التحكمي وتقديرات المعاملات لأجل رواسم بلوش - v التوافقية والرواسم شبه حافظات الزوايا – k التوافقية مع انحناء الحدية. قمنا بإعطاء الدوال الملساء والتجزئيات الى الوحدة والخطوط والفضاء والتجزئيات الى الوحدة على فضاءات باناخ الأكيدة. تم توضيح النظائم المحدبة والمستديرة المنتظمة الموضعية المزدوجة والبناء الى فضاءات باناخ WUR والنظائم الملساء والتقريب في فضاءات باناخ الى النوع C(k). تعاملنا مع مسائل نقطة – الشبكة الثلاث الى شتاينهاوس ومسألة قرمدة شتاينهاوس لأجل فضاءات باناخ.

Introduction

Pavlovic [14] proved that any quasiconformal and harmonic selfmapping F of the unit disk is bi-Lipschitz with respect to the Euclidean metric. We present some recent results on the topic of quasiconformal harmonic maps. The main result is that every quasiconformal harmonic mapping w of $C^{1;\mu}$ Jordan domain Ω_1 onto $C^{1,\mu}$ Jordan domain Ω is Lipschitz continuous, which is the property shared with conformal mappings.

A decade ago the late Professor Steinhaus sent a sequence of communications proposing certain problems about the number of points of the Cartesian lattice covered by congruent copies of a plane set S. We consider a set, L, of lines in R n and a partition of L into some number of sets: $L = L_1 \cup \ldots \cup L_p$. We seek a corresponding partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_p$ such that each line l in L_i meets the set S_i in a set whose cardinality has some fixed bound, ω_τ. We show several results related to a question of Steinhaus: is there a set $E \subset \mathbb{R}^2$ such that the image of E under each rigid motion of $\mathbb{R}^2$ contains exactly one lattice point

We present a lemma about partitions of unity. It is an open problem whether a non-separable Banach space with a C^k norm (or, more generally, a C^k "bump function") admits C^k partitions of unity, though many partial results in this direction are known. We show that the existence of an equivalent dual LUR norm on a dual Banach space can be characterized by a topological property similar to the fragmentability. We present an example of a Banach space E admitting an equivalent weakly uniformly rotund norm and such that there is no $\Phi : E \to c_0(\Gamma)$, for any set Γ, linear, one-to-one and bounded.

We establish that every K-quasiconformal mapping of w of the unit disk D onto a C^2 -Jordan domain Ω is Lipschitz provided that $\Delta w \in L^p\ (D)$ for some $p > 2$. We also show that if in this situation K → 1 with $\|\Delta w\|_{L^p(D)} \to 0$, and $\Omega \to D$ in $C^{1,\alpha}$-sense with $\alpha > 1/2$, then the bound for the Lipschitz constant tends to 1. For $f(z) = h(z) + \overline{g(z)}$ be a harmonic v-Bloch mapping defined in the unit disk $\mathbb{D}$ with $\|f\|_{B_v} \le M$, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in $\mathbb{D}$. We obtain the coefficient estimates for f as follows: $|a_n|^2 + |b_n|^2 \le A_n(v, M)$, where $A_n(v, M)$ is given. Furthermore, we show that for $v < 1$, $\lim_{n\to\infty} A_n(v, M) = 0$ and for $v \ge 1$, $A_n(v, M) \le O(n^{2v} - 2)$. We estimate the Jacobian of harmonic mapping of the unit disk onto a smooth and convex Jordan domain by the boundary function and the boundary curvature of the image domain.

We work in the theory ZFC; the usual axioms of set theory with the axiom of choice (AC). AC is used heavily in the main construction as we require, for example, an enumeration of the equivalence classes of the lattices under a certain equivalence relation. Recently, using Fourier transform methods, it was shown that there is no measurable Steinhaus set in $\mathbb{R}^3$, a set which no matter how translated and rotated contains exactly one integer lattice point. We show a new characterisation of the existence of smooth partitions of unity on a Banach space.

Results are proved about the Banach space $X = C(K)$, where K is compact and Hausdorff. We concern smooth approximation: let m be a positive integer or ∞; we show that if there exists on X a non-zero function of class C^m with bounded support, then all continuous real-valued functions on X can be uniformly approximated by functions of class C^m. Steinhaus proved that given a positive integer n, one may find a circle surrounding exactly n points of the integer lattice. This statement has been recently extended to Hilbert spaces by Zwoleński, who replaced the integer lattice by any infinite set that intersects every ball in at most finitely many points.

The Contents

Chapter 1
Bi-Lipschitz Type Inequalities with Quasi-Isometries and Quasiconformal

We find explicit estimations of bi-Lipschitz constants for F that are expressed by means of the maximal dilatation K of F and $|F^{-1}(0)|$. Under the additional assumption $F(0) = 0$ the estimations are asymptotically sharp as $K \to 1$, so F behaves almost like a rotation for sufficiently small K. We show versions of the Ahlfors–Schwarz lemma for quasiconformal euclidean harmonic functions and harmonic mappings with respect to the Poincaré metric. In addition, if - has $C^{2,\mu}$ boundary, then w is bi-Lipschitz continuous. These results have been considered.

Section (1.1): Quasiconformal Harmonic Mappings

Set $D{:} = \{z \in C{:}\, |z| < 1\}, T_r{:} = \{z \in C{:}\, |z| = r\}$ for $r > 0$ and $T{:} = T_1$. Given $K \geq 1$ and domains Ω_1 and $\Omega_2$in C write $QC(\Omega_1, \Omega_2; K)$ for the class of all K-quasiconformal mappings of Ω_1 onto Ω_2 and let $QC(\Omega_1, \Omega_2; K)$ be the class of all mappings in $QC(\Omega_1, \Omega_2; K)$ that are harmonic onΩ_1. In case $\Omega_1 = \Omega_2$we write shortly $QC(\Omega_1; K) \coloneqq QC(\Omega_1, \Omega_2; K)$ and $QCH(\Omega_1; K){:} = QCH(\Omega_1, \Omega_2; K)$.

There are a lot of results providing intrinsic characterizations of the boundary Valued mapping f for a mapping $F \in QC(D){:} = \bigcup_{K\leq 1} QC(D; K)$; cf. e.g. [6], [10] and S [18]. A similar problem may be posed in the case where $\in QCH(D)\ {:} = \bigcup_{K\leq 1} QC(D; K)$. In [9] and [11] several results were established that provide intrinsic characterizations of f in terms of the Cauchy and Cauchy–Stieltjes singular integrals involving f. The results also provide motivation for the further study of such integrals. We express the Cauchy singular integral of the derivative f' by means of two functions $V[f]$ and $V^*[f]$ defined in (13) and (14), respectively; cf. Theorem (1.1.2). It is done in the case where f is a sense-preserving homeomorphic self-mapping of T and f is absolutely continuous on T. The rest part with estimating $V[f]$ under the additional assumption that f is the boundary valued mapping of $F \in QC(D)$ and $F(0) = 0$; see Theorem (1.1.4) and Corollary (1.1.6). We gather a few results related to formal derivatives ∂F and $\bar{\partial} F$ of $F \in QC(D)$ in the context of Hardy spaces $H^1(D)$ and $H^\infty(D)$. They seem to be known, but we prove them for the sake of completeness of our considerations, where we present applications of Corollary (1.1.6). We prove Theorem (1.1.10) which gives asymptotically sharp estimations for $V[f]$ and $V^*[f]$ as $K \geq 1$ tends to 1, provided $F \in QCH(D; K){:} =$. We use them for the bi-Lipschitz type estimations for f (Theorem (1.1.11)) and F (Theorems (1.1.12) and (1.1.13)) under the additional assumption $F(0) = 0$. All the estimations are asymptotically sharp as $K \to 1$. These theorems combined with [11] essentially improve the eminent results by Pavlovic [14].

We recall that the Cauchy singular integral $C_T[f]$ of a function $f{:}\, T \to C$ Lebesgue integrable on T is defined for every $z \in T$ as follows:

$$C_T[f](z) \coloneqq PV \frac{1}{2\pi i} \int_T \frac{f(u)}{u-z} du \coloneqq \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{T/T(z,\varepsilon)} \frac{f(u)}{u-z} du \tag{1}$$

Whenever the limit exists and $C_T[f](z) := 0$ otherwise, where $T(e^{ix}, \varepsilon){:} = \{ e^{it} \in T{:}\, |t-x| < \varepsilon\}$. Here and subsequently, integration along any arc $I \subset T$ is understood under counterclockwise orientation and the limit operator is understood in C with the Euclidian distance. Given a function $f{:}\, T \to C$ and $z \in 2\ T$ we define

$$f'(z):=\lim_{u\to z}\frac{f(u)-f(z)}{u-z} \quad (2)$$

Provided the limit exists and $f'(z):=0$ otherwise. Write Hom+(T) for the class of all sense-preserving homeomorphic self-mappings of T. Each $f\in$ Hom+(T) defines a unique continuous function $\hat{f}$ satisfying $0\le \hat{f}(0)<2\pi$ and

$$f(e^{it})=e^{i\hat{f}(t)} \quad ,t\in R. \quad (3)$$

Actually, $\hat{f}$ is an increasing homeomorphism of R onto itself satisfying

$$\hat{f}(t+2\pi)-\hat{f}(t)=2\pi, \qquad t\in R, \quad (4)$$

Moreover, from (3) it follows that for every $t\in R$, f is differentiable at e^{it} iff $\hat{f}$ is differentiable at t, and for every such point t,

$$f'(e^{it})e^{it}=\hat{f}'(t)e^{i\hat{f}(t)}=|f'(e^{it})|f(e^{it}). \quad (5)$$

Thus by Lebesgue's classical theorem on the differentiation of a monotonic function, for each $f\in \mathrm{Hom}^+(T)$ the limit in (2) exists for a.e. $z\in T$.

Lemma (1.1.1)[1]: Suppose that $f\in \mathrm{Hom}^+(T)$ is absolutely continuous on T and that f is differentiable at a point $z\in T$. Then both the following limits exist and

$$\lim_{\varepsilon\to 0+} Re\left[\frac{\overline{zf(z)}}{\pi i}\int_{T/T(z,\varepsilon)}\frac{f'(u)}{u-z}du\right]=\lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)-f(z)|^2}{|u-z|^2}|du|. \quad (6)$$

Moreover, both the following limits simultaneously exist or not and in the first case

$$\lim_{\varepsilon\to 0+}\mathrm{Im}\left[\frac{\overline{zf(z)}}{\pi i}\int_{T/T(z,\varepsilon)}\frac{f'(u)}{u-z}du\right]=-\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)\overline{f(z)}|}{|u-z|^2}|du|. \quad (7)$$

Proof. Fix $z=e^{ix}\in T$ and $\varepsilon\in(0;\pi)$. Since f is absolutely continuous on T, we see, integrating by parts, that

$$\int_{T/T(z,\varepsilon)}\frac{f'(u)}{u-z}du=\int_{T/T(z,\varepsilon)}\frac{d}{du}[f(u)-f(z)]\frac{1}{u-z}du$$

$$=\frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon-z}-\frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon-z}+\int_{T/T(z,\varepsilon)}\frac{f(u)-f(z)}{(u-z)^2}du, \quad (8)$$

Where $z'_\varepsilon:=e^{i(x+\varepsilon)}$ and $z''_\varepsilon:=e^{i(x-\varepsilon)}$. Furthermore,

$$\frac{\overline{zf(z)}}{\pi i}\int_{T/T(z,\varepsilon)}\frac{f(u)-f(z)}{(u-z)^2}du=-\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{2-2\overline{f(z)}f(u)}{\bar{z}(u-z)^2\bar{u}}|du|. \quad (9)$$

$$=\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)|^2-2\overline{f(z)}f(u)+|f(z)|^2}{|u-z|^2}|du|.$$

Thus combining (8) and (9) we obtain

$$\frac{\overline{zf(z)}}{\pi i}\int_{T/T(z,\varepsilon)}\frac{f'(u)}{u-z}du=\frac{\overline{zf(z)}}{\pi i}\left[\frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon-z}-\frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon-z}\right] \quad (10)$$

$$+\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)-f(z)|^2}{|u-z|^2}|du|+\frac{1}{\pi i}\int_{T/T(z,\varepsilon)}\frac{Im|f(u)\overline{f(z)}|}{|u-z|^2}|du|.$$

Assume now that f is differentiable at z. Then

$$\lim_{\varepsilon\to 0+}\left[\frac{f(z''_\varepsilon)-f(z)}{z''_\varepsilon - z}-\frac{f(z'_\varepsilon)-f(z)}{z'_\varepsilon - z}\right] = f''(z)-f'(z) = 0 \qquad (11)$$

as well as

$$\lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)-f(z)|^2}{|u-z|^2}|du| = \frac{1}{2\pi}\int_T \frac{|f(u)-f(z)|^2}{|u-z|^2}|du| < +\infty \qquad (12)$$

Thus combining (10) with (11) and (12) we obtain the assertion of the lemma, which ends the proof.

Given a continuous function $f{:}\,T \to C$ and $z \in T$ set

$$V[f](z) := \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{T/T(z,\varepsilon)}\frac{|f(u)-f(z)|^2}{|u-z|^2}|du|, \qquad (13)$$

$$V^*[f](z) := -\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{T/T(z,\varepsilon)}\frac{\left|f(u)\overline{f(z)}\right|}{|u-z|^2}|du|, \qquad (14)$$

provided the limits exist as well as $V[f](z) := +\infty \; and \; V^*[f](z) := 0$ otherwise.

Theorem (1.1.2)[1]: *If* $f \in Hom^+(T)$ is absolutely continuous on T, then for $a.e. z \in T$ the limit in (1) with f replaced by f' and the limits in (13) and (14) exist, and

$$2C_T[f'](z) = \bar{z} f(z)\big(V[f](z) + iV^*[f](z)\big). \qquad (15)$$

Proof. Since $f(T) = T$ is a rectifiable curve, it follows that f' is a Lebesgue integrable function on T. Then by [4] we see that the limit

$$\frac{1}{2\pi}\lim_{\varepsilon\to 0+}\int_{\varepsilon<|t-x|\le\pi} f'\left(e^{it}\right) cot\,\frac{x-t}{2}\,dt \qquad (16)$$

exists for a.e. $z = e^{ix} \in T$. Moreover, as shown in the proof of [9], the following equality

$$\frac{1}{2\pi i}\int_{T/T(z,\varepsilon)}\frac{f'(u)}{u-z}du \qquad (17)$$

$$\frac{1}{4\pi}\int_{\varepsilon<|t-x|\le\pi} f'\left(e^{it}\right)dt + \frac{i}{4\pi}\int_{\varepsilon<|t-x|\le\pi} f'\left(e^{it}\right) cot\,\frac{x-t}{2}\,dt$$

Holds for all $z = e^{ix} \in T$ and$\varepsilon \in (0;\pi]$. Thus the limit in (1) with f replaced by f' exists for a.e. $z \in T$ and the theorem follows directly from Lemma (1.1.1).

We recall that for each $K > 0$ the Hersch–Pfluger distortion function Φ_K is defined by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K), 0 < r < 1;\; \Phi_K(0) := 0;\; \Phi_K(1) := 1, \qquad (18)$$

Where μ stands for the module of the Grötzsch extremal domain $D/[0;r]$; cf. [5] and [7].

Lemma (1.1.3)[1]: For every $K \ge 1$ the following inequalities hold:

$$1 \le M_K := \frac{4}{\pi}\int_0^{1/\sqrt{2}}\left(\frac{\Phi_K(r)}{r}\right)^{1+1/K}\frac{dr}{\sqrt{1-r^2}} \le K^2 2^{5(1-1/K^2)/2} \qquad (19)$$

and

$$1 \ge L_K := \frac{4}{\pi}\int_0^{1/\sqrt{2}}\left(\frac{\Phi_{1/K}(r)}{r}\right)^{1+1/K}\frac{dr}{\sqrt{1-r^2}} \ge \frac{K2^{5(1-1/K^2)/(2K)}}{K^2+K-1}. \qquad (20)$$

In particular, $L_K - 1$ and $M_K \to 1 \; as \; K \to 1^+$.

Proof. Fix $K \geq 1$. Substituting $r := sin\ t$ we have

$$M_K = \frac{4}{\pi}\int_0^{\pi/4} \left(\frac{\Phi_K(\sin t)}{\sin t}\right)^{1+1/K} dt \tag{21}$$

and

$$L_K = \frac{4}{\pi}\int_0^{\pi/4} \left(\frac{\Phi_{1/K}(\sin t)}{\sin t}\right)^{1+1/K} dt. \tag{22}$$

Since $\sin t \geq (4t)/(\pi\sqrt{2})$ for$t \in [0; \pi/4]$, we conclude from the Hübner inequality (cf. [2] or [7])

$$r^{1/K} \leq \Phi_K(r) \leq 4^{1-1/K}\, r^{1/K}, \qquad 0 \leq r \leq 1, K \geq 1, \tag{23}$$

That

$$1 \leq \frac{\Phi_K(\sin t)}{\sin t} \leq 4^{1-1/K}\left(\frac{4t}{\pi\sqrt{2}}\right)^{1/K-1}, 0 < t \leq \pi/4.$$

This together with (21) yields (19). From (18) it follows that the composition $\Phi_K \circ \Phi_{1/K}$ is the identity function on [0; 1]. Hence and by (23),

$$r^K \geq \Phi_{1/K}(r) \geq 4^{1-K}\, r^K, \qquad 0 \leq r \leq 1, K \geq 1. \tag{24}$$

Using once more the estimation $sint \geq (4t)/(\pi\sqrt{2})$ for $t \in [0; \pi/4]$ we conclude from (24) that

$$1 \geq \frac{\Phi_{1/K}(sin\ t)}{sin\ t} \geq 4^{1-1/K}\left(\frac{4t}{\pi\sqrt{2}}\right)^{K-1}, 0 < t \leq \pi/4.$$

This together with (22) yields (20). From the estimations (19) and (20) it easily follows that $L_K \to 1$ and $M_K \to 1$ as $K \to 1^+$, which ends the proof.

Given a continuous function $f: T \to C$ and $z \in T$ set

$$f^+(z) := \sup_{u \in T/\{z\}} \left|\frac{f(u) - f(z)}{u - z}\right| \in [0; +\infty], \tag{25}$$

$$f^-(z) := \inf_{u \in T/\{z\}} \left|\frac{f(u) - f(z)}{u - z}\right| \in [0; +\infty). \tag{26}$$

Theorem (1.1.4)[1]: Given $K \geq 1$ and $F \in QC(D; K)$ let f be the boundary valued function of F. If $F(0) = 0$, then

$$L_K\left(f^-(z)\right)^{1-1/K} \leq V[f](z) \leq MK\left(f^+(z)\right)^{1-1/K}, \quad z \in T. \tag{27}$$

Proof. Since $F \in QC(D; K)$ and$F(0) = 0$, we see by the quasi-invariance of the harmonic measure that for every $t \in [\theta - \pi; \theta + \pi]$,

$$\Phi_{1/K}\left(\cos\frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right) \leq \cos\frac{t-\theta}{4} \leq \Phi_K\left(\cos\frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right); \tag{28}$$

see e.g. [8]. Applying now the identity ([2])

$$\Phi_K(r)^2 + \Phi_{1/K}\left(\sqrt{1-r^2}\right)^2 = 1, 0 \leq r \leq 1, \tag{29}$$

We obtain for every$t \in [\theta - \pi; \theta + \pi]$,

$$\Phi_{1/K}\left(\sin\frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right) \leq sin\frac{t-\theta}{4} \leq \Phi_K\left(\sin\frac{|\hat{f}(t) - \hat{f}(\theta)|}{4}\right). \tag{30}$$

Given $\theta \in R$ and $t \in [\theta - \pi; \theta + \pi]$ set $\alpha := (t - \theta)/2$ and $\beta := \left(\hat{f}(t) - \hat{f}(\theta)\right)/2$. Then $|\alpha| \leq \pi/2$ and $|\beta| \in \pi$. from (28) and (30) it follows that

$$|\alpha| \leq |\beta| \Longrightarrow 1 \leq \frac{\sin\frac{|\beta|}{2}}{\sin\frac{|\alpha|}{2}} \leq \frac{\Phi_K\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}} \text{and} \frac{\Phi_{1/K}\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}} \leq \frac{\cos\frac{|\beta|}{2}}{\cos\frac{|\alpha|}{2}} \leq 1$$

and

$$|\alpha| \geq |\beta| \Longrightarrow \leq \frac{\Phi_{1/K}\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}} \leq \frac{\sin\frac{|\beta|}{2}}{\sin\frac{|\alpha|}{2}} \leq 1 \text{and } 1 \leq \frac{\cos\frac{|\beta|}{2}}{\cos\frac{|\alpha|}{2}} \leq \frac{\Phi_K\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}.$$

Hence

$$\min\left\{\frac{\Phi_{1/K}\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}}, \frac{\Phi_{1/K}\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}\right\} \leq \left|\frac{\sin\beta}{\sin\alpha}\right| \leq \max\left\{\frac{\Phi_K\left(\sin\frac{|\alpha|}{2}\right)}{\sin\frac{|\alpha|}{2}}, \frac{\Phi_K\left(\cos\frac{|\alpha|}{2}\right)}{\cos\frac{|\alpha|}{2}}\right\}. \tag{31}$$

From [2] it follows that for any fixed $K \geq 1$, $(0;1] \ni t \longmapsto \Phi_K(t)\, t^{1/K}$ is a decreasing function and $(0;1] \ni t \longmapsto \Phi_{1/K}(t)\, t^{-K}$ is an increasing function. Then (31) yields

$$\frac{\Phi_{1/K}\left(sin\frac{|\alpha|}{2}\right)}{sin\ \frac{|\alpha|}{2}} \leq \left|\frac{sin\beta}{sin\alpha}\right| \leq \frac{\Phi_K\left(sin\frac{|\alpha|}{2}\right)}{sin\ \frac{|\alpha|}{2}}. \tag{32}$$

Fix $z = e^{i\theta} \in T$. If $f^+(z) = +\infty$, then the second inequality in (27) is obvious. So we may assume that $f^+(z) < +\infty$. Applying (32), (13) and (25) we obtain

$$\begin{aligned} V[f](z) &= \frac{1}{2\pi}\int_T \left|\frac{f(u) - f(z)}{u - z}\right|^2 |du| \\ &\leq \frac{1}{2\pi}\int_T \left(f + (z)\right)^{1-1/K} \left|\frac{f(u) - f(z)}{u - z}\right|^{1+1/K} |du| \\ &= (f^+(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi}\left(\frac{sin\frac{\hat{f}(t) - \hat{f}(\theta)}{2}}{sin\ \frac{t-\theta}{2}}\right)^{1+1/K} dt \\ &\leq (f^+(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi}\left(\frac{\Phi_K\left(sin\frac{t-\theta}{2}\right)}{sin\ \frac{|t-\theta|}{2}}\right)^{1+1/K} dt. \end{aligned} \tag{33}$$

Thus substituting $s := \frac{t-\theta}{4}$ and using (21) we derive the second inequality in (27). Applying now (32), (13), (26) and following calculations from (33) we obtain

$$V[f](z) \geq (f^-(z))^{1-1/K}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi}\left(\frac{\Phi_{1/K}\left(sin\frac{|t-\theta|}{4}\right)}{sin\ \frac{|t-\theta|}{4}}\right)^{1+1/K} dt.$$

Thus substituting $s := \frac{t-\theta}{4}$ and using (22) we derive the first inequality in (27), which completes the proof.

Lemma (1.1.5)[1]: Suppose that $f \in \mathrm{Hom}^+(T)$ is absolutely continuous on T. Then

$$\sup_{z\in T} f^+(z) = e_f := \operatorname{ess}\ \sup_{z\in T}|f'(z)|. \tag{34}$$

as well as

$$\inf_{z\in T} f^-(z) = d_f \coloneqq \operatorname{ess}\ \inf_{z\in T}|f'(z)|. \tag{35}$$

Proof. From (25) and (26) it follows that

$$f^-(z) \le |f'(z)| \le f^+(z) \tag{36}$$

for each $z \in T$ such that the limit (2) exists. Hence

$$\inf_{z\in T} f^-(z) \le d_f \le e_f \sup_{z\in T} f^+(z). \tag{37}$$

Assume now that f is absolutely continuous on T. If $e_f = +\infty$, then (37) yields (34). Thus we may confine considerations to the case $e_f < +\infty$. Then

$$\left|\hat f(t) - \hat f(x)\right| = \left|\int_x^t \hat f'(s)\, ds\right| \le e_f|t-x|, \qquad t, x \in R. \tag{38}$$

Fix $u = e^{it}$; $z = e^{ix} \in T$. Since $e_f \ge 1$ and the function sin is increasing and concave on $[0;\pi/2]$, we conclude from (38) that

$$|f(u) - f(z)| = 2\sin\left|\frac{\hat f(t) - \hat f(x)}{2}\right| \le 2\sin e_f\left|\frac{t-x}{2}\right| \le 2e_f\sin\left|\frac{t-x}{2}\right| = e_f|u-z|$$

Provided $|t-x| \le \pi/e_f$. $If\ \pi/e_f \le |t-x| \le \pi$, then

$$e_f\ |u-z| = 2e_f\sin\left|\frac{t-x}{2}\right| \ge 2e_f\frac{2}{\pi}\left|\frac{t-x}{2}\right| \ge 2 \ge |f(u) - f(z)|.$$

Thus

$$|f(u) - f(z)| \le e_f|u-z|, \qquad u, z \in T. \tag{39}$$

Combining (39) with (37) we obtain (34).

If $d_f = 0$, then (37) yields (35). So we may assume that $d_f > 0$. Then

$$\left|\hat f(t) - \hat f(x)\right| = \left|\int_x^t \hat f'(s)\, ds\right| \ge d_f|t-x|, t, x \in R,$$

and so the inverse mapping f^{-1} is also absolutely continuous on T. Then for a.e. $z \in T$, $(f^{-1})'(z) = 1/f'\left(f^{-1}(z)\right)$ and, in consequence, $e_{f^{-1}} = 1/d_f$. Applying now (39) with f replaced by f^{-1} we get for any $u, z \in T$,

$$\begin{aligned} d_f|u-z| &= d_f\left|f^{-1}\big(f(u)\big) - f^{-1}\big(f(z)\big)\right| \le d_f e_{f^{-1}} j|f(u) - f(z)| \\ &= |f(u) - f(z)|. \end{aligned} \tag{40}$$

Combining (40) with (37) we obtain (35), which completes the proof.

Corollary (1.1.6)[1]: Given $K \ge 1$ and $F \in QC(D;K)$ let f be the boundary valued function of F. If $F(0) = 0$ and f is absolutely continuous on T, then

$$L_K d_f^{1-1/K} \le V[f](z) = 2\,Re\left[z\overline{f(z)}C_T[f'](z)\right] \le M_K d_f^{1-1/K} \tag{41}$$

for a.e. $z \in T$, where M_K, L_K, e_f and df are defined by (19), (20), (34) and (35), respectively.

Proof. The corollary follows directly from Theorems (1.1.4) and (1.1.2) and Lemma (1.1.5).

We collect results that seem to be known. However, we prove them for the sake of completeness of our considerations.

Lemma (1.1.7)[1]: Given $K \geq 1$ and a domain Ω in C let $F \in QCH(D, \Omega; K)$. If Ω is bounded by a rectifiable Jordan curve Γ, then

$$\sup_{0<r<1} \int_{T_r} |\partial F(z)||dz| \leq \frac{K+1}{2} |\Gamma|_1, \tag{42}$$

And

$$\sup_{0<r<1} \int_{T_r} |\partial F(z)||dz| \leq \frac{K-1}{2} |\Gamma|_1, \tag{43}$$

Where $|\Gamma|_1$ is the length of Γ. In particular, $\partial F, \overline{\partial F} \in H^1(D)$.

Proof. Write f for the boundary valued function of F. Then

$$F(z) = P[f](z) := \int_0^{2\pi} f(e^{is}) \, P_r(t-s) ds, \quad z = re^{it} \in D, \tag{44}$$

Where

$$P_r(\theta) := \frac{1}{\pi} Re \frac{1 + re^{i\theta}}{1 - re^{i\theta}}, 0 \leq r < 1, \theta \in R, \tag{45}$$

is the Poisson kernel function. Since the function P_r is symmetric, we get

$$\frac{\partial}{\partial t} P_r(t-s) = -\frac{\partial}{\partial s} P_r(t-s), \quad t, s \in R.$$

Then integrating by parts we conclude from (44) that

$$\frac{\partial}{\partial t} F\left(re^{it}\right) = \int_0^{2\pi} f\left(e^{is}\right) \frac{\partial}{\partial t} P_r(t-s) ds = -\int_0^{2\pi} f\left(e^{is}\right) \frac{\partial}{\partial s} P_r(t-s) ds \tag{46}$$

$$= \int_0^{2\pi} P_r(t-s) df\left(e^{is}\right), \ 0 \leq r < 1, t \in R,$$

Because the function $s \longmapsto f(e^{is})$ is of bounded variation on $[0; 2\pi]$; the last integral is regarded as the Stieltjes one. Fix $r \in (0; 1)$. Then by (46),

$$\frac{\partial}{\partial t} F\left(re^{it}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} P_r \left(t - 2\pi k/n\right)\left[f\left(e^{2\pi k i/n}\right) - f\left(e^{2\pi(k-1)i/n}\right)\right], t \in R. \tag{47}$$

Hence, applying Fatou's limiting integral lemma, we obtain

$$\int_0^{2\pi} \left|\frac{\partial}{\partial t} F\left(re^{it}\right)\right| dt$$

$$= \int_0^{2\pi} \left|\lim_{n\to\infty} \sum_{k=1}^{n} P_r \left(t - 2\pi k/n\right)\left[f\left(e^{2\pi k i/n}\right) - f\left(e^{2\pi(k-1)i/n}\right)\right]\right| dt \tag{48}$$

$$\leq \liminf_{n\to\infty} \int_0^{2\pi} \left|\sum_{k=1}^{n} P_r(t - 2\pi k/n) \left[f\left(e^{2\pi k i/n}\right) - f\left(e^{2\pi(k-1)i/n}\right)\right]\right| dt.$$

Since

$$\int_0^{2\pi}\left|\sum_{k=1}^{n}P_r(t-2\pi k/n)\left[f\left(e^{2\pi ki/n}\right)-f\left(e^{2\pi(k-1)i/n}\right)\right]\right|dt$$
$$\leq\sum_{k=1}^{n}\left[\left|f\left(e^{2\pi ki/n}\right)-f\left(e^{2\pi(k-1)i/n}\right)\right|\int_0^{2\pi}|P_r(t-2\pi k/n)|dt\right]$$
$$\leq\sum_{k=1}^{n}\left|f\left(e^{2\pi ki/n}\right)-f\left(e^{2\pi(k-1)i/n}\right)\right|\leq|\Gamma|_1$$

and since for $z=re^{it}$,

$$\frac{\partial}{\partial t}F\left(re^{it}\right)=i\left[z\partial F(z)-\bar{z}\bar{\partial}F(z)\right], \tag{49}$$

we conclude from (48) that

$$\int_{T_r}(|\partial F(z)|-\bar{\partial}F(z))|dz|\leq\int_0^{2\pi}\left|z\partial F(z)-\bar{z}\bar{\partial}F(z)\right|dt\leq|\Gamma|_1. \tag{50}$$

By the assumption, the mapping F is K-quasiconformal, which means that

$$(K+1)\left|\bar{\partial}F(z)\right|\leq(K-1)|\partial F(z)|, z\in D. \tag{51}$$

Hence by (50),

$$\int_{T_r}\left(\left|\bar{\partial}F(z)\right|+|\partial F(z)|\right)|dz|\leq K\int_{T_r}\left(\partial|F(z)|-\left|\bar{\partial}F(z)\right|\right)|dz|\leq K|\Gamma|_1.$$

Combining this with (50) and (51) leads to (42) and (43).

Corollary (1.1.8)[1]: Given $K\geq 1$ and a domain Ω in C let $F\in QCH(D,\Omega;K)$. If Ω is bounded by a rectifiable Jordan curveΓ, then the boundary valued function $f$of F is absolutely continuous.

Proof. From Lemma (1.1.7) it follows that $\partial F,\overline{\partial F}\in H^1(D)$. The classical result of Riesz [3] says that there exist functions $H,G:\overline{D}\to C$ continuous on $\overline{D}$, holomorphic on $\overline{D}$ and absolutely continuous on T and such that $H'(z)=\partial F(z)$ and $G'(z)=\overline{\partial F}(z)$, $z\in D$, i.e. H and G are primitive functions to ∂F and $\overline{\partial F}$ on D, respectively. Moreover, F has a continuous extension to $\overline{D}$. Hence for each $z\in T$,

$$f(z)-F(0)=\int_\gamma\partial F(u)du+\bar{\partial}F(u)\overline{du} \tag{52}$$
$$=\int_\gamma H'(u)du+\overline{G'(u)du}=H(z)-H(0)+\overline{G(z)-G(0)},$$

where $\gamma(t):=tz,t\in[0;1]$. From (52) we see that $f(z)=H(z)+\overline{G(z)}+F(0)-H(0)-\overline{G(0)}$ for $z\in T$. Thus f is an absolutely continuous function on T.

Modifying the proof of Lemma (1.1.7) we may easily derive the following lemma.

Lemma (1.1.9)[1]: Given $K\geq 1$ and a Jordan domain Ω in C let $F\in QCH(D,\Omega;K)$.
If the boundary valued function f of F satisfies the inequality

$$|f(u)-f(v)|\leq L|u-v|,\quad u,v\in T, \tag{53}$$

for some positive constant L, then

$$\sup_{z\in D}|\partial F(z)|\leq\frac{K+1}{2}L, \tag{54}$$

And

$$\sup_{z\in D}\left|\bar{\partial}F(z)\right| \leq \frac{K-1}{2}L. \tag{55}$$

In particular, $\partial F, \overline{\partial F} \in H^{\infty}(D)$.

Proof. From (53) it follows that Ω is bounded by a rectifiable Jordan curveΓ. Hence the function $s \longmapsto f(e^{is})$ is of bounded variation on $[0;2\pi]$ and, as in the proof of Lemma (1.1.7), the equality (47) holds. From (53) it also follows that for all $n \in N$ and $k = 1,2,\dots,n$,

$$\left|f\left(e^{2\pi ki/n}\right) - f\left(e^{2\pi(k-1)i/n}\right)\right| \leq L\left|e^{2\pi ki/n} - e^{2\pi(k-1)i/n}\right| \tag{56}$$
$$= \frac{2\pi}{n} = L(n/\pi)\sin(\pi/n).$$

Fix $r \in (0;1)$. Since$(n/\pi)\sin(\pi/n) \to 1$ $as\ n \to \infty$, we conclude from (47) and (56) that for every $t \in R$,

$$\left|\frac{\partial}{\partial t}F(re^{it})\right| \leq L \lim_{n\to\infty} L(n/\pi)\sin(\pi/n)\sum_{k=1}^{n} P_r\,(t-2\pi k/n)\frac{2\pi}{n} = L\int_0^{2\pi} P_r\,(t-s)ds$$
$$= L. \tag{57}$$

Since for $z = re^{it}$ the equality (49) holds, we conclude from (57) that

$$r\left(|\partial F(z)| - \left|\bar{\partial}F(z)\right|\right) \leq \left|z\partial F(z) - \bar{z}\,\bar{\partial}F(z)\right| \leq L. \tag{58}$$

By the assumption, the mapping F is K-qc., which means that (51) holds. Hence by (58),

$$r\left(\bar{\partial}|F(z)| - |\partial F(z)|\right) \leq rK\left(|\partial F(z)| - \left|\bar{\partial}F(z)\right|\right) \leq KL. \tag{59}$$

Combining the inequalities (58) and (59) with (51) we obtain the inequalities (54) and (55), because both the functions $\overline{\partial F}$ and ∂F are holomorphic on D.

Theorem (1.1.10)[1]: Given $K \leq 1$ and $F \in QCH(D;K)$ let f be the boundary valued function of F. Then for a.e. $z \in T$,

$$\left|V[f](z) + iV^*[f](z) - \frac{1}{2}\left(K + \frac{1}{K}\right)f'(z)\right| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)|f'(z)|. \tag{60}$$

In particular, for $a.e.\ z \in T$,

$$\frac{1}{K}\left||f'(z)| \leq V[f](z) \leq K|f'(z)| \text{ and } |V^*[f](z)| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)f'(z)\right|. \tag{61}$$

Proof. From [11] it follows that $f'(z) \neq 0$ for a.e. $z \in T$. By Corollary (1.1.8), f is absolutely continuous on$T$. Hence and by [9] we obtain

$$(K+1)\left|1 - 2\frac{C_T[f'](z)}{f'(z)}\right| \leq (K-1)\left|1 - 2\frac{C_T[f'](z)}{f'(z)}\right| \quad \text{for } a.e. z \in T,$$

which leads to

$$\left|2\frac{C_T[f'](z)}{f'(z)} - \frac{1}{2}\left(K + \frac{1}{K}\right)\right| \leq \frac{1}{2}\left(K - \frac{1}{K}\right) \text{ for } a.e. z \in T. \tag{62}$$

From Theorem (1.1.2) and (5) it follows that for a.e. $z \in T$,

$$2\frac{C_T[f'](z)}{f'(z)} = \frac{\bar{z}f(z)}{f'(z)}\left(V[f](z) + iV^*[f](z)\right) = \frac{1}{|f'(z)|}\left(V[f](z) + iV^*[f](z)\right)$$

This combined with (62) yields (60)). The inequalities (61) follow directly from (60)), which ends the proof.

Theorem (1.1.11)[1]: Given $K \geq 1$ and $F \in QCH(D;K)$ let f be the boundary valued function of F. If $F(0) = 0$, then for a.e. $z \in T$,

$$\frac{2^{5(1-K^2)/2}}{(K^2+K-1)^K} \le (L_K/K)^K \le |f'(z)| \le (M_K K)^K \le K^{3K} 2^{5(K-1/K)/2}, \qquad (63)$$

where M_K and L_K are defined by (19) and (20), respectively.

Proof. By Corollary (1.1.8), f is absolutely continuous on T. Then Corollary (1.1.6) and the first inequality in (61) show that for a.e. $z \in T$, $L_K d_f^{1-1/K} \le K|f'(z)|$ and $|f'(z)| \le KM_K e_f^{1-1/K}$, where e_f and d_f are defined by (34) and (35), respectively.

Hence $L_K d_f^{1-1/K} \le K d_f$ and $e_f \le KM_K e_f^{1-1/K}$, and consequently, we obtain the following implications

$$[0 < d_f \Rightarrow (L_K/K)^K \le d_f] \text{ and } [e_f < +\infty \Rightarrow e_f \le (M_K K)^K]. \qquad (64)$$

For any $n \in N$ let $D_n := \{z \in C: |z| < n/(n+1)\}$ and φ_n be the conformal mapping from D onto $F^{-1}(D_n)$ such that $\varphi_n(0) = 0$ and $\varphi_n'(0) > 0$. Then $F_n := (1+1/n)F \circ \varphi_n \in QCH(D;K)$ and $F_n(0) = 0, n \in N$. Fix $n \in N$. Since F is a C^2-diffeomorphic self-mapping of D we see that $F^{-1}(D_n)$ is a domain bounded by a C^2-Jordan curve. Applying Kellogg–Warschawski theorem ([16], [17]) we see that φ_n' has a continuous extension ψ_n to the closed disk $\bar{D}$ and $\psi_n(z) \ne 0$ for all $z \in \bar{D}$. Thus the boundary valued function f_n of F_n is a C^1-diffeomorphic self-mapping of T, and so $0 < d_{f_n} \le e_{f_n} < +\infty$. By (64) and Lemma (1.1.5) we see that for all $u, z \in T, u \ne z$,

$$(L_K/K)^K \le d_{f_n} \le \frac{|f_n(u) - f_n(z)|}{|u-z|} \le e_{f_n} \le (M_K K)^K, n \in N. \qquad (65)$$

Setting $F_0 := F$ we conclude from [7] that

$$|F_n(z) - F_n(\omega)| \le 16|z-\omega|^{1/K}, z, \omega \in \bar{D}, n = 0,1,2,\dots.$$

Hence for all $z \in T$ and $\omega \in D$,

$$|f_n(z) - f(z)| \le 32|z-\omega|^{1/K} + 16|\varphi_n(\omega) - \omega|^{1/K} + \frac{1}{n}, n \in N. \qquad (66)$$

From [15] it follows that $\varphi_n'(\omega) \to \omega$ as $n \to \infty$ for each $\omega \in D$. Thus given $\varepsilon > 0$ and $z \in T$ we can choose $\omega \in D$ and $\varepsilon_n \in N$ such that the right hand side in (66) is less than ε as $n > n_\varepsilon$. This means that for every $z \in T, f_n(z) \to f(z) as\ n \to \infty$. Since f is absolutely continuous on T, (65) and Lemma (1.1.5) then show that $(L_K/K)^K \le d_f \le e_f \le (M_K K)^K$. This and Lemma (1.1.3) yield (63), which ends the proof.

Theorem (1.1.12)[1]: Given $K \ge 1$ and $F \in QCH(D;K)$ assume that $F(0) = 0$.
Then for all $z, \omega \in D$,

$$|F(z) - F(\omega)| \le K(M_K K)^K |z-\omega| \le K^{3K+1} 2^{5(K-1/K)/2} |z-\omega| \qquad (67)$$

As well as

$$|F(z) - F(\omega)| \ge \frac{L_K^{3K}}{K^{4K+1} M_K^K} |z-\omega| \ge \frac{2^{5(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2+K-1)^{3K}} |z-\omega|, \qquad (68)$$

Where M_K and L_K are defined by (19) and (20), respectively.

Proof. Fix $z, \omega \in D$. Setting $\gamma(t) := z + t(\omega - z), t \in [0,1]$, we get

$$|F(z) - F(\omega)| = \left| \int_0^1 \frac{d}{dt} F(\gamma(t)) dt \right|$$

$$= \left| \int_0^1 \partial F(\gamma(t)) + \gamma'(t) + \bar{\partial} F(\gamma(t)) \overline{\gamma'(t)} dt \right| \qquad (69)$$

$$\leq \int_0^1 \left(\left|\partial F(\gamma(t))\right| + \gamma'(t) + \left|\bar{\partial} F(\gamma(t))\right|\right) dt\, |z-\omega|$$
$$\leq \sup_{u\in D}\left(|\partial F(u)| + \left|\bar{\partial} F(u)\right|\right) |z-\omega|.$$

From Corollary (1.1.8) and Lemmas (1.1.5) and (1.1.9) it follows that

$$\sup_{u\in D}\left(|\partial F(u)| + \left|\bar{\partial} F(u)\right|\right) |z-\omega| \leq K e_f, \tag{70}$$

Where f is the boundary valued function of F. Combining (69) and (70) we conclude from Theorem (1.1.11) that the estimation (67) holds. Setting now $\gamma^\circ(t) := F^{-1}(z + t(\omega - z)), t \in [0,1]$, We get

$$|z-\omega| = \int_0^1 \left|\frac{d}{dt} F(\gamma(t))\right| dt = \int_0^1 \left|\partial F(\gamma(t)) + \gamma'(t) + \bar{\partial} F(\gamma(t))\overline{\gamma'(t)}\right| dt$$
$$\geq \int_0^1 \left(\left|\partial F(\gamma(t))\right| |\gamma'(t)| - \left|\bar{\partial} F(\gamma(t))\right| \left|\overline{\gamma'(t)}\right|\right) dt \tag{71}$$
$$\geq \inf_{u\in D}\left(|\partial F(u)| - \left|\bar{\partial} F(u)\right|\right) \int_0^1 |\gamma'(t)|\, dt$$
$$\geq \inf_{u\in D} \frac{|\partial F(u)|^2 - \left|\bar{\partial} F(u)\right|^2}{|\partial F(u)| + \left|\bar{\partial} F(u)\right|} |F^{-1}(z) - F^{-1}(\omega)|.$$

From [12] it follows that $|\partial F(u)|^2 - \left|\bar{\partial} F(u)\right|^2 \geq d_f^3$for all $u \in D$.
Hence and by (71) and (70) we get

$$|\, F(z) - F(\omega)| \geq \frac{d_f^3}{K e_f} |z-\omega|. \tag{72}$$

Applying now Theorem (1.1.11) we obtain the estimation (68), which ends the proof.
Applying a variant of Heinz's inequality from [13] we derive an alternative estimation to (68) like below.

Theorem (1.1.13)[1]: Given $K \geq 1$ and $F \in QCH(D; K)$ assume that $F(0) = 0$.
Then for all$z, \omega \in D$,

$$|F(z) - F(\omega)| \geq \frac{1}{K} \max\left\{\frac{2}{\pi}, L_K^*\right\} |z-\omega|, \tag{73}$$

Where

$$L_K^* := \frac{2}{\pi} \int_0^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{dt}{\Phi_K(\sqrt{t})\Phi_{1/K}(\sqrt{1-t})}. \tag{74}$$

Proof. From (71), (51) and [13] we see that

$$|z-\omega| \geq \inf_{u\in D}\left(|\partial F(u)| - \left|\bar{\partial} F(u)\right|\right) \int_0^1 |\gamma'(t)|\, dt \geq \frac{2}{K+1} \inf_{u\in D} |\partial F(u)| |F^{-1}(z) - F^{-1}(\omega)|$$
$$\geq \frac{1}{K} \max\left\{\frac{2}{\pi}, L_K^*\right\} |F^{-1}(z) - F^{-1}(\omega)|,$$

which leads to (73).

Section (1.2): Harmonic Quasiconformal Mappings

It is convenient to give a few comments about the notation.

Let $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$denote the unit disc.

We write qc, qr,ρ-qch, and ρ-qrh instead of quasiconformal, quasiregular, ρ −harmonic quasiconformal and ρ-harmonic quasiregular, respectively and e-qch, h-qch instead of euclidean and hyperbolic harmonic quasiconformal. Basic definitions will be given.

The Schwarz lemma attracted a lot of attention and found numerous applications in geometric function theory.

It seems that investigations concerning the Schwarz lemma have been primarily concerned with the following question:

For our purpose the following are relevant.

Lemma (1.2.1)[19]: If $\rho > 0$ is a C^2 function (metric density)on $\mathbb{U}$ and the Gaussian curvature satisfies $K_\rho \leq -1$, then $\rho \leq \lambda$.

Sometimes we refer to this result as the Ahlfors–Schwarz lemma.

In [30], Yau mentioned that in order to draw a useful conclusion in the case of harmonic mappings between Riemannian manifolds, one has to assume the mapping is quasiconformal. Wan [29] was the first one who showed a result in a special situation concerning Yau's suggestion:

Lemma (1.2.2)[19]: (Wan). Every hyperbolic harmonic quasiconformal diffeomorphism from $\mathbb{U}$ onto itself is a quasi-isometry of the Poincaré disc.

In particular, the method of the proof is interesting. It provides at least a partial motivation to study this approach and raises the following question:

[24], proved an inequality of opposite type of the Ahlfors–Schwarz lemma:

Lemma (1.2.3)[19]: If $H > 0$ is a C^2 metric density on $\mathbb{U}$ for which the Gaussian curvature satisfies $K_H \geq -1$and if $H(z)$ tends to$+\infty$, when $|z|$ tends to $1-$, then $\lambda \leq H$.

We will use this lemma together with the Ahlfors–Schwarz lemma.

We prove an analogue of the Lemma (1.2.2) holds for quasiconformal euclidean harmonic mappings and we generalize it to quasiregular harmonic mappings with respect to the metric ρ, whose curvature is bounded from above by a negative constant.

It is interesting that we have a similar estimate of the hyperbolic distance for qc euclidean harmonic mappings and harmonic mappings with respect to the Poincaré metric, which are different in many respects.

Let f be aK-qc euclidean harmonic diffeomorphism from a domain D on to itself. We show that f is a$(1/K, K)$ quasi-isometry with respect to the Poincaré distance in the case where D is the isc or the upper-half plane. We refer to these results as the unit disc and the half plane euclidean-qch versions, respectively.

The proofs of these cases cannot be transferred to one another using conformal mappings because the euclidean metric is not invariant under them.

Theorem (1.2.4)[19]: (The half plane and the unit disc e-qch versions). Let f be aK-qc euclidean harmonic diffeomorphism from the upper half plane H(or the unit disc) onto itself. Then f is $a(1/K, K)$ quasi-isometry with respect to the Poincaré distance.

It is interesting that we use completely different techniques for the disc and the half plane. In the case of the unit disc we use a curvature estimate (see below). In the case of the upper half plane, the following known fact plays an important role:

Lemma (1.2.5)[19]: Let f be an euclidean harmonic 1-1 mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on$\overline{\mathbb{H}}$, normalized by $f(\infty) = \infty$and $v = Imf$. Then $v(z) = cIm\, z$, where c is apositive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.
This lemma is a corollary of the Herglotz representation of the positive harmonic function v(see for example [21]).
For information regarding the quasi-isometries, with respect to the hyperbolic metric for e-qch mappings with general codomains, see [27].
We extend Wan's result to qrρ-harmonic mappings:
Theorem (1.2.6)[19]: (ρ-qrh version). Let R be a hyperbolic surface with the Poincaré metric density λ,S a hyperbolic surface with metric density and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$be uniformly bounded from above on S by the negative constant $-a$. Then any ρ-harmonic k-quasiregular map f from R into S decreases distances up to a constant de-pending only on a and k.
The basic properties of ρ-harmonic functions will be briefly discussed.
A proof of the above result can be based on an application of the uniformization theorem with the fact that $\rho_0 = a(1-k^2)\rho \circ f|f_z|^2$ is an ultrahyperbolic metric density.
Using the conformal automorphisms $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{U}$,of $\mathbb{U}$, one can define the pseudo-hyperbolic distance on $\mathbb{U}$ by $\delta(a,b) = |\phi_a(b)|, a, b \in \mathbb{U}$.
The hyperbolic metric on the unit disc$\mathbb{U}$is$\lambda|dz|^2$, where
$\lambda(z) = \left(\frac{2}{1-|z|^2}\right)^2$.
We say that $\lambda$is the hyperbolic metric density. The hyperbolic distance on the unit disc $\mathbb{U}$ is

$$d_\lambda(z,\omega) = ln\frac{1+\delta(z,\omega)}{1-\delta(z,\omega)} = ln\frac{1+\left|\frac{z-\omega}{1-\bar{z}\omega}\right|}{1-\left|\frac{z-\omega}{1-\bar{z}\omega}\right|}.$$

We also use the notation $d_h$instead of d_λ.
The classical Schwarz lemma states: If $f: U \to U$ is an analytic function and if $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. Equality$|f(z)| = |z|$, with $z \neq 0$, or$|f'(0)| = 1$ can occur only for $f(z) = e^{i\alpha}z$, where α is a real constant.
It was noted by Pick that the result can be expressed in invariant form. See following result as the Schwarz–Pick lemma.
Theorem (1.2.7)[19](Schwarz–Pick lemma). Let F be an analytic function from the unit disc into itself. Then F does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.
A Riemannian metric given by the fundamental form

$$ds^2 = \rho(dx^2 + dy^2) = \rho|dz|^2$$

or $ds = \sqrt{\rho}|dz|, \rho > 0$, is conformal to the euclidean metric. We call ρ a metric density (scale) and denote by d_ρ the corresponding distance.
If $\rho > 0$ is a C^2 function on U, the Gaussian curvature of a Riemannian metric with density ρ on U is expressed by the formula

$$K = K_\rho = -\frac{1}{2\rho} ln\, \rho.$$

We also write $K(\rho)$ instead of K_ρ.If$s > 0$ is a constant, it is clear that $K(s\rho) = s^{-1}K(\rho)$.
A metric $\rho|dz|^2, \rho \ge 0$, is said to be ultrahyperbolic in a region $\Omega \subset \mathbb{C}$ if it has the following properties:

(a) ρ is upper semicontinuous; and
(b) at every z_0 with$\rho(z_0) > 0$ there exists a supporting metric density ρ_0,of class C^2 in a neighborhood $V$of z_0, such that $\rho_0 \leq \rho$ and $K_{\rho_0} \leq -1$inV, while $\rho_0(z_0) = \rho(z_0)$.
If a metric $\rho|dz|^2$ is ultrahyperbolic in a region $\Omega \subset \mathbb{C}$ we say that ρ is an ultrahyperbolic metric density.
Ahlfors (see [20]) proved a stronger version of the Schwarz–Pick lemma and of the Ahlfors–Schwarz lemma.
Theorem (1.2.8)[19] (Ahlfors–Schwarz lemma). Suppose ρ is an ultrahyperbolic metric on the unit disc$\mathbb{U}$.Then $\rho \leq \lambda$.

Sometimes we refer to this result as the Ahlfors–Schwarz lemma or the non-analytic form of the Schwarz lemma. If we wish to be more specific, we refer to this result as the Ahlfors ultrahyperbolic lemma.

Now, we can state Theorem (1.2.8) in the following form: If ρ is a metric density on $\mathbb{U}$ such that $K_\rho(z) \leq -a$, for some $a > 0$, then the metric a ρ is ultrahyperbolic and therefore a$\rho \leq \lambda$.
The notation of an ultrahyperbolic metric makes sense and the theorem remains valid if Ω is replaced by a Riemann surface.
In a plane region Ω whose complement has at least two points, there exists a unique maximal ultrahyperbolic metric and this metric has constant curvature of -1.
The maximal metric density is called the Poincarémetric (density) in Ω and we denote it by λ_Ω. It is maximal in the sense that every ultrahyperbolic metric density ρ satisfies $\rho \leq \lambda_\Omega$ throughout Ω.

Ultrahyperbolic metrics (without the name) were introduced by Ahlfors. They found many applications in the theory of several complex variables.
Let R and S be two surfaces. Let $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$ be metrics with respect to the isothermal coordinate charts on R and S, respectively, and let f be a C^2-map from R to S.
It is convenient to use the notation in local coordinates:

$$df = pdz + qd\bar{z}, \text{ where } p = f_z \text{ and } q = f_{\bar{z}} .$$

We also introduce the complex (Beltrami) dilatation

$$\mu f = Belt[f] = \frac{q}{p},$$

where it is defined.
We say that a C^2-map f from R to S is ρ-harmonic (harmonic with respect to the metric density P or, shortly, harmonic) if f satisfies the following equation:
$f_{\bar{z}z} + (log\rho)_w \circ f_{pq} = 0$.
For basic properties of harmonic maps and for further information see Jost [22] and Schoen and Yau [28].

Note that if R and S are domains in the complex plane and if σ and ρ are the euclidean metric densities (that is $\sigma = \rho = 1$), then f is euclidean harmonic.
If $f: \mathbb{U} \to \mathbb{U}$is a λ-harmonic mapping, we call f a hyperbolic harmonic or a harmonic mapping with respect to the Poincaré metric.

Let R and S be two Riemann surfaces and $f: R \to S$ be a C^2-mapping. If P is a point on $R, \bar{P} = f(P) \in S, \phi$ a local parameter on R defined near P and ψ a local parameter on S defined near $\bar{P}$, then the map $w = h(z)$ defined by$h = \psi \circ f \circ \phi^{-1}|_V$($V$is a sufficiently small neighborhood of P) is called a local representation of $f$at P.The map f is called k-

quasiregular if there is a on stant $k \in (0,1)$ such that for every representation h, at every point of R, $|h_{\bar{z}}| k |h_z|$.
If a k-qr apping is one-to-one, we call it a k-qc mapping. Also, if f is a k-qc mapping, we use the notation $K = \frac{1+k}{1-k}$, and we also write that $f$is K-qc.

We write:
$L_f = L_f(z) = |f_z(z)| + |f_{\bar{z}}(z)|$ and$l_f = l_f(z) = |f_z(z)| - |f_{\bar{z}}(z)|$,
If $f_z(z)$ and$f_{\bar{z}}(z)$ exist.
The following statements are useful in applications:

$$\frac{L_f(z)}{1-|f(z)|^2} \leq c_1 \frac{1}{1-|z|^2}, z \in \mathbb{U}, \tag{75}$$

Then $d_h(f(z_1), f(z_2)) \leq c_1 d_h(z_1, z_2)$.

$$\frac{l_f(z)}{1-|f(z)|^2} \geq c_2 \frac{1}{1-|z|^2}, z \in \mathbb{U}, \tag{76}$$

Then $d_h(f(z_1), f(z_2)) \geq c_2 d_h(z_1, z_2)$.
The proofs are straightforward. Note that in the proof of 3A it is convenient to consider the hyperbolic geodesic joining $z_1$and z_2 and in the proof of 3B the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$.
Proposition (1.2.9)[19]: (The unit disc euclidean-qch version). Let f be ak-quasiconformal Euclidean harmonic apping from the unit disc $\mathbb{U}$ into itself. Then for all $z \in \mathbb{U}$ we have

$$f_z(z) \leq \frac{1}{1-k} \frac{1-|f(z)|^2}{1-|z|^2}.$$

Notice that as a corollary we get $(1-|z|^2) L_f(z) 4K d_f(z)$, where $d_f(z) = dist(f(z), \partial U)$.
Proof. Let us define $\sigma(z) = (1-k)^2 \lambda(f(z)) |f_z(z)|^2, z \in \mathbb{U}$. Since f is harmonic in $\mathbb{U}$,i.e. $f_{\bar{z}z}(z) = 0, z \in \mathbb{U}$, then f_z is holomorphic in $\mathbb{U}$. By Lewy's theorem f_z does not vanish and hence the mapping $z \longmapsto log|f_z(z)|$ is harmonic in $\mathbb{U}$. Therefore, $(\Delta log\sigma)(z) = (\Delta log(\lambda \circ f))(z)$,for all $z \in \mathbb{U}$. A straightforward calculation gives

$$\begin{aligned}(\Delta log\sigma)(z) = log(\lambda \circ f)(z) &= 4(log(\lambda \circ f))_{\bar{z}z}(z) \\ &= \frac{8|f_z(z)|^2}{(1-|f(z)|^2)^2}\left(1 + |\mu(z)|^2 + 2Re\left(\frac{(f(z))^2 \overline{f_z(z) f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right)\right) \\ &= \frac{2\sigma(z)}{(1-k)^2}\left(1 + |\mu(z)|^2 + 2Re\left(\frac{(f(z))^2 \overline{f_z(z) f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right)\right).\end{aligned}$$

Hence, the Gaussian curvature of the conformal metric $ds^2 = \sigma(z)|dz|^2$ satisfies

$$K(\sigma)(z) = -\frac{1}{(1-k)^2}\left(1 + |\mu(z)|^2 + 2Re\left(\frac{(f(z))^2 \overline{f_z(z)} \overline{f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right)\right) \tag{77}$$

for all $z \in \mathbb{U}$. On the other hand we have

$$\left|Re\left(\frac{(f(z))^2 \overline{f_z(z)} \overline{f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right)\right| \leq \left|Re\left(\frac{(f(z))^2 \overline{f_z(z)} \overline{f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right)\right| \leq |\mu(z)|, \tag{78}$$

so we obtain

$$Re\left(\frac{(f(z))^2\overline{f_z(z)f_{\bar{z}}(z)|}}{|f_z(z)|^2}\right) \geq -|\mu(z)|.$$

Therefore,

$$K(\sigma)(z) \leq -\frac{1}{(1-k)^2}1+|\mu(z)|^2-2|\mu(z)| = -\frac{(1-|\mu(z)|)^2}{(1-k)^2} \leq -1,$$

and hence, using the Ahlfors–Schwarz lemma, we get $\sigma(z) \leq \lambda(z), z \in \mathbb{U}$, or equivalently

$$(1-k)^2\lambda\big(f(z)\big)|f_z(z)|^2 \leq \lambda(z) \qquad (79)$$

for all $z \in \mathbb{U}$. Now, the claim follows easily from (79).

Theorem (1.2.10)[19]: Let f be ak-quasiconformal euclidean harmonic mapping from the unit disc$\mathbb{U}$ into itself. Then for any two points $z_1$and $z_2$in $\mathbb{U}$ we have

$$d_h f(z_1), f\ (z_2) \leq \frac{1+k}{1-k} d_h(z_1, z_2),$$

where d_h is the hyperbolic distance function induced by the hyperbolic metric in $\mathbb{U}$.

Note that this statement follows from Proposition (1.2.9).

Notice that, in order to get the opposite inequality in Proposition (1.2.9), we need to assume that f is onto.

Theorem (1.2.11)[19]: Let f be ak-quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ onto itself. Then for all $z \in \mathbb{U}$ we have

$$|f_z(z)| \geq \frac{1}{1+k}\frac{1-|f(z)|^2}{1-|z|^2}$$

and $d_h f(z_1), f(z_2)) \geq \frac{1-k}{1+k} d_h(z_1, z_2)$.

Proof. By (77) and (78)

$$K(\sigma\ )(z) \geq -\frac{1}{(1-k)^2}(1+|\mu(z)|^2+2|\mu(z)|) = -\frac{(1+|\mu(z)|)^2}{(1-k)^2} \geq -K^2.$$

In [26], it has been proved that there is a constant $c > 0$ such that$|f_z| \geq c$ on $\mathbb{U}$. Hence, σ tends to $+\infty$, when $|z|$ tends to $1-$. Thus, by Lemma (1.2.3),$(1+k)^2\lambda(f(z))|f_z(z)|^2 \geq \lambda(z)$ and therefore, since $l_f(1-k)|f_z(z)|$,wehave $K^2\lambda(f(z))l_f^2 \geq \lambda$,i.e. $K\hat{\lambda}(f(z))l_f \geq \hat{\lambda}$, where $\hat{\lambda} = \sqrt{\lambda}$.

Now, an application of 3B immediately yields $d_h f(z_1), f\ (z_2) \geq \frac{1-k}{1+k} d_h(z_1, z_2)$.

For $a \in \mathbb{C}$ and $r > 0$ we define $B(a;r)z: |z-a| < r\}$. In particular, we write$\mathbb{U}_r$ instead of $B(0;r)$.

Theorem (1.2.12)[19] (The half plane euclidean-qch version). Let f be K-qc euclidean harmonic diffeo-morphism rom $\mathbb{H}$ onto itself. Then f is$(1/K, K)$ quasi-isometry with respect to the Poincaré distance.

We first show that, by precomposition with a linear fractional transformation, we can reduce the proof to the case $f(\infty) = \infty$.If $f(\infty) \neq \infty$, there is a real number a such that $f(a) = \infty$.

On the other hand, there is a conformal automorphism A of $\mathbb{H}$ such that $A(\infty) = a$. Since A is an isometry of $\mathbb{H}$ onto itself and $f \circ A$ is a K-qc euclidean harmonic diffeomorphism from $\mathbb{H}$ onto it self, the proof is reduced to the case $f(\infty) = \infty$.

It is well known that f has a continuous extension to$\overline{\mathbb{H}}$ (see [7]).

Hence, by Lemma (1.2.5),$f = u + ic\ Imz$, where c is a positive constant. Using the linear mapping B, defined by $B(w) = w/c$, and a similar consideration as the above, we can reduce the proof to the case $c = 1$. Therefore we can write f in the form $f = u + iImz =$

$\frac{1}{2}(F\ \ (z)+z+\overline{F(z)-z})$, where F is a holomorphic function in $\mathbb{H}$. Hence, $\mu_f(z)=\frac{F'(z)-1}{F'(z)+1}$ and $F'(z)=\frac{1+\mu_f(z)}{1-\mu_f(z)}, z\in\mathbb{H}$.
Define $w=S(\zeta)=\frac{1+\zeta}{1-\zeta}$. Then, $S(U_k)=B_k=B(a_k;R_k)$, Where $a_k=\frac{1}{2}(K+1/K)$ and $R_k=\frac{1}{2}(K-1/K)$.
Since f is k-qc, then $\mu_f(z)\in U_k$ and therefore $F'(z)\in B_k$, for $z\in\mathbb{H}$. This yields, first, $K+1\geq|F'(z)+1|\geq 1+1/K,\quad K-1|F'(z)-1|\geq 1-1/K$, and then, $1\leq L_f(z)=\frac{1}{2}(|F'(z)+1|+|F'(z)-1|)\leq K$.
So we have $l_f(z)L_f(z)/K\geq 1/K$. Thus, we find

$$\frac{1}{K}\leq l_f(z)\leq L_f(z)\leq K. \tag{80}$$

Since $\lambda(f(z))=\lambda(z), z\in\mathbb{H}$, using (80) and the corresponding versions of 3A and 3B for $\mathbb{H}$, we obtain

$$\frac{1-k}{1+k}d_h(z_1,z_2)\leq d_h\big(f(z_1),f(z_2)\big)\leq\frac{1+k}{1-k}d_h(z_1,z_2).$$

It also follows from (80) that

$$\frac{1}{K}|z_2-z_1|\leq f(z_2)-f(z_1)\leq K|z_2-z_1|,\qquad z_1,z_2\in\mathbb{H}.$$

This estimate is sharp (see also [23] for an estimate with some constant $c(K)$).

We first need some properties of harmonic mappings.

Let R and S be two surfaces. Let $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$ be metrics with respect to the isothermal coordinate charts on R and S, respectively, and let f be C^2-map from R to S.
We use the following notation:

$$\mu=Belt[f]=\frac{q}{p},|\partial f|^2=\frac{\rho}{\sigma}|f_z|^2,\left|\bar{\partial}f\right|^2=\frac{\rho}{\sigma}|f_{\bar{z}}|^2,J(f)=|\partial f|^2-\left|\bar{\partial}f\right|^2,$$

and the Bochner formula (see [28])

$$\Delta ln\,|\partial f|=-K_SJ(f)+K_R. \tag{81}$$

Let us briefly explain how we apply the Bochner formula: Let f be aρ-harmonic mapping,$\rho *=\rho f=\rho\circ f|p|^2$and $K^*=K_{\rho^*}$the Gaussian curvature of ρ^*. Recall, if σ is the Euclidean metric density (that is $\sigma=1$), it follows from the Bochner formula (81) that $K^*=K_S(1-|\mu|^2)$.

Note that the Bochner formula is useful tool (for ρ-harmonic mappings if the Gaussian curvature of ρ is negative), but it does not give new information for euclidean harmonic mappings.

Namely, if $\sigma$and ρ are euclidean metrics densities (that is $\sigma=\rho=1$), then f is Euclidean harmonic and application of the Bochner formula yields $\Delta ln|\partial f|=0$. Also, this is an easy consequence of the fact that ∂f is an analytic function.
Theorem (1.2.13)[19]: (Hyperbolic-qch version). Let f be ak-quasiconformal harmonic mapping from the unit disc$\mathbb{U}$ onto itself with respect to the Poincaré metric. Then for any two points $z_1$and z_2 in $\mathbb{U}$ we have

$$(1-k)d_h(z_1,z_2)\leq d_h\big(f(z_1,)f\ (z_2)\big)\leq\sqrt{\frac{1+k}{1-k}}d_h(z_1,z_2), \tag{82}$$

Where $d_h$is the hyperbolic distance induced by the hyperbolic metric in $\mathbb{U}$.
We now consider a generalization of Theorem (1.2.13). We are actually concerned with a generalization of the right inequality in (82) and we postpone a more general discussion.
We have the following lemma.
Lemma (1.2.14)[19]: Let σ and ρ be two metric densities on $\mathbb{U}$, which define the corresponding metrics $ds = \sigma(z)|dz|^2$and$ds = \rho(w)|dw|^2$, and let $f{:}\, \mathbb{U} \to \mathbb{U}$ be a C^1-mapping. If$\rho(f \quad (z))L_f^2(z)c\sigma(z), z \in \mathbb{U}$, then $d_\rho(f(z_2), f(z_1)) \le cd_\sigma(z_2, z_1)$, for all $z_2, z_1 \in \mathbb{U}$
The proof of this result, which is a generalization of 3A.
A version of the following result was announced in [25].
Theorem (1.2.15)[19]: (ρ-qrh version). Let R be a hyperbolic surface with the Poincaré metric density λ, S another with a metric density ρ and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$ e uniformly bounded from above on S by the negative constant $-a, a > 0$. Then any ρ-harmonic k-quasiregular map f from R into S decreases distances up to a constant depending only on a and k:

$$d_\rho\big(f(z_1,)f\ (z_2)\big) \le \frac{1}{\sqrt{a}}\sqrt{\frac{1+k}{1-k}}d_h(z_1, z_2), \tag{83}$$

where d_ρ is the corresponding distance induced by the metric ds 2=ρ(w)|dw|2 on S.
Proof. By the uniformization theorem we can suppose that R and S are the unit discs.
Let $\rho^* = \rho \circ f|p|^2, \rho_0 = a(1-k^2)\rho^*$and $K_0 = K(\rho_0)$ the Gaussian curvature of ρ_0. Set $K^* = K(\rho^*)$. First, we show thatρ_0 is an ultrahyperbolic metric density. Namely, if$\rho_0 = 0(that\ is\ p(z_0) = f_z(z_0) \neq 0)$, then there is a neighborhood W of z_0 such that f is one-to-one in W.
Using the fact that $K^* = K_S(1-|\mu|^2)$, we conclude that $K^* - a(1-k^2)$ and therefore $K_0 \le -1$on W. Thus, ρ_0 is an ultrahyperbolic metric on $\mathbb{U}$. Hence, by the Ahlfors ultrahyperbolic lemma, $a(1-k^2)\rho^* \le \lambda$ and $a(1-k^2)\rho f(z)L_f^2(z) \le K^2\lambda, z \in \mathbb{U}$. An application of Lemma (1.2.14) immediately yields the result.
Note that one can show that there is a qc mapping g and an analytic function F such that $f = F \circ g$.
Using the uniformization theorem, some results can be extended to a more general setting including Riemann surfaces, more general functions and metrics on both domains and codomains.

Section (1.3): Harmonic Mappings between Smooth Jordan Domains

For D and G be subdomains of the complex plane C: A homeomorphism $f{:}\, D \mapsto G$; where is said to be K-quasiconformal (K-q.c), $K \ge 1$, if f is absolutely continuous on almost every horizontal and almost every vertical line and

$$\left|\frac{\partial f}{\partial x}\right|^2 - \left|\frac{\partial f}{\partial y}\right|^2 \le \left(K + \frac{1}{K}\right)J_f\ a.e.on\ D; \tag{84}$$

Where J_f is the Jacobian of f (cf. [32]). Note that the condition (84) can be written as

$$|f_{\bar z}| \le k|f_z|\ a.e.\text{on } D \text{ where } k = \frac{K-1}{K+1} i.e. K\ = \frac{1+k}{1-k}:$$

A function ω is called harmonic in a region D if it is of the form $\omega = u + iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, there exist two analytic functions g and h defined on D such that w has the representation

$$\omega = g + \bar{h}.$$

If ω is a harmonic univalent function, then by Lewy's theorem (see [42]), ω has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

Let

$$P(r, x - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(x - \varphi) + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disk $U := \{z : |z| < 1\}$g has the representation

$$\omega(z) = P[f](z) = \int_0^{2\pi} P(r, x - \varphi) f(e^{ix})\, dx, \tag{85}$$

Where $z = re^{i\varphi}$ and f is a bounded integrable function defined on the unit circle S^1.

Suppose γ is a rectifiable, directed, differentiable curve given by its arc-length parametrization $g(s), 0 \le s \le l$, where l is the length of γ. Then $|g'(s)| = 1$ and $s = \int_0^s |g'(t)|\, dt$; for all $s \in [0, l]$.

If γ is a twice-differentiable curve, then the curvature of γ at a point $p = g(s)$ is given by $K_\gamma(p) = |g''(s)|$: Let

$$K(s,t) = Re\, [\overline{(g(t) - g(s))}. ig'(s)] \tag{86}$$

be a function defined on $[0, l] \times [0, l]$. By $K(s \pm l, t \pm l) = K(s,t)$ we extend it on $\mathbb{R} \times \mathbb{R}$. Note that i$g'(s)$ is the unit normal vector of γ at $g(s)$ and therefore, if γ is convex then

$K(s,t) \ge 0$ for every s and t: (87)

We say that $\gamma \in C^{1,\mu}, 0 < \mu \le 1$, if $g \in C^1$ and

$$\sup_{t,s} \frac{|g'(t) - g'(s)|}{|t - s|^\mu} < \infty.$$

Let $\gamma \in C^{1,\mu}$ be a Jordan curve such that the interior of ° contains the origin.

Let f be a $C^{1,\mu}$ function from the unit circle onto γ and let $F(x) = f(e^{ix}), x \in [0, 2\pi)$. Then the functions $\rho(x) = |F(x)|$ and $\theta(x) = arg\, F(x)$ mod 2π on $(0, 2\pi]$ have $C^{1,\mu}$ extension on $\mathbb{R}$. We will use f and F interchangeably and will write $f'(x)$ instead of $F'(x)$.

Suppose now that $f : \mathbb{R} \longmapsto \gamma$ is an arbitrary 2π periodic C^1 function such that $f|_{[0,2\pi)} : [0,2\pi) \longmapsto \gamma$ is an orientation preserving bijective function.

Then there exists an increasing continuous function $s : [0,2\pi] \longmapsto [0, l]$ such that

$$f(\varphi) = g(s(\varphi)). \tag{88}$$

Hence

$$f'(\varphi) = g'(s(\varphi)). s'(\varphi),$$

and therefore

$$|f'(\varphi)| = |g'(s(\varphi))|. |s'(\varphi)| = |s'(\varphi)|.$$

Along with the function K we will also consider the function K_f defined by

$$K_f\,(\varphi, x) = Re\,[(f(x) - f(\varphi)). if'(\varphi)].$$

It is easy to see that

$$K_f\,(\varphi, x) = s'(\varphi) Re\, \left[\overline{(g(s(x)) - g(s(\varphi)))}. ig'(s(\varphi))\right] = s'(\varphi) K(s(\varphi), s(x)). \tag{89}$$

The following lemma is a slight modiffications of the corresponding lemma in [37].

Lemma (1.3.1)[31]: Let γ be a $C^{1,\mu}$Jordan curve. Let $g:[0,l]\longmapsto\gamma$ be a natural parametrization and $f:[0,2\pi]\longmapsto\gamma$, be arbitrary parametrization of γ. Then

$$|K_f(s,t)|\leq C_\gamma s'(\varphi)\ \min\{|s(\varphi)-s(x)|^{1+\pi}\ (l-|s(\varphi)-s(x)|)^{1+\mu}\}, \tag{90}$$

and

$$|K_f(\varphi,x)|\leq C_\gamma s'(\varphi)\min\{|s(\varphi)-s(x)|^{1+\mu},(l-|s(\varphi)-s(x)|)^{1+\mu}\}, \tag{91}$$

Where

$$C_\gamma=\frac{1}{1+\mu}\sup_{0\leq t\neq s\leq l}\frac{|g'(t)-g'(s)|}{|t-\ s|^{1+\mu}}.$$

Here $d_\gamma\left(f(e^{i\varphi}),f(e^{ix})\right):=min\{|s(\varphi)-s(x)|,(l-|s(\varphi)-s(x)|\})$ is the distance (shorter) between $f(e^{i\varphi})$ and $f(e^{ix})$ along γwhich satisfies the relation

$$|f(e^{i\varphi})-f(e^{ix})|\ \leq\left(f(e^{i\varphi}),f(e^{ix})\right)\leq C_\gamma|f(e^{i\varphi})-f(e^{ix})|.$$

Moreover if γ has a bounded curvature then the relations (90) and (91) are true for

$$C_\gamma=\{sup\,|k_\gamma\ (g(s))|/2:s\in[0,l]\}$$

and $\mu=1$. In this case

$$\lim_{t\to s}\frac{K(s,t)}{(s-\ t)^2}=\frac{|k_\gamma(g(s))|}{2}$$

And

$$\lim_{x\to\varphi}\frac{K_f(\varphi,x)}{(s(x)-s(\varphi))^2}=\frac{|k_\gamma(g(s))|}{2}s'(\varphi),$$

and the constant C_γ is the best possible.

Proof. Note that

$$K(s,t)=Re\left[\overline{(g(t)-g(s))}.\,ig'(s)\right]$$

$$=Re\left[\overline{(g(t)-g(s))}\,.\,i\left(g'(s)-\frac{g(t)-g(s)}{t-s}\right)\right],$$

and

$$g'(s)-\frac{g(t)-g(s)}{t-s}=\int_s^t\frac{g'(s)-g'(\mathcal{T})}{t-s}d\mathcal{T}.$$

If γ has a bounded curvature then g'' is bounded and

$$\left|g'(s)-\frac{g(t)-g(s)}{t-s}\right|\leq\int_s^t\frac{|g'(s)-g'(\mathcal{T})|}{t-s}d\mathcal{T}$$

$$\leq\sup_{s\leq x\leq t}|g''(x)|\int_s^t\frac{g'(s)-g'(\mathcal{T})}{t-s}d\mathcal{T}=\frac{1}{2}\sup_{s\leq x\leq t}|g''(x)|\ (t-s).$$

On the other hand

$$|\overline{g(t)-g(s)}|\leq\sup_{s\leq x\leq t}\ |g'(x)(t-s)|=(t-s),$$

and thus

$$|K(s,t)| \le \frac{1}{2} \sup_{s \le x \le t} |g''(x)|(s-t)^2.$$

It follows that the inequality (90) holds for $C_\gamma = \sup_p |k_\gamma(p)|/2$ and $\mu = 1$.
From (90) and (89) we obtain (91). Since

$$\frac{\delta}{\delta s} K(s,t) = Re[\overline{(g(t)-g(s))}. ig''(s)],$$

it follows that

$$\lim_{t \to s} \frac{K_g(s,t)}{(s-t)^2} = \lim_{t\to s} \frac{Re\left[\overline{(g(t)-g(s))}. ig''(s)\right]}{2(s-t)}$$
$$= Re\,[-g'(s). ig''(s)]/2 = \varepsilon|g''(s)|/2 = k_\gamma(s)/2,$$

Here $\varepsilon = 1$ if $k_\gamma > 0$ and $\varepsilon = -1$ if $k_\gamma < 0$: Similarly we can prove the case $\circ\ \gamma \in C^{1,\mu}$.

Lemma (1.3.2)[31]: [37] Let $\omega = u + iv$ be a differentiable function defined on U. Then:

$$J_\omega(re^{i\varphi} = u_x v_y - u_y v_x = |\omega_z|^2 - |\omega_{\bar z}|^2 = \frac{1}{r}(u_r v_\varphi - u_\varphi v_r) \qquad (92)$$

And

$$D(\omega)(re^{i\varphi}) := |\omega_z|^2 + |\omega_{\bar z}|^2 = \frac{|\partial_r \omega|^2}{2} + \frac{|\partial_\varphi \omega|^2}{2r^2}. \qquad (93)$$

If in addition we suppose that $\omega = P[f](z)$, where $f \in C^{1,\mu}, f: S^1 \mapsto \gamma$, then there exist continuous functions J_ω and $D(\omega)$ on the unit circle de¯ned by:

$$J_\omega(e^{i\varphi}) = \lim_{r\to 1} J_\omega(re^{i\varphi}) \qquad (94)$$

And

$$D(\omega)(e^{i\varphi}) = \lim_{r\to 1} D(\omega)\,(e^{i\varphi}) = \lim_{r\to 1} \frac{|\partial_r \omega(re^{i\varphi})|^2}{2} + \frac{|f'(\varphi)|^2}{2r^2}. \qquad (95)$$

Proposition (1.3.3)[31]: (Kellogg). Let $\gamma \in C^{1,\mu}$ be a Jordan curve and let $\Omega = Int(\Gamma)$.

If ω is a conformal mapping of U onto Ω , then ω' and ln ω' are in Lip_μ. In particular, $|\omega'|$ is bounded from above and below by positive constants on U.

For the proof, see for example [41].

The following lemma is a generalization of Mori's Theorem, (cf. [32]).

Lemma (1.3.4)[31]: If ω is a K quasiconformal function between the unit disk and a Jordan domain Ω with $C^{1,\mu}$ boundaryγ, then there exists a constant C_K depending only on γ and on $\omega(0)$ such that

$$|\omega(z_1) - \omega(z_2)| \le C_K |z_1 - z_2|^\alpha, \alpha = \frac{1-k}{1+k}, z_1, z_2 \in U.$$

Note that the constant α is the best possible (in general case).

We give some estimates for the Jacobian of a harmonic univalent function. It is a slight improvement of [37].

Lemma (1.3.5)[31]: Let $\omega = P[f](z)$ be a harmonic function between the unit disk U and the Jordan domain Ω-, such that f is injective, $f \in C^{1,\mu}$, and $\partial\Omega = f(S^1) \in C^{1,\mu}$. Then for

$$C_1 = \frac{\pi}{4(1+\mu)} \sup_{s \ne t} \frac{|g'(s) - g'(t)|}{(s-t)^\mu}$$

one has

$$\lim_{r\to e^{i\varphi}} J_\omega(z)\, C_1|f'(\varphi)| \int_{-\pi}^{\pi} \frac{d_\gamma(f(e^{i(\varphi+x)}), f(e^{i\varphi}))^{1+\mu}}{x^2}\, dx \tag{96}$$

for all $e^{i\varphi} \in S^1$.

Proof. Since $f \in C^{1,\mu}$, by the proof of the Lemma (1.3.2) it follows that the partial derivatives of the function w have continuous extensions on the boundary. Since

$$F(x) = p(x)e^{i\varphi(x)};$$

we obtain

$u_r(e^{i\varphi}) = \lim_{z\to e^{i\varphi}} u_r(z)$, $v_r(e^{i\varphi}) = \lim_{z\to e^{i\varphi}} v_r(z)$,

$$\lim_{z\to e^{i\varphi}} u_\varphi(z) = Re\frac{\partial}{\partial\varphi}\left(\rho\left(\varphi e^{i\theta(\varphi)}\right)\right) = \rho'(\varphi)cos\,\theta(\varphi) - \rho(\varphi)\theta'(\varphi)sin\theta(\varphi)$$

and

$$\lim_{z\to e^{i\varphi}} v_\varphi(z) = Im\frac{\partial}{\partial\varphi}\left(\rho\left(\varphi e^{i\theta(\varphi)}\right)\right) = \rho'(\varphi)\, sin\theta(\varphi) - \rho(\varphi)\theta'(\varphi)cos\theta(\varphi).$$

Observe that $u(e^{i\varphi}) = \rho(\varphi)cos\,\theta(\varphi)$ and $v(e^{i\varphi}) = \rho(\varphi)sin\theta(\varphi)$. Thus:

$$\lim_{r\to e^{i\varphi}} J_\omega(z)\left(re^{i\varphi}\right) = \lim_{r\to 1}\frac{1}{r}\left(u_r v_\varphi - u_\varphi v_r\right)$$

$$= \lim_{r\to 1}\frac{\left(u\left(re^{i\varphi}\right) - u\left(e^{i\varphi}\right)\right)}{1-r}\rho'(\varphi)sin\theta(\varphi) + \rho(\varphi)\theta'(\varphi)\cos\theta$$

$$-\lim_{r\to 1}\frac{\left(v\left(re^{i\varphi}\right) - v\left(e^{i\varphi}\right)\right)}{1-r}\rho'(\varphi)\cos\theta(\varphi) + \rho(\varphi)\theta'(\varphi)sin\theta(\varphi)$$

$$= \lim_{r\to 1}\int_{-\pi}^{\pi} K_f(x,\varphi)\frac{P(r,\varphi - x)}{1-r}dx$$

$$= \lim_{r\to 1}\int_{-\pi}^{\pi} K_f(x+\varphi,\varphi)\frac{P(r,x)}{1-\ r}dx.$$

According to (91)

$$\left|K_f(x+\varphi,\varphi)\right| \le C_\gamma f'(\varphi)d_\gamma\left(f\left(e^{i(\varphi+x)}\right), f\left(e^{i\varphi}\right)\right)1+\mu.$$

On the other hand, using the inequality$|t| \le \pi/2\,|\,sin\,t|$ for $\pi/2 \le t \le \pi/2$, we obtain

$$P\frac{(r,x)}{1-\ r} = \frac{1+r}{2\pi(1+r^2-\ 2r\,cos\,x)} \le \frac{1}{\pi((1-\ r)^2+4r\,sin^2\ x/2)} \le \frac{\pi}{4rx^2}$$

For $0 < r \le 1\ and\ x \in [-\pi,\pi]$. Thus,

$$\lim_{r\to 1}\int_{-\pi}^{\pi} K(x,\varphi)\frac{P(r,\varphi-\ x)}{1-\ r}dx \le \frac{\pi C_\gamma}{4}|f'(\varphi)|\int_{-\pi}^{\pi}\frac{d_\gamma(f\left(e^{i(\varphi+x)}\right), f(e^{i\varphi}))^{1+\mu}}{x^2}dx.$$

The inequality now holds for

$$C_1 = \frac{\pi}{4(1+\mu)}\sup_{s\neq t}\frac{|g'(s) - g'(t)|}{(s-\ t)^\mu}$$

Using Lemma (1.3.2), Proposition (1.3.3), Lemma (1.3.4) and Lemma (1.3.5) we obtain:

Theorem (1.3.6)[31]: [37] Let $\omega = P[f](z)$ be a K q.c. harmonic function between the unit disk and a Jordan domainΩ, such that $\omega(0) = 0$. If $\gamma = \Omega \in C^{1,\mu}$, then there exists a constant $C' = C'(\gamma, K)$ such that

$$|\,f'(\varphi)| \le C' \text{ for almost every } \varphi \in [0,2\pi], \tag{97}$$

And

$$|\omega(z_1) - \omega(z_2)| \le KC'|z_1 - z_2| \text{ for } z_1, z_2 \in U. \quad (98)$$

Notice that Theorem (1.3.6) is a generalization of the corresponding result for the harmonic q.c. of the unit disk onto itself, see [14]. Theorem (1.3.6) has its extension to the class of q.c. mappings satisfying the differential inequality $|\Delta\omega| \le M|\omega_z||\omega_z|$ (see [40]).

Example (1.3.7)[31]: ([33]). Let P_n be a regular n¡polygon. Then the function

$$\omega(z) = \int_0^z (1 - z^n)^{-2/n}\, dz$$

is a conformal mapping of the unit disk onto the polygon P_n. However $\omega'(z) = (1 - z^n)^{-2/n}$ is an unbounded function on the unit disk and thus the condition $\gamma \in C^{1,\mu}$ in Theorem (1.3.6) is important.

Corollary (1.3.8)[31]: [37] Let w be a quasiconformal harmonic mapping between Jordan domains Ω andΩ_1, such that $\omega(0) = 0$. If $\gamma = \partial\Omega \in C^{1,\mu}$ and $\gamma_1 = \partial\Omega_1 \in C^{1,\mu_1}, 0 < \mu, \mu_1 \le 1$, then there exist the constants C and C_1 depending on and °1 such that

$$|\omega(z_1) - \omega(z_2)| \le C|z_1 - z_2| \quad (99)$$

and

$$D(\omega)(z) = |\omega_z(z)|^2 + |\omega_z(z)|^2 \le C_1. \quad (100)$$

The following theorem provides a necessary and sufficient condition for the q.c. harmonic extension of a homeomorphism from the unit circle to a $C^{1,\mu}$ convex Jordan curve. It is an extension of the corresponding theorem of Pavlovi¶c ([14]):

Theorem (1.3.9)[31]: [37] Let $f: S^1 \mapsto \gamma$be an orientation preserving absolutely continuous homeomorphism of the unit circle onto a convex Jordan curve ° 2C1;¹. Then $\omega = P[f]$ is a quasiconformal mapping if and only if

$$0 < \operatorname{ess\,inf}|f'(\varphi)|, \quad (101)$$

$$\operatorname{ess\,sup}|f'(\varphi)| < \infty \quad (102)$$

and

$$\operatorname*{ess\,sup}_{\varphi} \left| \int_0^\pi \frac{|f'(\varphi + t)||f'(\varphi - t)|}{\tan t/2} dt \right| < \infty. \quad (103)$$

We note that the hypothesis "absolutely continuous" in the previous theorem is needed, although this theorem appeared in [37] without this hypothesis.

Example (1.3.10)[31]: ([36]). Let

$$\theta(\varphi) = \frac{2 + b(\cos(\log|\varphi|) - \sin(\log|\varphi|))}{2 + b(\cos(\log\pi) - \sin(\log\pi))}\varphi, \varphi \in [-\pi, \pi],$$

where $0 < b < 1$. Then the function $\omega(z) = P[f](z) = P[e^{i\theta(\varphi)}](z)$ is a quasiconformal mapping of the unit disk onto itself such that $f'(\varphi)$ does not exist for $\varphi = 0$.

Hence a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case.

Corollary (1.3.11)[31]: [37] Let ω be a K quasiconformal harmonic function between a Jordan domain Ω and a convex Jordan domain Ω_1, such that $\omega(0) = 0$ and $\partial\Omega, \partial\Omega_1 \in C^{1,\mu}$. Then ω is bi-Lipschitz, i.e. there exists a constant $L \ge 1$ such that

$$L^{-1}|z_1 - z_2| < |\omega(z_1) - \omega(z_2)| < L|z_1 - z_2|, z_1, z_2. \quad (104)$$

Moreover, there exists $C_D = C(K, \Omega, \Omega_1) \ge 1$ such that

$$1/C_D \le D(\omega)(z) \le C_D, \quad \text{for } z \in \Omega \quad (105)$$

We have the following theorem. It is an extension of Corollary (1.3.11) for a nonconvex case.

Theorem (1.3.12)[31]: [38] Let $\omega = f(z)$ be a K quasiconformal harmonic mapping between a Jordan domain Ω with $C^{1,\mu}$ boundary and a Jordan domain Ω_1 with $C^{2,\mu}$ boundary. Let in addition $a \in \Omega$ and $b = f(a)$. Then ω is bi-Lipschitz. Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a, b) \geq 1$ such that

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, z_1, z_2 \in \Omega. \quad (106)$$

We write $L_f = L_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$ and $l_f = l_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$, if $\partial f(z)$ and $\bar{\partial} f(z)$ exist.

In [19], the following results have been obtained (see also [25]).

Theorem (1.3.13)[31]: Let f be a k-qc euclidean harmonic difeomorphism from the upper half-plane $\mathbb{H}$ onto itself and $\frac{1+k}{1-k}$. Then f is a $(1/K, K)$ quasi-isometry with respect to the Poincare distance d_h.

Outline of the proof: Precomposing f with a linear fractional transformation, we can suppose that $f(\infty) = \infty$ and therefore we can write f in the form $f = u + iy = \frac{1}{2}\left(F(z) + z + \overline{F(z) - z}\right)$, where F is a holomorphic function in $\mathbb{H}$.

Hence the complex dilatation

$$\mu_f = \frac{F'(z) - 1}{F'(z) + 1}, L_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|)$$

and

$$l_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|);$$

which yields

$$1 + 1/K \leq |F'(z) + 1| \leq K + 1, \quad 1 - 1/K \leq |F'(z) - 1| \leq K - 1$$

and therefore it follows

$$1 \leq L_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|) \leq K,$$

and consequently

$$l_f(z) \geq L_f(z)/K \geq 1/K.$$

Now using a known procedure, we obtain

$$\frac{1}{K}|z_2 - z_1| \leq |f(z_2) - f(z_1)| \leq K|z_2 - z_1| \;\; z_1, z_2 \in H, \quad (107)$$

$$\frac{1-k}{1+k} d_h(z_1, z_2) 6\, d_h\big(f(z_1), f(z_2)\big) \leq \frac{1+k}{1-k} d_h(z_1, z_2) z_1, z_2 \in H. \quad (108)$$

Both estimates are sharp (see also [23], [35] for an estimate with some constantc(K) in (107)).

The following generalization of Theorem (1.3.12) will appear in [43].

It is partially based on the results obtained in [38] and on Bochner formula for harmonic maps.

Theorem (1.3.14)[31]: [43] Let ω be a C^2 K quasiconformal mapping of the unit disk onto a $C^{2,\alpha}$ Jordan domain. Let ρ be a C^1 metric onΩ of non-negative curvature and w ½-harmonic, that is

$$\omega_{zz} + (log\ \rho)_\omega \omega_z \omega_z = 0.$$

Then $J_\omega \neq 0$ and ω is bi-Lipschitz.

Finally, notice that the proof of Theorem (1.3.9), which was published in [37], can be also based on the results presented in [26] and [27].

Chapter 2

Three Lattice-Point Problem and Partitions

We determine equivalences between the bounds on the size of the continuum, $2^{\omega} \leq \omega_{\theta}$, and some relationships between p, ω_{τ} and ω_{θ}. Assuming measurability we answer the analogous question in higher dimensions in the negative, and we improve on the known partial results in the two dimensional case. We also consider a related problem involving finite sets of rotations.

Section (2.1): Problems of Steinhaus

We shall use the following notation, μ will denote the Lebesgue measure of a set, plane or linear as appropriate; any metnion of μ will imply measurability of the set in question; μ_*, μ^* will denote inner, outer measure when we deal with non-measurable sets, in Theorem (2.1.9). p will be a rigid motion of the plane (without reflection); as in [57], it is sometimes convenient to regard the set S as moving and the points L of the Cartesian lattice as fixed, sometimes the opposite. Then $|p(S) \cap L| = |S \cap p^{-1}(L)|$ counts the number of points of L covered by the congruent copy $p(S)$ of S. Following [52] and [57], we denote the supremum, infimum of this function (taken over all such p) by $M(S), m(S)$ respectively. But if the p are restricted to be translations only, we write $M_T(S), m_T(S)$.

Then Steinhaus' problems may be shortly expressed thus:

(a) If $\mu(S)$ is finite, is $m(S)$ necessarily finite?

(b) If $\mu(S)$ is infinite, then is the supremum $M(S) = \infty$ necessarily attained for some p?

(c) Does there exist an S for which $M(S) = m(S)$ $(= n\ say)$? In fact (a) has already been solved: the idea, though without the concept of measure, goes back to Blichfeldt [46]; and Niven and Zuckerman [57] obtained a stronger relation, replacing $m(S)$ by $m_T(S)$, essentially restated as Theorem (2.1.1) below. The contributions here give partial solutions to (b) and (c); partial in the sense that we require extra conditions on S in each of the main Theorems (2.1.1) and (2.1.8). It seems that new and deeper ideas are needed to enable us to jettison these conditions: (b) and (c) as they stand are still resistant.

The generalization of the solution of (a) referred to above is:

Theorem (2.1.1)[44]: if $\mu(S) < \infty$, then $m_T(S) \leq [\mu(S)]$; and $M_T(S) \geq \{\mu(S)\}$; where $[\,], \{\,\}$ denote rounded down, rounded up, respectively, to the nearest integer.

The proof in [57] consists of the straightforward application of Fubini's Theorem to the characteristic function of S: we deduce that $\mu(S)$ is the 'average' of $|p(S) \cap L|$ over all translations, and, since m_T, M_T are integers, the result follows. A fortiori, we trivially deduce the same result for m, M.

We next note that the simple Fubini argument above applied to the case where $\mu(S) = \infty$ produces only $M_T(S)$ as a supremum, not necessarily attained. Indeed for translations it may not be so attained: a counterexample is given (in Cartesian coordinates) by the open set

$$S \equiv \bigcup_{m>0,n>0} \{(m, m + m^{-1}) \times (n, n + n^{-1})\}.$$

For rigid motions the best result I can prove in the direction of answering (b) is:

We need 2 lemmas.

Lemma (2.1.2)[44]: Let P be the set of points $Y > X^4$, where $Y = y - ax, X = x + \alpha y$, where α is a fixed irrational. Let L' be the set of lattice points $L \cap P$, with moduli $r_1, r_2, \ldots$, such that $r_1 \le r_2 \le r_3 \le \cdots$. Then $r_{i+1} - r_i \to 0$ as $i \to \infty$.

Proof. The result will follow if there is a sequence of points of L' with Y-coordinates $Y_i \to \infty$ with $Y_{i+1} - Y_i \to 0$. For then we have (since $L' \subset P$)

$$(1 + a^2)^{\frac{1}{2}}(r_{i+1}r_i) \le (Y_{i+1}^2 + Y_{i+1}^{\frac{1}{2}})^{\frac{1}{2}} - Y_i \to 0,$$

as desired.

We now suppose that $Y_{i+1} - Y_i \nrightarrow 0$, and deduce a contradiction. If this were so, then there would exist some $\varepsilon > 0$, such that there would be a sequence of rectangles, free of points of L', and all with width at least ε, and with lengths tending to ∞, all lying parallel to $Y = 0$. Any such rectangle. R_j may be translated any integer distance parallel to the original x- and y-axes and remain free of points of L, and in particular moved so that its longer axis cuts $X = 0$ and lies along $Y = \tau_j$, where $|\tau_j| < c$, some constant. By a compactness argument on the τ_j, we see that there must exist a strip of the plane of infinite length, of thickness at least $\frac{1}{2}\varepsilon$, parallel to $Y = 0$, and free of points of L. But this contradicts a well-known consequence of Kronecker's theorem, namely that any line of the form $y = \alpha x + \beta$ with a irrational passes arbitrarily close to points of L.

Corollary (2.1.3)[44]: If we consider the r_1 to refer not to points of $L' = P \cap L$, but to points of $L" = Q \cap L$, where Q is any fixed sector, given in polar coordinates by $\theta_1 > \theta > \theta_2$, say, then the conclusion $r_{i+1} - r_i \to 0$ remains valid.

For any fixed Q contains all the points of some such P (as in the lemma) which are sufficiently distant from the origin.

Lemma (2.1.4)[44]: If an S, satisfying (1), has lattice points $L_1, L_2 \ldots, L_k$ lying in S^0, then it is possible to make a rigid motion of S so small that each of these L_i remains in S^0, and yet also so that S^0 covers a further lattice point L_{k+1}.

Proof. Let the distance of L_i from $fr\, S$ be γ_i (> 0). We restrict consideration to rigid motions so small that they move each L_i by a distance at most 2^{-k-1} $(i = 1, 2, \ldots, k)$. Such motions certainly contain as a subset M all motions described in the following way (for some certain small $\varepsilon_1, \varepsilon_2$): a translation $t_{\xi,\eta}$ of magnitude $(\xi^2 + \eta^2)^{\frac{1}{2}} < \varepsilon_1$, followed by a rotation ρ, about 0, of magnitude $|\rho| < \varepsilon_2$. By the corollary to Lemma (2.1.2) above, in any sector $Q\colon \theta_1 > \theta > \theta_2$, so small that $\theta_1 - \theta_2 < \frac{1}{2}\varepsilon_2$, say, the set $\{\rho^{-1}(L)\}$, for

$|\rho| < \varepsilon_2$, contains a set of circular arcs, centre O, radii r_i with $r_i \uparrow \infty$ and $r_{i+1} - r_i \to 0$, and sweeping across the sector from $\theta = \theta_1$ to $\theta = \theta_2$. It follows that the set

$$\left\{t_{\xi,\eta}^{-1}\rho^{-1}(L): \left|t_{\xi,\eta}\right| < \varepsilon_1, |\rho| < \varepsilon_2\right\}$$

covers all sufficiently distant points in Q, and so (since the plane is covered by a finite number of sectors like Q) all sufficiently distant points of the plane; and in particular some point $A \in S^0$. Thus for some rigid motion $\rho \in M$, $\rho(A)$ is a lattice point L_{k+1}, as desired.

Theorem (2.1.5)[44]: Let S^0 the interior of S satisfy:

S^0 is unbounded; (1)

then $M(S) = \infty$ is attained.

Proof. We progress inductively using Lemma (2.1.4). Trivially the sum of the successive rigid motions performed to 'capture' successive new points of L has a limiting motion; and none of the points L_i once captured 'escapes' in later motions or in passing to the limit, since $\sum_{k=1}^{\infty} 2^{-k-1}\gamma_i < \gamma_i$. In the limit infinitely may L_i are captured,

We see from Lemma (2.1.4) above that the set of $p \in \Pi$ for which $|p(S) \cap L| \leq n$, for any integer n, is a nowhere dense set (for any neighbourhood contains a neighbourhood in the complementary set), and hence their union is a set of the first category in Π; that is, the set of rigid motions satisfying Theorem (2.1.5) is not only non-empty, but is a residual set in Π.

This last fact inclines one to the view that the whole phenomenon belongs to category theory rather than measure theory and suggests in particular that the appropriate generalization of condition (1), instead of being $\mu(S) = \infty$, might perhaps be: 'the part of S outside some large circle possesses the property of Baire and is of the second category'. See [58] especially ch. 4, 19, 20, 21, for such translations from measure to category. (The ambience of the result here is reminiscent of a previous 1-dimensional theorem of [48]: let $E \subset \mathbb{R}^+$ have unbounded interior, then there exists a dense set of values of h whose multiples $\{nh\}$ lie infinitely often in E. Professor J. F. C. Kingman pointed out that this was indeed a category result. It has been re-discovered several times; see [59], where it is characterized a 'folk-theorem'; and [53] for some applications and extensions. Here it is known that the measure-theoretic analogue is false: see Haight [50], [51], Lekkerkerker [54].)

The category theory aspect suggests faintly that the question (b) in its original measure-theoretic fomulation should have a negative answer.

Contrariwise, a full-blooded variant of (b) in measure-theoretic terms, much stronger (and so perhaps easier to disprove) would read:

Conjecture (2.1.6)[44]:$(\boldsymbol{b}^+)$. Let $\mu(S) < \infty$. Let M_∞ be the set of points in Π for which $|p(S) \cap L| = \infty$. Then its measure (in Π) $\pi(M_\infty) = 0$.

This we must also leave open. However, it is easy to prove:

Theorem (2.1.7)[44]: If $\mu(S) < \infty$, then $\pi(M_\infty) = 0$.

This result follows immediately on applying the Fubini technique of Theorem (2.1.1), but now in the 3 variables θ, X, Y.

We note that the results of Theorem (2.1.7) and (the category extension of) Theorem (2.1.5) overlap: if both S^0 is unbounded and $\mu(S) < \infty$, then M_∞, is a^* residual set but of measure 0 in the space Π.

The 'expanded rotated' lattice $L_{\lambda,\theta}$ denotes, for fixed $\lambda(> 0)$, θ, the set of points

$$\bigcup_{m,n} \{(\lambda[m \cos\theta \ + \ n \sin\theta], \lambda[-m \sin\theta \ + \ n \cos\theta])\}$$

The 'expanded translated' lattice $L_{\lambda,\xi,\eta}$ denotes, for fixed $A\ (> 0), \xi, \eta$, the set of points

$$\bigcup_{m,n} \{\lambda m + \xi, \lambda n + \eta\}$$

Then another variant of Theorem (2.1.5) is obtained by using the first of these:

Theorem (2.1.8)[44]: Let S^0 be unbounded. Then for some λ, θ, we have $|S \cap L_{\lambda,\theta}| = \infty$.

Proof. The analogous result to the Corollary to Lemma (2.1.2) is immediate in this case, and the arguments of Lemma (2.1.4) and after it, apply mutatis mutandis. The category-theoretic comments are similarly applicable.

However, we have, for the second variant above,

Theorem (2.1.9)[44]: There exists an open set S_i with $\mu(S) = \infty$, such that

$|S \cap L_{\lambda,\xi,\eta}|$ is finite for each λ, ξ, η.

Proof. The set of points whose Cartesian coordinates satisfy $0 < xy < 1$

clearly provides such an example.

Note the contrast between Theorem (2.1.8), with a positive conclusion, although it has 2 'degrees of freedom' and Theorem (2.1.9), negative yet it has 3.

It is interesting to see that the analogue of Conjecture (b) fails for the lattices $L_{\lambda,\theta}$.

Theorem (2.1.10)[44]: There exists a set S with $\mu(S) = \infty$, such that $|S \cap L_{\lambda,\theta}|$ is finite for all λ, θ.

Proof. Let $H \subset \mathbb{R}^+$ denote Haight's set of [51], constructed with his set G being $\{\sqrt{(m^2 \ + \ n^2)}: m, n \text{ integers}\}$. Let S be given in polar coordinates by $\{(r, \theta): r \in H\}$. Then $\mu(S) \ = \ \infty$, since $\mu(H) \ = \ \infty$. And we see by construction that $|S \cap L_{\lambda,\theta}|$ must be always finite, since Haight's condition (B) gives $\lambda \cdot \sqrt{(m^2 + n^2)} \in H$ has only finitely many solutions for any $\lambda > 0$.

For another possible variation of the lattice, see Macbeath [55].

Once we allow non-measurable S, other possibilities come alive. We can of course deduce trivial corollaries from Theorems (2.1.1) and (2.1.2) by the use of measurable kernels and measurable envelopes, but, as one might guess, there are 'paradoxical' results, not envisaged there; specifically:

Theorem (2.1.11)[44]: (i). There exists an S with $\mu_*(S) = 0$, but $m(S) = \infty$.

(ii) There exists an S with $\mu^*(S) = \infty$, but $M(S) = 1$.

Proof. (i) The exhibited S will be of the shape $J \times (-\infty, \infty)$. This will contain infinitely many lattice points on each non-vertical line of $p(L)$ (p a rigid motion), and hence in total, provided that J satisfies the condition:

J is a linear set that contains infinitely many members of every arithmetic progression $\{a + kb: a, b \text{ real}, k \text{ integral}\}$; we also need J non-measurable with $\mu_*(J) = 0$.

We construct first a set J_1 thus: either by considering a Hamel basis (see e.g. [62], 443-449) of the reals, with one member of it a rational, or by considering the reals as a vector space over the rationals and using the Axiom of Choice, we may decompose any real x into $x_q + x_r$ with $x_q \in \mathbb{Q}$, the rationals, and such that $x_q = x$ if $x \in \mathbb{Q}$ and $(x + y)_q = x_q + y_q$ for all x, y. Let $I = \bigcup_{J \text{ even}}[i, j + 1)$, and set $J_1 \equiv \{x: x_q \in I\}$.

We prove that J_1 has nearly all the properties claimed for J. First, if it were measurable, a standard argument, as in the usual demonstration of a non-measureable set, produces $\mu(J_1 \cap [—N, N]) \sim N$ as $N \to \infty$. But also J_1 has arbitrarily small periods, namely ty for any $t \in \mathbb{Q}$, and $y \neq 0$ any fixed real with $y_q = 0$. And we know that measurable sets with arbitrarily small periods have either full or empty measure. Hence J_1 is nonmeasurable, and, again using the periodicity, we find that we must have $\mu_*(J_1) = 0$. Also, for any real a, b, we have $k_a + b \in J_1$ if and only if $ka_q + b_q = I$, and this always occurs for infinitely many k (both positive and negative) except in the case a_q an even integer, and integer $[b_q]$ odd.

Similarly if J_2 be the complement of J_1 then $ka + b \in J_2$ for infinitely many k unless a_q be an even integer and $[b_q]$ even. Thus, finally, we see that all the desired properties are possessed by the set

$$J \equiv \big(J_1 \cap (0, \infty)\big) \cup \big(J_2 \cap (-\infty, 0)\big).$$

(ii) We proceed by transfinite induction, using the idea of [63] as a model.

The closed sets F of positive measure in the plane are of cardinality c, and may be well-ordered

$$F_0, F_1, F_2, \ldots, F_\alpha, \ldots, (\alpha < \Omega_0),$$

where Ω_0 is the smallest transfinite ordinal corresponding to the power of the continuum. We construct the points

$$p_1, p_2, \ldots, p_\alpha, \ldots, (a < \Omega_0)$$

that will constitute S by transfinite induction, as follows. Given any index $\alpha < \Omega_0$, we suppose that the points $p_\xi(\xi < \alpha)$ have been determined, and we show how to construct p_α. Since F_α has positive measure, there exists some line l for which $\mu(l \cap F_\alpha) > 0$, and so $l \cap F_\alpha$ has cardinality c.

Consider the set of points q on l which lie on some circle centre some $p_\xi(\xi < \alpha)$, radius $(m^2 + n^2)^{\frac{1}{2}}$ $(m, n$ some integers). Each such circle cuts l at most twice, and hence the set of such q has cardinal less than that of the continuum. Thus I may, and do, choose a $p_\alpha \in l \cap F_\alpha$ not on any such circle. Clearly S, the totality of points thus chosen has no pair of points p_β, p_γ ever at any distance $(m^2 + n^2)^{\frac{1}{2}}$ apart, so trivially $M(S) = 1$. And I say that since $S \cap F_\alpha \neq 0$ for any α, necessarily $\mu^*(S) = \infty$. For if $\mu^*(S)$ were finite, the measurable envelope of S would have a complementary set which would contain some closed set of positive measure.

Of course, the proofs of both (i), (ii) necessitate the Axiom of Choice, as they must.

We come now to consider Conjecture (c). The best result that we can prove is:

We need 2 lemmas and a definition of some independent interest.

Lemma (2.1.12)[44]: Let E be a non-empty plane set with $\mu(E) = 0$. Then there exists a rigid motion p such that $|p(E) \cap L| = 1$ exactly.

The proof will be by contradiction. We assume that there exists an E such that whenever E and $p^{-1}(L)$ have 1 common point they must have at least 2, and deduce that such an E must have positive plane measure.

As the last sentence indicates, it is convenient here to picture E as fixed and apply the p's to L.

Take first the origin O at an arbitrary point of E, and rotate L about it; by the supposition, at each orientation, E contains at least 1 of the other points of L. Since there are but countably many points of L (each of which we assume keeps its identity throughout the rotation), necessarily then E contains some set of positive measure of some circular arc (centre O) on which some lattice point, say L_1 moves. We may restrict ourselves by talcing a subset of this on an arc of small length; L_1 will denote a generic point of this set of positive linear measure. We now repeat the argument, with L_1 in place of O as centre for swinging the lattice round.

Thus, again, for each such L_1 there is a set of positive linear measure on some circular arc (centre L_1 radius some $\sqrt{(r^2 + s^2)}, r, s$ integers not both 0) lying in E. Let L_2 be a generic point of this set. By restricting consideration to one pair out of the countable number of combinations of r and s, we may take L_1L_2 of some constant length. Let OL_1, L_1L_2 be at angles θ, ϕ, respectively to some fixed line. To each fixed θ, there corresponds an $\varepsilon_0(\theta)$, such that there is a positive subset of ϕ of the above type satisfying also $|sin\ (\theta - \phi)| \geq \varepsilon_0(\theta)$. Thus restricting to further subsets if necessary, there is a set of θ of measure at least ε_1, to each of which corresponds a set of ϕ of measure at least ε_2, and such that each relevant $\varepsilon_0(\theta)$ is at least some fixed ε_3. As a final restriction, we choose one or other (whichever gives the larger measure of the set of L_2) of the choices: $\theta - \phi (\text{mod } 2\pi)$ lies in $(0, \pi)$ or $(\pi, 2\pi)$. Then we see that this means that we have a $(1,1)$-relation between the points L_2 and the pair (θ, ϕ). So finally the set of L_2 thus characterized has a real measure at least $\varepsilon_1 \varepsilon_2 \varepsilon_3$. So E has positive measure: the desired contradiction.

Definition (2.1.13)[44]: Let S be a measurable set. Then the set E of points at which the metric density of S is neither 0 nor 1 is the metrical boundary of S.

(The exact definition of density is of little consequence; we take it to be the limit, if it exists, of $\mu(S \cap Q)/\delta^2$ as $\delta \to 0$, where Q is a square of side δ, oriented along some appropriate Cartesian axes, and centred at the point in question.)

The following result is an immediate corollary of the standard 'density theorem': $\mu(E) = 0$, for any measurable S, bounded or not. See e.g. [61],(128-131); or simpler, observe that the proof of [47], (83), using Vitali's covering theorem, is valid in any number of dimensions.

We now have a result 'in the other direction'; this result may be known.

Lemma (2.1.14)[44]: Let E be the metrical boundary of a plane set S with $\mu(S) > 0$, $\mu(\mathbb{R}^2 \backslash S) > 0$; then $E \neq \emptyset$.

Proof. Let $Q_\delta(x)$ be the closed square of side δ and centre x. Then $D_\delta(x) = \mu(S \cap Q_\delta(x))/\delta^2$ is continuous in x, for fixed δ; and since, by the density theorem quoted above (see the same references), we have that there exist points y_1, y_2, and a δ_1 such that $D_{\delta_1}(y_1) > \frac{1}{2} > D_{\delta_1}(y_2)$, it follows by continuity that there is an x_1 such that $D_{\delta_1}(x_1) = \frac{1}{2}$. Now, working only in $Q_{\delta_1}(x_1)$, we observe that for δ/δ_1 sufficiently small, the average value of $Q_\delta(x)$ over $Q_{\delta_1 - \delta}(x_1)$ is near to $\frac{1}{2}$ (since the edge effect becomes insignificant as $\delta/\delta_1 \to 0$). Thus there exists a $\delta_2 (0 < \delta_2 < \delta_1)$ such that this average lies between $\frac{1}{2} \pm \varepsilon$ (for a given fixed ε). Since $D_{\delta_2}(x)$ is continuous in x, we find that for some $x_2 \in Q_{\delta_1 - \delta_2}(x)$, we have $|D_{\delta_2}(x_2) - \frac{1}{2}| < \varepsilon$.

Proceeding thus inductively, we obtain sequences x_i and δ_i such that

$x_{i+1} \in Q_{\delta_1 - \delta_{i+1}}$ and

$$\left| D_{\delta_{i+1}}(x_{i+1}) - \frac{1}{2} \right| < \varepsilon + \varepsilon^2 + \cdots + \varepsilon^i < \varepsilon/(1 - \varepsilon). \quad (2)$$

The squares $D_{\delta_i}(x_i)$ are a decreasing nested sequence, and, assuming, as we may, that $\delta_i \to 0$, we obtain, by compactness, $a\ z$ with $x_i \to z = \bigcap_1^\infty Q_{\delta_i}(x_i)$. I say that $z \in E$. For consider $D_{2\delta_i}(x)$. As $z \in Q_{\delta_i}(x_i)$, we have $Q_{2\delta_i}(z) \supset Q_{\delta_i}(x_i)$; so taking the two extreme possibilities of $S \cap (Q_{2\delta_i}(z) \backslash Q_{\delta_i}(x_i))$ having empty, full measure, we get

$$\mu(S \cap (Q_{\delta_i}(x_i))) \le \mu(S \cap Q_{2\delta_i}(z)) \le \mu(S \cap Q_{\delta_i}(x_i)) + \mu(Q_{2\delta_i}(z) \backslash Q_{\delta_i}(x_i)),$$

and so, on dividing by $4\delta_i^2$ and using (2), we obtain:

$$\frac{1}{4}\left(\frac{1}{2}-\varepsilon/(1-\varepsilon)\right) \le D_{2\delta_i}(z) \le \frac{1}{4}\left(\frac{1}{2}+\varepsilon/(1-\varepsilon)\right)+\frac{3}{4}.$$

Hence $D_\delta(z)$ cannot have limit 0 or 1 as $\delta \to 0$: either the limit is strictly between 0 and 1 or does not exist. This proves the lemma.

Theorem (2.1.15)[44]: Suppose that

S is measurable and essentially bounded, (3)

then it is impossible to have $M(S) = m(S) = n$ for any integer n.

Proof. Let S satisfy the given conditions, and also the negative of the conclusion, so that $|p(S) \cap L| = n$ for all rigid motions p;

we shall deduce a contradiction. The first step is to show the existence of some p for which

$$|p^{-1}(L) \cap E| = 1 \tag{4}$$

exactly, where E denotes the metrical boundary of S.

For, by a Fubini argument as in Theorem (2.1.1), we have that the (measurable) set S must have measure exactly n. Hence Lemma (2.1.14) is applicable to it, and S has a non-empty metrical boundary E that is of measure 0. And Lemma (2.1.12) applied to this E gives (4), as desired.

Finally, we start from the position of the lattice just guaranteed, i.e. satisfying (4). Let L_0 be the one point specified there. We consider making small translations $t_{\xi,n}$, say with $|\xi|, |\eta| \le \delta$, of the lattice from this position (with S remaining fixed throughout). It is now convenient to use the language of probability theory, supposing these translations to be made randomly with equal-area probability in the square $|\xi|, |\eta| \le \delta$.

Since S is essentially bounded, we may restrict consideration (by ignoring sets of measure 0 in (ξ, η)) to a finite number K of specified (moving) lattice points. Given small fixed $\varepsilon, \varepsilon' > 0$, the definitions of density and of the metrical boundary E of S imply:

(i) for each of the K lattice points mentioned, except L_0, the probability of $t_{\xi,\eta}$ shifting the status of the point from being in S to being not in S, or vice versa, is less than ε: this is true for all sufficiently small ε, and for all sufficiently small $\delta = \delta(\varepsilon)$;

(ii) for L_0 the probability of its shifting its status as above is greater than ε': this is true for all sufficiently small ε', and some small $\delta = \delta(\varepsilon')$.

Thus, if we arrange that $(1-\varepsilon)^{K-1} > 1-\varepsilon'$, as we may, then, by simple probability considerations, there is some ξ, η for which the $K-1$ lattice points each remain in or out of S, but for which L_0 is either 'captured', or else 'escapes'. For such a $t_{\xi,\eta}$ we may take p as its inverse to obtain:

$$|p(S) \cap L| = |S \cap p^{-1}(L)| = n \pm 1,$$

the desired contradiction with the supposition $|S \cap p^{-1}(L)| = n$ for all p.

On the other hand, jettisoning measurability renders the problem of a quite different nature. We inclined to believe that the Conjecture (c) has an affirmative answer, at least if we assume sufficient axioms of set theory, although a simple-minded attempt to build up such an S using transfinite induction fails.

We recall the (compact) space II, defined above, of rigid motions p, characterized by the 3 parameters θ, x, y; let π denote measure in this 3-dimensional space, and let $\mathcal{M}_n$ denote the set of pell for which $|p(S) \cap L| = n$ exactly. If $\mathcal{M}_n$ is non-empty, we say S represents the integer n; if $\pi(\mathcal{M}_n) > 0$, we say S represents n essentially.

Conjecture (2.1.16)[44]: Let S be a plane measurable set. If $\pi(\mathcal{M}_{n-r}) > 0$ and

$\pi(\mathcal{M}_{n+s}) > 0$ (some $, s \geq 1$) , then necessarily $\pi(\mathcal{M}_n) > 0$. That is, the (finite) set of integers essentially represented by S consists of a set of consecutive integers. (We might even hope for some inequalities between the $\pi(\mathcal{M}_n)$ for different n.)

Corollary (2.1.17)[44]: If $\mu(S) = n$ (integral), then there exists some rigid motion p such that $|p(S) \cap L| = n$.

Proof of Corollary (2.1.17) from Conjecture (2.1.16). Let π^* denote 2-dimensional measure in (x, y) of sets in subspaces $\theta = \theta_0$ of the space II. Suppose, if possible, that the corollary were false; then II consists of 2 sets $\mathcal{M}^+, \mathcal{M}^-$ being sets of points representing p's for which $|p(S) \cap L|$ is greater than, respectively less than, n. Fubini's Theorem gives us that the 'average' number of lattice points covered by $p(S)$ (with p now a translation, $\theta = \theta_0$ throughout) is n. So $\pi^*(\mathcal{M}^+) > 0, \pi^*(\mathcal{M}^-) > 0$, for each θ_0. Hence $\pi(\mathcal{M}^+) > 0, \pi(M^-) > 0$, and the conjecture is then untrue.

We observe that the corollary fails if 'rigid motion p' is replaced by 'translation t' in its statement. For a counter-example, we may take S to be a square (open or closed) of side $\sqrt{3}$ with axes parallel to those of L.

Further, Conjecture (2.1.16) fails even for 'nice' sets if the word 'essentially' is omitted: for S a closed disc of radius $\sqrt{5}$, the integers represented include 17 and 21, but not 18, 19, or 20; for S an open punctured disc of radius $\sqrt{5}$ with the centre removed, they include 12 and 14 but not 13.

One way to tackle Conjecture (2.1.16) is perhaps to show that the common metrical boundary of the 2 sets $\bigcup_{k<n} \mathcal{M}_k$, $\bigcup_{k>n} \mathcal{M}_k$ is 'large', in particular 2-dimensional.

(But we do not even know simpler results about metrical boundaries for sets in $\mathbb{R}^2$; e.g. what sets $E \subset \mathbb{R}^2$ of measure 0 are metrical boundaries of some set S? We might simple-mindedly hope, for example, that such an E is a regular, linearly-measurable set, in the sense of Besicovitch [45] (see also [60] 128-130, and [56]), and that its 'length' λ satisfied the isoperimetrical inequality $\lambda^2 \geq 4\pi\mu(S)$.)

A further conjecture not involving lattice points, but with a similar underlying theme is:

Conjecture (2.1.18)[44]: Let S be plane measurable with $0 < \mu(S) < \infty$. Then there exists a $\delta = \delta(S) > 0$ such that for each x $(0 < x < \delta)$, there is a line I such that the linear measure $\mu(S \cap l) = x$; and, moreover, we can choose l to vary continuously with x in the range $0 < x < \delta$.

The obvious approach here is to prove that $\mu(S \cap l)$ is continuous in l at $l = l_0$ if $\mu(E \cap l_0) = 0$ where E is the metrical boundary of S, and then show that $\mu(E \cap l_0) > 0$ for only a small set of l_0, and apply a connectivity argument in the space of lines l. Unfortunately the possibility of E's containing a Besicovitch set (i.e. one having plane measure 0 but containing a unit line-segment in every direction) or something similar, seems to render this approach at least complicated.

One curious aspect of Conjecture (2.1.18) is that the d (≥ 3) dimensional analogue (with hyperplanes replacing lines) has an easy affirmative answer.

This is an immediate consequence of some recent work of Falconer [49]: he proves, in $\mathbb{R}^d (d \geq 3)$, that if $F(\theta, t)$ is the 'sectional integral' of a measurable bounded function f of compact support, i.e. the $(d-1)$- dimensional integral of the function f over a hyperplane, at orientation θ, and distance t, then F is continuous in t, for p.p. θ. Letting f be the characteristic function of our d-dimensional set S, we obtain what we need by merely moving the hyperplane at constant orientation θ, providing we avoid a set of θ of measure 0.

This disparity between $d = 2$ and $d > 2$ dimensions is in sharp and curious contrast with the theorems and lemmas whose proofs generalize immediately to higher dimensions (although a little care is needed with Lemma (2.1.12)).

Finally, Dr. Falconer proposes the following

Conjecture (2.1.19)[44]: Let K_h be the set of vertical lines in the plane given by $\{(x, y): x = nh,\ n \in \mathbb{Z}, y \in \mathbb{R}\}$; let S have $0 < \mu(S) < \infty$. Suppose that for all rigid motions p, we have that $p(S) \cap K_1$ has linear measure c, where c is independent of p; then $c = 0$.

He observes that this would imply Theorem (2.1.15) above. Further, he believes that he can show, with the use of Fourier transforms, the equivalence of the hypotheses:

‘$p(S) \cap L$ is constant for all p’ and ‘$p(S) \cap K_r$ is constant for all p, for $r = 1/\sqrt{(m^2 + n^2)}$ for all $m, n \in \mathbb{Z}^+$’.

Section (2.2): Partitions of Lines and Space

In 1951, Sierpinski [77] showed that the continuum hypothesis is equivalent to the following: for the partition of the lines in $\mathbb{R}^3$ parallel to one of the coordinate axes into the disjoint sets L_1,L_2, and L_3, where L_i consists of all lines parallel to the ith axis, there is a partition of $\mathbb{R}^3$ into disjoint sets, S_1, S_2, and S_3, such that any line in L_i meets at most finitely many points in S_i. He also showed that the corresponding statement for $\mathbb{R}^4$, using L_1, L_2, L_3 and L_4 and four sets S_1, S_2, S_3 and S_4, is equivalent to $2^\omega \leq \omega_2$. Also, the corresponding statement for $\mathbb{R}^2$, using sets of lines L_1 and L_2 and sets S_1 and S_2, is false. He obtained analogous results by replacing “finite” by “countable”. Thus, CH is equivalent to the assertion that $\mathbb{R}^2$ can be divided into two disjoint sets S_1 and S_2 with each line in L_i meeting S_i in a countable set [76]. He showed that the countable version for

$\mathbb{R}^3$ with three sets is equivalent to $2^\omega \leq \omega_2$. These theorems were generalized by Kuratowski [75] and Sikorski [78]. Erd″os [69] raised the issue of whether these results could be further strengthened by considering partitions of all lines rather than just those lines parallel to some coordinate axis. Davies [67] showed that an analogous result is obtained if one partitions the lines in $\mathbb{R}^k$, $k \geq 2$, which are parallel to one of $L_1, \ldots, L_p$, where $L_1, \ldots, L_p$ are fixed pairwise non-parallel lines (and one partitions the lines according to which L_i it is parallel to). This result was extended by Simms [80], who considered translates of linear subspaces instead of just lines. Simms' result also generalizes Sikorski's result, and gives best possible bounds for the type of partitions it considers. Davies [68] later removed the restriction that the lines in $\mathbb{R}^k$ be partitioned in the special manner referred to above. Bagemihl [65] has also extended some of these results. See Simms [79] for an extensive historical survey.

We develop a general framework within which these theorems can be obtained as corollaries. Our framework deals with arbitrary partitions of all lines (or planes, or more general objects) and not necessarily special partitions or families of lines. The central issues are the number of sets of lines in the partition, the allowed size of the intersection of a line in a given set with the corresponding set in the decomposition of the space, and the value of the continuum. Galvin and Gruenhage [72], and independently Bergman and Hrushovski (cf. Proposition 19 of [66]), have previously obtained results which imply special cases of some of our results. In particular, those results yield (a)⇒(b) for the case $\theta = 0$ and $p = s + 2$. Corollary (2.2.21) also follows from [68] and unpublished results of [72]. We deal with some perhaps surprising phenomena arising from infinite partitions. In particular, we show that some interesting set-theoretic properties come into play.

We should mention that some of the key ideas of our arguments go back to combinatorial arguments of Erd″os and Hajnal [71.

If t is a positive integer, then $\text{card}(A) = |A| \leq \omega_{-t}$ means A is finite. If $\theta = \bar{\theta} + s$, where $\bar{\theta} > 0$ is a limit ordin and s is an integer, and t is an integer with $t > s$, then $|A| \leq \omega_{\theta-t}$ means $|A| < \omega_\theta$.

Before we prove Theorem (2.2.5), let us make some comments and derive some corollaries.

The first corollary yields Sierpiński's theorem as a special case and answers question a) in [69].

Corollary (2.2.1)[64]: The following are equivalent:

(i) CH , the continuum hypothesis, holds: $2^\omega = \omega_1$.

(ii) If the lines in $\mathbb{R}^3$ are decomposed into three sets L_i $(i = 1,2,3)$, then there exists a decomposition of $\mathbb{R}^3$ into three sets S_i such that the intersection of each line of L_i with the corresponding set S_i is finite.

Proof. Take $\theta = 1, n = 3$ and $p = 3$ in Theorem (2.2.5). Then each line in L_i meets S_i in a set of size at most $\omega_{\theta-p+1} = \omega_{-1}$, which by our convention means finite.

The second corollary yields the Bagemihl–Davies theorem [79] as a special case and notes that the condition that we be in $\mathbb{R}^3$ in Corollary (2.2.1) is not necessary. This also answers question b) in [69].

Corollary (2.2.2)[64]: The following are equivalent:

(i) $2^\omega = \omega_1$.

(ii) If the lines in $\mathbb{R}^2$ are decomposed into three sets L_i $(i = 1,2,3)$, then there exists a decomposition of $\mathbb{R}^2$ into three sets S_i such that the intersection of each line of L_i with the corresponding set Si is finite.

Proof. Take $\theta = 1, n = 2$ and $p = 3$ in Theorem (2.2.5).

The next corollary is a theorem of Kuratowski [75].

Corollary (2.2.3)[64]: Let $n \in \omega$, and $\bar{\theta}$ be a limit ordinal or zero. The following two statements are equivalent:

(i) $2^\omega < \omega_{\bar{\theta}}$.

(ii) There is a partition of $\mathbb{R}^{n+1}$, $\mathbb{R}^{n+1} = S_1 \cup \ldots \cup S_{n+1}$, such that $|l \cap S_i| < \omega_{\bar{\theta}}$ whenever l is parallel to the i −th axis.

Proof. As Kuratowski mentions, the case $n = 0$ is easy. If $n > 0$, take $\theta = \theta + (n - 1)$ and $p = n + 1$ in Theorem (2.2.5). Thus, $2^\omega \le \omega_\theta < \omega_{\bar{\theta}}$ if and only if $\mathbb{R}^{n+1} = \bigcup_{i=1}^{n+1} S_i$, where $|l \cap S_i| \le \omega_{\theta-p+1} = \omega_{\bar{\theta}}$ provided l is parallel to the ith axis. This means, by our convention, that $|l \cap S_i| < \omega_{\bar{\theta}}$, as required.

This yields Davies' theorem:

Corollary (2.2.4) (Davies). Let $n \ge 2$, and let $l_1, \ldots, l_p, p \ge 2$, be nonparallel lines in $\mathbb{R}^n$. Then the following are equivalent:

(i) $2^\omega \le \omega_\theta$.

(ii) There is a partition $\mathbb{R}^n = \bigcup_{i=1}^{p} S_i$ of the points in $\mathbb{R}^n$ such that for every line l parallel to l_i, $|l \cap S_i| \le \omega_{\theta-p+1}$.

Theorem (2.2.5)[64]: Let θ be an ordinal, $\theta = \bar{\theta} + s \ge 1$, where $\bar{\theta}$ is 0 or a limit ordinal, and let $s \in \omega$. The following statements are equivalent:

(a) $2^\omega \le \omega_\theta$.

(b) For e a c h $n \ge 2$ and for e a c h partition of L, the set of all lines in $\mathbb{R}^n$, into $p \ge 2$ disjoint sets, $L = L_1 \cup L_2 \cup \ldots \cup L_p$, there is a partition of $\mathbb{R}^n$ into p disjoint sets, $\mathbb{R}^n = S_1 \cup S_2 \cup \ldots \cup S_p$, such that each line in L_i meets S_i in a set of size $\le \omega_{\theta-p+1}$.

(c) For s o m e $n \ge 2$, s o m e p, with $s + 2 \ge p \ge 2$, and s o m e non-parallel lines $l_1, \ldots, l_p$ in $\mathbb{R}^n$, if we let L_i be the set of all lines in $\mathbb{R}^n$ parallel to l_i, then there is a partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_p$ such that every line in L_i meets S_i in a set of size $\leq \omega_{\theta-p+1}$.

Proof. We introduce an auxiliary Proposition (2.2.6) for integer $p \geq 2$.

Proposition (2.2.6)[64]: For each ordinal θ, if A is a set of lines and points in $\mathbb{R}^n$ of size at most ω_θ, and the set of lines in A, which we call L, is divided into k disjoint sets, $L = L_1 \cup \ldots \cup L_k$, where $k \geq p$, and if f is a function with domain S, the set of points in A, such that for all $x \in S, f(x) \subseteq \{1, \ldots, k\}$ and $|f(x)| \leq k - p$, then there is a partition of S into k sets, $S = S_1 \cup \ldots \cup S_k$, such that for each $x \in S$:

a) $x \notin S_a$, if $a \in f(x)$.

b) Each line l in L_i meets at most $\omega_{\theta-p+1}$ points in S_i .

We think of $f(x)$ as being forbidden "colors" for x. Thus, the hypothesis of $Q(p)$ requires there to be at least p non-forbidden colors for each point $x \in A$. Note that $Q(p)$ for all $p \geq 2$ yields $(a) \Rightarrow (b)$ of Theorem (2.2.5) by taking $k = p$ and f the function with constant value $\emptyset$.

We establish $Q(p)$, working in ZFC, by induction on p. So, assume first that $p = 2$. Let A be a set of points and lines in $\mathbb{R}^n$ of size $\leq \omega_\theta$, for some ordinal θ (we allow $\theta = 0$). Let $L = L_1 \cup \ldots \cup L_k$ be a partition of the lines in A with $k \geq p$, and let f be as in the statement of $Q(p)$. We define the partition $S = S_1 \cup \ldots \cup S_k$ of the points in A as required. Let $\{l_1^\alpha\}, \ldots, \{l_k^\alpha\}$ and $\{x^\alpha\}, \alpha < \omega_\theta$, enumerate the lines in $L_1, \ldots, L_k$, and the points of S, respectively. We inductively decide to which S_i we add x^α. Suppose we are at step $\alpha < \omega_\theta$ and we have decided for all $\beta < \alpha$ to which S_i we add x^β . Consider the following cases.

Case I. For some $1 \leq i \leq k$ such that $i \notin f(x\,\alpha)$, and all $\beta < \alpha, x\,\alpha \notin l_i^\beta$. In this case add x^α to S_i (choose i arbitrarily if the above is satisfied for more than one i).

Case II. For all $1 \leq i \leq k$ with $i \notin f(x\,\alpha), x^\alpha$ lies on some $l_i^{\beta(i)}$, with $\beta(i) < \alpha$. Let $\beta(i)$ in fact be the least such ordinal $< \alpha$. Let $i_0 \notin f(x^\alpha)$ be such that $\beta(i_0) \geq \beta(i)$ for all $i \notin f(x^\alpha)$. We then add x^α to S_{i0} .

Thus, we have defined a partition $S = S_1 \cup \ldots \cup S_k$. Fix now a line $l_i^\delta \in L$. We show that $|S_i \cap l_i^\delta| \leq \omega_{\theta-p+1} = \omega_{\theta-1}$ (this means, by our convention, that $|S_i \cap l_i^\delta| < \omega_\theta$). First, we need only consider those points x^α with $\alpha > \delta$, since there are $< \omega_\theta$ points x^α with $\alpha \leq \delta$. If x^α were put in S_i by virtue of Case I, then x^α would not lie on l_i^δ . Suppose then that $x^\alpha, \alpha > \delta$, is put in S_i by virtue of Case II. Thus, $\beta_j\,(\alpha)$ is defined for each $j \notin f(x\,\alpha)$. Since x^α is put into S_i , we have $\beta_i(\alpha) \geq sup\{\beta_j\,(\alpha) : j \notin f(x^\alpha)\}$. If $\beta_i(\alpha) > \delta$, then by Definition (2.2.7), $x^\alpha \notin l_i^\delta$. Thus, we need only consider x^α for which $\delta \geq \beta_i(\alpha) \geq sup\{\beta_j\,(\alpha) : j \notin f(x^\alpha)\}$. There are $< \omega_\theta$ possibilities for the set $\{\beta_j\,(\alpha)\}$. Since $k - f(x\,\alpha)$ has at least two elements, and two lines determine a point, it follows that the set of such x^α has size $< \omega_\theta$. This completes the proof of $Q(2)$.

Note, in particular, that $Q(2)$ holds when $\theta = 0$, that is, when A is countable. However, for countable A, $Q(2)$ easily implies $Q(p)$ for all $p \geq 2$ as well (since in this case $|l \cap S_i| \leq \omega_{\theta-p+1}$ means the same thing, i.e., $l \cap S_i$ is finite, for all $p \geq 2$).

Before giving the inductive step in the proof of $Q(p)$, we introduce a basic Definition (2.2.7).

Definition (2.2.7)[64]: If A is a collection of lines and points in $\mathbb{R}^n$, we call A good if it satisfies the following:

a) For any two distinct points $x, \mathcal{Y} \in A$, the line determined by x and y is also in A.

b) For any two distinct intersecting lines in A, the point of intersection is also in A.

Clearly, for any infinite set A of lines and points in $\mathbb{R}^n$, the good set generated by A has the same cardinality as A.

Assume now that $Q(p)$ holds, and we show $Q(p+1)$. Let A be a collection of lines L and points S in $\mathbb{R}^n$ with size ω_θ, and let $L = L_1 \cup \ldots \cup L_k$ be a partition of L with $k \geq p+1$. We may assume $\theta \geq 1$ by our note above. Without loss of generality, we may also assume that A is good. Let $f : S \to \{1, \ldots, k\}$ be given with $|f(x)| \leq k - (p+1) = k - p - 1$. Express A as an increasing union, $A = \bigcup_{\alpha < \omega_\theta} A_a$. where each A_α is good, and $|A_\alpha| \leq \omega_{\theta-1}$. We call a line $l \in A_\alpha$ "new" if $l \in A_\alpha \setminus \bigcup_{\beta < a} A_\beta$. and otherwise call l "old" (relative to α). We label the points of A_α as new and old in the same fashion. We define at step α the partition of S_α, the set of new points in $A_\alpha, S_\alpha = S_1^\alpha \cup \ldots \cup S_k^\alpha$. Suppose we are at step $\alpha < \omega_\theta$. Enumerate $L_{\alpha,i}$, the new lines of L_i in A_α, and points of A_α into type $\omega_\sigma(\alpha) < \omega_\theta$, say $l_{\alpha,i}^\beta, x_{\alpha,\beta}^\beta < \omega_{\sigma,1} \leq i \leq k$. Note that for each $\beta < \omega_{\sigma(\alpha)}$, each x_α^β lies on at most one old line, since each A_δ is good. Thus, for $\beta < \omega_{\sigma(\alpha)}$, set $f_\alpha(x_\alpha^\beta) = f(x_\alpha^\beta) \cup \{j\}$, where j is such that x_α^β lies on an old line in L_j if one exists, and otherwise set $f_\alpha(x_\alpha^\beta) = f(x_\alpha^\beta)$. Thus, fα maps the new points of A_α into $\{1, \ldots, k\}$ and $|f_\alpha(x)| \leq k - p$. Now, by the induction hypothesis $Q(p)$ applied to $\omega_{\sigma(\alpha)}$, we may partition the points in $S_\alpha, S_\alpha = S_1^\alpha \cup \ldots \cup S_k^\alpha$, so that any new line $l_{\alpha,i}^\beta$ intersects at most $\omega_{\sigma(\alpha)-p+1}$ points from S_i^α, and $x_\alpha^\beta \notin S_\alpha^a$ for any $a \in f_\alpha(x_\alpha^\beta)$. Note that $\omega_{\sigma(\alpha)-p+1} \leq \omega_{\theta-p}$.

This defines our partition of S. To show this partition works, fix a line l in A, say $l \in L_i$. Let α be the least such that $l \in L_{\alpha,i}$, so that l is a new line at step α. We must show that $\leq \omega_{\theta-p}$ points in $S_i = S_\alpha^\gamma$, in Si cannot lie on l, since then $i \in f_\gamma(x_\gamma^\delta)$, but, by construction, $x_\gamma^\delta \notin S_i$. So, we may assume $\gamma \leq \alpha$. Now, there is at most one point x_γ^δ for $\gamma < \alpha$ on the line l, since otherwise l would not be new at α. Thus, we need only consider points of the form $x_{\alpha,\delta}^\delta < \omega_{\sigma(\alpha)}$. However, from the Definition (2.2.7) of the set $S_i^\alpha, \leq \omega_{\sigma(\alpha)-p+1}$ of these points lie on $l \in L_{\alpha,i}$.

Thus, each line in L_i intersects $\leq \omega_{\theta-p}$ points of S_i . Since $f_\alpha(x_\alpha^\beta) \supset f(x_\alpha^\beta)$ for all $x_\alpha^\beta \in S$, we also have $x_\alpha^\beta \notin S_a$ if $a \in f(x_\alpha^\beta)$. This completes the proof of the proposition $Q(p+1)$ and, as mentioned, the proof that (a) implies (b).

Since (b) clearly implies (c), it only remains to prove (c) implies (a). Assume now (c) holds, with $\theta = \bar{\theta} + s$ and $2 \leq p \leq s+2$. Towards a contradiction, assume $2^\omega \geq \omega_{\theta+1}$. Let $l_1, \ldots, l_p$ and $L_1, \ldots, L_p$ and $S_1, \ldots, S_p$ be as in (c). For each $i, 2 \leq i \leq p$, let v_i be a vector parallel to l_i with $\|v_i\| = 1$. We construct sets $B_1, \ldots, B_p$ as follows. Let $B_1 \subseteq l_1 = \{x_0 + t_{vi} : t \in \mathbb{R}\}$ be any set of size $\omega_{(\theta-p+1)+1} \geq \omega_\theta$. Assume $1 \leq i \leq p-1$ and B_i has been defined with $|B_i| = \omega_{(\theta-p+1)+i} < 2^\omega$. Let D_i be the set of all distances between two distinct points of B_i . So, $|D_i| = |B_i|$. Let C_{i+1} be a subset of $\mathbb{R}$ such that $|C_{i+1}| = \omega_{(\theta-p+1)+(i+1)}$ and $(C_{i+1} - C_{i+1}) \cap D_i = \emptyset$ (where $A - B := \{a - b : a \in A, b \in B\}$). Let $B_{i+1} = \bigcup_{c \in C_{i+1}} [cv_{i+1} + B_i] = \bigcup_{x \in B_i} \left[x + x \in B_i \bigcup_{c \in C_{i+1}} cv_{i+1}\right]$. Thus, B_{i+1} consists of $\omega_{(\theta-p+1)+(i+1)}$ translates of B_i in the direction of l_{i+1}. Also, notice that these translates of B_i form a pairwise disjoint family. Finally, since $2^\omega \geq \omega_{\theta+1} = \omega_{(\theta-p+1)+p}, B_p$ is defined.

Consider first B_{p-1}. Since $|B_{p-1}| = \omega_\theta$, and since each line parallel to l_p through a point of B_{p-1} intersects S_p in at most $\omega_{\theta-p+1}$ points, $|S_p \cap B_p| \leq \omega_\theta$. But, since B_p consists of $\omega_{\theta+1}$ disjoint translates of B_{p-1}, there is some $c_p \in C_p$ such that $c_p v_p + B_{p-1} \subseteq S_1 \cup \ldots \cup S_{p-1}$. If $p = 2$, stop; otherwise, continue. So, in general, suppose $3 \leq j \leq p$ and we have produced numbers $c_i \in C_i$, for $j \leq i \leq p$, such that $e_j + B_{j-1} \subseteq S_1 \cup \ldots \cup S_{j-1}$, where $e_j = c_p v_p + c_{p-1} v_{p-1} + \ldots + c_j v_j$. Now, $e_j + B_{j-1} = \bigcup_{c \in C_{j-1}} [e_j + cv_{j-1} + B_{j-2}]$, the translates in this union being pairwise disjoint, and $|C_{j-1}| = \omega_\theta - p + j$. Since S_{j-1} contains at most $\omega_\theta - p + j^{-1}$ points of this union, there is some $c_{j-1} \in C_{j-1}$ such that $e_j + c_{j-1} v_{j-1} + B_{j-2} \subseteq S_1 \cup \ldots \cup S_{j-2}$. Finally, we have $\bar{B}_1 = e_2 + B_1 = c_p v_p + c_{p-1} v_{p-1} + \ldots + c_2 v_2 + B_1 \subseteq S_1$. As $\bar{B}_1$ is a translate of $B_1, |\bar{B}_1| = |B_1| = \omega_{\theta-p+2}$. But, also, $\bar{B}_1$ is a subset of the line through $x_0 + e_2$ parallel to l_1. Thus, $|\bar{B}_1| \leq \omega_{\theta-p+1}$. This is a contradiction.

Further generalizations are possible. The only properties of lines that were used in the preceding argument were that two distinct lines determine at most one point and two distinct points determine a line. We generalize this as follows. Definition (2.2.7). Let $H \subseteq \mathcal{P}(\mathbb{R}^n)$ be a family of subsets of $\mathbb{R}^n$. Let r and s be positive integers. We say that H is (r, s) finitely determined if the following are satisfied:

(a) The intersection of any r distinct elements of H is finite.

(b) For any s distinct points in $\mathbb{R}^n$, there are at most finitely many $h \in H$ which contain all those points.

Example (2.2.8)[64]: The set H of all circles in $\mathbb{R}^n$ $(n \geq 2)$ is (2, 3) finitely determined.

Example (2.2.9)[64]: The set H of all hyperplanes in $\mathbb{R}^n$ perpendicular to a coordinate axis is $(n, 1)$ finitely determined. Somewhat more generally still, we introduce the notion of a partition being (r, s) finitely determined.

Definition (2.2.10)[64]: Given $H \subseteq P(\mathbb{R}^n)$, we say a partition $H = H_1 \cup \ldots \cup H_k$ (k can be infinite here) is (r, s) finitely determined if:

(a) The intersection of r distinct elements of H lying in different H_i is finite.

(b) For any s distinct points in $\mathbb{R}^n$, there are at most finitely many $h \in H$ containing these s points.

Note that if $H \subseteq \mathcal{P}(\mathbb{R}^n)$ is an (r, s) finitely determined family of sets, then any partition of H is (r, s) finitely determined.

Theorem (2.2.5) may be generalized as follows, where our convention is still in force.

Theorem (2.2.11)[64]: Let $\theta \geq 1$ be an ordinal. The following are equivalent:

(i) $2^\omega \leq \omega_\theta$.

(ii) For each positive integer t, for each $n \geq 1$, and for any $r \geq 2, s \geq 1$, if $H = H_1 \cup \ldots \cup H_p$ is a_n (r, s) finitely determined partition of some $H \subseteq P(\mathbb{R}^n)$ into $p = t(r - 1) + 1$ disjoint sets, then there is a partition of $\mathbb{R}^n, \mathbb{R}^n = S_1 \cup \ldots \cup S_p$, such that $|h \cap S_i| \leq \omega_{\theta - t}$ for all $h \in H_i$,$1 \leq i \leq p$.

Proof. The proof that (i) implies (ii) is similar to that of Theorem (2.2.5). As there, we formulate an auxiliary Proposition (2.2.12), for $t \geq 1$, which we prove in ZFC by induction on t.

Proposition (2.2.12)[64]: For each ordinal θ, k, and integers $n \geq 1, r \geq 2, s \geq 1$, if $H = H_1 \cup \ldots \cup H_k$ is an (r, s) finitely determined partition of $H \subseteq P(\mathbb{R}^n)$ into $k \geq t(r - 1) + 1$ pieces, then if $A \subseteq H \cup \mathbb{R}^n$ is a set consisting of some elements of H and points, S, of $\mathbb{R}^n$ with $|A| \leq \omega_\theta$, and f is a function from S into $P(\{1, \ldots, k\})$ such that for all $x \in S$ we have $|f(x)| \leq k - [t(r - 1) + 1]$, then there is a partition of S into k sets, $S = S_1 \cup \ldots \cup S_k$, such that for each $x \in S, x \notin S_a$ if $a \in f(x)$, and if $h \in H_i \cap A$, then $|h \cap S_i| \leq \omega_{\theta - t}$, for $1 \leq i \leq k$.

The proof for $t = 1$ proceeds exactly as the proof of Theorem (2.2.5) for $p = 2$. Again, the determination of which set x^α should be placed into breaks into two cases. In the first case, for some $i \notin f(x^\alpha)$ we have $x^\alpha \notin h_i^\gamma$ for all $\gamma < \alpha$, and x^α is placed in some S_i with i in this set. For the x^α in the second case, one obtains a function $x^\alpha \to (\beta(i_1(\alpha)), \ldots, \beta(i_g(\alpha)))$, where $g \geq r$ and the $i_j(\alpha)$ list the i's such that x^α lies on some h_i^γ , with $\gamma < \alpha$, and $\beta(i_j(\alpha))$ is the least such γ. This function is not necessarily one-to-one as in Theorem (2.2.5), but, from the first condition of being (r, s) finitely determined, the function is finite-to-one. This is sufficient for the argument.

Note, as in Theorem (2.2.5), that if $\theta = 0$, then $R(1)$ easily implies $R(t)$ for all t. Thus, we may assume in the inductive step that $\theta \geq 1$.

The inductive step for obtaining $R(t+1)$ from $R(t)$ is similar to that for $Q(p)$. Perhaps it should be noted that in obtaining $R(t+1)$ from $R(t)$, one builds, as before, an increasing transfinite sequence of "good" sets, A_α, with $A = \bigcup_{\alpha<\omega_\theta} A_\alpha$ and $||A_\alpha| < \omega_\theta$ being good now means that if $h_1, \ldots, h_r$ are elements of distinct sets $A_\alpha \cap H_j$, then $\bigcap h_i \subseteq A_\alpha$, and for any s distinct points of A_α, the finitely many elements of H containing these points are in A_α. Since the partition of H is (r, s) finitely determined, the cardinality of the good set generated by an infinite set does not increase. The argument then proceeds as before.

To prove (ii) implies (i), take $t = 1$ and $n = 2$. Let H_i be the set of lines parallel to the ith axis. So, the partition is (2, 1) finitely determined. Applying (ii) to this family, we have $p = r = 2$ and $t = p - 1$. So, there is a partition $\mathbb{R}^2 = S_1 \cup S_2$ such that $|h \cap S_i| \le \omega_{\theta-t} = \omega_{\theta-p+1}$. Since (c) implies (a), $2^\omega \le \omega_\theta$.

Corollary (2.2.13)[64]: (Sikorski). The continuum hypothesis is equivalent to the following statement. The points in $\mathbb{R}^3$ can be partitioned into three sets S_1, S_2 and S_3 such that each plane perpendicular to the x_i axis meets S_i in at most countably many points.

Proof. If H = planes in $\mathbb{R}^3$ perpendicular to a coordinate axis, then H is (3, 1) finitely determined. Now, take $\theta = 1 = t$ in Theorem (2.2.11). The proof of the converse may be found in [78] or done directly. Of course, our proof also works for any partition of the planes in $\mathbb{R}^3$ which is $(3, s)$ finitely determined for some s.

As another example, consider the analog of Corollary (2.2.4) where "countable" is replaced by "finite". We first show that four "colors" are not sufficient (note: Lemma (2.2.14) and one direction of Corollary (2.2.16) follow of [79], [80], but are included here for the sake of completeness).

Lemma (2.2.14)[64]: There are four unit vectors, v_1, v_2, v_3 and v_4, in $\mathbb{R}^3$ such that if $H_i = \{h : h \text{ is a plane with normal } v_i\}$, then the partition $H_1 \cup \ldots \cup H_4$ is (3, 1) finitely determined, and yet there is no partition $\mathbb{R}^3 = S_1 \cup \ldots \cup S_4$ such that $|h \cap S_i| < \omega_0$ for all $h \in H_i$.

Proof. Let $v_i, i = 1, 2, 3$, be the canonical unit basis vectors for $\mathbb{R}^3$. Let $v_4 = 0, -\sqrt{2}/2, \sqrt{2}/2$. Let $A_1, A_2 \subseteq \mathbb{R}$ with $|A_1| = \omega_0$ and $|A_2| = \omega_1$, and let $A_3 = \mathbb{Q}$, the rationals. Let $G \subseteq \mathbb{R}$ be such that $|G| = \omega_1$ and $(G - G) \cap Q = \{0\}$. Let $W = \{(0, t, t) : t \in G\}$. Let $B = A_1 \times A_2 \times A_3$ and $E = B + W$. The following claim (2.2.15) suffices to finish the proof of the lemma.

Claim (2.2.15)[64]: For each $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $E + u \nsubseteq S_1 \cup \ldots \cup S_4$, where each S_i meets each plane with normal v_i in a finite set.

Proof. Fix u, and assume such sets S_i exist. For each $\mathcal{Y} \in A_2$, let $E_\mathcal{Y} = [A_1 \times \{\mathcal{Y}\} \times A_3] + W + u$. Then $E + u = \bigcup_{\mathcal{Y} \in A_2}$ Ey. To see that these sets are disjoint, notice that otherwise we would have $(a_1, \mathcal{Y}_1, q_1) + w_1 = (a_2, y_2, q_2) + w_2$, with $w_1 \neq w_2$. But this would imply $t_1 - t_2 \in \mathbb{Q}$ for some two distinct elements of G. Now, for each $x_1 \in A_1$, the plane $x = x_1 + u_1$ meets only finitely many points of S_1. Thus, $S_1 \cap (E + u)$ is countable and so there is some $\mathcal{Y}_0 \in A_2$ such that $E\mathcal{Y}_0 \subseteq S_2 \cup S_3 \cup S_4$. For each $(x, z) \in A_1 \times \mathbb{Q}$, the plane $h(x, \mathcal{Y}_0, z)$ passing through $(x, \mathcal{Y}_0, z) + u$ with normal v_4 meets only finitely many points of S_4. But $E_{y0} = \bigcup_{w\in W} [(A_1 \times \{y_0\} \times A_3) + w + u]$ and the sets in this union are disjoint. So, there is some $w_0 \in W$ such that $(A_1 \times \{y_0\} \times A_3) + w_0 + u \subseteq S_2 \cup S_3$. But this set lies in a plane with normal v_2. So, only finitely many points of this set are in S_2. Thus, there is some $z_0 \in \mathbb{Q}$ such that $(A_1 \times \{y_0\} \times \{z_0\}) + w_0 + u \subseteq S_3$. But this set is an infinite subset of a plane with normal v_3 and S_3 meets this plane in a finite set.

Corollary (2.2.16)[64]: The continuum hypothesis is equivalent to the following statement. If H is a set of planes in $\mathbb{R}^3$ and $H = H_1 \cup \ldots \cup H_5$ is a partition of H which for some s is $(3, s)$ finitely determined, then there is a partition $\mathbb{R}^3 = S_1 \cup \ldots \cup S_5$ such that each plane in H_i meets S_i in a finite set. More generally, the hypothesis $2^\omega \le \omega_n$ is equivalent to the above statement, where $H = H_1 \cup \ldots \cup H_5$ is replaced by $H = H_1 \cup \ldots \cup H_{2n+3}$.

Proof. If $2^\omega = \omega_1$, take $\theta = 1$ and $t = 2$ and apply Theorem (2.2.11). To prove the converse in this case, assume $2^\omega \ge \omega_2$. Let us follow the same notation used in the proof of Lemma (2.2.14). Let v_5 be a unit vector, $v_5 \ne v_i$, $1 \le i \le 4$, and $H_5 = \{h : h \text{ is a plane normal to } v_5\}$. Let $F \subseteq \{x : (x, v_5) = 0\}$ such that $(F - F) \cap (E - E) = \{0\}$ and $|F| = \omega_2$. Let $S_1, \ldots, S_5$ be the required partition of $\mathbb{R}^3$. Set $M = \bigcup_{f\in F} E + f = \bigcup_{e\in E} e + F$, the sets in each union being disjoint. For each $e \in E$, $|S_5 \cap (e + F)| < \omega_0. So$, $|S_5 \cap F| \le \omega_1$. Thus, there is a vector $f \in F$ such that $E + f \subseteq S_1 \cup \ldots \cup S_4$. This contradicts the Claim (2.2.15) in the proof of Lemma (2.2.14).

The argument for this direction can be strengthened slightly. We may take $F \subseteq \{\alpha x : \alpha \in \mathbb{R}\}$, where $(x, v_5) = 0$. Let $v_6 \ne v_1, \ldots, v_4$ be perpendicular to v_5, and define H_6 accordingly. Let $G \subseteq \{\alpha y : \alpha \in \mathbb{R}\}$, where $(y, v_6) = 0$, be such that $|G| = \omega_2$ and $(G - G) \cap (M - M) = \{0\}$. Set $N = \bigcup_{g\in G} M + g$. It is easy to check then that if $\mathbb{R}^3 = S_1 \cup \ldots \cup S_6$ is a partition of $\mathbb{R}^3$, for some $f \in F, g \in G$ we have $E + f + g \subseteq S_1 \cup \ldots \cup S_4$, a contradiction. Thus, the following statement implies the continuum hypothesis: for every partition $H = H_1 \cup \ldots \cup H_6$ of planes which is $(3, s)$ finitely determined for some s, there is a partition $\mathbb{R}^3 = S_1 \cup \ldots \cup S_6$ with each plane in H_i meeting S_i in a finite set.

If $2^\omega = \omega_n$, apply Theorem (2.2.11) with $\theta = n$ and $t = n + 1$ to obtain one direction. The converse direction (which follows from Simms) can be obtained by extending the above argument, assuming $2^\omega \ge \omega_{n+1}$, and using vectors $v_5, v_6, \ldots, v_{2n+4}, v_{2n+5}$. This, in fact, gives the stronger result that the stated partition property using $2n + 4$ sets H_i , S_i implies $2^\omega \le \omega_n$.

Theorem (2.2.11) may be refined in several different ways. For some families $H \subseteq \mathbb{R}^n$, the value of p in (ii) of Theorem (2.2.11) is not the best possible. For example, in $\mathbb{R}^4$, for each $\Lambda = \{i_1, i_2\} \subseteq \{1, 2, 3, 4\}$ with $i_1 \ne i_2$, let H_Λ consist of all planes of the form $x_{i1} = a_1$ and $x_{i2} = a_2$, where $a_1, a_2 \in \mathbb{R}$.

Notice that $H = \bigcup_\Lambda H_\Lambda$ is a (4, 3) finitely determined partition of some planes into 6 sets. Sikorski [78] showed, as a particular case of a general theorem, that there is a corresponding partition $\mathbb{R}^4 = \bigcup S_\Lambda$ such that if $h \in H_\Lambda$, then $h \cap S_\Lambda$ is finite. A direct application of Theorem (2.2.11) requires partitioning $\mathbb{R}^4$ into 7 sets. One can refine Theorem (2.2.11), however, to obtain Sikorski's theorem.

Theorem (2.2.28).9 of Simms [80] extends Sikorski's result by obtaining the best possible value of p in the case where H is the family of translates of a fixed finite number of subspaces of $\mathbb{R}^n$, and the elements h of H are partitioned according to which subspace they are a translate of. His results are stated in terms of the least integer n such that the collection of subspaces is "n −good". In fact, we may refine the argument of Theorem (2.2.11) to obtain Simms' result, and also allow general partitions of the family H. We briefly sketch the argument.

Let Π be a finite set of linear subspaces of $\mathbb{R}^n$, for some $n \geq 2$. Let H be the family of translates of these subspaces. That is, every $h \in H$ is of the form $h = V + u$, where $V \in \Pi$ and $u \in \mathbb{R}^n$. Following Simms, we say that Π is t −good if for every linear ordering $\prec$ of Π, there is a subset S of Π of size t such that for all $V \in \Pi, T\{V' \preccurlyeq V : \neg\exists V'' \in S$ such that $V' \prec V'' \prec V\}$ is finite. We thus have:

Corollary (2.2.17)[64]: Let $n \geq 2$, and $\theta \geq 1$ be an ordinal. The following are equivalent:

(a) $2^\omega \leq \omega_\theta$.

(b) For every non-empty set Π of size k of non-trivial linear subspaces V of $\mathbb{R}^n$ which is t −good, if $H = \{V + u : V \in \Pi, u \in \mathbb{R}^n\}$ is partitioned into k sets $H = H_1 \cup \ldots \cup H_k$, then there is a partition $\mathbb{R}^n = S_1 \cup \ldots \cup S_k$ such that for every $h \in H_i$, $|h \cap S_i| \leq \omega_{\theta-t}$.

(c) There is a non-empty set $\Pi = \{V_1, \ldots, V_k\}$ of non-trivial linear subspaces of $\mathbb{R}^n$ which is not $(t + 1)$ −good and for which there is a partition $\mathbb{R}^n n = S_1 \cup \ldots \cup S_k$ such that $\forall 1 \leq i \leq k \, \forall u \in \mathbb{R}^n \, |(Vi + u) \cap S_i| \leq \omega_{\theta-t}$. R e m a r k. The fact that (c) implies (a) is half of [80], and will not be proven here. The special case of (a)⇒(b) for the partition of (c) is the other half of that theorem.

Sketch of proof. Assume $2^\omega \leq \omega_\theta$, and let Π and $H = H_1 \cup \ldots \cup H_k$ be as in (b) above. As in the proof of Theorem (2.2.11), we prove in ZFC an auxiliary Proposition (2.2.12) (which suffices to prove the corollary).

Proposition (2.2.18)[64]: $R(t)$. Let θ be an ordinal, $n \geq 2, t \geq 1, k \geq 1$ be integers, $\Pi = \{V_1, \ldots, V_k\}$ be a set of non-trivial subspaces of $\mathbb{R}^n$ which is t-good, $H = H_1 \cup \ldots \cup H_k$ be a partition of $H = \{V + u : V \in \Pi, u \in \mathbb{R}^n\}$, and let $A \subseteq \mathbb{R}^n \cup H$ be a set of size $\leq \omega_\theta$. Then there is a partition $S = A \cap \mathbb{R}^n = S_1 \cup \ldots \cup S_k$ such that for all $h \in A \cap H_i$, $|h \cap Si| \leq \omega_{\theta-t}$.

If $t = 1$, then the hypothesis that Π is 1 −good simply says that $\bigcap_{i=1}^k V_i$ is finite. It follows that the intersection of any k distinct elements of H is also finite. Thus, the given partition of H is $(k, 1)$ finitely determined. Theorem (2.2.11) then finishes this case. Since Π being t-good implies Π is t' −good for all $t' \leq t$, we see that $R(t)$ also holds for all t when $\theta = 0$. So, we may assume $\theta \geq 1$. Likewise, in proving $R(t)$ we may assume that

$\theta = \theta + (t-1)$ for some ordinal θ. We call a set $A \subseteq \mathbb{R}^n \cup H$ good provided: (a) for any $h_1, \ldots, h_q \in A \cap H$, if $\bigcap_{i=1}^{q}$ hi is finite, then $\bigcap_{i=1}^{q} h_i \subseteq A$, and (c) for any $x \in \mathbb{R}^n \cap A$, the finitely many $h \in H$ which contain x also lie in A.

Without loss of generality, we may assume A is good, and $|A| = \omega_\theta$.

Write $A = \bigcup_{\alpha_1 < \omega_\theta} A_{\alpha_1}$, as an increasing union, where each A_{α_1} is good and has size $\leq \omega_{\theta-1}$. Similarly, we write each A_{α_1} as an increasing union $A_{\alpha_1} = \bigcup_{\alpha_2 < \omega_{\theta-1}} A_{\alpha_1,\alpha_2}$ where each A_{α_1,α_2} is good of size $\leq \omega_{\theta-2}$. Continuing, we define good sets $A_{\alpha_1,\alpha_2}, \ldots, \alpha_{t-1}$ for all $\alpha_1 < \omega_\theta, \ldots, \alpha_{t-1} < \omega_{\theta-(t-2)}$, such that each $A_{\alpha_1,\alpha_2}, \ldots, \alpha_{t-1}$ has size $\leq \omega_{\theta-t+1} = \omega_{\bar{\theta}}$. Write also $A_{\alpha_1,\ldots,\alpha_{t-1}} = \bigcup_{\alpha_t < \omega_{\bar{\theta}}} A_{\alpha_1,\ldots,\alpha_t}$, where each $A_{\alpha_1,\ldots,\alpha_t}$ has size $< \omega_{\bar{\theta}}$ but is not necessarily good (if $\bar{\theta} \geq 1$, then we may make these sets good as well). For each point x (or $h \in H$) in A and ordinals $\alpha_1, \ldots, \alpha_i$, $i \leq t$, we say that x (or h) is new relative to $\alpha_1, \ldots, \alpha_i$ provided for all $j \leq i, x \in A_{\alpha_1,\ldots,\alpha_j} - \bigcup_{\beta < \alpha_j} A_{\alpha_1,\ldots,\alpha_{j-1},\beta}$. There is clearly a unique sequence $\alpha_1 = \alpha_1(x), \ldots, \alpha_t = \alpha_t(x)$ such that x is new relative to $\alpha_1, \ldots, \alpha_t$.

For $x \in A$, we now describe the S_i into which we place x. Let $\alpha_1 = \alpha_1(x), \ldots, \alpha_t = \alpha_t(x)$. Let $h_1^1, \ldots, h_1^{a(1)}$ enumerate the $h \in H \cap A$ on which x lies which are old relative to α_1. Let $h_2^1, \ldots, h_2^{a(2)}$ be those $h \in H \cap A$ on which x lies which are new relative to α_1 but old relative to α_1, α_2, and continuing, $h_t^1, \ldots, h_t^{a(t)}$ those $h \in H \cap A$ on which x lies which are new relative to $\alpha_1, \ldots, \alpha_{t-1}$ but old relative to $\alpha_1, \ldots, \alpha_t$. Clearly, $a(1) + \ldots + a(t) \leq k$. If there is some "color" $1 \leq i \leq k$ not taken on by any of the h_j^l, put x into one such S_i. Note that this includes the case where $a(1) + \ldots + a(t) < k$.

Otherwise, let $h_1^1, \ldots, h_1^{a(1)}, h_2^1, \ldots, h_2^{a(2)}, \ldots, h_t^1, \ldots, h_t^{a(t)}$ correspond to the subspaces $W_1, \ldots, W_k$ of Π $(so, W_1, \ldots, W_k$ is a permutation of $V_1, \ldots, V_k)$. This determines a linear ordering $\prec = \prec(x)$ of Π. By t-goodness, there are $b(1) < \ldots < b(t)$ such that for all $0 \leq j < t, \bigcap_{m=b(j)}^{b(j+1)} W_m$ is finite (where we interpret $b(0)$ as 1). Note first that $b(1) > a(1)$, as otherwise $h_1^1 \cap \ldots \cap h_1^{a(1)}$ would be finite. This would contradict the fact that x is new relative to α_1, and all of the A_β are good. Without loss of generality, we may assume that $b(1) = a(1) + 1$. It then follows by similar reasoning that $b(2) > a(2)$, and again we may assume that $b(2) = a(2) + 1$. Continuing, we may assume that $b(t-1) = a(t-1) + 1$. Thus, $h_t^1 \cap \ldots \cap h_t^{a(t)}$ is finite. Also, by our above remarks, we may assume that $a(1) + \ldots + a(t) = k$, and each color $1 \leq i \leq k$ is taken on exactly once in the sequence $h_1^1, \ldots, h_t^{a(t)}$ (that is, for each i, there is exactly one h in this sequence which lies in H_i). For each $1 \leq j \leq a(t)$, let $\beta(h_t^j) < \omega_{\bar{\theta}}$ be the ordinal such that h_t^j is new relative to $\alpha_1, \ldots, \alpha_{t-1}, \beta(h_t^j)$. Finally, put x into S_i, where $h_t^l \in H_i$ and l is such that $\beta(h_t^l) \geq sup\{\beta(h_t^j) : 1 \leq j \leq a(t)\}$.

To show this works, fix an $h \in H_i \cap A$. We show that $|h \cap S_i| < \omega_{\bar{\theta}}$. Suppose $|h \cap S_i| \geq \omega_{\bar{\theta}}$. Let $\alpha_1 = \alpha_1(h), \ldots, \alpha_t = \alpha_t(h)$, i.e., h is new relative to $\alpha_1, \ldots, \alpha_t$. If $x \notin A_{\alpha_1}$, and x lies on h, then by Definition (2.2.7) of our coloring, $x \notin S_i$. There are no old (relative to α_1) points x which lie on h, since the A_β are good. Thus there must be $\geq \omega_{\overline{\theta}}$ points x $\in S_i$ which are new at α_1 which lie on h. Continuing, we see that $\geq \omega_{\overline{\theta}}$ points $x \in S_i$ which are new at $\alpha_1, \ldots, \alpha_{t-1}$ lie on h. There are $< \omega_{\overline{\theta}}$ points in $A_{\alpha_1}, \ldots, \alpha_t$, hence we need only consider x new at $\alpha_1, \ldots, \alpha_{t-1}, \beta$, where $\beta > \alpha_t$. If such an x lies on h, then the values of the $\beta(h_t^j), 1 \leq j \leq a(t)$, as computed for x, are all $\leq \alpha_t$ from the Definition (2.2.7)s of the $\beta(h_t^j)$ and our coloring. Since $h_t^1 \cap \ldots \cap h_t^{a(t)}$ is finite, it follows that there are $< \omega_{\overline{\theta}}$ such x, a contradiction.

We consider results related to partitions of lines and points into infinitely many pieces. The analog of Theorem (2.2.11) becomes the following.

Theorem (2.2.19)[64]: (ZFC) Let $n \geq 1$. For any $r \geq 2, s \geq 1$ and any (r, s) finitely determined partition $H = \bigcup_{k<\omega} H_k$ of $H \subseteq P(\mathbb{R}^n)$, there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that $|h \cap S_i| < \omega$ for all $i < \omega$ and $h \in H_i$.

Proof. First, one proves in ZFC, by induction on $\in ON$, the following proposition:

Proposition (2.2.20)[64]: If A is a collection of elements of H and points in $\mathbb{R}^n$, $|A| \leq \omega_\eta$, and $A \cap H = \bigcup_{n<\omega} A_n$, is a partition which is (r, s) finitely determined, and if f is a function with domain S = points in A such that $\forall x \in f(x) \subseteq \omega, |f(x)| < \omega$ then there is a partition $S = \bigcup_{n<\omega} S_n$ such that each $h \in A_n$ intersects S_n in a finite set, and, for all $l\ x \in S, x \notin S_a$ for any $a \in f(x)$.

Notice that $P(2^\omega)$ implies Theorem (2.2.19).

In proving $P(\eta)$, we may assume that A is good, that is, if $h_1, \ldots, h_r$ lie in different A_n, then $\bigcap_{i=1}^r h_i \subseteq A$ and if points $x_1, \ldots, x_s$ are in A, then so are the finitely many h in H which contain them. Note that $P(0)$ is essentially trivial (see the proof of Corollary (2.2.23) below). For $\eta \geq 1$, the proof that $P(\eta)$ holds is broken into cases depending on whether η is successor or limit. In each case, we write A as an increasing union of good sets, the argument then being essentially identical to those given earlier.

As a special case of Theorem (2.2.19), we have:

Corollary (2.2.21)[64]: (ZFC) If the lines L in $\mathbb{R}^n$ $(n \geq 2)$ are partitioned into ω disjoint pieces $L = \bigcup_{k<\omega} L_k$ then there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$. such that each line $l \in L_i$ meets S_i in a finite set, for all $i \in \omega$.

Still further generalizations are possible. For example, we may define $H \subseteq \mathcal{P}(\mathbb{R}^n)$ as being (r, s, κ) determined (or a partition being (r, s, κ) determined) where κ is an infinite cardinal as before, except that we now require that the intersection of r distinct elements of H (or the intersection of r elements of H lying in different H_n) has size $\leq \kappa$, and for any s distinct points at most κ many $h \in H$ contain these points. Then we have:

Theorem (2.2.22)[64]: (ZFC) Let $n \geq 1, r \geq 2, s \geq 1$ be integers, κ an infinite cardinal. Let $H \subseteq \mathcal{P}(\mathbb{R}^n)$ be (r, s, κ) determined. Then for any partition $H = \bigcup_{a<K} H_a$ into κ disjoint sets, there is a partition $\mathbb{R}^n = \bigcup_{a<k} S_a$ of $\mathbb{R}^n$ into κ disjoint sets such that $|h \cap S_\alpha| < \omega$ for all $h \in H_\alpha$.

The proof is a trivial generalization of that of Theorem (2.2.19); just start with good sets of size κ.

Of course, since the last theorem and previous corollary are proved in ZFC only, their conclusions imply no bound on the continuum.

Corollary (2.2.21) may be modified in a curious manner, which reintroduces set-theoretic connections. The case $p = 0$ follows from Davies [68].

Corollary (2.2.23)[64]: Suppose m is a positive integer and $2^{\omega} \leq \omega_m$. If the set L of lines in $\mathbb{R}^n, n \geq 2$, is partitioned into ω sets, $L = \bigcup_{k<\omega} L_k$. then there is a partition $\mathbb{R}^n = \bigcup_{k<\omega} S_k$. such that any line in L_k meets S_k in a set of size at most $m + 1$. More generally, if $2^{\omega} \leq \omega_m$, and the lines are partitioned into ω_p sets, $L = \bigcup_{\alpha<\omega} L_a$. for $p \in \omega$, then we may partition the points, $\mathbb{R}^n = \bigcup_{\alpha<\omega} S_a$. so that $|L_\alpha \cap S_\alpha| \leq m - p + 1$ for all $\alpha < \omega_p$.

Sketch of Proof. We show by induction on $m \geq 0$ (working in ZFC) that if A is a good set of lines and points in $\mathbb{R}^n$ $(n \geq 2)$ of size $\leq \omega_m, L = \bigcup_{\alpha<\omega} L_a$, is a partition of the lines in A, and f is a function which assigns to each $x \in S = A \cap \mathbb{R}^n$ a finite subset of ω_p, then we may partition the set S of points in $A, S = S$ write $A = \bigcup_{a<\omega} S_a$, so that each line $l \in L_\alpha$ intersects S_α in a set of size at most $m - p + 1$ and that for all $x \in S, x \notin S_\alpha$ for all $\alpha \in f(x)$. For $m \leq p$ the result is trivial (assign colors to the points of S in a one-to-one manner avoiding the forbidden colors). If A is a good set of size ωm, $m > p$, write $A = S \bigcup_{\beta<\omega_m} A_\beta$, with each $A\beta$ good of size $\leq \omega_{m-1}$. For $\beta < \omega_m$, consider the new points of A_β. Each such point lies on at most one old line. For each such point, let $\bar{f}(x) = f(x) \cup \{j\}$, where x lies on an On old line in L_j if one exists (otherwise set $\bar{f}(x) = f(x)$). By induction, we may partition the new points at β so that for any $x \in S_\alpha$ new at β, x lies on at most $m - p$ lines new at β in L_α, and also $\alpha \notin f(x)$. Since any line new at β has at most one old point which lies on it, it is easy to see that this partition of S works.

Considering the converse direction to Corollary (2.2.23) leads to some interesting questions. For example, assuming CH, given a partition $L = \bigcup_{k<\omega} L_k$ then we may partition the points, $\mathbb{R}^2 = \bigcup_{k<\omega} S_k$ so that for each $l \in L_i, |l \cap S_i| \leq 2$. Davies showed in [68], answering a question of [69], that we may not always get $|l \cap S_i| \leq 1$, even assuming CH. We will strengthen this result. It is natural to ask, then, whether this partition property implies CH, or has any strength beyond ZF at all.

Question. Is it true (or consistent) in ZFC that if the lines in the plane are partitioned into countably many sets, $L = \bigcup_{k<\omega} L_k$ then we may partition the points, $\mathbb{R}^2 = \bigcup_{k<\omega} S_k$ so that for each $l \in L_i, |l \cap S_i| \leq 2$? Is the analogous statement for $\mathbb{R}^n$ true (or consistent)? More generally, do the converse implications to Corollary (2.2.23) hold?

We begin by considering the question of whether the hypothesis $2^{\omega} \leq \omega_m$ is necessary in Corollary (2.2.23). Suppose that $2^{\omega} = \omega_2$. The argument of Corollary (2.2.23) shows that we still have the "two-point" partition property (i.e., for each line $l \in L_i, |l \cap S_i| \leq 2)$ provided we have the following:

$(*)$ For every set $A \subseteq L \cup \mathbb{R}^n$ of lines and points in $\mathbb{R}^n$ of size ω_1, and any partition $A \cap L = \bigcup_{k<\omega} L_k$ of the lines in A, there is a partition $A \cap \mathbb{R}^n = \bigcup_{k<\omega} S_k$ such that for each line $l \in L_k, |l \cap S_k| \leq 1$.

Our previous argument showed that $(*)$ fails assuming CH, but it is not immediately clear $(*)$ fails assuming just ZFC. We show below, however, that this is the case. We first reformulate $(*)$ into purely set-theoretic partition properties. Consider the following partition statements about ω_1 (F. Galvin pointed out to us that the properties $P(\omega_1), Q(\omega_1)$ were introduced earlier in [70], where they were shown to be false assuming CH):

$P(\omega_1)$ For every partition $P : (\omega_1)^2 \to \omega$, there is an $h : \omega_1 \to \omega$ such that for all $\alpha < \beta < \omega_1$, if $P(\alpha, \beta) = i$, then at least one of $h(\alpha), h(\beta) \neq i$.

Lemma (2.2.24)[64]: (ZFC) $(*) \Leftrightarrow P(\omega_1)$.

Proof. Assuming $(*)$, let $P : (\omega_1)^2 \to \omega$ be a partition. Let $B \subseteq \mathbb{R}^n$ be an independent set of size ω_1, i.e., no three points of A are colinear. Let $A = B \cup L$, where L is the set of lines through two points of B. Applying now $(*)$ to A produces an $h : \omega_1 \to \omega$ as required by $P(\omega_1)$, identifying ω_1 with B. Assuming $P(\omega_1)$, let A, L be as in the statement of $(*)$. Let $\{x_\alpha\}$ enumerate the points of A. Define $P : \omega_1 \to \omega$ by $P(\alpha, \beta) = i$ iff the line between x_α and x_β lies in L_i . Applying $P(\omega_1)$ then produces an $h : \omega_1 \to \omega$. This defines a corresponding partition of the x_α which easily works.

Note that it makes sense to consider $P(\omega_1)$ in just ZF. We reformulate $P(\omega_1)$ in a more suggestive manner of usual partition type properties:

$Q(\omega_1)$ For any partition $Q : (\omega_1)^2 \to \omega$, we may write $\omega_1 = \bigcup_{k<\omega} A_k$ so that for all $k, Q([A_k]^2)$ is co-infinite.

Note that in $Q(\omega_1)$ there is no loss of generality in assuming the A_k are disjoint.

Lemma (2.2.25)[64]: (ZF) $P(\omega_1) \Leftrightarrow Q(\omega_1)$.

Proof. Assume first $P(\omega_1)$, and let $Q : (\omega_1)^2 \to \omega$ be given. Let $r : \omega \to \omega$ be onto with $r^{-1}(i)$ infinite for all $i \in \omega$. Let $P(\alpha, \beta) = r(Q(\alpha, \beta))$. Let $h : \omega_1 \to \omega$ be as given by $P(\omega_1)$ for P. Let $A_k = \{\alpha < \omega_1 : h(\alpha) = k\}$. Then, for $\alpha, \beta \in A_k, r\big(Q(\alpha, \beta)\big) = k$, hence $Q(\alpha, \beta) \notin r^{-1}(k)$. Then, for $\alpha, \beta \in A_k, r\big(Q(\alpha, \beta)\big) \neq k$, hence $Q(\alpha, \beta) \notin r^{-1}(k)$. Assume now $Q(\omega_1)$, and let $P : (\omega_1)^2 \to \omega$ be given. Let $\{A_k : k \in \omega\}$ be as given by $Q(\omega_1)$ for the partition P. Let $n_0, n_1, \ldots$ be distinct integers such that $n_k \notin P([A_k]^2)$ for all k. Let $h(\alpha) = n_k$ for all $\alpha \in A_k$. This easily works.

The following theorem of Todorˇcević (see [81]) immediately implies that $Q(\omega_1)$ is false in ZFC.

Theorem (2.2.26)[64]: (Todorˇcević). Assume ZFC. There is a partition $c : [\omega_1]^2 \to \omega$ such that $c([C]^2) = \omega$ for all uncountable $C \subseteq \omega_1$.

Corollary (2.2.27)[64]: (ZFC) $(*), P(\omega_1), Q(\omega_1)$ are all false.

From the failure of $P(\omega_1)$, it follows (in ZFC) that there is a partition $L = \bigcup_{k<\omega} L_k$ of a set L of lines in $\mathbb{R}^2$, with $|L| = \omega_1$, such that for every partition R 2 = $\bigcup_{k<\omega} S_k$ we have $|l \cap S_n| \geq 2$ for some n and $l \in L_n$ (cf. the proof of Lemma (2.2.24)). This strengthens a result of Davies mentioned earlier. Todorˇcević's theorem is proved in ZFC, and thus it remains possible that $Q(\omega_1)$ (or, equivalently, $P(\omega_1)$) is consistent with ZF. We in fact show that $Q(\omega_1)$ is a theorem of AD, and thus holds in $L(\mathbb{R})$ assuming ZFC + large cardinal axioms. In fact, we show a much stronger version of $Q(\omega_1)$ under these hypotheses. Consider the following strengthening of $Q(\omega_1)$:

$Q^s(\omega_1)$ For every partition $Q : [\omega_1]^2 \to \omega$, we may write $\omega_1 = \bigcup_{k<\omega} A_k$where $Q([A_k]^2)$ is finite for all $k \in \omega$.

Theorem (2.2.28)[64]: $(ZF + AD + DC) Q^s(\omega_1)$ holds.

Proof. We sketch two proofs. The first uses only the theory of indiscernibles for $L(x), x \in \mathbb{R}$. The second uses the analysis of measures on ω_1 of [73]. The second proof, however, extends to cardinals other than ω_1.

Let $Q : [\omega_1]^2 \to \omega$ be given. From AD, there is an $x \in \mathbb{R}$ such that $Q \in L(x)$. Let $C = \{\xi_\alpha : \alpha \in ON\}$ be the canonical closed unbounded set of (Silver) indiscernibles for $L(x)$. Below, τ, σ denote terms in the language of set theory with x as a parameter. Let τ be a term such that $Q = \tau^L(x)(\xi_0, \ldots, \xi_n, \beta_0, \ldots, \beta_m)$, where $\xi_0 < \ldots < \xi_n < \omega_1 \leq \beta_0 < \ldots < \beta_m \in C$. For each $\alpha < \omega_1$ we canonically choose a representation $\alpha = \sigma(\alpha)\, L(x)\, (\theta_0(\alpha), \ldots, \theta_m(\alpha)\,(\alpha))$, for some term $\sigma(\alpha)$ and $\theta_0(\alpha) < \ldots < \theta_m(\alpha)\,(\alpha) < \omega_1$ in C. For each $\alpha < \omega_1$, let $p(\alpha) \in \omega$ be an integer which codes the term $\sigma(\alpha), m(\alpha)$, and the manner in which the two sequences of ordinals $(\xi_0, \ldots, \xi_n), (\theta_0(\alpha), \ldots, \theta_m(\alpha)\,(\alpha))$ are interlaced (including which of them are equal). For $\alpha, \beta < \omega_1$, let $q(\alpha, \beta) \in \omega$ be an integer which codes how the two sequences of ordinals $\sim\theta(\alpha), \sim\theta(\beta)$ are interlaced. Let $A_k = \{\alpha < \omega_1 : p(\alpha) = k\}$. To see this works, fix $k \in \omega$, and consider $Q\lceil [A_k]^2$. Note that $q([A_k]^2)$ is finite. It thus suffices to show that if $\alpha < \beta, \gamma < \delta$ are in A_k, and $q(\alpha, \beta) = q(\gamma, \delta)$, then $Q(\alpha, \beta) = Q(\gamma, \delta)$. However, from the fact that $\alpha, \beta, \gamma, \delta \in A_k$ and $q(\alpha, \beta) = q(\gamma, d)$, it follows that the manner in which $(\xi_0, \ldots, \xi_n), \vec{\theta}(\alpha)$, and $\vec{\theta}(\beta)$ are interlaced is the same as that for the sequences $(\xi_0, \ldots, \xi_n), \vec{\theta}(\gamma), \vec{\theta}(\delta)$. It thus follows by indiscernibility that

$$\begin{aligned} Q(\alpha, \beta) &= (\tau^L(x)\,(\xi_0, \ldots, \xi_n, \beta_0, \ldots, \beta_m))(\sigma^L(x)\,(\vec{\theta}(\alpha)), \sigma L(x)\,(\vec{\theta}(\beta))) \\ &= (\tau^L(x)\,(\xi_0, \ldots, \xi_n, \beta_0, \ldots, \beta_m))(\sigma^L(x)\,(\vec{\theta}(\gamma)), \sigma L(x)\,(\vec{\theta}(\delta))) \\ &= Q(\gamma, \delta). \end{aligned}$$

For the second proof, fix again $Q : [\omega_1]^2 \to \omega$. Let $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ be the countably additive ideal consisting of all $A \subseteq \omega_1$ such that $A \subseteq \bigcup_{k<\omega} S_k$ where each $S_k \subseteq \omega_1$ is such that $Q([S_k]^2)$ is finite. Assume by way of contradiction that I is a proper ideal (i.e., $\omega_1 \notin I$). By Kunen, from AD, any countably additive ideal on an ordinal $\kappa < \Theta$ can be extended to a measure (i.e., countably additive ultrafilter) on κ. (Proof: Let μ be the Martin measure on the Turing degrees $\mathcal{D}$. By the coding lemma, let $\pi : \mathbb{R} \to \mathcal{I}$ be onto. For

$d \in \mathcal{D}$, set $\varrho(d)$ = least $\alpha < \kappa$ not in $\bigcup_{x \in d} \pi(x)$. This is well-defined since $\mathcal{I}$ is proper. Then $\varrho(\mu)$ is a measure on κ giving measure 0 to all $I \in \mathcal{I}$, where $\varrho(\mu)(A) = 1$ iff $\mu(\{d \in \mathcal{D} : \varrho(d) \in A\}) = 1$.)

The claim of Section 2 of [73] analyzes, assuming AD, all measures ν on ω_1. The result (somewhat restated) is that there is a function $f : [\omega_1]\, m \to \omega_1$ for some $m \in \omega$ such that for all $B \subseteq \omega_1, \nu(B) = 1$ iff there is a c.u.b. $C \subseteq \omega_1$ such that $f(\delta_1, \ldots, \delta_m) \in B$ for all $\delta_1 < \ldots < \delta_m \in C$. By applying the finite exponent partition property on ω_1 (with exponent $2m$) finitely many times, we get a c.u.b. $C \subseteq \omega_1$ such that for all pairs of increasing sequences of length m from $C, (\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)$, the value $P(f(\alpha_1, \ldots, \alpha_m), f(\beta_1, \ldots, \beta_m))$ depends only on the manner of interlacing of the two sequences. This C, however, then defines a measure one set with respect to ν on which Q takes only finitely many values, a contradiction.

Corollary (2.2.29)[64]: $P(\omega_1)$, $]Q(\omega_1)$ are consistent with ZF.

Finally, we state without proof some extensions of the above ordinal partition properties. For cardinals κ, δ, let $P(\kappa, \delta)$ be the statement that for any partition $P : [\kappa]^2 \to \delta$, there is an $h : \kappa \to \delta$ such that for any $\alpha < \beta < \kappa$, at least one of $h(\alpha), h(\beta)$ is different from $P(\alpha, \beta)$. Let $Q(\kappa, \delta)$ be the statement that for any $Q : [\kappa]^2 \to \delta$, we may write $\kappa = \bigcup_{\lambda < \delta} A_\lambda$ where for each $\lambda, \delta - Q([A_\lambda]^2)$ is infinite. Let also $Q^s(\kappa, \delta)$ be as $Q(\kappa, \delta)$ except that we write $\kappa = \bigcup_{k<\omega}, A_k$, and we require each $Q([A_k]^2)$ to be finite.

The same argument given before shows that $\forall_\kappa, \delta\, (P(\kappa, \delta) \Leftrightarrow Q(\kappa, \delta))$. Also, in $Q(\kappa, \delta)$, we may replace "$\delta - Q([A_k]^2)$ is infinite" by "$\delta - Q([A_k]^2)$ has size δ". The second proof given above for $Q^s(\omega_1)$ when combined with the analysis of measures on δ^1_{2n+1} (see [74] for the case n = 1) yields:

Theorem (2.2.30)[64]: $(ZF + AD + DC)$ For all $\delta < \delta^1_{2n+1}, Q^s(\delta^1_{2n+1}, \delta)$ holds.

It is again easy to see that $Q^s(\kappa, \delta)$ fails in ZFC for all uncountable κ and infinite δ. We believe, however, that the Steel–Van Wesep–Woodin forcing [82] for recovering ω_1 −DC can be used to show the following: (ZFC + ADL(R)) There is a model of $ZF + \omega_1 - DC + \forall\delta < \delta^1_{2n+1}\, (Q^s(\delta^1_{2n+1}, \delta)$ holds). Thus, $Q^s(\kappa, \delta)$ is consistent with small amounts of choice.

As S. Todorˇcević pointed out to us, one can show that $Q(\omega_1)$, and hence $Qs\,(\omega_1)$, have consistency strength beyond ZFC. In fact, $Q(\omega_1)$ implies ω_1 is inaccessible to L. For if not, then for some $x \in \mathbb{R}, \omega_1 = \omega_1^{L(x)}$. Let $c : [\omega_1]^2 \to \omega$ be the Todorˇcević partition defined in $L(x)$. Applying Q, let $A \subseteq \omega_1, |A| = \omega_1$ be such that $c([A]^2)$ is co-infinite. The proof of Todorˇcević's theorem shows that in $L(x, A), c$ retains the property that $c([B]^2) = \omega$ for all $B \subseteq \omega_1$ of size ω_1. This contradicts $c([A]^2)$ being co −infinite.

The failure of $P(\omega_1)$ in ZFC rules out one approach for showing the "two-point" partition property (as in Corollary (2.2.23)) in ZFC, or even from $2^\omega = \omega_2$. The original question, stated at the end, however, remains. Note, however, that the consistency of $ZF + \neg CH + Q(\omega_1)$ shows that the "ordinal version" of the two-point partition problem is consistent with $ZF + \neg CH$. Here "lines" refers to subsets of ω_2 satisfying the usual properties, i.e., two ordinals less than ω_2 determine a line, and two distinct lines intersect in at most one ordinal.

Section (2.3): Steinhaus Tiling Problem

By a rotation and translation of a set $E \subset \mathbb{R}^d$ we mean of course a set of the form $\rho E + x$ for some $\rho \in SO(d)$ and $x \in \mathbb{R}^d$ It is natural to consider Steinhaus' question separately for measurable and nonmeasurable sets. Both the measurable and nonmeasurable cases are presently open, but will be concerned only with the measurable case, which leads to some attractive questions in harmonic analysis. Accordingly we define a Steinhaus set to be a measurable set $E \subset \mathbb{R}^d$ with the property that every rigid motion $\rho E + x$ contains exactly one lattice point. Croft [44] showed that a Steinhaus set cannot be bounded and Beck [84] gave a Fourier analysis proof of this result. One of the present authors showed in [91] that if E is a Steinhaus set (in $\mathbb{R}^2$), then $\int_E |x|^\alpha = \infty$ for all $\alpha > \frac{10}{3}$. The case of closed sets has also been considered; see [86] and [92].

For a given lattice Λ, the condition that every translate of E contain exactly one point of Λ is equivalent to requiring that the translates of E under the elements of Λ form a tiling. Note in particular that a Steinhaus set must have measure 1. More generally, one can consider tilings by functions instead of sets; we will say that an L^1 function f tiles with a lattice Λ if

$$\sum_{v \in \Lambda} f\,(x - u) \; is \; constant \; a.e. \,(dx)$$

One purpose is to solve the higher dimensional analogue of the (measurable) Steinhaus problem:

Theorem (2.3.1)[83]: Suppose that $d \geq 3$ and that $: \mathbb{R}^d \to \mathbb{R}$ is an L^1 function which tiles with every rotation of $\mathbb{Z}^d$, i.e.

$$\sum_{v \in \mathbb{Z}^d} f\,(x - \rho v)$$

is constant a.e. for each $\rho \in SO(d)$. Then f agrees a.e. with a continuous function.

In particular this means that f cannot be the indicator function of a set with positive measure, so we obtain

Corollary (2.3.2)[83]: There are no Steinhaus sets in three or more dimensions.

We have been unable to prove a similar result in $\mathbb{R}^2$ but we will improve on the bound in [91] in the following way:

Thus the result of [89] $\left(\beta = \frac{46}{73} + \varepsilon\right)$ implies that if E is Steinhaus then (28) holds for all $\alpha > \frac{46}{27}$; this is the best that we know unconditionally. The conjectured result ($\beta = \frac{1}{2} + \varepsilon$, see e.g. [90] or [93]) on (27) would imply (28) for all $\alpha > 1$. This same range $\alpha > 1$ also arises in another way- see the remark after the proof of Corollary (2.3.9).

Property (28) with $\alpha = 2$ can be proved by an argument similar to [91] but based on $L^2 \to L^2$ instead of $L^1 \to L^\infty$ estimates. We give this argument in Corollary (2.3.9) below. The relevant L^2 estimate, Corollary (2.3.8)(b), is quite simple and may be of some independent interest. Theorem (2.3.1) and Theorem (2.3.14) (in the case $\alpha < 2$) is proved. Both proofs use bounds for exponential sums.

We also consider a related problem for finite sets of rotations. It is natural to ask whether there are sets E which have the Steinhaus property relative to a large finite set of rotations $\{\rho_i\}$, i.e., whether it is possible to have $\sum_{v\in\mathbb{Z}^d} \chi_E\,(x - \rho_i v) = 1$ for each i. This question was answered in the affirmative in [92] for a more precise statement. We will prove an analogue of the Croft-Beck unboundedness result in this context and more generally for images of $\mathbb{Z}^d$ under linear maps with determinant 1 rather than just rotations:

It is based on uniform distribution modulo 1 and a theorem of Ronkin [94] and Berndtsson [85] on the real zeros of entire functions of exponential type in $\mathbb{C}^d$.

We let σ_t be the surface measure on the sphere in $\mathbb{R}^d$ of radius t, and will normalize the Fourier transform via $\hat{f}(\xi) = \int f\,(x)e^{-2\pi i x\cdot\xi}dx$. We note also that a (Schwarz function" will mean a function belonging to the Schwarz space as defined (say) in [88], p. 160, Definition 7.1.2.

Lemma (2.3.3)[83]: Assume $d \geq 2$. Let $q\colon \mathbb{R} \to \mathbb{R}$ be a C_0^∞ function supported in $\left[\frac{1}{2}, 2\right]$, and let $b \in (0,1]$. Define $K_N : \mathbb{R}^d \to \mathbb{C}$

$$K_N(x) = \sum_n \frac{1}{\sqrt{n+b}}\, q\left(\frac{\sqrt{n+b}}{N}\right) \widehat{\sigma_{\sqrt{n+b}}}(x)$$

Then for large N there is an estimate

$$|K_N(x)| \lesssim \begin{cases} (N|x|)^{-100} & \text{if } 1 \leq |x| \leq \dfrac{N}{2} \\ \left(\dfrac{N}{|x|}\right)^{\frac{d-2}{2}} & \text{if } |x| \geq \dfrac{N}{2} \end{cases}$$

Proof. This will follow from the asymptotics for the Fourier transform of surface measure and a simple form of the vander Corput method for estimating exponential sums. We remark that if $|x| \geq N^{\alpha}$ with $\alpha > 1$ then the bound can be improved by using exponent pairs, but Lemma (2.3.3) as stated is enough for the proof of Theorem (2.3.5).

It is well known (e.g. [95] p. 50) that $\widehat{\sigma_1}(x) = \mathrm{re}\big(B(|x|)\big)$ where $B(r) = a(r)e^{2\pi ir}$, with $a(r)$ being a complex valued function satisfying estimates

$$\left|\frac{d^k a}{dr^k}\right| \lesssim r^{-\frac{d-1}{2}-k} \tag{5}$$

Hence also $\widehat{\sigma_t}(x) = \mathrm{re}\left(t^{d-1}B(t|x|)\right)$. Define $t_+ = \max(t, 0)$,and let $r = |x|$.In the calculation below, we use that $q(t) = 0$ when $t < \frac{1}{2}$; this implies that various integrals may be taken interchangeably over $\mathbb{R}$ and over $(0, \infty)$. We have

$$\sum_{n\geq 0}\frac{1}{\sqrt{n+b}}\,q\left(\frac{\sqrt{n+b}}{N}\right)\left(\sqrt{n+b}\right)^{d-1}B\left(r\sqrt{n+b}\right)$$

$$= \sum_{n\in\mathbb{Z}}((n+b)_+)^{\frac{d-2}{2}}\,q\left(\frac{\sqrt{(n+b)_+}}{N}\right)a\left(r\sqrt{(n+b)_+}\right)e^{2\pi ir\sqrt{(n+b)_+}}$$

$$= \sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}((y+b)_+)^{\frac{d-2}{2}}\,q\left(\frac{\sqrt{(y+b)_+}}{N}\right)a\left(r\sqrt{(y+b)_+}\right)e^{2\pi ir\sqrt{(y+b)_+}}e^{-2\pi ivy}dy$$

$$= \sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}(Nz)^{d-2}\,q(z)a(rNz)e^{2\pi irNz}e^{-2\pi \mathrm{i}v(N^2z^2-b)}d(N^2z^2-b)$$

$$= r^{-\frac{d-1}{2}}N^{\frac{d+1}{2}}\sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}\varphi(z)e^{2\pi irNz}e^{-2\pi iv(N^2z^2-b)}dz \tag{6}$$

where $\varphi(z) = 2z^{d-1}(rN)^{\frac{d-1}{2}}a(rNz)q(z)$. We used the Poisson summation formula and then the change of variables $z = \frac{\sqrt{y+b}}{N}$. We note that the estimate (5) implies that the functions $\varphi = \varphi_{N,r}$ belong to a compact subset of C_0^∞; this means that the estimates below are uniform in r and N.

We rewrite the sum (6) isolating the $v = 0$ term and making some algebraic manipulations:

$$(6) = r^{-\frac{d-1}{2}}N\frac{d+1}{2}\int_{\mathbb{R}}\varphi(z)e^{2\pi irNz}dz$$
$$+\, r^{-\frac{d-1}{2}}N^{\frac{d+1}{2}}\sum_{v\in\mathbb{Z}\setminus\{0\}}e^{2\pi i\left(vb+\frac{r^2}{4v}\right)}\int_{\mathbb{R}}\varphi(z)e^{-2\pi ivN^2\left(z-\frac{r}{2Nv}\right)^2}dz \tag{7}$$

The first term in (7) is equal to $r-\frac{d-1}{2}N\frac{d+1}{2}\hat{\varphi}(-Nr)$, hence $\lesssim r-\frac{d-1}{2}N\frac{d+1}{2}(Nr)^{-k}$ for any k. In particular, it is $\lesssim (Nr)^{-100}$ if $r \geq 1$. The terms in the sum in (7) may be evaluated via the asymptotics for Gaussian Fourier transforms ([88], Lemma 7.7.3); the vth term is equal to

$$e^{2\pi i\left(vb+\frac{r^2}{4v}\right)}\sum_{k=0}^{m-1} c_k\,(vN^2)^{-k-\frac{1}{2}}\varphi_k\left(\frac{r}{2Nv}\right)+\mathcal{O}\left((vN^2)^{-m-\frac{1}{2}}\right) \tag{8}$$

for any m; here c_k are fixed constants and the φ_k are certain derivatives of φ. All the terms in the sum over k vanish if $v \notin \left[\frac{r}{4N},\frac{r}{N}\right]$ so that

$$(8) \lesssim \begin{cases}(vN^2)^{-\frac{1}{2}} & \text{if } v \in \left[\frac{r}{4N},\frac{r}{N}\right]\\ (vN^2)^{-m-\frac{1}{2}} & \text{if } v \notin \left[\frac{r}{4N},\frac{r}{N}\right]\end{cases}$$

Accordingly the sum in (7) is

$$\lesssim \text{card}\left(\mathbb{Z}\cap\left[\frac{r}{4N},\frac{r}{N}\right]\right)(rN)^{-\frac{1}{2}}+(rN)^{-m-\frac{1}{2}}$$

Taking m sufficiently large we obtain

$$(7) \lesssim r^{-\frac{d-1}{2}N\frac{d+1}{2}}\text{card}\left(\mathbb{Z}\cap\left[\frac{r}{4N},\frac{r}{N}\right]\right)(rN)^{-\frac{1}{2}}+(rN)^{-100}\sim< \begin{cases}\left(\frac{N}{r}\right)^{\frac{d-2}{2}} & \text{if } r \geq \frac{N}{2}\\ (rN)^{-100} & \text{if } 1 \leq r \leq \frac{N}{2}\end{cases}$$

The lemma follows since K_N is the real part of the quantity (7).

We need one more lemma, an easy consequence of the Poisson summation formula.

Lemma (2.3.4)[83]: Let $k \geq 2$ be an integer, let q be a fixed C_0^∞ function supported in $\left[\frac{1}{2},2\right]$, let $b \in [0,1)$ and let $h = h(t)$ be a function on the line satisfying the following estimate:

$$\left|\frac{d^j h}{dt^j}\right| \leq R$$

when $0 \leq j \leq k$ and $\frac{N}{100} \leq t \leq 100N$. Then for large N

$$\left|\sum_n \frac{1}{\sqrt{n+b}}\,q\left(\frac{\sqrt{n+b}}{N}\right)h\left(\sqrt{n+b}\right)-2\int q\left(\frac{t}{N}\right)h(t)dt\right| \lesssim RN^{-(k-1)} \tag{9}$$

where the implicit constant depends on q only.

Proof. Set $g(x) = \frac{h(\sqrt{x+b})}{\sqrt{x+b}}$ and $(x) = q\left(\frac{\sqrt{x+b}}{N}\right)$. Then a is supported in $x \approx N^2$ and derivatives of a satisfy

$$|\frac{d^j a}{dx^j}| \lesssim N^{-2j} \tag{10}$$

since the functions $q(\sqrt{x + bN - 2})$ belong to a compact subset of C_0^∞ and $a(x)$ is obtained from $q(\sqrt{x + bN^{-2}})$ by dilating by N^2 When $x \approx N^2$, derivatives of g satisfy

$$|\frac{d^j g}{dx^j}| \lesssim RN^{-(1+j)} \tag{11}$$

when $j \leq k$. Namely, it is easy to show by induction on j that the jth derivative of g is a sum of finitely many terms each of which has the form $\frac{h^{(i)}(\sqrt{x+b})}{(\sqrt{x+b})^\ell}$ where $h^{(i)} = i\text{th}$ derivative of h, with $i \leq j$ and $\ell \geq j + 1$. Estimate (11) is then obvious.

The left side of (9) is (make the change of variables $t = \sqrt{x + b}$) equal to

$$|\sum_n a\,(n) g(n) - \int a\,(x) g(x) dx|$$

By Poisson summation this is

$$\left|\sum_{v \neq 0} \widehat{ag}(v)\right| \tag{12}$$

and if we integrate by parts k times and use (10) and (11), we bound the $u\text{th}$ term in the sum (12) by

$$|v|^{-k} \int |\frac{d^k(ag)}{dx^k}| dx \lesssim |v|^{-k} \int_0^{2N^2} R\,N^{-(1+k)} dx$$

$$\lesssim |v|^{-k} R N^{-(k-1)}$$

Hence (12) $RN^{-(k-1)}$ and the proof is complete.

We prove Theorem (2.3.1). The argument is Fourier analytic and is based on the following observation: let f be a function satisfying the hypotheses of Theorem (2.3.1). Then $\hat{f}$ vanishes identically on any sphere centered at the origin which contains a point of $\mathbb{Z}^d$ When $d = 2$, this observation was made in [84] (and used also in [91]) and the proof extends immediately to higher dimensions. Since every integer is the sum of four squares and every integer congruent to 1 $\bmod 8$ is the sum of three squares, we see that it suffices to prove the following:

Theorem (2.3.5)[83]: Assume that $d \geq 3$ and let a and b be positive real numbers. Let $f: \mathbb{R}^d \to \mathbb{C}$ be an L^1 function such that $\hat{f}$ vanishes identically on the sphere centered at the origin with radius $\sqrt{am + b}$ for every positive integer m. Then f is continuous.

Proof : We may clearly assume that $a = 1$ and $b \leq 1$.

We let $q \in C_0^\infty(\mathbb{R})$ be supported in $\left[\frac{1}{2}, 2\right]$ and such that the functions $\{q_{2^j}\}_{-\infty}^{\infty}$ form a partition of unity on $(0, \infty)$; here we have defined $q_{2^j}(x) = q\left(\frac{x}{2j}\right)$. We define K_N as in Lemma (2.3.3) using this q.

Fix a ball D with radius 1; we will show that f is continuous on D. Let $\widetilde{D}$ be the concentric ball with radius 2, and let $f_i = \chi_{\widetilde{D}} f$ and $f_o = \chi_{\mathbb{R}^d \setminus \widetilde{D}} f$ where χ_E is the indicator function of the set E. By assumption, $\widehat{\sigma_{\sqrt{n+b}}} * f$ vanishes identically for any positive integer n and therefore $K_N * f$ vanishes identically for any N.

Claim (2.3.6)[83]: Suppose $\eta > 0$ is given. Then, provided k is large enough, we have

$$\sum_{j \geq k} |K_{2^j} * f_i(y)| \leq \eta \tag{13}$$

for all $y \in D$.

Namely, by the preceding remarks it suffices to prove this with f_i replaced by f_0. If $|y - z| \geq 1$, then Lemma (2.3.3) implies that

$$\sum_{j \geq 0} |K_{2j}(y - z)|_{\sim} < \sum_{j: 2^j \geq 2|y - z|} \left(2^j |y - z|\right)^{-100} + \sum_{j: 2^j \leq 2|y - z|} \left(\frac{2^j}{|y - z|}\right)^{\frac{d-2}{2}}$$

Since $d \geq 3$, it follows easily that for a suitable constant C_0

$$\sum_{j \geq 0} |K_{2j}(y - z)| \leq C_0 \tag{14}$$

for all $y \in D$ and $z \in \mathbb{R}^d \setminus \widetilde{D}$. Now fix a number $R \geq 2$ which is large enough that

$$\int_{\mathbb{R}^d \setminus D_R} |f| < \frac{\eta}{2C_0}$$

where D_R is the ball concentric with D and with radius R. Then, using Lemma (2.3.3) as in the proof of (14), if k is sufficiently large then

$$\sum_{j\geq k} |K_{2^j}(y-z)| < \frac{\eta}{2\|f\|_1}$$

for all $y \in D$ and $z \in D_R \backslash \widetilde{D}$. It follows that

$$\sum_{j\geq k} |K_{2^j} * f_o(y)| \leq \int_{D_R\backslash\widetilde{D}} \sum_{j\geq k} |K_{2^j}(y-z)||f(z)|dz + \int_{\mathbb{R}^d\backslash D_R} \sum_{j\geq k} |K_{2^j}(y-z)||f(z)|dz$$

$$< \frac{\eta}{2\|f\|_1}. \|f\|_1 + C_0 \cdot \frac{\eta}{2C_0} = \eta$$

as claimed.

We now fix $y \in D$ and define

$$h(r) \stackrel{\text{def}}{=} \int e^{2\pi i y\cdot\xi}\, \widehat{f_\iota}(\xi) d\sigma_r(\xi) = r^{d-1} \int_{|\xi|=1} \int_{\widetilde{D}} f(z) e^{2\pi i r(y-z)\cdot\xi} dz d\sigma_1(\xi)$$

The estimates below will be uniform in $y \in D$. Using Γourier inversion, we have

$$K_N * f_i(y) = \sum_n \frac{1}{\sqrt{n+b}} q\left(\frac{\sqrt{n+b}}{N}\right) \int e^{2\pi i y\cdot\xi}\, \widehat{f_\iota}(\xi) d\sigma_{\sqrt{n+b}}(\xi)$$

$$= \sum_n \frac{1}{\sqrt{n+b}} q\left(\frac{\sqrt{n+b}}{N}\right) h\left(\sqrt{n+b}\right)$$

If $y \in D$, then the second form of the definition of h shows that h and all its derivatives are $\mathcal{O}(N^{d-1})$ when $r \in \left[\frac{N}{100}, 100N\right]$. Accordingly, Lemma (2.3.4) with a large value of k implies

$$\int h(t) q\left(\frac{t}{N}\right) dt = \frac{1}{2} K_N * f_i(y) + \mathcal{O}(N^{-100}) \tag{15}$$

Now define $\psi_N: \mathbb{R}^d \to \mathbb{R}$ via

$$\widehat{\psi_N}(\xi) = q\left(\frac{|\xi|}{N}\right)$$

Then, using Γourier inversion and the definition of h, we have

$$\psi_N * f_i(y) = \int e^{2\pi i y\cdot\xi} q\left(\frac{|\xi|}{N}\right) \widehat{f_\iota}(\xi) d\xi$$

$$= \int h(t) q\left(\frac{t}{N}\right) dt$$

On the other hand ψ_N belongs to the Schwarz space, and $\sum_{j\geq k} \widehat{\psi_{2^j}}(\xi) = 1$ when $|\xi|$ is large. Accordingly, the function φ_{2^k} defined via

$$\widehat{\varphi_{2^k}}(\xi) = 1 - \sum_{j\geq k} \widehat{\psi_{2^j}}(\xi)$$

belongs to the Schwarz space. We have

$$\begin{aligned} f_i(y) - \varphi_{2^k} * f_i(y) &= \sum_{j\geq k} \psi_{2^j} * f_i(y) \\ &= \sum_{j\geq k} \int h(t) q\left(\frac{t}{2^j}\right) dt \\ &= \frac{1}{2} \sum_{j\geq k} K_{2^j} * f_i(y) + \mathcal{O}(2^{-100k}) \end{aligned}$$

by (15). We conclude using (13) that

$$\begin{aligned} |f_i(y) - \varphi_{2^k} * f_i(y)| &\lesssim \sum_{j\geq k} |K_{2j} * f_i(y)| + 2^{-100k} \\ &\lesssim 2\eta \end{aligned}$$

for any given η provided k is sufficiently large. Hence, on D, f is the uniform limit of the continuous functions $\varphi_{2^k} * f_i$ and therefore continuous.

If E is a nice enough set in $\mathbb{R}^d$ then it is well known that the indicator function χ_E cannot belong to the Sobolev space $W^{\frac{1}{2}}$, i.e. the integral $\int_{\mathbb{R}^d} |\xi||\widehat{\chi_E}(\xi)|^2 d\xi$ must be infinite. In fact, there is an asymptotic expression which implies in particular that

$$\int_{|\xi|\geq R} |\widehat{\chi_E}(\xi)|^2 d\xi \approx R^{-1} \tag{16}$$

as $R \to \infty$. This is often used in connection with irregularities of distribution; see e.g. [93].

We will not use (16), but we will need to know that the lower bound in (16) is valid without any regularity assumptions on the set E. This is not difficult but does not seem to be in the literature, so we prove it in Corollary (2.3.8) below.

Let φ be a Schwarz class function in $\mathbb{R}^d$ with $\hat{\varphi}(0) = 1$; φ will be kept fixed for the rest. Let φ_ε be the corresponding approximate identity defined by $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$.

Lemma (2.3.7)[83]: Suppose that E is a set in $\mathbb{R}^d$ with $|E| = 1$ and $|E \cap D| > 0$ for a certain ball D with radius 1. Let $\widetilde{D}$ be the concentric ball with radius C_d. Then

$$\left|\left\{x \in \widetilde{D}: \frac{1}{4} \leq \varphi_\varepsilon * \chi_E(x) \leq \frac{3}{4}\right\}\right| \gtrsim \varepsilon$$

provided that ε is sufficiently small; the implicit constants may depend on E.

Proof. We will use the following well-known fact:

$$\|\nabla(\varphi_\varepsilon * \chi_E)\|_\infty \lesssim \varepsilon^{-1} \tag{17}$$

To prove (17), let $\psi = \nabla\varphi$, let $C = \|\psi\|_1$ and define $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x)$. Differentiation under the integral sign leads to $\nabla(\varphi_\varepsilon * \chi_E) = \varepsilon^{-1}\psi_\varepsilon * \chi_E$. On the other hand, for any $x \in \mathbb{R}^d$, we have $|\psi_\varepsilon * \chi_E(x)| \leq \|\psi_\varepsilon\|_1\|\chi_E\|_\infty = \|\psi\|_1$, which proves that $\|\nabla(\varphi_\varepsilon * \chi_E)\|_\infty \leq C\varepsilon^{-1}$, as claimed.

It follows by the mean value theorem that if $\varphi_\varepsilon * \chi_E(x_0) = \frac{1}{2}$, then $\varphi_\varepsilon * \chi_E(x) \in \left[\frac{1}{4}, \frac{3}{4}\right]$ for all $x \in D(x_0, C^{-1}\varepsilon)$. We let σ be surface measure on S^{d-1}; here we take it to be normalized so that $\sigma(S^{d-1}) = 1$. We also let E^c be the complement of the set E.

Choose once and for all a point of density of $E \cap D$, which we may assume to be the origin. Let A be the set of all $\omega \in S^{d-1}$ such that the ray $\{r\omega : 1 < r < C_d\}$ contains a point of density of E^c. Since E has measure 1 it is clear that A must have measure $\geq \frac{3}{4}$ provided C_d is large enough. If $\omega \in A$ then we let $p_\omega = r_\omega w$ be the corresponding point of density of E^c In a similar way we can choose a small sphere centered at $0, x = \{\rho\omega : \omega \in S^{d-1}\}$, where $\rho < 1$ in such a way that $q_\omega = \rho\omega$ is a point of density of E for all $\omega \in B$ where $B \subset S^{d-1}$ is a set of measure $> \frac{3}{4}$.

By Egoroff's theorem, we can find subsets $A^* \subset A$ with measure $\geq \frac{2}{3}$ and $B^* \subset B$ with measure $\geq \frac{2}{3}$ and a number ε_0 such that if $\varepsilon < \varepsilon_0$ then

$$\frac{|E \cap D(p_\omega, \varepsilon)|}{|D(p_\omega, \varepsilon)|} < 10^{-6}\ for\ all\ \omega \in A^* \tag{18}$$

and

$$\frac{|E^c \cap D(q_\omega, \varepsilon)|}{|D(q_\omega, \varepsilon)|} < 10^{-6}\ for\ all\ \omega \in B^* \tag{19}$$

Note $|A^* \cap B^*| \geq \frac{1}{3}$.

Now fix $\varepsilon < \varepsilon_0$, let $\omega \in A^* \cap B^*$ and consider $\varphi_\varepsilon * \chi_E$ as a function on the line segment $\{tw: \rho \le t \le r_\omega\}$. Its value at ρ is$\ge 1 - 10^{-6}$ and its value at r_ω is $\le 10^{-6}$Accordingly, there must be a value of $t_\omega \in (\rho, r_\omega)$ where $\varphi_\varepsilon * \chi_E(t_\omega\omega) = \frac{1}{2}$.Then by the remarks at the beginning of the proof, $\varphi_\varepsilon * \chi_E(t\omega) \in \left(\frac{1}{4}, \frac{3}{4}\right)$ for all $\omega \in A^* \cap B^*$ and all t in the interval centered at t_ω with length $C^{-1}\varepsilon$. Using polar coordinates it now follows that the set $\left\{x : \varphi_\varepsilon * \chi_E(x) \in \left(\frac{1}{4}, \frac{3}{4}\right)\right\}$ has measure $\gtrsim \varepsilon$ where the constant is independent of ε provided ε is small.

Corollary (2.3.8)[83]: If $E \subset \mathbb{R}^d$ is a set with finite nonzero measure and if φ_ε is as in Lemma (2.3.7) then

(a) $\|\varphi_\varepsilon * \chi_E - \chi_E\|_2 \ge C_E^{-1}\varepsilon^{\frac{1}{2}}$ for small ε.

(b) $\int_{|\xi|\ge R} |\widehat{\chi_E}|^2 \ge (C_E R)^{-1}$ for a certain constant C_E depending on E and all sufficiently large R. In particular, $\chi_E \notin W^{\frac{1}{2}}$.

Proof. Part (a) is immediate from Lemma (2.3.7), since $\frac{1}{4} \le \varphi_\varepsilon * \chi_E(x) \le \frac{3}{4}$ implies $|\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \ge \frac{1}{4}$. Part (b) follows easily from (a). By (a) we have

$$\int_{\mathbb{R}^n} |\widehat{\chi_E}(\xi)|^2 |\hat{\varphi}(R^{-1}\xi) - 1|^2 d\xi \ge (C_E R)^{-1} \tag{20}$$

uniformly in R, and if φ has been chosen to be nonnegative, then $|\hat{\varphi}(R^{-1}\xi) - 1|$ is bounded away from zero when $|\xi| \ge R$.

From Corollary (2.3.8) we can obtain a form of Theorem (2.3.14) where $\alpha = 2$:

Corollary (2.3.9)[83]: If $E \subset \mathbb{R}^2$ is Steinhaus then $\int_E |x|^2 dx = \infty$.

Proof. As was done in [91], we use the elementary estimate (which is also the only known estimate) for the maximum gap between sums of two squares:

(G): If $r \in [1, \infty)$ then for a suitable fixed constant C_1 there is $u \in \mathbb{Z}^2$ such that $|r - |v|| \le C_1 r^{-\frac{1}{2}}$.

We also use the following form of the Poincare inequality, which is well-known.

(PI): Let Q be a square in the plane with side r and let γ be a Jordan arc contained in Q, such that the distance between the endpoints of γ is $\ge C_1^{-1}r$. Let f be a function which vanishes on γ. Then

$$\int_Q |f|^2 \le C_2 r^2 \int_Q |\nabla f|^2$$

where C_2 depends on C_1 only.

Fix a large number N and define $A_N \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2 : N \leq |\xi| \leq 2N\}$. Let C be a large enough constant and cover A_N with nonoverlapping squares Q of side $CN^{-\frac{1}{2}}$. If E is Steinhaus, $f = \widehat{\chi_E}$, then (G) implies that each square will satisfy the hypothesis of (PI). We conclude that

$$\int_Q |\widehat{\chi_E}|^2 \lesssim N^{-1} \int_Q |\nabla\widehat{\chi_E}|^2$$

for each Q and therefore

$$\int_{A_N} |\widehat{\chi_E}|^2 \lesssim N^{-1} \int_{A_N^*} |\nabla\widehat{\chi_E}|^2$$

where A_N^* is the union of the squares and is contained in $\{\xi \in \mathbb{R}^2 : N - 1 \leq |\xi| \leq 2N + 1\}$. Consequently

$$\int_{A_N} |\xi||\widehat{\chi_E}(\xi)|^2 d\xi \lesssim \int_{A_N^*} |\nabla\widehat{\chi_E}|^2$$

If we now sum over dyadic values of N and use that no point belongs to more than two A_N^*s, we obtain

$$\int_{\mathbb{R}^2} |\xi||\widehat{\chi_E}(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^2} |\nabla\widehat{\chi_E}|^2 d\xi + 1$$

Hence by Corollary (2.3.8)(b), $\int_{\mathbb{R}^2} |\nabla\widehat{\chi_E}|^2 = \infty$, i.e. $\int_E |x|^2 dx = \infty$ see [87].

If $1 < p < \infty$ and $\alpha > 0$, then we let $W^{p,\alpha}$ be the L^p Sobolev space with α derivatives. If E is any set with positive measure, then χ_E cannot belong to $W^{p,\frac{1}{p}}$. This is because Lemma (2.3.7) implies that $\|\chi_E - \varphi_\varepsilon * \chi_E\|_p \gtrsim \varepsilon^{\frac{1}{p}}$, which implies that χ_E cannot belong to any Besov space $\Lambda_{\frac{1}{p}}^{pq}$ with $q < \infty$. Since $\Lambda_{\frac{1}{p}}^{q}$ contains $W^{p,\frac{1}{p}}$ when $q \geq \max(p, 2)$ it follows that χ_E cannot belong to $W^{p,\frac{1}{p}}$.

We note that Croft's proof [44] that Steinhaus sets are unbounded was based on considering points which are density points neither of E nor of its complement. Corollary (2.3.8) is basically a quantitative version of existence of such points.

We now prove a further technical result, which we will need for the proof of Theorem (2.3.14). It says roughly that the lower bounds on $\|\varphi_\varepsilon * \chi_E - \chi_E\|_2$ obtained (as above) by considering large values are always sharp. If $E \subset \mathbb{R}^d$ is a set of finite measure, then we define

$$A_\varepsilon(E) = \|\varphi_\varepsilon * \chi_E - \chi_E\|_1$$

$$B_\varepsilon(E) = \|\varphi_\varepsilon * \chi_E - \chi_E\|_2^2$$

$$C_\varepsilon(E) = \left|\left\{x \in \mathbb{R}^d : |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}\right|$$

It is easy to see that

$$C_\varepsilon(E) \lesssim B_\varepsilon(E) \lesssim A_\varepsilon(E) \tag{21}$$

for any E and ε.

Lemma (2.3.10)[83]: For any given set $E \subset \mathbb{R}^d$ with $|E| < \infty$ there is a sequence $\varepsilon_j = 2^{-k_j} \to 0$ such that $A_{\varepsilon_j}(E) \lesssim C_{\varepsilon_j}(E)$; the constants here (and in (21)) depend only on d and φ.

Proof. We may assume that $|E| = 1$. If $D = D(x, \rho)$ is the ball with center x and radius ρ then we define

$$\alpha(D) = \min(|E \cap D|, |E^c \cap D|)$$

$$\beta(D) = \sum_{j=0}^{\infty} 2^{-10dj} \alpha(2^j D)$$

Here we have used the notation $E^c = \mathbb{R}^d \backslash E$ and $(x, \rho) = D(x, r\rho)$.

Let C_0 be a large constant. If D is any ball of radius $C_0^{-1}\varepsilon$ then we claim that the following are valid:

I. $\|\varphi_\varepsilon * \chi_E - \chi_E\|_{L^1(\mathrm{D})} \lesssim \beta(D)$

II. $\left|\left\{x \in D : |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}\right| \geq \alpha(D)$.

In fact, II follows easily from (17). Namely, if C_0 is large then (17) implies via the mean value theorem that the difference between the maximum and minimum values of $\varphi_\varepsilon * \chi_E$ on the ball D is less than $\frac{1}{2}$. It follows that one of the following must hold

(i) $\varphi_\varepsilon * \chi(x) \leq \frac{3}{4}$ for all $x \in D$, or

(ii) $\varphi_\varepsilon * \chi(x) \geq \frac{1}{4}$ for all $x \in D$.

In case (i) we have $\left|\left\{x \in D: |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}\right| \geq |E \cap D| \geq \alpha(D)$ and in case (ii) we have $\left|\left\{x \in D: |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}\right| E^c \cap D| \geq \alpha(D)$,i.e. II holds in either case.

To prove I, we express φ as a synthesis of C_0^∞ functions, say

$$\varphi = \sum_{j=0}^{\infty} a_j\, \varphi^j$$

where $\text{supp}\varphi^g \subset D\big(0, (2C_0)^{-1}2^j\big), \widehat{\varphi_j}(0) = 1$, $\|\varphi_j\|_1 \le C$ and $a_j \le C2^{-10dj}$. Let $\varphi_\varepsilon^j(x) = \varepsilon^{-d}\varphi^j(\varepsilon^{-1}x)$. It follows by Minkowski's inequality and the support properties that

$$\|\varphi_\varepsilon^j * \chi_E - \chi_E\|_{L^1(D)} \lesssim |E \cap \big(2^j D\big)|$$

and therefore also

$$\|\varphi_\varepsilon^j * \chi_E - \chi_E\|_{L^1(D)} \lesssim \alpha\big(2^j D\big)$$

since the left side is unchanged when E is replaced by E^c I now follows by summing over j.

Let $(\varepsilon) = \int_{\mathbb{R}^d} \alpha\left(D(x, C_0^{-1}\varepsilon)\right)dx$, $J(\varepsilon) = \int_{\mathbb{R}^d} \beta\left(D(x, C_0^{-1}\varepsilon)\right)dx$. Integrating I and II over $\mathbb{R}^d$ we get

$$\varepsilon^{-d}I(\varepsilon) \lesssim C_\varepsilon(E) \lesssim A_\varepsilon(E) \lesssim \varepsilon^{-d}J(\varepsilon) \tag{22}$$

Let k be a large positive integer and consider the sums

$$\mathcal{I}_k = \sum_{\ell=0}^{\infty} 2^{-5d\ell}\, I\big(2^{\ell-k}\big)$$

$$\mathcal{J}_k = \sum_{\ell=0}^{\infty} 2^{-5d\ell}\, J\big(2^{\ell-k}\big)$$

For any k, we have

$$\mathcal{J}_k = \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} 2^{-d(5\ell+10j)}\, I\big(2^{\ell+j-k}\big)$$

$$\lesssim \sum_{m=0}^{\infty} 2^{-5dm}\, I(2^{m-k}) = \mathcal{I}_k$$

On the next to last line, we set $m = j + \ell$ and used that $\sum_{j+\ell=m} 2^{-d(5\ell+10j)<} 2^{-5dm}$.

Now observe that $J(\varepsilon) \gtrsim \varepsilon^{d+1}$ for small ε, e.g. by (22) and Corollary (2.3.8)(a), and that $I(\varepsilon) \lesssim \varepsilon^{d}$ for any ε (even when $\varepsilon > 1$), e.g. by (22). It follows that $\mathcal{J}_k \gtrsim 2^{-(d+1)k}$ and that $\sum_{\ell > \frac{k}{2}} 2^{-5d\ell} I_{k-\ell}$ is small compared with $2^{-(d+1)k}$ Accordingly

$$\sum_{\ell \leq \frac{k}{2}} < 2^{-5d\ell} J(2^{\ell-k}) \lesssim \sum_{\ell \leq \frac{k}{2}} 2^{-5d\ell} I(2^{\ell-k})$$

which implies there is a value $2^{\ell-k} \leq \sqrt{2-k}$ with $J\left(2^{\ell-k}\right) \lesssim I\left(2^{\ell-k}\right)$. This and (22) prove the lemma.

We assume that the Schwarz function φ satisfies the following conditions:

$$supp\ \hat{\varphi} \subset D(0,1), \hat{\varphi}(\xi) = 1\ if\ \xi \in D\left(0, \frac{1}{2}\right) \tag{23}$$

We set $\varphi(x) = \varphi(x) - 2^{d}\varphi(2x)$; thus ψ is a Schwarz function with supp $\hat{\psi} \subset D(0,2) \setminus D\left(0, \frac{1}{2}\right)$. We define $\psi_{\varepsilon}(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x)$, so that $\sum_{j=0}^{\infty} \psi_{2-j\varepsilon} * f = \varphi_{\varepsilon} * f - f$ for any f and ε, as may be seen bytaking Fourier transforms. Property (23) implies that no point belongs to the support of $\widehat{\psi_J}$ for more that three values of j, so it follows by the Plancherel theorem that

$$\sum_{j=0}^{\infty} \| \psi_{2-j\varepsilon} * f \|^2 2 \gtrsim \| f - \varphi_{\varepsilon} * f \|_2^2 \tag{24}$$

Furthermore,

$$\| \psi_{2-j\varepsilon} * f \|_1 \lesssim \| f - \varphi_{\varepsilon} * f \|_1 \tag{25}$$

Namely, the support property (23) makes it possible to represent $\psi_{2-j} = g_j * (\delta - \varphi)$ with $\|g_j\|_1 \leq C$ (here δ is the Dirac delta function). Indeed if $j \geq 1$ then $\widehat{\psi_{2-J}}$ and $\hat{\varphi}$ have disjoint support so we can take $g_j = \psi_{2-j}$, and when $j = 0, \widehat{\psi_{2-J}} = \hat{\psi}$ is obtained from $1 - \hat{\varphi}$ by multiplication by the C_0^{∞} function m defined via $m(\xi) = -1$ when $|\xi| \leq 1$ and $\frac{\tilde{\psi}}{1-\varphi}$ when $|\xi| \geq 1$. It follows using dilations that $\psi_{2^{-j}\varepsilon} = g_{j,\varepsilon} * (\delta - \varphi_{\varepsilon})$ where $|\xi| \leq 1$ and when $||g_j, \varepsilon\|_1 = \|g \leq C$.

Accordingly $\|\psi_{2^{-j}\varepsilon} * f\|_1 = \|g_{j,\varepsilon} * (\delta - \varphi_{\varepsilon}) * f\|_1 = \|g_{j,\varepsilon} * (f - \varphi_{\varepsilon} * f)\|_1 \leq C\|f - \varphi_{\varepsilon} * f\|_1$ which is (25).

Corollary (2.3.11)[83]: Assume that φ satisfies (23) and define ψ as above. If E is a set of finite measure then there is a sequence $\varepsilon_j \to 0$ such that, for each j, (i) $\|\psi_{\varepsilon_j} * \chi_E\|_1 \lesssim \left(\log \frac{1}{\varepsilon_j}\right)^2 \|\psi_{\varepsilon_j} * \chi_E\|_2^2$ and (ii) $\|\psi_{\varepsilon_j} * \chi_E\|_2^2 \gtrsim \varepsilon_j$.

Proof Let ε be such that $A_\varepsilon(E) \lesssim B_\varepsilon(E)$. If $\eta_k = 2^{-k}\varepsilon$ then $\|\psi_{\eta_k} * \chi_E\|_1 \lesssim A_\varepsilon(E)$ by (25) and $\sum_{k\le 0} \|\, \psi_{\eta_k} * \chi_E\|_2^2 \gtrsim B_\varepsilon(E)$ by (24). Hence, for some k we must have

$$\max\left((k+1)^{-2}\|\psi_{\eta_k} * \chi_E\|_1, (k+1)^{-2}B_\varepsilon(E)\right) \lesssim \|\psi_{\eta_k} * \chi_E\|_2^2$$

Also $B_\varepsilon(E) > \varepsilon \sim$ by Corollary (2.3.8)(a), so $(k+1)^{-2}B_\varepsilon(E) \gtrsim \eta_k$, and $(k+1)^{-2} \gtrsim \left(\log \frac{1}{\eta_k}\right)^{-2}$ We conclude that

$$\max\left(\left(\log \frac{1}{\eta_k}\right)^{-2} \|\psi_{\eta_k} * \chi_E\|_1, \eta_k\right) \lesssim \|\psi_{\eta_k} * \chi_E\|_2^2$$

i.e. that there are arbitrarily small numbers ε_j such that (i) and (ii) hold.

The following fact will be used repeatedly below, so we formulate it as a lemma.

Lemma (2.3.12)[83]: If $N \geq 1$ then for any $\varepsilon > 0$ and $r > 0$

$$\sum_{v\in\mathbb{Z}^2} (1 + N|r - |v||)^{-100} \leq C_\varepsilon N^\varepsilon \max\left(\frac{r}{N}, 1\right)$$

Proof Because of the rapid decay of $(1 + Nt)^{-100}$ when $t \geq \frac{1}{N}$, it is easy to show that it suffices to prove the following estimate for all r:

$$n\left(r + \frac{1}{N}\right) - n(r) \lesssim N^\varepsilon \max\left(\frac{r}{N}, 1\right) \tag{26}$$

where $n(r)$ is as in (27). To prove (26), consider two cases.

(i) $r \leq N^3$ The number of lattice points on a circle is bounded by any given power of the radius, hence a circle of radius $\rho \in \left(r, r + \frac{1}{N}\right)$ contains $\lesssim r^{\frac{\varepsilon}{3}} \lesssim N^\varepsilon$ lattice points. There are $\lesssim \max(\frac{r}{N}, 1)$ values of ρ for which it contains some lattice point and (26) follows.

(ii) $r \geq N^3$. In this case we use (l) with the classical exponent $\beta = \frac{2}{3}$. Thus $n\left(r + \frac{1}{N}\right) - n(r) \lesssim \frac{r}{N} + r^{\frac{2}{3}} \approx \frac{r}{N}$.

The proof of Theorem (2.3.14) will be like the proof of Theorem (2.3.1) insofar as it is also based on using an appropriate “fundamental solution” However, we must replace the kernel in Lemma (2.3.3) by an analogous one involving a sum only over circles which contain lattice points. We will use the obvious choice where one counts each circle according to the number of lattice points it contains.

Let p be a nonnegative C^{∞} function of one variable supported in $t \leq 1$ and with $p(t) = 1$ when $t \leq \frac{1}{2}$. Define

$$K_N(x) = \sum_{v\in\mathbb{Z}^2, v\neq 0} \frac{1}{|v|} \widehat{\sigma_{|v|}}(r) p\left(\frac{|v|}{N}\right)$$

where $r = |x|$.

We will use complex notation when convenient and define operators T_p on $L^2(\mathbb{R}^2)$ via $T_p f(x) = \int f\left(x + \rho e^{i\theta}\right) \frac{d\theta}{2\pi}$, i.e. $T_p f$ is the circular mean over the circle of radius ρ.

Lemma (2.3.13)[83]: Let $E \subset \mathbb{R}^2$ be a Steinhaus set and let ψ be a Schwarz function in $\mathbb{R}^2$ with $\hat{\psi}(0) = 0$. Let $f = \chi_E$. Then

$$\psi * f(x) = -\sum_{v\in\mathbb{Z}^2, v\neq 0} T_{|v|}\left(\psi * f\right)$$

Proof The Steinhaus property gives after convolving with ψ that

$$\psi * f(x) = -\sum_{v\in\mathbb{Z}^2, v\neq 0} \psi * f(x + e^{i\theta} v)$$

for all θ and x. The lemma follows by integrating with respect to θ.

Theorem (2.3.14)[83]: Assume a bound of the form

$$n(r) = \pi r^2 + \mathcal{O}\left(r^{\beta}\right) \tag{27}$$

where $(r) = \mathrm{card}\left((\mathbb{Z}^2 \backslash \{0\}) \cap D(0, r)\right)$. Then any Steinhaus set $E \subset \mathbb{R}^2$ must satisfy

$$\int_E |x|^{\alpha} dx = \infty \tag{28}$$

for all $\alpha > \frac{\beta}{1-\beta}$.

Proof. We let β be such that (27) is true and assume toward a contradiction that E is Steinhaus and $\int_E |x|^{\alpha} < \infty$ for some $\alpha > \frac{\beta}{1-\beta}$.

Fix a Schwarz function φ satisfying (23) and set $(x) = \varphi(x) - 4\varphi(2x)$. Thus supp $\hat{\psi} \subset D(0,2)\backslash D\left(0,\frac{1}{2}\right)$. Let $\psi_R(x) = R^2\psi(Rx)$. Also fix a function p as in Lemma (2.3.22). Applying Lemma (2.3.13) with ψ_R, we get for any M

$$\psi_R * \chi_E = -A_M(\psi_R * \chi_E) - B_M(\psi_R * \chi_E) \tag{29}$$

where the operators A_M and B_M are defined by

$$A_M = \sum_{v\neq 0} p\left(\frac{|v|}{M}\right) T_{|v|}$$

$$B_M = \sum_{v} \left(1 - p\left(\frac{|v|}{M}\right)\right) T_{|v|}$$

Note that A_M and B_M are convolution operators and the convolution kernel of A_M is supported in $|x| \leq M$.

The strategy of the proof is to show that the right side of (29) is too small to be equal to the left side, and we start by making appropriate $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates for the operators A_M and B_M respectively. We state the estimates in a "localized" form for the sake of the application below.

Claim (2.3.15)[83]: Assume that $M < R$ and that $\operatorname{supp}(\hat{g}) \subset D(0,2R)\backslash D\left(0,\frac{R}{2}\right)$. Then, given (27), there is an estimate

$$\|A_M g\|_{L^2(D(a,M))} \lesssim MR^{-(1-\beta)}\|g\|_{L^2(D(a,10,M))} + R^{-100}\|g\|_2$$

for any $a \in \mathbb{R}^2$

Namely, let J_0^R and J^R be the annuli $\left\{\xi : \frac{R}{3} \leq |\xi| \leq 3R\right\}$ and $\left\{\xi : \frac{R}{4} \leq |\xi| \leq 4R\right\}$ respectively. The estimate

$$supp\ \hat{g} \subset J^R \Rightarrow \|A_M g\|_2 \lesssim MR^{-(1-\beta)}\|g\|_2 \tag{30}$$

is immediate from Lemma (2.3.22): A_M is a convolution operator, and the corresponding multiplier is the function K_M, whose L^∞ norm on $D(0,4R)\backslash D\left(0,\frac{R}{4}\right)$ is $\lesssim MR^{-(1-\beta)}$ by Lemma (2.3.22).

The localized form follows in a standard way using that the convolution kernel of A_M is supported in $|x| \leq M$: we may suppose $a = 0$, and we let $\rho \in C_0^\infty$ be such that $\rho = 1$ on $D(0,10)$. Define $\rho_M(x) = \rho(M^{-1}x)$. Let χ be a Schwarz function whose Fourier transform is supported in J^1 and equal to 1 on J_0^1 and define $\chi_R(x) = R^2\chi(Rx)$.

The support property of the convolution kernel implies that $A_M g(x) = A_M(g\rho_M)(x)$ when $\in D(0, M)$. Accordingly

$$\begin{aligned} \|A_M g\|_{L^2(D(0,M))} &\leq \|(A_M(\chi_R * (g\rho_M))\|_2 + \|A_M(g\rho_M - \chi_R * (g\rho_M))\|_2 \\ &\lesssim MR^{-(1-\beta)}\|g\rho_M\|_2 + M^2\|g\rho_M - \chi_R * (g\rho_M)\|_2 \end{aligned} \quad (31)$$

where we used (30), that $\|\chi_R\|_1 = \|\chi\|_1 \leq C$, and the trivial estimate $\|A_M f\|_{2_\sim} < M^2\|f\|_2$ (since A_M is convolution with a sum of $\mathcal{O}(M^2)$ probability measures) in the second term. On taking Fourier transforms we see that $\|g\rho_M - \chi_R * (g\rho_M)\|_2 = \|(1 - \widehat{\chi_R})\widehat{\rho_M} * \hat{g}\|_{2_\sim} < \|\widehat{\rho_M} * \hat{g}\|_{L^2(\mathbb{R}^2 \setminus J_0^R)} \lesssim (MR)^{-102}\|g\|_2$, where the last inequality follows since $\hat{g}$ is supported in $\frac{R}{2} \leq |\xi| \leq 2R$ and $|\widehat{\rho_M}(\eta)| \lesssim M^2(M|\eta|)^{-200}$. Claim (2.3.15) follows by substituting this bound into (31).

Claim (2.3.16)[83]: If $M < R$, supp $\hat{g} \subset D(0,2R)$ then for any $\varepsilon > 0$,

$$|BMg(x)| \lesssim R^\varepsilon \frac{R}{M}\|g\|_{L^1\left(D\left(x,\frac{M}{3}\right)^c\right)} + R^{-100}\|g\|_1$$

For this, we fix a Schwarz function ρ such that $\hat{\rho} = 1$ on $D(0,2)$ and define $\rho_R(x) = R^2\rho(Rx)$. Then $g = \rho_R * g$, so

$$\begin{aligned} B_M g &= \sum_v \left(1 - p\left(\frac{|v|}{M}\right)\right) T_{|v|}(\rho_R * g) \\ &= \sum_v \left(1 - p\left(\frac{|v|}{M}\right)\right) |v|^{-1}(\rho_R * \sigma_{|v|}) * g \end{aligned} \quad (32)$$

where $\sigma_{|v|}$ is arclength measure on the circle centered at 0 with radius $|v|$. We let H be the convolution kernel in (32), i.e. $H(x) = \sum_v \left(1 - p\left(\frac{|v|}{M}\right)\right) |v|^{-1}\rho_R * \sigma_{|v|}(x)$.

Uniformly in v we have

$$|\rho_R * \sigma_{|v|}(x)| \lesssim R\big(1 + R\big||v| - |x|\big|\big)^{-101} \quad (33)$$

This is well known and is easy to prove using that $\sigma_{|v|}(D(a, t)) \lesssim t$ uniformly in v, a and t. We now sum over v and use that $p(t) = 1$ when $t \leq \frac{1}{2}$. Thus

$$|H(y)| \lesssim \sum_{|v| \geq \frac{M}{2}} \frac{R}{|v|}(1 + R||v| - |y||)^{-101}$$

It is clear that

$$\sum_{\substack{|v|\geq\frac{M}{2} \\ ||v|-|y||\geq\frac{|v|}{100}}} \frac{R}{|v|}(1+R||v|-|y||)^{-101} \lesssim \sum_{|v|\geq\frac{M}{2}} \frac{R}{|v|}(R|v|)^{-101} \lesssim R^{-100}$$

Accordingly,

$$|H(y)| \lesssim R^{-100} + \sum_{\substack{|v|\geq\frac{M}{2} \\ ||v|-|y||\geq\frac{|v|}{100}}} \frac{R}{|v|}\left(1+R\big||v|-|y|\big|\right)^{-100}\big||v|-|y|\big| \leq \frac{|v|}{100} \tag{34}$$

If $|y| \leq \frac{M}{3}$ then the sum in (34) is empty, so

$$|H(y)| \lesssim R^{-100} \tag{35}$$

If $|y| > \frac{M}{3}$, then we observe that $|v| \geq \frac{|y|}{2}$ for all u in the sum (34), and then apply Lemma (2.3.12) with $r = |y|$ and $N = R$ obtaining

$$|H(y)| \lesssim R^{-100} + \frac{R}{|y|}\sum_{v}(1+R||v|-|y||)^{-100}$$

$$\lesssim \frac{R}{|y|}\cdot R^{\varepsilon} \max\left(\frac{|y|}{R},1\right)$$

$$\lesssim R^{\varepsilon}\frac{R}{M} \tag{36}$$

Claim (2.3.16) follows from formula (32) and the estimates (35), (36) for the convolution kernel H.

We now continue with the main proof. By Corollary (2.3.11), we can find arbitrarily large numbers R such that $\|\psi_R * \chi_E\|_2^2 \geq (\log R)^{-2}\|\psi_R * \chi_E\|_1$ and also $\|\psi_R * \chi_E\|_2^2 \gtrsim R^{-1}$ In the subsequent argument R is taken to be a sufficiently large number with these properties. We fix γ with $1-\beta > \gamma > \frac{1}{1+\alpha}$, and define

$$M = R^{\gamma} \tag{37}$$

To ease the notation we also define

$$g = \psi_R * \chi_E$$

Note that $\operatorname{supp}(\hat{g}) \subset D(0,2R)\setminus D\left(0,\frac{R}{2}\right)$; this fact will be used without mention below.

We subdivide $\mathbb{R}^2$ in squares Q of side $10^{-6}M$ taking one of them to be centered at the origin. We will denote the square centered at the origin by Q_0. Let $\tilde{Q}$ be the disc concentric with$Q$with radius $\frac{1}{10}M$ and $\tilde{Q}$ the concentric disc with radius M. Define a square Q *to be good* if $\|g\|_{L^2(Q)}^2 \geq (\log R)^{-4}\|g\|_{L^1(\tilde{Q})}$ and bad otherwise. The reason for making this definition is as follows:

Claim (2.3.17)[83]: If Q is a good square and $h : Q \to \mathbb{C}$ is a function on Q such that $\|h\|_\infty \leq \frac{1}{4}(\log R)^{-4}$ then

$$\|g + h\|_{L^2(Q)}^2 \gtrsim (\log R)^{-4}\|g\|_{L^2(\tilde{\tilde{Q}})}^2$$

Namely, let $Y = \{y \in Q: |g(y)| \geq 2\|h\|_\infty\}$. Then

$$\begin{aligned} \|g\|_{L^2(Q\setminus Y)}^2 &\leq \|g\|_{L^\infty(Q\setminus Y)}\|g\|_{L^1(Q\setminus Y)} \\ &\leq 2\|h\|_\infty\|g\|_{L^1(Q)} \\ &\leq 2(\log R)^4\|h\|_\infty\|g\|_{L^2(Q)}^2 \\ &\leq \frac{1}{2}\|g\|_{L^2(Q)}^2 \end{aligned}$$

so that $\|g\|_{L^2(Y)}^2 \geq \frac{1}{2}\|g\|_{L^2(Q)}^2$. If $y \in Y$, then $|g(y) + h(y)| \geq \frac{1}{2}|g(y)|$, so we have $|g + h\|_{L^2(Y)}^2 \geq \frac{1}{4}\|g\|_{L^2(Y)}^2 \geq \frac{1}{8}\|g\|_{L^2(Q)}^2$. Claim (2.3.17) now follows since $\|g\|_{L^2(Q)}^2 \geq (\log R)^{-4}\|g\|_{L^1(\tilde{\tilde{Q}})} \gtrsim (\log R)^{-4}\|g\|_{L^2(\tilde{\tilde{Q}})}^2$.

Next we have

Claim (2.3.18)[83]: There is a good square Q with the following two additional properties:

$$\|g\|_{L^1(\tilde{Q}^c)} \lesssim (\log R)^{100}M^{-\alpha} \tag{38}$$

$$\|g\|_{L^1(Q)} \geq R^{-50} \tag{39}$$

For this, we let $\mathcal{G}$ and B be the unions of the good and bad squares respectively and let $\tilde{\tilde{B}}$ be the union of the $\tilde{\tilde{Q}}$'s corresponding to bad Q's. We note that any given point y belongs to $\tilde{\tilde{Q}}$ for only a bounded number of Q's. We have

$$\begin{aligned} \|g\|_{L^1(\mathcal{G})} + \|g\|_{L^1(\tilde{\tilde{B}})} &\lesssim \|g\|_1 \\ &\lesssim (\log R)^2\|g\|_2^2 \\ &= (\log R)^2\|g\|_{L^2(\mathcal{G})}^2 + (\log R)^2\|g\|_{L^2(B)}^2 \end{aligned}$$

$$\lesssim (\log R)^2 \|g\|_{L^1(\mathcal{G})} + (\log R)^{-2} \|g\|_{L^1(\tilde{\tilde{B}})}$$

so that $\|g\|_{L^1(\tilde{\tilde{B}})} \lesssim (\log R)^2 \|g\|_{L^1(\mathcal{G})}$ and therefore

$$\|g\|_1 \lesssim (\log R)^2 \|g\|_{L^1(\mathcal{G})} \tag{40}$$

Next define $\mathcal{G}_*$ to be the union of all good squares Q which have property (38). We will show that

$$\|g\|_{L^1(\mathcal{G}_*)} \gtrsim (\log R)^{-2} \|g\|_1 \tag{41}$$

Namely, our decay assumption on the set E implies that

$$\|g\|_{L^1(Q_0^c)} \lesssim M^{-\alpha} \tag{42}$$

Now consider two cases:

(ii) $\|g\|_{L^1(Q_0)} \leq \frac{1}{2} (\log R)^{100} M^{-\alpha}$

(ii) $\|g\|_{L^1(Q_0)} > \frac{1}{2} (\log R)^{100} M^{-\alpha}$

In case (i), (42) implies that all squares Q satisfy (38) so (41) follows tautologically from (40). In case (ii), (42) implies that

$$\|g\|_{L^1(Q_0^c)} \lesssim (\log R)^{-100} \|g\|_{L^1(Q_0)} \tag{43}$$

If Q_0 were bad, then (43) would imply that $\|g\|_{L^1(\mathcal{G})} \lesssim (\log R)^{-100} \|g\|_1$, contradicting (40) if R is large enough So Q_0 must good, and therefore contained in $\mathcal{G}_*$ by (42).

Accordingly $\|g\|_{L^1(\mathcal{G}*)} \geq \|g\|_{L^1(Q_0)\sim} > \frac{(\log R)^{100}}{1+(\log R)^{100}} \|g\|_1$,where the last inequality follows from (43). This is stronger than (41), which has therefore been proved in both cases (i) and (ii).

Now let X be the union of all squares Q such that $\|g\|_{L^1(Q)} < R^{-50}$ Then, taking (say) $T = R^{10}$,

$$\begin{aligned}
\|g\|_{L^1(X)} &\leq \|g\|_{L^1(X \cap D(0,T))} + \|g\|_{L^1(X \cap D(0,T)^c)} \\
&\lesssim R^{-50} \left(\frac{T}{R}\right)^2 + T^{-\alpha} \\
&\leq R^{-10} \\
&\lesssim R^{-9} \|g\|_1
\end{aligned}$$

This and (41) imply that $\mathcal{G}_*$ cannot be contained in X, which gives Claim (2.3.18).

Let Q be the square in Claim (2.3.18). If $x \in Q$, then $D\left(x, \frac{M}{3}\right)^c$ is disjoint from $\tilde{Q}$. Accordingly, by Claim (2.3.16) and then (38) and (37), for any $\varepsilon > 0$

$$\|B_M(g)\|_{L^\infty(Q)} \leq R^\varepsilon \frac{R}{M} \|g\|_{L^1\left(\overline{Q}^c\right)} + R^{-100}\|g\|_1$$

$$\lesssim R^{1-\gamma+\varepsilon} (\log R)^{100} M^{-\alpha}$$

$$= (\log R)^{100} R^{1-\gamma-\gamma\alpha+\varepsilon}$$

If ε is small, then the exponent of R here is negative. It follows by Claim (2.3.17) that

$$\|g + B_M(g)\|_{L^2(Q)}^2 \gtrsim (\log R)^{-4} \|g\|_{L^2(\tilde{\tilde{Q}})}^2 \tag{44}$$

On the other hand,

$$\|g + B_M(g)\|_{L^2(Q)}^2 = \| - A_M(g)\|_{L^2(Q)}^2$$

$$\lesssim \left(MR^{-(1-\beta)}\right)^2 \|g\|_{L^2(Q)}^2 + R^{-200}$$

$$\lesssim R^{-\eta} \|g\|_{L^2(\tilde{\tilde{Q}})}^2 + R^{-200} \tag{45}$$

where $\eta = 2(1 - \beta - \gamma) > 0$. We used Claim (2.3.15) and (37). Combining (44) and (45) we get

$$(\log R)^{-4} \|g\|_{L^2(\tilde{\tilde{Q}})}^2 \lesssim R^{-\eta} \|g\|_{L^2(\tilde{\tilde{Q}})}^2 + R^{-200}$$

and therefore $\|g\|_{L^2(Q)}^2 \lesssim R^{-199}$ Since Q is good it follows that $\|g\|_{L^1(Q)} \leq R^{-198}$, which contradicts (39) so the proof of Theorem (2.3.14) is complete.

Before proving Theorem (2.3.19) we will make some further remarks about the question.

If $\Lambda \subset \mathbb{R}^d$ is a lattice then let $\Lambda^* = \{\xi \in \mathbb{R}^d : \xi \cdot x \in \mathbb{Z} \forall x \in \Lambda\}$ be the dual lattice. We note that a function f tiles with the lattice Λ precisely when f vanishes on $\Lambda^* \setminus \{0\}$.

The Steinhaus problem asks for a subset of $\mathbb{R}^d$ that tiles with all rotations of the lattice $\mathbb{Z}^d$ It seems reasonable instead to ask for a set $E \subset \mathbb{R}^d$ that tiles with a given finite collection of lattices, say $\Lambda_1, \ldots, \Lambda_n$. Γor lattices with volume 1 and with no nontrivial relation of the type

$$\lambda_1 + \cdots + \lambda_n = 0, \lambda_i \in \Lambda_i^*$$

it is shown in [92] that measurable such sets exist. The existence question is of course very easy if instead of trying to tile with a subset of $\mathbb{R}^d$ we try to find a function $f \in L^1(\mathbb{R}^d)$ that tiles simultaneously with a given collection of lattices, that is

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = Const_{\Lambda}, for \text{ a.e. } x\in\mathbb{R}^d, \tag{46}$$

and for all lattices Λ in the collection under consideration. Indeed, say we are dealing with the finite collection $\Lambda_1, \ldots, \Lambda_n$, assume that D_i is a fundamental parallelepiped for the lattice Λ_i, and write

$$f = \chi_{D_1} * \cdots * \chi_{D_n}. \tag{47}$$

Since tiling with a lattice Λ is equivalent with the vanishing of the Fourier Transform on $\Lambda^*\backslash\{0\}$, and since it is clear that χ_{D_i} tiles with the lattice Λ_i, it follows that the function f defined in (47) tiles with all Λ_i, $i = 1, \ldots, n$.

The problem becomes nontrivial if we try to find such a function f that tiles with $\Lambda_1, \ldots, \Lambda_n$ which has small support. It is easy to see that, whenever the Λ_i have volume 1, no matter what the choice of the D_i, the function f defined in (47) necessarily has support of diameter at least Cn, where C depends only on the dimension.

Theorem (2.3.19) gives a lower bound for the diameter of the support of a function $f\in L^1(\mathbb{R}^d)$ that tiles with a given finite number of trivially intersecting unimodular lattices.

Theorem (2.3.19)[83]: There is a constant $B = B(d)$ making the following true. Suppose that the lattices Λ_i, $i = 1, \ldots, n$, have volume 1 and that

$$\Lambda_i\cap\Lambda_j = \{0\}, for\ all\ i\neq j \tag{48}$$

Let $f\in L^1(\mathbb{R}^d)$ be a function which tiles with all the Λ_i, and assume that $\hat{f}(0)\neq 0$. Then the diameter of the support of f is at least $Bn^{\frac{1}{d}}$.

Proof. All constants below may depend only on the dimension d. We note that $\Lambda_1\cap\Lambda_2 = \{0\}$ implies that the lattice Λ_1^* is uniformly distributed mod Λ_2^*. This can be proved using Weyl's lemma-see for example [92].

We shall make use of a theorem of Ronkin [94] and Berndtsson [85] which concerns the zero set on the real plane of an entire function of several complex variables which is of exponential type. We formulate it as a lemma:

Lemma (2.3.20)[83]: ([94],[85]) Assume that $E\subset\mathbb{R}^d$ is a countable set with any two points having distance at least h and let

$$d_E = \lim_{r\to\infty}\sup\frac{|E\cap D(0,r)|}{|D(0,r)|}$$

be its “upper density” Assume that $g : \mathbb{C}^d \to \mathbb{C}$ is an entire function vanishing on E which is of exponential type

$$\sigma < A(d)h^{d-1}d_E.$$

Then g is identically 0. (Here $A(d)$ is an explicit function of the dimension d)

When $d = 1$ this is classical and follows from Jensen’s formula.

Assume that $: \mathbb{R}^d \to \mathbb{C}$ is as in Theorem (2.3.19). Then $\hat{f}$ vanishes on $(\cup_i \Lambda_i^*)\backslash\{0\}$. Write

$$\alpha = \ diam\ supp\ f$$

We may assume that supp f is contained in a disc of radius $\lesssim \alpha$ centered at the origin, since the assumptions are unaffected by a translation of coordinates. Then $\hat{f}$ can be extended to $\mathbb{C}^d$ as an entire function of exponential type $C\alpha$, in fact

$$|\hat{f}(x+iy)| \le C_f e^{C\alpha|y|}, for\ x+iy \in \mathbb{C}^d.$$

Furthermore, $\hat{f}$ vanishes on

$$Z = \bigcup_{i=1}^{n} \Lambda_i^*\backslash\{0\}.$$

Observe that, since every lattice Λ_i^* is uniformly distributed mod every Λ_j^*, $j \neq i$, the density of points in each Λ_i^* which are also in some Λ_j^* is 0 and therefore the density of the set Z is n.

In order to use Lemma (2.3.20) we have to select a large (in terms of upper density), well-separated subset of Z. Notice first that we can assume that for each i all points of Λ_i^* are at least distance $n^{-\frac{1}{d}}$ apart. For if $u, v \in \Lambda_i^*$ have $|u-v| < n^{-\frac{1}{d}}$ then for a suitable constant c, the one-dimensional version of Lemma (2.3.20) implies that the function $\hat{f}$ on the subs$_\wedge$pace $E= \mathbb{C}(u-v)$ cannot be of exponential type $\le cn^{\frac{1}{d}}$. Note also that the assumption $f(0) \neq 0$ precludes f vanishing identically on this subspace. But $\hat{f}$ restricted to E is the Fourier transform of $f_E : E \to \mathbb{C}$ defined by $f_E(x) = \int_{x+E^\perp} f\,(y)dy$ (here $E^\perp$ is the orthogonal complement of $E \cap \mathbb{R}^n$ in $\mathbb{R}^n$). Hence $\alpha \ge$ diam supp $f \ge Cn^{\frac{1}{d}}$, which is what we want to conclude about α.

Suppose now that we want to extract a subset of Z whose elements are at least h distance apart, for some $h > 0$ to be determined later. We shall say that point x of lattice Λ_i^* is *killed* by point y of lattice Λ_j^* if $|x-y| < h$. Then, we define the subset Z' of Z as those points of Z which are not killed by any point of the other lattices. This set clearly has all its points at distance at least h apart, provided that

$$h \le \frac{1}{2} \min_{u,v\in\Lambda_i^*} |u-v| \le Cn^{-\frac{1}{d}}, \tag{49}$$

so that no point of a lattice may kill a point of the same lattice. Let us see how many points of Λ_2^* are killed by some point of Λ_1^*. We use the uniform distribution of Λ_2^* mod Λ_1^*.

Fix a fundamental parallelepiped D_1 of Λ_1^*. It is clear that only a fraction $\rho(h) \le Ch^d$ of $D_1 = \mathbb{R}^d / \Lambda_1^*$ has distance from 0 that is less than h (this distance is measured on the *torus* D_1). As Λ_2^* is uniformly distributed mod Λ_1^* the subset of points of Λ_2^* which are killed by some point of Λ_1^* has density (h) . Hence the density of those points of Λ_2^* that are killed by *any* other lattice is at most $(n-1)\rho(h) \le Ch^d n$. We deduce that the density of Z' is at least $(1 - Cnh^d)n$. We now choose $h = cn^{-\frac{1}{d}}$, for a sufficiently small constant c, to ensure that the density of Z' is at least Cn. Applying Lemma (2.3.20) with $g = \hat{f}$ and $E = Z'$ we get

$$\alpha \ge CAh^{d-1}n \ge Cn^{\frac{1}{d}}.$$

We first let q be a nonnegative C_0^∞ function supported in the interval $\left[\frac{1}{2}, 2\right]$ and define a kernel J_N analogously to K_N replacing p by q:

$$J_N(x) = \sum_{v\in\mathbb{Z}^2} \frac{1}{|v|} \widehat{\sigma_{|v|}}(r) q\left(\frac{|v|}{N}\right) \tag{50}$$

Lemma (2.3.21)[83]: With notation as above there is a Schwarz function ψ, such that $\hat{\psi}$ vanishes in a neighborhood of the origin, and making the following true. Let $r = |x|$. If (say) $r \ge \frac{1}{2}$ and $N \ge \frac{1}{2}$ then

$$J_N(x) = Nr^{-\frac{1}{2}} \sum_{v\in\mathbb{Z}^2, v\neq 0} |v|^{-\frac{1}{2}} \psi(N(|v|-r)) + \mathcal{O}\left(N^{-(1-\varepsilon)} r^{-1} + N^{\frac{1}{2}} r^{-\frac{3}{2}}\right) \tag{51}$$

for any $\varepsilon > 0$.

Proof : First let $: \mathbb{R} \to \mathbb{R}$ be any C_0^∞ function supported in the interval $\left[\frac{1}{2}, 2\right]$. Define

$$I(T, \mu) = \int_0^{2\pi} \int_0^\infty \varphi(r) e^{-2\pi i T r(\mu - \cos\theta)} dr d\theta$$

We will show that there is a Schwarz function χ such that $\hat{\chi}$ vanishes in a neighborhood of 0 and such that, for $\mu > 0$ and $T \ge \frac{1}{2}$,

$$I(T, \mu) = T^{-\frac{1}{2}} \chi(T(\mu - 1)) + \mathcal{O}\left(T^{-\frac{3}{2}} (1 + T|\mu - 1|)^{-100}\right) \tag{52}$$

To prove (52), note first of all that $\hat{\varphi}$ is an entire function and satisfies

$$|\hat{\varphi}(x+iy)| \lesssim (1+|x|)^{-200} e^{\pi y}$$

when $y < 0$. Making a change of variable and using contour integration,

$$\begin{aligned} I(T,\mu) &= 2\int_0^{\pi} \hat{\varphi}\left(T(\mu - \cos\theta)\right) d\theta \\ &= 2\int_{-1}^{1} \hat{\varphi}\left(T(\mu - s)\right) \frac{ds}{\sqrt{1-s^2}} \\ &= I + II \end{aligned}$$

where

$$I = 2i\int_{t=0}^{\infty} \hat{\varphi}\left(T(\mu+1-it)\right) \frac{dt}{\sqrt{1-(-1+it)^2}}$$

$$II = -2i\int_{t=0}^{\infty} \hat{\varphi}\left(T(\mu-1-it)\right) \frac{dt}{\sqrt{1-(1+it)^2}}$$

Using that $\mu > 0$, we have

$$\begin{aligned} |I| &\leq 2\int_{t=0}^{\infty} |\hat{\varphi}\left(T(\mu+1-it)\right)| \frac{dt}{\sqrt{t}} \\ &\leq \left(1+T(1+\mu)\right)^{-200} \int_{t=0}^{\infty} e^{-\pi T t} \frac{dt}{\sqrt{t}} \\ &\lesssim T^{-\frac{1}{2}}\left(1+T(1+\mu)\right)^{-200} \lesssim T^{-\frac{3}{2}}(1+T|\mu-1|)^{-100} \end{aligned}$$

On the other hand,

$$II = \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty} \hat{\varphi}\left(T(\mu-1-it)\right) \frac{dt}{\sqrt{2t+it^2}}$$

$$= \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty} \hat{\varphi}\left(T(\mu-1-it)\right) \frac{dt}{\sqrt{2t}} + \mathcal{O}\left(\int_{t=0}^{\infty} |\hat{\varphi}\left(T(\mu-1-it)\right)| \sqrt{t}\,dt\right) \qquad (53)$$

since $\left|\frac{1}{\sqrt{2t+it^2}} - \frac{1}{\sqrt{2t}}\right| \lesssim \sqrt{t}$. The second term in (53) satisfies

$$\begin{aligned} \int_{t=0}^{\infty} |\hat{\varphi}\left(T(\mu-1-it)\right)| \sqrt{t}\,dt &\lesssim (1+T|\mu-1|)^{-100} \int_{t=0}^{\infty} e^{-\pi T t}\, t^{\frac{1}{2}} dt \\ &\approx T^{-\frac{3}{2}}(1+T|\mu-1|)^{-100} \end{aligned}$$

The first term in (53) is by change of variable $t \to Tt$ equal to $T^{-\frac{1}{2}}\chi\big(T(\mu-1)\big)$ where

$$\chi(x) = \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty}\hat{\varphi}\,(x-it)\frac{dt}{\sqrt{2t}}$$

χ is a Schwarz function, and the support of its inverse Γourier transform is contained in the support of φ- in fact $\check{\chi}(y)$ is a constant multiple of $\frac{\varphi(y)}{\sqrt{y}}$. This proves (52).

To prove Lemma (2.3.21) we use the first term in the asymptotic expansion of $\hat{\sigma}_1$: let $r = |x|$. Then

$$\hat{\sigma}_1(x) = 2\sqrt{2\pi}r^{-\frac{1}{2}}\cos\left(2\pi r-\frac{\pi}{4}\right)+\mathcal{O}\left(r^{-\frac{3}{2}}\right)$$

See e.g. [88], Theorem 7.7.14 or [96], Lemma IV. 3.11 and the preceding discussion relating Bessel functions to $\widehat{\sigma_1}$. It follows that

$$|v|^{-1}\widehat{\sigma_{|v|}}(r) = 2\sqrt{2\pi}(r|v|)^{-\frac{1}{2}}\cos\left(2\pi r|v|-\frac{\pi}{4}\right)+\mathcal{O}\left((r|v|)^{-\frac{3}{2}}\right) \tag{54}$$

Substituting (54) into the definition of J_N we find that

$$\begin{aligned}&\left(2\sqrt{2\pi}\right)^{-1}J_N(x)\\ &\qquad= \sum_{v\in\mathbb{Z}^2}(r|v|)^{-\frac{1}{2}}\cos\left(2\pi r|v|-\frac{\pi}{4}\right)q\left(\frac{|v|}{N}\right)+\mathcal{O}\left(\sum_{v\in\mathbb{Z}^2\setminus\{0\}}q\left(\frac{|v|}{N}\right)(r|v|)^{-\frac{3}{2}}\right)\end{aligned}$$

The second term here is $\lesssim N^{\frac{1}{2}}r^{-\frac{3}{2}}$ since there are $\mathcal{O}(N^2)$ lattice points v with $\frac{N}{2}\le|v|\le 2N$. We rewrite the first term using the Poisson summation formula, obtaining

$$\begin{aligned}\left(2\sqrt{2\pi}\right)^{-1}J_N(x) &= r^{-\frac{1}{2}}\sum_{v\in\mathbb{Z}^2}\mathrm{re}\left(e^{i\frac{\pi}{4}}\int_{\mathbb{R}^2}e^{2\pi i v\cdot y}\,|y|^{-\frac{1}{2}}e^{-2\pi i r|y|}q\left(\frac{|y|}{N}\right)dy\right)+\mathcal{O}\left(N^{\frac{1}{2}}r^{-\frac{3}{2}}\right)\\ &= N^{\frac{3}{2}}r^{-\frac{1}{2}}\sum_{v\in\mathbb{Z}^2}\mathrm{re}\left(e^{i\frac{\pi}{4}}\int_{\mathbb{R}^2}e^{-2\pi i N(r|y|-v\cdot y)}\,|y|^{-\frac{1}{2}}q(y)dy\right)+\mathcal{O}\left(N^{\frac{1}{2}}r^{-\frac{3}{2}}\right)\\ &= N^{\frac{3}{2}}r^{-\frac{1}{2}}\sum_{v\neq 0}\mathrm{re}\left(e^{i\frac{\pi}{4}}\int_{-\pi}^{\pi}\int_{0}^{\infty}\varphi\,(t)e^{-2\pi i N|v|t\left(\frac{r}{|v|}-\cos\theta\right)}dtd\theta\right)+\mathcal{O}\left(N^{\frac{1}{2}}r^{-\frac{3}{2}}\right)\end{aligned}$$

where $(t) = t^{\frac{1}{2}}q(t)$. Here the second line followed by change of variables $y \to Ny$, and on the last line we introduced polar coordinates with $\theta = \angle v0y$, and used that the contribution from $v = 0$ is equal to $\mathrm{re}\left(e^{i\frac{\pi}{4}}N^{\frac{3}{2}}r^{-\frac{1}{2}}\hat{\varphi}(Nr)\right)$ and therefore $\mathcal{O}((Nr)^{-100})$.

Now we apply (52) to the terms in the sum, with $= N|v|$, $\mu = \frac{r}{|v|}$. Letting $\psi(t) = \text{re}\left(e^{i\frac{\pi}{4}}\chi(t)\right)$ we conclude that

$$
\begin{aligned}
&(2\sqrt{2\pi})^{-1} J_N(x) \\
&\qquad = N^{\frac{3}{2}} r^{-\frac{1}{2}} \sum_{v\neq 0} (N|v|)^{-\frac{1}{2}} \psi\big(N(|v|-r)\big) \\
&\qquad + \mathcal{O}\left(N^{\frac{3}{2}} r^{-\frac{1}{2}} \sum_{v\neq 0} (N|v|)^{-\frac{3}{2}} (1 + N||v|-r|)^{-100}\right) + \mathcal{O}\left(N^{\frac{1}{2}} r^{-\frac{3}{2}}\right)
\end{aligned}
$$

The second term is $\lesssim r^{-2} N^{\varepsilon} \max\left(\frac{r}{N,\mathrm{d}}, 1\right)$, since the contribution to the sum from terms with $|v| \leq \frac{r}{2}$ is clearly very small and the contribution from $|v| \geq \frac{r}{2}$ can be estimated by Lemma (2.3.12). (51) follows from this on replacing ψ by $2\sqrt{2\pi}\psi$.

Lemma (2.3.22)[83]: Assume the bound (27). Then

$$|K_N(x)| \lesssim N|x|^{-(1-\beta)} \tag{55}$$

if $|x| \geq N \geq 1$.

Proof. We first prove the estimate (55) with K_N replaced by J_N. We define $(t) = t^{-\frac{1}{2}}\psi\big(N(t-r)\big)$, with ψ as in Lemma (2.3.21). Since ψ is in the Schwarz space it is easily seen using the product rule that for any fixed $\beta > 0$,

$$\int_{t=1}^{\infty} t^{\beta} |f'(t)| dt \lesssim r^{\beta - \frac{1}{2}} \tag{56}$$

uniformly in $N \geq \frac{1}{2}$ and $r \geq \frac{1}{2}$. Now consider the quantity $\left(r \geq \frac{1}{2}, N \geq \frac{1}{2}\right)$

$$
\begin{aligned}
\sum_{v\in\mathbb{Z}^2, v\neq 0} |v|^{-\frac{1}{2}} \psi\big(N(|v|-r)\big) &= \int_{t=0}^{\infty} f(t) dn(t) \\
&= \int_{t=0}^{\infty} 2\pi t f(t) dt + \int_{t=0}^{\infty} f(t) d(n(t) - \pi t^2) \\
&= \int_{t=0}^{\infty} 2\pi t f(t) dt + \int_{t=0}^{\infty} (n(t) - \pi t^2) f'(t) dt \qquad (57)
\end{aligned}
$$

The first term in (57) is easily seen to be very small:

$$\left|\int_{t=0}^{\infty} 2\pi t f(t) dt\right| = 2\pi \left|\int_{t=-r}^{\infty} (t+r)^{\frac{1}{2}} \psi(Nt)\, dt\right|$$

$$= 2\pi \left| \int_{-\infty}^{\infty} (t + r)^{\frac{1}{2}} \psi(Nt) dt \right| + \mathcal{O}((rN)^{-100})$$

$$\lesssim r^{-\frac{1}{2}} \int_{t=-\infty}^{\infty} |t| |\psi(Nt)| dt + (rN)^{-100}$$

$$\approx r^{-\frac{1}{2}} N^{-2}$$

Here the second line followed since ψ is in the Schwarz space and the third line followed since $(r+t)^{\frac{1}{2}} = r^{\frac{1}{2}} + \mathcal{O}\left(r^{-\frac{1}{2}}|t|\right)$ and $\psi(0) = 0$. The second term in (57) is $\lesssim \int_{t=1}^{\infty} t^{\beta} |f'(t)| dt + \int_{t=0}^{1} t^2 |f'(t)| dt \lesssim r^{\beta - \frac{1}{2}}$ by (56) and an obvious estimate for the contribution from $t < 1$. Now we use (51). Let $r = |x|$. We've assumed that $r \geq N$, so the error term in (51) is $\lesssim r^{-1}$ Hence

$$|J_N(x)| \lesssim N r^{-\frac{1}{2}} \left| \sum_{v \in \mathbb{Z}^2, v \neq 0} |v|^{-\frac{1}{2}} \psi\big(N(|v| - r)\big) \right| + r^{-1}$$

$$\lesssim N r^{-\frac{1}{2}} \cdot r^{\beta - \frac{1}{2}} + N r^{-\frac{1}{2}} \cdot r^{-\frac{1}{2}} N^{-2} + r^{-1}$$

$$\approx N r^{-(1-\beta)}$$

When $t > 0$ we can express p in the form $p(t) = \sum_{j \geq 0} q\left(2^j t\right)$ where q is supported in $\left[\frac{1}{2}, 2\right]$. Observe that if $\frac{N}{2^j} < \frac{1}{2}$ then the sum defining $J_{\frac{N}{2^j}}$ is empty. Hence $|K_N(x)| \leq \sum_j |J_{\frac{N}{2^j}}(x)| \lesssim \sum_j \frac{N}{2^j} |x|^{-(1-\beta)} \lesssim N|x|^{-(1-\beta)}$ and the proof is complete.

Corollary (2.3.23)[240]: Assume $\epsilon \geq 0$. Let $q: \mathbb{R} \to \mathbb{R}$ be a C_0^{∞} function supported in $\left[\frac{1}{2}, 2\right]$, and let $0 \leq \epsilon < 1$. Define $K_N: \mathbb{R}^{2+\epsilon} \to \mathbb{C}$

$$K_N(x) = \sum_n \frac{1}{\sqrt{n + 1 - \epsilon}} q\left(\frac{\sqrt{n + 1 - \epsilon}}{N}\right) \widehat{\sigma_{\sqrt{n+1-\epsilon}}}(x)$$

Then for large N there is an estimate

$$|K_N(x)| \lesssim \begin{cases} (N|x|)^{-100} & \text{if } 1 \leq |x| \leq \dfrac{N}{2} \\ \left(\dfrac{N}{|x|}\right)^{\frac{\epsilon}{2}} & \text{if } |x| \geq \dfrac{N}{2} \end{cases}$$

Proof: This will follow from the asymptotics for the Fourier transform of surface measure and a simple form of the vander Corput method for estimating exponential sums. We remark that if $|x| \geq N^{1+\epsilon}$ with $\epsilon > 0$ then the bound can be improved by using exponent pairs, but Corollary (2.3.23) as stated is enough for the proof of Theorem (2.3.5).

It is well known (e.g. [95] p. 50) that $\widehat{\sigma_1}(x) = \mathrm{re}(B(|x|))$ where $B(1+\epsilon) = a(1+\epsilon)e^{2\pi i(1+\epsilon)}$, with $a(1+\epsilon)$ being a complex valued function satisfying estimates

$$\left|\frac{(2+\epsilon)^k a}{(2+\epsilon)(1+\epsilon)^k}\right| \lesssim (1+\epsilon)^{-\frac{1+\epsilon}{2}-k} \tag{58}$$

Hence also $\widehat{\sigma_t}(x) = \mathrm{re}(t^{1+\epsilon}B(t|x|))$. Define $t_+ = \max(t, 0)$,and let $1+\epsilon = |x|$.In the calculation below, we use that $q(t) = 0$ when $t < \frac{1}{2}$; this implies that various integrals may be taken interchangeably over $\mathbb{R}$ and over $(0, \infty)$. We have

$$\sum_{n\geq 0}\frac{1}{\sqrt{n+1-\epsilon}}q\left(\frac{\sqrt{n+1-\epsilon}}{N}\right)\left(\sqrt{n+1-\epsilon}\right)^{1+\epsilon}B\left((1+\epsilon)\sqrt{n+1-\epsilon}\right)$$

$$= \sum_{n\in\mathbb{Z}}((n+1-\epsilon)_+)^{\frac{1+\epsilon}{2}}q\left(\frac{\sqrt{(n+1-\epsilon)_+}}{N}\right)a\left((1+\epsilon)\sqrt{(n+1-\epsilon)_+}\right)e^{2\pi i(1+\epsilon)\sqrt{(n+1-\epsilon)_+}}$$

$$= \sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}((1+x)_+)^{\frac{\epsilon}{2}}q\left(\frac{\sqrt{(1+x)_+}}{N}\right)a\left((1+\epsilon)\sqrt{(1+x)_+}\right)e^{2\pi i(1+\epsilon)\sqrt{(1+x)_+}}e^{-2\pi i v(x+\epsilon)}d(x+\epsilon)$$

$$= \sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}(N(x+2\epsilon))^{\epsilon}q(x+2\epsilon)a((1+\epsilon)N(x+2\epsilon))e^{2\pi i(1+\epsilon)N(x+2\epsilon)}e^{-2\pi \mathrm{i}v(N^2(x+2\epsilon)^2-(1-\epsilon))}d(N^2(x+2\epsilon)^2-(1-\epsilon))$$

$$= (1+\epsilon)^{-\frac{1+\epsilon}{2}}N^{\frac{3+\epsilon}{2}}\sum_{v\in\mathbb{Z}}\int_{\mathbb{R}}\varphi(x+2\epsilon)e^{2\pi i(1+\epsilon)N(x+2\epsilon)}e^{-2\pi i v\left(N^2(x+2\epsilon)^2-(1-\epsilon)\right)}d(x+2\epsilon) \tag{59}$$

where $\varphi(x+2\epsilon) = 2(x+2\epsilon)^{1+\epsilon}((1+\epsilon)N)^{\frac{1+\epsilon}{2}}a((1+\epsilon)N(x+2\epsilon))q(x+2\epsilon)$. We used the Poisson summation formula and then the change of variables $x+2\epsilon = \frac{\sqrt{1+x}}{N}$. We note that the estimate (58) implies that the functions $\varphi = \varphi_{N,1+\epsilon}$ belong to a compact subset of C_0^∞; this means that the estimates below are uniform in $(1+\epsilon)$ and N.

We rewrite the sum (59) isolating the $v = 0$ term and making some algebraic manipulations:

$$(59) = (1+\epsilon)^{-\frac{1+\epsilon}{2}} N \frac{3+\epsilon}{2} \left(\int_{\mathbb{R}} \varphi(x+2\epsilon) e^{2\pi i (1+\epsilon) N (x+2\epsilon)} d(x+2\epsilon) + \sum_{v \in \mathbb{Z} \setminus \{0\}} e^{2\pi i \left(v(1-\epsilon) + \frac{(1+\epsilon)^2}{4v} \right)} \int_{\mathbb{R}} \varphi(x+2\epsilon) e^{-2\pi i v N^2 \left((x+2\epsilon) - \frac{1+\epsilon}{2Nv} \right)^2} d(x+2\epsilon) \right) \quad (60)$$

The first term in (60) is equal to $1+\epsilon - \frac{1+\epsilon}{2} N \frac{3+\epsilon}{2} \hat{\varphi}(-N(1+\epsilon))$, hence $\lesssim 1+\epsilon - \frac{1+\epsilon}{2} N \frac{3+\epsilon}{2} (N(1+\epsilon))^{-k}$ for any k. In particular, it is $\lesssim (N(1+\epsilon))^{-100}$ if $\epsilon \geq 0$. The terms in the sum in (60) may be evaluated via the asymptotics for Gaussian Fourier transforms ([88], Lemma 7.7.3); the v th term is equal to

$$e^{2\pi i \left(v(1-\epsilon) + \frac{(1+\epsilon)^2}{4v} \right)} \sum_{k=0}^{m-1} c_k (vN^2)^{-k-\frac{1}{2}} \varphi_k \left(\frac{1+\epsilon}{2Nv} \right) + \mathcal{O}\left((vN^2)^{-m-\frac{1}{2}} \right) \quad (61)$$

for any m; here c_k are fixed constants and the φ_k are certain derivatives of φ. All the terms in the sum over k vanish if $v \notin \left[\frac{1+\epsilon}{4N}, \frac{1+\epsilon}{N} \right]$ so that

$$(61) \lesssim \begin{cases} (vN^2)^{-\frac{1}{2}} & \text{if } v \in \left[\frac{1+\epsilon}{4N}, \frac{1+\epsilon}{N} \right] \\ (vN^2)^{-m-\frac{1}{2}} & \text{if } v \notin \left[\frac{1+\epsilon}{4N}, \frac{1+\epsilon}{N} \right] \end{cases}$$

Accordingly the sum in (60) is

$$\lesssim \operatorname{card}\left(\mathbb{Z} \cap \left[\frac{1+\epsilon}{4N}, \frac{1+\epsilon}{N} \right] \right) ((1+\epsilon)N)^{-\frac{1}{2}} + ((1+\epsilon)N)^{-m-\frac{1}{2}}$$

Taking m sufficiently large we obtain

$$\lesssim (1+\epsilon)^{-\frac{1+\epsilon}{2}} N^{\frac{3+\epsilon}{2}} \operatorname{card}\left(\mathbb{Z} \cap \left[\frac{1+\epsilon}{4N}, \frac{1+\epsilon}{N} \right] \right) ((1+\epsilon)N)^{-\frac{1}{2}} + ((1+\epsilon)N)^{-100} \sim$$
$$< \begin{cases} \left(\frac{N}{1+\epsilon} \right)^{\frac{\epsilon}{2}} & \text{if } 1+\epsilon \geq \frac{N}{2} \\ ((1+\epsilon)N)^{-100} & \text{if } 1 \leq 1+\epsilon \leq \frac{N}{2} \end{cases}$$

The lemma follows since K_N is the real part of the quantity (60).

We need one more lemma, an easy consequence of the Poisson summation formula (see [83]).

Corollary (2.3.24)[240]: Let $\epsilon \geq 0$ be an integer, let q be a fixed C_0^∞ function supported in $\left[\frac{1}{2}, 2\right]$, let $0 \leq \epsilon < 1$ and let $h = h(t)$ be a function on the line satisfying the following estimate:

$$\left|\frac{d^j h}{dt^j}\right| \leq R$$

when $0 \leq j \leq 2 + \epsilon$ and $\frac{N}{100} \leq t \leq 100N$. Then for large N

$$\left|\sum_n \frac{1}{\sqrt{n+1-\epsilon}} q\left(\frac{\sqrt{n+1-\epsilon}}{N}\right) h\left(\sqrt{n+1-\epsilon}\right) - 2\int \ q\left(\frac{t}{N}\right) h(t)dt\right| \lesssim RN^{-(1+\epsilon)} \quad (62)$$

where the implicit constant depends on q only.

Proof Set $g(x) = \frac{h\left(\sqrt{x+1-\epsilon}\right)}{\sqrt{x+1-\epsilon}}$ and $(x) = q\left(\frac{\sqrt{x+1-\epsilon}}{N}\right)$. Then a is supported in $x \approx N^2$ and derivatives of a satisfy

$$\left|\frac{d^j a}{dx^j}\right| \lesssim N^{-2j} \quad (63)$$

since the functions $q\left(\sqrt{x + (1-\epsilon)N - 2}\right)$ belong to a compact subset of C_0^∞ and $a(x)$ is obtained from $q\left(\sqrt{x + (1-\epsilon)N^{-2}}\right)$ by dilating by N^2. When $x \approx N^2$, derivatives of g satisfy

$$\left|\frac{d^j g}{dx^j}\right| \lesssim RN^{-(1+j)} \quad (64)$$

when $j \leq 2 + \epsilon$. Namely, it is easy to show by induction on j that the jth derivative of g is a sum of finitely many terms each of which has the form $\frac{h^{(i)}\left(\sqrt{x+1-\epsilon}\right)}{\left(\sqrt{x+1-\epsilon}\right)^{\ell}}$ where $h^{(i)} = i$th derivative of h, with $i \leq j$ and $\ell \geq j + 1$. Estimate (64) is then obvious.

The left side of (62) is (make the change of variables $t = \sqrt{x + 1 - \epsilon}$) equal to

$$\left|\sum_n a(n)g(n) - \int \ a(x)g(x)dx\right|$$

By Poisson summation this is

$$\left|\sum_{v\neq 0} \widehat{ag}(v)\right| \quad (65)$$

and if we integrate by parts k times and use (63) and (64), we bound the uth term in the sum (65) by

$$|v|^{-k}\int |\frac{d^k(ag)}{dx^k}|dx \lesssim |v|^{-k}\int_0^{2N^2} R\,N^{-(1+k)}dx \lesssim |v|^{-k}RN^{-(k-1)}$$

Hence (65) $RN^{-(k-1)}$ and the proof is complete.

Corollary (2.3.25)[240]: Suppose that $\epsilon \geq 0$ and that $f\colon \mathbb{R}^{3+\epsilon} \to \mathbb{R}$ is an L^1 function which tiles with every rotation of $\mathbb{Z}^{3+\epsilon}$, i.e.

$$\sum_{v\in\mathbb{Z}^{3+\epsilon}} f\,(x-\rho v)$$

is constant a. e. for each $\rho \in SO(3+\epsilon)$. Then f agrees a. e. with a continuous function.

Proof: (see [83]). We may clearly assume that $a = 1$ and $\epsilon > 0$.

We let $q \in C_0^\infty(\mathbb{R})$ be supported in $\left[\frac{1}{2}, 2\right]$ and such that the functions $\{q_{2^j}\}_{-\infty}^{\infty}$ form a partition of unity on $(0,\infty)$; here we have defined $q_{2^j}(x) = q\left(\frac{x}{2j}\right)$. We define K_N as in Corollary (2.3.23) using this q.

Fix a ball D with radius 1; we will show that f is continuous on D. Let $\widetilde{D}$ be the concentric ball with radius 2, and let $f_i = \chi_{\widetilde{D}}f$ and $f_o = \chi_{\mathbb{R}^{2+\epsilon}\setminus\widetilde{D}}f$ where χ_E is the indicator function of the set E. By assumption, $\sigma_{\widehat{\sqrt{n+1-\epsilon}}} * f$ vanishes identically for any positive integer n and therefore $K_N * f$ vanishes identically for any N.

Corollary (2.3.26)[240]: Suppose that E is a set in $\mathbb{R}^{3+\epsilon}$ with $|E| = 1$ and $|E\cap D| > 0$ for a certain ball D with radius 1. Let $\widetilde{D}$ be the concentric ball with radius $C_{3+\epsilon}$. Then

$$|\left\{x\in\widetilde{D}: \frac{1}{4} \leq \varphi_\varepsilon * \chi_E(x) \leq \frac{3}{4}\right\}| \gtrsim \varepsilon$$

provided that ε is sufficiently small; the implicit constants may depend on E.

Proof We will use the following well-known fact:

$$\|\nabla(\varphi_\varepsilon * \chi_E)\|_\infty \lesssim \varepsilon^{-1} \quad (66)$$

To prove (66), let $\psi = \nabla\varphi$, let $C = \|\psi\|_1$ and define $\psi_\varepsilon(x) = \varepsilon^{-(3+\epsilon)}\psi(\varepsilon^{-1}x)$. Differentiation under the integral sign leads to $\nabla(\varphi_\varepsilon * \chi_E) = \varepsilon^{-1}\psi_\varepsilon * \chi_E$. On the other hand, for any $x \in \mathbb{R}^{3+\epsilon}$, we have $|\psi_\varepsilon * \chi_E(x)| \leq \|\psi_\varepsilon\|_1 \|\chi_E\|_\infty = \|\psi\|_1$, which proves that $\|\nabla(\varphi_\varepsilon * \chi_E)\|_\infty \leq C\varepsilon^{-1}$, as claimed.

It follows by the mean value theorem that if $\varphi_\varepsilon * \chi_E(x_0) = \frac{1}{2}$, then $\varphi_\varepsilon * \chi_E(x) \in \left[\frac{1}{4}, \frac{3}{4}\right]$ for all $x \in D(x_0, C^{-1}\varepsilon)$. We let σ be surface measure on $S^{2+\epsilon}$; here we take it to be normalized so that $\sigma(S^{2+\epsilon}) = 1$. We also let E^c be the complement of the set E.

Choose once and for all a point of density of $E \cap D$, which we may assume to be the origin. Let A be the set of all $\omega \in S^{2+\epsilon}$ such that the ray $\{(1+\epsilon)\omega : 1 < 1+\epsilon < C_{3+\epsilon}\}$ contains a point of density of E^c. Since E has measure 1 it is clear that A must have measure $\geq \frac{3}{4}$ provided $C_{3+\epsilon}$ is large enough. If $\omega \in A$ then we let $p_\omega = r_\omega \omega$ be the corresponding point of density of E^c In a similar way we can choose a small sphere centered at $0, x = \{\rho\omega : \omega \in S^{2+\epsilon}\}$, where $\rho < 1$ in such a way that $q_\omega = \rho\omega$ is a point of density of E for all $\omega \in B$ where $B \subset S^{2+\epsilon}$ is a set of measure $> \frac{3}{4}$.

By Egoroff's theorem, we can find subsets $A^* \subset A$ with measure $\geq \frac{2}{3}$ and $B^* \subset B$ with measure $\geq \frac{2}{3}$ and a number ε_0 such that if $\varepsilon < \varepsilon_0$ then

$$\frac{|E \cap D(p_\omega, \varepsilon)|}{|D(p_\omega, \varepsilon)|} < 10^{-6} \text{ for all } \omega \in A^* \tag{67}$$

and

$$\frac{|E^c \cap D(q_\omega, \varepsilon)|}{|D(q_\omega, \varepsilon)|} < 10^{-6} \text{ for all } \omega \in B^* \tag{68}$$

Note $|A^* \cap B^*| \geq \frac{1}{3}$.

Now fix $\varepsilon < \varepsilon_0$, let $\omega \in A^* \cap B^*$ and consider $\varphi_\varepsilon * \chi_E$ as a function on the line segment $\{tw : \rho \leq t \leq r_\omega\}$. Its value at ρ is $\geq 1 - 10^{-6}$ and its value at r_ω is $\leq 10^{-6}$Accordingly, there must be a value of $t_\omega \in (\rho, r_\omega)$ where $\varphi_\varepsilon * \chi_E(t_\omega\omega) = \frac{1}{2}$.Then by the remarks at the beginning of the proof, $\varphi_\varepsilon * \chi_E(t\omega) \in \left(\frac{1}{4}, \frac{3}{4}\right)$ for all $\omega \in A^* \cap B^*$ and all t in the interval centered at t_ω with length $C^{-1}\varepsilon$. Using polar coordinates it now follows that the set $\left\{x : \varphi_\varepsilon * \chi_E(x) \in \left(\frac{1}{4}, \frac{3}{4}\right)\right\}$ has measure $\gtrsim \varepsilon$ where the constant is independent of ε provided ε is small.

Corollary (2.3.27)[240]: If $E \subset \mathbb{R}^{3+\epsilon}$ is a set with finite nonzero measure and if φ_ε is as in Corollary (2.3.26) then

(a) $\|\varphi_\varepsilon * \chi_E - \chi_E\|_2 \geq C_E^{-1}\varepsilon^{\frac{1}{2}}$ for small ε.

(b) $\int_{|\xi|\geq 2+\epsilon}|\widehat{\chi_E}|^2 \geq (C_E(2+\epsilon))^{-1}$ for a certain constant C_E depending on E and all sufficiently large $2+\epsilon$. In particular, $\chi_E \notin W^{\frac{1}{2}}$.

Proof Part (a) is immediate from Corollary (2.3.26), since $\frac{1}{4} \leq \varphi_\varepsilon * \chi_E(x) \leq \frac{3}{4}$ implies $|\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}$. Part (b) follows easily from (a). By (a) we have

$$\int_{\mathbb{R}^n}|\widehat{\chi_E}(\xi)|^2|\hat{\varphi}((2+\epsilon)^{-1}\xi)-1|^2 d\xi \geq (C_E(2+\epsilon))^{-1} \quad (69)$$

uniformly in $2+\epsilon$, and if φ has been chosen to be nonnegative, then $|\hat{\varphi}((2+\epsilon)^{-1}\xi)-1|$ is bounded away from zero when $|\xi| \geq 2+\epsilon$.

From Corollary (2.3.27) we can obtain a form of Corollary (2.3.33) where $\epsilon = 1$:

Corollary (2.3.28)[240]: If $E \subset \mathbb{R}^2$ is Steinhaus then $\int_E |x|^2 dx = \infty$.

Proof As was done in [91], we use the elementary estimate (which is also the only known estimate) for the maximum gap between sums of two squares:

(G): If $0 < \epsilon < \infty$ then for a suitable fixed constant C_1 there is $u \in \mathbb{Z}^2$ such that $|1+\epsilon - |v|| \leq C_1(1+\epsilon)^{-\frac{1}{2}}$.

We also use the following form of the Poincare inequality, which is well-known.

(PI): Let Q be a square in the plane with side $1+\epsilon$ and let γ be a Jordan arc contained in Q, such that the distance between the endpoints of γ is $\geq C_1^{-1}(1+\epsilon)$. Let f be a function which vanishes on γ. Then

$$\int_Q |f|^2 \leq C_2(1+\epsilon)^2 \int_Q |\nabla f|^2$$

where C_2 depends on C_1 only.

Fix a large number N and define $A_N \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2 : N \leq |\xi| \leq 2N\}$. Let C be a large enough constant and cover A_N with nonoverlapping squares Q of side $CN^{-\frac{1}{2}}$. If E is Steinhaus, $f = \widehat{\chi_E}$, then (G) implies that each square will satisfy the hypothesis of (PI). We conclude that

$$\int_Q |\widehat{\chi_E}|^2 \lesssim N^{-1}\int_Q |\nabla\widehat{\chi_E}|^2$$

for each Q and therefore

$$\int_{A_N} |\widehat{\chi_E}|^2 \lesssim N^{-1} \int_{A_N^*} |\nabla \widehat{\chi_E}|^2$$

where A_N^* is the union of the squares and is contained in $\{\xi \in \mathbb{R}^2 : N - 1 \leq |\xi| \leq 2N + 1\}$. Consequently

$$\int_{A_N} |\xi||\widehat{\chi_E}(\xi)|^2 d\xi \lesssim \int_{A_N^*} |\nabla \widehat{\chi_E}|^2$$

If we now sum over dyadic values of N and use that no point belongs to more than two A_N^*s, we obtain

$$\int_{\mathbb{R}^2} |\xi||\widehat{\chi_E}(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^2} |\nabla \widehat{\chi_E}|^2 d\xi + 1$$

Hence by Corollary (2.3.27)(b), $\int_{\mathbb{R}^2} |\nabla \widehat{\chi_E}|^2 = \infty$, i.e. $\int_E \ |x|^2 dx = \infty$.

Corollary (2.3.29)[240]: For any given set $E \subset \mathbb{R}^{3+\epsilon}$ with $|E| < \infty$ there is a sequence $\varepsilon_j = 2^{-k_j} \to 0$ such that $A_{\varepsilon_j}(E) \lesssim C_{\varepsilon_j}(E)$; the constants here (and in (21)) depend only on $3 + \epsilon$ and φ.

Proof: We may assume that $|E| = 1$. If $D = D(x, \rho)$ is the ball with center x and radius ρ then we define

$$(1 + \epsilon)(D) = \min(|E \cap D|, |E^c \cap D|)$$

$$\beta(D) = \sum_{j=0}^{\infty} 2^{-10(3+\epsilon)j} (1 + \epsilon)(2^j D)$$

Here we have used the notation $E^c = \mathbb{R}^{3+\epsilon} \backslash E$ and $(x, \rho) = D(x, (1 + \epsilon)\rho)$.

Let C_0 be a large constant. If D is any ball of radius $C_0^{-1}\varepsilon$ then we claim that the following are valid:

I. $\|\varphi_\varepsilon * \chi_E - \chi_E\|_{L^1(\mathrm{D})} \lesssim \beta(D)$

II. $\left|\left\{x \in D : |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}\right| \geq (1 + \epsilon)(D)$.

In fact, II follows easily from (66). Namely, if C_0 is large then (66) implies via the mean value theorem that the difference between the maximum and minimum values of $\varphi_\varepsilon * \chi_E$ on the ball D is less than $\frac{1}{2}$. It follows that one of the following must hold

(i) $\varphi_\varepsilon * \chi(x) \leq \frac{3}{4}$ for all $x \in D$, or (ii)$\varphi_\varepsilon * \chi(x) \geq \frac{1}{4}$ for all $x \in D$.

In case (i) we have $|\left\{x \in D: |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}| \geq |E \cap D| \geq (1+\epsilon)(D)$ and in case (ii) we have $|\left\{x \in D: |\varphi_\varepsilon * \chi_E(x) - \chi_E(x)| \geq \frac{1}{4}\right\}| E^c \cap D| \geq (1+\epsilon)(D)$,i.e. II holds in either case.

To prove I, we express φ as a synthesis of C_0^∞ functions, say

$$\varphi = \sum_{j=0}^{\infty} a_j \, \varphi^j$$

where $\text{supp}\varphi^g \subset D(0, (2C_0)^{-1}2^j), \widehat{\varphi_j}(0) = 1$, $\|\varphi_j\|_1 \leq C$ and $a_j \leq C2^{-10(3+\epsilon)j}$. Let $\varphi_\varepsilon^j(x) = \varepsilon^{-(3+\epsilon)}\varphi^j(\varepsilon^{-1}x)$. It follows by Minkowski's inequality and the support properties that

$$\|\varphi_\varepsilon^j * \chi_E - \chi_E\|_{L^1(D)} \lesssim |E \cap (2^j D)|$$

and therefore also

$$\|\varphi_\varepsilon^j * \chi_E - \chi_E\|_{L^1(D)} \lesssim (1+\epsilon)(2^j D)$$

since the left side is unchanged when E is replaced by E^c I now follows by summing over j.

Let $(\varepsilon) = \int_{\mathbb{R}^{3+\epsilon}} (1+\epsilon)\left(D(x, C_0^{-1}\varepsilon)\right)dx$, $J(\varepsilon) = \int_{\mathbb{R}^{3+\epsilon}} \beta\left(D(x, C_0^{-1}\varepsilon)\right)dx$. Integrating I and II over $\mathbb{R}^{3+\epsilon}$ we get

$$\varepsilon^{-(3+\epsilon)}I(\varepsilon) \lesssim C_\varepsilon(E) \lesssim A_\varepsilon(E) \lesssim \varepsilon^{-(3+\epsilon)}J(\varepsilon) \qquad (70)$$

Let k be a large positive integer and consider the sums

$$\mathcal{I}_k = \sum_{\ell=0}^{\infty} 2^{-5(3+\epsilon)\ell}\, I(2^{\ell-k})$$

$$\mathcal{J}_k = \sum_{\ell=0}^{\infty} 2^{-5(3+\epsilon)\ell}\, J(2^{\ell-k})$$

For any k, we have

$$\mathcal{J}_k = \sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty} 2^{-(3+\epsilon)(5\ell+10j)}\, I(2^{\ell+j-k}) \lesssim \sum_{m=0}^{\infty} 2^{-5(3+\epsilon)m}\, I(2^{m-k}) = \mathcal{I}_k$$

On the next to last line, we set $m = j + \ell$ and used that $\sum_{j+\ell=m} 2^{-(3+\epsilon)(5\ell+10j)} < 2^{-5(3+\epsilon)m}$

Now observe that $J(\varepsilon) \gtrsim \varepsilon^{4+\epsilon}$ for small ε, e.g. by (70) and Corollary (2.3.27)(a), and that $I(\varepsilon) \lesssim \varepsilon^{3+\epsilon}$ for any ε (even when $\varepsilon > 1$), e.g. by (70). It follows that $\mathcal{J}_k \gtrsim 2^{-(4+\epsilon)k}$ and that $\sum_{\ell>\frac{k}{2}} 2^{-5(3+\epsilon)\ell} I_{k-\ell}$ is small compared with $2^{-(4+\epsilon)k}$ Accordingly

$$\sum_{\ell\leq\frac{k}{2}} < 2^{-5(3+\epsilon)\ell} J(2^{\ell-k}) \lesssim \sum_{\ell\leq\frac{k}{2}} 2^{-5(3+\epsilon)\ell} I(2^{\ell-k})$$

which implies there is a value $2^{\ell-k} \leq \sqrt{2-k}$ with $J(2^{\ell-k}) \lesssim I(2^{\ell-k})$. This and (70) prove the corollary.

We assume that the Schwarz function φ satisfies the following conditions:

$$\text{supp}\,\hat{\varphi} \subset D(0{,}1), \hat{\varphi}(\xi) = 1 \; if \; \xi \in D\left(0, \frac{1}{2}\right) \tag{71}$$

We set $\varphi(x) = \varphi(x) - 2^{3+\epsilon}\varphi(2x)$; thus ψ is a Schwarz function with supp $\hat{\psi} \subset D(0{,}2) \backslash D\left(0, \frac{1}{2}\right)$. We define $\psi_\varepsilon(x) = \varepsilon^{-(3+\epsilon)}\psi(\varepsilon^{-1}x)$, so that $\sum_{j=0}^{\infty} \psi_{2-j\varepsilon} * f = \varphi_\varepsilon * f - f$ for any f and ε, as may be seen bytaking Fourier transforms. Property (71) implies that no point belongs to the support of $\widehat{\psi_J}$ for more that three values of j, so it follows by the Plancherel theorem that

$$\sum_{j=0}^{\infty} \| \psi_{2-j\varepsilon} * f \|^2 2 \gtrsim \|f - \varphi_\varepsilon * f\|_2^2 \tag{72}$$

Furthermore,

$$\|\psi_{2-j\varepsilon} * f\|_1 \lesssim \|f - \varphi_\varepsilon * f\|_1 \tag{73}$$

Namely, the support property (71) makes it possible to represent $\psi_{2-j} = g_j * (\delta - \varphi)$ with $\|g_j\|_1 \leq C$ (here δ is the Dirac delta function). Indeed if $j \geq 1$ then $\widehat{\psi_{2-J}}$ and $\hat{\varphi}$ have disjoint support so we can take $g_j = \psi_{2-j}$, and when $j = 0, \widehat{\psi_{2-J}} = \hat{\psi}$ is obtained from $1 - \hat{\varphi}$ by multiplication by the C_0^∞ function m defined via $m(\xi) = -1$ when $|\xi| \leq 1$ and $\frac{\tilde{\psi}}{1-\varphi}$ when $|\xi| \geq 1$. It follows using dilations that $\psi_{2^{-j}\varepsilon} = g_{j,\varepsilon} * (\delta - \varphi_\varepsilon)$ where $|\xi| \leq 1$ and when $||g_j, \varepsilon\|_1 = \|g \leq C$.

Accordingly $\|\psi_{2^{-j}\varepsilon} * f\|_1 = \|g_{j,\varepsilon} * (\delta - \varphi_\varepsilon) * f\|_1 = \|g_{j,\varepsilon} * (f - \varphi_\varepsilon * f)\|_1 \leq C\|f - \varphi_\varepsilon * f\|_1$ which is (73).

Corollary (2.3.30)[240]: Assume that φ satisfies (71) and define ψ as above. If E is a set of finite measure then there is a sequence $\varepsilon_j \to 0$ such that, for each j, (i) $\|\psi_{\varepsilon_j} * \chi_E\|_1 \lesssim \left(\log \frac{1}{\varepsilon_j}\right)^2 \|\psi_{\varepsilon_j} * \chi_E\|_2^2$ and (ii) $\|\psi_{\varepsilon_j} * \chi_E\|_2^2 \gtrsim \varepsilon_j$.

Proof: Let ε be such that $A_\varepsilon(E) \lesssim B_\varepsilon(E)$. If $\eta_k = 2^{-k}\varepsilon$ then $\|\psi_{\eta_k} * \chi_E\|_1 \lesssim A_\varepsilon(E)$ by (73) and $\sum_{k\le 0} \| \psi_{\eta_k} * \chi_E\|_2^2 \gtrsim B_\varepsilon(E)$ by (72). Hence, for some k we must have

$$\max\left((k+1)^{-2}\|\psi_{\eta_k} * \chi_E\|_1, (k+1)^{-2}B_\varepsilon(E)\right) \lesssim \|\psi_{\eta_k} * \chi_E\|_2^2$$

Also $B_\varepsilon(E) > \varepsilon \sim$ by Corollary (2.3.27)(a), so $(k+1)^{-2}B_\varepsilon(E) \gtrsim \eta_k$, and $(k+1)^{-2} \gtrsim \left(\log \frac{1}{\eta_k}\right)^{-2}$ We conclude that

$$\max\left(\left(\log \frac{1}{\eta_k}\right)^{-2} \|\psi_{\eta_k} * \chi_E\|_1, \eta_k\right) \lesssim \|\psi_{\eta_k} * \chi_E\|_2^2$$

i.e. that there are arbitrarily small numbers ε_j such that (i) and (ii) hold.

Corollary (2.3.31)[240]: If $\epsilon > 0$

$$\sum_{v\in\mathbb{Z}^2} \left(1 + (1+\epsilon)\big|1+\epsilon - |v|\big|\right)^{-100} \le C_\varepsilon (1+\epsilon)^\varepsilon \max(1,1)$$

Proof: Because of the rapid decay of $(1+(1+\epsilon)t)^{-100}$ when $t \ge \frac{1}{1+\epsilon}$, it is easy to show that it suffices to prove the following estimate for all $1+\epsilon$:

$$n\left(\frac{2+2\epsilon+\epsilon^2}{1+\epsilon}\right) - n(1+\epsilon) \lesssim (1+\epsilon)^\varepsilon \max(1,1) \tag{74}$$

where $n(1+\epsilon)$ is as in (75). To prove (74), consider two cases.

(i) $r \le (1+\epsilon)^3$ The number of lattice points on a circle is bounded by any given power of the radius, hence a circle of radius $\rho \in \left(r, r + \frac{1}{1+\epsilon}\right)$ contains $\lesssim r^{\frac{\varepsilon}{3}} \lesssim (1+\epsilon)^\varepsilon$ lattice points. There are $\lesssim \max(\frac{r}{1+\epsilon}, 1)$ values of ρ for which it contains some lattice point and (74) follows.

(ii) $r \ge (1+\epsilon)^3$. In this case we use (l) with the classical exponent $\beta = \frac{2}{3}$. Thus $n\left(r + \frac{1}{1+\epsilon}\right) - n(r) \lesssim \frac{r}{1+\epsilon} + r^{\frac{2}{3}} \approx \frac{r}{1+\epsilon}$.

The proof of Corollary (2.3.33) will be like the proof of Corollary (2.3.25) insofar as it is also based on using an appropriate "fundamental solution" However, we must replace the kernel in Corollary (2.3.23) by an analogous one involving a sum only over circles which contain lattice points. We will use the obvious choice where one counts each circle according to the number of lattice points it contains.

Let p be a nonnegative C^∞ function of one variable supported in $t \leq 1$ and with $p(t) = 1$ when $t \leq \frac{1}{2}$. Define

$$K_{1+\epsilon}(x) = \sum_{v\in\mathbb{Z}^2, v\neq 0} \frac{1}{|v|}\widehat{\sigma_{|v|}}(r)p\left(\frac{|v|}{1+\epsilon}\right)$$

where $r = |x|$.

Corollary (2.3.32)[240]: Let $E \subset \mathbb{R}^2$ be a Steinhaus set and let ψ be a Schwarz function in $\mathbb{R}^2$ with $\hat{\psi}(0) = 0$. Let $f = \chi_E$. Then

$$\psi * f(x) = -\sum_{v\in\mathbb{Z}^2, v\neq 0} T_{|v|}(\psi * f)$$

Proof The Steinhaus property gives after convolving with ψ that

$$\psi * f(x) = -\sum_{v\in\mathbb{Z}^2, v\neq 0} \psi * f(x + e^{i\theta} v)$$

for all θ and x. The lemma follows by integrating with respect to θ.

Corollary (2.3.33)[240]: Assume a bound of the form

$$n(r) = \pi r^2 + \mathcal{O}(r^\beta) \tag{75}$$

where $n(r) = \mathrm{card}\big((\mathbb{Z}^2\backslash\{0\}) \cap D(0, r)\big)$. Then any Steinhaus set $E \subset \mathbb{R}^2$ must satisfy

$$\int_E |x|^{\frac{\beta}{1-\beta}+\epsilon} dx = \infty \tag{76}$$

for all $\epsilon > 0$.

Proof: We let β be such that (75) is true and assume toward a contradiction that E is Steinhaus and $\int_E |x|^{\frac{\beta}{1-\beta}+\epsilon} < \infty$ for some $\epsilon > 0$.

Fix a Schwarz function φ satisfying (71) and set $(x) = \varphi(x) - 4\varphi(2x)$. Thus supp $\hat{\psi} \subset D(0,2)\backslash D\left(0,\frac{1}{2}\right)$. Let $\psi_{2+\epsilon}(x) = (2+\epsilon)^2\psi((2+\epsilon)x)$. Also fix a function $1+\epsilon$ as in Corollary (2.3.36). Applying Corollary (2.3.32) with $\psi_{2+\epsilon}$, we get for any M

$$\psi_{2+\epsilon} * \chi_E = -A_M(\psi_{2+\epsilon} * \chi_E) - B_M(\psi_{2+\epsilon} * \chi_E) \tag{77}$$

where the operators A_M and B_M are defined by

$$A_M = \sum_{v\neq 0} p\left(\frac{|v|}{M}\right) T_{|v|}$$

$$B_M = \sum_{v} \left(1 - p\left(\frac{|v|}{M}\right)\right) T_{|v|}$$

Corollary (2.3.34)[240]: There is a constant $B = B(3+\epsilon)$ making the following true. Suppose that the lattices Λ_i, $i = 1, \ldots, n$, have volume 1 and that

$$\Lambda_i \cap \Lambda_j = \{0\}, \qquad \text{for all } i \neq j \tag{78}$$

Proof: (See [83]). All constants below may depend only on the dimension $3+\epsilon$. We note that $\Lambda_1 \cap \Lambda_2 = \{0\}$ implies that the lattice Λ_1^* is uniformly distributed $\bmod \Lambda_2^*$. This can be proved using Weyl's lemma-see for example [92].

Corollary (2.3.35)[240]: With notation as above there is a Schwarz function ψ, such that $\hat{\psi}$ vanishes in a neighborhood of the origin, and making the following true. Let $1+\epsilon = |x|$. If (say) $\epsilon \geq 0$ then

$$J_{\frac{1}{2}+\epsilon}(x) = \left(\frac{1}{2}+\epsilon\right)^{\frac{1}{2}} \sum_{v\in\mathbb{Z}^2, v\neq 0} |v|^{-\frac{1}{2}} \psi\left(\left(\frac{1}{2}+\epsilon\right)\left(|v| - \left(\frac{1}{2}+\epsilon\right)\right)\right) + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-(1-\varepsilon)-2}\right) \tag{79}$$

for any $\varepsilon > 0$.

Proof: First let $\varphi: \mathbb{R} \to \mathbb{R}$ be any C_0^∞ function supported in the interval $\left[\frac{1}{2}, 2\right]$. Define

$$I(T,\mu) = \int_0^{2\pi}\int_0^{\infty} \varphi\left(\frac{1}{2}+\epsilon\right) e^{-2\pi i T\left(\frac{1}{2}+\epsilon\right)(\mu - \cos\theta)} d\left(\frac{1}{2}+\epsilon\right) d\theta$$

We will show that there is a Schwarz function χ such that $\hat{\chi}$ vanishes in a neighborhood of 0 and such that, for $\mu > 0$ and $T \geq \frac{1}{2}$,

$$I(T,\mu) = T^{-\frac{1}{2}}\chi(T(\mu-1)) + \mathcal{O}\left(T^{-\frac{3}{2}}(1+T|\mu-1|)^{-100}\right) \tag{80}$$

To prove (80), note first of all that $\hat{\varphi}$ is an entire function and satisfies

$$|\hat{\varphi}(x+i(x+\epsilon))| \lesssim (1+|x|)^{-200}e^{\pi(x+\epsilon)}$$

when $x+\epsilon<0$. Making a change of variable and using contour integration,

$$I(T,\mu) = 2\int_0^{\pi}\hat{\varphi}\left(T(\mu-\cos\theta)\right)d\theta = 2\int_{-1}^{1}\hat{\varphi}\left(T(\mu-s)\right)\frac{ds}{\sqrt{1-s^2}} = I+II$$

where

$$I = 2i\int_{t=0}^{\infty}\hat{\varphi}\left(T(\mu+1-it)\right)\frac{dt}{\sqrt{1-(-1+it)^2}}$$

$$II = -2i\int_{t=0}^{\infty}\hat{\varphi}\left(T(\mu-1-it)\right)\frac{dt}{\sqrt{1-(1+it)^2}}$$

Using that $\mu>0$, we have

$$|I| \leq 2\int_{t=0}^{\infty}|\hat{\varphi}\big(T(\mu+1-it)\big)|\frac{dt}{\sqrt{t}} \leq \big(1+T(1+\mu)\big)^{-200}\int_{t=0}^{\infty}e^{-\pi Tt}\frac{dt}{\sqrt{t}}$$
$$\lesssim T^{-\frac{1}{2}}\big(1+T(1+\mu)\big)^{-200} \lesssim T^{-\frac{3}{2}}(1+T|\mu-1|)^{-100}$$

On the other hand,

$$II = \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty}\hat{\varphi}\left(T(\mu-1-it)\right)\frac{dt}{\sqrt{2t+it^2}}$$

$$= \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty}\hat{\varphi}\left(T(\mu-1-it)\right)\frac{dt}{\sqrt{2t}} + \mathcal{O}\left(\int_{t=0}^{\infty}|\hat{\varphi}\big(T(\mu-1-it)\big)|\sqrt{t}dt\right) \quad (81)$$

since $|\frac{1}{\sqrt{2t+it^2}}-\frac{1}{\sqrt{2t}}| \lesssim \sqrt{t}$. The second term in (81) satisfies

$$\int_{t=0}^{\infty}|\hat{\varphi}\big(T(\mu-1-it)\big)|\sqrt{t}dt \lesssim (1+T|\mu-1|)^{-100}\int_{t=0}^{\infty}e^{-\pi Tt}\,t^{\frac{1}{2}}dt$$
$$\approx T^{-\frac{3}{2}}(1+T|\mu-1|)^{-100}$$

The first term in (81) is by change of variable $t\to Tt$ equal to $T^{-\frac{1}{2}}\chi\big(T(\mu-1)\big)$ where

$$\chi(x) = \frac{-2i}{\sqrt{-i}}\int_{t=0}^{\infty}\hat{\varphi}\,(x-it)\frac{dt}{\sqrt{2t}}$$

χ is a Schwarz function, and the support of its inverse Fourier transform is contained in the support of φ- in fact $\check{\chi}(x+\epsilon)$ is a constant multiple of $\frac{\varphi(x+\epsilon)}{\sqrt{x+\epsilon}}$. This proves (80).

To prove Corollary (2.3.35) we use the first term in the asymptotic expansion of $\hat{\sigma}_1$: let $\frac{1}{2}+\epsilon = |x|$. Then

$$\hat{\sigma}_1(x) = 2\sqrt{2\pi}\left(\frac{1}{2}+\epsilon\right)^{-\frac{1}{2}} \cos\left(2\pi\left(\frac{1}{2}+\epsilon\right)-\frac{\pi}{4}\right)+\mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-\frac{3}{2}}\right)$$

See e.g. [88], Theorem 7.7.14 or [96], Lemma IV. 3.11 and the preceding discussion relating Bessel functions to $\widehat{\sigma_1}$. It follows that

$$\begin{aligned}|v|^{-1}\widehat{\sigma_{|v|}}\left(\frac{1}{2}+\epsilon\right) \\ &= 2\sqrt{2\pi}\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{1}{2}} \cos\left(2\pi\left(\frac{1}{2}+\epsilon\right)|v|-\frac{\pi}{4}\right) \\ &+\mathcal{O}\left(\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{3}{2}}\right) \qquad (82)\end{aligned}$$

Substituting (82) into the definition of $J_{\frac{1}{2}+\epsilon}$ we find that

$$\begin{aligned}(2\sqrt{2\pi})^{-1}J_{\frac{1}{2}+\epsilon}(x) \\ &= \sum_{v\in\mathbb{Z}^2}\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{1}{2}}\cos\left(2\pi\left(\frac{1}{2}+\epsilon\right)|v|-\frac{\pi}{4}\right)q\left(\frac{|v|}{\frac{1}{2}+\epsilon}\right) \\ &+\mathcal{O}\left(\sum_{v\in\mathbb{Z}^2\setminus\{0\}} q\left(\frac{|v|}{\frac{1}{2}+\epsilon}\right)\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{3}{2}}\right)\end{aligned}$$

The second term here is $\lesssim\left(\frac{1}{2}+\epsilon\right)^{-1}$ since there are $\mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{2}\right)$ lattice points v with $\frac{\frac{1}{2}+\epsilon}{2}\le|v|\le 2\left(\frac{1}{2}+\epsilon\right)$. We rewrite the first term using the Poisson summation formula, obtaining

$$\begin{aligned}(2\sqrt{2\pi})^{-1}J_{\frac{1}{2}+\epsilon}(x) \\ &= \left(\frac{1}{2}+\epsilon\right)^{-\frac{1}{2}}\sum_{v\in\mathbb{Z}^2}\mathrm{re}\left(e^{i\frac{\pi}{4}}\int_{\mathbb{R}^2} e^{2\pi i v\cdot(x+\epsilon)}\,|x\right. \\ &\left.+\epsilon|^{-\frac{1}{2}}e^{-2\pi i\left(\frac{1}{2}+\epsilon\right)|x+\epsilon|}q\left(\frac{|x+\epsilon|}{\frac{1}{2}+\epsilon}\right)d(x+\epsilon)\right)+\mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-1}\right)\end{aligned}$$

$$= \left(\frac{1}{2}+\epsilon\right)\sum_{v\in\mathbb{Z}^2} \operatorname{re}\left(e^{i\frac{\pi}{4}}\int_{\mathbb{R}^2} e^{-2\pi i\left(\frac{1}{2}+\epsilon\right)\left(\left(\frac{1}{2}+\epsilon\right)|x+\epsilon|-v\cdot(x+\epsilon)\right)}|x+\epsilon|^{-\frac{1}{2}}q(x+\epsilon)d(x+\epsilon)\right) + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-1}\right)$$

$$= \left(\frac{1}{2}+\epsilon\right)\sum_{v\neq 0} \operatorname{re}\left(e^{i\frac{\pi}{4}}\int_{-\pi}^{\pi}\int_0^{\infty}\varphi(t)e^{-2\pi i\left(\frac{1}{2}+\epsilon\right)|v|t\left(\frac{\frac{1}{2}+\epsilon}{|v|}-\cos\theta\right)}dtd\theta\right) + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-1}\right)$$

where $\varphi(t) = t^{\frac{1}{2}}q(t)$. Here the second line followed by change of variables $x+\epsilon \to \left(\frac{1}{2}+\epsilon\right)(x+\epsilon)$, and on the last line we introduced polar coordinates with $\theta = \angle v0(x+\epsilon)$, and used that the contribution from $v = 0$ is equal to $\operatorname{re}\left(e^{i\frac{\pi}{4}}\left(\frac{1}{2}+\epsilon\right)\hat{\varphi}\left(\left(\frac{1}{2}+\epsilon\right)^2\right)\right)$ and therefore $\mathcal{O}\left(\left(\left(\frac{1}{2}+\epsilon\right)^2\right)^{-100}\right)$. Now we apply (80) to the terms in the sum, with $= \left(\frac{1}{2}+\epsilon\right)|v|$, $\mu = \frac{\frac{1}{2}+\epsilon}{|v|}$. Letting $\psi(t) = \operatorname{re}\left(e^{i\frac{\pi}{4}}\chi(t)\right)$ we conclude that

$$\left(2\sqrt{2\pi}\right)^{-1}J_N(x) = \left(\frac{1}{2}+\epsilon\right)\sum_{v\neq 0}\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)\left(|v|-\left(\frac{1}{2}+\epsilon\right)\right)\right) + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)\sum_{v\neq 0}\left(\left(\frac{1}{2}+\epsilon\right)|v|\right)^{-\frac{3}{2}}\left(1+\left(\frac{1}{2}+\epsilon\right)\left||v|-\left(\frac{1}{2}+\epsilon\right)\right|\right)^{-100}\right) + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-1}\right)$$

The second term is $\lesssim \left(\frac{1}{2}+\epsilon\right)^{-2+\epsilon} \max\left(\frac{\frac{1}{2}+\epsilon}{\left(\frac{1}{2}+\epsilon\right),3+\epsilon}, 1\right)$, since the contribution to the sum from terms with $|v| \le \frac{\frac{1}{2}+\epsilon}{2}$ is clearly very small and the contribution from $|v| \ge \frac{\frac{1}{2}+\epsilon}{2}$ can be estimated by Corollary (2.3.31). (79) follows from this on replacing ψ by $2\sqrt{2\pi}\psi$.

Corollary (2.3.36)[240]: Assume the bound (75). Then

$$|K_{1+\epsilon}(x)| \lesssim (1+\epsilon)|x|^{-(1-\beta)} \qquad (83)$$

if $|x| \geq 1 + \epsilon \geq 1$.

Proof: We first prove the estimate (83) with $K_{\left(\frac{1}{2}+\epsilon\right)}$ replaced by $J_{\frac{1}{2}+\epsilon}$. We define $(t) = t^{-\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)\left(t-(\frac{1}{2}+\epsilon)\right)\right)$, with ψ as in Corollary (2.3.35). Since ψ is in the Schwarz space it is easily seen using the product rule that for any fixed $\beta > 0$,

$$\int_{t=1}^{\infty} t^{\beta}|f'(t)|dt \lesssim \left(\frac{1}{2}+\epsilon\right)^{\beta-\frac{1}{2}} \tag{84}$$

uniformly in $\epsilon \geq 0$. Now consider the quantity $(\epsilon \geq 0, \epsilon \geq 0)$

$$\sum_{v\in\mathbb{Z}^2, v\neq 0} |v|^{-\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)\left(|v|-\left(\frac{1}{2}+\epsilon\right)\right)\right) = \int_{t=0}^{\infty} f(t)dn(t)$$

$$= \int_{t=0}^{\infty} 2\pi t f(t)dt + \int_{t=0}^{\infty} f(t)d(n(t)-\pi t^2)$$

$$= \int_{t=0}^{\infty} 2\pi t f(t)dt + \int_{t=0}^{\infty} (n(t)-\pi t^2)f'(t)dt \tag{85}$$

The first term in (85) is easily seen to be very small:

$$\left|\int_{t=0}^{\infty} 2\pi t f(t)dt\right| = 2\pi\left|\int_{t=-\left(\frac{1}{2}+\epsilon\right)}^{\infty}\left(t+\frac{1}{2}+\epsilon\right)^{\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)t\right)dt\right|$$

$$= 2\pi\left|\int_{-\infty}^{\infty}\left(t+\frac{1}{2}+\epsilon\right)^{\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)t\right)dt\right| + \mathcal{O}\left(\left(\left(\frac{1}{2}+\epsilon\right)^{2}\right)^{-100}\right)$$

$$\lesssim \left(\frac{1}{2}+\epsilon\right)^{-\frac{1}{2}}\int_{t=-\infty}^{\infty}|t|\left|\psi\left(\left(\frac{1}{2}+\epsilon\right)t\right)\right|dt + \left(\left(\frac{1}{2}+\epsilon\right)^{2}\right)^{-100} \approx \left(\frac{1}{2}+\epsilon\right)^{-\frac{5}{2}}$$

Here the second line followed since ψ is in the Schwarz space and the third line followed since $\left(\frac{1}{2}+\epsilon+t\right)^{\frac{1}{2}} = \left(\frac{1}{2}+\epsilon\right)^{\frac{1}{2}} + \mathcal{O}\left(\left(\frac{1}{2}+\epsilon\right)^{-\frac{1}{2}}|t|\right)$ and $\psi(0) = 0$. The second term in (85) is $\lesssim \int_{t=1}^{\infty} t^{\beta}|f'(t)|dt + \int_{t=0}^{1} t^2|f'(t)|dt \lesssim \left(\frac{1}{2}+\epsilon\right)^{\beta-\frac{1}{2}}$ by (84) and an obvious estimate for the contribution from $t < 1$. Now we use (79). Let $\frac{1}{2}+\epsilon = |x|$. We've assumed that $\epsilon \geq 0$, so the error term in (79) is $\lesssim \left(\frac{1}{2}+2\epsilon\right)^{-1}$ Hence

$$
\begin{aligned}
|J_{\frac{1}{2}+\epsilon}(x)| &\lesssim \left(\frac{1}{2}+\epsilon\right)\left(\frac{1}{2}+2\epsilon\right)^{-\frac{1}{2}} \Big| \sum_{v\in\mathbb{Z}^2, v\neq 0} |v|^{-\frac{1}{2}}\psi\left(\left(\frac{1}{2}+\epsilon\right)\left(|v|-\left(\frac{1}{2}+2\epsilon\right)\right)\right)\Big| \\
&\quad + \left(\frac{1}{2}+2\epsilon\right)^{-1} \\
&\lesssim \left(\frac{1}{2}+\epsilon\right)\left(\frac{1}{2}+2\epsilon\right)^{-\frac{1}{2}} \cdot \left(\frac{1}{2}+2\epsilon\right)^{\beta-\frac{1}{2}} + \left(\frac{1}{2}+\epsilon\right)\left(\frac{1}{2}+2\epsilon\right)^{-\frac{1}{2}} \\
&\quad \cdot \left(\frac{1}{2}+2\epsilon\right)^{-\frac{1}{2}}\left(\frac{1}{2}+\epsilon\right)^{-2} + \left(\frac{1}{2}+2\epsilon\right)^{-1} \approx \left(\frac{1}{2}+\epsilon\right)\left(\frac{1}{2}+2\epsilon\right)^{-(1-\beta)}
\end{aligned}
$$

When $t > 0$ we can express p in the form $p(t) = \sum_{j\geq 0} q\left(2^j t\right)$ where q is supported in $\left[\frac{1}{2}, 2\right]$. Observe that if $\frac{\frac{1}{2}+\epsilon}{2^j} < \frac{1}{2}$ then the sum defining $J_{\frac{\frac{1}{2}+\epsilon}{2^j}}$ is empty. Hence $|K_{\frac{1}{2}+\epsilon}(x)| \leq \sum_j |J_{\frac{\frac{1}{2}+\epsilon}{2^j}}(x)| \lesssim \sum_j \frac{\frac{1}{2}+\epsilon}{2^j}|x|^{-(1-\beta)} \lesssim \left(\frac{1}{2}+\epsilon\right)|x|^{-(1-\beta)}$ and the proof is complete.

Chapter 3

Smooth Functions with Dual Locally and Structure

We show that the space $C_0[0, \Omega)$ admits C^∞ partitions of unity for every ordinal Ω. The compact spaces homeomorphic to weak* compact subsets of a dual LUR Banach space have the same properties as the class of Radon-Nikodym compact spaces.

Section (3.1): Partitions of Unity on Certain Banach Spaces

In [101], a method, based on the use of "Talagrand operators", for defining infinitely differentiable equivalent norms on the spaces $C_0(L)$ for certain locally compact, scattered spaces L. A special case of this result was that a C^∞ renorming exists on $C_0(L)$ for every countable locally compact L. Recently, H$\acute{a}$jek [100] extended this result by showing that a real normed space X admits a C^∞ renorming whenever there is a countable subset of the unit ball of X^* on which every element of X attains its norm, that is to say, a countable boundary. This suggested that the locally compact topology on L was perhaps not essential in [101], and we shall develop the methods of that work in a way that does not require such a topology. We obtain infinitely differentiable norms on certain (typically non-separable) Banach spaces X as well as on some certain injective tensor products $X \otimes_\epsilon E$.

We present a lemma about partitions of unity. It is an open problem whether a non-separable Banach space with a C^k norm (or, more generally, a C^k "bump function") admits C^k partitions of unity, though many partial results in this direction are known. Our lemma enables us to show that the answer is yes for classes of Banach spaces that admit projectional resolutions of the identity. In particular, we show that the space $C_0[0, \Omega)$ admits C^∞ partitions of unity for every ordinal Ω. Results from are used in [102] to give examples of Banach spaces admitting infinitely differentiable bump functions and partitions of unity but no smooth norms.

We have followed the conventions of [99]. Although that work contains everything that we will need in order to understand the present, we recall for convenience a few facts and definitions. It should be noted that we are concerned only with real, as opposed to complex, Banach spaces. When we refer to a function on a Banach space as being of class C^k , where k is a positive integer, it is the standard (Fr$\acute{e}$chet) notion of smoothness that we are employing. Making a mild abuse of language, we shall say that a norm $\|\cdot\|$ on a Banach space X is of class C^k if the function $x \longmapsto \|x\|$ is of that class on the set $X \setminus \{0\}$. (Of course, no norm is differentiable at 0.)

A bump function on a Banach space X is a function $\varphi : X \to \mathbb{R}$ with bounded, non-empty support. On finite-dimensional spaces, C^∞ bump functions are plentiful (and fundamental to the theory of distributions). The existence of a C^1bump function on an infinite-dimensional Banach space X is already a strong condition.

For a separable space X, it is equivalent to separability of the dual space X^*, and to the existence of an equivalent C^1 norm. More generally, the existence of a C^1 bump on X implies that X belongs to the important class of Asplund spaces. Whether every Asplund space admits a C^1 bump is an open problem, as is the relationship between existence of a

C^k bump and of a C^k norm on a separable space once k is greater than 1. On the other hand, the equivalences in the following proposition are very easy to establish.

Proposition (3.1.1)[97]: For a real Banach space X and $k \in \mathbb{N} \cup \{\infty\}$, the following are equivalent:

(i) X admits a C^k -bump function;

(ii) there exists a real number $R > 1$ and a function $\psi : X \to \mathbb{R}$, of class C^k , such that $0 \leq \psi \leq 1, \psi(x) = 0$ when $\|x\| \leq 0$ and $\|x\| = 1$ when $\|x\| \geq R$;

(iii) there is a function $\theta : X \to \mathbb{R}$, of class C^k , such that $\theta(x) \to \infty \; as \; \|x\| \to \infty$.

Proof. (i)$\Longrightarrow$ (ii):

Let φ be a C^k bump function with $\varphi(0) = 1$. There exist positive real numbers δ and M such that $\varphi(x) \geq \frac{2}{3}$ when $\|x\| \leq \delta$ and $\varphi(x) = 0$ when $\|x\| \geq M$. Let $\pi : \mathbb{R} \to [0,1]$ be a C^∞ function with $\pi(t) = 0$ for $t \geq \frac{2}{3}$ and $\pi(t) = 1$ for $t \leq \frac{1}{3}$. Then the function ψ defined by

$$\psi(x) = \pi(\varphi(\delta x))$$

has the required properties, with $R = \delta^{-1}M$.

(ii) $\Longrightarrow$ (iii):

Given R and ψ as in (ii), we may define

$$\theta(x) = \sum_{n=0}^{\infty} \psi(R^{-n}x) \, ,$$

noting that on each ball $\{x \in X : \|x\| < N\}$ the sum has only finitely many nonzero terms.

(iii) $\Longrightarrow$ (i):

Given θ, we define $\varphi(x) = \pi(\theta(x) - \theta(0))$, where $\pi : \mathbb{R} \to \mathbb{R}$ is the function already used above.

Many of our results concern spaces that are subspaces of the space $\ell_\infty(L)$ of bounded real-valued functions on a set L. For elements f of $\ell_\infty(L)$ we use "coordinate" notation, writing $(f_t)_{t\in L}$ and thinking of the f_t as coordinates. A certain class of very nice functions, already well-established, will be of particular importance. We shall say that a function φ, defined on a subset D of $\ell_\infty(L)$, depends locally on finitely many coordinates if, for each f^0 in D, there exist an open neighbourhood G of f^0 in D and a finite subset M of L such that, for $f \in G$, the value of $\varphi(f)$ depends only on f_t $(t \in M)$.

Theorem (3.1.2)[97]: Let L be a set and let $U(L)$ be the subset of the direct sum $\ell_\infty(L) \oplus c_0(L)$ consisting of all pairs (f, x) such that $\|f\|_\infty$ and $\|x\|_\infty$ are both strictly less than $\left\| |f| + \frac{1}{2}|x| \right\|_\infty$. The space $\ell_\infty(L) \oplus c_0(L)$ admits an equivalent norm $\|\cdot\|$ with the following properties:

(i) $\|\cdot\|$ is a lattice norm, in the sense that $\|(g,y)\| \le \|(f,x)\|$ whenever $|g| \le |f|$ and $|y| \le |x|$;

(ii) $\|\cdot\|$ is infinitely differentiable on the open set $U(L)$;

(iii) locally on $U(L)$, $\|(f,x)\|$ depends on only finitely many non-zero coordinates; that is to say, for each $(f^0, x^0) \in U(L)$ there is a finite $N \subseteq L$ and an open neighbourhood V of (f^0, x^0) in $U(L)$, such that for $(f,x) \in V$ the norm $\|(f,x)\|$ is determined by the values of f_t and x_t with $t \in N$ and such that $f_t \neq 0, x_t \neq 0$ for all such (f,x) and t.

We start the proof of Theorem (3.1.2) by fixing a strictly increasing $\mathcal{C}^\infty$ function $\varpi :$ $[0,2) \to [0,\infty)$ such that $\varpi(u) \to \infty$ as $u \uparrow 2$ and $\varpi(u) = 0$ for $u \le 1$. The inverse function ϖ^{-1} is $\mathcal{C}^\infty$ and strictly increasing from $(0,\infty)$ onto $(1,2)$. We define $\theta :$ $[0,\infty \to [0,\infty)$ by

$$\theta(c) = \int_0^c \frac{dv}{\varpi^{-1}(v)} ,$$

and start by recording some facts about this function.

Lemma (3.1.3)[97]:

(i) The function θ is strictly increasing and strictly concave from $[0,\infty)$ onto $[0,\infty)$. It is of class $\mathcal{C}^\infty$ on $(0,\infty)$, with $\theta'(c) = 1/\varpi^{-1}(c)$ $(c > 0)$, and differentiable at 0 with $\theta'(0) = \lim_{c\downarrow 0} 1/\varpi^{-1}(c) = 1$.

(ii) The composite function $\theta \circ \varpi$: $[0,2) \to [0,\infty)$ is infinitely differentiable, with

$$(\theta \circ \varpi)'(u) = \begin{cases} u^{-1}\varpi'(u) & (u \ge 0) \\ 0 & (u = 0) \end{cases}$$

(iii) We have $\frac{1}{2} c < c\theta'(c) < \theta(c) < c$ for all positive c.

The next lemma can be regarded as the finite-dimensional part of the proof of Theorem (3.1.2).

Lemma (3.1.4)[97]: Let N be a finite set, let η be a positive real number and and let W be the subset of $\mathbb{R}^N \times \mathbb{R}^N$ consisting of all $(\mathbf{f},\mathbf{x})$ such that $\left\| |\mathbf{f}| + \frac{1}{2}|\mathbf{x}| \right\|_\infty > max\{\|f\|_\infty, \|x\|_\infty\} + \eta$. Let the functions $F : \mathbb{R}^N \times \mathbb{R}^N \times$ $[0,\infty)^N \to \mathbb{R}, G : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(\mathbf{f},\mathbf{x},\mathbf{c}) = \exp\left[-\sum_{t\in N} c_t\right] \sum_{t\in N} [c_t|f_t| + \theta(c_t)|x_t|]$$

$$G(\mathbf{f}, \mathbf{x}) = \sup_{c \in [0,\infty)^N} F(\mathbf{f}, \mathbf{x}, \mathbf{c}).$$

If $(\mathbf{f}, \mathbf{x}) \in W$ then the supremum in the definition of $G(\mathbf{f}, \mathbf{x})$is attained at a unique **c**; this **c** has the property that $c_t = 0$ whenever $|f_t| \le \eta$ or $|x_t| \le \eta$. The function G is of class $\mathcal{C}^\infty$on W.

Proof. To start with, let us consider a fixed $(\mathbf{f}, \mathbf{x}) \in W$. We have

$$\frac{\partial F}{\partial c_s} = \exp\left[-\sum_{t \in N} c_t\right]\left[|f_s| + \theta'(c_s)|x_s| - \sum_{t \in N}[c_t|f_t| + \theta(c_t)|x_t|]\right]$$
$$\le exp\left[-\sum_{t \in N} c_t\right][(1 - c_s)|f_s| + (\theta'(c_s) - \theta(c_s))|x_s|],$$

which is non-positive when $c_s \ge 1$. Thus the supremum in the definition of $G(\mathbf{f}, \mathbf{x})$ may be taken over the compact set $[0,1]^N$ rather than over $[0,\infty)^N$; this supremum is thus attained at some **c**. By elementary calculus, any **c** at which the supremum is attained satisfies, for all s, either

(ia) $\quad c_s > 0 \quad and \quad |f_s| + \theta'(c_s)|x_s| = \nu$

(ib) $\quad or \quad c_s = 0 \quad and \quad |f_s| + \theta'(0)|x_s| \le \nu$

where $\nu = \sum_{t \in N}[c_t|f_t| + \theta(c_t)|x_t|]$. In case (ia) we have

(ii) $$c_s = \varpi\left(\frac{|x_s|}{\nu - |f_s|}\right)$$

because ϖ is the function inverse to $1/\theta'$. In fact, this equality holds in case 1b as well because then $|f_t| + |x_t| = |f_t| + \theta'(0)|x_t| \le \nu$, whence $|x_t|/(\nu - |f_t|) \le 1$ and $\varpi(|x_t|/(\nu - |f_t|)) = 0 = c_t$. Thus ν is a solution of

(iii) $$\nu = \sum_{t \in N}\left[\varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|f_t| + \theta \circ \varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|x_t|\right]$$

Since the right hand side of this equation is a decreasing function of ν it has only one solution. By equation (ii), we now see that c_s $(s \in N)$ are uniquely determined too.

Because $(\mathbf{f}, \mathbf{x}) \in W$, there is some s such that $|f_s| + \frac{1}{2}|x_s| > max\{\|f\|_\infty, \|f\|_\infty\} + \eta$; since $\theta'(c_s) > \frac{1}{2}$ we have $\nu \ge |f_s| + \frac{1}{2}|x_s|$ by (ia) and (ib). Thus $\nu > |f_t| + \eta$ and $\nu > |x_t| + \eta$ for all t. So if either $|x_t| \le \eta$ or $|f_t| \le \eta$ it must be that (ib) holds, with $c_t = 0$.

We now move on to consider the behaviour of $\nu = \nu(\mathbf{f}, \mathbf{x})$ and of $c_t = c_t(\mathbf{f}, \mathbf{x})$ as (**f**, **x**) varies over W. We consider the function H defined on the open set $V = \{(\mathbf{f}, \mathbf{x}, \nu) : (\mathbf{f}, \mathbf{x}) \in W \text{ and } \nu > max\{\|f\|_\infty, \|x\|_\infty\} + \eta\}$ of $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ by

$$H(\mathbf{f}, \mathbf{x}, \nu) = \sum_{t \in N}\left[\varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|f_t| + \theta \circ \varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|x_t|\right].$$

We have already noted that for each $(\mathbf{f},\mathbf{x}) \in W$ there is a unique $\nu = \nu((\mathbf{f},\mathbf{x})$ such that $H(\mathbf{f},\mathbf{x},\nu) = 0$. It is also easy to verify that $\frac{\partial H}{\partial \nu} \geq 1$ everywhere on V. Thus the infinite differentiability of $(\mathbf{f},\mathbf{x}) \longmapsto \nu(\mathbf{f},\mathbf{x})$ will follow from the Implicit Function Theorem provided we can show that H is itself infinitely differentiable. The absolute value signs appear to present a problem on a neighbourhood of a point where one of the variables f_t or x_t is zero. However, as soon as either $|f_t|$ or $|x_t|$ is smaller than η, the terms $\varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|f_t|$ and $\theta \circ \varpi\left(\frac{|x_t|}{\nu - |f_t|}\right)|x_t|$ vanish, showing that we do not have a problem at all.

Once we have shown that ν varies in an infinitely differentiable fashion with $(\mathbf{f},\mathbf{x})$, it follows from (ii) that the same is true for all the c_t and hence for G.

We now take up the proof of the theorem. We define a norm $\|\cdot\|$ on $\ell_\infty(L) \oplus_{c_0} (L)$ by

$$\|(f,x)\| = sup \left\{ e^{-\sum_{t\in L} dt} \sum_{t\in L} [d_t|f_t| + \theta(d_t)|x_t|] : dt \geq 0 \, for\ all\ t\ and\ d_t \right.$$
$$\left. = 0 \, for\ all\ but\ finitely\ many\ t \right\}.$$

and claim that this has the properties we require. It is clear that $\|\cdot\|$ is a lattice norm and that

$$e^{-1} max \left\{ \|f\|_\infty, \frac{1}{2}\|x\|_\infty \right\} \leq \|(f,x)\| \leq e^{-1}(\|f\|_\infty + \|x\|_\infty).$$

For $(f,x) \in U(L)$ we define

$$\xi(f,x) = \left\| |f| + \frac{1}{2}|x| \right\|_\infty - \max\{\|f\|_\infty, \|x\|_\infty\}$$

$$M(f,x) = \left\{ t \in L : |f(t)| + |x(t)| \geq \left\| |f| + \frac{1}{2}|x| \right\|_\infty \right\}$$

$$N(f,x) = \{ t \in L : |f(t)| + |x(t)| \geq \left\| |f| + \frac{1}{2}|x| \right\|_\infty - \frac{1}{2}\xi(f,x)\}$$

and note that $N(f,x)$ is a finite set, because $x \in c_0(L)$ and $N(f,x) \subseteq \{t : |x_t| \geq \frac{1}{2}\xi(f,x)\}$. We shall show first that in the definition of $\|(f,x)\|$ it is enough to take the supremum over families $d = (d_t)$ such that $d_t = 0$ for all $t \notin M(f,x)$. Indeed, let (f,x) be in $U(L)$, and suppose that $d = (d_t)_{t\in L}$ is such that $d_{t_1} > 0$ for some $t_1 \notin M(f,x)$. Let t_0 be chosen so that $|f(t_0)| + \frac{1}{2}|x(t_0)| = \left\| |f| + \frac{1}{2}|x| \right\|_\infty$ and let $d' = (d'_t)$ be defined by

$$d'_t = \begin{cases} d_t & if\ t \notin \{t_0, t_1\} \\ 0 & if\ t = t_1 \\ d_{t_0} + d_{t_1} & if\ t = t_0. \end{cases}$$

We note that $\sum_t d'_t = \sum_t d_t$ and that $\theta(d'_{t_0}) - \theta(d_{t_0}) > \frac{1}{2} d_{t_1}$, because θ' is everywhere greater than $\frac{1}{2}$.

$$\sum_{t\in L}[d'_t|f_t| + \theta(d'_t)|x_t|] - \sum_{t\in L}[d_t|f_t| + \theta(d_t)|x_t|]$$

$$= d_{t_1}[|f_{t_0}| - |f_{t_1}|] + \left(\theta(d'_{t_0}) - \theta(d_{t_0})\right)|x_{t_0}| - \theta(d_{t_1})|x_{t_1}|$$

$$\geq d_{t_1}\left[|f_{t_0}| + \frac{1}{2}|x_{t_0}| - |f_{t_1}| - |x_{t_1}|\right]$$

and this is strictly positive by our assumptions about t_0 and t_1. In this way, we may reduce to 0 all coordinates d_t with $t \notin M(f,x)$ while increasing the value of $[-\sum_t dt]\,[\sum_t d_t|f_t| + \theta(t)|x_t|]$.

We now set about finding the neighbourhoods V and finite sets N referred to in (iii). Given $(f^0, x^0) \in U(L)$, we set $N = N(f^0, x^0)$ and define V to be the open set

$$V = \{(f,x): \|f - f^0\|_\infty, \|x - x^0\|_\infty < \frac{1}{7}\xi(f^0, x^0)\}.$$

It is easy to check that if $(f,x) \in V$ then $\xi(f,x) > \frac{1}{2}\xi(f^0, x^0)$ and $M(f,x) \subseteq N$. By what we have already proved, this shows that on the open set V our norm depends only on coordinates in the finite set N.

Moreover, for $(f,x) \in V$ we have

$$\|(f,x)\| = \sup_{c\in[0,\infty)^N} F(\boldsymbol{c}, (f_t)_{t\in N}, (x_t)_{t\in N})$$

where $F: [0,\infty)^N \times \mathbb{R}^N \times \mathbb{R}^N$ is the function

$$F(\boldsymbol{d}, \mathbf{f}, \mathbf{x}) = exp\left(-\sum_{t\in N} dt\right)\sum_{t\in N}[d_t|f_t| + \theta(d_t)|x_t|].$$

We can thus apply Lemma (3.1.4) (with $\eta = \frac{1}{2}\xi(f^0, x^0)$) to conclude that $\|\cdot\|$ is infinitely differentiable on V.

In the following corollary to Theorem (3.1.2) we use the above remark to deduce a renorming result about injective tensor products. We recall some facts about such products. If X and E are Banach spaces and ξ, η are elements of the dual spaces X^*, E^* respectively, then a linear form $\xi \otimes \eta$ may be defined on the algebraic tensor product $X \odot E$ by

$$(\xi \otimes \eta)(\sum_{j=1}^{n} x_j \otimes e_j) = \sum_j \langle \xi, x_j \rangle \langle \eta, e_j \rangle.$$

The injective tensor product $X \otimes_\epsilon E$ is defined to be the completion of $X \odot E$ for the norm defined by

$$\|z\|_\epsilon = max\{|\langle(\xi \otimes \eta), z\rangle| : \xi \in ball\, X^* , \eta \in ball\, E^* , \}.$$

The elementary tensor forms $\xi \otimes \eta$ extend by continuity to $X \otimes_\epsilon E$ and $\{\xi \otimes \eta : \xi \in ball\, X, \eta \in ball\, E\}$ is a weak* compact subset of ball $(X \otimes_\epsilon E)^*$ on which every element of $X \otimes_\epsilon E$ attains its norm.

If $Q : X_1 \to X_2$ and $R : E_1 \to E_2$ are bounded linear operators then a bounded linear operator $Q \otimes R : X_1 \otimes_\epsilon E_1 \to X_2 \otimes_\epsilon E_2$ is determined by $(Q \otimes R)(x \otimes e) = (Qx) \otimes (Re)$. A special case is the so-called "slice map" $I_X \otimes \eta : X \otimes E \to X$ derived from an element η of E^* and given by $(I_X \otimes \eta)(x \otimes e) = \langle\eta, e\rangle x$. By our earlier remark about the attainment of norms on elementary tensor forms, we see that for any $z \in X \otimes_\epsilon E$ there is some $\eta \in ball\, E^*$ with $\|z\|_\epsilon = \| (I_X \otimes \eta)(z)\|$.

In the special case where X is a space $\ell_\infty(L)$, then $X \otimes_\epsilon E$ identifies isometrically with a subspace of the vector-valued function space $\ell_\infty(L; E)$ (the elementary tensor $(x_t)_{t\in L} \otimes e$ corresponding to $t \longmapsto x_t e$). The effect of a slice map on $z \in \ell_\infty(L) \otimes_\epsilon E$ regarded as an element $(z_t)_{t\in L}$ of $\ell_\infty(L; E)$ is simply

$$(I_{\ell_\infty}(L) \otimes \eta)(z) = (\langle\eta, z_t\rangle)_{t\in L}.$$

Corollary (3.1.5)[97]: Let X be a Banach space and let L be a set. Suppose that there exist a linear homeomorphic embedding $S : X \to \ell_\infty(L)$ and a linear operator $T : X \to c_0(L)$ with the property that $(S(x), T(x))$ is in the set $U(L)$ of Theorem (3.1.2) whenever x is a non-zero element of X. Then X admits an equivalent C^∞ norm. Moreover, whenever the Banach space E admits an equivalent C^k norm so does the injective tensor product $X \otimes_\epsilon E$.

Proof. It is clear that, using the norm on $\ell_\infty(L) \oplus c_0(L)$ that we defined in Theorem (3.1.2), we may set

$$\|x\| = \|(S(x), T(x))\|$$

and obtain an infinitely differentiable norm on X.

Now let E be a Banach space with a C^k norm $\|\cdot\|_E$. For $f \in \ell_\infty(L; E)$ we define $Nf \in \ell_\infty(L)$ by

$$(Nf)_t = \|f_t\|_E .$$

The operators S and T induce $S \otimes I_E$ and $T \otimes I_E$, taking the injective tensor product $X \otimes_\epsilon E$ to $\ell_\infty(L) \otimes_\epsilon E$ and $c_0(L) \otimes_\epsilon E$ respectively. Identifying $c_0(L) \otimes_\epsilon E$ with $c_0(L; E)$ and $\ell_\infty(L) \otimes_\epsilon E$ with a subspace of $\ell_\infty(L; E)$, we may define a norm on $X \otimes_\epsilon E$ by

$$\|z\| = \| (N((S \otimes I_E)(z)), (N((T \otimes I_E)(z)))\|.$$

The coordinate maps $x \longmapsto \left(N\left((S \otimes I_E)(x)\right)\right)_t = \| (S \otimes I_E)(x)_t\|$ and $x \longmapsto \left(N\left((T \otimes I_E)(x)\right)\right)_t = \| (T \otimes I_E)(x)_t\|$ are of class C^k except where they vanish. Thus, by the above remark, we shall be able to conclude that we have a C^k norm on $X \otimes_\epsilon E$ provided we can show that $(N((S \otimes I_E)(z)), (N((T \otimes I_E)(z)))$ is in

$U(L)$ whenever $z \in (X \otimes_\epsilon E) \setminus \{0\}$. This is not really difficult, being just a matter of disentangling tensor notation.

For such a z we choose some $\eta \in ball\, E^*$ with $\left\| I_{\ell_\infty(L)} \otimes \eta)((S \otimes I_E)(z)) \right\|_\infty = \|(S \otimes I_E)(z)\|_\infty$. We then note that $\left(I_{\ell_\infty(L)} \otimes \eta\right) \circ (S \otimes I_E) = S \circ (I_X \otimes \eta)$. Our hypothesis about S and , applied to $x = (I_X \otimes \eta)(z)$ tells us that there is some $t \in L$ with

$$\left|S\big((I_X \otimes \eta)(z)\big)_t\right| + \frac{1}{2}\,|T\left((I_X \otimes \eta)(z)\right)_t| > \left\|S\big((I_X \otimes \eta)(z)\big)\right\|_\infty.$$

Thus

$$\begin{aligned} N\big((S \otimes I_E)(z)\big)_t &+ \frac{1}{2} N\big((T \otimes I_E)(z)\big)_t \\ &= \| (S \otimes I_E)(z)_t\|_E + \frac{1}{2}\, \|(T \otimes I_E)(z)_t\|_E \\ &\geq |\langle \eta, (S \otimes I_E)(z)_t\rangle| + \frac{1}{2}\, |\langle \eta, (T \otimes I_E)(z)_t\rangle| \\ &= \left|\big(I_{\ell_\infty(L)} \otimes \eta\big)(S \otimes I_E)(z)_t\right| + \frac{1}{2}\left|\big(I_{c_0(L)} \otimes \eta\big)(T \otimes I_E)(z)_t\right| \\ &= \left|S\big((I_X \otimes \eta)(z)\big)_t\right| + \frac{1}{2}\left|T\left((I_X \otimes \eta)(z)\right)_t\right| \\ &> \left\|S\big((I_X \otimes \eta)(z)\big)\right\|_\infty \\ &= \| (S \otimes I_E)(z)\|_\infty \end{aligned}$$

A similar argument involving an η chosen so that $\left\|(I_{c_0(L)} \otimes \eta)((T \otimes I_E)(z)\right\|$ is equal to $\|(T \otimes I_E)(z)\|$ enables us to finish the proof.

The above corollary allows us to reprove Hájek's theorem from [100], though not, of course, the more recent, and very strong, result of [98], according to which any norm on a Banach space with countable boundary may be approximated by analytic norms.

Corollary (3.1.6)[97]: [Hájek]. Let X be a Banach space which admits a countable boundary. Then X admits a $\mathcal{C}^\infty$ renorming and $X \otimes_\epsilon E$ admits a $\mathcal{C}^k$ renorming whenever E does.

Proof. Let $\{\xi_n : n \in \mathbb{N}\}$ be a countable boundary for X and define $S : X \to \ell_\infty(\mathbb{N})$ and $T : X \to c_0(\mathbb{N})$ by $(Sx)_n = \langle \xi_n, x\rangle, (Tx)_n = 2^{-n}\langle \xi_n, x\rangle$. It is easy to see that $(S(x), T(x)) \in U(L)$ when x is a non-zero element of X, since for any x there exists n with $\langle \xi_n, x\rangle = \|x\|$.

Extending slightly the terminology of [101], we shall say that a bounded linear operator T from a subspace X of $\ell_\infty(L)$ into $c_0(L)$ is a Talagrand operator for X if for every non-zero x in x there exists $t \in L$ with $|x_t| = \|x\|_\infty$ and $(Tx)_t \neq 0$. It is clear that Corollary (3.1.5) is applicable to any such space, taking S to be the identity operator. In the particular case where L is equipped with a locally compact topology and X is the space $C_0(L)$ of continuous functions, vanishing at infinity, we retrieve the results of [101]. The classic example of a Talagrand operator is defined on the space $C_0([0,\Omega)$, where Ω is an ordinal, by

$$(Tf)_\alpha = f_\alpha - f_{\alpha+1}.$$

This has the required property since for any non-zero $f \in C_0([0,\Omega))$ there is a maximal α with $|f_\alpha| = \|f\|_\infty$. A non-linear version of a Talagrand operator is used in [101] to give an example of a space admitting a C^∞ bump function but no smooth norm. The earlier C^1 version of this result appears as Theorem VII.6.1 of [99]. The relevant application of our theorem is the following.

Corollary (3.1.7)[97]: Let X be a Banach space and let L be a set. Suppose that there exist continuous mappings $S : X \to \ell_\infty(L), T : X \to c_0(L)$ with the following properties:

(i) for all $x \in X$ the pair (Sx, Tx) is in $U(L) \cup \{0\}$;

(ii) the coordinates of S and of T are all C^k functions on the sets where they are non-zero;

(iii) $\|Sx\|_\infty \to \infty$ as $\|x\| \to \infty$.

Then X admits a C^k bump function.

Proof. Let $\theta : [0,\infty) \to [0,\infty)$ be a C^∞ function which vanishes on $[0,1]$ and which tends to infinity with its argument. The formula

$$\varphi(x) = \theta(\|(Sx, Tx)\|)$$

defines a C^k function on X which tends to infinity with $\|x\|$.

We do not know whether the results of Corollary (3.1.5) about injective tensor products extend to the non-linear set-up of Corollary (3.1.7). However, in the special case of spaces of continuous functions we have the following proposition.

Proposition (3.1.8)[97]: Let L be a locally compact space such that there exists a function $T : C_0(L) \to c_0(L)$ satisfying

(i) for all $f \in C_0(L)$ the pair (f, Tf) is in $U(L) \cup \{0\}$;

(ii) each coordinate of T is a C^k function depending locally on finitely many coordinates.

Let E be a Banach space admitting a C^k bump function. Then the space $C_0(L; E)$ also admits such a function. In particular, if each of the spaces $L_1, \ldots, L_n$ is homeomorphic to an ordinal then $C_0(L_1 \times \cdots \times L_n; E)$ admits a C^k bump function.

Proof. Let θ be a C^kfunction on E such that $\theta(x) \to \infty$ as $\|x\| \to \infty$. For $f \in C_0(L; E)$ the composition $\theta \circ f$ is in $C_0(L)$ and the pair $(\theta \circ f, T(\theta \circ f))$ is in $U(L) \cup \{0\}$. Moreover, for $t \in L$, the coordinate maps $f \longmapsto (\theta \circ f)_t$ and $f \longmapsto T(\theta \circ f)_t$ are of class C^kon $C_0(L; E)$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\rho(u) = 0$ for $u \leq 1$

and $\rho(u) \to \infty$ as $u \to \infty$. It is easy to check that the function $\varphi : \mathcal{C}_0(L;\, E) \to \mathbb{R}$ defined by

$$\varphi(f) = \rho\left(\|(\theta \circ f, T\,(\theta \circ f))\|\right)$$

is of class $\mathcal{C}^k$and tends to infinity with the norm of its argument.

When L is an ordinal Ω (identified with the set $[0, \Omega)$ of ordinals smaller than itself), an operator T of the type considered above does exist. Indeed, we may use the Talagrand operator $(Tf)_\alpha = f_\alpha - f_{\alpha+1}$. Thus $\mathcal{C}_0([0, \Omega);\, E)$ admits a $\mathcal{C}^k$ bump function whenever E does. Since $\mathcal{C}_0(L_1 \times \cdots \times L_n\,;\, E)$ may be identified with $\mathcal{C}_0(L_1; \mathcal{C}_0(L_2 \times \cdots \times L_n;\, E))$, an easy induction argument finishes the proof.

We say that a subset H of a Banach space X admits partitions of unity of class $\mathcal{C}^k$ if, for every open covering $\mathcal{V}$ of H, there is a locally finite partition of unity on H, subordinate to the covering $\mathcal{V}$, and consisting of the restrictions to H of functions that are of class $\mathcal{C}^k$on X. Once again, see [99], for the connection between partitions of unity and approximation by smooth functions and for Torunczyk's criterion: H admits $\mathcal{C}^k$ partitions of unity if and only if there is a σ-locally finite base for the topology of X consisting of $\mathcal{C}^k$-cozero sets (that is to say, sets of the form $\{x \in H : \varphi(x) \neq 0\}$ with $\varphi \in \mathcal{C}^k(X)$). It is not known whether $\mathcal{C}^k$ partitions of unity necessarily exist on every space that admits a $\mathcal{C}^k$ bump function, though many partial results. The following theorem has hypotheses that are involved but of fairly wide applicability.

Theorem (3.1.9)[97]: Let X be a Banach space, let Γ be a set and let k be a positive integer or ∞. Let $T : X \to c_0(\Gamma)$ be a function such that each coordinate $x \longmapsto T\,(x)_\gamma$ is of class $\mathcal{C}^k$ on the set where it is non-zero. For each finite subset F of Γ, let $R_F : X \to X$ be of class $\mathcal{C}^k$ and assume that the following hold:

(i) for each F, the image $R_F[X]$ admits $\mathcal{C}^k$ partitions of unity;

(ii) X admits a $\mathcal{C}^k$ bump function;

(iii) for each $x \in X$ and each $\epsilon > 0$ there exists $\delta > 0$ such that $\|x - R_F x\| < \epsilon$ if we set $F = \{\gamma \in \Gamma : |(Tx)(\gamma)| \geq \delta\}$.

Then X admits $\mathcal{C}^k$partitions of unity.

Proof. By Torunczyk's Criterion, it is enough to show that there is a σ-locally finite base for the topology of X, consisting of $\mathcal{C}^k$-cozero-sets. By hypothesis, each $R_F[X]$ admits a σ-locally finite base $\mathcal{V}_F$ consisting of $\mathcal{C}^k$-cozero-sets. Since X admits a $\mathcal{C}^k$- bump function, there is a neighbourhood base of 0 in X consisting of $\mathcal{C}^k$-cozero-sets, say $U_n (n \in \mathbb{N})$. We introduce the covering $\mathcal{W}$ of $c_0(\Gamma)$ consisting of $W_\emptyset = c_0(\Gamma)$, together with all sets

$$W_{F,q,r} = \{y \in c_0(\Gamma) : \min_{\gamma \in F} |y(\gamma)| > r \text{ and } \sup_{\gamma \in \Gamma \setminus F} |y(\gamma)| < q\}$$

with F a finite non-empty subset of Γ, and q, r positive rational numbers with $q < r$. We note that $\mathcal{W}$ is σ-locally finite, and that its members are $\mathcal{C}^\infty$-cozero-sets.

In X we consider the family of all sets of the form

$$T^{-1}[W_{F,q,r}] \cap R_F^{-1}[V] \cap (R_F - I)^{-1}[U_m]$$

with m a positive integer, F a finite subset of Γ, q, r positive rationals with $q < r$ and $V \in \mathcal{V}_F$. It is easy to check that this family is a σ-locally finite family of $\mathcal{C}^k$ cozero sets. We have to show that it forms a base for the topology of X.

Let x be in X and let $\epsilon > 0$ be given. We fix m so that

$$U_m \subseteq \frac{1}{3}\epsilon\, ball\, X,$$

and, using (iii), choose $\delta > 0$ so that

$$x - R_F(x) \in U_m$$

when we set $F = \{\gamma \in \Gamma : |(Tx)(\gamma)| \geq \delta\}$. Because $Tx \in c_0$ there exist rationals q, r with $0 < q < r < \delta$ such that $|(Tx)(\gamma)| < q$ whenever $\gamma \in \Gamma \setminus F$. Thus x is in $T^{-1}[W_{F,q,r}]$. Since $\mathcal{V}_F$ is a base for the topology of $R_F[X]$, there exists $V \in \mathcal{V}_F$ such that

$$R_F(x) \in V \subseteq R_F(x) + \frac{1}{3}\epsilon\, ball\, X.$$

It follows that x is in the set

$$T^{-1}\left[W_{F,n}\right] \cap R_F^{-1}[V] \cap (R_F - I)^{-1}[U_m].$$

If x' is any other member of this set, then we have

$$\|R_F(x) - RF(x')\| \leq \epsilon/3$$

because $R_F(x') \in V$, while

$$\|R_F(x') - x'\| \leq \epsilon/3,$$

because $(R_F(x') - x') \in U_m$. Thus $\|x - x'\| < \epsilon$, which is what we wanted to prove.

It should be noted that the mappings T and R_F in the theorem are not assumed to be linear; a non-linear T is used in [102] to give an example where $\mathcal{C}^\infty$ partitions of unity exist on a space with no smooth norm. However, the theorem offers some improvements on existing results even when only linear operators are involved. A special case of the corollary that follows occurs when the R_α form a "projectional resolution of the identity" on X. It is thus a result that is more general, as well as a bit simpler to prove, than the implication (vi) $\Longrightarrow$ (v) in Theorem VII.3.2 of [99].

Corollary (3.1.10)[97]: Let X be a Banach space admitting a $\mathcal{C}^k$ bump function. Let Ω be an ordinal and let R_α $(\alpha < \Omega)$ be a family of $\mathcal{C}^k$ functions from X to X having the property that, for every $x \in X$, the function $Rx : [0, \Omega] \to X$ defined by $(Rx)_\alpha = R_\alpha x$ $(\alpha < \Omega)$, $(Rx)_\Omega = x$ is continuous. If for each α the image of R_α admits $\mathcal{C}^k$ partitions of unity then so does X.

Proof. Since X admits a $\mathcal{C}^k$ bump function there exists a function $\varphi : X \to [0,1]$, of class $\mathcal{C}^k$ and such that $\varphi(x) = 0$ on some neighbourhood of 0 in X, while $\varphi(x) = 1$ whenever $\|x\| \geq 1$. We set $\Gamma = \Omega \times \mathbb{N}$ and define $T : X \to \ell_\infty(\Gamma)$ by

$$(Tx)(\alpha, n) = 2^{-n}\varphi(2^n (R_{\alpha+1}x - R_\alpha x)).$$

By construction, there is some $\eta > 0$ such that $\varphi(x) = 0$ whenever $\|x\| \leq \eta$. Given $x \in X$ and $\epsilon > 0$ we fix m such that $2^{-m} < \epsilon$ and then note that, because of the continuity of $\alpha \longmapsto R_\alpha x$, the quantity $\|R_{\alpha+1}x - R_\alpha x\|$ can exceed $2^{-m}\eta$ only for α in some finite set H. We thus have $|(Tx)_\gamma| \leq \epsilon$ except when $\gamma \in H \times \{0,1,2,\ldots,m-1\}$. This shows that T takes its values in $c_0(\Gamma)$.

To define the "reconstruction operators" R_F we set $R_\emptyset = R_0$ and $R_F = R_{\alpha(F)+1}$ where, for a finite non-empty subset F of Γ, $\alpha(F) = max\{\alpha : \exists n \; with \; (\alpha, n) \in F\}$. We shall show that Condition (iii) of Theorem (3.1.9) is satisfied. Given $x \in X$ and $\epsilon > 0$, it may be that $\|x - R_\alpha x\| < \epsilon$ for all $\alpha < \Omega$; in this case there is clearly no problem. Otherwise, by the continuity of $\alpha \longmapsto R_\alpha x$ on $[0,\Omega]$, there is a maximal $\beta < \Omega$ with $\|x - R_\beta x\| \geq \epsilon$. Again by the continuity of $\alpha \longmapsto R_\alpha x$, we know that there is some $\gamma > \beta$ such that $\|R_{\gamma+1}x - R_\gamma x\|$ takes a strictly positive value, η say. Now we fix n such that $2^n\eta \geq 1$, noting that $(Tx)(\gamma, n) = 2^{-n}$, and set $\delta = 2^{-n}$. If F is the set $\{(\alpha, m) \in \Omega \times \mathbb{N} : |(Tx)(\alpha, m)| \geq \delta\}$ then $(\gamma, n) \in F$ and so $\alpha(F) \geq \gamma > \beta$, whence $\|x - R_F\, x\| < \epsilon$, as required.

Corollary (3.1.11)[97]: Let Ω be an ordinal and let E be a Banach space admitting $\mathcal{C}^k$partitions of unity. Then the space $\mathcal{C}([0,\Omega];\, E)$ also admits $\mathcal{C}^k$ partitions of unity.

Proof. Proceeding by transfinite induction, we may suppose that $\mathcal{C}([0,\gamma];\, E)$ admits $\mathcal{C}^k$ partitions of unity whenever γ is an ordinal smaller than Ω. If we define $R_\gamma : \mathcal{C}([0,\Omega];\, E) \to \mathcal{C}([0,\Omega];\, E)$ by

$$\left(R_\gamma f\right)_\beta = \begin{cases} f_\beta & (\beta \leq \gamma) \\ f_\gamma & (\beta > \gamma) \end{cases}$$

then the range of R_γ is isomorphic to $\mathcal{C}([0,\Gamma];\, E)$ and so admits $\mathcal{C}^k$ partitions of unity. Moreover, the continuity hypothesis in the preceding corollary is certainly satisfied, so that the proof will be finished if we know that $\mathcal{C}([0,\Omega];\, E)$ admits a $\mathcal{C}^k$ bump function. This is true by Proposition (3.1.8), since $\mathcal{C}([0,\Omega];\, E)$ is isomorphic to $X = \mathcal{C}_0([0,\Omega);\, E) \oplus E$.

Section (3.2): Uniformly Rotund Norms

A Banach space X is said to be Asplund if every convex function on X is Fréchet differentiable on a dense $\mathcal{G}_\delta$-set. If a Banach space has an equivalent Fréchet differentiable norm then it is Asplund, but the converse is not true; see [106], for example. The Šmulyan criterion provides a method to construct an equivalent Fréchet differentiable norm on X: any equivalent norm on X is Fréchet differentiable provided that its dual norm on X^* is locally uniformly rotund (see [106]).

Definition (3.2.1)[104]: Let X be a Banach space endowed with a norm $\|.\|$ and let S_X denote its unit sphere. The norm $\|.\|$ is said to be locally uniformly rotund (LUR), if $\lim_k \|x - x_k\| = 0$ whenever $x, x_k \in S_X$ are such that $\lim_k \|x + x_k\| = 2$.

We study how close is the property of being the dual of an Asplund space to having an equivalent dual LUR norm. We shall need the following topological definitions. The first one has been introduced by Jayne and Rogers in [115].

Definition (3.2.2)[104]: Let (X, τ) be a topological space and let d be a metric on X. We say that X

(i) is fragmentable by d if for every $\varepsilon > 0$ and every nonempty $A \subset X$ there is $U \in \tau$ such that $A \cap U \neq \emptyset$ and $\operatorname{diam}(A \cap U) < \varepsilon$.
(ii) has property $P(d, \tau)$ if there is a sequence (A_n) of subsets of X, such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $\operatorname{diam}(A_n \cap U) < \varepsilon$.

Namioka and Phelps showed in [120] that a Banach space X is Asplund if and only if the unit ball of X^* endowed with the weak* topology is fragmented by the norm. They also showed [120] that if a dual Banach space X^* has an equivalent w^*-Kadec norm, that is, the weak* and the norm topologies agree on the unit sphere, then X is Asplund. Property P was introduced in [121] for pairs of topologies, but when stated as above it is equivalent to properties introduced and studied by Hansell [111] and Jayne, Namioka and Rogers [113]. The main result is the following theorem which says that dual LUR renormability of a dual space X^* is a nonlinear topological property.

Theorem (3.2.3)[104]: Let X^* be a dual Banach space. The following conditions are equivalent:

(i) X^* admits an equivalent dual LUR norm.
(ii) X^* admits an equivalent w^*-Kadec norm.
(iii) X^* has $P(\|.\|, w^*)$.

Statement (iii) above completes the characterizations of renormability given in [122]. Let us mention that there are no analogous results in Banach spaces for the weak topology. There exists a Banach space having a Kadec norm but with no equivalent strictly convex norm [102]. It is unknown whether every σ-fragmentable Banach space (in particular, if X has $P(\|.\|, w)$) has an equivalent Kadec norm [113].

We prove an interpolation result in the spirit of the results by Davis, Figiel, Johnson and Pelczyńfiski for Eberlein compaeta [108] and Namioka for Radon-Nikodý;m compacta [108], [119]. It can also be regarded as a "reciproque" of the transfer technique of Godefroy, Troyanski, Whitfield and Zizler for LUR renorming [110], [21].

Theorem (3.2.4)[104]: Let X be a Banach space, and let $K \subset X^*$ be a w^*-compact subset which has $P(\|.\|, w^*)$. Then there exists a Banach space Y such that Y^* has a dual LUR norm and a bounded linear operator $T: X \to Y$ with dense range such that $K \subset T^*(B_{Y^*})$.

A compact Hausdorff space is said to be a Radon-Nikodým compact if it is homeomorphic to a weak*-compact subset of a dual Banach space having the Radon

Nikodým property. A result of Namioka states that a weak*-compact subset of a dual Banach space X^* which is fragmented by the norm of X^* is a Radon- Nikodým compact. All these facts suggest that we introduce the following class of compact Hausdorff spaces.

Definition (3.2.5)[104]: A compact Hausdorff space K is called a Namioka-Phelps compact if it is homeomorphic to a weak*-compact subset of a dual Banach space having a dual LUR norm.

Clearly, any Namioka-Phelps compact space is Radon-Nikodým. Namioka characterizes internally the Radon-Nikodým compacta as those compact Hausdorff spaces which are fragmented by a lower semicontinuous metric. We will prove an analogous result.

Theorem (3.2.6)[104]: A compact Hausdorff space K is Namioka-Phelps if and only if it has property $P(d,\tau)$ with some τ-lower semicontinuous metric d.

If K is a Radon- Nikodým compact, then the space $C(K)$ is weak-Asplund, that is, every convex function on $C(K)$ is Gâteaux differentiable on a dense $\mathcal{G}_\delta$-set. Similarly, we obtain the following result.

Theorem (3.2.7)[104]: If K is a Namioka Phelps compact space, then $C(K)$ has an equivalent Gâteaux differentiable norm.

We study compact spaces having the property P with some metric showing the analogue with the properties of fragmentable compact spaces studied by Namioka in [119]. We prove the main result concerning the characterization of the existence of equivalent dual LUR norms in a dual Banach space. Finally, we study embedding properties of the Namioka-Phelps compact spaces.

A network of some topology is a family of subsets such that any open set is a union of subsets from that family. In [121] we introduced the property P for a couple of topologies. If X is a set and δ and τ topologies on X, we say that X has $P(\delta,\tau)$ if there is a sequence (A_n) of subsets of X such that for every $x \in X$ and every $V \in \delta$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U \subset V$. This property can be reformulated in terms of networks as follows: X has $P(\delta,\tau)$ if $\{A_n \cap U : n \in \mathbb{N}, U \in \tau\}$ is a network for δ. One can observe that this definition of property P extends Definition (3.2.2). We say that a topological space X has property $P(\delta,\tau)$ with τ -closed sets, if the sets $A_n \subset X$ can be taken τ -closed. The following is in [121].

Lemma (3.2.8)[104]: Suposse that a set X has $P(d,\tau)$ with a sequence of subsets (A_n). If the metric d is τ -lower semicontinuous, then X has $P(d,\tau)$ with the sequence $(\overline{A_n}^{\tau})$ In particular, X has $P(d,\tau)$ with τ -closed sets.

Proposition (3.2.9)[104]: Let X be a set, δ and τ two topologies on X. The following statements are equivalent:

(i) X has $P(\delta,\tau)$ with τ -closed sets.

(ii) There is a τ-1ower semicontinuous function $F: X \to \mathbb{R}$ such that for every net (x_ω) with $\tau\text{-}\lim_\omega x_\omega = x$ and $\lim\limits_{\omega} F(x_\omega) = F(x)$, then $\delta\text{-}\lim\limits_{\omega} x_\omega = x$.

A real function with the property stated in (ii) will be called a Kadec function.

Proof: (ii) ⇒ (i) For every $x \in X$ and every $V \in \delta$ with $x \in V$ there is $U \in \tau$ and $\varepsilon > 0$ such that if $y \in U$ and $|F(y) - F(x)| < \varepsilon$, then $y \in V$. Let (r_n) be an enumeration of the rational numbers. Define

$$A_n = \{y \in X: F(y) \leq r_n.\}.$$

The sets A_n are τ-closed because F is τ-lsc. We claim that X has $P(\delta, \tau)$ with the sequence A_n. Indeed, take rationals $r_m < F(x) < r_n$ and $r_n - r_m < \varepsilon$. Consider the τ-open set $U' = U \setminus A_m$. Then we have that

$$x \in A_n \cap U' \subset V,$$

which proves the claim.

(i) ⇒ (ii) Let Ξ_A be the characteristic function of the set A. Consider the series

$$F(x) = \sum_{n=1}^{\infty} 4^{-n} \Xi_{X \setminus A_n}(x).$$

It follows that F is τ-lsc. Let (x_ω) be a net with $\tau\text{-}\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$. We claim that $\delta\text{-}\lim_\omega x_\omega = x$. Indeed, a simple reasoning gives us that

$$\lim_\omega \Xi_{X \setminus A_n}(x_\omega) = \Xi_{X \setminus A_n}(x)$$

for every $n \in \mathbb{N}$. Now, for every δ-neighbourhood V of x there is n and $U \in \tau$ such that $x \in A_n \cap U \subset V$. Since $\Xi_{X \setminus A_n}(x_\omega)$ must be constant for ω big enough, we deduce that $x_\omega \in A_n$. Also, for ω big enough, $x_\omega \in U$. Thus $x_\omega \in V$. This shows the δ-convergence of (x_ω) to x. |

The following result compares with [119].

Corollary (3.2.10)[104]: Every weak compact subset of a Banach space has $P(\|\,.\,\|, \omega)$, and every Eberlein compact space is Namioka-Phelps.

Proof: Without loss of generality, we assume that $X = \overline{span}^{\|\,.\,\|}(K)$. Then the space X will have an equivalent LUR norm $|||.|||$, which is in particular a Kadec norm. Then apply Proposition (3.2.9). Any Eberlein compact space is isomorphic to a weak compact subset of a reflexive space, which has an equivalent LUR norm which clearly is dual.

A family of subsets of a topological space is said to be isolated if every point belonging to a subset of the family has a neighbourhood that does not meet another member of the family. A family of subsets is said to be σ-isolated if it is the union of a countable number of isolated families. Hansell studied in [111] the class of topological spaces having a σ-isolated network as a natural generalization of metrizable spaces; see also [118]. The following result is a consequence of the work of Hansell, and it shows the relation between fragmentability and property P.

Theorem (3.2.11)[104]: Let (K, τ) be a compact Hausdorff space and let d be a τ-lower semicontinuoas metric on K. The following are equivalent:

(i) K has (d, τ) .
(ii) d has a network which is σ-isolated with respect toτ.
(iii) τ has a σ-isolated network and d fragments (K, τ).

Proof: (i) ⇒ (iii) Any τ-lsc metric on K is finer than τ. Let $\mathfrak{B} = \bigcup_{m=1}^{\infty} \mathfrak{B}_m$ be a basis of d such that every family $\mathfrak{B}_m$ is d-discrete. Fix $n, m \in \mathbb{N}$ and $E \in \mathfrak{B}_m$. Define

$$H_E = \{x \in A_n: \exists U \in \tau\, s.t.\, x \in A_n \cap U \subset E\}.$$

It is easy to see that $\mathfrak{N}_{n,m} = \{H_E: E \in \mathfrak{B}_m\}$ is τ-isolated and $\mathfrak{N} = \bigcup_{n,m} \mathfrak{N}_{n,m}$ is a network of d. Since τ is coarser than d, we have that $\mathfrak{N}$ is a σ-isolated network of τ. On the other hand, it is shown in [121] that if K has $P(d, \tau)$, then (K, τ) is σ-fragmented by d. Since d is τ-Isc and (K, τ) is compact, a result from [114] states that (K, τ) is fragmented by d.

(iii)⇒ (ii) If (K, τ) has a σ-isolated network, then it is in particular hereditarily weakly θ-refinable, that is, every family of open sets in X has a σ-isolated (not necessary open) refinement. Hansell shows [111] that if a hereditarily weakly θ-refinable space is fragmented (or σ-fragmented) by some metric d, then the topology of d has a network $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ such that every $\mathfrak{N}_n$ is σ-isolated respect to τ.

(ii) ⇒ (i) If $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$, is a network of d such that every $\mathfrak{N}_n$ is σ-isolated with respect to τ, then it is easy to verify that K has $P(d, \tau)$ with the sequence of sets $A_n = \cup \mathfrak{N}_n$

Corollary (3.2.12)[104]: If X^* is a dual Banach space and $K \subset X^*$ is a w^*-compact subset having $P(\|.\|, w^*)$, then K is fragmentable by the norm. In particular, if X^* has $P(\|.\|, w^*)$, then X^* has the Radon- Nikodým property.

The following result compares with [119].

Theorem (3.2.13)[104]: Let (K_i, τ_i) be compact spaces for $i = 1, 2$ and let d_i be metrics on K_i. Suppose that there is a surjection $T: K_1 \to K_2$ such that T is $\tau_1 - \tau_2$ continuous and d_1-d_2 continuous. If K_1 has $P(d_1, \tau_1)$ with τ_1-closed sets, then K_2 has $P(d_2, \tau_2)$ with τ_2-closed sets.

Proof: If K_1 has $P(d_1, \tau_1)$ with τ_1-Closed sets, there is a τ_1-lsc function $F_1: K_1 \to [0, 1]$ with the Kadec property by Proposition (3.2.9). Define a function $F_2: K_2 \to [0, 1]$ as follows:

$$F_2(x) = inf\{F_1(x'): T(x') = x\}.$$

Since F_1 is τ_1-lsc, this infinmum is attained. We claim that F_2 is τ_2-1sc. Indeed, suppose that $\lim_{\omega} x_\omega = x$ in (K_2, τ_2) and $F_2(x_\omega) \leq r$ for every ω. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and $F_1(x'_\omega) = F_2(x_\omega)$. Let $x' \in K_1$ be a cluster point of (x'_ω). Since F_1 is τ_1-lsc we have that $F_1(x') \leq r$. On the other hand, by continuity, $T(x') = x$, so $F_2(x) \leq F_1(x')$. This shows that $F_2(x) \leq r$ and the claim is proved. We claim now that F_2 has the Kadec property and then the result will follow from Proposition (3.2.9). Suppose not, that is, there is a net (x_ω) in K_2 with τ_2-limit a point x such that $\lim_{\omega} F_2(x_\omega) = F_2(x)$, and there is $\varepsilon > 0$ such that $d_2(x_\omega, x) > \varepsilon$. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and $F_1(x'_\omega) = F_2(x_\omega)$. Let x' be a cluster point of (x'_ω).

Without loss of generality we can assume that (x'_ω) is τ_1-converging to x'. Clearly, we have that $T(x') = x$ and the following inequalities,

$$\lim_\omega F_2(x_\omega) = F_2(x) \leq F_1(x') \leq \lim_\omega F_1(x'_\omega) = \lim_\omega F_2(x_\omega).$$

We deduce that $\lim_\omega F_1(x'_\omega) = F_1(x')$. By the Kadec property of F_1 we have that $\lim_\omega d_1(x'_\omega, x') = 0$, and from the d_1-d_2 continuity of T we deduce that $\lim_\omega d_2(x_\omega, x) = 0$, which is a contradiction to our supposition.

Corollary (3.2.14)[104]: Let $T: X^* \to Y^*$ be a bounded linear operator between dual spaces which is $w^* - w^*$ continuous. If $K \subset X^*$ is a w^*-compact subset having $P(\|.\|, w^*)$, then $T(K)$ has $P(\|.\|, w^*)$ in Y^*.

The following result compares with [119].

Lemma (3.2.15)[104]: Let (K_n, τ_n) be compact spaces for $i \in \mathbb{N}$ and let d_n be a metric on K_n such that K_n has $P(d_n, \tau_n)$ with τ_n-closed sets for every $n \in \mathbb{N}$. Let τ be the topology product of the τ_n-topologies on $K = \prod_{n=1}^{\infty} K_n$ and let d be a metric compatible with the product of the d_n-topologies on K. Then K has $P(d, \tau)$ with τ-closed sets.

Proof: Take for every $n \in \mathbb{N}$ a τ_n,-lsc Kadec function $F_n: K_n \to [0,1]$. An easy lower semicontinuity argument shows that

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} F_n(x_n)$$

is a τ-lsc Kadec function on K linking d with τ, where $x = (x_n)$.

The following result can be regarded as a topological version of the transfer technique of Godefroy-Troyanski-Whitfield-Zizler [110], [106].

Theorem (3.2.16)[104]: Let (X, τ) be a topological space and let d be a τ-lower semicontinuous metric on X. Suppose that there exist τ-compacts sets $K_n \subset X$ having $P(d, \tau)$ such that $\overline{\bigcup_{n=1}^{\infty} K_n}^d = X$. Then X has $P(d, \tau)$.

Proof: We can suppose the sequence (K_n) increasing and the metric d bounded. By Proposition (3.2.9), for every $n \in \mathbb{N}$ there is a τ-lsc Kadec function $F_n: K_n \to [0,1]$. We define the functions f_n on X as follows,

$$f_n(x) = inf\{d(x, y) + F_n(y): y \in K_n\}.$$

Note that the infimum is attained. We claim that f_n is τ-lsc. Indeed, suppose that τ - $\lim_\omega x_\omega = x$ and $f_n(x_\omega) \leq r$ for every ω. Take points $y_\omega \in K_n$ such that $f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega)$. Let $y \in K_n$ be a cluster point of (y_ω). Then we have that

$$f_n(x) \leq d(x, y) + F_n(y) \leq r$$

because of the lower semicontinuity of d and F_n. Now we define a function F on X by the formula

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

We claim that F has the Kadec property. Indeed, suppose not. We can take a net (x_ω) in X with τ -limit a point x such that $\lim_\omega F(x_\omega) = F(x)$, and there is $\varepsilon > 0$ such that $d(x_\omega, x) > \varepsilon$. A standard argument of lower semicontinuity gives that $\lim_\omega f_n(x_\omega) = f_n(x)$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $1/n < \varepsilon/3$ and $d(x, K_n) < \varepsilon/3$. We can take points $y_\omega \in K_n$ such that

$$f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega).$$

Let $y \in K_n$ be a cluster point of (y_ω). Without loss of generality we can assume that τ-$\lim_\omega y_\omega = y$. Since

$$d(x,y) + F_n(y) \le \lim_\omega f_n(x_\omega) = f_n(x) \le d(x,y) + F_n(y)$$

we have that

$$\lim_\omega [d(x_\omega, y_\omega) + F_n(y_\omega)] = d(x,y) + F_n(y).$$

Using the lower semicontinuity, we deduce that $\lim_\omega d(x_\omega, y_\omega) = d(x,y) < \varepsilon/3$ and $\lim_\omega F_n(y_\omega) = F_n(y)$. The last equality gives that $\lim_\omega d(y_\omega, y) = 0$, thus for ω big enough we have that $d(x_\omega, y_\omega) < \varepsilon/3$ and $d(y_\omega, y) < \varepsilon/3$. Since $d(x,y) < \varepsilon/3$ we have that $d(x_\omega, x) < \varepsilon$, which is a contradiction.

Given a Banach space X, a bounded subset $Z \subset X^*$ is said to be norming if there is $\lambda > 0$ such that $\lambda\|x\| \le sup\{|x^*(x)|: x^* \in Z\}$ for all $x \in X$. Notice that the supremun defines an equivalent norm on X which is lower semicontinuous for the topology of convergence on Z, denoted $\sigma(X,Z)$. A linear subspace $Z \subset X^*$ is said to be norming if $B_{X^*} \cap Z$ is a norming subset.

The following result compares with [119].

Proposition (3.2.17)[104]: Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X^*$be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|, \sigma(X,Z))$. Then $\overline{span}^{\|.\|}(K)$ and $\overline{aco}^{\sigma(X,Z)}(K)$ have $P(\|.\|, \sigma(X,Z))$.

Proof: Let $I(n,m) = [-m,m] \times \ldots \times [-m,m]$ (n times) with the usual topology of $\mathbb{R}^n$. Let $K_{n,m} = I(m,n) \times K \times \ldots \times K$ (n times). If τ is the product topology when K is endowed with $\sigma(X,Z)$, then $K_{n,m}$ is τ-compact. If K is endowed with the norm topology, then the product topology is metrized by a metric that we call d. By Lemma (3.2.15), $K_{n,m}$ has $P(d,\tau)$. The map $T_{n,m}$ from $K_{n,m}$ to X defined by $T_{n,m}(\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$ is clearly τ-$\sigma(X,Z)$ continuous and d-$\|.\|$ continuous, thus every $\sigma(X,Z)$-compact set $T_{n,m}(K_{n,m})$ has $P(\|.\|, \sigma(X,Z))$ by Theorem (3.2.13). Since $\text{span}(K) = \bigcup_{n,m} T_{n,m}(K_{n,m})$, we have that $\overline{span}^{\|.\|}(K)$ has $P(\|.\|, \sigma(X,Z))$by Theorem (3.2.16). The result for the $\sigma(X,Z)$- closed absolutely convex hull follows from the fact that $\overline{aco}^{\sigma(X,Z)}(K) = \overline{aco}^{\|.\|}(K)$ because K is fragmentable by the norm; see [119].

The following result compares with [119].

Theorem (3.2.18)[104]: If K is a Namioka-Phelps compact, then $(B_{C(K)^*}, w^*)$ is also a Namioka-Phelps compact.

Proof. In the proof of [119] it is shown that if K is a Radon-Nikodým compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*-continuous linear operator $T: C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is moreover Namioka-Phelps, then $T(K)$ has $P(\|.\|, w^*)$. Then $T(B_{C(K)^*}) = \overline{aco}^{\|.\|}(T(K))$ has $P(\|.\|, w^*)$ by Proposition (3.2.17), and thus $T(B_{C(K)^*})$ is Namioka-Phelps.

A dual Banach space X^* having a dual LUR norm has the Radon- Nikodým property. The space $C[0, w_1]$ shows that the converse is not true. Fabian and Godefroy proved [FG] that a dual Banach space with the Radon- Nikodým property has an equivalent LUR norm (not necessarily dual, of course). The LUR norm in that case can be made a dual norm under additional hypothesis, e.g., if the predual X is WCD, or the space X^* is itself WCD; see [106]. Following Hansell [111], we say that a dual Banach space X^* is dual-descriptive if it has the Radon -Nikodým property and the weak* topology has a σ-isolated network. The class of dual-descriptive spaces coincides with the dual Banach spaces having a countable cover by sets of local small diameter in the sense of Jayne, Namioka and Rogers [113]. A dual Banach space with a w^*-Kadec norm is dual-descriptive [111]. Our main result states that the existence of an equivalent dual LUR norm is a topological property. Partial results in this direction were obtained in [122], in the spirit of the Moltó-Orihuela-Troyanski characterization of LUR renormability [117].

Theorem (3.2.19)[104]: Let X^* be a dual Banach space. The following conditions are equivalent:

(i) X^* admits an equivalent dual LUR norm.
(ii) X^* admits an equivalent norm such that weak topology and the weak* topology coincide on its unit sphere.
(iii) X^* is dual-descriptive.
(iv) X^* (resp. S_{X^*}) has $P(\|.\|, w^*)$.

Proof: (i) ⟺ (ii) It is proved in [122].

(i) ⇒ (iv) It follows from Proposition (3.2.9).

(iv) ⟺ (iii) It follows from Theorem (3.2.11).

(iv) ⇒ (ii) If a dual Banach space X^* has $P(\|.\|, w^*)$, then X^* has the Radon -Nikodým property, by Corollary (3.2.12). A result from [121] establishes that there is a w^*-lower semicontinuous real function F on $X^*$with $\|.\| \leq F(.) \leq 3\|.\|$ such that the norm and the w^*-topology coincide on the set $S = \{x^* \in X^*: F(x^*) = 1\}$. Let $= \{x^* \in X^*: F(x^*) \leq 1\}$. Since X^* has the Radon-Nikodým property, $\overline{co}^{\|.\|}(K)$ will be a w^*-compact set, symmetric and with nonempty norm interior, that is the unit ball of some equivalent dual norm on X^*.Without loss of generality we can suppose X^* endowed with that norm, namely $B_{X^*} = \overline{co}^{\|.\|}(K)$. We will show that the norm and the w^*-topology coincide on S_{X^*}.

Suppose not, that is, there is some $\varepsilon > 0$ and some net (x_ω^*) in S_{X^*} w^*- converging to a point $x^* \in S_{X^*}$ such that $\|x_\omega^* - x^*\| > \varepsilon$. Take Radon probabilities μ_ω on K such that $x_\omega^* = \int_K \mathbb{I} d\mu_\omega$ (integrals are taken in the sense of Bochner, see [107]). Without loss of generality we can suppose that (μ_ω) converges in $(C(K)^*, w^*)$ to a Radon probability μ on K. We must have that $x^* = \int_K \mathbb{I} d\mu$.

Since $\|x_\omega^*\| = \|x^*\| = 1$, we have that μ_ω and μ are supported by $S_{X^*} \cap K \subset S$. We can take disjoint norm compact sets $K_i \subset S$ for $i = 1, \ldots, n$ with diameter n less than $\varepsilon/7$ such that $\mu(\cup_{i=1}^n K_i) > 1 - \varepsilon/12$. We can take a norm compact set $K_0 \subset S$ disjoint from $\cup_{i=1}^n K_i$ such that $\mu(\cup_{i=0}^n K_i) > 1 - \varepsilon/12n$. Take disjoint norm open sets V_i for $i = 0, \ldots, n$ with $K_i \subset V_i$ and the diameter of V_i for $i = 1, \ldots, n$ less that $\varepsilon/6$. Since the norm and the w^*-topology coincide on S, we can take w^*-open sets U_i such that $U_i \cap S = V_i \cap S$.

By Urysohn's Lemma, we can take w^*-continuous functions f_i for $i = 0, \ldots, n$ from B_{X^*} to $[0,1]$ such that $f_i|_{K_i} = 1$ and $f_i|_{X^* \setminus U_i} = 0$. Since $\int_K f_i d\mu_\omega$ converges to $\int_K f_i d\mu \geq \mu(K_i)$ for $i = 0, \ldots, n$ we will have, for ω big enough, that $\mu_\omega(V_i) = \mu_\omega(U_i) \geq \int_K f_i d\mu_\omega > \mu(K_i) - \varepsilon/12n^2$ for $i = 0, \ldots, n$. On the other hand, we must have $\mu_\omega(V_i) < \mu(K_i) + \varepsilon/6n$. If not, then $\mu_\omega(V_j) \geq \mu(K_j) + \varepsilon/6n$ for some j. Summing the above inequalities for $i \neq j$ we will have $\mu_\omega(\cup_{i=0}^n V_i) > \mu(\cup_{i=0}^n K_i) + \varepsilon/6n - n\varepsilon/12n^2 > 1 - \varepsilon/12n + \varepsilon/6n - \varepsilon/12n = 1$ which is a contradiction. Thus we have that $|\mu_\omega(V_i) - \mu(K_i)| < \varepsilon/6n$ and $\mu_\omega(\cup_{i=1}^n V_i) > 1 - \varepsilon/6$.

Fix any $i = 1, \ldots, n$. We can take points $x_1^*, x_2^* \in \overline{co}^{\|.\|}(V_i)$ such that $\mu(K_i)x_1^* = \int_{K_i} \mathbb{I} d\mu$, and $\mu_\omega(V_i)x_2^* = \int_{V_i} \mathbb{I} d\mu_\omega$. Since the diameter of V_i is less than $\varepsilon/6$, then $\|x_1^* - x_2^*\| \leq \varepsilon/6$. We have that

$$\left\| \int_{V_i} \mathbb{I} d\mu_\omega - \int_{K_i} \mathbb{I} d\mu \right\| = \|\mu_\omega(V_i)x_2^* - \mu(K_i)x_1^*\|$$
$$\leq |\mu_\omega(V_i) - \mu(K_i)|. \|x_2^*\| + \mu(K_i)\|x_1^* - x_2^*\| \leq (1\backslash n + \mu(K_i)) \left(\frac{\varepsilon}{6}\right).$$

We will show that $\|x_\omega^* - x^*\| < \varepsilon$ to get the final contradiction

$$\|x_\omega^* - x^*\| = \left\| \int_K \mathbb{I} d\mu_\omega - \int_K \mathbb{I} d\mu \right\|$$
$$\leq \left\| \int_{K \setminus \cup_{i=1}^n V_i} \mathbb{I} d\mu_\omega - \int_{K \setminus \cup_{i=1}^n K_i} \mathbb{I} d\mu \right\| + \sum_{i=1}^n \left\| \int_{V_i} \mathbb{I} d\mu_\omega - \int_{K_i} \mathbb{I} d\mu \right\|$$
$$\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{12} + \sum_{i=1}^n \left(\frac{1}{n} + \mu(K_i)\right) \left(\frac{\varepsilon}{6}\right) < \varepsilon$$

This shows that the norm $\|.\|$ is ω^*-Kadec. .

Corollary (3.2.20)[104]: Let X be a Banach space such that its dual X^* satisfies any of the statements of Theorem (3.2.19). Then X has an equivalent Fr$\acute{e}$chet differentiable norrll.

Let X be an Asplund Banach space. We shall consider the following construction on its dual X^*. For any weak*-compact convex subset $B \subset X^*$ and $\varepsilon > 0$, take

$$(B)'_\varepsilon = \{x^* \in B: \forall U \in w^*, x^* \in U, diam(B \cap U) > \varepsilon\}.$$

Define by transfinite induction the sets (B_ε^α) as follows:

$$B_\varepsilon^0 = B_{X^*},$$

$$B_\varepsilon^{\alpha+1} = (B_\varepsilon^\alpha)'_\varepsilon$$

and, for α a limit ordinal,

$$B_\varepsilon^\alpha = \bigcap_{\beta<\alpha} B_\varepsilon^\beta .$$

Now take $Sz(X,\varepsilon) = inf\{\alpha: B_\varepsilon^\alpha = \emptyset$, and $Sz(X) = \sup_{\varepsilon>0}\delta_z(X,\varepsilon)\}$. The ordinal number $Sz(X)$ is called the Szlenk index of X. The following result was proved by Lancien [116] using a Kunen-Martin type argument.

Corollary (3.2.21)[104]: (Lancien): If X is a Banach space with $Sz(X) < \omega_1$, then X^* has an equivalent dual LUR norm.

The following is a transfer result for LUR renorming of Godefroy –Troyanski-Whitfield-Zizler [110], [106]. A topological version of it is Theorem (3.2.16).

Theorem (3.2.22)[104]: (Godefroy, Troyanski, Whitfield & Zizler): Let X be a Banach space, let $Z \subset X^*$ be a norming subset, let Y^* be a dual Banach space having a dual LUR norm and let $T: Y^* \to X$ be a bounded linear operator w^*-$\sigma(X,Z)$ continuous. Then X has an equivalent $\sigma(X,Z)$-1ower semicontinuous norm which is LUR at the points of $\overline{T(Y^*)}^{\|.\|}$.

We shall prove the following interpolation result in the spirit of the Davis-Figiel-Johnson-Pelczy$\acute{n}$ski Theorem, that can be regarded as a reciprocal of Theorem (3.2.22).

Theorem (3.2.23)[104]: Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X$ be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|,\sigma(X,Z))$. Then there exists a dual Banach space Y^* having a dual LUR norm and a bounded one-to-one linear operator $T: Y^* \to X$ which is w^*- $\sigma(X,Z)$ continuous such that $K \subset T(B_{Y^*})$.

Proof: After Lemma (3.2.17) we know that $K_0 = \overline{aco}^{\|.\|}(K)$ is an absolutely convex compact set with $P(\|.\|,\sigma(X,Z))$. Thus K_0 is fragmented by the norm. Following Namioka [119], there is an Asplund space Y and a bounded injective linear operator $T: Y^* \to X$ which is w^*- $\sigma(X,Z)$ continuous such that $K_0 \subset T(B_{Y^*}) \subset 2^n K_0 + B[0,1/2^n]$ for every $n \in \mathbb{N}$. By Theorem (3.2.16) we have that $T(B_{Y^*})$ will be a descriptive $\sigma(X,Z)$-compact subset of X. Since T is a homeomorphism when restricted to B_{Y^*}, we deduce that

(B_{Y^*}, w^*) has a σ-isolated network. Thus Y^* is dual-descriptive and it has an equivalent dual LUR norm by Theorem (3.2.19).

Corollary (3.2.24)[104]: Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X$ be a bounded $\sigma(X, Z)$-compact subset which has $P(\|.\|, \sigma(X, Z))$.

Then X has an equivalent $\sigma(X, Z)$-lower semicontinuous norm which is LUR at the points of $\overline{span}^{\|.\|}(K)$.

Proof: Apply Theorems (3.2.23) and (3.2.22). I

The following extends a well-known result of Deville [105] concerning the dual LUR renorming of $C(K)^*$, where K is a scattered compact space such that $K^{(\omega_1)} = \emptyset$; see also [106].

Corollary (3.2.25)[104]: Let K be a Hausdorff compact space. The following are equivalent:

(i) $C(K)^*$ has an equivalent dual LUR norm.
(ii) K is a countable union of relatively discrete subsets.

Proof: Suppose that $C(K)^*$ has an equivalent dual LUR norm. Then $C(K)^*$has $P(\|.\|, w^*)$ and, in particular, K has $P(\|.\|, w^*)$ with some sequence of subsets (A_n). The following sets

$$D_n = \{x \in A_n : \exists U \in w^*, x \in U, diam(A_n \cap U) < 1\}$$

are relatively discrete and cover K. Conversely, assume that K is a countable union of relatively discrete subsets. Then it is easy to see that K has $P(d, \tau)$ where d is the discrete metric. By Theorem (3.2.11), K is d-fragmentable, so K must be scattered. Regarding K as a subset of $C(K)^*$, it has $P(\|.\|, w^*)$ and $C(K)^* = \overline{span}^{\|.\|}(K)$ has an equivalent dual LUR norm by Corollary (3.2.24).

Proposition (3.2.26)[104]: Let K be Hausdorff compact space. The following statements are equivalent:

(i) K is Namioka-Phelps.
(ii) There is a lower semicontinuous metric d such that K has $P(d, \tau)$.
(iii) K is Radon-Nikodým and it has a σ-isolated network.

Proof: (i) ⟺ (iii) ⇒ (ii) It is clear after Theorem (3.2.11).

(ii) ⇒ (i) Let d be a lower semicontinuous metric on (K, τ) such that K has $P(d, \tau)$. There is a dual space X^* containing K as w^*-compact subset in such a way that the metric d is induced by the norm [112]. Then the result will follow from Theorem (3.2.23).

Theorem (3.2.27)[104]: Let K be a Namioka-Phelps compact space. Then $C(K)^*$has an equivalent W^*LUR norm. In particular, $C(K)$ has an equivalent Gâteaux differentiable norm.

Proof: The proof of [119] shows that if K is a Radon-Nikodým compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*- continuous linear operator

$T: C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is Namioka-Phelps, we have that $T(K)$ has $P(\|.\|, w^*)$. By Corollary (3.2.24), we can suppose that X^* is endowed with a dual norm which is LUR at the points of $T(C(K)^*)$. Define an equivalent dual norm $\|.\|$ on $C(K)^*$ by the formula $|||x|||^2 = \|x\|^2 + \|T(x)\|^2$. We claim that $\|.\|$ is W^*LUR. To see that, take points x, x_n in $C(K)^*$ with $|||x||| = |||x_n||| = 1$ and $\lim_n |||x_n + x||| = 2$. By a standard convexity argument [106], we have that $\lim_n \|T(x_n)\| = \|T(x)\|$ and $\lim_n \|T(x_n) + T(x)\| = 2\|T(x)\|$. Since $\|.\|$ is LUR at $T(x)$, we have that $\lim_n \|T(x_n) + T(x)\| = 0$. In particular, $T(x_n)$ is w^*-convergent to $T(x)$, and hence (x_n) is w^*-convergent to x because of the w^*-continuity of T^{-1} on $T(B_{C(K)^*})$.

Section (3.3): WUR Banach Spaces

Our motivation for the present work was two questions posed to us, in Paseky's conference (2004), by G. Godefroy and V. Zizler. We suspected that one of the examples of [125] is a possible candidate for answering both questions. Furthermore, discussing with S. Troyanski during his visit in Athens, we realized that Zizler's question is closely related to a problem posed by M. Fabian, G. Godefroy, P. Hájek and V. Zizler [130]. Thus we show that the second example of [125] answers negatively the following two questions.

Q1. Let X be a Banach space with a WUR norm. Does there exist a bounded, linear, one-to-one operator $\Phi : X \to c_0(\Gamma)$, for some set Γ?

Q2. Let X be a Banach space such that X is a subspace of a WCG and also there exists a norm-one projection $P : X^{**} \to X$. Is then X a WCG space?

The example from [125] answering the aforementioned questions is a subspace Y of a Banach space X with the following properties.

(i) The space X is WCG and it does not contain ℓ_1.
(ii) Both spaces X and Y are duals, $X^{**} = X \oplus \ell_2(\Gamma)$ and $Y^{**} = Y \oplus \ell_2(\Gamma)$. In particular X^{**} is WCG.
(iii) The space Y is not WCG and X/Y is reflexive.

The space X is of the form $(\sum_{n=1}^{\infty} \oplus J(T_n))_2$, where $(T_n)_n$ is the remarkable Rezničenko sequence of trees. This is a sequence of trees with each T_n of height ω and which satisfy a strong Baire property. The original construction of $(T_n)_n$ was based on a transfinite (for $\xi < \omega_1$) recursive argument. We provide a new construction with the use of a coding function σ. Each T_n consists of all σ-admissible sequences with first element the natural number n, ordered by the initial segment inclusion. It is worth pointing out that the space X is also a James tree space with $T = \bigcup_{n=1}^{\infty} T_n$, which shows that the WCG $J(T)$ spaces are not hereditarily WCG. The following is the key property for our results.

Proposition (3.3.1)[123]: Let Y be the subspace of X mentioned before. Then there is no $\Phi : Y^* \to c_0(\Gamma)$ linear, one-to-one and bounded.

This proposition in conjunction with the property that X^{**} is Hilbert-generated yields a negative answer to Q1. Let us recall that if E admits an equivalent WUR norm, then E^* is a subspace of a WCG ([128]). In particular, if E is isomorphic to Y^* for some Banach space , then Y could not contain ℓ_1. This actually shows that any example Y^* answering in negative Q1, should satisfy the following properties. First ℓ_1 does not embed in Y and second, Y^{**} is a non-WCG subspace of a WCG Banach space. Namely the space Y must satisfy the basic properties of the example presented here.

We start with the construction of the sequence $(T_n)_n$ mentioned above. First we fix a well ordering $\prec$ of the set $\mathbb{R}$ of real numbers.

Let $\{I_\alpha : \alpha < c\}$, with $|I_\alpha| = c$ for $\alpha < c$, be a disjoint family of subsets of the set $\mathbb{R} \setminus \mathbb{N}$, where c denotes the cardinality of the continuum. We denote by $\mathcal{L}$ the set of all sequences $\vec{s} = (s_1, s_2, \ldots)$ with the following properties:

(i) for every $k \in \mathbb{N}$, $s_k = (t_0, t_1, \ldots, t_{d_k})$, where $t_0 \in \mathbb{N}, d_k \geq 0, t_i \in \mathbb{R} \setminus \mathbb{N}$ for $1 \leq i \leq d_k$, $t_i \neq t_j$ for $1 \leq i < j \leq d_k$ and

(ii) $s_k \cap s_m = \emptyset$ for $k < m$.

Fix a one-to-one mapping $\sigma : \mathcal{L} \to [0, c)$, where $[0, c)$ is the interval of all ordinals smaller than c.

Definition (3.3.2)[123]: A finite sequence $(t_0, t_1, \ldots, t_d)$, where $t_0 \in \mathbb{N}, d \geq 1$, $t_i \in \mathbb{R} \setminus \mathbb{N}$ for $1 \leq i \leq d$, is said to be σ-admissible if $t_0 \prec t_1 \prec \cdots \prec t_d$ and for all $i = 1, 2, \ldots, d$, there exists $\vec{s}_i \in \mathcal{L}$ such that $(t_0, t_1, \ldots, t_{i-1}) \in \vec{s}_i$ and $t_i \in I_{\sigma(\vec{s}_i)}$.

Define for every $k \in \mathbb{N}$ a partial order $<_k$ in $\mathbb{R}$ as follows:

If $t, s \in \mathbb{R}$, then $t <_k s$ iff there exist a σ–admissible sequence $(t_0, t_1, \ldots, t_d)$ with $t_0 = k$ and $0 \leq i < j \leq d$ such that $t = t_i$ and $s = t_j$.

Set $T_k = \{t \in \mathbb{R} \setminus \mathbb{N} : k <_k t\} \cup \{k\}$ for $k \in \mathbb{N}$. Then the sequence of partially ordered sets $(T_k, <_k)$, $k \in \mathbb{N}$, has the properties of a sequence of Rezničenko trees (see also Proposition 3.2 in [125]). In fact we have the following

Theorem (3.3.3)[123]: (i) For every $k \in \mathbb{N}$, the partially ordered set $(T_k, <_k)$ is a tree of height ω with root k.

(ii) If $k_1 \neq k_2$ and I_i is a segment of T_{k_i}, $i = 1, 2$, then $|I_{k_1} \cap I_{k_2}| \leq 1$.

(iii) For every non empty subset M of $\mathbb{N}$ and I_n initial segment of T_n, $n \in M$, such that $I_n \cap I_m = \emptyset$ for $n \neq m$, there exist uncountable many $t \in \mathbb{R} \setminus \mathbb{N}$ such that $t \in S^n_{\max I_n}$, for all $n \in M$, (where for $t \in T_k$ we denote by S^k_t the set of all immediate successors of t in the tree T_k).

Proof: (i) Let us observe that the definition of the σ-admissible sequences yields that for any $k \in \mathbb{N}$ and every pair $(k = t_0, t_1, \ldots, t_{d_1}), (k = s_0, s_1, \ldots, s_{d_2})$ of σ- admissible sequences, there exists $0 \leq i_0 \leq min\{d_1, d_2\}$ such that for all $i \leq i_0$ we have $t_i = s_i$ and the sets $\{t_{i_0+1}, \ldots, t_{d_2}\}, \{s_{i_0+1}, \ldots, s_{d_2}\}$ are disjoint. This shows that $(T_k, <_k)$ is a tree of height ω.

(ii) By (i), it is enough to show the property only for initial segments. Let $k_1 \neq k_2$ and $(k_1, t_1, \ldots, t_{d_1}), (k_2, s_1, \ldots, s_{d_2})$ be σ-admissible sequences. Assume that

$|\{k_1, t_1, \ldots, t_{d_1}\} \cap \{k_2, s_1, \ldots, s_{d_2}\}| \geq 2$. Namely, there exist $1 \leq i_1 < i_2 \leq d_1$ and $1 \leq j_1 < j_2 \leq d_2$ such that $\{t_{i_1}, t_{i_2}\} = \{s_{j_1}, s_{j_2}\}$. Since $t_{i_1} \prec t_{i_2}$ and $s_{j_1} \prec s_{j_2}$ for the fixed well ordering $\prec$ of $\mathbb{R}$, we conclude that $t_{i_1} = s_{j_1}$ and $t_{i_2} = s_{j_2}$. This yields a contradiction since the σ-admissible sequences $(k_1, t_1, \ldots, t_{i_2-1}), (k_2, s_1, \ldots, s_{j_2-1})$ have common σ-extension although they are not disjoint. (iii) It follows immediately from the definitions of the function σ and the σ- admissible sequences.

Any sequence of trees $T_k, k \in \mathbb{N}$, satisfying the assertions (i) to (iii) of the above theorem is called a sequence of Rezni$\check{c}$enko trees. As it is shown in [125] (Proposition 3.3), any sequence of Rezni$\check{c}$enko trees satisfies a sort of Baire category property. To this end we need the following definition.

Definition (3.3.4)[123]: Let T be a tree. A subset D of T is said to be successively dense in T if there exists $t_0 \in T$ such that for every $t \in T$ with $t_0 \leq t$ we have $D \cap S_t \neq \emptyset$.

Let us point out that if T has the additional property that for each $t \in T$ $S_t \neq \emptyset$, then every successively dense subset D of T must contain an infinite segment. Under the above terminology we have the following fundamental property of Rezni$\check{c}$enko sequences of trees.

Theorem (3.3.5)[123]: Let $T_n, n \geq 1$ be any sequence of Rezni$\check{c}$enko trees, so that each T_n has as a root the number $n \in \mathbb{N}$ and $T = \bigcup_{n=1}^{\infty} T_n$. If $D_n, n \geq 1$ is any sequence of subsets of T with $T = \bigcup_{n=1}^{\infty} D_n$ then there exists $k_0 \in \mathbb{N}$ such that the set D_{k_0} is successively dense in T_{k_0}. In particular, there exists $t_0 \in S_{k_0}^{k_0}$ such that for every $t \in T_{k_0}$ with $t_0 \leq_{k_0} t$ we have $|S_t^{k_0} \cap D_{k_0}| \geq \omega_1$.

The proof follows the arguments of [125], [124].

Theorem (3.3.6)[123]: There exists a WCG Banach space X such that X^{**} is also WCG not containing ℓ_1. Moreover there exists a closed subspace Y of X such that:

(a) the spaces Y and Y^{**} are not WCG;

(b) the quotient X/Y is a reflexive space.

We first recall the definition of a James space $J(T)$, for a given tree $(T, \leq)$. So $J(T)$ is the completion of the linear space $c_{00}(T)$ of finitely supported real functions on T under the norm

$$\|x\|_{J(T)} = sup\left\{\sum_{i=1}^{n}\left(\sum_{t\in S_i} x(t)^2\right)^{1/2} : S_1, \ldots, S_n \; are\; disjoint\; segments\; of\; (T, \leq)\right\}.$$

The space X in the above theorem is of the form $(\sum_{m=1}^{\infty} \oplus X_m)_2$, where X_m is the James space $J(T_m \times \{m\})$ and $T_m, m \geq 1$, is a sequence of Rezni$\check{c}$enko trees. Since each tree T_m is of height ω, each X_m has the following properties:

(i) X_m is a WCG, $X_m \cong Z_m^*$ and $X_m^*/Z_m \cong \ell^2(B_m)$, where Z_m is the closed linear span of the set $\{e_{(t,m)}^* : t \in T_m\}$ in X_m^* and $\mathcal{B}_m$ the set of branches of the tree T_m (clearly Z_m is a WCG, since the set $\{e^*_{(t,m)} : t \in T_m\} \cup \{0\}$ is weakly compact in X^*_m).

Using properties of Dixmier's projection $P_m : Z^{***}_m \to Z^*_m$ we find that,

(ii) $X^{**}_m \cong X_m \oplus \ell_2(\mathcal{B}_m)$ (cf.[129]).

Set $Z = (P\infty\, m = 1 \oplus Zm)2$. Then using properties (i) and (ii) (and Dixmier 's projection $P : Z *** \to Z *$) we get that,

(iii) $X \cong Z^*$,$X^*/Z \cong \ell_2(\mathcal{B})$ and $X^{**} \cong X \oplus \ell_2(\mathcal{B})$, where $B = \bigcup_{m=1}^{\infty} \mathcal{B}_m$. It follows in particular that X is complemented in X^{**} by Dixmier' s projection $P : X^{**} \to X$.

We notice that, it follows for the definition of X and properties (i) and (iii) that both of the spaces X and $X^{**}$are WCG. These spaces have the additional property to be Hilbert-generated. We recall that a Banach space Z is Hilbert generated if there exists a bounded linear operator from a Hilbert space onto a dense subspace of Z (see [130]).

Lemma (3.3.7)[123]: The spaces X and $X^{**}$are Hilbert generated.

Proof: It clearly follows from the definition of X and property (iii) that it is enough to show that each James space $Z = J(T)$, where T is any tree of height ω, is Hilbert-generated. Indeed, let $T(n)$ be the n-th level of $T, n \geq 0$. Then $T = \bigcup_{n=0}^{\infty} T(n)$ and each of the subspaces $Z_n = \overline{span}\{e_t : t \in T(n)\}$ of Z is isometric to the Hilbert space $\ell_2(T(n))$. Since the union of $\bigcup_{n=0}^{\infty} Z_n$ generates Z, it is easily verified that the operator $F : \ell_2(T) \to Z$ defined by $F(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^n}$, where $x_n = x|_{T(n)}$ for $x \in \ell_2(T)$ and $n \geq 0$, makes Z a Hilbert-generated space.

The space Y is defined as follows: for every $t \in T = \bigcup_{m=1}^{\infty} T_m$, set

$$D_t = \{m \in \mathbb{N} : t \in T_m\} \;\; and \;\; x_t = \sum_{m \in D_t} \frac{1}{2^{m/2}} e_{(t,m)} \, .$$

Finally set, $Y = \overline{span}\{x_t : t \in T\} \subset X$. Then the following facts can be proved (see [125]).

(i) There exists a family $\{f_t : t \in T\} \subset Y^*$ so that the family $\{(x_t, f_t) : t \in T\}$ is an M-basis for Y , where for $t \in T$ and $m \in D_t$, $f_t = 2^{\frac{m}{2}} I^*(e^*_{(t,m)})$ and $I : Y \to X$ is the natural embedding of Y into X.

(ii) Let $m \in \mathbb{N}$ and $b = \{t_1 < \ldots < t_n < \ldots\}$ be any branch of the tree T_m . Then the series $\sum_{k=1}^{\infty} f_{t_k}$ is weak$*$ convergent in Y^* .

Facts (i) and (ii) together imply that Y is not WCG.

(iii) $Y^{**} \cong Y \oplus \ell_2(\mathcal{B})$.

Since Y is not a WCG, it clearly follows from fact (iii) that neither Y^{**} is WCG. The following lemma is the analogue for trees T of height ω of a known property of the James tree space [132].

Lemma (3.3.8)[123]: The space Y is complemented in Y^{**} by a norm-one projection and hence it is a dual space of a WCG space Y_0 (having a shrinking M-basis).

Proof: Let $P : X^{**} \cong X \oplus \ell_2(\mathcal{B}) \to X$ be Dixmier's projection and $y^{**} \in Y^{**} \subset X^{**}$. Then $y^{**} = y + w$, where $y \in X$ and $w \in \ell_2(\mathcal{B})$. Since from fact (iii), $Y^{**} \cong Y \oplus \ell_2(\mathcal{B})$ we find that $X \cap Y^{**} = Y$, so $y = y^{**} - w \in X \cap Y^{**} = Y$. Therefore the restriction of P on the subspace Y^{**} of X^{**} is a norm-one projection of Y^{**} onto Y.

We define Y_0 to be the closed linear span of the set $\{f_t : t \in T\}$ in the space Y^*. We shall show that $Y_0^{**} \cong Y$. So we define the operator $F : Y \to Y_0^*$ by $(y) = y_{|Y_0}$. It is clear that F is a well defined linear bounded ($\|F(y)\| \leq \|y\|$) operator and since the family $\{f_t : t \in T\}$ separates the points of Y it is also one-to-one.

Let $g \in Y_0^*$. Then by Hahn-Banach theorem there exist $\hat{g} \in Y^{**} : \hat{g}_{|Y_0} = g$ and $\|g\| = \|\hat{g}\|$. Set $P(\hat{g}) = y$; then clearly $y \in Y$ and

$$w = \hat{g} - y \in \ell_2(\mathcal{B}).$$

So we have that for all $t \in T$,

$$g(f_t) = \hat{g}(f_t) = (y + w)(f_t) = y(f_t) + w(f_t) = y(f_t),$$

because $w(f_t) = 0$, for every $t \in T$ (recall that, $f_t = 2^{\frac{m}{2}} I^*(e^*_{(t,m)})$, for $m \in D_t$). It follows that $g(y^*) = y(y^*)$ for all $y^* \in Y_0$, which implies that $F(y) = g$. Therefore the operator F is surjective and thus an isomorphism between the spaces Y and Y_0^*.

It is obvious from the above that the family $\{(f_t, x_t) : t \in T\}$ is a shrinking M-basis for Y_0.

Now we are able to prove the main result.

Proposition (3.3.9)[123]: There is no bounded linear one-to-one operator $F : Y^* \to c_0(\Gamma)$ for any set Γ.

Proof: Assume, for the purpose of contradiction, that there exists a bounded linear one-to-one operator $F : Y^* \to c_0(\Gamma)$ for some set Γ. Let $F^* : \ell_1(\Gamma) \to Y^{**}$ be the dual operator of F. Then we may assume without loss of generality that $F^*(e^*_\gamma) \neq 0$ for all $\gamma \in \Gamma$ and note that the set $\{F^*(e^*_\gamma) : \gamma \in \Gamma\} \cup \{0\}$ is a weak$*$ compact (and weak$*$ total) in Y^{**}, so that for every sequence $(\gamma_n)_n$ of distinct points of Γ we have that $w^* - \lim_{n\to\infty} F^*(e^*_{\gamma_n}) = 0$. By Lemma (3.3.8), the M-basis $\{(f_t, x_t) : t \in T\}$ of the predual Y_0 of Y is shrinking, therefore the set

$$\Omega = \left\{\frac{f_t}{\|f_t\|} : t \in T\right\} \textit{ is weakly discrete and the set } \Omega \cup \{0\} \textit{ is weakly compact in } Y_0.$$

We consider the map

$$\Phi : T \times \Gamma \to \mathbb{R} : \Phi(t,\gamma) = F^*(e^*_\gamma)(f_t) \quad for \quad (t,\gamma) \in T \times \Gamma.$$

It follows that there exist partitions $\{T_\delta : \delta \in \Delta\}$ and $\{\Gamma_\delta : \delta \in \Delta\}$ of T and Γ into countable sets, such that for every $\delta_1, \delta_2 \in \Delta$ with $\delta_1 \neq \delta_2$ and for every $t \in T_{\delta_1}, \gamma \in \Gamma_{\delta_2}$, we have that $\Phi(t,\gamma) = 0$ (see [127] and [125]).

We enumerate each Γ_δ and T_δ as $\{\gamma_n^\delta : n \geq 1\}, \{t_n^\delta : n \geq 1\}$ and for $n, m \in \mathbb{N}$ we put

$$D_{n,m} = \{t \in T : t = t_n^\delta \ for\ some\ \delta \in \Delta\ and\ there\ exists\ \gamma \in \Gamma_\delta : |\Phi(t,\gamma)| \geq \frac{1}{m}\}$$

$$\Gamma_{n,m} = \left\{\gamma \in \Gamma : \gamma = \gamma_n^\delta \ for\ some\ \delta \in \Delta\ and\ there\ exists\ t \in T_\delta : |\Phi(t,\gamma)| \geq \frac{1}{m}\right\}.$$

Set $D_m = \bigcup_{n=1}^{\infty} D_{n,m}$ and $\Gamma_m = \bigcup_{n=1}^{\infty} \Gamma_{n,m}$ for $m \in \mathbb{N}$. Then we have

(a) $T = \bigcup_{m=1}^{\infty} D_m$;

(b) if $(t,\gamma) \in T \times \Gamma$ and $\Phi(t,\gamma) \neq 0$ then there exists $m \in \mathbb{N}$ such that $(t,\gamma) \in D_m \times \Gamma_m$ and

(c) for every $m \in \mathbb{N}$ and $x \in D_m \cup \Gamma_m$ there exists $y \in D_m \cup \Gamma_m$ such that,

$$either\ \ x \in D_m,\ y \in \Gamma_m\ \ and\ \ |\Phi(x,y)| \geq \frac{1}{m}$$

$$or\ \ x \in \Gamma_m,\ y \in D_m\ \ and\ \ |\Phi(y,x)| \geq \frac{1}{m}.$$

We get from fact (iii) that for every $\gamma \in \Gamma$ there exists a unique pair $y_\gamma \in Y$ and $w_\gamma \in \ell_2(\mathcal{B})$ such that $F^*(e_\gamma^*) = y_\gamma + w_\gamma$.

Let $m_0 \in \mathbb{N}$ be such that D_{m_0} is successively dense in the tree T_{m_0} (see Theorem (3.3.5) and also (a)). Using this fact and also properties (a)–(c) above, we can choose by induction sequences $(\gamma_n)_n \subset \Gamma_{m_0}$ and $(t_n)_n \subset T_{m_0}$ such that:

(d) $\{t_1 < \ldots < t_n < \ldots\}$ is an infinite segment of the tree T_{m_0} ;

(e) for every $n \geq 1, |\Phi(t_{n+1},\gamma_{n+1})| \geq \frac{1}{m_0}$ and $t_{n+1} \notin b$ for all branches $b \in \mathcal{B}$ with $w_{\gamma_n}(b) \neq 0$. Note that $w_\gamma \in \ell_2(\mathcal{B})$ thus the set $\{b \in \mathcal{B} : w_\gamma(b) \neq 0\}$ is at most countable.

Fact (ii) and (d) above imply that the series $\sum_{k=1}^{\infty} f_{t_k}$ is weak$*$ -convergent in Y^*, say $x^* = w^* - \sum_{k=1}^{\infty} f_{t_k}$. It also follows from (e) that $w_{\gamma_n}(x^*) = 0$ for all $n \geq 1$. We shall show that the sequence $\left(F^*(e_{\gamma_n}^*)\right)_n$ is not weakly* null. Indeed

$$F^*(e^*_\gamma)(x^*) = (w_{\gamma_n} + y_{\gamma_n})(x^*) = y_{\gamma_n}(x^*) = \lim_{\ell\to\infty}\sum_{k=1}^{\ell} f_{t_k}(y_{\gamma_n}) = f_{t_n}(y_{\gamma_n})$$
$$= (w_{\gamma_n} + y_{\gamma_n})(f_{t_n}) = F^*(e^*_{\gamma_n})(f_{t_n}) = \Phi(t_n,\gamma_n).$$

Therefore

$$|F^*(e^*_{\gamma_n})(x^*)| = |\Phi(t_n,\gamma_n)| \geq \frac{1}{m_0} \quad for\ all \quad n \geq 1,$$

which proves the claim and so the proof of the theorem is complete, [99], [135].

We first recall that a norm $\|\cdot\|$ of a Banach space X is said to be weakly uniformly rotund (WUR for sort) if $w - lim(x_n - y_n) = 0$ whenever $\|x_n\| = \|y_n\| = 1$ for all n and $lim\|x_n + y_n\| = 2$. Fabian, Hájek, and Zizler have proved that if X is a WUR Banach space, then its dual X^* is a subspace of a WCG. More exactly, they proved that the space X admits an equivalent WUR norm if and only if the bidual unit ball $B_{X^{**}}$ of X^{**} in its weak* topology is a uniform Eberlein compact space ([128]). The following result is an easy consequence of the theorem of Fabian, Hájek and Zizler.

Corollary (3.3.10)[123]: Let E be a Banach space such that E^* is a subspace of a Hilbert generated F. Then E admits a WUR renorming.

Proof: We simply observe that $(B_{E^{**}}, w^*)$ is a continuous image of a uniform Eberlein compact space (i.e., of the ball of (B_{F^*}, w^*) of F^*), hence a well-known result of Benyamini, Rudin and Wage yields that the space $(B_{E^{**}}, w^*)$ is a uniform Eberlein compact ([126]). Now by the above mentioned result of Fabian, Hájek and Zizler we get the conclusion.

Summing up all the previous results, we get a negative answer to the problem of Fabian, Godefroy, Hájek and Zizler mentioned in the introduction as question Q1.

Theorem (3.3.11)[123]: There exists a WUR renormable Banach space E that does not admit any bounded, linear, one-to-one operator into some $c_0(\Gamma)$.

Proof: Set $E = Y^*$, where Y is the space of Proposition (3.3.9), so there is no bounded, linear, one-to-one operator from E to $c_0(\Gamma)$. On the other hand, $E^* = Y^{**}$ is a subspace of the Hilbert generated space X^{**} (see Lemma (3.3.7)) and hence, by the above corollary, E admits a WUR renorming. The proof of the theorem is completed.

The following describes a peculiar property of James tree spaces.

Proposition (3.3.12)[123]: Let T be a tree. Then the following are equivalent.

- (i) $J(T)$ is weakly countably determined.
- (ii) There exists a sequence $(A_n)_{n\in\mathbb{N}}$ such that each A_n is an antichain of T and $T = \bigcup_{n=1}^{\infty} A_n$.
- (iii) $J(T)$ is Hilbert generated (hence it is WCG).

Proof: (i)⇒(ii) Let us observe that every branch b of T is at most countable (otherwise the ordinal ω_1 will be subset of $B_{J(T)^*}$ yielding a contradiction) and moreover the set

$$\mathcal{D} = \{S^* : S \text{ is a segment of } T\}$$

is a w*-compact subset of $B_{J(T)^*}$. Hence $\mathcal{D}$ is a Gulko compact subset of $\Sigma\{0,1\}^T$. Clearly the adequate closure of $\mathcal{D}$,

$$\widehat{\mathcal{D}} = \{A \subseteq T : \exists S \in \mathcal{D} \text{ with } A \subseteq S\}$$

remains Gulko compact. This follows from Theorem 3.6 [135], [134] yields that $T = \bigcup_{n=1}^{\infty} A_n$ with each A_n an antichain of T.

(ii)⇒(iii) As we have mentioned in Lemma (3.3.7), for A antichain of T, the space $\overline{span}\{e_t : t \in A\}$ is isometric to $\ell_2(A)$. The result follows from arguments similar to the proof of Lemma (3.3.7).

(iii)⇒(i) Obvious.

Chapter 4

Quasiconformal Maps and Coefficients Estimate for Harmonic ν-Bloch Mappings with Curvature of the Boundary

We provide a quasiconformal analogue of the Smirnov absolute continuity result over the boundary. Moreover, if f is a harmonic K-quasiconformal self-mapping of $\mathbb{D}$, then $|a_n| + |b_n| \leq B_n(K)$, where $B_n(K)$ is given such that $\lim_{n\to\infty} B_n(K) = 0$ and $B_n(1) = 4\,n\pi$. We make some asymptotically sharp estimates of constant of quasiconformality for harmonic diffeomorphisms between the unit disk and the convex domains by their boundary mappings.

Section (4.1): Controlled Laplacian

The map $w : \mathbb{D} \to \mathbb{C}$ of the unit disc to the complex plane is quasiconformal if it is a sense preserving homeomorphism that has locally L^2-integrable weak partial derivatives, and it satisfies for almost every $z \in \mathbb{D}$ the distortion inequality $|w_{\bar{z}}| \leq k|w_z|$, where $k < 1$. In this situation we say that w is K-quasiconformal, with $K := (1 + k)/(1 - k)$.We refer to [32] and [139] for basic notions and results of the quasiconformal theory. Quasiconformal mappings, even when C^2-smooth, can be far from being Lipschitz maps.

However, in the situation where $w : \mathbb{D} \to \mathbb{D}$ is a quasiconformal homeomorphism that is also harmonic Pavlović [14] proved that f is bi-Lipschitz.

Many generalisations of this result for harmonic maps heve been proven see [149] and [140].

The addresses the following question: how much one can relax the condition of harmonicity of the quasiconformal map w, while still being able to deduce the Lipschitz property of w - in this situation it is less natural to inquire w to be bi-Lipschitz. Answers to this kind of questions ought to be useful also in applications to non-linear elasticity. A natural measure for the deviation from harmonic functions is to consider $\|\Delta w\|_{L^p(\mathbb{D})}$ for some $p \geq 1$ and ask whether finiteness of this quantity enables one to make the desired conclusion. Our first main result yields the following:

The second main result shows that in the setting of Theorem (4.1.3) the Lipschitz constant of f becomes arbitrarily close to 1 if the image domain Ω approaches the unit disc in a suitably defined $C^{1,\alpha}$-sense, and if both K close to one is and the deviation from harmonicity are small enough. Below we identify $[0, 2\pi)$ and $\mathbb{T}$ in the usual way.

Theorem (4.1.1)[137]: Let $p > 2$ and assume that $w_n : \mathbb{D} \to n$ is a K_n-quasi-conformal normalised map normalised by $w(0) = 0$, and with

$$\lim_{n\to\infty} K_n = 1 \; and \; \lim_{n\to\infty} \|\Delta w\|_{L^p(\mathbb{D})} = 0.$$

Moreover, we assume that for each $n \geq 1$ the bounded Jordan domain Ω_n approaches the unit disc in the $C^{1,\alpha}$-bounded sense. More precisely, there is a parametrisation

$$\partial\Omega_n = \{f_n(\theta) \mid \theta \in \mathbf{T}\},$$

where f_n satisfies for some $\alpha > 1/2$

$$\left\|f_n(\theta) - e^{i\theta}\right\|_{L^\infty(\boldsymbol{T})} \to 0 \; as \; n \to \infty \; and \; \sup_{n\geq 1}\|f_n(\theta)\|_{C^{1,\alpha}} < \infty.$$

Then for large enough n the function w_n is Lipschitz, and moreover its Lipschitz constant tends to 1 as $n \to \infty$:

$$\lim_{n\to\infty} \|\nabla w\|_{L^\infty(\mathbb{D})} = 1. \quad (1)$$

This result will be obtained as a corollary of slightly more general results. Together, our Theorems (4.1.3) and (4.1.1) considerably improve the main result of the first author and Pavlović from [151], where it was instead assumed that $\Delta w \in C(\overline{\mathbb{D}})$. Other related results are contained in [150], see [141] for other type of connections between quasiconformal and Lipschitz maps.

In order to state our last theorem, we recall the result of V. I. Smirnov, stating that a conformal mapping of the unit disk $\boldsymbol{U}$ onto a Jordan domain Ω with rectifiable boundary has a absolutely continuous extension to the boundary. This implies in particular that if $E \subset \boldsymbol{T}$ is a set of zero 1-dimensional Hausdorff measure then its image $f(E)$ is a set of zero 1-dimensional Hausdorff measure in $\partial\Omega$. Further, this result has been generalized for the class of q.c. harmonic mapping by several authors (see e.g. [1], [147]). On the other hand if we assume that f is merely quasiconformal, then its boundary function need not be in general an absolutely continuous function. We prove the following generalization of Smirnov's theorem for quasiconformal mappings, subject again to an size condition on their Laplacian:

Further comments, generalisations and open questions related to the above results are included.

Lipschitz-property of qc-solutions to $\Delta f = g$.

In what follows, we say that a bounded Jordan domain $\Omega \subset \mathbb{C}$ has C^2-boundary if it is the image of the unit disc $\mathbb{D}$ under a C^2-diffeomorphism of the whole complex plane onto itself. For planar Jordan domains this is well-known to be equivalent to the more standard definition, that requires the boundary to be locally isometric to the graph of a C^2-function on $\mathbb{R}$. In what follows, Δw always refers to the distributional Laplacian. We shall make use of the following well-known fact, whose proof we recall:

Lemma (4.1.2)[137]: Assume that $w \in C(\overline{\mathbb{D}})$ is such that $\|\Delta w\|_{L^p(\mathbb{D})} < \infty$ with $p > 1$.

(i) In case $p > 2$ one has $\|\nabla w\|_{L^\infty(B(0,r))} < \infty$ for any $r < 1$. Moreover, if $w_{|\partial D} = 0$, then there is $C_p < \infty$ so that

$$\|\nabla w\|_{L^\infty(\mathbb{D})} \le C_p \|\Delta w\|_{L^p(\mathbb{D})}.$$

(ii) If $w_{|\partial\mathbb{D}} = 0$, and $1 < p < 2$, then $\|\nabla w\|_{2p/(2-p)} < \infty$.

Proof. By the classical representation we have for $|z| < 1$

$$w(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\varphi}) w(e^{i\varphi}) d\varphi + \int_U G(z,\omega)\Delta w(\omega) dA(\omega), \quad (2)$$

where P stands for the Poisson kernel and $G(z,\omega) := \frac{1}{2\pi}\log\left|\frac{1-z\overline{\omega}}{z-\omega}\right|$ for the Green's function of $\mathbb{D}$. We observe first that since G is real-valued, $|\nabla G| = 2|\partial_z G|$ so that

$$|\nabla G(z,\omega)| = \frac{1}{2\pi}\frac{-\overline{\omega}}{1 - z\overline{\omega}} - \frac{1}{z - \omega} \le \frac{1}{\pi|z - \omega|}. \quad (3)$$

Hence an application of Hölder's inequality shows that the second term in (2) has uniformly bounded gradient in $\mathbb{D}$. To conclude part (i) it suffices to observe that the first term vanishes if $w_{|\partial\mathbb{D}} = 0$, and in the general case it has uniformly bounded gradient in compact subsets of $\mathbb{D}$. Finally, part (ii) follows immediately from (3) by the standard mapping properties of the Riesz potential I_1 with the kernel $|z - \omega|^{-1}$, see [154].

Theorem (4.1.3)[137]: Assume that $g \in L^p(\mathbb{D})$ and $p > 2$. If w is a K-quasiconformal solution of $\Delta w = g$, mapping the unit disk onto a bounded Jordan domain $\Omega \subset \mathbb{C}$ with C^2-boundary, then w is Lipchitz continuous. The result is sharp since it fails in general if $p = 2$.

Proof. It would be natural to try to generalise the ideas in [150] where differential inequalities were applied while treating related problems.

However, it turns out that by introducing appropriate new ideas the approach of [148], where the use of distance functions was initiated, is flexible enough for further development.

By our assumption on the domain, we may fix a diffeomorphism $\psi : \overline{\Omega} \to \mathbb{D}$ that is C^2 up to the boundary. Denote $H := 1 - |\psi|^2$, whence H is C^2-smooth in and vanishes on $\partial\Omega$ with $|\nabla H| \approx 1$ in a neighbourhood of $\partial\Omega$.

We may then define $h : \mathbb{D} \to [0,1]$ by setting

$$h(z) := H \circ w(z) = 1 - \left|\psi\big(w(z)\big)\right|^2 \text{ for } z \in \mathbb{D}.$$

The quasiconformality of f and the behaviour of ∇H near $\partial\Omega$ imply that there is $r_0 \in (0,1)$ so that the weak gradients satisfy

$$|\nabla h(x)| \approx |\nabla w(x)| \text{for } r_0 \le |x| < 1. \quad (4)$$

Moreover, by Lemma (4.1.2)(i) we have $|\nabla h(x)| \lesssim |\nabla w(x)| \le C$ for $|x| \le r_0$. It follows that for any $q \in (1,\infty]$ we have that

$$\nabla h \in L^q(\mathbb{D}) \text{if and only if } \nabla w \in L^q(\mathbb{D}). \quad (5)$$

A direct computation, simplified by the fact that H is real valued, yields that

$$\begin{aligned} \Delta h &= \Delta(H \circ w) \\ &= (\Delta H)(w)(|w_z|^2 + |w_{\overline{w}}|^2) + 2\mathrm{Re}(H_{zz}(w)w_z w_z + H_z(w)\Delta w. \end{aligned} \quad (6)$$

Especially, since $H \in C^2(\overline{D})$ and the function w is bounded we have

$$|\Delta h| \lesssim |\nabla w|^2 + |g|. \quad (7)$$

The higher integrability of quasiconformal self-maps of $\mathbb{D}$ makes sure that $\nabla(\psi \circ w) \in L^q(\mathbb{D})$ for some $q > 2$, which implies that $\nabla w \in L^q(\mathbb{D})$.

By combining this with the fact that $g \in L^p(\mathbb{D})$ with $p > 2$, we deduce that $\Delta h \in L^r(\mathbb{D})$ with $r = \min(p, q/2) > 1$. This information is not enough to us in case $q \le 4$, but we will actually show that one may improve the situation to $q > 4$ via a bootstrapping argument based on the following observation: in our situation

$$\text{if } \nabla w \in L^q(\mathbb{D}) \text{ with } 2 < q < 4, \text{ then actually } \nabla w \in L^{2q/(4-q)}(\mathbb{D}). \quad (8)$$

In order to prove (8), assume that $\nabla w \in L^q(\mathbb{D})$ for an exponent $q \in (2,4)$. Then (7) and our assumption on g verify that $\Delta h \in L^{q/2}(\mathbb{D})$. Since h vanishes continuously on the boundary $\partial\mathbb{D}$, we may apply Lemma (4.1.2)(ii) to obtain that $\nabla h \in L^{2q/(4-q)}(\mathbb{D})$ which yields the claim according to (5).

We then claim that in our situation one has case $\nabla w \in L^q(\mathbb{D})$ with some exponent $q > 4$. For that end, fix an exponent $q_0 > 2$ obtained from the higher integrability of the quasiconformal map w so that $\nabla w \in L^{q_0}(\mathbb{D})$. By diminishing q_0 if needed, we may well assume that $q_0 \in (2,4)$ and

$$q_0 \notin \left\{\frac{2^n}{2^{n-1} - 1} \quad \text{for} \quad n = 3,4,\dots\right\}.$$

Then we may iterate (8) and deduce inductively that $\nabla w \in L^{q_k}(\mathbb{D})$ for $k = 0,1,2 \ldots k_0$, where the indexes q_k satisfy the recursion $q_{k+1} = \frac{2q_k}{4-q_k}$ and k_0 is the first index such that $q_{k_0} > 4$. Such an index exists since we may explicitly solve for $k \geq 0$

$$q_k = \frac{2}{1 - 2^k(1 - 2/q_0)}.$$

Thus we may assume that $\nabla w \in L^q(\mathbb{D})$ with $q > 4$. At this stage (7) shows that $\Delta h \in L^{p\wedge(q/2)}(\mathbb{D})$. As $p \wedge (q/2) > 2$, Lemma (4.1.2)(ii) verifies that ∇h is bounded. Finally, by (5) we have the same conclusion for ∇w, and hence w is Lipschitz as claimed.
In order to verify the sharpness of the result, consider the following map

$$w_0(z) = z \log^a\left(\frac{e}{|z|^2}\right),$$

where $a \in (0,1/2)$ is fixed. Then w_0 is a self-homeomorphism of $\mathbb{D}$ that is quasiconformal with even continuous Beltrami-coefficient since we may easily compute $w_0(z) = z \log^{a-1}\left(\frac{e}{|z|^2}\right) \log\left(\frac{e^{1-a}}{|z|^2}\right)$ and $(w_0)_{\bar{z}} = -a\frac{z}{\bar{z}} \log^{a-1}\left(\frac{e}{|z|^2}\right)$ so that the complex dilatation of w0 satisfies

$$\left|\mu_{w_0}(z)\right| = \left|-a\frac{z}{\bar{z}}\left(\log\left(\frac{e^{1-a}}{|z|^2}\right)\right)^{-1}\right| \leq \frac{a}{1-a} < 1.$$

In addition, we see that $\Delta w_0 \in L^2(\mathbb{D})$ since

$$|\Delta w_0(z)| = \left|4\frac{d}{\bar{z}}(w_0)_z(z)\right| = \left|\frac{4a}{\bar{z}} \log^{a-2}\left(\frac{e}{|z|^2}\right)\left((a-1) - \log\left(\frac{e}{|z|^2}\right)\right)\right|$$
$$\lesssim |z|^{-1}\left(\log\left(\frac{e}{|z|^2}\right)\right)^{a-1}.$$

Finally, it remains to observe that w is not Lipschitz at the origin.

We start with an auxiliary lemma.

Lemma (4.1.4)[137]: There exists a function $\psi : (1,2) \to \mathbb{R}^+$ with the following property: If $w : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal self-map normalised with $\psi(0) = 0$, then

$$\| |w_z|^2 + |w_{\bar{z}}|^2 - 1 \|_{L^3(\mathbb{D})} \leq \psi(K).$$

Moteover, $\lim_{K\to 1^+} \psi(K) = 0$.

Proof. By the sharp area distortion $\|\nabla w\|_{L^6(\mathbb{D})} < 8$ for $K < 3/2$. By reflecting w over the boundary $\partial\mathbb{D}$ we may also assume that w extends to a K-quasiconformal map (still denoted by w) to the whole plane. By rotation of needed, we may also impose the condition that $w(1) = 1$. Furthermore, we may even assume that $w_{C\setminus B(0,e^{3\pi})}$ is the identity map, since we may use standard quasiconformal surgery (choose $k = (K-1)/(K+1)$ and $\_ = 2k$ in [139] to produce $\frac{3K-1}{3-K}$-quasiconformal modification (still denoted by w) that equals to w in $\mathbb{D}$ and is satisfies $w(z) = z$ for $|z| \geq e^{3\pi}$.
Especially, it is a principal solution. Since $\frac{3K-1}{3-K} \to 1$ as $K \to 1$, and we are interested only on small values of K, it is thus enough to prove the corresponding claim for principal solutions with complex dilatation supported in $B(0,e^{3\pi})$.
Denote by M the norm of the Beurling operator on $L^6(\mathbb{C})$. Fix $R_0 > 0$ and consider a principal solution w to the Beltrami equation $w_{\bar{z}} = \mu w_z$ with $|\mu| \leq k < 1/2M$. Then we have the standard Neumann-series representation

$$w_{\bar z} = \mu + \mu T\mu + \mu T\mu T\mu + \dots \text{ and } \quad w_z - 1 = Tw_{\bar z}.$$
We thus obtain that

$$\|w_{\bar z}\|_{L^6(\mathbb{C})} \le \|\mu\|_{L^6(\mathbb{C})}\left(1 + \frac{k}{2M} + \left(\frac{k}{2M}\right)^2 + \dots\right) \le 2\|\mu\|_{L^6(\mathbb{C})} \le Ck^{1/6}$$

and, a fortiori $\|w_z - 1\|_{L^6(\mathbb{C})} \le MCk^{1/6} = C'k^{1/6}$. We obtain the desired L^3-estimate for $|f_{\bar z}|^2$ since $k \to 0$ as $K \to 1$. The estimate for $|f_z|^2 - 1$ follows by noting that $||f_z|^2 - 1| \le |f_z - 1|(|f_z - 1| + 2)$ and applying Hölder's inequality.

Before proving the more general convergence result stated in the introduction it is useful to consider first the case where the image domain is fixed, and in fact equals $\mathbb{D}$.

Proposition (4.1.5)[137]: Assume that $p > 2$. There exist a function

$$[1,\infty) \times [0,\infty) \ni (K,u) \to \bar{C}_p(K,u)$$

with the property: if $w : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal self map of the unit disc, normalised by $w(0) = 0$, and with $\Delta w \in L^p(\mathbb{D})$, then one has

$$\|\nabla w\|_{L^\infty(\mathbb{D})} \le \bar{C}_p(K, \|\Delta w\|_p).$$

Moreover, the function $\bar{C}_p$ satisfies

$$\lim_{K\to 1^+, u\to 0^+} \bar{C}_p(K,u) = 1. \tag{9}$$

Proof. We follow the line to the proof of Theorem (4.1.3), in particular we employ its notation, but this time we strive to make the conclusion quantitative.

We may well assume that $p \le 3$. Let us then assume that w is as in the assumption of the Proposition with $K < 1 + 1/100$, say. In addition, we may freely assume that $w(1) = 1$. As the image domain is D, the function h from the proof of Theorem (4.1.3), takes the form

$$h(z) = 1 - |w(z)|^2.$$

Let us write $h_0(z) = 1 - |z|^2$, which corresponds to h when w is the identity map. An application of (6) and Lemma (4.1.4) allow us to estimate

$$\begin{aligned}\|\Delta(h - h_0)\|_{L^p(\mathbb{D})} &= \|4(|w_z|^2 - 1) + 4|w_{\bar z}|^2 + 2\mathrm{Re}(\bar w g)\|_{L^p(\mathbb{D})}\\ &\le 4\|4(|w_z|^2 - 1) + 4|w_{\bar z}|^2\|_{L^3(\mathbb{D})} + \|g\|_{L^p(\mathbb{D})}\\ &\le 4\psi(K) + \|g\|_{L^p(\mathbb{D})}.\end{aligned} \tag{10}$$

Lemma (4.1.2) implies that

$$\|\nabla h - \nabla h_0\| \le c_p\big(\psi(K) + \|g\|_{L^p(\mathbb{D})}\big).$$

The quasiconformality of w verifies a.e. that

$|\nabla h(z)| \ge K^{-1}|(\nabla h_0)(w(z)||\nabla w(z)|$.

Since $|\nabla h_0(z)| = 2|z|$, we obtain by considering the annulus $1 - \varepsilon \le |z| < 1$ with arbitrarily small $\varepsilon > 0$ that

$$\begin{aligned}\limsup_{|z|\to 1^-} |\nabla w(z)\| &\le \frac{K}{2}\limsup_{|z|\to 1^-}(|\nabla h - \nabla h_0| + |\nabla h_0|)\\ &\le \frac{c_pK}{2}\big(\psi(K) + \|g\|_{L^p(\mathbb{D})}\big) + K.\end{aligned} \tag{11}$$

Let us then write w in terms of the standard Poisson decomposition $w = u + f$, where u is harmonic with $u_{|\partial\mathbb{D}} = w_{|\partial\mathbb{D}}$, the term f has vanishing boundary values and it satisfies $\Delta f = \Delta w = g$ in $\mathbb{D}$. Then maximum principle applies to the subharmonic function $|\nabla u| = |u_z| + |u_{\bar z}| = |a'| + |b'|$, where a and b are analytic functions such that $u = a + b$, together with 1 shows that $|\nabla w|$ is bounded by $c\|g\|_{L^p(\mathbb{D})}$. All, in all combing these observations with (11) we deduce that

$$\sup_{|z|<1}|\nabla w(z)| \leq \limsup_{|z|\to 1^-}|\nabla u| + \sup_{|z|<1}|\nabla f(z)| \leq \limsup_{|z|\to 1^-}|\nabla w| + 2\sup_{|z|<1}|\nabla f(z)|$$
$$\leq \frac{c_p K}{2}\left(\psi(K) + \|g\|_{L^p(\mathbb{D})}\right) + K + 2c_p\|g\|_{L^p(\mathbb{D})}. \quad (12)$$
We may thus choose for small enough K
$$\bar{C_p}(K,t) = K + \frac{c_p K}{2}\psi(K) + \frac{c_p(K+4)}{2}t,$$
which has the desired behavior.

Below Id stands for the identity matrix $\mathrm{Id} := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Definition (4.1.6)[137]: Let $p > 2$. We say that the sequence of bounded Jordan domains $\Omega_n \subset \mathbb{C}$ such that $0 \in \Omega_n$ for each $n \geq 1$ converge in $W^{2,p}$-controlled sense to the unit disc $\mathbb{D}$ if there exist sense-preserving diffeomorphisms $\Psi_n : \mathbb{D} \to \Omega_n$, normalized by $\Psi_n(0) = 0$, such that for some $M_0 < \infty$ it holds that
$$\lim_{n\to\infty}\|D\Psi_n - \mathrm{Id}\|_{L^\infty(\mathbb{D})} = 0,$$
$$\text{and } \|\Psi_n\|_{\&\overline{W}^{2,p}(\mathbb{D})} \leq M_0 \; for\; all\; n \geq 1, \quad (13)$$
together with
$$\|\Delta\Psi_{\mathrm{n}}\|_{L^p(\mathbb{D})} \to 0 \quad as \quad n \to \infty. \quad (14)$$
One should observe that above since $\Psi_n \in W^{2,p}(D)$ with $p > 2$, it follows automatically that $\nabla\Psi_n$ is continuous, so asking Ψ_n to be a diffeomorphism makes perfect sense in terms and, in particular, Ψ_n is a bi-Lipschitz map for large enough n. Also, each Ω_n is a bounded C^1-Jordan domain in the plane.

It turns out that the above condition is in a sense symmetric with respect to the domains $\mathbb{D}$ and Ω:

Lemma (4.1.7)[137]: Assume that Ω_n tends to $\mathbb{D}$ in a controlled sense and (Ψ_n) is the associated sequence of diffeomorphisms satisfying the conditions of Definition (4.1.6). Then the inverse maps $\Phi_n := \Psi_n^{-1} : \Omega_n \to \mathbb{D}$ satisfy
$$\lim_{n\to\infty}\|D\Phi_n - \mathrm{Id}\|_{L^\infty(\Omega_n)} = 0,$$
$$and\; \|\Phi_n\|_{W^{2,p}(\Omega_n)} \leq M_0' \; for\; all\; n \geq 1, \quad (15)$$
together with
$$\|\Delta\Phi_n\|_{L^p(\Omega_n)} \to 0 \; as\; n \to \infty. \quad (16)$$
Proof. Conditions (15) follows easily by employing the formulas for the derivatives implicit function, after first approximating by smooth functions.

Note, in regards condition (16), that in general the inverse of a harmonic diffeomorphism needs not to be harmonic, so (16) is not a direct consequence of (14). However, the first condition in (13) tells us the our functions are asymptotically conformal, the maximal complex dilatation k_n of Ψ_n tends to 0 as $n \to \infty$, so that Ψ_n is asymptotically conformal. This makes (16) more plausible, and indeed a direct computations shows that for C^2-diffeo $\Psi : \mathbb{D} \to \Omega$ with maximal dilatation k and controlled derivative $|D\psi|, |(D\psi)^{-1}| \leq C$, it holds that
$$\Delta\Phi = A \circ \Phi,$$
where (recall that the Jacobian has the formula $J_\Psi = |\Psi_z|^2 - |\Psi_{\bar{z}}|^2)$
$$A = \frac{4}{(J_\Psi)^3}\Big[-\Psi_{\bar{z}}\big(\overline{\Psi_{z\bar{z}}}J_\Psi - \overline{\Psi_z}(\overline{\Psi_z}\Psi_{zz} + \Psi_z\Psi_{z\bar{z}} - \overline{\Psi_{\bar{z}}}\Psi_{z\bar{z}} - \Psi_{\bar{z}}\overline{\Psi_{\overline{zz}}})\big)$$
$$+ \Psi_z\big(\overline{\Psi_{zz}}J_\Psi - \overline{\Psi_z}(\overline{\Psi_z}\Psi_{z\bar{z}} + \Psi_z\overline{\Psi_{zz}} - \overline{\Psi_{\bar{z}}}\Psi_{\overline{zz}} - \Psi_z\overline{\Psi_{z\bar{z}}})\big)\Big]$$

We next recall that Ψ_z is bounded and $|\Psi_{\bar{z}}| \le k\Psi_z$, and observe that above in the right hand side the terms that do not directly contain either $\Psi_{z\bar{z}}$ or $\Psi_{\bar{z}}$ as a factor sum up to

$$\overline{\Psi_{zz}}(J_\Psi - |\Psi_z|^2) = -\overline{\Psi_{zz}}|\Psi_{\bar{z}}|^2,$$

We obtain that

$$|A| \lesssim |kD^2\Psi| + |\Delta\Psi,$$

and (16) follows by applying this on Ψ_n.

We may now generalize Proposition (4.1.5) to include variable image domains that converge to the unit disc in controlled sense.

Theorem (4.1.8)[137]: Let $p > 2$ and assume that the planar Jordan domains Ω_n converge to $\mathbb{D}$ in $W^{2,p}$-controlled sense. Moreover, assume that $w_n : \mathbb{D} \to \Omega_n$ is a K_n-quasi-conformal normalised map normalised by $w(0) = 0$, and with

$$\lim_{n\to\infty} K_n = 1 \; and \; \lim_{n\to\infty} \|\Delta w_n\|_{L^p(\mathbb{D})} = 0.$$

Then for large enough n the function w_n is Lipschitz, and moreover its Lipschitz constant tends to 1 as $n \to \infty$:

$$\lim_{n\to\infty} \|\nabla w\|_{L^\infty(\mathbb{D})} = 1. \tag{17}$$

Proof. Let $\Psi_n : \mathbb{D} \to \Omega_n$ be the maps as in Definition (4.1.6). By renumbering, if needed, we may assume that that $|\Psi_n'(z) - 1| < 1/2$ for all n and $z \in \mathbb{D}$. Write $\Phi_n = \Psi_n^{-1}$ and define

$$\overline{w}_n := \Psi^{-1} \circ w_n = \Phi_n \circ w_n : \mathbb{D} \to \mathbb{D}.$$

Then $\widetilde{w}_n$ is $\widetilde{K}_n$-quasi-conformal, with $\widetilde{K}_n \to 0$ as $n \to \infty$ by the first condition in (13). Fix an index $q \in (2, p)$. Again by the just mentioned condition, in order to prove (17) Proposition (4.1.5) shows that we just need to verify that

$$\lim_{n\to\infty} \|\Delta\widetilde{w}_n\|_{L^q(\mathbb{D})} = 0. \tag{18}$$

A simple computation yields that

$$\begin{aligned}\Delta\widetilde{w}_n &= (\Delta\Phi_n)(w_n)(|(w_n)_z|^2 + |(w_n)_{\bar{z}}|^2) \\ &\quad + 4\big((\Phi_n)_{zz}(w)(w_n)_z(w_n)_{\bar{z}} + (\Phi_n)_{\overline{zz}}(w_n)\overline{(w_n)_z(w_n)_{\bar{z}}}\big) \\ &\quad + ((\Phi_n)_z(w_n)\Delta w_n + (\Phi_n)_{\bar{z}}(w_n)\overline{\Delta w_n}) \\ &=: A + B + C,\end{aligned} \tag{19}$$

Since $|D\Phi_n|$ remains uniformly bounded and we know that $\|\Delta w_n\|_{L^p(\mathbb{D})} \to 0$, we see that $\|C\|_{L^p(\mathbb{D})}$ tends to zero as $n \to \infty$, whence the same is true for the L^q-norm. Set $\tilde{q} := \sqrt{qp}$ so that $q < \tilde{q} < p$. Since $\widetilde{w}_n$ is a normalized K_n-quasiconformal self-map of the unit disc $\mathbb{D}$, and $K_n \to 0$, we may assume, again by discarding small values of n and relabeling, if needed, by the higher integrability of quasiconformal maps that $\int_{\mathbb{D}} |\nabla w_n|^{2(eq/q)'} < C$ and $\int_\Omega \big(J_{w_n^{-1}}\big)^{(p/eq)'} dA(z) < C$ for all n. Here $(eq/q)'$ stands for the dual exponent.

Denoting $k_n = (K_n - 1)/(K_n + 1)$ we thus obtain for any measurable function F on Ω

$$\int_{\mathbb{D}} |F \circ w_n|(w_n)_z|^2|^q dA(z) \le \left(\int_{\mathbb{D}} |F \circ w_n|^{\tilde{q}} dA(z)\right)^{\frac{q}{\tilde{q}}} \left(\int_{\mathbb{D}} |\nabla w_n|^{2\left(\frac{\tilde{q}}{q}\right)'} dA(z)\right)^{\frac{\frac{1}{\tilde{q}'}}{q}}.$$

$$\lesssim \left(\int_{\mathbb{D}} |F \circ w_n|^{\tilde{q}} dA(z)\right)^{\frac{q}{\tilde{q}}}$$

$$\leq \left(\int_{\Omega} |F|^{\tilde{q}} J_{w_n^{-1}} dA(z)\right)^{\frac{q}{\tilde{q}}}$$

$$\lesssim \left(\int_{\Omega} |F|^{P} dA(z)\right)^{\frac{q}{P}} \left(\int_{\Omega} \left(J_{w_n^{-1}}\right)^{\left(\frac{p}{\tilde{q}}\right)'} dA(z)\right)^{q/(\tilde{q}(p/\tilde{q})')} \leq \left(\int_{\Omega} |F|^{P} dA(z)\right)^{\frac{q}{P}}.$$

By employing this formula and Lemma (4.1.7) we obtain immediately that

$$\|A\|_{L^q(\mathbb{D})} \lesssim \|\Delta\Phi_n\|_{L^p(\Omega)} \to 0 \ as \ n \to \infty.$$

Moreover,

$$\|B\|_{L^q(\mathbb{D})} \lesssim k_n \to 0 \quad as \quad n \to \infty.$$

This ends the proof of the Theorem.

We next examine what kind of convergence of the boundaries $\partial\Omega_n \to \partial\mathbb{D}$ imply $W^{2,p}$-controlled convergence of the domains itself. First of all, given $\psi_n : \mathbb{D} \to \Omega$ as in Definition (4.1.6) we have $\Psi_n \in W^{2,p}(\mathbb{D})$, so that by the trace theorem of the Sobolev spaces the induced map on the boundary satisfies $\Psi_{n|\partial\mathbb{D}} \in B_{p,p}^{2-\frac{1}{p}}(\mathbb{D})$. On the other hand, for $p > 2$ we may pick $\alpha, \alpha' \in (1/2, 1)$ so that

$$C^{1,\alpha'}(\partial\mathbb{D}) \subset B_{p,p}^{2-\frac{1}{p}}(\mathbb{D}) \subset C^{1,\alpha}(\partial\mathbb{D}),$$

see [156]. Hence about the best one can hope is to have a theorem where the boundary converges if $C^{1,\alpha}$ for some $\alpha > 1/2$. In fact, this can be realised:

Theorem (4.1.9)[137]: Let (Ω_n) be a a sequence of bounded Jordan domains in $\mathbb{C}$ such that there is the parametrisation

$$\partial\Omega_n = \{f_n(\theta) \mid \theta \in (0, 2\pi)\}.$$

for each n, where f_n satisfies for some $\alpha > 1/2$

$$\left\|f_n(\theta) - e^{i\theta}\right\|_{L^\infty(\boldsymbol{T})} \to 0 \, as \, n \to \infty \, and \sup_{n\geq 1}\|f_n(\theta)\|_{C^{1,\alpha}} < \infty. \qquad (20)$$

Then the sequence (Ω_n) converges to $\mathbb{D}$ in $W^{2,p}$-controlled manner. In particular, the conclusion of Theorem (4.1.8) holds true for the sequence (Ω_n).

Proof. Let us first observe that instead of (20) we may fix $\alpha' \in (1/2, \alpha)$ and assume that

$$\left\|f_n(\theta) - e^{i\theta}\right\|_{C^{1,\alpha'}} \to 0 \, as \, n \to \infty.$$

Namely this follows applying interpolation on (20). Write $g_n(\theta) = f_n(\theta) - e^{i\theta}$. By relabeling, if needed, we may assume that for all $n \geq 1$ we have $\left\|f_n(\theta) - e^{i\theta}\right\|_{C^{1,\alpha}} \leq 1/10$, say. Since $\mathrm{Id} : \mathbb{T} \to \mathbb{C}$ is 1-bi-Lipschitz, and $Lip(g) \leq 1/5$, we obtain that $f_n : \mathbb{T} \to \partial\Omega_n$ is a diffeomorphism. We simply define Ψ_n as the harmonic extension

$$\Psi_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it}) f_n(e^{it}) dt = z + \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it}) g_n(e^{it}) dt$$
$$= z + G_n(z), \qquad z \in \mathbb{D}.$$

Since $\|g_n'\|_\infty \to 0$ and $\|Hg_n'\|_\infty \to 0$ (recall that the Hilbert transform H is continuous in C^α), we may also assume that $|DG_n(z)| \leq 1/2$ for all n, and we have $\lim_{n\to\infty}\|DG_n\|_{L^\infty(\mathbb{D})} = 0$. Especially, $\Psi_n : \overline{\mathbb{D}} \to \overline{\Omega_n}$ is bi-Lipschitz, hence diffeomorphism. The first condition in (13) follows immediately, and condition (14) is immediate since Ψ_n is harmonic. It remains to verify the second condition in (13). For that end observe that by [154] the fact that $\|g_n\|_{C^{1,\alpha}(\boldsymbol{T})} \le C$ for all n implies (actually is equivalent to) that the Poisson extension satisfies

$$\|D^2 G_n(z)\| \le \frac{C'}{(1 - |z|)^{1-\alpha}},$$

which obviously yields the desired uniform bound for $\|D^2 G_n(z)\|_{L^p(\mathbb{D})}$ if $p < (1 - \alpha)^{-1}$.

Another condition is obtained by specialising to Riemann maps – the proof of the preceding theorem could also be based on the following lemma:

Lemma (4.1.10)[137]: Let $p > 2$. The sequence of bounded Jordan domains $\Omega_n \subset \mathbb{C}$ converges in $W^{1,p}$-controlled sense to the unit disc $\mathbb{D}$ if the Riemann maps $F_n : \mathbb{D} \to \Omega_n$ (normalized by $F_n(0) = 0$ and $\arg F_n'(0) > 0$) satisfy

$$\lim_{n\to\infty} \|F_n' - 1\|_{L^\infty(\mathbb{D})} = 0, \textit{and } \|F_n''\|_{L^p(\mathbb{D})} \le M_0 \textit{ for all } n \ge 1, \tag{21}$$

with some $M_0 < \infty$.

Proof. Obvious after the definition of controlled convergence.

Theorem (4.1.11)[137]: Assume that f is a quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary such Δw is locally integrable and satisfies

$$|\Delta f(z)| \le C(1 - |z|)^{-a}$$

for some constants $a < 1$, and $C < \infty$. Then $f_{|T}$ is an absolutely continuous function. The result is optimal: there is a quasiconformal self-map of $w : \mathbb{D} \to \mathbb{D}$, with non-absolutely continuous boundary values, and such that $f \in C^\infty(\mathbb{D})$ and with $|\Delta f(z)| \le C(1 - |z|)^{-1}$ in $\mathbb{D}$.

Proof. We first assume that f is as in the theorem so that $\Delta f(z) \le (1 - |z|)^{-a}$ with $a \in (0, 1)$. Then we are to show that the boundary map induced by w is absolutely continuous. For that end we need two simple lemmas.

Lemma (4.1.12)[137]: Assume that $u \in C(\overline{D})$ is a harmonic mapping of the unit disk into $\boldsymbol{C}$ such that the $f := u_{|\boldsymbol{T}}$ is a homeomorphism and $f(\mathbf{T}) = \Gamma$ is a rectifiable Jordan curve. Then $|\Gamma_r| := \int_{\mathbf{T}} |\partial_\theta u(re^{i\theta})| d\theta$ is increasing in r so that $|\Gamma_r| \le |\Gamma|$. Especially, the angular derivative of u satisfies $\partial_\theta u(z) \in h^1$.

Proof. By differentiating the Fourier-series representation

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{g}_n r^{|n|} e^{in\theta}$$

we see immediately that $\partial_\theta u(z)$ is the harmonic extension to U of the distributional derivative $\partial_\theta g$. By assumption, g is of bounded variation, and hence $\partial_\theta g$ is a finite (signed) Radon measure, which implies that $\partial_\theta u \in h^1$.

It is well-known (see [153]) that for functions in h^1 the integral average $\int_{\mathbf{T}} |\partial_\theta u(re^{i\theta})| d\theta$ is increasing in r.

Lemma (4.1.13)[137]: Let $g \in L^p(\boldsymbol{U})$ with $p > 1$. Then there is a unique solution to the Poisson equation $\Delta v = g$ such that $v \in C(\overline{U})$ and $v_{|\boldsymbol{T}} = 0$. Moreover, the weak derivative Dv can be modified in a set of measure zero so that

$$\int_0^{2\pi} |Du(re^{i\theta})| d\theta \le C(g) < \infty \quad \textit{for } r \in (1/2, 1).$$

Proof. The classical regularity theory for elliptic equations (see [138], [144] yields a quick approach, as it guarantees that our Poisson equation has a unique solution v in the Sobolev space $W^{2,p}(U)$, and continuity up to the boundary follows from the inclusion $W^{2,p}(U) \subset C(\overline{U})$. Then the derivatives satisfy $\partial_z, \partial_{\bar z} \in W^{1,p}(U)$. Especially, we then have $\|Dv\|_{W^{1,p}}(B(0,r)) \leq C'$ for any $r \in (1/2, 1)$. At this stage the trace theorem (see e.g. [156]) for the space $W^{1,p}(U)$ and a simple scaling argument shows that for a suitable representative of Dv it holds that

$$\|(Dv)_r\|_{W^{1-1/p,p}(\boldsymbol{T})} \leq C' \text{ for } r \in (1/2, 1).$$

Here $(Dv)_r$ stands for the function $\boldsymbol{T} \ni \theta \mapsto v(re^{i\theta})$. The claim follows by observing the continous imbeddings $W^{1-1/p,p}(\boldsymbol{T}) \subset L^p(\boldsymbol{T}) \subset L^1(\boldsymbol{T})$.

Recall also that any analytic (or anti-analytic) function in h^1 can be represented as the Poisson integral of an L^1-function, see [153] or [145]. In order to proceed towards the absolute continuity of boundary values of f, write $f = a + \bar{b} + v$, where v solves $\triangle v = g := \Delta f$ with $v_{|\boldsymbol{T}} = 0$ and a and b are analytic in the unit disk. Since $u := a + \bar{b} = \mathcal{P}[f_{|\boldsymbol{T}}]$, where $f_{|\boldsymbol{T}}$ is a homeomorphism, it follows from Lemma (4.1.12) that $\partial_t u = i(za' - \overline{zb'}) \in h^1(\boldsymbol{U})$, because $f(\boldsymbol{T})$ is a rectifiable curve. Further, the weak derivatives satisfy

$$f_z = a' + vz, \qquad f_{\bar z} = \overline{b'} + v_{\bar z}$$

Now we use that

$$|f_{\bar z}| \leq k|f_z|, \qquad k = \frac{K-1}{K+1}$$

which implies that

$$|a' + v_z| \leq k|b' + \overline{v_{\bar z}}|.$$

As

$$b' = \frac{\bar z}{z}a' - \frac{i}{z}\overline{u_t},$$

we obtain for $z \neq 0$ that

$$|a'| \leq k\left|\frac{\bar z}{z}\overline{a'} - \frac{i}{z}\overline{u_t} + \overline{v_{\bar z}}\right| + |v_z|.$$

This yields for $|z| \geq 1/2$ the inequality, valid almost everywhere

$$|a'| \leq \frac{1}{1-k}(2|\overline{u_t}| + |\overline{v_{\bar z}}| + |v_z|).$$

Our assumption on the size of the Laplacian of f yields that $\Delta f \in L^p(\mathbb{D})$ for some $p > 1$. By combining this with above inequality, and noting that
$\overline{u_t} \in h^1$ by Lemma (4.1.12), we infer from (and simple argument that uses Fubini as the above inequality holds only for a.e. z) that $a' \in H^1$. Then the relation $b' = \frac{\bar z}{z}a' - \frac{i}{z}\overline{u_t}$ verifies that also $b \in H^1$. Thus $\partial_t u$ is the Poisson integral of an L^1 function, and we conclude that $f_{|\boldsymbol{T}} = u_{|\boldsymbol{T}}$ is absolutely continuous.

In order to prove the optimality of Theorem (4.1.11), we are to construct quasiconformal maps with non-absolutely continuous boundary values, but with not too large Laplacian. For that end it is easier to work in the upper half space $\mathbb{C}^+ := \{z : \operatorname{Im} z > 0\}$. We need to produce quasisymmetric functions on $\mathbb{R}$ which can be quasiconformally extended to the upper half plane with not too large Laplacian, so somehow the function itself should be as smooth as possible while its derivative still possessing a singular part.

We will produce the desired functions with the help of Zygmund measures.

Recall first that a bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$ is Zygmund if

$$|g(x+t) + g(x-t) - 2g(x)| \leq C|t| \text{ for all } x, t \in \mathbb{R}.$$

The smallest possible C above is the Zygmund norm of g. If g is increasing, its derivative is a positive finite Borel measure, $g' = \mu$, on $\mathbb{R}$ and we call g a singular Zygmund function if, in addition, μ is singular. It is well-known that there exists singular Zygmund measures, see [152] or [146]. [142].

We next recall a modified version of the Beurling-Ahlfors extension, due to Fefferman, Kenig and Pipher [143]. For that end denote the Gaussian density by $\psi(x) := (2\pi)^{-1/2}e^{-x^2/2}$, and notice that $-\psi'(x) = -x\psi(x)$. As usual, for $t > 0$ we define the dilation $\psi_t(x) := t^{-1}\psi(x/t)$, and ψ_t' is defined in analogous way. Then the extension u of and (at most polynomially) increasing homeomorphism $g : \mathbb{R} \to \mathbb{R}$ is defined by setting

$$u(x+it) := (\psi_t * g)(x) + i(-\psi_t' * g)(x), \qquad \text{for all } x + it \in \mathbb{C}^+. \tag{22}$$

Obviously, u is smooth in $\mathbb{C}^+$ and it has the right boundary values. We have:

Lemma (4.1.14)[137]: ([143]). If $g : \mathbb{R} \to \mathbb{R}$ is quasisymmetric, then the extension u defined via (22) defines a quasiconformal homeomorphism of
$\mathbb{C}^+$ whose boundary map coincides with g.

We need one more auxiliary result:

Lemma (4.1.15)[137]: Assume that $g : \mathbb{R} \to \mathbb{R}$ is Zygmund. Then the extension (22) of g satisfies

$$|\Delta u(x+it)| \le Ct^{-1} \, for\ all\ x \in \mathbb{R}, \quad t > 0, \quad and$$
$$|\nabla u(x,it)| \le C\log\big(max(e,t^{-1})\big) \ \ for\ all\ x \in \mathbb{R}, \quad t > 0,$$

where $C > 0$ is a constant.

Proof. Let us first observe that if g is Zygmund, then for any $\varphi \in W^{2,1}(\mathbb{R})$ (i.e. $\varphi, \varphi'' \in L^1(\mathbb{R})$) we have

$$\left\|\frac{d^2}{dx^2}\varphi_t * g\right\|_{L^\infty(\mathbb{R})} = O(t^{-1}), \qquad for\ all\ t > 0. \tag{23}$$

We note that this follows easily from the mere definition of Zygmund functions if φ is even, but for general φ we shall use the fact that g can be decomposed as the sum $g = \sum_{j=0}^{\infty} g_j$, where $\|g_j\|_{L^\infty(\mathbb{R})} = O(2^{-j})$ and $\|g_j''\|_{L^\infty(\mathbb{R})} = O(2^j)$ for all $j \ge 0$, see [154]. We may compute in two ways

$$\frac{d^2}{dx^2}\big(\varphi_t * g(x)\big) = t^{-1}\int_{-\infty}^{\infty}\varphi_t(x-y)g''(y)dy$$
$$= t^{-3}\int_{-\infty}^{\infty}\varphi_t''(x-y)g(y)dy.$$

By assuming first that $t \le 1$ with $t \sim 2^{-k}$ we apply the first formula above to the sum $g = \sum_{j=0}^{k} g_j$, and the second one to the remainder $g = \sum_{j=k+1}^{\infty} g_j$. By noting that $\int_{-\infty}^{\infty}|\varphi_t(y)|dy = O(t)$ and $\int_{-\infty}^{\infty}|\varphi_t''(y)|dy = O(t)$, we obtain

$$\left|\frac{d^2}{dx^2}(\varphi_t * g(x))\right| = O\left(t^{-1}\cdot t\sum_{j=1}^{k}2^j + t^{-3}\cdot t\sum_{j=k+1}^{\infty}2^{-j}\right) = O(t^{-1}),$$

which proves (23) for $t \in (0,1]$. If $t > 1$ we simply apply the second formula directly on the bound $\|g\|_{L^\infty(\mathbb{R})} < \infty$ and obtain $\left\|\frac{d^2}{dx^2}\varphi_t * g\right\|_{L^\infty(\mathbb{R})} \le O(t^{-2}) = O(t^{-1})$ for $t > 1$.

We then consider the Laplacian of the extension u of g. Since ψ, $\psi' \in W^{2,1}(\mathbb{R})$, we obtain immediately from (23) that $\left|\frac{d^2}{dx^2}u(x,t)\right| = O(t^{-1})$ uniformly in $x \in \mathbb{R}$. In turn, to consider differentiation with respect to t, assume that $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and $(1 + |t|^2)\phi(t)$ is integrable. Then

$$\frac{d}{dt}\varphi_t * g(x) = \int_{-\infty}^{\infty}\left(-t^{-2}\varphi_t(x-y) - t^{-3}(x-y)\varphi_t'(x-y)\right)g(y)dy$$

$$= \int_{-\infty}^{\infty} g(y)\frac{d}{dy}(t^{-2}(x-y)\varphi_t(x-y)))dy$$

$$= -t^{-1}\int_{-\infty}^{\infty}$$

$$= (\varphi_1)_t * g'(x),$$

where $\varphi_1(y) := -y\varphi(y)$. An iteration gives, by denoting $\varphi_2(y) := y^2\varphi(y)$,

$$\frac{d^2}{dt^2}\left(\varphi_t * g(x)\right) = (\varphi_2)_t * g''(x)$$

$$= \frac{d^2}{dx^2}\left((\varphi_2)_t * g(x)\right). \tag{24}$$

Since all the functions $t\psi(t), t^2\psi(t), t\psi(t), t^2\psi(t)$ and their second derivatives are integral, we may apply (24) and obtain as before the desired estimate for $\frac{d^2}{dt^2}u(x,t)$.

The stated estimate for ∇u is proven in a similar way. We use the fact that for in the decomposition $g = \sum_{j=0}^{\infty} g_j$, one may in addition demand that $\|g_j'\|_\infty \leq C$ for all $j \geq 1$ (see [155]), which yields as before for $t \sim 2^{-k} < 1$

$$\left|\frac{d}{dx}\left(\varphi_t * g(x)\right)\right| = O\left(t^{-1} \cdot t\sum_{j=1}^{k} 1 + t^{-2} \cdot t\sum_{j=k+1}^{\infty} 2^{-j}\right)$$

$$= O(\log(t^{-1}).$$

The case $t \geq 1$ is trivial, and the case of the t-derivative is reduced to estimating the x-derivative as before.

After these preparations it is now a simple matter to produce the desired example. Let g_0 be a singular Zygmund function which is constant outside $[-1,1]$ so that Set $g(x) = x + g_0(x)$ for $x \in \mathbb{R}$. Then, as g_0 is Zygmund, the function g is quasi symmetric. Then its Fefferman-Kenig-Pipher extension $u : \mathbb{C}^+ \to \mathbb{C}^+$ is quasiconformal with non-absolutely continuous boundary values over $[-1,1]$. Since the extension of the linear function $x \mapsto x$ is linear, we see that the Laplacian of u equal that of the extension of g_0, and by the previous lemma we obtain the estimate

$$|\Delta u(x + it)| \leq Ct^{-1}\, for\, all\, x + it \in \mathbb{C}^+.$$

Let $h : \mathbb{D} \to \Omega'$ be conformal, where Ω' is a bounded and smooth Jordan domain that is contained in the upper half space $\mathbb{C}^+$ and contains $[-2,2]$ as a boundary segment. Next, set $\Omega = u(\Omega')$ so that Ω is smooth by our construction. Finally, let $\tilde{h} : \Omega \to \mathbb{D}$ be conformal end set $f := u \circ h$. Then f satisfies all the requirements, as the main terms in the formula for the Laplacian of f (compare to (19)) are $|\Delta u|$ and $|\nabla u|^2$, and the previous lemma also yields suitable bounds for the gradient term.

Corollary (4.1.16)[137]: If f is a quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary such that $\Delta f \in L^p(\mathbb{D})$ for some $p > 1$, then $f_{|T}$ is an absolutely continuous function. The claim fails in general if $p < 1$.

Proof. The counterexample constructed above obviously works also for the Corollary. In a similar vain, hand, the proof of the positive direction of the Theorem also applies as such for the Corollary since it in fact used as a starting point the fact that $\Delta u \in L^p(\mathbb{D})$ for some $p > 1$.

However, it is an open problem whether Corollary (4.1.16) is true for the exponent $p = 1$, as merely implementing the Kahane measures described above into our proof seems not to give enough extra decay for the Laplacian.

Section (4.2): Harmonic K-Quasiconformal Mappings

A complex-valued function $f(z)$ of class C^2 is said to be a harmonic mapping, if it satisfies $f_{z\bar{z}} = 0$. Assume that $f(z)$ is a harmonic mapping defined in a simply connected domain $\Omega \subseteq \mathbb{C}$. Then $f(z)$ has the canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in Ω. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk; we consider harmonic mappings $f(z)$ in $\mathbb{D}$.

For $z \in \mathbb{D}$, let

$$\Lambda_f(z) = \max_{0\le\theta\le2\pi}\left|f_z(z) + e^{-2i\theta}f_{\bar{z}}(z)\right| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\le\theta\le2\pi}\left|f_z(z) + e^{-2i\theta}f_{\bar{z}}(z)\right| = \left||f_z(z)| - |f_{\bar{z}}(z)|\right|.$$

It is well known that f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies

$$J_f(z) = \lambda_f(z)\Lambda_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0 \text{ for } z \in \mathbb{D}.$$

Let

$$\beta_h = \sup_{z,w\in\mathbb{D},z\neq w}\frac{|f(z) - f(w)|}{\rho(z,w)}$$

be the Bloch constant of f, where ρ denotes the hyperbolic distance in $\mathbb{D}$, and $\rho(z,w) = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)$ where r is the modulus of $\frac{z-w}{1-\bar{z}w}$. In [163], we see that the Bloch constant of $f = h + \bar{g}$ can be expressed in terms of the modulus of the derivatives of h and g as follows:

$$\beta_h = \sup_{z\in\mathbb{D}}(1 - |z|^2)(|h'(z)| + |g'(z)|).$$

For the extensive discussions on harmonic Bloch mappings, see [158]–[162], [167].

For $v \in (0,\infty)$, a harmonic mapping f is called a harmonic v-Bloch mapping if and only if

$$\|f\|_{B_v} := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^v\Lambda_f(z) < \infty. \tag{25}$$

Harmonic mappings are nature generalizations of analytic functions. Many classical results of analytic functions under some suitable restrictions can be extended to harmonic mappings. One of the well-known results is the Landau-type theorems for harmonic mappings. Many have considered such an active topic.

In [168], Liu proved the following theorems.

Theorem (4.2.1)[157]: Suppose that f is a harmonic mapping of $\mathbb{D}$ with $f(0) = \lambda_f(0) - 1 = 0$.

If $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$, then

$$|a_n| + |b_n| \le \frac{\Lambda^2 - 1}{n\Lambda} \quad n = 2, 3, \ldots. \tag{26}$$

The above estimates are sharp for all $n \ge 2$ with extremal functions $f_n(z) = \Lambda^2 z - \int_0^z \frac{(\Lambda^3 - \Lambda)dz}{\Lambda + z^{n-1}}$.

Theorem (4.2.2)[157]: Let f be a harmonic mapping of $\mathbb{D}$ with $f(0) = \lambda_f(0) - 1 = 0$, and $\Lambda_f(z) \le \Lambda$ for all $z \in \mathbb{D}$. Then f is univalent in the disk D_{r_1} with $r_1 = \frac{1}{1+\Lambda-\frac{1}{\Lambda}}$ and $f(D_{r_1})$ contains a schlicht disk D_{σ_1} with

$$\sigma_1 = \begin{cases} 1 + \left(\Lambda - \frac{1}{\Lambda}\right) \ln \frac{\Lambda - \frac{1}{\Lambda}}{1 + \Lambda - \frac{1}{\Lambda}} & \Lambda > 1 \\ 1 & \Lambda = 1. \end{cases}$$

The result is sharp when $\Lambda = 1$.

Subsequently, in 2011, Chenetal. [161] proved the following theorems.

Theorem (4.2.3)[157]: Let $f = h + \bar{g}$ be a harmonic v-Bloch mapping, where h and g are analytic in $\mathbb{D}$ with the expansions

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \ and \ g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{27}$$

If $\lambda_f(0) = \alpha$ for some $\alpha \in (0, 1)$ and $\|f\|_{B_v} \le M$ for $M > 0$. Then for $n \ge 2$,

$$|a_n| + |b_n| \le A_n(\alpha, v, M) = \inf_{0<r<1} \mu(r)$$

where

$$\mu(r) = \frac{M^2 - \alpha^2(1 - r^2)^{2v}}{nr^{n-1}(1 - r^2)^v M}.$$

Particularly, if $v = M = \alpha = 1$, then $A_2(1,1,1) = 0$, $A_3(1,1,1) = \frac{1}{3}$ and for $n \ge 4$, $A_n(1,1,1) < \frac{(n+1)eM}{2n}$. The above results are sharp for $n = 2$ and $n = 3$.

Theorem (4.2.4)[157]: Let f be a harmonic mapping with $f(0) = \lambda_f(0) - \alpha = 0$ and $\|f\|_{B_v} \le M$, where M and $\alpha \in (0, 1]$ are constants. Then f is univalent in $\mathbb{D}_{\rho_0}$, where

$$\rho_0 = \psi(r_0) = \max_{0<r<1} \psi(r),$$

$$\psi(r) = \frac{\alpha r(1 - r^2)M}{\alpha M (1 - r^2)^v - \alpha^2(1 - r^2)^{2v} + M^2}.$$

Moreover, $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 = r_0[\alpha + \frac{M^2 - \alpha^2 (1 - r^2)^{2v}}{M(1 - r_0^2)^v} \log \frac{M^2 - \alpha^2(1 - r^2)^{2v}}{\alpha M (1 - r_0^2)^v - \alpha^2(1 - r_0^2)^{2v} + M^2}.$$

The coefficient estimates are crucial in obtaining Landau-type theorems. By using Parseval equation, we first obtain the coefficient estimates for harmonic v-Bloch mappings, and then for $0 < v < \frac{1}{2}$, we obtain its Landau-type theorems.

Assume that

$$f(z) = P[F](z) = \int_0^{2\pi} P(r, x - \varphi)F(x)dx$$

is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ with the boundary function $F(x) = e^{i\gamma(x)}$ where

$$P(r, x - \varphi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(x - \varphi) + r^2)}$$

is the Poisson kernel and $z = re^{i\varphi} \in \mathbb{D}$. Then $f(z)$ is called a harmonic K-quasiconformal mapping if there exists a constant k such that

$$\sup_{z\in\mathbb{D}} \left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \leq k = \frac{K - 1}{K + 1}.$$

For harmonic K-quasiconformal mappings defined in $\mathbb{D}$, there are many interesting results (See [164], [166], [14] and [171]–[174]). In [1], Partyka and Sakan proved the following theorem:

Theorem (4.2.5)[157]: Given $K \geq 1$ and let $f(z) = P[F](z)$ be a harmonic K-quasiconformal mapping of $\mathbb{D}$ onto itself, with the boundary function $F(t)$. If $f(0) = 0$, then for a.e. $z = e^{it} \in \partial\mathbb{D}$

$$(\frac{2^{5(1-K^2)/2}}{(K^2 + K - 1)^K} \leq |F'(t)| \leq K^{3K} 2^{5\left(1-\frac{1}{k}\right)/2}. \quad (28)$$

Using this theorem, we obtain the coefficient estimates for $f = P[F]$ as follows:

$$|a_n| + |b_n| \leq B_n(K) = \frac{4}{n\pi} K^{3K} 2^{5\left(1-\frac{1}{k}\right)/2}, \qquad n = 1, 2, \ldots.$$

Theorem (4.2.6)[157]: Assume that $f(z) = h(z) + \overline{g(z)}$ is a harmonic v-Bloch mapping such that $f(0) = 0$ and $\|f\|_{B_v} \leq M$ for some constants $M > 0$, where $h(z)$ and $g(z)$ are given by (27). Then the following inequality

$$|a_n|^2 + |b_n|^2 \leq A_n(v, M)$$

holds for all $n = 1, 2, 3, \ldots$, where

$$A_n(v, M) = \begin{cases} \frac{M^2}{n} \inf_{0<t<1} \frac{1 - (1 - t^2)^{1-2v}}{t^{2n}(1 - 2v)} & v \neq \frac{1}{2} \\ \frac{M^2}{n} \inf_{0<t<1} \frac{-\ln(1 - t^2)}{t^{2n}} & v = \frac{1}{2} \end{cases}.$$

Furthermore, if $0 < v < 1$, then $\lim_{n\to\infty} A_n(v, M) = 0$. If $v \geq 1$, then $A_n(v, M) \leq \frac{M^2}{2v-1} \frac{(n+1)^{2v-1}-1}{n} \left(1 + \frac{1}{n}\right)^n$.

Proof. Using the assumption that $f(0) = 0$ and $\|f\|_{B_v} \leq M$, according to (25), we have

$$\Lambda_f(z) = |h'(z)| + |g'(z)| \leq \frac{M}{(1 - |z|^2)^v} := \Lambda_r$$

holds for any $z = re^{i\theta} \in \mathbb{D}$. Using $f_\theta(z) = i\left[zh'(z) - \overline{zg'(z)}\right]$ and applying Parseval equation, then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f_\theta\left(re^{i\theta}\right)\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=1}^{\infty} na_n r^n e^{in\theta} - \sum_{n=1}^{\infty} n\overline{b_n} r^n e^{in\theta}\right|^2 d\theta$$

$$= \sum_{n=1}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2\right) r^{2n}.$$

Applying $|f_\theta(z)| \leq |z|\Lambda_f(z) \leq r\Lambda_r$, we have

$$\sum_{n=1}^{\infty} n^2 \, (|a_n|^2 \;+\; |b_n|^2) r^{2n} \le\; r^2 \Lambda_r^2 \le \frac{r^2 M^2}{(1 \;-\; r^2)^{2v}}.$$

This implies that

$$\sum_{n=1}^{\infty} n^2 \, (|a_n|^2 \;+\; |b_n|^2) r^{2n-1} \le \frac{r M^2}{(1 \;-\; r^2)^{2v}}.$$

For any $0 \;<\; t \;<\; 1$, integrals from both sides give

$$\sum_{n=1}^{\infty} n(|a_n|^2 \;+\; |b_n|^2) \frac{r^{2n}}{2} \le\; M^2 \int_0^t \frac{r}{(1 \;-\; r^2)^{2v}} dr :=\; M^2 \varphi(t). \tag{30}$$

(i) For $v \;=\frac{1}{2}$. In this case, $\varphi(t) \;=\frac{-\ln(1-t^2)}{2}$. It follows from (30) that

$$|a_n|^2 \;+\; |b_n|^2 \;\le \frac{M^2}{n} \frac{-\ln(1 \;-\; t^2)}{t^{2n}}.$$

If $n \;=\; 1$, then $\min_{0<t<1} \frac{M^2}{n} \frac{-\ln(1-t^2)}{t^2} \;=\; M^2$. For $n \;>\; 1$, since $\lim_{t \to 0} \frac{-\ln(1-t^2)}{t^{2n}} \;= \infty =$ $\lim_{t \to 1} \frac{-\ln(1-t^2)}{t^{2n}}$, we see that $\inf_{0<t<1} \frac{-\ln(1-t^2)}{t^{2n}}$ exists. Hence,

$$|a_n|^2 \;+\; |b_n|^2 \le\; A_n\left(\frac{1}{2}, M\right) \;= \frac{M^2}{n} \inf_{0<t<1} \frac{-\ln(1-t^2)}{t^{2n}}.$$

Let $t_0 \;= \sqrt{\frac{n}{n+1}}$. Then

$$A_n\left(\frac{1}{2}, M\right) \le \frac{M^2}{n} \frac{-\ln(1-t^2)}{t_0^{2n}} = \frac{M^2 \ln(n \;+\; 1)}{n} \left(1 \;+ \frac{1}{n}\right)^n. \tag{31}$$

This implies that $\lim_{n\to\infty} A_n\left(\frac{1}{2}, M\right) \;=\; 0$.

(ii) For $v \;\neq \frac{1}{2}$. In this case, $\varphi(t) \;= \frac{1-(1-t^2)^{1-2v}}{2(1-2v)}$. It follows from (30) that

$$|a_n|^2 + |b_n|^2 \le \frac{M_2}{n} \frac{1 \;-\; (1 \;-\; t^2)^{1-2v}}{(1 \;-\; 2v) t^{2n}} := \frac{M^2}{n} m(t).$$

If $v \;< \frac{1}{2}$, then $\inf_{0<t<1} m(t) \;=\; \frac{1}{1-2v}$. Hence,

$$A_n(v, M) \le \frac{M^2}{n(1 \;-\; 2v)}, \qquad \left(v \;< \frac{1}{2}\right). \tag{32}$$

For $v > \frac{1}{2}, m(t) \;= \frac{1-(1-t^2)^{2v-1}}{(1-t^2)^{2v-1}(2v-1)t^{2n}} > 0$. If $n \;=\; 1$, then $\inf_{0<t<1} m(t) \;=\; 2v - 1$. Else if $n \;>\; 1$, then since $\lim_{t\to 0} m(t) = \infty = \lim_{t \to 1} m(t)$ we see that $\inf_{0<t<1} m(t)$ exists. Therefore $A_n(v, M) = \frac{M^2}{n} \inf_{0<t<1} m(t)$ and

$$A_n(v, M) \le \frac{M^2}{n} m(t_0) = \frac{M^2}{2v \;-\; 1} \frac{(n \;+\; 1)^{2v-1} \;-\; 1}{n} \left(1 \;+ \frac{1}{n}\right)^n, \qquad \left(v \;> \frac{1}{2}\right). \tag{33}$$

It follows from (31), (32) and (33) that if $v \;<\; 1$, then $\lim_{n\to\infty} A_n(v, M) \;=\; 0$. If $v \;=\; 1$, then $A_n(1, M) \;\le\; M^2 \left(1 \;+ \frac{1}{n}\right)^n$. If $v \;>\; 1$, then $A_n(v, M) \le \frac{M^2}{2v-1} \frac{(n+1)^{2v-1}-1}{n} \left(1 \;+ \frac{1}{n}\right)^n =$ $O(n^{2v-2})$.

This completes the proof.

Example (4.2.7)[157]: For $v \;=\; 1$, we consider harmonic function:

$$f(z) = \sum_{n=1}^{\infty} z^{2^n}.$$

Then

$$\frac{|zf'(z)|}{1 - |z|} \leq \sum_{n=1}^{\infty}\left(\sum_{2^k \leq n} 2^k\right)|z|^n \leq \sum_{n=1}^{\infty} 2n|z|^n = \frac{2|z|}{(1 - |z|)^2},$$

Hence,

$$(1 - |z|^2)|f'(z)| \leq 4 \quad (|z| < 1).$$

It follows from (25) that $f(z)$ is a 1-Bloch harmonic function. Moreover, its coefficients do not tend to 0.

Theorem (4.2.8)[157]: Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic v-Bloch mapping of $\mathbb{D}$ satisfying $f(0) = \lambda_f(0) - 1 = 0$ and $0 < v < \frac{1}{2}$. Then f is univalent in the disk $\mathbb{D}_{r_*} := \{z : |z| < r_*\}$, where r_* is the root of the following equation:

$$1 - M\sqrt{\frac{2}{1 - 2v}}\Phi(r) = 0$$

and $\Phi(r) := \sum_{n=1}^{\infty} \sqrt{n + 1}r^n$.

Proof. Let $z_1 = r_1 e^{i\theta_1} \in \mathbb{D}_r$ and $z_2 = r_2 e^{i\theta_2} \in \mathbb{D}_r$, where $0 < r < r_*$ and $z_1 \neq z_2$. For $0 < v < \frac{1}{2}$, applying Theorem (4.2.6), we have

$$|a_n| + |b_n| \leq \sqrt{2(|a_n|^2 + |b_n|^2)} \leq \sqrt{\frac{2}{1 - 2v}}\frac{M}{\sqrt{n}}.$$

Then

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq \lambda_f(0)|z_1 - z_2| - |z_1 - z_2|\sum_{n=2}^{\infty}(|a_n| + |b_n|)nr^{n-1} \\ &\geq |z_1 - z_2|\left(1 - M\sqrt{\frac{2}{1 - 2v}}\sum_{n=2}^{\infty}\sqrt{n}r^{n-1}\right) \\ &= |z_1 - z_2|\left(1 - M\sqrt{\frac{2}{1 - 2v}}\Phi(r)\right) \\ &:= |z_1 - z_2|\varphi(r). \end{aligned}$$

Since $\varphi(r)$ is a continuous decreasing function satisfying $\varphi(0) = 1, \lim_{r\to 1^-}\varphi(r) = -\infty$, we see that equation $\varphi(r) = 0$ has the root $0 < r_* < 1$. Then for any $0 < r < r_*$, we have $|f(z_1) - f(z_2)| > 0$. This shows that $f(z)$ is univalent in the disk D_{r_*}. The proof is completed.

For $M = 1$ and some constants $v \in \left(0, \frac{1}{2}\right)$, when calculated by computer, we obtain some r_* which were shown by the following table:

M	v	r_*

1	1/5	0.264534
1	1/4	0.248227
1	1/3	0.214222
1	49/100	0.0650995

Theorem (4.2.9)[157]: Given $K \geq 1$, let $f(z) = P[F](z) = h(z) + \overline{g(z)}$ be a harmonic K-quasiconformal self-mapping of $\mathbb{D}$ satisfying $f(0) = 0$ with the boundary function F, where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \ and \ g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbb{D}$. Then

$$|a_n| + |b_n| \leq B_n(K) := \frac{4}{n\pi} K^{3K} 2^{5(K-1/K)/2} \quad n = 1, 2, \ldots. \tag{35}$$

In particular, if $K = 1$ then $|a_n| + |b_n| \leq B_n(1) = \frac{4}{n\pi}$.

Proof. For every $z = re^{i\theta} \in \mathbb{D}$,

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

We find that

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta, \qquad n = 1, 2, \ldots,$$

$$\overline{b_n} r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{in\theta} d\theta, \qquad n = 1, 2, \ldots.$$

For every n (see [169] and [170]), we set $a_n = |a_n| e^{i\alpha_n}$, $b_n = |b_n| e^{i\beta_n}$ and $\theta_n = \frac{\alpha_n + \beta_n}{2n}$.

Then

$$(|a_n| + |b_n|) r^n = \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \left[e^{-in\theta} e^{-i\alpha_n} + e^{i\beta_n} e^{in\theta} \right] d\theta \right|$$
$$= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \left[e^{-in(\theta+\theta_n)} + e^{in(\theta+\theta_n)} \right] d\theta \right|$$
$$= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \cos n(\theta + \theta_n) \, d\theta \right|.$$

Integrating by parts, we have

$$(|a_n| + |b_n|) r^n = \left| \frac{1}{n\pi} \int_0^{2\pi} f_\theta(re^{i\theta}) \sin n(\theta + \theta_n) \, d\theta \right|. \tag{36}$$

In [165], Kalaj proved that the radial limits of f_θ and f_r exist almost everywhere and

$$\lim_{r \to 1^-} f_\theta(re^{i\theta}) = F(\theta),$$

for almost every $z = re^{i\theta} \in \mathbb{D}$. Here F is the boundary function of f. Hence, tending $r \to 1^-$ in (36) and also using (28), we obtain:

$$|a_n| + |b_n| \le \frac{1}{n\pi}\int_0^{2\pi} |F(\theta)||\sin n(\theta + \theta_n)|d\theta$$
$$\le \frac{4K^{3K}\,2^{5\left(K-\frac{1}{K}\right)/2}}{n\pi}.$$

This completes the proof.

Section (4.3): Quasiconformal Harmonic Mappings

One of central questions on harmonic mapping theory is under what condition a homeomorphism F of the unit circle onto a Jordan curve γ generates, via Poisson integral a harmonic diffeomorphism. A fundamental result in this direction is the Rado–Choquet–Kneser theorem which asserts that, if γ is convex and F is a homeomorphism, then $w = P[F]$ is a diffeomorphism. Further, an interesting question is that, under what condition on F and γ, $w = P[F]$ is quasiconformal. O. Martio was the first to observe such a question. Pavlović in [14] solved this problem for γ being the unit disk. Kalaj solved this problem for γ being a convex Jordan curve of class $C^{1,\alpha}$ in [37] and for Dini's smooth Jordan curve in [182]. Zhu in [174] considered this problem for general convex Jordan curve. For some different approaches in the plane concerning the class of q.c. harmonic mappings see [177], [180], [19]–[1], [187], [157]. Some recent optimal results on the generalization of this class has been done in [176], [137], [188]. We focus our attention in some quantitative estimates of quasiconformal constant of a mapping via its trace F mapping the unit circle onto a strictly convex Jordan curve γ. This is done in Theorems (4.3.9), (4.3.10) and (4.3.11). One of main tools in the proof is Lemma (4.3.6), which makes itself an interesting result.

The function

$$P(r,t) = \frac{1 - r^2}{2\pi(1 - 2r\cos t + r^2)}, \qquad 0 \le r < 1, \qquad t \in [0,2\pi],$$

is called the Poisson kernel. The Poisson integral of a complex-valued function $F \in L^1(\mathbf{T})$ is a complex-valued harmonic function given by

$$w(z) = u(z) + iv(z) = P[F](z)$$
$$= \int_0^{2\pi} P(r, -\tau)F(e^{it})dt, \tag{37}$$

where $z = re^{i\tau} \in \boldsymbol{U}$. Here $\boldsymbol{U} := \{z \in \boldsymbol{C} : |z| < 1\}$ and $\boldsymbol{T} := \{z \in \boldsymbol{C} : |z| = 1\}$. On the other hand the following claim holds:

Claim (4.3.1)[175]: If w is a bounded harmonic function, then there exists a function $F \in L^\infty(\mathbf{T})$, such that $w(z) = P[F](z)$ (see e.g. [21]).

See Axler, Bourdon and Ramey [21] for good setting of harmonic functions.

The Hilbert transformation of a function $\chi \in L^1(\mathbf{T})$ is defined by the formula

$$\tilde{\chi}(\tau) = H(\chi)(\tau) = -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\chi(\tau + t) - \chi(\tau - t)}{2\tan(t/2)}dt.$$

Here $\int_{0^+}^{\pi}\Phi(t)dt := \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\Phi(t)dt$. This integral is improper and converges for a.e. $\tau \in [0,2\pi]$; this and other facts concerning the operator Hused can be found in the book of Zygmund [189]. If f is a complex-valued harmonic function then a complex-valued harmonic function $\tilde{f}$ is called the harmonic conjugate of f if $f + i\tilde{f}$ is an analytic function. Notice that such a $\tilde{f}$ is uniquely determined up to an additive constant. Let $\chi, \tilde{\chi} \in L^1(\mathbf{T})$. Then

$$P[\tilde{\chi}] = \widetilde{P[\chi]}, \tag{38}$$

where $\widetilde{P[\chi]}$ is the harmonic conjugate of $P[\chi]$ (see e.g. [186]).

Assume that $z = x + iy = re^{i\tau} \in \boldsymbol{U}$. The complex derivatives of a differential mapping $w: \boldsymbol{U} \to \boldsymbol{C}$ are defined as follows:

$$w_z = \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right)$$

and

$$w_{\bar{z}} = \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

The derivatives of win polar coordinates can be expressed as

$$w_\tau(z) := \frac{\partial w(z)}{\partial \tau} = i(zw_z - \bar{z}w_{\bar{z}})$$

and

$$w_r(z) := \frac{\partial w(z)}{\partial r} = (e^{i\tau}w_z + e^{-i\tau}w_{\bar{z}}).$$

The Jacobian determinant of w is expressed in polar coordinates as

$$J_w(z) = |w_z|^2 - |w_{\bar{z}}|^2 = \frac{1}{r}\operatorname{Im}(w_\tau \overline{w_r}) = \frac{1}{r}\operatorname{Re}(iw_r \overline{w}_\tau). \tag{39}$$

Assume that $w = P[F](z)$ is a harmonic function defined on the unit disk $\boldsymbol{U}$. Then there exist two analytic functions hand k defined in the unit disk such that $w = h + \overline{k}$. Moreover $w_\tau = i(zh'(z) - \overline{zk'(z)})$ is a harmonic function and $rw_r = zh'(z) + \overline{zk'(z)}$ is its harmonic conjugate.

Assume now that F is Lipschitz continuous. Then $F' \in L^1(\mathbf{T})$ and by (37), using integration by parts, it follows that w_τ equals the Poisson integral of F':

$$w_\tau(re^{i\tau}) = \int_0^{2\pi} P(r, \tau - t)F'(t)dt.$$

Let $0 < \alpha < \pi/2$ and define

$$\Gamma_\alpha = \{z : \arg z \in [\pi - \alpha, \pi + \alpha]\}$$

and

$$\Gamma_\alpha(s) = \boldsymbol{U} \cap e^{is}(\Gamma_\alpha + 1).$$

That is, $\Gamma_\alpha(s)$ is the wedge inside the unit disk with angle 2α: whose axis passes between eisand zero. We say that a function $f: \boldsymbol{U} \to \boldsymbol{C}$ has a nontangential limit at e^{is}: if for $0 < \alpha < \pi/2$ the following limit exists

$$g(s) = \lim_{\Gamma_\alpha(s) \ni z \to e^{is}} f(z)$$

and do not depends on α.

We now recall Fatou's theorem [21]:

Claim (4.3.2)[175]: If $G \in L^1(\boldsymbol{T})$, then the Poisson extension $W(z) = P[G](z)$ has non-tangential limit at almost every $\zeta \in \mathbf{T}$.

By using Fatou's theorem we have that the radial limits of w_τ exist a.e. and
$$\lim_{r\to 1^-} w_\tau(re^{i\tau}) = F'(\tau) \ (a.e.). \tag{40}$$
If F is Lipschitz continuous, then $\Phi = F' \in L^\infty(\mathbf{T})$, and by famous Marcel Riesz theorem (see e.g. [4]), for $1 < p < \infty$ there is a constant A_p such that
$$\|H(F')\|_{L^p(\mathbf{T})} \le A_p \|F'\|_{L^p(\boldsymbol{T})}.$$
It follows that $\widetilde{\Phi} = H(F') \in L^1$. Since rw_r is the harmonic conjugate of w_τ, according to (38), we have $rw_r = P[H(F')]$, and by using again the Fatou's theorem we have
$$\lim_{r\to 1^-} w_r(re^{i\tau}) = H(F')(\tau) \ (a.e.). \tag{41}$$
Suppose that γ is a rectifiable Jordan curve in the complex plane $\boldsymbol{C}$. Denote by l the length of γ and let $g: [0, l] \mapsto \gamma$ be an arc length parameterization of γ, i.e. a parameterization satisfying the condition:
$$|g'(s)| = 1 \text{ for all } s \in [0, l].$$
We will say that γ is of class $C^{1,\alpha}, 0 < \alpha \le 1$, if g is of class C^1 and
$$\sup_{t,s} \frac{|g'(t) - g'(s)|}{|t - s|^\alpha} < \infty.$$
Definition (4.3.3)[175]: Let $f: [a, b] \to \boldsymbol{C}$ be a continuous function. The modulus of continuity of f is
$$\omega(t) = \omega_f(t) = \sup_{|x-y|\le t} |f(x) - f(y)|.$$
The function f is called Dini continuous if
$$\int_{0^+} \frac{\omega_f(t)}{t} dt < \infty. \tag{42}$$
Here $\int_{0^+} := \int_0^k$ for some positive constant k. A smooth Jordan curve γ with the length $l = |\gamma|$, is said to be Dini smooth if g' is Dini continuous. Observe that every smooth $C^{1,\alpha}$ Jordan curve is Dini smooth.

Let
$$K(s,t) = \mathrm{Re}[\overline{(g(t) - g(s))} \cdot ig'(s)] \tag{43}$$
be a function defined on $[0, l] \times [0, l]$. By $K(s \pm l, t \pm l) = K(s,t)$ we extend it on $\mathbf{R} \times \mathbf{R}$. Note that $ig'(s)$ is the inner unit normal vector of γ at $g(s)$ and therefore, if γ is convex then
$$K(s,t) \ge 0 \text{ for every } s \text{ and } t. \tag{44}$$
Suppose now that $F: \boldsymbol{R} \mapsto \gamma$ is an arbitrary 2π periodic Lipschitz function such that $F_{|[0,2\pi)} : [0, 2\pi) \mapsto \gamma$ is an orientation preserving bijective function. Then there exists an increasing continuous function $f: [0, 2\pi] \mapsto [0, l]$ such that
$$F(\tau) = g(f(\tau)). \tag{45}$$
We will identify $[0, 2\pi)$ with the unit circle $\boldsymbol{T}$, and $F(s)$ with $F(e^{is})$. In view of the previous convention we have for a.e. $e^{i\tau} \in \boldsymbol{T}$ that
$$F'(\tau) = g'(f(\tau)) \cdot f'(\tau),$$
and therefore
$$|F'(\tau)| = |g'(f(\tau))| \cdot |f'(\tau)| = f'(\tau).$$

Along with the function K we will also consider the function K_F defined by

$$K_F(t,\tau) = \mathrm{Re}[\overline{(F(t) - F(\tau))} \cdot iF'(\tau)].$$

It is easy to see that

$$K_F(t,\tau) = f'(\tau)K(f(t), \quad f(\tau)). \tag{46}$$

Now we prove the following subtle lemma which can be of interest for its own right.

We need the following lemma

Lemma (4.3.4)[175]: For $y \in [0,1]$ and $x \in [0,\pi]$ we have $\sin(xy) - y\sin x \geq 0$.

Proof. Let $h(x) = \sin(xy) - y\sin x$. Then $h'(x) = y\cos(xy) - y\cos x = y(\cos(yx) - \cos x)$ and so $h'(x) \geq 0$, because $\cos$ is decreasing on $[0,\pi]$. Thus his an increasing function on $[0,1]$. Since $h(0) = 0$, we obtain that $h(x) \geq 0$.

Lemma (4.3.5)[175]: For every bi-Lipschitz diffeomorphism $\phi : [0,\pi] \to [0,\pi]$, we have

$$\operatorname{ess\,inf} \phi'(x) \leq \frac{\sin\phi(x)}{\sin x} \leq \operatorname{ess\,sup} \phi'(x). \tag{47}$$

Proof. Let

$$h(x) = \frac{sin\ \phi(x)}{\sin x}. \tag{48}$$

Then his differentiable in $[0,\pi]$. Then

$$h'(x) = -\frac{\cot x}{\sin x}\sin\phi(x) + \frac{\cos\phi(x)}{\sin x}\phi'(x).$$

The stationary points of h satisfy the equation

$$\phi'(x)\frac{\cos\phi(x)}{\sin x} - \frac{\cos x}{\sin x}h = 0, \tag{49}$$

i.e.

$$h^2(x)\cos^2 x = \phi'(x)^2\cos^2\phi(x).$$

Since from (47) we have

$$h^2(x)\sin^2 x = sin^2\ \phi(x),$$

we obtain

$$h^2(x) = \phi'(x)^2\cos^2\phi(x) + \sin^2\phi(x). \tag{50}$$

Since

$$\pi = \phi(\pi) - \phi(0) = \int_0^\pi \phi'(x)dx$$

and $\phi'(x) \geq 0$, we have that $\min_x(\phi'(x)) \leq 1 \leq \max_x(\phi'(x))$. Now in view of (50), it follows that

$$\min_x(\phi'(x))^2 \leq h^2(x) \leq \max_x(\phi'(x))^2.$$

Lemma (4.3.6)[175]: Assume that $f: [0,2\pi] \to [0,2\pi], f(0) = 0, f(2\pi) = 2\pi$ is a diffeomorphism such that $f'(0) = f'(2\pi)$ and

$$\int_0^{2\pi} e^{if(t)}dt = 0 \tag{51}$$

and let $m = \min f'(x)$ and $M = \max f'(x)$. Then the double inequality

$$m\left|\int_y^x \sin(t - y)\,dt\right| \leq \left|\int_y^x \sin\big(f(t) - f(y)\big)\,dt\right|$$

$$\leq M\left|\int_y^x \sin(t\ \ y)\,dt\right| \quad (52)$$

holds.

Proof. Extend first the mapping f to $\boldsymbol{R}$ by $f(x \pm 2k\pi) = f(x) \pm 2k\pi$. For $y > 0$ define the mapping $g(t) = f(t + y) - f(y)$, and observe that $g(0) = 0$ and $g(2\pi) = 2\pi$. Then we need to show that

$$m\left|\int_0^{x-y} \sin t\,dt\right| \leq \left|\int_0^{x-y} \sin g(t)\,dt\right| \leq M\left|\int_0^{x-y} \sin t\,dt\right|.$$

For simplicity use instead of $x - y$ the notation x. We have $\int_0^{2\pi} e^{ig(t)}dt = 0$. Further, assume that $g(\pi) = a \geq \pi$. In the contrary define $h(t) = 2\pi - g(2\pi - t)$, and then $g(\pi) = 2\pi - a > \pi$, and denote it as well by g. Assume first that $x \leq \pi$. Observe that $M = \max_x f'(x) = \max_x g'(x)$. Let $\phi(t) = \frac{\pi}{a} g(t)$. Then $\phi(0) = 0$ and $\phi(\pi) = \pi$. Thus the conditions of Lemma (4.3.5) are satisfied. It follows that

$$\sin\phi(t) \leq \frac{\pi}{a} M \sin t$$

and consequently

$$\frac{a}{\pi}\sin\phi(t) \leq M \sin t.$$

By Lemma(4.3.5) we have

$$\sin g(t) \leq \frac{a}{\pi}\sin\frac{\pi}{a}\,g(t) = \frac{a}{\pi}\sin\phi(t).$$

Combining we obtain that for $0 \leq t \leq x \leq \pi$,

$$\sin g(t) \leq M \sin t.$$

By integrating the previous inequality we obtain

$$\int_0^x \sin g(t)dt \leq M\int_0^x \sin t dt. \quad (53)$$

Since $g(\pi) = a > 0$, it follows that $\sin g(t) < 0$ for $t \in (\pi, 2\pi)$. Further let $a' \in (0, \pi)$ such that $g(a') = \pi$. This implies that

$$\int_0^x \sin g(t)dt \geq \int_0^\pi \sin g(t)\,dt = -\int_\pi^{2\pi} \sin g(t)\,dt \geq 0, \qquad x \in (a', \pi).$$

Having in mind the fact that for $x \in (0, a')$, $\int_0^x \sin g(t)dt \geq 0$, in view of (53) we have

$$\left|\int_0^x \sin g(t)\,dt\right| \leq M\left|\int_0^x \sin t\,dt\right|, \qquad x \in (0, \pi). \quad (54)$$

If $\pi < x < 2\pi$ then $2\pi - x < \pi$ and then we use (51). Namely

$$\int_0^x \sin g(t)\,dt = -\int_x^{2\pi} \sin g(t)\,dt = -\int_{x-2\pi}^0 \sin g(t)\,dt$$

$$= \int_0^{2x-x} \sin(-g(-t))\,dt,$$

and the function $h(t) = -g(-t)$ satisfies as well the condition of the lemma. Then we have

$$\int_0^{2x-x} \sin(-g(-t))\,dt \le M \int_0^{2x-x} \sin t\,dt,$$

and so

$$\left|\int_0^x \sin g(t)\,dt\right| \le M \left|\int_0^x \sin t\,dt\right|.$$

The second part of (52), i.e. the part

$$m\left|\int_0^x \sin t\,dt\right| \le \left|\int_0^x \sin f(t)\,dt\right|$$

can be proved similarly, but this case we assume that $g(\pi) = a \le \pi$. Let $\phi(t) = \frac{\pi}{a} g(t)$. Then $\phi(\pi) = \pi$ and by Lemmas (4.3.5) and (4.3.4),

$$m\frac{\pi}{a} \sin x \le \sin \phi(x) = \sin\frac{\pi}{a} g(x) \le \frac{\pi}{a} \sin g(x), \qquad x \in (0,\pi).$$

The rest is similar to the previous proof.

We consider quasiconformal harmonic mappings between the unit disk and strictly convex domains. Let D be a convex domain with C^2 Jordan boundary γ. Let in addition κ_z be the curvature of γ at $z \in \gamma$ and $\kappa_0 = \min\{\kappa_p : p \in \gamma\}$, $\kappa_1 = \max\{\kappa_p : p \in \gamma\}$, then $0 \le \kappa_0 < \kappa_1 < \infty$.

Lemma (4.3.7)[175]: Let γ be a C^2 strictly convex Jordan curve and let F be an arbitrary parametrization. Let $m = \min_{\tau\in[0,2\pi]}|F'(\tau)|$ and
$M = \max_{\tau\in[0,2\pi]}|F'(\tau)|$. Then we have the following double inequalities

$$\kappa_0 \le \frac{K(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \kappa_1, \tag{55}$$

and

$$\kappa_0 m^3 \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \kappa_1 M^3. \tag{56}$$

Proof. Let $\tilde{g}$ be arc length parametrization of the curve $\tilde{\gamma} = \frac{1}{|\gamma|}\gamma$, where $|\gamma|$ is the length of γ. Let $\tilde{\kappa}_0 = \min_{z\in\tilde{\gamma}} \tilde{\kappa}_z$ and $\tilde{\kappa}_1 = \max_{z\in\tilde{\gamma}} \tilde{\kappa}_z$, where $\tilde{\kappa}_z$ is the curvature of $\tilde{\gamma}$ at z. It is clear that

$$|\gamma|\kappa_{|\gamma|z} = \tilde{\kappa}_z. \tag{57}$$

Let

$$G(\sigma,\varsigma) := \frac{\langle \tilde{g}(\sigma) - \tilde{g}(\varsigma), i\tilde{g}'(\varsigma)\rangle}{2\sin^2\frac{\sigma-\varsigma}{2}}.$$

Since $\tilde{g}'(\varsigma)$ is a unit vector and γ is a C^2 strictly convex curve, there exists a diffeomorphism $\beta: R \to R, \beta(0) = 0, \beta(2\pi + \sigma) = 2\pi + \beta(\sigma)$ such that

$$\tilde{g}'(\sigma) = e^{i\beta(\sigma)}. \tag{58}$$

Therefore

$$G(\sigma, \varsigma) = \frac{\int_{\varsigma}^{\sigma} \sin\big(\beta(\tau) - \beta(\varsigma)\big)\, d\tau}{2\sin^2 \frac{\sigma - \varsigma}{2}}. \tag{59}$$

On the other hand from

$$\tilde{g}''(\tau) = i\beta'(\tau)e^{i\beta(\tau)}$$

it follows that

$$\kappa_{\tilde{g}(\tau)} = \beta'(\tau). \tag{60}$$

According to (58), we obtain first that

$$\int_0^{2\pi} e^{i\beta(\sigma)} d\sigma = \tilde{g}(0) - \tilde{g}(2\pi) = 0. \tag{61}$$

Thus

$$\int_0^{2\pi} \sin(\beta(\sigma)) d\sigma = \int_0^{2\pi} \cos(\beta(\sigma)) d\sigma = 0. \tag{62}$$

Therefore

$$\int_{\varsigma}^{\sigma} \sin(\beta(\tau) - \beta(\varsigma)) d\tau = \int_{[0,2\pi]\setminus[\varsigma,\sigma]} \sin(\beta(\varsigma) - \beta(\tau)) d\tau$$

and using Lemma (4.3.6) we obtain that

$$\min_{\tau} \beta'(\tau) \le G(\sigma, \varsigma) \le \max_{\tau} \beta'(\tau). \tag{63}$$

From (63) we obtain

$$\tilde{\kappa}_0 \le G(\sigma, \varsigma) \le \tilde{\kappa}_1. \tag{64}$$

On the other hand there exists a homeomorphism $\sigma: [0, 2\pi] \to [0, 2\pi]$ such that

$$F(\tau) = |\gamma|\tilde{g}(\sigma(\tau)).$$

Thus

$$F'(\tau) = |\gamma|\sigma'(\tau)\tilde{g}'(\sigma(\tau)) \tag{65}$$

and

$$|F'(\tau)| = |\gamma|\sigma'(\tau). \tag{66}$$

Thus

$$\begin{aligned} K_F(t,\tau) &= \langle \overline{F(t) - F(\tau)}, iF'(\tau)\rangle \\ &= |\gamma|^2 \sigma'(\tau) \langle \overline{\tilde{g}\big(\sigma(\tau)\big) - \tilde{g}\big(\sigma(t)\big)}, i\tilde{g}'\big(\sigma(\tau)\big)\rangle \\ &= |\gamma|^2 \sigma'(\tau) G\big(\sigma(t), \sigma(\tau)\big) \\ &\cdot 2\sin^2 \frac{\sigma(\tau) - \sigma(t)}{2}. \end{aligned} \tag{67}$$

By applying again Lemma (4.3.5) we obtain

$$\min_t \big(\sigma'(t)\big)^2 \le \frac{2\sin^2 \frac{\sigma(\tau) - \sigma(t)}{2}}{2\sin^2 \frac{\tau - t}{2}} \le \max_t \big(\sigma'(t)\big)^2. \tag{68}$$

Combining (64), (67) and (68) we obtain

$$|\gamma|^2\tilde{\kappa}_0\sigma'(\tau)\min_t(\sigma'(t))^2 \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le |\gamma|^2\tilde{\kappa}_1\sigma'(\tau)\max_t(\sigma'(t))^2. \quad (69)$$

Combining (69), (57) and (66) we obtain

$$\kappa_0 m^3 \le \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le \kappa_1 M^3.$$

This yields (56). In particular, if $F = g$, where g is natural parametrization of γ we obtain (55).

By using (39) in [181] has been obtained the following

Lemma (4.3.8)[175]: Let γ be a Dini smooth Jordan curve, denote by g its arc-length parameterization and assume that $F(t) = g(f(t))$ is a Lipschitz homeomorphism from the unit circle onto γ. If $w(z) = u(z) + iv(z) = P[F](z)$ is the Poisson extension of F, then for almost every $\tau \in [0,2\pi]$ exists the limit

$$J_w(e^{i\tau}) := \lim_{r\to 1^-} J_w(re^{i\tau})$$

and there holds the formula

$$J_w(e^{i\tau}) = \int_0^{2\pi} \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}}\frac{dt}{2\pi}. \quad (70)$$

From Lemma (4.3.7) and Lemma (4.3.8) we obtain following theorem.

Theorem (4.3.9)[175]: If $w = P[F]$ is a harmonic diffeomorphism of the unit disk onto a convex Jordan domain $D = \operatorname{int}\gamma \in C^2$, such that F is (m, M) bi-Lipschitz, then

$$\kappa_0 m^3 \le J_w(e^{i\tau}) \le \kappa_1 M^3. \quad (71)$$

Proof. From (70) we obtain

$$J_w(e^{i\tau}) = \int_0^{2\pi} \frac{K_F(t,\tau)}{2\sin^2\frac{\tau-t}{2}}\frac{dt}{2\pi}. \quad (72)$$

From (72) and (56) we obtain (71).

By using an approach of Jost, by constructing an one parameter family [179], see as well [178], and previous theorem and [36] we obtain

Theorem (4.3.10)[175]: If F is an a.c. homeomorphism of the unit disk $\boldsymbol{U}$ onto a convex Jordan domain bounded by the Jordan curve γ such that $|\partial_t F(e^{it})| \ge m$ and if $w = P[F]$ is a harmonic mapping defined on the unit disk, then for $z \in \boldsymbol{U}$ we have

$$J_w(z) \ge \mathcal{J} := \max\left\{\frac{m \cdot \operatorname{dist}^2(\gamma, w(0))}{4}, \kappa_0 m^3\right\}.$$

Theorem (4.3.11)[175]: a) Assume that Ω is a bounded convex domain containing 0 and let $\gamma = \partial\Omega \in C^2$ and assume that κ_ζ be the curvature of γ at $\zeta \in \gamma$. Further let $\kappa_0 = \min_{\zeta\in\gamma}\kappa_\zeta$ and $\kappa_1 = \max_{\zeta\in\gamma}\kappa_\zeta$ and let F be an absolutely continuous homeomorphism of the unit circle onto γ. Then $w = P[F] : \boldsymbol{U} \to \Omega$ is a quasiconformal mapping if and only if

$$0 < m = \operatorname{ess\,inf}|F'(\tau)|, \quad (73)$$

$$M := \operatorname{ess\,sup}|F'(\tau)| < \infty \quad (74)$$

and

$$\mathcal{H} := \operatorname{ess\,sup}_\tau |H(F')(\tau)| < \infty. \quad (75)$$

b) If F satisfies the conditions (73), (74) and (75), then $w = P[F]$ is K quasiconformal, where

$$K \leq \frac{M^2 + H^2 + \sqrt{(M^2 + H^2)^2 - 4\mathcal{J}^2}}{2J}. \tag{76}$$

The constant K is the best possible in the following sense, if w is the identity or it is a mapping close to the identity, then $K = 1$ or K is close to 1 (respectively).

c) Moreover, under conditions of a) the mapping w is bi-Lipschitz with the bi-Lipschitz constant L satisfying the inequality

$$L \leq K \max\left\{M, \frac{1}{m}\right\}. \tag{77}$$

In particular L is asymptotically sharp.

Proof. The part a) of this theorem coincides with [37]. Prove the part b). We have to prove that under the conditions (73), (74) and (75) w is K quasiconformal, where K is given by (76). This means that, we need to prove that the function

$$K(z) = \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|} = \frac{1 + |\mu|}{1 - |\mu|} \tag{78}$$

is bounded by K.

It follows from (40) and (41) that

$$\lim_{r\to 1^-} F_r\left(re^{i\tau}\right) = H(F')(\tau)(a.e.),$$

and

$$\lim_{r\to 1^-} F_\varphi\left(re^{i\tau}\right) = F'(\tau).$$

As

$$|w_z|^2 + |w_{\bar{z}}|^2 = \frac{1}{2}\left(|w_r|^2 + \frac{|F_\varphi|^2}{r^2}\right),$$

it follows that

$$\lim_{r\to 1^-}(|w_z|^2 + |w_{\bar{z}}|^2) \leq \frac{1}{2}(M^2 + \mathcal{H}^2). \tag{79}$$

On the other hand, by (71)

$$\lim_{r\to 1^-}(|w_z|^2 + |w_{\bar{z}}|^2) \geq \mathcal{J} := \max\left\{\kappa_0 m^3, \frac{d(0,\gamma)^2}{4}\right\}. \tag{80}$$

From (79) and (80) we obtain

$$\lim_{r\to 1^-} \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|} \leq C := \frac{M^2 + \mathcal{H}^2}{2\mathcal{J}}, \tag{81}$$

i.e.

$$\lim_{r\to 1^-} \frac{|w_{\bar{z}}|}{|w_z|} \leq \sqrt{\frac{C - 1}{C + 1}}. \tag{82}$$

By Lewy's theorem, $|w_{\bar{z}}|/|w_z|$ is a subharmonic function bounded by 1. From (82) it follows that

$$\frac{|w_{\bar{z}}|}{|w_z|} \leq \sqrt{\frac{C - 1}{C + 1}}.$$

Further

$$K = \frac{\sqrt{C+1}+\sqrt{C-1}}{\sqrt{C+1}-\sqrt{C-1}} = C + \sqrt{C^2-1}$$
$$= \frac{M^2 + \mathcal{H}^2 + \sqrt{(M^2+\mathcal{H}^2)^2 - 4\mathcal{J}^2}}{2\mathcal{J}}.$$

The last quantity is equal to 1 for F being identity because all the constants appearing at the quantity are 1 in this special case. Moreover, if F is close to identity in C^2 norm, then the quantity is close to 1 (cf. Example (4.3.12)).

It remains to prove the part c). As $|F'(t)| \le L$, since $\partial_\tau w(re^{i\tau}) = P[F'](re^{i\tau})$, by maximum principle it follows that $|\partial_\tau w(re^{i\tau})| \le M$. On the other hand, since w is $K-$quasiconformal, it follows that $|w_z| + |w_{\bar z}| \le K(|w_z| - |w_{\bar z}|)$. So $|w_z| + |w_{\bar z}| \le KM$, and thus w is KM-Lipschitz. Similarly we obtain that w is $K\max\{M, 1/m\}$ bi-Lipschitz.

Example (4.3.12)[175]: If F is the arc-parametrization of a C^2 convex Jordan curve γ, then $m = \|F\|_\infty = 1$. We assume w.l.g. that the length of γ is 2π. Further since $F'(\tau + t) = e^{i\beta(\tau+t)}$, by applying Lemma (4.3.5) again we obtain

$$|H[F'](\tau)| = \left| -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{F'(\tau+t) - F'(\tau-t)}{2\tan(t/2)}dt \right|$$
$$\le \frac{1}{\pi}\int_{0^+}^{\pi} \frac{\left|e^{i\beta(\tau+t)} - e^{i\beta(\tau-t)}\right|}{2\tan(t/2)}dt$$
$$= \frac{1}{\pi}\int_{0^+}^{\pi} \frac{2\left|\sin\frac{\left(\beta(\tau+t)-\beta(\tau-t)\right)}{2}\right|}{2\tan(t/2)}dt$$
$$\le \sup|F''(s)|\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin t}{\tan(t/2)}dt = \kappa_1.$$

So

$$K \le \frac{1+\kappa_1^2 + \sqrt{(1+\kappa_1^2)^2 - 4\kappa_0^2}}{2\kappa_0}.$$

If γ is the unit circle, then $\kappa_0 = 1 = \kappa_1$. So the estimate (76) is asymptotically sharp; if the curve γ approaches in C^2 topology to the unit circle centered at origin, then the quasiconformal constant tends to 1.

In particular if γ is the ellipse $\gamma = \{(x,y) : x^2/a^2 + y^2/b^2 = 1\}, a \le b, |\gamma| = 2\pi$, then $\kappa_0 = 1/b$ and $\kappa_1 = 1/a$ and

$$K \le \frac{1}{2}\left(1 + \frac{1}{a^2} + \sqrt{\left(1+\frac{1}{a^2}\right)^2 - \frac{4}{b^2}}\right)b.$$

Corollary (4.3.13)[240]: For $y \in [0,1]$ and $x \in [0,\pi]$ we have $\sin(xy) - y\sin x \ge 0$.

Proof. Let $h_r(x) = \sin(xy) - y\sin x$. Then $h_r'(x) = y\cos(xy) - y\cos x = y(\cos(yx) - \cos x)$ and so $h_r'(x) \ge 0$, because $\cos$ is decreasing on $[0,\pi]$. Thus his an increasing function on $[0,1]$. Since $h_r(0) = 0$, we obtain that $h_r(x) \ge 0$.

Corollary (4.3.14)[240]: For every bi-Lipschitz diffeomorphism $\phi^r : [0,\pi] \to [0,\pi]$, we have

$$\text{ess inf}\, \phi^{r\prime}(x) \le \frac{\sin \phi^r(x)}{\sin x} \le \text{ess sup}\, \phi^{r\prime}(x). \quad (83)$$

Proof. Let

$$h_r(x) = \frac{\sin \phi^r(x)}{\sin x}. \quad (84)$$

Then his differentiable in $[0,\pi]$. Then

$$h_r'(x) = -\frac{\cot x}{\sin x}\sin \phi^r(x) + \frac{\cos \phi^r(x)}{\sin x}\phi^{r\prime}(x).$$

The stationary points of h_r satisfy the equation

$$\phi^{r\prime}(x)\frac{\cos \phi^r(x)}{\sin x} - \frac{\cos x}{\sin x}h_r = 0, \quad (85)$$

i.e.

$$h_r^2(x)\cos^2 x = \phi^{r\prime}(x)^2\cos^2 \phi^r(x).$$

Since from (83) we have

$$h_r^2(x)\sin^2 x = sin^2\, \phi^r(x),$$

we obtain

$$h_r^2(x) = \phi^{r\prime}(x)^2\cos^2 \phi^r(x) + \sin^2 \phi^r(x). \quad (86)$$

Since

$$\pi = \phi^r(\pi) - \phi^r(0) = \int_0^\pi \phi^{r\prime}(x)dx$$

and $\phi^{r\prime}(x) \ge 0$, we have that $\min_x\left(\phi^{r\prime}(x)\right) \le 1 \le \max_x\left(\phi^{r\prime}(x)\right)$. Now in view of (86), it follows that

$$\min_x\left(\phi^{r\prime}(x)\right)^2 \le h_r^2(x) \le \max_x\left(\phi^{r\prime}(x)\right)^2.$$

Corollary (4.3.15)[240]: Assume that $f^r: [0,2\pi] \to [0,2\pi], f^r(0) = 0, f^r(2\pi) = 2\pi$ is a diffeomorphism such that $f^{r\prime}(0) = f^{r\prime}(2\pi)$ and

$$\int_0^{2\pi}\sum_r e^{if^r(t)}dt = 0 \quad (87)$$

and let $m = \min f^{r\prime}(x)$ and $M = \max f^{r\prime}(x)$. Then the double inequality

$$m\left|\int_y^x \sin(t-y)\,dt\right| \le \left|\int_y^x \sum_r \sin\left(f^r(t) - f^r(y)\right)dt\right| \le M\left|\int_y^x \sin(t-y)\,dt\right| \quad (88)$$

holds.

Proof. Extend first the mapping f^r to $\boldsymbol{R}$ by $f^r(x \pm 2k_r\pi) = f^r(x) \pm 2k_r\pi$. For $y > 0$ define the mapping $g^r(t) = f^r(t+y) - f^r(y)$, and observe that $g^r(0) = 0$ and $g^r(2\pi) = 2\pi$. Then we need to show that

$$m\left|\int_0^{x-y} \sin t\, dt\right| \le \left|\int_0^{x-y} \sum_r \sin g^r(t)\, dt\right| \le M\left|\int_0^{x-y} \sin t\, dt\right|.$$

For simplicity use instead of $x-y$ the notation x. We have $\int_0^{2\pi}\sum_r e^{ig^r(t)}dt = 0$. Further, assume that $g^r(\pi) = a \ge \pi$. In the contrary define $h_r(t) = 2\pi - g^r(2\pi - t)$, and then $g^r(\pi) = 2\pi - a > \pi$, and denote it as well by g^r. Assume first that $x \le \pi$. Observe that $M = \max_x f^{r\prime}(x) = \max_x g^{r\prime}(x)$. Let $\phi^r(t) = \frac{\pi}{a} g^r(t)$. Then $\phi^r(0) = 0$ and $\phi^r(\pi) = \pi$. Thus the conditions of Corollary (4.3.14) are satisfied. It follows that

$$\sin \phi^r(t) \le \frac{\pi}{a} M \sin t$$

and consequently

$$\frac{a}{\pi} \sin \phi^r(t) \le M \sin t.$$

By Lemma1.3we have

$$\sin g^r(t) \le \frac{a}{\pi} \sin \frac{\pi}{a} g^r(t) = \frac{a}{\pi} \sin \phi^r(t).$$

Combining we obtain that for $0 \le t \le x \le \pi$,

$$\sin g^r(t) \le M \sin t.$$

By integrating the previous inequality we obtain

$$\int_0^x \sin g^r(t) dt \le M \int_0^x \sin t dt. \tag{89}$$

Since $g^r(\pi) = a > 0$, it follows that $\sin g^r(t) < 0$ for $t \in (\pi, 2\pi)$. Further let $a' \in (0,\pi)$ such that $g^r(a') = \pi$. This implies that

$$\int_0^x \sum_r \sin g^r(t) dt \ge \int_0^\pi \sum_r \sin g^r(t)\, dt = -\int_\pi^{2\pi} \sum_r \sin g^r(t)\, dt \ge 0,$$
$$x \in (a', \pi).$$

Having in mind the fact that for $x \in (0, a')$, $\int_0^x \sin g^r(t) dt \ge 0$, in view of (89) we have

$$\left|\int_0^x \sum_r \sin g^r(t)\, dt\right| \le M\left|\int_0^x \sin t\, dt\right|, \qquad x \in (0,\pi). \tag{90}$$

If $\pi < x < 2\pi$ then $2\pi - x < \pi$ and then we use (87). Namely

$$\int_0^x \sum_r \sin g^r(t)\, dt = -\int_x^{2\pi} \sum_r \sin g^r(t)\, dt = -\int_{x-2\pi}^0 \sum_r \sin g^r(t)\, dt$$
$$= \int_0^{2x-x} \sum_r \sin\left(-g^r(-t)\right) dt,$$

and the function $h_r(t) = -g^r(-t)$ satisfies as well the condition of the lemma. Then we have

$$\int_0^{2x-x} \sum_r \sin\left(-g^r(-t)\right) dt \le M \int_0^{2x-x} \sin t\, dt,$$

and so

$$\left|\int_0^x \sum_r \sin g^r(t)\,dt\right| \le M \left|\int_0^x \sin t\,dt\right|.$$

The second part of (88), i.e. the part

$$m\left|\int_0^x \sin t\,dt\right| \le \left|\int_0^x \sum_r \sin f^r(t)\,dt\right|$$

can be proved similarly, but this case we assume that $g^r(\pi) = a \le \pi$. Let $\phi^r(t) = \frac{\pi}{a} g^r(t)$. Then $\phi^r(\pi) = \pi$ and by Corollaries (4.3.14) and (4.3.13),

$$m\frac{\pi}{a}\sin x \le \sin\phi^r(x) = \sin\frac{\pi}{a}g^r(x) \le \frac{\pi}{a}\sin g^r(x), \qquad x \in (0,\pi).$$

The rest is similar to the previous proof.

Corollary (4.3.16)[240]: Let γ be a C^2 strictly convex Jordan curve and let F_r be an arbitrary parametrization. Let $m = \min_{\tau\in[0,2\pi]}|F_r'(\tau)|$ and
$M = \max_{\tau\in[0,2\pi]}|F_r'(\tau)|$. Then we have the following double inequalities

$$(\kappa_r)_0 \le \frac{K(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le (\kappa_r)_1, \tag{91}$$

and

$$(\kappa_r)_0 m^3 \le \frac{K_{F_r}(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \le (\kappa_r)_1 M^3. \tag{92}$$

Proof. Let $\widetilde{g^r}$ be arc length parametrization of the curve $\tilde{\gamma} = \frac{1}{|\gamma|}\gamma$, where $|\gamma|$ is the length of γ. Let $\widetilde{(\kappa_r)}_0 = \min_{z\in\tilde{\gamma}}\widetilde{(\kappa_r)}_z$ and $\widetilde{(\kappa_r)}_1 = \max_{z\in\tilde{\gamma}}\widetilde{(\kappa_r)}_z$, where $\widetilde{(\kappa_r)}_z$ is the curvature of $\tilde{\gamma}$ at z. It is clear that

$$|\gamma|(\kappa_r)_{|\gamma|z} = \widetilde{(\kappa_r)}_z. \tag{93}$$

Let

$$G_r(\sigma,\varsigma) := \sum_r \frac{\langle \widetilde{g^r}(\sigma) - \widetilde{g^r}(\varsigma), i\widetilde{g^r}'(\varsigma)\rangle}{2\sin^2\frac{\sigma-\varsigma}{2}}.$$

Since $\widetilde{g^r}'(\varsigma)$ is a unit vector and γ is a C^2 strictly convex curve, there exists a diffeomorphism $\beta^r: R \to R, \beta^r(0) = 0, \beta^r(2\pi+\sigma) = 2\pi + \beta^r(\sigma)$ such that

$$\widetilde{g^r}'(\sigma) = e^{i\beta^r(\sigma)}. \tag{94}$$

Therefore

$$G_r(\sigma,\varsigma) = \frac{\int_\varsigma^\sigma \sum_r \sin\left(\beta^r(\tau) - \beta^r(\varsigma)\right)d\tau}{2\sin^2\frac{\sigma-\varsigma}{2}}. \tag{95}$$

On the other hand from

$$\widetilde{g^r}''(\tau) = i\beta^{r\prime}(\tau)e^{i\beta^r(\tau)}$$

it follows that

$$(\kappa_r)_{\widetilde{g^r}(\tau)} = \beta^{r\prime}(\tau). \tag{96}$$

According to (94), we obtain first that

$$\int_0^{2\pi} \sum_r e^{i\beta^r(\sigma)}d\sigma = \sum_r \left(\widetilde{g^r}(0) - \widetilde{g^r}(2\pi)\right) = 0. \tag{97}$$

Thus

$$\int_0^{2\pi} \sum_r \sin(\beta^r(\sigma))d\sigma = \int_0^{2\pi} \sum_r \cos(\beta^r(\sigma))d\sigma = 0. \quad (98)$$

Therefore

$$\int_\varsigma^\sigma \sum_r \sin(\beta^r(\tau) - \beta^r(\varsigma))d\tau = \int_{[0,2\pi]\setminus[\varsigma,\sigma]} \sum_r \sin(\beta^r(\varsigma) - \beta^r(\tau))d\tau$$

and using Corollary (4.3.15) we obtain that

$$\min_\tau \beta^{r\prime}(\tau) \le G_r(\sigma,\varsigma) \le \max_\tau \beta^{r\prime}(\tau). \quad (99)$$

From (99) we obtain

$$\widetilde{(\kappa_r)}_0 \le G_r(\sigma,\varsigma) \le \widetilde{(\kappa_r)}_1. \quad (100)$$

On the other hand there exists a homeomorphism $\sigma: [0,2\pi] \to [0,2\pi]$ such that

$$F_r(\tau) = |\gamma|\widetilde{g^r}(\sigma(\tau)).$$

Thus

$$F_r'(\tau) = |\gamma|\sigma'(\tau)\widetilde{g^r}'(\sigma(\tau)) \quad (101)$$

and

$$|F_r'(\tau)| = |\gamma|\sigma'(\tau). \quad (102)$$

Thus

$$\begin{aligned} K_{F_r}(t,\tau) &= \sum_r \langle \overline{F_r(t) - F_r(\tau)}, iF_r'(\tau)\rangle \\ &= |\gamma|^2\sigma'(\tau)\sum_r \langle \overline{\widetilde{g^r}(\sigma(\tau)) - \widetilde{g^r}(\sigma(t))}, i\widetilde{g^r}'(\sigma(\tau))\rangle \\ &= |\gamma|^2\sigma'(\tau)G_r(\sigma(t),\sigma(\tau)) \cdot 2\sin^2\frac{\sigma(\tau) - \sigma(t)}{2}. \end{aligned} \quad (103)$$

By applying again Corollary (4.3.14) we obtain

$$\min_t(\sigma'(t))^2 \le \frac{2\sin^2\frac{\sigma(\tau) - \sigma(t)}{2}}{2\sin^2\frac{\tau - t}{2}} \le \max_t(\sigma'(t))^2. \quad (104)$$

Combining (100), (103) and (104) we obtain

$$\begin{aligned} |\gamma|^2\widetilde{(\kappa_r)}_0\sigma'(\tau)\min_t(\sigma'(t))^2 &\le \frac{K_{F_r}(t,\tau)}{2\sin^2\frac{\tau - t}{2}} \\ &\le |\gamma|^2\widetilde{(\kappa_r)}_1\sigma'(\tau)\max_t(\sigma'(t))^2. \end{aligned} \quad (105)$$

Combining (105), (93) and (102) we obtain

$$(\kappa_r)_0 m^3 \le \frac{K_{F_r}(t,\tau)}{2\sin^2\frac{\tau - t}{2}} \le (\kappa_r)_1 M^3.$$

This yields (92). In particular, if $F_r = g^r$, where g^r is natural parametrization of γ we obtain (91).

Corollary (4.3.17)[240]: If $w^r = P[F_r]$ is a harmonic diffeomorphism of the unit disk onto a convex Jordan domain $D = \text{int}\gamma \in C^2$, such that F_r is (m, M) bi-Lipschitz, then

$$(\kappa_r)_0 m^3 \le J_{w^r}(e^{i\tau}) \le (\kappa_r)_1 M^3. \quad (107)$$

Proof. From (106) we obtain

$$J_{w^r}(e^{i\tau}) = \int_0^{2\pi} \sum_r \frac{K_{F_r}(t,\tau)}{2\sin^2\frac{\tau-t}{2}} \frac{dt}{2\pi}. \quad (108)$$

From (108) and (92) we obtain (107).

Corollary (4.3.18)[240]: a) Assume that Ω is a bounded convex domain containing 0 and let $\gamma = \partial\Omega \in C^2$ and assume that $(\kappa_r)_\zeta$ be the curvature of γ at $\zeta \in \gamma$. Further let $(\kappa_r)_0 = \min_{\zeta\in\gamma}(\kappa_r)_\zeta$ and $(\kappa_r)_1 = \max_{\zeta\in\gamma}(\kappa_r)_\zeta$ and let F_r be an absolutely continuous homeomorphism of the unit circle onto γ. Then $w^r = P[F_r] : \boldsymbol{U} \to \Omega$ is a quasiconformal mapping if and only if

$$0 < m = \text{ess inf}\,|F_r'(\tau)|, \quad (109)$$

$$M := \text{ess sup}\,|F_r'(\tau)| < \infty \quad (110)$$

and

$$\mathcal{H} := \operatorname*{ess\,sup}_\tau |H(F_r')(\tau)| < \infty. \quad (111)$$

b) If F_r satisfies the conditions (109), (110) and (111), then $w^r = P[F_r]$ is K quasiconformal, where

$$K \leq \frac{M^2 + H^2 + \sqrt{(M^2 + H^2)^2 - 4\mathcal{J}^2}}{2J}. \quad (112)$$

The constant K is the best possible in the following sense, if w^r is the identity or it is a mapping close to the identity, then $K = 1$ or K is close to 1 (respectively).

c) Moreover, under conditions of a) the mapping w^r is bi-Lipschitz with the bi-Lipschitz constant L satisfying the inequality

$$L \leq K \max\left\{M, \frac{1}{m}\right\}. \quad (113)$$

In particular L is asymptotically sharp.

Proof. The part a) of this theorem coincides with [37]. Prove the part b). We have to prove that under the conditions (109), (110) and (111) w^r is K quasiconformal, where K is given by (112). This means that, we need to prove that the function

$$K(z) = \frac{|w_z^r| + |w_{\bar z}^r|}{|w_z^r| - |w_{\bar z}^r|} = \frac{1 + |\mu|}{1 - |\mu|} \quad (114)$$

is bounded by K.

It follows from (40) and (41) that

$$\lim_{\epsilon\to 0} F_r\big((1-\epsilon)e^{i\tau}\big) = H(F_r')(\tau)\ (a.e.),$$

and

$$\lim_{\epsilon\to 0}((F_r)_{1-\epsilon})_\varphi\big((1-\epsilon)e^{i\tau}\big) = F_r'(\tau).$$

As

$$|w_z^r|^2 + |w_{\bar z}^r|^2 = \frac{1}{2}\left(|w_{1-\epsilon}^r|^2 + \frac{|(F_r)_\varphi|^2}{(1-\epsilon)^2}\right),$$

it follows that

$$\lim_{\epsilon\to 0}(|w_z^r|^2 + |w_{\bar z}^r|^2) \leq \frac{1}{2}(M^2 + \mathcal{H}^2). \quad (115)$$

On the other hand, by (107)

$$\lim_{\epsilon\to 0}(|w_z^r|^2 + |w_{\bar z}^r|^2) \geq \mathcal{J} := \max\left\{(\kappa_r)_0 m^3, \frac{d(0,\gamma)^2}{4}\right\}. \quad (116)$$

From (115) and (116) we obtain

$$\lim_{\epsilon\to 0}\frac{|w_z^r| + |w_{\bar z}^r|}{|w_z^r| - |w_{\bar z}^r|} \le C := \frac{M^2 + \mathcal{H}^2}{2\mathcal{J}}, \tag{117}$$

i.e.

$$\lim_{\epsilon\to 0}\frac{|w_{\bar z}^r|}{|w_z^r|} \le \sqrt{\frac{C-1}{C+1}}. \tag{118}$$

By Lewy's theorem, $|w_{\bar z}^r|/|w_z^r|$ is a subharmonic function bounded by 1. From (118) it follows that

$$\frac{|w_{\bar z}^r|}{|w_z^r|} \le \sqrt{\frac{C-1}{C+1}}.$$

Further

$$K = \frac{\sqrt{C+1}+\sqrt{C-1}}{\sqrt{C+1}-\sqrt{C-1}} = C + \sqrt{C^2-1}$$
$$= \frac{M^2 + \mathcal{H}^2 + \sqrt{(M^2+\mathcal{H}^2)^2 - 4\mathcal{J}^2}}{2\mathcal{J}}.$$

The last quantity is equal to 1 for F_r being identity because all the constants appearing at the quantity are 1 in this special case. Moreover, if F_r is close to identity in C^2 norm, then the quantity is close to 1.

It remains to prove the part c). As $|F_r'(t)| \le L$, since $\partial_\tau w^r((1-\epsilon)e^{i\tau}) = P[F_r']((1-\epsilon)e^{i\tau})$, by maximum principle it follows that $|\partial_\tau w^r((1-\epsilon)e^{i\tau})| \le M$. On the other hand, since w^r is $K-$quasiconformal, it follows that $|w_z^r| + |w_{\bar z}^r| \le K(|w_z^r| - |w_{\bar z}^r|)$. So $|w_z^r| + |w_{\bar z}^r| \le KM$, and thus w^r is KM-Lipschitz. Similarly we obtain that w^r is $K\max\{M, 1/m\}$ bi-Lipschitz.

Chapter 5

Lattice Problem and Comments of Steinhaus

We show that the argument cannot generalize to any lattice and, on the other hand, give some lattices to which this method applies. We also show there is no measurable Steinhaus set for a special honeycomb lattice, the standard tetrahedral lattice in $\mathbb{R}^3$. We show a slight generalisation of some as well as a very easy recovery of most of the known results using a unified treatment.

Section (5.1): On a Lattice Problem

Sometime in the 1950's, Steinhaus posed the following problem. Do there exist two sets A and S in the plane such that every set congruent to A has exactly one point in common with S? The trivial case where one of the sets is the plane and the other consists of a single point is ruled out. The first appearance of this problem seems to be in a 1958 of Sierpiński [200]. We showed the answer is yes, a result later rediscovered by Erdös [193]. There are many variants of this problem. For example, one could specify the set A. In this direction, Komjáth showed that such a set exists if $A = \mathbb{Z}$, the set of all integers [199]. Steinhaus also asked about the specific case where $A = \mathbb{Z}^2$. This problem also seems to be Sierpiński's 1958 where he mentions that in this case there is no set S which is bounded and open or else bounded and closed. This specific problem has been widely noted (see [191], [192]), but has remained unsolved until now. We answer this question.

Theorem (5.1.1)[190]: There is a set $S \subseteq \mathbb{R}^2$ such that for every isometric copy L of the integer lattice $\mathbb{Z}^2$ we have $|S \cap L| = 1$.

We work in the theory ZFC; the usual axioms of set theory with the axiom of choice (AC). AC is used heavily in the main construction as we require, for example, an enumeration of the equivalence classes of the lattices under a certain equivalence relation. By "lattice" we mean a set in the plane which is isometric with the integer lattice $\mathbb{Z}^2$ (a brief exception occurs in Lemmas (5.1.5), (5.1.6) where we consider scaled versions).

We point out that there are several things proven which are stronger than what is needed to prove Theorem (5.1.1). Stronger forms of our two main technical lemmas, Lemma (5.1.13) (Lemma (5.1.3)) and Lemma (5.1.29) (Lemma (5.1.29)), are proven here than is required for the main theorem. In [196] a shorter argument is given for the main theorem. For example, a shorter proof of Lemma (5.1.13) is given there. Here we give a more involved induction argument. This argument, which uses only basic number theory and combinatorics, shows something much stronger and interesting in its own right. We feel that these stronger results may be useful in resolving whether the main theorem holds for other lattices and other dimensions.

We note that the geometric Lemma (5.1.29) is also stronger than what is required for a proof of the main theorem. A weaker alternative is also indicated. It is also quite possible that something like Lemma (5.1.29) may be needed to resolve the problem for other lattices. We note that theorems similar to Lemma (5.1.29) may be found in the theory of mechanical linkages [197]. Recall a four-bar linkage may be described as two circles C_1, C_2, and a rigid "bar" connecting two points p_1, p_2 constrained to lie on C_1, C_2 respectively. If we consider a third point p_3, and require that the triangle $\Delta p_1 p_2 p_3$ be rigid, then the locus of points traced out by p_3 is called a coupler curve for the linkage. We say the coupler point p_3 is non-trivial if it is not one of the endpoints p_1, p_2. In this terminology Lemma (5.1.29) is the statement that the curve traced out by a non-trivial coupler point of a four-bar linkage has, except in the degenerate case noted, a finite intersection with any circle. In particular, Lemma (5.1.29) is implicit in the analysis of Gibson and Newstead [195]. Their analysis uses a fair amount of machinery from algebraic geometry. However, since we were not able to find the precise statement of the lemma and as it is crucial to our methods, we give two very different elementary proofs of it.

We call a set S as in Theorem (5.1.1) a Steinhaus set and note that whether there can be a Lebesgue measurable Steinhaus set remains unsolved. (We also do not know whether a Steinhaus set can be connected although one can prove that if it is measurable, then it is totally disconnected.) Concerning measurable Steinhaus sets, H. T. Croft [44] and, independently, J. Beck showed that there is no bounded measurable Steinhaus set [84] and Koulountzakis obtained some further refinements [198]. Also, Kolountzakis and Wolff showed that there is no measurable Steinhaus set for the higher dimensional version of Steinhaus' problem [83]. It is relatively easy to see that no Steinhaus set can be a Borel set or even have the Baire property if one follows the arguments given by Croft. We briefly sketch this argument. Suppose S has the Baire property. Since $\mathbb{R}^2 = \bigcup_{z \in \mathbb{Z}^2}(S + z)$, S cannot be meager. Fixing a ball with respect to which S is comeager and noting that the gaps between successive lattice distances converge to 0, we see that there is some ball M such that the part of S outside this ball is meager. Let E be the set of points where neither S nor $\mathbb{R}^2 \setminus S$ is meager in any neighborhood. Then E is a nonempty closed nowhere dense set and following the proof of Croft's paper, we see that there is an isometric copy L of $\mathbb{Z}^2$ which meets E in exactly one point, p. Thus, there is a ball $B(p, d)$ such that neither S nor $\mathbb{R}^2 \setminus S$ is meager in that ball but $\mathbb{R}^2 \setminus S$ is comeager in $B(x, d)$ for every $x \in L$ with $x \neq p$. But, this would mean there is a small translation of L which would entirely miss S. We also note that the question of whether there is a bounded Steinhaus set remains unsolved. Steinhaus' problem and variants were discussed in some detail by Croft [44] and have been updated in [191]. In particular, Steinhaus also asked about sets meeting each copy of the lattice points in exactly n points. The fact that the answer to this question is yes follows directly from our main theorem and is discussed in our concluding remarks.

We say a lattice distance is a real number of the form $\sqrt{n^2 + m^2}$ where $n, m \in \mathbb{Z}$. Theorem (5.1.1) is clearly equivalent to the existence of a set $S \subseteq \mathbb{R}^2$ satisfying the following two properties:

(i) For every isometric copy L of $\mathbb{Z}^2$, $S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S$, $\rho(z_1, z_2)$ is not a lattice distance, where ρ denotes the usual Euclidean distance.

In fact, we prove a slight strengthening of Theorem (5.1.1):

Theorem (5.1.2)[190]. There is a set $S \subseteq \mathbb{R}^2$ satisfying:

(i) For every isometric copy L of $\mathbb{Z}^2$ we have $S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S, \rho(z_1, z_2)^2 \notin \mathbb{Z}$.

We call a set $S \subseteq \mathbb{R}^2$ satisfying (ii) of Theorem (5.1.2) a partial Steinhaus set.

The Steinhaus problem has a natural interpretation for smaller sets of lattices. Namely, given an arbitrary set $\mathcal{L}$ of lattices (each of which is an isometric copy of $\mathbb{Z}^2$), we may ask whether there is a partial Steinhaus set S such that $S \cap L \neq \emptyset$ for all $L \in \mathcal{L}$. Indeed, establishing this restricted version of the problem for the case where $\mathcal{L}$ is the (countable) family of rational translations of $\mathbb{Z}^2$ is a central step toward proving Theorem (5.1.2). Actually, we need a slight technical strengthening of this "rational translation" case, which we state below.

In proving Theorem (5.1.2), it is natural to proceed inductively. That is, we build the desired set S in (transfinitely many) stages. At limit stages, we take unions, and at successor stages we enlarge S_α to $S_{\alpha+1}$ so as to meet a new lattice, while at the same time keeping property (ii). Note that (ii) is then trivially satisfied at limit stages. If we can meet every lattice L along the way, then the final set $S = \bigcup_\alpha S_\alpha$ will be as desired. While this is our general plan, there are several steps that must be taken to ensure its success. For example, we do not simply enumerate the lattices $\mathcal{L}$ in type 2^ω. To appreciate the difference, we note that there does exist a "finite obstruction". That is, there is a finite set of points $F \subseteq \mathbb{R}^2$ (in fact $F \subseteq \mathbb{Q}^2$) which forms a partial Steinhaus set, but which cannot be extended to meet even the integer lattice $\mathbb{Z}^2$ and remain a partial Steinhaus set. For example the following set of 17 points forms such an obstruction (this set was constructed by considering a partial good permutation of 65 of size 17 which cannot be extended to a good permutation of 65; these concepts are explained):

(216/5,2/5)	(107/5,4/5)	(283/5,1/5)	(174/5,3/5)
(677/13,5/13)	(340/13,10/13)	(744/13,2/13)	(407/13,7/13)
(70/13,12/13)	(474/13,4/13)	(137/13,9/13)	(541/13,1/13)
(204/13,6/13)	(712/13,11/13)	(271/13,3/13)	(779/13,8/13)
(2601/65,57/65)			

Rather, it is important that we use the "hull construction" which has played an important role in several other theorems of this general character (see [64], [194]). The idea, described abstractly, is to consider a continuous elementary chain $\{M_\alpha\}_{\alpha<2^\omega}$, of substructures (say of some large V_κ) with each M_α of size $< 2^\omega$, but $\mathbb{R} \subseteq \bigcup_{\alpha<2^\omega} M_\alpha$. Let $\mathcal{L}_\alpha$ denote the isometric copies of $\mathbb{Z}^2$ which are in M_α. At successor steps, we now enlarge S_α to $S_{\alpha+1}$ which meets all lattices $L \in \mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha$, while of course keeping property (ii). While this gives us more to do at each successor step, it also provides us with a powerful inductive assumption, namely, the closure of $\mathcal{L}_\alpha$under various operations. For the reader unfamiliar with the set-theoretic terminology, we may describe the idea as follows. We write the collection of lattices $\mathcal{L}$ as an increasing union of sets $\mathcal{L}_\alpha$ where at limit stages we take unions, and we require each $\mathcal{L}_\alpha$ to be closed under certain finitary functions $F_k : (\mathcal{L})^{<\omega} \to \mathcal{L}$. We could specify in advance which functions F_k we need the $\mathcal{L}_\alpha$ to be closed under, but it is more convenient not to. We note that when the continuum is greater than ω_1 , the actual construction we will use will be a bit more complicated, essentially an iteration of this hull construction.

We now state precisely two lemmas, which we call Lemma (5.1.13) and Lemma (5.1.29), which we will need to carry out the plan sketched above. The first of these is the "rational translation" case mentioned above.

Lemma (5.1.3)[190]. (A). Let $\mathcal{L}_{\mathbb{Q}}$ denote the set of rational translations of $\mathbb{Z}^2$, that is, lattices of the form $\mathbb{Z}^2 + (r, s)$ where $r, s \in \mathbb{Q}$. Then there is a set $S \subseteq \mathbb{R}^2$ satisfying the following:

(i) For every lattice $L \in \mathcal{L}_{\mathbb{Q}}, S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S, \rho(z_1, z_2)^2 \notin \mathbb{Z}$.

Actually, we require a slight technical strengthening of Lemma (5.1.13), which we call Lemma (5.1.12) . In this lemma, and for the rest, we adopt the following terminology. If $L \subseteq \mathbb{R}^2$ is a lattice, then by a "rational translation" of L we mean a lattice of the form $L + r\vec{u} + s\vec{v}$ where $r, s \in \mathbb{Q}$, and $\vec{u}, \vec{v}$ are the unit basis vectors for L. In other words, we are always referring to the coordinate system of the lattice L.

The second lemma is a result in pure plane geometry, which arises in carrying out the hull construction mentioned above.

We give the proof of Theorem (5.1.2) assuming Lemmas (5.1.12) and (5.1.29). We prove Lemma (5.1.12) , and we prove Lemma (5.1.29). We are self-contained and may be read independently.

We prove Theorem (5.1.2) assuming Lemmas (5.1.12) and (5.1.29). Throughout, "lattice" will mean an isometric copy of $\mathbb{Z}^2$. ω denotes the first infinite ordinal, which we identify with the set of natural numbers.

Recall that by a "rational translation" of a lattice L we are referring to the coordinate system of the lattice L. By a rational rotation of $\mathbb{Z}^2$ we mean an operation of the form $\mathbb{Z}^2 \to R(\mathbb{Z}^2)$, where R is a rotation of the plane whose corresponding matrix $M_R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ has rational entries. In this case, M_R must be of the form $\begin{pmatrix} \frac{a}{d} & -\frac{b}{d} \\ \frac{b}{d} & \frac{a}{d} \end{pmatrix}$, where a, b, d are integers and $a^2 + b^2 = d^2$. For a general lattice L, a rational rotation means a rotation about a point of L which is rational in the coordinate system of L.

Definition (5.1.4)[190]. Two lattices are equivalent, $L_1 \sim L_2$, if L_2 can be obtained from L_1 by rational rotations and translations.

This is equivalent to saying that in the coordinate system determined by L_1, the isometry moving L_1 to L_2 is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_5 \\ q_6 \end{pmatrix},$$

where all of the q_i are rational. Equivalently, $L_1 \sim L_2$ iff all of the points of L_2 have rational coordinates with respect to the coordinate system determined by L_1 (and vice-versa). This is easily an equivalence relation, with each equivalence class countable.

We first prove a lemma which will help us deal with rotations.

Lemma (5.1.5)[190]. Let L_1be a lattice, and let L_2 be obtained from L_1 by a rational rotation. Let $S \subseteq \mathbb{R}^2$ satisfy the following:

(i) For every lattice L which is a rational translation of $L_1, S \cap L \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S, \rho(z_1, z_2)^2 \notin \mathbb{Z}$.

Then for every lattice L' which is a rational translation of L_2 we have $S \cap L' \neq \emptyset$.

Proof. Without loss of generality we may assume $L_1 = \mathbb{Z}^2$. Let the rational rotation R correspond to the matrix $M = \begin{pmatrix} \frac{a}{d} & -\frac{b}{d} \\ \frac{b}{d} & \frac{a}{d} \end{pmatrix}$, where $a, b, d \in \mathbb{Z}, d > 1$, and $a^2 + b^2 = d^2$. $L_2 = R(\mathbb{Z}^2)$ has standard basis vectors $\vec{u} = (\frac{a}{d}, \frac{b}{d})$ and $\vec{v} = (-\frac{b}{d}, \frac{a}{d})$. It suffices to show, for any positive integer e such that $d|e$ and any rationals of the form $r = \frac{m}{e}, s = \frac{n}{e}$ (m, n integers), that $S \cap L'_{r,s} \neq \emptyset$, where $L'_{r,s} = L_2 + r\vec{u} + s\vec{v}$ is the rational translation of L_2 by (r, s). Fix a positive integer e with $d|e$. Consider the e^2 set of points of the form $\frac{m}{e}\vec{u} + \frac{n}{e}\vec{v}$, where $0 \leq m, n < e$. For each such point p, we must show that there are integers $k = k_p, l = l_p$ such that $p + k\vec{u} + l\vec{v} \in S$.

We require the following technical lemma whose proof we give below.

Lemma (5.1.6)[190]. Let e be a positive integer, and let R be the rational rotation with matrix $M = \begin{pmatrix} \frac{a}{d} & -\frac{b}{d} \\ \frac{b}{d} & \frac{a}{d} \end{pmatrix}$, where $d|e$. Let $L'_2 = \frac{1}{e}R(\mathbb{Z}^2)$. Then there are exactly e^2 points of the scaled lattice L'_2 which are of the form (x, y) with $0 \leq x, y < 1$.

Granting the lemma, we finish the proof of Lemma (5.1.5). Let T denote the e^2 set of points in L'_2 of the form (x, y) with $0 \leq x, y < 1$. Note that each of these points has coordinates (x, y) with x and y rational (in fact, their denominators can be taken to be de). By property (i) of S, for each such point (x, y) there are integers (k', l') such that $(x, y) + (k', l') \in S$. For each such (x, y), let $(x', y') = (x, y) + (k', l')$ denote the corresponding point in S. Clearly the map $f(x, y) = (x', y')$ from T into S is one-to-one. Thus $f[T]$ is a subset of S of size exactly e^2. Note also that in the coordinate system determined by L_2, each point of $f[T]$ has coordinates in $\frac{1}{e}\mathbb{Z}^2$ (since this is true of the points in T, and (k', l') has coordinates with respect to L_2 which have denominators d and $d|e$). For each point $(x', y') \in f[T]$, let k'', l'' be integers such that $(x'', y'') .= (x', y') + k''\vec{u} + l''\vec{v}$ has coordinates with respect to L_2 of the form $(\frac{m}{e}, \frac{n}{e})$, where $0 \leq m, n < e$. Let g be the function defined on $f[T]$ sending (x', y') to (x'', y''). Note that g is one-to-one, or else we would violate property (ii) of S. Thus, $(g \circ f)[T]$ consists of e^2 points which in the L_2 coordinate system all have coordinates of the form $(\frac{m}{e}, \frac{n}{e})$ where $0 \leq m, n < e$. Since there are only e^2 such points, $(g \circ f)[T]$ exhausts this set. By definition of g, we thus have for any point p having L_2 coordinates of the form $(\frac{m}{e}, \frac{n}{e})$, $0 \leq m, n < e$, that there are integers $k = -k'', l = -l''$ such that $p + k\vec{u} + l\vec{v} \in S$. This completes the proof of Lemma (5.1.5).

Proof . Scaling by e, the lemma follows immediately from the following well-known more general fact about lattices: Suppose $v_1, \ldots, v_d$ are linearly independent vectors in $\mathbb{Z}^d$. Let $D = det(v_1, \ldots, v_d)$. Then there are exactly D points of $\mathbb{Z}^d$ of the form $a_1v_1 + \cdots + a_dv_d$ where $0 \leq a_1, \ldots, a_d < 1$. To see this, let R be the fundamental domain for the lattice determined by the v_i. That is, $R = \{a_1v_1 + \cdots + a_dv_d : 0 \leq a_1, \ldots, a_d < 1, a_i \in \mathbb{R}\}$. Suppose there are D' points of $\mathbb{Z}^d$ in R. Clearly any translation of R of the form $R + n_1v_1 + \cdots + n_dv_d$, where the n_i are integers, also contains exactly D' points of $\mathbb{Z}^d$. Thus, $nR = \{a_1v_1 + \cdots + a_dv_d : 0 \le a_1, \ldots, a_d < n, a_i \in \mathbb{R}\}$ contains exactly $(D')n^d$ points of $\mathbb{Z}^d$. On the other hand, a volume argument shows this number to be of the form $(D + o(1))n^d$.

Lemma (5.1.7)[190]. Let L be a lattice and $z \in \mathbb{R}^2$. Suppose z has coordinates (x, y) with respect to the lattice L, where at least one of x, y is irrational. Then there is a line $l = l(z, L)$ such that if w has rational coordinates with respect to L and $w \notin l$, then $\rho(w, z)^2 \notin \mathbb{Q}$.

Proof. Without loss of generality, suppose $L = \mathbb{Z}^2$. Suppose $z = (x, y)$ with at least one of x, y irrational and $w = (a, b) \in \mathbb{Q}^2$. If $\rho(w, z)^2 \in \mathbb{Q}$, then $(x - a)^2 + (y - b)^2 \in \mathbb{Q}$, and so

$$x^2 + y^2 - 2ax - 2yb \in \mathbb{Q}.$$

If $w_1 = (a_1, b_1)$ and $w_2 = (a_2, b_2)$ were two such points, then subtracting the corresponding equations we would have

$$2(a_1 - a_2)x + 2(b_1 - b_2)y \in \mathbb{Q}. \quad (1)$$

If $w_3 = (a_3, b_3)$ were a third such point, then we likewise have

$$2(a_1 - a_3)x + 2(b_1 - b_3)y \in \mathbb{Q}. \quad (2)$$

If w_1, w_2, w_3 were not collinear, then we could solve equations (1), (2) for x and y, and these numbers would both be rational, a contradiction. Thus, all such points w (if any) must lie on a single line.

Lemma (5.1.8)[190]. Let L_1, L_2 be lattices which are not equivalent. Then there is at most one point which has rational coordinates with respect to both L_1 and L_2.

Proof. Assume without loss of generality that $L_1 = \mathbb{Z}^2$. If there were two points in $\mathbb{Q}^2$ having rational coordinates with respect to L_2, then the standard basis vectors $\vec{u}, \vec{v}$ of L_2 would also have rational coordinates. Since one point of L_2 has rational coordinates, it follows that all of the points of L_2 have rational coordinates, that is, $L_1 \sim L_2$.

We now turn to the proof of Theorem (5.1.2).

If $L \subseteq \mathbb{R}^2$ is an isometric copy of $\mathbb{Z}^2$, let $[L]$ denote the equivalence class of L under the equivalence relation ~ of Definition (5.1.4). Let $\mathfrak{L}$ denote the family of all equivalence classes. By AC, let $\mathcal{L} \to L(\mathcal{L})$ be a function which picks for each equivalence class $\mathcal{L}$ a member $L(\mathcal{L}) \in \mathcal{L}$.

To carry out the main construction, we first describe a particular enumeration of the equivalence classes of the lattices. Let $\kappa(\emptyset) = 2^\omega$, and let $\{M_{\alpha_0} : \alpha_0 < \kappa(\emptyset)\}$ be a continuous increasing chain of elementary substructures of a large V_κ ($V_{\omega+1}$ will actually suffice) with $|M_{\alpha_0}| < \kappa(\emptyset)$ for all $\alpha_0 < \kappa(\emptyset)$ and such that every equivalence class of lattices is in some M_{α_0}. Assume also $M_0 = \emptyset$. Let $N_{\alpha_0} = M_{\alpha_0+1} - M_{\alpha_0}$. In general, suppose that $M_{\vec{\alpha}}$ is defined for $\vec{\alpha}$ in a certain subtree of $ON^{<\omega}$. If $M_{\alpha_0,\cdots\alpha_k}$ is defined, we assume also that $\kappa(\alpha_0, \cdots, \alpha_{k-1})$ has been defined and is an uncountable cardinal. Furthermore, we assume in this case that $M_{\alpha_0,\cdots\alpha_{k-1},\beta}$ is defined $\beta < \kappa(\alpha_0, \cdots, \alpha_{k-1})$. We let $N_{\alpha_0,\cdots,\alpha_k}$ denote $M_{\alpha_0,\cdots\alpha_{k+1}} - M_{\alpha_0,\cdots\alpha_k}$.

Suppose now that $M_{\alpha_0,\cdots\alpha_k}$ is defined. If $N_{\alpha_0,\cdots\alpha_k}$contains only countably many equivalence classes of lattices, let $\mathcal{L}_{\alpha_0,\cdots\alpha_k;n}$enumerate them. In this case, $(\alpha_0,\cdots,\alpha_k)$ is a terminal node in the tree of indices $\vec{\alpha}$ for which $M_{\vec{\alpha}}$ is defined. Otherwise, let $\kappa(\alpha_0,\cdots,\alpha_k) = |N_{\alpha_0,\cdots\alpha_k} \cap \mathfrak{L}|$ and write

$$N_{\alpha_0,\cdots\alpha_k} = \bigcup_{\alpha_{k+1}<\kappa(\alpha_0,\cdots,\alpha_k)} M_{\alpha_0,\cdots\alpha_k,\alpha_{k+1}}$$

as a continuous, increasing union, where each $M_{\alpha_0,\cdots\alpha_k,\alpha_{k+1}}$ is the intersection of $N_{\alpha_0,\cdots\alpha_k}$ with an elementary substructure of V_κ, and each $M_{\alpha_0,\cdots\alpha_k,\alpha_{k+1}}$ contains fewer than $\kappa(\alpha_0,\cdots,\alpha_k)$ many equivalence classes of lattices. Assume also $M_{\alpha_0,\cdots\alpha_k;0} = \emptyset$;. Easily, the tree of indices is well founded (since the $\kappa_{\vec{\alpha}}$ are decreasing along any branch).

If $\vec{\alpha}$ is incompatible with $\vec{\beta}$, then $N_{\vec{\alpha}}$ and $N_{\vec{\beta}}$ have no equivalence class of lattices in common. Furthermore, every equivalence class occurs as some $\mathcal{L}_{\alpha_0,\cdots\alpha_k;n}$. Thus, the $\mathcal{L}_{\alpha_0,\cdots\alpha_k;n}$ precisely enumerate the equivalence classes of lattices. We consider the indices to be (well) ordered lexicographically.

The following simple lemma will be used.

Lemma (5.1.9)[190]. *Suppose $\vec{\alpha}$ is an index for which $M_{\vec{\alpha}}$ is de_ned. Let $a_1,\cdots a_m \in M_{\vec{\alpha}}$ and suppose b is definable from $a_1,\cdots a_m$ in V_κ. Then $b \in \bigcup_{\vec{\beta}\le\vec{\alpha}} M_{\vec{\beta}}$.*

Proof. Let $\vec{\alpha} = (\alpha_0,\cdots,\alpha_k)$ and assume $b \notin \bigcup_{\vec{\beta}\le\vec{\alpha}} M_{\vec{\beta}}$.. Since $M_{\alpha_0,\cdots\alpha_k}$ is relatively closed under the skolem functions of V_κ inside of $N_{\alpha_0,\cdots\alpha_{k-1}}$, it follows that $b \notin N_{\alpha_0,\cdots\alpha_{k-1}}$. Since $b \notin M_{\alpha_0,\cdots\alpha_{k-1}}$ by assumption, we thus have $b \notin M_{\alpha_0,\cdots\alpha_{k-1}+1}$. Continuing, we eventually have $b \notin M_{\alpha_0+1}$, a contradiction since M_{α_0+1} is a substructure of V_κ containing the a_i.

Now fix a terminal index $\vec{\alpha} = (\alpha_0,\cdots,\alpha_k)$. Assume inductively we have defined for each terminal index $\vec{\beta} < \vec{\alpha}$ a set $S_{\vec{\beta}} \subseteq \mathbb{R}^2$ which satisfies the following:

(i) If $\vec{\beta}_1 < \vec{\beta}_2 < \vec{\alpha}$, then $S_{\vec{\beta}_1} \subseteq S_{\vec{\beta}_2}$.

(ii) For every terminal index $\vec{\beta}$ less than $\vec{\alpha}, S_{\vec{\beta}}$ meets every lattice in every equivalence class $\mathcal{L}_{\vec{\beta};n}$.

(iii) Every point of $S_{\vec{\beta}} - \bigcup_{\vec{\gamma}<\vec{\beta}} S_{\vec{\gamma}}$ lies on some lattice of the form $\mathcal{L}_{\vec{\beta};n}$.

(iv) For all distinct $z_1, z_2 \in S_{\vec{\beta}}$, $\rho(z_1; z_2)^2 \notin \mathbb{Z}$.

(v) Suppose $\vec{\beta}_1 < \vec{\beta}_2 < \vec{\alpha}, x \in S_{\vec{\beta}_1}$, and $y \in S_{\vec{\beta}_2} - \bigcup_{\vec{\gamma}<\vec{\beta}_2} S_{\vec{\gamma}}$. Then, if $\rho(x,y)^2 \in \mathbb{Q}, x, y$ both have rational coordinates with respect to some lattice of the form $\mathcal{L}_{\vec{\beta}_2;n}$.

Let $S_{<\vec{\alpha}} = \bigcup_{\vec{\beta}<\vec{\alpha}} S_{\vec{\beta}}$. We show how to extend $S_{<\vec{\alpha}}$ a set $S_{\vec{\alpha}}$ also satisfying (iv), (v) and such that $S_{\vec{\alpha}}$ meets every lattice in each equivalence class $\mathcal{L}_{\vec{\alpha},n}$. This suffices to prove Theorem (5.1.2).

To ease notation, let $\mathcal{L}_n = \mathcal{L}_{\vec{\alpha},n}$, and let $L_n = L(\mathcal{L}_n)$. From Lemma (5.1.5), it. Suffices to maintain property (iv), to have property (v) when $\vec{\beta}_2 = \vec{\alpha}$, and to have $S_{\vec{\alpha}}$ meet every rational translation of each L_n (recall a rational translation of L_n refers to a motion which is a translation in the coordinate system of L_n).

For integers $n, d, i, j,$ let $L_n^{d,i,j,}$ denote the translation of L_n by the amount $(\frac{i}{d}, \frac{j}{d})$ (in the coordinate system of L_n).

Note for the following the simple fact that if two distinct points y, z lie on a lattice L, then L is definable from y and z. In fact, there are only finitely many lattices containing both y and z. More generally, if y, z both have rational coordinates with respect to L, then L is definable from y and z.

Claim (5.1.10)[190]. *For each n and rationals $\frac{i}{d}, \frac{j}{d}$, there is a finite set of lines $G_n(\frac{i}{d}, \frac{j}{d})$ with the following property: if $c \in S_{<\vec{\alpha}}$ does not have rational coordinates with respect to L_n, if $z \in L_n^{d,i,j,}$, and if $\rho(c,z)^2 \in \mathbb{Q}$, then $z \in \bigcup G_n(\frac{i}{d}, \frac{j}{d})$.*

Suppose there is a $z_1 \in L_n^{d,i,j,}$ and a $c_1 \in S_{<\vec{\alpha}}$ not rational with respect to L_n such that $\rho(z_1, c_1)^2 \in \mathbb{Q}$ (otherwise there is nothing to prove). Let $l_1 = l(c_1, L_n^{d,i,j,})$ be the line (necessarily through z_1) given by Lemma (5.1.7). Suppose there is a $z_2 \notin l_1, z_2 \in L_n^{d,i,j,}$, and a $c_2 \in S_{<\vec{\alpha}}$ not rational with respect to L_n with $\rho(z_2, c_2)^2 \in \mathbb{Q}$ (necessarily $c_2 \neq c_1$). Let $l_2 = l(c_2, L_n^{d,i,j,})$ be given by Lemma (5.1.7). Continuing, construct $z_m \in L_n^{d,i,j,}, c_m \in S_{<\vec{\alpha}}$ if possible so that $z_m \neq l_1 \cup \cdots \cup l_{m-1}$ and $\rho(z_m; c_m)^2 \in \mathbb{Q}$. If the construction fails at some point, then the claim is proved. Assume toward a contradiction that we continue to produce an infinite sequence $z_1, c_1, z_2, c_2, \cdots$. Note that the c_i are distinct. Let $\vec{\beta}^m = (\vec{\beta}_0^m, \cdots, \vec{\beta}_l^m)$ be the terminal index (where l depends on m) such that $c_m \in N_{\vec{\beta}^m}$. Thus, $\vec{\beta}^m < \overrightarrow{\alpha}$. Easily, there is a $k' \leq k$ such that for infnitely many m we have $\beta_0^m = \alpha_0, \cdots \beta_{k'-1}^m = \alpha_{k'-1}$, and $\beta_{k'}^m < \alpha_{k'}$ (we allow $k' = 0$, in which case we have $\beta_0^m < \alpha_0$). Let $\vec{\gamma} = (\alpha_0, \cdots, \alpha_{k'})$. Thus $\vec{\gamma} \leq \vec{\alpha}$, and for these infnitely many m we have $c_m \in M_{\vec{\gamma}}$. Let m_1, m_2, m_3 be three such m. Let $r_1 = \rho(c_{m_1}, z_{m_1})$, and similarly for r_2, r_3. We apply Lemma (5.1.29) to the circles with centers at c_{m_i} of radii r_i and the points z_{m_i}. Note that we are not in the exceptional case of Lemma (5.1.29), as otherwise we would have $\rho(z_{m_1}, z_{m_2}) = \rho(c_{m_1}, c_{m_2})$. This contradicts the fact that $\rho(c_{m_1}, c_{m_2})^2 \notin \mathbb{Z}$ as they lie in $S_{<\vec{\alpha}}$ (note that $\rho(z_{m_1}, z_{m_2})^2 \in \mathbb{Z}$ as z_{m_1}, z_{m_2} lie in $L_n^{d,i,j,}$). From Lemma (5.1.29), the points z_{m_1}, z_{m_2}, and z_{m_3} are definable from c_{m_1}, c_{m_2}, and c_{m_3}. Since L_n is definable from z_{m_1}, z_{m_2}, and z_{m_3} (in fact, from any two of them), L_n is definable from c_{m_1}, c_{m_2}, and c_{m_3}. It follows from Lemma (5.1.9) that L_n lies in some $M_{\vec{\beta}}$, for $\vec{\beta} \leq \vec{\alpha}$. This contradicts $L_n \in N_{\vec{\alpha}}$.

We next construct a sequence of points $\{x_m\}_{m\in\omega}$, which we view ''potential points'' to be added to the set $S_{<\vec{\alpha}}$ to form $S_{\vec{\alpha}}$. We will in fact have $S_{\vec{\alpha}} - S_{<\vec{\alpha}} \subseteq \{x_m : m \in \omega\}$.

Let $(n, d, i, j, a, b, p) \to (n, d, i, j, a, b, p) \in \omega$ be a fixed bijection between ω^7 and ω. For $m \in \omega$, let $(m)_0, (m)_1, \cdots$ be the ''decoding functions'' for our bijection, that is, $m = \langle (m)_0, (m)_1, \cdots (m)_6 \rangle$. If the integer m is understood, we will write n for $(m)_0, d$ for $(m)_1$, etc. Let $M_n^{d,i,j,a,b} \subseteq L_n^{d,i,j}$ be the sublattice of points whose coordinates in the L_n system are of the form $(\frac{i}{d} + k, \frac{j}{d} + l)$, where $k \equiv a, l \equiv b \bmod d$.

We inductively construct the x_m to satisfy the following (here n denotes $(m)_0, d$ denotes $(m)_1$, etc.):

(i) $x_m \in M_n^{d,i,j,a,b}$

(ii) If $m_1 \neq m_2$, then $x_{m_1} \neq x_{m_2}$.

(iii) Suppose $m_1 < m_2$. If x_{m_1} does not have rational coordinates with respect to L_{n_2} ($= L(m_2)_0$), then $x_{m_2} \notin l(x_{m_1}, L_{n_2})$, where $l(x_{m_1}, L_{n_2})$ is as in Lemma (5.1.7).

(iv) $x_m \notin \bigcup G_n(\frac{i}{d}, \frac{j}{d})$.

Since at each step there are only finitely many points and lines to avoid, there is no problem defining the sequence $\{x_m\}$.

Claim (5.1.11)[190]. *For each n, there is at most one point in $S_{<\vec{\alpha}} \cup \{x_m | (m)_0 \neq n\}$ having rational coordinates with respect to L_n.*

Proof. Suppose y and z were two such points. Suppose first both y and z were in $S_{<\vec{\alpha}}$. Say $y \in S_{\vec{\beta}_1} - \bigcup_{\vec{\gamma}<\vec{\beta}_1} S_{\vec{\gamma}}$, $z \in S_{\vec{\beta}_2} - \bigcup_{\vec{\gamma}<\vec{\beta}_2} S_{\vec{\gamma}}$ where $\vec{\beta}_1 \leq \vec{\beta}_2$. If $\vec{\beta}_1 = \vec{\beta}_2$, then each of y, z lies on a lattice in $N_{\vec{\beta}_2}$. Since L_n is definable from y and z, L_n is definable from two points which lie in some $M_{\vec{\beta}}$ for some $\vec{\beta} \leq \vec{\alpha}$. From Lemma (5.1.9) it follows that $L_n \in \bigcup_{\vec{\gamma} \leq \vec{\alpha}} M_{\vec{\gamma}}$, a contradiction. If $\vec{\beta}_1 < \vec{\beta}_2$, then from inductive property (v) we have either $\rho(y,z)^2 \notin \mathbb{Q}$ which is impossible (as both y, z have rational coordinates with respect to L_n), or else y, z both have rational coordinates with respect to some lattice L in $N_{\vec{\beta}_2}$. This would again imply that $L_n \in \bigcup_{\vec{\gamma} \leq \vec{\alpha}} M_{\vec{\gamma}}$, a contradiction. Suppose next that $y \in S_{<\vec{\alpha}}$ and $z = x_m$ where $(m)_0 \neq n$. Since y and z are rational with respect to L_n we have $\rho(y,z)^2 \in \mathbb{Q}$. Since $x_m \notin \bigcup G_{(m)_0}(\frac{i}{d}, \frac{j}{d})$ (where $d = (m)_1$, $i = (m)_2$, $j = (m)_3$), we must have that y is rational with respect to $L(m)_0$ (as otherwise $\rho(y,z)^2 \notin \mathbb{Q}$). Thus, both y and z have rational coordinates with respect to both L_n and $(m)_0$, a contradiction to Lemma (5.1.8). Suppose now $y = x_{m_1}$, $z = x_{m_2}$, where $(m_1)_0, (m_2)_0 \neq n$. Let $n_1 = (m_1)_0$, $n_2 = (m_2)_0$, and assume without loss of generality that $m_1 < m_2$. Again, $\rho(y,z)^2 \in \mathbb{Q}$, as both are rational with respect to L_n. From the definition of x_{m_2} , we must have that x_{m_1} is rational with respect to L_{n_2} (as otherwise $\rho(y,z)^2 \notin \mathbb{Q}$).
Thus, both y and z are rational with respect to L_n and L_{n_2} , a contradiction.

Let w_n, if it exists, be the unique point having rational coordinates with respect to L_n which is either in $S_{<\vec{\alpha}}$ or of the form x_m for some m with $(m)_0 \neq n$.

By induction on n we define sets $T_n \subseteq \{x_m : (m)_0 = n\}$. Assume $T_0, \cdots, T_{n-1}$ have been defined, and for $i < n$, $T_i \subseteq \{x_m : (m)_0 = i\}$. Let $P_1 = \{x_m : (m)_0 = n\}$. Let $P_2 = P_1 - \{w_i : i < n\}$. If w_n exists and $w_n \in S_{<\vec{\alpha}} \cup \bigcup_{i<n} T_i$, let $w = w_n$ and $P = P_2 \cup \{w\}$. If w_n exists, but $w_n \notin S_{<\vec{\alpha}} \cup \bigcup_{i<n} T_i$, let $P = P_2 - \{w_n\}$ and let w be some point in P. If w_n does not exist, let $P = P_2$ and let w be some point in P. Now apply Lemma (5.1.12) to the lattice L_n, the set P, and the point w. Let T_n be the set produced from Lemma (5.1.12).

Let $S_{\vec{\alpha}} = S_{<\vec{\alpha}} \cup \bigcup_n T_n$. Clearly $S_{\vec{\alpha}}$ meets each lattice in each $\mathcal{L}_n$, and $S_{\vec{\alpha}} \subseteq \bigcup_{\vec{\beta} \leq \vec{\alpha}, k} \bigcup \mathcal{L}_{\vec{\beta},k}$. Thus, inductive property (ii) is still satisfied. Properties (i) and (iii) are trivially satisfied. By construction, if $z \in S_{\vec{\alpha}} - S_{<\alpha}$ (say $z \in T_n - \bigcup_{m<n} T_m$) and

$y \in S_{<\vec{\alpha}}$, then either $\rho(y,z)^2 \notin \mathbb{Q}$ or y,z are both rational with respect to L_n. Thus property (v) continues to hold.

To complete the proof, we show that $\rho(y,z)^2 \notin \mathbb{Z}$ for any $y,z \in S_{\vec{\alpha}}$. By induction, we may assume y,z do not both lie in $S_{<\vec{\alpha}}$. Suppose first that $y \in S_{<\vec{\alpha}}$ and $z \in T_n - \bigcup_{i<n} T_i$. Say $z = x_m$. Note that $(m)_0 = n$ as otherwise $z = w_n$, and this is impossible since from the construction $w_n \in T_n$ implies $w_n \in \bigcup_{i<n} T_i$. If y does not have rational coordinates with respect to L_n, then since $x_m \in P$ (P as in the definition of T_n) and $P \cap (\bigcup G_n(\frac{i}{d}, \frac{j}{d})) = \emptyset$;, we would have $\rho(y,z)^2 \notin \mathbb{Q}$.
So, assume y is rational with respect to L_n, and hence $y = w_n$. In defining T_n in this case, we took $w = w_n$ in applying Lemma (5.1.12). Since $z \in T_n$, we therefore have $\rho(y,z)^2 \notin \mathbb{Z}$. Suppose next that y first appears in T_{n_1}, and z first appears in T_{n_2}. From the construction it again follows that $y = x_{m_1}$ where $(m_1)_0 = n_1$, and $z = x_{m_2}$ where $(m_2)_0 = n_2$ (in fact, $y \neq w_{n_1}$ and $z \neq w_{n_2}$). If $n_1 = n_2$, then from the definition of T_{n_1} we have $\rho(y,z)^2 \notin \mathbb{Z}$. Assume without loss of generality that $n_1 < n_2$. If $x_{m_1} = w_{n_2}$, then by definition of T_{n_2} we have $\rho(y,z)^2 \notin \mathbb{Z}$, so assume $y = x_{m_1} \neq w_{n_2}$. By construction, $z = x_{m_2} \neq w_{n_1}$, as $n_1 < n_2$ (w_{n_1} cannot first get into T_{n_2} as $n_1 < n_2$; recall the definition of P_2). Thus, y does not have rational coordinates with respect to L_{n_2}, and z does not have rational coordinates with respect to L_{n_1}. If say $m_1 > m_2$ (the other case being identical), it now follows from the definition of x_{m_1} that $\rho(x_{m_1}, x_{m_2})^2 \notin \mathbb{Q}$.

This completes the proof of Theorem (5.1.2), assuming Lemmas (5.1.12) and (5.1.29).

Lemma (5.1.12)[190]: Let L be a lattice, and let w be a point having rational coordinates with respect to L. Let P be a (countable) set of points containing w, all of which have rational coordinates with respect to L, and satisfying the following: for all integers d,i,j,a,b, there are infinitely many points of P which have coordinates with respect to L of the form $(\frac{i}{d} + k, \frac{j}{d} + l)$, where k,l are integers with $k \equiv a \bmod d, l \equiv b \bmod d$. Then there is a set S satisfying:

(i) For every rational translation L' of L we have $S \cap L' \neq \emptyset$.

(ii) For all distinct $z_1, z_2 \in S$ we have $\rho(z_1,z_2)^2 \notin \mathbb{Z}$.

(iii) $w \in S$.

(iv) $S \subseteq P$.

Note that Lemma (5.1.12) immediately implies Lemma (5.1.13) taking P to be the set of all points having rational coordinates with respect to L.

Proof : Our goal is to prove Lemma (5.1.12). Actually, we concentrate on proving Lemma (5.1.13), as a minor adjustment to this proof will prove Lemma (5.1.12).

Throughout we use the following notation. For $a,b \in \mathbb{Z}$ we write $a|b$ for $''a$ divides b''. If $b > 0$, we write $a \bmod b$ for the unique $0 \leq a' < b$ with $a' \equiv a \bmod b$. For rationals r,s, let $L_{r,s} = \mathbb{Z}^2 + (r,s)$ be the rational translation of $\mathbb{Z}^2$ by (r,s).

Recall the statement of Lemma (5.1.13):

Lemma (5.1.13)[190]: *Then there is a set $S \subseteq \mathbb{R}^2$ satisfying the following:*

(i) *For every rational r,s, $S \cap L_{r,s} \neq \emptyset$;.*

(ii) *For all distinct $z_1,z_2 \in S$, $\rho(z_1,z_2)^2 \notin \mathbb{Z}$.*

Let $R = \mathbb{Q}^2 \cap([0,1) \times [0,1))$. For each positive integer d let $R_d \subseteq R$ be defined by $R_d = \{(\frac{i}{d}, \frac{j}{d}) : 0 \le i; j < d\}$.

We may reformulate Lemma (5.1.13) as follows. For all $(r,s) \in R$, there are integers $k = k(r,s)$ and $l = l(r,s)$ such that if $S = \{(r + k(r,s), s + l(r,s)) : r, s \in \mathbb{Q}\}$, then for all distinct $z_1, z_2 \in S$, $\rho(z_1, z_2)^2 \notin \mathbb{Z}$ (property (ii) of Lemma (5.1.13)). Thus, our problem is to define the integer valued functions $k(r,s)$ and $l(r,s)$ satisfying property (ii).

Our plan for defining these functions is to proceed inductively as follows. Assume we have defined the values $k(r,s), l(r,s)$ for all $(r,s) \in R_d$ for some $d > 1$. Assume that the partial functions k, l so far defined satisfy property (ii), more precisely, assume:

$(*)_d$: For any distinct $\left(\frac{i_1}{d}, \frac{j_1}{d}\right), \left(\frac{i_2}{d}, \frac{j_2}{d}\right) in\ R_d, if\ z_1 = \left(\frac{i_1}{d} + k_1, \frac{j_1}{d} + l_1\right), z_2 = \left(\frac{i_2}{d} + k_2, \frac{j_2}{d} + l_2\right)$ where $k_1 = k(\frac{i_1}{d}, \frac{j_1}{d}), l_1 = l(\frac{i_1}{d}, \frac{j_1}{d}))$, and similarly for k_2, l_2, then $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.

Let p be a prime and $d' = pd$. We then show that we can extend the k, l functions to rational pairs in $R_{d'}$, maintaining property (ii). This clearly suffices to prove Lemma (5.1.13).

We note that in this inductive step of the proof, it is important that we assume that the k, l functions are defined on *all* of the points $(\frac{i}{d}, \frac{j}{d})$ in R_d (and satisfy property (ii), of course). It is not true in general that functions k, l which are defined on a subset of $R_{d'}$ (and satisfy property (ii)) can be extended to functions defined on all of $R_{d'}$ also satisfying property (ii).

We make the following simple general observation. If $x = (\frac{i_1}{d} + k_1, \frac{j_1}{d} + l_1)$ and $y = (\frac{i_2}{d} + k_2, \frac{j_2}{d} + l_2)$, then $\rho(x,y)^2 \in \mathbb{Z}$ iff

$$(i_1 - i_2)^2 + (j_1 - j_2)^2 + 2d[(i_1 - i_2)(k_1 - k_2) + (j_1 - j_2)(l_1 - l_2)] \in d^2\mathbb{Z}. \quad (3)$$

We use this frequently below. We will also frequently let a denote $i_1 - i_2$ and let b denote $j_1 - j_2$, in which case our equation becomes

$$(a^2 + b^2) + 2d[a(k_1 - k_2) + b(l_1 - l_2)] \in d^2\mathbb{Z}. \quad (4)$$

Since the general inductive step is somewhat technical, we feel it helps to illustrate the main points involved by considering a special case. Thus, we first show how to define the k, l functions on the points in R_{p^n}, for p a prime, and then show how to extend the functions from R_{p^n} to $R_{p^{n+1}}$. [We could start with $n = 1$, but this does not really simplify the argument, and would cause us to repeat part of the argument.] These arguments are not necessary for the general case, and we may choose to skip down to the general argument.

So, let $d = p^n$. Consider two points of the form $z_1 = (\frac{i_1}{p^n} + k_1, \frac{j_1}{p^n} + l_1)$ and $z_2 = (\frac{i_2}{p^n} + k_2, \frac{j_2}{p^n} + l_2)$, where $0 \le i_1, i_2, j_1, j_2 < p^n$ and k_1, k_2, l_1, l_2 are integers. Substituting into equation (4), we see that $\rho(z_1, z_2)^2 \notin \mathbb{Z}$ unless

$$(a^2 + b^2) + 2p^n[a(k_1 - k_2) + b(l_1 - l_2)] \equiv 0 \ mod\ p^{2n}. \quad (5)$$

First note that if $p = 2$ or $p \equiv 3$ mod 4, then we may define the k, l values arbitrarily and equation (5) will have no solutions. For clearly if equation (4) holds, then we must have $p^n | a^2 + b^2$. Since $0 \leq i_1, i_2 < p^n$, p^n does not divide a, and likewise p^n does not divide b. Say $a = p^e u$, $b = p^f u$, where $e, f < n$ and u, v are prime to p. Suppose w.l.o.g. that $e \leq f$. Dividing equation (5) through by p^{2e} we get $u^2 + p^{2f-2e} v^2 \equiv 0$ mod p. This implies $e = f$. Hence, $u^2 + v^2 \equiv 0$ mod p. Thus, $\left(\frac{v}{u}\right)^2 \equiv -1$ mod p, a contradiction if $p \equiv 3$ mod 4, since -1 is not a square mod p in this case. If $p = 2$, then since u, v are both odd, $u^2 + v^2 \equiv 2$ mod 4. Dividing equation (5) through by p^{2e} gives

$$(u^2 + v^2) + 2p^{n-e}[u(k_1 - k_2) + v(l_1 - l_2)] \equiv 0 \ mod \ p^{2(n-e)}.$$

This is impossible, however, as 4 divides the remaining terms in this equation.
Thus, if $p = 2$ or $p \equiv 3$ mod 4, we may define the k, l functions arbitrarily on R_{p^n} and property (ii) will be satisfied. For the rest of the special case we therefore assume $p \equiv 1$ mod 4.

Recall that if $p \equiv 1$ mod 4, then there are exactly two square roots of -1 mod p^m for any m. Let λ, μ, with $0 < \lambda, \mu < p^n$, be the two square roots of -1 mod p^n. Note that $\lambda \equiv -\mu$ mod p^n. Note also that for any $k < n$, $(\lambda \ mod \ p^k)$ and $(\mu \ mod \ p^k)$ are the two square roots of -1 mod p^k.

As we remarked above, if equation (5) holds, we must have $p^n | a^2 + b^2$. In this case, if $(p, a) = 1$ (and hence also $(p, b) = 1$), this gives $\left(\frac{b}{a}\right)^2 \equiv -1$ mod p^n, and hence either $b \equiv \lambda a$ mod p^n, or $b \equiv \mu a$ mod p^n. Suppose now $p | a$ (and hence $p | b$, or else equation (5) cannot hold). Say $a = p^e u$, $b = p^f v$, where $e, f < n$, and $(p, u) = (p, v) = 1$. Assuming $e \leq f$ (the other case being similar), putting these into equation (5), and dividing through by p^{2e} we have

$$\left(u^2 + p^{2f-2e} v^2\right) + p^{n-e}\left[u(k_1 - k_2) + p^{f-e} v(l_1 - l_2)\right] \in p^{2n-2e}\mathbb{Z}.$$

This clearly implies $e = f$. Also, using a previous remark, $v \equiv \lambda u$ mod p^{n-e} or $v \equiv \mu u$ mod p^{n-e}. Multiplying through byp^e, we conclude that in all cases for equation (5) to hold, we must have either $b \equiv \lambda a$ mod p^n, or $b \equiv \mu a$ mod p^n.

Suppose, for example, that equation (5) holds and $b \equiv \lambda a$ mod p^n. Let $\tilde{j}$ be the integer, $0 \leq \tilde{j} < p^n$, such that $\tilde{j} + \lambda i_1 \equiv j_1$ mod p^n. Note that $\tilde{j} + \lambda i_2 \equiv j_2$ mod p^n as well. Let $\bar{J}_1 = \tilde{j} + \lambda i_1$, and let m_1 be such that $\bar{J}_1 = j_1 + p^n m_1$. Likewise define $\bar{J}_2$ and m_2. Note that $\bar{J}_1 - \bar{J}_2 = \lambda(i_1 - i_2)$. Also, we may express the points z_1, z_2 now as

$$z_1 = \left(\frac{i_1}{p^n} + k_1, \frac{\bar{J}_1}{p^n} + (l_1 - m_1)\right), \qquad z_2 = \left(\frac{i_2}{p^n} + k_2, \frac{\bar{J}_2}{p^n} + (l_2 - m_2)\right).$$

Substituting into equation (3), and dividing through by p^n we obtain:

$$(i_1 - i_2)^2 \left(\frac{1 + \lambda^2}{p^n}\right) + 2(i_1 - i_2)[(k_1 - k_2) + \lambda(l_1 - l_2 - m_1 + m_2)] \equiv 0 \quad mod p^n.$$

Note that this makes sense as $p^n | (1 + \lambda^2)$. Let $r < n$ be such that $i_1 - i_2 = p^r u$, where $(p, u) = 1$. This equation is then equivalent to

$$(i_1 - i_2)\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{p^n}\right) + [(k_1 - k_2) + \lambda(l_1 - l_2 - m_1 + m_2)] \equiv 0 \quad mod p^{n-r}.$$

Rearranging, this becomes

$$(k_1 + \lambda l_1) + i_1\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{p^n}\right) - \lambda m_1$$
$$\equiv (k_2 + \lambda l_2) + i_2\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{p^n}\right) - \lambda m_2 \quad mod\ p^{n-r} \tag{6}$$

This suggests the following definition.

Definition (5.1.14)[190]. A *good* permutation $\pi = (\pi(0), \pi(1), \cdots, \pi(p^n - 1))$ of length p^n is a permutation of the integers $(0, 1, \cdots, p^n - 1)$ such that for all $i_1 \neq i_2$ with $0 \leq i_1, i_2 < p^n$, $if\ i_1 - i_2 = p^r u$ where $(p, u) = 1$, then $\pi(i_1) - \pi(i_2) \not\equiv 0 \bmod p^{n-r}$.

We use the following simple fact.

Fact 1. *There is a good permutation of length p^n.*

Proof. If $n = 1$, let $\pi = (0, 1, \cdots, p - 1)$. For $n > 1$, suppose $i = b_0 + b_1 p + b_2 p^2 + \cdots + b_{n-1} p^{n-1}$ where $0 \leq b_i < p$. Set $\pi(i) = b_0 p^{n-1} + b_1 p^{n-2} + \cdots + b_{n-1}$. This easily works.

With the above arguments as motivation, we are now in a position to state precisely and prove two lemmas which complete the analysis for the special case $d = p^n$ that we are considering.

Lemma (5.1.15)[190]. Let p be a prime and $n \geq 1$. There are integer functions k, l defined on R_{p^n} such that for all distinct $(\frac{i_1}{p^n}, \frac{j_1}{p^n})$, $(\frac{i_2}{p^n}, \frac{j_2}{p^n}) \in R_{p^n}$ we have $\rho(z_1, z_2)^2 \notin \mathbb{Z}$, where $z_1 = \left(\frac{i_1}{p^n} + k_1, \frac{j_1}{p^n} + l_1\right)$, $z_2 = (\frac{i_2}{p^n} + k_2, \frac{j_2}{p^n} + l_2)$, and $k_1 = k\left(\frac{i_1}{p^n}, \frac{j_1}{p^n}\right)$, $l_1 = l(\frac{i_1}{p^n}, \frac{j_1}{p^n})$, and similarly for k_2, l_2.

Proof. If $p = 2$ or $p \equiv 3 \bmod 4$, the result is trivial (that is, we may define the k, l functions arbitrarily) as shown above. So assume $p \equiv 1 \bmod 4$, and let λ, μ be the two square roots of $-1 \bmod p^n$. Let $\pi = (\pi(0), \pi(1), \cdots, \pi(p^n - 1))$ be a good permutation of length p^n.

Suppose now $0 \leq i, j < p^n$, and we define the k, l values for the corresponding point $(\frac{i}{p^n}, \frac{j}{p^n})$. Let $\tilde{j}$ be such that $\tilde{j} + \lambda i \equiv j \bmod p^n$, and $0 \leq \tilde{j} < p^n$. Let $\bar{j} = \tilde{j} + \lambda i$. Let m be the integer such that $\bar{j} = j + p^n m$. Consider then the equation

$$k + \lambda l \equiv \pi(i) + \lambda m - \frac{1}{2}\left(\frac{1+\lambda^2}{p^n}\right) i \quad mod\ p^n. \tag{7}$$

Similarly, let $\tilde{\tilde{j}}$ be such that $\tilde{\tilde{j}} + \mu i \equiv j \bmod p^n$, and let $\bar{\bar{j}} = \tilde{\tilde{j}} + \mu i$. Let m' be such that $\bar{\bar{j}} = j + m' p^n$. Consider also the equation

$$k + \mu l \equiv \pi(i) + \mu m' - \frac{1}{2}\left(\frac{1+\mu^2}{p^n}\right) i \quad mod\ p^n. \tag{8}$$

Equations (7) and (8) form a non-singular system mod p^n, and we let (k, l) be a solution (to be specific, say the unique solution with $0 \leq k, l < p^n$). This completes the definition of the k, l functions on R_{p^n}.

Suppose now that $(\frac{i_1}{p^n}, \frac{j_1}{p^n})$ and $(\frac{i_2}{p^n}, \frac{j_2}{p^n})$ are given with $0 \leq i_1, i_2, j_1, j_2 < p^n$. Let (k_1, l_1) and (k_2, l_2) be the corresponding values as defined above. Let $z_1 = (\frac{i_1}{p^n} + k_1, \frac{j_1}{p^n} + l_1)$ and similarly for z_2. We must show that $\rho(z_1, z_2)^2 \notin \mathbb{Z}$.

Again let $a = i_1 - i_2$ and $b = j_1 - j_2$. From equation (4), we must show that

$$(a^2 + b^2) + 2p^n[a(k_1 - k_2) + b(l_1 - l_2)] \not\equiv 0 \quad mod\, p^{2n}.$$

As we have already noted, this inequality is immediate unless $b \equiv \lambda a$ mod p^n or $b \equiv \mu a$ mod p^n. Assume $b \equiv \lambda a$ mod p^n, the other case being similar. Let $\tilde{j}$ be such that $\tilde{j} + \lambda i_1 \equiv j_1$ mod p^n. Let $\bar{j}_1 = \tilde{j} + \lambda i_1$, and let m_1 be such that $\bar{j}_1 = j_1 + p^n m_1$. Since $b \equiv \lambda a$ mod p^n, we also have that $\tilde{j} + \lambda i_2 \equiv j_2$ mod p^n. Let $\bar{j}_2 = \tilde{j} + \lambda i_2$, and let m_2 be such that $\bar{j}_2 = j_2 + p^n m_2$. Note that $\bar{j}_1 - \bar{j}_2 = \lambda(i_1 - i_2)$. If we let $r < n$ be such that $i_1 - i_2 = p^r u$ where $(p, u) = 1$, then as we showed above, this equation reduces to

$$(k_1 + \lambda l_1) + i_1\left(\frac{1}{2}\right)\left(\frac{1 + \lambda^2}{p^n}\right) - \lambda m_1 \qquad (9)$$
$$\not\equiv (k_2 + \lambda l_2) + i_2\left(\frac{1}{2}\right)\left(\frac{1 + \lambda^2}{p^n}\right) - \lambda m_2 \quad mod\, p^{n-r}.$$

Substituting in the definitions of k_1, l_1, k_2, l_2 (cf. equation (7); note that this equation holds mod p^n, and so mod p^{n-r}) this becomes $\pi(i_1) \not\equiv \pi(i_2)$ mod p^{n-r}. This, however, follows immediately from the definition of r and the fact that π is good.

The following remark on the proof just given will be used in the following arguments.

Lemma (5.1.16)[190]. Suppose to all $0 \leq i, j < p^n$ we have assigned a pair of integers $(k, l) = (k(i, j), l(i, j))$ such that for any pair of distinct points of the form $z_1 = (\frac{i_1}{p^n} + k(i_1, j_1), \frac{j_1}{p^n} + l(i_1, j_1)), z_2 = (\frac{i_2}{p^n} + k(i_2, j_2), \frac{j_2}{p^n} + l(i_2, j_2))$ we have $\rho(z_1, z_2)^2 \notin \mathbb{Z}$. For each of the two square roots λ, μ of -1 mod p^n, for each $0 \leq \tilde{j} < p^n$, , and for each $0 \leq i < p^n$, define $0 \leq \pi_{\tilde{j}}^{\lambda}(i) < p^n$ to be the integer such that

$$\pi_{\tilde{j}}^{\lambda}(i) \equiv \big(k(i, j) + \lambda l(i, j)\big) - \lambda m + \frac{1}{2}\left(\frac{1 + \lambda^2}{p^n}\right)i \qquad mod\, p^n.$$

Here $0 \leq j < p^n$. is the integer such that $\tilde{j} + \lambda i \equiv j$ mod p^n, and also $\tilde{j} + \lambda i = j + mp^n$. Then, $\pi_{\tilde{j}}^{\lambda}$ is a good permutation of p^n.

Proof. Fix one of the roots, say λ, and a value of $\tilde{j}$. Let i_1 and i_2 be distinct integers with $0 \leq i_1, i_2 < p^n$. Let j_1 and j_2 be as in the statement of the lemma for i_1 and i_2 respectively. Let $z_1 = (\frac{i_1}{p^n} + k(i_1, j_1), \frac{j_1}{p^n} + l(i_1, j_1))$, and $z_2 = (\frac{i_2}{p^n} + k(i_2, j_2), \frac{j_2}{p^n} + l(i_2, j_2))$. Note that if $a = i_1 - i_2$ and $b = j_1 - j_2$, then we are in the case where $b \equiv \lambda a$ mod p^n. Let $\bar{j}_1 = \tilde{j} + \lambda i_1$ and $\bar{j}_2 = \tilde{j} + \lambda i_2$. Since $\rho(x, y)^2 \notin \mathbb{Z}$, equation (3) becomes:

$$(i_1 - i_2)^2 + \big(\lambda(i_1 - i_2) - p^n(m_1 - m_2)\big)^2$$
$$+ 2p^n\big[(i_1 - i_2)(k_1 - k_2) + \big(\lambda(i_1 - i_2) - p^n(m_1 - m_2)\big)(l_1 - l_2)\big]$$
$$\not\equiv\ 0 \quad mod\, p^{2n}.$$

Dividing through by p^n, this is equivalent to:

$$(i_1 - i_2)^2\left(\frac{1 + \lambda^2}{p^n}\right) - 2\lambda(i_1 - i_2)(m_1 - m_2)$$
$$+ 2[(i_1 - i_2)(k_1 - k_2) + \lambda(i_1 - i_2)(l_1 - l_2)] \not\equiv 0 \quad mod\, p^n.$$

Suppose now $i_1 - i_2 = p^r u$ where $r < n$ and $(p, u) = 1$. Dividing through by $2(i_1 - i_2)$ we have:

$$(i_1 - i_2)\,\frac{1}{2}\left(\frac{1 + \lambda^2}{p^n}\right) - \lambda(m_1 - m_2) + [(k_1 - k_2) + \lambda(l_1 - l_2)] \not\equiv 0 \quad mod\, p^{n-r}.$$

Using the definitions of $\pi_{\tilde{j}}^{\lambda}(i_1)$ and $\pi_{\tilde{j}}^{\lambda}(i_2)$, this becomes $\pi_{\tilde{j}}^{\lambda}(i_1) \not\equiv \pi_{\tilde{j}}^{\lambda}(i_2)$ mod p^{n-r}, and we are done.

Suppose now the k, l functions have been defined at all points of R_{p^n} and satisfy $(*)_{p^n}$. We now show how to extend these functions to $R_{p^{n+1}}$ satisfying $(*)_{p^{n+1}}$. We again assume $p \equiv 1$ mod 4, as otherwise the extension is arbitrary. Again let λ, μ denote the square roots of -1 mod p^n. Let λ', μ' denote the square roots of -1 mod p^{n+1}, chosen so that $\lambda \equiv \lambda'$ mod p^n and $\mu \equiv \mu'$ mod p^n. For each $0 \le \tilde{j} < p^n$, let $\pi_{\tilde{j}}^{\lambda}, \pi_{\tilde{j}}^{\mu}$ be the good permutations of length p^n from Lemma (5.1.16).

For each $0 \le \tilde{j} < p^{n+1}$ we define good permutations $\sigma_{\tilde{j}}^{\lambda'}, \sigma_{\tilde{j}}^{\mu'}$ of length p^{n+1}. If p does not divide $\tilde{j}$, let these be arbitrary good permutations of length p^{n+1}. It remains to define the permutations $\sigma_{p\tilde{j}}^{\lambda'}, \sigma_{p\tilde{j}}^{\mu'}\ for\ 0 \le \tilde{j} < p^n$.

First, for any $0 \le i < p^n$, we define $\sigma_{p\tilde{j}}^{\lambda'}(pi)$. This is defined as in the statement of Lemma (5.1.16), using p^{n+1}. To be specifc, let $0 \le \sigma_{p\tilde{j}}^{\lambda'}(pi) < p^{n+1}$ be such that

$$\sigma_{p\tilde{j}}^{\lambda'}(pi) \equiv (k + \lambda' l) - \lambda' m' + \left(\frac{1}{2}\right)\left(\frac{1 + \lambda'^2}{p^{n+1}}\right)pi \qquad mod\, p^{n+1}; \tag{10}$$

where k, l are the values of the functions at the point $\left(\frac{pi}{p^{n+1}}, \frac{pj}{p^{n+1}}\right)$, $pj \equiv p\tilde{j} + \lambda'(pi)$ mod p^{n+1}, and $p\tilde{j} + \lambda'(pi) = pj + p^{n+1}m'$. Since we also have $j \equiv \tilde{j} + \lambda i$ mod p^n, we also have

$$\pi_{\tilde{j}}^{\lambda}(i) \equiv (k + \lambda l) - \lambda m + \left(\frac{1}{2}\right)\left(\frac{1 + \lambda^2}{p^n}\right)i \qquad mod\, p^n,$$

where these are the same k, l values, and $\tilde{j} + \lambda i = j + p^n m$. Say $\lambda' = \lambda + ep^n$. Then $pj + p^{n+1}m' = p\tilde{j} + \lambda'(pi) = p(\tilde{j} + \lambda i + ep^n i) = pj + p^{n+1}m + ep^{n+1}i$. Hence, $m' = m + ei$. Thus we have

$$\sigma_{p\tilde{j}}^{\lambda'}(pi) \equiv (k + \lambda l) - \lambda(m + ei) + \left(\frac{1}{2}\right)\left(\frac{1 + \lambda^2 + 2e\lambda p^n}{p^{n+1}}\right)pi \quad mod\, p^n$$
$$\equiv (k + \lambda l) - \lambda m + \left(\frac{1}{2}\right)\left(\frac{1 + \lambda^2}{p^n}\right)i \quad mod\, p^n \tag{11}$$
$$\equiv \pi_{\tilde{j}}^{\lambda}(i) \quad mod\ p^n.$$

We say a map σ from the integers $i, 0 \leq i < p^{n+1}$, which are divisible by p to the integers mod p^{n+1} is a *partial good* permutation if whenever $0 \leq i_1, i_2 < p^{n+1}$ are distinct integers with $i_1 - i_2 = p^r u$ and $(p, u) = 1$, then $\sigma(i_1) \not\equiv \sigma(i_2)$ mod p^{n+1-r}. Since $\pi_{\tilde{j}}^{\lambda}$ is a good permutation of length p^n, it follows now easily from the above equation that $\sigma_{p\tilde{j}}^{\lambda'}$ is a partial good permutation.

Lemma (5.1.17)[190]. If σ is a partial good permutation on p^{n+1}, then there is a good permutation of length p^{n+1} extending σ.

Proof. For i of the form $i = i_0 + pm$ where $0 \leq i_0 < p$, extend σ by defining $\sigma(i) = \sigma(pm) + i_0 p^n$. This easily works.

Now extend each $\sigma_{\tilde{j}}^{\lambda'}$ to a good permutation of length p^{n+1}. Likewise we define the good permutations $\sigma_{\tilde{j}}^{\mu'}$. Using these good permutations, we may define k, l functions on $R_{p^{n+1}}$ which satisfy $(*)_{p^{n+1}}$. Furthermore, for points of the form $(\frac{pi}{p^{n+1}}, \frac{pj}{p^{n+1}})$, we may take the k, l values already defined on R_{p^n}, since by definition of the (partial) permutations $\sigma_{p\tilde{j}}^{\lambda'}, \sigma_{p\tilde{j}}^{\mu'}$, these values will be a solution to the two equations for $k + \lambda' l$ and $k + \mu' l$ (the equation defining $k + \lambda' l$, for example, is just equation (10) rearranged). Thus, we have extended k, l functions satisfying $(*)_{p^n}$ to functions defined on all of $R_{p^{n+1}}$ and satisfying $(*)_{p^{n+1}}$. This completes the arguments for the special case $d = p^n$.

We now give the general proof of Lemma (5.1.13), and note at the end how the proof also shows Lemma (5.1.12). The following lemma, whose proof occupies the rest, embodies what must be shown.

Lemma (5.1.18)[190]. Let $d > 1$, and suppose functions k, l have been defined on R_d and satisfy $(*)_d$. Let p be a prime and $d' = pd$. Then these functions may be extended to $R_{d'}$ so as to satisfy $(*)_{d'}$.

The proof will use the following definition and lemma, which generalize Definition (5.1.14)and Lemma (5.1.17).

Definition (5.1.19)[190]. Let $d > 1$, and let $d = p_1^{a_1} \cdots p_n^{a_n}$ be its prime decomposition. We say a permutation $\pi = (\pi(0), \ldots, \pi(d-1))$ of the set $(0, 1, \ldots, d-1)$ is a d-goodpermutation if whenever $0 \leq i_1, i_2 < d$ are distinct and $i_1 - i_2 = p_1^{b_1} \cdots p_n^{b_n} v$ where $(v, d) = 1$, then $\pi(i_1) \not\equiv \pi(i_2)$ mod $p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)}$. Here, $\eta(m)$ is defined to be m if $m \geq 0$, and 0 otherwise.

Note that the goodness conditions equivalent to saying that if $i_1 - i_2 = uv$ where u is a product of powers of primes dividing d and $(v, d) = 1$, then $\pi(i_1) \not\equiv \pi(i_2)$ mod $\frac{d}{u}$, where in writing $\frac{d}{u}$ we adopt the convention that if any prime divides u to a higher power than d, then that prime is removed completely from both the numerator and denominator. We also adopt this convention for the proof of the following lemma.

Suppose $d > 1, p$ is a prime, and $d' = pd$. Suppose $0 \leq i_d < p$, and by the distinguished class we mean those $0 \leq i < d'$ with $i \equiv i_d$ mod p. If $\pi(i)$ is defined on the distinguished class and satisfies $\pi(i_1) \not\equiv \pi(i_2)$ mod $\frac{d'}{u}$ whenever $i_1 \neq i_2$ are in the distinguished class (recall here our convention above) and $i_1 - i_2 = uv$ where $(v, d') = 1$, then we say π is partially d'-good.

The next lemma is a general extension lemma which allows us to partially extend d'-good permutations to good permutations.

Lemma (5.1.20)[190]. Let $d > 1$, let p be a prime, and let $d' = pd$. Let $0 \leq i_d < p$ represent a distinguished class mod p. Let π be defined on the distinguished class and be partially d'-good. Let u be defined by $d' = up^n$, where $(u, p) = 1$. Let $s: d' \to d'$ be a function satisfying the following:

(i) If $i_1 \equiv i_2$ mod p, then $s(i_1) = s(i_2)$.

(ii) $s(i)$ is divisible by u for all i.

(iii) For i in the distinguished class, $s(i) = 0$.

(iv) For all $i, i + s(i) \equiv i_d$ mod p.

Define σ by $\sigma(i) = \pi(i + s(i) \ mod\ d') + \frac{s(i)}{u} d \ mod\ d'$. Then σ extends π and is d'-good.

Proof. From (iii) it is clear that σ extends π. To show goodness, suppose $0 \leq i_1, i_2 < d'$. Let $i_1' = i_1 + s(i_1) \ mod\, d', i_2' = i_2 + s(i_2) \ mod\, d'$. Suppose first that $i_1 \equiv i_2$ mod p. Then by (i), $i_1' - i_2' \equiv i_1 - i_2$ mod d'. Also, from the definition of $\sigma, \sigma(i_1) - \sigma(i_2) \equiv \pi(i_1') - \pi(i_2')$ mod d'. Since π is partially d'-good, the result follows.

Suppose now $i_1 - i_2$ is not divisible by p. Say, $i_1 - i_2 = u_1 v$ where $(v, d') = 1$ and $(u_1, p) = 1$. Consider first the case where $i_1' = i_2'$, with i_1', i_2' as above. Then $\sigma(i_1) - \sigma(i_2) \equiv \frac{s(i_1) - s(i_2)}{u} d$. Since $s(i_1) - s(i_2) \not\equiv 0$ mod p in this case, we have $\sigma(i_1) \not\equiv \sigma(i_2)$ mod p^n (note: $d = up^{n-1}$). Since p^n divides $\frac{d'}{u_1}$ (using our conventions), the result follows. Suppose finally that $i_1' \neq i_2'$. From (ii) it follows that $u_1 | (i_1' - i_2')$. Also, $p | (i_1' - i_2')$. So by partial goodness, $\sigma(i_1') \not\equiv \sigma(i_2')$ mod $\frac{d'}{pu_1} = \frac{d}{u_1}$. Since $\sigma(i_1) \equiv \sigma(i_1')$ mod d, and likewise for i_2, it follows that $\sigma(i_1) \not\equiv \sigma(i_2)$ mod $\frac{d'}{pu_1}$, and hence are not equivalent mod $\frac{d'}{u_1}$.

We say that a prime is trivial if $p = 2$ or $p \equiv 3$ mod 4. Otherwise, we say p is non-trivial. The next lemma shows that we need only consider the non-trivial primes.

Lemma (5.1.21)[190]. If Lemma (5.1.18) holds for all d which are divisible by only non-trivial primes, then the lemma holds for all d.

Proof. Let $d = p_1^{a_1} \cdots p_n^{a_n} q_1^{c_1} \cdots q_m^{c_m}$, where the p_i are non-trivial and the q_i are trivial. We assume the k, l functions are defined on R_d and satisfy $(*)_d$. Let $d' = pd$, and assume first that p is non-trivial. Let $P = p_1^{a_1} \cdots p_n^{a_n}$, $P' = pP$, and $Q = q_1^{c_1} \cdots q_m^{c_m}$. Let G be the subgroup of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ of elements of the form $(\frac{i}{d} + \mathbb{Z}, \frac{j}{d} + \mathbb{Z})$, and likewise define G' using d'. Let H be the subgroup of G consisting of elements of the form $(\frac{i}{P} + \mathbb{Z}, \frac{j}{P} + \mathbb{Z})$, and likewise define H' using P'. Let K be the subgroup of elements of the form $(\frac{i}{Q} + \mathbb{Z}, \frac{j}{Q} + \mathbb{Z})$. Note that the given k, l functions may be viewed as selector functions on the group G, that is, functions on G with $(k(r+\mathbb{Z},s+\mathbb{Z}),l(r+\mathbb{Z},s+\mathbb{Z})) \in (r+\mathbb{Z},s+\mathbb{Z})$. We extend these selector functions to the group G'. The cosets of H' in G' are exactly enumerated as $H'+(r+\mathbb{Z},s+\mathbb{Z})$, where $r+\mathbb{Z},s+\mathbb{Z} \in K/\mathbb{Z}$. Consider such a coset of H' in G', say $C' = H'+(r+\mathbb{Z},s+\mathbb{Z})$. The k,l functions are already defined on the corresponding coset $C = H+(r+\mathbb{Z},s+\mathbb{Z})$ of H. Since C, C' are translations of H, H', we may by assumption extend the k,l functions from C to functions k', l' on C' so as to satisfy $(*)$ on C' (that is, for any distinct cosets $x=(r_1+\mathbb{Z},s_1+\mathbb{Z}), y=(r_2+\mathbb{Z},s_2+\mathbb{Z}) \in C', \rho(z_1,z_2)^2 \notin \mathbb{Z}$, where $z_1=(k'(x),l'(x)), z_2=(k'(y),l'(y))$). Doing this for each coset of H' in G' defines the k', l' functions on G'.

To see that this works, let $x=(r_1+\mathbb{Z},s_1+\mathbb{Z}), y=(r_2+\mathbb{Z},s_2+\mathbb{Z})$ be distinct elements of G'. Let $z_1=(k'(x),l'(x)), z_2=(k'(y),l'(y))$, and we show that $\rho(z_1,z_2)^2 \notin \mathbb{Z}$. We may assume that x,y are in distinct cosets of H'. Thus $(a,b) \doteq z_1 - z_2 \notin H'$. The result now follows from the fact that if $a,b \in \mathbb{Q}$ and $a^2+b^2 \in \mathbb{Z}$, then a,b (when reduced) have denominators divisible only by non-trivial primes.

The case where p is trivial is similar but easier. Briefly, view G' now as a union of cosets of H, with H as above. For those cosets which are subsets of G, the k,l functions are already defined, and for the other cosets they are defined easily using the fact that these cosets are translations of H (in this case we do not use our assumption that the result holds for d divisible by only non-trivial primes). As above, the resulting k', l' functions satisfy $(*)_{d'}$.

Returning to the proof of Lemma (5.1.18), by Lemma (5.1.21) be may assume that d and d' are divisible by only non-trivial primes. We make this standing assumption for the remainder of the proof of Lemma (5.1.18).

Let $d = p_1^{a_1} \cdots p_n^{a_n}$, where all of the p_i are non-trivial primes. We prove two lemmas which characterize the existence of the functions k,l on R_d satisfying $(*)_d$ in terms of the existence of a family of permutations satisfying certain properties.

Suppose k,l functions are given on R_d. Since all of the p_i are non-trivial primes, there are exactly 2^n classes λ mod d such that $\lambda^2 \equiv -1$ mod d. We refer to such a λ as a d-root. For each d-root λ, each $0 \le \tilde{j} < d$, and each $0 \le i < d$, define

$$\pi_{\tilde{j}}^{\lambda}(i) = (k+\lambda l) - \lambda m + \frac{1}{2}\left(\frac{1+\lambda^2}{d}\right)(i) \quad mod\ d, \tag{12}$$

where (k,l) are the values associated to $(\frac{i}{d},\frac{j}{d})$, where $0 \le j < d$, and j,m are defined by

$$j = \tilde{j} + \lambda i - md.$$

We introduce two conditions on the $\pi_{\tilde{j}}^{\lambda}$.

(d-goodness) For each $0 \le \tilde{j} < d$ and each d-root $\lambda, \pi_{\tilde{j}}^{\lambda}$ is a d-good permutation.

(d-consistency) Suppose $0 \le \tilde{j}_1, \tilde{j}_2 < d$ and λ_1, λ_2 are both d-roots. Suppose p^a is one of the prime factors $p_1^{a_1}, \ldots, p_n^{a_n}$ and $\lambda_1 \equiv \lambda_2$ mod p^a. Then

$$\pi_{\tilde{j}_1}^{\lambda_1}(i) - \pi_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d} \quad mod\ p^a \tag{13}$$

for any $0 \le i < d$ such that

$$i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2) \quad mod\ d \tag{14}$$

(in equation (13),λ could be either λ_1 or λ_2; note that this expression makes sense since $p^a | (\tilde{j}_1 - \tilde{j}_2)$).

Note that the values of i satisfying equation (14) are precisely those $0 \le i < d$ such that if we define $0 \le j_1, j_2 < d$ (and m_1, m_2) by

$$j_1 = \tilde{j}_1 + \lambda_1 i - m_1 d,$$
$$j_2 = \tilde{j}_2 + \lambda_2 i - m_2 d,$$

then $j_1 = j_2$.

Lemma (5.1.22)[190]. Let $d = p_1^{a_1}, \ldots, p_n^{a_n}$ where each p_i is non-trivial. Assume the k, l functions are defined on R_d and satisfy $(*)_d$, and the $\pi_{\tilde{j}}^{\lambda}$ are defined by equation (12). Then the $\pi_{\tilde{j}}^{\lambda}$ satisfy the d-goodness and d-consistency conditions.

Proof. Fix $0 \le \tilde{j} < d$ and a d-root λ. We show that $\pi_{\tilde{j}}^{\lambda}$ defined by equation (12) is d-good. Let $0 \le i_1, i_2 < d$ be distinct. Let $0 \le j_1, j_2 < d$ and m_1, m_2 be defined by

$$j_1 = \tilde{j} + \lambda i_1 - m_1 d,$$
$$j_2 = \tilde{j} + \lambda i_2 - m_2 d, \tag{15}$$

Let k_1, l_1 be the values associated to the point $w_1 = (\frac{i_1}{d}, \frac{j_1}{d})$, and k_2, l_2 the values associated to $w_2 = (\frac{i_2}{d}, \frac{j_2}{d})$. If $z_1 = w_1 + (k_1, l_1)$ and $z_2 = w_2 + (k_2, l_2)$, then since , $\rho(z_1, z_2)^2 \notin \mathbb{Z}$ we have

$$(i_1 - i_2)^2 + (j_1 - j_2)^2 + 2d[(i_1 - i_2)(k_1 - k_2) + (j_1 - j_2)(l_1 - l_2)] \not\equiv 0 \quad mod\ d^2.$$

Substituting from equation (15) we have

$$(i_1 - i_2)^2(1 + \lambda^2) - 2(i_1 - i_2)(m_1 - m_2)d\lambda$$
$$+2d[(i_1 - i_2)(k_1 - k_2) + \lambda(i_1 - i_2)(l_1 - l_2)] \not\equiv 0 \quad mod\ d^2. \tag{16}$$

Since d divides $1 + \lambda^2$, we may divide through by d to get

$$(i_1 - i_2)^2 \left(\frac{1 + \lambda^2}{d}\right) - 2(i_1 - i_2)(m_1 - m_2)\lambda$$
$$+2[(i_1 - i_2)(k_1 - k_2) + \lambda(i_1 - i_2)(l_1 - l_2)] \not\equiv 0 \quad mod\ d. \tag{17}$$

Say $i_1 - i_2 = p_1^{b_1} \cdots p_n^{b_n} u$, where $(u, d) = 1$. Dividing through by $2(i_1 - i_2)$ we have

$$(i_1 - i_2)^2 \left(\frac{1 + \lambda^2}{2d}\right) - (m_1 - m_2)\lambda + [(k_1 - k_2) + \lambda(l_1 - l_2)]$$
$$\not\equiv 0 \quad mod p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)},$$

where we recall $\eta(r) = r$ if $r \ge 0$,and $\eta(r) = 0$ for $r < 0$. Since $p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)}$ divides d, we have

$$\pi_{\tilde{j}}^{\lambda}(i_1) \equiv (k_1 + \lambda l_1) - \lambda m_1 + \left(\frac{1 + \lambda^2}{2d}\right) i_1 \quad mod\ p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)}$$
$$\not\equiv (k_2 + \lambda l_2) - \lambda m_2 + \left(\frac{1 + \lambda^2}{2d}\right) i_2 \quad mod\ p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)} \tag{18}$$
$$\equiv \pi_{\tilde{j}}^{\lambda}(i_2) \qquad mod\ p_1^{\eta(a_1 - b_1)} \cdots p_n^{\eta(a_n - b_n)}.$$

Thus, $\pi_{\tilde{j}}^{\lambda}$ is d-good.

To verify d-consistency, suppose λ_1 and λ_2 are both d-roots, and $\lambda_1 \equiv \lambda_2$ mod p^a, where p^a is one of the prime powers occurring in d. Let $0 \le \tilde{j}_1, \tilde{j}_2 < d$, and let $0 \le i <$

d be such that $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2) \bmod d$. If we let $0 \leq j_1, j_2 < d$ and m_1, m_2 be defined by

$$j_1 = \tilde{j}_1 + \lambda_1 i - m_1 d,$$
$$j_2 = \tilde{j}_2 + \lambda_2 i - m_2 d,$$

then $j_1 = j_2$, which we now denote by j. Say $\lambda_2 = \lambda_1 + ep^a$. Thus,

$$\tilde{j}_1 - \tilde{j}_2 = -i(\lambda_1 - \lambda_2) + d(m_1 - m_2) = iep^a + d(m_1 - m_2).$$

Let k, l be the values associated to the point $(\frac{i}{d}, \frac{j}{d})$. From the definition of the $\pi_{\tilde{j}}^{\lambda}$ we have:

$$k + \lambda_1 l \equiv \pi_{\tilde{j}_1}^{\lambda_1}(i) + \lambda_1 m_1 - \frac{1}{2}\left(\frac{1 + \lambda_1^2}{d}\right) i \quad mod\, p^a,$$

$$k + \lambda_2 l \equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_2 m_2 - \frac{1}{2}\left(\frac{1 + \lambda_2^2}{d}\right) i \quad mod\, p^a$$

$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_1 m_2 - \frac{1}{2}\left(\frac{1 + \lambda_2^2}{d}\right) i \quad mod\, p^a$$

$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_1 m_1 + \lambda_1\left(\frac{iep^a}{d} - \frac{\tilde{j}_1 - \tilde{j}_2}{d}\right) - \frac{1}{2}\left(\frac{1 + (\lambda_1 + ep^a)^2}{d}\right) i \quad mod\, p^a$$

$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_1 m_1 - \frac{1}{2}\left(\frac{1 + \lambda_1^2}{d}\right) i - \frac{\lambda_1(\tilde{j}_1 - \tilde{j}_2)}{d} \quad mod\, p^a.$$

Note that p^a divides $\tilde{j}_1 - \tilde{j}_2$, so the last two equations make sense. Thus, we have:

$$\pi_{\tilde{j}_1}^{\lambda_1}(i) - \pi_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda_1(\tilde{j}_1 - \tilde{j}_2)}{d} \quad mod \;\; p^a.$$

This verifies d-consistency.

We now establish a converse to Lemma (5.1.22). Suppose that for each d-root λ and each $0 \leq \tilde{j} < d$,a d-good permutation $\pi_{\tilde{j}}^{\lambda}$ is given, and these permutations satisfy the d-consistency condition. We show how to define the k, l functions on R_d so as to satisfy $(*)_d$. Fix a point $(\frac{i}{d}, \frac{j}{d})$, where $0 \leq i, j < d$, and define the values of k, l associated to that point. Let p^a be one of the prime powers occurring in d. For any d-root λ, $\lambda_{p^a} .= \lambda \bmod p^a$ is one of the two square roots of $-1 \bmod p^a$. Fix for the moment such a λ and λ_{p^a}. Define $0 \leq \tilde{j} < d$ and m by

$$j = \tilde{j} + \lambda i - md.$$

Consider the following mod p^a equation:

$$k + \lambda_{p^a} l \equiv \pi_{\tilde{j}}^{\lambda}(i) + \lambda_{p^a} m - \frac{1}{2}\left(\frac{1 + \lambda^2}{d}\right) i \quad mod\, p^a. \tag{19}$$

We claim that the right-hand side of this equation depends only on λ_{p^a}. For, let $\lambda_1 = \lambda$, and suppose λ_2 is also a d-root with $\lambda_2 \equiv \lambda_1 \bmod p^a$. Say,$\lambda_2 = \lambda_1 + ep^a$. Let $\tilde{j}_1, m_1$ be the values using λ_1, and $\tilde{j}_2, m_2$ the values using λ_2. Since

$$j = \tilde{j}_1 + \lambda_1 i - m_1 d = \tilde{j}_2 + \lambda_2 i - m_2 d,$$

we have $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2) \bmod d$. Therefore, by consistency we have

$$\pi_{\tilde{j}_1}^{\lambda_1}(i) - \pi_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda_1(\tilde{j}_1 - \tilde{j}_2)}{d} \quad mod \;\; p^a.$$

Thus

$$\pi_{\tilde{j}_1}^{\lambda_1}(i) + \lambda_{p^a} m_1 - \left(\frac{1+\lambda_1^2}{2d}\right) i$$
$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) - \frac{\lambda_1(\tilde{j}_1 - \tilde{j}_2)}{d} + \lambda_{p^a}\left(m_2 + \frac{\tilde{j}_1 - \tilde{j}_2}{d} + i\frac{\lambda_1 - \lambda_2}{d}\right)$$
$$- \left(\frac{1+(\lambda_2 + ep^a)^2}{2d}\right) i \quad mod\, p^a \tag{20}$$
$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_{p^a} m_2 - i\lambda_{p^a}\frac{ep^a}{d} - \left(\frac{1+(\lambda_2 + ep^a)^2}{2d}\right) i \quad mod\, p^a$$
$$\equiv \pi_{\tilde{j}_2}^{\lambda_2}(i) + \lambda_{p^a} m_2 - \left(\frac{1+\lambda_2^2}{2d}\right) i \quad mod\, p^a.$$

This verifies the claim. Thus, for each of the two square roots $\lambda_{p^a}, -\lambda_{p^n}$ of -1 mod p^a we have unambiguous values, say v_1 and v_2, for the right-hand sides of equation (19). For each prime factor p^a, and each of the two roots $\pm\lambda_{p^a}$ mod p^a, we solve the system

$$k + \lambda_{p^a} l \equiv v_1 \quad mod\, p^a,$$
$$k - \lambda_{p^a} l \equiv v_2 \quad mod\, p^a.$$

From the Chinese remainder theorem, we may choose (k, l) so that all of these systems for the various p^a are simultaneously satisfied. This completes the definition of the k, l functions.

To verify $(*)_d$, let $0 \le i_1, j_1, i_2, j_2 < d$, and let $w_1 = (\frac{i_1}{d}, \frac{j_1}{d})$, $w_2 = (\frac{i_2}{d}, \frac{j_2}{d})$. let $z_1 = w_1 + (k_1, l_1), z_2 = w_2 + (k_2, l_2)$, where k_1, l_1 are the values as defined above for w_1, and similarly for k_2, l_2. We must show $\rho(z_1, z_2)^2 \notin \mathbb{Z}$. Toward a contradiction,assume $\rho(z_1, z_2)^2 \in \mathbb{Z}$, which becomes as usual

$$(i_1 - i_2)^2 + (j_1 - j_2)^2 + 2d[(i_1 - i_2)(k_1 - k_2) + (j_1 - j_2)(l_1 - l_2)]$$
$$\equiv 0 \quad mod\, d^2. \tag{21}$$

Consider for the moment one of the prime powers p^a of d such that if p^e is the exact power of p dividing $i_1 - i_2$, then $e < a$ (such a factor must clearly exist since $|i_1 - i_2| < d$). Write $i_1 - i_2 = p^e u$ where $(u, p) = 1$. Let f be the exact power of p dividing $j_1 - j_2$, and write $j_1 - j_2 = p^f v$, where $(v, p) = 1$. Since $< a$, it follows easily from equation (21) that $e = f$. Dividing through by p^{2e} shows that $u^2 + v^2 \equiv 0$ mod p^{a-e}. Thus, there is a square root $\bar{\lambda}$ of -1 mod p^{a-e} such that $v \equiv \bar{\lambda}u$ mod p^{a-e}. There is a square root λ_{p^a} of -1 mod p^a such that $\lambda \equiv \bar{\lambda}$ mod p^{a-e}. Thus,$v \equiv \lambda_{p^a} u$ mod p^{a-e} as well. Hence $j_1 - j_2 \equiv \lambda_{p^a}(i_1 - i_2)$ mod p^a.

If p^a is a prime power occurring in d for which $e \ge a$, equation (21) implies that $f \ge a$ as well (using the notation above). Thus, for any square root λ_{p^n} of -1 mod p^a the equation $j_1 - j_2 \equiv \lambda_{p^a}(i_1 - i_2)$ mod p^a holds trivially.

Now let λ be a d-root such that for any prime power p^a occurring in $d, \lambda \equiv \lambda_{p^n}$ mod p^a, with λ_{p^a} as in the cases above. It follows that $j_1 - j_2 \equiv \lambda(i_1 - i_2)$ mod d.

Let $0 \le \tilde{j} < d$ and m_1 be defined by

$$j_1 = \tilde{j} + \lambda i_1 - m_1 d. \tag{22}$$

Since $j_1 - j_2 \equiv \lambda(i_1 - i_2)$ mod d, it follows that there is an m_2 such that

$$j_2 = \tilde{j} + \lambda i_2 - m_2 d. \tag{23}$$

From the definitions of k_1, l_1 (in which we use the above values of $\tilde{j}, \lambda$; this is permissible by d-consistency) we have

$$k_1 + \lambda l_1 \equiv \pi_{\tilde{j}}^{\lambda}(i_1) + \lambda m_1 - \left(\frac{1+\lambda^2}{2d}\right) i_1 \quad mod\ d, \tag{24}$$

since this equation holds mod each prime power $p_i^{a_i}$ occurring in d. Likewise,

$$k_2 + \lambda l_2 \equiv \pi_{\tilde{j}}^{\lambda}(i_2) + \lambda m_2 - \left(\frac{1+\lambda^2}{2d}\right) i_2 \quad mod\ d. \tag{25}$$

Substituting equations (22), (23) into equation (21) and dividing through by $2d$ we obtain

$$(i_1 - i_2)^2 \left(\frac{1+\lambda^2}{2d}\right) - \lambda(i_1 - i_2)(m_1 - m_2) + [(i_1 - i_2)(k_1 - k_2) + \lambda(i_1 - i_2)(l_1 - l_2)] \equiv 0 \quad mod\ d. \tag{26}$$

Dividing through by $i_1 - i_2$ gives

$$(i_1 - i_2)\left(\frac{1+\lambda^2}{2d}\right) - \lambda(m_1 - m_2) + [(k_1 - k_2) + \lambda(l_1 - l_2)] \equiv 0 \quad mod\ p_1^{\eta(a_1-b_1)} \cdots p_n^{\eta(a_n-b_n)}, \tag{27}$$

where $i_1 - i_2 = p_1^{b_1} \cdots p_n^{b_n} u$ and $(u, d) = 1$. Substituting equations (24) and (25) now gives $\pi_{\tilde{j}}^{\lambda}(i_1) - \pi_{\tilde{j}}^{\lambda}(i_2) \equiv 0$ mod $p_1^{\eta(a_1-b_1)} \cdots p_n^{\eta(a_n-b_n)}$. This, however, contradicts the assumed d-goodness of the $\pi_{\tilde{j}}^{\lambda}$.

Summarizing, we have shown the following converse to Lemma (5.1.22).

Lemma (5.1.23)[190]. Let $d = p_1^{a_1} \cdots p_n^{a_n}$ be a product of non-trivial primes. Assume that for each $0 \le \tilde{j} < d$ and each d-root λ a d-good permutation $\pi_{\tilde{j}}^{\lambda}$ is given, and these permutations satisfy the d-consistency condition. Then we may associate to each $(\frac{i}{d}, \frac{j}{d})$, $0 \le i, j < d$, integer values k, l such that for all d-roots λ and all $0 \le \tilde{j} < d$ satisfying $j \equiv \tilde{j} + \lambda i$ mod d (say $j = \tilde{j} + \lambda i - md$), we have

$$k + \lambda l \equiv \pi_{\tilde{j}}^{\lambda}(i) + \lambda m - \left(\frac{1+\lambda^2}{2d}\right) i \quad mod\ d.$$

Furthermore, these k, l functions satisfy $(*)_d$.

To unify notation, let us now write $d = p_1^{a_1} \cdots p_n^{a_n}$ and $d' = p_1^{a_1+1} p_2^{a_2} \cdots p_n^{a_n}$(thus we do not assume these primes are in increasing order, and we allow $a_1 = 0$). The case $a_1 = 0$ differs in only trivial notational ways from the case $a_1 \ge 1$, so we assume below all of the a_i are positive. We are assuming the k, l functions have been defined on R_d and satisfy $(*)_d$, and we must extend them to functions k', l' on $R_{d'}$ satisfying $(*)_{d'}$.

For each $0 \le \tilde{j} < d$, and each d-root λ, let $\pi_{\tilde{j}}^{\lambda}$ be as in Lemma (5.1.22) using the given k, l functions. Thus, each $\pi_{\tilde{j}}^{\lambda}$ is a d-good permutation, and this family satisfies the d-consistency condition.

For each $\tilde{j}$ with $0 \le p_1\tilde{j} < d'$, each d'-root λ' (that is, $\lambda'^2 \equiv -1$ mod d'), and each i with $0 \le p_1 i < d'$, define

$$\sigma_{p_1\tilde{j}}^{\lambda'}(p_1 i) = (k + \lambda' l) - \lambda' m + \frac{1}{2}\left(\frac{1+\lambda'^2}{d'}\right)(p_1 i) \quad mod\ d' \tag{28}$$

where (k, l) are the values already assigned to the pair $(\frac{i}{d}, \frac{j}{d}) = (\frac{p_1 i}{d'}, \frac{p_1 j}{d'})$, and j, m are defined by

$$p_1 j = p_1\tilde{j} + \lambda'(p_1 i) - md'. \tag{29}$$

This makes sense since the right-hand side is divisible by p_1. Thus, each $\sigma^{\lambda'}_{p_1\tilde{j}}$ is a partial function in that it is only defined on the $0 \le i < d'$ which are divisible by p_1. We will momentarily extend these to fully d'-good permutations satisfying the d'-consistency condition, but first we catalog the properties satisfied by these partial functions.

First note that if $\lambda = \lambda'$ mod d, then $\sigma^{\lambda'}_{p_1\tilde{j}}(p_1 i) \equiv \pi^{\lambda}_{j}(i)$ mod d. To see this, let $\lambda' = \lambda + ed$. Thus, $j = \tilde{j} + \lambda' i - md = \tilde{j} + \lambda i - (m - ei)d$. Hence, if k, l are the values associated to the pair $(\frac{i}{d}, \frac{j}{d})$, then

$$\begin{aligned}
\sigma^{\lambda'}_{p_1\tilde{j}}(p_1 i) &= (k + \lambda' l) - \lambda' m + \frac{1}{2}\left(\frac{1+\lambda'^2}{d'}\right)(p_1 i) \quad mod\ d' \\
&\equiv (k + \lambda l) - \lambda m + \frac{1}{2}\left(\frac{1+\lambda'^2}{d'}\right)(p_1 i) \quad mod\ d \qquad (30) \\
&\equiv (k + \lambda l) - \lambda(m - ei) - ie\lambda + \frac{1}{2}\left(\frac{(1+(\lambda + ed)^2)}{d}\right) i \quad mod\ d \\
&\equiv (k + \lambda l) - \lambda(m - ei) + \frac{1}{2}\left(\frac{1+\lambda^2}{d}\right) i \quad mod\ d \\
&\equiv \pi^{\lambda}_{j}(i) \quad mod\ d.
\end{aligned}$$

We introduce now the following "partial" goodness and consistency conditions for the $\sigma^{\lambda'}_{p_1\tilde{j}}$.

(partial d'-goodness) If $0 \le p_1\tilde{j} < d'$, $0 \le p_1 i_1, p_1 i_2 < d'$, and $(p_1 i_1 - p_1 i_2) = p_1^{b_1} \cdots p_n^{b_n} v$, where $(v, d') = 1$, then

$$\sigma^{\lambda'}_{p_1\tilde{j}}(p_1 i_1) \not\equiv \sigma^{\lambda'}_{p_1\tilde{j}}(p_1 i_2) \quad mod\ p_1^{\eta(a_1+1-b_1)} \cdots p_n^{\eta(a_n-b_n)}.$$

(partial d'-consistency) If $0 \le p_1\tilde{j}_1, p_1\tilde{j}_2 < d'$ and λ'_1, λ'_2 are d'-roots with $\lambda'_1 \equiv \lambda'_2$ mod p^a where p^a is one of the prime factors $p_1^{a_1+1} \cdots p_n^{a_n}$ of d', then for any $0 \le p_1 i < d'$ with $(p_1 i)(\lambda'_1 - \lambda'_2) \equiv -(p_1\tilde{j}_1 - p_1\tilde{j}_2)$ mod d' we have

$$\sigma^{\lambda'_1}_{p_1\tilde{j}_1}(p_1 i) - \sigma^{\lambda'_2}_{p_1\tilde{j}_2}(p_1 i) \equiv -\frac{\lambda'(p_1\tilde{j}_1 - p_1\tilde{j}_2)}{d'} \quad mod\ p^a.$$

Lemma (5.1.24)[190]. The partial functions $\sigma^{\lambda'}_{p_1\tilde{j}}$ satisfy the d'-partial goodness and d' partial consistency conditions.

Proof. The proof is essentially identical to that of Lemma (5.1.22). For example, to verify partial d'-consistency, let j, m'_1, m'_2 be defined by

$$\begin{aligned}
p_1 j &= p_1\tilde{j}_1 + \lambda'_1(p_1 i) - m'_1 d' \\
&= p_1\tilde{j}_2 + \lambda'_2(p_1 i) - m'_2 d'.
\end{aligned}$$

Let (k, l) be the values associated to $(\frac{p_1 i}{d'}, \frac{p_1 j}{d'})$, and let $\lambda'_2 = \lambda'_1 + ep^a$. Then we have:

$$\sigma^{\lambda_1'}_{p_1\tilde{j}_1}(p_1 i) \equiv (k + \lambda_1' l) - \lambda_1' m_1 + \frac{1}{2}\left(\frac{1+\lambda_1'^2}{d'}\right)(p_1 i) \quad mod\ p^a$$

$$\equiv (k + \lambda_2' l) - \lambda_2' m_2 - \lambda_2' \frac{p_1(\tilde{j}_1 - \tilde{j}_2) + (\lambda_1' - \lambda_2')(p_1 i)}{d'} + \frac{1}{2}\left(\frac{1 + (\lambda_2' - ep^a)^2}{d'}\right)(p_1 i) \quad mod\ p^a$$

$$\equiv (k + \lambda_2' l) - \lambda_2' m_2 + \frac{1}{2}\left(\frac{1+\lambda_2'^2}{d'}\right)(p_1 i) - \frac{\lambda_2'(p_1\tilde{j}_1 - p_1\tilde{j}_2)}{d'} \quad mod\ p^a$$

$$\equiv \sigma^{\lambda_2'}_{p_1\tilde{j}_2}(p_1 i) - \frac{\lambda_2'(p_1\tilde{j}_1 - p_1\tilde{j}_2)}{d'} \quad mod\ p^a.$$

We now define the permutations $\sigma^{\lambda'}_{\tilde{j}}(i)$ for all $0 \le \tilde{j} < d'$, all d'-roots λ', and all $0 \le i < d'$, and which extend the partial permutations so far defined (the $\sigma^{\lambda'}_{p_1\tilde{j}}\ (p_1 i)$). Since we do not need to refer to the d-roots anymore, we will henceforth use λ to refer to the d' roots. Also, we refer to the $i, j, \tilde{j}$ which are divisible by p_1 as “old”, and the other $i, j, \tilde{j}$ as “new”. Thus, $\sigma^{\lambda'}_{\tilde{j}}(i)$ is currently defined for the old $\tilde{j}$ and i, and we wish to extend to the new values.

We introduce two families of functions, $r^{\lambda}_{\tilde{j}}$ and $s^{\lambda}_{\tilde{j}}$, from d' to d'. These “shift” functions will tell us how to extend certain partially defined permutations to fully good permutations. These functions are defined for each d' root λ. The r functions are defined for old $\tilde{j}$, and the s functions for new $\tilde{j}$. Actually, for the construction below it su□ces (though it is not necessary) to take $r^{\lambda}_{\tilde{j}}$ functions which are independent of $\tilde{j}$ and λ, that is, we have a single function $r : d' \to d'$. In general, the properties we desire of the r and s functions are described in the following lemma.

Definition (5.1.25)[190]. Let λ be a root mod d', and let $0 \le \tilde{j} < d'$. By the $\lambda, \tilde{j}$-distinguished class we mean the equivalence class mod p_1 of $0 \le i < d'$ satisfying $i(\lambda - \bar{\lambda}) \equiv -\tilde{j}$ mod p_1, where $\bar{\lambda}$ is a root not equivalent to λ mod p_1 (so, $\bar{\lambda} \equiv -\lambda$ mod p_1).

Note that for a given $\tilde{j}$, there are really only two distinguished classes, one for each of the two possible values of a root mod p_1, and each of these classes is the negative of the other, mod p_1.

Lemma (5.1.26)[190]. There are functions $r, s^{\lambda}_{\tilde{j}} : d' \to d'$ satisfying the following:

(i) For each $0 \le i < d'$, $i + r(i)$ is divisible by p_1. Further, if $p_1 | i$, then $r(i) = 0$.

(ii) For each root λ, new $\tilde{j}$, and $0 \le i < d'$, $i + s^{\lambda}_{\tilde{j}}(i)$ is in the $\lambda, \tilde{j}$-distinguished class. Further, if i is in the $\lambda, \tilde{j}$-distinguished class, then $s^{\lambda}_{\tilde{j}}(i) = 0$.

(iii) $r(i)$, $s^{\lambda}_{\tilde{j}}(i)$ only depend on the classes of $\tilde{j}$ and i mod p_1.

(iv) $s^{\lambda}_{\tilde{j}}(i)$ depends only on the class of λ mod p_1.

(v) $r(i)$, $s^{\lambda}_{\tilde{j}}(i)$ are divisible by u (recall $= p_2^{a_2} \cdots p_n^{a_n}$).

For the remaining statements we fix some notation. Let $0 \leq \tilde{j}_1, \tilde{j}_2 < d'$, with $\tilde{j}_1, \tilde{j}_2$ new. Let λ_1, λ_2 be d' roots with $\lambda_1 \equiv -\lambda_2$ mod p_1. Let $0 \leq i < d'$. Suppose $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2)$ mod p_1.

(vi) $s_{\tilde{j}_1}^{\lambda_1}(i) + r(i + s_{\tilde{j}_1}^{\lambda_1}(i)) = s_{\tilde{j}_2}^{\lambda_2}(i) + r(i + s_{\tilde{j}_2}^{\lambda_2}(i))$ mod d'.

(vii) $s_{\tilde{j}_1}^{\lambda_1}(i) = r(i + s_{\tilde{j}_2}^{\lambda_2}(i))$.

With the notation as fixed in the statement of the lemma, if we let s_1 abbreviate $s_{\tilde{j}_1}^{\lambda_1}(i)$, $r_2 = r(i + s_{\tilde{j}_1}^{\lambda_1}(i))$, $s_2 = s_{\tilde{j}_2}^{\lambda_2}(i)$, and $r_1 = r(i + s_{\tilde{j}_2}^{\lambda_2}(i))$, then the last two statements become

(vi) $s_1 + r_2 = s_2 + r_1$.

(vii) $s_1 = r_1$.

Of course, we also have in this case that $s_2 = r_2$.

Proof. We give an algorithm for constructing the $r, s_{\tilde{j}}^{\lambda}$ functions. First, let $r(i) = (-\frac{i}{u} \; mod \, p_1)u$, where $u = p_2^{a_2} \cdots p_n^{a_n}$. Clearly (i) is satisfied.

Suppose that λ is a root and $\tilde{j}$ is new. Let $0 \leq i_d < p_1$ represent the $\lambda, \tilde{j}$ distinguished class. Let $s_{\tilde{j}}^{\lambda}(i)$ be the unique value in $\{r(0), \ldots, r(p_1 - 1)\}$ such that $i + s_{\tilde{j}}^{\lambda}(i) \equiv i_d \; mod \, p_1$.

This completes the definition of the r and $s_{\tilde{j}}^{\lambda}$ functions. Property (ii) is clear, and (iii) is also since the $\lambda, \tilde{j}$-distinguished class depends on the class of $\tilde{j}$ mod p_1. Likewise, this class depends only the value of λ mod p_1, and so (iv) follows. (v) is immediate from the definitions.

To see (vi), fix $i, \tilde{j}_1, \tilde{j}_2, \lambda_1, \lambda_2$ with $\lambda_1 \equiv -\lambda_2$ mod p_1 and $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2)$ mod p_1. Let s_1, r_2, s_2, r_1 be as above. Let $i_1 = i + s_1$ mod d', so i_1 is in the $\lambda_1, \tilde{j}_1$-distinguished class. Likewise, let $i_2 = i + s_2$ mod d', which is in the $\lambda_2, \tilde{j}_2$ distinguished class. Since i_1 is in the distinguished class we have $i_1(\lambda_1 - \lambda_2) \equiv -\tilde{j}_1$ mod p_1, and likewise we have $i_2(\lambda_2 - \lambda_1) \equiv -\tilde{j}_2$. Subtracting these equations gives

$$(i_1 + i_2)(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2) \equiv i(\lambda_1 - \lambda_2) \qquad mod \, p_1.$$

Thus, $i \equiv i_1 + i_2$ mod p_1. Also, by definition of the r function we have $r_2 \equiv -i_1$ mod p_1 and $r_1 \equiv -i_2$ mod p_1. Thus, $i + r_2 \equiv i - i_1 \equiv i_2$ mod p_1. From the definition of s_2 it now follows that $s_2 = r_2$. Similarly, $r_1 \equiv -i_2$ mod p_1 and so $i + r_1 \equiv i - i_2 \equiv i_1$ mod p_1 from which it follows that $s_1 = r_1$. This verifies (vii) as well.

We now define the $\sigma_{\tilde{j}}^{\lambda}$. First assume that $\tilde{j}$ is old. In this case, $\sigma_{\tilde{j}}^{\lambda}(i)$ is already defined for the old i. We extend the partial function $\sigma_{\tilde{j}}^{\lambda}$ to all values of i using Lemma (5.1.20) and the r function. Thus,

$$\sigma_{\tilde{j}}^{\lambda}(i) = \sigma_{\tilde{j}}^{\lambda}(i + r(i) \quad mod \, d') + \left(\frac{r(i)}{u}\right) d \qquad mod \, d'.$$

It is immediate from Lemma (5.1.20) that $\sigma_{\tilde{j}}^{\lambda}$ is d'-good.

Suppose now $\tilde{j}$ is new. Let i_d represent the congruence class mod p_1 of the distinguished class. We first define $\sigma_{\tilde{j}}^{\lambda}(i)$ for $i \equiv i_d$ mod p_1, that is, in the distinguished class. Fix such an i, and define $\sigma_{\tilde{j}}^{\lambda}(i)$ by defining its congruence class mod $p_1^{a_1+1}, \ldots, p_n^{a_n}$. Consider one of these prime powers p^a, and suppose first that $p \neq p_1$. Let λ_2 be a root with $\lambda_2 \equiv \lambda \bmod p^a$ and $\lambda_2 \equiv -\lambda \bmod p_1$. Define $\tilde{j}_2$ by $i(\lambda - \lambda_2) \equiv -(\tilde{j} - \tilde{j}_2) \bmod d'$. Note that since i is in the distinguished class, $p_1 | \tilde{j}_2$, that is, $\tilde{j}_2$ is old. Then define

$$\sigma_{\tilde{j}}^{\lambda}(i) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i) - \frac{\lambda(\tilde{j} - \tilde{j}_2)}{d'} \quad mod\ p^a.$$

We check that this is well defined, that is, it does not depend on the choice of λ_2. Suppose λ_3 is another root with $\lambda_3 \equiv \lambda \bmod p^a$ and $\lambda_3 \equiv -\lambda \bmod p_1$, so $\lambda_3 \equiv \lambda_2 \bmod p_1^{a_1+1}$ as well. Let $\tilde{j}_3$ be such that $i(\lambda - \lambda_3) \equiv -(\tilde{j} - \tilde{j}_3) \bmod d'$. Since

$$i(\lambda - \lambda_2) \equiv -(\tilde{j} - \tilde{j}_2) \quad and \quad i(\lambda - \lambda_3) \equiv -(\tilde{j} - \tilde{j}_3) \quad mod\ d',$$

it follows that

$$i(\lambda_2 - \lambda_3) \equiv -(\tilde{j}_2 - \tilde{j}_3) \quad mod\ d'.$$

Let $i' = i + r(i) \bmod d'$. Then we also have $i'(\lambda_2 - \lambda_3) \equiv -(\tilde{j}_2 - \tilde{j}_3) \bmod d'$ since $(i - i')(\lambda_2 - \lambda_3)$ is divisible by d' (recall $r(i)$ is divisible by u). Since $i', \tilde{j}_2, \tilde{j}_3$ are old, by partial d'-consistency we therefore have

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i') - \sigma_{\tilde{j}_3}^{\lambda_3}(i') \equiv -\frac{\lambda(\tilde{j}_2 - \tilde{j}_3)}{d'} \quad mod\ p^a.$$

Since $\sigma_{\tilde{j}_2}^{\lambda_2}(i) - \sigma_{\tilde{j}_3}^{\lambda_3}(i) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i') - \sigma_{\tilde{j}_3}^{\lambda_3}(i') \bmod d'$, it follows that

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i) - \sigma_{\tilde{j}_3}^{\lambda_3}(i) \equiv -\frac{\lambda(\tilde{j}_2 - \tilde{j}_3)}{d'} \quad mod\ p^a.$$

Consequently, $\sigma_{\tilde{j}_2}^{\lambda_2}(i) - \frac{\lambda(\tilde{j} - \tilde{j}_2)}{d'} \equiv \sigma_{\tilde{j}_3}^{\lambda_3}(i) - \frac{\lambda(\tilde{j} - \tilde{j}_3)}{d'} \bmod p^a$, and we are done.

For i still in the $\lambda, \tilde{j}$-distinguished class, we now define $\sigma_{\tilde{j}}^{\lambda}(i) \bmod p_1^{a_1+1}$. Let π be a fixed good permutation of length $p_1^{a_1}$. For i in the distinguished class let $i' = \frac{i - (i\ mod\ p_1)}{p_1}$.

Then define $\sigma_{\tilde{j}}^{\lambda}(i) \equiv \pi(i') - \frac{\lambda\left(\tilde{j} - \left(\tilde{j}\ mod\ p_1^{a_1+1}\right)\right)}{d'} \quad mod\ p_1^{a_1+1}$.

This defines $\sigma_{\tilde{j}}^{\lambda}(i)$ for i in the distinguished class. We extend this to a full permutation using the $s_{\tilde{j}}^{\lambda}$ function. Thus,

$$\sigma_{\tilde{j}}^{\lambda}(i) \equiv \sigma_{\tilde{j}}^{\lambda}\left(i + s_{\tilde{j}}^{\lambda}(i) \quad mod\ d'\right) + \left(\frac{s_{\tilde{j}}^{\lambda}(i)}{u}\right)d \quad mod\ d'.$$

This completes the definition of the $\sigma_{\tilde{j}}^{\lambda}$ functions. It remains to verify that they satisfy the goodness and consistency conditions.

Lemma (5.1.27)[190]. The $\sigma_{\tilde{j}}^{\lambda}$ satisfy the d'-goodness condition.

Proof. We have already observed that this is the case for old $\tilde{j}$, so assume $\tilde{j}$ is new. It is enough to check that $\sigma_{\tilde{j}}^{\lambda}$ restricted to the distinguished class is a partially good function. To see this, suppose i_1, i_2 are in the distinguished class (in particular, $i_1 \equiv i_2 \bmod p_1$). Let $i_1 - i_2 = p_1^{b_1} \ldots p_n^{b_n} v$ where $(v, d') = 1$. Suppose first that $b_1 < a_1 + 1$. Let i_1', i_2' correspond to i_1, i_2 as in the definition of $\sigma_{\tilde{j}}^{\lambda} \bmod p_1^{a_1+1}$. So, $i_1' - i_2' = \frac{(i_1 - i_2)}{p_1}$. By goodness of π, $\pi(i_1') \not\equiv \pi(i_2') \bmod p_1^{a_1-(b_1-1)} = p_1^{(a_1+1)-b_1}$, and so $\sigma_{\tilde{j}}^{\lambda}(i_1) \not\equiv \sigma_{\tilde{j}}^{\lambda}(i_2) \bmod p_1^{a_1+1-b_1}$, and thus also inequivalent mod $p_1^{\eta(a_1+1-b_1)} \cdots p_n^{\eta(a_n-b_n)}$.

Assume next that $b_1 \geq a_1 + 1$. We must show that $\sigma_{\tilde{j}}^{\lambda}(i_1) \not\equiv \sigma_{\tilde{j}}^{\lambda}(i_2)$ mod $\doteq p_2^{\eta(a_2-b_2)} \cdots p_n^{\eta(a_n-b_n)}$. Let λ_2 be the root with $\lambda_2 \equiv -\lambda$ mod $p_1^{a_1+1}$, but $\lambda_2 \equiv \lambda$ mod $p_i^{a_i}$ for $i \geq 2$. Let $\tilde{j}_2$ be defined by $i_1(\lambda - \lambda_2) \equiv -(\tilde{j} - \tilde{j}_2)$ mod d'. Note then that we also have $i_2(\lambda - \lambda_2) \equiv -(\tilde{j} - \tilde{j}_2)$ mod d', as $p_1^{a_1+1}$ divides $i_1 - i_2$. By the well-definedness noted above, we may use λ_2 and $\tilde{j}_2$ in the definitions of both $\sigma_{\tilde{j}}^{\lambda}(i_1)$ and $\sigma_{\tilde{j}}^{\lambda}(i_2)$ modulo any of the powers $p_i^{a_i}, i \geq 2$. Let p^a denote one of these powers. From the definition of the $\sigma_{\tilde{j}}^{\lambda}$ we have

$$\sigma_{\tilde{j}}^{\lambda}(i_1) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_1) - \frac{\lambda(\tilde{j} - \tilde{j}_2)}{d'} \qquad mod\ p^a$$

and

$$\sigma_{\tilde{j}}^{\lambda}(i_2) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_2) - \frac{\lambda(\tilde{j} - \tilde{j}_2)}{d'} \qquad mod\ p^a,$$

and thus

$$\sigma_{\tilde{j}}^{\lambda}(i_1) - \sigma_{\tilde{j}}^{\lambda}(i_2) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_1) - \sigma_{\tilde{j}_2}^{\lambda_2}(i_2) \qquad mod\ p^a.$$

Since this is true for each of the prime powers p^a, we also have

$$\sigma_{\tilde{j}}^{\lambda}(i_1) - \sigma_{\tilde{j}}^{\lambda}(i_2) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_1) - \sigma_{\tilde{j}_2}^{\lambda_2}(i_2) \qquad mod\ u,$$

where $= p_2^{a_2} \cdots p_n^{a_n}$. Hence it is enough to show that $\sigma_{\tilde{j}_2}^{\lambda_2}(i_1) \not\equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_2)$ mod w. Since $i_1 \equiv i_2$ mod p_1, $r(i_1) = r(i_2)$. If i_1^* denotes $i_1 + r(i_1)$ mod d' and likewise for i_2^*, then $i_1^* - i_2^* \equiv i_1 - i_2$ mod d', and also $\sigma_{\tilde{j}_2}^{\lambda_2}(i_1) - \sigma_{\tilde{j}_2}^{\lambda_2}(i_2) \equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_1^*) - \sigma_{\tilde{j}_2}^{\lambda_2}(i_2^*)$ mod d' from the definition of $\sigma_{\tilde{j}}^{\lambda}$ for the old $\tilde{j}$. So, it is enough to show that $\sigma_{\tilde{j}_2}^{\lambda_2}(i_1^*) \not\equiv \sigma_{\tilde{j}_2}^{\lambda_2}(i_2^*)$ mod w. This, however, follows immediately from the partial goodness of $\sigma_{\tilde{j}_2}^{\lambda_2}$ and the fact that $i_1^* - i_2^* \equiv i_1 - i_2$ mod d'.

We have now shown that $\sigma_{\tilde{j}}^{\lambda}$ restricted to the distinguished class is partially good. The goodness of the full function $\sigma_{\tilde{j}}^{\lambda}$ now follows immediately from the extension Lemma (5.1.20).

Lemma (5.1.28)[190]. The $\sigma_{\tilde{j}}^{\lambda}$ functions satisfy the d'-consistency conditions.

Proof. Fix $i, \tilde{j}_1, \tilde{j}_2, \lambda_1, \lambda_2$ with $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2)$ mod d'. Let p^a be a prime power with $\lambda_1 \equiv \lambda_2$ mod p^a. We may assume that $\tilde{j}_1, \tilde{j}_2$ are not both old, and without loss of generality that $\tilde{j}_1$ is new. For if $\tilde{j}_1, \tilde{j}_2$ are both old, then as in an argument above we would have $i'(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2)$ mod d' and $\sigma_{\tilde{j}_1}^{\lambda_1}(i') - \sigma_{\tilde{j}_2}^{\lambda_2}(i') \equiv \sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i)$ mod d', where $i = i + r(i)$ mod d'. The result then follows.

Assume first that $\tilde{j}_2$ is old. In this case we must have i is new and $\lambda_2 \equiv -\lambda_1$ mod p_1. In particular, $p \neq p_1$. From well-definedness, we may use $\tilde{j}_2, \lambda_2$ in the definition of $\sigma_{\tilde{j}_1}^{\lambda_1}(i)$ mod p^a. However, it is then immediate that

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'} \qquad mod\ p^a,$$

where λ denotes either λ_1 or λ_2.

Assume henceforth that $\tilde{j}_1, \tilde{j}_2$ are both new. Consider first the case $p = p_1$. Thus, $\lambda_1 \equiv \lambda_2 \mod p_1^{a_1+1}$, and so $\tilde{j}_1 \equiv \tilde{j}_2 \mod p_1^{a_1+1}$. Thus, $s_{\tilde{j}_1}^{\lambda_1} = s_{\tilde{j}_2}^{\lambda_2} = s$, say. Let $i' = i + s(i) \bmod d'$. Then i' is in the $\lambda_1, \tilde{j}_1$-distinguished class, which is the same as the $\lambda_2, \tilde{j}_2$-distinguished class. From the definition of the permutation extension, it follows that

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv \sigma_{\tilde{j}_1}^{\lambda_1}(i') - \sigma_{\tilde{j}_2}^{\lambda_2}(i') \qquad mod\ d'.$$

Thus, it su□ces to show that $\sigma_{\tilde{j}_1}^{\lambda_1}(i') - \sigma_{\tilde{j}_2}^{\lambda_2}(i') \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'} \mod p_1^{a_1+1}$. Let $i^* = \frac{i-(i\ mod\ p_1)}{p_1}$. Then

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i') \equiv \pi(i^*) - \frac{\lambda\left(\tilde{j}_1 - \left(\tilde{j}_1 \quad mod\ p_1^{a_1+1}\right)\right)}{d'} \qquad mod\ p_1^{a_1+1},$$

where again λ denotes either λ_1 or λ_2. Likewise

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i') \equiv \pi(i^*) - \frac{\lambda\left(\tilde{j}_2 - \left(\tilde{j}_2 \quad mod\ p_1^{a_1+1}\right)\right)}{d'} \qquad mod\ p_1^{a_1+1},$$

and so $\sigma_{\tilde{j}_1}^{\lambda_1}(i') - \sigma_{\tilde{j}_2}^{\lambda_2}(i') \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'}\ mod\ p_1^{a_1+1}$.

Consider finally the case $p \neq p_1$. First, we argue that we may assume $\lambda_1 \not\equiv \lambda_2 \mod p_1^{a_1+1}$. For assume we can prove consistency in this case, and suppose $\lambda_1 \equiv \lambda_2 \mod p_1^{a_1+1}$. Let $\lambda_3 \equiv \lambda_1 \equiv \lambda_2 \mod p^a$, but $\lambda_3 \equiv -\lambda_1 \equiv -\lambda_2 \mod p_1$. Define $\tilde{j}_3$ by $i(\lambda_1 - \lambda_3) \equiv -(\tilde{j}_1 - \tilde{j}_3) \mod d'$. Since $i(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_2) \mod d'$, it also follows that $i(\lambda_2 - \lambda_3) \equiv -(\tilde{j}_2 - \tilde{j}_3) \mod d'$. By assumption we can show that

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_3}^{\lambda_3}(i) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_3)}{d'} \qquad mod\ p^a,$$

and also

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i) - \sigma_{\tilde{j}_3}^{\lambda_3}(i) \equiv -\frac{\lambda(\tilde{j}_2 - \tilde{j}_3)}{d'} \qquad mod\ p^a.$$

Subtracting, it follows that

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'} \qquad mod\ p^a.$$

So, we may assume $\lambda_1 \equiv -\lambda_2 \mod p_1$. Consider first the definition of $\sigma_{\tilde{j}_1}^{\lambda_1}(i)$. Let $s_1 = s_{\tilde{j}_1}^{\lambda_1}(i)$. Let $i_1 = i + s_1 \bmod d'$. Thus,

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) \equiv \sigma_{\tilde{j}_1}^{\lambda_1}(i_1) + \left(\frac{s_1}{u}\right) d \qquad mod\ d'.$$

Recall i_1 is in the $\lambda_1, \tilde{j}_1$-distinguished class. In defining $\sigma_{\tilde{j}_1}^{\lambda_1}(i_1) \bmod p^a$, we may use the root λ_2 as $\lambda_2 \equiv \lambda_1 \mod p^a$ and $\lambda_2 \equiv -\lambda_1 \mod p_1^{a_1+1}$. Let $\tilde{j}_3$ be defined by $i_1(\lambda_1 - \lambda_2) \equiv -(\tilde{j}_1 - \tilde{j}_3) \mod d'$. We then have

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i_1) - \sigma_{\tilde{j}_3}^{\lambda_2}(i_1) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_3)}{d'} \qquad mod\ p^a,$$

where again λ denotes either λ_1 or λ_2. Note that $\tilde{j}_3$ is old. Let $r_2 = r(i_1)$. Let $i' = i_1 + r_1 \bmod d'$. Then again by definition we have

$$\sigma_{\tilde{j}_3}^{\lambda_2}(i_1) - \sigma_{\tilde{j}_3}^{\lambda_2}(i') \equiv \left(\frac{r_2}{u}\right) d \qquad mod\ d'.$$

Combining these, we get

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) \equiv \sigma_{\tilde{j}_3}^{\lambda_2}(i') + \left(\frac{s_1 + r_2}{u}\right)d - \frac{\lambda(\tilde{j}_1 - \tilde{j}_3)}{d'} \quad mod\ p^a.$$

Now consider $\sigma_{\tilde{j}_2}^{\lambda_2}(i)$. Let $s_2 = s_{\lambda_2}^{\tilde{j}_2}(i)$ and $i_2 = i + s_2$ mod d'. So,

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i_2) \equiv \left(\frac{s_2}{u}\right)d \quad mod\ d'.$$

In defining $\sigma_{\tilde{j}_2}^{\lambda_2}(i_2)$, we may use λ_1 as the auxiliary root. Let $\tilde{j}_4$ be defined by $i_2(\lambda_2 - \lambda_1) \equiv -(\tilde{j}_2 - \tilde{j}_4)$ mod d'. Thus we have

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i_2) - \sigma_{\tilde{j}_4}^{\lambda_1}(i_2) \equiv -\frac{\lambda(\tilde{j}_2 - \tilde{j}_4)}{d'} \quad mod\ p^a.$$

Let $r_1 = r(i_2)$. Let $i'' = i_2 + r_1$ mod d'. Since $i' \equiv i + s_1 + r_2$ mod d' and $i'' \equiv i + s_2 + r_1$ mod d', from (vi) of Lemma (5.1.26) it follows that $i' = i''$. We therefore have

$$\sigma_{\tilde{j}_4}^{\lambda_1}(i_2) - \sigma_{\tilde{j}_4}^{\lambda_1}(i') \equiv \left(\frac{r_1}{u}\right)d \quad mod\ d'.$$

Combining, we get

$$\sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv \sigma_{\tilde{j}_4}^{\lambda_1}(i') - \frac{\lambda(\tilde{j}_2 - \tilde{j}_4)}{d'} + \left(\frac{s_2 + r_1}{u}\right)d \quad mod\ p^a.$$

Thus,

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv \sigma_{\tilde{j}_3}^{\lambda_2}(i') - \sigma_{\tilde{j}_4}^{\lambda_1}(i') + \frac{\lambda(\tilde{j}_3 - \tilde{j}_4)}{d'} - \frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'} \quad mod\ p^a.$$

We now claim that i' satisfies the hypothesis of the consistency condition for $\lambda_2, \tilde{j}_3$ and $\lambda_1, \tilde{j}_4$, that is, we claim that $i'(\lambda_2 - \lambda - 1) \equiv -(\tilde{j}_3 - \tilde{j}_4)$ mod d'. If so, then by partial consistency (note: $i', \tilde{j}_3, \tilde{j}_4$ are old) we have

$$\sigma_{\tilde{j}_3}^{\lambda_2}(i') - \sigma_{\tilde{j}_4}^{\lambda_1}(i') \equiv -\frac{\lambda(\tilde{j}_3 - \tilde{j}_4)}{d'} \quad mod\ p^a.$$

It then follows that

$$\sigma_{\tilde{j}_1}^{\lambda_1}(i) - \sigma_{\tilde{j}_2}^{\lambda_2}(i) \equiv -\frac{\lambda(\tilde{j}_1 - \tilde{j}_2)}{d'} \quad mod\ p^a,$$

and we are done.

It remains to show the claim. Collecting the above definitions we have (all the following equations are mod d'):

$$\begin{aligned}
i(\lambda_1 - \lambda_2) &\equiv -(\tilde{j}_1 - \tilde{j}_2),\\
i_1(\lambda_1 - \lambda_2) &\equiv -(\tilde{j}_1 - \tilde{j}_3),\\
i_2(\lambda_2 - \lambda_1) &\equiv -(\tilde{j}_2 - \tilde{j}_4),\\
i_1 &\equiv i + s_1,\\
i_2 &\equiv i + s_2,\\
i' &\equiv i + s_1 + r_2 \equiv i + s_2 + r_1.
\end{aligned}$$

Thus,

$$i'(\lambda_2 - \lambda_1) \equiv (i + s_2 + r_1)(\lambda_2 - \lambda_1) \equiv (\tilde{j}_1 - \tilde{j}_2) + (s_2 + r_1)(\lambda_2 - \lambda_1).$$

On the other hand,

$$\begin{aligned}
-(\tilde{j}_3 - \tilde{j}_4) &\equiv -[\tilde{j}_1 + i_1(\lambda_1 - \lambda_2) - \tilde{j}_2 - i_2(\lambda_2 - \lambda_1)]\\
&\equiv -(\tilde{j}_1 - \tilde{j}_2) - (i_1 + i_2)(\lambda_1 - \lambda_2)\\
&\equiv -(\tilde{j}_1 - \tilde{j}_2) - (2i + s_1 + s_2)(\lambda_1 - \lambda_2) \quad mod\ d'.
\end{aligned}$$

Thus,

$$\begin{aligned}
&i'(\lambda_2 - \lambda_1) + (\tilde{j}_3 - \tilde{j}_4)\\
&\qquad \equiv 2(\tilde{j}_1 - \tilde{j}_2) - (s_2 + r_1)(\lambda_1 - \lambda_2) + (2i + s_1 + s_2)(\lambda_1 - \lambda_2)
\end{aligned}$$

$$\equiv -2i(\lambda_1 - \lambda_2) - (s_2 + r_1)(\lambda_1 - \lambda_2) + (2i + s_1 + s_2)(\lambda_1 - \lambda_2)$$
$$\equiv (\lambda_1 - \lambda_2)(s_1 - r_1) \quad mod\ d'.$$

From (vii) of Lemma (5.1.26) we have $s_1 = r_1$, and so $i'(\lambda_2 - \lambda_1) + (\tilde{j}_3 - \tilde{j}_4) \equiv 0$ mod d', which gives the claim.

We now summarize and finish the proof of Lemma (5.1.18). Let $d = p_1^{a_1} \cdots p_n^{a_n}$ and $d' = p_1 d$. Assume the k, l functions are defined on R_d and satisfy $(*)_d$. From Lemma (5.1.21), we may assume all of the p_i are non-trivial (congruent to 1 mod 4). By Lemma (5.1.22), we get d-good permutations $\pi_{\tilde{j}}^{\lambda}$ for all $0 \le \tilde{j} < d$ and d-roots λ which satisfy the d-consistency condition. From Lemmas (5.1.27), (5.1.28), the family $\sigma_{\tilde{j}}^{\lambda'}$ for $0 \le \tilde{j} < d', \lambda'$ a d'-root, satisfies the d'-goodness and d'-consistency conditions. From Lemma (5.1.23), we have functions k', l' defined on $R_{d'}$ which satisfy $(*)_{d'}$. Finally, without loss of generality, we may assume the k', l' functions extend the k, l functions. This follows from d'-consistency, since for points of the form $(\frac{i}{d}, \frac{j}{d}) = (\frac{p_1 i}{d'}, \frac{p_1 j}{d'})$, by definition of the $\sigma_{p_1\tilde{j}}^{\lambda'}$ the given values of k, l for this point satisfy the defining equations for $k' + \lambda' l'$ mod each prime power of d'. More specifically, for any d' root λ', the definition of the $\sigma_{p_1\tilde{j}}^{\lambda'}$, equation (28), rewritten becomes

$$(k + \lambda' l) \equiv \sigma_{p_1\tilde{j}}^{\lambda'}(p_1 i) + \lambda' m - \frac{1}{2}\left(\frac{1 + \lambda'^2}{d'}\right)(p_1 i) \quad mod\ d', \tag{31}$$

where $\tilde{j}$ and m are such that $0 \le \tilde{j} < d$ and $p_1 j = p_1 \tilde{j} + \lambda' p_1 i - m d'$. If p^a is a primepower occurring in d' and $\lambda'_{p^a} = \lambda'$ mod p^a, then

$$\left(k + \lambda'_{p^a} l\right) \equiv \sigma_{p_1\tilde{j}}^{\lambda'}(p_1 i) + \lambda'_{p^a} m - \frac{1}{2}\left(\frac{1 + \lambda'^2}{d'}\right)(p_1 i) \quad mod\ p^a, \tag{32}$$

and this is precisely equation (19) (with $\sigma_{p_1\tilde{j}}^{\lambda'}$ replacing $\pi_{\tilde{j}}^{\lambda}$, and λ' replacing λ), which is a typical defining equation for k', l'.

This completes the proof of Lemma (5.1.18), and of Lemma (5.1.13). We now indicate the minor adjustments necessary to get Lemma (5.1.12). There are two di□erences between Lemma (5.1.13) and Lemma (5.1.12). First, in Lemma (5.1.12) there is a distinguished point $(r, s) \in \mathbb{Q}^2 \cap R$ for which there are prescribed values for the k, l functions. Secondly, in Lemma (5.1.12) we must arrange that all of the points $z + (k(z), l(z))$ for $z \in \mathbb{Q}^2 \cap R$ lie in the set P as in the statement of Lemma (5.1.12).

Fix i, j, d such that $r = \frac{i}{d}$, $s = \frac{j}{d}$. Let k_d, l_d be functions on R_d satisfying $(*)_d$. If we add constant values k_0, l_0 to the k_d, l_d functions respectively, the new functions k'_d, l'_d also satisfy $(*)_d$. We choose k_0, l_0 so that k'_d, l'_d take the prescribed values at (r, s). Inspecting equation (3), we see that if functions k''_d, l''_d satisfy $k''_d(z) \equiv k'_d$ mod d, $l''_d(z) \equiv l'_d$ mod d for all $z \in R_d$, then k''_d, l''_d also satisfy $(*)_d$. From the assumed property of P, we may choose k''_d, l''_d so that $z + (k''_d(z), l''_d(z)) \in P$ for all $z \in R_d$. Similarly, at each step when we extend the k, l functions from R_d to $R_{d'}$, only the values of the extended functions mod d' matter in determining $(*)_{d'}$. We may therefore adjust these values mod d' so that $z + (k(z), l(z)) \in P$ for all $z \in R_{d'}$. This completes the proof of Lemma (5.1.12).

Lemma (5.1.29)[190]. Let c_1, c_2, c_3 be three distinct points in the plane, and let $r_1, r_2, r_3 > 0$ be real numbers. Let C_1 be the circle in the plane with center at c_1 and radius r_1, and likewise for C_2 and C_3. Let a, b, c be three distinct points in the plane. Then, except for the exceptional case described below, there are only finitely many triples of points (p_1, p_2, p_3) in the plane such that

(i) $p_1 \in C_1, p_2 \in C_2$, and $p_3 \in C_3$.

(ii) The triangle p_1, p_2, p_3 is isometric with the triangle abc (we allow the degenerate case where the points a, b, c are collinear).

The exceptional case is when $r_1 = r_2 = r_3$ and the triangle abc is isometric with $c_1 c_2 c_3$.

Proof: We prove Lemma (5.1.29), which completes the proof of Theorem (5.1.2). First, we note that a weaker version of Lemma (5.1.29) due to Komjáth (Lemma 1.1 of [199]) would su□ce for our main theorem. Specifically,

Lemma (5.1.30)[190]. (Komjáth). There is a bound $s \in \omega$ such that if $c_1, \ldots, c_s$ are points in the plane with $\rho(c_i, c_j)^2 \notin \mathbb{Z}$ for distinct c_i, c_j, and if $z_1, \ldots, z_s$ are colinear points with $\rho(c_i, z_i)^2 \in \mathbb{Q}$ and $\rho(z_i, z_j)^2 \in \mathbb{Z}$, then the z_i are definable from $\{c_1, \ldots, c_s\}$; in fact, for fixed $c_1, \ldots, c_s$, distances $\rho(c_i, z_i)$ and $\rho(z_i, z_j)$ there are only finitely many such $\{z_1, \ldots, z_s\}$.

To see this su□ces, consider (in the notation of Claim (5.1.10)) the set E_n of points z having rational coordinates with respect to L_n such that $\rho^2(c, z) \in \mathbb{Q}$ for some $c \in S_{<\tilde{\alpha}}$, where c is not rational respect to L_n. Using Lemma (5.1.30) it is easy to see that E_n is semi-small with respect to L_n. By this we mean that for any rational translation L of L_n, there is a finite set F of lines such that for any line $l \notin F, l \cap L \cap E_n$ is finite. Then at each stage in the construction of the points x_m (following Claim (5.1.10)) we must have x_m avoid a certain semi-small set, which is no problem.

Lemma (5.1.29) is implicit in the analysis of Gibson-New stead [195], although it is not explicitly stated there. New stead (private communication) pointed out the following argument. Consider the coupler curve traced out by the point p_3, where triangle $\Delta p_1 p_2 p_3$ is rigid and p_1, p_2 are constrained to lie on circles C_1, C_2 respectively. From [195], the complexification of this curve is a degree 6 curve C in the complex projective plane. They show it is the projection of a higher dimensional curve (the "residual curve") also of degree 6, whose singularities they analyze. Thus, the irreducible components of C precisely correspond to those of R. The components of R are analyzed in [195]. We gives two cases where R (and thus C) can have a component of degree two, namely:

(i) $c_1 c_2 p_2 p_1$ is a parallelogram.

(ii) $p_1 = c_2$ or $p_2 = c_1$.

The second case forces $c_3 = c_2$ or $c_3 = c_1$, which is forbidden as we require c_1, c_2, c_3 to be distinct. The first case is our exceptional case of Lemma (5.1.29).

We now present two elementary proofs of Lemma (5.1.29). The first is a short algebraic proof using some computer algebra, and the second is a purely geometric argument.

The following algebraic computations were performed using Maple.

We assume without loss of generality that C_1 is the circle centered at $c_1 = (0, 0)$ of radius 1, C_2 is the circle centered at $c_2 = (a, 0)$ of radius r, and C_3 is the circle centered at

(b, c) of radiuss. Let $p_1 = (x, y)$ be a point on C_1. If we let d denote the fixed distance between p_1 and the point p_2 on C_2, then we may coordinatize $p_2 = (x_2, y_2)$ by

$$x_2 = x + d\cos(\theta),$$

$$y_2 = y + d\,sin(\theta), \tag{33}$$

where θ denotes the angle that p_1p_2 makes with the horizontal, measured in the usual way. Let α denote the fixed angle of the triangle $p_1p_2p_3$, and let $e = \rho(p_1, p_3)$. Thus, the coordinates of p_3 are of the form

$$x_3 = x + e\,cos(\alpha + \theta) = x + u\,cos(\theta) - v\,sin(\theta),$$

$$y_3 = y + e\,sin(\alpha + \theta) = y + v\,cos(\theta) + u\,sin(\theta), \tag{34}$$

where we let $u = e\,cos(\alpha)$ and $v = e\,sin(\alpha)$. Since p_1, p_2, p_3 lie on C_1, C_2, C_3, we have

$$\begin{gathered} x^2 + y^2 - 1 = 0\,, \\ (x_2 - a)^2 + y_2^2 - r^2 = 0\,, \\ (x_3 - b)^2 + (y_3 - c)^2 - s^2 = 0\,. \end{gathered} \tag{35}$$

Subtracting the second and third equations from the first gives two linear equations for x, y in terms of θ:

$$\begin{gathered} -1 - 2\,x\,d\,cos(\theta) + 2\,x\,a + 2\,d\,cos(\theta)\,a - a^2 - 2\,y\,d\,sin(\theta) - d^2 + r^2 = 0\,, \\ -1 - 2\,cos(\theta)\,u\,x + 2\,cos(\theta)\,u\,b - v^2 + 2\,sin(\theta)\,v\,x - 2\,sin(\theta)\,v\,b + 2\,x\,b - b^2 \\ - u^2 - 2\,sin(\theta)\,u\,y + 2\,sin(\theta)\,u\,c - 2\,cos(\theta)\,v\,y + 2\,cos(\theta)v\,c + 2\,y\,c \\ - c^2 + s^2 = 0\,. \end{gathered} \tag{36}$$

Solving these two equations for x, y gives:

$$\begin{aligned} x = -\frac{1}{2}(&-d\,sin(\theta) - cos(\theta)\,v\,r^2 + cos(\theta)\,v\,d^2 + cos(\theta)\,v\,a^2 - c - v^2\,d\,sin(\theta) \\ &+ sin(\theta)\,u + cos(\theta)\,v - sin(\theta)\,u\,r^2 + sin(\theta)\,u\,d^2 + sin(\theta)\,u\,a^2 \\ &+ s^2\,d\,sin(\theta) - c^2\,d\,sin(\theta) - u^2\,d\,sin(\theta) - b^2\,d\,sin(\theta) + 2\,c\,d\,cos(\theta)\,a \\ &- c\,a^2 - c\,d^2 + c\,r^2 - 2\,v\,d\,a + 2\,cos(\theta)\,v\,c\,d\,sin(\theta) \\ &- 2\,sin(\theta)\,u\,d\,cos(\theta)\,a + 2\,v\,d\,a\,sin^2(\theta) - 2\sin^2(\theta)\,v\,b\,d \\ &+ 2\sin^2(\theta)\,u\,c\,d + 2\,cos(\theta)\,u\,b\,d\,sin(\theta))/\,(c\,a + b\,d\,sin(\theta) \\ &- sin(\theta)\,u\,a - cos(\theta)\,v\,a - c\,d\,cos(\theta) + v\,d\,), \end{aligned} \tag{37}$$

$$\begin{aligned} y = \frac{1}{2}(&-d\,cos(\theta) + a - b + cos(\theta)\,u - sin(\theta)\,v - v\,d^2 sin(\theta) + v\,r^2 sin(\theta) \\ &- v\,a^2 sin(\theta) - cos(\theta)\,u\,r^2 + d^2 cos(\theta)\,u + cos(\theta)\,u\,a^2 - d\,cos(\theta)\,c^2 \\ &- d\,cos(\theta)\,b^2 - d\,cos(\theta)\,u^2 + d\,cos(\theta)\,s^2 - d\,cos(\theta)\,v^2 \\ &- 2\,a\,sin(\theta)\,u\,c - 2\,u\,b\,d\sin^2(\theta) - 2\,v\,c\,d\sin^2(\theta) + 2\sin^2(\theta)\,u\,d\,a \\ &- 2\,d\,cos(\theta)\,sin(\theta)\,v\,b - 2\,a\,cos(\theta)\,u\,b - d^2\,b + a\,v^2 + a\,u^2 + a\,c^2 \\ &- a\,s^2 - a^2\,b + r^2\,b + a\,b^2 - 2\,u\,d\,a + 2\,u\,b\,d + 2\,v\,c\,d \\ &+ 2\,d\,cos(\theta)\,sin(\theta)\,u\,c + 2\,d\,cos(\theta)\,v\,a\,sin(\theta) + 2\,a\,sin(\theta)\,v\,b \\ &- 2\,a\,cos(\theta)\,v\,c + 2\,d\,cos(\theta)\,a\,b)/(c\,a + b\,d\,sin(\theta) - sin(\theta)\,u\,a \\ &- cos(\theta)\,v\,a - c\,d\,cos(\theta) + v\,d\,). \end{aligned} \tag{38}$$

Substituting these expressions back into the equation $x^2 + y^2 - 1$ now gives a large rational function of $sin(\theta), cos(\theta)$. Setting the numerator of this expression to 0 now gives an equation of the form

$$Z_{00} + z_{01}\, sin(\theta) + z_{10}\, cos(\theta) + z_{11}\, cos(\theta) sin(\theta) + z_{20} \cos^2(\theta) + z_{21} \cos^2(\theta) sin(\theta) + z_{30} \cos^3(\theta) = 0\,, \tag{39}$$

where all of the z_{ij} are polynomials in a, b, c, d, u, v, r, and s.

The exceptional case of Lemma (5.1.29) corresponds to a motion of p_1, p_2, p_3 where θ remains constant. Assuming we are not in this case, there will be infinitely many values of θ satisfying equation (39). Thus, the function of equation (39) is identically 0. Since the trigonometric polynomials of equation (39) are linearly independent, this implies that all of the z_{ij} are 0.

In fact, just the last two equations $z_{21} = 0, z_{30} = 0$ su□ce to finish the proof. These two expressions are:

$$\begin{aligned} z_{21} &= 8\, v\, a^2\, d^2\, b - 16\, v\, a^2\, u\, b\, d + 16\, a\, u\, c\, d^2\, b - 8\, v^2\, d\, c\, a^2 + 8\, u^2\, d\, c\, a^2 \\ &\quad - 8\, b^2\, d^2\, v\, a - 16\, c\, u^2\, d\, a\, b - 8\, u\, a^2 c\, d^2 + 8\, c^2\, d^2 v\, a - 16\, d\, u\, c^2 a\, v \\ &\quad + 16\, d\, v\, b^2 a\, u + 16\, v^2 d\, b\, a\, c, \end{aligned} \tag{40}$$

$$\begin{aligned} z_{30} &= -32\, v\, a\, u\, c\, b\, d + 16\, c\, a\, v\, d^2 b + 8\, d\, c^2 a\, u^2 - 8\, v\, a^2 c\, d^2 - 8\, d\, v^2 a^2 b \\ &\quad + 16\, u\, a^2 v\, c\, d + 8\, d\, v^2 a\, b^2 - 8\, d\, c^2 a\, v^2 - 8\, u\, a^2 d^2 b + 8\, a^2 u^2 d\, b \\ &\quad - 8\, c^2 d^2 u\, a - 8\, d\, u^2 a\, b^2 + 8\, d^2\, b^2 u\, a. \end{aligned}$$

Computing a list of reduced Gröbner bases for this pair of equations yields the following (this means that the variety determined by the system $z_{21} = z_{30} = 0$ is the union of the subvarieties determined by the polynomials in each basis listed):

$$\begin{aligned} &[d], [a], [c\, u\, d\, a - a\, d\, v\, b - 2\, c\, u\, b\, d + 2\, d\, v\, b^2 - u^2 a\, c + 2\, u\, a\, v\, b + v^2 a\, c \\ &\quad + 2\, c\, u^2 b - 4\, v\, b^2 u - 2\, v^2 b\, c, a\, u\, b\, d + c\, a\, v\, d - 2\, u\, b^2 d - 2\, b\, v\, c\, d \\ &\quad - b\, a\, u^2 - 2\, v\, a\, u\, c + b\, a\, v^2 + 2\, u^2 b^2 + 4\, v\, u\, c\, b - 2\, v^2 b^2, c^2 \\ &\quad + b^2], [v, c, b], [d - 2\, u, c, b], [u\, d\, a - v\, cd + 2\, a\, v^2 + 2\, u\, v\, c, v\, d\, a \\ &\quad + d\, u\, c - 2\, u\, v\, a + 2\, v^2\, c, u^2 + v^2, b\,], [d - u, v, b], [u\, d\, a - a\, u^2 \\ &\quad + u\, v\, c, v\, d\, a - u\, v\, a + v^2 c, -u\, c - a\, v + c\, d, c^2 + a^2, b\,], [c\, u\, d\, a \\ &\quad - c\, u\, b\, d - c^2\, d\, v + 2\, v^2 a\, c + 2\, u\, c^2 v - 2\, v^2 b\, c, a\, u\, b\, d - u\, b^2\, d \\ &\quad - b\, v\, c\, d + 2\, b\, a\, v^2 + 2\, v\, u\, c\, b - 2\, v^2\, b^2, v\, d\, a + d\, u\, c - d\, v\, b - 2\, u\, v\, a \\ &\quad + 2\, v\, u\, b + 2\, v^2 c, u^2 + v^2], [u, v, c], [a - b, v, c], [d - u, v, c], [d \\ &\quad - u, v], [a\, u\, b\, d - u\, b^2 d - b\, a\, u^2 + v\, u\, c\, b + u^2\, b^2, v\, d\, a - d\, v\, b \\ &\quad + v\, u\, b - u\, v\, a + v^2\, c, -u\, c + v\, b - a\, v + c\, d, b^2 - 2\, a\, b + c^2 + a^2]. \end{aligned} \tag{41}$$

Recalling that $u^2 + v^2 = e^2$, inspecting the bases in this list shows that they imply, in succession: $d = 0, a = 0, b = c = 0, b = c = 0, b = c = 0, e = 0, e = 0, b = c = 0, e = 0, e = 0, b = a$ and $c = 0, e = 0, e = 0, b = a$ and $c = 0$. We have used here the fact that the equations $d = u$ and $v = 0$ imply that $p_3 = p_2$, and hence $e = 0$. Since the centers c_1, c_2, c_3 are distinct, all of these cases are forbidden. This completes the algebraic proof of Lemma (5.1.29).

Let C_1 be the circle with center c_1 and radius r_1, and C_2 the circle with center c_2 and radius r_2. Let p_1, p_2 be distinct points with $p_1 \in C_1$ and $p_2 \in C_2$. Let $f = \rho(p_1, p_2)$. By a "motion" of (p_1, p_2) we mean continuous functions $p_1(t), p_2(t)$ for $0 \leq t \leq 1$ such that $p_1(0) = p_1, p_2(0) = p_2$, and for all t from 0 to 1 we have $p_1(t) \in C_1, p_2(t) \in C_2$, and $\rho(p_1(t), p_2(t)) = f$. We say (q_1, q_2) is in the motion of (p_1, p_2) if there is a motion from (p_1, p_2) to (q_1, q_2) We will also say q_1 is in the motion of p_1 (and likewise for p_2, q_2) if there is a motion from p_1, p_2 to some pair (q_1, q_2) For a given motion, let $\theta_1(t)$ (and likewise for $\theta_2(t)$) be the continuous function such that $\theta_1(0) \in [0, 2\pi)$, and $\theta_1(t)$ mod 2π is the angle θ such that $p_1(t) = c_1 + (r_1 \cos(\theta), r_1 \sin(\theta))$.

We say a motion $(p_1(t), p_2(t))$ is analytic if the coordinate functions $p_1(t) = (x_1(t), y_1(t)), p_2(t) = (x_2(t), y_2(t))$ are analytic functions of t.

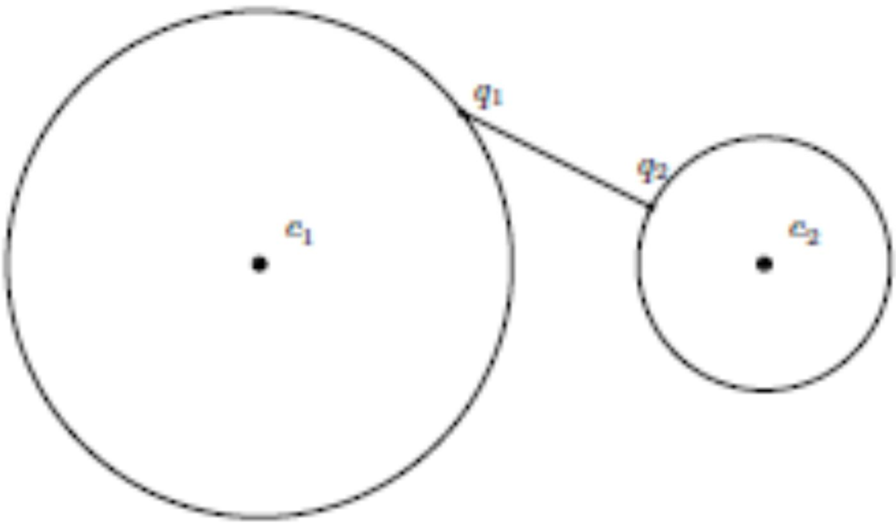

Figure (1)[190]:

Definition (5.1.31)[190]. We say (q_1, q_2) is an extreme point in the motion of (p_1, p_2) for q_1 (and likewise for q_2) if it is in the motion of (p_1, p_2), and any motion of (q_1, q_2) has, for su□ciently small t, q_1 moving in at most one of the two possible tangential directions on C_1 (we refer to this side as the allowable side of q_1). We will also refer to q_1 as being an extreme point in the motion of p_1. We say an extreme point (q_1, q_2) is non-trivial if there is a non-constant motion from (q_1, q_2).

If (q_1, q_2) is an extreme point in the motion of (p_1, p_2) for q_1, then q_1q_2 must pass through c_2. In fact, the non-trivial extreme points can be characterized as those points (q_1, q_2) such that q_1q_2 passes through one of the centers c_1, c_2, but not the other.

Figure 1 illustrates a possible extreme configuration (it is also possible that q_2 lies on the other side of c_2 from q_1).

The following lemma is not required for the proof of Lemma (5.1.29), but it helps to put the above definition in perspective.

Lemma (5.1.32)[190]. Suppose c_2 lies outside of the circle C_1, or c_1 lies outside C_2. Then except for the exceptional case where $r_1 = r_2$ and $\rho(p_1, p_2) = \rho(c_1, c_2)$, there must be an extreme point in the motion of (p_1, p_2).

Proof. Without loss of generality we may assume $c_1 = (0,0)$, and $c_2 = (c,0)$ is on the x-axis and to the right of C_1 $(c > r_1)$. First assume $r_1 > r_2$. We show there is an extreme point in the motion of p_1. If not, then there is a motion of p_1 to the point $(-r_1, 0)$, and also a motion to the point $(r_1, 0)$. Note that C_2 lies entirely to the right of the line $x = 0$. The fact that p_1 can be moved to $(-r_1, 0)$ shows that $f \geq c + r_1 - r_2$. The fact that p_1 can be moved to $(r_1, 0)$, however, shows that $f \leq c + r_2 - r_1$, a contradiction. Assume next that $r_1 < r_2$, and we show there is an extreme point in the motion of p_2. Suppose not, so p_2 can be moved to both $(c + r_2, 0)$ and $(c - r_2, 0)$. From the first fact it follows that $f \geq c + r_2 - r_1$. If $c - r_2 \leq 0$, then the second fact implies $f \leq r_1 + r_2 - c$. Hence $c \leq r_1$,a contradiction. If $c - r_2 > 0$, the second fact implies $f \leq c - r_2 + r_1$. Hence $r_2 \leq r_1$, also a contradiction. Finally, if $r_1 = r_2$, then the argument of the first case also gives a contradiction unless $f = c$, that is,$\rho(c_1, c_2) = \rho(p_1, p_2)$. This is the exceptional case of Lemma (5.1.29).

Definition (5.1.33)[190]. We say a point (q_1, q_2) in the motion of (p_1, p_2) is a double point for q_1 iff or all q_1' in a one-sided neighbourhood of q_1 on C_1 (which we call an allowable side; this may include both sides) except perhaps for $q_1$itself, there are two distinct points q_2', q_2'' on C_2 such that $\rho(q_1', q_2') = f, \rho(q_1', q_2'') = f$ and there is an analytic motion from $(q_1', q_2')$to (q_1', q_2'').

If (q_1, q_2) is a non-trivial extreme point for q1 in the motion of (p_1, p_2), then it is a double point for q_1. For if $q_1' \neq q_1$ is su□ciently close to q_1 and on the allowable side of q_1, then there will be two distinct q_2', q_2'' such that $\rho(q_1', q_2') = f$, $\rho(q_1', q_2'') = f$, with q_2', q_2'' close to q_2 and lying on opposite sides of q_2. If $q_2(t)$ is an analytic function moving from q_2' to q_2'' along C_2, then the corresponding motion of q_1 is also described by an analytic function $q_1(t)$. [In general, if $q_2(t)$ is an analytic motion along C_2 and $q_1(t)$ is a motion along C_1 such that $\rho(q_1(t), q_2(t)) = f$ for all t, then $q_1(t)$ is necessarily analytic provided $q_1(t)q_2(t)$ does not pass through c_1 for all t.]

Note that in the definition of a double point, we do not require that in the analytic motion from (q_1', q_2') to (q_1', q_2'') the function $q_1(t)$ stay in a small neighborhood of q_1'. This is the case, however, if $(q_1, q_2)$is an extreme point in the motion of q_1, as the above argument shows.

We turn now to the proof of Lemma (5.1.29). Fix circles C_1, C_2 with centers at c_1, c_2 and radii r_1, r_2, and assume $c_1 \neq c_2$. Fix $p_1 \in C_1, p_2 \in C_2$, and let $f = \rho(p_1, p_2)$ (we assume $f > 0$). Fix a triangle abc with $f = \rho(a, b)$. We henceforth assume we are not in the exceptional case of Lemma (5.1.29), so either $r_1 \neq r_2$ or $f \neq \rho(c_1, c_2)$. It su□ces to show that for any analytic motion $p_1(t), p_2(t)$ of (p_1, p_2), the corresponding motion $p_3(t)$ does not lie entirely on a circle C_3. Here $p_3(t)$ is the point such that the triangle $p_1(t)p_2(t)p_3(t)$ is congruent to abc. To see this, suppose (p_1^n, p_2^n, p_3^n) were infinitely many triples with $p_i \in C_i$ and $p_1^n p_2^n p_3^n$ congruent to abc. Let $p_1 \in C_1, p_2 \in C_2, p_3 \in C_3$ be such that (p_1, p_2, p_3) is a limit of a subsequence of the (p_1^n, p_2^n, p_3^n). Consider an analytic motion $p_1(t)$ on C_1 nearby p_1. If p_1p_2 does not pass through c_2, then the corresponding motions $p_2(t)$, $p_3(t)$ are uniquely determined and also analytic. Since $\rho(p_3(t), c_3)^2$ is analytic and has infinitely many zeros in a neighborhood of $t = 0$ (we assume $p_1(0) = p_1$); this function must then be identically zero, and thus $p_3(t)$ lies entirely on C_3. Suppose p_1p_2 passes through c_2. Let $p_1(t)$ be an analytic motion on C_1 nearby p_1 moving in a direction from p_1 such that there are infinitely many p_1^n in any interval $[p_1(0), p_1(t))$ for any $t > 0$. There at two analytic functions $p_2(t), p_2'(t)$ such that $p_2(0) = p_2$ and $p_2(t) \in C_2, \rho\,(p_1(t), p_2(t)) = f$ for all t. Furthermore, all (q_1, q_2) close enough to (p_1, p_2) with q_1 on the appropriate side of p_1 and such that $q_1 \in C_1, q_2 \in C_2$, and $\rho(q_1, q_2) = f$ must be of the form $(p_1(t), p_2(t))$ or $(p_1(t), p_2'(t))$ for some t. Without loss of generality, assume for infinitely many n that $(p_1^n, p_2^n) = (p_1(t_n), p_2(t_n))$. Let $p_3(t)$ be the analytic function corresponding to $p_1(t), p_2(t)$. Considering the function $\rho(p_3(t), c_3)^2$ as before now shows that $p_3(t)$ lies entirely on C_3.

We will consider several cases in the proof of Lemma (5.1.29).

<u>Case I.</u> There is a double point (q_1, q_2) in the motion of (p_1, p_2).

If $z_1 \in C_1$ is su□ciently close to q_1 and on an allowable side of q_1, then there are two points z_2, z_2' which lie on C_2 and satisfy $\rho(z_1, z_2) = \rho(z_1, z_2') = f$. Furthermore, there is an analytic motion from (q_1, q_2) to either (z_1, z_2) or (z_1, z_2'). Note that z_2, z_2' are symmetrical with respect to the line from z_1 to c_2. See Figure 2. Let z_3, z_3' denote the corresponding values of z_3. Since z_3, z_3' both lie on C_3, clearly the

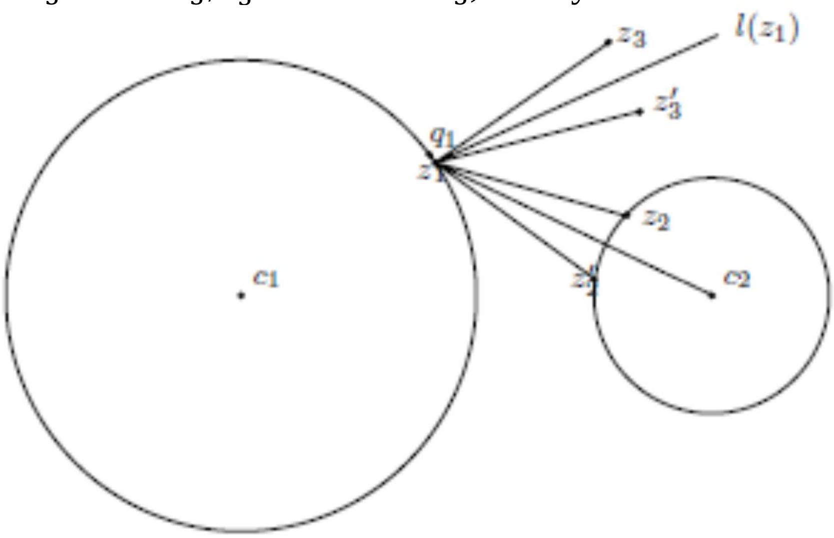

Figure (2)[190]:

line through z_1 which bisects the segment $z_3\, z_3'$ passes through c_3. In other words, if $l(z_1)$ denotes the line through z_1 such that the angle between $l(z_1)$ and z_1c_2 is $\alpha \doteq$ the angle cab, then $l(z_1)$ must pass through c_3. To express this analytically, we coordinatize the circles by letting (without loss of generality) $c_1 = (0, 0), c_2 = (a, 0)$, and $r_1 = 1$. Let $c_3 = (c, d)$ and $\gamma = tan(\alpha)$. Let β be the angle between the segment z_1c_2 and the horizontal line from z_1. Let $z_1 = (cos(\theta), sin(\theta))$. Thus, $(\beta) = \frac{sin(\theta)}{a - cos(\theta)}$. Note that $\alpha - \beta$ is the angle between the horizontal and the segment z_1c_3. If $m(\theta)$ denotes the slope of the line through z_1 and c_3, then we have

$$m(\theta) = \tan(\alpha - \beta) = \frac{tan(\alpha) - \dfrac{sin(\theta)}{a - cos(\theta)}}{1 + tan(\alpha)\left(\dfrac{sin(\theta)}{a - cos(\theta)}\right)}$$

$$= \frac{\gamma\big(a - cos(\theta)\big) - sin(\theta)}{\big(a - cos(\theta)\big) + \gamma sin(\theta)}.$$

Thus, the equation of the line $l(z_1)$ is

$$y = \frac{\gamma(a-cos(\theta)) - sin(\theta)}{(a-cos(\theta)) + \gamma sin(\theta)} x + \left[sin(\theta) - (cos(\theta)) \frac{\gamma(a-cos(\theta)) - sin(\theta)}{(a-cos(\theta)) + \gamma sin(\theta)} \right].$$

Since all of these lines pass through (c, d), it follows that

$$\frac{\gamma(a-cos(\theta)) - sin(\theta)}{(a-cos(\theta)) + \gamma sin(\theta)} (c - cos(\theta)) + sin(\theta) - d$$

is identically 0 for θ in some interval. This simplifies to

$$(\gamma + \gamma ac - ad) + (a - c - \gamma d) sin(\theta) + (-\gamma a - \gamma c + d) cos(\theta) = 0 .$$

Since $1, sin(\theta), cos(\theta)$ are linearly independent, we have

$$\begin{aligned} c\gamma a - ad + \gamma &= 0 , \\ -c + a - \gamma d &= 0 , \\ -c\gamma + d - \gamma a &= 0 . \end{aligned} \qquad (42)$$

From the first and third equations it follows that either $\gamma = 0$ or $a = 1$. If $\gamma = 0$, then from the second equation we have $c = a$. Since $\alpha = 0$ or π in this case, we must therefore have $d = 0$. That is, $c_3 = c_2$, a contradiction.

Assume now that $a = 1$. Solving the second and third equations for c and d gives $c = \frac{1-\gamma^2}{1+\gamma^2}, d = \frac{2\gamma}{1+\gamma^2}$. Thus, $c_3 = (c, d)$ lies on the circle C_1 of radius 1.

Since $a = 1$, c_2 also lies on C_1. Recall $f = \rho(p_1, p_2)$, and let $e = \rho(p_1, p_3)$. Let $r = r_2$ be the radius of the second circle, and $s = r_3$ the radius of the third. Using the same coordinatization and notation as above, except β now denotes the angle $\angle z_2 z_1 c_2 = \angle z_3 z_1 c_3$, the law of cosines gives

(43)

$$r^2 = f^2 + \sin^2(\theta) + (cos(\theta) - a)^2 - 2f\sqrt{sin^2(\theta) + (cos(\theta) - a)^2} \cos(\beta),$$

$$s^2 = e^2 + sin(\theta - d)^2 + cos(\theta - c)^2 - 2e\sqrt{(sin(\theta) - d)^2 + (cos(\theta - c)^2)} cos(\beta).$$

This becomes

$$\frac{u + acos(\theta)}{f\sqrt{sin^2(\theta) + (cos(\theta) - a)^2}} = \frac{v + d\, sin(\theta) + c\, cos(\theta)}{e\sqrt{(sin(\theta) - d)^2 + (cos(\theta - c)^2)}}, \qquad (44)$$

where $2u = r^2 - f^2 - a^2 - 1$ and $2v = s^2 - e^2 - c^2 - d^2 - 1$. Substituting $a = 1$,cross-multiplying and squaring, this becomes

$$h_1 + h_2\, cos(\theta) + h_3 \cos^2(\theta) + h_4 \cos^3(\theta) + h_5\, sin(\theta) + h_6\, sin(\theta) cos(\theta) + h_7\, sin(\theta) \cos^2(\theta) = 0 ,$$

where

$$\begin{aligned} h_1 &= e^2u^2 d^2 + e^2u^2 c^2 - 2f^2 v^2 - 2f^2 d^2 + e^2u^2, \\ h_2 &= 2f^2v^2 + 2\, e^2\, u + 2\, e^2\, u\, c^2 + 2e^2\, u\, d^2 + 2f^2\, d^2 - 4f^2\, vc - 2e^2\, u^2\, c, \\ h_3 &= 2f^2\, d^2 - 4e^2\, uc - 2f^2\, c^2 + e^2c^2 + d^2\, e^2 + 4f^2\, vc + e^2, \\ h_4 &= -2f^2d^2 - 2e^2\, c + 2f^2c^2, \\ h_5 &= -4f^2\, vd - 2e^2\, u^2\, d, \\ h_6 &= -4e^2\, ud - 4f^2\, dc + 4f^2\, v\, d, \\ h_7 &= -2e^2\, d + 4f^2\, dc. \end{aligned} \qquad (45)$$

By linear independence, $h_1 = \cdots = h_7 = 0$. From $h_7 = 0$ we have either $d = 0$, a contradiction as then $c_3 = c_2$, or $e^2 = 2f^2c$. Substituting into the fourth equation we have $f^2(c^2 + d^2) = 0$, hence $f = 0$, a contradiction.

This completes the proof of Lemma (5.1.29) in Case I.

Case II. There is no point (q_1, q_2) in the motion of (p_1, p_2) such that $q_1 q_2$ passes through both c_1 and c_2.

We may assume by Case I that there is no double point, and hence no extreme point in the motion of (p_1, p_2). Let p_1', p_2' denote the reflections of p_1, p_2 about the x-axis, where we again assume $c_1 = (0, 0)$ and $c_2 = (a, 0)$. Let α denote the acute angle between $p_1 p_2$ and the ray $c_1 p_2$. See Figure 3.

Consider an analytic $p_2(t)$ where $p_2(t)$ moves from p_2 to p_2'. Note that in any motion of (p_1, p_2) to a point (q_1, q_2), $q_1 q_2$ cannot pass through either c_1 or c_2. For if it passed through exactly one of these, (q_1, q_2) would be a (non-trivial) extreme point in the motion of (p_1, p_2). Also, by the assumption of the case, $q_1 q_2$ cannot pass through both centers. This implies that there is a uniquely determined analytic function $p_1(t)$ describing the corresponding motion of p_1. Let $\alpha(t)$ denote the angle between $p_2(t) p_1(t)$ and $c_1 p_2(t)$ (so $\alpha(0) = \alpha$). Thus, $\alpha(t) \neq 0$ for all $t \in [0,1]$. It follows that the terminal value of p_1, namely $p_1(1)$, is not the reflected point p_1', but rather the point p_1'' which is the reflection of p_1' about the line $c_1 p_2'$. Thus, p_1'' is obtained from p_1 by two reflections, first about the x-axis, and then about the

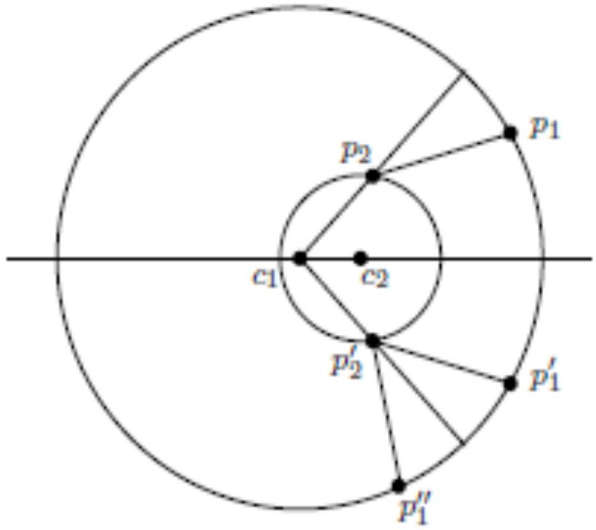

Figure (3)[190]:

line $c_1 p_2'$. Let $p_3(t)$ be the analytic function corresponding to $p_1(t), p_2(t)$. Since the composition of two reflections is orientation preserving, it follows that $p_3(1)$ is obtained from $p_3(0)$ by the same two reflections. In particular, this shows that $p_3(0), p_3(1)$ are equidistant from c_1. Let $l = l(p_1, p_2)$ be the perpendicular bisector of $p_3(0) p_3(1)$. Thus, l passes through $c_1 = (0, 0)$ as well as through c_3.

Consider now another point (q_1, q_2) in the motion of (p_1, p_2), and let $l(q_1, q_2)$ be the corresponding line. If $l(q_1, q_2) \neq l(p_1, p_2)$, then $c_3 = c_1$, a contradiction. Thus, $l(q_1, q_2) = l$ is independent of (q_1, q_2). This can be seen to be impossible. For example, we may argue as follows. By taking a motion of (p_1, p_2), we may assume p_2 is on the x-axis. It follows that l is the line through the origin and p_3. Since the composition of the two reflections described above is just a rotation about the origin, it follows that if we move (p_1, p_2) to any (q_1, q_2), then the angle that c_1q_3 makes with l is the same as q_2c_1 makes with the x-axis. Thus, if we rotate triangle $q_1q_2q_3$ about the origin by this angle, the resulting triangle $q_1'q_2'q_3'$ will be such that q_2' is on the x-axis, q_1' is on C_1, and q_3' is on l. This implies (for su□ciently small non-zero motions) that $(q_1', q_2', q_3') = (p_1, p_2, p_3)$. In other words, (q_1, q_2, q_3) is obtained from (p_1, p_2, p_3) by a rotation about the origin. This shows that $c_2 = c_1 = (0,0)$, a contradiction.

<u>Case III.</u> There is a point (q_1, q_2) in the motion of (p_1, p_2) such that q_1q_2 passes through both c_1 and c_2.

Again, we may assume that in any analytic motion of (p_1, p_2), thereis no extreme point. Thus, as we take an analytic motion of p_1 to the point $q_1 = (1,0)$, p_2 moves in an analytic manner to a point of intersection q_2 of C_2 with the x-axis. It su□ces to show that no analytic motion $(q_1(t), q_2(t))$ of (q_1, q_2) can have the corresponding $q_3(t)$ lying entirely on a circle C_3. In fact, it clearly su□ces to show that if $(q_1(t), q_2(t))$ is an analytic motion in which $q_1(t)$ moves at a uniform rate (say, $q_1(t) = (cos(\pi t), sin(\pi t)))$ to the opposite point $(-1,0)$, then $q_3(t)$ cannot lie entirely on C_3. We can also check that the only case where there is not an obvious extreme point in the motion of (q_1, q_2) occurs when $q_2 = a - r_2$ and $a - r_2 < 0$.

The analytic motion $(q_1(t), q_2(t))$ can be extended to $t < 0$ so that $(q_1(-t), q_2(-t))$ is the reflection of $(q_1(t), q_2(t))$ about the x-axis for $0 \leq t \leq 1$. Thus, for $0 \leq t \leq 1$, $q_1(t)$ moves counter-clockwise from $(1,0)$ to $(-1,0)$, and for t from 0 to −1, $q_1(t)$ moves clockwise from $(1,0)$ to $(-1,0)$. The two terminal positions of $q_2(t)$, namely, $q_2(1)$ and $q_2(-1)$, lie on C_2 and are reflections of each other about the x-axis. By continuity, for each q_1' near $(-1,0)$, there are points q_2', q_2'' on C_2 with $\rho(q_1', q_2') = \rho(q_1', q_2'') = f$, and such that there is an analytic motion from (q_1', q_2') to (q_1', q_2'') (note that this motion involves moving q_1' a full revolution around C_1). This shows that $(q_1(1), q_2(1))$ is a double point for $q_1(1)$, contrary to hypothesis.

An immediate consequence of the existence of a Steinhaus set is the existence of an "n-point" Steinhaus set.

Theorem (5.1.34)[190]. For each integer $n \geq 1$ there is a set $S_n \subseteq \mathbb{R}^2$ such that for every isometric copy L of $\mathbb{Z}^2$ we have $|S_n \cap L| = n$.

Proof. Let $S_1 = S$ be the Steinhaus set from Theorem (5.1.1). Let $z_1, \ldots, z_n$ be n distinct points in $\mathbb{Z}^2$. Let $S_n = \bigcup_{i=1}^{n} S + z_i$. Since S is a Steinhaus set, the sets $S + z_i$ are pairwise disjoint. Each lattice L clearly meets each $S + z_i$ in exactly one point, and the result follows.

There are many problems about Steinhaus sets that remain open. As we mentioned, a Steinhaus set $S \subseteq \mathbb{R}^2$ cannot be both bounded and measurable.

Section (5.2): Comments about the Steinhaus Tiling Problem

The Steinhaus tiling problem, first proposed by Steinhaus in 1957, is whether there exists a set in the plane which, under any isometry, contains exactly one point of $\mathbb{Z}^2$. Recently, Jackson and Mauldin [190] have constructed such a set. The question of whether such a subset of $\mathbb{R}^2$ can be measurable remains open although there are several partial results [84], [44], [83]; in [83] it is shown that such a set cannot have the Baire property.

Kolountzakis and Papadimitrakis [202] considered a variation of this problem: Does there exist a measurable subset E of $\mathbb{R}^d$ such that for almost every $x \in \mathbb{R}^d$ and almost every isometry S, the set $(SE + x)$ contains exactly one point of $\mathbb{Z}^d$? They showed the answer to this question is no, for $d \geq 3$. This result had been shown earlier by Kolountzakis and Wolff [83] by more complicated means which also yield some stronger results. One purpose is to examine how far the argument given in [202] might extend. We begin by repeating the key aspect of this argument, by generalizing from the lattice $\mathbb{Z}^d$, for $d \geq 3$, to the lattice $B\mathbb{Z}^d$, where $B \in GL(d, \mathbb{R})$.

We call the above condition the "almost sure" Steinhaus property on B or on the lattice $B\mathbb{Z}^d$. Specifically, a set E is said to have the almost sure Steinhaus property on B or on the lattice $B\mathbb{Z}^d$, where B is an invertible matrix, provided that under almost every isometry S and almost every point x, $|(SE + x) \cap (B\mathbb{Z}^3)| = 1$. For the remainder we shall suppress the words almost sure. Observe that this property may be described as follows:

$$\sum_{n \in B\mathbb{Z}^d} 1_{SE}(x - n) = 1 \ , a.e. x \in \mathbb{R}^d, a.e. isometry\, S. \qquad (46)$$

Let $\Lambda_A = A\mathbb{Z}^d \subset \mathbb{R}^d$, for $A \in GL(d, \mathbb{R})$, be the lattice induced by A, and let $\Lambda_A^* = A^{-T}\mathbb{Z}^d$ be its dual lattice. From elementary harmonic analysis, we have that if f is an L^1 function, then

$$\sum_{\lambda \in \Lambda_A} f(x - \lambda) = C, \quad a.e. x, \qquad (47)$$

if and only if its Fourier transform satisfies:

$$\hat{f}(\lambda) = 0, \ \forall \lambda : \lambda \in \Lambda_A^* \setminus \{0\}. \qquad (48)$$

By integrating both sides of (47) over the parallelepiped spanned by the columns of A, we find that the constant C equals the integral of f times $|det(A^{-1})|$.

It follows from this that a measurable set E has the almost sure Steinhaus property on A if and only if $\mu(E) = |det(A)|$, where $\mu(E)$ is the Lebesgue measure of E, and $\widehat{1_E}$ vanishes on all points x, such that $\|x\| = \|\lambda\|$ for some $\lambda \in \Lambda_A^*$, $\lambda \neq 0$. In view of this, given a matrix M, let $\mathcal{D}(M) = \{\|Mx\|^2 \mid x \in \mathbb{Z}^d\}$ be the set of possible square distances between points of the lattice $M\mathbb{Z}^d$.

We are now in a position to give sufficient conditions under which there is no measurable set with the almost sure Steinhaus property on B. To this end, suppose we can find a matrix A such that $\mathcal{D}(A^{-T}) \subseteq \mathcal{D}(B^{-T})$, and such that $det(A^{-T})/det(B^{-T}) = det(B)/det(A)$ is not an integer. Now suppose, by way of contradiction, that a measurable set E has the Steinhaus property on B. Then $\widehat{1_E}$ vanishes on all nonzero points with norm square in $\mathcal{D}(B^{-T})$. So, $\widehat{1_E}$ vanishes on all nonzero points with norm square in $\mathcal{D}(A^{-T})$. This means $\widehat{1_E}$ vanishes on $\Lambda_A^* \setminus \{0\}$. This gives us

$$\sum_{\lambda \in \Lambda_A} 1_E(x - \lambda) = \frac{|det(B)|}{|det(A)|}, \quad a.e. x. \qquad (49)$$

It is easy to see that the left side must be an integer, and we have supposed that the right side is not. We adopt a notation with which to state this result.

If A and B are matrices such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, we say B norm dominates A, and write $B \succ A$ or $A \prec B$. If $B \succ A$ and we have that $det(A)/det(B)$ is irrational, we say B strongly norm dominates A, and write $B \succ_s A$. If $B \succ A$ and we have $det(A)/det(B)$ not an integer, we say B weakly norm dominates A, and write $B \succ_w A$. Finally, if $B \succ A$ and $det(A)/det(B) \in \mathbb{Z}$, we say B trivially norm dominates A, and write $B \succ_t A$. With this terminology in place, we have proven the following theorem.

Theorem (5.2.1)[201]: Let $B \in GL(d,\mathbb{R})$ and suppose there exists a matrix $A \in GL(d,\mathbb{R})$, where $B^{-T} \succ_w A$. Then there is no measurable set with the almost sure Steinhaus property on B.

We deal with the question of when a matrix A exists such that $B \succ_w A$. To this end, it is useful to note the following two-part strategy: if we can find a matrix C such that $C \succ_t B$ and $B \succ_s A$, then $C \succ_s A$ and, of course, $C \succ_w A$. Kouluntzakis and Papadimitrakis [202] have resolved some issues about this strategy for the case $B = I$. They show that in case $d = 2$, there is no such A so that this strategy cannot be applied. In case $d > 2$, there is such an A and so the strategy applies. We will not complete their proof here, but mention that, for $d = 3$, their proof concludes by showing (under different terminology) that

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \succ_w \begin{bmatrix} \sqrt{2} & & \\ & \sqrt{6} & \\ & & \sqrt{18} \end{bmatrix}.$$

We prove that if $A^T A$ has rational entries, then there is a diagonal matrix B such that $A \succ_t B$ and $B^T B$ has rational entires. Thus, we can carry out the first part of the strategy. We demonstrate the limitation of the general method by exhibiting a class of diagonal matrices such that if $A \succ B$, then $A \succ_t B$. We provide an example of a matrix H for which the strategy works. Our proof uses some special quadratic forms and the method of descent.

Let us refer to an invertible matrix A as norm rational if $A^T A$ has only rational entries. This is equivalent to saying that the inner product of any two columns of A is rational and also equivalent to saying that $||Ax||^2$ is rational for every $x \in \mathbb{Z}^d$.

Theorem (5.2.2)[201]: Let $A \in GL(d,\mathbb{R})$ be norm rational. Then there is a diagonal norm rational matrix B, also in $GL(d,\mathbb{R})$, such that $A \succ_t B$.

Note that this theorem gives a halfway point toward showing that Theorem (5.2.1) applies to all norm rational matrices.

We prove the theorem in the case of $d = 3$, indicating how to generalize where appropriate.

Proof . First, if R is any linear isometry, then $B \succ A$ if and only if $RB \succ A$, since $\mathcal{D}(B) = \mathcal{D}(RB)$. Also, A is norm rational if and only if RA is norm rational. Therefore, we may assume A is lower triangular and norm rational. Let us say

$$A = \begin{bmatrix} a_{1,1} & & \\ a_{2,1} & a_{2,2} & \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} .$$

We show that general $a_{i,j}$ may be written as $q_{i,j}\sqrt{n_i}$, for some $q_{i,j} \in \mathbb{Q}, n_i \in \mathbb{Z}$. To do this, we note that any number x with $x^2 \in \mathbb{Q}$ may be written uniquely as $q\sqrt{n}$, where q is rational and n is a square free integer. This implies that if $x = q_1\sqrt{n} \neq 0$ and $xy \in \mathbb{Q}$, then $y = q_2\sqrt{n}$, for some rational q_2.

We denote the ith column of A as A_i. We have $\langle A_3, A_3\rangle = a_{3,3}^2 \in \mathbb{Q}$, so we may write $a_{3,3} = q_{3,3}\sqrt{n_3}$. Next, we have $\langle A_1, A_3\rangle = a_{3,1}a_{3,3} \in \mathbb{Q}$ and $\langle A_1, A_2\rangle = a_{3,1}a_{3,2} \in \mathbb{Q}$, so by the second above property, we may also write $a_{3,1} = q_{3,1}\sqrt{n_3}, a_{3,2} = q_{3,2}\sqrt{n_3}$. This is valid since $det(A) \neq 0$ implies $a_{3,3} \neq 0$. Note that this gives $a_{3,i}a_{3,j} \in \mathbb{Q}$ for all i, j.

We now proceed to row 2. We have $\langle A_2, A_2\rangle = a_{2,2}^2 + a_{3,2}^2 \in \mathbb{Q}$. Since we have $a_{3,2}^2 \in \mathbb{Q}$, this gives $a_{2,2}^2 \in \mathbb{Q}$, so we may write $a_{2,2} = q_{2,2}\sqrt{n_2}$. We have $\langle A_1, A_2\rangle = a_{2,1}a_{2,2} + a_{3,1}a_{3,2} \in \mathbb{Q}$, which gives $a_{2,1}a_{2,2} \in \mathbb{Q}$. Again, we have $a_{2,2} \neq 0$, so we may write $a_{2,1} = q_{2,1}\sqrt{n_2}$.

Finally, we have $\langle A_1, A_1\rangle = a_{1,1}^2 + a_{1,2}^2 + a_{1,3}^2 \in \mathbb{Q}$, which gives $a_{1,1}^2 \in \mathbb{Q}$, so we may write $a_{1,1}$ in the desired form.

The method for general dimension d is similar to the above. We proceed row by row upwards, beginning with the diagonal element, which gives all other elements in the row.

We let v_i be a common denominator of the entries $q_{i,j}$ in row i, letting us write $q_{i,j} = \frac{u_{i,j}}{v_i}$, where $u_{i,j}$ and v_i are integers. Now we examine

$$\mathcal{D}(A) = \{\|Ax\|^2 : x \in \mathbb{Z}^3\}$$

$$= \left\{(a_{1,1}x)^2 + (a_{2,1}x + a_{2,2}y)^2 + (a_{3,1}x + a_{3,2}y + a_{3,3}z)^2 : \vec{x} \in \mathbb{Z}^3\right\}$$

$$= \{\frac{n_1}{v_1^2}(a_{1,1}x)^2 + \frac{n_2}{v_2^2}(a_{2,1}x + a_{2,2}y)^2 + \frac{n_3}{v_3^2}(a_{3,1}x + a_{3,2}y + a_{3,3}z)^2 : \vec{x} \in \mathbb{Z}^3\}.$$

If x_1, y_1, and z_1 are nonzero integers, we can consider the subset of the above when $x_1|x$, and so on, and consider $x = x_1\hat{x}, y = y_1\hat{y}\, z = z_1\hat{z}$. The above continues as:

$$\supseteq \left\{\frac{n_1}{v_1^2}(a_{1,1}x_1\hat{x})^2 + \frac{n_2}{v_2^2}(a_{2,1}x_1\hat{x} + a_{2,2}y_1\hat{y})^2 \right. \tag{50}$$

$$\left. + \frac{n_3}{v_3^2}(a_{3,1}x_1\hat{x} + a_{3,2}y_1\hat{y} + a_{3,3}z_1\hat{z})^2 : \vec{\hat{x}} \in \mathbb{Z}^3\right\}. \tag{51}$$

$$= \{\frac{n_1 a_{1,1}^2 x_1^2}{v_1^2}(\hat{x})^2 + \frac{n_2 a_{2,2}^2 y_1^2}{v_2^2}\left(\frac{a_{2,1}x_1}{a_{2,2}y_1}\hat{x} + \hat{y}\right)^2 + \frac{n_3 a_{3,3}^2 z_1^2}{v_3^2}\left(\frac{a_{3,1}x_1}{a_{3,3}z_1}\hat{x} + \frac{a_{3,2}y_1}{a_{3,3}z_1}\hat{y} + \hat{z}\right)^2 : \vec{\hat{x}} \in \mathbb{Z}^3\}.$$

Suppose nonzero integers $x_1, y_1,$ and z_1 have been chosen such that $\frac{a_{2,1}x_1}{a_{2,2}y_1}$, $\frac{a_{3,1}x_1}{a_{3,3}z_1}$, and $\frac{a_{3,2}y_1}{a_{3,3}z_1}$ are integers. This is clearly possible; one method would be to let z_1 be one, then choose y_1 such that the third is an integer, and then choose x_1 so that the first two are integers. (In the higher dimensional case, we have $\frac{d(d-1)}{2}$ fractions which we wish to force to be integers. In that fraction which involves the ith and jth variable, with $i < j$, we will have the ith variable on top, so that we may assign the parameters in decreasing order as in the case of $d = 3$.) This allows a transformation which gives us

$$= \left\{\frac{n_1 a_{1,1}^2 x_1^2}{v_1^2}(\hat{x})^2 + \frac{n_2 a_{2,2}^2 y_1^2}{v_2^2}(\hat{y})^2 + \frac{n_3 a_{3,3}^2 z_1^2}{v_3^2}(\hat{z})^2 : \vec{\hat{x}} \in \mathbb{Z}^3\right\} = \mathcal{D}(B) \quad (52)$$

where B is defined by

$$B = \mathcal{D}\left(\begin{bmatrix} \frac{a_{1,1}x_1\sqrt{n_1}}{v_1} & & \\ & \frac{a_{2,2}y_1\sqrt{n_2}}{v_2} & \\ & & \frac{a_{3,3}z_1\sqrt{n_3}}{v_3} \end{bmatrix}\right).$$

We have shown $A \succ B$, and it is easy to check that $det(B)/det(A)$ is $x_1 y_1 z_1$, and so is an integer, giving us $A \succ_t B$.

Here, we should clear up some possible confusion over the application of Theorem (5.2.1). Theorem (5.2.1) will give us that there is no measurable set with the Steinhaus property over B, if we have that B^{-T} weakly norm dominates another matrix. For this reason, it is useful to note that B is a norm rational matrix if and only if B^{-T} is a norm rational matrix, since $B^T B$ is rational if and only if its inverse, $B^{-1}B^{-T} = (B^{-T})^T B^{-T}$, is rational. This means that Theorem (5.2.2) is useful towards showing a matrix B yields no set having the Steinhaus property on B only if B is norm rational. This will be examined further, where we will use this theorem to show that there is no Steinhaus set on the honeycomb lattice in $\mathbb{R}^3$.

Here, we show that the result or technique given cannot always be used to show that there is no measurable set with the Steinhaus property over any A in $\mathbb{R}^3$. For the lattices described in the next theorem we do not know whether there can be a measurable almost sure Steinhaus set.

Theorem (5.2.3)[201]: There exists a matrix $B \in GL(d, R)$ with the property that for any matrix A if $A \prec B$, then $A \prec_t B$.

Proof . Let the diagonal matrix B have the form

$$B = \begin{bmatrix} \sqrt{\alpha_1} & & \\ & \cdots & \\ & & \sqrt{\alpha_d} \end{bmatrix},$$

where $\alpha_1, \ldots, \alpha_d$ are independent over $\mathbb{Q}$. Suppose also that $A \prec B$. We express A as a quadratic form:

$$\|A\vec{x}\|^2 = f(\vec{x}) = \sum_{i=1}^{d} a_{i,i} x_i^2 + \sum_{i=1}^{d} \sum_{j=i+1}^{d} a_{i,j} x_i x_j . \quad (53)$$

From norm domination, we know that for any $\vec{x} \in \mathbb{Z}^d, f(\vec{x}) = \alpha_1 \hat{x}_1^2 + \ldots + \alpha_d \hat{x}_d^2$ for some $\vec{\hat{x}} \in \mathbb{Z}^d$. It is clear that each of the $a_{i,j}$ are in the rational span of $\{\alpha_1, \ldots, \alpha_d\}$. Thus, we may write $a_{i,j} = a_{i,j,1}\alpha_1 + \ldots + a_{i,j,d}\alpha_d$, where each of the $a_{i,j,k}$ are rational. Thus, we may write

$$f(\vec{x}) = \alpha_1 \Big(\sum_{i=1}^{d} a_{i,i,1} x_i^2 + \sum_{i=1}^{d} \sum_{j=i+1}^{d} a_{i,j,1} x_i x_j \Big) + \ldots + \alpha_d \Big(\sum_{i=1}^{d} a_{i,i,d} x_i^2 + \sum_{i=1}^{d} \sum_{j=i+1}^{d} a_{i,j,d} x_i x_j \Big).$$

We denote the components of this representation as

$$f(\vec{x}) = \alpha_1 f_1(\vec{x}) + \ldots + \alpha_d f_d(\vec{x}) .$$

Since the f_i are rational valued, since $f(\vec{x}) = \alpha_1 \hat{x}_1^2 + \ldots + \alpha_d \hat{x}_d^2$ for some $\vec{\hat{x}} \in \mathbb{Z}^d$, and since $\alpha_1, \ldots, \alpha_d$ are independent over $\mathbb{Q}$, then the f_i are always integer square valued. Here we state a lemma to be proven later.

Lemma (5.2.4)[201]: Any quadratic form which is always integer square valued is the square of an integer linear form.

This means that the f_i are squares of linear forms. We may then write

$$f(\vec{x}) = \alpha_1 \left(\pm\sqrt{a_{1,1,1}}\, x_1 + \ldots \pm \sqrt{a_{d,d,1}}\, x_d \right)^2 + \ldots + \alpha_d \left(\pm\sqrt{a_{1,1,d}}\, x_1 + \ldots \pm \sqrt{a_{d,d,d}}\, x_d \right)^2$$
$$= \left\| \begin{bmatrix} \pm\sqrt{\alpha_1 a_{1,1,1}} & \pm\sqrt{\alpha_1 a_{2,2,1}} & \cdots & \pm\sqrt{\alpha_1 a_{d,d,1}} \\ \pm\sqrt{\alpha_2 a_{1,1,2}} & \pm\sqrt{\alpha_2 a_{2,2,2}} & \cdots & \pm\sqrt{\alpha_2 a_{d,d,2}} \\ \cdots & \cdots & \cdots & \cdots \\ \pm\sqrt{\alpha_d a_{1,1,d}} & \pm\sqrt{\alpha_d a_{2,2,d}} & \cdots & \pm\sqrt{\alpha_d a_{d,d,d}} \end{bmatrix} \vec{x} \right\|^2 .$$

We can see that the $\sqrt{a_{i,i,j}}$ are integers, which gives us that the determinant of the above matrix is $\sqrt{\alpha_1 \ldots \alpha_d}$ times the determinant of an integer matrix. Since this must also be the determinant of A, we have $det(A)/det(B)$ is an integer.

We now prove Lemma (5.2.4).

Proof . We prove by induction on d, the number of variables in our quadratic form, beginning with our base case of $d = 2$.

Suppose $f(x,y) = ax^2 + bxy + cy^2$ is integer square valued on $\mathbb{Z}^2$. Our goal is to show that this is a linear form or, equivalently, to show that $b^2 - 4ac = \Delta = 0$. Assume, by way of contradiction that $\Delta \neq 0$. Considering $f(1,0)$ gives that a is a square. We assume $a \neq 0$, as the alternative case is trivial. We have that $f(-b, 2a) = a(-\Delta)$ is a square, and since a is a nonzero square, we have $-\Delta$ is a square. We also have that $f(2a\Delta - b, 2a) = -\Delta a(4a^2(-\Delta) + 1)$ is a square. Since $-\Delta a$ is a nonzero square, we then have $(4a^2(-\Delta) + 1)$ is a square. On the other hand, we have that $4a^2(-\Delta)$ is a square. This gives us two squares whose difference is one, which implies that $4a^2(-\Delta) = 0$, which yields a contradiction.

We now assume that any quadratic form in $n-1$ variables which is integer square valued on $\mathbb{Z}^{n-1}$ is the square of a linear form, and show the case of n. Assume that f is integer square valued, where f is given by:

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_{i,i}x_i^2 + \sum_{i=1}^{n-1} n \sum_{j=i+1}^{d} a_{i,j}x_ix_j .$$

By considering the case of all but two of the x_i are equal to zero and applying the case of $d = 2$, we find that each of the $a_{i,j} = \pm 2\sqrt{a_{i,i}a_{j,j}}$. We now write f as

$$f(x_1, \ldots, x_n) = (\sum_{i=1}^{n-1} a_{i,i}x_i^2 + \sum_{i=1}^{n-2} n \sum_{j=i+1}^{d-1} a_{i,j}x_ix_j) + \sum_{i=1}^{n} a_{i,n}x_ix_n.$$

The parenthetical of the above is $f(x_1, \ldots, x_n - 1, 0)$, which is a binary form on $n - 1$ variables, and which also is integer square valued. We apply our induction hypothesis to write:

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n-1} b_ix_i\right)^2 + 2\sqrt{a_{n,n}}x_n(\sum_{i=1}^{n-1} c_ix_i) + a_{n,n}x_n^2,$$

where each of the $b_i = \pm\sqrt{a_{i,i}}$ and each of the $c_i = \pm\sqrt{a_{i,i}}$.

Now, if for all i we have $b_i = c_i$, or if for all i we have $b_i = -c_i$, then the above is the square of a linear form, and we are done. Suppose then, by way of contradiction, that $b_j = c_j \neq 0$, for some j, and $b_k = -c_k \neq 0$, for some $k \neq j$. Now consider the case of $x_j = b_kt, x_k = b_jt$, and $x_i = 0$ for all i not equal to j or k. The above then reduces to

$$= 4b_j^2\, b_k^2 t^2 + a_{n,n}x_n^2,$$

which is a binary quadratic form on t and x_n and, again, remains square valued. The base case gives us that $\Delta = -16b_j^2\, b_d^2 a_{n,n} = 0$. We have assumed $b_j \neq 0 \neq b_k$, so we must have $a_{n,n} = 0$, which gives us that our form on n variables is actually a form on $n-1$ variables. Our induction hypothesis then completes the proof.

This result and the result of the previous seem to indicate that the ability of a matrix A to weakly norm dominate any matrix is related to the dimension of the entries of $A^T A$ in the rationals. A reasonable conjecture might be that, for $d = 3$, A weakly norm dominates another matrix if and only if A is a constant times a norm rational matrix. This conjecture cannot hold in general dimension, however, by a counterexample in dimension 6. (Consider a diagonal matrix which has its diagonal comprised of two diagonal matrices from dimension 3 which do weakly norm dominate, one of which is rational, the other multiplied by a transcendental.)

Here we apply Theorem (5.2.1) to show that there is no measurable set with the Steinhaus property over the 3-dimensional standard tetrahedral lattice. The vectors which generate this honeycomb lattice may be visualized by considering three edges of a regular tetrahedron which have a vertex in common. That is, they are three unit vectors, each pair of which has an angle of sixty degrees.

Theorem (5.2.5)[201]: There is no measurable set with the Steinhaus property over the honeycomb lattice H, where H is given by:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{6} \\ & & \frac{\sqrt{6}}{3} \end{bmatrix}, H^{-T} = \begin{bmatrix} 1 & & \\ -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} & \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{6} \end{bmatrix}.$$

Proof .

From Theorem (5.2.1), our goal is to find a matrix A, such that $A \prec_w H^{-T}$. To do this, we actually find two matrices A and B, such that $A \prec_s B$ and $B \prec_t H^{-T}$. We begin by going through the steps of the proof of Theorem (5.2.2) to find B.

$$\begin{aligned} \mathcal{D}(H^{-T}) &= \{x^2 + \frac{1}{3}(-x + 2y)^2 + \frac{1}{6}(-x - y + 3z)^2 : \vec{x} \in \mathbb{Z}^3\} \\ &\supseteq \{x_1^2(\hat{x})^2 + \frac{4y_1^2}{3}\left(-\frac{x_1}{2y_1}\hat{x} + \hat{y}\right)^2 + \frac{3z_1}{2}\left(-\frac{x_1}{3z_1}\hat{x} - \frac{y_1}{3z_1}\hat{y} + \hat{z}\right)^2 \\ &: \vec{\hat{x}} \in \mathbb{Z}^3\}, \end{aligned}$$

if x_1, y_1, and z_1 are integers. We must choose x_1, y_1, and z_1 such that $-\frac{x_1}{2y_1}$, $-\frac{x_1}{3z_1}$, and $\frac{y_1}{3z_1}$ are integers. The simplest such choice is $(x_1, y_1, z_1) = (6, 3, 1)$. This gives us

$$= \left\{36(\hat{x})^2 + 12(-\hat{x} + \hat{y})^2 + \frac{3}{2}(-2\hat{x} - \hat{y} + \hat{z})^2 : \vec{\hat{x}} \in \mathbb{Z}^3\right\}$$

$$= \mathcal{D}\left(\begin{bmatrix} 6 & & \\ & 2\sqrt{3} & \\ & & \frac{1}{2}\sqrt{6} \end{bmatrix}\right) = \frac{3}{2}\{24x^2 + 8y^2 + z^2 : \vec{x} \in \mathbb{Z}^3\}.$$

We now have shown:

$$H^{-T} = \begin{bmatrix} 1 & & \\ -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} & \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{6} \end{bmatrix} \succ_t \begin{bmatrix} 6 & & \\ & 2\sqrt{3} & \\ & & \frac{1}{2}\sqrt{6} \end{bmatrix} = B.$$

Now, we need a matrix A, such that $B \succ_s A$. From this we will have that $H^{-T} \succ_s A$ and the proof will be complete. Let us define A as follows:

$$A = \begin{bmatrix} 2\sqrt{3} & & \\ & \frac{\sqrt{102}}{2} & \\ & & 6 \end{bmatrix}.$$

We have that $det(A)/det(B) = \sqrt{17}/2$ is irrational. We need to show that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, or that

$$\{12x^2 + \frac{51}{2}y^2 + 36z^2 : \vec{x} \in \mathbb{Z}^3\} \subseteq \{36x^2 + 12y^2 + \frac{3}{2}z^2 : \vec{x} \in \mathbb{Z}^3\}.$$

Multiply both sides by $\frac{2}{3}$:

$$\{8x^2 + 17y^2 + 24z^2 : \vec{x} \in \mathbb{Z}^3\} \subseteq \{24x^2 + 8y^2 + z^2 : \vec{x} \in \mathbb{Z}^3\}.$$

It has been shown (see [203]) that those positive integers which cannot be expressed as $24x^2 + 8y^2 + z^2$ are exactly those integers of the form $4n + 2, 4n + 3$, or $4^k(8n + 5)$. By way of contradiction, let us suppose that $8x^2 + 17y^2 + 24z^2 = 4n + 2, 4n + 3$, or $4^k(8n + 5)$. We can immediately rule out $4n + 2$ and $4n + 3$ as we have $8x^2 + 17y^2 + 24z^2 \equiv y^2$ (mod 4), and the quadratic residues mod 4 are 0 and 1.

We consider the remaining case $8x^2 + 17y^2 + 24z^2 = 4^k(8n + 5)$. We consider the values of k.

$k = 0$: Taking the equation mod 8 gives $y^2 \equiv 5$ (mod 8), which is a contradiction, as the only squares mod 8 are 0, 1, and 4.

$k = 1$: We have $8x^2 + 17y^2 + 24z^2 = 4(8n + 5)$. Since the left side is then even, y must be even. Write $y = 2y_1$. Then, dividing by 4 gives $2x^2 + .y_1^2 + 6z^2 \equiv 5$ (mod 8). The only squares mod 8 are 0, 1, and 4, and checking all cases shows that this is a contradiction.

$k \geq 2$: We have $8x^2 + 17y^2 + 24z^2 = 4^k(8n + 5)$. We must have y divisible by 4 for the left side to be divisible by 8, and we write $y = 4y_1$. This gives $x^2 + 34y_1^2 + 3z^2 = 4^{k-2}(16n + 10)$. We see that x and z are both odd, or both even. If they are both even, then we write $x = 2x_1, z = 2z_1$, and $y = 2y_2$ and arrive at $8x_1^2 + 17y_2^2 + 24z_1^2 = 4^{k-1}(8n + 5)$, from which we may repeat the argument and descend until $k < 2$ or x, z are odd.

Assume then that x and z are odd. We write $x = 2x_1 + 1, z = 2z_1 + 1$. This gives us $2x_1(x_1 + 1) + 17y_1^2 + 6z_1(z_1 + 1) + 2 = 4_{k-2}(8n + 5)$. If $k > 2$, then y_1 is even, which gives a contradiction, as 4 divides the right side, but not the left. So, assume $k = 2$. We then have $2x_1(x_1 + 1) + y_1^2 + 6z_1(z1 + 1) \equiv 3 \pmod 8$. For the left side to be odd, we must have y_1 odd, which gives $y_1^2 \equiv 1$, giving $2x_1(x_1 + 1) + 6z_1(z_1 + 1) \equiv 2 \pmod 8$. This yields $x_1(x_1 + 1) + 3z_1(z_1 + 1) \equiv 1 \pmod 4$, which gives a contradiction, as the left side is even and the right is odd.

We now have $A \prec_s B$ and $B \prec_t H^{-T}$, which gives $A \prec_w H^{-T}$, which, by Theorem (5.2.1), completes the proof.

It is important to note that, in the above proof, we relied heavily on having a simple expression for those integers not of the form $x^2 + 8y^2 + 24z^2$. In [203], a ternary quadratic form of the form $ax^2 + by^2 + cz^2$ is called regular if the set of positive integers not represented by it can be written as a union of arithmetic sequences. It is stated that there are exactly 102 regular forms when $gcd(a, b, c) = 1$, which indicates that proofs like this one will not apply to general B.

One of the simplest irregular forms is $x^2 + y^2 + 7z^2$. There is empirical evidence to suggest that

$$\{x^2 + 8y^2 + 28z^2 : \vec{x} \in \mathbb{Z}^3\} \subset \{x^2 + y^2 + 7z^2 : \vec{x} \in \mathbb{Z}^3\}.$$

Specifically, the subset relation holds when the images are restricted to the first 2000 integers. This would give an example of strong norm domination of an irregular form by an irregular form, but no means of proving that the subset holds in general is obvious to us.

Section (5.3): Smooth Partitions of Unity

Smooth partitions of unity are an important tool in the theory of smooth approximations (see [208]), smooth extensions, theory of manifolds, and other areas. Clearly a necessary condition for a Banach space to admit smooth partitions of unity is the existence of a smooth bump function. The sufficiency of this condition for a general Banach space is still an open problem. A positive answer was established in many cases, the most important of which are the following (i.e. if one of the conditions below is fulfilled, then the existence of a smooth bump function on X implies that X admits smooth partitions of unity):

(i) X has an SPRI (separable “projectional resolution of the identity”), [110].
(ii) X belongs to a $\bar{\mathcal{P}}$-class, [97].
(iii) $X = C(K)$ for K compact, [207].

(iv) X has a subspace Y isomorphic to $c_0(\Gamma)$ such that X/Y admits smooth partitions of unity, [205].

(v) X^* is weakly compactly generated (WCG), [211].

For the definition and basic properties of an SPRI see [127] or [209]; for the definition of a $\bar{\mathcal{P}}$-class.

The original proofs of the results (i), (iv), and (v) use Toruńczyk's characterisation of the existence of smooth partitions of unity by non-linear homeomorphic embedding into $c_0(\Gamma)$ with smooth component functions (see e.g. [208]). The other two results use the following theorem of Richard Haydon:

Theorem (5.3.1)[204]: ([97], see also [208]). Let X be a normed linear space that admits a C_k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Γ be a set and $\Phi: X \to c_0(\Gamma)$ a continuous mapping such that for every $\gamma \in \Gamma$ the function $e_\gamma^* \circ \Phi$ is C^k-smooth. For each finite $F \subset \Gamma$ let $P_F \in C^k(X; X)$ be such that the space span $P_F(X)$ admits locally finite C^k-partitions of unity. Assume that for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - P_F(x)\| < \varepsilon$ if we set $F = \{\gamma \in \Gamma; |\Phi(x)(\gamma)| \geq \delta\}$. Then X admits locally finite and σ-uniformly discrete C^k-partitions of unity.

While pondering the applicability of Haydon's theorem we were led to another characterisation of the existence of smooth partitions of unity. This characterisation allows very easy recovery of all the results above except for the $C(K)$ case. In fact, an immediate consequence is a (at least formal) generalisation of (i), (ii), and (v) given in Corollary (5.3.6), which puts all these results under a common roof (this is either obvious or shown in Theorem (5.3.10) and Corollary (5.3.9)). There is also another tiny advantage for the insight into the problem when using Theorem (5.3.1): All the original proofs that use Toruńczyk's characterisation (of course they all come from the same workshop) at some point invoke the completeness of the underlying space, but as we shall see here, the completeness is completely irrelevant to the problem.

Before we start, we fix some notation. By $U(x,r)$, resp. $B(x,r)$ we denote the open, resp. closed ball centred at x with radius r. For a function $f: X \to \mathbb{R}$ we denote $supp_o f = f^{-1}(\mathbb{R} \setminus \{0\})$. For other unfamiliar notation or terminology see [208] or [206].

Now, the reason that Haydon's theorem can be successfully used to prove the wonderful result (iii) is that there is a rich supply of projections of norm one on an Asplund $C(K)$ space (formed by restrictions to clopen subsets of K). So what do we have on an arbitrary Banach space? The projections onto one-dimensional subspaces, of course. This observation leads to the following characterisation:

We make a short technical intermission. Applications of Theorem (5.3.1) involve constructions of continuous mappings into $c_0(\Gamma)$. To avoid repeating the same argument in several of these constructions we will make use of the following simple lemma.

Lemma (5.3.2)[204]: Let X be a topological space, Γ a set, and $\Phi: X \to \mathbb{R}^{\mathbb{N}\times\Gamma}$. Suppose that all the component functions $x \mapsto \Phi(x)(n,\gamma)$ are continuous, $\lim_{n\to\infty} \Phi(x)(n,\gamma) = 0$ locally uniformly in $x \in X$ and uniformly in $\gamma \in \Gamma$, and for each fixed $n \in \mathbb{N}, x \in X$, and $\varepsilon > 0$ there are a neighbourhood U of x and a finite $F \subset \Gamma$ such that $|\Phi(y)(n,\gamma)| < \varepsilon$ whenever $y \in U$ and $\gamma \in \Gamma \setminus F$. Then Φ is a continuous mapping into $c_0(\mathbb{N} \times \Gamma)$.

Proof. Fix $x \in X$ and $\varepsilon > 0$. There are $n_0 \in \mathbb{N}$ and a neighbourhood U of x such that $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $n > n_0, y \in U$, and $\gamma \in \Gamma$. For each $n \in \mathbb{N}, n \leq n_0$ there are a neighbourhood $V_n \subset U$ of x and a finite $F_n \subset \Gamma$ such that $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $y \in V_n$ and $\gamma \in \Gamma \setminus F_n$. Put $F = \bigcup_{n \leq n_0} \{n\} \times F_n$ and $V = \bigcap_{n \leq n_0} V_n$. Then F is finite and $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $y \in V$ and $(n,\gamma) \in \mathbb{N} \times \Gamma \setminus F$. This shows that Φ maps into $c_0(\mathbb{N} \times \Gamma)$. The continuity of Φ follows from the fact that $|\Phi(y)(n,\gamma) - \Phi(x)(n,\gamma)| < \varepsilon$ whenever $y \in V$ and $(n,\gamma) \in \mathbb{N} \times \Gamma \setminus F$, and from the continuity of the functions $y \longmapsto \Phi(y)(n,\gamma), (n,\gamma) \in F$.

Theorem (5.3.3)[204]: Let X be a normed linear space and $k \in \mathbb{N} \cup \{\infty\}$. The following statements are equivalent:

(i) X admits locally finite and σ-uniformly discrete C^k-partitions of unity.

(ii) X admits a C^k-smooth bump and there are a set Γ, a continuous $\Phi: X \to c_0(\Gamma)$ such that $e_\gamma^* \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$, and vectors $\{x_{\gamma_n}\}_{\gamma \in \Gamma, n \in \mathbb{N}} \subset X$ such that $x \in \overline{span}\{x_{\gamma_n};\ \Phi(x)(\gamma) \neq 0, n \in \mathbb{N}\}$ for every $x \in X$.

Notice that the condition (ii) resembles a property of a strong Markushevich basis.

Proof . For the purpose of the proof let us consider the following intermediate statement:

(ii)' X admits a C^k -smooth bump and there are a set Λ, a continuous $\Psi: X \to c_0(\Lambda)$ such that $e_\lambda^* \circ \Psi \in C^k(X)$ for every $\lambda \in \Lambda$, and vectors $\{x_\lambda\}_{\lambda \in \Lambda} \subset X$ such that $x \in \overline{\{x_\lambda;\ \Psi(x)(\lambda) \neq 0\}}$ for every $x \in X$.

(ii)'⇒(i) Since X admits a smooth bump, there are functions $h_n \in C^k(X;\ [0,1])$ such that $supp_o\, h_n \subset U(0,\frac{1}{n})$ and $h_n(0) > 0$. Set $\Gamma = \mathbb{N} \times \Lambda$ and define $\Phi: X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(n,\lambda) = \frac{1}{n}\, h_n(x - x_\lambda)\Psi(x)(\lambda).$$

Then Φ is a continuous mapping into $c_0(\Gamma)$ by Lemma (5.3.2). Clearly, $e_{(n,\lambda)}^* \circ \Phi \in C^k(X)$ for each $(n,\lambda) \in \Gamma$. Next, for each finite non-empty subset $F \subset \Gamma$ let us set $m(F) = max\{n \in \mathbb{N}; (n,\lambda) \in F\ for\ some\ \lambda \in \Lambda\}$, let $\alpha(F) \in \Lambda$ be chosen arbitrarily such that $(m(F), \alpha(F)) \in F$, and let $P_F : X \to X$ be the linear projection onto span$\{x_{\alpha(F)}\}$ of norm at most one. We also set $P_\emptyset = 0$. We show that the assumptions of Theorem (5.3.1) are satisfied. Each one-dimensional subspace of X admits locally finite C^k-partitions of unity ([208]). Given $x \in X$ and $\varepsilon > 0$ find $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \frac{\varepsilon}{2}$. By the assumption there is $\alpha \in \Lambda$ such that $\Psi(x)(\alpha) \neq 0$ and x_α is so close to x that $h_m(x - x_\alpha) > 0$. If we set $\delta = |\Phi(x)(m,\alpha)|$ and $F = \{(n,\lambda) \in \Gamma;\ |\Phi(x)(n,\lambda)| \geq \delta\}$, then $(m,\alpha) \in F$ and hence $m(F) \geq m$. Further, $|\Phi(x)\,(m(F),\alpha(F))| \geq \delta > 0$, and in particular $h_{m(F)}(x - x_{\alpha(F)}) > 0$. It follows that $\|x - x_{\alpha(F)}\| < \frac{1}{m(F)} \leq \frac{1}{m} \leq \frac{\varepsilon}{2}$.

Note that $P_F(x_{\alpha(F)}) = x_{\alpha(F)}$ and therefore $\|x - P_F(x)\| \le \|x - x_{\alpha(F)}\| + \|P_F(x_{\alpha(F)}) - P_F(x)\| < \varepsilon$.

(ii)⇒(ii)' Put $\Lambda = c_{00}^{\mathbb{Q}}(\Gamma \times \mathbb{N})$, i.e. the set of all vectors in $c_{00}(\Gamma \times \mathbb{N})$ with rational coordinates. For each $\lambda \in \Lambda$ set $x_\lambda = \sum_{\gamma\in\Gamma, n\in\mathbb{N}} \lambda(\gamma,n) x_{\gamma_n}$. Clearly, $\{x_\lambda;\ \lambda \in \Lambda\} = span_{\mathbb{Q}}\{x_{\gamma_n};\ \gamma \in \Gamma, n \in \mathbb{N}\}$. Further, let $q : \mathbb{Q} \to \mathbb{N}$ be some one-to-one mapping with $q(0) = 1$ and put $m(\lambda) = max\{n \in \mathbb{N};\ \lambda(\gamma,n) \neq 0\ for\ some\ \gamma \in \Gamma\}$ for $\lambda \in \Lambda \setminus \{0\}$ and $m(0) = 1$. Finally, define $\Psi\colon X \to \mathbb{R}^\Lambda$ by

$$\Psi(x)(\lambda) = \frac{1}{m(\lambda)\prod_{\gamma\in\Gamma,n\in\mathbb{N}} q\left(\lambda(\gamma,n)\right)} \prod_{\gamma\in\Gamma:\ \exists n, \lambda(\gamma,n)\neq 0} \Phi(x)(\gamma)\ .$$

We claim that Ψ is actually a continuous mapping into $c_0(\Lambda)$.

Indeed, fix $x \in X$ and $\varepsilon > 0$. Since Φ is continuous, there are a neighbourhood U of x and a finite set $H \subset \Gamma$ such that $\|\Phi(y)\| < \|\Phi(x)\| + 1$ and $|\Phi(y)(\gamma)| < 1$ for each $y \in U$ and $\gamma \in \Gamma \setminus H$. Note that $\prod_{\gamma\in\Gamma:\ \exists n,\lambda(\gamma,n)\neq 0} |\Phi(y)(\gamma)| \le (\|\Phi(x)\| + 1)^{|H|}$ for any $y \in U$ and $\lambda \in \Lambda$, and the same holds if we omit any one of the factors in the product. Next, there are a neighbourhood V of x, $V \subset U$, and a finite set $E \subset \Gamma$ such that $|\Phi(y)(\gamma)| < \varepsilon/(\|\Phi(x)\| + 1)^{|H|}$ for each $y \in V$ and $\gamma \in \Gamma \setminus E$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon/(\|\Phi(x)\| + 1)^{|H|}$. Put

$$F = \{\lambda \in \Lambda;\ supp\,\lambda \subset E \times \{1,\dots,N\}\ and\ q(\lambda(\gamma,n)) \le N\ for\ all\ \gamma \in \Gamma, n \in \mathbb{N}\}$$

and note that F is finite. Now if $y \in V$ and $\lambda \in \Lambda \setminus F$, then $|\Psi(y)(\lambda)| < \varepsilon$. It easily follows that Ψ is a continuous mapping into $c_0(\Lambda)$.

Clearly, $e_\lambda^* \circ \Psi \in C^k(X)$ for every $\lambda \in \Lambda$. Finally, given $x \in X$ and a neighbourhood U of x, by the assumption there is $\lambda \in \Lambda$ such that $x_\lambda \in U$ and $\Phi(x)(\gamma) \neq 0$ if $\lambda(\gamma,n) \neq 0$ for some $n \in \mathbb{N}$. Consequently, $\Psi(x)(\lambda) \neq 0$.

(i)⇒(ii) The existence of a C^k-smooth bump is clear (just take a partition of unity subordinated to a covering of X by $U(0,2)$ and $X \setminus B(0,1)$). Next, for each $n \in \mathbb{N}$ let $\{\phi_{n\lambda}\}_{\lambda\in\Lambda}$ be a locally finite C^k -partition of unity on X subordinated to the uniform covering of X by open balls of radius $\frac{1}{n}$ (clearly $\{\phi_{n\lambda}\}$ can be constructed by scaling the domains of $\{\phi_{1\lambda}\}$ so that the index set is always the same). Without loss of generality we may assume that all the functions $\phi_{n\lambda}$ are non-zero. We put $\Gamma = \mathbb{N} \times \Lambda$ and define $\Phi\colon X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(n,\lambda) = \frac{1}{n}\,\phi_{n\lambda}(x).$$

Then Φ is a continuous mapping into $c_0(\Gamma)$ by Lemma (5.3.2). To finish, choose any $x_{n\lambda}$ in each $supp_o\,\phi_{n\lambda}$. Fix $x \in X$ and $\delta > 0$. Let $n \in \mathbb{N}$ be such that $\frac{2}{n} < \delta$. There is $\lambda \in \Lambda$ such that $x \in supp_o\,\phi_{n\lambda}$. Then $\Phi(x)(n,\lambda) > 0$ and $\|x - x_{n\lambda}\| < \frac{2}{n} < \delta$. It follows that $x \in \overline{\{x_{n\lambda};\ \Phi(x)(n,\lambda) \neq 0\}}$.

As a first application we show how the above characterisation can be used to rather easily obtain the result (iv). Not only that our proof is substantially shorter than the original, but it also does not use any fancy tools like lifting, Bartle–Graves selectors, etc. The stripped-down proof clearly exposes the three main ideas behind it: the use of linear functionals on the subspace , so that they can be extended to the whole space; the use of a fundamental biorthogonal system in Y , which allows to link these extensions to functionals on X/Y ; and the crucial property of the norm on $c_0(\Gamma)$: if we drop all small coordinates, the vector stays close.

Corollary (5.3.4)[204]: ([205]). Let X be a normed linear space and $Y \subset X$ a subspace isomorphic to $c_0(\Gamma)$ for some Γ. If the quotient X/Y admits locally finite C^k-partitions of unity for some $k \in \mathbb{N} \cup \{\infty\}$, then X admits locally finite and σ-uniformly discrete C^k-partitions of unity.

Proof. By extending the equivalent norm from Y we may assume without loss of generality that Y is actually isometric to $c_0(\Gamma)$. Let $Q: X \to X/Y$ be the canonical quotient mapping. Let $\{(e_\gamma;\ f_\gamma)\}_{\gamma\in\Gamma}$ be the canonical basis of $c_0(\Gamma)$ and further assume that each f_γ is actually a norm-one functional on X (use the Hahn–Banach theorem). For each $n \in \mathbb{N}$ let $\{\psi_{n\lambda}\}_{\lambda\in\Lambda}$ be a locally finite C^k-partition of unity on X/Y subordinated to the uniform covering of X/Y by open balls of radius $\frac{1}{n}$ (clearly $\{\psi_{n\lambda}\}$ can be constructed by scaling the domains of $\{\psi_{1\lambda}\}$ so that the index set is always the same). Without loss of generality we may assume that all the functions $\psi_{n\lambda}$ are non-zero. Choose $z_{n\lambda} \in supp_o\, \psi_{n\lambda}$ and $x_{n\lambda} \in X$ such that $Q(x_{n\lambda}) = z_{n\lambda}$. Let $\theta_n \in C^\infty(\mathbb{R}; \left[0, \frac{1}{n}\right]), n \in \mathbb{N}$ be Lipschitz functions satisfying $\theta_n(t) = 0$ if and only if $|t| \le \frac{2}{n}$. We define a mapping $\Phi: X \to \ell_\infty(\mathbb{N} \times \Lambda \times \Gamma \cup \mathbb{N} \times \Lambda)$ by

$$\Phi(x)(n,\lambda,\gamma) = \theta_n\Big(f_\gamma(x - x_{n\lambda})\Big)\,\psi_{n\lambda}\big(Q(x)\big),$$

$$\Phi(x)(n,\lambda) = \frac{1}{n}\,\psi_{n\lambda}\big(Q(x)\big).$$

First we show that Φ is actually a continuous mapping into $c_0(\mathbb{N} \times \Lambda \times \Gamma \cup \mathbb{N} \times \Lambda)$. Fix $x \in X$ and $\varepsilon > 0$. Clearly, $0 \le \Phi(y)(n,\lambda,\gamma) < \varepsilon$ and $0 \le \Phi(y)(n,\lambda) < \varepsilon$ for $n > \frac{1}{\varepsilon}$ and all $\lambda \in \Lambda, \gamma \in \Gamma, y \in X$. Now fix $n \in [1, \frac{1}{\varepsilon}]$. Since $\{\psi_{n\lambda}\}_{\lambda\in\Lambda}$ is locally finite, there is a neighbourhood V of $Q(x)$ and a finite $F \subset \Lambda$ such that $\psi_{n\lambda} = 0$ on V for $\lambda \in \Lambda \setminus F$. Further, there is a neighbourhood U of x such that $(U) \subset V$. Then clearly $\Phi(y)(n,\lambda,\gamma) = \Phi(y)(n,\lambda) = 0$ for $y \in U$ and $\lambda \in \Lambda \setminus F, \gamma \in \Gamma$. Now fix $\lambda \in F$. Assume that $\psi_{n\lambda}\big(Q(x)\big) \neq 0$. Then $\|Q(x) - z_{n\lambda}\| < \frac{2}{n}$. Put $H = \{\gamma \in \Gamma;\ \left|f_\gamma(x - x_{n\lambda})\right| \ge \frac{2}{n}\}$. We claim that H is finite. Indeed, if H is infinite, then by the w∗-compactness there is a w∗-accumulation point $f \in B_{X^*}$ of $\{f_\gamma\}_{\gamma\in H}$. Then $f|_Y = 0$, since $\{(e_\gamma;\ f_\gamma)\}$ is a fundamental biorthogonal system in Y . In particular, f can be considered also as a member of $\left(\frac{X}{Y}\right)^*$, and so $\frac{2}{n} \le |f(x - x_{n\lambda})| = |f(Q(x) - z_{n\lambda})| \le \|Q(x) - z_{n\lambda}\| < \frac{2}{n}$, a contradiction. Thus $\Phi(x)(n,\lambda,\gamma) = 0$ for $\gamma \in \Gamma \setminus H$. Since the family of functions

$y \mapsto \Phi(y)(n, \lambda, \gamma), \gamma \in \Gamma$, is equi-continuous, there is a neighbourhood W of x such that $|\Phi(y)(n, \lambda, \gamma)| < \varepsilon$ whenever $y \in W$ and $\gamma \in \Gamma \setminus H$. Thus we may apply Lemma (5.3.2).

Next, we set $x_{n\lambda\gamma} = e_\gamma$. Fix any $x \in X$ and $\varepsilon > 0$. There is $n \in \mathbb{N}, n \geq \frac{8}{\varepsilon}$, and $\lambda \in \Lambda$ such that $\psi_{n\lambda}(Q(x)) > 0$ and $\|Q(x) - z_{n\lambda}\| < \frac{\varepsilon}{4}$. Thus there is $u \in Y$ such that $\|x - x_{n\lambda} - u\| < \frac{\varepsilon}{4}$. Put $F = \{\gamma \in \Gamma;\ |f_\gamma(u)| > \frac{\varepsilon}{2}\}$, which is a finite set (possibly empty), and $v = \sum_{\gamma \in F} f_\gamma(u) e_\gamma$. Then $\|u - v\| \leq \frac{\varepsilon}{2}$ (we have the supremum norm here) and so $\|x - (x_{n\lambda} + v)\| \leq \|x - x_{n\lambda} - u\| + \|u - v\| < \varepsilon$. Note that $|f_\gamma(x - x_{n\lambda})| \geq |f_\gamma(u)| - |f_\gamma(x - x_{n\lambda} - u)| > \frac{\varepsilon}{2} - \|x - x_{n\lambda} - u\| > \frac{\varepsilon}{4} \geq \frac{2}{n}$ for $\gamma \in F$. Consequently, $\Phi(x)(n, \lambda, \gamma) > 0$ for $\gamma \in F$. It follows that $x \in \overline{span}\{x_{n\lambda};\ \Phi(x)(n, \lambda) \neq 0\} \cup \{x_{n\lambda\gamma};\ \Phi(x)(n, \lambda, \gamma) \neq 0\}$.

Each component of Φ is clearly $\mathcal{C}^k$-smooth and the space X admits a $\mathcal{C}^k$--smooth bump by [205]. Thus we may conclude the proof by using Theorem (5.3.3).

Before going further, we review some notions useful in the study of the (linear) structure of non-separable Banach spaces. Let $\mathcal{X}$ be a class of Banach spaces. We say that $\mathcal{X}$ is a $\mathcal{P}$-class if for every non-separable $X \in \mathcal{X}$ there exists a projectional resolution of the identity $\{P_\alpha\}_{\alpha \in [\omega, \mu]}$ on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{X}$ for all $\alpha < \mu$. We say that $\mathcal{X}$ is a $\bar{\mathcal{P}}$-class if for every non-separable $X \in \mathcal{X}$ there exists a projectional resolution of the identity $\{P_\alpha\}_{\alpha \in [\omega, \mu]}$ on X such that $P_\alpha(X) \in \mathcal{X}$ for all $\alpha < \mu$. Note that if a class $\mathcal{X}$ admits PRI and is closed under complemented subspaces, then $\mathcal{X}$ is both $\mathcal{P}$-class and $\bar{\mathcal{P}}$-class. Therefore reflexive, WCG, WCD, and WLD are all both $\mathcal{P}$-classes and $\bar{\mathcal{P}}$-classes, as are 1-Plichko spaces ([209]; proof of [210] combined with [210]), spaces with a 1-projectional skeleton (Ondřej Kalenda, private communication; [210]), and duals of Asplund spaces ([99]). Recall that any space from a $\mathcal{P}$-class has an SPRI ([209]), and any space with an SPRI has a strong Markushevich basis (see also [209]or the proof of Theorem (5.3.8)).

Although the characterisations of the existence of smooth partitions of unity are inherently non-linear, in all the results from the introduction, except for the $C(K)$case, the constructions are based on the linear structure in a substantial way. Naturally suggests the following definition:

Definition (5.3.5)[204]: Let X be a normed linear space. We say that a system $\{(x_\gamma;\ f_\gamma)\}_{\gamma \in \Gamma} \subset X \times X^*$ is a fundamental coordinate system if $T(x) = (f_\gamma(x))_{\gamma \in \Gamma}$ is a bounded linear operator from X to $c_0(\Gamma)$ and $x \in \overline{span}\{x_\gamma;\ f_\gamma(x) \neq 0\}$ for each $x \in X$.

Note that the operator T from the definition is necessarily one-to-one and $\{f_\gamma\}_{\gamma \in \Gamma}$ is bounded (by $\|T\|$). The following corollary of Theorem (5.3.3) is now obvious.

Corollary (5.3.6)[204]: Let X be a normed linear space with a fundamental coordinate system and such that it admits a $\mathcal{C}^k$-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$. Then X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.

If X has a strong Markushevich basis, then it also has a fundamental coordinate system (take normalised coordinate functionals). Thus we have an immediate generalisation of the result (i). On the other hand, as we shall see below (Corollary (5.3.9)), the space $JL_p, 1 < p < \infty$, has a fundamental coordinate system but it does not have a Markushevich basis ([209]).

In connection with Corollary (5.3.6) and Corollary (5.3.4) we remark that the space $C(K)$, where K is a Ciesielski-Pol compact, does not continuously linearly inject into any $c_0(\Gamma)$ (and so it does not have a fundamental coordinate system), although it has a subspace Y isometric to $c_0(\Gamma_1)$ such that the quotient $C(K)/Y$ is isomorphic to $c_0(\Gamma_2)$, [99].

Concerning the result (i) we note that for spaces with an SPRI we do not have to rely on the full construction of a strong Markushevich basis (which is rather hard): Let $\{P_\alpha\}_{\alpha\in[\omega,\mu)}$ be an SPRI on X. For each $\alpha \in [\omega,\mu)$ put $Q_\alpha = P_{\alpha+1} - P_\alpha$, let $\{y_{\alpha n}\}_{n\in\mathbb{N}}$ be a dense subset of the separable space $Q_\alpha(X)$, and let $\{g_{\alpha k}\}_{k\in\mathbb{N}} \subset Q_\alpha(X)^*$ be separating for $Q_\alpha(X)$. Put $f_{\alpha k} = g_{\alpha k} \circ Q_\alpha/(\|g_{\alpha k} \circ Q_\alpha\| + 1), \Gamma = [\omega,\mu) \times \mathbb{N} \times \mathbb{N}$, and define $T : X \to \ell_\infty(\Gamma)$ by $T(x)(\alpha,k,n) = \frac{1}{kn} f_{\alpha k}(x)$. Then T is clearly a bounded linear operator.

Further, set $x_\gamma = y_{\alpha n}$ for $\gamma = (\alpha,k,n) \in \Gamma$. Fix $x \in X$. Since $Q_\alpha(x) \neq 0$ if and only if there is $k \in \mathbb{N}$ such that $g_{\alpha k}(Q_\alpha(x)) \neq 0$, which is equivalent to $f_{\alpha k}(x) \neq 0$, we have

$$x \in \overline{span}\{Q_\alpha(x);\ \alpha \in [\omega,\mu)\} = \overline{span}\{Q_\alpha(x);\ \alpha \in [\omega,\mu), Q_\alpha(x) \neq 0\}$$

$$\subset \overline{span}\{\overline{\{y_{\alpha n}\}_{n\in\mathbb{N}}};\ \alpha \in [\omega,\mu), \exists k \in \mathbb{N}: f_{\alpha k}(x) \neq 0\}$$

$$= \overline{span}\{y_{\alpha n};\ n \in \mathbb{N}, \alpha \in [\omega,\mu), \exists k \in \mathbb{N}: T(x)(\alpha,k,n) \neq 0\}$$

$$= \overline{span}\{x_\gamma;\ T(x)(\gamma) \neq 0\}.$$

Note that $Q_\beta \circ Q_\alpha = 0\ for\ \beta \neq \alpha$. Hence, given $\alpha \in [\omega,\mu)$ and $n \in \mathbb{N}$, we have $T(y_{\alpha n})(\beta,k,m) = \frac{1}{km} f_{\beta k}(y_{\alpha n}) = \frac{1}{km} f_{\beta k}(Q_\alpha(y_{\alpha n})) = \frac{1}{km} g_{\beta k}(Q_\beta(Q_\alpha(y_{\alpha n})))\ /\ (\|g_{\beta k} \circ Q_\beta\| + 1) = 0$ for $\beta \neq \alpha$. Also, $|T(y_{\alpha n})(\alpha,k,m)| = \frac{1}{km} |f_{\alpha k}(y_{\alpha n})| \leq \frac{1}{km} \|y_{\alpha n}\|$. Thus $T(y_{\alpha n}) \in c_0(\Gamma)$. Since we have seen above that $X = \overline{span}\{y_{\alpha n};\ \alpha \in [\omega,\mu), n \in \mathbb{N}\}$, it follows that T maps into $c_0(\Gamma)$ and so X has a fundamental coordinate system.

We proceed by deducing the result (v) from Corollary (5.3.9) and the result (ii) from Theorem (5.3.10). We start with an easy observation.

Fact (5.3.7)[204]: Let X be a normed linear space and $\{(x_\gamma;\ f_\gamma)\}_{\gamma\in\Gamma} \subset X \times X^*$. Then $x \in \overline{span}\{x_\gamma;\ f_\gamma(x) \neq 0\}$ for every $x \in X$ if and only if $f \in \overline{span}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\}$ for every $f \in X^*$.

Proof. ⇒ Assume that it is not true for some $f \in X^*$. Denote $A = \{\gamma \in \Gamma;\ f(x_\gamma) \neq 0\}$. By the separation theorem there is $x \in X$ such that $f(x) \neq 0$ and $f_\gamma(x) = 0$ for each

$\gamma \in A$. It follows that $x \in \overline{span}\{x_\gamma;\ f_\gamma(x) \neq 0\} \subset \overline{span}\{x_\gamma;\ \gamma \in \Gamma \setminus A\} \subset \{f\}_\perp$, a contradiction.

$\Leftarrow$ Assume that it is not true for some $x \in X$. Denote $A = \{\gamma \in \Gamma;\ f_\gamma(x) \neq 0\}$. By the separation theorem there is $f \in X^*$ such that $f(x) \neq 0$ and $f(x_\gamma) = 0$ for each $\gamma \in A$. It follows that $f \in \overline{span}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\} \subset \overline{span}^{w^*}\{f_\gamma;\ \gamma \in \Gamma \setminus A\} \subset \{x\}^\perp$, a contradiction.

The first part of the next theorem is probably folklore among experts.

Theorem (5.3.8)[204]: Let X be a WCG Banach space and let $K \subset X$ be a weakly compact convex symmetric set that generates the space X. Then X has a strong Markushevich basis $\{(x_\gamma;\ f_\gamma)\}_{\gamma\in\Gamma} \subset K \times X^*$. Such a basis has the following properties: $T(f) = \left(f(x_\gamma)\right)_{\gamma\in\Gamma}$ is a bounded linear operator from X^* to $c_0(\Gamma)$ and $f \in \overline{span}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\}$ for each $f \in X^*$.

Proof. We prove the first part by transfinite induction on dens X. Suppose first that X is separable. Let $\{z_n\}_{n\in\mathbb{N}} \subset K$ be a dense set in K and $\{h_n\}_{n\in\mathbb{N}}$ a norming set in X^*. Note that $\overline{span}\{z_n\} = X$. By [206] there is a Markushevich basis $\{(y_n;\ g_n)\}_{n\in\mathbb{N}}$ of X such that $span\{y_n\} = span\{z_n\}$ and $span\{g_n\} = span\{h_n\}$. In particular, this basis is norming. Hence by [209] there is a strong Markushevich basis $\{(x_n;\ f_n)\}_{n\in\mathbb{N}}$of X such that $\{x_n\} \subset span\{y_n\} = span\{z_n\}$. Since $span\{z_n\} \subset \bigcup_{n\in\mathbb{N}} nK$, by scaling we may assume that $\{x_n\} \subset K$.

Now assume that dens $X > \omega$ and the statement is true for all WCG spaces of density less than dens X. By [206] there is a PRI $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ on X such that $P_\alpha(K) \subset K$ for each $\alpha \in [\omega,\mu]$. Denote $Q_\alpha = P_{\alpha+1} - P_\alpha$. For each $\alpha \in [\omega,\mu)$ the space $Q_\alpha(X)$ is of density at most card $\alpha <$ *dens* X and is generated by the weakly compact convex symmetric set $\frac{1}{2}Q_\alpha(K)$. Thus by the inductive hypothesis $Q_\alpha(X)$ has a strong Markushevich basis $\{(x_\gamma^\alpha;\ g_\gamma^\alpha)\}_{\gamma\in\Gamma_\alpha}$ such that $\{x_\gamma^\alpha\}_{\gamma\in\Gamma_\alpha} \subset \frac{1}{2}Q_\alpha(K) \subset K$. Put $f_\gamma^\alpha = g_\gamma^\alpha \circ Q_\alpha$. We claim that $\{(x_\gamma^\alpha;\ f_\gamma^\alpha)\}_{\alpha\in[\omega,\mu),\gamma\in\Gamma_\alpha}$ is a strong Markushevich basis of X.

Indeed, $Q_\alpha(x_\eta^\beta) = Q_\alpha(Q_\beta(x_\eta^\beta)) = 0$ and hence $f_\gamma^\alpha(x_\eta^\beta) = g_\gamma^\alpha(Q_\alpha(x_\eta^\beta)) = 0$ for $\alpha \neq \beta$. Further, $f_\gamma^\alpha(x_\eta^\alpha) = g_\gamma^\alpha(Q_\alpha(x_\eta^\alpha)) = g_\gamma^\alpha(x_\eta^\alpha) = \delta_{\gamma,\eta}$ (the Kronecker delta). Fix any $x \in X$. Then $Q_\alpha(x) \in \overline{span}\{x_\gamma^\alpha;\ \gamma \in \Gamma_\alpha : g_\gamma^\alpha(Q_\alpha(x)) \neq 0\} = \overline{span}\{x_\gamma^\alpha;\ \gamma \in \Gamma_\alpha : f_\gamma^\alpha(x) \neq 0\}$.

Hence $x \in \overline{span}\{Q_\alpha(x);\ \alpha \in [\omega,\mu)\} \subset \overline{span} \bigcup_{\alpha\in[\omega,\mu)} \overline{span}\{x_\gamma^\alpha;\ \gamma \in \Gamma_\alpha : f_\gamma^\alpha(x) \neq 0\} \subset \overline{span}\{x_\gamma^\alpha;\ \alpha \in [\omega,\mu), \gamma \in \Gamma_\alpha : f_\gamma^\alpha(x) \neq 0\}$. Finally, note that this strongness property implies that the biorthogonal system is total.

To prove the second part of the theorem, denote by τ the topology on X^* given by the uniform convergence on K. Put $T(f) = \left(f(x_\gamma)\right)_{\gamma\in\Gamma}$ for $f \in X^*$. Then T is clearly a bounded linear operator from X^* to $\ell_\infty(\Gamma)$. Since $\|T(f)\| = \sup_{\gamma\in\Gamma}|f(x_\gamma)| \leq$

$\sup_{x\in K}|f(x)|$, the operator T is moreover $\tau - \|\cdot\|$ continuous. Further, $T(f_\alpha) \in c_{00}(\Gamma)$ for every $\alpha \in \Gamma$. By the Mackey–Arens theorem, $\overline{span}^{\tau}\{f_\alpha\} = \overline{span}^{w^*}\{f_\alpha\} = X^*$. Consequently, $T(X^*) = T(\overline{span}^{\tau}\{f_\alpha\}) \subset \overline{span}\{T(f_\alpha)\} \subset c_0(\Gamma)$. The rest follows from Fact (5.3.7).

We remark that the heart of the construction of a strong Markushevich basis lies in the separable case and is seriously difficult. The strongness of the PRI then arranges the rest. However, for our purpose (the second part of the previous theorem), the full strongness (and even the biorthogonality) of the Markushevich basis is not necessary. It would be sufficient to carry the required properties through the transfinite induction and use just the strongness provided by the PRI. The weak compactness is indispensable though.

Corollary (5.3.9)[204]: Let X be a normed linear space such that X^* is WCG. Then X has a fundamental coordinate system.

Proof. Let $\{(f_\gamma;\, F_\gamma)\}_{\gamma\in\Gamma} \subset X^* \times X^{**}$ be a Markushevich basis from Theorem (5.3.8). Note that $\{f_\gamma\}_{\gamma\in\Gamma}$ is bounded. Fix $\gamma \in \Gamma$. Then by the Goldstine theorem $F_\gamma \in \bar{B}^{w^*}$ for some ball $B \subset X$. Since X^* has the property C ([206]), by [206] there is a countable set of vectors $\{x_{\gamma n}\}_{n\in\mathbb{N}} \subset B$ such that $F_\gamma \in \overline{conv}^{w^*}\{x_{\gamma n}\}_{n\in\mathbb{N}}$. We claim that $\left\{\left(x_{\gamma n};\frac{1}{n} f_\gamma\right)\right\}_{\gamma\in\Gamma, n\in\mathbb{N}}$ is a fundamental coordinate system.

Indeed, $T(x) = \left(\frac{1}{n} f_\gamma(x)\right)_{\gamma\in\Gamma, n\in\mathbb{N}}$ is a bounded linear operator from X to $c_0(\Gamma \times \mathbb{N})$, as $\left(f_\gamma(x)\right)_{\gamma\in\Gamma} \in c_0(\Gamma)$ by Theorem (5.3.8). Fix $x \in X$ and denote $A = \{x_{\gamma n};\, f_\gamma(x) \neq 0, n \in \mathbb{N}\}$. Theorem (5.3.8) implies that $F \in \overline{span}^{w^*}\{F_\gamma;\, F(f_\gamma) \neq 0\}$ for any $F \in X^{**}$ and so

$$x \in \overline{span}^{w^*}\{F_\gamma;\, f_\gamma(x) \neq 0\} \subset \overline{span}^{w^*} \bigcup_{\gamma\in\Gamma:\, f_\gamma(x)\neq 0} \overline{conv}^{w^*}\{x_{\gamma n}\}_{n\in\mathbb{N}} = \overline{span}^{w^*} A.$$

But since $x \in X$ and $span\, A \subset X$, this means that $x \in \overline{span}^{w} A = \overline{span} A$.

We note that there is a Banach space X such that it is a second dual space, it has an equivalent C^1-smooth norm, X^* is a subspace of a Hilbert-generated space (in particular a subspace of a WCG space), and there is no bounded linear one-to-one operator from X to $c_0(\Gamma)$, [123]. Therefore there is no hope for generalising the result (v) beyond the dual being WCG using the approach above (or the original proof as well – both result in a linear injection into $c_0(\Gamma)$).

Theorem (5.3.10)[204]: Every Banach space that belongs to a $\bar{\mathcal{P}}$-class has a fundamental coordinate system.

Proof. Let $\mathcal{X}$ be a $\bar{\mathcal{P}}$-class and $\in \mathcal{X}$. We use transfinite induction on dens X. If X is separable, then we can use the existence of a strong Markushevich basis. However, this difficult result is not necessary. A direct construction is as follows: Let $\{y_n\}_{n\in\mathbb{N}} \subset X$ be dense in X and let $\{g_n\}_{n\in\mathbb{N}} \subset X^*$ be such that it separates the points of X and $\|g_n\| \leq \frac{1}{n}$.

For $k, n \in \mathbb{N}$ put $x_{kn} = y_n$ and $f_{kn} = \frac{1}{n} g_k$. Then $\{(x_{kn}, f_{kn})\}_{k,n\in\mathbb{N}}$ is a fundamental coordinate system: Fix $x \in X$. Then $|f_{kn}(x)| \le \frac{1}{nk} \| x\|$. Also, there is $m \in \mathbb{N}$ such that $g_m(x) \neq 0$ and so $x \in \overline{span}\{y_n;\ n \in \mathbb{N}\} = \overline{span}\{x_{mn};\ n \in \mathbb{N}\} \subset \overline{span}\{x_{kn};\ f_{kn}(x) \neq 0\}$.

Now assume that dens $X > \omega$ and every space in $\mathcal{X}$ of density less than dens X has a fundamental coordinate system. Let $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ be a PRI on X such that $P_\alpha(X) \in \mathcal{X}$ for $\alpha \in [\omega, \mu)$. Put $Q_\alpha = P_{\alpha+1} - P_\alpha$. By the inductive hypothesis, for each $\alpha \in [\omega, \mu)$ there is a fundamental coordinate system $\left\{\left(x_\gamma^\alpha\ ; g_\gamma^\alpha\ \right)\right\}_{\gamma\in\Gamma_\alpha}$ $on\ P_\alpha(X)$ and there is $K_\alpha > 0$ such that $\left\{g_\gamma^\alpha\ \right\}_{\gamma\in\Gamma_\alpha} \subset B(0, K_\alpha)$. Since $Q_\alpha(X) \subset P_{\alpha+1}(X)$, we may set $f_\gamma^{\alpha+1} = \frac{1}{K_{\alpha+1}} g_\gamma^{\alpha+1} \circ Q_\alpha$ and note that $\left\|f_\gamma^{\alpha+1}\right\| \le 2$. We claim that $\left\{\left(x_\gamma^{\alpha+1}\ ; f_\gamma^{\alpha+1}\ \right)\right\}_{\alpha\in[\omega,\mu),\gamma\in\Gamma_{\alpha+1}}$ is a fundamental coordinate system on X.

Indeed, the formula $T(x) = \left(f_\gamma^{\alpha+1}(x)\right)_{\alpha\in[\omega,\mu),\gamma\in\Gamma_{\alpha+1}}$ clearly defines a bounded linear operator from X to $\ell_\infty(\Gamma)$, where $\Gamma = \bigcup_{\alpha\in[\omega,\mu)}\{\alpha\} \times \Gamma_{\alpha+1}$. Now fix $x \in X$ and $\varepsilon > 0$. Then the set $A = \{\alpha \in [\omega, \mu);\ \|Q_\alpha(x)\| > \varepsilon\}$ is finite. So, $\left|f_\gamma^{\alpha+1}(x)\right| \le \frac{1}{K_{\alpha+1}} \left\| g_\gamma^{\alpha+1} \right\| \|Q_\alpha(x)\| \le \varepsilon$ whenever $\alpha \in [\omega, \mu) \setminus A$ and $\gamma \in \Gamma_{\alpha+1}$. On the other hand, if $\alpha \in A$, then the set $\left\{\gamma \in \Gamma_{\alpha+1};\ \left|f_\gamma^{\alpha+1}(x)\right| > \varepsilon\right\} = \{\gamma \in \Gamma_{\alpha+1};\ |g_\gamma^{\alpha+1}(Q_\alpha(x))| > K_{\alpha+1}\varepsilon\}$ is finite by the definition of a fundamental coordinate system. Finally, as $Q_\alpha(x) \in P_{\alpha+1}(X)$, theassumption gives $Q_\alpha(x) \in \overline{span}\{x_\gamma^{\alpha+1}; g_\gamma^{\alpha+1}\left(Q_\alpha(x)\right) \neq 0\} = \overline{span}\left\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\right\}$. Therefore $x \in \overline{span}\{Q_\alpha(x);\ \alpha \in [\omega, \mu)\} \subset \overline{span} \bigcup_{\alpha \in [\omega,\mu)\}} \overline{span}\left\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\right\} \subset \overline{span}\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\}$.

Chapter 6

Smooth Norms with Banach Spaces and their Duals

We show that if X admits a norm, equivalent to the supremum norm, with locally uniformly convex dual norm, then X also admits an equivalent norm that is of class C^∞ (except at 0). It is shown that a Banach space with locally uniformly convex dual admits an equivalent norm that is itself locally uniformly convex. We investigate Banach spaces satisfying a property, which we call (S), and characterise them by means of a new geometric property of the unit sphere which allows us to show, e.g., that all strictly convex norms have (S), there are plenty of non-strictly convex norms satisfying (S). We also study the corresponding renorming problem.

Section (6.1): Approximation in Banach Spaces of the Type $\mathcal{C}(K)$

We shall show two results about smoothness in Banach spaces of the type $\mathcal{C}(K)$. Both build upon earlier of [213], [214].We first establish a special case of a conjecture that remains open for general Banach spaces and concerns smooth approximation. We recall that a bump function on a Banach space X is a function $\beta : X \to \mathbb{R}$ which is not identically zero, but which vanishes outside some bounded set. The existence of bump function of class $\mathcal{C}^1$ implies that the Banach space X is an Asplund space, which, in the case where $X = \mathcal{C}(K)$, is the same as saying that K is scattered. It is a major unsolved problem to determine whether every Asplund space has a $\mathcal{C}^1$ bump function. Another open problem is whether the existence of just one bump function of some class $\mathcal{C}^m$ on a Banach space X implies that all continuous functions on X may be uniformly approximated by functions of class $\mathcal{C}^m$. It is to this question that we give a positive answer (Theorem (6.1.7)) in the special case of $X = \mathcal{C}(K)$.

Our second result represents some mild progress with a conjecture made by the second author in [102]. The analysis of compact spaces constructed using trees suggests that for a compact space K, the existence of an equivalent norm on $\mathcal{C}(K)$, which is of class $\mathcal{C}^1$ (except at 0 of course), might imply the existence of such a norm that is of class $\mathcal{C}^\infty$. Certainly, this is what happens with norms constructed using linear Talagrand operators as in [101], [97], [102]. The other important (and older) method of obtaining $\mathcal{C}^1$ norms is to construct a norm with locally uniformly rotund (LUR) dual norm. What we show in Theorem (6.1.5) is that whenever $\mathcal{C}(K)$ admits an equivalent norm with LUR dual norm, there is also an infinitely differentiable equivalent norm on $\mathcal{C}(K)$.

For background on smoothness and renormings in Banach spaces, including an account of Asplund spaces, see [99]. In particular, an account is given there of the connection between smooth approximability of continuous functions and the existence of smooth partitions of unity. Following what seems to be standard practice in the literature, we have chosen to state the formal version of our first theorem (Theorem (6.1.5)) in terms of partitions of unity, rather than approximation.

Formalizing a definition that appears implicitly in [97], [102], we shall say that a mapping $T : \mathcal{C}(K) \to c_0(K \times \Gamma)$ is a (nonlinear) Talagrand operator of class $\mathcal{C}^m$ if

(i) for each non-zero $f \in \mathcal{C}(K)$, there exist $t \in K, u \in \Gamma$ such that $|f(t)| = \|f\|_\infty$ and $(Tf)(t,u) \neq 0$;

(ii) each coordinate function $f \longmapsto (Tf)(t,u)$ is of class $\mathcal{C}^m$ on the set where it is not zero.

It follows from [97] that if $\mathcal{C}(K)$admits a Talagrand operator of class $\mathcal{C}^m$, then $\mathcal{C}(K)$has a bump function of the same class. It is shown in [97] or [101], that if $\mathcal{C}(K)$admits a linear Talagrand operator, then $\mathcal{C}(K)$admits an equivalent $\mathcal{C}^\infty$ norm. Although there certainly exist examples of compact K such that $\mathcal{C}(K)$ has a $\mathcal{C}^\infty$ renorming but no linear Talagrand operator (for instance, the Cieselski–Pol space [99]), it is worth noting that, by the first of the theorems, a nonlinear Talagrand operator exists whenever there is a bump function.

We consider a non-empty compact scattered space K. The derived set K' is defined as usual to be the set of points t in K that are not isolated in K. Successive derived sets $K^{(\alpha)}$ are defined by the transfinite recursion

$$K^{(0)} = K, \qquad K^{(\beta)} = \bigcap_{\alpha<\beta} (K^{(\alpha)})',$$

There is an ordinal δ such that $K^{(\delta)}$ is non-empty and finite (so that $K^{(\delta+1)} = \emptyset$). For each $t \in K$, there is a unique ordinal $\alpha(t) \leq \delta$ such that $t \in K^{(\alpha(t))} \setminus K^{(\alpha(t)+1)}$. Since t is an isolated point of $K^{(\alpha(t))}$, there is a compact open subset V of K such that $V \cap K^{(\alpha(t))} = \{t\}$; we choose such a V and call it V_t. For finite subsets B of K we set $V_B = \bigcup_{t\in B} V_t$.

Lemma (6.1.1)[208]: Let B be a non-empty finite subset of K and let $\alpha = \alpha(B)$ be maximal subject to $B \cap K^{(\alpha)} \neq \emptyset$. Then $V_B \cap K^{(\alpha)} = B \cap K^{(\alpha)}$ and hence $V_B \cap K^{(\alpha+1)} = \emptyset$.

Proof . Let t be in B. If $\alpha(t) < \alpha$, then $V_t \cap K^{(\alpha)} = \emptyset$, whereas if $\alpha(t) = \alpha$, then $V_t \cap K^{(\alpha)} = \{t\}$. Thus $V_B \cap K^{(\alpha)} = B \cap K^{(\alpha)}$, as claimed.

We shall say that a finite subset A of K is admissible if $s \notin V(t)$ whenever s and t are distinct elements of A.

Lemma (6.1.2)[208]: Let K be a compact scattered space and let H be a non-empty, closed subset of K. There is a unique admissible set A with the property that $A \subseteq H \subseteq V(A)$.

Proof . We start by describing a recursive procedure that constructs one possible admissible A with the required property. Let $\alpha_0 = max\{\alpha : H \cap K^{(\alpha)} \neq \emptyset\}$; thus, $H \cap K^{(\alpha_0)}$ is a non-empty finite set, which we shall call A_0. If $\subseteq V_{A_0}$, we set $A = A_0$ and stop. Otherwise, we set $H_1 = H \setminus V_{A_0}, \alpha_1 = max\{\alpha : H_1 \cap K^{(\alpha)} \neq \emptyset\}, A_1 = H_1 \cap K^{(\alpha_1)}$ and continue. In this way, we construct a decreasing (and so, necessarily finite) sequence $\alpha_0 > \alpha_1 > \cdots > \alpha_l$ of ordinals, and finite sets $A_j = H \cap K^{(\alpha_j)} \setminus$

$V_{A_0 \cup \cdots \cup A_{j-1}}$, in such a way that $H \subseteq V_{A_0 \cup \cdots \cup A_l}$. By construction, $A = A_0 \cup \cdots \cup A_l$ is an admissible set.

We now show uniqueness. It will be convenient to proceed by transfinite induction on α_0. Let B be admissible and suppose that $B \subseteq H \subseteq V_B$. By Lemma (6.1.1), $\alpha(B) = \alpha_0$ and $B \cap K^{(\alpha_0)} \subseteq C \cap K^{(\alpha_0)} \subseteq V_B \cap K^{(\alpha_0)} = B \cap K^{(\alpha_0)}$. Thus $A_0 \subseteq B$. We now have a closed set $H_1 = H \setminus V_{A_0}$ and an admissible set $B_1 = B \setminus A_0$ with $B_1 \subseteq H_1 \subseteq V_{B_1}$. Since $\alpha_1 = max\{\alpha : H_1 \cap K^{(\alpha)} \neq \emptyset\} < \alpha_0$, we may use our inductive hypothesis to deduce that $B_1 = A \setminus A_0$, whence $B = A$.

Let X be a Banach space that admits a bump function of class $\mathcal{C}^m$; so there is a function $\alpha \in \mathcal{C}^m(X)$ such that $\alpha(0) = 1$, whereas $\alpha(x) = 0$ for $\|x\| \geq 1$. By forming β, where $\beta(x) = \varphi(\alpha(x/R))$, with $R > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ suitably chosen, we obtain a function of class $\mathcal{C}^m$, taking values in [0, 1] satisfying

$$\beta(x) = \begin{cases} 1 & when\ \|x\| \leq 1, \\ 0 & when\ \|x\| \geq R. \end{cases}$$

Of course, if X admits partitions of unity of class $\mathcal{C}^m$, then (starting with a partition of unity each of whose members has support of diameter at most ϵ) we easily obtain a function β satisfying the above conditions, with $R = 1 + \epsilon$ and ϵ an arbitrarily small positive real number. In general, we do not know whether a bump function can always be 'improved' in this way. We devoted to showing how to achieve such an improvement in the case where X is a space $\mathcal{C}(K)$ equipped with the supremum norm. We start with an elementary and no doubt well-known exercise in calculus, in which we use the notation to be found for instance in [212].

Lemma (6.1.3)[208]: Let K be a compact space and let $\theta : \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^m$. Then the mapping $\Theta: \mathcal{C}(K) \to \mathcal{C}(K)$ given by $\Theta(f) = \theta \circ f$ is of class $\mathcal{C}^m$.

Proof. We proceed by induction on m. For $m = 0$, we are merely assuming θ to be continuous, and continuity Θ of follows from the uniform continuity of θ on bounded subsets of $\mathbb{R}$.

If $m \geq 1$, we consider f, h in $\mathcal{C}(K)$and apply the mean value theorem point by point to obtain

$$\theta(f(t) + h(t)) - \theta(f(t)) = \theta'(f(t) + \zeta(t)h(t))h(t),$$

with $0 < \zeta(t) < 1$. The uniform continuity of θ' on bounded subsets of $\mathbb{R}$ now tells us that the right-hand side of the above equality equals $\theta'(f(t)) + o(\|h\|_\infty)$. So Θ is differentiable with

$$D\Theta(f) \cdot h = (\theta' \circ f) \times h.$$

The linear mapping $D\Theta(f) : \mathcal{C}(K) \to \mathcal{C}(K)$ is thus the operator $M_{\theta' \circ f}$ of multiplication by $\theta' \circ f$. Thus the derivative $D\Theta : \mathcal{C}(K) \to \mathcal{L}(\mathcal{C}(K))$ may be factored as follows:

$$\mathcal{C}(K) \to \mathcal{C}(K) \to \mathcal{L}(\mathcal{C}(K)),$$

where the first factor is $f \mapsto \theta' \circ f$ and the second is the linear isometry $g \mapsto M_g$. Our inductive hypothesis tells us that the first of these is of class $\mathcal{C}^{m-1}$. So $D\Theta$ is of class $\mathcal{C}^{m-1}$ and Θ of class $\mathcal{C}^{m}$.

Proposition (6.1.4)[208]: Let K be a compact space such that $\mathcal{C}(K)$ admits a bump function of class $\mathcal{C}^m$. Then, for all real numbers $\eta > \xi > 0$, there is a function $\beta_{\xi,\eta} : \mathcal{C}(K) \to [0,1]$ of class $\mathcal{C}^m$ such that

$$\beta_{\xi,\eta}(f) = \begin{cases} 1 & when\ \|f\|_\infty \le \xi, \\ 0 & when\ \|f\|_\infty \ge \eta. \end{cases}$$

Proof . By hypothesis, there exists a function $\alpha : \mathcal{C}(K) \to \mathbb{R}$, of class $\mathcal{C}^m$, such that $\alpha(0) = 1$, whereas $\alpha(f) = 0$ for $\|f\|_\infty \ge 1$. As in our introductory remarks, we may assume that α takes values in [0, 1]. We define $\beta_{\xi,\eta}$ by

$$\beta_{\xi,\eta}(f) = \alpha(\theta \circ f),$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ chosen so that

$$\theta(x) = \begin{cases} 0 & when\ |x| \le \xi, \\ 1 & when\ |x| \ge \eta. \end{cases}$$

We devoted to a proof of the following theorem.

Theorem (6.1.5)[208]: Let K be a compact space and let m be a positive integer or ∞. The following are equivalent:

(i) $\mathcal{C}(K)$ admits a bump function of class $\mathcal{C}^m$;

(ii) $\mathcal{C}(K)$ admits a Talagrand operator of class $\mathcal{C}^m$;

(iii) $\mathcal{C}(K)$ admits partitions of unity of class $\mathcal{C}^m$.

It will be enough to prove that (i) implies both (ii) and (iii). We start by showing how to construct a Talagrand operator, starting with a bump function on $\mathcal{C}(K)$. As we remarked, the existence of a smooth bump function forces K to be scattered. So we can use the notion of admissible sets as developed above. We let Q be the set of all triples (ξ,η,ζ) in $\mathbb{Q}^3$ with $0 < \xi < \eta < \infty$. and write $\mathcal{A}$ for the set of all admissible subsets of K. Choose positive real numbers $c(\xi,\eta,\zeta)$ with $\sum_{(\xi,\eta,\zeta)\in Q} c(\xi,\eta,\zeta) < \infty$. For $0 < \xi < \eta$, let $\beta_{\xi,\eta}$ be as in Proposition (6.1.4), and, finally, for $0 < \eta < \eta$, let $\beta_{\xi,\eta}$ be as in Proposition (6.1.4), and, finally, for $0 < \eta < \zeta$, let $\varphi_{\eta,\zeta} : \mathbb{R} \to [0,1]$ be of class $\mathcal{C}^\infty$ with

$$\varphi_{\eta,\zeta}(x) = \begin{cases} 0 & when\ |x| \le \eta, \\ 1 & when\ |x| \ge \zeta. \end{cases}$$

We define $T : \mathcal{C}(K) \to \ell^\infty(K \times Q \times \mathcal{A})$ by

$$(Tf\,)(s,\xi,\eta,\zeta,A) = c(\xi,\eta,\zeta\,)\beta_{\xi,\eta}(f \times \chi_{K\setminus V_A})\prod_{t\in A}\varphi_{\eta,\zeta}\,(f\,(t))\chi_A(s).$$

We notice that, for this expression to be non-zero, we need $A \subseteq F \subseteq V_A$, where F is the closed set $\{t \in K : |f\,(t)| \geq \eta\}$. Now, we know by Lemma (6.1.2) that, for given η and , there is just one A for which this is true. It follows easily that T takes values in $c_0(K \times Q \times \mathcal{A})$. It is also clear that each coordinate of T, that is to say each mapping $f \longmapsto (Tf\,)(s,\xi\,,\eta,\zeta,A)$, is of class $\mathcal{C}^m$.

To show that T has the Talagrand property, we consider $f \neq 0$ and set $F = \{t \in K : |f\,(t)| = \|f\|_\infty\}$. Let A be the admissible set for which $A \subseteq F \subseteq V_A$ and choose rationals $0 < \xi < \eta < \xi$such that $\xi < \|f\|_\infty$ and $\xi > \left\|f \times \chi_{K\setminus V_A}\right\|_\infty$. For any $s \in A$ we have $|f\,(s)| = \|f\|_\infty$ and $(Tf\,)(s,\xi\,,\eta,\zeta\,) \neq 0$.

We now pass to the construction of partitions of unity. We shall proceed by transfinite recursion on the derived length of K. Recall from that there is an ordinal δ such that $K^{(\delta)}$ is finite and non-empty, so that $K^{(\delta+1)}$ is the first empty derived set of K. We assume inductively that if V is a compact space with $V^{(\delta)} = \emptyset$ and such that $\mathcal{C}(V\,)$ has a bump function of class $\mathcal{C}^m$, then $\mathcal{C}\,(V\,)$ admits $\mathcal{C}^m$ partitions of unity. We need to show that $\mathcal{C}(K)$also admits $\mathcal{C}^m$ partitions of unity. To do this, it will be enough to construct partitions of unity on the finite-codimensional subspace $X = \{f \in \mathcal{C}(K) : f\,(t) = 0\, for\ all\ t \in K^{(\delta)}\}$. We shall use the following result.

Proposition (6.1.6)[208]: [97] Let X be a Banach space, let L be a set and let m be a positive integer or ∞. Let $T : X \to c_0(L)$ be a function such that each coordinate $x \longmapsto T\,(x)_\gamma$ is of class $\mathcal{C}^m$ on the set where it is non-zero. For each finite subset F of L, let $R_F : X \to X$ be of class $\mathcal{C}^m$ and assume that the following hold:

(i) for each F, the image $R_F\,[X]$ admits $\mathcal{C}^k$ partitions of unity;

(ii) X admits a $\mathcal{C}^k$ bump function;

(iii) for each $x \in X$ and each $\epsilon > 0$, there exists $\lambda > 0$ such that $\|x - R_F\,x\| < \epsilon$ if we set $F = \{u \in L : |(T\,x)(u)| \geq \lambda\}$.

Then X admits $\mathcal{C}^m$ partitions of unity.

In applying this result, we shall take L to be $K \times Q \times \mathcal{A}_0$, where $\mathcal{A}_0$ consists of the admissible subsets A such that $A \cap K^{(\delta)} = \emptyset$. The operator T is the Talagrand operator constructed above (though with the argument A restricted to lie in $\mathcal{A}_0$). We have already shown that T takes values in $c_0(L)$ and that the coordinates of T are of class $\mathcal{C}^m$. We define the reconstruction operators R_F as follows: if $F \subset L$ has elements $(s_i,\xi_i,\eta_i,\zeta_i,A_i)$ $(0 \leq i < n)$, we set $V\,(F\,) = \bigcup_{i<n} V_{A_i}$ and define $R_F\,(f\,) = f \times \chi_{V\,(F)}$. So $R_F : X \to X$ is a bounded linear operator and the image R_F may be identified with $\mathcal{C}(V\,(F\,))$ which, by our inductive hypothesis, admits partitions of unity of class $\mathcal{C}^m$.

It only remains to check that (iii) holds, so let $f \in \mathcal{C}(K)$ and $\epsilon > 0$ be given. Let H be the set $\{t \in K : |f\,(t)| \geq \epsilon\}$ and let A be the admissible set such that $A \subseteq H \subseteq V_A$. For suitably chosen $0 < \xi < \eta < \zeta < \epsilon$ we have

$$(Tf)(s,\xi,\eta,\zeta,A) = c(\xi,\eta,\zeta) > 0$$

for all $s \in A$. We set $\lambda = c(\xi,\eta,\zeta)$ and note that $V(F) \supseteq V_A$. So

$$\|f - R_F f\|_\infty = \left\|f \times \chi_{K\setminus V(F)}\right\|_\infty$$

$$\leq \left\|f \times \chi_{K\setminus V_A}\right\|_\infty < \epsilon .$$

We shall now prove the second theorem.

Theorem (6.1.7)[208]: Let K be a compact space such that $C(K)$admits an equivalent norm with locally uniformly rotund dual norm. Then $C(K)$admits an equivalent which is of class C^∞ on $X \setminus \{0\}$.

The norm which we construct will be a generalized Orlicz norm, defined on the whole of $\ell^\infty(K)$, which we shall show to be infinitely smooth on the subspace $C(K)$. We recall some definitions. Suppose that, for each $t \in K$, we are given a convex function $\varphi_t = \varphi(t,\cdot) : [0,\infty) \to [0,\infty)$ satisfying $\varphi(t,0) = 0, \lim_{x\to\infty} \varphi(t,x) = \infty$ (that is, to say an Orlicz function). The generalized Orlicz space, $\ell_{\varphi(\cdot)}(K)$, is defined to be the space of all functions $f : K \to \mathbb{R}$ such that $\sum_{t\in K} \varphi(t,|f(t)|/\rho) < \infty$ for some $\rho \in (0,\infty)$. The generalized Orlicz norm of such a function is defined to be

$$\|f\|_{\varphi(\cdot)} = inf\left\{\rho > 0 : \sum_{t\in K} \varphi\left(\frac{t,|f(t)|}{/\rho}\right) \leq 1\right\}.$$

The first of the following lemmas is elementary and the second uses the familiar idea of 'local dependence on finitely many coordinates'.

Lemma (6.1.8)[208]: Suppose that there exist positive real numbers $R < S$ *such that* $\varphi(t,R) = 0$ *and* $\varphi(t,S) \geq 1$ *for all* $t \in K$. *Then* $\ell_{\varphi(\cdot)}(K) = \ell^\infty(K)$ *and*

$$R\|f\|_{\varphi(\cdot)} \leq \|f\|_\infty \leq S\|f\|_{\varphi(\cdot)} .$$

Lemma (6.1.9)[208]: *Let* $\varphi(\cdot), R$ *and* S *be as in Lemma (6.1.8) and let* X *be a linear subspace of* $\ell^\infty(K)$. *Suppose that, whenever* $f \in X$ *and* $\|f\|_{\varphi(\cdot)} = 1$, *there exist a positive real number* δ *and a finite subset* F *of* K *such that* $g(t) = 0$ *whenever* $g \in X, \|f - g\|_\infty < \delta$ *and* $t \notin F$. Assume further that each of the functions $\varphi(t,\cdot)$ is of class C^∞. Then the generalized Orlicz norm $\|\cdot\|_{\varphi(\cdot)}$ is of class C^∞ on $X \setminus \{0\}$.

Proof . If f, F and δ are as in the statement of the lemma, then the function Φ defined by

$$\Phi(g) = \sum_{t\in K} \varphi(t,|g(t)|)$$

is of class C^∞ on $\{g \in X : \|f - g\|_\infty < \delta\}$, since it coincides with the finite sum $\sum_{t\in F} \varphi(t,|g(t)|)$ of given C^∞ functions. Thus our hypothesis says that there is an open subset U of X containing the set $\{f \in X : \|f\|_{\varphi(\cdot)} = 1\}$ and such that Φ is C^∞ on U. We define $V = \{h,\rho) \in (X \setminus \{0\}) \times (0,\infty) : \rho^{-1}h \in U\}$, an open set in X.

On V we define $\Psi(h,\rho) = \Phi(\rho^{-1}h)$, which is of class C^∞. For each $h \in X \setminus \{0\}$, there is a unique $\rho = \|h\|_{\varphi(\cdot)}$ such that $(h,\rho) \in V$ and $\Psi(h,\rho) = 1$. Moreover, we may calculate the partial derivative

$$D_2\Psi(h,\rho) = -\rho^{-2}\sum_t \varphi_t'(\rho^{-1}|h(t)|)$$

and note that this is non-zero when $\rho = \|h\|_{\varphi(\cdot)}$, since $\varphi_t'(x) > 0$ whenever $\varphi_t(x) > 0$. Thus, the implicit function theorem may be applied to conclude that $\|\cdot\|_{\varphi(\cdot)}$ is of class C^∞ on $X \setminus \{0\}$.

To choose suitable Orlicz functions φ_t in our theorem, we shall need to use the special properties of K. The assumption that $C(K)^*$ has an equivalent LUR dual norm implies (and, by a theorem of Raja [104], is actually equivalent to) the compact space K being σ-discrete. So we may assume that there are pairwise disjoint subsets D_i of K, each one discrete in its subspace topology, with $K = \bigcup_{i\in\omega} D_i$. We note in passing that we are not assuming D_i to be closed, merely that D_i has empty intersection with its derived set D_i'. We fix positive real numbers $r_i < 1$ with $\prod_{i\in\omega} r_i > 0$ and, for $t \in K$, define two real numbers

$$\alpha(t) = \prod\{r_i : t \in \overline{D}_i\},$$

$$\beta(t) = \prod\{r_i : t \in D_i'\}.$$

We notice that $(t) = \alpha(t) \times r_j$, where j is the (unique) natural number such that $t \in D_j$. (Note that it is here, and only here that we use the discreteness hypothesis that $D_j \cap D_j' = \emptyset$.). In particular, therefore, $0 < \alpha(t) < \beta(t) < 1$. So we may choose an infinitely differentiable Orlicz function φ_t such that

$$\varphi_t(x) = \begin{cases} 0 & when\, x \le \alpha(t), \\ > 1 & when\, x \ge \beta(t). \end{cases}$$

We are going to show that Lemma (6.1.9) may be applied to these Orlicz functions and the subspace $C(K)$ of ℓ^∞ . It is convenient to state one of the ingredients of this proof as a property of the functions α and β.

Lemma (6.1.10)[208]: Let (t_n) be a sequence of distinct elements of K which converges to some $t \in K$. Then $\beta(t) \le \lim\inf \alpha(t_n)$.

Proof. By taking subsequences and diagonalizing, we may assume that $\alpha(t_n)$ tends to a limit as $n \to \infty$ and also that, for each $i \in \omega$, either all of $t_{i+1}, t_{i+2}, \dots$ are in $\overline{D}_i$ or else none is. Let M be the set of natural numbers i such that $t_{i+1}, \dots$ are in $\overline{D}_i$ .Then for each n we have

$$\prod_{i<n, i\in M} r_i \times \prod_{i\ge n} r_i \le \alpha(t_n) \le \prod_{i\in M} r_i ,$$

whence $\alpha(t_n) \to \prod_{i\in M} r_i$ as $n \to \infty$. On the other hand, since the t_n are distinct and $t_n \in D_i$ whenever $n > i \in M$, it must be that the limit point t is in the derived set D_i' whenever $i \in M$. Thus

$$\beta(t) = \prod\{r_i : t \in D_i'\} \le \prod_{i\in M} r_i.$$

To complete the proof of the theorem, we consider $f \in \mathcal{C}(K)$ with $\|f\|_{\varphi(\cdot)} = 1$. If no δ and F exist with the property of Lemma (6.1.9), there exist a sequence (f_n) in $\mathcal{C}(K)$converging uniformly to f and a sequence of distinct elements (t_n) of K such that $\varphi(t_n, |f_n(t_n)|) > 0$ for all n. For this to be the case it must be that $|f_n(t_n)| \ge \alpha(t_n)$. Extracting a subsequence, we may suppose that the sequence (t_n) converges to some $t \in K$. Now by uniform convergence and the continuity of f, we have $f(t) = \lim_n f_n(t_n)$, so that $|f(t)| \ge \beta(t)$ by Lemma (6.1.10). Thus $\varphi(t, |f(t)|) > 1$ and $\|f\|_{\varphi(\cdot)} > 1$, a contradiction, which ends the proof.

Section (6.2): Locally Uniformly Convex Norm

If we consider a real Banach space Z under a norm $\|\cdot\|$ and its dual space Z^*, equipped with the dual norm $\|\cdot\|^*$, there are important and well-established connections between convexity properties of $\|\cdot\|^*$ and smoothness properties of $\|\cdot\|$. Indeed, strict convexity of $\|\cdot\|^*$ implies Gâteaux-smoothness of $\|\cdot\|$, locally uniform convexity of $\|\cdot\|^*$ implies Fréchet-smoothness of $\|\cdot\|$ and uniform convexity of $\|\cdot\|^*$ is equivalent to uniform smoothness of $\|\cdot\|$. On the other hand, there would seem to be, a priori, no reason why a convexity condition in the dual space Z^* should imply any sort of convexity in Z. However, it is a consequence of the Enflo–Pisier renorming theorem [219], [226], or [99] that uniform convexity of $\|\cdot\|^*$ implies that there exists a norm $|||\cdot|||$ on Z, equivalent to the given norm, which is itself uniformly convex. One can even arrange that this new norm be both uniformly convex and uniformly smooth.

It is natural to ask whether a similar result about equivalent norms holds for the weaker properties of strict convexity and locally uniform convexity. A counterexample to one of these questions was given in [102]: there is a Banach space Z, $\|\cdot\|$ with strictly convex dual, but such that no equivalent norm on Z is strictly convex. That the situation may be better for the third property, locally uniform convexity, was suggested by a theorem of Kenderov and Moors [223]. This states that a Banach space with locally uniformly convex dual has the topological property of being σ -fragmentable. The main result is an affirmative answer to the full question about locally uniform convexity.

Theorem (6.2.1)[216]: Let Z, $\|\cdot\|$ be a Banach space such that the dual norm $\|\cdot\|^*$ on Z^* is locally uniformly convex. There exists an equivalent norm $|||\cdot|||$ on Z which is locally uniformly convex. Moreover, $|||\cdot|||$ may be chosen to have locally uniformly convex dual norm $|||\cdot|||^*$.

The "moreover" statement in Theorem (6.2.1) is an immediate consequence of the technique of Asplund averaging, see [99]. Now it is known [99] that a Banach space with a norm which is locally uniformly convex and has locally uniformly convex dual norm admits $\mathcal{C}^1$-partitions of unity: equivalently, on such a space every continuous real-valued function may be uniformly approximated by functions of class $\mathcal{C}^1$. We thus have the following corollary.

Corollary (6.2.2)[216]: Let Z be a Banach space with locally uniformly convex dual. Every continuous realvalued function on X may be uniformly approximated by functions of class $\mathcal{C}^1$.

We note that for general Banach spaces Z it is still not known whether the existence on Z of an equivalent Fréchet-smooth norm (or, more generally, a "bump function" of class $\mathcal{C}^1$) implies $\mathcal{C}^1$ approximability as in the above corollary. In the special case of spaces $Z = \mathcal{C}(K)$, this implication has been established in [207]. It is also unknown whether Fréchet-renormability of Z implies LUR renormability.

Spaces of the type $\mathcal{C}(K)$ play an important part in our proof of Theorem (6.2.1). It is of course always the case that we may identify Z with a subspace of $\mathcal{C}(K)$, where K is the unit ball of the dual space Z^*, equipped with the $weak^*$ topology. When the dual norm $\|\cdot\|^*$ is locally uniformly convex, this K belongs to what Raja [104] has called the class of Namioka–Phelps compacts. Theorem (6.2.1) will thus follow from the following $\mathcal{C}(K)$-renorming theorem.

Theorem (6.2.3)[216]: Let K be a Namioka–Phelps compact. Then there is a norm on $\mathcal{C}(K)$, equivalent to the supremum norm, which is locally uniformly convex.

The rest is devoted to a proof of (a mild generalization of) Theorem (6.2.3). The definition of a Namioka–Phelps compact, as well as of the various other topological and renorming properties with which we are concerned, will be given. We then move on to develop some topological machinery before defining a norm. The remaining contain the proof that this norm is locally uniformly convex. We note the crucial role played by general topology in the proof that follows: though Theorem (6.2.1) clearly has some kind of geometrical content, there is actually surprisingly little geometry in the proof. The key is the topological concept of a descriptive space, due to Hansell [222], and a careful analysis of the σ -isolated networks which exist in such spaces. I see Hispano–Bulgarian school of geometric functional analysis, and [118], [104], [227].

Let Z be a real vector space and let φ be a non-negative real-valued convex function on Z. When $f \in Z$ and $f_r \in Z\ (r \in \omega)$, we shall say that the LUR hypothesis holds for φ (and f , and the sequence (f_r)) if

$$\frac{1}{2}\varphi(f)^2 + \frac{1}{2}\varphi(f_r)^2 - \varphi\left(\frac{1}{2}(f + f_r)\right)^2 \to 0.$$

When the function φ is positively homogeneous, this statement is equivalent to saying that both $\varphi(f_r)$ and $\varphi(\frac{1}{2}(f + f_r))$ tend to $\varphi(f)$ as $r \to \infty$. This is recorded as Fact II.2.3 in [99], where it is also noted that, if the function φ is an ℓ^2-sum $\varphi^2 = \sum_{n=1}^{\infty}\varphi_n^2$ of non-negative convex functions and if the LUR hypothesis holds for φ, then it holds for each of the φ_n. We shall make repeated use of this observation. We say that a norm $\|\,.\,\|$ is locally uniformly rotund at a given element f if, whenever the LUR hypothesis holds for $\|\,.\,\|, f$ and a sequence (f_r), we necessarily have $\|f - f_r\| \to 0$. This brings us back to a completely standard definition: we say that a norm on X is locally uniformly convex (the term "locally uniformly rotund" and its abbreviation LUR are also used) if it has this property at each $f \in X$.

We introduce the topological properties that are relevant to our results. Most of these ideas are due to Hansell [222]. Our terminology follows [225], [227], where succinct accounts can be found of all the results that we need. A crucial notion is that of a network for a topology: a collection $\mathcal{S}$ of subsets of X is said to be a network for the topology $\mathcal{T}$ if every set in $\mathcal{T}$ is a union of sets in $\mathcal{S}$: that is to say, whenever $x \in U \in \mathcal{T}$, there exists $N \in \mathcal{S}$ such that $x \in N \subseteq U$. A family of sets $\mathcal{I}$ is said to be isolated for a topology $\mathcal{T}$ if, for each $N \in \mathcal{I}$, there exists $U \in \mathcal{T}$ such that $N \subseteq U$ and $U \cap M = \emptyset$ for all $M \in \mathcal{I} \setminus \{N\}$; equivalently, $N \cap \overline{\bigcup \mathcal{I} \setminus \{N\}} = \emptyset$. A family $\mathcal{S}$ is said to be σ -isolated if it can be expressed as $\mathcal{S} = \bigcup_{n \in \omega} \mathcal{I}_n$ with each $\mathcal{I}_n$ isolated.

Let $(X, \mathcal{T})$ be a topological space and let d be a metric on X inducing a topology finer than . We say that the property $P(d, \mathcal{T})$ holds if there is a sequence $(B_n)_{n \in \omega}$ of subsets of X such that the topology generated by $\mathcal{T} \cup \{B_n : n \in \omega\}$ is finer than the topology $\mathcal{T}_d$ induced by the metric d. An equivalent formulation is that there exists a sequence $(A_n)_{n \in \omega}$ of subsets of X such that the intersections $A_n \cap U$, with $U \in \mathcal{T}$, form a network for $\mathcal{T}_d$. When $P(d, \mathcal{T})$ holds, there is a network $\mathcal{S}$ for the metric topology $\mathcal{T}_d$ which is σ -isolated for the topology . An equivalent formulation of this statement is that, for each $\epsilon > 0$, there is a covering $\mathcal{S}$ of X, which is σ -isolated for $\mathcal{T}$ and which consists of sets with d-diameter at most ϵ. A compact topological space $(K, \mathcal{T})$ which has property $P(d, \mathcal{T})$ for some metric d is said to be descriptive. There is an intrinsic characterization of this property: K is descriptive if and only if there is a network for $\mathcal{T}$ which is $\mathcal{T}$ -σ -isolated. Hansell's general notion of descriptive space [222] is a space X which is $\check{C}$ech-analytic and has a σ -isolated network: we are only concerned with descriptive compact spaces in. Raja [227] shows that the unit ball of a dual Banach space Z^* is descriptive for its weak$*$ topology if and only if Z admits an equivalent norm with "weak$*$ LUR" dual norm.

If $(K, \mathcal{T})$ is compact and has $P(d, \mathcal{T})$ for some $\mathcal{T}$ -lower semicontinuous metric d, then K is called a Namioka–Phelps compact. Raja [104] has shown that unit ball of a dual Banach space Z^* is a Namioka–Phelps compact (in the weak$*$ topology) if and only if Z admits an equivalent norm with LUR dual norm. The hard part of this theorem is the "only if" implication. We just use the easy "if" implication.

As has already been mentioned, we shall obtain our main theorem from a renorming result for $C(K)$ where K is a Namioka–Phelps compact. In fact we prove something slightly more general.

Theorem (6.2.4)[216]: Let $(K, \mathcal{T})$ be a (descriptive) compact space which has property $P(d, \mathcal{T})$ for a metric d. There is a norm $\|\cdot\|$ on $C(K)$, equivalent to the supremum norm, which is locally uniformly rotund at f , whenever f is both $\mathcal{T}$ -continuous and d-uniformly continuous.

Theorem (2.4.6) shows that there is a LUR norm on $C(K)$ provided the metric d can be chosen in such a way that all $\mathcal{T}$ -continuous functions are d-uniformly continuous. A metric with this property has been called a Reznichenko metric. It is easy to see that a lower semi-continuous metric is Reznichenko, which is why Theorem (6.2.4) implies Theorem (6.2.3).

Consider a topological space $(X, \mathcal{T})$, equipped with a metric inducing a topology finer than . The space X is said to be fragmented by the metric d if, for every non-empty subset Y of X and every $\epsilon > 0$, there exists $U \in \mathcal{T}$ such that the intersection $Y \cap U$ is non-empty and of d-diameter at most ϵ. Theorem 2.2 of [225] shows that if $(X, \mathcal{T})$ is a descriptive compact that is fragmented by a metric d then property $P(d, \mathcal{T})$ holds.

If X is compact and is fragmented by some lower semicontinuous metric, we say that X is a Radon–Nikodym compact. A compact space is Namioka–Phelps if and only if it is both descriptive and Radon–Nikodym. see [127], [119] for more about this interesting class of spaces. The outstanding open problem is whether every continuous image of a Radon–Nikodým compact is again Radon–Nikodým. If we relax the definition of a Radon–Nikodým compact by asking only that X be fragmented by some Reznichenko metric we have Reznichenko's defi- nition of a strongly fragmented compact space. Every continuous image of a Radon–Nikodým compact is strongly fragmented. There are other definitions [217], [221] which have recently been shown to be equivalent to strong fragmentability [218], [224] and Arvanitakis's terminology in which strongly fragmented compacta are called quasi-Radon–Nikodým seems to have become standard. It is not known whether every quasi-Radon–Nikodým compact is Radon–Nikodým; a positive answer would of course settle the problem of continuous images. see [220] for a survey of all this material.

As has already been remarked, Theorem (6.2.4) leads to a LUR renorming of $\mathcal{C}(K)$when K has property P for some Reznichenko metric, or equivalently when K is descriptive and quasiRadon–Nikodým. We do not know whether such spaces are necessarily Namioka–Phelps, but at any rate we may state a theorem which may (or may not!) be a generalization of Theorem (6.2.3) as follows.

Theorem (6.2.5)[216]: If K is descriptive and is a continuous image of a Radon–Nikodym compact then $\mathcal{C}(K)$admits a LUR renorming.

We do not know whether $\mathcal{C}(K)$is LUR-renormable for all descriptive compacta K. By Raja's results the corresponding question about Banach spaces would be whether a space Z for which the dual norm on Z^* is w^*LUR has itself an equivalent LUR norm. The most we can get in this direction (using Theorem (6.2.5), Raja's Theorem (6.2.1) and Theorem 1.5.6 of [220]) is the following.

Corollary (6.2.6)[216]: Let Z be a Banach space such that the dual norm on Z^* is w^*LUR. If, in addition, Z is a subspace of an Asplund-generated space then Z admits an equivalent LUR norm.

We develop some additional structure in a descriptive compact space. We start by making some general observations about isolated and σ -isolated families, which are valid without any compactness assumption. Let K be a topological space and let $\mathcal{I}$ be an isolated family of subsets of K. Then, by definition, we have

$$N \cap \overline{\bigcup \mathcal{I} \setminus \{N\}} = \emptyset,$$

for all $N \in \mathcal{I}$. If we set

$$\widetilde{N} = \overline{N} \setminus \overline{\bigcup \mathcal{J}\setminus\{N\}},$$

and $\tilde{I} = \{\widetilde{N}: N \in \mathcal{J}\}$ then it is clear that $\tilde{I}$ is again an isolated family. If $N = \widetilde{N}$ for all $N \in \mathcal{J}$, we shall say that $\mathcal{J}$ is a regular isolated family.

We shall now introduce some notation for regular isolated families, which will be employed consistently in all that follows. If $\mathcal{J}$ is a regular isolated family we write I for the union of the family $\mathcal{J}$, that is

$$I = \bigcup \mathcal{J},$$

and we define

$$J = \{t \in K: each\ neighbourhood\ of\ t\ meets\ at\ least\ two\ members\ of\ \mathcal{J}\}.$$

By virtue of its definition, J is a closed set. Moreover, the closure $\bar{I}$ is the union of its disjoint subsets I and ; that is to say, $J = \bar{I} \setminus I$.

We consider a space with a covering $\mathcal{S}$, which is the union of countably many regular isolated families $\mathcal{J}(i)$ $(i \in \omega)$. In accordance with the notation above, we write

$$I(i) = \bigcup \mathcal{J}(i), \qquad J(i) = \bar{I}(i) \setminus I(i).$$

We now make a recursive definition of further families $\mathcal{J}(i) = \mathcal{J}(i_0, \ldots, i_k)$, together with the associated sets $J(i)$, when $i = (i_0, \ldots, i_k) \in \omega^{<\omega}$ is a finite sequence of natural numbers.

$$I(i_0, \dots, i_k) = \bigcup \mathcal{J}(i_0, \dots, i_k),$$

$$J(i_0, \dots, i_k) = \bar{I}(i_0, \dots, i_k)\setminus I(i_0, \dots, i_k),$$

$$\mathcal{J}(i_0, \ldots i_k, i_{k+1}) = \{ N \cap J(i_0, \ldots, i_k): N \in \mathcal{J}(i_{k+1}) \}.$$

Lemma (6.2.7)[216]: If $i = (i_0, \dots, i_k)$ and $0 \leq l < k$ then

$$I(i_0, \dots, i_k) \subseteq J(i_0, \dots, i_l) \subseteq J(i_l).$$

If the natural numbers $i_0, i_1, \ldots, i_k$ are not all distinct then $I(i_0, \ldots, i_k) = \emptyset$.

Proof. By definition

$$I(i_0, \ldots, i_{m+1}) = I(i_{m+1}) \cap J(i_0, \ldots, i_m),$$

so that

$$I(i_0, \ldots, i_{m+1}) \subseteq J(i_0, \ldots, i_m).$$

Now $J(i_0, \ldots, i_m)$ is a closed set, so we have

$$J(i_0, \ldots, i_{m+1}) \subseteq \bar{I}(i_0, \ldots, i_{m+1}) \subseteq J(i_0, \ldots, i_m).$$

Since this is true for all m, we easily obtain

$$I\,(i_0,\dots,i_k)\ \subseteq\ J\,(i_0,\dots,i_l)$$

for $0\ \le l < k$.

To see that $J\,(i_0,\dots,i_l)\ \subseteq\ J\,(i_l)$, consider $t\ \in\ J\,(i_0,\dots,i_l)$. Every neighbourhood of t meets at least two members of the family $\mathcal{I}(i_0,\dots,i_l)$, and hence at least two members of the family $\mathcal{I}(i_l)$, so that $t\ \in\ J\,(i_l)$.

Finally, suppose that $i_m\ =\ i_l$ for some $0 \le\ m\ < l \le\ k$. We have

$$I\,(i_0,\dots,i_l)\ \subseteq\ I\,(i_l)\ \cap\ J\,(i_0,\dots,i_m)\ \subseteq\ I\,(i_l)\ \cap\ J\,(i_m),$$

which is empty since $I\,(i)\ \cap\ J\,(i)\ =\ \emptyset$ for all i.

We shall be concerned especially with the sets $I(i)$ when the sequence i is strictly increasing. We shall write Σ for the set of all such sequences $i\ = (i_0,\dots,i_k)$ with $k \ge\ 0$ and $i_0\ <\ i_1\ < \cdots <\ i_k$. We equip Σ with a total order $\prec$, defined by saying that $i\ =\ (i_0,\dots,i_k)\ \prec\ j\ =\ (j_0,\dots,j_l)$ if either

(i) there exists $r \le\ min\{k,l\}$ such that $i_s\ =\ j_s$ for $0 \le\ s < r$,and $i_r < j_r$ or

(ii) $k > l$ and $j_s\ =\ i_s$ for $0\ \le s \le\ l$.

Rephrasing this definition, we may say that $i\ \prec\ j$ if either $i\ <\ j$ for the lexicographic order, or i is a proper extension of j. I am grateful to Gilles Godefroy who pointed out that the order $\prec$ may be regarded as just the usual lexicographic order if we think of our finite sequences as infinite sequences terminating in a long run of $\infty's$.

Lemma (6.2.8)[216]: Let $j\ =\ (j_0,\dots,j_l)\ \in\ \Sigma$ and write

$$A_1\ =\ \bigcup_{\substack{0\le r\le l\\ j_{r-1}<i<j_r}} \bar{I}(j_0,\dots,j_{r-1},i),$$

$$A_2\ =\ \bigcup_{\substack{k>l\\ i_k>i_{k-1}>\cdots>i_{l+1}>j_l}} \bar{I}(j_0,\dots,j_l,i_{l+1},\dots,i_k)\ .$$

Then

$$\bigcup_{i\prec j}\bar{I}(i)\ =\ A_1\ \cup\ A_2\ =\ A_1\ \cup\ J\,(j).$$

In particular $\bigcup_{i\prec j}\bar{I}(i)$ is a closed subset of K.

Proof. It is clear that A_2 is exactly the union of the sets $\bar{I}(i)$ where i satisfies clause (ii) in the definition of the relation $\prec$. If $i\ =\ (i_0,\dots,i_k)$ satisfies clause (i) of that definition, then we have

$$\bar{I}(i_0,\dots,i_k)\ \subseteq\ \bar{I}(i_0,\dots,i_r)\ =\ \bar{I}\,(j_0,\dots,j_{r-1}i_r)\ \subseteq\ A_1.$$

It follows that A_1 is exactly the union of the sets $I\ (i)$ where i satisfies (i).

It is clear from the definitions that $A_2 \subseteq J\ (j)$, so, to prove the second equality, it will be enough to show that $J\ (j) \subseteq A_1 \cup A_2$. Suppose then that $t \in J\ (j)$; for some i, we have $t \in I\ (i)$, and i is not equal to any of the j_s, since $J\ (j) \subseteq J\ (j_s)$ and $I\ (i) \cap J\ (i) = \emptyset$. There are now two cases. If $i > j_l$ then

$$t \in I\ (j_0, \ldots, j_l, i) \subseteq A_2.$$

If $i < j_l$ we choose r minimal with respect to $i < j_r$, noting that $i > j_{r-1}$, and observe that

$$t \in I(j_0, \ldots, j_{r-1}, i) \subseteq A_1.$$

It is immediate that our set is closed, since we have shown it to be the union of the closed set $J\ (j)$ with finitely many closures $\overline{I}\ (i)$.

Given j and a finite subset $\mathcal{M}$ of $\mathcal{I}(j)$ we shall write

$$G(j, \mathcal{M}) = K \setminus \left(\bigcup_{i \prec j} \bar{I}(i) \cup \overline{\bigcup \mathcal{I}(j) \setminus \mathcal{M}} \right),$$

noting that this is an open subset of K.

Lemma (6.2.9)[216]: Let t be any element of K. There is a $\prec$-minimal element j^* of Σ with $t \in \bar{I}\ (j^*)$.

Proof. Since $\mathcal{S} = \bigcup_{i\in\omega} \mathcal{I}(i)$ covers K, there is some $i \in \omega$ with $t \in I\ (i)$; let i^* be the minimal such i. Now let j_0 be minimal subject to $t \in \overline{I}\ (j_0)$. Certainly $j_0 \leq i^*$, and if $j_0 = i^*$ we are finished: indeed any $i \in \Sigma$ with $t \in \overline{I}\ (i)$ and $i \prec (j_0)$ must be a proper extension of (j_0), by minimality of j_0; but $\bar{I}(i) \subseteq J(j_0) = J\ (i^*)$ for any such i and $t \notin J(i^*)$ since $I\ (i^*) \cap J\ (i^*) = \emptyset$; thus $j^* = (j_0) = (i^*)$ is the minimal element of Σ satisfying $t \in \bar{I}(j^*)$. Otherwise, $t \in I\ (j_0, i^*)$ and we let j_1 be minimal subject to $t \in \bar{I}\ (j_0, j_1)$, then continue in a similar fashion. Eventually we obtain $j_0 < j_1 < \cdots < j_k = i^*$, where, for each $r \leq k, j_r$ is minimal subject to $t \in \overline{I}\ (j_0, \ldots, j_r)$. Arguing as above, we see that $j^* = (j_0, j_1, \ldots, j_k)$ is the minimal element of Σ satisfying $t \in \bar{I}\ (j^*)$.

Finally, we have a lemma which needs compactness of the space K.

Lemma (6.2.10)[216]: Let K be a compact space and let $\mathcal{S} = \bigcup_{i\in\omega} \mathcal{I}(i)$ be a covering of K which is the union of regular isolated families $\mathcal{I}(i)$. Let H be a non-empty closed subset of K. Then there exists a minimal $j \in \Sigma$ with $H \cap \overline{I}\ (j) \neq \emptyset$. Moreover, $H \cap \overline{I}\ (j) \subseteq I\ (j)$ and there is a non-empty, finite $\mathcal{M} \subseteq \mathcal{I}(j)$ such that $H \cap M \neq \emptyset$ for all $M \in \mathcal{M}$ and $H \subseteq G(j, \mathcal{M})$.

Proof. Let $\mathcal{J} = \{j \in \Sigma : H \cap \overline{I}\ (j) \neq \emptyset\}$ and for $j \in \mathcal{J}$ define $H\ (j) = H \cap \bigcup_{i \prec j} \bar{I}(i)$. By Lemma (6.2.8), each $H\ (j)$ is a closed set, and the sets $H\ (j)$ form a downward directed family because the set Σ is totally ordered by $\prec$. I claim that $\bigcap_{j \in \mathcal{J}} H\ (j) = \emptyset$: indeed, otherwise let t be in this intersection and let j^* be as in Lemma

(6.2.9); since $t \in H \cap \bar{I}(j^*)$ we have $j^* \in \mathcal{J}$, and so $t \in \bigcup_{i \prec j^*} \bar{I}(i)$, contradicting minimality of j^*. By compactness, we now see that there is some $j \in \mathcal{J}$ such that $H(j) = \emptyset$. For this j we have $H \cap \bar{I}(j) \neq \emptyset$ and $H \cap \bar{I}(i) = \emptyset$ whenever $i \prec j$.

Continuing to work with our minimal j, we have $\bar{I}(j) = I(j) \cup J(j)$ and by Lemma (6.2.8), $J(j) \subseteq \bigcup_{i \prec j} \bar{I}(i)$. Thus $H \cap J(j) = \emptyset$ and so $H \cap \bar{I}(j) = H \cap I(j)$. The compact set $H \cap \bar{I}(j)$ is thus covered by the family $\mathcal{I}(j)$, the elements of which are disjoint and open, relative to $\bar{I}(j)$. Thus, if we define $\mathcal{M} = \{M \in \mathcal{I}(j): M \cap H \neq \emptyset\}$, it must be that $\mathcal{M}$ is finite and $H \cap \overline{I(j)} \subseteq \bigcup \mathcal{M}$. Finally, to see that $H \subseteq G(j, \mathcal{M})$, we need to show that $H \cap \bigcup_{i \prec j} \bar{I}(i) = H \cap \overline{\bigcup \mathcal{I}(j) \setminus \mathcal{M}} = \emptyset$. The first of these is just the minimality of j again; the second is immediate when we recall that $H \cap \bar{I}(j) \subseteq \bigcup \mathcal{M}$ and that $\bigcup \mathcal{M} \cap \overline{\bigcup \mathcal{I}(j) \setminus \mathcal{M}} = \emptyset$ because the family $\mathcal{I}(j)$ is isolated.

When K is a descriptive compact space having property P with some metric d then there exists, for each natural number l, a σ -isolated covering $\mathcal{S}^l = \bigcup_{i \in \omega} \mathcal{I}^l(i)$ of K, consisting of sets that are of d-diameter at most 2^{-l} . When d is lower semi-continuous, the sets $\widetilde{N}$ defined at the start are also of diameter at most 2^{-l} . In general, this is not the case: however, each $\widetilde{N}$ is contained in the $\mathcal{T}$ -closure of some set (namely N) of d-diameter at most 2^{-l} . We may summarize the situation in the form of a proposition.

Proposition (6.2.11)[216]: Let $(K, \mathcal{T})$ be a compact space equipped with a metric d such that property $P(d, \mathcal{T})$ holds. Then, for each $l \in \omega$, there is a covering $\mathcal{S}^l$ of K, which is the union $\bigcup_{i \in \omega} \mathcal{I}^l(i)$ of regular isolated families $\mathcal{I}^l(i)$, such that each $N \in \mathcal{S}^l$ is contained in the $\mathcal{T}$ -closure of some set of d-diameter at most 2^{-l} .

From now on, we shall assume $\mathcal{S}^l = \bigcup_{i \in \omega} I^L(i)$ to be as above, and shall construct the associated $\mathcal{I}^l(i), I^l(i), J^l(i)$ and $G^l(i, \mathcal{M})$ as described.

We now set about constructing a norm on $\boldsymbol{C}(K)$when K is a descriptive compact space. As well as the topological machinery set up, we shall need one more ingredient. Let L be a closed subset of K, let l be a natural number, let m, n be positive integers and let $i, j \in \Sigma$; we write $\mathcal{B}(L, l, i, j, m, n)$ for the set of all pairs $(\mathcal{M}, \mathcal{N})$ of finite subsets of $\mathcal{I}^l(i), \mathcal{I}^l(j)$, respectively, which satisfy $\#\mathcal{M} = m, \#\mathcal{N} = n, M \cap L \neq \emptyset$ for all $M \in \mathcal{M}, N \cap L \neq \emptyset$ for all $N \in \mathcal{N}$, and

$$\overline{\bigcup \mathcal{M}} \cap \overline{\bigcup \mathcal{N}} = \emptyset.$$

If $f \in \boldsymbol{C}(K)$ and $L, \mathcal{M}, \mathcal{N}$ are as above, we set

$$\Phi(f, L, \mathcal{M}, \mathcal{N}) = \frac{1}{2}\left(n^{-1} \sum_{N \in \mathcal{N}} \max f[L \cap \bar{N}] - m^{-1} \sum_{M \in \mathcal{M}} \min f[L \cap \bar{M}]\right)^{+},$$

noticing that Φ is a non-negative, positively homogeneous, convex function of its argument f and that

$$\Phi(f, L, \mathcal{M}, \mathcal{N}) \leq \frac{1}{2}\, osc(f|L) \leq \| f|L\|_{\infty}.$$

Whenever $(\mathcal{M},\mathcal{N})$ is a pair of finite sets as above, satisfying

$$\overline{\bigcup \mathcal{M}} \cap \overline{\bigcup \mathcal{N}} = \emptyset,$$

we fix, once and for all, a pair of closed subsets $(X(\mathcal{M},\mathcal{N}), Y(\mathcal{M},\mathcal{N}))$ such that

$$X(\mathcal{M},\mathcal{N}) \cup Y(\mathcal{M},\mathcal{N}) = K, \qquad X(\mathcal{M},\mathcal{N}) \cap \bigcup \mathcal{N} = Y(\mathcal{M},\mathcal{N}) \cap \bigcup \mathcal{M} = \emptyset.$$

In the definition of our norm, we shall also need to fix positive real numbers $c(i)$ $(i \in \Sigma)$ with $\sum_{i\in\Sigma} c(i) \leq 1$. We could, for instance, take

$$c(i_0, i_1, \dots, i_k) = 2^{-2^{i_0}-2^{i_1}-\dots-2^{i_k}}.$$

Proposition (6.2.12)[216]: There are unique non-negative real-valued functions $\Omega(f,L,l), \Theta(f,L,l,i,j,m,n), \Theta_p(f,L,l,i,j,m,n), \Theta_p(f,L,l,\mathcal{M},\mathcal{N})$ *and* $\Psi(f,L,l,\mathcal{M},\mathcal{N})$, defined for functions $f \in \mathcal{C}(K)$, , closed subsets L of K, natural numbers l,m,n,p, elements i,j of Σ, and $(\mathcal{M},\mathcal{N}) \in B(L,l,i,j,m,n)$, which are convex in their argument f , and which satisfy the inequalities

$$\Omega(f,L,l), \Theta(f,L,l,i,j,m,n), \Theta_p(f,L,l,i,j,m,n),$$

$$\Theta_p(f,L,l,\mathcal{M},\mathcal{N}), \Psi(f,L,l,\mathcal{M},\mathcal{N}) \leq \|f|L\|_\infty,$$

as well as the relations

$$6\Omega(f,L,l)^2 = \|f \mid L\|_\infty^2 + osc(f \mid L)^2$$

$$+ \sum_{i,j\in\Sigma} c(i)c(j) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2^{-m-n}\Theta(f,L,l,i,j,m,n)^2,$$

$$\Theta(f,L,l,i,j,m,n)^2 = \sum_{p=1}^{\infty} 2^{-p}\Theta_p(f,L,l,i,j,m,n)^2,$$

$$\Theta_p(f,L,l,i,j,m,n) = \sup_{(\mathcal{M},\mathcal{N})\in\mathcal{B}(L,l,i,j,m,n)} \Theta_p(f,L,l,\mathcal{M},\mathcal{N}),$$

$$2\Theta_p(f,L,l,\mathcal{M},\mathcal{N})^2 = \Phi(f,L,l,\mathcal{M},\mathcal{N})^2 + p^{-1}\Psi(f,L,l,\mathcal{M},\mathcal{N})^2,$$

$$3\Psi(f,L,l,\mathcal{M},\mathcal{N})^2 = \Omega(f,L \cap X(\mathcal{M},\mathcal{N}),l)^2 + \Omega(f,L \cap Y(\mathcal{M},\mathcal{N}),l)^2.$$

We may define a norm $\|.\|$ on $\mathcal{C}(K)$, equivalent to the supremum norm, by setting

$$\|f\|^2 = \sum_{l=1}^{\infty} 2^{-l-1}\Omega(f,K,l)^2.$$

Proof. The functions Θ and Θ_p are defined in terms of Ψ and the known function Φ defined earlier. Hence all we have to show is that the mutual recursion in the definitions of

Ω and Ψ really does define something. We do this by applying a fixed-point theorem, as in [102].

Let Z be the set of all tuples $(f, L, l, \mathcal{M}, \mathcal{N})$ with $f \in \boldsymbol{C}(K)$, L a closed subset of K, l a positive integer and $(\mathcal{M}, \mathcal{N}) \in \bigcup_{i,j,m,n} B(L, l, i, j, m, n)$. Let $\mathcal{Z}$ be the set of all pairs (Ω, Ψ) of non-negative real-valued functions $\Omega(f, L, l), \Psi(f, L, l, \mathcal{M}, \mathcal{N})$, which are convex, symmetric and positively homogeneous in their argument , and which satisfy the inequalities

$$\Omega(f, L, l), \Psi(f, L, l, \mathcal{M}, \mathcal{N}) \leq \| f\|_\infty.$$

Define a metric ρ on $\mathcal{Z}$ by setting

$$\rho\,(\Omega, \Psi), (\Omega', \Psi') = \sup \max\{\, |\Omega(f, L, l)^2 - \Omega'(f, L, l)^2|\,, |\,\Psi(f, L, l, \mathcal{M}, \mathcal{N})^2 - \Psi'(f, L, l, \mathcal{M}, \mathcal{N})^2|\},$$

where the supremum is taken over all $L, l, \mathcal{M}, \mathcal{N}$ and all f with $\|f\|_\infty \leq 1$. It is clear that this makes $\mathcal{Z}$ a complete metric space.

Now define a mapping $F : \mathcal{Z} \to \mathcal{Z}$ by setting $F\,(\Omega, \Psi) = (\tilde{\Omega}, \tilde{\Psi})$, where

$$3\tilde{\Psi}\,(f, L, l, \mathcal{M}, \mathcal{N})^2 = \Omega(f, L \cap X(\mathcal{M}, \mathcal{N}), l)^2 + \Omega(f, L \cap Y(\mathcal{M}, \mathcal{N}), l)^2,$$

and

$$6\vec{\Omega}(f, L, l)^2 = \| f \mid L\|_\infty^2 + osc(f \mid L)^2 + \sum_{i,j \in \Sigma} c(i)c(j) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-m-n} \Theta(f, L, l, i, j, m, n)^2,$$

the function Θ being obtained from Ψ via the formulae in the statement of the proposition. It may be noted that, though the function Θ is not symmetric in , we do have

$$\Theta(-f, L, l, i, j, m, n) = \Theta(f, L, l, i, j, n, m),$$

so that $\tilde{\Omega}$ is symmetric.

It is easy to check that $\rho(F\,(\Omega, \Psi), F\,(\Omega', \Psi')) \leq \frac{2}{3}\rho((\Omega, \Psi), (\Omega', \Psi'))$, so that F has a unique fixed point, by Banach's fixed point theorem. This fixed point yields the functions that we want, and hence enables us to define the norm $\|\cdot\|$.

It is the norm defined in Proposition (6.2.12) that we shall show to be locally uniformly rotund in the case where d is a lower semi-continuous (or, more generally, Reznichenko) metric fragmenting the descriptive compact space K. By the discussion at the end it will be enough to prove the following theorem.

Theorem (6.2.13)[216]: Let $(K, \mathcal{T})$ be a descriptive compact space and let d be a metric on K such that property $P(d, \mathcal{T})$ holds. Let the norm $\|\cdot\|$ be defined as in Proposition (6.2.12). If f be a function in $\mathcal{C}(K)$ which is d-uniformly continuous then the norm $\|\cdot\|$ is locally uniformly convex at .

The proof of this theorem will occupy the remainder. We shall consider a sequence (f_r) in $C(K)$ which satisfies

$$\frac{1}{2}\| f\|^2 + \frac{1}{2} \|f_r\|^2 - \left\|\frac{1}{2} (f + f_r)\right\|^2 \to 0,$$

as $r \to \infty$. In the language introduced earlier, we are assuming that the LUR hypothesis holds for$\| \cdot\|$ (and our given f and f_r). We have to prove that f_r converges to f uniformly on K. Given $\epsilon > 0$, we may use uniform continuity of f to choose a positive integer l such that

$$d(t,u) \le 2^{-l} \Rightarrow | f (t) - f (u)| \le \frac{1}{3}\epsilon .$$

Lemma (6.2.14)[216]: If $N \in \mathcal{S}^l$ then the oscillation of r on N is at most $\frac{1}{3}\epsilon$.

Proof. As in Proposition (6.2.10), we are supposing that for each $N \in \mathcal{S}^l$ there is some set M of d-diameter at most 2^{-l} such that N is contained in the $\mathcal{T}$ -closure of M. The uniform continuity estimate tells us that the oscillation of r on M is at most $\frac{1}{3}\epsilon$ and the $\mathcal{T}$ -continuity of f enables us to extend this to N.

The definition of our norm as an ℓ^2-sum

$$\|f \|^2 = \sum_{k=1}^{\infty} 2^{-k-1}\Omega(f,K,k)^2$$

implies, thanks to an observation we made earlier, that the LUR hypothesis holds for each of the functions $\Omega(\cdot,K,k)$ and in particular for $\Omega(\cdot,K,l)$. This is all we shall use in our proof that $\|f - f_r\|_\infty$ is eventually smaller than ϵ.

Let L be a closed subset of K, let m,n be positive integers and let $i,j \in \Sigma$. (Recall that f , ϵ and l are now fixed.) For a pair $(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}) \in B(L,l,i,j,m,n)$, we define the following real numbers:

$$A = \min f [L],$$

$$a = \max_{M\in\widetilde{\mathcal{M}}} inf f [L \cap \overline{M}],$$

$$\alpha = \min f\left[L \setminus G^l \left(i,\widetilde{\mathcal{M}}\right)\right] ,$$

$$\beta = \max f\left[L \setminus G^l \left(j,\widetilde{\mathcal{N}}\right)\right],$$

$$b = min_{N\in\widetilde{\mathcal{N}}} sup f [L \cap \overline{N}],$$

$$B = max f [L].$$

Of course, we have $a \ge A, \alpha \ge A, b \le B$ and $\beta \le B$. We shall say that the pair $(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}})$ is a good choice (of type (i,j,m,n)) on L if

$$n^{-1}(B - \beta) > (B - b) + (a - A) and$$

$$m^{-1}(\alpha - A) > (B - b) + (a - A).$$

Lemma (6.2.15)[216]: If L is a closed subset of K and the oscillation of f on L is at least ϵ then there is at least one good choice on L.

Proof. Let $H_1 = \{t \in L: f(t) = max\, f[L]\}$ and apply Lemma (6.2.10). There exist $j \in \Sigma$ and a finite subset $\widetilde{\mathcal{N}}$ of $\mathcal{J}^l(j)$ such that $H_1 \cap N \neq \emptyset$ for all $N \in \widetilde{\mathcal{N}}$ and $H_1 \subseteq G^l(j, \widetilde{\mathcal{N}})$. It follows that, in the notation just established, we have $B = b$ and $B > \beta$. A similar argument applied to the set $H_2 = \{t \in L: f(t) = min\, f[L]\}$ yields i and $\widetilde{\mathcal{M}}$ such that $A = a$ and $A < \alpha$. To finish showing that $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is a good choice, we need to check that $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is in $B(L, l, \boldsymbol{i}, \boldsymbol{j}, m, n)$, and what remains to be proved is that $\bar{M} \cap \overline{N} = \emptyset$ for all $M \in \widetilde{\mathcal{M}}$ and all $N \in \widetilde{\mathcal{N}}$.

Our choice of l ensures that the oscillation of f on each $M \in \widetilde{\mathcal{M}}$ and on each $N \in \widetilde{\mathcal{N}}$ is at most $\epsilon/3$, and, by continuity of f, the same holds for each $\bar{M}$ and each $\bar{N}$. Hence

$$max\, f[\bar{M}] \leq A + \epsilon/3,$$

$$min\, f[\bar{N}] \geq B - \epsilon/3,$$

for all such M, N. Since we are assuming that the oscillation $B - A$ of f on L is at least ϵ, we deduce that $\bar{M} \cap \overline{N} = \emptyset$ as claimed. _

Lemma (6.2.16)[216]: *Let $L_1, L_2, \ldots$ be a decreasing sequence of non-empty closed subsets of K with intersection L. If $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is a good choice on L, then it is a good choice on L_s for all sufficiently large values of s.*

Proof. Let us define $A, B, \alpha, \beta, b, B$ as above and set

$$A_s = min\, f[L_s],$$

$$a_s = \max_{M \in \widetilde{\mathcal{M}}} inf f\, [L_s \cap \bar{M}],$$

$$\alpha_s = \min f\left[\, L_s \setminus G^l\left(i, \widetilde{\mathcal{M}}\right)\right],$$

with analogous definitions for β_s, b_s, B_s. Standard compactness arguments show that $A_s \to A$ as $s \to \infty$, and so on. Hence the inequalities defining a good choice for L_s do hold for all sufficiently large s.

The third lemma reveals why good choices are so named: it is a “rigidity condition” of a type that occurs commonly in LUR proofs. It will be convenient to state it in terms of “strong attainment” of a certain supremum, a notion with which most will be familiar, but which we shall nonetheless define explicitly. If $(\gamma_i)_{i \in I}$ is a bounded family of real numbers, we shall say that the supremum $\sup_{i \in I} \gamma_i$ is strongly attained at j if $\sup_{i \in I \setminus \{j\}} \gamma_i < \gamma_j$. This of course implies that if (i_r) is a sequence in I and $\gamma i_r \to \sup_{i \in I} \gamma_i$ *as* $r \to \infty$, then $i_r = j$ for all large enough r.

Lemma (6.2.17)[216]: Let L be a closed subset of K and suppose that there exists a good choice $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ of type (m, n, i, j) on L. Then the supremum $\sup\{\Phi(f, L, \mathcal{M}, \mathcal{N}): (\mathcal{M}, \mathcal{N}) \in B(L, l, i, j, m, n)\}$ is strongly attained at $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$.

Proof. Let us write $A, a, \alpha, \beta, b, B$ for the quantities associated with $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ in the definition of a good choice. So we have $min\, f[L \cap \bar{M}] \leq a$ and $max\ f\ [L \cap \bar{N}] \geq b$ for all $M \in \widetilde{\mathcal{M}}$ and all $N \in \widetilde{\mathcal{N}}$. Thus $2\Phi(f, L, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}) \geq b - a$.

Now suppose that $(\mathcal{M}, \mathcal{N})$ is in $B(L, l, i, j, m, n)$ and that $\mathcal{M} \neq \widetilde{\mathcal{M}}$. Since $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have the same number of elements, namely m, $\mathcal{M}$ must have at least one element M_0 which is not in $\widetilde{\mathcal{M}}$. It follows from the definition of $G(i, \widetilde{\mathcal{M}})$ that $\bar{M}_0 \cap G(i, \widetilde{\mathcal{M}}) = \emptyset$ so that $minf[L \cap \bar{M}_0] \geq \alpha$. For the other $m - 1$ elements of $\mathcal{M}$, we certainly have $min\, f[L \cap \bar{M}_0] \geq A$, and of course $max\, f[L \cap \bar{N}] \leq B$ for all $N \in \mathcal{N}$. Hence

$$2\Phi[f, L, \mathcal{M}, \mathcal{N}] \leq \frac{1}{n} nB - \frac{1}{m} (\alpha + (m - 1)A) = B - A - \frac{1}{m}(\alpha - A).$$

By the definition of a good choice, this is strictly smaller than $b - a$, Similarly, we show that if $N \neq \mathcal{N}$ then

$$2\Phi[f, \mathcal{M}, \mathcal{N}] \leq B - A - \frac{1}{n} (B - \beta),$$

another quantity which is known to be smaller than $b - a$.

We record for convenience the following version of [99].

Lemma (6.2.18)[216]: Let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be two pointwise-bounded families of non-negative, realvalued, convex functions on a real vector space Z. For $i \in I$ and positive integers p define functions $\theta_{i,p}, \theta_p$ and θ by setting

$$2\theta_{i,p}(x)^2 = \varphi_i(x)^2 + p^{-1}\psi_i(x)^2,$$

$$\theta_p(x) = \sup_{i\in I} \theta_{i,p}(x),$$

$$\theta\,(x)^2 = \sum_{p=1}^{\infty} 2^{-p}\theta_p(x)^2\,.$$

Let x and x_r $(r \in \omega)$ be elements of Z and assume that

$$\frac{1}{2}\,\theta(x)^2 + \frac{1}{2}\,\theta(x_r)^2 - \theta\left(\frac{1}{2}\,(x + x_r)\right)^2 \to 0$$

as $r \to \infty$. Then there is a sequence (i_r) of elements of I such that

$$\varphi i_r\,(x) \to \sup_{i\in I} \varphi_i(x) \quad and$$

$$\frac{1}{2}\,\psi i_r\,(x)^2 + \frac{1}{2}\,\psi i_r\,(x_r)^2 - \psi i_r\left(\frac{1}{2}\,(x + x_r)\right)^2 \to 0$$

as $r \to \infty$.

Corollary (6.2.19)[216]: If, in addition to the hypotheses of Lemma (6.2.18), we assume that the supremum $\sup_{i\in I} \varphi_i(x)$ is strongly attained at j , then we may conclude that

$$\frac{1}{2}\psi_j(x)^2 + \frac{1}{2}\psi_j(x_r)^2 - \psi_j\left(\frac{1}{2}(x + x_r)\right)^2 \to 0.$$

Proof. This is of course automatic, since the assumptions imply that a sequence (i_r) for which

$$\varphi i_r(x) \to \sup_{i\in I}\varphi_i(x)$$

as $r \to \infty$ must necessarily satisfy $i_r = j$ for all large enough j .

We may rephrase the statement of this corollary by saying that if the LUR hypothesis holds for θ and the supremum $\sup_{i\in I}\varphi_i(x)$is strongly attained at , then the LUR hypothesis holds for ψ_j . It is precisely this formulation that we shall be applying in the next result, where we return to the proof of Theorem (6.2.13) and where of course we are still dealing with fixed f , f_r, ϵ and l.

Proposition (6.2.20)[216]: Let L be a closed subset of K and assume that the LUR hypothesis holds for $\Omega(\cdot, L, l)$. If $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is a good choice on L then the LUR hypothesis holds for $\Omega(\cdot, L \cap X(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}), l)$ and $\Omega(\cdot, L \cap Y(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}), l)$.

Proof. Let $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$be of type (i, j, m, n). The expression for $\Omega(\cdot, L, l)$ as an ℓ^2-sum implies that the LUR hypothesis holds for $\Theta(\cdot, L, l, i, j, m, n)$, which is readily recognizable as a function to which we may apply Deville's lemma. Moreover, by Lemma (6.2.17), we are in the situation where the supremum $\sup_{(\mathcal{M},\mathcal{N})\in B(m,n,i,j)}\Phi(f, L, l, \mathcal{M}, \mathcal{N})$ is strongly attained at $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$. So, by the above corollary, the LUR hypothesis holds for $\Psi(\cdot, L, l, \widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$. The formula for this as an ℓ^2 sum now shows that the LUR hypothesis holds for $\Omega(\cdot, L \cap X(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}), l)$ and $\Omega(\cdot, L \cap Y(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}), l)$, as claimed.

Lemma (6.2.21)[216]: Let L be a closed subset of K on which the oscillation of f is smaller than ϵ. If the LUR hypothesis holds for $\Omega(\cdot, L, l)$ then $\|(f - f_r)| L\|_\infty < \epsilon$ for all large enough r.

Proof. From the formula for Ω as an ℓ^2-sum, we see that the LUR hypothesis holds for the convex functions $g \longmapsto \|g| L\|_\infty$ and $g \longmapsto osc(g| L)$. So in particular, $\|f_r | L\|_\infty \to \|f| L\|_\infty$, $\left\|\frac{1}{2}(f + f_r)| L\right\|_\infty \to \|f| L\|_\infty$ and $osc(f_r| L) \to osc(f | L)$ as $r \to \infty$. The required result follows from a fairly standard argument. Let us write $osc(f| L) = \epsilon - 4\eta$ and suppose that r is large enough for us to have

$$\|f_r | L\|_\infty < \|f| L\|_\infty + \eta,$$

$$\left\|\frac{1}{2}(f + f_r)| L\right\|_\infty > \|f| L\|_\infty - \eta,$$

$$osc(f_r| L) < \epsilon - 3\eta.$$

There exists $t \in K$ with $|\frac{1}{2}(f + f_r)(t)| > \|f| L\|_\infty - \eta$, and we may assume that $(f + f_r)(t) > 0$. It follows that

$$f_r(t) > 2\|f|L\|_\infty - 2\eta - \|f|L\|_\infty$$
$$= \|f|L\|_\infty - 2\eta,$$
$$f(t) > 2\|f|L\|_\infty - 2\eta - \|f_r|L\|_\infty$$
$$> \|f|L\|_\infty - 3\eta.$$

Now for any $u \in L$ we have

$$f(u) \geq f(t) - osc(f|L)$$
$$> \|f|L\|_\infty - 3\eta - \epsilon + 4\eta$$
$$= \|f|L\|_\infty - \epsilon + \eta,$$
$$f(u) \leq \|f|L\|_\infty,$$
$$f_r(u) \geq f_r(t) - osc(f_r|L)$$
$$> \|f|L\|_\infty - 2\eta - \epsilon + 3\eta$$
$$= \|f|L\|_\infty - \epsilon + \eta,$$
$$f_r(u) \leq \|f|L\|_\infty$$
$$< \|f|L\|_\infty + \eta.$$

It follows immediately that $|f(u) - f_r(u)| < \epsilon$.

Proposition (6.2.22)[216]: There is a finite covering $\mathcal{L}$ of K with closed subsets such that the LUR hypothesis holds for $\Omega(\cdot, L, l)$ and the oscillation of f on L is smaller than ϵ, for each $L \in \mathcal{L}$.

Proof. We shall define a tree Υ whose elements will be certain pairs (L, s) with L a closed subset of K and s a natural number. We shall give a recursive definition which will specify which such pairs are nodes of our tree, and shall define the tree ordering by saying which (if any) nodes are the immediate successors of a given (L, s). To do this, we shall need to fix a mapping $\tau : \omega \to \Sigma \times \Sigma \times \omega \times \omega$ with the property that each quadruple (i, j, m, n) occurs as $\tau(s)$ for infinitely many $s \in \omega$.

It will be ensured during the construction that, whenever $(L, s) \in \Upsilon$, the LUR hypothesis holds for $\Omega(\cdot, L, l)$. We start by declaring that there is one minimal node $(K, 0)$. (Notice that our hypotheses do ensure that the LUR hypothesis holds for $\Omega(\cdot, K, l)$.) If (L, s) is a node of our tree then there are three possibilities:

(i) if the oscillation of f on L is smaller than ϵ then (L, s) has no immediate successors in the tree (that is to say, (L, s) is a maximal element);

(ii) if the oscillation of f on L is at least ϵ and there is a good choice $(\mathcal{M}, \mathcal{N})$ of type $\tau(s)$ on L then we introduce into Υ two immediate successors, $(L \cap X(\mathcal{M}, \mathcal{N}), s + 1)$ and $(L \cap X(\mathcal{M}, \mathcal{N}), s + 1)$, of (L, s) (notice that, by Proposition (6.2.20), the LUR hypothesis holds for the Ω functions associated with these two new nodes);

(iii) if the oscillation of f on L is at least ϵ but no good choice of type $\tau(s)$ exists, then we introduce just one immediate successor $(L, s+1)$ of (L, s) into the tree.

We shall now show that the tree Υ we have just constructed has only finitely many elements. By König's lemma, it will be enough to show that Υ has no infinite branch. So suppose, if possible, that there is a sequence $(L_s)_{s\in\omega}$ of closed subsets of K such that the pairs (L_s, s) are nodes of Υ and such that, for each s, $(L_{s+1}, s+1)$ is an immediate successor of (L_s, s) in Υ. The sets L_s form a decreasing sequence of closed subsets of K; let us write L for their intersection. By a compactness argument, the oscillation $osc(f\,|L_s)$ tends to $osc(f\,|\,L)$ as $s\to\infty$. Since each (L_s, s) has successors in , we have $osc(f\,|L_s)\geq s$ for each s, and we can thus deduce that $osc(f\,|\,L)\geq\epsilon$. So, by Lemma (6.2.15), there is a good choice $(\mathcal{M},\mathcal{N})$ on L, of type (i,j,m,n) say. By Lemma (6.2.16), $(\mathcal{M},\mathcal{N})$ is also a good choice on L_s for all sufficiently large s. Recalling that $\tau(s) = (i,j,m,n)$ for infinitely many values of s, we see that we can choose s such that $(\mathcal{M},\mathcal{N})$ is a good choice on L_s of type $\tau(s)$. The way we constructed the tree Υ means that L_{s+1} is one or other of the two sets $L_s\cap X(\mathcal{M},\mathcal{N})$ and $L_s\cap Y(\mathcal{M},\mathcal{N})$. So one or other of $L_{s+1}\cap\bigcup\mathcal{N}$ and $L_{s+1}\cap\bigcup\mathcal{M}$ is empty. But this is absurd, since $L_{s+1}\supseteq L$ and the sets $L\cap M, L\cap N$ are non-empty for all $M\in\mathcal{M}$ and $N\in\mathcal{N}$.

Having proved that Υ is finite, we define $\mathcal{L}$ to be the set of all L such that there is a maximal element of Υ of the form (L, r). Our construction ensures that the LUR hypothesis holds for $\Omega(\cdot, L, l)$ for each such L, and, by maximality, the oscillation of f on any such L is smaller than ϵ. We just need to show that $\bigcup\mathcal{L} = K$. This is most easily proved by induction: for each s let $\mathcal{L}_s = \{L: (L,s)\in\Upsilon\}$; I claim that, for all s, $\bigcup\mathcal{L}\cup\bigcup\mathcal{L}_s = K$. Certainly this is true for $s = 0$ since $\mathcal{L}_0 = \{K\}$. To deal with the inductive step let $t\in\bigcup\mathcal{L}\cup\bigcup\mathcal{L}_s$ be given. If $t\in\bigcup\mathcal{L}$ there is no problem, so assume that $t\in L$ for some $L\in\mathcal{L}_s$. By the construction of Υ, one of (i), (ii) and (iii) occurs for the pair (L, s). If it is (i) then $t\in L\in\mathcal{L}$. If it is (ii) then t is one or other of the two sets $L\cap X$ and $L\cap Y$ which themselves are members of $\mathcal{L}_{s+1}$. Finally if it is (iii) then $t\in L\in\mathcal{L}_{s+1}$. In all cases, we have $t\in\bigcup\mathcal{L}\cup\bigcup\mathcal{L}_{s+1}$, which completes our proof by induction. Since Υ is finite, $\mathcal{L}_s$ is empty for large enough s, which shows that $\bigcup\mathcal{L} = K$.

We can now finish the proof of Theorem (6.2.13). Indeed, by Proposition (6.2.22), K is the union of finitely many subsets L, for each of which $\sup_{t\in L}|f_r(t) - f(t)|$ is eventually smaller than ϵ. So $\|f_r - f\|_\infty$ is eventually smaller than ϵ, which is what we wanted to prove.

Section (6.3): Steinhaus'Lattice-Point Problem

The following feature of the integer lattice in the Euclidean plane was probably first observed by Steinhaus [238]: for any natural number n one may find a circle surrounding exactly n lattice points. Zwoleński [239] generalised this fact to the setting of Hilbert spaces in the following manner. He replaced the set of lattice points by a more general quasi-finite set, i.e., an infinite subset A of a metric space X such that each ball in X contains only finitely many elements of A. His result then reads as follows.

Theorem (6.3.1)[228]: ([239]). Let A be a quasi-finite subset of a Hilbert space X. Then there exists a dense subset $Y \subset X$ such that for every $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.

We distill the property that we will term Steinhaus' property (S). A metric space X has this property if, by definition,

(S) for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for all $y \in Y$ and $n \in \mathbb{N}$ there exists a ball B centred at y with $|A \cap B| = n$.

We translate condition (S), formulated above, into three equivalent statements concerning the geometry of the unit ball of a Banach space. They require that, locally, the unit sphere of X does not look the same at any two distinct points. This approach will be particularly beneficial, as it will allow us to identify spaces that share that property with Hilbert spaces, yet of a very different nature. Our first main result then reads as follows.

We employ the announced equivalence to extend Zwoleński's result to strictly convex Banach spaces (Corollary (6.3.4)). It is well-known that not every Banach space admits a strictly convex renorming, just to mention the examples of $\ell_\infty(\Gamma)$ for any uncountable set Γ (see [231] and [232]) or the quotient space ℓ_∞/c_0 ([230]). This motivates the question of whether strict convexity and property (S) are equivalent at the level of renormings, and a negative answer is a part of our next result.

Solovay ([237]) proved that the assertion that the continuum is a real-valued cardinal is equiconsistent with the existence of a two-valued measurable cardinal number, therefore its consistency cannot be proved in ZFC alone (assuming of course that ZFC itself is consistent). Interestingly, our construction in this universe is possible because the real-valued measurability of the continuum implies the failure of the Continuum Hypothesis ([229]) and we take advantage not only of pleasant measure-theoretic properties of the continuum but also of the existence of an uncountable cardinal number below it.

It seems unlikely that real-measurability of the continuum is really necessary to show that there exist Banach spaces with (S) but which do not have a strictly convex renorming. This leaves the question of possibility of such constructions in ZFC open.

Theorem (6.3.2)[228]: Let X be a Banach space. The following assertions are equivalent:

(S) X has Steinhaus' property;

(S_1) for any quasi-finite set $A \subset X$ there exists a dense set $Y \subset X$ such that for every $y \in Y$ there exists a ball B centred at y with $|A \cap B| = 1$;

(S') for all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ and each $\delta > 0$ there exists a $z \in X$ with $\|z\| < \delta$ such that one of the vectors $x + z$ and $y + z$ has norm greater than 1, whereas the other has norm smaller than 1;

(S") for all $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ and each $\delta > 0$ there exists a $z \in X$ with $\|z\| < \delta$ such that $\|x + z\| \neq \|y + z\|$.

In other words, condition (S") means exactly that one cannot find a 'neighbourhood' of parallel line segments on the unit sphere of equal length. This seems to be a new geometric property which, as we will see, is essentially weaker than strict convexity. Notice that, in contrast to many other classical properties, property (S) is not inherited by subspaces and, in a sense, is neither local nor global.

Properties (S') and (S") are related to another (weaker) property of 'non-flatness' of the unit sphere:

(F) the unit sphere S_X of X does not contain any flat faces, that is to say, there is no non-empty subset of S_X, open in the relative norm topology, that is contained in a hyperplane.

Here by a hyperplane of X we understand a translation of a subspace of X of codimension 1, i.e., a set of the form $x + ker(x^*)$ for some $x \in X$ and $x^* \in X^*$. Note, however, that (F) does not imply (S") that is witnessed by the norm $\|(x, y, z)\| = max\{\sqrt{x^2 + y^2}, |z|\} for (x, y, z) \in \mathbb{R}^3$ (consider the points $(1, 0, 0)$ and $(1, 0, \frac{1}{2})$). However, whether every Banach space admits a renorming satisfying (F) seems to be an attractive open problem.

Proof . Since the implications (S) ⇒ (S_1) and (S') ⇒ (S") hold true trivially, it is enough to prove that (S_1) ⇒ (S'), (S') ⇒ (S) and (S") ⇒ (S').

(S_1) ⇒ (S'): Suppose that (S_1) holds. Fix any $\delta > 0$ and $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$. Consider any quasi-finite set $A \subset X$ such that $A \cap (1 + \delta)B_X = \{x, y\}$, where B_X stands for the closed unit ball of X. According to (S_1), there is a $u \in X$, $\|u\| < \delta/2$, such that for some $r > 0$ the open ball $B(u, r)$ contains exactly one element of A. Suppose there is an $a \in A \setminus \{x, y\}$ belonging to $B(u, r)$. Then

$$r > \|a - u\| \geq \|a\| - \|u\| > (1 + \delta) - \frac{\delta}{2} = 1 + \frac{\delta}{2},$$

hence $\|x - u\| < r$, that is $x \in B(u, r)$; a contradiction. Consequently, $B(u, r)$ contains exactly one of the points x and y, say $x \in B(u, r)$ and $y \notin B(u, r)$. Then

$$1 - \frac{\delta}{2} < \|x - u\| < r \leq \|y - u\| < 1 + \frac{\delta}{2}.$$

Suppose that $r \leq 1, r = 1 - \varepsilon$ with some $\varepsilon \in [0, \delta/2)$ and take any number ρ satisfying

$$0 < \rho < min\left\{r - \|x - u\|, \frac{\delta}{2} - \varepsilon\right\}.$$

Obviously, we may find $v \in X$ with $\|v\| \leq \varepsilon + \rho$ such that $\|y - (u + v)\| \geq r + \varepsilon + \rho > 1$. Then we also have

$$\|x - (u + v)\| \leq \|x - u\| + \|v\| < r - \rho + \|v\| \leq 1.$$

Therefore, setting $z = -(u + v)$ completes the proof of our claim, since we have the estimate $\|u + v\| < \varepsilon + \rho + \delta/2 < \delta$. We proceed similarly in the case where $r > 1$ so the proof of (S_1) ⇒ (S') is then complete.

(S') ⇒ (S): Let X be a Banach space X that satisfies (S') and let $A \subset X$ be a quasi-finite set. For any $n \in \mathbb{N}$ set

$$G_n = \{x \in X : |A \cap B(x,r)| = n \, for\, some\, r > 0\}\,.$$

It is evident, in view of the definition of a quasi-finite set, that each G_n is an open subset of X. We shall prove that it is also dense.

Assume, in search of a contradiction, that there is an open ball $U = B(x_0, r_0)$ in X not intersecting G_n. Rescaling U if necessary, we may suppose that $A \cap U = \emptyset$. With any point $x \in U$ we associate two integers $m(x) < n$ and $k(x) \geq 2$ defined as follows: Since $x \notin G_n$, there is the largest non-negative integer $m(x) < n$ for which there exists $q > 0$ with $|A \cap B(x,q)| = m(x)$. Then, for every $s > q$ we have either $|A \cap B(x,s)| = m(x)\, or\, |A \cap B(x,s)| > n$. Define

$$s = \inf\{t > 0: |A \cap B(x,t)| > n\}\,.$$

Then exactly $m(x)$ points $a_1, \ldots, a_m(x) \in A$ lie in the ball $B(x,s)$, whereas at least two such points lie on the boundary of $B(x,s)$; let us call them $b_1, \ldots, b_k$, where $k \geq 2$. In this way we define $k(x) = k$.

Now, we shall use an infinite descent argument to obtain a desired contradiction. Let $a_1, \ldots, a_m, b_1, \ldots, b_k$ be as above for $x = x_0$, where $m = m(x_0)$ and $k = k(x_0)$. Pick any $\delta > 0$ such that

$$\{a_i : 1 \leq i \leq m\} \subseteq B(x_0 + u, s) \cap A \subseteq \{a_i, b_j : 1 \leq i \leq m, 1 \leq j \leq k\}\, for\, every\, u \in X\, with\, \|u\| < \delta.$$

Define $\rho = max\{\|a_i - x_0\|: 1 \leq i \leq m\} < s$ and set $\gamma = s - \rho$. Each of the vectors $(b_j - x_0)/s$ $(j = 1, \ldots, k)$ lies in the unit sphere. Applying the hypothesis (S') to any two of them (e.g., to $j = 1, 2$), we obtain a point $z \in X$ with

$$\|sz\| < min\{\delta, \gamma/2\}$$

such that one of the vectors: $b_j - x_0 - sz$ $(j = 1, 2)$ has norm greater than s, whereas the other has norm smaller than s. By decreasing δ, if necessary, we may also assume that the point $x := x_0 + sz$ still lies in U. Therefore the ball $B(x,s)$ with the centre in U contains all a_i's $(1 \leq i \leq m)$ and at least one but not all among b_j's $(1 \leq j \leq k)$. Observe also that by our choice of z, we have

$$\|a_i - x\| \leq \|a_i - x_0\| + \|sz\| < \rho + \frac{\gamma}{2} = s - \frac{\gamma}{2}\, for\, each\, 1 \leq i \leq m$$

and

$$\|b_j - x\| \geq \|b_j - x_0\| + \|sz\| > s - \frac{\gamma}{2}\, for\, each\, 1 \leq j \leq k.$$

Therefore, by suitably rescaling the ball $B(x,s)$, we obtain a new ball centred at x which contains all of a_i's and whose boundary contains some but not all of b_j's. This shows that we have either $m(x) > m(x_0)$ or $k(x) < k(x_0)$. This construction (with x_0 replaced by

x) will ultimately lead to a contradiction, as we finally arrive at a point $u \in U$ with $m(u) \geq n$ or $k(u) < 2$. Therefore, all the sets G_n $(n \in \mathbb{N})$ are open and dense.

By the Baire Category Theorem, the set $Y = \bigcap_{n=1}^{\infty} G_n$ is dense in X and, obviously, for each $y \in Y$ and $n \in \mathbb{N}$ there is a ball B centred at y with $|A \cap B| = n$. This completes the proof of (S).

(S") ⇒ (S'): Assume the negation of (S') and choose distinct unit vectors $x, y \in X$ and $\delta > 0$ so that there is no vector $z \in X$ with $\|z\| < \delta$ for which exactly one of the vectors $x + z$ and $y + z$ lies inside the unit ball of X. For every $u \in S_X$ define

$$V_u = \{ z \in S_X : \|u + \alpha z\| < 1 \, for \; some \; \alpha > 0\}$$

and

$$\lambda_u(z) = \min\{\delta, inf\{\alpha > 0: \|u + \alpha z\| \geq 1\}\} \quad (z \in V_u).$$

By the assumption, we have $V_x = V_y$ and $\lambda_x(z) = \lambda_y(z)$ for every $z \in V_x$, which means that the unit sphere looks locally the same at x and y (via the translation by $y - x$), namely,

$$y - x + (B(x,\delta) \cap S_X) = B(y,\delta) \cap S_X. \tag{1}$$

Pick $\eta > 0$ so small that

$$\left\|x - \frac{x + z}{\|x + z\|}\right\| < \delta \;\; and \;\; \left\|y - \frac{y + z}{\|y + z\|}\right\| < \delta \;\; if \;\; \|z\| < \eta. \tag{2}$$

Now, using (S"), choose a vector $z \in X$ with $\|z\| < \eta$ so that $\|x + z\| \neq \|y + z\|$. We have then two possibilities: either $\|x + z\| \leq 1$ and $\|y + z\| \leq 1$, or $\|x + z\| \geq 1$ and $\|y + z\| \geq 1$. We shall consider the former case; for the latter one the argument is similar.

With no loss of generality we can assume that $\|x + z\| > \|y + z\|$. Consider the function $g : [0,\infty) \to [0,\infty)$ given by

$$g(\alpha) = \|x + z + \alpha(y - x)\|$$

which is convex, as can be easily verified. In view of (1) and (2), we have

$$\left\|y - x + \frac{x + z}{\|x + z\|}\right\| = 1,$$

that is, $g(\|x + z\|) = \|x + z\|$. We have also $g(0) = \|x + z\|$and $g(1) = \|y + z\| < \|x + z\|$. This is a contradiction with the convexity of g, as the arguments: $0, \|x + z\|$ and 1 lie in this order on the real line.

We will demonstrate some applications of Theorem (6.3.2) in concrete situations. We begin with a strengthening of Zwoleński's result.

Given two elements x, y in a real vector space X, we denote by $\overline{xy}$ the line segment between x and y, i.e., $\overline{xy} = \{\lambda x + (1 - \lambda)y : \lambda \in [0,1]\}$.

Proposition (6.3.3)[228]: Let X be a Banach space and suppose that $x, y \in X$ are distinct unit vectors. If $\overline{xy} \nsubseteq S_X$, then for each $\delta > 0$ there is $z \in X$ with $\|z\| < \delta$ such that one of the vector $x + z, y + z$ has norm greater than 1 whereas the other one has norm strictly less than 1.

Proof. Let $\delta > 0$ and $x, y \in X$ with $x \neq y, \|x\| = \|y\| = 1$ be given. Then each point inside the segment $\overline{xy}$, joining x and y, has norm smaller than 1, whereas each point lying on the straight line passing through x and y, but outside $\overline{xy}$, has norm larger than 1. Therefore, any point $z \in X$ satisfying $0 < \|z\| < \delta$ and $x + z \in \overline{xy}$ does the job.

Corollary (6.3.4)[228]: Every strictly convex Banach space X satisfies (S).

Now, we will see that strictly convex spaces do not exhaust the whole class of Banach spaces satisfying Steinhaus' condition. In fact, these two classes differ already in dimension three. The following construction will also serve as a base for the proof of Theorem (6.3.10).

Example (6.3.5)[228]: We claim that there exists a norm $|||\cdot|||$ in $\mathbb{R}^3$ such that $(\mathbb{R}^3, |||\cdot|||)$ contains ℓ_∞^2 isometrically (and hence is not strictly convex), nonetheless it satisfies condition (S). We are indebted to the referee for suggesting the following example which significantly simplified our original construction.

First, observe that the negation of (S”) easily implies that there are two different points x and y on the unit sphere and $\delta > 0$ so that $\|x + z\| = \|w + z\|$ whenever $\|z\| < \delta$ and $w \in \overline{xy}$. In other words, if a given Banach space fails Steinhaus' condition, then there must be a 'neighbourhood' of segments on the unit sphere. Having this in mind we set

$$B = \{(x_1, x_2, x_3) \in [-1,1]^3 : |x_3| \leq f(x_1, x_2)\},$$

where $f : [-1,1]^2 \to [0,1]$ is any continuous function satisfying the equations $f(0,0) = 1$ and $f(-x_1, -x_2) = f(x_1, x_2)$ which vanishes on the boundary of $[-1,1]^2$ and is strictly concave on $(-1,1)^2$. For example, we can take $(x_1, x_2) = (1 - |x_1|)^p(1 - |x_2|)^p$ with $0 < p < \frac{1}{2}$. Then, let $|||\cdot|||$ be the norm on $\mathbb{R}^3$ defined as the Minkowski functional of B. Since there are only four segments lying on the unit sphere (the edges of the square $[-1,1]^2 \times \{0\}$), the Banach space $(\mathbb{R}^3, |||\cdot|||)$ satisfies Steinhaus' condition due to the remark above.

It is worth noticing a simple geometrical feature of B which makes $|||.|||$ satisfy condition (S'). Namely, considering any two different points $x = (t, 1, 0)$ and $y = (u, 1, 0)$ with $0 \leq t < u < 1$ we see that the curve lying on B that starts at x and is parallel to the x_2x_3-plane is flatter at the point x than its counterpart at the point y. Therefore, for a given $\delta > 0$, one can take a vector $z \in \mathbb{R}^3$ with $\|z\| < \delta$ of the form $z = (0, v, w)$ to guarantee that exactly one (more precisely: the latter one) of the vectors $x + z, y + z$ goes outside of B. For any other two points our claim is either trivial or analogous. The upper part of the ball B defined as above with $= \frac{1}{3}$, as well as some contour lines illustrating the above-mentioned flattening effect, are depicted in the two figures below.

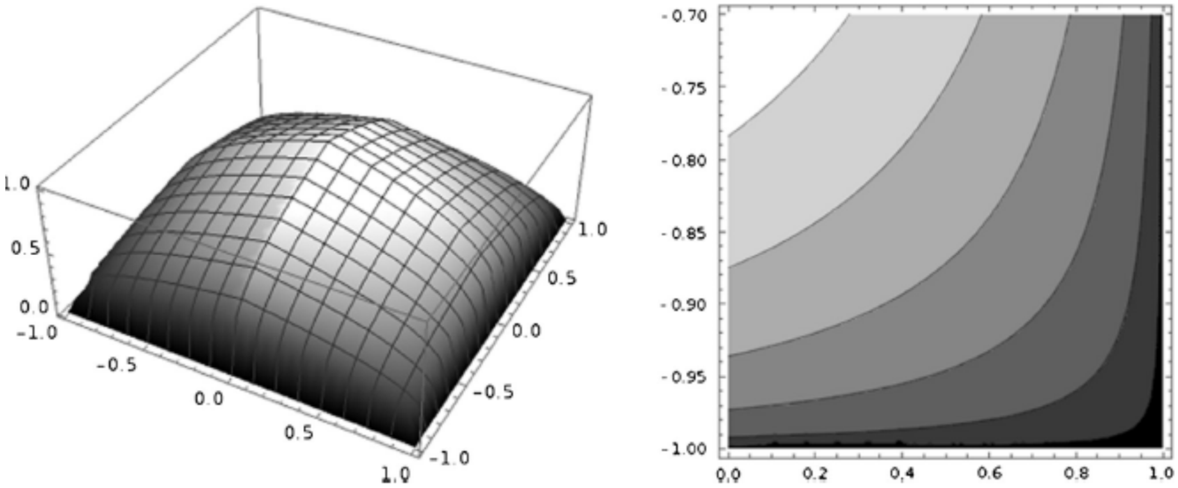

Figure (1)[228]:

The next corollary demonstrates that the classical $L_1(\mu)$ −spaces for atomless measures μ also satisfy Steinhaus' condition, giving thus another class of examples of non-strictly convex spaces with this property. Recall that a set A in a measure space is called an atom if $\mu(A) > 0$ and $\mu(B) \in \{0, \mu(A)\}$ for every measurable subset B of A.

Proposition (6.3.6)[228]: Let (Ω, Σ, μ) be a measure space. Then, the space $L_1(\mu)$ satisfies (S) if and only if Ω contains at most one atom (up to measure-zero sets).

Proof. By Luther's theorem [235], there is a decomposition $\mu = \mu_1 + \mu_2$ with μ_1 being semi-finite (i.e., for each $A \in \Sigma$ with $\mu_1(A) = \infty$, there is a subset $B \in \Sigma$ of A such that $0 < \mu_1(B) < \infty$) and μ_2 being degenerate (i.e., the range of μ_2 is contained in $\{0, \infty\}$). The space $L_1(\mu)$ is then isometrically isomorphic to $L_1(\mu)$ (for any $f \in L_1(\mu)$ we have $\mu_2(\{x: f(x) \neq 0\}) = 0$, thus the identity map yields the desired isometry). Therefore, we consider only the case where μ is semi-finite.

First, suppose that (Ω, Σ, μ) is atomless. Fix two functions $f, g \in L_1(\mu)$ with $f \neq g$ and $\|f\| = \|g\| = 1$, and let $\delta > 0$ be given. Interchanging f and g, if necessary, we may assume that there is a set $F \in \Sigma$ such that $0 < \mu(F) < \infty$ and $f(\omega) > g(\omega)$ for $\omega \in F$. Since

$$F = \bigcup_{n=1}^{\infty} \{\omega \in F : f(\omega) > g(\omega) + \frac{1}{n}\} \quad ,$$

we may also suppose that for some $\varepsilon > 0$ and all $\omega \in F$ we have $f(\omega) > g(\omega) + \varepsilon$. Approximating f and g by step functions we may find a measurable set $F' \subset F$ with $\mu(F') > 0$ and some $c_f, c_g \in \mathbb{R}$ such that

$$|f(\omega) - c_f| < \frac{\varepsilon}{5} \quad and \quad |g(\omega) - c_g| < \frac{\varepsilon}{5} \quad (\omega \in F').$$

Hence, $c_f > c_g + \frac{3}{5}\varepsilon$ and $m_f > M_g + \frac{1}{5}\varepsilon$, where $m_f = ess\,inf f(F')$ and $M_g = ess\,sup\, g(F')$. We have three possibilities:

(i) $m_f > 0$ and $M_g \geq 0$,

(ii) $m_f > 0$ and $M_g < 0$,
(iii) $m_f \leq 0$ and $M_g < 0$.

With no loss of generality suppose that either (i) or (ii) occurs (the case (iii) is analogous to (i)). Then there is a positive number d such that $|m_f - d| < m_f$ and $|M_g - d| > |M_g|$; indeed, in the former case we shall take any $d \in (2M_g, 2m_f)$, while in the latter one any sufficiently small d does the job.

Now, observe that for almost all $\omega \in F'$ we have

$$|f(\omega) - d| < |f(\omega)| \qquad and \qquad |g(\omega) - d| > |g(\omega)|. \tag{3}$$

Indeed, for the first inequality note that in the case where $f(\omega) \geq d > 0$ it holds trivially true, while in the opposite case we have

$$|f(\omega) - d| = d - f(\omega) \leq d - m_f \leq |d - m_f| < m_f \leq |f(\omega)|.$$

For the other one observe that since $M_g < d$ (recall $|M_g - d| > |M_g|$), we have $g(\omega) < d$, thus in the case where $g(\omega) \geq 0$ we have

$$|g(\omega) - d| = d - g(\omega) \geq d - M_g = |d - M_g| > |M_g| \geq g(\omega) = |g(\omega)|,$$

whereas in the case where $g(\omega) < 0$ this inequality is trivial.

By the Darboux property of finite atomless measures ([236], see also [233]), there is a measurable set $H \subset F'$ with $0 < \mu(H) < \delta/d$. Then $\|d \cdot 1_H\| < \delta$, where 1_H stands for the characteristic function of H, while inequalities (3) imply that $\|f - d \cdot 1_H\| < 1$ *and* $\|g - d \cdot 1_H\| > 1$. This proves assertion (S'), and hence also (S).

In the case where there is exactly one atom $A \subset \Omega$ (up to measure-zero sets), either $\mu(\Omega \setminus A) = 0$, which means that $L_1(\mu) \cong \mathbb{R}$ isometrically, or there exists an atomless part $B \subset \Omega$ of positive measure so that $\Omega = A \cup B$. In the latter case, fix any $f, g \in L_1(\mu)$ with $f \neq g$ and $\|f\| = \|g\| = 1$. First, assume that $f|_B = g|_B$ outside a set of measure zero. Both f and g are constant almost everywhere on A; denote those constant values as c_f and c_g, respectively. Since $\|f\| = |c_f|\mu(A) + \int_B |f|\, d\mu$ and $\|g\| = |c_g|\mu(A) + \int_B |f|\, d\mu$, we have $|c_f| = |c_g|$ and $c_f \neq c_g$. So, assuming that $c_f > 0$ and $c_g < 0$, for any given $\delta > 0$ we have $\|f + \delta \cdot 1_A\| > 1$ and $\|g + \delta \cdot 1_A\| < 1$. In the case where $f|_B$ and $g|_B$ do not coincide almost everywhere, we repeat the argument from the first part of the proof for the atomless measure space $(B, \Sigma', \mu|_{\Sigma'})$, where $\Sigma' = \{B \cap C : C \in \Sigma\}$. Consequently, $L_1(\mu)$ has property (S) whenever the underlying measure space contains at most one atom.

Finally, suppose Ω contains two disjoint atoms, say A_1 and A_2. Consider the functions $f = \frac{1}{2}(1_{A_1} + 1_{A_2})$ and $= \frac{1}{4}1_{A_1} + \frac{3}{4}1_{A_2}$. Obviously, for $\delta \in (0, \frac{1}{4})$ there is no function h with $\|h\| < \delta$ so that exactly one of $f + h$ and $g + h$ has norm larger than 1, thus in this case (S) fails to hold.

Theorem (6.3.2) gives an immediate answer to the question about Steinhaus' property for $C_0(K)$-spaces, and it is unsurprisingly negative except the trivial case where the considered space is one-dimensional.

Corollary (6.3.7)[228]: Let K be a locally compact Hausdorff space that contains at least two points. Then the space $C_0(K)$consisting of scalar-valued functions on K vanishing at infinity does not have property (S).

Proof. Pick any two distinct points $u, v \in K$, and their disjoint neighbourhoods U and V. Since K is completely regular, there is a continuous map $\phi: K \to [0,1]$ such that $\phi(u) = 1$ and $\phi|_{K\setminus U} = 0$. Similarly, since $K \setminus U$ is also completely regular, there is a continuous map $\phi_1 : K \setminus U \to [0,1/2]$ such that $\phi_1(v) = 1/2$ and $\phi_1|_{K\setminus(U\cup V)} = 0$. Then the mapping $\psi: K \to [0,1]$ defined by

$$\psi(x) = \begin{cases} \phi(x) & for\ x \in U, \\ \phi_1(x) & for\ x \in K \setminus U, \end{cases}$$

is continuous and, of course, $\phi \neq \psi$. So, both functions ϕ and ψ belong to the unit sphere of $C_0(K)$, but for any $\delta \in (0,1/2)$ condition (S') is violated.

Theorem (6.3.8)[228]: Assume that c is a real-valued cardinal number and let Γ be a set with cardinality less than c. Then the Bochner space $X = L_1(\mu, \ell_\infty(\Gamma))$ has property (S) for some atomless, probability measure μ.

Proof. As c is assumed to be a real-valued cardinal number, there exists an atomless probability measure space $(\Omega, \wp(\Omega), \mu)$, where Ω is a set with the cardinality of the continuum and μ is c-complete. Then μ is the required measure.

Let $f \neq g$ be two norm-one elements of X. Since members of X are equivalence classes of the relation of equality almost everywhere, let us work with concrete representatives $f, g : \Omega \to \ell_\infty(\Gamma)$. Fix $\delta > 0$. There exists $n_0 \in \mathbb{N}$ such that $\mu(F_{n_0}) > 0$, where

$$F_{n_0} = \{\omega \in \Omega: \|f(\omega) - g(\omega)\|_{\ell_\infty(\Gamma)} > \frac{1}{n_0}\} .$$

For each $\gamma \in \Gamma$ let

$$G_\gamma = \left\{\omega \in F_{n_0} : |f(\omega)(\gamma) - g(\omega)(\gamma)| > \frac{1}{n_0}\right\}.$$

Since μ is defined on the power set of Ω, there is no problem with measurability of the sets G_γ $(\gamma \in \Gamma)$. Also, as μ is c-complete and $|\Gamma| < c$, the set G_{γ_0} has positive measure for some $\gamma_0 \in \Gamma$. Interchanging f with g, if necessary, we may suppose that the set

$$F = \{\omega \in G_{\gamma_0} : f(\omega)(\gamma_0) > g(\omega)(\gamma_0) + \frac{1}{n_0}\}$$

has positive measure. Now, we proceed as in the proof of Proposition (6.3.6).

Approximating the functions $\omega \mapsto f(\omega)(\gamma_0)$ and $\omega \mapsto g(\omega)(\gamma_0)$ $(\omega \in \Omega)$ by step functions we may find a set $F' \subset F$ with $\mu(F') > 0$ and some $c_f, c_g \in \mathbb{R}$ such that for almost all $\omega \in F'$ we have

$$|f(\omega)(\gamma_0) - c_f| < \frac{1}{5n_0} \quad and \quad |g(\omega)(\gamma_0) - c_g| < \frac{1}{5n_0}.$$

Hence, $c_f > c_g + \frac{3}{5n_0}$ and $m_f > M_g + \frac{1}{5n_0}$, where $m_f = ess\,inf\, f(F')$ and $M_g = ess\,sup\, g(F')$. We have then three possibilities:

(i) $m_f > 0\ and\ M_g \geq 0$,
(ii) $m_f > 0\ and\ M_g < 0$,
(iii) $m_f \leq 0\ and\ M_g < 0$,

which we tackle completely analogously as in the proof of Proposition (6.3.6) (here $f(\omega)(\gamma_0)$ and $g(\omega)(\gamma_0)$ play the rôle of $f(\omega)$ and $g(\omega)$, respectively). Therefore (assuming either (i) or (ii) holds true), we observe that for some $d > 0$ and almost all $\omega \in F'$ we have

$$|f(\omega)(\gamma_0) - d| < |f(\omega)(\gamma_0)| \quad and \quad |g(\omega)(\gamma_0) - d| > |g(\omega)(\gamma_0)|. \tag{4}$$

Since μ is atomless, there is a measurable set $H \subset F'$ with $0 < \mu(H) < \delta/d$. Then $\|d \cdot \delta_{\gamma_0} \cdot 1_H\|_X < \delta$, where $\delta_{\gamma_0} \in \ell_\infty(\Gamma)$ stands for the element that is zero apart from the γ_0^{th} coordinate where it assumes value 1. Hence, (4) imply that $\|f - d \cdot \delta_{\gamma_0} \cdot 1_H\|_X < 1$ and $\|g - d \cdot \delta_{\gamma_0} \cdot 1_H\|_X > 1$, as desired.

Corollary (6.3.9)[228]: Under the assumptions of Theorem (6.3.8), for every uncountable set Γ with cardinality less than the continuum, the Banach space $X = L_1(\mu, \ell_\infty(\Gamma))$ has (S), yet it lacks a strictly convex renorming.

We have thus proved the first assertion of Theorem (6.3.10); clause (i) has been also already observed. It remains to prove clause (ii).

Theorem (6.3.10)[228]: Assuming that the continuum is a real-valued measurable cardinal, there exists a non-strictly convexifable Banach space whose norm satisfies (S). Moreover, for any Banach space X we have:

(i) if dim $X \leq 2$, then X has property (S) if and only if X is strictly convex;
(ii) if dim $X > 2$ and X admits a renorming with property (S), then it also admits a non-strictly convex renorming with property (S).

Proof.

Here, we shall construct a Banach space with property (S) but without any strictly convex renorming. Assume that the continuum c is a real-valued cardinal number. This implies that there is an atomless, c-complete probability measure μ defined on the power set of a set Ω with the cardinality of the continuum (see, e.g., [234])—here, by a λ-complete measure μ (λ is an uncountable cardinal) we understand a measure satisfying the following condition: for every cardinal $\kappa < \lambda$ and for every family $(A_\alpha)_{\alpha<\kappa}$ of measurable sets, their union A is measurable and

$$\mu(A) = \sup\{\mu(\bigcup_{\alpha\in F} A_\alpha) : F \subset \kappa \; finite\}.$$

The statement that the continuum is a real-valued cardinal is equiconsistent with the existence of a two-valued measurable cardinal ([237]), which is stronger than the consistency of ZFC alone. Banach and Kuratowski ([229]) proved that if such a measure exists, then the Continuum Hypothesis fails to hold, hence there exists at least one uncountable cardinal below the continuum. We will show that for any set Γ with $|\Gamma| < c$, the Bochner space $L_1(\mu, \ell_\infty(\Gamma))$ satisfies Steinhaus' condition. In particular, if Γ is uncountable, such space does not have a strictly convex renorming as it contains $\ell_\infty(\Gamma)$ embedded via constant functions and this space does not have such a renorming by a result of Day ([231], see also [232]).

Assume that a Banach space X with $\dim X > 2$ has a norm $\|\cdot\|$ satisfying (S); we can assume that this norm is in fact strictly convex, as otherwise we are done. In the case where $\dim X = 3$, the assertion is proved by Example (6.3.5), so assume that $\dim X > 3$. Choose any subspace $Y \subset X$ of codimension 2 so that we have $X = Y \oplus \mathbb{R} \oplus \mathbb{R}$ and every element $x \in X$ may be typically written as (y, α, β) with $y \in Y$, $\alpha, \beta \in \mathbb{R}$. (In fact, the symbols $\mathbb{R}$ formally stand for some fixed one-dimensional subspaces of X.) Note that Y is a strictly convex space of dimension at least 2. Let $\|\cdot\|'$ be a new norm on X given by the decomposition $X = \left(Y \oplus_{\ell_1} \mathbb{R}\right)_{\oplus_{\ell_2}} \mathbb{R}$, that is

$$\|x\|' = \sqrt{(\|y\| + |\alpha|)^2 + |\beta|^2} \qquad (x = (y, \alpha, \beta)).$$

As any two finite direct sums of the same normed spaces are isomorphic, this defines an equivalent norm on X which obviously fails to be strictly convex. Next, we shall show that it has property (S).

For, suppose $x_1 = (y_1, \alpha_1, \beta_1)$ and $x_2 = (y_2, \alpha_2, \beta_2)$ are two distinct points from the unit sphere of $(X, \|\cdot\|')$. If $\beta_1 \neq \beta_2$, then $(\|(y_1, \alpha_1)\|, \beta_1)$ and $(\|(y_2, \alpha_2)\|, \beta_2)$ are two distinct points on the unit circle, where the norm symbol stands for the ℓ_1-norm on $Y \oplus \mathbb{R}$. Thus, by manipulating the coordinates α and β we obtain a vector z of the form $(0, \alpha, \beta)$, and of arbitrarily small length, so that $\|x + z\|' \neq \|y + z\|'$.

Now, suppose that $\beta_1 = \beta_2$ and hence $\|y_1\| + |\alpha_1| = \|y_2\| + |\alpha_2|$. If $y_1 = y_2$, then it must be $\alpha_1 = -\alpha_2 \neq 0$, whence we easily find a desired vector z being of the form $(0, \alpha, 0)$. So, assume we have $y_1 \neq y_2$. In this case, we can find z with the aid of following simple observation:

Claim (6.3.11). Since Y is strictly convex and $\dim Y \geq 2$, for every pair of distinct vectors $y_1, y_2 \in Y$ and every $\delta > 0$ there exists $z \in Y$ such that $\|z\| < \delta$ and $\|y_1 + z\| - \|y_1\| \neq \|y_2 + z\| - \|y_2\|$.

Indeed, if the vectors y_1 and y_2 are linearly independent, we take $z = \eta y_1$ for suitably small $\eta > 0$. Then, $\|y_1 + z\| - \|y_1\| = \eta\|y_1\|$ and this is equal to $\|y_2 + z\| - \|y_2\|$ if and only if $\|y_2 + z\| = \|y_2\| + \|z\|$, which is impossible as the norm is strictly convex. In the case where $y_2 = \gamma y_1$ for some $\gamma \in \mathbb{R}$, we pick any vector z that is linearly independent of y_1 and satisfies $\|z\| < \delta$. Then, assuming with no loss of generality that $|\gamma| \geq 1$, the required condition becomes $\|\gamma y_1 + z\| \neq (|\gamma| - 1)\|y_1\| + \|y_1 + z\|$ which again follows from the strict convexity of . The claim (6.3.11) has been thus proved.

Now, take a vector $z \in Y$ as in the above claim (6.3.11). Then we have:

$$\|y_1 + (z, 0, 0)\|' \neq \|y_2 + (z, 0, 0)\|' \Longleftrightarrow$$

$$\|y_1 + z\| + |\alpha_1| \neq \|y_2 + z\| + |\alpha_2| \Longleftrightarrow$$

$$(because\ \|y_1\|' = \|y_2\|' and\ \beta_1 = \beta_2)$$

$$\|y_1 + z\| - \|y_1\| \neq \|y_2 + z\| - \|y_2\|,$$

which is true. Therefore, we have checked that $(X, \|\cdot\|')$ satisfies condition (S”).

List of Symbols

symbol		page
H^1	Hardy Space	1
H^∞	Essential Hardy Space	1
Hom	Homeomorphism	2
Im	Imaginary	2
Re	Real	2
a.e	almost everywhere	2
Sup	supremum	4
inf	infinmum	4
max	maximum	11
qc	quasi conformal	12
qr	quasi regular	12
P - qch	P – harmonic quasi conformal	12
P - qrch	P – harmonic quasi regular	12
e - qch	euclidean	12
h – qch	hyperbolic harmonic quasi conformal	12
Belt	Beltrami	14
dist	distance	15
arg	argument	19
mod	modular	20
min	minimum	20
Int	Interior	21
Lip	Lipschitz	21
CH	Continuous Hypothesis	34
ZFC	Zermelo – Fraenkel with the axioms of choice	45
ZF	Zermelo – Fraenkel axioms	47
AD	Adjiont action	48
DC	Axioms of dependent choice	48
L^1	Lebesgue an the read line	50
L^2	Hilbert Space	50
L^∞	Essential Lebesgue space	50
Card	Cardinality	53
$W^{p,\alpha}$	Sopolev space	60
L^p	Lebesgue space	60
supp	Support	61
diam	diameter	72
LUR	Locally uniformly rotund	95
$\otimes$	Tensor product	95
L_∞	essential Banach space	96
$\oplus$	Direct sum	96
$\odot$	Algebraic Tensor product	100
WCD	Weakly Countably Determined	113
co	closure	113

WUR	Weakly uniformly rotund	117
WCG	Weakly compactly generator	117
L^q	Dual of Lebesgue space	127
ess	essential	145
AC	Axiom of choice	159
det	determinate	163
SPRI	Separable projectional resolution of the Identity	206
PRI	projectional resolution of the Identity	211
osc	oscillation	230

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