**Sudan University of Science and Technology**
**College of Graduate Studies**

# Galois Correspondence for Free Actions of Compact Abelian Groups on $C^*$-Algebras and Generic Points of Invariant Measures

# تقابل جالوا للأفعال الحرة للزمر الابيلية المتراصة على جبريات- $C^*$ والنقاط النوعية للقياسات اللامتغيرة

**A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics**

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# Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla of Sudan University of Science and Technology.

# Abstract

We deal with algebras of sphericl functions associated with covariant systems over a compact group with locally compact group action on $C^*$-algebras and compact subgroups and duality theory for nonergodic actions. The quasi product actions of compact abelian group on a $C^*$ - algebra and freeness of actions of finite abelian groups on $C^*$-algebras and free of compact quantum groups on unital $C^*$ - algebras are considered. The Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, for compact quantum group actions and $C^*$-algebras are studied.  The homoclinic groups and expansive algebraic actions are presented. The invariant measures for homeomorphisms with weak specification and orbit equivalence for generalized Toeplitz subshifts are introduced. The generic points of invariant measures for an amenable residually finite group actions with the weak specification property for ergodic group automorphisms of abelian groups are characterized.

# الخلاصة

تعاملنا مع الجبريات للدوال الدائرية المشاركة مع الانظمة المتغيرة فوق الزمرة المتراصة مع افعال الزمرة المتراصة الموضعية علي جبريات $C^*$ والزمرة الجزئية المتراصة ونظرية الثنائية للأفعال غير الارجوديه . قمنا باعتبار شبه حاصل الضرب والأفعال الحرة للزمر الابيلية المتراصة علي جبريات - $C^*$ والأفعال الحرة لزمر الكم المتراصة علي جبريات - $C^*$ الوحيدة . تمت دراسة تقابلات جالوا لزمر التراص لأوتو مورفيزمات جبريات فون نيومن مع التعميم الي جبريات كاك ولأفعال زمر الكم المتراصة وعلي جبريات - $C^*$ .قمنا بتقديم زمر هومو كلينيك وأفعال الجبر التوسعي. تم ادخال القياسات اللا متغيرة للهوميومورفيزمات مع التخصيص الضعيف وتكافؤ المدار للازاحات الجزئية تبوليتز المعممه. تم تشخيص النقاط النوعية للقياسات اللا متغيرة الأفعال للزمرة المنتهية المتبقية القابلة مع خاصية التخصيص الضعيفة تقريبا لاوتومورفيزمات الزمرة الارجودية للزمر الابيلية .

# Introduction

If $M$ is a $C^*$-algebra, $K$ a compact group and $\varrho : K \to \text{Aut}\ (M)$ a homomorphism, one can form the covariance algebra $K \times_{\varrho} M$. We show that the classification of the factor representations of $K \times_{\varrho} M$ (in particular the irreducible ones) can be reduced toe the classification of all factor (or irreducible) representations of the algebras $M \otimes B\big(X(D)\big)^{\varrho \otimes \text{ad}\, D}$ $(D \in \widehat{K})$ which can be considered as generalizations of the algebras of subgroup. For $(A, G, \alpha)$ be a $C^*$-dynamical system and $K \subset G$ a compact subgroup. We give necessary and suffkient conditions in order that the crossed product $G \times_{x}\ A$ be simple. Several conditions on an action of a compact abelian group on a separable prime $C^*$-algebra to be equivalent.

Generalizing work by Pinzari and Roberts, we characterize actions of a compact quantum group G on $C^*$-algebras in terms of what we call weak unitary tensor functors from Rep G into categories of $C^*$-correspondences. We discuss the relation of our construction of a $C^*$-algebra from a functor to some well-known crossed product type constructions, such as cross-sectional algebras of Fell bundles and crossed products by Hilbert bimodules. Let F be a field, Γ a finite group, and Map(Γ, F) the Hopf algebra of all set-theoretic maps Γ → F. If E is a finite field extension of F and Γ is its Galois group, the extension is Galois if and only if the canonical map E ⊗F E → E ⊗F Map(Γ, F) resulting from viewing E as a Map(Γ, F)-comodule is an isomorphism. Similarly, a finite covering space is regular if and only if the analogous canonical map is an isomorphism. We extend this point of view to actions of compact quantum groups on unital $C^*$-algebras.

We show that the results of K. Sigmund hold for homeomorphisms satisfying weak specification. We show that every automorphism of a compact metric abelian group is ergodic under the Haar measure.

For $M$ be a factor with separable predual and G a compact group of automorphisms of M whose action is minimal, i.e., $M^{G'} \cap M = C$, where $M^G$ denotes the G-fixed point subalgebra. Then every intermediate von Neumann algebra MG/N/M has the form $N = M^H$ for some closed subgroup H of G.

We establish a Galois correspondence for a minimal action of a compact quantum group G on a von Neumann factor $M$. This extends the result of Izumi, Longo and Popa who treated the case of a Kac algebra. We show a Galois correspondence for compact group actions on $C^*$-algebras in the presence of a commuting minimal action.

We study and classify free actions of compact quantum groups on unital $C^*$-algebras in terms of generalized factor systems. We study free actions of compact groups on unital $C^*$-algebras. For F be a field, Γ a finite group, and Map(Γ, F) the Hopf algebra of all set-theoretic maps Γ → F. If E is a finite field extension of F and Γ is its Galois group, the extension is Galois if and only if the canonical map E ⊗F E → E ⊗F Map(Γ, F) resulting from viewing E as a Map(Γ, F)-comodule is an isomorphism. Similarly, a finite covering space is regular if and only if the analogous canonical map is an isomorphism. We extend this point of view to actions of compact quantum groups on unital $C^*$-algebras. We show that such an action is free if and only if the canonical map (obtained using the underlying Hopf algebra of the compact quantum group) is an isomorphism.

We show that for every metrizable Choquet simplex K and for every group G, which is infinite, countable, amenable and residually finite, there exists a Toeplitz G-subshift whose set of shift-invariant probability measures is affine homeomorphic to K. We give algebraic characterizations for expansiveness of algebraic actions of countable groups. The notion of p-expansiveness is introduced for algebraic actions, and we show that for countable amenable groups, a finitely presented algebraic action is 1-expansive exactly when it has finite entropy. We also study the local entropy theory for actions of countable amenable groups on compact groups by automorphisms, and show that the IE group determines the Pinsker factor for such actions. We show that every measure invariant for an amenable residually finite group action satisfying the weak specification property has a generic point.

# The Contents

# Chapter 1

# Algebras of Sphericl Functions Associated with Locally Compact Group and Quasi Product Actions

We show that as a corollary we get that if $K$ is abelian then $K \times_{\varrho} M$ is liminal or postliminal if and only if $M^{\varrho}$ has the same property. The conditions are in terms of the dual $\widehat{K}$ of the compact subgroup $K$. We also consider the similar problem for prime crossed products. We improve that some results are subsatantially.

## Section (1.1): Covariant Systems over a Compact Group

We classify the most fundamental building blocks, for instance to find all irreducible or factor representations of $a^*$ - algebras or a group. In the theory of representations of group this classification often can bedone by a reduction process . For instance if a locally compact group $G$ has a closed normal subgroup $N$ the classification of the irreducible unitary representations of $G$(this set is called $\widehat{G}$) can be reduced to the classification of $\widehat{N}$ andcertain projctive representations of some subgroups of $G/N$ , c.f. [12]. This theory has later been extended to more general systems c.f. [6], [12], [14] and [17].

If the group $G$ contains a " large " compact subgroup $K$ , there is another method of classifying $\widehat{G}$first studied by R. Godement in [7].Here the classification of $\widehat{G}$ is reduced to that of determining $\widehat{K}$ and to find all irreducible representations of certain algebras of " spherical functions". We try to extend this method to covariant systems over acompact group .

For $M$ be a $C^*$ - algebra, $K$ acompact group and $\varrho$ ahomorphism of $K$into the group Aut (M)of * - automorphisms of $M$such that the map $K \longrightarrow {}_{\varrho k}(a)$
Is norm-continuous for all $a \in M$. Let $L^1(K, M)$ be all integrable functions from $K\ to\ M$.With the following definitons $L^1(K, M)$ is a Banach$^*$-algebra :

$$fg(x) = \int_K f(y)_{\varrho y}\left(g(y^{-1}x)\right)dy$$
$$f^*(x) = {}_{\varrho x}(f(x^{-1}))^*$$
$$\|f\| = \int_K \|f(x)\|dx.$$

Let $\mathfrak{A}$ be the enveloping $C^*$-algebra of $L^1(K, M)$ $\mathfrak{A}$ is called the convariance algebra of the convariant sysetm $(M, {}_{\varrho}, K)$ and is also denoted $K \times_{\varrho} M$. These concepts were introduced by S.Doplicher , D.Kastler and D.Robinsonin [5] and they showed that there is a one –to-one correspondance between non-degnenerate * - representations $T$ of $\mathfrak{A}$ and pairs $(U, \pi)$ with $U$ a continuous unitary representation of $K, \pi$ a non-degnenerate * - representation of $M$ such that $\pi$ and $U$ acts on the same Hilbert space and

$$U_k \pi(a) U_{k^{-1}} = \pi\left({}_{\varrho k}(a)\right) \quad \text{for } k \in K, a \in M. \tag{1}$$

In fact if $(U, \pi)$ is geven , that then $T$ is defined by

$$T_f = \int \pi\left(f(k)\right)U_k dk \quad for\ \ f \in L^1(K, M).$$

The starting point in the Mackey-Takesaki theory is now to study the representation $\pi$ of $M$. $\pi$ is not necessarily irreducible even if $T$ is and one looks at $\pi$'s decomposition into irreducible representations . (This may not be possible so one has to make certain assumptions about $M$). The group $K$ acts naturally on the set $\widehat{M}$ of irreducible representations of $M$, and if this actinon is "nice" it is possible to describe $\mathfrak{A}$ by $\widehat{M}$ and certain projective representations of some subgroups of $K$.

We are instead going to look at the decomposition of $U$ into irreducible representations of $K$, and we shall classify all pairs $(\pi, U)$ as above having a given $D \in \widehat{K}$ as a subrepresentation of $U$.

For $D \in \widehat{K}$, let $\psi_D$ be its character , i.e.

$$\psi_D(k) = \dim(\mathrm{D})\mathrm{tr}\,(\mathrm{D}_{\mathrm{k}^{-1}}) \qquad \text{for } k \in K.$$

$L^1(K)$ can be embedded in $M\big(L^1(K,M)\big)$ = the algebra of multipliers of $L^1(k,M)$ be defining

$$\varphi f(x) = \int_K \varphi(k)_{\varrho k}\left(\mathrm{f}(k^{-1}x)\right)dk$$

$$f\varphi(x) = \int_K \varphi(k)f\,(xk^{-1})dk, \qquad for\ \varphi \in L^1(K), \qquad f \in L^1(K,M), x \in K.$$

$K$ can also be embedded in $M\big(L^1(K,M)\big)$ by

$$Kf(x) = {}_{\varrho k}\big(f(k^{-1}x)\big)$$

$$fk(x) = \ f(k^{-1}x) \qquad \text{for } f \in L^1(K,M), k, x \in K.$$

Look at the two sided ideal

$$A(D) = \text{colsed lin span}\,\{f\psi_{\mathrm{D}}\mathrm{g}|\mathrm{f},\mathrm{g} \in L^1(K,M)\}$$

of $L^1(K,M)$. The importance of this ideal is seen from

**Lemma (1.1.1)[1]:** if $T$ is a non - degnerate * - representationof $L'(K,M) and U\ and$ л are the repersentations of K and M such that (1) holds , then a given $D \in K$ occurs in U if and only if the restriction of Tto the two – sided ideal A(D) is non zero.

**Proof:** Let $P_D = \int k\psi_D(K)U_k dk, so\ P_D$is aprojection. Now $D$ occurs in $U$ if and only if $P_D \neq 0$ . Since $T$ is non – degenerate $P_D \neq 0$ if and only if there are $f$ and $g$ in $L^1(K,M)$ with $T_g P_D T_f \neq 0$. Using that $P_D T_f = T_{\psi_D f}$ it follows that $P_D \neq 0$ if and only if $T\big(A(D)\big) \neq 0$ .

$$_{\varrho k}\big(f(K^{-1}xK)\big) = f(x) \qquad for\ almost\ all\ k\ , x \in K\ , \tag{2}$$

$$\psi_D f = f \tag{3}$$

So $A(D)$ and $L_D^0\,(M)$ are strongly Morita equivalent . (We have here used the following : if $A$ is an algebra and its multiplier algebra $M(A)$ contains a sub algebra $B$ isomorphic to a full matrix algebra , then $A \cong B \otimes C$ where $C = (A \cap B')$.

The algebra $L_D^0\,(M)$ is the analogue of the algebra $L^0(d)$ considered in [7].

We shall also prove that $L_D^0\,(K,M)$ is isomorphic to algebra

$$B(D) = \ \{a \in M \otimes B(X(\overline{D}\ ))|\,(e_x \otimes ad\overline{D}_x)(a) = a\ for\ all\ x \in K\}.$$

($\overline{D}is$ the conjugate repersentation of $D$.)

So the classificaton of irreducible representations of the covariant system $(M, {}_{\varrho}, K)$ has been redused to the classificaton of $\widehat{K}$ and the irreducible representations of $B(D)$ for different $D \in \widehat{K}$ .If $K$ in particular is abelian,this classificaton becomes very simple , then $B(D) \cong\ M^{\varrho}$.

So the representations theory of $(M, {}_{\varrho}, K)$ is determined by the fixpoint algebraof $M$,and by $\widehat{K}$. We revised and updated version of [11] which originated during.

We shall keep all definitions. In addition we make the following convenntion :If $S$ is a repressentation, we let $X(S)$ denote the corresponding Hillbert space. If $X$ is a Hillbert space, $B(X)$ is the algebra of all bounded operator on $X$ and $CC(X)$ is the subalegbra of all compact opertors. For a covariant system $\left(M, _{\varrho}, K\right), M^{\varrho}$ denotes its fixpont-alegbra, i.e.

$$M^{\varrho} = \left\{a \in M\middle|_{\varrho x}(a) = a \text{ for all } x \in X\right\}.$$

If $D$ is a continuous representation of $K$, ad $D$ is the map from $K$ to Aut $B(X(D))$ defined by

$$\text{ad } D_x(R) = D_x R D_{x^{-1}} \text{ for } x \in K, R \in B\big(X(D)\big).$$

**Lemma (1.1.2)[1]:**let $A$ be $a$ Banach *- algebra , J a closed self- adjoint two- sided ideal in $A$ Then there is a bijectivecorrspondance between non-degenetate factor reperesentations $S$ of $J$ and non-degenetate factor reperesentations $T$ of $A$ with $T(J) \neq \{0\}$. If $T$ is gevev, $S$ is the restriction of $T$ to $j$ . If $S$ is geven , $T$ is defined by

$$T_a(S_b\xi) = S_{ab}\xi \quad for\ a \in A, \quad b \in J, \xi \in X(S). \tag{4}$$

Furthermore, $T(A)'' = S(J)''$, so $S$ and $T$ are of the same type , and $S$ is irreducible if and only if $T$ is .

**Proof :**If $T$ is given, let $S$ be the restriction of $T$ to $j$.Then the closure $S(J)''$ is a tow – sided ideal in $T(A)''$ , so $S(J)'' = T(A)'', S\ is$ non- dengenerate , and $S$ is also a factor representation.

Converesly , suppose $S$ is a non- degenerate *- representation of $J$(not necessarily a factor representation), and without loss of generality we may assume that $S$ has acyclic vector $\xi_0$ . On the dense subsace $X_0 = \{S_b\xi_0 | b \in J\}$ we define $T$ by

$$T_a(S_{ab}\xi_0) = S_{ab}\xi_0 \text{ for } a \in A, b \in J.$$

Now ,

$$\|S_{ab}\xi_0\|^2 = (S_{a^*ab}\xi_0\,, S_b\xi_0) \leq \|S_b\xi_0\|\|S_{a^*ab}\xi_0\| \leq \cdots \leq \|S_b\xi_0\|^{2-2^{-n}}\|S_{(a^*a)^{2^n}b}\xi_0\|^{2^{-n}}$$
$$\leqq \|S_b\xi_0\|^{2-2^{-n}}\|a^*a\|\,(\|b\|\ \|\xi_0\|)^{2^{-n}} \quad \text{for } n = 1, 2, \ldots,$$

So $\|S_{ab}\xi_0\| \leq \|a\|\|S_b\xi_0\|$ for all $a \in A, b \in J$.

Hence $T_a$ is well defined over $X_0$and extends to abounded operator over $X(S).T$ will obvioslyof be a non – degenerate *- representation of $A$and (4) wil lhold for all $a \in A, b \in J, \xi \in X(x) obviously\ S(J)'' \subset T(A)''$. Converselyif$R \in S(J)', a \in A, b \in J$then

$$RT_aS_b\xi_0 = RS_{ab}\xi_0 = S_{ab}R\xi_0 = T_aS_bR\xi_0 = T_aRS_b\xi_0$$

Thus $RT_a = T_aR$ for all $a \in A$, so$R \in T(A)'$. Hence $T(A)'' = S(J)''$ , and in particular $S$ is a factor representation if $T$ is . To conclude ,it is easy to check that the maps $S \longrightarrow T$ and $T \longrightarrow S$ are each others inverses .

**Lemma (1.1.3)[1]:** Define a linear map $\phi :\ L^1(K\,, M) \longrightarrow M \otimes B\ \big(X\ (\overline{D}\ )\big)by$

$$\phi(f) = \int_k f(k^{-1}) \otimes \overline{D}_K dK$$

Then $\phi$ is $a * -$ isomorphism of $L_D^0(K, M)$ onto

$B(D) = M \otimes \mathrm{B}(\mathrm{X}(\overline{D}))^{\varrho \otimes \mathrm{ad}\overline{D}}$

**Proof :** Note that $\phi$ is not $a * -$ homomorphism of $L^1(K, M)$, but using that elments of $\boldsymbol{L_D^0}(K, M)$ satisfy(2) and (3), staright forward calculations show that $\phi$ is $a * -$ homomorphism of $\boldsymbol{L_D^0}(K, M)$ into $B(D)$ .

Let $Q: M \otimes B(X(\overline{D})) \longrightarrow M$be defined by

$Q(s \otimes a) = \dim(D)\ tr(a)s$ for $s \in M, a \in B\big(X(\overline{D})\big)$.

Then define $\psi: M \otimes B(X\big(X(\overline{D})\big) \longrightarrow L^1(K, M)$ by

$\psi(b)(x) = Q(b(I \otimes \overline{D}_x))$ for $b \in M \otimes B(X(X(\overline{D}))$, $x \in K$.
It is now not difficult to see that $\psi$ maps $B(D)$ into $L^0_D(K,M)$ and that $\psi|B(D)$ and $\phi|L^0_D(K,M)$ are each others inverses.

In the introdution it was shown that $A(D)$ and $B(D)$ are Morita equivalent, both being equivalent to the algebra $A_1(D)$. How to find animprimitivity bimodule connectting $A(D)$ and $A_1(D)$ is described. Using that $A_1(D) \cong B(X(D)) \otimes L^0_D(K,M)$ and a procedure similar to the one can get an imprimitivity bimodule connecting $A_1(D)$ and $L^0_D(K,M)$. However ,we shalldirectly give an imprimitivity bimodule $L$ between $A(D)$ and $B(D)$, i.e $A(D)$ acts to the left on $L$ and $B(D)$ acts to therighton $L$ such that the axioms in [6] are sattisfied .

The following definitions should be rather natural .
$L = M \otimes X(D)$ and we define actions of $L^1(K,M)$ and $M \otimes B(X(X(\overline{D}))$ on $L$ by

$$f(a \otimes \xi) = \int f(x)\, \varrho_x(a) \otimes D_x \xi dx \qquad (5)$$

for $f \in L^1(K,M), a \in M, \xi \in X(D)$,

$$(a \otimes \xi)(b \otimes s) = ab \otimes J^* s^* j\xi \qquad (6)$$

for $a, b \in M, s \in B(X(\overline{D}))$, $\xi \in X(D)$,

Straight forward computations show that this really defines actions and that $f(rb) = (fr)b$ for $f \in L^1(K,M), r \in L$ and $b \in B(D)$ ,but not for all $b \in M \otimes B(X(\overline{D}))$.
If $\xi$ and $\eta$ are vectors in a Hilbert space $H$, let $E(\xi,\eta)$ be the rank one operator in $B(H)$ defined by

$$E(\xi,\eta)\zeta = (\zeta,\eta)\xi.$$

The $A(D)-$ rigging and $B(D)-$ rigging are now defined by

$$[a \otimes \xi, b \otimes \eta]_A(X) = \langle \xi, D_x \eta \rangle a \varrho_x(b^*)$$

$$[a \otimes \xi, b \otimes \eta]_B = \int \varrho_x(a^* b) \otimes \overline{D}_x E(j\xi, j\eta) \overline{D}_{x^{-1}} dx$$

for $a, b \in M$ and $\xi, \eta \in X(D)$.

Obviously $[r,s]_B \in B(D)$ for $r, s \in L$ .To see that $[r,s]_A \in A(D)$ note that if $\{\xi_i\}$ is an orthonormal basis in $X(D)$ ,

$$[a_i \otimes \xi_{i,} a_j \xi_j]_A(X) = dinD \int \langle \xi_{i,} D_y \xi_I \rangle a_i\ \varrho_x(\langle \xi_j D_{x^{-1}} \xi_I \rangle a_j)^*\ dy = f_i\, f_{j^*}\,(x) \qquad (7)$$

Where

$$f_i(x) = dimD^{\frac{1}{2}} \langle \xi_{i,} D_i \xi_i \rangle a_i\ . \qquad (8)$$

Since $f_i \psi_D = f_i$, it follows that $[r,s]_A \in A(D)$ for all $r, s \in L$ .
It is more or less straightforward now to check the formulas(1) –(5) on page 72 of [6]. As an example we take (5):

$$[a \otimes \xi, b \otimes \eta]_A(c \otimes \xi) = \int \langle \xi D_x \eta \rangle a \varrho_x(c) \otimes D_x \zeta dx$$

$$= \int a \varrho_x(b^* c) \otimes j^*\, \overline{D}_x E(J\zeta, J\eta) \overline{D}_{x^{-1}} j \xi dx$$

$$= (a \otimes \xi)(\varrho_x(b^* c)) \otimes \overline{D}_x E(j\eta, j\zeta) \overline{D}_{x^{-1}}) dx = (a \otimes \xi)[b \otimes \eta, c \otimes \zeta]_B.$$

So$(L, [.,.]_A, [.,.]_B)$ is an imprimtivity bimodule. Since

$$linspan\{a^* b \otimes E(j\xi, j\zeta | a, b \in M, \xi, \eta \in X(D)\}$$

Is norm dense in $M \otimes B(X(\overline{D})$ it should be obvious that lin span $\{[r,s]_B | r, s \in L\}$ is norm dense in $B(D)$ .

If $f \in L^1(K), a \in M$ define $f \otimes$ a $\in L^1(K, M)$ by

$$(f \otimes a)(x) = f(x)\text{a} .$$

Then with $\xi_i$ as above

$$(f \otimes a)\psi_D(g \otimes b)^* = dimD \sum_i [a \otimes D_f\xi_i, b \otimes D_g\xi_i]_A,$$

Where

$$D_f = \int f(x)D_x\, dx .$$

Form this it follows that lin span $\{[r, s]_A | r, s \in L\}$ is norm dense in $A(D)$ ,so our imprimitivity bimodule is topologically strict as defined in [6].

**Lemma (1.1.4) [1]:** Every * - representation of A(D) (respectiveiy B(D)) is L-positive. (cf. [6]).

**Proof:** Let $\{\xi_i\}$ be an orthonormal basis of $X(D)$ as before , and let $r = \sum a_i \otimes \xi_i \in L$ with $a_i \in M$ . Then

$$[r, r]_B = \int \sum_{i,k} \varrho_x(a_i{}^* a_k) \otimes \bar{D}_x E(j\xi_i, j\xi_k)\bar{D}_{x^{-1}} dx,$$

So $[r, r]_B \geq 0$ as an element of the $C^* -$ algebra B (D). Take $f_i$ as in ( 8) and $f = \sum f_i$ . Then (7) gives That $[r, r]_A = ff^*$ Hence $S_{[r,r]_B} \geq 0$ and $T_{[r,r]_A} \geq 0$ for any *-representations $S$ o $f$ $B(D)$ and $T$ of $A(D.)$ It should now be clear that $(L, [\ ,\ ]_A, [\ \ ,\ ]_B)$ satisfies

**CoroLllary(1.1.5)[1]:** $K \times_\varrho M\, is$ postliminal ifand only if $M \otimes B(X(D))^{\varrho \otimes adD}$ $(X(D))$ $=B(\bar{D})$ is postliminal forall $D \in \widehat{K}$ in particular this is the case if $M$ itsel $f$ is postliminal .

**Proof :** the first part follows from the fact that $a$ $C^* -$ algebra is postliminal if and only if all its factor representations are of type $I$. If $M$ is postliminal, then so is $M \otimes B(X(D))$ and its $C^* -$ subalgebra $(\bar{D})$ **.**

**Corollary (1.1.6)[1]:** $K \times_\varrho M$ is liminalif and only if $M \otimes B(X(D))^{\varrho \otimes adD}$ $(X(D))$ $=B(\bar{D})is$ liminal for all $D \in K$.In particular this is case it $M$itsel is liminal .

**Proof :** it follows that $A(D)$ is liminal if and onlyif $B(D)$ is liminal . if $K \times_\varrho M$ is liminal then so is $A(D)$ . Conversely , suppose

For each $D$ occurring in the restrictionof $T$ to $K$ $(and\, T^D = 0$ other wise) .Now $f = \sum_{D\in\widehat{K}} f\psi_D$ (convergence in norm) for each $f \in L^1(K, M)$ so to prove that $T_f$ is compact it suffices to prove that $T_{f\psi_D}$ is compact for all $D \in \widehat{K}$ ..Now $f^*\psi_D f \in A(D)$ so $(T_{f\psi_D})^*\ T_{f\psi_D} = T^D_{f^*\psi_D f}$ is compact for all $D \in \widehat{K}$ and we have proved that $K \times_\varrho M$ is liminal .

If $M$ itself is liminal , then obviously $M \otimes B(X(D))$ together with its sub algebra $B(\bar{D})$ also will be liminal .

The next result is a slight generalization of a result in [16].

**Corollary (1.1.7)[1]:** If $K \times_\varrho M$ is simple then so are all $M \otimes B(X(D))^{\varrho \otimes adD}$. in particular $M^\varrho$ is simple.

**Proof :** if $K \times_\varrho M$ is simple , then $B(D)$ also must be simple beign morita equivalent to the ideal $A(D)^-$ in $K \times_\varrho M$ .

**Corollary (1.1.8)[1]:** If $K$ is a compact abelian group $K \times_\varrho M$ is liminal (postliminal) if and Only if $M^\varrho$ is liminal (postliminal) .

Our algebras $B(D)$ should be considered as analogues of the algebras $L^0(d)$ of spherical functions defined in [7]. The following should therefore come.

**Proposition(1.1.9)[1] :** if dim $S \leq n$ for every irreducible representation $S$ of $B(D) = M \otimes B(X(\bar{D}))^{e\otimes ad\bar{D}}$ and $T$ is an irreducible * - representation of $K \times_{\varrho} M$ , then $D$ occurs at most $n$ times in the restriction of $T$ to $K$ .

**Proof:** Let $T$ be an irreducible *- representation of $K \times_{\varrho} M$ and let $X_0$ be a non – zero subspace of $X(T)$ invariant under the restriction of $T$ to $K$ and such that

$$\int \psi_D(X) T_X \, \alpha dx = \alpha \text{ for } \alpha \in X_0$$

Then the indused representation $S$ of $B(D)$ corresponding to $T$

$$\langle a \otimes \xi \otimes \alpha, b \otimes \eta \otimes \beta \rangle = \langle T\, [b \otimes \eta, a \otimes \xi]_A \, \alpha, \beta \rangle$$

$$= \int \langle \eta, D_x \xi \rangle b_{\varrho x} \, (a^*) \langle T_x \alpha, \beta \rangle \, dx \quad for \ \ a, b \in \bar{L} \otimes XM, \xi, \eta \in X(D), \alpha, \beta \in X(T). \quad (9)$$

Take $X(S)$ to be the separated completion of with this inner product. $S$ is then defined by

$$S_b(a \otimes \xi \otimes \alpha) = (a \otimes \xi) b * \otimes \alpha \text{ for } b \in B(D)$$

See [6]

Let $Y_0$ be the closed linear span in $X(S)$ of $\{r \otimes \alpha | r \in \bar{L}, \alpha \in X_0\}$ .Then $Y_0 \neq \{0\}$ , because the expression (9) always is 0 we will have

$$\int \langle \eta, D_x, \xi \rangle \langle T_x, \alpha, \beta \rangle \, dx = 0 \ ,$$

so

$$\langle \alpha, \beta \rangle = \int \psi_D \, (x) \langle T_x \alpha, \beta \rangle dx = 0 \text{ for all } \alpha, \beta \in X_0 \ ,$$

acontradiction .

Furthermore, if $X_1 \perp X_0$is another subspace of $X(T)$ with the same properties as $X_0$, define $Y_1$ to be the corresponding subspace of $X(S)$ .(9) then shows that $Y_0 \perp Y_1$,so$D$ can occur at most dim $X(S)$ times in the restriction of $T$ to $K$.

**Proposition (1.1.10)[1]:** Suppose$(M, \varrho, K)$ is $\alpha$ covariant system with $M$ $a$ von Neumann algebra and $\varrho$ $\sigma$- weakly continuous . If each $D \in \bar{K}$ occurs only finitely many times in 0 , then $K \times_{\varrho} M$ $(W^* - crossedproduct)$ is $a$ type 1 von Neumann algebra .

**Proof:** suppose $M \subseteq B(H)$ and let

$$M_0 = \{a \in M | x \longrightarrow \varrho_x(a) is \ norm \ continuous \} \ .$$

This is a $\sigma -$ weakly dense $C^*$ -subalgebra of $M$ so $(M_0, \varrho, K)$is a $C^*$ - covariant system . For $D, E \in \hat{G}$ let

$$M(E) = \{a \in M | \int \psi_E \, (x) \varrho_x(a) dx = a\}$$

and

$$B(X(D), E) = \left\{b \in B\big(X(D)\big) \middle| \int \psi_E(x) \, D_x b D_{x^{-1}} dx = b\right\}.$$

$M(E)$ is by assumption a finite dimensional subspace of $M$ ,and $M(E) \subset M_0$ for a given $D$, $B(X(D), E)$ is non – zero for only finitely many $E$in $\widehat{K}$ .it then follows from the theory of tensor – product representations of compact groups (c.f .[8]) that

$$B(\bar{D}) = M_0 \otimes B(X(D))^{\varrho \otimes adD} \subset \textstyle\sum_{E \in \hat{G}} M(\bar{E}) \otimes B(X(D), E) \ .$$

So $B(\bar{D})$is finite dimensional for all $D \in \widehat{K}$,thus $K \times_{\varrho} M_0$ is a type $IC^*$- algebra .

Since $K \times_{\varrho} M$ is the $\sigma$ - weak closure of $K \times_{\varrho} M_0$ over $L^2(K, H), K \times_{\varrho} M$is type 1.

**Corollary (1.1.11)[1]:** If $(M, \varrho, K)$ is as in proposition(1.1.10) and $\varrho$ is ergodic (i.e$M^{\varrho} = CI$),then $K \times_{\varrho} M$ is type 1 von Neumann algebra .

**Proof:** It is proved in [9] that ergodicity implies finite multiplicity , so the result follows form Proposition (1.1.10) .

**Example (1.1.12)[1]:** M. Takeesaki showed in [18] that if is a uniformly hyper finite $C^*$-algebra there is a compact abelian group $K$ and $\varrho : K \longrightarrow Aut\ A$ such that $K \times_\varrho M$ is liminal. It is not difficult to show that in this case $A^\varrho$ is abelian so Corollary (1.1.8) give a different proof that $K \times_\varrho M$ is liminal .

**Example (1.1.13)[1]:** let $H$ be Mautners 5- dimensional non- type I solvable lie group (for details , see [17],)Ten one can form a semi- direct product $G$ of $H$ and the circle group $T$ exatly as in [17], except that we use $T$ instead of the real numbers . The action of $T$ on $H$ induces $\varrho$ on $C^*(H)$ such that

$$C^*(G) \cong T \times_\varrho C^*(H)\ .$$

Also in this case one can show that $C^*(H)^\varrho$is liminal so by Corollary (1.1.8) $C^*(G)$ (thus $G$) is liminal .

Corollary (1.1.8) tells us that there is a close correspondence between the representation theories of $K \times_\varrho M$ and $M^\varrho$ when $K$is abelian . They are however not Morita equivalent (e.g. take $M = C$), but $A$. Kishimoto and . Takai has shown in [10] that under certain assumptions $K \times_\varrho M \cong M^\varrho \otimes CC(L^2(K))$ so $K \times_\varrho M$and $M^\varrho$ are Morita equivalent.

## Section (1.2): $C^*$ Algebras and Compact Subgroups

For $(A, G, a)$ be a $C$*-dynamical system with A a C*-algebra, G a locally compact group, and $\alpha: G \longrightarrow Aut(A)$ a homomorphism such that the mapping $g \longrightarrow \alpha_g\ (a)$ is continuous from $G$ to A for every $a \in A$. We denote by $A^x$ the fixed point algebra of $\alpha$.

Let also $K \in G$ be a compact subgroup. We shall give necessary and sufficient conditions in order that the crossed product $G \times_\alpha A$ be simple or prime. Our conditions are in terms of the dual $k$of the compact subgroup $K$.

We summarize some definitions and results about continuous Banach representations of compact groups. We contain the definitions of the subspaces of spherical functions $S_{\pi_1.\pi_2}(\pi_{1\,,}.\pi_{2\,,} \in G)$ those of the algebras $S_\pi\,, I_\pi$ and the relations between these subspaces.

Contains a study of saturated actions of $G$ on $A$ in terms of the ideals $S_{\pi 1.} * S_{I\pi.}$ We mention that for compact groups the notion of saturated action has been defined by Rieffel (see [36]). We contain conditions in order that the crossed product be simple. For compact abelian groups G, our result reduces to the following: $G \times_\alpha A$ is simple if and only if $(a)\ Sp(\alpha) = \hat{G}$ (here $Sp(x)$ stands for the Arveson spectrum of a) and $(b) A^x$.

It is then shown that similar results hold for prime C*-crossed products.

Let $K$ be a compact group. We shall denote by $\widehat{K}$ the dual of $K, i.e.$, the set of all unitary equivalence classes of irreducible representations of $K$. Fo each $\pi \in \widehat{K}$ we denote also by $\pi$ a representative of that class and by $H_\pi$ the (finite-dimensional) Hilbert space of $\pi$. We let $i$ be the trivial one-dimensional representation of $K$.

If $\pi \in \widehat{K}\ let\ X_\pi$ be its normalized character $X_\pi(K) = d(\pi\,)tr(\pi_k^{-1})$, where $d(\pi\,)$ is the dimension of $H_\pi$ Let $B$ be a Banach space and $\beta$ a continuous representation of $K\ on\ B$. Associated with a $\pi \in \widehat{K}$ are the following continuous operators on $B$:

$$P^\beta(\pi\,): a \longrightarrow \int_k X_\pi\,(K)\beta_k(a)\; d$$

and

$P_{ij}^\beta(\pi\,): a \longrightarrow \int_k \overline{\pi_{ji}}\,(K)\beta_k(a)\; dk\ 1 \le i,j \le d(\pi),$

where $[\pi_{ij}(k)]$ is the matrix of $\pi_k$ in a fixed orthonormal basis of $H_\pi$

For standard properties of these operators we refer the reader to [29], [8]. Here we note some of them for further use:

**Remark (1.2.1)[19]:**(i) $P^\beta(\pi)P_{ij}^\beta(\pi) = P_{ij}^\beta(\pi)P^\beta(\pi) = P_{ij}^\beta(\pi)$.

(ii) If $B$ is a $C^*$-algebra and $\beta_k$ are $C^*$-automorphisms then $P^\beta\bar{\pi}(B) = [P^\beta(\pi)]^*$; where $\bar{\pi}$ is the conjugate representation of $\pi$

(iii) $\beta_k\left(P_{IJ}(\pi)(a)\right) = \sum_i \pi_{ij}(K)P_{ij}^\beta(a)$.

If we denote by $[P_{ij}^\beta(\pi)(a)]$ the matrix in $B \otimes B(H_\pi)$ with entries $P_{ij}^\beta(\pi):(a)$, then (iii) may be written:

(iv) $(\beta_k \otimes i)\left(\left[P_{ij}^\beta(\pi)(a)\right]\right) = \left[P_{ij}^\beta(\pi)(a)\right].(I_B \otimes \pi_K)$.

We now make the following notations:

$$B_1(\pi) = \left(a \in B | P^\beta(\pi)(a) = a\right\}$$

and

$$B_2(\pi) = \{[a_{ij}] \in B \otimes B(H_\pi) | (\beta_K \otimes i)([a_{ij}]) = [a_{ij}].(I_B \otimes \pi_K).k \in K\}$$

Clearly $B_1(\pi) \subset B \; and \; B_2(\pi) \subset B \otimes B(H_\pi)\pi \in \widehat{K}$. Using the above Remarks we also have

$$B_1(\pi) = \{P^\beta(\pi)(a) | \; a \in B\}$$

and

$$B_2(\pi) = \left\{[a_{ij}] \in B \otimes B(H_\pi) | \beta_K(a_{ij}) = \sum_i \pi_{ij}(K)a_{ij}\right\}.$$

By Remark (iii) above $[P_{ij}^\beta:(\pi)(a)] \in B_2(\pi)$ for every $a \in B, \pi \in \widehat{K}$ The next lemma shows that every element of $B_2(\pi)$ is of this form.

**Lemma (1.2.2)[19]:** $B_2(\pi) = \left\{\left[P_{ij}^\beta(\pi)(a)\;\right] | a \in B\right\}$

**Proof.** The above discussion shows that $\left\{\left[P_{ij}^\beta(\pi)(a)\;\right] | a \in B\right\} \subset B_2(\pi)$ Conversely let $[a_{ij}] \in B_2(\pi)$ . Denote $\sum_i a_{ij}$Then a standard calculation using orthogonality relations shows that$P_{ij}^\beta(\pi)(a) = a_{ij}$ .

The proof of the following lemma is similar to the one given in [8] for Hilbert spaces

**Lemma (1.2.3)[19]**: $\sum_{\pi \in \widehat{K}} B_1(\pi) \; is \; dense \; in \; B$.

**Lemma (1.2.4)[19]**: Let $a \in B$ and suppose that $P^\beta(\pi)(a) = 0$ for all $\pi \in \widehat{K}$ Then $a = 0$.

II. Spaces of Spherical Functions inside the Crossed Product Let $(A, G, \alpha)$ be a C*-dynamical system. We denote by $C(G)$ the set of all continuous, compactly supported functions from $G$ to $A$. Let $K \subset G$ be a compact subgroup. It is known that $L'(K)$ can be imbedded in the multiplier algebra $M(G \times_\alpha A)$ of the crossed product.

If $\emptyset E \; LL^|(K) and \; Y \in C_C\,(G, A) then$

$$(\emptyset * y)(g) = \int_K \emptyset(K)\alpha_K\left(y(K^{-1}g)\right)dk$$

and

$$( y * \emptyset)(g) = \int_K \emptyset(K)\alpha_K \left(y(K^{-1} g )\right)dk$$

In particular, if $\emptyset = X_n$ then $\emptyset$ is a projection in $M(G \times_\alpha A)$ Let us define the following representation z of K on the (Banach space $C_C (G, A)$:

$$(t_k y)(g) = x_k\left(y(K^{-1} g)\right).$$

Then $t$ extends to a continuous representation of $K$ on (the Banach space) $G \times_\alpha A$).

**Lemma (1.2.5)[19]**: (i) $p^t(\pi)f = x_\pi * f$ for $\pi \in \widehat{K}, f \in G \times_x A$.

(ii) If $X_\pi * f = 0$ for all $\pi \in \widehat{K}$ then $f = 0$.

(iii) $\sum_{\pi\in\widehat{K}} X_\pi * (G \times A)\, ne\&\, xII\ \ is\ dense\ in G \times_x A$ .

(iv) If $\pi_1 \neq \pi_2,\ in\widehat{K}$, then the projections $X_{\pi 1}$ and $X_{\pi 2}\ of M(G \times_\alpha A)$ are orthogonal.

(v) $\sum_{\pi\in\widehat{K}} X_\pi = 1$ (us projections in $M(G \times_\alpha A)$).

**Proof:** (i) follows from definitions, (ii) is a consequence of (i) and Lemma(1,2,4), (iii) follows from (i) and Lemma(1,2,3)(iv) is an easy consequence of orthogonality relations, and (v) follows from (iv) and (iii).

Let us denote $S_{\pi_1.\pi_2} = X_{\pi 1} * (G \times_x A) * X_{\pi 2}(\pi_1.\pi_2 \in \widehat{K})$. Then $S_{\pi_1.\pi_2}$ are closed subspaces of $G \times_x A$.

If $\pi_1 = \pi_2 = \pi$ we put $S_\pi = S_{\pi\ .\pi}$ S,,. Then $S_\pi$ , is a hereditary C*-subalgebra of $G \times_x A$ (that is, the set of positive elements of $S_\pi$ is a hereditary subcone of the cone of positive elements in $G \times_x A$ ; see [35] for definitions and properties of hereditary subalgebras),

**Lemma (1.2.6)[19]:** (i) $If \pi_1 ,, \pi_2, \pi_3, \pi_4 \in \widehat{K}\ \ and \pi_2 \neq \pi_3\ then$

$$S_{\pi_1.\pi_2} * S_{\pi_3,\pi_4} = 0.$$

(ii) $S_{\pi_1.\pi_2} * S_{\pi_2.\pi_3} = S_{\pi_1.\pi_3}$.

(iii) $(S_{\pi_1.\pi_2})^* = S_{\pi_2.\pi_1}$.

(iv) $S_{\pi_1.\pi_2} * S_{\pi_2.\pi_1}\ is\ a\ two-sided\ ideal\ of\ S_{\pi_1}$ .

(v) Assume that $G = K$ is compact. For each $a \in A$ denote by$\tilde{a} \in C$ , $(G, A)$ the constant function $\tilde{a}(g) = a$. In this case $S_{\pi 1} = \{\tilde{a} | a \in A_1(\pi).\}$

Proof: Part (i) follows from Lemma (1.2.5)(iv). Parts (ii), (iii), (iv), and (v) from definitions.

We shall assume for simplicity that the group $G$ is unimodular. This assumption is not essential but we make it in order to simplify our computations.

We now define the algebra I generated by the $K$-central functions in $G \times_x A. A$ function $y \in C,. (G, A)$ is called K-central if

$$x_k\left(y(K^{-1} gK )\right) = Y( g ) for\ all\ k \in K, g \in G$$

Then $I$ is a $C$*-subalgebra of $G \times_x A$. We make the following notations:

$$I_\pi = I \cap S_\pi$$

$$\mathrm{I}(\pi) = \mathrm{C(K)} * \mathrm{X}_\pi$$

**Proposition (1.2.7)[19]:** The bilinear map $(f, y) \longrightarrow f * y$ from $I(\pi) \times I_\pi$ into $G \times_x A$ when l$ted to a linear map of $I(\pi) \times I_\pi$ into $G \times A$ establishes a *-algebra isomorphism of the tensor product $I(\pi) \times I_\pi$ with $S_\pi$

For the theory of imprimitivity bimodules and Morita equivalence of $C$*-algebras we refer the reader to [14], [37].

We note the following corollary of the above proposition

**Corollary (1.2.8)[19]:** $I_\pi$ is strongly Morita equivaIent with $S_\pi$Therefore
(i) $I_\pi$ is simple if and only if- $S_\pi$, is simple,
(ii) $I_\pi$ is prime if and ordy if $S_\pi$ is prime,
(iii) $I_\pi$is type I if and only if $S_\pi$ is type $I$.
**Proof**: $I_\pi$ is strongly Morita equivalent with $S_\pi$ by Proposition(1.2.7) (i) and (ii) follows from and (iii) We consider the following map $P$ from $C,(G,A)$ into itself:

$$(Py)(g) = \int_k \alpha_k(y(\ K^{-1}gK)\, dk. \tag{10}$$

It is then easily seen that $P(C,(G,A)) \subset I$ and $P$ can be extended to a projection of $G \times_x, A$ onto $I$. Another fact that will be used is the following observation: $If\ f \in I\ and\ X_\pi * f = f$, then also $f = f * X_\pi$ Thus, the map $f \longrightarrow X_\pi * f$ is a projection of $I$onto$I_\pi$

Let us now specialize the above notions to the case when $G = K$. The following result is due to Landstad.

**Lemma (1.2.9)[19]:** Define a linear map $\emptyset\ : C,(G,A) \longrightarrow A \otimes B(H_\pi)$ by $\emptyset(f) = \int F\,(g^{-1}`) \otimes \overline{\pi_g}\ dg$. Then $\emptyset$ is $a\ *$ isomorphism of $I_\pi$ onto the fixed point algebra $A \otimes B(H_\pi\ )^{\alpha\otimes ad\overline{\pi}})$ The inverse of this isomorphism is the map $\psi$ from $A \otimes B(H_{\overline{\pi}})^{\alpha\otimes Ad\pi}$ defined by

$$\psi(b)(g) = d(\overline{\pi})tr(b(1 \otimes \overline{\pi}(g)).$$

From the above lemma and its proof it follows that $I_\pi$ consists only of continuous functions.

**Lemma (1.2.10)[19]:**$P\overline{\left(S_{\pi,\iota},* S_{\iota,\pi}\right)} = \psi\overline{(A_2(\overline{\pi}) * A_2(\overline{\pi})\ )}$.

**Proof**: By Lemma (1.2.6)(v), $S_{\pi,i}$ is the closure of the set of all $\tilde{\alpha}$ with $a \in A_1(\pi)$. By Lemma (1.2.6)(iii), $S_{i,\pi}$ is the closure of all $(\tilde{b})^*$with $b \in A_1(\pi)$ . Since, by definition of the involution in $G \times_x\ A, (\tilde{b}) * (g) = x_g a,(b^*)$, it follows that $S_{\pi,i} * \overline{S_{\iota,\pi}}$ is generated by all functions of the form

$$\tilde{a} * \ (\widetilde{b})^*(g) = a x_g(\ b^*), \qquad a,b \in A_1(\pi).$$

We have

$$P(\tilde{a} * \ (\widetilde{b})^*(g) = \int_K \alpha_k\,(\tilde{a} * \ (\widetilde{b})^*\ (K^{-1}gK)dk = \int_K \alpha_k\,(\alpha\alpha_k^{-1}gK(\ b^*)dk$$
$$= \int_K \alpha_k\,(a)\alpha_{gk}(\ b^*)dk.$$

We now use the fact that $a,b \in A_1\ A(\pi)$, Therefore

$$a = P^X(\overline{\pi})(a) = \sum\nolimits_i P_{ij}^x\,(\pi)(a)$$

and

$$b^* = p^x(\overline{\pi})(b^*) = \sum\nolimits_j P_{ij}^x\,(\overline{\pi})(b^*).$$

For simplicity, during this proof we shall denote $P_{ij}^x(\pi)(a) = a_{ij}$ and $P_{ij}^x(\overline{\pi})(b^*) = b_{ij}^*$. With these notations, we have

$$p\left(\tilde{a} * \tilde{b}\,(g)\right) = \int x_k\,(a)\alpha_g(b^*)\,dk = \sum_i \sum_j \int \alpha_k\left(a_{ij}\right)\alpha_{gk}\left(b_{ij}^*\right)dk.$$

Using Remark (1.2.1)(iii) we further have

$$p\left(\tilde{a} * \widetilde{(b)}^* (g)\right) = \sum_{i,j,r,i} (\int \pi_{ri} (k) \bar{\pi}_{ij} (gk)\, dk)\, a_{ir} b_{ji}^*$$

$$= \sum_{i,j,r,i} (\bar{\pi})_{is}(g) \left( \int \pi_{ri}(K)\, (\bar{\pi})_{si}(K) dK \right) a_{ir} b_{ji}^*$$

Taking into account the orthogonality relations we have

$$p\left(\tilde{a} * \widetilde{(b)}^* (g)\right) = \frac{1}{d(\pi)} \sum_{i,j,s} \bar{\pi}_{is}(g)\, a_{is} b_{ij}^*$$

$$= \frac{1}{d(\pi)} \sum_{i,j,s} a_{is}(g) \left( \sum_{i,j,s} \bar{\pi}_{is}(g)\, b_{ij}^* \right) = \frac{1}{d(\pi)} \sum_{i,s} a_{is} a_j (b_{ij}^*)$$

$$= \frac{1}{d(\pi)} \sum_{s,i} a_{is} a_j (b_{ij}^*) = \frac{1}{d(\pi)} tr([C_{ij}]\, [(b_{ij}^*](I \times \bar{\pi}_g)$$

(by Remark (1.2.1)(iv)) $= \frac{1}{d(\pi)^2} \psi([C_{ij}][(b_{ij}^*])$.

Where $C_{ij} = a_{ji}$ and the matrix $[C_{ij}]$ belongs to $A_2(\bar{\pi})$ and $[b_{ij}^*]$ belongs to $A_2(\bar{\pi})$ Therefore $P(S_{\pi,i} * \overline{S_{\iota,\pi}}) \subset \psi(A_2 ( \overline{(\pi * A_2(\bar{\pi})})$. The reverse inclusion is obtained by reversing the arguments.

Let $(A, G, a)$ be a $C$ *-dynamical system. There are several approaches to questions concerning the ideal structure of the crossed products $G \times_\alpha A$.

We shall review them following [27].

The first involves the Effros-Hahn conjecture for amenable $G$ [28]. The second, of abelian groups of automorphisms, involves the Connes spectrum [33], [34].

The third involves the notion of outer automorphism groups [26] and the fourth the notion of partly inner automorphism [38]. In particular, Olesen and Pedersen [33] proved that if G is discrete, then $G \times_\alpha A$ is simple if and only if (a) the Cannes spectrum $\Gamma(\alpha) = \hat{G}$ and $(b)$ $A$ is G-simple (that is, A contains no G-invariant two-sided ideals).

This result is false if $G$ is not discrete. Bratteli [3] has given a counterexample for $G = T$ (the circle group) and $A$ the $UHF$ algebra of type $2^\infty$.In an appendix to Bratteli's, Rosenberg [16] has "explained" somewhat Bratteli's counterexample. He proved that if $G$ is compact and $G \times_\alpha A$ is simple then the fixed point algebra $A^\alpha$ shows that if $G$ is compact abelian then $G \times_\alpha A$ is simple if and only if $(a)$ the Arveson spectrum $Sp(\alpha) = \hat{G}$ *and* $(b) A_\alpha$ is simple.

For a different approach to the above problem for G compact we send to [36]. We shall consider a compact subgroup $K \subset G$ and give conditions for simplicity $G \times_\alpha A$ in terms of the subspaces $S_{\pi I, \pi 2}$ introduced above.

First, we shall exhibit a left$-S_{\pi I,}$-right-$G \times_\alpha A$ bimodule $X$ which in some cases will be imprimitivity bimodule in the sense of [14].

Let $X = (G \times_\alpha A) * X_i$. Then, the convolution turns $X$ into a left-$S_i$-right- $G \times_\alpha A$ bimodule. We define the following S,-valued (respectively $G \times_\alpha A$ valued) inner products on $X$:

$$\langle x, y \rangle s_i = y^* * x$$
$$(respectively \langle x, y \rangle_{G x_2 A} = x * y^*).$$

It is then a matter of standard computations in $(G \times_{\alpha} A)$ that $X$ is aleft $S_i$- . Thus $X$ may fail to be an $S_i$- $G \times_{\alpha} A$ imprimitivity bimodule in the sense ofonly in that the range of the $G \times_{\alpha} A$ valued inner product need not be dense.

This can be fixed by making the following definition, which for the case $G = K$ has been given by Rieffel.

**Definition(1.2.11)[19]:** Let $(A, G, a)$ and $K$ be as above. We say that a is $K$-saturated if $X$ is an $S_i - G \times_{\alpha} A$ imprimitivity bimodule. By reformulating the denseness of the range of the $G \times_{\alpha} A$-valued inner product we obtain:

**Lemma (1.2.12)[19]:** The action $x(is\ K -$saturated if and only if the two-sided ideal generated by $S_i$ coincides with $G\ x, A$ (i.$e., S, is\ , full\ in\ G \times_{\alpha} A$

**Proof**: Plainly the range of the $G \times_{\alpha} A$ -valued inner product is the closed two-sided ideal$(G \times_{\alpha} A) * (G \times_{\alpha} A)$ generated by $X_i$ and this last ideal coincides with the closed two-sided ideal generated by $S_i$ Therefore $(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A) = G \times_{\alpha} A$ if and only if $S_i$ is full.

**Theorem(1.2.13)[19]:** Let $(A, G, a)$ be a $C$*-dynamical system and $K \subset G\ a$ compact subgroup. Then $\alpha\ is\ K$-saturated if and only if $S_{\pi,i} * \overline{S_{\iota,\pi}}$for all $\pi \in \widehat{K}$ .

**Proof**: Assume that $\alpha$ is saturated and let $\pi \in \widehat{K}$. ByLemma(1.2.12), $(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A) = G \times_{\alpha}$.By multiplying the above equality on both sides with $X_{\pi}$ we obtain

$$\overline{X_{\pi} * (G \times_{\alpha} A) * X_i * (G \times_{\alpha} A) * X_{\pi}} = X_{\pi} * (G \times_{\alpha} A)X_{\pi}.$$

Hence $\overline{S_{\pi,\iota} * S_{\iota,\pi}} = S_{\pi}$. Conversely assume that $S_{\pi,i} * \overline{S_{\iota,\pi}} = S_{\pi}$ for all $\pi \in$ K.

We shall prove that the ideal $(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A)$ (is dense in $G \times_{\alpha} A$. By Lemma (1.2.14)(iii), $\sum_{\pi \in \widehat{K}} X_{\pi} * (G \times A)$ is dense in $G\alpha A$. Therefore, in order to prove that $(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A)$ (is dense, it is suflicient to every element in), $\sum_{\pi \in \widehat{K}} X_{\pi} * (G \times_{\alpha} A)$Ccan be approximated (in norm) by elements of$(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A)$.

Let $\pi \in \widehat{K}$ and $f \in G \times A$ such that $f = X_{\pi} * f \neq 0$. Since $S_{\pi,i} * S_{i,\pi}$ is a dense ideal of $S_{\pi}$ then there exists an approximate identity $(e_{\lambda}\ of\ S_{\pi}$ contained in that ideal. We claim that (norm) $lim\ e_{\lambda} f = f$. Indeed

$$\begin{aligned} \|e_{\lambda} * f - f\|^2 &= \|(e_{\lambda} * f - f) * f - f) * (f^* * e_{\lambda} - f^*)\| \\ &= \|(e_{\lambda} * f * f^* * e_{\lambda} * ff^* - ff^* * e_{\lambda} + f * f^*\| . \end{aligned}$$

Taking into account that $f * f^* \in S_{\pi}$ and $(e_{\lambda})$ is an approximate identity of $S_{\pi}$ the claim follows. Since $e_{\lambda} \in S_{\pi,i} * S_{i,\pi}$ it immediately follows that $e_{\lambda} * f \in (G \times_{\alpha} A) * X_i * (G \times_{\alpha} A)$. So this ideal is dense and $xis\ K$-saturated.

We can now prove the result about simplicity of crossed products.

**Theorem(1.2.14)[19]:** *Let* $(A, G, \alpha)$ be a $C$*-dynamical system and $K \subset G$ compact subgroup. Then the following conditions are equivalent:

(i) $G \times_{\alpha} A$ is simple and

(ii) $(a)$ $S_{\pi,i} \neq (0)$ f or all $\pi \in \widehat{K}$ and $(b)$ $I_{\pi}$ is simple for all $\pi \in \widehat{K}$ (equivalently, $S_{\pi}$ is simple for all $\pi \in \widehat{K}$).

In this case $G \times_{\alpha} A$ is strongly Morita equivalent with $S_{\pi,} \pi \in \widehat{K}$.

**Proof**: Assume (i). Suppose by contradiction that $S_{\pi,i} = (0)$ for some $\pi \in \widehat{K}$. Then $S_{\pi,i} = X_{\pi} * (\overline{G \times_{\alpha} A) * X_{\iota}} = (0)$. From this, it follows that the ideal $(G \times_{\alpha} A) * X_i * (G \times_{\alpha} A)$. is proper and therefore $(G \times_{\alpha} A)$ is not simple. Hence (i) implies (ii)(a).

We now prove (ii)(b). Since $G \times_{\alpha} A$ is simple, then by every hereditary subalgebra of $G \times_{\alpha} A$, in particular $S_{\pi}$ is simple. By Corollary (1.2.8)(i) this is equivalent with $I_{\pi}$ being simple.

Conversely assume (ii). Then because $S_{\pi,}$ are simple and $S_{\pi,i} * S_{i,\pi}$ are two-sided ideals of S,, it follows that $\overline{S_{\pi,\iota} * S_{\iota,\pi}} = S_\pi$ By Theorem (1.2.13) $G \times_\alpha A$ is strongly Morita equivalent with $S_{\pi,i}$ which is simple b hypothesis. Applying it follows that $G \times_\alpha A$ is simple.
The last part of the theorem follows from the (easily verified) fact that $S_{\pi,i}$ is an $S_\pi$ - $S_i$ imprimitivity bimodule.

We shall specialize the above notions and give equivalent formulations of the preceding results in the case $G = K$. In this case, $f \in C(G, A), then\ (f * X_i)(g) = \int_k F(k)\, dk = a \in A$ (for some $\tilde{a}$)Therefore $f * X_i = a$ for some $a \in A^k$. It follows that the bimodule $X$ is the closure of$\{ci\ (\tilde{a} \in A >$. In the notation of [36], $X = \bar{A}$ On the other hand, in this case ($G$ compact) $S_i$ may be identified with $A^\alpha$ [16]. It is then easy to see that the above left-$S_i$ -right- $G \times_\alpha A$ bimodule structure of $X$ coincides with that defined by Rieffel .

**Corollary(1.2.115)[19]:**$Let$ $(A, G, a)$ be $a$ $C$*-dynamical system with compact. Then a is saturated tf and only if $\overline{A_2(\pi) * A_2(\pi)} = A \times B(H_\pi)^{\alpha\otimes\alpha d\pi}$ for all $\pi \in \hat{G}$.

**Proof:** From Lemmas (1.2.10)and (1.2.9)it follows that $\overline{A_2(\pi) * A_2(\pi)} = A \times B(H_\pi)^{\alpha\otimes\alpha d\pi}$ and only if $S_{\pi,i} * \overline{S_{\iota,\pi}} = S_\pi$ The corollary follows from Theorem(1.2.15).

If $G$ is compact abelian, then for each $\pi \in \widehat{K}$ we have

$$A_1(\pi) = A_2(\pi) = \{a \in A | \alpha_g(a) = \langle g, \pi\rangle a\}.$$

The following result is due to Rieffel.

**Corollary(1.2.16)[19]:** Let $(A, G, a)$ be a $C$*-dynamical system with $G$ compact abelian. Then a is saturated tf and only if $A\ (\pi) = \overline{A\ (\pi)}$ for all $\pi \in \widehat{K}$. Further, we shall derive consequences of Theorem (1.2.14)for compact groups .

**Corollary(1.2.17)[19]:** Let $(A, G, a)$ be a $C$*-dynamical system with $G$ compact. The following conditions are equivalent

(i) $G \times_\alpha A$ is simple and

(ii) $(a) A_1(\pi) \neq (0) for all \pi \in \widehat{K} (b) A \otimes B(H_\pi)^{\alpha\otimes\alpha d\pi}$ is simple for $\pi \in \widehat{K}$.

**Proof:** From Lemma (1.2.6)(v) it follows that $A_1(\pi) \neq (0))$ if and onlyif $S_1(\pi) \neq (0))$. On the other hand, by Lemma(1.2.9), $A \times B(H_\pi)^{\alpha\otimes\alpha d\pi}$ is *-isomorphic with $I_\pi$ . Therefore $A \otimes B(H_\pi)^{\alpha\otimes\alpha d\pi}$ is simple for all $\pi \in G\ if$ and only if $I_\pi$ is simple for all $\pi \in \hat{G}$. It then follows that the condition (ii) of Corollary (1.2.17) is equivalent with the condition (ii) of Theorem (1.2.14)(for compact $G$) so the corollary follows from Theorem (1.2.14) .

If $G$ is compact abelian then the set $\{ \pi \in G\ |\ A_1(\pi) \neq (0))$ is the Arveson spectrum $sp(\alpha)$ [20], [35]. Therefore:

**Corollary(1.2.18)[19]:** Let $(A, G, a)$ be a $C$*-dynamical system with $G$ compact abelian. The following conditions are equivalent:

(i) $G \times_\alpha A$ is simple and

(ii) $(a)\ sp(\alpha) = \hat{G}\ and\ (b)\ A^\alpha$ is simple .

**proof:** Obviously, if $G$ is abelian we have $A \otimes B(H_\pi)^{\alpha\otimes\alpha d\pi} = A^\alpha$ for all $\pi \in \widehat{K}$. The corollary follows from the preceding one.

If $G$ is compact not necessarily abelian for $\pi = I$, we obviously have $A \otimes B(H_i)^{\alpha\otimes\alpha d\pi} = A^\alpha$. One may ask whether in Corollary (1.2.17)the simplicity of $A^\alpha$ together with the condition (ii)(a) still implies the simplicity of the crossed product. The answer is no as shown by the following.

**Example(1.2.19)[19]:** Let $G = S_3$be the permutation group on three elements.
Then $G$ has order 6 and is generated by two elements $r, s$ satisfying the relations $r^3 3 = e, s^2 = e, rs = sr^2$ . The dual $\hat{G}$ of $G$ consists of (the classes of) three representations $\pi_{1,} \pi_{2,} \pi_3$ $with\ d(\pi_1) = d(\pi_2) = I\ and\ d(\pi_3) = 2$ , where

$$\pi_1(r) = 1 \qquad \pi_1(s) = 1$$

$$\pi_2(r) = 1 \qquad \pi_2(s) = -1$$

$$\pi_3(r) = \begin{bmatrix} -1/2 & \sqrt{3/2} \\ -\sqrt{3/2} & -1/2 \end{bmatrix} \qquad \pi_3(r) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\pi_3$ the only element in $\hat{G}$ with $d(\pi_3) = 2$, we have $\pi_3 = \overline{\pi_3}$ $A$ simple application of shows that $\pi_3 = \pi_1 + \pi_2 + \pi_3$ .

Let $A = M_2$ be the algebra of 2 x 2 matrices and define $\alpha = ad\pi_3$. Then $\alpha$ is ergodic on $A$ so $A^\alpha = C.1$ is simple. Since a is equivalent with $\pi_3 \otimes \overline{\pi_3}$ we have that $A_1(\pi) \neq (0)$ for all $X \in G$. However, $A \otimes B(H_{\pi_3})^{\alpha\otimes\alpha d\pi_3} = (M_2 \otimes M_2)^{\alpha\otimes\alpha d\pi_3\otimes\alpha d\pi_3}$ is not simple because its centre is not trivial.

We consider now the similar question for prime crossed products.
Recall that a $C*$-algebra A is called prime if for all nonzero two-sided ideals $J_1, J_2$ of A, their product is not zero, or equivalently if for every two (positive) elements $a, b$ in the multiplier algebra $M(A) of\ A, we\ have\ a.A.b \neq (0)$.

For actions of an abelian group on a $W*$-algebra $A$, the crossed product is $a$ factor if and only if the action is ergodic on the center of $A$ and th Connes spectrum $T(a)$ equals $G$ [21], [22].

Research into the $A*$ analog of this result was started in [30], [32] with the study of compact abelian actions on prime or simple $C*$-algebras. $A$ result, proved independently in $is\ that\ if$
$A$ is prime and $G$ is compact abelian then $A^\alpha$is prime if and only if $Sp(\alpha) = \Gamma(\alpha)$
One of our results (Corollary(1.2.21)) is that if $G$ is compact abelian then $G \times_\alpha A$ is prime if and only if $A^\alpha$ is prime and $sp(\alpha) = \hat{G}$. This is an alternate characterization of primeness of $G \times_\alpha A$ (for compact abelian $G$) of that given by Olesen and Pedersen [33] for every locally compact abelian $G$.

**Remarks (1.2.20)[19]:** (i) If $B \subset A$ is a hereditary $C*$-subalgebra of $A\ then\ \overline{BAB} = B$.
(ii) If $A$ is a prime $C*$-algebra then every hereditary $C*$-subalgebra $B\ of\ A$ is prime.
(iii) Let $A$ be a $C*$-algebra. If $A$ contains an essential ideal J that is also a prime $C*$-algebra then $A$ is prime.

**Proof:** Part (i) follows from $E$ffros' characterization of hereditary $C*$- subalgebras [25].
(ii) Assume by contradiction that $B$ is not prime. Then there exist two non-zero positive elements $a, b \in B$ such that $aBb = (0)$. By (i) above $aA \subset B$. Therefore $a^2 Ab^2 = a(aAb)\ b \subset aBb = (0)$ which contradicts the fact that $A$ is prime.
(iii) is straightforward.

**Theorem(1.2.21)[19]:** Let $(A, G, \alpha)$ be $a\ C*$-dynamical system and $K \subset G\ a$ compact subgroup. The following conditions are equivalent:
(i) $G \times_\alpha A$ is prime und

(ii) $(a)\ S_{\pi,i} \neq (0)$ for all $\pi \in \widehat{K}$ and $(b) S_\pi$ are prime ,for all $\pi \in \widehat{K}$.

*Prof* : Assume that $G \times_\alpha A$ is prime. Since $X_\pi, X_i$ are non-zero elements of $M(G \times_\alpha A)$ we have $S_{\pi,i} = X_\pi * (G \times_\alpha A) * X_i \neq (0)$. On the other hand, $S_\pi$ are hereditary $C*$-subalgebras of the prime algebra $G \times_\alpha A, so$ by Remark (1.2.19)(ii) they are prime. Conversely, assume (ii), and let $I = (G \times_\alpha A) * X_i * (G \times_\alpha A)$ be the two-sided ideal of $G \times_\alpha Ag$ generated by $X_i$.

We claim that $I$ is an essential ideal of $G \times_\alpha A$. In order to prove the claim it is enough to prove that ,$f * I \neq (0)$ for all $f \in G \times_\alpha A, f \neq 0$. Let $f \in G \times_\alpha A . f \neq O$. By Lemma (1.2.5)(ii) there exists $\pi \in K$ such that ,$f * X_\pi \neq (0)$. The

$$X_\pi * f^* * f * X_\pi \neq O. Now, X_\pi * I * X_\pi = X_\pi * (G \times_\alpha A) * X_\pi * (G \times_\alpha A) * X_\pi = S_{\pi,i} * X_{i,\pi}$$

Therefore $X_\pi * I * X_\pi \neq (0)$ by (ii)(b). On the other hand, $S_{\pi,i} * X_{i,\pi}$ (and hence $X_\pi * I * X_\pi$) is a two-sided ideal of $S_\pi$ by Lemma (1.2.6)(iv). Since $S_\pi$ is prime by our hypothesis (ii)(b), it follows that $X_\pi * f^* * f * X_\pi * I * X_\pi \neq (0)$. From this, it follows that $f * X_\pi * I \neq (0)$ so.$f * I \neq\neq (0)$. Therefore $I$ is an essential ideal .

We now claim that $I$ is a prime $C*$-algebra. Indeed, by $I$ is strongly Morita equivalent with $X_i * (G \times_\alpha A) * X_i$ which is prime by assumption. By [37] $I$ is prime. Therefore $G \times_\alpha A$ contains an essential ideal $I$ which is prime. By Remark (1.2.19)(ii), $G \times_\alpha A$ is prime.

Using arguments similar to the ones used in the proofs of Corollaries (1.2.17)and (1.2.18)we obtain the following consequences of the above theorem:

**Corollary (1.2.22)[19]:** Let $(A, G, \alpha)$ be a $C*$-dynamical system with $G$ compact. Then the following conditions are equivalent:

(i) $G \times_\alpha A$ is prime and$(ii)\ (a) A_i(\pi) \neq (0)$for all $\pi \in \hat{G}$ and $(b) A \otimes B(H_\pi)^{\alpha \otimes \alpha d\pi}$ is prime for all $\pi \in \hat{G}$ .

**Corollary (1.2.23)[19]:** Let $(A, G, \alpha)$ be $a\ P$-dynamical system with $G$ compact abelian. Then the following conditions are equivalent:

(i) $G \times_\alpha A$ is prime and

(ii) $(a)\ Sp(\alpha) = \hat{G}\ and\ (b)\ A^\alpha$ prime.

**Example (1.2.24)[19]:**shows that if G is compact (even finite) not necessarily abelian, the primeness of $A^\alpha$ together with the condition (ii)(a) of does not necessarily imply that $G \times_\alpha A$ is prime.

An easyapplication of the above results shows that the corresponding result for von Neumann algebras also holds.

## Section (1.3): Compact Abelian Group on a $C^*$ - Algebra

For $\alpha$ be an action of a compact abelian group on a separable prime $C*$-algebra $A,$ such that also the fixed point subalgebra, $A^\alpha$, is prime. Several conditions on α are shown to be equivalent, among which are the following:

for each $g\ e\ G$, either $\alpha_g = 1\ or\ \alpha_g$ is properly outer;

there exists a faithful irreducible representation of $A$ which is also irreducible on $A^\alpha$ there exists a faithful irreducible representation of $A$ which is covariant.

An example of a nontrivial action satisfying these conditions is the infinite tensor product action on $M_{2^\infty} = \otimes_{n=1}^\infty M_2$ obtained from a sequence of nontrivial inner actions on $M_2$ each one appearing infinitely often. This example was shown to be, in a certain sense, typical of nontrivial actions satisfying the third condition. This fact is the key to deducing the first two conditions from the third.

The second condition is noteworthy in two respects. First, it involves only the fixed point subalgebra$A^{\alpha} \subseteq A$ , not the action $\alpha$ itself. (This is not evident in the case of the other two conditions.) Second, while a representation verifying the third condition is required to be co variant, a representation verifying the second condition must in fact be as far as possible form begin covariant.

1. In [32], Olesen, Pedersen, and Stormer obtained results concerning the system consisting of a prime $C*$-algebra and a compact abelian group of automorphisms such that the fixed point subalgebra is prime. They showed that if the group is either the circle group or is finite of prime order, then

(i) the only multipliers commuting with the fixed point subalgebra are the scalars, and

(ii) the only automorphism in the group that is determined by a multiplier is the identity.

In addition, assuming that the group is finite but not necessarily of prime order, and that the $C*$-algebra is simple, they showed that the properties (i) and (ii), which need no longer hold, are equivalent. (The nontrivial implication is $(ii) \to (i)$.)

We shall improve these two results substantially. Our methods require that the C*-algebra be separable

We shall formulate properties (i)' and (ii)' (14 and 15 below) which are stronger than (i) and (ii), but reduce to these in the case that the $C*$-algebra is simple. We shall show that the stronger properties still hold if the group is the circle group or is finite of prime order, and that, in any case, they are equivalent. The latter result is new even in the case that the $C*$-algebra is simple, there being no restriction on the compact abelian group. Furthermore, and in fact as part of the proof, we shall show that the properties (i)' and (ii)' are equivalent to a number of other properties (1 to 13 below).

Properties (i)' and (ii)' are stated in terms of the limit multiplier $C*$-algebra, which was used in [53], after a suggestion by G. K. Pedersen, and was considered further by Pedersen in [66]. (See also [32].) Recall that the limit multiplier $C*$-algebra $M^{\infty}(A))$, of a $C*$-algebra $A$ is defined as the inductive limit of the net of multiplier $C*$- algebras of essential closed two-sided ideals of $A$. In this connection, note that if$I \supseteq J$are two such ideals, then $M(I) \subseteq M(J)$and that $if\ I$ and $J$are any two such ideals, then also $I\ \cap$J is such.

Two of the properties (4 and 15) involve the proper outerness of certain automorphisms (either $\hat{\alpha}_y, y \neq 0\ or\ \alpha_g, g \neq 0$). Proper outerness of an automorphism of a $C*$-algebra $A$ was defined in [26] to mean that the restriction to any nonzero invariant closed two-sided ideal is at distance two from any automorphism of that ideal determined by a multiplier. It was shown in [53] and [26] that, at least in the case that $A$ is an $AF$ algebra ($i.e., a$ separable approximately finite-dimensional $C$ *-algebra), the condition for an automorphism of $A$ to be properly outer fails—and, moreover, with respect to an essential ideal—if, and only if, the canonical extension of the automorphism $to\ M^{\infty}(A)$ is inner. Various other reformulations of proper outerness in the case of $AF$ algebras were also given in [53] and [26], and most of these are now known to be valid for any separable $C*$-algebra, as a result of work of Kishimoto in [56] and Brown in [47]—see also [57] and [63] (a complete summary is given in of [63]). The reformulation in terms of $M^{\infty}(A)$ follows from a result of Pedersen in [66]—see we also establish other facts concerning $M^{\infty}(A)$that we shall need.

Two other properties (1 and 12) refer to the action on the algebra of the unitary group of the fixed point subalgebra; the first is topological transitivity of this action, in the sense of [60], and the second is strong topological transitivity, in the sense of [44]. (It is open in general whether these two properties are equivalent.)

Another property (13) is an analogue of Tannaka duality. It is stated for automorphisms of $M^{\infty}(A)$), instead of just for automorphisms of $A$, in order to deduce the other properties from it. (Stated just for $A$, it is already known to follow from the property 1—see [60].)

A more general form of this analogue of Tannaka duality is given, in which some of the automorphisms are allowed to be inner. (In 13, none of the automorphisms can be inner, as follows from 13 $\rightarrow$ 15.)

We show that strong topological transitivity and ergodicity are equivalent notions for an action on a von Neumann algebra. This is used for proving the implication 10 → 12 in **Theorem(1.3.1)[41]:** This result also yields a new proof of the Tannaka duality theorem for von Neumann algebras given in [43].

In the following theorem, $G$ denotes the dual group of $G$, G(α) denotes the Connes spectrum of the action α ([35]), πω denotes the cyclic representation defined by the state $\omega$, and $\int_G^{\otimes} \pi\alpha_g \, dg$ is viewed in the canonical way as a representation on $H_\pi \otimes L^2(G)$, where $H_\pi$ is the Hilbert space of the representation $\pi$.

**Theorem(1.3.2)[41]:** *Let A be a separable $C^*$-algebra, and let α be a faithful action of a compact abelian group $G$ on $A$. Suppose that $G \neq 0$.*

*The following fifteen conditions are equivalent.*

1. $If x, y e A\backslash\{0\}, then x A^{\alpha} y \neq 0$.

*2. Any sub-$C^*$-algebra of $A$ containing $A^{\alpha}$ is prime.*

3. *For any closed subgroup $H of G$ such that $G/H \cong\ T$ or $G/H \cong Z|nZ$ for some $n = 1, 2, \ldots,$ the fixed point algebra $A^{H}$ is prime.*

*4. $A^{\alpha}$ is prime and the dual automorphisms $\hat{\alpha}_y, \gamma \in \hat{G}\backslash\{0\}$, of the crossed product $C^*$-algebra $A \rtimes_{\alpha} G$ are properly outer.*

5. *$A^{\alpha}$ is prime and there exists an $\alpha$-invariant pure state $\omega$ of $A$ such that $\pi_{\omega}$ is faithful.*

6. *$\hat{G}(\alpha) = \hat{G}$ and there exists an en-invariant pure state $\omega$ of $A$ such that $\pi_{\omega}$ is faithful.*

*1 . For any sequence $(\xi_n)$ of finite-dimensional unitary representations of $G$ thereexists an $\alpha$ -invariant sub-$C^*$ -algebra $B$ of $A$ and a closed $\alpha^{**}$ -invariant projection $q$ in the bidual $A^{**}$ of $A$ such that*

(i) q∈ $B^{/}$

(ii) $gAg = Bg$

(iii) $q \in J^{**} \subseteq A^{**}$ *for any nonzero closed two-sided ideal $J$ of $A$,*

(iv) *the $C^*$ -dynamical system $(Bq, G, \alpha^{**} \mid Bq)$ is isomorphic to the product system$\left(\otimes_{n=1}^{\infty} M_{dim \xi_n} G \otimes_{n=1}^{\infty} Ad\xi_n\right)$.*

8. *$B$ and $q$ exist as in 7 in the case that dim $\xi_n = 2$ and $\xi_n = l \oplus \chi_n$ where $(\chi_n)$ is a sequence in $\hat{G}$ in which each element of $G$ appears infinitely many times.*

9. *There exists an $\alpha$-invariant state $\omega$ of $A$ such that $\pi_{\omega}$ is faithful and*

$$\pi_{\omega}(A^{\alpha})' \cap \pi_{\omega}(A^{\alpha})'' = C.$$

*10 .There exists a faithful representation $\pi$ of $A$ such that*

$$\pi(A^{\alpha})' \cap \pi(A)" = C.$$

11. *There exists a faithful irreducible representation* $\pi$ *of* $A$ *such that*

$$\pi(A^{\alpha})'' = \pi(A)".$$

11 '. *There exists a faithful irreducible representation* $\pi$ *of* $A$ *such that*

$$(\left(\int_G^{\otimes} \pi\alpha_g dg)(A)\right)^{"} = \pi(A)^{"} \otimes L^{\infty}(G).$$

12. *For each pair* $(x_1, \dots, x_n)$ *and* $(y_1, \dots, y_n)$ *of finite sequences in* $M^{\infty}(A)$ *such that* $\sum_{i=1}^{n} x_i \otimes y_i \neq 0$ *there exists* $a \in A^{\alpha}$ *such that* $\sum_{i=1}^{n} x_i a y_i \neq 0$ .

13. *A and* $A^{\alpha}$ *are prime, and if* $\beta$ *is an automorphism of in* $M^{\infty}(A)$*) such that* $\beta \mid A^{\alpha} = 1$ *then* $\beta = \alpha_g$ *for some* $g \in G$.

14. $A^{\alpha}$ *is prime, and* $(A^{\alpha})' \cap M^{\infty}(A) = C$.

15. *A and* $A^{\alpha}$ *are prime, and* $\alpha_g$ *is properly outer for each* $g \in G\backslash\{0\}$.

*Furthermore, ifG is the circle group or a finite cyclic group of prime order, then these conditions are equivalent to the following one.*

16. *A and* $A^{\alpha}$ *are prime.*

(If $G = 0$, the equivalence of all the conditions, with the exception of 7 and 8, remains valid, but the theorem then reduces to the well-known fact that a separable $C*$- algebra is primitive if and only if it is prime.)

**Lemma(1.3.3)[41]:** *Let G be a compact abelian group, let α be a faithful action of G on a* $C*$*-algebra A, and let H be a closed subgroup of* $G$. *The following two conditions are equivalent:*

1. $A^H$ *is prime.*

2. $(A \rtimes_{\alpha} G) \rtimes_{\alpha} H^{\perp}$ *is prime.*

**Proof:** Note that $(A \rtimes_{\alpha} G) \rtimes_{\alpha} H^{\perp}$ is the fixed point subalgebra *of* $(A \rtimes_{\alpha} G) \rtimes_{\alpha} G$ under $\alpha_H^{\wedge}$. By [32], the system *((A XΛ* G) xά G, α) is isomorphic to the system $(A \rtimes_{\alpha} K(L^2(G)), \alpha(x)A)$, where $K(L^2(G))$ denotes the algebra of compact operators on $L^2(G)$, and $\lambda$ is the representation of $G$ *on* $K(L^2(G))$ determined by the left regular representation. Since $G$ is compact, we have a canonical system of matrix units

$$(e_{xx'})_{x,y' \in G} \; for \; K\big(L^2(G)\big), with \; e_{xx'} \in K^{\lambda}(\chi - \chi')9 \; i.e. \alpha_g(e_{xx'})$$
$$= \langle \chi - \chi', g \rangle e_{xx'} \; Clearly$$
$$(1 \otimes e_{00})(A \otimes K)^H (1 \otimes e_{00}) = A^H \otimes e_{00} .$$

Denote by / the closed two-sided ideal of $(A \otimes K)^H$ generated by $A^H \otimes e_{00}$ . By subalgebra $A^H \otimes e_{00}$ have the same spectrum,and, in particular, one is prime if and only if the other is.

The implication 2 → 1 is now immediate: if $(A \otimes K)^H$ is prime, then / is prime and hence $A^H$ is prime.

To prove the implication 1 → 2, moreover, it is now enough to show that if $A^H$ is prime, then the ideal /of $(A \otimes K)^H$ is essential. Suppose that $A^H$ is prime, and denote the largest closed two-sided ideal of $(A \otimes K)^H$ orthogonal to $A^H \otimes e_{00}$ by $J$. We must show that $J = 0$.

Since $G$ is abelian, both $(A \otimes K)^H$ and $A^H \otimes e_{00}$ are invariant under $\alpha \otimes ƛ$, and therefore $J$ is invariant. Hence (as $G$ is compact), if there exists a nonzero element in $J$ then there exists one of the form $a \otimes e_{xx'} > \; with a \in A^{\alpha}\Lambda(\chi' - \chi)$. below, $A^{\alpha}*$ contains an approximate unit for $A$. The tensor product of this with finite sums $\Sigma e_{\chi\chi}$ is an approximate unit for $A \otimes K$, invariant under $\alpha \otimes ƛ$. Hence, if $0 \neq x \in J^{\alpha \otimes ƛ}$, then, after multiplication on the left by an element of $A^{\alpha} \otimes e_{\chi\chi}$ for some $\chi$, and similarly on the right, $x$ has the desired form.) Fix $\chi, \chi' \in \hat{G}$ and $a \in A^{\alpha}\chi' - \chi)$ with $a \otimes e_{\chi\chi'} \in J$, and let us show that $a = 0$. Since $A^{\alpha}(\chi + H^{\perp}) \otimes e_{0x} \subseteq (A \otimes K)^{H}$ we have

$$0 = (A^{H} \otimes e_{00})(A^{\alpha}(X + H^{\perp}) \otimes e_{0x}) = A^{H}A^{\alpha}(X + H^{\perp})a \otimes e_{0x}$$

Since $H$ is compact, $A^{H}$ contains an approximate unit for $A$.

$$A^{\alpha}(X + H^{\perp})a = 0\,, \qquad i.e.,$$

$$A^{\alpha}(X + H^{\perp})^{*}A^{\alpha}(X + H^{\perp})aa^{*} = 0.$$

But $A^{\alpha}(\chi + H^{\perp})^{*}A^{\alpha}(\chi + H^{\perp})\}$ is a two-sided ideal of $A^{H}$, and $aa*$ belongs to $A^{H}$ (in fact to
$A^{\alpha}$) Hence, since $A^{H}$ is prime, either$A^{\alpha}(\chi + H^{\perp}) = 0$, or $a = 0$. But by Lemma(1.3.3), below,
with $(A, H, \alpha \mid H)$ and $\chi + H^{\perp} \in \hat{H}$ in place of $(A, G, \alpha)$ and $\chi$, since α is faithful and $A^{H}$ is
prime, $A^{\alpha}(\chi + H^{\perp}) \neq 0$. Therefore α = 0, as desired.

**Lemma (1.3.4)[41]:** *Let $G$ be a compact abelian group, let α be a faithful action of $G$ on a
$C*$-algebra $A$, and suppose that $A^{\alpha}$ is prime. It follows that $Sp\alpha = \hat{G}$ i.e., for every$X \in \hat{G}, A^{\alpha}(X) \neq 0$.*

**Proof:** First, let us show that Sp $\alpha$ is a subgroup of $\hat{G}$. If $A^{\alpha}(\chi) \neq 0$ and $A^{\alpha}(X') \neq \wedge 0$, Then

$$A^{\alpha}(X - X') \supseteq A^{\alpha}(X)A^{\alpha}(X')^{*}$$

and $A^{\alpha}(\chi)^{*}A^{\alpha}(\chi), A^{\alpha}(\chi')^{*} A^{\alpha}(\chi')$ are nonzero two-sided ideals of the prime algebra $A^{\alpha}$, so
have nonzero product. This shows that$A^{\alpha}(X - X') \neq 0$

Second, as Sp $\alpha$ is a subgroup of $\hat{G}$, we have Sp $\alpha = H^{\perp}$ where H= $(SP\alpha)^{\perp} \subseteq G Hence, in \hat{H},$

$$Sp(\alpha\,|H^{\perp}) = (Sp\,\alpha)|H^{\perp} = H^{\perp}|H^{\perp} = 0\,.$$

In other words, α | $H$ is trivial. Since α is faithful, $H = 0, i.e. Sp\alpha = \hat{G}$.

Returning to the proof of the implication 3 $\longrightarrow$ 4, we now have that $(A \rtimes_{\alpha} G) \rtimes_{\alpha} H^{\perp}$ is prime. Hence by [33] (with $A \rtimes_{\alpha} G$ *in place of* $A, H^{\perp}$ inplace $G$, and $\alpha|H^{\perp}$*inplace of* $\alpha$),

$$(H^{\perp})^{\wedge}\ (\alpha|H^{\perp}) = (H^{\perp})^{\wedge}\ .$$

Since $H^{\perp}$ is the cyclic subgroup of $G$ generated by $y$, it follows, either that άy is properly outer, as desired If $\beta$ is an automorphism of a $C^{*}$-algebra which is not properly outer, then to show that the Connes spectrum $T(\beta)$ (or the Borchers spectrum $TB(\beta)$) is equal to $\{1\}$, it is enough by 1.3 (*or* 2.1) *of* [57] to consider the case $\beta = exp\delta$ where $\delta$ is a derivation. Since $Sp\,\beta" = (Sp\,\beta)"$, we have $T(\beta^{n}) \subseteq T(\beta)^{n}$ (*and* $T_{B}(\beta^{n}) \subseteq T_{B}(\beta^{n})$), and so to prove that $T(\beta) = \{1\}$ we may replace $\delta$ *by* $n^{-1}\delta$ and suppose that $||\beta — 1|| < |e^{\frac{2\pi\iota}{3}} - 1\,|$, so that $Sp\beta$ does not contain any nontrivial subgroup of $T$. But then $T(\beta)$ equals $\{1\}$ because it is a subgroup of $T$([35], 8.8.4; *to get* $T_{B}(\beta) = \{1\}$ *use* [35], 8.8.5). and using compactness of 7, that every derivation is close to zero on some invariant hereditary sub-$C$*-algebra. However, this proof does not seem to give a subalgebra which is invariant under all automorphisms commuting with the derivation, as does that in [63].)

$Ad$ 4 $\longrightarrow$ 5. (We prove this implication by combining ideas from the proof of Theorem (1.3.2)in [56] and the argument on page 161 in [58].) Assume 4. In particular, $A^*$ is prime, and it follows by Lemma(1.2.3), with $G$ in place $of\ H$, that $A \rtimes_\alpha G$ is prime. (This does not use the hypothesis of proper outerness.)

Since $A \rtimes_\alpha G$ is separable, there exists a sequence $(J_n)$ of nonzero closed two-sided ideals of $A \rtimes_\alpha G$ such that every nonzero closed two-sided ideal contains some $J_n$ ([49], 3.3.4). $(If\ (x_n)$ is a dense sequence in the unit sphere of $A \rtimes_\alpha G\ A$, we may take $J_n n$ to be the

smallest closed two-sided ideal of $A \rtimes_\alpha G$ such that $\|x_n + J_n\| \leq {}^1/_2$ for if $J$ is any nonzero closed two-sided ideal there is some $n$ such that $||x_n + J|| \leq \|x_n\|/2 = 1/2.)$ Since $A \rtimes_\alpha G$ is prime, we may replace $J_n\ by\ J_1 \cap \ldots \cap J_n$ and suppose that the sequence $(J_n)$ is decreasing.

Denote by $T$ the set of $a \in A \rtimes_\alpha G$ such that $\alpha \leq O, ||a|| = 1$, and there exists $0 \neq b \in A \rtimes_\alpha GA\ with\ ab = b$. By spectral theory, $T$ is not empty, and if $a \in T$ then there exists $b \in T$ such that $ab = b$.

Choose a dense sequence of unitaries $(U_M)$ in $(A \rtimes_\alpha G)^{\sim}$,, the $C^*$-algebra $A \rtimes_\alpha G$ with unit adjoined, and let $(\ \sigma n\ )$ be an enumeration of the automorphisms $(AdU_M)\hat{\alpha}_y y, m = 1,2,\ldots, y \in \hat{G}\backslash\{0\}$.

Construct as follows a sequence $(e_n)\ in\ T$ such that

$$e_n e_{n+1} = e_{n+1}, e_n \in J_n, and\ \|e_n \sigma_n(e_n)\| \leq n^{-1}.$$

Suppose that we have constructed $e_k \in T\ for\ 1 \leq k < n\ such\ that\ e_{k-1}e_k = e_k, e_k \in J_k, and \|e_k(ek)\| \leq K^{-1}\ Choose\ x \in T\ such\ that\ e_{n-1}x = x. (If\ n = l$ , just choose $x \in T.)$

applied to the properly outer automorphism $\sigma_n$ and the hereditary sub- $C^*$ -algebra /„ $J_n \cap (x(A \rtimes_\alpha G).x)^-$, there exists $e_n$ in this subalgebra such that $0 \leq n, \| e_n\| = 1, and\ \|e_n\sigma_n(e_n)\| \leq n^{-1}. Necessarily, e_{n-1}e_n = e_n$ , and modifying $e_n$ slightl using spectral theory ensures that, in addition, $e_n \in T$, as desired.

As $(e_n)$ is a decreasing sequence of positive elements of $A \rtimes_\alpha G$ of norm one, the set of states of $A \rtimes_\alpha G$ with value 1 on $e_n$for all $n$ is a nonempty compact face in the state space of $A \rtimes_\alpha G$. Therefore, there exists a pure state $\varphi_0\ of A \rtimes_\alpha G$ such $that\ \varphi_0(e_n) = 1$ for all $n$. Denote by $\varphi$ the unique $\alpha$-invariant extension of $\varphi_0$ to a state of $(A \rtimes_\alpha G) \rtimes_\alpha \hat{G}$.

We shall show that $\pi_\varphi$ is faithful and that $\varphi$ is pure.

Since $\varphi_0(e_n) = 1\ and\ e_n \in J_n, J_n$ is not contained in Ker $\pi_{\varphi_0}$ for any $n$. Hence Ker $\pi_{\varphi_0}$ = 0. Since Ker $\pi_{\varphi_0}\ is\ \alpha^{\wedge}$-invariant, if it were nonzero its intersection with the fixed point subalgebra $A \rtimes_\alpha G$ would be nonzero, but this intersection is clearly contained in Ker $\pi_{\varphi_0}$ , which is zero. Therefore Ker $\pi_\varphi$ = 0.

Since $\varphi_0$ is pure, to show that $\varphi$ is pure, it suffices to show that $\varphi$ is the unique extension of $\varphi_0$ to a state of $(A \rtimes_\alpha G) \rtimes_\alpha \hat{G}$ Let $\psi$ be a state of $(A \rtimes_\alpha G) \rtimes_\alpha \hat{G}$ such that $\psi|A \rtimes_\alpha G = \phi_0$. To show that $\psi = \phi$ , we must show that $\psi\ is\ \alpha^{\wedge}$-invariant, i.e. that

$\psi(bu(y)) = O$ for any $b \in A \rtimes_\alpha G$ and any $O \neq y \in \hat{G}$. Here $u(y)$ denotes the unitary multiplier of the crossed product by $G$ canonically associated with $\gamma \in \hat{G}$. Since any $C^*$-algebra is spanned linearly by its unitary elements, it is enough to suppose that $b$ is unitary, and then of course $b$ can be approximated by $a$ subsequence of $(u_m)$. In the enumeration, above, of $(Ad\, u_m)\alpha_y^{\wedge}$ as $\sigma_n$, let us denote by $\sigma_{m\,y}$ and $e_{my}$ the $\sigma_n$ and $e_n$ corresponding to $(Adu_m)\alpha_y^{\wedge}$ Thus, for fixed $O \neq y \in G$,

$$e_{m,y}\sigma_{m,y}(e_{m,y}) \longrightarrow 0 \qquad (m \longrightarrow \infty).$$

In other words, with $y \in \hat{G}\backslash\{0\}$ fixed, we have

$$e_{m,y}U_m(y)e_{m,y} = e_{m,y}\sigma_{m,y}(e_{m,y})u_m u(y) \longrightarrow 0.$$

Since $\psi(e_{m,y}) = \phi_0(e_{m,y}) = 1$ for all m, we have
whence $\psi(bu(y)) = 0$. Therefore, $\psi = \phi$. This shows that $\phi$ is pure.

Let us again identify$(A \rtimes_\alpha G) \rtimes_\alpha \hat{G}$ with $A \otimes K(L^2(G))$, and $\alpha^{\wedge}$ with $\alpha \otimes \lambda$. Then, with
$e_{yy}$ as above, in the proof of Lemma(1.3.1), for any $y \in \hat{G}$ the positive functional $\phi_y = (1 \otimes e_{yy})\phi(1 \otimes e_{yy})$, if nonzero, is $a$ scalar multiple of an $\alpha^{\wedge}-$invariant pure state of $A \otimes e_{yy} \leq A \otimes K$, i.e. an $\alpha$-invariant pure state, say $\omega_y, of\ A$. Furthermore, for some $y \in \hat{G}, \phi_y$ is nonzero, and then $\pi_{\phi_y}$ is faithful, since $\pi_\phi$ is. Since $\pi_{\phi_y}(a \otimes e_{yy}) = \pi_{\omega y}(\alpha) \otimes e_{yy}\ it$ follows that, for such $y, \pi_{\omega\gamma}$ is a faithful representation of $A$. This shows that, with $\omega = \omega_\gamma$ for such a $7, \omega$ is an $\alpha$-in variant pure state of $A, and\ \pi_\omega$ is faithful, as desired.

$Ad$ 5 $\longrightarrow$ 6. Condition 5 implies that both $A\ and\ A*$ are prime, whence by 8.10.4 of [35], $\hat{G}(\alpha) = Sp\,\alpha$. In particular, as $G(\alpha)$ is group ([35], 8.8.4), so also $is\ Sp\,\alpha$, and since $\alpha$ is faithful this implies $Sp\,\alpha = \hat{G}$.

$Ad$ 6 $\longrightarrow$ 7. Except for the property (iii), this is exactly Theorem (1.3.1) of [46]. Referring to the proof of that theorem, we ensure that $B$ and $q$ have the extra property as follows.

Since $A$ is separable and prime there exists a decreasing sequence $(J_n)$ of nonzero closed two-sided ideals of $A$ such that any nonzero closed two-sided ideal of $A$ contains some $J_n$ $(see\ proof\ of\ 4 \longrightarrow 5\ above)$. Also, since $A$ is prime and $G$ is compact we have

$$\bigcap_{g \in G} \alpha_g(J) \neq 0$$

for any nonzero closed two-sided idea $J\ of\ A$. (First, by strong continuity of $\alpha$, for any $h \in G$ there is a neighbourhood $U_h\ of\ h\ in\ G\ such\ that\ I_h = \cap_{g\in U_h} \alpha_g(J) \neq 0\ by\ compactness$ of $G$, there are $h_1, \ldots, h_k \in G$ such that $U_{h_1} \cup \ldots \cup U_{hk} = G$, finally, byprimeness of $A, I_{h_1} \cap \ldots \cap I_{h_1} \neq O, i.e.\ \cap\ \alpha_g(J) \neq 0) \cap \alpha_g(J) \neq 0$

$$g \in G$$

It follows that $J \cap A^\alpha \neq 0$
for any nonzero closed two-sided idea $J\ /\ of\ A$. In particular,$(J \cap A^\alpha)$is a decreasing sequence of nonzero closed two-sided ideals of $A^*$.

Now note that the quasimatrix system $(e_n), (v_{n,i})_i^{d_n} = 1$ constructed so that $e_n \in J_n \cap A^\alpha$
$n = 1, 2,$. Then the projection $q_n \in A^{**}$ defined in the proof of Theorem (1.3.1)of [46] is contained $in\ J^{**}$, and hence the limit $q = lim\, q_n$ is contained in $\cap J_n^{**}$. Since any nonzero
closed two-sided ideal $J$ contains some $J_n$ (iii) holds.

$Ad$ 7 + 8. 8 is a special case of 7.

$Ad\ 8+9$. Assume $8$. Denote by $\tau$ the unique tracial state of the $C^*$-algebra $qAq$, which is isomorphic to the $G$limm algebra $M_{2^\infty}$, and denote by $\omega$ the corresponding state of $A$,

$$A \ni a \mapsto \tau(qaq).$$

Since $q\ is\ \alpha^{**}$-invariant, and $\tau$ is unique, $\omega\ is\ \alpha$-in variant.

Since $\omega(q) = l,\ q \notin Ker(\pi_\omega^{**}) \supseteq (Ker\pi_\omega)^{**}$, and hence by $8(iii), Ker\pi_\omega\ =\ 0$.

That $\pi_\omega(A^\alpha)' \cap \pi_\omega(A)"\ =\ C$

$Ad\ 9\ \longrightarrow\ 10$. This is evident.

$Ad\ 8\ \longrightarrow\ 1\ 1$. Assume 8. Let $(\phi_n)$ be any sequence of pure states of $M_2$ such that, for each $\chi \in \hat{G}$, the (infinite) subsequence $(\phi_n)_{\chi_n} = \chi$ contains a subsequence that converges to a nondiagonal pure state, i.e. to a pure state with density matrix not equal to $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$or $\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$I. For instance, $\phi_n$ may be taken to be a fixed pure state with density matrix different from$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ and$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$ (e.g. the pure state with density matrix $\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$

Denote by $\phi$ the pure state *ofqAq* obtained from $\otimes_{n=1}^{\infty} \phi_n$ by identifying $qAq$ with $\otimes_{n=1}^{\infty} M_2$ as in 8, and denote also by $\phi$ the corresponding pure state of $A$,

$$A \in a \longmapsto\ \phi(qaq)\ = \otimes_{n=1}^{\infty} \phi_n(qaq)\,.$$

Since $\phi\ (q) = 1\ q \notin Ker(\pi_\phi^{**}) \supseteq (Ker\pi_\phi)^{**}$, and hence by 8(iii), $Ker\pi_\phi\ =\ 0$.

We shall show that $\pi_\phi(A^\alpha)"\ = \pi_\phi\ (A)"$. We shall show this, or, equivalently, that $\pi_\phi$ $(A^\alpha)'\ =\ C$, in two steps: first, we shall show that $\phi$, the canonical cyclic vector for $\pi_\phi(A)$, is also cyclic for $\pi_\phi(A")$, and thus separating for $\pi_\phi(A^\alpha)'$ and, second, we shall show that

$\pi_\phi\ (A^\alpha)'\phi\ =\ C\phi.$

Let us show that $\phi$ is cyclic for $\pi_\phi(A^\alpha)$To do this, we shall show that, for each$X \in \hat{G}$

$$\phi \in \pi_\phi\big(A^\alpha(X)\big)^{-}\phi,$$

where the bar denotes ultra weak closure. Then, for each $\chi\ \in\ G$,

$\pi_\phi\big(A^\alpha(-\chi)\big)\phi\ \subseteq\ \pi_\phi\big(A^\alpha\ (-X\ )\big)\pi_\phi\big(A^\alpha(X)\big)^{-}\phi, \subseteq \pi_\phi\big(A^\alpha(-\chi)\big)A^\alpha\ (X))^{-}\phi \subseteq \pi_\phi$ $(A^\alpha)^{-}\phi \subseteq (\pi_\phi(A^\alpha)\phi)^{-}$

*where the last bar denotes weak* closure in $H_\phi$, which on a linear subspace is the same as norm closure. Since the closed linear span of $(\cup_{\chi\in G}\ A^\alpha(-\!\!-\ X)$ is equal to $A$ and $\phi$is cyclic for $\pi_\phi(A)$, this shows that is cyclic for $\pi_\phi(A^\alpha)$

Let, then, $\chi$ be an element of $G$ and let us show that $\phi \in \pi_\phi(A^\alpha(\chi))^{-}\phi.(If\ \chi\ =\ 0$ this follows from the fact that $A^\alpha$ contains an approximate unit for $A$ For each $k$ such that $\chi_K\ =\ \chi$ denote by *ck* the image of

$$\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \in M_2$$

let $M_2$ under the K-th embedding of $M_2$ in $qAq\ = \otimes_{n=1}^{\infty} M_2$. Then, by the choice of $\phi_n$

$$\phi C_K = \phi_k \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \nrightarrow 0$$

(Here $k$ is such that $\chi_K =\ \chi$.) Passing to a subsequence of *($C_K ck$)* we may suppose that

$$\phi(C_K) \longrightarrow \lambda \neq 0$$

and then, since $(C_K)$ is a central sequence in $qAq$, and $\pi_\phi(qAq)$ is irreducible on $\pi_\phi(q)H_\phi$,
$\pi_\phi(C_K) \longrightarrow \lambda\pi_\phi(q)$ ultra weakly .
Since $q \in B'$, the map

$$B \ni b \longmapsto b(q) \in Bq = qAq$$

is a morphism, and so we can choose $b_K \in B$ with $b_K q = C_K$ , and $||b_K|| \leq 2$ (even with $\|b_K\| = 1$). Replacing $b_K$ *by* $\int_G \langle \overline{X,q} \rangle\, \alpha_g(b_K)dg$, we may suppose that $b_K \in B^\alpha(\chi)$. If now $b$ is any ultra weak limit point of the sequence $\pi_\phi\ (b_K)$ we have

$$b\pi_\phi(q) = \pi_\phi(q)b = \lambda\pi_\phi(q).$$

And as $\pi_\phi(q)\ \phi = \phi\ wehave$

$$\phi = \lambda^{-1}\phi \in \pi_\phi\big(A^\alpha(X)\big)^{-}\phi,$$

as desired.

Now let us show that $\pi_\phi(A^\alpha)'\phi = C\phi$. Since $q$ is an $\alpha$**-invariant projection in $A^{**}$ and also is closed, we have $q \in (A^\alpha)^{**} \subseteq A^{**}$. ($1 - q$ is the unit of the ultraweak closure in $A^{**}$ of an $\alpha$-invariant hereditary sub- $C*$ -algebra of $A$, and so is the limit of an approximate unit of this subalgebra; this approximate unit may, as remarked above, be chosen to be $\alpha$-invariant.) In particular, $\pi_\phi(q) \in \pi_\phi\ (A^\alpha)'' * y$ . Moreover, for each $\chi \in \hat{G}$, it was shown in the preceding paragraph that $\pi_\phi(q) \in \pi_\phi(A^\alpha((\chi))^-$ (where the bar denotesultraweak closure). Hence, for each$X \in \widehat{G}$.

$$\pi_\phi(A^\alpha(-\chi)q) \subseteq \pi_\phi\big(A^\alpha(-\chi)\big)\pi_\phi(A^\alpha((\chi))^- \subseteq \pi_\phi(A^\alpha(-\chi)A^\alpha\ (\chi))^- \subseteq \pi_\phi(A^\alpha)^-.$$

Since the closed linear span of $\cup_{X\in\hat{G}} A^\alpha(—\chi)$ is equal to $A$, and $\pi_\phi(A)$ is irreducible, this shows that $\pi_\phi(qA^\alpha q)$ is irreducible on $\pi_\phi(q)H_\phi$. Now we have $\pi_\phi(q) \in \pi_\phi(A^\alpha)''$ and
$\pi_\phi(A^\alpha)'(q) = \pi_\phi(qA^\alpha q)'\ \pi_\phi(q) = C\pi_\phi(q)$ .
Since $\pi_\phi(q)\phi = \phi$ it follows that $\pi_\phi(A^\alpha)'\phi = C\phi$ .

*Ad* 11 ⟶ 10. This is evident.

*Ad* 10 ⟶ 12.

**Remark(1.3.5)[41]:** It is interesting to inquire whether the implications 9 ⟶ 1 1 *and* 11 ⟶ 9 (which are now established, since 12 ⟶ 1 is evident) can be proved directly. Certainly, our proof of 8⟶11 yields an alternative proof of 8 ⟶9, since the pure states $\phi$ constructed in the proof of 8 ⟶ 1 1 constitute a direct integral decomposition of the invariant state ω constructed in our proof of 8 ⟶ 9 above. Given a state $\omega$ as in 9, and adirect integral decomposition of $\pi_\omega$ as $\int^\otimes \pi\ (\zeta)d\mu(\zeta)$, must almost every $\pi(\zeta)$ verify 1 1 ?

*Ad* 1 1 ↔ 1 1′ Γ. If $\pi$ is an irreducible representation of $A$, then, as we shall show, $\pi$ verifies 11 if, and only if, it verifies 1 1′ . (See also [59].)
1 1 ↔ 1 1′, for $\pi$,
Assume1 1′. Set $\int_G^\otimes \pi\alpha_g dg = p. p\ is\ \alpha$-co variant, and $\alpha$ is implemented by the right regular representation of $G$ *on* $H_\pi \otimes L^2(G)$ (we do not need here that $G$ is abelian). Since $G$ is compact, $P(A^\alpha)'' = (P(A''))^\alpha = (\pi(A)''\phi L^\infty)(G))^\infty \otimes I$.
But $p(A^\alpha) = \pi(A^\alpha) \otimes I$, so (We have not used here that $\pi$ is irreducible.)

*Ad* 12 ⟶ 13. Assume 12. Let $\beta$ be an automorphism of $M^\infty(A)$ such that $\beta \setminus A^\alpha = 1$.

Let us prove that $\beta = \alpha_g$ for some $g \in G$. below, $M^\infty(A)^\alpha \subseteq M^\infty(A^\alpha)$, and it follows that $\beta \backslash M^\infty(A)^\alpha = I$.
We now note that, except for continuity of $\alpha$ from $G$ into Aut $M^\infty(A)$, all the hypotheses of Theorem (1,3,2) of [44] are fulfilled, with$M^\infty(A)$ in place of $A$, and $(U(M^\infty(A)^\alpha \backslash Ad)$ in place of $(H, \tau)$. The proof of Theorem (1.3.2) of [44], which is valid without continuity of $\alpha$ until the very last line — provided that $M^\infty(A)_F$ is defined as the set of all $x \in M^\infty(A)$ such that the linear span of $\alpha_G(x)$ is finite-dimensional — , yields that, for some $g \in G$,
$\beta(x) = a_g(x)\, for\, all\, x \in M^\infty(A)_F$ .
In particular, this holds for all $x \in A_F$, and since $A_F$ is dense in $A$ (continuity is known for $\alpha : G \longrightarrow AutA$), this shows that $\beta = \alpha_g$. (Here we have not used that $G$ is abelian. A proof in the case that $G$ is abelian can also be obtained by modifying, in $a$ somewhat less trivial way, the proof of Theorem (1.3.3)of [60].)

*Ad* 13 ⟶ 14. Assume 13. Then for each unitary $u \in (A^\alpha)' \cap M^\infty(A)dw = \alpha_g$ for some $g \in G$as $A$ is prime, Centre$M^\infty(A) = C$. By commutativity of $G$, it follows that $\alpha_g(u) \in Tu$ for every $U \in (A^\alpha)' \cap M^\infty(A)$ and every $g \in G$. But, for fixed such $u$, and fixed $g \in G$, it follows from the fact that $\alpha_g(v) \in T_V$ for every unitary $v$ in the $C$*-algebra generated by $u$, that $\alpha_g(u) = u$. Since $g$ is arbitrary, it follows that $u$ is in $M^\infty(A)^\alpha$, which is contained in $M^\infty(A)^\alpha$. But $U \in (A^\alpha)'\, so\, u$ belongs to Centre $M^\infty(A)^\alpha$. as $A^*$is prime, this is equal to C. Since any $C^*$- algebra is spanned linearly by its unitary elements we have

$$(A^\alpha)' \cap M^\infty(A) = C.$$

We should like to point out that if $G$ is not abelian, then the implication 13 ⟶ 14 may fail. For example, it fails if $A = M_n, n = 2, 3, , and\, G = Aut\, A$. However, this may be essentially the only case in which the implication fails.

*Ad* 14 ⟶ 15. Assume 14. Let us first show that $A$ is prime. Clearly, Centre $M^\infty(A) = C$;, this just says that $A$ is prime.

Let $0 \in G$, and suppose that $ag$ is not properly outer there is a unitary $u$ in $M^\infty(A)$ such that $\alpha_g = Ad\, u$. Since $ue(A^\alpha)' \cap M^\infty(A)$, by 14 we have $\alpha_g =$ 1. Since α is faithful, $g = 0$.

*Ad* 15 ⟶ 4. Assume 15. In particular, $A^*$ is prime. Hence by Lemma (1.3.1)(as in the proof of 4 ⟶ 5), with $H = G$, also $A \rtimes_\alpha G$ is prime.

Let $y \in \hat{G}$, and suppose that ά$y$ is not properly outer. We must prove that $\gamma = 0$. As $A \rtimes_\alpha G$ is prime, there exists a unitary $u \in M^\infty(A \rtimes_\alpha G)$ such that $ά_y = Ad\, u$. Centre $M^\infty(A \rtimes_\alpha G) = C$. Therefore $u$ is unique up to a scalar multiple. By commutativity of $G$, it follows that $u^{-1}ά_\xi(u) \in T$ for every $\xi \in G$. Therefore the map $\xi \longmapsto u^{-1}ά_\xi(u)$ is a character of $\hat{G}$, and so there exists $0 \in G = \hat{G}$ with

$$ά_\xi(u) = \langle \xi, g \rangle u\,, \quad \xi \in \hat{G}.$$

Since also

$$ά_\xi(\lambda(g)) = \langle \xi, g \rangle \lambda(g) \quad , \xi \in \hat{G}.$$

where$\lambda(g)$ is the canonical unitary multiplier of $A \rtimes_\alpha G$corresponding to $g$, it follows that, with $v = \lambda(g)u^*$,

$$v \in M^\infty(A \rtimes_\alpha G)^\alpha$$

By the choice of $u\, \lambda(t)u\lambda(t)^{-1} = \langle -y, t \rangle u, t \in G$ and hence as $G$ is abelian, $\lambda(t)v\lambda(t)^{-1} = \langle y, t \rangle v, t \in G. Since\, uAu^{-1} = A, also\, vAv^{-1} = A,$

$$v \in M^{\infty}(A) \subseteq M^{\infty}(A \rtimes_{\alpha} G)$$

Since $(Adv)|A = \alpha_g, g = 0$. This shows that

Recall that $\lambda(t)u\lambda(t)^{-1} = \langle -y, t\rangle u, t \in G, and\ uAu^{-1} = A$. Hence

$$u \in M^{\infty}(A) \subseteq M^{\infty}(A \rtimes_{\alpha} G).$$

By the choice of $w, Adu = \acute{\alpha}_y$, and in particular, $(Adu)\backslash A = I$. Therefore, we $u \in$ Centre $M^{\infty}(\Lambda)$.

$As\ A$ is prime, Centre $M^{\infty}(\Lambda) = C$, and so $u \in$T. Hence $\acute{\alpha}_y = Ad\ u = 1$, and $y = 0$ as desired.

$Ad4 \longrightarrow 1$. This follows from $4 \longrightarrow 12$, established above, since 12 is clearly stronger than 1. Let us, however, give a more direct proof of $4 \longrightarrow 1$.

Assume 4. Let $x\ and\ y$ be nonzero elements of $A$ and let us find $a \in A^{\alpha}$ such that $xay \neq 0$. Replacing $x$ and $y$ by $x^*x\ and\ yy^*$, we may suppose that $x$ and $j$ are positive.

As in the proof of Lemma(1.3.1), above, we shall identify the systems $((A \rtimes_{\alpha} G) \rtimes_{\alpha} \hat{G}, \alpha^{\mathbb{A}})\ and\ (A \otimes K(L^2G)\backslash\ \alpha \otimes \lambda)$ ,and the subalgebras $A \rtimes_{\alpha} G \subseteq (A \rtimes_{\alpha} G) \rtimes_{\alpha} G$ and $(A \otimes K)^{\alpha\otimes\lambda} \subseteq A \otimes K$. Recall that

$$(I \otimes e_{00})(A \otimes K)^{\alpha\otimes\lambda}(I \otimes e_{00}) = A^{\alpha} \otimes e_{00}\ ,$$

where $e_{00}$ is the projection onto the one-dimensional subspace of $L^2(G)$ generated by the trivial character.

Since $x\ and\ y$ are positive and nonzero, so also are $x_0 = \int_G \alpha_g\,(x)dg\ and\ y_0 = \int_G \alpha_g\,(y)dg$.

We shall prove the following inequality, which is stronger than what is needed:

$$sup_{a\in A^{\alpha}}, \|a\| \leq 1 \quad \|xay\| \geq \|x_0\|\|y_0\|.$$

Let us identify $x, y\ with\ x \otimes e_{00}, y \otimes e_{00} \in A \otimes e_{00}$. Recalling that $(A \rtimes_{\alpha} G) \rtimes_{\alpha} \hat{G}$, we may replace $x$ and $y$ by finite approximating sums

$$\sum_{y\in F} x_y u(y)\ , \sum_{y\in F} y_y u(y),$$

where $x_y, y_y \in A \rtimes_{\alpha} G$ for each $y$ in the finite set $F \subseteq \hat{G}$. Here we cannot insist that all $x_y$ and $y_y$ be the canonical (Fourier) coefficients of (the original) $x\ and\ y$, since the Fourier series only converges in the Cesaro mean in general. However, we may certainly assume that $x_0$ and $y_0$ are as defined above (the zeroth Fourier coefficients). We may also assume that all $x_y$ and $y_y$ lie in $A^{\alpha} \otimes e_{00}$ , the cutdown of $A \rtimes_{\alpha} G\ by$ I$\otimes e_{00}$ From now on we shall suppose that $x\ and\ y$ are equal to such finite sums. We shall also suppose that $\|x\| = \|y\| = 1$

We shall now use with $(A \otimes K)^{\alpha\otimes\lambda} = A \rtimes_{\alpha} G$in place of $A, (\acute{\alpha}_y)_{y\in\backslash\{0\}}$ in place of $\alpha_1, .. \alpha_n$, and, successively, $(x_y)_{y\in F}\ and\ (Y_y)_{y\in F}$ in place of $\alpha_0, a_1, \ldots, a_n$. This $y$ields, for each $\varepsilon > 0$, elements $w\ and\ z$ of $A \rtimes_{\alpha} G$, of norm one, such that $\|wx_0w\| \geq \|x_0\| - \varepsilon, \|zy_0z\| \geq \|y_0\| - \varepsilon$ , and $\|wx_y\acute{\alpha}_y(w)\| \leq \varepsilon, \|zy_y\acute{\alpha}_y(z)\| \leq \varepsilon, y \in F\backslash\{0\}$. sub- $C^*$ -algebras generated by $x_0$ and $y_0$, which are contained in $A^{\alpha} \otimes e_{00}$, and so we may suppose that $w, Z \in A^{\alpha} \otimes e_{00}$

Since $A^{\alpha} \otimes e_{00}$ is prime, there exists $b \in A^{\alpha} \otimes e_{00}$ such that $||b|| = 1$ and

$$\|wx_0wbzy_0z\| \geq \|wx_0w\|\ \|zy_0z\| - \varepsilon\ .$$

Hence, with $a = wbz, we\ have\ ||\alpha|| < 1, a \in A^{\alpha} \otimes e_{00}$, and

$$\|xay\| \geq ||wxwbzyz||$$

$$\geq \|wx_0wbzy_0\, z\| - \sum_{y,y'\in F\setminus\{0\}} \|wx_y u(y)wbzy_y > u(y')z\|$$
$$\geq \|wx_0w\|\|zy_0\, z\| - \varepsilon - \sum_{y,y'\in F\setminus\{0\}} \|wx_y \acute{\alpha}_{y'}(w)\|\|zy_{y'}\acute{\alpha}_{y'}(z)\|$$
$$\geq \|x_0\|\|y_0\| - 3\varepsilon - n^2\varepsilon^2$$
where $n = card(F\setminus\{0\})$. Since $\varepsilon > 0$ is arbitrary, the desired inequality is proved.

Finally, suppose that $G = T\ or\ G = Z/pZ\ with\ p$ prime, and let us show that 16 is equivalent to 1 to 15.

$Ad$ 3 $\longrightarrow$ 16. This is evident.

$Ad$ 16 $\longrightarrow$ 3. In the case $G = Z/pZ$ with $p$ prime, this is evident, as $G$ is simple.

In the case $G = T$, there are nontrivial proper closed subgroups, but these are all finite. Assume 16. Let $H$ be a closed subgroup of $G$, where now $G = T. If G/H \cong Z/nZ for$ some $n = 1, 2, \ldots$ then necessarily $n = 1$ and $H = G$, and so $A^H$ is equal to $A^\alpha$, which is prime by $16. If\ G/H \cong$T, then $H$ is finite and cyclic; choose an element $h$ generating $H$. We must show that $A^H$ is prime.

By $16, A$ is prime. Therefore, by Theorem (1.3.1) of [32], it is equivalent to show that, if $\beta$ denotes the restriction of α to the subgroup $H \subseteq G$, then the Connes spectrum of $\beta$ is equal to the Arveson spectrum of $\beta — i.e., \widehat{H}(\beta) = Sp\beta$. In terms of the automorphism $\alpha_h$, this says that, for every nonzero hereditary sub- $C^*$ -algebra $B\ of\ A$ which is invariant
under $\alpha_h$,

(1) $$Sp(\alpha_h/\, B) = sp\alpha_h$$

By $16, A^\alpha$ is prime, and so by Theorem (1.3.1) of [32], $G(\alpha) = Sp\ \alpha$. In particular, (1) holds if $B\ is\ \alpha$-in variant. applied to αh, and as simplified using that $A$ is prime, there exists a canonical nonzero closed two-sided ideal $J\ of\ A$, invariant under $\alpha_h$, such that (1) holds when both sides are restricted to /, i.e.
$$sp(\alpha_h|B\,) = sp\,(\alpha_h|\,J).$$
That $J$ is canonical entails that $J$ is invariant under $\alpha$. ($J$ is in fact constructed to contain all other such ideals. Therefore (as $G(\alpha) = Sp\alpha), (1)$ holds for /. Hence, for any $\alpha_h$-invariant $B$,
$$sp\alpha_h \supseteq sp(\alpha_h|B\,) \supseteq p(\alpha_h|B\cap J\,) = sp\,(\alpha_h|\,J) = sp\alpha_h,$$
i.e. (1) hold for $B$,as desired .

**Proposition (1.3.6)[41]:** *Let $A$ be a $C^*$-algebra. The following four properties are equivalent.*

(i) *$A$ is prime.*

(ii) *$M^\infty(A)$ is prime.*

(iii) Centre $M^\infty(A) = C$.

(iv) Centre$M\ (I) = C$ *for every nonzero closed two-sided ideal $I$ of $A$.*

**Proof:** $Ad \longrightarrow (ii)$. As pointed out on page 303 of [66], this follows from the fact that each nonzero closed two-sided ideal of $M^\infty(A)$ has a nonzero intersection with $A$.

$Ad\ (ii) \longrightarrow (iii)$. This is evident.

$Ad\ (iii) \longrightarrow (iv)$. Assume (iii). Let $I$be $a$ nonzero closed two-sided ideal of $A$. To show that Centre $M(I) = C$ it is sufficient to do this with $I$ replaced by $I + J$ where $IJ = 0$.

Therefore, we may suppose that $I$ is essential, so that $M(I) \subseteq M^{\infty}(A)$. If $.J$ is any essential closed two-sided ideal of $I$, then Centre $M(I) \subseteq$ $Centre\ M(J)$ as follows by considering a faithful representation of $I$ which is nondegenerate on $J$. Hence

$$Centre\ M(I)\ c\ Centre\ M^{\infty}\ (A).$$

In particular, from (iii) follows Centre $M(A) = C$.

A natural question arises here: is $Centre\ \ M^{\infty}\ (A)$ the inductive limit of Centre $M(I)$ ($I$ an essential ideal)?

$Ad\ (iv) \longrightarrow (i)$. If $A$ is not prime, then there exist nonzero closed two-sided ideals $I_1$ and $I_2$ of $A$ with $I_1 I_2 = 0. Set\ I_1 + I_2$ = I. Then $I$ is nonzero and Centre $M(I) \neq C$.

**Proposition (1.3.7)[41]:** *Let A be a separable prime $C^*$-algebra, and let $\alpha$ be an automorphism of A. The following three properties are equivalent.*

(i) *$\alpha$ is not properly outer.*

(ii) *$\alpha$ is inner in $M^{\infty}(A)$.*

(iii) *$\alpha$ is weakly inner in every faithful factor representation of $A$.*

**Proof:** $Ad\ (i) \longrightarrow (ii)$. Assume (i). By definition, there is a nonzero invariant closed two-sided ideal $I$ of $A$ such that for some unitary $u$ in $M(I), \|\alpha\ |I— \ (Ad\ u)|\ I\ \| < 2$. By the Kadison-Ringrose theorem ([55]), there exists a derivation $\delta$ of $I$ such that

$$\alpha|I = (Adu)exp\delta.$$

as $A$ is separable, $\delta$ is inner in $M^{\infty}(I)$. Since $A$ is prime, $(I) =\longrightarrow$, so $\alpha$ is inner in $M^{\infty}(A)$, as desired.

$Ad\ (ii) \longrightarrow\ (iii)$. This follows from the fact, that any faithful factor representation of $A$ extends to a representation of $M^{\infty}(A)$. (Here we do not need $A$ to be separable. Also, the implication $(ii) \longrightarrow\ (i)$ holds for any $C^*$-algebra.)

$Ad\ (iii)\ \longrightarrow (i)$. Assume that $\alpha$ is properly outer. By the proof of Theorem (1.3.1) of [56], with Lemma (1.3.2) of [56] replaced by Proposition (1.3.10) of [63] (see also Proposition (1.3.11)of [63]), there exists a pure state $\phi$ of $A$ such that $\phi_{\alpha}$ is disjoint from $\phi$ . $A$ modification of the proof of Theorem (1.3.1) of [56], using that $A$ is separable and prime in the same way as in the proof of 4 $\longrightarrow$ 5 of Theorem 1, above, shows that $\phi$ may be chosen so that $\pi_{\phi}$ is faithful. Thus, $\pi_{\phi}$ is a faithful factor representation in which $\alpha$ is not weakly inner.

**Proposition (1.3.8)[41]:** *Let Abe a $C^*$ -algebra, let G be a compact group, and let $\alpha$ be an action of G on A. Then $M^{\infty}(A)^{a} \subseteq M^{\infty}A^{a}$.*

**Proof:** As shown in the proof of 6 $\longrightarrow$ 7 of Theorem (1.3.1), above, if $I$ is a nonzero closed two-sided ideal of $A$, then (as $G$ is compact) /contains $a$ nonzero $\alpha$-in variant closed two-sided ideal; the largest such is of course $\cap_{g\in G}\ \alpha_g(I)$It follows easily that if $I$ is essential, then also$\cap_{g\in G}\ \alpha_g(I)$ is essential.

This shows that, in the definition of $M^{\infty}(A)$ , as the inductive limit of multiplier algebras $M(I)$ over all essential closed two-sided ideals $I$, we may restrict / to be α-invariant without changing the definition (or, at least, without changing the resulting algebra). Thus,

$$M^{\infty}(A)\ = Lim_{/invarant}\ M(I).$$

Hence, using $a$ second time that $G$ is compact, we have

$$M^{\infty}(A)^{a}\ = Lim_{/invarant}\ M(I)^{a}.$$

Next, let us show that for invariant $I, M(I)^{a} = M(I^{a})$. We have

$$M(I)^a \subseteq M(I^a) \subseteq M(I).$$

where the first inclusion is evident, and the second holds as $I^\alpha$ contains an approximate unit for $I$. (This uses again that $G$ is compact: If $(e_i)$ is an approximate unit for $I$, then so also is $(\int_G \alpha_g(e_i)\, dg$. To see this just note that $e_i {\alpha_g}^{-1}(a) \longrightarrow {\alpha_g}^{-1}(a)$ uniformly in $g$ since $G$ is compact, or, in other words, $\alpha_g(e_i)a \longrightarrow a$ uniformly in $g$ .) Hence, immediately, $M(I^a) = M(I)^a$.

We now have

$$M^\infty(A)^a = Lim_{/invarant}\ M((I^a) \subseteq M^\infty(A^a)\ .$$

**Proposition (1.3.9)[41]:**. *Let Abe a $C^*$ -algebra, let G be a compact abelian group, and let α be an action of G on A. It follows that*

*Assume that A is separable and prime, that $A^\alpha$ is prime, and that G is separable, and let u be a unitary element of $M^\infty (A \rtimes_\alpha G)^{\hat{\alpha}}$such that $uAu^{-1} = A$, and, for some $\gamma \in \hat{G}$,*
*$\lambda(g)u\lambda(g)^{-1} = \langle y, g\rangle u, \quad g \in G.$*
*It follows that $u \in M^\infty(A)$.*

**Proof:** First, using only that $G$ is compact, let us show that $M^\infty(A) \subseteq M^\infty(A \rtimes_\alpha G)$.

If $I$ is an $\alpha$-in variant closed two-sided ideal of $A$, then $I \subseteq M(I \rtimes_\alpha G)$, and since an approximate unit for $I$acts also as one on $I \rtimes_\alpha G, also\ M(I) \subseteq M(I \rtimes_\alpha G)$. (This uses only that $G$ is locally compact.)

Since $G$ is amenable, for each $\alpha$-invariant essential closed two-sided ideal $I$of $A$, the crossed product ideal on $I \rtimes_\alpha G$ is essential in $A \rtimes_\alpha G$. (By 7.7.8 of [35], for any faithful representation $\pi$ of $A$, the representation of $A \rtimes_\alpha G\ on\ H_\pi \otimes L^2(G)$ induced by $\pi$ is faithful. If $\pi$ is chosen to be nondegenerate on $I$, so that / and $A$ have the same weak closure in the representation $\pi$, then $I \rtimes_\alpha G$ and $A \rtimes_\alpha G$ have the same weak closure in the induced representation, and since this is faithful it follows that $I \rtimes_\alpha G$ is essential.)

Hence by compactness of $G$, as in the proof of Proposition (1.3.6).

$$M^\infty(A) = lim_{/\ essential}\ M(I) - lim_{/\ essential\ and\ invariant} M(I)$$
$$\subseteq lim_{/\ essential\ and\ invariant}\ M(I \rtimes_\alpha G) \subseteq M^\infty(A \rtimes_\alpha G)\ .$$

It of course follows, as $G$ is abelian, that

$$M^\infty(A) \subseteq M^\infty(A \rtimes_\alpha G)^\alpha.$$

Now, assume that $A$ is separable and prime, and let $u \in M^\infty(A \rtimes_\alpha G)^\alpha\ and\ y \in \hat{G}$ be such that $u$ is unitary, $uAu^{-1} = A, and\ \lambda(g)u\lambda(g)^{-1} = \langle y, g\rangle u, g \in G$. Let us show that $u \in M^\infty(A)$ (If, in addition, $u \in M(A \rtimes_\alpha G)$, then it follows from 7.8.9 [35] of that $u \in M(A)$.What we are establishing is a very limited generalization of 7.8.9 of [35] to the limit multiplier algebra. In particular, the assumption that $A$ and $M^\infty(A)$ are separable and prime may be superfluous.)

First, let us show that there exists $v \in M^\infty(A)$ such that $Ad\ v$ agrees on $A$ with $Ad\ u$. By Proposition (1.3,5), for this it is sufficient to show that the automorphism $\beta = (Adu) \mid A$ is weakly inner in every faithful factor representation of $A$. Let $\pi$ be $a$ faithful factor representation of $A$, and denote by $p$ the representation of $A \rtimes_\alpha G$ induced by $\pi$ on $H_\pi \otimes L^2(G)$. Note that ά, which extends to $p(A \rtimes_\alpha G)$", acts ergodically on the centre of $p(A \rtimes_\alpha G)$" (as $\pi$ is factorial). It follows, as we shall show below, that $p$ can be extended from $A$ to $p(A \rtimes_\alpha G)^\alpha$, mapping this algebra into $(p(A \rtimes_\alpha G)^\alpha)''$, and commuting with $AdU\lambda(g)$ for each $g \in G$. Since $\lambda(g)p(u)\lambda(g)^{-1} = \langle y, g\rangle p(u), g \in G$, it follows that $\rho(u) = V \otimes y\ with\ V \in \pi(A)". (p(u)(I \otimes y)^{-1}$ commutes with $I \otimes \xi$ and $I \otimes \lambda)$ for all

$\xi \in G$ and $\xi \in \hat{G}$, and therefore with $I \otimes B(L^2(G))$. By construction, $\rho(A \rtimes_\alpha G)$" is contained in $\pi(A)$"$B(L^2(G).)$ We now have, for each $a \in A$,

$$p(\beta(a)) = p(uau^{-1}) = p(u)p(a)p(u)^{-1} = (V \otimes y)p(a)(V \otimes y)^{-1}$$

and since $p(\alpha)$ is just the function $t \longrightarrow \pi(\alpha_t(a))$, evaluating at $t = 0$ we get

$$\pi\beta = \big(A(dv)\big)\pi,$$

and so $\beta$ is weakly inner in $\pi$, as desired.

Before proceeding to modify $u$ using u, let us show as announced that if $p$ is a faithful representation of $A \rtimes_\alpha G$ such that the restriction of $p$ to any $\hat{\alpha}$-invariant essential closed two-sided ideal of $A \rtimes_\alpha G$ is nondegenerate, then $p\,|A$ can be extended to $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$. (It was pointed out earlier in the proof of this theorem that $p$ as defined in the preceding paragraph is faithful; the second property also holds for that p, since $\hat{\alpha}$ extends to an action on $p(A \rtimes_\alpha G)^{"}$ which is ergodic on the centre.) Let $\in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$, and let $(J_n)$ be a sequence of essential closed two-sided ideals of $A \rtimes_\alpha G$ such that there exists $x_n \in M(J_n)$ with $\|x — x_n\| = \varepsilon_n \longrightarrow 0$. Then, for any $m$ and $n$, and any $\xi, \eta \in \hat{G}$,

$$\|\hat{\alpha}_\xi(x_m) - \hat{\alpha}_\eta(x_n)\| \leq \varepsilon_m + \varepsilon_n$$

(this uses the triangle inequality and $\hat{\alpha}_\xi(x) = x = \hat{\alpha}_\eta(x_n)$. Denote by $e^n$ the unit of $p(J_n)^{"}$, a central projection in $p(A \rtimes_\alpha G)$" . For each$n$, the representation $p|\,J_n$, on the Hilbert space $e^n H_p$, has $a$ unique extension to $a$ representation of $M(J_n)$, which we could denote by $(p \mid J_n)^{**}$, but will denote by $p^n$ for brevity. Let $\xi_{1,}\ \xi_2, \dots$ be an enumeration of $\hat{G}$, which is countable since $G$ is compact and separable. Fix $n$, and define projections $p_1^n, p_2^n, \dots$ in Centre $\rho(A \rtimes_\alpha G)$" by orthogonalizing the units of $\rho(\hat{\alpha}_{\xi_1}(J_n)^{"}\rho(\hat{\alpha}_{\xi_2}(J_n)^{"} \dots$, which we shall denote by $e_1^n, e_2^n$, .... Thus,

$$p_1^n = e_{1,}^n p_2^n = (1 - p_1^n)e_2^n, p_3^n = (1 - p_1^n \vee p_2^n)e_3^n.$$

Then $V_k p_k^n = V_k\, e_k^n = 1$ since $V_k\, e_k^n$ is the unit of $p(J_n)$" where $J_n$ is the smallest closed two-sided ideal of $A \rtimes_\alpha G$ containing $\hat{\alpha}_{\xi_1}(J_n), \hat{\alpha}_{\xi_2}(J_n), \dots$ and $J_n\ is\ \hat{\alpha}$ -invariant and essential (so by hypothesis p is nondegenerate on $J_n$). For each $k$ denote by $p_k^n\ k$ the unique extension of $p\,|\hat{\alpha}_{\xi_k}(J_n)$ to a representation of $M(\hat{\alpha}_{\xi_k}(J_n)) = \hat{\alpha}_{\xi_k}(M(J_n))$ on the Hilbert space $e_k^n H_p$. (Thus, $p_k^n = (p|\hat{\alpha}_{\xi_k}(J_n))^{**}$. Set

$$\sum\nolimits_k p_k^n p_k^n(\hat{\alpha}_{\xi_k}(x_n)) = y_n$$

Then $y_n \in p(A \rtimes_\alpha G)$". Furthermore, the sequence $(y_n)$ is Cauchy:

$\|y_m - y_n\| = sup_{k,i}\|p_k^m p_i^n(y_m - y_n)\| = sup_{k,i}\left\|p_k^m p_i^n(p_k^m\left(\hat{\alpha}_{\xi_k}(x_m)\right) - p_k^n\left(\hat{\alpha}_{\xi_k}(x_n)\right))\right\| \leq sup_{k,i}\left\|(p|\hat{\alpha}_{\xi_k}(J_m) \cap \left(\hat{\alpha}_{\xi_k}(J_n)\right)^{**}\ (\hat{\alpha}_{\xi_k}(x_m) - \left(\hat{\alpha}_{\xi_k}(x_n)\right))\right\| = sup_{k,i}\|\hat{\alpha}_{\xi_k}(x_m) - \hat{\alpha}_{\xi_i}(x_n)\| \leq \varepsilon_m + \varepsilon_n.$

Here we have used that $p_k^m, p_i^n \leq e_k^m e_i^n$ and that $e_k^m e_i^n$ is the unit of $p(\hat{\alpha}_{\xi_k}(J_m) \cap \left(\hat{\alpha}_{\xi_k}(J_n)\right)^{**}$. Set$Lim y_n = p(x)$.

From what we have shown, namely, that

$$\|y_m - y_n\| \leq \|\chi_m - x\| + \|X_n - x\|,$$

it is clear that $p(x)$ is independent of any choices made in the construction. Hence, in particular, $p(x)$ depends additively and multiplicatively on $x$, and $p(x^*) = p(x)^*$.

Furthermore, $p$ defined on $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$ in this way agrees with the unique extension of $p$ to a representation of $M(A \rtimes_\alpha G)$ (or to $M(J)$ for any closed two-sided ideal $J$ of

$(A \rtimes_\alpha G)$on which $p$ is nondegenerate). Finally, for use at the end of this proof, let us notethat, by construction, $p$ is isometric on $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$.

Now let us return to the proof that $u \in M^\infty(A)$. As we have shown, there exists a unitary $v \in M^\infty(A)$ such that $(Adv)|A = (Adu)|A$. Since $(Adw)|$ A commutes with $(Ad\,\lambda(g))|\,A = \alpha_g$ for each $g \in G$, it follows that $v^{-1}\alpha_g(v)$ belongs to Centre $M^\infty(A)$ for each $g \in G$. By Proposition(1.3.4), as $A$ is prime, Centre$M^\infty(A) = C$. Hence, by Proposition (1.3.8), below, the map $g \longrightarrow v^{-1}\alpha_g(v)$ is continuous. This map is clearly multiplicative.

Therefore, there exists $\xi \in \hat{G}$ such that

$$\lambda(g)v\lambda(g)^{-1} = \langle \xi, g\rangle v\,, \qquad g \in G\,.$$

Replacing $u$ by $uv^*$, and $\gamma$ by $y — \xi$, we then have that $u$ fulfills the hypotheses of the proposition and, in addition, $uau^{-1} = a$ for all $a \in A$. In other words, we now have that $u \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$ and, replacing $y$ by $-y$,

$\hat{\alpha}_y = Adu$ .

Using only the hypothesis that $A$ and $A^\alpha$ have faithful irreducible representations, $(\hat{\alpha})^\perp$ we shall now deduce that $y = 0$, and hence that $u$ is a scalar multiple of $I$.

First, let us show that $y = 0$. Since $A$ is prime,] we have $G(\hat{\alpha}) = G$. To show that $y = 0$, therefore, it is sufficient to show that $y \in G$. By Proposition (1.3.7)of [63], for this it is sufficient to find $a$ nonzero $\hat{\alpha}$-invariant hereditary sub-$C^*$-algebra $B$ of $A \rtimes_\alpha G$ such that $\hat{\alpha}_y\,|\,B = exp\,\delta$ for some $\hat{\alpha}$-invariant derivation of $B$. If $B$ is $\hat{\alpha}$-invariant and $|\,Sp(\hat{\alpha}_y\,|\,B) — 1\,| \leq 1$ then this of course holds, with $\delta = log(\hat{\alpha}_y\,|\,B)$.

Since $\hat{\alpha}_y = Adu$ with $u \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$ $\hat{\alpha}_y$ is not properly outer. (The implication $(ii) \longrightarrow (i)$ of Proposition (1.3.5) holds for any $C^*$-algebra; just note that if an automorphism

$\beta$ of a $C^*$-algebra is, when restricted to a not necessarily invariant closed two-sided ideal $I$, strictly within distance two of an automorphism of $I$, then $\beta$ leaves $I$invariant.)

Hence, by $(viii) \longrightarrow (i)$ (this implication does not use separability), there exists a nonzero $\hat{\alpha}_y$ -invariant hereditary sub- $C^*$ -algebra $B_0\ of\ A \rtimes_\alpha G$ such that $|\,Sp(\hat{\alpha}_y\,|\,BQ) — 1| \leq 1$ . Using that $u$ is $\hat{\alpha}$-invariant, we shall show that if $B$ denotes the $\hat{\alpha}$--invariant hereditary sub-$C^*$-algebra of $A \rtimes_\alpha G$ generated by $B_0$, then also $|Sp(\hat{\alpha}_y|B) - 1\,| \leq 1$, as desired.

We shall in fact show that $Sp(\hat{\alpha}_y|B) = Sp(\hat{\alpha}_y|B_0)$. To do this we shall proceed in two steps, using a faithful irreducible representation $\pi$ of $A \rtimes_\alpha G$. (Recall that by Lemma (1.3.2), with $H = G$, the hypothesis that $A^\alpha$ is prime implies that $A \rtimes_\alpha G$ is prime.) Since $\pi$ extends to $M^\infty(A \rtimes_\alpha G)$ (being both faithful and factorial), and $\hat{\alpha}_y = Ad\,u$, we may extend $\hat{\alpha}_y\ to\ \pi(A \rtimes_\alpha G)''$, writing $\hat{\alpha}_y = Ad\pi(u)$.

We shall prove first that

$$Sp(\hat{\alpha}_y|B_0) = Sp(\hat{\alpha}_y|\pi(B_0)''),$$

and second that

$$Sp\big(\hat{\alpha}_y\big|\pi(B_0)\big) = Sp(\hat{\alpha}_y|\pi(B)\big).$$

Since, for single automorphisms, spectrum and Arveson spectrum, and therefore also point spectrum, coincide — see [35], 8.1.14 — , we have

$$Sp(\hat{\alpha}_y|B_0) \subseteq Sp(\hat{\alpha}_y|(B) \subseteq Sp(\hat{\alpha}_y|\pi(B_0)'')$$

and the desired equality,

$$Sp(\hat{\alpha}_y|B_0) = \mathrm{Sp}(\hat{\alpha}_y|(\mathrm{B}),$$

follows.

Let us show that $Sp(\hat{\alpha}_y|B_0) = Sp(\hat{\alpha}_y|\pi(B_0)")$. By 8. 1 .9 of [35], $\lambda \in Sp\,\beta$, where $\beta$ is an automorphism of a $C^*$-algebra or a von Neumann algebra, if and only if, for each $f \in I^1(Z)\, with \zeta(\lambda) \neq 0, \Sigma f(n)\beta^n \neq 0$. Applying this first with $\beta = \hat{\alpha}_y|B_0$ and then with $\beta = (\hat{\alpha}_y|(B_0)"$, we see that $Sp(\hat{\alpha}_y|(B_0)" = Sp(\hat{\alpha}_y|\pi(B_0)")$, as desired.

Let us show that $Sp(\hat{\alpha}_y|\pi(B_0)") = Sp(\hat{\alpha}_y|\pi(B)")$. As above, the inclusion of the spectrum on the smaller domain in the spectrum on the larger domain holds since the spectrum is point spectrum. Conversely, let $\lambda \in Sp(\hat{\alpha}_y|\pi(B)")$, and let us show that $\lambda \in Sp(\hat{\alpha}_y|\pi(B_0)")$. $\lambda = \lambda_1 {\lambda_2}^{-1}$ with $\lambda_1 \lambda_2 \in Sp\pi(u)$. We shall show that $\lambda_1\lambda_2 \in Spe_0\pi(u)$ where $e_0$ is the unit of $\pi(B_0)"$, using that $u$ is $\hat{\alpha}$-in variant. Since $B$ is the smallest ά-invariant hereditary sub- $C^*$ -algebra of $A \rtimes_\alpha G$ containing $B_0$, the unit of $\pi(B)"$, say e, is the smallest projection containing the unit of $\pi(\hat{\alpha}_y|\pi(B_0))", say\, e_\xi$, for every $\xi \in \hat{G}$. For each $\xi \in \hat{G}$, since $\hat{\alpha}_\xi(u) = u$, we have $u\hat{\alpha}_\xi(B_0)u^{-1} = \hat{\alpha}_\xi(B_0$, and hence $e_\xi\pi(u) = \pi(u)e_\xi$.

Let us show that, for each $\xi \in \hat{G}, Sp\, e_\xi\pi(u) = Sp\, e_0\pi(u). Since\, \hat{\alpha}_y|B_0 = (Ad\, u) \mid B_0$ with $u \in M^\infty(A \rtimes_\alpha G)$, and $B_0$ is a hereditary sub-$C^*$-algebra of $A \rtimes_\alpha G$, so that every faithful factor representation of $B_0$ extends to a faithful factor representation of $A \rtimes_\alpha G$, (and hence of $M^\infty(A \rtimes_\alpha G)$on a larger Hilbert space, by Proposition (1.3,5)there exists $w \in M^\infty(B_0)$ such that $\hat{\alpha}_y|B_0 = (Ad\, w) \mid B_0$. Since $\pi$ is irreducible, also the restriction of $\pi$ to $B_0$ is irreducible on the Hilbert space $e_0H_\pi$. It follows that $\pi(w)$ is a scalar multiple of $e_0\pi(u)$, and so we may modify $w$ so that $\pi(w) = e_0\pi(u$. Hence, for any $a, b \in B_0, awb = aub$. It follows that, for any $\xi \in \hat{G}$, on considering the irreducible representation $\pi\hat{\alpha}_\xi$ of $A \rtimes_\alpha G$, and its restriction to $B_0$, which is irreducible on the Hilbert space $e_\xi H_\pi$, we have $\pi\hat{\alpha}_\xi(w) = e_\xi\pi\hat{\alpha}_\xi(u) = e_\xi\pi(u)$. Since $\pi\hat{\alpha}_\xi$ is faithful on $B_0$ and therefore on $M^\infty(B_0)$, we have $Sp\,\pi\hat{\alpha}_\xi(w) = Sp\, w$. This shows that $Spe_\xi\pi(u)$ is independent of $\xi$, as desired.

Now let us show, as announced, that $\lambda_1$ and $\lambda_2$ belong to $Spe_0\pi(u)$. Note that, since $e = V_{\xi\in\hat{G}}e_\xi$, the homomorphism

$$C^*(e\pi(u)) \ni x \longmapsto (e_\xi x) \in \textstyle\prod_{\xi\in\hat{G}} C^*(e_\xi\pi(u)).$$

is injective, so that $Sp \in \pi(u) = (\cup_{\xi\in\hat{G}} Spe_\xi\pi(u))^-. Since\, Spe_\xi\pi(u) = Spe_0\pi(u)$ for each $\xi$, this shows that $Sp\, e\pi(u) = Sp\, e_0\pi(u)$. In particular, $\lambda_1, \lambda_2 \in Spe_0\pi(u)$.

Since $\lambda = \lambda_1, {\lambda_2}^{-1}$ , and $\hat{\alpha}_y \mid \pi(B_0)" = (Ad\, e_0\pi(u)) \mid \pi(B_0)"$, [21] that $\lambda \in Sp\hat{\alpha}_y \mid \pi(B_0)"$, as asserted.

This completes the proof that, after $w$ is modified as above, $y = 0$. Let us now show that w, thus modified, is a scalar multiple of 1. Let $\pi$ be a faithful factor representation of $A$, so that, as noted above, the induced representation $p$ of $A \rtimes_\alpha G$ is also faithful, and, moreover, extends from $A \subseteq M^\infty(A \rtimes_\alpha G)$ to $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$, and is faithful there. What we must show, then, is that $p(u)$ is a scalar multiple of 1. As shown above, $p(u) = V \otimes \gamma$ with $\in \pi(A)". Since\, y = 0 \in \hat{G}$, by which we mean that $y$ is the trivial character 1 , we have $u = V \otimes 1$ . As shown above, $\pi((Ad\, u)|A) =$

$(Ad\, V)\pi$. Since $(Ad\, u) \mid A = 1$ and $\pi$ is factorial, it follows that Kis a scalar multiple of 1, and therefore also $u$ is. In particular, $u \in M^{\infty}(A)$.

We do not know if all the assumptions made in the second half of the proposition are necessary.

**Proposition (1.3.10)[41]:** *Let $A$ be a prime $C^{*}$-algebra and let $\alpha$ be an action of a compact group $G$ on $A$. Let $\phi$ be a pure state of $A$ such that $\pi_{\phi}$ is faithful, so that $\phi$ extends uniquely to a pure state of $M^{\infty}(A)$. It follows that for any $a, b, c \in M^{\infty}(A)$ the map*

$$G \in g \longmapsto b\, \phi c(\alpha_{g}(\alpha) b)$$

*is continuous.*

**Proof:** As shown in the proof of Proposition(1.3.6), we have $M^{\infty}(A) = lim_{/\, invariant}\, M(I)$. Therefore it is sufficient to consider the case that $\alpha \in M(I)$, where /is a nonzero $\alpha$-invariant closed two-sided ideal of $A$. Again as shown in the proof of Proposition (1.3.6), $I^{\alpha}$ contains an approximate unit $(e_{i})$ for $I$ Then $\|\phi - \phi e_{i}\| \longrightarrow 0$, and the same holds with $b\phi c$ in place of $\phi$. The conclusion follows as $g \longmapsto \alpha_{g}(e_{i}a)$ is continuous

Duality for a partially inner action.

**Theorem(1.3.11)[41]:** *Let $A$ be a separable prime $C^{*}$-algebra, and let α be an action of a compact abelian group $G$ on $A$. Set*

*$H = \{t \in G;\ \alpha_{t}$, is not properly outer$\}$ ,*

*and suppose that $A^{G}$ and $A^{H}$ are prime.*

*If $\beta$ is an automorphism of $A$ such that $\beta \mid A^{G} = 1$ and $\beta\alpha_{t} = \alpha_{t}\beta, t \in H$, then there exists $g \in G$ such that $\beta = \alpha_{g}$.*

**Proof:** We may suppose that $\alpha$ is faithful.

By Proposition (1.3.5), for each $t \in H$ there exists a unitary $u(t) \in M^{\infty}(A)$ such that $\alpha_{t} = Adu(t)$ furthermore, this holds only for $t \in H$ By Proposition (1.3.5), Centre $M^{\infty}(A) = C$, and so $u(t)$ is unique up to a phase factor.

It follows in particular that $H$ is a subgroup of $G$. Let us equip $H$ with the discrete topology. Since $G$ is abelian we have $\alpha_{t} = Ad\, \alpha_{g}(u(t))$ for each $t \in H$ and $g \in G$, and by uniqueness of $u(t)$ *we have* $u(t)^{-1}(u(t)) \in T$. Hence, for each fixed $t \in H$, *by* Proposition (1.3.8), the map $g \longmapsto u(t)^{-1}\alpha_{g}(u(t))$ is continuous. This map is clearly multiplicative, and is therefore a character of $G$, say $\psi(t)$. Clearly, also, $\psi: H \longrightarrow \widehat{G}$ is a homomorphism.

Denoting by $\chi :\ G \longrightarrow \widehat{H}$ the dual of $\psi$, we have

$$\alpha_{g}\big(u(t)\big) = \langle g\, , \psi(t)\rangle\, u(t) = \langle X(g), t\rangle u(t), \qquad g \in G \quad t \in H$$

Let $N$ denote $Ker\chi = (Im\psi)^{\perp}$. We shall establish the following five assertions.

1 . $X\left(\widehat{H}\right) = \widehat{H}$

2. $\chi|\ H\ is\ injective.$

3. $N\overline{H} = G.$

4. $A^{N}$ is prime.

5. $B(A^{N}) = A^{N}$.

**Proof (1):** Since $X\left(\widehat{H}\right)$is a compact subgroup of $\widehat{H}$, it suffices to show that $\chi(H)$ is dense in $H$. Let $t \in H$be such that $\langle \chi(h), t\rangle = 1$ for all $h \in H, i.e. \alpha_{h}$ (u(t)) = $u(t), h \in H$.

Hence, by continuity of $g \mapsto u(t)^{-1}\alpha_g(u(t))$ (see above), $\alpha_h(u(t)) = u(t)$ for all $h \in \bar{H}$.
Therefore,
$u(t) \in (A^{\bar{H}})' \cap M^{\infty}(A)^{\bar{H}}$. By Proposition (1.3.6)then $u(t) \in (A^{\bar{H}})' \cap M^{\infty}(A^{\bar{H}})$ $i.e., u(t) \in\ Centre\ M^{\infty}(A^{\bar{H}})$. Since $A^{\bar{H}}$ is prime, by Proposition (1.3.4), Centre $M^{\infty}(A^{\bar{H}}) = C$. This shows that $\alpha_t = 1$ , and so $t = 0$.

**Proof (2):** If $h, t \in H$, then

$$\langle X(h), t\rangle = u(h)u(t)u(h)^{-1}u(t)^{-1} = \langle X(t), h\rangle^{-1}.$$

It follows that if $t \in H\ and\ \chi(t) = 0$ then $t \in \chi(H)^{\perp} = 0$.

**Proof (3):** Since $\chi(\bar{H}) = \hat{H}$ and Ker $\chi = N$, we have $N\bar{H} = G$.

**Proof (4):** By definition of $N = Ker\ \chi$,

$$\alpha_s\big(u(t)\big) = u(t\ )\ ,\qquad s \in N\quad ,\quad t \in H\ ,$$

i.e. $u(t) \in M^{\infty}(A)^N\ t \in H\ .Ry$ Proposition (1.3.6), it follows that $u(t) \in M^{\infty}(A)^N\ t \in H. Since\ G$ is abelian, $A^N$ is $\alpha$-invariant. Suppose that $A^N$has nonzero closed two-sided ideals $I$and $I$ such that $IJ = Q$, and let us deduce an absurdity. We may suppose that $I + J$ is essential, and then $M^{\infty}(A)^N = M^{\infty}(I) + M^{\infty}(J)$, where $M^{\infty}(I)M^{\infty}(J) = 0$. Since $\alpha_t = Adu(t), t \in H$, it follows from u( $s), u(t) \in M^{\infty}(A)^N$ for $s, t \in H$ that $\alpha_s(I)\alpha_t(J) = 0$, for any $s, t \in H$ and hence for any $s.t \in H$. Denote by /0 and /0 the smallest closed two-sided ideals of $A^N$containing $I$ and $J$ and invariant under $\alpha_{\bar{H}}$. Then $I_0\,J_0 = 0$, and since $(A^N)^{\bar{H}} = A^{N\bar{H}} = A^G$, $I_0 \cap A^G\ and\ J_0 \cap A^G$ are orthogonal nonzero ideals of $A^G$. (Note that $I_0 \cap A^G = I_0^{\bar{H}}, J_0 \cap A^G = J_0^{\bar{H}}$) This contravenes the hypothesis that $A^G$ is prime.

**Proof (5):** Denote by $\sigma$ the action of $\bar{H}$ on $A^N$ obtained by restricting $\alpha$. Denote by $\psi_1$the composition of $\psi : H \longrightarrow \hat{G}$ and the restriction map $\hat{G} \longrightarrow \hat{H}$. For each fixed $t \in H$ we have, as shown in the proof of $4, u(t) \in M^{\infty}(A)^N \subseteq M^{\infty}(A^N)$. Furthermore,

$$\alpha_h\big(u(t)\big) = \langle h, \psi_1(t)\rangle\, u(t),\quad h \in \bar{H}$$

and it follows, as we shall now show, that $\psi_1(t) \in Sp\sigma$. As shown in the proof of Proposition (1.3.6),

$$M^{\infty}(A^N) = Lim_{I\sigma-invariant}M(I),$$

and so there exist sequences $(J_n)\ and\ (a_n), I_n$ a nonzero $\sigma$-invariant closed two-sided ideal
of $A^N$ and $a_n \in M(I_n)$, such that $a_n$ converges to $u(t)$. Then with $b_n = \int_{\bar{H}} dh\, \langle \overline{h, \Psi_1(t)}\rangle(\sigma_h(a_n)$, the integral converging in the strict topology of $M(I_n)$, we have

$$(\sigma_h(b_n) = \langle h, \Psi_1(t)\rangle b_n\ ,$$

and, as we shall show, $b_n \longrightarrow u(t)$, and in particular, $b_n \neq 0$, at least for large $n.To$ see that $b_n \longrightarrow u(t)$, note that for each $c \in I_n$ and for each $c \in I_n$ invariant under $\sigma$,
$(b_n - u(t))c = \int_{\bar{H}} dh\, \langle \overline{h, \Psi_1(t)}\rangle(\sigma_h\Big(\big(a_n - u(t)\big)c\Big),$,
the integral converging in norm, and hence, if $\|c\| \leq 1$,

$$\big\|\big(b_n - u(t)\big)c\big\| \leq \big\|\big(a_n - u(t)\big)c\big\| \leq \big\|\big(a_n - u(t)\big)\big\|.$$

Since $A^N$ is prime (by 4), and is separable, there is a faithful irreducible representation of $A^N$, necessarily nondegenerate on $I_n$, and extending to a faithful representation of $M^{\infty}(A^N)$. Hence

$$\|(b_n - u(t))\| = sup_{c\in I_{n\|c\|\le 1,}} \|(b_n - u(t))c\| \le \|(a_n - u(t))\| \longrightarrow 0 \ ,$$

as desired. This shows that, at least for large $n, b_n \neq 0$ whence, for some $\sigma$-invariant $c_n \in I_n, b_n c_n \neq 0$. As $b_n c_n$ belongs to the spectral subspace of $I_n$ for the action $\sigma$ of $\overline{H}$ corresponding to $\psi_1(t) \in \hat{\overline{H}}, I_n^{\sigma}(\psi_1(t)\ ),$ and therefore to the spectral subspace $(A^N)^{\sigma}(\psi_1(t))$ of $A^N$, this shows that $(A^N)^{\sigma}(\psi_1(t)) \neq 0, i.e. \psi_1(t) \in Sp\sigma,$ as asserted.

We have shown that $\psi_1(H) \subseteq Sp\sigma$. Let us show that $\psi_1(H) = Sp\sigma$. Let $h \in \overline{H} be$ an element of $\psi_1 \ (H)^{\perp}(H)L. Then\ \alpha_h(u(t)) = u(t), t \in H, and\ so\ \chi(h) = 0,\ i.e. h \in N$. This shows that $\overline{H} \cap N \supseteq \psi_1(H)^{\perp}$, or, in other words, $(\overline{H} \cap N)^{\perp} \cap \hat{\overline{H}} \subseteq \psi_1(H)$. Since $\sigma\,|\,\overline{H} \cap N$ is trivial, one has $Sp\sigma \subseteq (\overline{H} \cap N)^{\perp} \cap \hat{\overline{H}}$ . This shows that $Sp\sigma \subseteq \psi_1 H), and\ so\ Sp\sigma = \psi_1 H.$

Since $\beta a_t = (a_t \beta\ fort \in H,$ and Centre $M^{\infty}(A) = C$ (Proposition 3.1), there is a $p \in \hat{H}$ such that

$$\beta(u(t)) = \langle t, p\rangle u(t) \ \ , \quad t \in H.$$

For each $t \in H$ , and each $a \in (A^N)^{\sigma}(-\psi_1(t)), one\ has\ au(t) \in M^{\infty}(A)^G$ (since, by 3, $N\overline{H} = G,$ and since $\alpha_h(u(t)) = \langle h, \psi_1(t)\rangle u(t\ ) forh \in \overline{H} and\ \alpha_s(u(t)) = u(t)\ S \in N$). By hypothesis, $\beta|\ A^G = 1$, and it follows that $\beta|\ M^{\infty}(A)^G = 1$. (This can be seen by examining the proof of Proposition (1.3.6), which identifies $M^{\infty}(A)^G$ with a subalgebra of $M^{\infty}(A^G)$: since each $\alpha$- invariant closed two-sided ideal $I\ of\ A$ has an approximate unit consisting of elements that are α-invariant, and therefore $\beta$-$m$ variant, $I$ is also $\beta$-in variant; hence $\beta(M(I) = M(I)$ and therefore $\beta|\ M(I)^G = 1$; it follows in the limit that $\beta|\ M^{\infty}(A)^G = 1$.) From this we obtain

$$\beta(\alpha u(t)) = \alpha u(t)\ , t \in H\ ,$$

and as $\beta(u(t)) = \langle t\ , p\rangle u(t)$ it follows that $\beta(a) = \langle \overline{t\ , p}\rangle a$. This shows that $\beta((A^N \gamma(-\psi_1(t))) = (A^N)^{\sigma}(-\psi_1(t))$ for each $t \in H$ , and since $\overline{H}$ is compact and $Sp\sigma = \psi_1(H)$, it follows that $\beta(A^N) = A^N$ as desired.

Now let us show that $\beta = \alpha_g$ for some $g \in G$. First, we shall show that $\beta\ |\ A^N = \sigma_h$ for some $h \in H,$ where, as in the proof of 5, $\sigma$ denotes the action of $\overline{H}$ by $\alpha$ on $A^N$. What we showed in the proof of 5 is that there exists $p \in \hat{H}$ such that, for each $t \in H$

$$\beta\ |(A^N)^{\sigma}\psi_1(t) = \langle t\ , p\rangle.$$

In particular, $\langle t\ , p\rangle$depends only on $\psi_1(t)$; that is, there exists a character $h_0$ of $\psi_1(H) \subseteq \hat{\overline{H}}$ such that

$$\langle h_0\ , \psi_1(t)\rangle = \langle t\ , p\rangle\ , \qquad t \in H\ .$$

Extending$h_0$ to a character on $\hat{\overline{H}}$, we have $h \in \overline{H} such\ that\ \langle h, \psi_1(t)\rangle = \langle t\ , p\rangle\ t \in H$. Then

$$\beta\ |(A^N)^{\sigma}\psi_1(t) = \langle h, \psi_1(t)\rangle \ = \sigma_h|(A^N)^{\sigma}\psi_1(t),$$

$t \in H$ , and since $\psi_1(H) = Sp\sigma$ (this was shown in the proof of 5), it follows that

$$\beta|A^N = \sigma_h = \alpha_h|A^N.$$

Set $\alpha_h^{-1}\ \beta = \beta_1$. Then $\beta_1\ |A^N = 1$, and we wish to show that $\beta_1 = \alpha_s$ for some $s \in N$. By 2, $N \cap H = Q$. In other words, αs is properly outer for every $s \in N\backslash\{0\}$. Since $A$ is separable and prime, and (by 4) also $A^N$ is prime, this shows that Condition 15 of Theorem (1.3.1) is verified with $N$ in place of $G$ (and $\alpha\ |\ N$ in place of $\alpha$). Hence Condition 1 3 of Theorem (1.3.1) is also verified, with $\beta_1$ in place of $\beta$, and so $\beta_1 = \alpha_S$ for some $s \in N$. This shows that $\beta = \alpha_g, with\ g = hs \in G,$ as desired.

**Remark (1.3.12)[41]:** If $\alpha$ is ergodic under the assumptions of the theorem, then by [42] (see also [64]) $H$ is dense in $G$. In particular, in this case $A^H = A^G$, and so the assumption that $A^H$ is prime follows from the assumption that $A^G$ is prime.

In general, the hypothesis that $A^H$is prime does not follow from the other hypotheses, and is necessary for the conclusion of the theorem. This is seen from the following example.

**Example(1.3.13)[41]:** Let $\sigma$ be an outer automorphism of the Glimm $C^*$-$algebra\ A = M_{2^\infty}$ with period two, and define an action $\alpha$ of $Z/2Z \times Z/2Z$ on $M_2 \otimes A$ by

$$\alpha_{(1,0)} = Ad\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes 1\,,$$

$$\alpha_{(0,1)} = Ad\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes \sigma\,,$$

Then $A^\alpha = \{(a\,\sigma(a), a \in A)\} \cong A\,, H = Z/2Z \times 0, A^H = A \times A$ , and $\beta = Ad\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$Verifies the conditions $\beta\backslash A^\alpha = 1\,, \beta\alpha_t = \alpha_t\beta \quad, t \in H$.

We give a new proof of the following von Neumann algebra analogue of the Tannaka duality theorem, given in [43], [33],

**Theorem(1.3.14)[41]:** (Araki, Haag, Kastler, Takesaki). *Let $M$ be a von Neumann algebra, and let $\alpha$ be an action of a compact group $G$ on $M$. Let $H$ be another group and $\tau$ an action of $H$ on $M$ such that $[\alpha,\tau] = 0$ $(i.e. \alpha_g\tau_h = \tau_h\alpha_g\ for\ all g \in G, h \in H)$. Suppose that $\tau$ is ergodic$(i.e. M^\tau = C$, where $M^\tau$ denotes the fixed point subalgebra for $\tau)$. It follows that for any automorphism $\beta$ of $M$ such that $\beta \setminus M^\alpha = 1$ and $[\beta,\tau] = 0$, there exists a $g \in G$ such that*

$$\beta = \alpha_g\,.$$

For example, if $(M^\alpha)' \cap M = C$, then $H$ could be taken to be the unitary group of $M^\alpha$, and $\tau$ to be the adjoint mapping, $h \longmapsto (Adh) \mid M$.

We shall deduce this theorem from the $C^*$-algebra analogue, given later in [44], in which the hypothesis of ergodicity of $\tau$ is replaced by a stronger condition called strong topological transitivity. To do this, we shall show that for an action of a group on a von Neumann algebra, the two conditions are equivalent: ergodicity implies strong topological transitivity.

One way in which our proof is new is that it does not depend on the type of the von Neumann algebra. The original proof consists of first reducing to the infinite case, and then using Roberts's construction of Hilbert spaces in the algebra ([67]). Our proof does not use Hilbert spaces in the algebra.

**Theorem(1.3.15)[41]:***Let $M$ be a von Neumann algebra, $H$ a group, and $\tau$ an action of $H$ on $M$.*

*The following three conditions are equivalent.*

1. *$\tau$ is ergodic$(i.e. M^\tau = C$.*
2. *$\tau$ is topologically transitive, $i.e.$*
3. *$\tau$ is strongly topologically transitive, $i.e.$*

$\sum_{(\text{finite})} x_i\,\tau_h\,(y_i) = 0\ \forall\, h \in H\ \Sigma x_i \otimes y_i = 0.$

**Proof:** The implications $3 \longrightarrow 2$ and $2 \longrightarrow 1$ hold in any $C^*$-algebra, the firsttrivially, and the second by spectral theory. (If $M^\tau \neq C$ then there exist (positive) nonzero $x, y \in M^\tau$ *with* $xy = 0$, whence $x\tau_h(h) = xy = Q$ for all $h \in H$.)

$Ad\ 1 \longrightarrow 3$. We may suppose that $M$ is represented covariantly on a Hilbert space, for example by taking the crossed product by $\tau$. In other words, we may suppose that $\tau$ is determined by a unitary representation $U$ of $H$ : $\tau_h = (Ad\ U(h))\ |M, h \in H$.

Assume that $\tau$ is ergodic, and let $(x_i), (y_i)$ be finite sequences in $M$ such that, for each $h \in H$,

$$\sum x_i \tau_h (y_i) = 0\,, \quad i.e, \sum x_i U(h) y_i = 0\,,$$

It follows that

$$\sum x_i U(h) z' y_i = 0\,,\ \ h \in H\ , z' \in M'.$$

Hence,

$$\sum x_i b y_i = 0\,, \tag{11}$$

for any $b$ in the weakly closed linear span of $U(H)M'$. But since $U(h)M'U(h)^* = M'$ for each $h \in H$, the linear span of $U(H)M'$ is a *-algebra, and so by the bicommutant theorem its weak closure is

$$(M' \cup U(H)''\gamma = (M \cap U(H)')' = (M^\tau)' = c'\,,$$

$i.e.$ the algebra of all bounded operators on the Hilbert space.

In particular, (11) holds with $b$ an operator of rank one, i.e. with $b = \xi \otimes \eta^* : \xi \mapsto (\xi|\eta)\xi$ and from

$$\sum x_i (\xi \otimes \eta^*) y_i = 0\,, \qquad \text{i.e.} \sum x_i \xi \otimes (y_i^* \eta)^* = 0\,,$$

Follows

$$\sum x_i \xi \otimes y_i^* \eta = 0\,,\ \text{i.e.}\ \ (\sum x_i \otimes y_i^*)(\xi \otimes \eta) = 0\,.$$

Since $\sum x_i \otimes y_i^*$ is a bounded linear operator and the vectors $\xi$ and $\eta$ are arbitrary, this shows that $\sum x_i \otimes y_i^* = 0$. Therefore, $\sum x_i \otimes y_i = 0$,

**Corollary(1.3.16)[41]:** Let Abe a $C^*$ -algebra, and let $\tau$ be an action of a group H on A. Suppose that there exists a faithful $\tau$-covariant representation $\pi$ of A such that the extension of $\tau$ to $\pi(A)''$ is ergodic (i.e. $(\pi(A)'')^\tau = C$). It follows that $\tau$ is strongly topologically transitive.

**Corollary(1.3.17)[41]:** (special case of 5.3). Let Abe a $C^*$ - algebra, and let B be a sub - $C^*$-*álgebra* of $A$. Suppose that there exists a faithful representation $\pi$ of $A$ such that

$$\pi(B)' \cap \pi(A)'' = C\,.$$

*It follows that the unitary group of B (with unit adjoined, if necessary) acts strongly* topologically transitively on A. (Compare 10 $\longrightarrow$ 12 of Theorem (1.3.1).

It follows that the unitary group of B (with unit adjoined, if necessary) acts strongly topologically transitively on $A$. (Compare 10 $\longrightarrow$ 12 of Theorem (1.3.1)

(using [44]). Let $\beta$ be an automorphism of $M$ such that $\beta|\ A^\alpha = 1$ and $[\beta, \tau] = 0$. All the hypotheses of Theorem (1.3.1) of [44] are now verified, except that the system $(M, G, \alpha)$ is assumed only to be a $W^*$ -dynamical system, not a $C^*$-dynamical system. It is straightforward, however, to modify the proof of Theorem (1.3.1) of [44] by putting the ultra weak topology of $M$ in place of the norm topology. The conclusion $\beta = \alpha_g$ for some $g \in G$ follows.

Alternatively, as in the proof of 12 ⟶ 13 of Theorem (1.3.1) above we may note that the proof of Theorem (1.3.1) of [44] is valid without any assumption of continuity of α at all until the last line — provided that $M_F$ is defined as the set of all $x \in M$ such that the linear span of $\alpha_g(X)$ is finite-dimensional. This yields that, for some $\in G, \beta = (\alpha_g\ on$ $M_F$. By the Peter- Weyl theorem generalized to boundedly complete locally convex spaces (including Banach space duals, and therefore $W^*$ −algebras), $M_F$ is ultraweakly dense in $M$, and hence $\beta = \alpha_g$.

We note, finally, that the condition that the relative commutant of the fixed point sub algebra be trivial appears in recent work of Doplicher and Roberts ([50], [51], [52]).

# Chapter 2
# Duality Theory and Free Action Compact Quantum Group

We relate our setting to recent work of De Commer and Yamashita by showing that any object in a module $C^*$-category over Rep G produces a weak unitary tensor functor, and, as a consequence, actions can also be described in terms of (Rep G)-module $C^*$-categories. As an application we discuss deformations of $C^*$-algebras by cocycles on discrete quantum groups. We show that an action is free if and only if the canonical map (obtained using the underlying Hopf algebra of the compact quantum group) is an isomorphism. In particular, we are able to express the freeness of a compact Hausdorff topological group action on a compact Hausdorff topological space in algebraic terms. As an application, we show that a field of free actions on unital $C^*$-algebras yields a global free action.

## Section (2.1): Nonergodic Actions

Category theory has, from the early beginning, played an important role in quantum groups. In the operator algebraic approach to quantum groups the key result connecting the two areas is due to Woronowicz [146]. Generalizing the classical Tannaka-Krein duality he showed that by associating to a compact quantum group its representation category together with the canonical fiber functor, we get a duality between compact quantum groups on one side, and $C^*$-tensor categories with conjugates and a unitary fiber functor, on the other. Therefore, in principle, all properties of a compact quantum group $G$ can be formulated entirely in terms of its representation category Rep $G$ and canonical fiber functor. Remarkably, a lot of properties depend on Rep$G$ alone. $A$ systematic study of such properties was made possible by Bichon, De Rijdt and Vaes [132], who showed how, given two monoidally equivalent (that is, having equivalent representation categories) compact quantum groups, to construct a linking $C^*$-algebra connecting the two. The linking algebra is equipped with ergodic actions of both quantum groups, and by fixing one quantum group $G$ and varying the other, or in other words, by considering all possible unitary fiber functors on Rep $G$, we get all ergodic actions of $G$ of full quantum multiplicity. This categorical point of view on actions, together with the construction of linking algebra, has been extremely successful. It has been applied to a variety of seemingly unrelated problems, from a study of random walks on discrete quantum groups [136] to $K$-theoretic computations [145].

Results of Bichon, De Rijdt and Vaes were later generalized by Pinzari and Roberts [143], who showed how to describe all ergodic actions of $G$ in terms of Rep$G$. Namely, for every ergodic action they constructed a "spectral functor" from Rep$G$ into the category of Hilbert spaces, and then gave an abstract characterization of such functors. Their result implies, in particular, that isomorphism classes of ergodic actions of monoidally equivalent compact quantum groups are in canonical correspondence with each other. Soon afterwards De Rijdt and Vander Vennet [136] showed that the same is true even for nonergodic actions. Their argument bypasses category theory altogether and is based on induction using the linking algebra. $A$ natural problem completing this circle of ideas is nevertheless to find a description of actions of $G$ entirely in terms of Rep$G$. Our goal is to do exactly that. By modifying the definition of a spectral functor and the axioms of Pinzari and Roberts, we show that actions of a compact quantum group G correspond to a class of functors, which we call weak unitary tensor functors, from Rep$G$ into categories CorrA of

$C^*$-correspondences over $C^*$-algebras $A$. It should become apparent from our results, and is not difficult to show directly, that in the case $A = C$ our definitions/axioms are equivalent to the ones given by Pinzari and Roberts. Overall, the construction of a $C^*$-algebra from a functor $RepG \rightarrow CorrA$ follows familiar lines going back to Woronowicz [146]. Since some of the maps involved are not adjointable, we just have to be more careful not to overuse various dualities.

A different solution to the same problem is suggested by recent work of De Commer and Yamashita [134]. Complementing the results of Pinzari and Roberts, they showed that ergodic actions of $G$ can also be described in terms of semisimple $(RepG) - moduleC^*$ −categories with a fixed simple generating object. In fact, a significant part of their arguments does not involve ergodicity/semisimplicity in any way, and we show that indeed by discarding these assumptions we get a characterization ofgeneral actions in terms of module categories. The relation between two categorical pictures can be described as follows. Given a $right\ (RepG) - module\ C^*$ −category and an object $M$ in it, the functor Mor$(M, M \otimes \cdot)$ has a canonical structure of a weak unitary tensor functor. Therefore, using the analogy with representation theory, we can say that the relation between two pictures is as between a cyclic representation and its matrix coefficient defined by the cyclic vector. From this point of view weak unitary tensor functors are categorifications of positive definite functions. It is interesting that the module category approach, being less economical than the approach via weak tensor functors, seems, nevertheless, more suitable for classification of actions, at least for representation categories described by simple universal properties [135].

We discuss some examples and applications of our general results.

The construction of a $C^*$ -algebra from a weak unitary tensor functor is reminiscent of various crossed product type constructions. To make the connection more explicit, we reformulate this construction in a category-free way. This will make it clear that for duals of discrete groups it generalizes such constructions as cross-sectional algebras of Fell bundles or crossed products by Hilbert bimodules. We also show that categorical point of view on actions naturally leads to a construction of deformation of $C^*$-algebras by 2-cocycles on discrete quantum groups.

We follow the same conventions as in [142]. Consider a compact quantum group $G$. The Hopf $*$-algebra of matrix coefficients of finite dimensional representations of $G$ is denoted by $(\mathbb{C}[G], \Delta)$. $A$ finite dimensional representation $U\ of\ G$ is an invertible element of $B(H_U) \otimes C(G)\ such\ that\ (\iota \otimes \Delta)(U) = U_{12}U_{13}$ . The tensor product of two representations $U\ and\ V$ is denoted by U × V and is defined by $U \times V = U_{13}V_{23}$.

The contragredient representation to a finite dimensional representation $U$ is defined by

$$U^c = (j \otimes \iota)(U^{-1}) \in B(H_U^*) \otimes C(G),$$

where j is the canonical anti-isomorphism $B(H_U) \cong B(H_U^*)$. When HU is a Hilbert space, we identify the dual space$H_U^*$ with the complex conjugate Hilbert space $H_U$.

We denote the Woronowicz character $f_1 \in C[G] *\ by\ \rho$. For every finite dimensional representation $U\ of\ G$ we have a representation $\pi U$ of the algebra $C[G]\ ^*$ on HU defined by $\pi U(\omega) = (\iota \otimes \omega)(U)$. Given a finite dimensional unitary representation $U\ of\ G$, the conjugate re resentationis defined by

$$_{\overline{U}} = (j(\pi U\ (\rho)1/2) \otimes 1)Uc(j(\pi U(p)^{\frac{-1}{2}} \otimes 1)\ \ B(H_U) \otimes C(G).$$

This is a unitary representation equivalent to $U^c$, and $\pi^{-}U\ (\rho) = j(\pi U\ (\rho) - 1)$. Unitarity of $^{-}U$ essentially characterizes $\rho$: $if\ U$ is an irreducible unitary representation, then $\pi U(\rho)$

is the unique strictly positive operator in $B(H_u)$ such that the above definition of $\bar{U}$ gives a unitary element and such that $Tr(\pi U(\rho)) = Tr(\pi U(p)^{-1})$. We will usually suppress πU and simply $write\ \rho\xi\ for\ \xi \in HU\ instead\ of\ \pi U(\rho)\xi$.

Denote by Rep$G$ the $C^*$-tensor category of finite dimensional unitary representations of $G$. In this category $\bar{U}$ is conjugate to $U$, in the sense that there exist morphisms $R_U$ : $1 \to \bar{U} \times U\ and\ R_U : 1 \to U \times \bar{U}$, where 1 is the trivial representation of $G$ on the one-dimensional space $C$, such that the compositions

$$U \xrightarrow{\otimes H_u} U \otimes \bar{U} \otimes \xrightarrow{A_i \otimes_i} and \bar{U} \xrightarrow{\otimes R_u} \bar{U} \otimes \mathrm{U} \otimes \bar{U} \xrightarrow{H_i \otimes_i} \bar{U}$$

are the identity morphisms. Using the Woronowicz character $\rho$ we can define such morphisms $by$

$$R_u(1) = \sum_i \bar{\xi_\iota} \otimes p^{\frac{-1}{2}}, \overline{R_u}(1) = \sum_i p^{\frac{1}{2}}\xi_i \otimes \bar{\xi_\iota} \qquad (1)$$

where { $\xi_i$ }i is an orthonormal basis in $HU$. Note that the above expressions do not depend on the choice of an orthonormal basis, and

$$[\![R_u]\!] = [\![\overline{R_u}]\!] = (dim_q U)^{\frac{1}{2}},$$

where dimq $U = Tr\ \pi U(\rho)$ is the quantum dimension of $U$.

Consider now a continuous left action $\theta\ of\ G\ on\ aC^* -$algebra $B, so\ \theta : B \to C(G) \otimes B\ is\ an\ injective *$ -homomorphism such that $(\Delta \otimes \iota)\theta = (\iota \otimes \theta)\theta$ and $(C(G) \otimes 1)\theta(B)\ is\ dense\ in\ C(G) \otimes B$. Consider the $*$-subalgebra $B \subset B$ consisting of elements $x \in B$ such that $\theta(x)$ lies in the algebraic tensorproduct $C[G] \otimes B$. Equivalently, B is the linear span of elements of the form $(h \otimes \iota)\big((a \otimes 1)\theta(x)\big)$,where $a \in C[G], x \in B$ and h is the Haar state on $G$. We call B the algebra of regular elements in $B$. It is a dense $*$-subalgebra of $B$, and θ defines a left coaction of the Hopf $*$-algebra $(C[G], \_)$ on it. As follows from [133], the positive map $E = (h \otimes \iota)\theta : B \to BG$ is faithful on B, in the sense that $E(x^*x)\ 6 = 0$ for every nonzero $x \in B$. Conversely, assume we have a left coaction θ of the Hopf $* -$algebra $(C[G], \Delta)\ on\ a\ * -$algebra $B$. By slightly extending the definition in [134] we say that $\theta$ is an algebraic action of $G$ if the following conditions are satisfied:

(i) the fixed point algebra $A = B^G = \{x \in B \mid \theta(x) = 1 \otimes x\}\ is\ a\ C^* -$algebra;

(ii) the projection $E = (h \otimes \iota)\theta : B \to A$ is positive and faithful, so $E(x^*x) \geq 0$ and $E(x^*x) \neq 0$ for $x \neq 0$;

(iii) $E(x^*a^*ax) \leq [\![a]\!]^2\, E(x^*x) for\ all\ a \in A\ and\ x \in B$.

Note that condition (iii) follows from (i) and (ii) if B is unital with unit $1 \in A$. Note also that conditions (ii) and (iii) can be formulated by saying that $B$ is a right pre-Hilbert $A$-module with inner product $(x, y) = E(x^*y)$, and the operators of multiplication on the left by elements of A are bounded.

Under the above conditions (i)-(iii) it is not difficult to show that the $*$-algebra B admits a unique $C^*$-completion $B$ such that $\theta$ extends to a continuous left action of the reduced form of $G\ on\ B$ , Namely, $B$ is faithfully represented by operators of multiplication on the left on the right pre-Hilbert A-module B with inner product$(x, y) = E(x^*y)$, and this defines a norm on B.

Note that in general the subalgebra of regular elements in the completion $B\ of\ B$ can be strictly larger than $B$. Given a finite dimensional unitary representation U of G, we can consider HU as a left comodule over $(C[G], \Delta)$ by defining

$$\delta_U : H_U \to C[G] \otimes H_U by \delta_U(\xi) = U_{21}^*(1 \otimes \xi)$$

Then, if $\theta$ is a continuous left action of $G$ on a $C^*$-algebra B, we can consider comodule maps $H_U \rightarrow B$. The linear span of images of all such maps is denoted by $B(U)$ and is called the spectral subspace of $B$ corresponding to $U$. Choosing representatives $U_\alpha$ of isomorphism classes of irreducible unitary representations of $G$, for the subalgebra $B \subset B$ of regular elements we get

$$B = \oplus_\alpha B(U_\alpha).$$

Consider the tensor product comodule $H_U \otimes B$. We denote by $(H_U \otimes B)^G$ the subcomodule of invariant vectors, so

$$(H_U \otimes B)^G = \{X \in H_U \otimes B \mid U_{12}X_{13} = (\iota \otimes \theta)(X)\}.$$

In other words, if $\{\xi_i\}$i is an orthonormal basis in $H_U$ and $U = (uij)i,j$ is written in the matrix formwith respect to this basis, then

$(H_U \otimes B)^G = \{X = \sum_i \xi_i \otimes x_i \mid \theta(x_i) = \sum_i U_{ij} \otimes x_j \, for\ all\ i\}$. Note that using Frobenius reciprocity we can identify $(H_U \otimes B)^G$with Hom$G(H_U, B)$, but we are not going to do this. The spectral subspaces can be recovered from$(H_U \otimes B)^G$ using the canonical surjective maps

$$\overline{H_U} \otimes (H_U \otimes B)^G \rightarrow B(\overline{U}), \bar{\xi} \otimes X \mapsto (\bar{\xi} \otimes \iota)(X),$$

which are isomorphisms for irreducible $U$.

The spaces $(H_U \otimes B)^G$ is our main object of interest. Clearly, if $A = B^G$ then these spaces are $A$-bimodules. Furthermore, if $X = \sum_i \xi_i \otimes x_i \; and \; Y = \sum_i \xi_i \otimes y_i$ are vectors in$(H_U \otimes B)^G$, then the element $\sum_i \xi_i x_i^* y_i is \; G$-invariant. Hence$(H_U \otimes B)^G$ is a right Hilbert $A$-module with inner product(X , Y) = $\sum_i x_i^* y_i$ This inner product is independent of the choice of an orthonormal basis, and by slightly it can be written as (X , Y) $= X^*Y$. Given two finite dimensional unitary representation $U$ and $V \; of \; G$, we have a map

$$(H_U \otimes B)^G \otimes (H_V \otimes B)^G, \rightarrow (H_{U\times V} \otimes B)^G, X \otimes Y \mapsto X_{13}Y_{23}$$

In other words, if we fix orthonormal bases $\{\xi_i\}$i in HU and $\{\zeta_j\}$j in $H_V$, then for $X = \sum_i \xi_i \otimes x_i \; and \qquad Y = \sum_i \zeta_J \otimes Y_j \; we \; have$

$$X \otimes Y \mapsto \sum_i \xi_i \otimes \zeta_j \otimes x_i y_j .$$

It is obvious that this map defines an isometric map

$$(H_U \otimes B)^G \otimes A \, (H_V \otimes B)^G \rightarrow (H_{U\times V} \otimes B)^G$$

We are now ready to define spectral functors.

**Definition (2.1.1)[130]:** Given a continuous left action of a compact quantum group $G$ on a $C^*$-algebra $B$ with fixed point algebra $A$, the associated spectral functor is the unitary functor F from Rep$G$ into

the $C^*$- tensor category CorrA of $C^*$ -correspondences over A defined by

$F(U) = (H_U \otimes B)^G$ with inner product $\langle X,Y\rangle = X^*Y$

for representations $U$, and $F(T) = T \otimes \iota$ for morphisms, together with the $A$-bilinear isometries

$$F_2, U, V : F(U) \otimes_A F(V) \rightarrow F(U \times V), X \otimes Y \mapsto X_{13}Y_{23}$$

A few comments are in order. By a$C^*$ -correspondence over $A$ we mean a right Hilbert $A$-modul together with a *nondegenerate* left action of $A$ on it. We have to explain why the left action on $(H_U \otimes B)^G$is nondegenerate in the nonunital case. This is a consequence of the following simple lemma.

**Lemma (2.1.2)[130]:** *If $\theta$ is a continuous left action of a compact quantum group $G$ on a $C^*$ -algebra $B$,then the fixed point algebra $A$ $B^G$ = is a nondegenerate $C^*$ -subalgebra of $B$.*

**Proof:** Let $\{e_s\}_s$ be an approximate unit in $A$. Define an $A$-valued inner product on $B$ by $\langle X,Y\rangle = E(x^*y)$. Then $xe_s \to x$ in the norm defined by this inner product for every $x \in B$. By [133], on every spectral subspace $B(U) \subset B$ the norm defined by the inner product is equivalent to the $C^*$-norm. Therefore $xe_s \to x$ in the $C^*$ -norm for every $x \in B$, hence for every $x \in B$.

$C^*$-correspondences over $A$ form a $C^*$ -tensor category CorrA with adjointable $A$-bilinear maps as morphisms. We emphasize that the isometries $F_2, U, V$ in the definition of the spectral functor are not claimed to be adjointable, and therefore formally they are not morphisms in CorrA.

Finally, recall that two natural notions of isometry between Hilbert modules coincide: if $M$ and $N$ are right Hilbert $A$-modules, and $T : M \to N$ is an A-linear map such that $\|TX\| = \|X\|$ for all $X \in M$, then$\langle TX,TY\rangle = \langle X,Y\rangle$for all $X, Y \in M$.

We give an abstract characterization of spectral functors. Here is the main definition.

**Definition (2.1.3)[130]**: Given a $C^*$-algebra $A$ and a strict$C^*$ -tensor category $C$ with unit object 1, *by a* weak unitary tensor functor $C \to CorrA$ we mean a linear functor $F : C \to CorrA$ together with natural $A$-bilinear isometries $F_2 = F_2, U, V : F(U) \otimes A\, F(V) \to F(U \otimes V)$ such that the following conditions are satisfied:

(i) F(1) = A;

(ii) $F = F(T)^*$ *for any morphism* $T$ *in* $C$;

(iii) $F_2 \colon A \otimes_A F(U) \to F(1 \otimes U) = F(U)$ *maps* $a \otimes X$ *into* $aX$, *and similarly* $F_2 : F(U) \otimes_A A \to F(U)$ *maps* $X \otimes a$ *into* $Xa$;

(iv) the diagrams

$$\begin{array}{ccc} F(U)\otimes_A F(V)\otimes_A F(W) & \xrightarrow{F_2\otimes\iota} & F(U\otimes V)\otimes_A F(W) \\ \downarrow{\scriptstyle \iota\otimes F_2} & & \downarrow{\scriptstyle F_2} \\ F(U)\otimes_A F(V\otimes W) & \xrightarrow{F_2} & F(U\otimes V\otimes W) \end{array}$$

(v) for all objects U and V in C and every vector $X \in F(U)$, the right A-linear map $S_X = S_{XV},\colon F(V) \to F(U \otimes V)$ mapping $Y \in F(V)$ into $F_2(X \otimes Y)$ is adjointable, and the diagrams

$$\begin{array}{ccc} F(U\otimes V)\otimes_A F(W) & \xrightarrow{F_2} & F(U\otimes V\otimes W) \\ \downarrow{\scriptstyle S_X^*\otimes\iota} & & \downarrow{\scriptstyle S_X^*} \\ F(V)\otimes_A F(W) & \xrightarrow{F_2} & F(V\otimes W) \end{array}$$

commute.

Note that any unitary tensor functor $C \to CorrA$ defines a weak unitary tensor functor. In othe words, if conditions (i)-(iv) are satisfied and the maps $F_2$ are surjective, then condition (v) is also satisfied. Indeed, the map $S_X : F(V) \to F(U \otimes V)$ is adjointable, because by assumption $F_2$ is unitary and the map $Y \longmapsto X \otimes Y$ is adjointable, with adjoint given by $X' \otimes Y' \longmapsto hX, X'iY'$. Since

$S_X F_2 = F_2(S_X \otimes \iota) : F(V) \otimes_A F(W) \to F(U \otimes V \otimes W)$, by taking the adjoints we get $F_2^* = (S_X^* \otimes \iota) F_2^*$. This is equivalent to commutativity of the diagram in (v) by unitarity of $F_2$ Note also that if we consider $F$ simply as a functor into the category of vector spaces, then $F_2$is a natural transformation from F to $F(U \otimes \cdot)$, *and so* $S_X^*$ is a

$$S_X^* F(\iota \otimes T) = F(T) S_X^* \tag{2}$$

for morphisms $T$ *in* $C$.

Given a continuous left action of a compact quantum group $G$ on a $C^*$-algebra B with fixed point algebra $A$, the associated spectral functor $RepG \to CorrA$ is a weak unitary tensor functor. Indeed, properties (i)-(iv) are immediate, while (v) follows by observing that the adjoint of the map

$$S_X : (H_V \otimes B)^G \to (H_{U\times V} \otimes B)^G, Y \longmapsto X_{13}Y_{23}$$

is given by $S_X^* Z = X_{13}^* Z$. In other words, if $X = \sum_{ij} \xi_i \otimes x_i$ and $Z = \sum_{ij} \xi_i \otimes z_{ij}$ for orthonormal bases $\{\xi_i\}$i in $H_U$ and $\{\xi_i\}$j in$H_V$ , then

$$S_X^* Z = \sum_{ij} \zeta_j \otimes x_i^* z_{ij} \in (H_V \otimes B)^G \tag{3}$$

The following is our main result.

$$xy = \pi(x \cdot y) for\, x, y \in B_F.$$

**Lemma(2.1.4)[130]:** *The map* π : $\bar{B}_F \to B_F$ *is a homomorphism, hence the product on* $B_F$ *is associative.*

**Proof:** We have to check that $\pi(\pi(x) \cdot \pi(y)) = \pi(x \cdot y)$ for all $x, y \in \overline{B_F}$. Take $x = \bar{\xi} \otimes X \in \overline{H_U} \otimes F(U), y = \bar{\zeta} \otimes Y \in \overline{H_V} \otimes F(V)$ and choose isometries $U_I \in Mor(U_{\alpha i}, U), V_J \in Mor(U_{\alpha J}, V)$ and $wijk\ W_{ijk} \in Mor(u_{\alpha K}, u_{\alpha i} \times, u_{\alpha j})$ defining decompositions $of\ U, V\ and\ U_{\alpha I} \times U_{\alpha J}$ into irreducibles. Then

$$\pi(\pi(x) \cdot \pi(y)) = \sum_{ijk} \overline{W_{\iota jk}^*(U_I^* \xi \otimes V_J^* \zeta)} \otimes F(W_{ijk}^*) F_2(F(U_I^*)X \otimes F(V_j^*)Y),$$

While

$$\pi(x \cdot y) = \sum_{ijk} \overline{W_{\iota jk}^*(U_I^* \xi \otimes V_J^* \zeta)} \otimes F\left(W_{ijk}^*(U_I^* \otimes V_j^*)\right) F_2(X \otimes Y).$$

By naturality of$F_2$ these expressions are equal.

We can identify the space $\overline{He} \otimes F(\overline{Ue}) = \bar{\mathbb{C}} \otimes A \subset B_F$ with $A$. Under this identification, the space $A$, with its original product, becomes a subalgebra of $B_F$. Furthermore, the left and right multiplications on $\overline{H_\alpha} \otimes F(U\alpha)$ by elements of $A \subset B_F$ are defined by the $A$-bimodule structure *on* $F(U\alpha)$, that is, the product on BF has the property

$$a(\bar{\xi} \otimes X) = \bar{\xi} \otimes aX, (\bar{\xi} \otimes X)a = \bar{\xi} \otimes Xa.$$

Our next goal is to define an involution on $B_F$. For a finite dimensionalunitary representation $U$ of $G$ consider the standard solution $(R_U, \overline{R_U})$ of the conjugate equations for $U$ defined by (2.1.1).

**Lemma(2.1.5)[130]**: *For every* $X \in F(U)$ *there exists a unique element* $X^* \in F(\bar{U})$ *such that*

$$\langle X^*, Y \rangle = F(R_U^*) F_2(X \otimes Y)\ for\ all\ Y \in F(\bar{U}).$$

*If the* $C^*$*-algebra* $A$ *is unital, then* $X^* = S_X^* F(\overline{R_U})(1)$. *We also have*

$$\langle X^*, Y \rangle = F(R_U^*) F(X^* \otimes Y)\ for\ all\ Y \in F(U).$$

**Proof:** The uniqueness is clear. In order to prove the existence assume first that $A$ is unital. We then have

$\langle\, S_X^* F(\overline{R_U})(1), Y \,\rangle = \langle F(\overline{R_U})(1), F_2(X \otimes Y)\rangle = F(\overline{R_U^*})F_2(X \otimes Y)$,
so $X^* = \langle\, S_X^* F(\overline{R_U})(1)$. If A is nonunital, then a similar computation shows that for any $X \in F(U)$ and $a \in A$ the element$(aX)^*$exists and $(aX)^* = \langle\, S_X^* F(\overline{R_U})(a^*)$. But this is enough, since by Cohen's factorization theorem any element of F(U) has the form aX. To prove the last statement in the formulation, assume once again that $A$ is unital, the nonunital case requires only a minor modification. For $Y \in F(U)$ we compute

$$\begin{aligned} F(R_U^*)F_2(X^* \otimes Y) &= F(R_U^*)F_2(S_X^* F(\overline{R_U})(1) \otimes Y) \\ &= F(R_U^*)S_X^* F_2\,(F(\overline{R_U})(1) \otimes Y) \\ &= S_X^* F(\iota \otimes R_U^*)F(\overline{R_U} \otimes \iota)Y \; by\ (1) \\ &= S_X^* Y. \end{aligned}$$

$Here\ S_X\ is\ the\ map\ A \to F(U), a \mapsto F_2(a \otimes X) = aX, so\ S_X^*\, Y = \langle X, Y\rangle$

This lemma implies that the correspondences $F(U)\ and\ F(\bar{U})$ are, in some sense, dual to each other. In general, this is not the duality in the $C^*$-categorical sense. Already the simplest examples, such as the spectral functor associated with the action of $T$ by rotations on the unit disk, show that the objects $F(U)$ do not necessarily have conjugates in $CorrA$.

Similarly, for every vector $\xi \in H_U$ define a vector $\xi^* \in H_{\bar{U}}$ by

$$\xi^* = \left(\iota \otimes \bar{\xi}\right) R_U\,(1) = \overline{P^{-1/2}\xi}, so\ (\zeta, \xi^*) = R_U^*(\zeta \otimes \xi) for\ all\ \zeta \in H_{\bar{U}}.$$

Define an anti-linear $map\ \bar{B}_F \to \bar{B}_F\ , x \mapsto x^*, by$

$$(\bar{\xi} \otimes X)^* = \xi^* \otimes X^*.$$

For $x \in B_F\ put\ x^* = \pi(x^*). On\ A \subset B_F$ this clearly coincides with the involution on A. Although we will not need this, we remark that it is not difficult to show that the particular choice of solutions $(R_U, \overline{R_U})$ was not important for defining the involution on $B_F$, in the sense that for every $x \in \overline{B_F}$ the element $\pi(x^*)$ is independent of any choices

**Lemma (2.1.6)[130]**: *The map $x \mapsto x^*$ defines an involution on the algebra$B_F$, and for every $x \in \bar{B}_F$ we have $\pi(x)^* = \pi(x^*)$*

**Proof:** We start by proving the second part. We have to show that$\pi(\pi(x)^*) = \pi(x^*)$ Take an element $x = \bar{\xi} \otimes X \in \bar{H}_U \otimes F(U)$. Choose isometries $W_i \in Mor(U_{\alpha i}, U)$ defining a decomposition of $U$ into irreducibles. Write $R_i$ for $R_{U\alpha_i}$ and$\bar{R_\iota}$ for $\bar{R}_{U\alpha_i}$. Then $R_U = \sum_i(\,\overline{w_\iota} \otimes wi)R_i$ and $\bar{R}_U = \sum_i(\,w_i \otimes \bar{w}_i)\bar{R}_i$. For any $Y \in F(\bar{U})$ we have

$$F\left(\bar{R}_U^*\ F_2(X \otimes Y\,) = \sum\nolimits_i F\,(\bar{R}_U^*)F_2(F(W_i^*\,)X \otimes F(\bar{W}_i^*\,)Y\,)\right.$$

$$= \sum\nolimits_i \langle\langle F\,(W_i^*)X \Big)\ ^*, F(\,\bar{W}_i^*)Y\,,$$

so $X^\bullet = \sum_i F\,(\overline{W_\iota})(F(W_i)X)\ ^\bullet$. We also have $\xi^\bullet = \sum_i \bar{W}_i\,(W_I^*\xi)\ ^\bullet$. Therefore

$$x^\bullet = \sum\nolimits_i \bar{\bar{W}}_i\,\overline{(W_I{}^*\xi)\ ^\bullet} \otimes F(\bar{W}_i)F(W_i)X)\ ^\bullet.$$

Applying π and using (3) we get

$$\pi(x^\bullet) = \sum\nolimits_i \pi\ \overline{(W_I{}^*\xi)\ ^\bullet} \otimes\ (F(W_i)X)\ ^\bullet = \pi(\pi(x)\ ^\bullet).$$

We next prove anti-multiplicativity of the map $*$ on BF. For this it suffices to check that for all
$x, y \in \bar{B}_F$ we have$\pi((x.y)^\bullet) = \pi(y^\bullet \cdot x^\bullet)$. Take $x = \bar{\xi} \otimes X \in \bar{H}_U \otimes F(U) and\ y = \bar{\zeta} \otimes Y \in \bar{H}_V \otimes F(V)$.

The unitary $\sigma : H_{\bar V} \otimes H_{\bar U} \to H_{\overline{U\times V}}$ mapping $\bar\eta \otimes \bar\vartheta$ into $\overline{\vartheta \otimes \eta}$ defines an equivalence between $\bar V \times \bar U$ and $\overline{U \times V}$ , and we have

$R_{U\times V} = (\sigma \otimes \iota \otimes \iota)(\iota \otimes R_U \otimes_i)R_V$ and $\bar R_{U\times V} = (\iota \otimes \iota \otimes \sigma)(\iota \otimes \bar R_V \otimes \iota)\bar R_U$.

Assuming that A is unital we compute:

$$F_2(X \otimes Y)^\bullet = S^*_{F2\ (X\times Y)^F} F(\bar R_{U\times V})(1) \quad \text{by Lemma (2.1.6)}$$
$$= S^*_Y S^*_X\ F(\iota \otimes \iota \otimes \sigma)F(\iota \otimes \bar R_V \otimes \iota)F(\bar R_U)(1) \text{as } SF_2(X \otimes Y) = S_X S_Y$$
$$= F(\sigma)S^*_Y F(\bar R_V \otimes \iota)S^*_X F(\bar R_U)(1) \qquad by\ (1)$$
$$= F(\sigma)S^*_Y F(\bar R_V \otimes \iota)(X^\bullet)$$
$$= F(\sigma)S^*_Y F_2(F(\bar R_V)(1) \otimes X^\bullet)$$
$$= F(\sigma)F_2 S^*_Y( F(\bar R_V)(1) \otimes X^\bullet)$$
$$= F(\sigma)F_2(Y^\bullet \otimes X^\bullet).$$

In the nonunital case we get the same identity by replacing $X$ and $Y$ by elements of the form $aX$ and b$Y$ , see the proof of Lemma (2.1.6)We also have $(\xi \otimes \zeta)^\bullet = \sigma(\zeta^\bullet \otimes \xi^\bullet)$. Therefore

$$(x \cdot y)^\bullet = (\bar\sigma \otimes F(\sigma))\,(\overline{\zeta^\bullet \otimes \xi^\bullet}) \otimes F_2(Y^\bullet \otimes X^\bullet)_- = (\bar\sigma \otimes F(\sigma))(y\bullet \cdot x\bullet).$$

Applying π we get $\pi((x.y)^\bullet) = \pi(y^\bullet \cdot x^\bullet)$.

It remains to show that the *map* $x \mapsto x^*$ on $B_F$ is involutive. Equivalently, we have to show that $\pi(x^{\bullet\bullet}) = \pi(x)$ for all $x \in \bar B_F$ . Take an element $x = \bar\xi \otimes X \in \bar H_U \otimes F(U)$. Consider the unitary

$u: H_U \to H_{\bar{\bar U}}$ *mapping* $\zeta$ *into* $\bar\zeta$. *Then* $\bar R_{\bar U} = (\iota \otimes u)R_U$. *Hence, for any* $Y \in F(\bar U)$,

$$\langle X^{\bullet\bullet}, Y\rangle = F(\bar R^*_U)F_2(X^\bullet \otimes Y) = F(R^*_U)F_2(X^\bullet \otimes F(U^*)Y) = \langle X, F(U^*)Y\rangle,$$

where the last equality follows from Lemma (2.1.6)Thus $X^{\bullet\bullet} = F(u)X$. We also have $\xi^{\bullet\bullet} = \bar{\bar{\bar\xi}} = u\xi$. Therefore

$$x^{\bullet\bullet} = \big(\bar u \otimes F(u)\big)x,$$

and applying π we get $\pi(x^{\bullet\bullet} = \pi(x)$.

We next define a linear map $\theta_F : B_F \to C[G] \otimes B_F$ by

$$\theta_F\,(\bar\xi \otimes X) = (U^C_\alpha)_{\ 21})^* 1(1 \otimes \bar\xi \otimes X)\ for\ \bar\xi \otimes X \in \bar H_\alpha \otimes F(U_\alpha).$$

In other words, if we fix an orthonormal *basis* $\{\xi_i\}_i$ in $H_\alpha$ and write $U_\alpha$ as a matrix $(U_{ij})_{ij}$ , then

$$\theta_F\,(\bar\xi_i \otimes X) = \sum\nolimits_i U_{IJ} \otimes \xi_i \otimes X.$$

**Lemma (2.1.7)[130]:** *The map $\theta_F$ defines a left algebraic action of $G$ on $B_F$ with fixed point algebra $A$.*

**Proof:** Clearly, the map $\theta_F$ turns BF into a comodule over $(C[G],\Delta)$ with fixed point subcomodule $A$.

In order to show that $\theta_F$ is a homomorphism, observe first that we have a left comodule structure map $\theta_F : \bar B_F \to C[G] \otimes \bar B_F$ on $\bar B_F$ defined in the same way as for $B_F$ :$\bar\theta_F$, so $(\xi \otimes X) = (U^C)^*_{21}\big(1 \otimes \bar\xi \otimes X\big)$

for $\bar\xi \otimes X \in \bar H_U \otimes F(U)$. Then $\pi : \bar B \to B_F$ is a comodule map, since if $w \in Mor(U,V)$, *then* $U^{C*}(\bar w \otimes 1) = (\bar w \otimes 1)V^{C*}$. Using that $(U\times V)^{C*} = (U^C)^*_{13}(V^C)^*_{23}$, modulo identification of $\overline{H_U \otimes H_V}$. with $\bar H_U \otimes \bar H_V$ , it is easy to see that $\bar\theta_F$ is a homomorphism. Hence $\theta_F$ is also a homomorphism

We check that $\theta_F$ is $*$-preserving. It suffices to show that $\bar{\theta}_F\,(x)^{*\otimes\bullet}) = \bar{\theta}_F\ (x^\bullet)\, for\ x \in \bar{H}_U \otimes F(U) \subset \bar{B}_F$. $F$ ixing an orthonormal basis $\{U_i\}_i$ in $H_U$ and identifying $\overline{H}_U$ with, $H_U$ we get

$\theta_F\,(\bar{\xi}_i \otimes X)^{*\otimes\bullet} = \sum_i U_{ij}^* \otimes \rho^{-\frac{1}{2}}\xi_j \otimes X^\bullet$ and $\bar{\theta}_F((\xi_j \otimes X)^\bullet = (U^c)\ {}^*_{21}\,(1 \otimes \rho^{-\frac{1}{2}}\xi_j \otimes X^\bullet)$.

Since $\overline{U}^{c*} = (\rho^{-\frac{1}{2}} \otimes 1)U^*(\rho^{\frac{1}{2}} \otimes 1)$, these expressions coincide.

It remains to show that $B_F$ is a right pre-Hilbert $A$-module with inner product $\langle x,y\rangle = E(x^*y)$, where $E = (h \otimes \iota)\,\theta_F$ , and the left action of $A$ on $B_F$ by multiplication is bounded. This will follow immediately, if we can show that the spaces $\bar{H}_\alpha \otimes F(U_\alpha)$ are mutually orthogonal and

$$\langle\bar{\xi} \otimes X, \bar{\zeta} \otimes Y\rangle = \frac{1}{dim_g U_\alpha}\,(p^{-1}\xi,\zeta)\langle X,Y\rangle\ for\ \xi,\zeta \in H_\alpha, X,Y \in F(U_\alpha).$$

Note that if $z = \bar{\eta} \otimes Z \in \bar{H}_U \otimes F(U)$, then $E\big(\pi(z)\big) = \sum_i \overline{w_i^*\,\eta} \otimes F(W_i^*\,i\,)$, where $w_i \in Mor(1,U) are$ isometries such $that\ \sum_i w_i\,w_i^*$ is the projection onto the isotypic component of U corresponding to the trivial representation. This clearly implies mutual orthogonality of the spaces $\bar{H}_\alpha \otimes F(U_\alpha)$. If $U = \bar{U}_\alpha \times U_\alpha$, then the only isometry in Mor$(1,U)$, up to a phase factor, is $(dim_g\,u^\alpha)^{-1/2}R_\alpha$, where $R_\alpha = R_{U\alpha}$ . Therefore for $\xi,\zeta \in H_\alpha$ and $X,Y \in F(U_\alpha)$ we have

$$\langle\bar{\xi} \otimes X, \bar{\zeta} \otimes Y\rangle = \frac{1}{dim_g U_\alpha}\overline{R_\alpha^*(\xi^\bullet \otimes \zeta)}F(R_\alpha^*\,)F_2(X^\bullet \otimes Y\,).$$

By Lemma (2.1.5)we have $F(R_\alpha^*)F_2(X^\bullet \otimes Y) = \langle X,Y\rangle$. Using that $\xi^\bullet = \overline{p^{-1/2}\xi}$ it is also straightforward to check that $R_\alpha^*(\xi^\bullet \otimes \zeta) = (\zeta, p^{-1}\xi)$. This finishes the proof of the lemma.

As we discussed in the previous, an algebraic action of $G$ on $B$ uniquely defines a completion $B_F$ of $B_F$ carrying a continuous action of the reduced form of $G$. Therefore the previous lemma finishes our construction of a continuous action from a weak unitary tensor functor.

**Theorem(2.1.8)[130]:** *Assume $G$ is a reduced compact quantum group and $A$ is a $C^*$ −algebra. Then by associating to an action of $G$ on a$C^*$ -algebra its spectral functor we get a bijection between iso- morphism classes of triples $(B,\theta,\psi)$, where $\theta$ is a continuous left action of $G$ on a$C^*$ -algebra $B$ and $\psi$: $A \rightarrow B$ is an embedding such that $B^G = \psi(A)$, and natural unitary monoidal isomorphism classes of weak unitary tensor functors $RepG \rightarrow CorrA$.*

In the proof we will identify A with ψ(A) and simply talk about actions with fixed point algebra $A$.

The main part of the proof is, of course, a construction of an action from a weak unitary tensor functor $F : RepG \rightarrow CorrA$ . We will define this action in a series of lemmas.Choose representatives $U_\alpha$ of isomorphism classes of irreducible unitary representations of $G$, and write$H_\alpha$ instead of H$U_\alpha$ for the underlying Hilbert spaces. We assume that there exists an index e such that $U_e = 1$.Consider the space

$$B_F = \otimes_\alpha H_\alpha \otimes F(U_\alpha).$$

It will also be convenient to consider a much larger space. Choose aa small $C^*$-tensor subcategory $C \subset RepG$ containing the objects $U_\alpha$, and then put

$$B_F = \otimes_U \; H_U \otimes F(U),$$

where the summation is over all objects in C. We have a canonical linear $map\, \pi : \tilde{} B_F \to B_F\, defined\, as$ follows. For a finite dimensional unitary representation $U\, of\, G$, choose isometries $W_I \in Mor(U_{\alpha I}, U)$such that $\sum_i w_i w_i^* = \iota$. Then put

$$\pi(\bar{\xi} \otimes X) = \sum_i w_i^* \, \bar{\xi} \otimes F(w_i^*)X,$$

where $\overline{w_\iota}\zeta = wi\zeta, so\, \overline{w_\iota^*} \;\; \bar{\xi} = w_i^* \xi$. This definition is independent of the choice of isometries $w_i$, since for any other choice vj there exists a unitary matrix $(u_{ij)ij}$ $suchthatw_i = \sum_i u_i\, v_j$. One property of $\pi$ that we will regularly use, is that if $\bar{\xi} \otimes X \in$ $\bar{} \otimes H_U\, F(U)\, and\, w \in Mor(U, V)$ is an isometry, then

$$\pi\, w\xi \otimes F(w)X = \pi(\bar{\xi} \otimes X) \tag{4}$$

.

Define a product on ˜ BF by

$$(\bar{\xi} \otimes X) \cdot (\bar{\zeta} \otimes Y) = (\xi \otimes \zeta) \otimes F_3(X \otimes Y).$$

It is immediate that this product is associative. Considering BF as a subspace of ˜ $B_F$, we define a product on $B_F$ by

**Proof:** It is clear that isomorphic actions produce naturally unitarily monoidally isomorphic weak unitary tensor functors, and naturally unitarily monoidally isomorphic weak unitary tensor functors produce isomorphic actions. It remains to show that up to isomorphisms the constructions are inverse to each other.

Assume θ is a continuous left action of $G$ on a $C^*$-algebra B with fixed point algebra $A$. Let $F$ be the associated spectral functor and $B \subset B$ be the subalgebra of regular elements. Consider the algebraic action $\theta_F$ of $G$ on BF defined by $F$ as described above. We have a linear isomorphism

$$B_F \cong B\, mapping\, \pi(\bar{\xi} \otimes X) \in B_F\, into\, (\bar{\xi} \otimes \iota)(X) \in B$$

for $\bar{\xi} \otimes X \in \bar{H}_U \otimes (H_U \otimes B)^G$. It is easy to see that this is a $G$-equivariant isomorphism of algebras. It is a bit less obvious that this isomorphism is ∗-preserving. In order to show this, fix an irreducible representation $U_\alpha$ and an orthonormal basis $\{\xi_i\}_i$ in $H_\alpha$. Consider an element $X = \sum_i \xi_i \otimes x_i \in (H_\alpha \otimes B)^G$. Writing $\bar{R}_\alpha$ for $\bar{R}_{U_\alpha}$, assuming for simplicity that $A$ is unital and using Lemma (2.1.. 6) and identity (2.2) for $S_X^*$, we get

$$X^\bullet = S_X^* \; F(\bar{R}_\alpha)(1) = S_X^* \left( \sum_i p^{1/2} \otimes \xi_i \otimes \bar{\xi}_j \otimes 1 \right) = \sum_{i,j} (p^{1/2}\xi_j \xi_i)\bar{\xi}_j \otimes x_j^*$$

$$= \sum_i \overline{p^{1/2}\xi_\iota} \otimes x_i^* .$$

From this we see that the image of the element $(\bar{\xi} \otimes X)^* = \pi\,(p^{-1/2}\xi \otimes X^\bullet) \in$ $B_F\, in\, B\, equals$

$$\sum_I \left(p^{-1/2}\xi_i p^{1/2}\xi_j\right) x_j^* = (\sum_i (\xi_i, \xi) x_i)^* ,$$

so the isomorphism $B_F \cong B$ is indeed ∗-preserving

Now conversely, assume we start with a weak unitary tensor functor $F$, consider the action $\theta_F$ of $G$ on $B_F$, and define the corresponding spectral functor $F'$. It is easy to see that if we fix an irreducible representation $U_\alpha$ and an orthonormal basis $\{\xi_i\}_i$in $H_\alpha$, then the dense subspace $(H_\alpha \otimes B_F)G\, of \;\; F'(U_\alpha) = (H_\alpha \otimes B_F)^G$ consists of vectors of the form $\sum_i(\xi_i \otimes \bar{\xi}_i \otimes X), with\, X \in F(U_\alpha)$. We have the obvious $A$-bilinear map $F(U_\alpha) \to F'(U_\alpha)$ with dense image, mapping $X$ into $\sum_i(\xi_i \otimes \bar{\xi}_i \otimes X)$. Let us check that this map is isometric. Taking vectors $X' = \sum_i(\xi_i \otimes \bar{\xi}_i \otimes X)\, \bar{\xi}_i \otimes X)$ and $Y' = \sum_i(\xi_i \otimes \bar{\xi}_i \otimes Y)$ in $F'(U_\alpha)$, and writing $R_\alpha$ for $R_{U_\alpha}$, we compute:

$$\langle X', Y'\rangle = \sum_i (\bar{\xi}_i \otimes X)^* (\bar{\xi}_i \otimes Y) = \pi\left(\sum_i \left(\overline{p^{-1/2}\xi_\iota} \otimes X^\bullet\right) \cdot (\bar{\xi}_i \otimes Y)\right)$$
$$= \pi\left(\overline{R_\alpha)(1)} \otimes F_2(X^\bullet \otimes Y)\right)$$

Since, up to a scalar factor, $R_\alpha$ is an isometry in $\mathrm{Mor}(1, \bar{U}_\alpha \times U_\alpha)$, the last expression equals $F(R_\alpha^*)F_2(X^\bullet \otimes Y) = \langle X, Y\rangle$
by Lemma(2.1.6). Thus we get unitary isomorphisms $F(U_\alpha) \cong F'(U_\alpha)$. These isomorphisms for all $\alpha$ extend uniquely to a natural unitary isomorphism between the functors $F$ and. It isstraightforward to check that this isomorphism is monoida.
We will give a different categorical description of actions in terms of module categories. Recall that given a $C^*$-tensor category $C$, a right $C$-module $C^*$ -category is a $C^*$ -category $M$ equipped with a bilinear unitary functor $\otimes : M \times C \to M$ together with natural unitary isomorphisms $\varphi: (M \otimes U) \otimes V \to M \otimes (U \otimes V)\, and\, e: M \otimes 1 \to M$ satisfying certain coherence relations, see [134] for details. If $C$ is strict, a module category $M$ is called strict if $\varphi$ and e are the identity morphisms. Any module category over a strict $C^*$ -tensor category is equivalent to a strict one. In the following discussion we will tacitly assume that the $C^*$ -categories that we consider have
subobjects, meaning that for every projection $p$ in End($M$) there exists an object N and an isometry $v \in Mor(N, M)$ such that $vv^* = p$. This is a very mild assumption, as we can always complete a $C^*$-category with respect to subobjects.

Assume we are given a continuous left action of a reduced compact quantum group $G$ on a *unital* $C^*$-algebra $B$. Following [134], consider the category $D_B$ of unitary $G$-equivariant finitely generated right Hilbert $B$-modules. By definition, the morphisms in $D_B$ are $G$-equivariant maps of Hilbert B-modules. Since we consider only finitely generated Hilbert modules, such maps are automatically adjointable, so $D_B$ is a $C^*$ -category. It is a strict right (Rep$G$)-module $C^*$ -category: given a right Hilbert $B$-module $M$ with the action of $G$ given by an isometry $\delta M : M \to C(G) \otimes M$, we define $M \otimes U$ as the Hilbert $B$-module $M \otimes H_U$ with the action of $G$ given by $x \otimes \xi \mapsto U_{31}^*(\delta_M(x) \otimes \xi)$. Note that for $M = B$ the module $B \otimes U$ is, up to identification of $H_U \otimes B$ with $B \otimes H_U$, the same equivariant module $H_U \otimes B$ that we considered. The module $B$ generates the category $D_B$, in the sense that any object $D_B$ in is a subobject of $B \otimes U$ for some $U$. In other words, any $G$-equivariant finitely generated right Hilbert $B$-module $M$ is isomorphic to a direct summand of $B \otimes H_U$ for some $U$,
Let $F$ be the spectral functor associated with the action of $G$ on $B$. We have canonical isomorphisms

$$F(U) = (H_U \otimes B)^G \cong Mor(B, B \otimes U)$$

that map $\sum_i(\xi_i \otimes x_i) \in (H_U \otimes B)^G$ into the morphism $x \mapsto \sum_i(x_i x \otimes \xi_i)$. For ergodic actions, these isomorphisms, modulo some identifications in terms of $F$ robenius reciprocity, were already to identify algebras constructed in [134] with those defined by Pinzari and Roberts [143]. In other words, the key relation between the results in [134] and

[143] can be described by saying that given a (Rep$G$)-module$C^*$ -category $M$ and a simple object $M$ in $M$, the functor $U \mapsto Mor(M, M \otimes U)$ has all the properties of a spectral functor. With our characterization of spectral functors this becomes almost immediate. Specifically, and more generally, we have the following.

**Proposition(2.1.9)[130]:** *Assume* M *is a strict right module $C^*$ -category over a strict $C^*$ tensor category $C$. Take an object* M ∈M *and consider the unital $C^*$ -algebra $A = End(M)$. Then the following defines a weak unitary tensor functor $C \to CorrA$:*

$$F(U) = Mor(M, M \otimes U),$$

*with the right $A$-module structure on $F(U)$ given by composition of morphisms, the left $A$-module structure by $aX = (a \otimes \iota)X$ and the inner product by $\langle X, Y\rangle = X^*Y$ , the action of $F$ on morphisms is defined by* $\mathrm{F}(T)X = (\iota \otimes T)X$, *and*

$$F_2 : F(U) \otimes_A F(V) \to F(U \otimes V)$$

*is given by* $\mathrm{X} \otimes Y \mapsto (X \otimes \iota)Y$.

**Proof:** This is a routine verification. We only remark that the adjoint of the map

$S_X : F(V) \to F(U \otimes V) \mapsto Y \ (X \otimes \iota)Y,$

is obviously given by $S_Z^* Z = (X^* \otimes \iota)Z$.

If the object M happens to be generating, then we can reconstruct the whole category $M$ from the functor $F$. For $C = RepG$ this gives the following result.

**Proposition(2.1.10)[130]:** *Assume $G$ is a reduced compact quantum group and $M$ is a strict right $(RepG)$- module $C^*$-category generated by an object $M$. Put $A = End(M)$ and consider the weak unitary tensor functor $F : RepG \to CorrA$ defined by the object* M *as described in the previous proposition.*

*Let $\theta : B \to C(G) \otimes B$ be the continuous action corresponding to this functor by Theorem(2.1.4). Then $D_B$ is unitarily equivalent, as a $(RepG)-$module $C^*$ -category, to $M$, via an equivalence that maps the generator $B \in D_B$ into $M$.*

**Proof:** Consider the functor $F' : D_B \to CorrA$ defined by the object $B \in D_B$. By the above discussion, it is naturally unitarily monoidally isomorphic to the spectral functor associated with the action of $G$ on $B$, hence to $F$. *Let* $\psi: F' \to F$ be such an isomorphism. Note that we automatically have that $\psi: A = F'(1) \to F(1) = A$ is the identity map, since it is a bimodule map such that

$$\psi F_2 = F'_2(\psi \otimes \psi).$$

Consider the full subcategories $\bar{D}_B \subset D_B$ and $\bar{M} \subset M$ consisting of objects $B \otimes U$ and $M \otimes U$,

respectively. We want to define a functor $E: \bar{D}_B \to \bar{M}$. On objects we put $E(B \otimes U) = M \otimes U$. For morphisms $T \in \mathrm{Mor}(B, B \otimes U)$ we put $E(T) = \psi(T)$. More generally, given two finite dimensional unitary representations U and V , we have Frobenius reciprocity isomorphisms Mor $(B \otimes U, B \otimes V) \to Mor(B, B \otimes V \otimes \bar{U}), T \mapsto (T \otimes \iota)(\iota \otimes \bar{R}_U)$, with inverse $S \mapsto (\iota \otimes \iota \otimes R_U^*)(S \otimes \iota)$. We also have similar isomorphisms in $\bar{M}$. Hence we can define linear isomorphisms

$$E: Mor(B \otimes U, B \otimes V) \to Mor(M \otimes U, M \otimes V)$$

$$\text{by } E(T) = (\iota \otimes \iota \otimes R_U^*)\,(\psi\,((T \otimes \iota)(\iota \otimes \bar{R}_U)) \otimes_i.$$

Before we turn to the proof that E is indeed a functor, let us make two observations. The first one is that given a morphism $T : B \otimes U \to B \otimes V$ and a finite dimensional unitary representation $W$ of G, for the morphism $T \otimes \iota W = T \otimes \iota : B \otimes U \otimes W \to B \otimes V \otimes W$ we have

$$E(T \otimes \iota) = E(T) \otimes \iota. \quad (5)$$

The second observation is that given a morphism T : $B \otimes U \to B \otimes V$ and a morphism $S : V \to W$, we have

$$E\big((\iota \otimes S)T\big) = (\iota \otimes S)E(T). \quad (6)$$

Both claims follow easily from naturality of ψ, which means that for any morphisms $T : B \to B \otimes U$ and $S : U \to V$ we have $\psi((\iota \otimes S)T) = (\iota \otimes S)\psi(T)$.

Consider now morphisms $R: B \otimes U \to B \otimes V$ *and* $T : B \otimes V \to B \otimes W$, and define the morphisms $P = (R \otimes \iota)(\iota \otimes \bar{R}_U) : B \to B \otimes V \otimes \bar{U}$ and $S = (T \otimes \iota)(\iota \otimes \bar{R}_V) : B \to B \otimes W \otimes \bar{V}$. We then have

$$TR = (\iota B \otimes \iota W \otimes R_V^*)(S \otimes \iota V)(\iota B \otimes \iota V \otimes R_U^*)(P \otimes \iota U)$$
$$= (\iota B \otimes \iota W \otimes R_V^*)(\iota B \otimes \iota W \otimes \iota \bar{V} \otimes \iota V \otimes R_U^*)((S \otimes \iota \otimes \iota)P \otimes \iota U)$$
$$= (\iota B \otimes \iota W \otimes R_V^* \otimes R_U^*)(F'_2(S \otimes P) \otimes \iota U),$$

where $F'_2(S \otimes P) = (S \otimes \iota \otimes \iota)P : B \to B \otimes W \otimes \bar{V} \otimes V \otimes \bar{U}$. A similar computation gives

$$E(T)E(R) = (\iota M \otimes \iota W \otimes R_V^* \otimes R_U^*)\big(F_2\big(\psi(S) \otimes \psi(P)\big) \otimes \iota U\big).$$

From this we immediately get that $E(TR) = E(T)E(R)$ *using* (6), (5) and monoidality of $\psi$, which means that $\psi F'_2(S \otimes P) = F_2(\psi(S) \otimes \psi(P))$. Therefore $E$ is a functor. Since it is surjective on objects and fully faithful, it is an equivalence of the linear categories ˜DB and $\widetilde{M}$. Let us show next that the equivalence E is unitary, that is, $E(T)^* = E(T^*)$ on morphisms. Let us check this first for $T \in Mor(B, B \otimes U)$. Since $\psi$ is unitary, for any $S \in Mor(B, B \otimes U)$ we have

$$E(T)^*E(S) = \psi(T)^*\psi(S) = \langle\psi(T), \psi(S)\rangle = \langle T, S\rangle = T^* S = E(T^*S)$$
$$= E(T^*)E(S).$$

Since this is true for all S, we conclude that $E(T)^* = E(T)^*$. By virtue of (5) we then also get $E(T \otimes \iota)^* = E((T \otimes \iota)^*)$. But any morphism in $\bar{D}_B$ is a composition of such a morphism $T \otimes \iota W$ and a morphism of the form $\iota M \otimes S$ for some morphism $S$ in Rep$G$. Since as a particular case of (6) we have $E(\iota \otimes S)^* = \iota \otimes S^* = E((\iota \otimes S)^*)$, it follows that $E$ is unitary. Next, from (5) and (6) we see that if we define

$$E_2 = E_2, B \otimes U, V : E(B \otimes U) \otimes V \to E(B \otimes U \otimes V)$$

to be the identity maps, then we get a natural isomorphism of bilinear functors $E(\cdot) \otimes \cdot$ and $E(\cdot \otimes \cdot)$. Therefore the pair $(E, E_2)$ defines a unitary equivalence of $(RepG)$-module categories $\bar{D}_B$ and $\bar{M}$. Finally, since $D_B$ *and* $M$ are completions of these categories with respect to subobjects, the equivalence between $\bar{D}_B$ and $\bar{M}$ extends uniquely, up to a natural unitary isomorphism, to a unitary equivalence between the $(RepG)$-module $C^*$-categories $D_B$ *and* $M$.

This leads to the main theorem, a generalization of results of De Commer and Yamashita [134] to the nonsemisimple/nonergodic case.

**Theorem (2.1.11)[130]:** *Assume $G$ is a reduced compact quantum group. Then by associating to an actionof $G$ on a unital $C^*$-algebra $B$ the* (R*epG*)*-module category $D_B$with generator* B, *we get a bijec- tion between isomorphism classes of continuous left actions of $G$ on unital $C^*$-algebras and unitary equivalence classes of pairs* (M,M), *where* M *is a right* (RepG)*-module $C^*$-category and $M$ is a generating object in $M$.*

**Proof:** In view of the above proposition we only have to show that two actions of $G$ on unital $C^*$- algebras $B$ and $C$ are isomorphic if and only if the pairs $(D_B, B)$ *and* $(D_C, C)$ are unitarily equivalent. Given such an equivalence, we first of all get an isomorphism $B^G = End\, D_B(B) \cong EndD_C\,(C) = C^G$. Modulo the identification of $B^G$ with $C^G$ using this isomorphism, we then also get a natural unitary isomorphism between the spectral functors associated with our actions. Hence the actions are isomorphic by the easy part of Theorem (2.1.4). Conversely, it is clear that isomorphic actions produce unitarily equivalent pairs.

As in [134], this result can also be formulated in terms of Morita equivalent actions.

**Corollary (2.1.12)[130]:** *For any reduced compact quantum group $G$, there is a bijection between Morita equivalence classes of continuous left actions of $G$ on unital $C^*$-algebras and unitary equivalence classes of singly generated right $(RepG)$-module $C^*$-categories.*

**Proof:** It suffices to show that two actions of G on unital $C^*$-algebras B and C are Morita equivalent if and only if the $(RepG)-$module $C^*$-categories $D_B\ and\ D_C$ are unitarily equivalent. In one direction this is obvious: if a $C-B$-bimodule M defines the Morita equivalence, then $D_C$ and DB are unitarily equivalent, via an equivalence that maps $N \in D_C\ into\ N \otimes C_M$. Conversely, assume we have a unitary equivalence $E: D_C \to D_B\ of\ (RepG)$ $-$module categories. Consider the right Hilbert BmoduleM $= E(C)$. Since it is a generating object in $D_B$, and therefore the right Hilbert $B$ $-$module $B$ can be isometrically embedded into $M \otimes H_U$ for some representation $U$, the module $M$ must be full. Thus the action of $G$ on $B$ is Morita equivalent to the action of $G\ on\ C' = EndB(M).The\ (RepG)$ $-$module $C^*$-categories $D_B$and $D_{C'}$ are unitarily equivalent, via an equivalence that maps $M \in D_B$ into $C' \in D_{C'}$. It follows that the $(RepG)-module\ C^*$ $-$categories $D_C$and$D_{C'}$ are unitarily equivalent, via an equivalence that maps $C\ into\ C'$. By Theorem (2.1.11)this implies that the actions of $G\ on\ Cand\ C'$ are isomorphic, so the actions of $G$ on $B$ and $C$ are Morita equivalent.

**Remark (2.1.13)[130]:**In the above proof we used that if an equivariant right Hilbert B-module M is a generating object in $D_B$, then it is full. The proof implies that the converse is also true, since if $M$ is full, then $M$ is the image of the generating object $C$ in DC, where $C = End_B(M), under$ the equivalence of categories $D_C$ and $D_B$defined by $M$, hence $M$ is a generating object in$D_B$. Somewhat more explicitly this can also be proved as follows. It suffices to show that $B \in D_B$ is a subobject of $M \otimes V$ for some finite dimensional unitary representation $V\ of\ G$. Replacing $M$ by $M^n$ we may assume that there exists a vector $X \in M$ such that the element $\langle X, X\rangle \in B$ is invertible. Furthermore, since the union of spectral subspaces of $M$ is dense in $M$, we may assume that $X$ lies in a spectral subspace of $M$ corresponding to some representation $U$. In other words, there exist an orthonormal basis $\{\xi_i\}_i H_U$ and vectors $X_i \in M$ such that $\delta_M(X_i) = \sum_i u^*_{IJ} \otimes X_j$ and one of the inner products $\langle X_i, X_i\rangle \in B$ is invertible. Consider the vector

$$Y = \sum_i X_i \otimes \overline{p^{1/2}\xi_\iota} \in M \otimes H_{\bar{U}}.$$

Then $Y$ is invariant and $\langle Y, Y\rangle = \sum_{i,j} X_i \langle X_{i,} X_j\rangle\, (p\xi_{i,}\xi_j)$ Since the matrix $(p\xi_{i,}\xi_j)_{ij}$ is positive and invertible, and the matrix $(\langle X_i, X_j\rangle)_{ij}$ is positive, there exists a constant $c > 0$ such that

$$\langle Y, Y\rangle \geq c \textstyle\sum_i \langle X_i\ , X_i\rangle.$$

It follows that $\langle Y, Y\rangle$ i is invertible, so the map $B \ni x \mapsto Y \langle Y, Y\rangle^{-1/2} x$ gives an equivariant isometric

The data provided by a weak unitary tensor functor and the construction of the corresponding algebra are reminiscent of various crossed product type constructions. To make the connection more explicit, let us give an equivalent description of weak unitary tensor functors in terms of collections of Hilbert module maps satisfying a system of quadratic relations.

Assume $G$ is a compact quantum group and $F : RepG \to CorrA$ is a weak unitary tensor functor.

As fix representatives $U_\alpha$of irreducible unitary representations of $G$, and assume $U_e$ is the trivial representation. Consider the correspondences $M_\alpha = F(U_\alpha)$ and the linear maps $\varphi^\gamma_{\alpha\beta}$ from Mor$(U_\alpha \times U_\beta, U_\gamma)$ into the space of bounded $A$-bilinear maps $M_\alpha \otimes_A M_\beta \to M_\gamma$ defined by $\varphi^\gamma_{\alpha\beta}(T) = F(T)F_2$. We then have the following:

(i) $M_e = A$;

(ii) if a morphism of the form $(T_1, \dots, T_n) : U_\alpha \times \beta \to \oplus_{i=1}^{n}= U_{\gamma i}$ is unitary, then the map $\left(\varphi(T_1), \dots, \varphi(T_n)\right): M_\alpha \otimes_A M_\alpha \to \oplus_{i=1}^{n} M_{\gamma i}$ *is isometric*;

(iii) the image of the identity map $U_\beta \to U_\beta$ under $\varphi^\beta_{e\beta}$ is the map $A \otimes_A M_\beta \to M_\beta$ such that $a \otimes X \longmapsto aX$, and similarly the image of the identity map $U_\alpha \to U_\alpha$ under $\varphi^\alpha_{\alpha e}$ e is the map $M_\alpha \otimes_A A \to M_\alpha$ such that $X \otimes a \mapsto Xa$;

(iv) if a morphism $U_\alpha \times U_\beta \times U_\gamma \to U_\delta$ is written as $\sum_i S_i (T_i \otimes \iota) = \sum_i S'_j (\iota \otimes T'_j)$ for some morphisms $T_i : U_\alpha \times U_\beta \to U_{\alpha i}$ , $S_i : U_{\alpha i} \times U_\gamma \to U_\delta$ $T'_j : U_\beta \times U_\gamma \to U_{\beta j}$ *and* $S'_j : U_\alpha \times U_{\beta j} \to U_\delta$ Then

$$\sum_i \delta (S_i)(\delta(T_i) \otimes \iota) = \sum_i \delta \left(S'_j\right)\left(\iota \otimes \delta\left(T'_j\right)\right) as\ maps\ M_\alpha \otimes_A M_\delta \to M_\eta.$$

(v) for every vector $X \in M_\alpha$ and every morphism $T : U_\alpha \times U_\beta \to U_\gamma$ , the right $A-$linear map $S_X[T] : M_\beta \to M$ mapping $Y$ into $\varphi(T)(X \otimes Y)$ is adjointable, and if a morphism $U_\gamma \times U_\eta \to U_\alpha \times U_\eta$ is written *as* $\sum_i (\iota \otimes S_i)(T_i^* \otimes \iota) = \sum_j p_j^* R_j$ for somemorphism

$$T_i : U_\alpha \times U_{\beta i} \to U_\gamma S_i : \times U_{\beta i}\ U_\delta \to U_\eta, R_j : U_\gamma \times U_\delta \to U_{\gamma j}\ and\ p_j : U_\alpha \times U_\gamma \to U_{\gamma j}\ , then$$

$$\sum_i \varphi (S_i)(S_X[T_i]^* \otimes \iota) = \sum_i S_X\left[P_j\right]^* \varphi\left(R_j\right) as\ maps\ M \otimes_A M_\delta \to M_\eta.$$

Properties (i)-(iv) follow immediately by definition. As will become clear from the proof of the following proposition, the last property, in the presence of the other four, is equivalent to condition (v) in Definition (2.1.3) In particular, if in (ii) we have unitary maps instead of isometric maps, then (v) is a consequence of properties (i) (iv).

Proposition (2.1.13). *Assume we are given correspondences $M_\alpha \in CorrA$ and linear maps $\varphi^\gamma_{\alpha,\beta}$ from* Mor( $U_\alpha \times U_\beta, U_\gamma$) *into the space of bounded $A-$ bilinear maps $M_\alpha \otimes_A M_\beta \to M_\gamma$ such that the above*

*conditions (i)-(v) are satisfied. Then there exists a unique, up to a natural unitary monoidal iso-*

*morphism, weak unitary tensor functor $F : RepG \to CorrA$ such that $\varphi^\gamma_{\alpha,\beta}(T) for\ T \in Mor\left(U_\alpha \times U_\beta, U_\gamma\right)$.*

**Proof:** By virtue of semisimplicity of Rep$G$, there exists a unique, up to a natural unitary isomorphism, unitary functor $F : RepG \to CorrA$ such that $F(U_\alpha) = M_\alpha$. We then define

$$F_2 : F(U_\alpha) \otimes_A F\left(U_\beta\right) \to F\left(U_\alpha \times U_\beta\right) by\ F_2 = \textstyle\sum_i F\left(W_i^*\right)\varphi(W_i),$$

where $W_i : U_\alpha \times U_\beta \rightarrow U_{\gamma i}$ are coisometric morphisms such that $\sum_i W_i^* W_i = \iota$. It is easy to see that this definition does not depend on the choice of $w_i$. By condition (ii) the map $F_2$ is isometric. Note also that $F(W_i)F_2 = (W_i), whence\, F(T)F_2 = \varphi(T)\, for\, all\, T \in Mor(U_\alpha \times U_\beta, U_\gamma)$. By semisimplicity of RepG the isometries $F_2 : F(U_\alpha) \otimes_A F(U_\beta) \rightarrow F(U_\alpha \times U_\beta)$ uniquely define a family of natural isometries $F_2 : F(U) \otimes_A F(V) \rightarrow F(U \times V)$. The only not entirely obvious property left to check is commutativity of two diagrams in Definition(2.1.3).

For the first diagram, we have to check that $F_2(F_2 \otimes \iota) = F_2(\iota \otimes F_2)$ as maps $F(U_\alpha) \otimes_A F(U_\beta) \otimes_A\ ,\ F(U_\gamma) \rightarrow F(U_\alpha \times U_\beta \times U_\gamma)$ . It suffices to check that $F(w)F_2(F_2 \otimes \iota) = F(w)F_2(\iota \otimes F_2)$ for any morphism $w\!: \rightarrow U_\alpha \times U_\beta \times U_\gamma \longrightarrow U_\delta$. Any such morphism can be written as $\sum_i S_i\,(T_i \otimes \iota) = \sum_i S'_j\ (\iota \otimes T'_j)$. By naturality of $F_2$ we then have

$$F(w)F_2(F_2 \otimes \iota) = \sum_i F\ (S_i)F_2(F(T_i)F_2 \otimes \iota) = \sum_i \varphi\,(S_i)(\varphi(T_i) \otimes \iota),$$

and similarly $F(w)F_2(\iota \otimes F_2) = \sum_i \varphi\ (S'_j)(\iota \otimes \phi(T'_j))$. By condition (iv) these expressions are equal.

It remains to show that for every X ∈ F(U) the maps $S_X = S_X, V : F(V) \rightarrow F(U \otimes V)$ are adjointable and $F_2(S_X^* \otimes \iota) = S_X^*\, F_2$. For the adjointability it suffices to show that the map

$S_X\!: F(U_\beta) \rightarrow F(U_\alpha \times U_\beta)$ is adjointable for every $X \in F(U_\alpha)$. Decomposing $U_\alpha \times U_\beta$ into irreducible representations, we see that adjointability of $S_X$ is equivalent to adjointability of $F(T)S_X$ for all morphisms $T : U_\alpha \times U_\beta \rightarrow U_\gamma$. Since $F(T)S_X(Y) = F(T)F_2(X \otimes Y) = \varphi\,(T)(X \otimes Y)$ , we have $F(T)S_X = S_X[T]$, so adjointability of $F(T)S_X$ is part of condition (v).

Finally, we have to show that $F_2(S_X^* \otimes \iota) = S_X^* F_2$ as maps $F(U_\alpha \times U_\beta) \otimes_A F(U_\delta) \rightarrow F(U_\eta)$

for $X \in F(U_\alpha)$. This is equivalent to

$$F(S)F_2(S_X^* \otimes \iota)(F(T^*) \otimes \iota) = F(S)S_X^*\, F_2(F(T^*) \otimes \iota) as\, maps\, F(U_\gamma) \otimes_A F(U_\delta) \rightarrow F(U_\eta)$$

for all morphisms $S : U_\beta \times U_\delta \rightarrow U_\eta\ and\, T : U_\alpha \times U_\beta \rightarrow U_\gamma$. The left hand side of the above identity equals $\varphi(T)(S_X[T]^* \otimes \iota)$, while the right hand side, by (2.1), equals

$$S_X^*\, F(\iota \otimes S)F_2(F(T^*) \otimes \iota) = S_X^*\ F\big((\iota \otimes S)(T^* \otimes \iota)\big)F_2.$$

Writing the morphism $(\iota \otimes S)(T^* \otimes \iota)\, as \sum_j p_j^*\, R_j$ for some $R_j : U_\gamma \times U_\delta \rightarrow U_{\gamma j}\ and\, p_j : U_\alpha \times U_\delta \rightarrow U_{\gamma j}$ , we can write the last expression as

$$\sum_j S_X^*\, p_j^*\, F(R_j)F_2 = \sum_j S_X\,[P_J\,]^* \varphi(R_j\,).$$

We thus see that the identity $F_2(S_X^* \otimes \iota) = S_X^*\, F_2$ follows from condition (v). Furthermore, from the proof we see that it is equivalent to that condition, since any morphism $U_\gamma \times U_\delta \rightarrow U_\alpha \times U_\eta$ can be written as $\sum_i(\iota \otimes S_i)\,(T_i^* \otimes \iota)$ for appropriate morphisms $T_i : U_\alpha \times U_{\beta i} \rightarrow U_\gamma\ and\, S_i : U_{\beta i} \times U_\delta \rightarrow U_\gamma$ using Frobenius reciprocity.

Given data $\{M_\alpha\}_\alpha\, and\, \{\varphi_{\alpha\beta}^\gamma\}\{\varphi_{\alpha\beta}^\gamma\}_{\alpha,\beta,\gamma}$ as above, the construction of the corresponding $C^*$-algebra $B_F$ from goes as follows. For every α define a new scalar product on $\bar{H}_\alpha = \bar{H}_{U\alpha}$ by

$$(\bar{\xi},\bar{\zeta}) = \frac{1}{dim_g U_\alpha}(\zeta, p^{-1}\xi).$$

Consider the right Hilbert $A$-module

$$M = \varrho^{2\oplus}{}_{\alpha} - \bar{H}_\alpha \otimes M_\alpha$$

For every $x = \bar{\xi} \otimes X \in \bar{H}_\alpha \otimes M_\alpha$ define an operator $L_x$ $on\ M$ by

$$L_x(\bar{\zeta} \otimes Y) = \sum_i W_i(\xi \otimes \zeta) \otimes \varphi^{\gamma i}_{\alpha,\beta}(W_i)(X \otimes Y) for\ \bar{\zeta} \otimes Y \in \bar{H}_\beta \otimes M_\beta,$$

where $W_i \in Mor(U_\alpha \times U_\beta \times U_{\gamma i})$ are coisometries such that $\sum_i W_i^* W_i = \iota$Then the can be summarized by saying that the operators $L_{\bar{\xi}\otimes X}$ for all $\xi \in H_\alpha, X \in M_\alpha$ and all indices α, span a ∗-algebra of bounded operators on the Hilbert $A$-module $M, and\ B_F$ is the norm closure of this algebra.

**Example.(2.1.14)[130]:** Assume $G$ is the dual of a discrete groupΓ. We identify the set of isomorphism classes of irreducible representations of G with Γ. Then, up to equivalence, a weak unitary tensor functor $F : RepG \to CorrA$ is the same as a collection of $C^*$-correspondences $M_\alpha, \alpha \in \Gamma$, over $A$, together with $A$-bilinear isometries $\varphi_{\alpha,\beta} :$ $M_\alpha \otimes_A M_\beta \to M_{\alpha\beta}$ such that

(a)$M_e = A$;

(b) $\varphi_{e,\alpha} : A \otimes_A M_\alpha \to M_\alpha\ and \varphi_{e,\alpha} : M_\alpha \otimes_A A \to M_\alpha$ are the maps $a \otimes X \mapsto aX\ and\ X \otimes a \mapsto Xa$, respectively;

(c) $\varphi_{e\beta,\gamma}(\varphi_{\alpha,\beta} \otimes \iota) = \varphi_{\alpha,\beta\gamma}(\iota \otimes \varphi_{\beta,\gamma})$;

(d) for every vector $X \in M_\alpha\ and\ \beta \in \Gamma$, the map $S_X : M_\beta \to M_{\alpha\beta}\ Y \mapsto \varphi_{\alpha,\beta\gamma}(X \otimes Y)$, is adjointable, and $\varphi_{\beta,\gamma}(S_X^* \otimes \iota) = S_X^* \varphi_{\alpha\beta,\gamma}$ as maps $M_{\alpha\beta} \otimes_A M_\gamma \to M_{\beta\gamma}$

This is similar to the definition of product systems of $C^*$-correspondences [138]. The difference is that instead of semigroups we consider groups, the maps $\varphi_{\alpha,\beta}$ are not assumed to be unitary, but then the additional assumption (d) is required. We remind again that if the maps $\varphi_{\alpha,\beta}$ are unitary, condition (d) is not needed.

Since conditions (a)-(d) describe spectral subspaces of an arbitrary coaction of Γ, our results for $G = \hat{\Gamma}$ simply mean that these conditions give an equivalent characterization of Fell bundles over Γ [137]. Explicitly, the ∗-structure on the bundle $\{M_\alpha\}_\alpha \in \Gamma$ is given by the operation • defined in Lemma(2.1.6), so $X^\bullet = S_X^*(1) \in M_{\alpha^{-1}}\ for\ X \in M_\alpha$ if A is unital, and in general $X^\bullet$ is characterized by $\langle X^\bullet, Y\rangle = \varphi_{\alpha,\alpha^{-1}}(X \otimes Y)\ for\ Y \in M_{\alpha^{-1}}$. Clearly, $B_F$ is nothing else than the cross-sectional $C^*$-algebra of this bundle.

Assume $G = T$. Let $M$ be a Hilbert $A$-bimodule, meaning thatM carries the structures of a right Hilbert $A$-module with inner product $L\langle .,. \rangle$ and $X\langle Y, Z\rangle_R$ of a left Hilbert A-module with inner product $L\langle .,. \rangle$ and $X\langle Y, Z\rangle_R = L\langle X,Y\rangle Z\ for\ all\ X,Y,Z \in M$. Consider the complex conjugate Hilbert $A$-bimodule $\bar{M}, so\ a\bar{X} = Xa^*, \bar{X}a = \overline{a^* X}, L\langle \bar{X},\bar{Y}\rangle =$ and $\langle X, Y\rangle_R$and$\langle \bar{X}, \bar{Y}\rangle_R = L\langle X,Y\rangle$. Define $C^*$-correspondences $M_n, n \in Z$, over A by $M_0 = A, M_n = M \otimes\otimes_{A^n}\ for\ n \geq 1$ and $M_n = \bar{M}^{\otimes_{A^{|n|}}}$ for $n \leq -1$.We have obvious isometries $\varphi_{m,n}: M_m \otimes_A M_n \to M_{m+n}$. In order to show that they define a weak unitary tensor functor $F : Rep\ T \to CorrA$, we have to check conditions (a)-(d) from the previous example. Conditions (a) and (b) are obviously satisfied. $A$ moment's reflection shows that since the maps $\varphi_{mn}$are surjective for m and n of the same sign, it suffices to check the other two conditions only for $\alpha = \pm 1$.For such α conditions (c) and (d) easily follow from the identity $X\langle Y, Z\rangle_R = L\langle X, Y\rangle Z$ The corresponding $C^*$-algebra

BF is the algebra $A \rtimes M^{\mathbb{Z}}$, the crossed product of $A$ by the Hilbert $A$-bimodule $M$, defined in [131], where it was shown directly that $\{M_n\}_n \in Z$ forms a Fell bundle over $\mathbb{Z}$. Recall that the $C^*$-algebra $A \rtimes M^{\mathbb{Z}}$ is canonically isomorphic to the Cuntz-Pimsner algebra $O_M$.
Let us return to the case of a general compact quantum group $G$. Recall that a unitary 2-cocycleon the dual discrete quantum group $\hat{G}$ is a unitary element

$$\Omega \in W^*(G) \overline{\otimes} W^*(G) \subset (C[G] \otimes C[G])^*$$

such that $(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega)$. Any such cocycle defines a unitary fiber functor $E : RepG \rightarrow Hilbf$ that is identity on objects and morphisms, while the tensor structure $E_\Omega(U) \otimes E_\Omega(V) \rightarrow E_\Omega(U \times V)$ is given by the action of $*$. By Woronowicz's Tannaka-Krein duality, this functor defines a new deformed compact quantum group $G$. More concretely, we have $W^*(\hat{G}_\Omega) = W^*(\hat{G})$ as von Neumann algebras, while the new coproduct is given $by\, \hat{\Delta}_\Omega(W) = \Omega\hat{\Delta}\,(\mathrm{W})\,\Omega^*$ Equivalently, $\mathbb{C}[G_\Omega] = \mathbb{C}[G]$ as coalgebras, while the new product and involution are
obtained by dualizing $\left(W^*(G_\Omega), \hat{\Delta}_\Omega\right)$.
Assume now that we have a continuous left action θ of the reduced form of $G$ on a $C^*$-algebra $B$ with fixed point algebra $A$. Consider the corresponding spectral functor $F : RepG \rightarrow CorrA$. Since by construction the categories RepG and RepG are equivalent, we can consider $F$ as a weak unitary tensor functor $RepG \rightarrow CorrA$. It defines a $C^*$-algebra B carrying a continuous left action of the reduced form of $G$. These algebras were defined and studied in greater generality in [136] and [141] (in particular, see [147] for the case of the dual of a discrete group). But as we will see in a moment, the categorical picture provides a very simple and concrete approach.
First of all we will need a special element $u \in U(G) = [G]^*$ defined by $u = m(\iota \otimes \hat{S})(\Omega)$, where m: $(\mathbb{C}[G] \otimes \mathbb{C}[G])^* \rightarrow \mathbb{C}[G]^*$ is the product map, which is by definition dual to the coproduct on $\mathbb{C}[\mathrm{G}]_\Omega$.The element u is invertible, with inverse given by

$$U^{-1} = m(\hat{S} \otimes \iota)(\Omega^*) = \hat{S}(U^*).$$

The antipode on the dual of $G$ is given by $\hat{S} = u\,\hat{S}(\cdot)U^{-1}$, and correspondingly the involution † on $\mathbb{C}\,[G_\Omega]$ is given by

$$a^\dagger = [(U^{-1})^* \otimes \iota \otimes U^*)\Delta^2(a)]^2 = (U^* \otimes \iota \otimes (U^{-1})^*\Delta^{(2)}(a^*),$$

. It is easy to check that the element u can also be characterized by the identities

$$\Omega\, R_U = (u \otimes \iota)R_U \; as\; maps\; \mathbb{C} \rightarrow H_{\bar{U}} \otimes H_U\,, \qquad (5)$$

Next, consider the subalgebra $B \subset B$ of regular elements. Then the map $B \otimes U(G) \rightarrow B,\; x \otimes \omega \longmapsto x \rhd \omega = (\omega \otimes \iota)\theta(x)$, defines a right $U(G)$-module structure on $B$.
**Proposition(2.1.15)[130]:** *With the above notation, the following formulas define a new* $*$-*algebra* B *withunderlying space* $B$, *product* $\star$ *and involution*† :

$$x \star y = m((x \otimes y) \rhd), x \dagger = x^* \rhd u^*,$$

*where* $m: B \otimes B \rightarrow B$ *is the original product map. Furthermore, the map* $\theta$, *considered as a map* $B_\Omega \rightarrow \mathbb{C}[G_\Omega] \otimes B_\Omega$, *defines a left algebraic action of* $G_\Omega$ *on* $B_\Omega$.
**Proof:** By multiplying by a phase factor we may assume that is counital, that is, $(\hat{\varepsilon} \otimes \iota)(\Omega) = (\iota \otimes \hat{\varepsilon})(\Omega) = 1$.
For every finite dimensional unitary representation $U \in B(H_U) \otimes \mathbb{C}\,[G]$ denote by $U$ the same element $U$ considered as an element of $B(H_U) \otimes \mathbb{C}\,[G]$. Then we have a unitary monoidal equivalence ofcategories $E^\Omega : RepG_\Omega \rightarrow RepG\; such\; that E^\Omega\,(U^\Omega) = U, E$ is the identity map on morphisms,

and $E_2^{\Omega}$ : $E^{\Omega}(U^{\Omega}) \otimes E^{\Omega}(V^{\Omega}) \to E^{\Omega}(U^{\Omega} \times V^{\Omega})$ is given by $\Omega$: $H_U \otimes H_V \to H_U \otimes H_V = H_{U\times V}$.
we claim that with the above setup the $*$-algebra $B_{FE^{\Omega}}$ corresponding to the weak unitary tensor functor $FE^{\Omega}$ is exactly$B_{\Omega}$, and the map $\theta_{FE^{\Omega}}$ coincides with $\theta$. Note that counitality of $\Omega$ is needed for condition (iii) in Definition (2.1.3)to be satisfied by the functor $FE^{\Omega}$.
As linear spaces, we have

$$B_{FE^{\Omega}} = \oplus_{\alpha} \bar{H}_{U\alpha^{\Omega}} \otimes FE^{\Omega}(U_{\alpha}^{\Omega}) = \oplus_{\alpha} \overline{H}_{\alpha} \otimes F(U_{\alpha}) = B.$$

Denote by $\star$ the product on $B_{FE^{\Omega}}$. Note that if $w$: $H_{\gamma} \to H_{\alpha} \otimes H_{\beta}$ is a morphism $U_{\gamma} \to U_{\alpha} \times U_{\beta}$, then $\Omega\, w$ is a morphism $U_{\gamma}^{\Omega} \to U_{\alpha}^{\Omega} \times U_{\beta}^{\Omega}$. From this we get that if $x = \bar{\xi} \otimes X \in \bar{H}_{\alpha} \otimes F(U_{\alpha})$ and $y = \bar{\zeta} \otimes Y \in \bar{H}_{\beta} \otimes F(U_{\beta})$, then

$$x \star y = \sum_i \overline{W_i^* \,\Omega^*(\xi \otimes \zeta)} \otimes F(W_i^*)F_2(X \otimes Y),$$

where $W_i \in Mor(u_{\gamma i}, U_{\alpha} \times U_{\beta})$ are isometries such that $\sum_i W_i W_I^* = \iota$. Since the right $U(G)$-module structure on $B = B_F$ is given by $(\bar{\eta} \otimes Z) \rhd \omega = \overline{\omega * \eta} \otimes Z$, the above identity means exactly that

$$x \star y = m\big((x \otimes y) \rhd \Omega\big).$$

Denote the involution on $B_{FE^{\Omega}}\ by$ $\dagger$. Take $x = \bar{\xi} \otimes X \in \bar{H}_{\alpha} \otimes F(U_{\alpha})$. We may assume that $\bar{U}_{\alpha} = U_{\bar{\alpha}}$ for some index $\bar{\alpha}$. Then, by definition,

$$x^{\dagger} = \overline{\xi^{\#}} \otimes X^{\#}\, s,$$

where $\xi^{\#} \in H_{\bar{\alpha}}$ is such that $(R_{\alpha}^{\Omega})(\zeta \otimes \xi) = (\zeta, \xi^{\#})$ for all $\zeta \in H_{\bar{\alpha}}\ and\ X^{\#} \in F(U_{\bar{\alpha}})$ is such that$(FE^{\Omega})(\bar{R}_{\alpha}^{\Omega})^*(FE^{\Omega})_2(X \otimes Y) = \langle X^{\#}, Y\rangle$ for all $Y \in F(U_{\bar{\alpha}})$, where $R_{\alpha}^{\Omega}$ and $\bar{R}_{\alpha}^{\Omega}$ solve the conjugate equations for $U_{\alpha}^{\Omega}$ and $U_{\bar{\alpha}}^{\Omega}$. Note that by irreducubility the operation # depends on the choice of such a solution, but$\overline{\xi^{\#}} \otimes X^{\#}$ does not. Taking the solutions $R_{\alpha}\ and\ \bar{R}_{\alpha}$ of the conjugate equations for $U_{\alpha}\ and\ U_{\bar{\alpha}}$ defined by (1.1), we can take $R_{\alpha}^{\Omega} = \Omega R_{\alpha}\ and\ \bar{R}_{\alpha}^{\Omega} = \Omega\bar{R}_{\alpha}$. In this case $X^{\#} = X^{\bullet}$, while for $\xi^{\dagger}$, using (4.1), we get

$$(R_{\alpha}^{\Omega})^*(\zeta \otimes \xi) = R_{\alpha}^*\ (U^*\zeta \otimes \xi) = (U^*\zeta, \xi^{\bullet}),$$

$so\ \xi^{\#} = u\xi^{\bullet}. Therefore$

$$x^{\dagger} = u\xi^{\bullet} \otimes X^{\bullet} = (\xi^{\bullet} \otimes X^{\bullet}) \rhd U^* = x^* \rhd U^*.$$

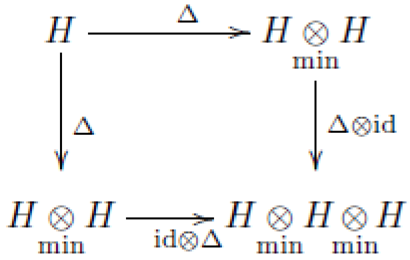
Finally, the maps $\theta = \theta_F\ and \theta_{FE^{\Omega}}$ coincide on$B_{FE^{\Omega}}$, since they both define the same right $U(G)$-module structure, given by $(\bar{\eta} \otimes Z) \rhd \omega = \overline{w^*\eta} \otimes Z$.
The construction of a new product on a module algebra using a cocycle on a Hopf algebra is, of course, well-known. The point of the above proposition is that it effortlessly gives not only the new product, but also the $*$-structure and the existence of a unique $C^*$-completion of the algebra carrying an action of the reduced deformed quantum group.

## Section (2.2): Unital $C^*$ - Algebras

A compact quantum group [176], [177] is a unital $C^*$ -algebra H with a given unital injective $*$-homorphism $\Delta$ (referred to as comultiplication)

$$\Delta: H \to H \underset{min}{\otimes} H \tag{7}$$

that is coassociative, i.e. it renders the diagram



commutative, and such that the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{cls} = H \underset{min}{\bigotimes} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{cls}. \tag{8}$$

Here $\underset{min}{\otimes}$ denotes the spatial tensor product of $C^*$ -algebras and cls denotes the closed linear span of a subset of a Banach space.

Let $A$ be a unital $C^*$ -algebra and $\delta : A \to A \underset{min}{\otimes} H$ an injective unital $*$-homomorphism. We call δ a coaction (or an action of the compact quantum group $(H, \Delta)$ *on* $A, cf$.

(i) $(\delta \otimes id) \circ \delta = (id \otimes \Delta) \circ \delta$ (coassociativity),

(ii) $\{\delta(a)(1 \otimes h) \mid a \in A, h \in H\}^{cls} = A \underset{min}{\otimes} H$ (counitality).

We shall consider three properties of coactions.

**Definition (2.2.1)[148]:** The coaction $\delta : A \to A \underset{min}{\otimes} H$ is free iff

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{cls} = A \underset{min}{\otimes} H.$$

Given a compact quantum group $(H, \Delta)$, we denote by $O(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of its irreducible unitary representations [177], [166]. This is Woronowicz's Peter-Weyl theory in the case of compact quantum groups. Moreover, denoting by $\otimes$ the purely algebraic tensor product over the field C of complex numbers, we define the Peter-Weyl subalgebra of $A$ (cf. [167], [173]) as

$$P_H(A) := \{ a \in A \mid \delta(a) \in A \otimes O(H) \}. \tag{9}$$

Using the coassociativity of the coaction δ, one can check that $P_H(A)$ is a right $O(H)$ −comodule algebra. In particular, $P_H(A) = O(H)$. The assignment $A \mapsto P_H(A)$ is functorial with respect to equivariant unital $*$-homomorphisms and comodule algebra maps. We call it the Peter-Weyl functor.

**Definition (2.2.2)[148]:** The coaction $\delta : A \to A \underset{min}{\otimes} H$ satisfies the Peter-Weyl-Galois (PWG) condition iff the canonical map can: $P_H(A) \overset{\otimes}{_B} P_H(A) \to P_H(A) \otimes O(H)$,

$$can: x \otimes y \mapsto (x \otimes 1)\delta(y), \tag{10}$$

is bijective. Here $B := A^{coH} := \{a \in A \mid \delta(a) = a \otimes 1\}$ is the unital $C^*$-subalgebra of coaction invariants (fixed-point subalgebra).

Throughout the tensor product over an algebra denotes the purelyalgebraic tensor product over that algebra. Note that $P_H(A) \otimes_B P_H(A)$ is not in general an algebra, and even if we lift the canonical map to

$$\widetilde{can}: P_H(A) \otimes P_H(A) \ni x \otimes y \longmapsto (x \otimes 1)\delta(y) \in P_H(A) \otimes O(H), \tag{11}$$

it is not an algebra homomorphism, and cannot as such be completed into a continuous map between C*-algebras. However, it can be defined on the level of Hilbert modules (see [133]).

**Definition (2.2.3)[148]:** The coaction $\delta : A \to A \underset{min}{\otimes} H$ is strongly monoidal iff for all left $O(H)$-comodules V and W the map

$$\beta : (P_H(A) \boxdot V) \otimes B (P_H(A) \boxdot W) \to P_H(A) \boxdot (V \otimes W),$$

$$(\sum_i a_i \otimes v_i) \otimes (\sum_i b_j \otimes w_j) \mapsto (\sum_{i,j} a_i b_j \otimes w_j) \otimes (v_i \otimes w_j),$$

is bijective.

In the above definition, we have used the cotensor product

$$P_H(A) \boxdot V := \{t \in P_H(A) \otimes V \mid (\delta \otimes id)(t) = (id \otimes_V \Delta)(t)\}, \quad (12)$$

where ${}_V\Delta : V \to O(H) \otimes V$ is the given left coaction of $O(H)$ on $V$. The coaction of $O(H)$ on $V \otimes W$ is the diagonal coaction.

**Theorem(2.2.4)[148]:** Let $A$ be a unital $C^*$-algebra equipped with an action of a compact quantum group $(H, \Delta)$ given by $\delta : A \to A \underset{min}{\otimes} H$. Then the following are equivalent:

(i) The action of $(H, \Delta)$ on $A$ is free.

(ii) The action of $(H, \Delta)$ on $A$ satisfies the Peter-Weyl-Galois condition.

(iii) The action of $(H, \Delta)$ on A is strongly monoidal.

Note that of the three equivalent conditions, the first uses functional analysis, the second is algebraic, and the third is categorical. The difficult implication, which is the core of the theorem, is $(i) \Rightarrow (ii)$. It proves that, for any free action, there exists a strong connection, a key technical device for index-pairing computations (e.g. [164]). In the spirit of Woronowicz's Peter-Weyl theory, our result states that the original functional-analysis formulation of free action is equivalent to the much more algebraic $PWG$-condition

We now proceed to explain our main result in the classical setting. Let $G$ be a compact Hausdorff topological group acting on a compact Hausdorff topological space $X$ by a continuous right action $X \times G \to X$. It is immediate that the action is free, i.e. $xg = x \Rightarrow g = e$ (where e is the identity element of $G$), if and only if

$$(x, g) \longmapsto (x, xg) \quad , (13)$$

is a homeomorphism. Here $X \underset{X/G}{\times} X$ is the subset of $X \times X$ consisting of pairs $(x_1, x_2)$ such that $x_1$ and $x_2$ are in the same $G$-orbit.

This is equivalent to the assertion that the $*$-homomorphism

$$C\left(X \underset{\frac{X}{G}}{\times} X\right) \to C(X \times G) \quad (14)$$

obtained from the above map $(x, g) \mapsto (x, xg)$ is an isomorphism. Here, as usual, $C(Y)$ denotes the commutative $C^*$-algebra of all continuous complexvalued functions on a compact Hausdorff space $Y$.

In turn, the assertion that the $*$-homomorphism (14) is an isomorphism is readily proved equivalent to

$$\{(x \otimes 1)\delta(y) \mid x, y \in C(X)\}^{cls} = C(X) \underset{min}{\otimes} C(G), \quad (15)$$

where

$$\delta : C(X) \longrightarrow C(X) \underset{min}{\otimes} C(G), \big(\delta(f)(g)\big)(x) := f(xg) \quad (16)$$

is the $*$-homomorphism obtained from the action map $X \times G \to X$. Hence, in the case of a compact Hausdorff group acting on a compact Hausdorff space, freeness in the usual sense agrees with freeness as defined in the setting of a compact quantum group acting on a unital $C^*$-algebra. Thus Theorem (2.2.4) provides the following characterization of free actions in the classical case.

**Theorem (2.2.5)[148]:** Let $G$ be a compact Hausdorff group acting continuously on a compact Hausdorff space $X$. Then the action is free if and only if the canonical map

$$can: P_{C(G)}\big(C(X)\big) \underset{C\left(\frac{X}{G}\right)}{\otimes} P_{C(G)}\big(C(X)\big) \to P_{C(G)}\big(C(X)\big) \otimes O\big(C(G)\big) \quad (17)$$

is an isomorphism.

Observe that even in the above special case of a compact Hausdorff group acting on a compact Hausdorff space, a proof is required for the equivalence of freeness of the action and the bijectivity of the canonical map ($PWGcondition$). Theorem (2.2.5)brings a new algebraic tool (strong connection) to the realm of compact Hausdorff principal bundles.

In this classical setting, the Peter-Weyl algebra $PC(G)(C(X))$ is the algebra of continuous global sections of the associated bundle of algebras $X \times_G O(C(G))$:

$$P_{C(G)}\big(C(X)\big) = \Gamma\,(X \times_G O(C(G))) \quad (18)$$

Here $O(C(G))$ is the subalgebra of $C(G)$ spanned by the matrix coefficients of irreducible unitary representations of $G$. We view $O(C(G))$ as a representation space of $G$ via the formula

$$\big(\varrho(g)(f)\big)(h) := f(g^{-1}h). \quad (19)$$

The algebra $O(C(G))$ is topologized as the direct limit of its finite-dimensional subspaces. Multiplication and addition is pointwise. Note that, since $O(C(G))$ is cosemisimple, it belongs to the category of representations of $G$ that are purely algebraic direct sums of finite-dimensional representations of $G$. We denote this category by $F\mathrm{Rep}^{\oplus}(G)$. Due to the cosemisimplicity of $O(C(G))$, the following formula for the left coaction of $O(C(G))\ on\ V$ $\ \big(v\,\Delta(v)\big)(g) := \varrho(g^{-1})(v), where\ \varrho : G \to GL(V)$ is a representation, establishes an equivalence of $F\mathrm{Rep}^{\oplus}(G)$ with the category of all left $O(C(G)) -$ comodules. As with the special case $V = O(C(G))$, all vector spaces in this category are topologized as the direct limits of their finite-dimensional subspaces.

Theorem (2.2.5)unifies continuous free actions of compact Hausdorff groups on compact Hausdorff spaces and principal actions of affine algebraic groups on affine schemes [159], [170]. Thus the main result of might be viewed as continuing the Atiyah-Hirzebruch program of transferring ideas (e.g. K-theory) from algebraic geometry to topology [150], [151]. In the same spirit, our main theorem (Theorem(2.2.4)) unifies the $C^*$-algebraic concept of free actions of compact quantum groups [161] with the Hopf-algebraic concept of principal coactions [163]. Theorem (2.2.4)implies the existence of strong connections [162] for free actions of compact quantum groups on unital $C^*$ -algebras (connections on compact quantum principal bundles) thus providing a theoretical foundation for the plethora of concrete constructions studied over the past two decades within the general framework noncommutativegeometry [156]. We apply Theorem (2.2.4)to fields of $C^*$ -algebras. We prove the key part of our main theorem, that is the equivalence of freeness and the Peter-Weyl-Galois condition., we consider the general algebraic setting of principal coactions. Following Ulbrich [175] and Schauenburg [170], we prove thatthe principality of a comodule algebra P over a Hopf algebra $\mathcal{H}$ is equivalent to the exactness and strong monoidality of the cotensor product functor $P \boxdot^{\mathcal{H}}$. In particular, this proves the equivalence of the Peter-Weyl-Galois condition and strong monoidality for actions of compact quantum groups, thus completing the proof of the main theorem.

Although Theorem (2.2.5)is a special case of Theorem(2.2.4), the proof we give of Theorem (2.2.5) is not a special case of the proof of Theorem (2.2.4), Therefore, we treat Theorem(2.2.5) separately, and prove it The proof uses the trong monoidality (i.e. the preservation of tensor products) of the Serre-Swan equivalence and a general algebraic argument, we give a vector-bundle interpretation of the aforementioned general algebraic argument. This provides a much desired translation between the algebraic and topological settings.

We prove that if a unital $C^*$-algebra $A$ equipped with an action of a compact quantum group can be fibred over a compact Hausdorff space $X$ with the $PWG$-condition valid on each fibre, then the $PWG$-condition is valid for the action on $A$. We end with an appendix discussing the well-known fact that regularity of a finite covering is equivalent to bijectivity of the canonical map (19).

A.Equivalence of freeness and the Peter-Weyl-Galois condition

The implication "$PWG\text{-}condition \Rightarrow freeness$" is proved as follows. The PWG-condition immediately implies that

$$(P_H(A) \otimes \mathbb{C})\delta\big(PH(A)\big) = P_H(A) \otimes O(H). \tag{20}$$

As the right-hand side is a dense subspace of A $\otimes_{min}$ H, we obtain the density condition defining freeness.

For the converse implication "$PWG$-condition $\Leftarrow$ freeness" we need some preparations. If $(V, \delta_V)$ is a finite-dimensional right $H$-comodule, we write $H_V$ for the smallest vector subspace of H such that $\delta_V(V) \subseteq V \otimes H_V$. We write

$$A_V := \{a \in A \mid \delta(a) \in A \otimes H_V\}. \tag{21}$$

Note that in the case $(A, \delta) = (H, \Delta)$, we have $A_V = H_V$. Thus HV is a coalgebra. One can define a continuous projection map $E_V$ from $A$ onto $A_V$ as follows. Let us call two finite-dimensional comodules of $H$ disjoint if the set of morphisms between them only contains the zero map.

Then $E_V$ is the unique endomorphism of $A$ which is the identity on $A_V$ and which vanishes on $A_W$ for any finite-dimensional comodule W that is disjoint from $V$. In the special case of $(A, \delta) = (H, \Delta)$, we use the notation $e_V$ instead of $E_V$. The equivariance property

$$\delta \circ E_V = (id \otimes e_V) \circ \delta \tag{22}$$

is proved by a straightforward verification. When $V$ is the trivial representation, we write $E_V = E_B$ $and$ $eV = \varphi H, where$ $B := A^{coH}$ is the algebra ofcoaction invariants and $\varphi H$ is the invariant state on $H$. Then the formula (3.16) specializes to

$$E_B = (id \otimes \varphi H) \circ \delta. \tag{23}$$

The key lemma in the proof of Theorem (2.2.4)is:

**Lemma (2.2.6)[148]:** in [133]). Let $\delta : A \to A \otimes_{min} H$ be a free coaction, and let $V$ be a finite-dimensional $H$-comodule. Then $A_V$ is finitely generated projective as a right $B$-module.

Note that in the classical case $X \times G \to X$, we have $H = C(G)$ and $B = C(X/G)$. The $B$-module $A_V$is then $\Gamma(X \times_G H_V)$, and thus it is finitely generated projective.

Define a $B$-valued inner product on$A_V$ by

$$\langle a, b\rangle_B := E_B(a^*b). \tag{24}$$

**Lemma (2.2.7)[148]:** The $B$-valued inner product(19) makes $A_V$ a right Hilbert $B$-module [139]. The Hilbert module norm $\|a\|_B := \left\|\langle a, a\rangle_B\right\|^{1/2}$ is equivalent to the $C^*$ −norm of A restricted to $A_V$.We will need the following lemma concerning the interior tensor product of Hilbert modules.

**Lemma (2.2.8)[148]**: Let $C$ *and* $D$ be unital $C^*$-algebras, and let $(\mathcal{E}, \langle \cdot \ , \cdot \rangle_C)$ be a right Hilbert $C$-module that is finitely generated projective as a right $C$-module. Let $(\mathcal{g}\langle \cdot , \cdot \rangle_D)$ be an arbitrary right Hilbert D-module, and $\pi : C \to L(\mathcal{g})$ be a unital $*$-homomorphism of $C$ into the $C^*$ -algebra of adjointable operators on $F$. Then the algebraic tensor product $\mathcal{E} \ \otimes_C \ F$ is a right Hilbert $D$-module with respect to the inner product given by

$$\langle x \otimes y , z \otimes w \rangle := \langle y, \pi(\langle x, z\rangle_C) w\rangle_D.$$

**Proof:** We need to prove that the semi-norm $\|z\| = \|\langle z, z\rangle_D\|^{1/2} on\ \mathcal{E} \otimes_C \mathcal{g}$ is in fact a norm with respect to which $\mathcal{E} \otimes_C \mathcal{g}$ is complete. The statement obviously holds for $\mathcal{E} = C^n$ , the n-fold direct sum of the standard right $C$module $C$. Since $E$ is finitely generated projective, $E$ can be realized as a direct summand of $C^n$.o that the conclusion also applies in this case.

We are now ready to prove the implication "$PWG$-condition ⇐ freeness". By the freeness assumption, the image of can is dense in $A \otimes H$. In particular, for a given finite-dimensional comodule $V$ and any $h \in H_V$ , we can find a sequence $K_n \in N$ and elements $p_{n.i}$and $q_{n.i}$ in $p_H$(A) with $1 \leq i \leq K_n$ such that

$$\sum_{i=1}^{K_n} ( p_{n.i} \otimes 1)\delta(q_{n.i}) \underset{n\to\infty}{\longrightarrow} 1 \otimes h \tag{25}$$

in the $C^*$-norm. Applying id $\otimes e_V$ to this expression, and using (25), we seethat we can take $q_{n,i} \in A_V$.

Applying δ to the first leg of (19) and using coassociativity, we obtain

$$\sum_{i=1}^{K_n} \delta\big( p_{n.i)} \otimes 1\big)(id \otimes \Delta)(\delta(q_{n,i}) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes h. \tag{26}$$

Observe now that, since $q_{n.i} \in A_V$ , by (15) we obtain $(id \otimes \Delta)\big(\delta(q_{n.i})\big) \in A_V \otimes H_V \otimes H_V$. Hence the left-hand side of (26) belongs to the tensor product $(A \otimes_{min} H) \otimes H_V.AsH_V$ is finite dimensional, the restriction of the antipode $S$ of $O(H)$ to $H_V$ is continuous. Therefore, we can apply S to the third leg of
(25) to conclude

$$\sum_{i=1}^{K_n} (\delta(p_{n.i}) \otimes 1)(id \otimes (id \otimes S) \circ \Delta)\left(\delta\big(q_{n,i}\big)\right) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes S(h) \tag{27}$$

Again by the finite dimensionality of $H_V$, multiplying the second and third legs is a continuous operation, so that

$$\sum_{i=1}^{K_n} \delta(p_{n.i})(q_{n.i} \otimes 1) \underset{n\to\infty}{\longrightarrow} 1 \otimes S(h). \tag{28}$$

Since $S(h) \in H_{\bar{V}}$ , where $\bar{V}$ is the contragredient of $V$ , applying $id \otimes e_{\bar{V}}$ to the above limit and using the equivariance property (16) shows that in the above limit we can choose $p_{n.i} \in A_{\bar{V}}$.

Consider now the right B-module map

$$G_V : A_{\bar{V}} \overset{\otimes}{_B} A_V \to A_{\bar{V}\otimes V} \otimes H_{\bar{V}} , a \otimes b \mapsto \delta(a)(b \otimes 1). \tag{29}$$

By Lemma (2.2.6) and Lemma(2.2.8), the algebraic tensor product on the left hand side becomes an interior tensor product of right Hilbert $B$-modules. The inner product for $A_{\bar{V}} \otimes_B A_V$ is

$$\langle c \otimes d, a \otimes a \otimes b\rangle_B = E_B(d^* E_B(C^* a)b). \tag{30}$$

On the other hand, equipping $H_{\overline{V}}$ with the Hilbert-space structure $\langle h,k\rangle = \varphi_H(h^*k)$, the right-hand side is a right Hilbert $B$-module with inner product

$$\langle b \otimes h, a \otimes g\rangle = \varphi_H(h^*g)E_B(b^*a). \tag{31}$$

It follows from these formulas and (3.17) that $G_V$is an isometry between these Hilbert modules. Hence the range of $G_V$ is closed.

From (22) and the equivalence of $C^*$-module and Hilbert $C^*$-module norms in Lemma(2.2.7), we infer that the range of $G_V$ contains $1 \otimes S(h)$. Therefore, as the domain of $G_V$is an algebraic tensor product, we can find a finite number of elements $p_i, q_i \in P_H(A)$ such that

$$\sum_i \delta\,(p_i)(q_i \otimes 1) = 1 \otimes S(h). \tag{32}$$

Now applying the map $a \otimes g \longmapsto (1 \otimes S^{-1}(g))\delta(a)$ to both sides yields

$$\sum_i (p_i \otimes 1)\,\delta(q_i) = 1 \otimes h. \tag{33}$$

As h was arbitrary in $O(H)$, it follows that can is surjective.

Finally, as the Hopf algebra $O(H)$ is cosemisimple, according to, bijectivity of the canonical map can follows from its surjectivity.This completes the proof of the implication "$PWG$-condition ⇐ freeness".

The framework of principal comodule algebras unifies in one category many algebraically constructed noncommutative examples and classical compact Hausdorff principal bundles.

Definition (2.2.9)Let $H$ be a Hopf algebra with bijective antipode, and let $\Delta_p : P \to P \otimes \mathcal{H}$ be a coaction making P an H-comodule algebra. We call $P$ principal if and only if:

(i) $P \otimes_B P \ni p \otimes q \longmapsto can(p \otimes q) := (p \otimes 1)\Delta_p(q) \in P \otimes \mathcal{H}$ is bijective, where $B := P^{coH} := \{p \in P \mid \Delta_p(p) = p \otimes 1\}$;

(ii) there exists a left $B$-linear right $\mathcal{H}$ -colinear splitting of the multiplication map $B \otimes P \to P$.

Here (i) is the Hopf-Galois condition and (ii) is the right equivariant left projectivity of $P$.

Alternately, one can approach principality through strong connections:

**Definition (2.2.10)[148]:** Let $\mathcal{H}$ be a Hopf algebra with bijective antipode $S$, and $\Delta_p : P \to P \otimes H$ be a coaction making P a right $\mathcal{H}$ -comodule algebra.

A strong connection ℓ on P is a unital linear map ℓ : $\mathcal{H}$ → P ⊗ P satisfying:

(i) $(id \otimes \Delta_p) \circ \ell = (\ell \otimes id) \circ \Delta$;

(ii) $(\Delta_p \otimes id) \circ \ell = (id \otimes \ell) \circ \Delta, where\, \Delta_p := (S^{-1} \otimes id) \circ flip \circ \Delta_p$;

(iii) $\widetilde{can} \circ \ell = 1 \otimes id, where\, gcan: P \otimes P \ni p \otimes q \longmapsto (p \otimes 1)\Delta_p(q) \in P \otimes \mathcal{H}$.

One can prove that a comodule algebra is principal if and only if it admits a strong connection.

If $\Delta_M : M \to M \otimes C$ is a coaction making $M$ a right comodule over a coalgebra $C$ and $N$ is a left $C$-comodule via a coaction ${}_N\Delta: N \to C \otimes N$, then we define their cotensor product as

$$M_{\square}^{C}\; N := \{t \in M \otimes N \mid (\Delta M \otimes id)(t) = (id \otimes N\Delta)(t)\}.$$

In particular, for a right $\mathcal{H}$ -comodule algebra P and a left $\mathcal{H}$ -comodule $V$ , we observe that $P \,\square^H V$ is a left $P$ coH- module in a natural way. One of the key properties of principal comodule algebras is that, for any finite-dimensional left $\mathcal{H}$ -comodule $V$ , the left $P^{\mathrm{coH}}$ -

module $P \square^H V$ is finitely generated projective Here P plays the role of a principal bundle and $P\square^H V$ plays the role of an associated vector bundle. Therefore, we call $P \square^H V$ an associated module.
Principality can also be characterized by the exactness and strong monoidality of the cotensor functor. This characterisation uses the notion of coflatness ofa comodule: right comodule is coflat if and only if cotensoring it with leftcomodules preserves exact sequences.
**Theorem(2.2.11) [148]:** Let H be a Hopf algebra with bijective antipode and P be a right H-comodule algebra. Then P is principal if and only if $P$ is right $\mathcal{H}$ -coflat and for all left $\mathcal{H}$ -comodules $V$ $and$ $W$ the map

$$\beta : (P \square V) \underset{B}{\otimes} (P \square W) \to P\square(V \otimes W),$$

$$\left(\sum_i a_i \otimes v_i\right) \otimes \left(\sum_j b_j \otimes w_j\right) \mapsto \sum_{i,j} a_i b_j \otimes (v_i \otimes w_j),$$

is bijective. In other words, $P$ is principal if and only if the cotensor product functor is exact and strongly monoidal with respect to the above map $\beta$.
**Proof.** The proof relies on putting together.
First assume that P is principal. Then $P$ is right equivariantly projective, and it follows from that P is faithfully flat. Now we can apply to conclude that $\beta$ is bijective. Furthermore, by, the faithful flatness of $P$ implies the coflatness of $P$. Conversely, assume that cotensoring with P is exact and strongly monoidal with respect to β. Then substituting $\mathcal{H}$ for $V$ and $W$ yields the Hopf-Galois condition. Now implies the equivariant projectivity of $P$.
**Corollary(2.2.12)[148]:** Let $A$ be a unital $C^*$-algebra equipped with an action of a compact quantum group (H,Δ) given by $\delta : A \to A \otimes_{min}$ H. Then the following are equivalent:
(i)The action of $(H, \Delta)$ $on$ $A$ satisfies the Peter-Weyl-Galois condition.
(ii)The action of $(H, \Delta)$ $on$ $A$ is strongly monoidal.
**Proof**: The Hopf algebra $O(H)$ always has bijective antipode. It follows from that any comodule over this Hopf algebra is coflat. Hence implies that the equivariant projectivity condition is valid for any O(H)-comodule algebra such that the canonical map is bijective. The corollary now follows from Theorem one can use the combination.

i.e. we prove Theorem(2.2.5). As in the proof of the general noncommutative case, we rely on the fact that the module of continuous of an associated vector bundle is finitely generated projective. However, unlike in the proof, herein we first prove strong monoidality, and then conclude the $PWG$-condition. Anentirely different proof of Theorem (2.2.5), using local triviality, can be found in [152].
To be consistent with general notation, we should only use $C^*$-algebras $C(G)$, $C(X)$, etc., rather than spaces themselves. However, this would make formulas too cluttered, we consistently omit writing $C()$ in the subscript and the argument of the Peter-Weyl functor.
The implication "$PWG$-condition ⇒ freeness" is proved as follows. The $PWG$-condition immediately implies that

$$(P_G(X) \otimes \mathbb{C})\delta\big(P_G(X)\big) = P_G(X) \otimes O(G). \quad (34)$$

As the right-hand side is a dense subspace of $C(X) \otimes_{min} C(G)$, we obtain the density condition (34 ) The latter is equivalent to freeness, as explained in the introduction. For the converse implication "$PWG$-condition ⇐ freeness", we shall use the Serre-Swan theorem.

**Theorem (2.2.13)[148]:** ([174]). Let $Y$ be a compact Hausdorff topological space. Then a $C(Y)$ −module is finitely generated and projective if and only if it is isomorphic to the module of continuous global of a vector bundle over $Y$.

For a compact Hausdorff topological space $Y$, we denote by Vect($Y$) the category of $C$ vector bundles on $Y$. An object in Vect($Y$) is a $C$ vector bundle $E$ with base space $Y$. The projection of $E$ onto $Y$ is denoted by $\pi_E : E \to Y$.

A section of $E$ is a continuous map

$$s: Y \to E \text{ with } \pi_E \circ s = idY. \tag{35}$$

A morphism in Vect($Y$) is a vector bundle map, $i.e.$ $a$ continuous map

$$\varphi: E \to F \text{ such that } \pi_F \circ \varphi = \pi_E \tag{36}$$

and, for all $y \in Y$, the restriction-corestriction map $\varphi y : \pi^{-1}{}_E(y) \to \pi^{-1}{}_F(y)$ is a linear map between finite-dimensional vector spaces.

View the commutative $C^*$-algebra $C(Y)$ as a commutative ring with unit. Denote by FProj($C(Y)$) the category of finitely generated projective $C(Y)$ −modules. $An$ object in the category FProj($C(Y)$) is a finitely generated projective $C(Y)$-module. $A$ morphism in FProj($C(Y)$) is a map of $C(Y)$ −modules $\psi: M \to N$.

If E is a $\mathbb{C}$ vector bundle on $Y$, then $\Gamma(E)$ denotes the $C(Y)$ −module consisting of all continuous sections of $E$. The module structure is pointwise. According to the Serre-Swan theorem, the functor $\Gamma$

$$Vect(Y) \to FProj\big(C(Y)\big), E \mapsto \Gamma(E), \tag{37}$$

is an equivalence of categories and preserves all the basic properties of the two categories. In particular, $E\ \Gamma(E)\ preserves\ \oplus\ and\ \otimes$:

$$\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F),$$

$$\Gamma(E \otimes F) = \Gamma(E) \otimes C(Y)\Gamma(F). \tag{38}$$

Let $X$ be a compact Hausdorff space equipped with a continuous free action of a compact Hausdorff group $G$. Next, let FRep($G$) denote the category of representations of $G$ on finite-dimensional complex vector spaces. Due to the freeness asumption, we can define the functor

$$FRep(G) \to Vect\left(\frac{X}{G}\right), V \mapsto X \underset{G}{\times} V, \tag{39}$$

preserving $\oplus$ $and$ $\otimes$:

$$X \underset{G}{\times} (V \oplus W) = (X \underset{G}{\times} V) \oplus (X \underset{G}{\times} W),$$

$$X \underset{G}{\times} (V \otimes W) = (X \underset{G}{\times} V) \otimes (X \underset{G}{\times} W). \tag{40}$$

Combining the functor $\Gamma$ with the functor $X \times_G$ yields the functor

$$FRep(G) \to FProj\left(C\left(\frac{X}{G}\right)\right), V \mapsto \Gamma(X \underset{G}{\times} V). \tag{41}$$

Furthermore, note that the $C(X/G)$ −module $CG(X,V)$ of all continuous G-equivariant functions from $X\ to\ V$ is naturally isomorphic with $\Gamma(X \times G\ V)$. Here $G$-equivariance means

$$\forall x \in X, g \in G : f(xg) = \varrho(g^{-1})\big(f(x)\big), \qquad \varrho : G \to GL(V). \tag{42}$$

Hence we can replace the above $\otimes$-preserving functor with the $\otimes$-preserving functor

$$FRep(G) \to FProj\left(C\left(\frac{X}{G}\right)\right), V \mapsto CG(X,V). \quad (43)$$

The following elementary observation is key in translating from the topologicalto the algebraic setting.

**Lemma(2.2.14)[148]:** Let $X$ be a compact Hausdorff space equipped with a continu- ous action of a compact Hausdorff group $G$, and let $V$ be a finite-dimensional representation of $G$. Then the evident identification $C(X,V) = C(X) \otimes V$ determines an equivalence of tensor functors:

$$C_G(X,V) = P_G(X)\square V.$$

**Proof.** Let $\{e_i\}_{i=1}^{n}$ be a basis of $V$ and $\{e_i\}_{i=1}^{n}$ be the basis of $V*$ dual to$\{e_i\}_{i=1}^{n}$. Given $f \in C(X,V)$, we note that

$$\sum_{i=1}^{n}(e^i \circ f) \otimes e^i \in P_G(X)\square V$$

$$\Updownarrow$$

$$\sum_{i=1}^{n}\delta\,(e^i \circ f) \otimes e^i = \sum_{i=1}^{n}(e^i \circ f) \otimes_V \Delta(e^i)$$

$$\Updownarrow$$

$$\forall\, x \in X, g \in G : f(xg) = \varrho(g^{-1})\big(f(x)\big). \quad (44)$$

The second equivalence is an immediate consquence of the definitions of δ and␣see (17) and (21)). The first equivalence follows directly from the definition of cotensor product (see (12)) and the fact that

$$\sum_{i=1}^{n}\delta\left(e^i \circ f\right) \otimes_V \Delta\left(e^i\right) \in C(X) \otimes O(G) \otimes V. \quad (45)$$

Thus the evident identification yields $C_G(X,V) = P_G(X)\square V$.

Finally, let $\beta$ be the map defined in Theorem(2.2.11), and let

$$diag: C_G(X,V) \underset{C\left(\frac{X}{G}\right)}{\otimes} C_G(X,W) \to C_G(X, V \otimes W),$$

$$diag: f_1 \otimes f_2 \longmapsto \big(x \longmapsto f_1(x) \otimes f_2(x)\big). \quad (46)$$

The commutativity of the diagram

$$\begin{array}{ccc} C_G(X,V)) \underset{C\left(\frac{X}{G}\right)}{\otimes} C_G(X,W) & \overset{dig}{\longrightarrow} & C_G(X, V \otimes W \\ \downarrow & & \downarrow \\ (p_G(X)\square V) \underset{C\left(\frac{X}{G}\right)}{\otimes} (p_G(X)\square W) & \beta\,/ & p_G(X)\square(V \otimes W) \end{array} \quad (47)$$

proves that the identification $C_G\ (X,V) = p_G(X)\square V$ defines an equivalence of tensor functors.

Assume now that the action of $G$ on $X$ is free. Then, by the Serre-Swan theorem, the functor $\Gamma(X \times_G)$ is strongly monoidal. Since it is equivalent as a tensor functor to $C_G(X,)$, we conclude from Lemma (2.2.14)that the cotensor product functor

$$FRep(G) \to FProj\left(C\left(\frac{X}{G}\right)\right), V \longmapsto PG(X)V, \quad (48)$$

is also strongly monoidal.

Next, since $O(G)$ is cosemisimple, any $O(G)$ −comodule is a purely algebraic direct sum of finite-dimensional comodules. Furthermore, as the cotensor product is defined as the kernel of a linear map, it commutes with such direct sums. As it is also clear that the map $\beta$ commutes with such direct sums, we infer that the extended cotensor product functor

$$FRep^{\oplus}(G) \to FProj^{\oplus}\left(C\left(\frac{X}{G}\right)\right), V \longmapsto P_G(X)V, \tag{49}$$

is strongly monoidal. Here $FProj^{\oplus}(C(X/G))$ is the category of projective modules over $C(X/G)$ that are purely algebraic direct sums of finitely generated projective $C(X/G)$ −modules, and $FRep^{\oplus}(G)$ is the category of representations of G defined above (48). (One can think of these categories as the indcompletions in the sense of [149].) Combining this with allows us to conclude the proof of the implication "$PWG$condition⇐ freeness".

We now give a vector-bundle interpretation of the proof. To this end, we need to extend the functor $C_G(X,)$ to the category $FRep^{\oplus}(G)$, which includes the representation $O(G)$. Let V be a purely algebraic direct sum of finite-dimensional representations of $G$. We topologize V as the direct limitof its finite-dimensional subspaces, and denote by $C(X,V)$ the space of all continuous maps from X to $V$. An elementary topological argument shows that the image of any continuous map from $X$ to $V$ is contained in a finite dimensional subspace of $V$. Therefore, Lemma (2.2.14)generalizes to:

**Corollary(2.2.15)[148]:** Let $V$ be an object in the category $FRep^{\oplus}(G)$. Then the evident identification $C(X,V) = C(X) \otimes V$ determines an equivalence of tensor functors:

$$C_G(X,V) = P_G(X)V.$$

Taking $V = O(G)$ topologized with the direct limit topology, we immediately obtain the following presentation of the Peter-Weyl algebra:

$$C_G\big(X,O(G)\big) = P_G(X)O(G) = P_G(X). \tag{50}$$

Assume now that the action of $G$ on $X$ is free. Then $X \times_G O(G)$ is a vector bundle in the sense that it is a direct sum of ordinary (i.e. with finite-dimensional fibers) vector bundles, and

$$\Gamma\big(X \times_G O(G)\big) = C_G\big(X,O(G)\big) = P_G(X). \tag{51}$$

Moreover, arguing as for the cotensor product functor, we conclude that the functor

$$FRep^{\oplus}(G) \to FProj^{\oplus}P\left(C\left(\frac{X}{G}\right)\right), V \longmapsto C_G(X,V), \tag{52}$$

is strongly monoidal. Hence, taking advantage of (51), we obtain

$$C_G\big(X,O(G) \otimes O(G)\big) = P_G(X) \otimes C\left(\frac{X}{G}\right) P_G(X). \tag{53}$$

Next, denote by $O(G)^{trivial}$ the vector space $O(G)$ with the trivial action of $G$, i.e. every $g \in G$ is acting by the identity map of $O(G)$. Then, as before, we obt

$$C_G\big(X,O(G) \otimes O(G)^{trivial}\big) = P_G(X) \otimes C\left(\frac{X}{G}\right) C\left(\frac{X}{G}\right) \otimes O(G)$$

$$= P_G(X) \otimes O(G). \tag{54}$$

**Lemma (2.2.16)[148]:** The $G$-equivariant homeomorphism

$$W : G \times G^{trivial} \to G \times G, W((g,g')) := (g,gg'),$$

gives an isomorphism of representations of $G$

$$O(G) \otimes O(G)^{trivial} \cong O(G) \otimes O(G).$$

Here $G \times G^{trivial}$ and $G \times G$ are right $G$-spaces via the formulas

$$(g,g')h := (h^{-1}g,g') \text{ and } (g,g')h := (h^{-1}g,h^{-1}g'),$$

respectively.

**Proof.** Since $O(G)$ is a Hopf algebra, the pullback of W restricts and corestricts to

$$W^* : O(G) \otimes O(G) \dashrightarrow O(G) \otimes OG^{trivial}. \tag{55}$$

We infer that $W^*$ is the required intertwining operator.

Combining Lemma (2.2.16) with (53) and (54) gives

$$P_G(X) \underset{C\left(\frac{X}{G}\right)}{\otimes} P_G(X) \cong P_G(X) \otimes O(G). \tag{56}$$

Finally, to see that this isomorphism is indeed the canonical map, we explicitly put together all identifications used on the way. First, we observe that, since the isomorphism

$$P_G(X) \to P_G(X)O(G) \tag{57}$$

is given by the coaction δ, the identification (52) is implemented by the maps

$$p_G(X) \overset{E}{\underset{\overset{F}{\leftarrow}}{\rightarrow}} C_G(X,O(G)),$$

$$(E(f)(x)(g) := f(xg), F(\alpha)(x) := \alpha(x)(e), E \circ F = id, F \circ E = id. \tag{58}$$

We can now easily check that the following composition of isomorphisms

$$P_G(X) \underset{C\left(\frac{X}{G}\right)}{\otimes} P_G(X) \xrightarrow{E\otimes E} C_G(X,O(G)) \underset{C\left(\frac{X}{G}\right)}{\otimes} C_G(X,O(G))$$

$$\xrightarrow{dig} C_G(X,O(G)\otimes O(G)^{trivial}) \xrightarrow{\Sigma_i (id\otimes e^i)\otimes e_i} C_G(X,O(G)) \otimes O(G) \xrightarrow{F\otimes id}$$

$$\to P_G(X) \otimes O(G)$$

is the canonical map, as desired.

Let $A$ be a unital $C^*$ -algebra with center $Z(A)$, let $X$ be a compact Hausdorff space and let $\theta : C(X) \to Z(A)$ be a unital inclusion. The triple $(A, C(X), \theta)$ is called a unital $C(X)$ −algebra ([165]). In the following, we simply consider $C(X)$ as a subalgebra of $A$. For $x \in X$, let $J_x$ be the closed two-sided ideal in $A$ generated by the functions $f \in C(X)$ that vanish at $x$. Then we have quotient $C^*$-algebras $A_x := A/J_x$ with natural projection *maps* $\pi_x: A \to A_x$ , and the triple $(X, A, \pi_x)$ is a field of $C^*$ algebras. For any $a \in A$, the map $n_x : X \to \mathbb{R}, x \longmapsto \|\pi_x(a)\|$is upper semi-continuous [160] (see also [168]). If the latter map is continuous, the field is called continuous, but this property will not be necessary to assume for our purposes.

**Lemma(2.2.17)[148]:** Let $X$ be a compact Hausdorff space, $A$ a unital $C(X)$ −algebra, and $(H, \Delta)$ a compact quantum group acting on $A$ via $\delta : A \to A \otimes_{min} H$. $A$ ssume that $C(X) \subseteq AcoH$ . Then for each $x \in X$ there exists a unique coactions $\delta_x : A_x \to A_x \otimes_{min} H$ such that for all $a \in A$

$$\delta x(\pi_x(a)) = (\delta_x \pi_x \otimes id)(\delta(a)). \tag{59}$$

**Proof:** Let $x \in X$ *and* $f \in C(X)$ *with* $f(x) = 0$. *As* $\delta(f) = f \otimes 1$ by assumption, it follows that $(\pi_x \otimes id)(\delta(f)) = 0$. *Hence* $(\pi_x \otimes id)(\delta(a)) = 0$ *for* $a \in J_x$ , so that $\delta_x$ can be defined by (59). It is straightforward to check that each $\delta_x$ satisfies the coassociativity and counitality conditions. Finally, to see that $\delta_x$ is injective, assume that $\delta_x(\pi_x(a)) = 0$.Then

$$(\pi_x \otimes id)(\delta(a)) = 0, \tag{60}$$

whence (id ⊗ ω)(δ(a)) $\in J_x$ for all $\omega \in A^*$. In particular, if $(g_\alpha)_\alpha$ is a bounded positive approximate unit for $C_0(X \setminus \{x\})$, then

$$g_\alpha(id \otimes \omega)(\delta(a)) \xrightarrow[\alpha]{norm} \alpha\,(id \otimes \omega)(\delta(a)). \qquad (61)$$

Hence we obtain

$$(g_\alpha s \otimes 1)\delta(a) weakly \underset{\alpha}{\overset{weakly}{\longrightarrow}} \alpha\,\delta(a). \qquad (62)$$

$$\alpha$$

However, as $(g_\alpha \otimes 1)\delta(a) = \delta(g_\alpha a)$ and δ is injective, we find that

$$g_\alpha a \underset{\alpha}{\overset{weakly}{\longrightarrow}} a. \qquad (63)$$

Consequently, $\pi_x(a) = 0$, and we conclude that $\delta_x$ is injective.

**Theorem(2.2.18)[148]:** Let $X$ be a compact Hausdorff space, $A$ a unital $C(X)$ −algebra, and $(H,\Delta)$ a compact quantum group acting on $A$ via $\delta : A \to A \otimes_{min} H$. Assume that $C(X) \subseteq A^{coH}$. Then, the coaction δ is free if and only if the coactions $\delta_x$ are free for each $x \in X$.

**Proof:** First note that $A \otimes_{min} H$ is again a $C(X)$ −algebra in a natural way.

We will denote the quotient $(A \otimes_{min} H)/(\mathrm{J}x \otimes_{min} H)$ by $A_x \otimes_x H$. This will be a $C^*$ −completion of the algebraic tensor product algebra $A_x \otimes H$ (not necessarily the minimal one). We will denote the quotient map at $x$ by $\pi_x \otimes_x id : A \otimes min\, H \to A_x \otimes_x H$.

The implication "$\delta$ is free ⇒ the coactions $\delta_x$ are free for each $x \in X$" follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} A \otimes A & \overset{can}{\longrightarrow} & A \overset{\otimes}{_{min}} H \\ \pi_x \otimes \pi_x \downarrow & & \downarrow \pi_x \overset{\otimes}{_x} id \\ A_x \otimes A_x & \to & A_x \overset{\otimes}{_x} H \end{array} \qquad (64)$$

Here the upper horizontal arrow is given by the formula $a \otimes a' \longmapsto (a \otimes 1)\delta(a')$, and the lower horizontal arrow is given by $a \otimes a' \longmapsto (a \otimes 1)\delta_x(a')$. Assume now that each $\delta_x$ is free. Fix $\varepsilon > 0$, and choose $h \in O(H)$. By Theorem (2.2.4)for each $x \in X$ we can find an element $z_x \in (A \otimes C)\delta(A)$ such that $(\pi_x \otimes x\, id)(z_x) = 1 \otimes h\, in\, A_x \otimes_x H$. Consider the function

$$f_x : X \ni y \longmapsto \|(\pi_y \otimes_y id)(z_x) - 1 \otimes h\| = \|(\pi_y \otimes_y id)(z_x) - 1 \otimes h\|\, \mathbb{R} \quad (65)$$

As the norm on the field $y \longmapsto A_y \otimes_y$ H is upper semi-continuous, the function $y \longmapsto f_x(y)$ is upper semi-continuous. Since $f_x(x) = 0$, we can find an open neighborhood $U_x$ of $x$ such that for all $y \in U_x$

$$f_x(y) = \|(\pi_y \otimes_y id)(z_x) - 1 \otimes h\| A_y \otimes_y H < \varepsilon. \qquad (66)$$

Let$\{f_i\}i$ be a partition of unity subordinate to a finite subcover $\{U_{xi}\}i$. An easy estimate shows that for $z := \sum_i (f_i \otimes 1) z_{xi}$ and all $y \in X$

$$\|(\pi_y \otimes_y id)\,(z - 1 \otimes h\| A_y \otimes_y H < \varepsilon \qquad (67)$$

Taking the supremum over all y, we conclude by [160] and the compactness of $X\, that\, [\![z - 1 \otimes h]\!] < \varepsilon$. Hence $(A \otimes \mathbb{C})\delta(A)$ is dense in $A \otimes H, i.e. the\, coaction\, \delta$ is free.

Combining Theorem(2.2.4) and Theorem(2.2.16), we obtain: Corollary (2.2.17) Let $X$ be a compact Hausdorff space, A a unital $C(X)$ −algebra, and $(H,\Delta)$ a compact quantum group acting on $A$ $via$ $\delta : A \to A \otimes_{min} H$. Assume that $C(X) \subseteq A^{coH}$. Then, the coaction $\delta$ satisfies the $PWG$-condition if and only if the coactions $\delta_x$ satisfy the $PWG$-condition for each $x \in X$.

We consider:

**Definition (2.2.19)[148]:**(cf. [157]). Let $(H,\Delta)$ be a compact quantum group acting on a unital $C^*$-algebra $A$ via $\delta : A \to A \otimes_{min} H$. We call the unital $C^*$-algebra

$$A \circledast^{\delta} H := \{f \in C([0,1], A \otimes_{min} H) | f(0) \in \mathbb{C} \otimes H, f(i) \in \delta(A)\}$$

the equivariant noncommutative join of $A$ and $H$.

The $C^* - algebra\ A \circledast^{\delta} H$ is obviously a $C([0,1])$ − algebra with $(A \circledast^{\delta} H)x = A \otimes_{min} H\ for\ x \in (0,1), (A \circledast^{\delta} \mathrm{H})_0 = H\ and\ (A \circledast^{\delta} H)_1 \cong A$ . The following lemma shows that A $\circledast^{\delta}$ H carries a natural action of (H,Δ).

Lemma (2.2.19) The compact quantum group $(H,\Delta)$ acts on the unital $C^*$ − algebra $A \circledast^{\delta} H$ via

$$\delta_{A \circledast^{\delta} H} : A \circledast^{\delta} H \ni f \mapsto (id \otimes \Delta) \circ f \in \left(A \circledast^{\delta} H\right) \otimes_{min} H.$$

**Proof**: We first show that the range of $\delta_{A \circledast^{\delta} H}$ is contained in$\left(A \circledast^{\delta} H\right) \otimes_{min}$ H.

To this end, we take any function $f \in A \circledast^{\delta} H$ and identify $\left(A \circledast^{\delta} H\right) \otimes_{min} H$ as a subalgebra of $C([0,1], A \otimes_{min} H \otimes_{min} H)$. Since $f$ is uniformly continuous and $P_H(A)$ is dense in $A$ by [173], an elementary partition of unity argument shows that f can be approximated by finite sums of functions of three kinds:

(i) $F_1 : [0,1] \ni t \mapsto \xi_0 (t)(1 \otimes h) \in \mathbb{C} \otimes O(H), where\ \xi_0 \in C([0,1],[0,1]),$ $\xi_0(1) = 0, and\ h\ is\ a\ fixed\ element\ of\ O(H);$

(ii) $F_2 : [0,1] \ni t \mapsto \xi(t)(a \otimes h) \in P_H (A) \otimes O(H), where\ \xi \in C([0,1],[0,1])$ $with\ \xi(0) = 0 = \xi(i), and\ a\ and\ h\ are\ respectively\ fixed\ elements\ of$ $P_H(A) and\ O(H);$

$(iii)$ $F_3 : [0,1] \ni t \mapsto \xi_1(t)\delta(a) \in \delta(P_H(A)), where\ \xi_1 \in C([0,1],[0,1]),$ $\xi_1(0) = 0, and\ a\ is\ a\ fixed\ element\ of\ P_H (A).$

It is clear that $(id \otimes \Delta) \circ F_i \in C([0,1], A \otimes_{min} H) \otimes H\ for\ all\ i$. As the rightmost tensor product is algebraic, evaluations commute with $id \otimes \Delta$, and $\delta$ is coassociative, we infer that $(id \otimes \Delta) \circ F_i \in \left(A \circledast^{\delta} H\right) \otimes H$ for i (cf. [157]). Furthermore, since $\delta_{A \circledast^{\delta} H}$ viewed as a map into $C([0,1], A \otimes_{min} H \otimes_{min} H\_ is\ a *$ − homomorphism, it is continuous, so that

$(id \otimes \Delta) \circ f \in \left(A \circledast^{\delta} H\right) \otimes_{min} H. Hence \delta_{A \circledast^{\delta} H}$ has range in $\left(A \circledast^{\delta} H\right) \otimes_{min} H$

The injectivity and coassociativity of $\delta_{A \circledast^{\delta} H}$ are immediate respectively from the injectivity and coassociativity of $\Delta$. The counitality condition follows from the same approximation argument as above.

**Corollary(2.2.20)[148]:** If the coaction $\delta : A \to A \otimes_{\min}$ H is free, then so is the coaction $\delta_{A \circledast^{\delta} H} : A \circledast^{\delta} H \to \left(A \circledast^{\delta} H\right) \otimes_{\min}$ H.

**Proof.** The $C^*$ −algebra $A \circledast^{\delta} H$ is a unital $C([0,1])$ −algebra with $C([0,1]) \subseteq (A \circledast^{\delta} H)^{coH}$. With the notation of Lemma(2.2.17), we have:

(i) $(,(A * H)_0 \delta_0) = (H,\Delta),$

(ii) $((A * H)_x, \delta_x) = (A \otimes_{min} H, id \otimes \Delta)\ for\ x \in (0,1),$

As each of the above actions are free, we infer from Theorem(2.2.16) that δA∗H is free. Alternatively, one can use a direct approximation argument as in Lemma (2.2.15).
Let $\pi: X \to Y$ be $a$ covering map of topological spaces. As usual, this means that given any $y \in Y$ there exists an open set $U$ in $Y$ with $y \in U$ such that $\pi^{-1}(U)$ is a disjoint union of open sets each of which $\pi$ maps homeomorphically onto $U.A$ deck transformation is a homeomorphism $h: X \to X$ with $\pi \circ h = \pi$.

**Proposition (2.2.21)[148]:** Let $X$ and $Y$ be compact Hausdorff topological spaces. Let $\pi: X \to Y$ be a covering map, and let $\Gamma$ be the group of deck transformations of this covering map. Assume that $\Gamma$ is finite. Then $X$ is a principal $\Gamma$-bundle over $Y$ if and only if the canonical map

$$can: C(X) \otimes_{C(Y)} \longrightarrow C(X)\, C(X) \otimes C(\Gamma),$$
$$can: f_1 \otimes f_2 \mapsto (f_1 \otimes 1)\delta(f_2),$$

is an isomorphism. Here δ is given by (3.11).

**Proof.** If $X$ is a principal $\Gamma$-bundle over $Y$ , then $C(Y) = C\left(\frac{X}{\Gamma}\right) = C(X)^{coC(\Gamma)}$ and, by (51), can is surjective. Furthermore, since $C(\Gamma)$ is cosemisimple, by the result of $H.-J.$ Schneider, the surjectivity of can implies its bijectivity.
Assume now that can is bijective. The local triviality assumption in the definition of a covering map implies that for any continuous function f on $X$ one has a continuous function $\Theta(f)$ on $Y$ given by the formula

$$\big(\Theta(f)\big)(y) := \frac{1}{\#\pi^{-1}(y)} \sum_{x\in\pi^{-1}(y)} f(x). \tag{68}$$

Note that the fibres are finite due to the compactness of $X$. Also, one immediately sees that $\Theta$ is a unital $C(Y)-$linear map from $C(X) to\, C(Y)$.
Now it follows from the bijectivity of can and that $C(Y) = C(X)^{co\, C(\Gamma)} = C(X/\Gamma)$. Hence the fibres of the covering map $\pi: X \to Y$ are the orbits of $\Gamma$. Finally, the freeness of the action of $\Gamma$ on $X$ follows from the surjectivity of can and (51). If $X$ is connected, then it is always the case that the group of deck transformations $\Gamma$ is finite and that the action of $\Gamma$ on $X$ is free. The issue is then whether or not the action of $\Gamma$ on each fiber of $\pi$ is transitive. Thus we conclude from Proposition (2.2.20):

**Corollary (2.2.22)[148]**: Let $X$ and $Y$ be connected compact Hausdorff topological spaces, and let $\pi: X \to Y$ be a covering map. Denote by $\Gamma$ the group of deck transformations. Then the action of $\Gamma$ on each fiber of $\pi$ is transitive if and only if the canonical map

$$can: C(X) \underset{C(Y)}{\otimes} C(X) \longrightarrow C(X) \otimes C(\Gamma)$$

is an isomorphism.

**Remark (2.2.23)[148]:** An alternative proof of Proposition (2.2.20) is as follows. Consider the commutative diagram

$$\begin{array}{ccc} C(X) \underset{C(Y)}{\otimes} C(X) & \overset{can}{\longrightarrow} & C(X) \otimes C(\Gamma) \\ \downarrow & & \downarrow \\ C(X \underset{Y}{\times} X) & \longrightarrow & C(X \times \Gamma) \end{array} \tag{69}$$

in which each vertical arrow is the evident map and the lower horizontal arrow is the $*$-homomorphism resulting from the map of topological spaces

$$X \times \Gamma \longrightarrow X \underset{V}{\times} X , \quad (x, y) \longmapsto (x, xy). \tag{70}$$

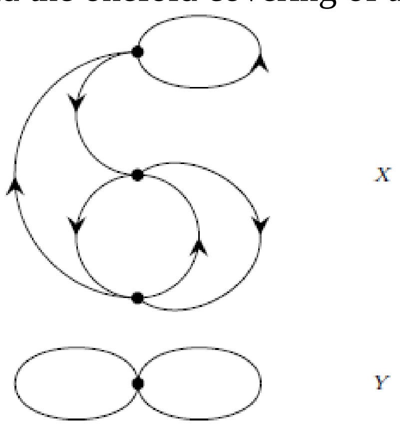
Note that $X$ is a (locally trivial) principal $\Gamma$ bundle on $Y$ if and only if this map of topological spaces is a homeomorphism, and the latter is equivalent to bijectivity of the lower horizontal arrow.
Hence to prove Proposition (2.2.20), it will suffice to prove that the two vertical arrows are isomorphisms. The right vertical arrow is an isomorphism because $\Gamma$ is a finite group, so $C(\Gamma)$ is a finite-dimensional vector space over the complex numbers $\mathbb{C}$
For the left vertical arrow, let $E$ be the vector bundle on $Y$ whose fiber at $y \in Y$ is $Map(\pi^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from $\pi^{-1}(y)$ to $\mathbb{C}$. $As\ \pi^{-1}(y)$ is a discrete subset of the compact Hausdorff space $X$, it is finite. Let $S(E)$ be the algebra consisting of all the continuous sections of $E$. Then S(E) = C(X).
Similarly, define

$$\pi^{(ii)}: X \underset{Y}{\times} X \longrightarrow Y \quad by \quad \pi^{(ii)}: (x_1, x_2) \longmapsto \pi(x_1) = \pi(x_2). \tag{71}$$

Let $F$ be the vector bundle on $Y$ whose fiber at $y \in Y$ is $Map((\pi^{(ii)})^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from$((\pi^{(ii)})^{-1}(y)$ to $\mathbb{C}$. Then $S(F) = C(X \times_Y X)$,where $S(F)$ is the algebra consisting of all the continuous sections of $F$. Since $F = E \otimes E$ as vector bundles on $Y$ , we conclude $S(F) = S(E) \otimes C(Y)\, S(E)$, which proves bijectivity for the left vertical arrow. Without connectivity, the group of deck transformations can be infinite. For example, let $Y$ be the Cantor set and let $\pi: Y \times \{0,1\} \rightarrow Y$ be the trivial twofold covering. Let $U$ be a subset of $Y$ which is both open and closed. Define $\gamma U : Y \times \{0,1\} \rightarrow Y \times \{0,1\}$ by

$$\gamma_U(y,t) := \begin{cases} (y,t) & \text{for } y \notin U \\ (y.1-t) & \text{for } y \in U. \end{cases} \tag{72}$$

Then $\gamma U$ is a deck transformation and there are infinitely many closed and open subsets $U$.
The following example is a threefold covering $X$ of the one point union of two circles $Y$. Here the preimage of the left circle of the base space is the usual threefold covering of the circle. The preimage of the right circle of the base space is the disjoint union of the usual twofold covering of the circle and the onefold covering of the circle.



In this example, the group of deck transformations is trivial. Indeed, let $\gamma$ be a deck transformation. Consider $\gamma$ restricted to the preimage of the right circle of the base space. This preimage has two connected components. Since $\gamma$ is a deck transformation of this preimage, it must map each connected compenent to itself. This implies that $\gamma$ has a fixed point. Hence, as $X$ is connected, $\gamma = id$. In particular, this shows that the group of deck transformations need not act transitively on fibers of a covering map. The canonical map is surjective but not injective.

# Chapter 3
# Invariant Measures and Almost Weak Specification Property

We consider the space of measures provided with the weak topology. We show a property for ergodic group automorphisms of abelian groups.

## Section (3.1): Homeomorphisms with Weak Specification

We consider the space of measures provided with the weak topology. In [185], [186], K. Sigmund discussed some categories in the space of invariant measures for homeomorphisms satisfying specification. The ingredient of his proofs is in the densely periodic property of homeomorphisms with specification. It is known that weak specification for homeomorphisms is strictly weaker than specification.

We show that the results of K. Sigmund hold for homeomorphisms satisfying weak specification (Theorems (3.1.3) and (3.1.5)). The idea of proofs is in constructing the property ”smallest sets“that is found in the weak specification property.

For $X$ be a compact metric space with metric $d$ and $\mathfrak{M}(X)$ be the space of Borel probability measures of $X$ with metric $\bar{d}$ which is compatible with the weak topology, where $\bar{d}$ is defined by

$$\bar{d}(\mu,\nu) = \inf\{\varepsilon; \mu(B) \leqq \nu(\{x \in X; d(x,B) \leqq \varepsilon\}) + \varepsilon \text{ and } \nu(B) \leqq \mu(\{x \in X; d(x,B) \leqq \varepsilon\}) + \varepsilon \text{ for all Borel sets } B\}$$

(p. 9 of [183] or p. 238 of [181]).

Define a point measure $\delta(x)$ by $\delta(x)(B) = 1$ if $x \in B$ and $\delta(x)(B) = 0$ if $x \notin B$ (Borel sets B ), and denote by $B(x,\varepsilon)$ an $\varepsilon$-closed ball about $x$ in $X$. For arbitrary finite points $x_i \in X$ and $\mu_i \in \mathfrak{M}(X)$ $(1 \le i \le n)$ with card $\{1 \le i \le n;\ \mu_i(B(x,\varepsilon)) < 1\}/n < \varepsilon$, we get easily $\bar{d}(1/n\sum_{i=1}^{n}\delta(x_i), 1/n\sum_{i=1}^{n}\mu_i) < \varepsilon$. It is clear that the map $x \to \delta(x)$ $x \in X$ is a homeomorphism from $X$ onto a subset of $\mathfrak{M}(X)$.

Let $\sigma$ be a self-homeomorphism of $X$ Then $\sigma$ induces a homeomorphism $\sigma: \mathfrak{M}(X) \to \mathfrak{M}(X)$ by $\sigma\mu(B) = \mu(\sigma^{-1}B)$ (Borel sets $B$ and $\mu \in \mathfrak{M}(X)$) such that $\delta(\sigma x) = \sigma\delta(x)$ for all $x \in X$. Hence we can consider that $(X,\sigma)$ is a subsystem of $(\mathfrak{M}(X),\sigma)$. It is known (p. 17 of [183]) that the set $\mathfrak{M}_\sigma(X)$of $\sigma$ -invariant measures is a compact convex set.

Let $\mathcal{g}(X)$ denote the set of ergodic measures in $\mathfrak{M}_\sigma(X)$. Then 2 $\mathcal{g}(X)$ is a nonempty $G_\delta$-set in $\mathfrak{M}_\sigma(X)$ (p. 25 of [183]). Let $g(x)$ denote the set of strongly mixing measures in $\mathfrak{M}_\sigma(X), \mathcal{D}(X)$ denote the set of measures positive on all nonempty open sets in $\mathfrak{M}_\sigma(X)$, and $\mathcal{N}(X)$ denote the set of non-atomic measures in $\mathfrak{M}_\sigma(X)$. We denote by $V_\sigma(x)$ the set of $\omega$-limits of the sequence $\{1/\sum_{j=0}^{n-1}\delta(\sigma^j x)\}_{n=1}^{\infty}$ for $x \in X$. Then we know (p. 18 of [183]) that for every $x \in X$ , $V_\sigma(x)$ is a nonempty compact connected subset of $\mathfrak{M}_\sigma(X)$.

Let $X$ and $\sigma$ be as above. Then $(X,\sigma)$ is said to satisfy weak specification if for $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that for every $k \ge 1$, $k$ points $x_1,\cdots,x_k \in X$ and for every set of integers $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$ with $a_i - b_{i-1} \ge M(\varepsilon)$ $(2 \le i \le k)$, the set $\hat{B} = \bigcap_{i=1}^{k}\bigcap_{j=\sigma_i}^{b_i}\sigma^{-j}B(\sigma^j x_i,\varepsilon)$ is nonempty. Since $\emptyset \ne \bigcap_{r=1}^{\infty}\bigcap_{n=-r}^{r}\bigcap_{i=1}^{k}\bigcap_{j=a_i+nq}^{b_i+nq}\sigma^{-j}\ B(\sigma^{j-nq}x_i,\varepsilon) \subset \hat{B}$ for all $\ge b_k - a_1 + M(\varepsilon)$ , we get easily that $\hat{B}$ contains a $\sigma^q$-invarant subset. When $(X,\sigma)$ obeys weak specification and has the following additional condition; for every $q \ge b_k - a_1 + M(\varepsilon)$ there is an $x \in B$ with $\sigma^q x = x$ , we say $(X,\sigma)$ to satisfy specification.

In order to solve whether every zero-dimensional ergodic automorphism satisfies specification, in [180] N. Aoki constructs a zero-dimensional ergodic automorphism without densely periodic property. This implies that such an automorphism obeys weak specification, but not specification. For the class of all solenoidal automorphisms, it is proved in [179] that the class of automorphisms with weak specification is wider than the class of automorphisms with specification.

The following theorems are proved for the class of homeomorphisms with weak specification of compact metric spaces.

We show two results which are used in the proof of the theorems. Hereafter $X$ is a compact metric space with metric $d$ and $\sigma$is a self-homeomorphism of $X$.

A nonempty closed subset $\Delta$ is said to be a smallest set if there is an integer $q \geq 1$ such that $\sigma^q \Delta = \Delta$ and $\Delta$ contains no completely $\sigma^q$-invariant closed proper subsets. We call the least positive integer in the set of such $q \geq 1$ the period of $\Delta$, and we denote it by per $(\Delta)$. Obviously, $\sigma^i \Delta \cap \Delta = \emptyset$ for $i$with $1 \leq i \leq$per $(\Delta) - 1$. Let $\Delta$ be a smallest set. Then $\tilde{\Delta} = \bigcup_{i=1}^{per(\Delta)-1} \sigma^i \Delta$.is a minimal set under $\sigma$ ; i.e., $\tilde{\Delta}$ contains no completely $\sigma$ –invariant closed proper subsets. Since $\tilde{\Delta}$ is compact and $\tilde{\Delta} = \tilde{\Delta}$ , as before we can consider the space $\mathfrak{M}_\sigma(\tilde{\Delta})$ of $\sigma$ -invariant Borel probability measures of $\tilde{\Delta}$. Then every $\mu \in \mathfrak{M}\,(\tilde{\Delta})$ defines a measure $\bar{\mu} \in \mathfrak{M}_\sigma(X)$by $\bar{\mu} \in (B) = \mu(B \cap \tilde{\Delta})$ for Borel sets $B$ of $X$ . It is clear that if $\mu \in \mathfrak{M}_\sigma(\tilde{\Delta})$ is ergodic, then $\bar{\mu} \in \mathcal{g}(X)$. We remark that $\bar{\mu}\left(\sigma^j \Delta\right) = 1/\text{per}(\Delta)$ $(0 \leq j \leq \text{per}(\Delta) - 1)$ for all $\mu \in \mathfrak{M}_\sigma\,(\tilde{\Delta})$. Define $\bar{\mu}_j \in \mathfrak{M}\,(X)$ $(j \geq 0)$ by $\bar{\mu}_j\,(B) = \text{per}\,(\Delta)\bar{\mu}(B \cap \sigma^j \Delta)$ for Borel sets $B$ of $X$.Then we have $\bar{\mu} = (1/\text{per}(\Delta)) \bigcup_{j=0}^{\text{per}(\Delta)-1} \bar{\mu}_j$.We say that $x \in X$ is a generic point for $\mu \in \mathfrak{M}_\sigma(X)$ if $V_\sigma(x) = \{\mu\}$. Every $\mu \in \mathcal{g}(X)$ has generic points and the set of generic points for $\mu$ has $\sigma$-measure one (c.f. see p. 25 of [183]).

**Proposition (3.1.1)[178]:** If $(X, \sigma)$ satisfies weak specification, then $\mathcal{g}(X)$ is dense in $\mathfrak{M}_\sigma\,(X)$.

**Proof:** It is clear that $\mathcal{g}(X) \neq \emptyset$. First we prove that for every $\mu_1, \mu_2 \in \mathcal{g}(X)$, every $t \in [0,1]$ and every $\varepsilon > 0$, there exists $v \in \mathcal{g}(X)$ with $\bar{d}\,(v, t\mu_1 + (1-t)\mu_2) < \varepsilon$.

Take an integer $m > 4/\varepsilon$, then there exists an integer $m_1$ with $1 \leq m_1 \leq m - 1$ such that $|m_1/m - t| \leq 1/m$. It follows from the definition of $\bar{d}$ that

$$\bar{d}\left(t\mu_1 + (1-t)\mu_2, \frac{m_1}{m}\,\mu_1 + \frac{m - m_1}{m}\mu_2\right) < \varepsilon/2.$$

Let $x_1$ and $x_2$ be generic points for $\mu_1$ and $\mu_2$, respectively and choose $M = M(\varepsilon/4)$ as in the definition of weak specification. Since $x_i$ is a generic point for $\mu_i (i = 1,2)$, we can find an $N_0 \geq 4M/\varepsilon$ such that for all $n \geq N_0, \bar{d}\,\left(\mu_i\,, (1/n)\sum_{j=0}^{n-1} \delta\left(\sigma^j x_i\right)\right) < \varepsilon/4$ $(i = 1,2)$.

Put $N_1 = m_1 N_0 - M$ and $N_2 = (m - m_1)N_0 - M$. Then we can calculate easily that

$$\bar{d}\left(\frac{m_1}{m}\mu_1 + \frac{m - m_1}{m}\mu_2, (N_1 + N_2 2M)^{-1} \sum_{i=1}^{2} \sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i)\right)$$
$$= \bar{d}\left(\frac{m_1}{m}\mu_1 + \frac{m - m_1}{m}\mu_2, \frac{m_1}{m}\left(\frac{1}{N_1 + M}\sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i)\right)\right.$$
$$\left.+ \frac{m - m_1}{m}\left(\frac{1}{N_2 + M}\sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i)\right)\right) < \varepsilon/4.$$

To use the weak specification property, we put $a_1 = 0, b_1 = N_1, a_2 = b_1 + M, b_2 = a_2 + N_2$ , $q = b_2 + M$, $y_1 = x_1$ and $y_2 = \sigma^{-a_2} x_2$. Since $X$ is compact, it follows that there is a smallest set $\Delta$ such that

$$\sigma^q \Delta = \Delta \subset \bigcap_{i=1}^{2} \bigcap_{j=a_i}^{b_i} \sigma^{-j} B(\sigma^{-j} y_i, \varepsilon/3).$$

Take an ergodic measure $\in \mathfrak{M}_\sigma(\tilde{\Delta})$. , then $\bar{\nu}_j\left(B(\sigma^j y_i, \varepsilon/4)\right) = 1$ $(a_i \leqq j \leqq b_i,\ i = 1,2)$ and so $\sum_{i=1}^{2} \operatorname{Card}\left\{a_i \leqq \mathrm{j} \leqq b_i + M - 1 ;\ \bar{\nu}_j\left(B(\sigma^j y_i, \varepsilon/4)\right) < 1\right\}/q < 2M/< \varepsilon/4$. We remark that $\bar{\nu} = (1/q)\sum_{j=0}^{q-1} \bar{\nu}_j$ since $q$ is divided by per $(\Delta)$. Then

$$\bar{d}\left(\bar{\nu}, \frac{1}{q}\left(\sum_{i=1}^{2} \sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i)\right)\right)$$

$$= \bar{d}\left(\frac{1}{q}\sum_{j=0}^{q-1} \bar{\nu}_j, \frac{1}{q}\sum_{i=1}^{2} \sum_{j=a_1}^{b_i+M-1} \delta(\sigma^j y_i)\right) \leqq \varepsilon/4.$$

Hence

$$\bar{d}(\bar{\nu}, t\mu_1 + (1-t)\mu_2) \leqq \bar{d}\left(\bar{\nu}, \frac{1}{q}\sum_{i=1}^{2} \sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i)\right)$$

$$+\bar{d}\left(\frac{1}{q}\sum_{i=1}^{2} \sum_{j=0}^{N_i+M-1} \delta(\sigma^j x_i), \frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2\right)$$

$$+\bar{d}\left(\frac{m_1}{m}\mu_1 + \frac{m-m_1}{m}\mu_2, t\mu_1 + (1-t)\mu_2\right) < \varepsilon$$

(since $= N_1 + N_2 + 2M$ ).

We use induction to get the conclusion. Take $\mu \in \mathfrak{M}_\sigma(X)$, then for every $\varepsilon > 0$ there exist $k \geqq 1, \mu_1, \cdots, \mu_k \in g(X)$ and $t_1, \cdots, t_k \geqq 0$ with $t_1 + t_2 \ldots + t_k = 1$ such that $\bar{d}\left(\mu, \sum_{i=1}^{k} t_i \mu_i\right) < \varepsilon/2$ p. 25 of [183]). By the first part of the proof, there is a $\nu_1 \in g(X)$ such that $\bar{d}(t_1/(t_1+t_2)\mu_1 + t_2/(t_1+t_2)\mu_2, \nu_1) < \varepsilon/4$. Also there is a$\nu_2 \in g(X)$ such that $\bar{d}((t_1+t_2)/(t_1+t_2+t_3)\nu_1 + t_3/(t_1+t_2+t_3)\mu_3, \nu_2) < \varepsilon/8$. Put $t^{(i)} \sum_{j=1}^{i} t_j$ for $1 \leqq i \leqq k$, then it follows from definition of $\bar{d}$ that

$$\bar{d}\left(\sum_{j=1}^{3} \frac{t_j}{t^{(3)}} \mu_j, \nu_2\right)$$

$$\leqq \bar{d}\left(\frac{t^{(2)}}{t^{(3)}}\left(\frac{t_1}{t^{(2)}}\mu_1 + \frac{t_2}{t^{(2)}}\mu_2\right) + \frac{t_3}{t^{(3)}}\mu_3, \frac{t^{(2)}}{t^{(3)}}\nu_1 + \frac{t_3}{t^{(3)}}\mu_3\right)$$

$$+\bar{d}\left(\frac{t^{(2)}}{t^{(3)}}\nu_1 + \frac{t_3}{t^{(3)}}\mu_3, \nu_2\right) < \varepsilon/4 + \varepsilon/8.$$

When $\nu_i \in g(X)$ $(2 \leqq i \leqq k-2)$ is already defined, by the above way we can find $\nu_{i+1} \in g(X)$ such that

$$\bar{d}\left(\frac{t^{(i+1)}}{t^{(i+)}}\nu_i+\frac{t_{i+2}}{t^{(i+2)}}\mu_{i+2},\nu_{i+2}\right)<\varepsilon/2^{i+1}.$$

Since $\nu_{k-1}\in g(X)$and $\bar{d}\left(\sum_{i=1}^{k} t_i\,\mu_i,\nu_{k-1}\right)\leqq\sum_{i=1}^{k-1}1/2^{i+1}<\varepsilon/2,$ the proof is completed.

Let us put $Z(\Delta,\delta)=\{0\leqq j<per(\Delta)$; diam $(\sigma^j\Delta)<\delta\}$ for a smallest set $\Delta$ and $\delta>0$. Denote by $A(\delta)$ the collection of smallest sets $\Delta$ with prime period satisfying the conditions;

$$\text{per }(\Delta)>\delta^{-1}\text{ and card }(Z(\Delta,\delta))/\text{per }(\Delta)>1-\delta.$$

It is easy to check that $A(\delta_1)\subset A(\delta_2)$ when $\delta_1\leqq\delta_2$.

**Proposition (3.1.2)[178]:** If $(X,\sigma)$ (card $(X)>1$ ) satisfies weak specification, for every $\delta>0$ with $\delta<\text{diam}(X)/4$ and for every $\mu\in\mathfrak{M}_\sigma(X)$ there exists $a$ $\Delta\in A(\delta)$such that every measure $\nu$ in $\mathfrak{M}_\sigma(\tilde{\Delta})$ holds $\bar{d}(\mu,\bar{\nu})<\delta$. Consequently the set $\bigcup_{\Delta\in A(\delta)}\{\bar{\nu}\in\mathfrak{M}_\sigma(X);\ \nu\in\mathfrak{M}_\sigma(\tilde{\Delta})\}$ is dense in $\mathfrak{M}_\sigma(X)$for all $\delta>0$.

**Proof:** Since $g(X)$ is dense in $\mathfrak{M}_\sigma(X)$ by Proposition (3.1.1), there is an $\mu_1\in g(X)$ such that $\bar{d}(\mu,\mu_1)<\delta/3$. Choose $M=M(\delta/3)$ as in the definition of weak specification. Let $x_1$ be a generic point for $\mu_1$. Then there is an $N_0>6M/\delta$ such that $\bar{d}\left((1/n)\sum_{j=1}^{n-1}\delta(\sigma^j x_i),\mu_1\right)<\delta/3$ $(n\geqq N_0)$.Take a prime $p$ with $p>N_0+2M$ and put $N=p-2M$. For $x_2\in X$ with $d(\sigma^{N+M}x_2,x_1)>2\delta$ , putting $a_1=0$, $b_1=N$ and $a_2=b_2=N+M$. As before we have that there is a smallest set $\Delta$ such that $\sigma^p\Delta=\Delta\subset\bigcap_{i=1}^{2}\bigcap_{j=a_i}^{b_i}\sigma^{-j}B(\sigma^j x,\delta/3)$.

Since $\Delta\cap\sigma^{N+M}\Delta\subset B(x_1,\delta/3)\cap B(\sigma^{N+M}x_2,\delta/3)=\emptyset$, we get per $(\Delta)\neq1$ and per $(\Delta)$ divides $p$. But $p$ is prime so that per $(\Delta)=p>\delta^{-1}$. Since $\{0,1,\cdots,N\}\subset Z(\Delta,\delta)$ and card $(Z(\Delta,\delta))/p>1-2M/p>1-\delta/3$ , we get $\Delta\in A(\delta)$. Since $\bar{\nu_j}(B(\sigma^j x_1,\delta/3))=1\leqq j\leqq N)$ for all $\nu\in\mathfrak{M}_\sigma(\tilde{\Delta})$ , it follows that

$$\text{card }\{0\leqq j\leqq p;\bar{\nu}_j(B(\sigma^j x_1,\delta/3))<1\}<\frac{p-(N+1)}{p}<2M/p<\delta/3.$$

Since $\bar{\nu}=(1/p)\sum_{j=0}^{p-1}\bar{\nu}_j$ , we get easily that $\bar{d}\left((1/p)\sum_{j=0}^{p-1}\delta(\sigma^j x_1),\bar{\nu}\right)=\bar{d}\left((1/p)\sum_{j=0}^{p-1}\delta(\sigma^j x_1),(1/p)\sum_{j=0}^{p-1}\bar{\nu}_j\right)<\delta/3$. Therefore

$$\bar{d}(\mu_1,\bar{\nu})\leqq\bar{d}\left(\mu_1,\frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1)\right)+\bar{d}\left(\frac{1}{p}\sum_{j=0}^{p-1}\delta(\sigma^j x_1),\bar{\nu}\right)<2\delta/3$$

$$\left(\nu\in\mathfrak{M}_\sigma(\tilde{\Delta})\right)$$

and the proof is completed.

We prove Theorems (3.1.3), (3.1.4), and (3.1.5) that are mentioned in (I).

**Theorem (3.1.3)[178]:** Let $X$ be a compact metric space (card $(X)>1$ ), and a be a self-homeomorphism of $X$ . If $(X,\sigma)$ satisfies weak specification, then $g(X),\mathcal{D}(X)$ ,and $\mathcal{N}(X)$ are dense $G_\delta$-sets of $\mathfrak{M}_\sigma(X)$ , and $g(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

**Proof:** Since $g(X)$ is dense in $\mathfrak{M}_\sigma(X)$ by Proposition (31.1), $g(X)$ is a dense $g_\delta$ -subset of $\mathfrak{M}_\sigma(X)$. Let $\mathcal{U}=\{U_i\}_{i=1}^{\infty}$ be a countable open basis of . Since $(X,\sigma)$ satisfies weak specification, we can find a smallest set $\Delta_i$with $\Delta_i\subset U_i$ for $U_i\in\mathcal{U}$. For every $\geqq1$ , take $\mu_i\in\mathfrak{M}_\sigma(\tilde{\Delta_\iota})$ , then $\mu_i\in(U_i)\geqq$ per $(\Delta_i)^{-1}$.Hence $\mu=\sum_{i=1}^{\infty}(1/2^i)\,\mu_i$ is a measure positive on all nonempty open sets; i.e., $\mu\in\mathcal{D}(X)$. It follows from that $\mathcal{D}(X)$ is a dense

$g_\delta$ -subset of $\mathfrak{M}_\sigma(X)$ unless $\mathcal{D}(X)$ is empty. For every integer $r > 0, K_r = \{\mu \in \mathfrak{M}_\sigma(X); \mu(x) < 1/r$ for all $x \in X\}$ is open in $\mathfrak{M}_\sigma(X)$. Using Proposition (3.1.2), we have that $K_r$ is a dense in $\mathfrak{M}_\sigma(X)$ for all $r \geqq 1$ . Since $\mathcal{N}(X) = \bigcap_{r=1}^{\infty} K_r$ , $\mathcal{N}(X)$ is a dense $g_\delta$ -subset of $\mathfrak{M}_\sigma(X)$.

Since $\mathcal{D}(X)$ is a dense $G_\delta$-subset of $\mathfrak{M}_\sigma(X)$, it is enough to show that $L(X) \cap \mathcal{D}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

Since card $(X) > 1$ , there is two nonempty disjoint closed neighbor-hoods $F_1$ and $F_2$ in $X$ . For $n \geqq 2$, put $S(n) = \{\mu \in L(X); \mu(F_1) \geqq 1/n$ and $\mu(F_2) \geqq 1/n$ },then $L(X) \cap \mathcal{D}(X) \subset \bigcup_{n=2}^{\infty} S(n)$. Let $V_m$ be an $1/m$ open neighbourhood of $F_1$ for every $m \geqq 2$, then $S(n) \subset \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} E[m,r]$ wher$E[m,r] = \bigcap_{j=1}^{\infty}\{\mu \in \mathfrak{M}_\sigma(X), \mu\left(V_m \cap \sigma^j V_m\right) - \mu(F_1)^2 \leqq 1/2r^2$, $\mu(F_1) \geqq 1/n$ , and $\mu(F_2) \geqq 1/n\}$. Since $V_m (m \geqq 1)$ is open and $F_1$ and $F_2$ are closed, it is easy to check that each $E[m,r]$ is closed.

We show that for every $m \geqq 1$ and $r \geqq 1$, $E[m,r]$ is a nowhere dense subset of $\mathfrak{M}_\sigma(X)$. For fixed , take $n \geqq 1$ such that $m \leqq 2r^2$. For every $\Delta \in A(2r^2)$, define a set $Z = \{ 0 \leqq j < \text{per}(\Delta); \sigma^j \Delta \cap F_1 \neq \emptyset$ and $\sigma^j \Delta \not\subset V_m\}$. Then by the definition of $A(1/2r^2)$ , we have card $(Z)/\text{per}\,(\Delta) < 1/2r^2$. For every $\nu \in \mathfrak{M}_\sigma(\tilde{\Delta})$, $\bar{\nu}\left(V_m \cap \sigma^{j\, per(\Delta)} V_m\right) > \bar{\nu}(F_1 - 1/2r^2$ $(j \geqq 1)$, , and so $\bar{\nu}(V_m \cap \sigma^{j\, per(\Delta)} V_m) - \bar{\nu}(F_1)^2 > \bar{\nu}(F_1)\left(1 - \bar{\nu}(F_1)\right) - 1/2r^2$.This shows that $\bar{\nu} \notin E[m,r]$. By Proposition (3.1.2), $\bigcup_{\Delta \in A(1/2r^2)}\left\{\bar{\nu} \in \mathfrak{M}_\sigma(X), \nu \in \mathfrak{M}_\sigma\left(\tilde{\Delta}\right)\right\}$ is dense in $\mathfrak{M}_\sigma(X)$. Hence $(L)(X) \cap \mathcal{D}(X)$ contained in a countable union of nowhere dense closed sets, and so $(L)(X) \cap \mathcal{D}(X)$ is a set of first category in $\mathfrak{M}_\sigma(X)$.

**Theorem(3.1.4)[178]:** Let $X$ and $\sigma$ be as in Theorem (3.1.1). If $(X,\sigma)$ satisfies weak specification, then $(\mathfrak{M}(X), \sigma)$ has the specification property.

**Proof :** Let $\varepsilon > 0$ be given and $\mathfrak{M}(\varepsilon/2)$ be as in the definition of weak specification. Let $\mu_1, \cdots, \mu_k \in \mathfrak{M}(X)$ be given, as well as integers $a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots < a_k \leqq b_k$ and $q$ with $a_i - b_{i-1} \geqq M(\varepsilon/2)$ and $q \geqq M(\varepsilon/2) + b_k - a_1$ . Since $\sigma: \mathfrak{M}(X) \to \mathfrak{M}(X$ is uniformly continuous, there exists an $\eta > 0$ such that $\bar{d}(\mu,\nu) < \eta$ implies $\bar{d}(\sigma^j\mu, \sigma^j\nu) < \varepsilon/2$ for $a_1 \leqq j \leqq b_k$. For some integer $n > 0$ there exist $x_r^i \in X(1 \leqq r \leqq n, 1 \leqq i \leqq k)$such that putting $\nu_i = 1/n \sum_{r=1}^{n} \delta\left(x_r^i\right)$ $(1 \leqq i \leqq k), \bar{d}(\mu_i, \nu_i) < \eta$ holds for $1 \leqq i \leqq k$ .Since $\sigma: X \to X$ satisfies weak specification, there exist smallest sets $\Delta_r$ with $\sigma^q \Delta_r = \Delta_r$ and $\Delta_r \subset \bigcap_{i=1}^{k} \bigcap_{j=a_i}^{b_i} \sigma^j B(\sigma^j x_r^i, \varepsilon/2$ for $1 \leqq r \leqq n$ .Take $\rho^r \in \mathfrak{M}_\sigma\left(\tilde{\Delta}_r\right)$ and put $\rho = (1/n) \sum_{r=1}^{n} \bar{\rho}_0^{\,r}$ where $\bar{\rho}_0^{\,r}(B)$ =per $(\Delta_r)\bar{\rho}^r(B \cap \Delta_r)$for Borel sets $B$. Obviously $\sigma^q \rho = \rho$ and $\bar{d}\left(\sigma^j \rho, \sigma^j \nu_i\right) = \bar{d}\left((1/n) \sum_{r=1}^{n} \sigma^j \bar{\rho}_0^{\,r}, (1/n) \sum_{r=1}^{n} \sigma^j \bar{\rho}_0^{\,r}\, \delta(\sigma^j x_r^i)\right) \leqq \varepsilon/2$ $(a_i \leqq j \leqq b_i, i = 1, \cdots, k)$ . Hence $\bar{d}\left(\sigma^j \rho, \sigma^j \mu_i\right) < \varepsilon$ for $a_i \leqq j \leqq b_i$ , $i = 1, \cdots, k$. The proof is completed.

**Theorem (3.1.5)[178]:** Let $X$ and $\sigma$ be as in Theorem (3.1.1). If $(X,\sigma)$ satisfies weak specification, then for every nonempty compact connected subset $V$ of $\mathfrak{M}_\sigma(X)$, there is an $x \in X$ such that $V_{\sigma^r}(x) = V$ for all $r \geqq 1$ and the set of such points $x$ is a dense set in $X$.

**Proof:** Since $V$ is compact and connected, by Proposition (3.1.2), there exist a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers with $\varepsilon_n \searrow 0$ and a sequence $\{\Delta_n\}_{n=1}^{\infty}$ in $A(\varepsilon_n)$ such that for some $\mu_n \in \mathfrak{M}_\sigma\left(\tilde{\Delta}_n\right)$ the followings hold;

**(i)** $B_n \cap B_{n+1} \cap V \neq \emptyset$,

**(ii)** $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B_n = V$

where $B_n (n \geqq 1)$ is the $\varepsilon_n$-closed neighborhood of $\bar{\mu}_n$ in $\mathfrak{M}(X)$. We have to show that for every $x_0 \in X$ and $\delta > 0$ there exists an $x \in B(x_0, \delta)$ such that $V_{\sigma^r}(x) = V$ for all

$r \geqq 1$ . For every $n \geqq 1$ , take an $x_n \in \Delta_n$. Since $(X, \sigma)$ satisfies weak specification, there exist positive integers $M_n (n \geqq 0)$ such that for every set of integers $a_0 \leqq b_0 < a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots$ with $a_n - b_{n-1} \geqq M_{n-1}$ $(n \geqq 1)$, there exists an $x \in X$ such that $d(\sigma^j x, \sigma^j x_n) \leqq \varepsilon_n$ $(a_n \leqq j \leqq b_n,\ n > 0)$ and $d(\sigma^j x, \sigma^j x_0) \leqq \delta$ $(a_0 \leqq j \leqq b_0),\ n > 0)$ (c.f. see Orbit specification lemma in [186]). With the above notations, take $a_n$ and $b_n (n \geqq 0)$ as follows;

**(I)** $a_0 = b_0 = 0$,

**(II)** $a_n$ is divided by $n!$ and $b_{n-1} + M_{n-1} \leqq a_n < b_n + M_{n-1} + n!$ $(n \geqq 1)$ and

**(III)** $b_n = a_n + (n+1)!$ $(a_n + M_n)$ per $(\Delta_n)$per $(\Delta_{n-1})$ $(n \geqq 1)$.

Then, we have an $x \in B(x_0, \sigma)$ with $d(\sigma^j x, \sigma^j x_n) \leqq \varepsilon_n$ $(a_n \leqq j \leqq b_n,\ n \geqq 1)$.

We have to show that $V_{\sigma^r}(x) = V$ for all $r \geqq 1$. Though the proof is similar to that in [186], we sketch it for completeness .

It is clear that for $r \geqq 1$ there is $N_0 \geqq r$ such that per $(\Delta_n) > r$ for all $n \geqq N_0$. Now we fix the integers $r,\ n$ with $n \geqq N_0$ and $k$ with $b_n/r < k \leqq b_{n+1}/r$, , and write.

$$A_1 = A \cap \left[\frac{a_n}{r}, \frac{b_n}{r}\right)$$

where $A = \{0 \leqq j \leqq k;\ j \text{ is an integer}\}$. Take $\acute{k}$ with $k$-per $(\Delta_{n+1}) < \acute{k} \leqq k$ such that $\acute{k} - a_{n+1}/r$ is divided by per $(\Delta_{n+1})$.

Then it is easy to see that $A_2 = A \cap [a_{n+1}/r, \acute{k})$ is nonempty when $k \geqq a_{n+1}/r +$ per $(\Delta_{n+1})$ and $A_2$ is empty when $k < a_{n+1}/r +$ per $(\Delta_{n+1})$.

Obviously per $(\Delta_{n+1})$ divides card $(A_2)$ . By (III), per $(\Delta_n)$ divides card $(A_1)$. Remark that per $(\Delta_n)$ and per $(\Delta_{n+1})$ are prime numbers. Since $n \geqq N_0$, per $(\Delta_n)$ and per $(\Delta_{n+1})$ are both prime to the integer $r$, so that

$$\bar{d}(\mathrm{card}(A_1)^{-1} \sum_{j \in A_1} \delta\left(\sigma^{jr} x_n\right), \bar{\mu}_n) \leqq \varepsilon_n$$

and

$$\bar{d}\left(\mathrm{card}(A_2)^{-1} \sum_{j \in A_2} \delta\left(\sigma^{jr} x_{n+1}\right), \bar{\mu}_{n+1}\right) \leqq \varepsilon_{n+1}.$$

By the definition of metric $\bar{d}$ , we get that

$$\bar{d}\left(\frac{1}{\mathrm{k}} \sum_{j \in A} \delta\left(\sigma^{jr} x\right), \quad \mathrm{card}\ (A_1 \cup A_2)^{-1} \sum_{j \in A_1 \cup A_2} \delta\left(\sigma^{jr} x\right)\right)$$
$$< 2\mathrm{card}(A_1)^{-1}\{k - \mathrm{card}\ (A_1 \cup A_2)\}$$
$$\leqq \frac{4}{(n+1)!} + 2\varepsilon_n.$$

Since $d(\sigma^{jr} x, \sigma^{jr} x_n) \leqq \varepsilon_n$ $(j \in A_1)$ and $d(\sigma^{jr} x, \sigma^{jr} x_n) \leqq \varepsilon_{n+1}$ $(j \in A_2)$, it is easy to check that

$$\bar{d}\left(\frac{1}{\mathrm{k}} \sum_{j \in A} \delta\left(\sigma^{jr} x\right), \quad \mathrm{card}\ (A_1 \cup A_2)^{-1} \sum_{j \in A_1} \delta\left(\sigma^{jr} x_n\right) + \sum_{j \in A_2} \delta\left(\sigma^{jr} x_{n+1}\right)\right)$$

$$< \frac{4}{(n+1)!}2\varepsilon_n + \bar{d}(\mathrm{card}(A_1 \cup A_2)^{-1} \sum_{j \in A_1 \cup A_2} \delta\left(\sigma^{jr}x\right),$$

$$(\mathrm{card}(A_1 \cup A_2)^{-1}(\sum_{j \in A_1} \delta\left(\sigma^{jr}x_n\right) + \sum_{j \in A_2} \delta\left(\sigma^{jr}x_{n+1}\right)))$$

$$< \frac{4}{(n+1)!} + 3\varepsilon_n + \varepsilon_{n+1}.$$

Thus we can compute that

$$\bar{d}\left(\frac{1}{\mathrm{k}}\sum_{j \in A} \delta\left(\sigma^{jr}x\right), \quad \mathrm{card}\ (A_1 \cup A_2)^{-1}(\mathrm{card}\ (A_1)\bar{\mu}_n + (\mathrm{card}\ (A_2)\bar{\mu}_{n+1})\right)$$

$$< \frac{4}{(n+1)!}3\varepsilon_n + \varepsilon_{n+1}$$

$$+\bar{d}(\mathrm{card}(A_1 \cup A_2)^{-1}\left(\sum_{j \in A_1} \delta\left(\sigma^{jr}x_n\right) + \sum_{j \in A_2} \delta\left(\sigma^{jr}x_{n+1}\right)\right)$$

$$\mathrm{card}\ (A_1 \cup A_2)^{-1}\big(\mathrm{card}\ (A_1)\bar{\mu}_{n+1} + (\mathrm{card}\ (A_2)\bar{\mu}_{n+1})\big)$$

$$< \frac{4}{(n+1)!}3\varepsilon_n + 2\varepsilon_{n+1}\ .$$

Since $\bar{d}(\bar{\mu}_n, \bar{\mu}_{n+1}) \leqq \varepsilon_n + \varepsilon_{n+1}$ by (i), we have that

$$\bar{d}\left(\frac{1}{\mathrm{k}}\sum_{j \in A} \delta\left(\sigma^{jr}x\right), \bar{\mu}_n\right) < \frac{4}{(n+1)!} + 5\varepsilon_n + 3\varepsilon_{n+1}.$$

Since $n \geqq N_0$ and $b_n/r < k \leqq b_{n+1}/r$ are arbitrary, $V_{\sigma^r}(x)$ coincides with the $\omega$-limit set of the sequence $\{\bar{\mu}_n\}_{n+1}^{\infty}$ and so $V_{\sigma^r}(x)$ coincides with $V$ by (ii). The proof is completed.

## Section (3.2): For Ergodic Group Automorphisms of Abelian Groups

The property of specification plays an important role in classifying the class of invariant probability measures preserved under a homeomorphism (see K. Sigmund [185], [186] and [178]. In [191] B. Marcus introduced the notion of almost weak specification weaker than that of specification by using toral automorphisms.

The purpose is to prove that every automorphism of a compact metric abelian group is ergodic under the Haar measure if and only if it satisfies almost weak specification (Corollary of Theorem (3.2.2)).

For $X$ be a compact metric space with metric $d$ and $\sigma$ be a homeomorphism from $X$ onto itself. Then $\sigma$ satisfies almost weak specification if for every $\varepsilon > 0$ there is a function $M_\epsilon: Z^+ \to Z^+$($Z^+$denotes the set of non-negative integers) with $M_\epsilon(n)/n \to 0$ as $n \to \infty$ such that for every $k \geqq 1$ and $k$ points $x_1, \cdots, x_k \in X$ and for every sequence of integers $a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots < a_k \leqq b_k$ with $a_i - b_{i-1} \geqq M_\epsilon(b_i - a_i)(2 \leqq i \leqq k)$ there is an $x \in X$ with $d(\sigma^n x, \sigma^n x_i) \leqq \varepsilon$ $(a_i \leqq n \leqq b_i, 1 \leqq i \leqq k)$. A homeomorphism satisfies weak specification if it has the proPerty of almost weak specification with some constant function $M_\epsilon$. It is clear from definition that if $(X, \sigma)$ satisfies almost weak specification, then it is topologically mixing. Almost weak specification is preserved under direct products and homeomorPhic images. A shift of compact metric state space satisfies almost weak specification [183]).

Hereafter let $X$ be a compact metric abelian group and $\sigma$ be an automorphism of $X$. $G$ denotes the dual group of $X$. Define the dual automorphism $\gamma$ of $G$ by $(\gamma g)(x) = g(\sigma x)(x)(g \in G, x \in X)$. Group operations of $X$ and $G$ will be denoted by addition. If $X$ is connected, then $G$ is torsion free ( i.e., $ng \neq 0$ for all $0 \neq g \in G$ and $0 \neq n \in Z$). When $X$ is connected, $(X, \sigma)$ is said to satisfy condition (A) if for every $0 \neq g \in G$ there is $0 \neq P(x) \in Z[x]$ ($Z[x]$ denotes the ring of polynomials with integral coefficients) such that $p(\gamma)g = 0$, and $(X, \sigma)$ is said to satisfy condition (B) if one has $p(\gamma)g \neq 0$ for every $0 \neq g \in G$ and $0 \neq P(x) \in Z[x]$.

(L.1) [188]: Let $X$ be a group as above; then $X$ splits into a sum $X = X_1 + X_2 + X_3$ of $\sigma$-invariant subgroups $(\sigma(X_i) = (X_i,\ i = 1,2,3)$ such that (i) $X_i$ is totally disconnected, (ii) $X_2$ is connected and $(X_2, \sigma)$ satisfies condition (A), and (iii) $X_3$ is connected and $(X_3, \sigma)$satisfies condition (B). If in particular $(X, \sigma)$ is ergodic under the Haar measure, then $X_i (i = 1,2,3)$ can be chosen such that $(X_i, \sigma)$ is ergodic under the Haar measure.

(L.2) Let $\{X_n\}_{n \geqq 0}$ be a sequence of $\sigma$-invariant subgroups such that $X = X_0 \supset X_1 \supset \cdots \supset \bigcap_{n \geqq 0} X_n = \{0\}$ and assume that for $n \geqq 1, (X/X_n, \sigma)$ satisfies almost weak specification. Then $(X, \sigma)$ satisfies almost weak specification.

Indeed, define an invariant metric $d_n$ on $X/X_n, \sigma (n \geqq 1)$ by

$$d_n(x + X_n, y + X_n) = \min_{z \in X_n} d(x, y + z) \quad (x, y \in X).$$

Take and fix $\varepsilon > 0$. Choose $n \geqq 1$ with diam $(X_n) < \varepsilon/2$. Since , $(X/X_n, \sigma)$ satisfies almost weak specification, there is a function $M_{\varepsilon/2}^{(n)}: Z^+ \to Z^+$ with $M_{\varepsilon/2}^{(n)}(m)/m \to 0$ as $m \to \infty$ such that for every $k \geqq 1$, $k$ points $x_1, \cdots, x_k \in X$, and a sequence of integers $a_1 \leqq b_1 < \cdots < a_k \leqq b_k$ with $a_i - b_{i-1} \geqq M_{\varepsilon/2}^{(n)}(b_i - a_i)$ $(2 \leqq i \leqq k)$, there is an $x \in X$ with $d_n(\sigma^j x + X_n, \sigma^j x_i + X_n) \leqq \varepsilon/2$ $(a_i \leqq j \leqq b_i, 1 \leqq i \leqq k)$. Obviously $d(\sigma^j x, \sigma^j x_i) \leqq d_n(\sigma^j x + X_n, \sigma^j x_i + X_n) +$ diam $(X_n) \leqq \varepsilon$ $(a_i \leqq j \leqq b_i, 1 \leqq i \leqq k)$. Letting $M_\epsilon = M_{\varepsilon/2}^{(n)}$ for simplicity, we can easily check that $(X, \sigma)$ satisfies almost weak specification.

(L.3): Let $X_3$ be as in (L.1). If $(X_3, \sigma)$ is ergodic under the Haar measure, then it satisfies almost weak specification.

This follows from the proof of [188] together with (L.2).

(L.4) [189]: Let $X_1$ be as in (L.1). If $(X_1, \sigma)$ is ergodic under the Haar measure, then it satisfies almost weak specification.

Let $X$ be as above. Then $X$ is said to be solenoidal if $X$ is connected and finite dimensional. Clearly every finite-dimensional torus is solenoidal.

(L.5) [188]: Let $X_2$be as in (L.1). Then there is a sequence

$$X_2 = X_{2,0} \supset X_{2,1} \supset \cdots \supset \bigcap_{n \geqq 0} X_{2,n} = \{0\}$$

of $\sigma$-invariant subgroups such that each $X_2/X_{2,n}$ is solenoidal.

For the following statements $(L.6) \sim (L.14)$, let $X$ be $r$-dimensional solenoidal. Since $\mathrm{rank}(G) = r < \infty$ and $G$ is torsion free, there exists an into isomorphism $\varphi: G \to Q^r$ ( $Q^r$ denotes the vector space over ), so that $\bar{\gamma} = \varphi^0 \gamma^0 \varphi^{-1}$ is extended to $Q^r$ and further to $R^r$. We denote again by $\gamma$ the extension to $R^r$.

(L.6) [179]: Under the above notations there are a homomorphism $\psi: R^r \to X$ and a totally disconnected subgroup $F$ such that (i) $\psi^0\gamma = \sigma^0\psi$, (ii) $X = \psi(R^r) + F$, and further (iii) there is a small closed neighborhood $U$ of 0 in $R^r$such that $\psi(U) \cap F = \{0\}$ and $U \times F$ is homeomorphic to $\psi(U + F$ and $\psi(U) + F$ is a closed neighborhood of 0 in $X$ (we write $\psi(U) \oplus F$ for such a neighborhood $\psi(U) + F$).

(L.7) [179]: Let $F$ be as in (L.6). Then $F$ contains subgroups $F^{-1}, F^{+1}$, and $H$ such that (i) $(H) = H$ ,(ii) $F^{-1} \supset \sigma^{-1}F^{-} \supset \cdots \supset \bigcap_0^\infty \sigma^{-n} F^{-} = \{0\}$, (iii) $F^{+} \supset \sigma F^{+} \supset \cdots \supset \bigcap_0^\infty \sigma^{n} F^{+} = \{0\}$ , and (iv) $F = F^{-} \oplus F^{+} \oplus H$.

Let $(R^r, \gamma)$ be a lifting system of $(X, \sigma)$ by . Then $R^r$splits into a direct sum $\boldsymbol{R}^r = E^u \oplus E^s \oplus E^c$ of $\sigma$-invariant subspaces $E^u, E^s$, and $E^c$such that the eigenvalues of $\gamma_{1E^u}$ have modulus $> 1$, the eigenvalues of $\gamma_{1E^s}$ modulus $< 1$ and the eigenvalues of $\gamma_{1E^c}$ modulus one. We call $(\boldsymbol{R}^r, \gamma)$ hyperbolic if $E^c = \{0\}$; i.e., $\boldsymbol{R}^r = E^u \oplus E^s$. If $E^c \neq \{0\}$, by using Jordan's normal form in the real field for $(E^c, \gamma)$ the subspace $E^c$ splits into a finite direct sum $E^c = E^{c_0} \oplus E^{c_1} \oplus \cdots \oplus E^{c_k}$ of subspaces of $E^c$ satisfying the following three conditions; (a) for $0 \leqq i \leqq k$, the dimension of $E^{c_i}$ is 1 or 2, .

(b)
$$\gamma_{1E^c} = \begin{pmatrix} \gamma_0 & I_0 & & 0 \\ & \gamma_1 & \ddots & I_k \\ 0 & \ddots & & \gamma_k \end{pmatrix}$$

where $\gamma_i: E^{c_i} \to E^{c_i}$ is an isometry under some norm $||.||_{c_i}$ of $E^{c_i}$ and each $I_i: E^{c_i} \to E^{c_{i-1}}$ is either a zero-map or a map corresponding to the identity matrix. We call that $(\boldsymbol{R}^r, \gamma)$ has central spin if $E^c \neq \{0\}$ and each $I_i: E^{c_i} \to E^{c_{i-1}}$ is a zero-map. If $(\boldsymbol{R}^r, \gamma)$ has central spin, then each $E^{c_i}$ is $\sigma$-invariant. Let $I$ denote the identity map of $R^r$ . For every $m > 0$ , the eigenvalues of $I - \gamma^m$ on $E^{c_i}$ are $1 - \lambda_i^m$ where $\lambda_i$ is are eigenvalues of $\gamma_{1E^{c_i}}$ .It is easily proved that there is a constant $c_{(i)}$ such that $||(I - \gamma^m)x||_{c_i} < c_{(i)}|1 - \lambda_i^m| \, ||x||_{c_i} (x \in E^{c_i}, m > 0)$. We define a norm $||.||_c$ of $E^c$ by $||x||_c \max_{0 \leqq i \leqq k} \left\{ ||x^i||_{c_i} \right\} = (x = x^0 + \cdots + x^k \in E^{c_0} \oplus \cdots \oplus E^{c_k})$. There are $0 < \lambda_0 < 1$ and norms $||\ \ ||_u$ and $||\ \ ||_s$ on $E^u$ and $E^s$ respectively such that $||\gamma^n x||_u \leqq \lambda_0^n ||x||_u (n \leqq 0, x \in E^u)$and $||\gamma^n x||_s \leqq \lambda_0^n ||x||_u (n \leqq 0, x \in E^s)$. Define a norm $||\ \ ||$ on $R^r$ by

$$||x|| = \max \{||x_u||_u, ||x_s||_s, ||x_c||_c\} \quad (x = x_u + x_s + x_c \in E^u \oplus E^s \oplus E^c)$$

and define a metric $d_0$ on $\boldsymbol{R}^r$ by

$$d_0(x, y) = \|x - y\| \quad (x, y \in R^r).$$

(L.8) [179]: There is $\alpha_1 > 0$ such that (i) for $\varepsilon \in (0, \alpha_1], B(\varepsilon) = \{x \in \boldsymbol{R}^r ; d_0(x, 0) \leqq \varepsilon\}$ splits into a direct sum $B(\varepsilon) = B^u(\varepsilon) \oplus B^s(\varepsilon) + B^c(\varepsilon)$ where $B^u(\varepsilon) = B(\varepsilon) \cap E^s, B^s(\varepsilon) = B(\varepsilon) \cap E^s$ and $B^c(\varepsilon) = B(\varepsilon) \cap E^c$ , (ii) $B(\alpha_1) \oplus F^- \oplus F^+ \oplus H$ is a closed neighborhood of 0 in $X$.

(L.9) [179]: There is an invariant metric $d$ on $X$ and a positive number $\alpha_0$ with $\alpha_0 < \alpha_1$ such that (i) for $\varepsilon \in (0, \alpha_0]$, $W(\varepsilon) = \{x \in X; d(x, 0) \leqq \varepsilon\}$is expressed as $W(\varepsilon) = W^u(\varepsilon) \oplus W^s(\varepsilon) \oplus W^c(\varepsilon)$ where $W^u(\varepsilon) = W(\varepsilon) \cap \{\psi B^u(\varepsilon) \oplus F^-\}$, $W^s(\varepsilon) = W(\varepsilon) \cap \{\psi B^s(\varepsilon) \oplus F^+\}$ and $W^c(\varepsilon) = W(\varepsilon) \cap \{\psi B^c(\varepsilon) \oplus H\}$, and (ii) for $\varepsilon \in (0, \alpha_0]$ $W(\varepsilon) \cap H$ is a subgroup of $H$ and there is an $n \geqq 0$ such that, $W(\varepsilon) \cap F^- = \sigma^n F^-$ and $W(\varepsilon) \cap F^+ = \sigma^n F^+$ and moreover that $W(\varepsilon) \cap F^- = \sigma^{-n} F^-$and $W(\varepsilon) \cap F^+ = \sigma^n F^+$.

(L.10) [179]: There is $0 < \lambda_0 < 1$ such that for $\varepsilon \in (0, \alpha_0]$, and $x = x_u + x_s + x_c \in W^u(\varepsilon) \oplus W^s(\varepsilon) \oplus W^c(\varepsilon$ the following hold (i) $d(x, 0) = \max \{d(x^u, 0), d(x^s, 0),$

$d(x^c, 0)\}$, (ii) $d(\sigma^n x, 0) \leqq \lambda_0^{-n} d(x, 0)$ $(x \in W^u(\varepsilon), n \leqq 0)$, (iii) $d(\sigma^n x, 0) \leqq \lambda_0^n d(x, 0)$ $(x \in W^s(\varepsilon), n \geqq 0)$ and (iv) $d(\sigma^n x, 0) = d(x, 0)$ $(x \in W^c(\varepsilon) \cap H, n \in Z)$.

$X$ is said to have property $(*)$ is finitely generated under $\gamma$; i.e., there is a finite set $\Lambda$ in $G$ such that $G = gp \bigcup_{-\infty}^{\infty} \gamma^j \Lambda$ (the notation $gpE$ means the subgroup generated by a set $E$). We say $(X, \sigma)$ is hyperbolic if $(X, \sigma)$ has property $(*)$ and $(\boldsymbol{R}^r, \gamma)$ is hyperbolic, and we say $(X, \sigma)$ has central spin if either $(\boldsymbol{R}^r, \gamma)$ is hyperbolic and $X$ does not have property $(*)$, or $(\boldsymbol{R}^r, \gamma)$ has central spin.

(L.11) [179] : Assume that $(X, \sigma)$ is ergodic under the Haar measure. Then $W^u(\alpha_0) \neq \{0\}$ and $W^s(\alpha_0) \neq \{0\}$.

(L.12): For $\varepsilon \in (0, \alpha_0]$, the following hold:

(i) $\sigma^{-1} W^u(\alpha) \subset W^u(\lambda_{0\varepsilon})$,

(ii) $\sigma W^s(\alpha) \subset W^s(\lambda_{0\varepsilon})$,

(iii) $\sigma W^c(\alpha) \subset W^c(\lambda_{0\varepsilon})$ if $(X, \sigma)$ has central spin,

(iv) $W^u(\alpha), W^s(\alpha)$ and $W^c(\alpha)$ are symmetric sets at 0 in $X$.

(i) and (ii) are the consequences of (L.10). (iii) follows from the definition of central spin together with (L.10). (iv) is clear from the definition.

For $\varepsilon \in (0, \alpha_0]$, put $K(\varepsilon) = W^s(\alpha) \oplus (W^c(\alpha) \cap H)$ and $W_n^u(\varepsilon) = \{x \in \sigma^n W^u(\alpha); x + W^u(\alpha) \subset \sigma^n W^u(\alpha)\}$ $(n \geqq 1)$. Then we have the following:

(L.13): For every $\varepsilon \in (0, \alpha_0]$,

(i) $\sigma K(\varepsilon) \subset K(\varepsilon)$,

(ii) $K(\varepsilon) \oplus \psi B^c(\varepsilon) = W^s(\alpha) \oplus W^c(\alpha)$,

(iii) $W_n^u(\varepsilon) \subset \sigma W_n^u(\varepsilon) \subset W_{n+1}^u(\varepsilon)$ $(n \geqq 1)$.

(L.14): Assume that either $(X, \sigma)$ is hyperbolic, or ergodic and has central spin. Then, for every $\varepsilon \in (0, 2\alpha_0/3)$ there is $M = M(\varepsilon) > 0$ such that for every $n \geqq M, W_n^u(\varepsilon) + K(\varepsilon) \oplus \psi B(\varepsilon) = X$.

From the proof (Step 1.2.1) for every $\varepsilon \in (0, 2\alpha_0/3)$ there is $M = M(\varepsilon) > 0$ such that for every $n \geqq M$ and $x \in X$, $W_n^u(\varepsilon) \cap (x + K(\varepsilon) \oplus \psi B^c(\varepsilon) \neq \emptyset)$. Since $K(\varepsilon) \oplus \psi B^c(\varepsilon) = W^s(\alpha) \oplus W^c(\alpha)$ is symmetric, we have $x \in W_n^u(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon)$.

(L.15): Assume that $(X, \sigma)$ is ergodic; then there exists a finite sequence $X = X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{+1} = 0$ of $\sigma$-invariant subgroups such that each $X_i$ is connected and either $(X_i/X_{i+1}, \sigma)$ is hyperbolic or ergodic and has central spin.

By [179] there is a finite sequence $X = X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{+1} = 0$ of $\sigma$-invariant subgroups such that each $X_i$ is connected and $(X_i/X_{i+1}, \sigma)$ satisfies weak specification. By ([179], Theorems (3.2.2), if $(X_i/X_{i+1}, \sigma)$ satisfies weak specification, then $(X_i/X_{i+1}, \sigma)$ is either hyperbolic or ergodic and has central spin.

(L.16): For $s \geqq 1$, $(a_1, \cdots, a_s) \in R^s$ and $N \geqq 1$, there is an integer $n$ with $1 \leqq n \leqq N^s$ such that $|na_i - m_i| < 1/N (1 \leqq i \leqq s)$ for some $(m_1, \cdots, m_s) \in Z^s$.

This is shown as follows. For every $1 \leqq n \leqq N^s$there is $(m_1^{(n)}, \cdots, m_s^{(n)}) \in Z^s$ such that $na_i - m_i^{(n)} \in [0,1)$ $(1 \leqq i \leqq s)$. If there is an $n$ with $1 \leqq n \leqq N^s$ such that for every $1 \leqq i \leqq s, na_i - m_i^{(n)} \in [0, 1/N)$, (L.16) holds. For otherwise, we can find $u$ and $v$ with $1 \leqq u < v \leqq N^s$ and $j_i$ with $0 \leqq j_i \leqq N - 1$ such that for $1 \leqq i \leqq s, ua_i - m_i^{(n)}, va_i - m_i^{(n)} \in [j_i/N, (j_i + 1)/N)$. Put $n = v - u$ and $m_i = m_i^{(v)} - m_i^{(u)} (1 \leqq i \leqq s)$.Then we have $|na_i - m_i| < 1/N (1 \leqq i \leqq s)$.

The following is the main result.

**Corollary (3.2.1)[187]:** Let $X$ be a compact metric abelian group and $a$ be an automorphism of $X$. Then $(X,\sigma)$ is ergodic (under the Haar measure) if and only if $(X,a)$ satisfies almost weak specification.

If we established Theorem (3.2.2), then the corollary is shown as follows. Clearly$(X,\sigma)$ is ergodic if $(X,\sigma)$ satisfies almost weak specification. Assume that $(X,\sigma)$ is ergodic. Then $X$ splits into a sum $X = X_1 + X_2 + X_3$ and $\sigma$-invariant subgroups with the notation of (L.1) And so $(X_1,\sigma)$ satisfies almost weak specification and$(X_3,\sigma)$ satisfies almost weak specification by (L.3). Use (L.5) for $(X_2,\sigma)$. Then there is a sequence $X_2 = X_{2,0} \supset X_{2,1} \supset \cdots \supset \bigcap_{n\geqq 0} X_{2,n} = \{0\}$ of $\sigma$-invariant subgroups such that each $X_2/X_{2,n}$ is solenoidal. Since $(X_2,\sigma)$ is ergodic, each $(X_2/X_{2,n},\sigma)$ is ergodic. By Theorem (3.2.2), $(X_2/X_{2,n},\sigma)$ satisfies almost weak specification. Since the product system $(X_1 \times X_2 \times X_3, \sigma\times\sigma\times\sigma)$ satisfies almost weak specification, $(X,\sigma)$ satisfies almost weak specification.

**Theorem (3.2.2)[187]:** Let $X$ be a solenoidal group and $\sigma$ be an automorphism of $X$. Then $(X,\sigma)$ is ergodic (under the Haar measure) if and only if $(X,\sigma)$ satisfies almost weak specification.

Theorem (3.2.2) derives the following.

**Proof:** AS before let $(G,\sigma)$ be the dual of $(X,\sigma)$. Since $X$ is solenoidal, we have rank $(G) = \dim(X) = r < \infty$ . Then $X$ is expressed as

$$X = \psi(E^u \oplus E^s \oplus E^c) + \{F^- \oplus F^+ \oplus H\}.$$

We prepare a sequence of lemmas leading to the proof of Theorem (3.2.2).

**Lemma (3.2.3)[187]:** If $E^c \neq \{0\}$ and $a = dim(E^c)$, then $||\gamma^n x||_c \leqq (n+1)^{\alpha-1}\big||x|\big|_c (x \in E^c, n \geqq 0)$.

**Proof:** $E^c$ splits into a finite direct sum $E^c = E^{c_0} \oplus \cdots \oplus E^{c_k}$ of 1 or 2-dimensional subspaces which satisfy (a) and (b) in (3.2.2). For $x \in E^{c_0} \oplus \cdots \oplus E^{c_k}$ and for $n \geqq 0, \gamma^n_{1E^c}x$ splits into $\gamma^n_{1E^c}x = x^n_0 + \cdots + x^n_k$ with $x^n_i \in E^{c_i}(1 \leqq i \leqq k)$ . By (b) we get $||x^n_i||_{c_i} \leqq ||x^{n+1}_i||_{c_i} + \left\|x^{n-1}_{i+1}\right\|_{c_{i+1}} (0 \leqq i \leqq k-1\,, n \geqq 0)$ and $||x^n_k||_{c_k} = ||x^0_k||_{c_k}(n \geqq 0)$. It is checked that for every $n \geqq 0, ||x^n_i||_{c_i} \leqq (n+1)^{k-i}||x||_c$ $(1 \leqq i \leqq k)$. Indeed, $||x^0_i||_{c_i} \leqq ||x||_c$ when $n = 0$ .Assume that the inequality is true for $n-1$. Then $||x^n_i||_{c_i} \leqq ||x^{n-1}_i||_{c_i} + \left\|x^{n-1}_{i+1}\right\|_{c_i} \leqq n^{k-i}||x||_c + n^{k-i-1}||x||_c \leqq (n+1)^{k-i}||x||_c$ for $0 \leqq i \leqq k-1$. Since $k \geqq a-1$, the conclusion is obtained.

**Lemma(3.2.4)[187]:** Assume that $(X,\sigma)$ is either hyperbolic or ergodic and has central spin. Then for every $\varepsilon \in (0, \alpha_0/3)$ there is a sequence $\{N_\varepsilon(n)\}_{n=1}^\infty$ of non-negative integers such that $N_\varepsilon(n^p)/n \to 0$ as $n \to \infty$ for all $p \geqq 1$, and $W^u_m(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon/n) \supset \psi B^c(\varepsilon)$ for $n \geqq 1$ and $m \geqq N_\varepsilon(n)$.

**Proof:** If $(R^r,\gamma)$ is hyperbolic $(\text{i.e.}, E^c = \{0\})$, by putting $N_\varepsilon(n) = 0$ for $\varepsilon \in (0, \alpha_0/3)$ and $\geqq 1$ , the lemma holds.

It only remains to prove the lemma for the case when $(R^r,\gamma)$ has central spin. To see this, use (L.14). Then there is $M > 0$ such that $W^u_M(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon) \supset \psi B^c(2\varepsilon)$. Since $a = \dim(E^c)$, we can find points $t_1, \cdots t_a \in B^c(2\varepsilon)$ such that $\{t_1, \cdots t_a\}$ is linearly independent over $R$ and

$$\psi(t_i) \in W^u_M(\varepsilon) + K(\varepsilon) \quad (1 \leqq i \leqq a). \tag{1}$$

Since $\gamma$ has central spin, $\gamma$ is an isometry on $(E^c, ||.||_c)$.By Dirichlet's theorem there is $L > 0$ such that

$$\|(\gamma^L - I)x\|_c \leqq \tfrac{1}{2}\|x\|_c \qquad (x \in E^c) \tag{2}$$

where $I$ denotes the identity map.

Since $\gamma$ is aperiodic (by ergodicity of $\sigma$), $\gamma^L - I$ is one-to-one and so for some $\mu$ with $0 < \mu < 1/2$

$$\|(\gamma^L - I)x\|_c > \mu\|x\|_c \qquad (x \in E^c). \tag{3}$$

Notice that $\{(\gamma^L - I)^n t_i \,;\ 1 \leqq i \leqq a\}(n > 0)$ is linearly independent over $R$. Define $A = \{s \in E^c\,;\ s = \sum_{i=1}^a a_i\, t_i, a_i \in Z,\ 1 \leqq i \leqq a\}$ and put $\delta = \min\{\|s\|_c;\ \ 0 \neq s \in A\}$. Then by (3)

$$\mu^n \delta < \min\{\|t\|_c;\ \ 0 \neq t \in (\gamma^L - I)^n A\} \qquad (n > 0). \tag{4}$$

Since $\Theta = (\gamma^L - I)^n A \cap B^c(3\varepsilon/2)$ is non-trivial, every element of $\Theta$is expressed as $= \sum_{i=1}^a a_i\, n_i (\gamma^L - I)^n t_i$ . Put $C_1 = 2^a a (3a\varepsilon/2\delta)^a$ and $C_2 = \mu^{-a}$. Then we have the following Step 1.

**Step1**. $\sum_{i=1}^a |n_i| \leqq C_1 C_2^n \qquad (n \geqq 2).$

Indeed, put $C_2 = 3a\varepsilon/2\delta\mu^n$. For $\in \Theta(t = \sum_{i=1}^a a_i\, n_i (\gamma^L - I)^n t_i$ , if we have

$$|n_i| \leqq (c_n + 1)^a \qquad (1 \leqq i \leqq a), \tag{5}$$

then $\sum_{i=1}^a |n_i| \leqq a(c_n + 1)^a = a \sum_{i=1}^a \binom{a}{j} c_n^j \leqq C_1 C_2^n$.

We must prove (5) to get Step 1. Assume that $n_1 \geqq (c_n + 1)^a$ for some $n \geqq 2$ and $t = \sum_{i=1}^a n_i (\gamma^L - I)^n t_i \in \Theta$. We write $s = a - 1$ and put $a_i = n_{i+1}/n_1$ $(1 \leqq i \leqq a - 1 = s)$. Choose $N \in \mathbb{N}$ with $c_n < N \leqq c_{n+1}$ .Then by (L.16) we get an integer $m_1$ with $1 \leqq m_1 \leqq N^{a-1}$ such that $|m_1 n_i/n_1 - m_i| < 1/N < c_n^{-1}$ for $2 \leqq i \leqq a$ and for some $(m_2, \cdots, m_a) \in Z^{a-1}$ . Since $m_1 \neq 0$ , we get $0 \neq m_i(\gamma^L - I)^n t_i$ and since $t = \sum_{i=1}^a n_i (\gamma^L - I)^n t_i\,, m_i(\gamma^L - I)^n t_i$ and since $t = \sum_{i=1}^a n_i (\gamma^L - I)^n t_i$,

$$\left\|\sum_i m_i (\gamma^L - I)^n t_i\right\| \leqq \left\|\sum_i m_i (\gamma^L - I)^n t_i - (m_1/n_1)t\right\|_c + \left\|\sum_i (m_1/n_1)t\right\|_c$$
$$< \sum_i c_n^{-1} \|(\gamma^L - I)^n t_i\|_c + (m_1/n_1)\|t\|_c.$$

Since $t \in B^c(3\varepsilon/2)$, clearly $||t||_c \leqq 3\varepsilon/2$. Since $\|(\gamma^L - I)^n t_i\|_c < (1/2^n)\|t_i\|$ (by (3)) and $m_1/n_1 < c_n^{-1}$ ( because $m_1 < N^{a-1}$ and $n_1 > N^a$), we have $\|\sum_i m_i(\gamma^L - I)^n t_i\|_c < \delta\mu^n/2^n + \delta\mu^n/a < \delta\mu^n$ (because $a \geqq 2$ by ergodicity). Comparing this inequality with (4), we have $t = 0$, which is impossible. Therefore $n_1 \leqq (c_n + 1)^a$ . Repeat the same argument for $n_i$ .Then we get $n_1 \leqq (c_n + 1)^a$ for $1 \leqq i \leqq a$.

To get the conclusion of Lemma (3.2.4), we prepare the following Step 2.

**Step 2**. Let $\psi$ be as in (L.6). For every $n \geqq 2$, we can find $D(n), C(n) \in \boldsymbol{Z}^+$ such that $\sup_n D(n)/n < \infty$, $C(n) < D(n)$ and for every $m \geqq D(n)$

$$W_m^u(\varepsilon) + K(\varepsilon) \supset \sigma^{c(n)}\psi\{(\gamma^L - I)^n A \cap B^c(3\varepsilon/2)\}.$$

Indeed, let $\lambda_0$ be as in (L.10) and let $C_1$ and $C_2$ be as in Step 1. Choose positive integers $D_1, D_2(n)$ satisfying $D_1 \geqq -(\log\lambda_0)^{-1}\log C_1$ and $D_2(n) \geqq -n(\log\lambda_0)^{-1}\log 2C_2$, and put $C(n) = D_1 + D_2(n)$ for $n \geqq 2$. Fix $n \geqq 2$ and take $t \in (\gamma^L - I)^n A \cap B^c(3\varepsilon/2)$. Since $t = \sum_{i=1}^a n_i(\gamma^L - I)^n t_i$, we can easily calculate

$$\sigma^{c(n)}\psi(t) = \sigma^{c(n)}\psi\left(\sum_{i=1}^a n_i (\gamma^L - I)^n t_i\right) = \sum_{i=1}^a n_i (\gamma^L - I)^n \sigma^{c(n)}\psi(t_i)$$

$$= \sum_{i=1}^{a} n_i \sum_{j=0}^{a} \binom{n}{j} \sigma^{Lj+c(n)} \psi(t_i).$$

Since $\sum_{i=1}^{a} |n_i| \leqq C_1 C_2^n$ (by Step1) and $\sum_{j=0}^{a} \binom{n}{j} = 2^n$, and since $W_M^u(\varepsilon) \subset \sigma W_M^u(\varepsilon)$ and $K(\varepsilon) \supset \sigma K(\varepsilon)$, we have $\sigma^{Lj+c(n)}\psi(t_i) \in \sigma^{Ln+c(n)} j_M^0 + j^1$, for $0 \leqq j \leqq n$ and $1 \leqq i \leqq a$, and $\sigma^{c(n)}\psi(t) \in \sigma^{Ln+c(n)} j_M^0 + j^1$, where

$$j_M^0 = \underbrace{W_M^u(\varepsilon) + \cdots + W_M^u(\varepsilon)}_{C_1(2C_2)^n} \quad and \quad j^1 = \underbrace{K(\varepsilon) + \cdots + K(\varepsilon)}_{C_1(2C_2)^n}.$$

By (L.12, i) we get $\sigma^{-c(n)} W^u(\varepsilon) \subset W^u\left(\lambda_0^{c(n)} \varepsilon\right) = W^u(\varepsilon/(C_1(2C_2)^n))$, and so $\sigma^{-c(n)} W_M^u(\varepsilon) + \sigma^{-c(n)} W^u(\varepsilon) \subset \sigma^M W^u(\varepsilon/(C_1(2C_2)^n))$. Therefore $\sigma^{-c(n)} j_M^0 + \sigma^{-c(n)} W^u(\varepsilon) \subset \sigma^M W^u(\varepsilon)$, i.e., $j_M^0 \subset W_{M+c(n)}^u(\varepsilon)$. Since $\sigma^{n-m} W_M^u(\varepsilon) \subset W_n^u(\varepsilon) (n \leqq m)$ by (L.13, iii), we have $\sigma^{Ln+c(n)} j_M^0 \subset W_{M+Ln+2c(n)}^u(\varepsilon)$. Using (L.12, ii), we get $\sigma^{c(n)} j^1 \subset K(\varepsilon)$.

Put $D(n) = M + Ln + 2C(n)$ for $n \geqq 2$. Then from the above facts, we have $\sigma^{c(n)}\psi(t) \in W_M^u K(\varepsilon) + K(\varepsilon)$ for every $m \geqq D(n)$ $(n \geqq 2)$ .The conclusion of Step 2 is obtained.

Now we are ready to prove the lemma. Let $J(n)$ be the integer part of $(\log 2)^{-1}(\log 3a + \log n) + 1$ for $n \geqq 2$ .Since $||t_i||_c < 3\varepsilon$ , by (2) we have $||(\gamma^L - I)^{J(n)} t_i|| < \varepsilon/an$. Remark that $E^c = \operatorname{span}\{(\gamma^L - I)^{J(n)} t_i\, ; 1 \leqq i \leqq a\}$. for $n \geqq 2$ .For fixed $n \geqq 2$, $x \in E^c$ is expressed as $x = \sum_1^a (a_i + n)(\gamma^L - I)^{J(n)} t_i$ where $a_i \in [0,1)$ and $n_i \in \boldsymbol{Z}$.Recall that $A = \{\, s \in E^c;\ s = \sum_1^a n_i t_i \ \ , n_i \in \boldsymbol{Z},\ 1 \leqq i \leqq a\}$ .Then we have

$$\min_{s \in (\gamma^L - I)^{J(n)} t_i} \|x - s\|_c < \sum_{1}^{a} a_i \left\|(\gamma^L - I)^{J(n)} t_i\right\|_c < \varepsilon/n, \tag{6}$$

and so $\{(\gamma^L - I)^{J(n)} A \cap B^c\big((1 + 1/n)\varepsilon\big)\} + B^c(\varepsilon/n) \supset B^c(\varepsilon)$. This follows from the fact that for every $x \in B^c(\varepsilon)$ there is $t \in (\gamma^L - I)^{J(n)} A$ such that $||x - t||_c < \varepsilon/n$ by (6), and then $t \in (\gamma^L - I)^{J(n)} A \cap B^c\big((1 + 1/n)\varepsilon\big)$. Let $C(n)$ be as in Step 2. Then $\sigma^{c(J(n))}\psi\{(\gamma^L - I)^n A \cap B^c(3\varepsilon/2) + B^c(\varepsilon/n) \supset \psi B^c(\varepsilon)$. From this and Step 2, we have $W_M^u(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon/n) \supset \psi B^c(\varepsilon)$ for $m \geqq D(J(n))$ .We put $N_\varepsilon(n) = D(J(n))$ for $n \geqq 2$ and in particular $N_\varepsilon(1) = N_\varepsilon(2)$ . Since $J(n^p)/n \to 0$ as $n \to \infty$ for all $p \geqq 1$ and $\sup_n D(n)/n < \infty$ (by Step 2), clearly $N_\varepsilon(n^p)/n \to 0$ as $n \to \infty$ for all $p \geqq 1$. The proof of Lemma (3.2.4) is completed.

**Lemma (3.2.5)[187]:** Assume that $(X, \sigma)$ is either hyperbolic or ergodic and has central spin. Then for every $\varepsilon \in (0, 2\alpha_0/3)$, there is a sequence $\{M_\varepsilon(n)\}_{n=1}^{\infty}$ of positive integers such that for $p \geqq 1$ $M_\varepsilon(n^p)/n \to 0$ as $n \to \infty$and for all $n \geqq 1$ and $m \geqq M_\varepsilon(n)$

$$W_M^u(\varepsilon) + K(\varepsilon)) \oplus \psi B^c(\varepsilon/n) = X.$$

**Proof:** Take and fix $\varepsilon \in (0, 2\alpha_0/3)$ .From (L.14) we have $W_M^u(\varepsilon/2) + K(\varepsilon/2)) \oplus \psi B^c(\varepsilon/2) = X$ for some $M = M(\varepsilon/2) > 0$. Let $\{N_\varepsilon(n)\}_{n=1}^{\infty}$ be as in Lemma (3.2.4). Then for $m \geqq N_{\varepsilon/2}(n)$,

$$\{W_M^u(\varepsilon/2) + K(\varepsilon/2)\} + \{W_m^u(\varepsilon/2) + K(\varepsilon/2) \oplus \psi B^c(\varepsilon/n)\} = X \tag{7}$$

by Lemma (3.2.4). Let $\lambda_0 \in (0,1)$ be as before, $N$ be an integer with $\lambda_0^{-N} > 2$ and put $M_\varepsilon(n) = N + \max\{M, N_{\varepsilon/2}(n)\}$ .Clearly $M_\varepsilon(n^p)/n \to 0$ as $n \to \infty$ for all $p \geqq 1$ .Since

$\sigma^n W^u(\varepsilon) \supset W^u(\varepsilon)$, , we have $W_m^u(\varepsilon/2) + W_m^u(\varepsilon/2) \subset W_{m+N}^u(\varepsilon)$ for all $m \geqq 1$. From (7) we have $X = W_m^u(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon/n)$ for $m \geqq M_\varepsilon(n)$.

**Lemma (3.2.6)[187]:** If $(X, \sigma)$ is ergodic, then for every $\varepsilon \in (0, 2\alpha_0/3)$, there is a function $L_\varepsilon : \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ such that for $p \geqq 1$, $L_\varepsilon(n^p)/n \to 0$ as $n \to \infty$ and such that for $n \geqq 1, m \geqq L_\varepsilon(n)$ and $x, y \in X$ there is $z \in y + K(\varepsilon) \oplus \psi B^c(\varepsilon/n+1)$ such that $z + W^n(\varepsilon) \subset \sigma^m\{x + W^u(\varepsilon)\}$.

**Proof:** Since $X$ is solenoidal, there are $n_0 > 0$ and a sequence $X = X_0 \supset X_1 \supset \cdots \supset X_{n_0-1} \supset X_{n_0} = \{0\}$ of $\sigma$-invariant subgroups which satisfy all the conditions of (L.15). Take and fix $\varepsilon(0, \alpha_0]$. For $0 \leqq i \leqq n_0$ we put $W_n^u(\varepsilon)_i = W_n^u(\varepsilon) \cap X_i$ for $n \geqq 1$, $K(\varepsilon)_i = K(\varepsilon) \cap X_i$ and $\psi B^c(\varepsilon)_i = \psi B^c(\varepsilon) \cap X_i$ . Since $(X_i/X_{i+1}, \sigma)$ is either hyperbolic, or ergodic and has central spin, we can use Lemma (3.2.5) for $(X_i/X_{i+1}, \sigma)$. Then there is a sequence $\{M_\varepsilon^{(i)}(n)\}_{n+1}^\infty$ of positive integers such that for $p \geqq 1$, $M_\varepsilon^{(i)}(n^p)/\to 0$ as $n \to \infty$ , and for $m \geqq M_\varepsilon^{(i)}(n)$

$$W_m^u(\varepsilon)_i + K(\varepsilon)_i) \oplus \psi B^c(\varepsilon/n)_i + X_{i+1} = X_i.$$

Then for $n \geqq 1$ and $m \geqq W_{\varepsilon/n_0}^{(i)}(n+1)$

$$W_m^u(\varepsilon/n_0)_i K(\varepsilon/n_0)_i) \oplus \psi B^c(\varepsilon/n_i(n+1))_i + X_{i+1} = X_i \quad (8)$$

Choose $C > 0$ with $\lambda_0^{-c} > n_0$ and put $L_\varepsilon(n) = C + \max\{W_{\varepsilon/n_0}^{(i)}(n+1);\ 0 \leqq i \leqq n_0\}$ for $n \geqq 1$ .Clearly $L_\varepsilon(n^p)/n \to 0$ as $n \to \infty$ for all $p \geqq 1$ .Since a $\sigma^c W^u(\varepsilon/n_0) \supset W^u(\varepsilon)$,

we have $W_m^u(\varepsilon) \supset \sum_{i=0}^{n_{0-1}} W_{m-c}^u(\varepsilon/n_0)_i$ for $m \geqq L_\varepsilon(n)$. It is clear that $K(\varepsilon) \supset \sum_{i=0}^{n_{0-1}} K(\varepsilon/n_0)_i$and $\psi B^c(\varepsilon/(n+1)) \supset \sum_{i=0}^{n_{0-1}} \psi B^c(\varepsilon/(n+1))_i$ , and so by (8)

$$W_m^u(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon/(n+1))$$
$$\supset \sum_{i=0}^{n_{0-1}} \{W_{m-c}^u(\varepsilon/n_0)_i + K(\varepsilon/n_0)_i \oplus \psi B^c(\varepsilon/(n+1))_i\} = X.$$

Hence for $n \geqq 0$, $m \geqq L_\varepsilon(n)$ and $x, y \in X$, there is $z \in y + K(\varepsilon) \oplus \psi B^c(\varepsilon/(n+1))$ with $z + W^u(\varepsilon) \subset \sigma^m\{x + W^u(\varepsilon)\}$ since $X = \sigma^m x + W_m^u(\varepsilon) + K(\varepsilon) \oplus \psi B^c(\varepsilon/(n+1)) \ni y$ and $K(\varepsilon) \oplus \psi B^c(\varepsilon/(n+1))$ is symmetry (by (L.12, iv)).

Now we are ready to prove Theorem (3.2.7).

**Theorem (3.2.7)[187]:** Let $X$ be a solenoidal group and $\sigma$ be an automorphism of $X$ .Then $(X, \sigma)$ is ergodic (under the Haar measure) if and only if $(X, \sigma)$ satisfies almost weak specification .

Theorem (3.2.7) derives the following.

**Proof:** Let $\varepsilon \in (0, 2\alpha_0/3)$ and $L_\varepsilon: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ as in Lemma (3.2.6). With the notation $a = \dim(E^c)$, we define a function $M_\varepsilon: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ by $M_\varepsilon(n) = L_\varepsilon(n^a)$ $(n \geqq 1)$ .Clearly $M_\varepsilon(n)/n \to 0$ as $n \to \infty$, and for every $n > 0, m \geqq M_\varepsilon(n)$ and $x, y \in X$, there is $z \in y + K(\varepsilon) \oplus \psi B^c(\varepsilon/(n+1)^a)$ with $z + W^u(\varepsilon) \subset \sigma^m\{x + W^u(\varepsilon)\}$ (by Lemma (3.2.6)). Applying Lemma (3.2.3), we get $\gamma^i B^c(\varepsilon/(n+1)^a) \subset B^c(\varepsilon(i+1)^{a-1}/(n+1)^a)$ for $i \geqq 0$, so that for $0 \leqq i \leqq n$

$$\sigma^i z \in \sigma^i y + \sigma^i K(\varepsilon) \oplus \psi B^c(\varepsilon(i+1)^{a-1}/(n+1)^a) \subset \sigma^i y + K(\varepsilon) \oplus \psi B^c(\varepsilon). \quad (9)$$

For every $k \geqq 1$, $x_1, \cdots, x_k \in X$ and a sequence of integers $a_1 \leqq b_1 < \cdots < a_k \leqq b_k$ with h $a_i - b_{i-1} \geqq M_\varepsilon(b_i - a_i)$ $(2 \leqq i \leqq k)$. Letting $z_1 = \sigma^{a_1} x_1$ ,we can find a sequence of $k-1$ points $z_2, \cdots, z_k$ such that for $1 \leqq i \leqq k-1, z_{i+1} \in \sigma^{a_{i+1}} x_i + K(\varepsilon) \oplus \psi B^c(\varepsilon/(b_{i+1} - a_{i+1} + 1)^a)$ and $z_{i+1} + W^u(\varepsilon) \subset \sigma^{i+1^{-b_i}}\{\sigma^{b_i - a_i} z_i + W^u(\varepsilon)\}$ .This is easily obtained using (9). We now have for $1 \leqq i \leqq k-1$,

$$\sigma^{-a}\{z_i + \sigma^{-(b_i-a_i)}W^u(\varepsilon)\} \supset \sigma^{-a_{i+1}}\{z_{i+1} + W^u(\varepsilon)\}z \in \sigma^i y$$
$$\supset \sigma^{-a_{i+1}}\{z_{i+1} + \sigma^{-(b_{i+1}-a_{i+1})}W^u(\varepsilon)\},$$
which yields
$$\bigcap_1^k \sigma^{-a_i}\{z_i + \sigma^{-(b_i-a_i)}W^u(\varepsilon)\} = \sigma^{-a_k}\{z_k + \sigma^{-(b_k-a_k)}W^u(\varepsilon)\}.$$
Take a point $x$ from the last set. Then for $a_i \leqq j \leqq b_i$ $(1 \leqq i \leqq k)$ we get
$$\sigma^i x \in \sigma^{j-a_i}z_i + \sigma^{-(b_i-a_i)}W^u(\varepsilon)$$
$$\subset \sigma^j x_i + \{\sigma^{j-a_i}(K(\varepsilon) \oplus \psi B^c((j-a_i+1)^a \varepsilon/(b_i-a_i+1)^a) \oplus \sigma^{-(b_i-a_i)}W^u(\varepsilon)\}.$$
This shows that $(X, \sigma)$ satisfies almost weak specification.

# Chapter 4
# Galois Correspondence for Compact Groups

We show that an extension of the result to the case of actions of compact Kac algebras on factors is also presented. No assumptions are made on the existence of a normal conditional expectation onto N. We show that there exists a one-to-one correspondence between the lattice of left coideals of $G$ and that of intermediate subfactors of $M^G \subset M$. We show that there is a one-to-one correspondence between the $C^*$-subalgebras that are globally invariant under the compact action and the commuting minimal action, that in addition contain the fixed point algebra of the compact action and the closed, normal subgroups of the compact group.

## Section (4.1): Automorphisms of von Neumann Algebras with a Generalization to Kac Algebras

A classical theme in Operator Algebras is the Galois correspondence between groups of automorphisms of a von Neumann algebra and von Neumann subalgebras.

To be more specific, let $M$ be a von Neumann algebra and to each group G of automorphisms of $M$ let associate $M^G$, the von Neumann subalgebra of the $G$-fixed elements

$$G \to M^G. \tag{1}$$

In a dual way to each von Neumann subalgebra $N$ of $M$ we may associate the group $G_N$ of the automorphisms of $M$ leaving $N$ pointwise fixed

$$N \to G_N, \tag{2}$$

These two maps are in general not one another inverse, but restricting to (closed) subgroups of a given group $G$ and to intermediate von Neumann subalgebras $M^G \subset N \subset M$ they may actually become one another inverse.

Such a Galois correspondence was shown to hold by Nakamura and Takeda [251] and Suzuki [262] in the case $M$ be $II_1$ -factor and $G$ a finite group whose action on $M$ is minimal, namely $M^{G'} \cap M = C$.

A different Galois correspondence, between normal closed subgroups of a compact (minimal) group $G$ and globally $G$-invariant intermediate von Neumann algebras, was obtained by Kishimoto [240], following methods in the analysis of the chemical potential in Quantum Statistical Mechanics [43]. Generalizations of this result concerning dual actions of a locally compact group $G$ were dealt with by Takesaki, in case of $G$ abelian, and by Nakagami more generally, see [252].

Another kind of Galois correspondence was provided by $H$. Choda [224]. It concerns inparticular the crossed product of a factor by an outer action of a discrete group and characterizes the intermediate von Neumann subalgebras that are crossed product by a discrete subgroup. An important assumption here is the existence of a normal conditional expectation onto the intermediate subalgebras.

We consider any compact group $G$ of automorphisms of a (separable) factor $M$, whose action is minimal, and show that any intermediate von Neumann algebra $M^G N \subset M$ is the fixed-point algebra $N = M^H$for some closed subgroup $H$ of $G$, namely the general Galois correspondence holds in the compact minimal case. Indeed as a corollary the two maps (1) and (2) are one another inverse.

A particular case of our result concerning the action of the periodic modular group with maximal spectrum on a type $III_\lambda$ factor, $0 < \lambda < 1$, has been recently obtained in [234].

Concerning the ingredients in our proof, we mention the spectral analysis for compact group actions, endomorphisms and index theory for infinite factors, arguments based on modular theory, injective subfactors, and averaging techniques.

We emphasise that the main step in the proof of our result is to show the existence of a (necessarily unique) normal conditional expectation of $M$ onto any intermediate subfactor between $M^G$ and $M$.

We also obtain a Galois correspondence for inter- mediate von Neumann algebras in the case of crossed products of factors by outer actions of discrete groups (again, without the a priori existence of normal conditional expectation).

This poses the following question: If $M_1 \subset M_2 \subset M_3$ are von Neumann algebras such that $M_1^{/} \cap M_3 = C$, with a normal expectation $\varepsilon: M_3 \to M_1$, does there exist a normal expectation of $M_3$ onto $M_2$? In other words, does = factor through $M_2$? Besides the case dealt, we know a (positive) answer in some cases (for example, if $M_1 \subset M_3$ has finite index or if $M_3$ is semifinite), but no counter-example is known.

We briefly comment on the super-selection structure in particle physics, which partly motivated our work. It is well-known that the group of the internal symmetries in a Quantum Field Theory is the dual of the tensor $C^*$-category defined by the super-selection sectors [227]. Our result classifies the extensions of the net of the observable algebras made up by field operators. An analysis of further aspects of this structure goes beyond the purpose. However we notice that in low dimensional Quantum Field Theory the internal symmetry is realized by a more general, not yet understood, quantum object and this suggests to be of interest to extend our result to a wider class of ``quantum groups."

We take a first step in this direction by providing a version of our result of actions of compact Kac algebras on factors that turns out to be new even in the finite-dimensional case.

For the theory of operator valued weights and basic construction, See [231], [232], [241].

For $M \subset N$ be an inclusion of von Neumann algebras. We denote by $\mathcal{P}(M, N), \mathcal{E}(M, N)$ the set of normal semifinite faithful (abbreviated as n.s.f.), operator valued weights, and that of normal faithful conditional expectations respectively. We denote by $\mathcal{P}_0(M, N)$ the set of $T \in \mathcal{P}(M, N)$ whose restriction to $M \cap N'$ is semifinite, (thus $T$ is called regular in [264], [265]). Note that $\mathcal{P}_0(M, N)$ is either empty or $\mathcal{P}(M, N)$ [232]. For $T \in \mathcal{P}(M, N)$, we use the following standard notations:

$$n_T = \{x \in M; T(x^*x) < \infty\},$$
$$m_T = n_T^* n_T.$$

For a n.f.s. weight $\varphi$ on $M, H_\varphi.$ and $\Lambda_\varphi.$ denote the GNS Hilbert space and the canonical injection $\Lambda_\varphi: n_\varphi \to H_\varphi$.

For $M \subset N$ with $E \in \mathcal{E}(M, N)$, we fix a faithful normal state $\omega$ on $N$ and set $.\varphi := \omega.E$. We regard M as a concrete von Neumann algebra acting on $H_\varphi$. Let $e_N$ be the Jones projection defined by $e_N \Lambda_\varphi(x) = \Lambda_\varphi.(E(x))$, which does not depend on $\omega$ but only on the natural cone of H. [242]. The basic extension of $M$ by $E$ is the von Neumann algebra generated by $M$ and $e_N$, which coincides with $J_M N^{\prime} J_M$, where $J_M$ is the modular conjugation for $M$. For $x \in B(H_\varphi)$, we set $j(x) = J_M x^* J_M$. The dual operator valued weight $\hat{E} \in \mathcal{P}(M_1, M)$ of $E$ is defined by $j.E^{-1}.j\Big|_{M_1}$, where $E^{-1} \in \mathcal{P}(N', M')$ is characterized by spatial derivatives [222]:

$$\frac{d(\psi, E)}{d\varphi'} = \frac{d\psi}{d(\varphi'.E^{-1})}, \qquad \psi \in \mathcal{P}(N, C), \quad \varphi' \in \mathcal{P}(M', \mathbb{C}).$$

Since $\hat{E}$ satisfies $E^{-1}(e_N) = 1$, $Me_NM \subset m_{\hat{E}}$. In [241], Kosaki defined the index of $E$ by $I$ and $E = E^{-1}(1)$ in the case where $M$ and $N$ are factors, which is known to coincide with the probabilistic index defined in [257].

First, we consider a Pimsner -Popa push-down lemma in our setting (cf. [257]).

**Lemma (4.1.1)[219]:** Let $M$ be a von Neumann algebra and . a n.f.s. weight on $M$. Suppose $\mathcal{A}$ is $a$ *-subalgebra of $n_\varphi^* \cap n_\varphi$. which is dense in $M$ in weak topology, and globally invariant under the modular automorphism group. Then $\Lambda_\varphi(\mathcal{A})$ is dense in $H_\varphi$.

**Proof:** Let $p$ be the projection onto the closure of $\Lambda_\varphi(\mathcal{A})$. Then $p \in \acute{\mathcal{A}} = \acute{M}$. Thanks to $\sigma_t^\varphi(\mathcal{A}) = \mathcal{A}, p$ commutes with $\Delta_t^\varphi$, and consequently we have $\Delta_\varphi^{1/2} p \supset p\Delta_\varphi^{1/2}$. Since $\mathcal{A} \subset n_\varphi \cap n_\varphi^*, \Lambda_\varphi(x), x \in \mathcal{A}$ is in the domains of $S_\varphi$ and $\Delta_\varphi^{1/2}$. Thus we get the following:

$$J_\varphi \Lambda_\varphi(x) = J_\varphi S_\varphi \Lambda_\varphi(x^*) = \Delta_\varphi^{1/2} \Lambda_\varphi(x^*) = p\Delta_\varphi^{1/2} \Lambda_\varphi(x^*) = pJ_\varphi \Lambda_\varphi(x).$$

This means that $p$ commutes with $J_\varphi$, and $p \in M \cap \acute{M}$. So we get $\Lambda_\varphi((1-p)\, x) = 0\ for\ x \in \mathcal{A}$. Since . is faithful, this implies $(1-p)\, x = 0$,which shows $p = 1$ because $\mathcal{A}$ is dense in $M$ in weak topology.

**Proposition (4.1.2)[219]:** (Push Down Lemma). Let $M \supset N$ be an inclusion of factors with $E \in \mathcal{E}(M, N)$, and $M_1$ be the basic extension of $M$ by $E$. Then for all $x \in n_{\hat{E}}, e_N \hat{E}\,(e_N x) = e_N x$ holds.

**Proof:** Let $\varphi$ be as above and $\varphi_1 = \varphi.\hat{E}$. Then $e_N x$, and $e_N \hat{E}\,(e_N x)$ belong to $n_\varphi$. So we get the following:

$$\begin{aligned} &\left\|\Lambda_{\varphi_1}(e_N x) - \Lambda_{\varphi_1} e_N \hat{E}(\,(e_N x))\right\|^2 \\ &= \varphi_1(x^* e_N x) - \varphi_1\left(x^* e_N \hat{E}\,(e_N x)\right) - \varphi_1(\hat{E}\,(e_N x)^* e_N x) \\ &\quad + \varphi_1(\hat{E}\,(e_N x)^* e_N \hat{E}\,(e_N x) x^* e_N \hat{E}\,(e_N x)) \\ &= \left\|\Lambda_{\varphi_1}(e_N x)\right\|^2 - \left\|\Lambda_{\varphi_1}(e_N \hat{E}(e_N x))\right\|^2. \end{aligned}$$

So, we can define a bounded operator $V$ on $e_N H_{\varphi_1}$ by

$$Ve_N \Lambda_{\varphi_1}(x) = \Lambda_{\varphi_1}\left((e_N \hat{E}(e_N x)\right), \qquad x \in n_{\hat{E}}.$$

By simple computation, one can show that $V$ is the identity on $e_N \Lambda_{\varphi_1}(Me_NM)$. So to prove the statement, it suffices to show that $\Lambda_{\varphi_1}(Me_NM)$ is dense in $H_{\varphi_1}$. We set $\mathcal{A} = Me_NM$ and show that $A$ satisfies the assumption of the previous lemma. Indeed, since $M_1$ is the weak closure of $Me_NM + M$, the weak closure of $\mathcal{A}$ is a closed two-sided ideal of $M_1$, and coincides with $M_1$. From the definition of $\varphi_1$, we have $\sigma_t^{\varphi_1}(Me_NM) = \sigma_t^\varphi(M)\, \sigma_t^{\varphi_1}(e_N) \sigma_t^\varphi(M) = M\sigma_t^\varphi(e_N)M$. Thanks to $j\,.E^{-1}.j = (\,j\,.E.\ j)^{-1}$, we get

$$\frac{d\varphi_1}{d(\omega.j)} = \frac{d(\varphi.(\,j\,.E.\ j)^{-1}}{d(\omega.j)} = \frac{d\varphi}{d(\omega.E.j)} = \frac{d\varphi}{d(\varphi.j)} = \Delta_\varphi.$$

Since $\Delta_\varphi$ commutes with $e_N$, we get $\sigma_t^{\varphi_1}(e_N) = e_N$.

**Lemma (4.1.3)[219]:** Under the same assumption, assume that $R$ is a factor including $M$ and satisfying the following:

**(i)** There is a projection $e \in R$ such that $R$ is generated by $e$ and $M$, and $exe = E(x)\, e$ holds for $x \in M$.

**(ii)** There is $T \in \mathcal{P}(R, M)$ satisfying $T(e) = 1$, and $e \in (R \cap N')_{E\,.\,T}$ .Then there is an isomorphism $\pi: M_1 \to R$ satisfying $\pi\Big|_M = id_M, \pi(e_N) = e$, and $T.\pi = \pi\,.\hat{E}$.

**Proof:** Let $\psi = \varphi.T$. For the same reason as before, $\Lambda_\psi(MeM)$ is dense in $H_\psi$. So we can define a surjective isometry $U: H_{\varphi_1} \to H_\psi$ and an isomorphism $\pi: M_1 \to R$ by

$$U\Lambda_{\varphi_1}\left(\sum x_i\, e_N y_i\right) = \Lambda_\psi\left(\sum x_i\, e y_i\right), \qquad x_i\,, y_i \in M,$$

$$\pi(x) = UxU^*, \qquad x \in \pi\,.$$

Clearly, $\pi$ satisfies $\pi\Big|_M = id_M, \pi(e_N) = e$. Thanks to $\sigma_t^{E.\,T}(e) = e$, the modular automorphism groups of $\varphi_1$ and $\psi\,.\pi$ coincide on $Me_NM$ (and on $M_1$). Since $M_1$ is a factor, this implies that $\varphi_1$ is a scalar multiple of $\psi\,.\pi$, and consequently that $\hat{E}$ is a scalar multiple of $\pi^{-1}.T.\pi$. From $\hat{E}\,(e_N) = 1$ and $T(e) = 1$, we get the result.

**Lemma (4.1.4)[219]:** The following hold:

**(i)** Let $\{p_i\}_{i\in I} \subset M \cap \acute{N}$ be a family of mutually orthogonal projections, and $p = \sum p_i$ . If $\mathcal{P}_0(p_iMP_i\,, p_iN) \neq \emptyset$ for every $i \in I$, then $\mathcal{P}_0(pMp\,, pN) \neq \emptyset$.

**(ii)** Let $p \in M \cap \acute{N}$ be a projection. If $\mathcal{P}_0(M\,, N) \neq \emptyset$ then $\mathcal{P}_0(pMp\,, pN) \neq \emptyset$.

**(iii)** Let $p \in M \cap \acute{N}$ be a projection satisfying $\mathcal{P}_0(pMp\,, pN) \neq \emptyset$ and $c(p)$ the central support of $p$ in $M \cap \acute{N}$. Then $\mathcal{P}_0(c(p)\,Mc(p), c(p)\,N) \neq \emptyset$,

**(iv)** Let $\{p_i\}_{i\in I} \subset M \cap \acute{N}$ be a family of projections, and $p_0 = \vee\, p_i$ . If $\mathcal{P}_0(p_iMp_i\,, p_iM) \neq \emptyset$ for every $i \in I$, then $\mathcal{P}_0(p_0Mp_0\,, p_0N) \neq \emptyset$.

**Proof: (i)** This follows from the following easy facts:

$$\mathcal{P}_0(pMp\,, \oplus\, p_iMp_i) \neq \emptyset, \qquad \mathcal{P}_0(\oplus\, p_iMp_i\,, \oplus\, p_iN) \neq \emptyset,$$

$$\mathcal{P}_0(\oplus\, p_iN, pN) \neq \emptyset.$$

**(ii)** Since $\mathcal{P}_0(M\,, N) \neq \emptyset$ there is a separating family of normal conditional expectations from $M$ to $N$ $\{E_\propto\}$. Then $\{E_\propto(p.p)\, p\}$ is a separating family of bounded normal operator valued weights from $pMp$ to $pN$, and $\mathcal{P}_0(pMp\,, pN) \neq \emptyset$.

**(iii)** Let $\{e_j\} \in M \cap \acute{N}$ be a family of projections satisfying $p \succ e_j\,, \sum e_j = c(p)$ .Then $\mathcal{P}_0\big(e_jMe_j\,, e_jN\big) \neq \emptyset$. So using (i), we get $\mathcal{P}_0(c(p)Mc(p), c(p)N) \neq \emptyset$.

**(iv)** Let $z_i = c(p_i)$ and $z_0 = \vee\, z_i$ . Then thanks to (i), (ii), and (iii), $\mathcal{P}_0(z_0Mz_0\,, z_0N) \neq \emptyset$ . Since $z_0$ is the central support of $p_0$ , we get the statement by using (i).

**Lemma(4.1.5)[219]:** Let $M \supset N$ be an inclusion of von Neumann algebras (not necessarily with separable predual). Then, there is a unique central projection $z$ of $M \cap \acute{N}$ satisfying the following two conditions:

**(i)** $\mathcal{P}_0(pMp\,, pN) = \emptyset$ holds for every projection $p \in M \cap N', p \leq 1 - z$.

**(ii)** $\mathcal{P}_0(zMz\,, zN) = \mathcal{P}(zMz\,, zN)$.

Moreover, if $\mathcal{P}(M,N)$ is not empty, then $(1-z)(M\cap N')\cap m_T=\{0\}, z\in (M\cap \acute{N})_T$, and $T|_{z(M\cap N')}$ is semifinite for every $T\in\mathcal{P}(M,N)$.

To prove the lemma, we need the following:

**Proof:** Let z be the supremum of the projections $p\in M\cap N'$ Satisfying $\mathcal{P}_0(pMp,pN)=\emptyset$. Then thanks to Lemma (4.1.4)(iii), (iv), $z$ is a central projection satisfying (i) and (ii). It is easy to show the uniqueness of such a projection. If $T\in\mathcal{P}(M,N)$, $\sigma_t^T(z)$ also satisfies (i) and (iii), and we get $z\in(M\cap\acute{N})_T$. This implies that $zT(z\,.\,z)$ belongs to $\mathcal{P}(zMz,zN)$. So, due to, $T|_{z(M\cap N')}$ is semifinite. Suppose $x$ is a nonzero positive element in $m_T\cap(1-z)(M\cap N')$. Then there is a nonzero spectral projection $p$ of $x$ satisfying $T(p)<\infty$. This implies $\mathcal{E}(pMp,pN)=\emptyset$, that contradicts Lemma (4.1.4)(iii).

To analyze local structure of the inclusions obtained by basic construction in the infinite index case, we need the following:

**Lemma (4.1.6)[219]:** ([241]. Let $M\supset N$ be an inclusion of factors. Then the following hold:

**(i)** Let $T\in\mathcal{P}(M,N)$, and $p\in m_T\cap(M\cap N')_T$ a non-zero projection. Then $I$nd $T_p=T(p)\,T^{-1}(p)$, where $T_p\in\mathcal{E}(pMp,pN)=\emptyset$ is defined by $T_p(x)=pT(x)/T(p), x\in pMp$.

**(ii)** If $\mathcal{P}_0(M,N)\neq\emptyset, \mathcal{P}_0(N',M')\neq\emptyset$ then $M\cap N'$ is a direct sum of type $I$ factors and $pMp\supset pN$ has finite index for every finite rank projection in $M\cap N'$.

**Proposition (4.1.7)[219]:** Let $M\supset N$ be an inclusion of factors with $E\in\mathcal{E}(M,N)$, and $M_1$ the basic extension. Then $M_1\cap\acute{N}$ is direct sum of four subalgebras,

$$M_1\cap\acute{N}=A\oplus B_1\oplus B_2\oplus C,$$

satisfying the following:

**(i)** Each of the four subalgebras is globally invariant under $\{\sigma_t^{E\,.\hat{E}}\}$.

**(ii)** $j(A)=A, j(B_1)=B_2, j(B_2)=B_1, j(C)=C$.

**(iii)** $\hat{E}\big|_{A\oplus B_1}$ is semifinite.

**(iv)** $m_{\hat{E}}\cap(B_2\oplus C)=\{0\}$.

**(v)** A is direct sum of type $I$ factors and $pM_1\,p\supset pN$ has finite index for every finite rank projection $p\in A$.

**Proof:** First, we show $\sigma_t^{E\,.\hat{E}}.j=\sigma_t^{E\,.\hat{E}}$ on $M_1\cap N'$. Indeed, for $x\in M_1\cap N'$ we get the following as :

$$\begin{aligned} j.\sigma_t^{E\,.\hat{E}}(j(x)) &= J_M\left(\frac{d(\varphi.\hat{E})}{d(\omega.j)}\right)^{it} J_M x J_M\left(\frac{d(\varphi.\hat{E})}{d(\omega.j)}\right)^{-it} J_M\\ &= J_M\Delta_\varphi^{it} J_M x J_M \Delta_\varphi^{-it} J_M=\Delta_\varphi^{it}x\Delta_\varphi^{-it}\\ &=\left(\frac{d(\varphi.\hat{E})}{d(\omega.j)}\right)^{it} x\left(\frac{d(\varphi.\hat{E})}{d(\omega.j)}\right)^{-it}\\ &=\sigma_t^{E\,.\hat{E}}(x). \end{aligned}$$

Now, let $z$ be the central projection of $M_1\cap\acute{N}$ determined by Lemma (4.1.5) for $M_1\supset N$. We set

$$A=zj(z)(M_1\cap N'),\qquad C=(1-z)\,j(1-z)(M_1\cap N'),$$

$$B_1 = zj(1-z)(M_1 \cap N'), \qquad B_2 = (1-z)j(z)(M_1 \cap N').$$

Then by construction (ii), (iii), and (iv) hold. Since j commutes with $\sigma_t^{E.\hat{E}}$, $j(z) \in (M_1 \cap N')_{E.\hat{E}}$, and we get (i). Note that for a projection $p \in M_1 \cap N'$, $J_M(pM_1 p)'J_M = j(p)\, N, J_M(pN)'\, J_M = j(p)\, M_1\, j(p)$. So $(pN)' \supset (pM_1 p)'$ is anti conjugate to $j(p)\, M_1\, j(p) \supset j(p)\, N$. Thus thanks to Lemma (4.1.6), we get (V).

Our basic references for the theory of sectors are [246], [247], [237]. Let $M$ be an infinite factor. We denote by $End(M)$ and $Sect(M)$ the set of unital endomorphisms of $M$ and that of sectors, which is the quotient of $End(M)$ by the unitary equivalence. Note that every element in $End(M)$ is automatically normal for $M$ with separable predual. For $\rho_1, \rho_2 \in End(M)$, $(\rho_1, \rho_2)$ denotes the set of intertwiners between $\rho_1$ and $\rho_2$, i.e.,

$$\rho_1, \rho_2) = \{v \in M; v\rho_1(x) = \rho_2(x)v, x \in M\}.$$

If $\rho_1$ is irreducible, i.e., $M \cap \rho_1(M)^{/} = C$, $(\rho_1, \rho_2)$ is a Hilbert space with the following inner product:

$$\langle V|W\rangle\, 1 = W^*V, \qquad V, W \in (\rho_1, \rho_2).$$

We define the dimension $d(\rho)$ $of$ $\rho$ $by$ $d(\rho) = [M: \rho(M)]_0^{1/2}$, where $[M: \rho\,(M)]_0$ is the minimum index of $M \supset \rho\,(M)$. For $\rho$ with $d(\rho) < \infty$ we denote by $E_\rho$ and $\phi_\rho$ the minimal conditional expectation onto $\rho$ (M) and the standard left inverse of $\rho$, i.e., $\phi_\rho = \rho^{-1}.E_\rho$.

There are three natural operations in Sect $(M)$: the sum, the product, and the conjugation. For simplicity, we denote by $\bar{\rho}$ one of the representatives of the conjugate sector $\overline{[\rho]}$ of $[\rho]$. When $d(\rho)$ is finite, it is known that there are two isometries $R_\rho \in (id, \bar{\rho}\rho), \bar{R}_\rho \ni (id, \bar{\rho}\rho)$ satisfying

$$\bar{R}_\rho^* \rho(R_\rho) = R_\rho^* \bar{\rho}(\bar{R}_\rho) = \frac{1}{d(\rho)}. \tag{3}$$

Although such a pair is not unique, we fix it once and forever. Unless $\rho$ is a pseudoreal sector [246], we can take $\bar{R}_\rho$equal to $R_{\bar{\rho}}$. If it is, we set $\bar{R}_\rho = -R_\rho$.

Let ${}_M X_M$ be a $M-M$ bimodule, and $\rho \in End(M)$. Then we define a new Bimodule ${}_M(X_\rho)_M$ (respectively ${}_M({}_\rho X\ )_M$ by

$$x.\acute{\xi}.y := x.\xi.\rho(y) \qquad \text{(respectively } x.\acute{\xi}.y := \rho(x).\xi.y) \qquad x, y \in M,$$

where $\acute{\xi} = \xi$ as an element of Hilbert space $X$. It is known that there is one- to-one correspondence between Sect $(M)$ and the set of equivalence classes of $M-M$ bimodules. The correspondence is given by $[\rho] \to [\ {}_M(L^2(M)_\rho)_M]$, which preserves the three operations. The conjugate sector of $[\rho]$ is characterized by

$${}_M(L^2(M)_\rho)_M \simeq {}_M\left({}_{\bar{\rho}}(L^2(M)\ )\right)_M.$$

Let $\phi$ be a unital normal completely positive map from $M$ to $M$. Following Connes [223], there is a natural way to associate a $M-M$ bimodule with $\phi$. Let $\Omega$ be a separating and cyclic vector of $M$. We introduce a positive Semidefinite sesquilinear form on the algebraic tensor product $M \otimes_{alg} M$ as follows:

$$\langle \sum_i x_i \otimes y_i, \sum_j z_i \otimes w_i \rangle = \sum_{i,j} \langle \phi(z_j^* x_i) J_M w_j y_j^* J_M \Omega | \Omega \rangle$$

We denote by $H_\phi$ the Hilbert space completion of the quotient of $M \otimes_{alg} M$ by the kernel of the sesquilinear form, and by $\Lambda_\phi$ the natural map $\Lambda_\phi: M \otimes_{alg} M \to H_\phi . H_\phi$ is naturally a $M - M$ bimodule by the following action:

$$x.\Lambda_\phi\left(\sum_i z_i \otimes w_i\right).y = \Lambda_\phi\left(\sum_i xz_i \otimes w_i\, y\right).$$

Thanks to the one-to-one correspondence stated above, there is an endomorphism $\rho_\phi$ satisfying ${}_M(H_\phi)_M \simeq {}_M\left({}_{\rho_\phi}L^2(M)\right)_M$. Actually, $\rho_\phi$ is Steinspring type dilation of $\phi$. Indeed, let $W: H_\phi \to L^2(M)$ be the intertwining surjective isometry, and set $\xi_0 = W\Lambda_\phi(1 \otimes 1)$. Then we get

$$\langle \phi(x).\Omega.y|\Omega\rangle = \langle x.\Lambda_\phi(1 \otimes 1).y, \Lambda_\phi(1 \otimes 1)\,\rangle$$

$$= \langle \rho_\phi(x).\xi_0.y|\xi_0\rangle.$$

We define an isometry $v$ by $v(\Omega, y) = \xi_0.y$. Then by definition, $v$ commutes with the right action of $M$. So $v$ belongs to $M$ and satisfies $\phi(x) = v^*\rho_\phi(x)\,v, x \in M$. Note that the support of $vv^*$ in $M \cap \rho_\phi(\acute{M})$ is $1$. Indeed, suppose $z \in M \cap \rho_\phi(\acute{M})$ satisfying $z\xi_0\,.y = 0$, for all $y \in M$. Then $z\rho_\phi(x)\xi_0.y = 0$ for all $x, y \in M$. Since $\overline{\rho_\phi(M).\xi_0.M} = WH_\phi = L^2(M)$, we get $z = 0$.

**Proposition (4.1.8)[219]:** Let $M$ and , be as above. Then the following hold:

**(i)** Let $\sigma \in End(M)$, and $v_1 \in M$ be an isometry satisfying $(x) = v_1^*\sigma(x)\,v_1$ . If the support of $v_1v_1^*$ in $M \cap \sigma(\acute{M})$ is 1, then $[\rho_\phi] = [\sigma]$.

**(ii)** The equivalence class of $H_\phi$ does not depend on the choice of the cyclic separating vector $\Omega$.

**(iii)** Let $\mu$ be another unital normal completely positive map from $M$ to $M$. If there is a positive constant $c$ such that $c\mu + \phi$ is completely positive, then $[\rho_\mu]$ contains $[\rho_\phi]$.

**Proof:** (i) Let $\xi_0$ be as before. Then by assumption, we get the following:

$$\langle \sigma(x)v_1\Omega.y|v_1\Omega\rangle = \langle \rho_\phi(x)\xi_0.y|\xi_0\rangle,$$
$$\overline{\sigma(M)v_1\Omega.M} = L^2(M).$$

So we can define a unitary $u \in M$ by $u\sigma(x)v_1\Omega\,.y = \rho_\phi(x)\xi_0.y$, and get $\rho_\phi(x) = u\sigma(x)u^*$

**(ii)** follows from (i).

**(iii)** Since $c\mu - \phi$ is completely positive, we can define a bounded map $T: H_\mu \to H_\phi$ by

$$T\Lambda_\mu\left(\sum_i x_i \otimes y_i\right) = \Lambda_\phi\left(\sum_i x_i \otimes y_i\right).$$

Then T is an $M - M$ bimodule map whose image is dense in $H_\phi$. Let $T = U\,|T|$ be the polar decomposition of $T$ . Then $U$ is a co-isometry belonging to Hom $\left({}_M(H_\mu)_M, {}_M(H_\phi)_M\right)$. Thus $[\rho_\mu]$ contains $[\rho_\phi]$.

[248], proved that for an arbitrary infinite factor $M$ (with separable predual), there exists an injective subfactor $R \subset M$ satisfying $R' \cap J_MR'J_M = C$. A subfactor $R$ of $M$ is called simple if $R' \cap J_MR'J_M = C$. A simple subfactor $R$ determines the automorphisms of $M$ in the following sense; if $\alpha, \beta \in Aut(M)$ satisfying $\alpha|_R = \beta|_{R'}$ then $\alpha = \beta$. Indeed,

let $u$ be the canonical implementation of $\alpha^{-1}.\beta$. Then $u \in R'$, and $u$ commutes with $J_M$. So $u$ is a scalar, that means $\alpha = \beta$. We can generalize this to some class of endomorphisms as follows:

**Proposition(4.1.9)[219]:** Let $M$ be an infinite factor and $R$ a simple subfactor. For every $\rho \in End(M)$ with $E \in \mathcal{E}(M, \rho(M))$, the following holds:

$$\{T \in M; Tx = \rho(x)\,T, x \in R\} = (id, \rho). \tag{4}$$

**Proof:** First, we show that the general case can be reduced to the case where $(id, \rho) = \{0\}$. Indeed, let $\{V_i\}_i$ be an orthonormal basis of $(id, \rho)$, and $W$ an isometry in $M$ satisfying $WW^* = 1 - \sum V_i V_i^*$. Then $\rho(x) = \sum V_i\, x\, V_i^* + W\sigma(x)W^*$, where $\sigma \in End(M)$ is defined by $\sigma(x) = W^*\rho(x)\,W$. Note that $(id, \sigma) = \{0\}$ by construction. If $T$ is in the left-hand side of (4), then $c_i := V_i^* T \in \acute{R} \cap M = C$, and $W^*T$ satisfies $W^*Tx = \sigma(x)W^*T, x \in R$. Since $T = \sum V_i\, V_i^* T + WW^*T$, if the statement is true for $\sigma$, i.e., $W^*T = 0$; we get $T = \sum c_i V_i \in (id, \sigma)$.

Secondly, we construct the ``canonical implementation" of $\sigma$ as follows.

Let $\Omega$ be a separating and cyclic vector for $M$, and $L^2(M, \Omega)_+$ the natural cone with respect to $\Omega$. Then there are unique vectors $\xi_0, \xi_1 \in L^2(M, \Omega)_+$ satisfying

$$\langle E(x)\Omega | \Omega\rangle = \langle x\xi_0 | \xi_0\rangle.$$

$$\langle \rho(x)\Omega | \Omega\rangle = \langle x\xi_1 | \xi_1\rangle.$$

Note that $\xi_0, \xi_1$ are cyclic because they belong to the natural cone and implement faithful states. So we can define an isometry $V_\rho\ by\ V_\rho x\xi_1 = \rho(x)\,\xi_0$. We set $e_\rho = V_\rho V_\rho^*$, which is the Jones projection of $E. V_\rho$ satisfies $V_\rho x = \rho(x)\,V_\rho$ and $J_M V_\rho = V_\rho J_M$. Indeed, the first equality is obvious. By identifying $e_\rho L^2(M)$ with $L^2(\rho(M), \xi_0)$, we get $J_{\rho(M)} V_\rho = V_\rho J_M$. On the other hand, since $e_\rho$ is the Jones projection, we have $e_\rho J_M = J_M e_\rho = J_{\rho(M)}$ .So $V_\rho$ commutes with $J_M$.

Now suppose that $(id, \sigma) = \{0\}$ and there exists a nonzero element $T$ in the left-hand side of (4). Since $T^*T \in M \cap R' = C$, we may assume that $T$ is an isometry. We set $\tilde{T} = TJ_M TJ_M$, which commutes with $J_M$ and satisfies
$\tilde{T} = \rho(x)\tilde{T},\ x \in R$. Then $V_\rho^*\tilde{T}, \in \acute{R} \cap J_M \acute{R} J_M = C$. Let $\lambda = V_\rho^*\tilde{T}$, which is not zero because

$$\begin{aligned}\langle V_\rho^*\tilde{T}\xi_0 | \xi_1\rangle &= \langle\, \tilde{T}\xi_0 | \xi_0\rangle = \langle\, T\xi_0 |\, J_M T^*\xi_0\rangle \\ &= \langle T\xi_0 \big| \Delta_\varphi^{1/2} T\xi_0\rangle = \left\|\, \Delta_\varphi^{1/4} T\xi_0 \right\|^2,\end{aligned}$$

where $\varphi(x) = \langle E(x)\,\Omega \mid \Omega\rangle,\ x \in M$. We define a unital completely positive map $\phi: M \to M\ by\ , \phi(x) = T^*\rho(x)\,T, x \in M$, which equals to $\tilde{T}^*\rho(x)\,\tilde{T}$. By construction, $[\rho]$ contains $[\rho_\phi]$. So we show that $[\rho_\phi]$. contains $[id]$ and get contradiction. Thanks to Proposition (4.1.8), it suffices to show that $\phi - |\lambda|^2\, id$ is completely positive. In fact,

$$\begin{aligned}\phi(x) = \tilde{T}^*\rho(x)\tilde{T} &= \tilde{T}^* e_\rho \rho(x)\tilde{T} + \tilde{T}^*(1 - e_\rho)\rho(x)\tilde{T} \\ &= \tilde{T}^* V_\rho V_\rho^*\, \rho(x)\tilde{T} + \tilde{T}^*(1 - e_\rho)\rho(x)\tilde{T} \\ &= \tilde{T}^* V_\rho x\, V_\rho^*\, \tilde{T} + \tilde{T}^*(1 - e_\rho)\rho(x)\tilde{T} \\ &= |\lambda|^2 x + \tilde{T}^*(1 - e_\rho)\rho(x)\tilde{T}.\end{aligned}$$

Since $e_\rho$ commutes with $\rho(M), x \mapsto \tilde{T}^*(1-e_\rho)\rho(x)\tilde{T}$ is a complete positive map. So $[\rho]$ contains $[id]$ and we get contradiction.

**Corollary (4.1.10)[219]:** Let $M, R, \rho$ be as above, and $\sigma \in End(M)$ with $d(\sigma) < \infty$. Then the following hold:

**(i)** $\{T \in M;\ T\sigma(x) = \rho(x)\, T, x \in R\} = (\sigma\,, \rho)$.

**(ii)** If $\sigma|_R = \rho|_R$, then $\sigma = \rho$.

**Proof:** (i) Let $T$ be in the left-hand side of (i), and set $X = \bar{\sigma}\,(V)\, R_\sigma$, where $R_\sigma$ is the isometry in (3). Then $X$ satisfies $Xx = \bar{\sigma}.\rho(x)\, X\,, x \in R$. So thanks to Proposition (4.1.9), we get $X \in (id, \bar{\sigma}.\rho\,)$. By simple computation using (3), we obtain $V = d(\sigma)\, \bar{R}_\sigma^* \sigma(X)$, and $V \in (\sigma, \rho)$.

(ii) Thanks to (i), $1 \in (\sigma, \rho)$, that means $\sigma = \rho$.

Let $\psi$ be a dominant weight on $M$ [225]. Since every dominant weight is unitary equivalent, for every $\alpha \in Aut(M)$ there is a unitary $u \in M$ satisfying $\alpha, \psi \ \in Ad(u) = \psi$ This fact is used to define the Connes-Takesaki module of $\alpha$. The endomorphism version is given as follows, which will be used.

**Lemma (4.1.11)[219]:** Let $M$ be an infinite factor. Then the following hold:

**(i)** For every $\rho \in End(M)$ with $d(\rho) < \infty$, there exist a dominant weight $\psi_\rho$ and a unitary $u \in M$ such that

$$\psi_\rho \cdot \rho \cdot Ad(u) = d(\rho)\psi_\rho, \qquad \psi_\rho \cdot E_\rho = \psi_\rho.$$

**(ii)** Let $\psi$ be a dominant weight. Then for every $[\rho] \in Sect(M)$ with $d(\rho) < \infty$, there exists a representative $\rho$ satisfying

$$\psi \cdot \rho = d(\rho)\psi, \qquad \psi \cdot E_\rho = \psi.$$

**Proof:** (i) Let $\psi_0$ be a dominant weight on $\rho(M)$. Since both $d(\rho)\, \psi_0\ \cdot E_\rho$ and $\psi_\rho \cdot \rho$ are dominant weights on $M$, there exists a unitary $u \in M$ satisfying $d(\rho)\psi_0\ \cdot E_\rho = \psi_0 \cdot \rho \cdot Ad(u)$. So $\psi_\rho \coloneqq \psi_0 \cdot E_\rho$is the desired weight.

(ii) follows from (i) and the fact that every dominant weight is unitary equivalent.

We investigate the structure of irreducible inclusions offactors with normal conditional expectations. We present the ultimate form of the Galois correspondence of outer actions of discrete groups and minimal actions of compact groups on factors, which has been studied [43], [224], [240], [250], [251]. The key argument is how to show the existence of a conditional expectation for every intermediate subfactor.

Let $M \supset N$ be an irreducible inclusion, i.e., $M \cap N' = C$, of infinite factors with a conditional expectation $E \in \mathcal{E}(M, N)$. For $\rho \in End(N)$, we set

$$\mathcal{H}_\rho = \{V \in M;\ Vx = \rho(x)\, V, x \in N\}.$$

Then thanks to the irreducibility of $M \supset N, \mathcal{H}_\rho$ is a Hilbert space with inner product $\langle V\, |W\rangle 1 = W^*V$ as usual. We denote by $s(\mathcal{H}_\rho)$ the support of $\mathcal{H}_\rho$, that is $\sum_i V_i V_i^*$ where $\{V_i\}_i$ is an orthonormal basis of $\mathcal{H}_\rho$. Let $M_1$ be the basic extension of $M$ by $N$, and $e_N$ the Jones projection of $E$. Then $\mathcal{H}_\rho^*\, e_N \mathcal{H}_\rho \subset M_1 \cap \grave{N}$.

Let $y\colon M \to N$ be the canonical endomorphism [246], [247], [248]. Then it is known that ${}_N L^2(M)_N \simeq {}_N(y|_N L^2(N)\,)_N$. When $Ind\ E < \infty$ it is easy to show that an irreducible sector $[\rho] \in Sect(N)$ is contained in $[\lambda|_N]$ if and only if $\mathcal{H}_\rho \neq 0$ (Frobenius reciprocity) [238]. First, we establish the infinite index version of this statement. For this purpose, it is convenient to giveexplicit correspondence between submodules of ${}_N L^2(M)_N$ and subsectors of $\lambda|_N$. Let $p \in M_1 \cap \grave{N}$ be a nonzero projection. Since both $e_N$ and $\rho$ are infinite projection in $M_1$, there is a partial isometry $W \in M_1$ satisfying $WW^* = e_N, W^*W = p$. Due to $e_N M_1 e_N = e_N N$, we can define $\rho \in End(N)$ by $WxW^* = e_N \rho(x), x \in N$.

**Lemma(4.1.12)[219]:** Under the above assumption and notation, the following holds:

$$_N(pL^2(M))_N \simeq {}_N\big(\rho L^2(N)\big)_N.$$

**Proof:** We regard $W$ as a surjective isometry from $pL^2(M)$ to $e_N L^2(M) = L^2(N)$ .Since $M_1 = J_M \grave{N} J_M$,Wcommutes with $J_M N J_M$. So for $\xi \in pL^2(M),\ x, y \in N$, we obtain

$$W(x \cdot \xi \cdot y) = WxJ_M\, y^* J_M \xi = \rho(x) W J_M\, y^* J_M \xi = \rho(x) J_M\, y^* J_M W \xi.$$

By using $e_N J_M = J_M eN = J_N$, we get $W(x \cdot \xi \cdot y) = \rho(x) J_N\, y^* J_N W \xi$.

**Proposition(4.1.13)[219]:** Let $M \supset N$ be an irreducible inclusion of infinite factors with $E \in \mathcal{E}(M, N)$, and $\gamma: M \to N$ the canonical endomorphism. Then for $\rho \in End(M)$, the following two statements are equivalent:

**(i)** $\mathcal{H}_\rho \neq 0$ and the support of $E(s(\mathcal{H}_\rho))$ is 1.

**(ii)** $\mathcal{E}\big(N, \rho(N)\big)$ is nonempty and $[\rho]$ is contained in $[\lambda|_N]$ up to multiplicity, i.e., there is decomposition $[\rho] = \oplus[\rho_a]$ such that each $[\rho_a]$ is contained in $[\lambda|_N]$.

**Proof:** (i)⟹(ii). Assume that \ satisfies (i). By a simple argument, one can show that there is decomposition $[\rho] = \oplus[\rho_a]$ such that for every $a$ there exists $V_a \in \mathcal{H}_{\rho_a}$ satisfying $E(V_a V_a^*) \geq 1$. We set $W_a = e_N E(V_a V_a^*)^{-1/2} V_a$. Then $W_a$ satisfies $W_a W_a^* = e_N, p_a := W_a^* W_a \in M_1 \cap N'$. Since $W_a x W_a^* = e_N \rho_a(x), x \in N, [\rho_a]$ is contained in $[\lambda|_N]. \hat{E}(p_a) = V_a^* E(V_a V_a^*)^{-1} V_a < \infty$ implies $\mathcal{E}(p_a M_1 p_a, p_a N) \neq \emptyset$ and consequently $\mathcal{E}\big(N, \rho_a(N)\big) \neq \emptyset$.

(ii) ⟹ (i). It is easy to show that if $[\rho] = \oplus[\rho_a]$ and each \a satisfies (i), then so does $\rho$. Assume that $[\rho]$ is contained in $[\lambda|_N]$ and $\mathcal{E}\big(N, \rho(N)\big) \neq \emptyset$. Let $p \in M_1 \cap N'$ be the projection corresponding to $[\rho]$. Then $\mathcal{E}\big(pM_1, p, p(N)\big) \neq \emptyset$.. which implies $p \in A \oplus B$ where $A$ and $B$ are as in Lemma (4.1.6). Let $z$ be the central support of $p$ in $M_1 \cap N'$. Since $\sigma_t^{E \cdot \hat{E}}$ is trivial on the center of $A \oplus B, \hat{E}\big|_{z(M_1 \cap N')}$ is semifinite. So there are two families of projections $\{p_a\}, \{q_a\}$ in $z(M_1 \cap \acute{N})$ such that $p = \sum_a p_a\ , p_a \sim q_a$ in $z(M_1 \cap N')$ and $q_a \in m_E$ . Let $W_a$ be a partial isometry satisfying $W_a W_a^* = e_N, W_a^* W_a = q_a$, and $\rho_a \in End(N)$ defined by $W_a x W_a^* = e_N \rho(x),\ x \in N$. Then $[\rho] = \oplus[\rho_a]$. Since $W_a = e_N W_a q_a \in m_E$ , due to Lemma (4.1.2), there exists $V_a \in M$ satisfying $W_a = e_N V_a$ . It is easy to check $V_a \in \mathcal{H}_{\rho_a}$ and $E(V_a V_a^*) = 1$. So $\rho_a$ satisfies (i).

Let $\{[\rho_\xi]\}_{\xi \in \Xi}$ be the set of irreducible sectors with finite dimension contained in $[\lambda|_N]$. We arrange the index set $\Xi$ such that $\overline{[\rho_\xi]} = [\rho_{\bar{\xi}}]$ holds for every $\xi \in \Xi$. For simplicity, we use notations $R_\xi, \bar{R}_\xi, \mathcal{H}_\xi, d(\xi), E_\xi$ instead of $R_{\rho_\xi}, \bar{R}_{\rho_\xi}$, etc. We define the Frobenius maps $c_\xi: \mathcal{H}_\xi \to \mathcal{H}_{\bar{\xi}}, \bar{c}_\xi : \mathcal{H}_{\bar{\xi}} \to \mathcal{H}_\xi$ by

$$c_\xi(V) = \sqrt{d(\xi)}\ V^* \bar{R}_\xi, \qquad V \in \mathcal{H}_\xi,$$

$$\bar{c}_\xi(\bar{V}) = \sqrt{d(\xi)}\ \bar{V}^* R_\xi, \qquad \bar{V} \in \mathcal{H}_{\bar{\xi}},$$

Then thanks to (3), $\bar{c}_\xi\, c_\xi = 1_{\mathcal{H}_\xi}, \bar{c}_\xi\, c_\xi = 1_{\mathcal{H}_{\bar{\xi}}}$ . So in particular, both $c_\xi$ and $\bar{c}_\xi$ are invertible. We introduce a new inner product to $\mathcal{H}_\xi$ by

$$(V_1, V_2)1 = d(\xi) E(V_1, V_2^*) \in \big(\rho_\xi, \rho_\xi\big) = C, \quad V_1, V_2 \in \mathcal{H}_\xi.$$

Due to the estimate $|V_1, V_2| \leq d(\xi)\| V_1 \| \| V_2 \|$, there is a nonsingular positive operator $a_\xi \in B(\mathcal{H}_\xi)$ satisfying

$$(V_1, V_2) = \langle a_\xi V_1 | V_2 \rangle.$$

Let $\{V_i\}_i \subset \mathcal{H}_\xi$ be an orthonormal basis of $\mathcal{H}_\xi$. Since $\sum V_i V_i^* = s(\mathcal{H}_\xi) \leq 1$, we get

$$Tr(a_\xi) = \sum (V_i, V_i)\, d(\xi)E(s(\mathcal{H}_\xi)) \leq d(\xi)$$

So $a_\xi$ is a trace class operator. By simple computation one can show the following:

$$\langle c_\xi (V_1) | c_\xi (V_2) \rangle = (V_2, V_1) = \langle a_\xi V_2 | V_1 \rangle,$$

$$\langle \bar{c}_\xi (\bar{V}_1) | \bar{c}_\xi (\bar{V}_2) \rangle = (\bar{V}_2, \bar{V}_1) = \langle \bar{a}_\xi \bar{V}_2 | \bar{V}_1 \rangle.$$

Thus we get $c_\xi^* c_\xi = a_\xi$, $\bar{c}_\xi^* \bar{c}_\xi = \bar{a}_\xi$. This shows that $a_\xi$ is an invertible trace class operator, that implies $n_\xi = \dim \mathcal{H}_\xi < \infty$. Thanks to $\bar{c}_\xi = c_\xi^{-1}$, we obtain

$$Tr(\bar{a}_\xi) = Tr(\bar{c}_\xi^*, \bar{c}_\xi) = Tr(\bar{c}_\xi, \bar{c}_\xi^*) = Tr(a_\xi^{-1}).$$

This implies

$$\frac{1}{d(\xi)} \leq a_\xi \leq d(\xi), \qquad n_\xi \leq d(\xi)^2.$$

If $a_\xi = 1$ (this is the case if, for instance, $Ind\ E < \infty$), then $n_\xi \leq d(\xi)$. On the other hand if $n_\xi = d(\xi)$, then it is easy to show that $a_\xi = 1$ and $E(s(\mathcal{H}_\xi)) = 1$, i.e., $s(\mathcal{H}_\xi) = 1$.

**Theorem (4.1.14)[219]:** Let $M \supset N$ be an irreducible inclusion of infinite factors with $E \in \mathcal{E}(M, N)$, and $M_1 \cap \acute{N} = A \oplus B_1 \oplus B_2 \oplus C$ the decomposition described in Proposition (4.1.7). Then with the same notation as above, the following hold:

**(i)** $A = \oplus_{\xi \in \Xi} A_\xi$, where $A_\xi = \mathcal{H}_\xi^* e_N \mathcal{H}_\xi \simeq M(n_\xi, C)$.

**(ii)** $B_1$ and $B_2$ are of type $I$.

**(iii)** For $V_1, V_2 \in \mathcal{H}_\xi$, $\sigma_t^{E\mathrm{O}\hat{E}} (V_1^* e_N V_2) = V_1^* a_\xi^{-it} e_N a_\xi^{it} V_2$.

**(iv)** For $V_1, V_2 \in \mathcal{H}_\xi$, $j(V_1^* e_N V_2) = c_\xi(a_\xi^{1/2} V_2)^* e_N c_\xi(a_\xi^{1/2} V_1)$.

**Proof:** (i) Thanks to Proposition (4.1.7), $A$ is direct sum of type I factors. By using the one-to-one correspondence as described just before Lemma (4.1.12), we can parametrize the direct summands of $A$ by $\Xi$ such that $A = \oplus A_\xi$ and $A_\xi \supset \mathcal{H}_\xi^* e_N \mathcal{H}_\xi$ hold. So it suffices to show that $A_\xi$ is of type $\mathrm{I}_{n_\xi}$. If $A_\xi$ is finite, then $A_\xi \subset m_{\hat{E}-}$ because $\hat{E}\big|_{A_\xi}$ is semifinite. So we can take matrix units $\{e_i, j\}_{1 \leq i,j}$ of $A_\xi$ (with $\sum e_{i,i} = \mathrm{I}_{A_\xi}$) such that $\hat{E}(e_{i,j}) = b_i \delta_{i,j}$. We may assume that there is a partial isometry $W_1 \in M_1$ satisfying $W_1 W_1^* = e_N, W_1^* W_1 = e_{1,1}$ and $W_1 x W_1^* = e_N \rho_\xi(x)$ for $x \in N$. We set $W_i = W_1 e_{1,i}$. Then there exists $V_i \in M$ such that $W_i = \sqrt{b_i}\, e_N V_i$. $\{V_i\}$ is an orthonormal basis of $\mathcal{H}_\xi$. Indeed, it is easy to show that it is an
orthonormal system. Suppose $V \in \mathcal{H}_\xi$ is perpendicular to $\{V_i\}$. Since $e_{i,j} = W_i^* e_N W_j = \sqrt{b_i b_j} V_i^* e_N V_j$, $V^* e_N V$ is an element in $A_\xi$ satisfying $e_{i,j} V^* e_N V = 0$. This means $V^* e_N V = 0$ and $0 = \hat{E}(V^* e_N V) = V^* V$, i.e., $V = 0$. So $\{V_i\}$ is an orthonormal basis of $\mathcal{H}_\xi$ and the rank of $A_\xi$ coincides With $n_\xi$. Now suppose $A_\xi$ is of type $\mathrm{I}_\infty$. Since $\hat{E}\big|_{A_\xi}$ is semifinite, there is a matrix unit $\{e_{i,j}\}_{1 \leq i,j < \infty}$ (not necessarily $\sum e_{i,i} = 1$), such that $\hat{E}(e_{i,i}) < \infty, \hat{E}(e_{i,j}) = 0$ for $i \neq j$. Then we can define $W_i$ and $V_i$ as before. However, $\{V_i\}_{1 \leq i < \infty}$ is an orthonormal system of $\mathcal{H}_\xi$, that contradicts the fact $\dim \mathcal{H}_\xi = n_i < \infty$.

(ii) Since $\hat{E}\Big|_{B_1}$ is semifinite and $(B_1) = B_2$, it suffices to show that $pB_1\, p$ is of type I for every $p \in B_1$ with $\hat{E}(\,p) < \infty\_$. Let $W \in M_1$ be a partial isometry with $WW^* = e_N, W^*W = p$, and define $\rho \in End(M)$ by $WxW^* = e_N\rho(x),\ x \in N$ as before. there exists an isometry $V \in \mathcal{H}_\rho$ satisfying $W = \sqrt{c}\ \ e_N V, c = \hat{E}(\,p)$. So $E(VV^*) = 1/c$ and we get $1/c \leq E(s(\mathcal{H}_\rho)) \leq 1$. Let $P = N \cap \rho(\acute{N})$ . Then in the same way as in the proof of Lemma (4.1.12), we can show that $p\ B_1\ p$ is isomorphic to $P$. So we show that $P$ is of type I. Thanks to $\mathcal{H}_\rho = \mathcal{H}_\rho$ , we can define a normal representation of $P$ on $\mathcal{H}_\rho$ by $\pi(x)\ V = xV, x \in P, V \in \mathcal{H}_\rho$ . Note that $1/c \leq E(s(\mathcal{H}_\rho))$ implies that $\pi$ is faithful. Thus to prove that $P$ is of type I, we show that there exists a normal conditional expectation from $B(\mathcal{H}_\rho)$ to $\pi(P)$. For $\omega \in P_*$ we can define a bilinear form on $\mathcal{H}_\rho$ by $\omega(E(V_1V_2^*)), V_1, V_2 \in \mathcal{H}_\rho$ with an estimate $\left|\omega\big(E(V_1V_2^*)\big)\right| \leq \|\omega\|\ \|V_1\|\ \|V_2\|$. So there exists a unique bounded operator $h_\omega$ satisfying

$$\omega\big(E(V_1V_2^*)\big) = \langle h_\omega V_1 \mid V_2\rangle.$$

For $x, y \in P, \omega\ \in P_*, h_\omega$ satisfies $h_{x.\omega.y} = \pi(x)\ h_\omega\pi(\,y)$. Indeed, by definition we get

$$\begin{aligned} x\cdot\omega\cdot y\big(E(V_1V_2^*)\big) &= \omega(yE(V_1V_2^*)x) = \omega(E(\pi(y)V_1(\pi(x)^*V_2)^*)) \\ &= \langle h_\omega\pi\ (y)V_1|\pi(x)^*V_2\rangle = \langle\pi\ (x)h_\omega\pi(y)V_1|V_2\rangle. \end{aligned}$$

If $\omega \in P_*$ is positive, we have

$$Tr(h_\omega) = \omega\left(E\left(s\big(\mathcal{H}_\rho\big)\right)\right) \leq \omega(1) = \|\omega\|,$$

so by using polar decomposition of linear functionals and the fact just proved above, we get

$$\|h_\omega\|_1 \coloneqq Tr(|h_\omega|) = Tr\big(h_{|\omega|}\big) \leq \|(|\omega|)\| = \|\omega\|, \quad \omega \in P_*.$$

Hence we can define a bounded order preserving linear map $\theta: P_* \to B(\mathcal{H}_\rho)_*$ by $\theta(\omega)(a) = Tr(h_\omega a), a\ \in B(\mathcal{H}_\rho)$ . Note that $\theta$ satisfies $(x\ ,\omega\ ,y) = \pi(x)\cdot\theta(\omega)\cdot\pi(y), x, y \in P$ . Let $F_0$ be the transposition of $\theta$. Then $F_0$ is a positive normal map $F_0 :\ B(\mathcal{H}_\rho) \to P$ satisfying $F_0\big(\pi(x)a\pi(\,y)\big) = xF_0(a)y, x, y \in P, a \in B(\mathcal{H}_\rho)$ . Note that $F_0(1) = E(s(\mathcal{H}_\rho))$ is a central element of $P$ because us $(\mathcal{H}_\rho)\ u^* = s(\mathcal{H}_\rho)$ holds for every unitary $u \in P$ . Since $E(s(\mathcal{H}_\rho))$ is invertible, we can define a normal conditional expectation $F: B(\mathcal{H}_\rho) \to \pi(P)$ by

$$F(a) = \pi\big(E(s(\mathcal{H}_\rho)\big)^{-1/2}F_0(a)E\left(s\big(\mathcal{H}_\rho\big)\right)^{-1/2}), \quad a \in B\big(\mathcal{H}_\rho\big).$$

Therefore, $P$ is of type I.

(iii) By a simple argument one can show that unitary perturbation of $\rho_\xi$ does not have any effect on the formulae in (iii) and (iv). So thanks to Lemma (4.1.11), we assume that there is a dominant weight $\psi$ on $N$ satisfying $\psi\ \cdot\rho_\xi = d(\xi)\ \psi, \psi\ \cdot E_\xi = \psi$ for every $\psi \in \Xi$. Then $\sigma_t^\psi$ commute with $\rho_\xi$ and we get $\sigma_t^{\psi.E}(\mathcal{H}_\xi) = \mathcal{H}_\xi$ . So we show $\sigma_t^{\psi.E}\ (V) = a_\xi^{it}V$ for $\in \mathcal{H}_\xi$ , that implies the statement. Indeed, since $\dim\mathcal{H}_\xi < \infty$, every element in $\mathcal{H}_\xi$ is analytic for $\left\{\ \sigma_t^{\psi.E}\right\}$. Let $V \in \mathcal{H}_\xi$ and $x \in m_\psi$. Then by using the KMS condition, we obtain

$$\psi\cdot E(VxV^*) = \psi\cdot E\left(xV^*\sigma_{-i}^{\psi.E}(V)\right) = \langle\sigma_{-i}^{\psi.E}(V)|V\rangle\psi(x).$$

On the other hand, from $E(VxV^*) = E(\rho_\xi(x)\ VV^*) = (1/d(\xi))(V, V)\ \rho_\xi(x)$ we get

$$\psi \cdot E(VxV^*) = \frac{1}{d(\xi)}(V,V)\psi \cdot \rho_\xi(x) = \langle a_\xi V|V\rangle\, \psi(x)$$

So we obtain $\sigma_t^{\psi.E}(V) = a_\xi^{it} \cdot V$.

(iv) Let $z_\xi$ be the unit of $A_\xi$. Then by using the correspondence between sub-bimodules of ${}_N L^2(M)_N$ and subsectors of $\gamma_N$, we get $j(A_\xi) = A_{\bar{\xi}}$ and $j(z_\xi) = z_{\bar{\xi}}$. Let $\psi$ be as before. Then due to (i), it is easy to show that $\mathcal{H}_\xi^* \Lambda_{\psi.E}(n_\psi \cap n_\psi^*)$ is dense in $z_\xi H_\psi E$. Since both $j(V_1^* e_N V_2)$ and $c_\xi(a_\xi^{1/2} V_2)^* e_N c_\xi(a_\xi^{1/2} V_1)$ belong to $A_{\bar{\xi}}$, it suffices to show the equality on $\mathcal{H}_{\bar{\xi}}^* \Lambda_{\psi.E}(n_\psi \cap n_\psi^*)$. Let $a \in n_\psi \cap n_\psi^*$ and $X \in \mathcal{H}_{\bar{\xi}}$. Since $V_1, V_2$ are analytic elements for $\left\{\sigma_t^{\psi.E}\right\}$, we get

$$\begin{aligned} j(V_1^* e_N V_2)\Lambda_{\psi.E}(X^* a) &= J_M V_2^* J_M e_N J_M V_1 J_M \Lambda_{\psi.E}(X^* a) \\ &= J_M V_2^* J_M\, e_N\, J_M\, \Lambda_{\psi.E}(X^* a \sigma_{i/2}^{\psi.E}(V_1)^*) \\ &= J_M V_2^* J_M\, e_N\, \Lambda_{\psi.E}(X^* \sigma_{i/2}^{\psi.E}(V_1)^* \rho_\xi(a)) \\ &= J_M V_2^* J_M\, e_N\, \Lambda_{\psi.E}(E(X^* \sigma_{i/2}^{\psi.E}(V_1)^* \rho_\xi(a)) \\ &= \Lambda_{\psi.E}(E(X^* \sigma_{i/2}^{\psi.E}(V_1)^* \rho_\xi(a) \sigma_{i/2}^{\psi.E}(V_2)) \\ &= \Lambda_{\psi.E}\left(E\left(X^* \sigma_{i/2}^{\psi.E}(V_1)^*\right)\sigma_{-i/2}^{\psi.E}(V_2) a\right). \end{aligned}$$

By using $X = c_\xi\left(\bar{c}_\xi(X)\right) = \sqrt{d(\xi)}\, \bar{c}_\xi(X)^* \bar{R}_\xi$, we get

$$\begin{aligned} j(V_1^* e_N V_2)\Lambda_{\psi.E}(X^* a) &= \sqrt{d(\xi)}\ \Lambda_{\psi.E}\left(\bar{R}_\xi^* E\left(\bar{c}_\xi(X) \sigma_{i/2}^{\psi.E}(V_1)^*\right)\sigma_{-i/2}^{\psi.E}(V_2) a\right) \\ &= \frac{1}{\sqrt{d(\xi)}}(\bar{c}_\xi(X), \sigma_{i/2}^{\psi.E}(V_1))\ \Lambda_{\psi.E}(\bar{R}_\xi^* \sigma_{-i/2}^{\psi.E}(V_2) a) \\ &= \frac{1}{d(\xi)}\left(\bar{c}_\xi(X), a_\xi^{-1/2} V_1\right)\ \Lambda_{\psi.E}\left(c_\xi\left(a_\xi^{-1/2} V_2\right)^* a\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &c_\xi\, (a_\xi^{1/2} V_2)^* e_N c_\xi(a_\xi^{1/2} V_1)\ \Lambda_{\psi.E}(X^* a) \\ &\quad = \frac{1}{d(\xi)}(c_\xi(a_\xi^{1/2} V_1), X)\ \Lambda_{\psi.E}\left(c_\xi\left(a_\xi^{1/2} V_2\right)^* a\right), \end{aligned}$$

so it suffices to show $\left(\bar{c}_\xi(X), a_\xi^{-1/2} V_1 = (c_\xi(a_\xi^{1/2} V_1\right), X)$. Actually,

$$(c_\xi(a_\xi^{1/2} V_1), X) = \langle \bar{c}_\xi(X) \left| a_\xi^{1/2} V_1 \right\rangle = \langle \bar{c}_\xi(X), a_\xi^{-1/2} V_1 \rangle.$$

**Remark (4.1.15) [219]:** Let $V_1, V_2 \in \mathcal{H}_\xi$. Then we get

$$\begin{aligned} \hat{E}(V_1^* e_N V_2) &= \langle V_2 | V_1 \rangle. \\ \hat{E}(j(V_1^* e_N V_2)) &= \langle c_\xi(a_\xi^{1/2} V_1) \left| c_\xi(a_\xi^{1/2} V_2) \right\rangle \\ &= (a_\xi^{1/2} V_2, a_\xi^{1/2} V_1) = \langle\, a_\xi^2 V_2) | V_1 \rangle. \end{aligned}$$

So $\hat{E} \cdot j|_A = \hat{E}\big|_A$ if and only if $a_\xi = 1$ for all $\xi \in \Xi$. It is also easy to show that $\hat{E}\big|_A$ is a trace if and only if $a_\xi$ is a scalar for all $\xi \in \Xi$.

There is no known example which Violates $a_\xi = 1$. However, the following example shows that $\hat{E} \cdot j|_{M_1 \cap N'} = \hat{E}\big|_{M_1 \cap N'}$ does not hold in.

**Example (4.1.16)[219]:** (i) Let $G$ be a discrete group and $H$ a subgroup, and let $\alpha$ be an outer action of $G$ on a factor $L$. We set $N = L \times_\alpha H, M = L \times_\alpha G$. Then $M \supset N$ is an irreducible inclusion of factors with a unique conditional expectation $E$. We identify $M$ and $N$ with $(L \otimes C) \times G$ and $(L \otimes C) \times H$ acting on $L^2(L) \otimes \ell_2(G/H) \otimes \ell_2(G)$ in an obvious sense. Let $f$ be the ortho- gonal projection onto $C\delta_{\dot{e}} \subset \ell_2(G/H)$, where $\delta$ stands for the $\delta$ -function and $\dot{e}$ the class of the neutral element e, and $F_0 = id \otimes Tr \in \mathcal{P}(L \otimes \ell^\infty(G/H), L \otimes C)$. Then we can identify $M_1$ with $(L \otimes \ell^\infty(G/H) \times G$ where the action of $G$ on $\ell^\infty(G/H)$ is the translation, $e_N$ with $1 \otimes f \otimes 1$ and $\hat{E}$ with the natural extension of $F_0$ to $(L \otimes \ell^\infty(G/H) \times G$. So under this identification we get $M_1 \cap \acute{N} = \ell^\infty(H\backslash G/H)$. For $\dot{g} \in G/H$, we denote by $p_{\dot{g}} \in \ell^\infty(H\backslash G/H)$ the projection corresponding to the $H$-orbit of $\dot{g}$. Then $\hat{E}\ (p_{\dot{g}})$ is exactly the length of the orbit, i.e., $\hat{E}\ (p_{\dot{g}}) = [H : H_g]$ where $H_g := gHg^{-1} \cap H \cdot j(p_{\dot{g}})$ can be computed by using bimodules as in [244], and we have $j(p_{\dot{g}}) = p_{\dot{g}^{-1}}$. So for example if $g^{-1}Hg \subset H$ and $g^{-1}Hg \neq H$, then $\hat{E}\ (p_{\dot{g}}) = 1$ although $\hat{E}\ (j\, p_{\dot{g}})) \neq 1$. Let $G$ be the group generated by the finite permutations of $Z$ and g where $g$ is the translation of $Z$, and $H$ the finite permutations of $N \cup \{0\}$. Then $gHg^{-1}$ is the finite permutation of $N$ and we get $gHg^{-1} \subset H, [H : H_g] = \infty$. So we obtain $\hat{E}\ (p_{\dot{g}}) = \infty, \hat{E}\ (p_{\dot{g}^{-1}}) = 1$. This means $B_i \neq \{0\}, i = 1, 2$ in this example.

(ii) Let $G \supset H$ be a pair of discrete groups with the following property: for every $g \neq e \in G$ $\{hgh^{-1}; h \in H\}$ is an infinite set. Let $M := L(G)$ be the group von Neumann algebra of $G$ and $N := L(H)$ the subfactor of $M$ generated by $H$. Then in exactly the same way as one proves that $M$ is a factor, one can show $M \cap \acute{N} = C$. Although this example looks similar to the previous one, these two have essentially different natures. As before we can identify $N, M$ and $M_1$ with $C \times H, C \times G$ and $\ell^\infty(G/H) \times G$ acting on $\ell^\infty(G/H) \otimes \ell^\infty(G)$. However, we can conclude only $\ell^\infty(H\backslash G/H) \subset M_1 \cap \acute{N}$ because the action of $G$ on $G/H$ is not necessarily free. In fact the equality does not hold in general. For example, let $G = F_3$ be the free group generated by $g_1, g_2, g_3$ and $H = F_2 = \langle g_1, g_2 \rangle$. Then the $N - N$ bimodule ${}_N X_N$ generated by $\delta_{g_3} \in \ell^2(F_3)$ is equivalent to ${}_N\ell^2(F_2) \otimes \ell^2(F_2)_N$ where $\otimes$ is the usual tensor product and the left and the right actions act on each tensor component respectively. So $End({}_N X_N) \simeq N^{op} \otimes N$. This means that $M_1 \cap \acute{N}$ has a type II summand. Actually, a little more effort shows that $A = Ce_N, B_1 = B_2 = 0$, and $C$ is of type II where $A, B_1, B_2$, and $C$ are as in Proposition (4.1.7).

**Definition (4.1.17)[219]**: An inclusion of factors is called discrete if and only if $\mathcal{E}(M, N)$ is nonempty and $\hat{E}\big|_{M_1 \cap N'}$ is semifinite for some (and equivalently all) $E \in \mathcal{E}(M, N)$.

In what follows we assume that $M \supset N$ is an irreducible discrete inclusion of infinite factors. Note that discreteness is equivalent to $M_1 \cap \acute{N} = A$ in the decomposition given in Proposition (4.1.7), and to $[\lambda|_N] = \oplus n_\xi[\rho_\xi], d(\xi) < \infty$.

For each $\xi \in \Xi$ choose an orthogonal basis $\{V(\xi)i\}_{i=1}^{n_\xi}$ consisting of eigenvectors of $a_\xi$ belonging to $a_{\xi,i}$. For $x \in M$ we define the ``Fourier coefficient" $x(\xi)_i$ by

$$x(\xi)_i = \frac{d(\xi)}{a_{\xi,i}} E(V(\xi)_i x).$$

Then $x$ has the following formal expansion:

$$x=\sum_{\xi\in\Xi}\sum_{i=1}^{n_\xi}V(\xi)_i^*x(\xi)_i.$$

Although the above sum does not converge even in weak topology in general, we can give justification of the expansion as follows. We define $p_{\xi,i}\in M_1\cap \acute{N}$ by

$$p_{\xi,i}=\frac{d(\xi)}{a_{\xi,i}}V(\xi)_i^*e_NV(\xi)_i.$$

Then $a_{\xi,i}$ is a projection with $z_\xi=\sum_{i=1}^{n_\xi}p_{\xi,i}$ , where $z_\xi$ is the unit of $A_\xi$. By discreteness assumption we have $\sum_{\xi\in\Xi}z_\xi=1$ . Let $\omega$ be a faithful normal state on $N$ and set $\varphi=\omega\cdot E$. Since $p_{\xi,i}\Lambda_\varphi(x)=\Lambda_\varphi(V(\xi)_i^*\,x(\xi)_i)$ and $\Lambda_\varphi(x)=\sum_{\xi,i}p_{\xi,i}\,,\Lambda_\varphi(x)$, the sum converges in Hilbert space topology. Note that $\{x(\xi)_i\}$ uniquely determines $x$ while it is difficult to tell when a series $\{x(\xi)_i\}$ is actually the Fourier coefficient of some element $x\in M$.

Although the following lemma might sound trivial, we need to prove it because the expansion does not make sense in any decent operator algebra topology.

**Lemma (4.1.18)[219]:** Under the above assumption, assume that there is an assignment of subspaces $\mathcal{K}_\xi\subset\mathcal{H}_\xi$ satisfying the following conditions.

**(i)** $a_\xi\mathcal{K}_\xi\subset\mathcal{K}_\xi$.

**(ii)** $\mathcal{K}_\xi^*\subset N\,\mathcal{K}_{\bar{\xi}}$.

**(iii)** Let $\eta,\in\Xi$ and set $\Xi_{\eta,\zeta}=\{\xi\in\Xi;\ \rho_\eta\rho_\zeta\succ\rho_\zeta\}$. Then, $\mathcal{K}_\eta\mathcal{K}_\zeta\subset\sum_{\xi\in\Xi_{\eta,\zeta}}N\,\mathcal{K}_\xi$ .

Let $L$ be the *von* Neumann algebra generated by $N$ and $\{\mathcal{K}_\xi\}_{\xi\in\Xi}$. Then there exists $E_L\mathcal{E}\,(M,L)$, and $L$ is characterized by

$$L=\left\{x\in M;E\left(\mathcal{K}_{\bar{\xi}}^{\frac{1}{}}x\right)=0,\xi\in\Xi\right\},$$

Where $\mathcal{K}_{\bar{\xi}}^{\frac{1}{}}$ is the orthogonal complement of $\mathcal{K}_\xi$ with respect to $\langle\ |\ \rangle$.

**Proof:** Let $L_0$ be the direct sum of $\mathcal{K}_\xi^*N$. Thanks to (ii) and (iii), $L_0$ is the *-algebra generated by $N$ and $\{\mathcal{K}_\xi\}$, which is dense in $L$. Let $L_1=\{x\in M;\ E(\mathcal{K}_{\bar{\xi}}^{\frac{1}{}}x)=0,\xi\in\Xi\}$ and $K$ the closure of $\Lambda_\varphi(L)$ in $H_\varphi$. First, we claim $L_1=[x\in M;\ \Lambda_\varphi(x)\in K]$. Indeed, due to (i) we may arrange $\{V(\xi)_i\}$ such that $\{V(\xi)_i\}_{i=1}^{m_\xi}$ is an orthonormal basis of $\mathcal{K}_\xi$ . Then we get $K=\oplus_{\xi\in\Xi}\oplus_{i=1}^{m_\xi}H_{\xi,i}$ where $H_{\xi,i}=p_{\xi,i}\,H_\varphi$, and so

$$L_1=\{x\in M;p_{\xi,i}\,\Lambda_\varphi(x)=0,i>m_\xi\}.$$

Thus we get the claim. Secondly, we show that there exists $E_L\in\mathcal{E}(M,L)$ with $\varphi\cdot E_L=\varphi$. Thanks to the Takesaki theorem on conditional expectations $[S]$, it suffices to prove $\sigma_t^\varphi(L)=L$, or in our case $\sigma_t^\varphi(\mathcal{K}_\xi)\subset N\mathcal{K}_\xi$. As before we may and do assume that there is a dominant weight $\psi$ on $N$ satisfying $\psi\cdot E_\xi=\psi,\psi\cdot\rho_\xi=d(\xi)\,\psi$, so we have $\sigma_t^{\psi.E}(V)=a_\xi^{it}V$ for $V\in\mathcal{H}_\xi$. We set $u_t^\xi=[D\omega:D\psi]_t\,\rho_\xi\backslash!\,([D\omega:D\psi]_t^*)\in N$, where $[D\omega:D\psi]_t$ is the Connes cocycle derivative. Then we get

$$\begin{aligned}\sigma_t^\varphi(V)&=Ad([D\omega\cdot E:D\psi\cdot E]_t)\cdot\sigma_t^{\psi.E}(V)\\&=Ad([D\omega:D\psi\cdot E]_t)\big(a_\xi^{it}V\big)=u_i^\xi a_\xi^{it}V,\end{aligned}$$

so due to (i) we get $\sigma_t^\varphi(\mathcal{K}_\xi)\subset N\mathcal{K}_\xi$. Now let $e_L$ be the Jones projection for $E_L$, i.e., $e_L\Lambda_\varphi(x)=\Lambda_\varphi(E_L(x)),x\in M$. Then $e_L$ is the orthogonal projection onto $K$. Since $L$ is characterized by $L=\{x\in M;\ e_L\Lambda_\varphi(x)=\Lambda_\varphi(x)\}$, we get $L=L_1$ .

The following is the main technical result.

**Theorem (4.1.19)[219]:** Let $M \supset N$ be an irreducible inclusion of infinite factors with $E \in \mathcal{E}(M,N)$. We assume that the inclusion is of discrete type and $\sigma_t^{E.\,\hat{E}}$ is trivial. Let $L$ be an intermediate subfactor and $\mathcal{K}_\xi = L \cap \mathcal{H}_\xi$ . Then $\{\mathcal{K}_\xi\}$ satisfies the assumption of Lemma (4.1.18) and $L$ is generated by $N$ and $\{\mathcal{K}_\xi\}$.Consequently, there exists $E_L \in \mathcal{E}(M,N)$.

**Proof:** First, we show that the statement can be reduced to the case where $N$ is of type III. Suppose that the statement holds for type III factors. Then we apply the statement to $\widehat{M} = M \otimes P, \widehat{N} = N \otimes P$ and $\hat{L} = L \otimes P$ where $P$ is a type III factor, and get that $\hat{L}$ is generated by $\widehat{N}$ and $(\mathcal{H}_\xi \otimes C) \cap \hat{L} = \mathcal{K}_\xi \otimes C.\{\mathcal{K}_\xi\}$ satisfies the assumption of Lemma (4.1.18) because so does $\{[\mathcal{K}_\xi \otimes]\}$ by assumption. Thanks to Lemma (4.1.18)we get

$$\hat{L} = \left\{x \in M \otimes P; (E \otimes id)((\mathcal{K}_{\xi}^{\frac{1}{\xi}} \otimes 1)x) = 0, \xi \in \Xi\right\},$$

and so we obtain

$$L = \left\{x \in M; E(\mathcal{K}_{\xi}^{\frac{1}{\xi}} x)x = 0, \xi \in \Xi\right\}.$$

Therefore, the statement holds for $L$ as well. Now, we assume that $N$ is of type III. Let $\{V(\xi)_i\}$ be as in the proof of Lemma (4.1.18). Thanks to $\mathcal{H}_\xi^* \subset N\mathcal{H}_{\bar{\xi}}, \mathcal{H}_\eta \mathcal{H}_\zeta \subset \sum_{\xi \in \Xi_{\eta.\zeta}} N\mathcal{H}_\xi$ and the Fourier decomposition, to prove that $\{\mathcal{K}_\xi\}$ satisfies the assumption of Lemma (4.1.18)it suffices to show $x(\xi)_i = 0$ for $x \in L, \xi \in \Xi, i > m_\xi$ , which is actually enough for the statement due to Lemma (4.1.18).

Suppose the converse; there exists $x \in L$ such that $x(\xi)_i \neq 0$ for some $\xi \in \Xi$ and some $, i > m_\xi$. Let $y = axb, a, b \in N$. Then $E_\xi(\, y(\xi)_i) = \rho_\xi(a)\, E_\xi(\, x(\xi)_i b)$ since $N$ is a type III factor, we can choose $a, b$ such that $E_\xi(\, y(\xi)_i) = 1$, so we assume $E_\xi(\, x(\xi)_i) = 1$ from the beginning. Let $R$ be a simple injective subfactor of $N$ and $U(R)$ the unitary group of $R$. We set $\mathfrak{C} = \overline{conv\{ux\rho_\xi(u^*);\ u \in\ U(R)\}}^{w}$and define an action $\theta$ of $U(R)$ on $\mathfrak{C}$ by $\theta_u(w) = uw\rho_\xi(u^*), u \in U(R), w \in \mathfrak{C}$. We claim that the set of fixed points of $\mathfrak{C}$ under $\theta$, which is the same as $\{w \in \mathfrak{C};\ aw = w\rho_\xi(a), a \in R\}$, is nonempty. Indeed, since $R$ is AFD, there exists an increasing sequence of finite dimensional unital von Neumann-subalgebras $\{R_n\}_{n=1}^{\infty}$ generating $R$. Let $\mathfrak{C}_n$ be the fixed points of $\mathfrak{C}$ under$|_{U(R_n)}$ , that is a nonempty compact set because $U(R_n)$ is a compact group. Then $\{\mathfrak{C}_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty compact sets, and so $\mathfrak{C}_\infty := \bigcap_{n=i}^{\infty} \mathfrak{C}_n$ is nonempty as well. Let $w \in \mathfrak{C}_\infty$. Then w satisfies $aw = w\rho_\xi(a)$ for $\in \bigcup_n R_n$ , and for $a \in R$ because $\bigcup_n R_n$ is dense in $R$. Thus $\mathfrak{C}_\infty$ is the set of the fixed points. From the definition of the Fourier coefficient of $w \in \mathfrak{C}_\infty$ we get $\rho_\eta(a)\, w(\eta)_j = w(\eta)_j\, \rho_\xi(a)$ for $a\ \in R, \eta \in \Xi$ . Applying Corollary (4.1.10) we obtain $w(\eta)_j = 0$ for $\eta \neq \xi$ and $w(\eta)_j \in C$ , that means $w^* \in \mathcal{H}_\xi \cap N\ = \mathcal{K}_\xi$ . On the other hand, $E_\xi(\, x(\xi)_i) = 1$ implies $E_\xi(\left(ux\rho_\xi(u^*)(\xi)_i\right) = \rho_\xi(u)E_\xi(\, x(\xi)_i)\rho_\xi(u^*) = 1$ for $u \in U(R)$ and so $E_\xi(\, w(\xi)_i) = 1$ by continuity. Since $w(\xi)_i$ is a scalar$(\, w(\xi)_i) = 1$. Hence $w^* \notin \mathcal{K}_\xi$ , that is contradiction. Therefore we get $x(\xi)_i = 0$ for $x \in L, \xi \in \Xi, i > m_\xi$.

**Corollary (4.1.20)[219]:** Let $M \supset N$, $\Xi$ be as above and $\Xi_1$ a self-conjugate subset of $\Xi$ with the following properties; whenever $\xi, \eta \in \Xi_1$ , $\Xi_{\xi,\eta} \subset \Xi_1$ .Then there exists a unique intermediate subfactor $L$ such that if we denote by $\acute{\gamma}$ the canonical endomorphism $\acute{\gamma}: L \to N$, then

$$[\lambda'|_N] = \bigoplus_{\xi \in \Xi_1} n_\xi \, [\rho_\xi],$$

**Proof:** Set $= N \vee \{\mathcal{H}_\xi\}_{\xi \in \Xi_1}$.

**Corollary (4.1.21)[219]:** Let $M \supset N$ be an irreducible inclusion of factors ($N$ is not necessarily infinite) with $E \in \mathcal{E}(M, N)$. We assume that the inclusion is of discrete type and $\sigma_t^{E.\hat{E}}$ is trivial. Then for every intermediate subfactor $L, \mathcal{E}(M, L)$ is not empty.

**Proof**: It is enough to prove the statement when $N$ is finite and $M$ is infinite. Let $F$ be a type $I_\infty$ factor. Then thanks to Theorem (4.1.19), $\mathcal{E}(M \otimes F, L \otimes F)$ is not empty. Since we can identify $M \supset N$ with $e(M \otimes F)e \supset e(L \otimes F)e$ where $e$ is a minimal projection of $F, \mathcal{E}(M, L)$ is not empty.

**Theorem (4.1.22)[219]:** Let $G$ be a discrete group and $\alpha$ an outer action of $G$ on a factor $N$. Then the map $H \mapsto N \times_\alpha H$ gives one-to-one correspondence between the lattice of all subgroups of $G$ and that of all intermediate subfactors of $N \subset N \times_\alpha G$.

**Proof:** Let $\{\lambda(\mathrm{g})\}$ denote the implementing unitaries of $\alpha$ in $M := N \times_\alpha G$. Then it is easy to see $\Xi = G$ and $\mathcal{H}_\mathrm{g} = C\lambda(\mathrm{g})$ where $\Xi$ and $H\mathrm{g}$ are as in Theorem (4.1.19). (Note that the argument in Theorem (4.1.19) makes sense even when $N$ is finite as far as $\{\rho_\xi\}$ are automorphisms.) Let $\{\mathcal{K}_\mathrm{g}\}_{\mathrm{g} \in G}$ be a system of subspaces satisfying (ii) and (iii) of Theorem (4.1.19). Then there exists a subgroup $H \subset G$ such that $\mathcal{K}_\mathrm{g} = C\lambda(\mathrm{g})$ if $\mathrm{g} \in H$ and $\mathcal{K}_\mathrm{g} = 0$ if $\mathrm{g} \notin H$. This means that for every intermediate subfactor $L$ there exits a subgroup $H$ with $L = N \times_\alpha H$.

Let $G$ be a compact group. We call an action : of $G$ on a factor $M$ minimal if $\alpha$ is faithful and $M \cap M^{\acute{G}} = C$ where $M^G$ is the fixed point algebra under $\alpha$. It is known that if $\alpha$ is minimal the crossed product $M \times_\alpha G$ is always a factor. We fix a complete system of representatives of the equivalence classes of the irreducible representations of $G$ and denote it by $\hat{G}$. If $\alpha$ is minimal and the fixed point algebra $MG$ is infinite, using the same type of argument as in [43], one can show that for every $\pi \in \hat{G}$ there exists a Hilbert space $\mathcal{H}_\pi \in M$ with support 1 such that $\mathcal{H}_\pi$ is globally invariant under $\alpha$ and $\alpha|_{\mathcal{H}_\pi}$ is equivalent to $\pi$. This means that $M$ is the crossed product of $M^G$ and the dual object of $G$ by the corresponding Roberts action [260]. We fix such a $\mathcal{H}_\pi$ for each $\pi \in \hat{G}$ and choose an orthonormal basis $\{V(\pi)_i\}_{i=1}^{d(\pi)}$ of $\mathcal{H}_\pi$ where $d(\pi)$ is the dimension of $\mathcal{H}_\pi$. Let $N = M^G$ and $E$ the unique element in $\mathcal{E}(M, N)$ obtained by

$$E(x) = \int_G \alpha_\mathrm{g}(x) d\mathrm{g}, \qquad x \in M.$$

We define an endomorphism $\rho_\pi \in End(N)$ by

$$\rho_\pi(x) = \sum_{i=1}^{d(\pi)} V(\pi)_i x V(\pi)_i^*, \qquad x \in N.$$

Thanks to the minimality of $\alpha, \rho_\pi$ is always irreducible with $d(\rho_\pi) = d(\pi)$. It is routine to show that $\rho_\pi$ does not depend on the choice of the basis and that the sector of $\rho_\pi$ does not depend on the choice of $\mathcal{H}_\pi$. Note that $\mathcal{H}_\pi$ is characterized by

$$\mathcal{H}_\pi = \{V \in M;\ Vx = \rho_\pi(x)\, V, x \in N\}.$$

Let $e_N$ be the Jones projection for $E$. Then using Peter-Weyl theorem we can show

$$\sum_{\pi\in\hat{G}}\sum_{i=1}^{d(\pi)} d(\pi)\, V(\pi)_i^*\, e_N\, V(\pi)_i = 1.$$

This means that we can identify $\varXi$ in Theorem (4.1.14) with $\hat{G}$, and when $\xi \in \varXi$ and $\pi \in \hat{G}$ are identified we can identify $\mathcal{H}_\xi$ with $\mathcal{H}_\pi$ as well. Note that $a_\pi = 1$ because

$$\begin{aligned}\left(V(\pi)_i, V(\pi)_j\right) &= \int_G \alpha_g\left(V(\pi)_i V(\pi)_j^*\right)dg, \qquad x \in M.\\ &= d(\pi)\sum_{k,l}\left(\int_G \pi(g)_{k,i}\,\overline{\pi(g)_{l,J}}\,dg\right)V(\pi)_k\, V(\pi)_l^*\\ &= \delta_{i,j}\sum_k V(\pi)_k\, V(\pi)_k^* = \delta_{i,j}1.\end{aligned}$$

**Lemma(4.1.23)[219]:** Let $G$ be a compact group and $Rep(G)$ the category of finite dimensional unitary representations of $G$. For $\pi \in Rep(G), H_\pi$ denotes the representation space of $\pi$. Suppose we have a Hilbert subspace $K_\pi \subset H_\pi$ for each $\pi \in Rep(G)$ satisfying the following:

$$K_\pi \oplus K_\sigma \subset K_{\pi\oplus\sigma}, \qquad \pi,\sigma \in Rep(G),$$
$$K_\pi \otimes K_\sigma \subset K_{\pi\otimes\sigma}, \qquad \pi,\sigma \in Rep(G),$$
$$\overline{K_\pi} = K_{\bar{\pi}}, \qquad \pi \in Rep(G),$$

where $\bar{\pi}$ is the complex conjugate representation and $\overline{K_\pi}$ is the image of $K_\pi$ under the natural map from $H_\pi$ to its complex conjugate Hilbert space. Then there exists a closed subgroup $H \subset G$ such that

$$K_\pi = \{\xi \in H_\pi\ ;\ \pi(h)\xi = \xi, h \in H\}.$$

**Proof:** Let $B_0$ be the linear span of

$$\{\langle\pi(\cdot)\xi|\eta\rangle \in C(G);\ \xi \in K_\pi, \eta \in H_\pi, \pi \in Rep(G)\},$$

where $C(G)$ is the $C^*$-algebra of the continuous functions on $G$. Then by assumption, $B_0$ is a unital *-subalgebra of $C(G)$ that is globally invariant under the left translation by $G$. Let $B$ be the norm closure of $B_0$. Then thanks to [43], there exists a closed subgroup $H \subset G$ such that $B = C(G/H)$. This implies that $K_\pi$ is a subspace of the set of $H$ invariant vectors $L_\pi$. Suppose $\xi \in L_\pi \ominus K_\pi$ and set $f_\eta(g) = \langle\pi(g)\xi|\eta\rangle$ for $\eta \in H_\pi, g \in G$. Then $f_\eta \in C(G/H)$. On the other hand, the Peter-Weyl theorem shows that $f_\eta$ is perpendicular to $C(G/H)$ in $L^2(G)$ because $B_0$ is dense in $C(G/H)$ in uniform norm and consequently in $L^2(G)$. Thus $f_\eta = 0$ for all $\eta \in H_\pi$ and $\xi = 0$. This proves the statement.

**Theorem (4.1.24):** Let $G$ be a compact group and $\alpha$ a minimal action of $G$ on $M$. Then the map $H \mapsto M^H$ gives one-to-one correspondence between the lattice of all closed subgroups of $G$ and that of all intermediate subfactors of $M \supset M^G$.

To prove the theorem, which is essentially contained in [261].

**Proof:** We may assume that $M^G$ is infinite because after getting the result for $M\otimes B(\ell^2(N))$ we can remove $B(\ell^2(N))$. It easily follows from the existence of $\{\mathcal{H}_\pi\}_{\pi\in\hat{G}}$ that the map is infective. Let $L$ be an intermediate subfactor and set $\mathcal{K}_\pi = L \cap \mathcal{H}_\pi$. We arrange the orthonormal basis $\{V(\pi)_i\}_{i=1}^{d(\pi)}$ such that $\{V(\pi)_i\}_{i=1}^{m_n}$ is an orthonormal basis of $\mathcal{K}_\pi$. Thanks to Lemma (4.1.18) and Theorem (4.1.19), $L$ is characterized by

$$L = \left\{x \in M;\ E(\mathcal{K}\frac{1}{n}x) = 0, \pi \in \hat{G}\right\} = \{x \in M;\ x(\pi)_i = 0, i > m_\pi, \pi \in \hat{G}\}.$$

Thus it is enough to show that there exists a closed subgroup $H \subset G$ such that

$$\mathcal{K}_\pi = [V \in \mathcal{H}_\pi;\alpha_h h(V) = V, h \in H].$$

Indeed, since $\{\mathcal{K}_\pi\}_{\pi\in\hat{G}}$ satisfies the assumption of Lemma (4.1.18), it is routine to show that one can extend the assignment $\pi \mapsto \mathcal{K}_\pi$ to the whole category of representations such that the assumption of Lemma (4.1.23) is fulfilled. Thus Lemma (4.1.23) captures the desired closed subgroup $H$.

We generalize Theorem (4.1.22) and Theorem (4.1.24) to the case minimal actions of compact Kac algebras. It turns out that the Galois correspondence holds between the lattice of intermediate subfactors and that of left coideal von Neumann subalgebras. We also prove a bicommutant type theorem between the left coideal von Neumann subalgebras of a compact Kac algebra and right coideal von Neumann subalgebras of its dual Hopf algebras.

Let $\mathcal{A}$ be a compact Kac algebra [229], [221] with coproduct $\delta$, antipode $k$, and normalized Haar measure $h$, which is a normal trace state. We regard $\mathcal{A}$ as a concrete von Neumann algebra represented on the G.N.S. Hilbert space $L^2(\mathcal{A})$ of $h$ with the G.N.S. cyclic vector $\Omega_h$ . The multiplicative unitary associated with $\mathcal{A}$ is defined by

$$V(x\Omega_h \otimes \xi) = \delta(x)\Omega_h \otimes \xi, \qquad \xi \in L^2(\mathcal{A}), \quad x \in \mathcal{A}. \tag{5}$$

Following [221], we adopt the dual Hopf algebra [221] rather than the dual Kac algebra [229] as the dual object of $\mathcal{A}$; the dual Hopf algebra $\hat{\mathcal{A}}$ of $\mathcal{A}$ is the von Neumann algebra generated by

$$\{(id\otimes\omega)(V);\ \omega \in B(L^2(\mathcal{A}))_*\}$$

with the comultiplication and the antipode,

$$\hat{\delta}\,(y) = V^*(1\otimes y)V, \qquad \hat{\kappa}\,(y) = J_{\mathcal{A}}\, y^* J_{\mathcal{A}}, \qquad y \in \hat{\mathcal{A}}, \tag{6}$$

where $J_{\mathcal{A}}$ is the canonical conjugation of $\mathcal{A}$ with respect to $\Omega_h$ . Let $U \in B(L^2(\mathcal{A}))$ be the unitary operator defined by

$$Ux\Omega_h = \kappa(x)\Omega_h \quad ,x \in \mathcal{A}$$

and set

$$\hat{V} = F(U\otimes 1)\,V(U\otimes 1)\,F \in \mathcal{A}\otimes\hat{\hat{\mathcal{A}}}, \tag{7}$$

$$\hat{V} = F(1\otimes U)\,V(1\otimes U)\,F \in \acute{\mathcal{A}}\otimes\hat{\mathcal{A}}, \tag{8}$$

as in [221] where $F$ is the flip operator of $L^2(\mathcal{A})\otimes L^2\,(\mathcal{A})$. $\hat{V}$ *and* $\tilde{V}$ are multiplicative unitaries satisfying

$$\hat{V}^*\,(\xi\otimes x\Omega_h) = \delta(x)(\xi\otimes\Omega_h), \qquad \xi \in L^2(\mathcal{A})\,, \qquad x \in \mathcal{A}, \tag{9}$$

$$\tilde{V}\,(y\otimes 1)\tilde{V}^* = \hat{\delta}(y), \qquad y \in \hat{\mathcal{A}}.$$

A finite dimensional unitary corepresentation $\pi$ is a pair of a finite dimensional Hilbert space $H_\pi$ and a linear map $\Gamma_\pi:\ H_\pi \to H_\pi\otimes\mathcal{A}$ satisfying

$$(\Gamma_\pi\otimes id)\cdot\Gamma_\pi = (id\otimes\delta)\cdot\Gamma_\pi$$

and the following unitarity condition: If $\{e(\pi)_i\}$ is an orthonormal basis of $H_\pi$ and

$$\Gamma_\pi(e(\pi)_j) = \sum_i e(\pi)_i \otimes u(\pi)_{i,j}\,,$$

then $u(\pi) = (u(\pi)_{i,j})$ is unitary as an element in $M(d(\pi), C)\otimes\mathcal{A}$ , where $d(\pi)$ is the dimension of $H_\pi$ . We abuse the notation and call $u(\pi)$ a unitary corepresentation as well. Basic notions such as tensor product, direct sum, complex conjugate corepresentations, and irreducibility are defined by a standard procedure. Note that since $\mathcal{A}$ is a Kac algebra the complex conjugate corepresentation $u(\bar{\pi}) = (u(\bar{\pi}\,)_{i,j} = u(\pi)^*_{i,j})$ of $u(\pi)$ is always unitary [176]. Let $\pi,\sigma$ be unitary corepresentations of $\mathcal{A}$ . Then the following orthogonality relation holds:

$$h\left(u(\pi)_{i,j}^{*}\, u(\sigma)_{\kappa,l}\right) = \frac{1}{d(\pi)}\delta_{i,\kappa}\delta_{j,l}\delta_{\pi,\sigma}.$$

Let $\Xi$ be a complete system of representatives of the irreducible corepresentations of $\mathcal{A}$. Then the linear span of $\{u(\pi)_{i,j}\}_{1\le i,j\le d(\pi),\pi\in\Xi}$ is a dense in $\mathcal{A}$ in weak topology. For $x\in\mathcal{A}$ we define $x(\pi)_{i,j}$ by

$$x(\pi)_{i,j} = d(\pi)\, h(u(\pi)_{i,j}^{*}\, x).$$

$\{x(\pi)_{i,j}\}$ determines $x$ in the sense that $x=\sum x(\pi)_{i,j}\, u(\pi)_{i,j}$ holds in Hilbert space topology in $L^2(\mathcal{A})$.

**Definition (4.1.25)[219]:** A unital von Neumann subalgebra $\mathcal{B}$ of a $K$ac algebra $\mathcal{A}$ is called a left (right) coideal von Neumann subalgebra if and only if $\delta(\mathcal{B})\subset\mathcal{A}\otimes\mathcal{B}$ (respectively$\delta(\mathcal{B})\subset\mathcal{B}\otimes\mathcal{A}$) holds.

Let $Corep\,(\mathcal{A})$ be the category of finite dimensional unitary corepresentations of $\mathcal{A}$.

**Proposition (4.1.26)[219]:** Let $\mathcal{A}$ be a compact $K$ac algebra. Then there exists one-to-one correspondence between the following two sets.

**(i)** The sets of left coideal von Neumann subalgebras of $\mathcal{A}$.

**(ii)** The set of systems of Hilbert subspaces $K_\pi\subset H_\pi, \pi\in Corep\,(\mathcal{A})$ satisfying the following:

$$K_\pi\oplus K_\sigma\subset K_{\pi\oplus\sigma}\,,\qquad \pi,\sigma\in Corep(\mathcal{A}).$$
$$K_\pi\otimes K_\sigma\subset K_{\pi\otimes\sigma}\,,\qquad \pi,\sigma\in Corep(\mathcal{A}).$$
$$\overline{K_\pi}=K_{\overline{\pi}}\,,\qquad \pi\in Corep(\mathcal{A}).$$

The correspondence is given as follows. Let $\{K_\pi\}$ be a system of subspaces satisfying the condition in (ii) and $\{e(\pi)_i\}_{i=1}^{d(\pi)}$ an orthonormal basis of $H_\pi$ such that $\{e(\pi)_i\}_{i=1}^{m_\pi}$ is an orthonormal basis of $K_\pi$ . Then the corresponding left coideal von Neumann subalgebra $\mathcal{B}$ is the weak closure of the linear span of $\{u(\pi)_{i,j}\}\ 1\le i\le d(\pi), 1\le j\le m_\pi\,, \pi\in Corep(\mathcal{A})$

**Proof:** First we note that two distinct von Neumann subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ give rise to distinct Hilbert subspaces $\overline{\mathcal{B}_1\Omega_h}\,,\overline{\mathcal{B}_2\Omega_h}$ because $h$ is a faithful normal trace. It is easy to show that the weakly closed linear sub-space $\mathcal{B}$ defined in the statement is actually a left coideal von Neumann subalgebra, so it suffices to prove that every left coideal von Neumann subalgebra $\mathcal{B}$ arises in this way. $\{e(\pi)_i\}_{i=1}^{d(\pi)}$ be an orthonormal basis of $H_\pi$ and we set

$$K_\pi = span\left\{\sum_{j=1}^{d(\pi)} x(\pi)_{i,j}e(\pi)_j\,; x\in\mathcal{B}\,, 1\le i\le d(\pi)\right\}.$$

Since $K_\pi$ does not depend on the choice of the basis, we may and do assume that $\{e(\pi)_i\}_{i=1}^{m_\pi}$
is an orthonormal basis. Thus $x(\pi)_{i,j}=0$ for $x\in\mathcal{B}\,, j>m_\pi$.We show that $u(\pi)_{i,j}\in\mathcal{B}$ for $1\le i\le d(\pi), 1\le j\le m_\pi$ . By the definition of $K_\pi$ , for $j$ with $1\le j\le m_\pi$ there exist $x^1, x^2,\dots, x^{d(\pi)}\in\mathcal{B}$ such that $\sum_{i=1}^{d(\pi)} x^i\,(\pi)_{i,\kappa}=\delta_{j,k}$ . Using unitarity of $u(\pi)$ and $\delta\left(u(\pi)_{p,q}\right)=\sum_r u(\pi)_{p,r}\otimes u(\pi)_{r,q}$, we get

$$u(\pi)^*_{i,\kappa}\otimes 1=\sum_p\big(1\otimes u(\pi)_{\kappa,p}\big)\,\delta\big(u(\pi)^*_{i,p}\big).$$

Since $\mathcal{B}$ is a left coideal we obtain

$$\begin{aligned}\mathcal{B}\ni\sum_i(h\otimes id)\,((u(\pi)^*_{i,\kappa}\otimes 1)\delta(x^i))\\ =\sum_{i,p}(h\otimes id)\Big(\big(1\otimes u(\pi)_{\kappa,p}\big)\delta\big(u(\pi)^*_{i,p}x^i\big)\Big)\\ =\sum_{i,p}x^i(\pi)_{i,p}\,u(\pi)_{\kappa,p}=u(\pi)_{\kappa,j}.\end{aligned}$$

Thus $\mathcal{B}$ is characterized as

$$\mathcal{B}=\{x\in\mathcal{A};\ x(\pi)_{i,j}=0,\pi\in\ \Xi\,,j>m_\pi\}.$$

Since $\mathcal{B}$ is a *-subalgebra, the natural extension of $\{K_\pi\}_{\pi\in\Xi}$ to the whole category of unitary corepresentations satisfies the three conditions of (ii).

**Definition (4.1.27)[219]:** Let $\Gamma\colon M\to M\otimes\mathcal{A}$ be an action of a compact $K$ac algebra $\mathcal{A}$ on a factor $M$.

**(i)** $\Gamma$ is called minimal if and only if the linear span of $\{(\omega\otimes id)\cdot\Gamma(M);\ \omega\in M_*\}$ is dense in A and the relative commutant of the fixed point algebra $M^\Gamma=\{x\in M;\ \Gamma(x)=x\otimes 1\}$ in $M$ is trivial.

**(ii)** Let $\mathcal{B}$ be a left coideal von Neumann subalgebra of $\mathcal{A}$. The intermediate subalgebra $M(\mathcal{B})$ of $M^\Gamma\subset M$ associated to $\mathcal{B}$ is defined by

$$M(\mathcal{B})=\{x\in M;\ \Gamma(x)\in M\otimes\mathcal{B}\}.$$

**Theorem (4.1.28)[219]:** Let $\Gamma\colon M\to M\otimes\mathcal{A}$ be a minimal action of a compact $K$ac algebra $\mathcal{A}$ on a factor $M$. Then the map $\mathcal{B}\to M(\mathcal{B})$ gives one-to-one correspondence between the lattice of left coideal von Neumann subalgebras of $\mathcal{A}$ and that of the intermediate subfactors of $M^\Gamma\subset M$.

**Proof:** For the same reason as in the proof of Theorem (4.1.24) we may assume that $M^\Gamma$ is infinite. Note that there exists a normal conditional expectation $E\in\mathcal{E}\,(M,M^\Gamma)$ given by

$$E(x)\otimes 1=(id\otimes h)\cdot\ \delta(x),\qquad x\in M.$$

In exactly the same way as in the case of compact group actions, for each $\pi\in\Xi$ one can find a Hilbert space $\mathcal{H}_\pi$ in $M$ with support 1 and its basis $\{V(\pi)\}_{i=1}^{d(\pi)}$ satisfying

$$\delta(V(\pi)_i)=\sum_j V(\pi)_j\otimes u(\pi)_{j,i}.$$

Thus, as before, we can identify our $\Xi$ with that in Theorem (4.1.14) and we get $a_\pi=1$ thanks to the orthogonality relation. Let $L$ be an intermediate subfactor and $\mathcal{K}_\pi=L\cap\mathcal{H}_\pi$. Thanks to Lemma (4.1.18) and Theorem (4.1.19), $L$ generated by $M^\Gamma$ and $\{\mathcal{K}_\pi\}_{\pi\in\Gamma}$, and is characterized by

$$L=\Big\{x\in M;\ E(\mathcal{K}\tfrac{1}{n}x)=0,\pi\in\Xi\Big\}.$$

Therefore as we can conclude $L=M(\mathcal{B})$ by using Proposition (4.1.26), where $\mathcal{B}$ is the left coideal von Neumann subalgebra corresponding to $\{\mathcal{H}_\pi\}_\pi$. The map is injective because two distinct systems of subspaces satisfying the assumption of Proposition (4.1.26) (ii) give rise to two distinct intermediate subfactors.

**Remark (4.1.29)[219]:** The crossed product $M\times\Gamma^{\hat{\mathcal{A}}}$ is the von Neumann algebra generated by $\Gamma(M)$ *and* $C\otimes\hat{\mathcal{A}}$. As is expected, we can identify the basic extension $M_1$

with $M \times \Gamma^{\hat{\mathcal{A}}}$ if the action is minimal as follows. Let $e_0$ be the projection in $\hat{\mathcal{A}}$ corresponding to the trivial corepresentation of $\mathcal{A}$ and we set $e = 1 \otimes e_0$ . Since we have the dual operator valued weight of the crossed product whose restriction to $\hat{\mathcal{A}}$ is a semifinite trace (Plancherel weight), if $MeM$ is dense in $M \times \Gamma^{\hat{\mathcal{A}}}$ we can apply Lemma (4.1.3) and get the result. Indeed, it is known [221] that $\hat{\mathcal{A}}$ is a direct sum of type $I_{d(\pi)}$ factors $\hat{\mathcal{A}}_\pi$ , $\pi \in \Xi$ and the multiplicative unitary $V$ can be expanded as

$$V = \sum_{\pi \in \Xi} \sum_{1 \le i,j \le d(\pi)} e(\pi)_{i,j} \otimes u(\pi)_{i,j} \,,$$

where $\{e(\pi)_{i,j}\}$ are matrix units of $\hat{\mathcal{A}}_\pi$ . Thanks to (5), we have

$$e(\pi)_{i,j}\, u(\sigma)_{\kappa,l}\Omega_h = \delta_{\pi,\sigma}\delta_{j,l}u(\pi)_{\kappa,i}\, \Omega_h \,.$$

Now, we show

$$d(\pi)\ \delta(V(\pi)_i^*)\ e\delta(V(\pi)_j) = 1 \otimes \hat{\kappa}\ (e(\pi)_{j,i}).$$

From $\delta(V(\pi)_i) = \sum_k V\,(\pi)_k \otimes u(\pi)_{k,i}$ . We get

$$\delta(V(\pi)_i^*)e\delta\big(V(\pi)_j\big) = \sum_k : 1 \otimes u(\pi)_{k,i}^*\, e_0 u(\pi)_{k,j}.$$

Thanks to the orthogonality relation, we obtain

$$d(\pi) \sum_k u(\pi)_{k,i}^*\, e_0 u(\pi)_{k,j} u(\sigma)_{p,q}^* \Omega_h = \delta_{\pi,\sigma}\delta_{j,q} u(\pi)_{p,i}^*,$$

where we use the fact that $e_0$ is the projection onto the space spanned by $\Omega_h$ . On the other hand,

$$\begin{aligned} \hat{\kappa}\big(e(\pi)_{j,i}\big)u(\sigma)_{p,q}^*\Omega_h &= J_{\mathcal{A}} e(\pi)_{i,j} J_{\mathcal{A}} u(\sigma)_{p,q}^* \Omega_h = J_{\mathcal{A}} e(\pi)_{i,j} u(\sigma)_{p,q} \Omega_h \\ &= \delta_{\pi,\sigma}\delta_{j,q}\, J_{\mathcal{A}} u(\pi)_{p,i}\Omega_h = \delta_{\pi,\sigma}\delta_{j,q} u(\pi)_{p,i}^*\Omega_h. \end{aligned}$$

Thus $\delta(M)\ e\delta(M)$ is dense in $M \times \Gamma^{\hat{\mathcal{A}}}$.

The above theorem suggests that it is worth while to study the structure of the lattice of the left coideal von Neumann subalgebras of $K$ac algebras. For compact and discrete $K$ac algebras we have the following:

**Theorem (4.1.30)[219]:** Let $\mathcal{A}$ be a compact $K$ac algebra and $\hat{\mathcal{A}}$ its dual Hopf algebra represented on $L^2(\mathcal{A})$. Let $\mathcal{B} \subset \mathcal{A}$ be a left coideal von Neumann subalgebra and $\mathcal{C} \in \hat{\mathcal{A}}$ a right coideal von Neumann subalgebra. Then the following hold:

**(i)** $\acute{\mathcal{B}} \cap \hat{\mathcal{A}}$ is a right coideal von Neumann subalgebra of $\hat{\mathcal{A}}$ and $(\acute{\mathcal{B}} \cap \hat{\mathcal{A}}\,)' \cap \mathcal{A} = \mathcal{B}$.

**(ii)** $\acute{\mathcal{C}} \cap \mathcal{A}$ is a left coideal von Neumann subalgebra of $\mathcal{A}$ and $(\acute{\mathcal{C}} \cap \mathcal{A}\,)' \cap \hat{\mathcal{A}} = \mathcal{C}$.

**(iii)** Set $\tilde{\mathcal{B}} = \hat{\kappa}\ (\acute{\mathcal{B}} \cap \hat{\mathcal{A}}\ )$. Then the map given by $\mathcal{B} \mapsto \tilde{\mathcal{B}}$ is a lattice anti-isomorphism between the set of left coideal von Neumann subalgebras of $\mathcal{A}$ and that of $\hat{\mathcal{A}}$.

**Proof:** (i) Let $E_{\mathcal{B}}$ be the h preserving conditional expectation in $\mathcal{E}(\mathcal{A},\mathcal{B})$ and $e_{\mathcal{B}}$ its Jones projection, i.e., $e_{\mathcal{B}}$ is the projection defined by $e_{\mathcal{B}} x\Omega_h = E_{\mathcal{B}}(x)\Omega_h, x \in \mathcal{A}$ . Note that $e_{\mathcal{B}} \in \hat{\mathcal{A}}$ and $\{e_{\mathcal{B}}\}' \cap \mathcal{A} = \mathcal{B}$ hold. Thus to prove $(\acute{\mathcal{B}} \cap \hat{\mathcal{A}}\,)' \cap \mathcal{A} = \mathcal{B}$ it suffices to show $e_{\mathcal{B}} \in \acute{\mathcal{B}} \cap \hat{\mathcal{A}}$ . First, we prove $(id \otimes E_{\mathcal{B}}\,) \cdot \delta = \delta \cdot E_{\mathcal{B}}$. Let $\{\mathcal{K}_\pi\}_{\pi\in\Xi}$ be the system of Hilbert subspaces corresponding to $\mathcal{B}$ and $\{e(\pi)_i\}_{i=1}^{d(\pi)}$ an orthonormal basis of $\mathcal{H}_\pi$ such that $\{e(\pi)_i\}_{i=1}^{m_n} = 1$ is a basis of $\mathcal{K}_\pi$ . As we saw in the proof of Proposition (4.1.26), the linear span of $\{u(\pi)_{i,j}\}$, $\pi \in \Xi\,, 1 \le i \le d(\pi), 1 \le j \le m_n$ is dense in $\mathcal{B}$ in weak topology. Let $x \in \mathcal{A}, 1 \le j \le m_n$ . Then we get

$$
\begin{aligned}
(id\otimes h)\left(\left(1\otimes u(\pi)_{i,j}\right)\delta(x)\right) &= \sum_{\kappa=1}^{d(\pi)} u(\pi)_{k,i}^{*}\ (id\otimes h)\left(\delta\left(u(\pi)_{k,j}x\right)\right) \\
&= \sum_{\kappa=1}^{d(\pi)} u(\pi)_{k,i}^{*}\ h(u(\pi)_{k,j}x) \\
&= \sum_{\kappa=1}^{d(\pi)} u(\pi)_{k,i}^{*}\ h(u(\pi)_{k,j}E_{\mathcal{B}}(x)) \\
&= (id\otimes h)\left(\left(1\otimes u(\pi)_{i,j}\right)\delta\left(E_{\mathcal{B}}(x)\right)\right),
\end{aligned}
$$

which implies $(id\otimes E_{\mathcal{B}}) \cdot \delta(x) = \delta \cdot E_{\mathcal{B}}(x)$. Let $\hat{V}$ be the multiplicative unitary defined in (7) for $\xi \in L^2(\mathcal{A})$ and $x \in \mathcal{A}$ we get

$$
\begin{aligned}
\hat{V}^*(1\otimes e_{\mathcal{B}})(\xi\otimes x\Omega_h) &= \hat{V}^*(\xi\otimes E_{\mathcal{B}}(x)\Omega_h) = \delta\left(E_{\mathcal{B}}(x)\right)(\xi\otimes\Omega_h) \\
&= (id\otimes E_{\mathcal{B}}) \cdot \delta(x)\ (\xi\otimes\Omega_h) = (1\otimes e_{\mathcal{B}})\ \hat{V}^*(\xi\otimes x\Omega_h),
\end{aligned}
$$

and so $(1\otimes e_{\mathcal{B}})$ commutes with $\hat{V}$. Since $\left\{\omega\otimes id)(\hat{V}); \omega \in B(L^2(\mathcal{A}))_*\right\}$ is dense in $\hat{\hat{\mathcal{A}}}\ \delta, e_{\mathcal{B}} \in \hat{\mathcal{A}}$. Let $x \in \mathcal{B}, y \in \acute{\mathcal{B}} \cap \hat{\mathcal{A}}$. Then

$$
\begin{aligned}
\hat{\delta}\,(y)(x\otimes 1) &= V^*(1\otimes y)\,V(x\otimes 1) = V^*(1\otimes y)\,\delta(x)\,V = V^*\delta(x)(1\otimes y)\,V \\
&= (x\otimes 1)\,V^*(1\otimes y)\,V = (x\otimes 1)\,\hat{\delta}(y).
\end{aligned}
$$

Thus $\acute{\mathcal{B}} \cap \hat{\mathcal{A}}$ is a right coideal von Neumann subalgebra of $\hat{\mathcal{A}}$.

**(ii)** As above, in a similar way one can show that $\mathcal{C} \cap \mathcal{A}$ is a left coideal von Neumann subalgebra of $\mathcal{A}$. Let $\mathcal{C}_0 = (\acute{\mathcal{C}} \cap \mathcal{A})' \cap \hat{\mathcal{A}}$. Then it is easy to show $\acute{\mathcal{C}}_0 \cap \mathcal{A} = \acute{\mathcal{C}} \cap \mathcal{A}$. Thus to prove $\mathcal{C}_0 = \mathcal{C}$, it suffices to prove that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct right coideal von Neumann subalgebras of $\hat{\mathcal{A}}$, then $\acute{\mathcal{C}}_1 \cap \mathcal{A}$ and $\acute{\mathcal{C}}_2 \cap \mathcal{A}$ are distinct. Since the Plancherel weight of $\hat{\mathcal{A}}$ is
the restriction of the usual trace on $B(L^2(\mathcal{A}))$, there exists a trace preserving conditional expectation $F \in \mathcal{E}(B(L^2(\mathcal{A})), \hat{\mathcal{A}})$. Note that one can identify $F$ with the dual weight of the crossed product of $\hat{\mathcal{A}}$ and $\hat{\hat{\mathcal{A}}} = \acute{\mathcal{A}}$ when $\hat{\delta}$ is regarded as an action of $\hat{\mathcal{A}}$ on itself. Thus the restriction of $F$ to $\acute{\mathcal{A}}$ is a trace. We claim that $F((\acute{\mathcal{C}} \cap \mathcal{A})') = \mathcal{C}$ for every right coideal von Neumann subalgebra $\subset \hat{\mathcal{A}}$. To prove the claim it is enough to show that $\mathcal{C} \cdot \acute{\mathcal{A}}$ is weakly dense in $(\mathcal{C} \cup \acute{\mathcal{A}})''$ because of $(\mathcal{C} \cap \mathcal{A})' = (\mathcal{C} \cup \acute{\mathcal{A}})''$. Let $\tilde{V}$ be as in (8). Thanks to (8) and (9), for $c \in \mathcal{C}$ and $\omega \in B(L^2(\mathcal{A}))_*$ we get

$$
(id \otimes \omega)(\tilde{V})\,c = (id \otimes \omega)(\tilde{V}\,(c \otimes 1)) = (id \otimes \omega)(\hat{\delta}\,(c)\,\tilde{V}) \in \overline{\mathcal{C} \cdot \acute{\mathcal{A}}^w},
$$

which shows $\overline{\mathcal{C} \cdot \acute{\mathcal{A}}^w} = (\mathcal{C} \cup \acute{\mathcal{A}})''$. Using the claim, now we can show that if $\mathcal{C}_1 \neq \mathcal{C}_2$ are right coideal von Neumann subalgebras of $\hat{\mathcal{A}}, (\acute{\mathcal{C}}_1 \cap \mathcal{A})' \neq (\acute{\mathcal{C}}_2 \cap \mathcal{A})'$, and so $(\acute{\mathcal{C}} \cap \mathcal{A})' \cap \hat{\mathcal{A}} = \mathcal{A}$.

**(iii)** This is a direct consequence of (i) and (ii).

In what follows, we assume $n := \dim \mathcal{A} < \infty$. Let $\varepsilon$ and $\hat{\varepsilon}$ be the counit of $\mathcal{A}$ and $\hat{\mathcal{A}}$, and e and $\hat{e}$ the integrals of $\mathcal{A}$ and $\hat{\mathcal{A}}; e$ and $\hat{e}$ are the minimal central projections satisfying $ex = e\varepsilon(x), x \in \mathcal{A}$, and $\hat{e}\,y = \hat{e}\hat{\varepsilon}(y), y \in \hat{\mathcal{A}}$. It is known that the G.N.S. cyclic vector $\Omega_{\hat{h}}$ of the normalized Haar measure $\hat{h}$ of $\hat{\mathcal{A}}$ can be identified with $\sqrt{n}\, e\Omega_h$

and we have $\sqrt{n}\,\hat{e}\Omega_{\hat{h}} = \Omega_h$ as well [245]. The dual pairing between $\mathcal{A}$ and $\hat{\mathcal{A}}$ can be written in terms of the Hilbert space inner product as follows:

$$\langle x, y\rangle = \sqrt{n}\,\langle x\Omega_h \mid y^*\Omega_{\hat{h}}\rangle, \qquad x \in \mathcal{A} \quad y \in \hat{\mathcal{A}}\,. \tag{10}$$

The following is a space-free description of the anti-isomorphism of the two lattices.

**Proposition (4.1.31)[219]:** Let $\mathcal{A}$ be a finite dimensional $K$ac algebra and $\mathcal{B}$ a left coideal van Neumann subalgebra of $\mathcal{A}$. We set

$$\tilde{B} = \{y \in \hat{\mathcal{A}}\,;\ \langle xb, y\rangle = \varepsilon(b)\,\langle x, y\rangle, x \in \mathcal{A}, b\ \in \mathcal{B}\}.$$

Then the following hold:

**(i)** $\tilde{B}$ is $a$ left coideal von Neumann subalgebra of $\hat{\mathcal{A}}$ with $dim\,\mathcal{B}\cdot\ dim\,\tilde{B} = dim\,\mathcal{A}$.

**(ii)** $\tilde{\tilde{B}}\ = \mathcal{B}$.

**(iii)** $\tilde{B} = \hat{\kappa}\,(\acute{\mathcal{B}} \cap \hat{\mathcal{A}}\,)$.

**Proof:** (i) It is routine to show that $\tilde{B}$ is a left coideal von Neumann subalgebra of $\hat{\mathcal{A}}$ . Using (10) for $x \in \mathcal{A}, b \in \mathcal{B}$, and $y \in \hat{\mathcal{A}}$ we get the following:

$$\langle xb, y\rangle = \sqrt{n}\,\langle xb\Omega_h \mid y^*\Omega_{\hat{h}}\rangle = \sqrt{n}\,\langle J_{\mathcal{A}}b^*x^*\Omega_h \mid y^*\Omega_{\hat{h}}\rangle$$
$$= \sqrt{n}\,\langle\, bJ_{\mathcal{A}}\,y^*\Omega_{\hat{h}} \mid x^*\Omega_h\,\rangle = \sqrt{n}\,\langle\, b\hat{\kappa}\,(\,y)\,\Omega_{\hat{h}} \mid x^*\Omega_h\,\rangle.$$

On the other hand we have

$$\varepsilon(b)\langle x, y\rangle = \varepsilon(b)\sqrt{n}\,\langle x\Omega_h|y^*\Omega_{\hat{h}}\rangle = \varepsilon(b)\sqrt{n}\,\langle\hat{\kappa}\,(\,y)\Omega_{\hat{h}} \mid x^*\Omega_h\rangle,$$

and so $\tilde{B}$ is characterized as

$$\tilde{B} = \{y \in \hat{\mathcal{A}}\,;\ b\hat{\kappa}\,(\,y)\Omega_{\hat{h}} = \varepsilon(b)\,\hat{\kappa}\,(\,y)\,\Omega_{\hat{h}}, b \in \mathcal{B}\,\}.$$

Let $E_{\mathcal{B}}$ and $E_{\tilde{\mathcal{B}}}$ be the $h$ and $\hat{h}$ preserving conditional expectations onto $\mathcal{B}$ and $\tilde{\mathcal{B}}$ respectively, and $e_{\mathcal{B}}$ and $e_{\tilde{\mathcal{B}}}$ the corresponding Jones projections. The above characterization shows

$$\tilde{B} \supset \hat{\kappa}(\acute{\mathcal{B}} \cap \hat{\mathcal{A}}) \ni J_{\mathcal{A}}\,e_{\mathcal{B}}\,J_{\mathcal{A}} =\ e_{\mathcal{B}}.$$

More specifically we show $e_{\mathcal{B}} = n\varepsilon\cdot E_{\mathcal{B}}(e)E_{\tilde{\mathcal{B}}}\,(\hat{e}\,)$. Indeed, using $\hat{\kappa}\,(e_{\mathcal{B}}) = J_{\mathcal{A}}\,e_{\mathcal{B}}\,J_{\mathcal{A}} = e_{\mathcal{B}}$ we get the following for $\tilde{b} \in \tilde{B}$ :

$$\hat{h}\left((e_{\mathcal{B}}\tilde{b}\right) = \hat{h}\left(\hat{\kappa}\left(\tilde{b}\,\right)e_{\mathcal{B}}\right) = \langle\hat{\kappa}\,(\tilde{b}\,)\,e_{\mathcal{B}}\,\Omega_{\hat{h}} \mid \Omega_{\hat{h}}\,\rangle$$
$$= \sqrt{n}\,\langle\, e_{\mathcal{B}}e\Omega_h\,|\hat{\kappa}\,(\tilde{b}^*)\,\Omega_{\hat{h}}\,\rangle = \sqrt{n}\,\langle E_{\mathcal{B}}(e)\,\Omega_h\,|\hat{\kappa}\,(\tilde{b}^*)\ \Omega_{\hat{h}}\,\rangle$$
$$= \sqrt{n}\,\langle\Omega_h\,|E_{\mathcal{B}}(e)\hat{\kappa}(\tilde{b}^*)\,\Omega_{\hat{h}}\,\rangle =\ \varepsilon\cdot E_{\mathcal{B}}(e)\sqrt{n}\ \langle\Omega_h|\,\hat{\kappa}\,(\tilde{b}^*)\ \Omega_{\hat{h}}\,\rangle$$
$$= \varepsilon\cdot E_{\mathcal{B}}(e)\sqrt{n}\ \langle\hat{\kappa}\left(\tilde{b}\,\right)\Omega_h|\,\Omega_{\hat{h}}\,\rangle =\ \varepsilon\cdot E_{\mathcal{B}}(e)\hat{\varepsilon}(\hat{\kappa}(\tilde{b}))\sqrt{n}\langle\Omega_h \mid \Omega_{\hat{h}}\,\rangle$$
$$=\ \varepsilon\cdot E_{\mathcal{B}}(e)\hat{\varepsilon}\left(\hat{\kappa}\left(\tilde{b}\right)\right) = n\hat{h}\,\left(\hat{e}\tilde{b}\right)\varepsilon\cdot E_{\mathcal{B}}(e).$$

Thus we obtain the claim. Note that $\hat{h}$ is the restriction of the normalized trace of $B(L^2(\mathcal{A}))$, and so $\hat{h}(e_{\mathcal{B}}) = dim\ \mathcal{B}/n$. Thus we get

$$\varepsilon\cdot E_{\mathcal{B}}(e) = \frac{\dim\mathcal{B}}{n}\,, \qquad e_{\mathcal{B}} = dim\,\mathcal{B}E_{\tilde{\mathcal{B}}}\,(\hat{e}\,).$$

In the same way we can get

$$\hat{\varepsilon}\cdot\ E_{\tilde{\mathcal{B}}}(\hat{e}\,) = \frac{\dim\tilde{B}}{n}.$$

Since $e_{\mathcal{B}}$ is a projection,

$$e_{\mathcal{B}} = e_{\mathcal{B}}^2 = (dim\,\mathcal{B})^2\ E_{\tilde{\mathcal{B}}}\,(\hat{e}\,E_{\tilde{\mathcal{B}}}(\hat{e}\,)) = (dim\,\mathcal{B})^2\hat{\varepsilon}\cdot E_{\tilde{\mathcal{B}}}(\hat{e}\,)E_{\tilde{\mathcal{B}}}(\hat{e}\,)$$
$$= dim\,\mathcal{B}\hat{\varepsilon}\cdot E_{\tilde{\mathcal{B}}}(\hat{e}\,)e_{\mathcal{B}}.$$

Therefore, $dim\,\mathcal{B}\ dim\,\tilde{B} = n$.

**(ii)** It is easy to show $\mathcal{B} \subset \tilde{\tilde{B}}$. Since $\tilde{B}$ is a left coideal von Neumann subalgebra of $\hat{\mathcal{A}}$ we also have $dim\,\tilde{B}\ dim\,\tilde{\tilde{B}} = n$, and so $\mathcal{B} = \tilde{\tilde{B}}$.

**(iii)** In a similar way as in (i), one can show $dim\,\mathcal{B}\ \cdot dim(\acute{\mathcal{B}} \cap \hat{\mathcal{A}}) = n$. Since we have the inclusion $\tilde{B} \supset \hat{\kappa}\,(\acute{\mathcal{B}} \cap \hat{\mathcal{A}})$ we get the equality.

Related to Theorem (4.1.14) and Remark (4.1.15), we give here an example of an irreducible inclusion of factors $N \subset M$ with a normal conditional expectation $E$ such that $N \subset M$ is discrete, $E^{-1}$ is a semifinite trace on $\acute{N} \cap M_1$ ( so that $B_1 = B_2 = C = \{0\}$) yet $E^{-1} \circ j \neq E^{-1}$ on $\acute{N} \cap M_1$ . In fact, our factors $N, M$ are hyperfinite of type $\mathrm{II}_1$ with $E \in \mathcal{E}(M, N)$ being the unique normal conditional expectation preserving the trace $\tau$ on $M$, and $E^{-1}$ being a semifinite trace on $\acute{N} \cap M_1$ . Thus, while the irreducibility of an inclusion of (type $\mathrm{II}_1$) factors $N \subset M$ with $[M : N] < \infty$ automatically entails its extremality ( thus, the trace-preservingness of $j = J_M \cdot J_M$ on $\acute{N} \cap M_1$); this is no longer the case when $[M : N] = \infty$, even if $N \subset M$ is discrete.

Our construction is based on Powers binary shifts and their properties ([255], [256]).

**Lemma (4.1.32)[219]:** [256]. Let $\sigma$ be a bilateral Powers binary shift acting on $\{u_n\}_{n\in\mathbb{Z}}$ as in [255], such that each half-line bitstream of $\sigma$ is aperiodic. Let $P = vN\{u_n\}_{n\in\mathbb{Z}}$ and $N = vN\{u_n\}_{n\geq 0}$. Then the following hold true:

**(i)** $N$ and $P$ are factors;

**(ii)** $\sigma(N) \subset N$ and $[N : \sigma(N)] < \infty$.

**(iii)** $\sigma^n(\acute{N}) \cap N = C, \forall n \geq 1$.

**(iv)** $\bigcap_{n\geq 1} \sigma^n(N) = C1$.

**(v)** $\bigcup_{n\geq 1} \sigma^{-n}\,(N)$ is a dense *-subalgebra of $P$.

**Proof:** All these are well known properties from [255], [256].

**Proposition (4.1.33)[219]:** Let $P$ be a type $\mathrm{II}_1$ factor with an aperiodic auto-morphism $\sigma \in Aut\,P$ and a subfactor $N \subset P$ such that $P, \sigma, N$ satisfy the conditions (i)-(v) of the previous Lemma.

Let $M = P \rtimes_\sigma \mathbb{Z}$. Then we have:

**(a)** $N' \cap M = C1$.

**(b)** $N \subset M$ is discrete, i.e., $L^2(M_1, Tr)$ is generated by $N - N$ sub-bimodules which have finite dimensions both as left and right N-modules, where $Tr$ is the unique semifinite trace on $M_1 = \langle N, M\rangle$ such that $Tre_N = 1$.

**(c)** $J_M \cdot J_M$ is not $Tr$-preserving on $N' \cap M_1$ , equivalently, there are irreducible $N - N$ sub-bimodules of $L^2(M_1, Tr)$ for which the left dimension over $N$ does not coincide with the right dimension over $N$.

**Proof:** (a) By property (iii) we have $N' \cap \sigma^{-n}(N) = C1$. Thus, if $a \in \acute{N} \cap P$ then $\left\|E_{\sigma^{-n}(N)}(a) - a\right\|_2 \to 0$ (by (ii) and (v)) and $E_{\sigma^{-n}(N)}(a) \in \acute{N} \cap \sigma^{-n}(N) = C$ (by commuting squares). Thus $a \in C1$, showing that $\acute{N} \cap P = C$. Similarly $\sigma^n(\acute{N}) \cap P = C, \forall n \geq 1$.

Assume now that $a = \sum_{n\in\mathbb{Z}} b_n u^n$ satisfies $ax = xa, \forall x \in N$. Thus, if $b_n \neq 0$ for some $n$ then $xb_n = b_n\sigma^n(x), \forall x \in N$. By using $\sigma^n(N) \subset N$, it follows that $xb_nb_n^* = b_n\sigma^n(x)\, b_n^* = b_nb_n^*\, x, \forall x \in N$. Thus $b_nb_n^* \in C1$ so that $b_n$ is a (multiple of a) unitary element $v \in P$ satisfying

$$xv = v\sigma^n(x), \quad \forall x \in N. \tag{11}$$

In particular, we have

$$\sigma^n(x)v = v\sigma^{2n}(x), \quad \forall x \in N \tag{12}$$

Also, by applying $\sigma^n$ to both sides of (11) we get

$$\sigma^n(x)\sigma^n(v) = \sigma^n(v)\sigma^{2n}(x), \qquad \forall x \in N. \tag{13}$$

From (12) and (13) we get:

$$v^*\sigma^n(x)v = \sigma^n(v^*)\sigma^n(x)\sigma^n(v), \qquad \forall x \in N. \tag{14}$$

Thus, $\sigma^n(v) = \alpha v$ for some $\alpha \in C_1$. Let then $m_k \nearrow \infty$ be such that $\alpha^{m_k} \to 1$.Then $\|\alpha^{nm_k}(v) - v\| \to 0$ as $k \to \infty$. But $\bigcap_{m\geq 0} \sigma(N) = C1$ clearly implies $\sigma$ is mixing, i.e., $\lim_{u\to\infty} \tau(\sigma^n(x)\, y) = \tau(x)\,\tau(y)$, $\forall x, y \in P$. A contradiction unless $v \in C_1$, showing that $N' \cap M = C$.

**(b)** Let $K_{n,m} = u^{-1}L^2(N)u^{n+m}, for\ n \geq 0, m \in Z, n \geq -m$. It is trivial to see that $K_{n,m} \nearrow L^2(P)\,u^m, as\ n \nearrow \infty$. Thus $\{K_{n,m}\,|n \geq -m, n \geq 0, m \in \mathbb{Z}\} = L^2(M)$ ,with all $K_{n,m}$, m being $N - N$ bimodules.

Also, since as a left $N$ module $K_{n,m} = L^2(\sigma^{-n}(N))u^m$ is isomorphic to $L^2(\sigma^{-n}(N))$, we have $dim({}_NK_{n,m}) = [N : \sigma(N)]^n < \infty$. Furthermore, as a right $N$-module $K_{n,m} = u^mL^2(\sigma^{-n-m}(N))$ is isomorphic to $L^2(\sigma^{-n-m}(N))$, so that we have $dim(K_{n,mN}) = [N{:}\,\sigma(N)]^{n+m} < \infty$.

This shows that $N \subset M$ is discrete.

**(c)** This part is now clear, since we showed above that there exist sub-bimodules $K \subset L^2(M_1, Tr)$ which are finitely generated both as left and right modules, but with different corresponding dimensions (e.g., just take $K = K_{n,m}$, for some $n \geq m, n \geq 0, m \neq 0$).

**Corollary (4.1.34)[219]:**There exist irreducible discrete inclusions of hyper finite type $\text{II}_1$ factors $N \subset M$ for which $J_M \cdot J_M$ is not trace preserving on $\acute{N} \cap M_1$; Equivalently for which $Tr_{M_1}$ and $Tr_{\acute{N}}$ do not agree on $\acute{N} \cap M_1$, or, further, for which the local indices $[pM_1\, p : Np]$ are not equal to $(\mathrm{T}r\ p)^2$ for all $p \in \acute{N} \cap M_1$.

## Section (4.2): Compact Quantum Group Actions

We present a Galois correspondence for compact quantum group actions. The theory of Galois correspondences for group actions on von Neumann algebras was initiated by M. Nakamura and Z. Takeda [270], [251] and has been studied extensively in various settings by many researchers. The Galois correspondence for a group action $G$ on a von Neumann algebra $M$ refers to a one-to-one correspondence between the subgroups $H$ of $G$ and the intermediate von Neumann subalgebras $M^G \subset M^H \subset M$, where $M^H$ denotes the fixed point algebra by the $H$-action.

In [251], the Galois correspondence was established for a minimal action of a finite group on a $\text{II}_1$ factor. For compact group actions, A. Kishimoto obtained a Galois correspondence between normal closed subgroups and intermediate von Neumann subalgebras that are globally invariant under the compact group, assuming a certain condition on actions which is satisfied by minimal ones [240].

Another kind of Galois correspondence was provided in [270] for crossed product von Neumann algebras in the case of a discrete group $G$ acting freely on a finite factor $M$. Their result again shows a one-to-one correspondence between the lattice of subgroups and that of intermediate subfactors of $M \subset H \rtimes G$. In [273], M. Takesaki studied a generalization of this result for a locally compact abelian group action. Y. Nakagami strengthened the result to the case of general locally compact group actions [250]. In [224], H. Choda investigated the crossed product by free actions of discrete groups on a factor of arbitrary type and obtained the Galois correspondence for intermediate von Neumann subalgebras which are the ranges of normal conditional expectations.

M. Izumi, R. Longo and S. Popa [219] have further developed theory of Galois correspondence to compact group minimal actions and discrete group free actions on factors of arbitrary type. Moreover, they obtained the Galois correspondence for minimal actions of compact $K$ac algebras, which unifies the results for compact groups and discrete groups.

Therefore, it is natural to explore a generalization of the Galois correspondence of [219] to minimal actions of compact quantum groups. In [267], M. Enock focused on this problem in a more general setting, but there is a flaw in his proof. We point out the reason why the proof of [219] does not work in the quantum case. The main step of their proof is to show the existence of a normal conditional expectation from an ambient von Neumann algebra onto any intermediate subfactor. However, this is no longer true in general for compact quantum group actions. Indeed, we consider a minimal action of $SU_q(2)$ $(0 < q < 1)$ on a factor, whose existence has been shown by Y. Ueda [276]. The intermediate subfactor associated with a Podles´ sphere $S^2_{q,\theta}$ [271], [167] is not the range of a normal conditional expectation.

We proceed to study an irreducible inclusion of discrete type in the sense of [219]. Let $N \subset M$ be an irreducible inclusion of discrete type. In [219], assuming a technical condition on the modular automorphism groups, it is shown that an intermediate subfactor $N \subset H \subset M$ is generated by $N$ and Hilbert spaces in $L$ which implement irreducible endomorphisms on $N$. We will strengthen the result by dropping assumptions on the modular automorphism groups.

**Theorem (4.2.1)[266]:** Let $N \subset M$ be an irreducible inclusion of discrete type with the faithful normal conditional expectation $E : M \to N$. Let $N \subset L \subset M$ be an intermediate subfactor. Then the following statements hold.

**(i)** The subfactor $N \subset L$ is discrete.

**(ii)** Suppose $N$ is infinite. Let $\gamma_N^M : M \to N$ and $\gamma_N^L : L \to N$ be the canonical endomorphisms for $N \subset M$ and $N \subset L$, respectively. Then $\left[\gamma_N^M\big|_N\right]$ contains $\left[\gamma_N^L\big|_N\right]$ in Sect $(N)$.

The second statement of the above theorem is equivalent to saying that the bimodule ${}_NL^2(M)_N$ contains ${}_NL^2(L)_N$. This statement might sound trivial as in the case of finite $M$, but indeed we need a little more efforts to prove it (see Remark (4.2.10)).
Next we apply this result to minimal actions of compact quantum groups and prove the following Galois correspondence (Theorem (4.2.18)) which generalizes the correspondence presented in [219].

**Theorem (4.2.2)[266]:** Let $\mathbb{G}$ be a compact quantum group and $M$ a factor. Let $\alpha : M \to M \otimes L^\infty(\mathbb{G})$ be a minimal action. Then there exists a one-to-one correspondence between the lattice of left coideals in $L^\infty(\mathbb{G})$ and that of intermediate subfactors of $\mathrm{M}^\alpha \subset \mathrm{M}$.

We always assume separability of von Neumann algebras. Let $M$ be a von Neumann algebra with predual $M_*$. For a weight $\phi$ on $M$, we set $n_\phi = \{x \in M | \phi(x^*x) < \infty\}$. The GNS representation of $M$ with respect to a faithful normal semifinite weight f is denoted by the pair $\{H_\phi, \Lambda_\phi\}$, where $\Lambda_\phi : n_\phi \to H_\phi$ is the canonical injection to the GNS Hilbert space $H_\phi$. We always regard $M$ as a von Neumann subalgebra in $B(H_\phi)$.

Let $N \subset$ M be an inclusion of von Neumann algebras. We denote by $\mathcal{P}(M,N)$ the set of faithful normal semifinite operator valued weights from $M$ to $N$. For theory of operator valued weights, readers are referred to [231], [232].

For a subset $X \subset M$, we denote by $\bar{X}^w$ , $\overline{\mathrm{co}}^w(X)$ and $\overline{\mathrm{span}}^w(X)$ the weak closure of $X$, the weak closure of the convex hull of $X$ and the weak closure of the linear space spanned by $X$, respectively.

We denote by $\otimes$ the minimal tensor product for $C^*$-algebras and the spatial tensor product for von Neumann algebras.

We say that $\mathcal{H}$ is a Hilbert space in $M$ if $\mathcal{H} \subset M$ is a s-weakly closed subspace and $\eta^*\xi \in \mathbb{C}$ for all $\xi,\eta \in \mathcal{H}$ [67]. The smallest projection $e \in M$ with $e\mathcal{H} = H$ is called the support of $\mathcal{H}$.

We denote by $\mathrm{End}(M)$ and $\mathrm{Sect}(M)$ the set of endomorphisms and sectors on $M$, that is, Sect $(M)$ is the set of equivalence classes of endomorphisms on $M$ by unitary equivalence. For the sector theory, See [237], [246], [247].

We recall the notion of discreteness introduced in [219] for an inclusion of factors and summarize the basic properties.

Let $N \subset M$ be an inclusion of factors with a faithful normal conditional expectation $E_N^M : M \to N$. Fix a faithful state $\omega \in N_*$ . We set $\varphi : \omega\, o E_N^M \in M_*$ . Let $\{H_\varphi, \Lambda_\varphi\}$ be the GNS representation of $M$ associated with the state $\varphi$. We define the Jones projection $e_N \in B(H_\varphi)$ by $e_N\Lambda_\varphi(x) = \Lambda_\varphi(E_N^M(x))$ for $x \in M$ and set the basic extension $M_1 := M \vee \{e_N\}'' \subset B(H_\varphi)$. The dual operator valued weight of $E_N^M$ is denoted by $\hat{E}_M^{M_1}$ which is an element of $\mathcal{P}(M_1, M)$ [241]. By definition, we have $\hat{E}_M^{M_1}(ae_Nb) = ab$ for $a,b \in M$. Define the faithful normal semifinite weight $\varphi_1 =: \varphi \circ \hat{E}_M^{M_1}$ on $M_1$

In [219] the discreteness of an inclusion of factors is introduced as follows.

**Definition (4.2.3)[266]:** An inclusion of factors $N \subset M$ is said to be discrete when there exists a faithful normal conditional expectation $E_N^M : M \to N$. such that its dual operator valued weight $\hat{E}_M^{M_1}$ is semifinite on $\acute{N} \cap M_1$.

Note that the discreteness is equivalent to saying that the $N-N$-bimodule $L^2(M)$ is the direct sum of $N-N$-bimodules of finite index.

Let $N \subset M$ be an irreducible inclusion of discrete type. Then $N' \cap M_1$ can be decomposed into a direct sum of matrix algebras as [219] :

$$\acute{N} \cap M_1 = \bigoplus_{\xi \in \Xi} A_\xi,$$

where $A_\xi$ is a type $\mathrm{I}_{n_\xi}$ factor for some $n_\xi \in \mathbb{N}$.

Consider the case that $N$ is infinite. Let $\gamma_N^M : M \to N$ be the canonical endomorphism of the inclusion $N \subset M$ ([246], [247], [248]). By definition, we have the isomorphism of $M-N$ –bimodules

$${}_M\gamma_N^M L^2(N)_N \cong {}_M L^2(M)_N,$$

where $L^2(M)$ and $L^2(N)$ denote the standard Hilbert space for $M$ and $N$, respectively. Since $\acute{N} \cap M_1 = \mathrm{End}_N L^2(M)_N$ , a minimal projection $e_\xi$ in $A_\xi$ corresponds to an irreducible endomorphism with finite index, $\rho_\xi \in \mathrm{End}(N)$, that is, we have the following isomorphism of $N-N$-bimodules ( [219]):

$${}_N\rho_\xi L^2(N)_N \cong {}_N e_\xi L^2(M)_N.$$

By definition, the sector $\left[\gamma_N^M\Big|_N\right]$ contains the sector $[\rho_\xi]$ with multiplicity $n_\xi$ in Sect $(N)$.

We define the subspace $\mathcal{H}_\xi \subset M$ by

$$H_\xi = \{V \in M \big| Vx = \rho_\xi(x)V \text{ for all } x \in N\}.$$

Then by the inner product $\langle V|W\rangle 1 = W^*V, \mathcal{H}_\xi$ is a Hilbert space in $M$. We note that the support projection of $\mathcal{H}_\xi$ may not be equal to 1. We prepare another inner product defined by $(V,W)1 = d(\xi)E_N^M(VW^*)$ for $V,W \in \mathcal{H}_\xi$, where $d(\xi)$ is the square root of the minimal index of $\rho_\xi(N) \subset N$ [268]. By [219] we have dim $\mathcal{H}_\xi = n_\xi$ and $A_\xi = \mathcal{H}_\xi^*\, e_N \mathcal{H}_\xi$.

Let $N \subset M$ be an irreducible inclusion of discrete type with infinite $N$. Let $N \subset L \subset M$ be an intermediate subfactor. We denote by $E_N^L$ the restriction of $E_N^M$ on L. For $\xi \in \Xi$, we define the Hilbert space $\mathcal{K}_\xi$ in $L$ by

$$\mathcal{K}_\xi := \mathcal{H}_\xi \cap L.$$

We set $\Xi_L := \{\, \xi \in \Xi \big| \mathcal{K}_\xi \neq 0\}$ and $m_\xi :=$ dim $(\mathcal{K}_\xi)$ for $\xi \in \Xi_L$.

For $\xi \in \Xi$, we take a basis $\{V_{\xi_i}\}_{i\in I_\xi}$ in $\mathcal{H}_\xi$ such that $\left(V_{\xi_i}, V_{\xi_j}\right) = d(\xi)_{\delta_{i,j}}$. If $\xi \in \Xi_L$, we may assume that the family $\{V_{\xi_i}\}_{i\in I_\xi}$ contains a basis of $\mathcal{K}_\xi$, which we denote by $\{V_{\xi_i}\}_{i\in I_\xi^L}$ .Then the family $\{V_{\xi_i}^* e_N V_{\xi_i}\}_{i,j\in I_\xi}$ is a system of matrix units of $A_\xi$. We prepare the following projections in $N' \cap M_1$,

$$(z_L)_\xi := \sum_{i\in I_\xi^L} V_{\xi_i}^* e_N V_{\xi_i} \quad \text{for all } \xi \in \Xi_L, \qquad z_L := \sum_{i\in \Xi_L} (z_L)_\xi\,.$$

In the following lemma, we determine the subfactor $L \subset M$ at the GNS Hilbert space level. Our argument is essentially the same as the one given in [219], but the assumption there is different from ours. We present a proof for the completeness of our discussion.

**Lemma (4.1.4)[266]:** With the above settings, $(z_L\mathcal{H}_\varphi = \overline{\Lambda_\varphi\,(L)}$ holds. In particular, one has $z_L \in \acute{L} \,\cap M_1$ and $e_N \leqq z_L$.

**Proof:** First we note that the following holds:

$$z_L H_\varphi = \overline{\text{span}\{\,\Lambda_\varphi(V_{\xi_i}^* N) |\, \xi \in \Xi_L, \imath \in I_\xi^L\,\}}. \tag{15}$$

Indeed, let $x \in M$. Then we have

$$z_L\Lambda_\varphi(x) = \sum_{\xi\in\Xi_L}\sum_{i\in I_\xi^L} V_{\xi_i}^* e_N V_{\xi_i}\,\Lambda_\varphi(x) = \sum_{\xi\in\Xi_L}\sum_{i\in I_\xi^L} (V_{\xi_i}^* E_N^M\left(V_{\xi_i}(x)\right).$$

Hence the left-hand side of (15) is contained in the right-hand one. The converse inclusion follows from $E_N^M\left(V_{\xi_i}V_{\xi\eta_i}^*\right) = \delta_{\xi\eta}\delta_{ij}$ for $\xi\eta \in \Xi$ and $i \in I_\xi\,, j \in I_\eta$.

In particular, this yields $z_L H_\varphi \subset \overline{\Lambda_\varphi(L)}$. We will prove the equality by using the averaging technique presented in the proof of [219], as shown below. To prove it, we may and do assume that $N, L$ and M are factors of type III by tensoring with a type III factor. Assume that there would exist $x \in L$ such that $\Lambda_\varphi(x) \notin z_L H_\varphi$. By the following equality:

$$(1-z_L)\Lambda_\varphi(x) = \sum_{\xi\in\Xi\backslash\Xi_L}\sum_{i\in I_\xi}\Lambda_\varphi(V_{\xi_i}^* E_N^M(V_{\xi_i}x))_+ = \sum_{\xi\in\Xi_L}\sum_{i\in I_\xi\backslash I_\xi^L}\Lambda_\varphi(V_{\xi_i}^* E_N^M(V_{\xi_i}x),$$

the following two cases could occur: (I) there exists $\xi\in\Xi\backslash\Xi_L$ such that $E_N^M(V_{\xi_i}x)\neq 0$for some $i\in I_\xi$ or (II) there exists $\xi\in\Xi_L$ such that $E_N^M(V_{\xi_i}x)\neq 0$for some $i\in I_\xi\backslash I_\xi^L$ . In case (I), we set $I_\xi^L=\emptyset$ and then proceed as with case (II). Assume that case (II) would occur. Take $\xi\in\Xi$ and $i\in I_\xi\backslash I_\xi^L$ such that $E_N^M(V_{\xi_i}x)\neq 0$ . Let $E_\xi : N \to \rho_\xi(N)$ be the faithful normal conditional expectation with respect to $\rho_\xi$. By using the equality $E_N^M(V_{\xi_i}axb) = \rho_\xi(a)\,E_N^M(V_{\xi_i}x)\,b$ for $a,b\in N$, we may assume that $E_\xi(E_N^M(V_{\xi_i}x)) = 1$ since $N$ is of type III.

We take a hyperfinite subfactor $R\subset N$ which is simple in the sense of [248]. Then consider the weakly closed convex set

$$C := \overline{co}^w\{ux\rho_\xi(u^*)|u\in U(R)\}\subset L,$$

where $U(R)$ is the set of all unitaries in R. The hyperfiniteness of $R$ assures that there exists a point $w^*\in C$ such that w satisfies $wx=\rho_\xi(x)wx$ for all $x\in R$ and hence for all $x\in N$ by [219]. This shows $w\in L\cap\mathcal{H}_\xi=\mathcal{K}_\xi$ . Since $i\in I_\xi\backslash I_\xi^L, V_{\xi_i}$ is orthogonal to $\mathcal{K}_\xi$, that is, $E_N^M(V_{\xi_i}w^*) = d(\xi)^{-1}(V_{\xi_i}w)=0$. However $E_\xi\,(E_N^M(V_{\xi_i}C))=\{1\}$, and this is a contradiction. Therefore the cases (I) and (II) never occur, and for any $x\in L$, $(1-z_L)\Lambda_\varphi(x)=0$. This implies that $\overline{\Lambda_\varphi(L)}\subset z_L H_\varphi$.

By the previous lemma, we can describe the corner subalgebra $z_L M_1 z_L$ in $M_1$.

**Lemma (4.2.5)[266]:** One has $z_L M_1 z_L = \overline{Le_N\,L}^w = Lz_L\vee\{e_N\}''$ .

**Proof:** Recall that $e_N M_1 e_N = Ne_N$. For $\xi,\eta\in\Xi$ , $i\in I_\xi^L$ and , $j\in I_\eta^L$ , we have

$$V_{\xi_i}^* e_N V_{\xi_i} M_1 V_{\eta_j}^* e_N V_{\eta_j}\subset V_{\xi_i}^* e_N M_1 e_N V_{\eta_j}\subset V_{\xi_i}^* N e_N V_{\eta_j}\subset Le_N L.$$

This implies that $z_L M_1 z_L\subset\overline{Le_N\,L}^w$. By the previous lemma, $z_L\in\acute{L}\cap M_1$ and $z_L e_N = e_N$ . Since $M_1$ contains $L$ and $e_N$, we have $z_L M_1 z_L\supset Lz_L\vee\{e_N\}''\supset Le_N L$. Hence we have $z_L M_1 z_L = \overline{Le_N\,L}^w = Lz_L\vee\{e_N\}''$.

Next we will show that the two-step inclusion $Nz_L\subset Lz_L\subset z_L M_1 z_L$ is identified with the basic extension of $N\subset L$. One might be able to prove this by using the abstract characterization of the basic extension [219], Lemma (4.2.6). To apply that result, we need to show that the restriction $\hat{E}_M^{M_1}$ on $z_L M_1 z_L$ is an operator valued weight from $z_L M_1 z_L$ to $Lz_L\cong L$, but we do not have a proof for such a statement yet. We avoid using this method and directly compare the basic extension of $N\subset L$ with $Nz_L\subset Lz_L\subset z_L M_1 z_L$ instead.

We set $\psi := \omega\circ E_N^L\in L_*$ .Then $\varphi\big|_L = \psi\big|_L$ holds trivially. Let $\{H_\psi,\Lambda_\psi\}$ be the GNS representation of $L$ associated with the state $\psi$. Let $f_N\in B(H_\psi)$ be the Jones projection defined by $f_N\Lambda_\psi(x) = \Lambda_\psi(E_N^L(x))$ for $x\in L$. We set $L_1 := L\vee\{f_N\}''\subset B(H_\psi$. Then we obtain

the Jones' basic extension $N\subset L\subset L_1$ associated with $E_N^L$ .The dual operator valued weight of $E_N^L$ is denoted by $\hat{E}_L^{L_1}$. Note that we do not know whether $\hat{E}_L^{L_1}$ is semifinite on $\acute{N}\cap L_1$ or not. Set a weight $\psi_1 := \psi\circ\hat{E}_L^{L_1}\in\mathcal{P}(L_1,\mathbb{C})$. Let $\{H_{\psi_1},\Lambda_{\psi_1}\}$ be the GNS representation of $L_1$ associated with the weight $\psi_1$. Recall the weight $\varphi_1=\varphi\circ\hat{E}_M^{M_1}$ on

$M_1$. Let $\{H_{\varphi_1}, \Lambda_{\varphi_1}\}$ be the GNS representation of $M_1$ associated with the weight $\varphi_1$. Then the following holds [219]:

$$H_{\psi_1} = \overline{\Lambda_{\psi_1}(Lf_NL)}, \qquad H_{\varphi_1} = \overline{\Lambda_{\varphi_1}(Me_NM)}.$$

We introduce an isometry $U : H_{\psi_1} \to H_{\varphi_1}$ satisfying

$$U\Lambda_{\psi_1}(xf_Ny) = \Lambda_{\varphi_1}(xe_Ny), \qquad for\ x, y \in L.$$

The well-definedness is verified as follows. For $x, y, a, b \in L$, we have

$$\begin{aligned}\langle\Lambda_{\varphi_1}(xe_Ny), \Lambda_{\varphi_1}(ae_Nb)\rangle &= \varphi_1(a^*e_Nb^*xe_Ny) = \varphi_1(b^*E_N^M(a^*x)e_Ny)\\ &= \varphi \circ \hat{E}_M^{M_1}(b^*E_N^M(a^*x)e_Ny) = \varphi(b^*E_N^M(a^*x)y)\\ &= \psi(b^*E_N^L(a^*x)y) = \langle\Lambda_{\psi_1}(xf_Ny), \Lambda_{\psi_1}(af_Nb)\rangle.\end{aligned}$$

**Lemma (4.2.6)[266]:** One has $xU = Ux$ for $x \in L$ and $e_NU = Uf_N$.

**Proof:** Since the subspace $\Lambda_{\psi_1}(Lf_NL) \subset H_{\psi_1}$ is dense, it suffices to show the equalities on $\Lambda_{\psi_1}(Lf_NL)$. Let $x, a, b \in L$. Then we have

$$xU\Lambda_{\psi_1}(af_Nb) = x\Lambda_{\varphi_1}(ae_Nb) = \Lambda_{\varphi_1}(xae_Nb) = U\Lambda_{\psi_1}(xaf_Nb) = Ux\Lambda_{\psi_1}(af_Nb).$$

Hence $xU = Ux$. Next $e_NU = Uf_N$ is verified as follows:

$$\begin{aligned}e_NU\Lambda_{\psi_1}(af_Nb) &= e_N\Lambda_{\varphi_1}(ae_Nb) = \Lambda_{\varphi_1}(e_Nae_Nb)\\ &= \Lambda_{\varphi_1}(E_N^M(a)e_Nb) = \Lambda_{\varphi_1}(E_N^L(a)e_Nb)\\ &= U\Lambda_{\psi_1}(E_N^L(a)f_Nb) = Uf_N\Lambda_{\psi_1}(af_Nb).\end{aligned}$$

Set the range projection $p_L := UU^* \in B(H_{\varphi_1})$. It is clear that $p_LH_{\varphi_1} = \overline{\Lambda_{\varphi_1}(Le_NL)}$. By the previous lemma or the definition of $p_L$, $p_L$ commutes with $L$ and $e_N$. In particular, $p_L \in (Lz_L)' \cap \{e_N\}' \subset B(H_{\varphi_1})$.

The subspace $z_LH_{\varphi_1}$ plays a similar role to the GNS Hilbert space of $z_LM_1z_L$ associated with the restricted weight $\varphi_1\big|_{z_LM_1z_L}$,but $p_LH_{\varphi_1}$ may not coincide with the closure of $\Lambda_{\varphi_1}(n_{\varphi_1} \cap z_LM_1z_L)$ because the function $t \in \mathbb{R} \mapsto \sigma_t^{E_N^M \circ \hat{E}_N^{M_1}}(z_L) \in N' \cap M_1$ may not extend
to the bounded analytic function on the strip $\{z \in \mathbb{C} | 0 \leqq \mathrm{Im}\,(z) \leqq 1/2\}$. However, the following lemma is sufficient for our purpose.

**Lemma (4.2.7)[266]:** *In* $B(H_{\varphi_1}), p_L \leqq z_L$ holds.

**Proof:** Using $z_L \in L' \cap M_1$ and $z_Le_N = e_N$, we have

$$z_L\, p_LH_{\varphi_1} = z_L\, \overline{\Lambda_{\varphi_1}(Le_NL)} = \overline{\Lambda_{\varphi_1}(Lz_Le_NL)} = \overline{\Lambda_{\varphi_1}(Le_NL)} = p_LH_{\varphi_1}.$$

Hence $p_L \leqq z_L$.

**Lemma (4.2.8)[266]:** There exists an isomorphism $\Psi_L : z_LM_1z_L \to L_1$ such that:

**(i)** $\Psi_L(xz_L) = x$ for $x \in L$.

**(i)** $\Psi_L(e_N) = f_N$.

In particular, the inclusions $Nz_L \subset Lz_L \subset z_LM_1z_L$ and $N \subset L \subset L_1$ are isomorphic.

**Proof:** We define the normal positive map $\Psi_L: z_LM_1z_L \to B(H_{\psi_1})$ by

$$\Psi_L(x) = U^*xU \qquad for\ x \in z_LM_1z_L.$$

Since $p_L$ commutes with $Lz_L \vee \{e_N\}'' = z_L M_1 z_L$ as is remarked after Lemma (4.2.6), we see that $\Psi_L$ is multiplicative. By the previous lemma, we have

$$\Psi_L(z_L) = U^* z_L U = U^* z_L p_L U = U^* p_L U = 1,$$

that is, $\Psi_L$ is unital. Hence $\Psi_L$ is a unital _-homomorphism. By Lemma (4.2.6), the range of $\Psi_L$ is equal to $U^*(Lz_L \vee \{e_N\}'')U = L \vee \{f_N\}'' = L_1$. Also we have $\Psi_L(x\, z_L) = x$ for $x \in L$ and $\Psi_L(e_N) = f_N$. Since $z_L M_1 z_L$ is a factor, $\Psi_L$ is an isomorphism onto $L_1$.

**Theorem (4.2.9)[266]:** Let $N \subset M$ be an irreducible inclusion of discrete type. Let $N \subset L \subset M$ be an intermediate subfactor. Then one has the following:

**(i)** The inclusion $N \subset M$ is discrete.

**(ii)** Suppose that $N$ is infinite. Let $\gamma_N^M$ and $\gamma_N^L$ be the canonical endomorphisms for $N \subset M$ and $N \subset L$, respectively. Then $\left[\gamma_N^M\big|_N\right]$contains $\left[\gamma_N^L\big|_N\right]$ in Sect $(N)$.

**(iii)** Suppose that $N$ is infinite and $\left[\gamma_N^L\big|_N\right]$ has in Sect $(N)$ the following decomposition into irreducible sectors $[\rho_\xi], \xi \in \Xi_L$ :

$$\left[\gamma_N^L\big|_N\right] = \bigoplus_{\xi\in\Xi_L} m_\xi\, [\rho_\xi]$$

Set $\mathcal{K}_\xi = \{V \in L \,|\, Vx = \rho_\xi(x)\, V \text{ for all } x \in N\}$ for $\xi \in \Xi_L$ . Then one has $m_\xi = \dim(\mathcal{K}_\xi)$ and $L$ is weakly spanned by $\mathcal{K}_\xi^*\, N, \xi \in \Xi$.

**Proof: (i)** We may and do assume that $N$ is infinite by tensoring with an infinite factor if necessary. For $\xi \in \Xi_L$, we define the matrix algebra $B_\xi \subset N' \cap M_1$ by $B_\xi\, \mathcal{K}_\xi^* e_N \mathcal{K}_\xi$ .Then it is easy to see that $z_L A_\xi z_L = (z_L)_\xi A_\xi (z_L)_\xi = B_\xi$. Hence we have

$$z_L (N' \cap M_1)_{z_L} = \bigoplus_{\xi\in\Xi_L} B_\xi.$$

Let $\Psi_L : z_L M_1 z_L \to L_1$ be the isomorphism constructed in the previous lemma. Using the equalities

$$\Psi_L(z_L(N' \cap M_1)_{z_L}) = \Psi_L((Nz_L)' \cap z_L M_1 z_L) = N' \cap L_1 \text{ and } \Psi_L(B_\xi) = \mathcal{K}_\xi^* f_N \mathcal{K}_\xi,$$

We have

$$N' \cap L_1 = \bigoplus_{\xi\in\Xi_L} \mathcal{K}_\xi^* f_N \mathcal{K}_\xi.$$

Since $\hat{E}_L^{L_1}$ is finite on each matrix algebra $\mathcal{K}_\xi^* f_N \mathcal{K}_\xi, \hat{E}_L^{L_1}$ is semi-finite on $N' \cap L_1$. Therefore the inclusion $N \subset L$ is discrete.

**(ii)** Take $V \subset \mathcal{K}_\xi$ such that $V^* f_N V$ is a minimal projection in $\mathcal{K}_\xi^* f_N \mathcal{K}_\xi$ .Note that $E_N^L(V^*V) = 1$. The projection $V^* f_N V$ corresponds to an irreducible sector in Sect $(N)$. The sector is actually equal to $[\rho_\xi] \in Sect(N)$ as seen below. Set $W := f_N V \in L_1$. Using $WW^* = f_N E_N^L(V^*V) f_N = f_N$ and $W^*W = V^* f_N V$ , we have

$$f_N \rho_\xi(x) = WW^* \rho_\xi(x) = WV^* f_N \rho_\xi(x) = WV^* \rho_\xi(x) f_N = WxV^* f_N = WxW^*.$$

By [219] the minimal projection $V^* f_N V$ corresponds to $[\rho_\xi]$, and the canonical endomorphism $\gamma_N^L : L \to N$ has the following decomposition in Sect $(N)$:

$$\left[\gamma_N^L\big|_N\right] = \bigoplus_{\xi\in\Xi_L} \dim(\mathcal{K}_\xi)[\rho_\xi].$$

From this, we see that $\left[\gamma_N^L\middle|_N\right]$ is contained in $\left[\gamma_N^M\middle|_N\right]$ because each irreducible is contained in $\left[\gamma_N^M\middle|_N\right]$ and we trivially have dim $(\mathcal{K}_\xi) \leqq \dim(\mathcal{H}_\xi)$.

**(iii)** Apply [219] to the discrete inclusion $N \subset L$.

We apply Theorem (4.2.9) to inclusions of factors coming from minimal actions of compact quantum groups, and we obtain the Galois correspondence (Theorem (4.2.18)).

We briefly explain compact quantum groups and their actions. We adopt the definition of a compact quantum group that is introduced in [177], Definition (4.2.3) as follows:

**Definition (4.2.10)[266]:** A compact quantum group $\mathbb{G}$ is a pair $(C(\mathbb{G}), \delta)$ that satisfies the following conditions:

**(i)** $C(\mathbb{G})$ is a separable unital $C^*$-algebra.

**(ii)** The map $\delta: C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a coproduct, i.e. it is a faithful unital *-homomorphism satisfying the coassociativity condition,

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta.$$

**(iii)** The vector spaces $\delta(C(\mathbb{G}))(\mathbb{C} \otimes C(\mathbb{G}))$ and $\delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes C)$ are dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

Let $h$ be the Haar state on $C(\mathbb{G})$, which satisfies the invariance condition:

$$(\mathrm{id} \otimes h)\big(\delta(a)\big) = h(a)1 = (h \otimes \mathrm{id})\big(\delta(a)\big) \quad \text{for all } a \in C(\mathbb{G}).$$

We always assume that the Haar state is faithful. If the Haar state is tracial, we say that the compact quantum group is of $K$ac type [229].

Let $\{L^2(\mathbb{G}), \Lambda_h\}$ be the GNS representation of $C(\mathbb{G})$ associated with $h$. We define the von Neumann algebra $L^\infty(\mathbb{G}) = \overline{C(\mathbb{G})^w} \subset B(L^2(\mathbb{G}))$. We can extend the coproduct $\delta$ to $L^\infty(\mathbb{G})$ by the standard procedure [269]. The extended coproduct is also denoted by $\delta$. Then the pair $(L^\infty(\mathbb{G}), \delta)$ is a von Neumann algebraic compact quantum group in the sense of [269].

Let $H$ be a Hilbert space. We say that a unitary $v \in L^\infty(\mathbb{G}) \otimes B(H)$ is a (left) unitary representation on $H$ when it satisfies $(\delta \otimes \mathrm{id})(v) = v_{13}\, v_{23}$. The unitary representation v is said to be irreducible if $\{T \in B(H) | (1 \otimes T)v = v(1 \otimes T) = \mathbb{C}\}$. The set of the equivalence classes of all irreducible unitary representations is denoted by $\mathrm{Irr}(\mathbb{G})$. For $\pi \in \mathrm{Irr}(\mathbb{G})$, take a representative $v_\pi \in L^\infty(\mathbb{G}) \otimes B(K_\pi)$. Then it is well-known that $K_\pi$ is finite dimensional. Set $d_\pi := \dim(K_\pi)$. We denote by $L^\infty(\mathbb{G})_\pi$ the subspace of $L^\infty(\mathbb{G})$ that is spanned by the entries of $v_\pi$. Then the subspace $A(\mathbb{G}) = span\{L^\infty(\mathbb{G})_\pi | \pi \in \mathrm{Irr}(\mathbb{G})\}$ is a weakly dense unital *-subalgebra of $L^\infty(\mathbb{G})$.

We also use the modular objects of $L^\infty(\mathbb{G})$. Let $\{f_z\}_{z \in \mathbb{G}}$ be the Woronowicz characters on $A(\mathbb{G})$. For its characterization, readers are referred to [177]. Then the modular automorphism group $\{\sigma_t^h\}_{t \in \mathbb{R}}$ on $A(\mathbb{G})$ is given by

$$\sigma_t^h(x) = (f_{it} \otimes \mathrm{id} \otimes f_{it})\left((\delta \otimes \mathrm{id})(\delta(x))\right) \qquad \text{for all } t \in \mathbb{R}, x \in A(\mathbb{G}).$$

We define the following map $\tau_t : A(\mathbb{G}) \to A(\mathbb{G})$ by

$$\tau_t(x) = (f_{it} \otimes \mathrm{id} \otimes f_{-it})\left((\delta \otimes \mathrm{id})(\delta(x))\right) \qquad \text{for all } t \in \mathbb{R}, x \in A(\mathbb{G}).$$

Then $\{\tau_t\}_{t \in \mathbb{R}}$ is a one-parameter automorphism group on $A(\mathbb{G})$ and it is called the scaling automorphism group. Since the Haar state h is invariant under the *-preserving maps $\sigma_t^h$ and $\tau_t$, we can extend them to the maps on $C(\mathbb{G})$, and on $L^\infty(\mathbb{G})$. By definition, we have

$$\sigma_t^h(x) = (f_{2it} \otimes \tau_{-t})\left(\delta(x)\right) \quad \text{for all } x \in A(\mathbb{G}). \tag{16}$$

We recall the notion of a left coideal von Neumann algebra introduced in [219] . We simply call it a left coideal.

**Definition (4.2.11)[266]:** Let $B \in L^\infty(\mathbb{G})$ be a von Neumann subalgebra. We say that $B$ is a left coideal if $\delta(B) \subset L^\infty(\mathbb{G}) \otimes B$.

If $G$ comes from an ordinary compact group, it is known that any left coideal is of the form $L^\infty(\mathbb{G})/\mathbb{H}$ for a closed subgroup $\mathbb{H} \subset \mathbb{G}$ [43]. Therefore a left coideal can be considered as an object like a ''closed subgroup'' of $\mathbb{G}$. Even in quantum case, we can also introduce the notion of a closed quantum subgroup $\mathbb{H} \subset \mathbb{G}$ as [167]. However when G is not a compact group, not all the left coideals of $\mathbb{G}$ have quotient forms as $L^\infty(\mathbb{G})/\mathbb{H}$ ( [271], [167], [274]). For a compact quantum group satisfying a certain condition, we have a necessary and sufficient condition so that a left coideal is of the form $L^\infty(\mathbb{G})/\mathbb{H}$ ( [275].

Now let $B$ be a left coideal, and we put $B_\pi := B \cap L^\infty(\mathbb{G})_\pi$. Since $B$ admits the left $\mathbb{G}$-action $\delta, B$ is weakly spanned by $B_\pi, \pi \in \operatorname{Irr}(\mathbb{G})$ .

**Lemma (4.2.12)[266]:** Let $B \subset L^\infty(\mathbb{G})$ be a left coideal and $\pi \in \operatorname{Irr}(\mathbb{G})$. Then there exist a unitary representation $u_\pi = (u_{\pi_{i,j}})_{i,j \in I_\pi}$ and a subset $I_\pi^B \subset I_\pi$ such that:

**(i)** $u_\pi$ is equivalent to $v_\pi$.

**(ii)** $B_\pi = \operatorname{span}\left\{(u_{\pi_{i,j}} \mid i \in I_\pi, j \in I_\pi^B\right\}$.

**Proof:** Let $\operatorname{Hom}_\mathbb{G}(K_\pi, L^\infty(\mathbb{G}))$ be the set of $\mathbb{G}$ -linear maps from $K_\pi$ into $L^\infty(\mathbb{G})$, that is, it consists of linear maps $S: K_\pi \to L^\infty(\mathbb{G})$ such that $\delta \circ S = (id \otimes S) \circ v_\pi$ , where $v_\pi$ is regarded as a map from $K_\pi$ to $L^\infty(\mathbb{G})_\pi \otimes K_\pi$ . Similarly we define $\operatorname{Hom}_\mathbb{G}(K_\pi, B)$ , which is a subspace of $\operatorname{Hom}_\mathbb{G}\left(K_\pi, L^\infty(\mathbb{G})\right)$ .Let $(\mathcal{E}_i)_{i \in I_\pi}$be an orthonormal basis of $K_\pi$. We prepare the inner product of $\operatorname{Hom}_\mathbb{G}\left(K_\pi, L^\infty(\mathbb{G})\right)$ defined by

$$\langle S|T \rangle 1 := \sum_{i \in I_\pi} T(\varepsilon_i)^* S(\varepsilon_i).$$

Then $\operatorname{Hom}_\mathbb{G}\left(K_\pi, L^\infty(\mathbb{G})\right)$ is a Hilbert space of dimension $d_\pi$. We take an orthonormal basis $\{S_i\}_{i \in I_\pi}$ of $\operatorname{Hom}_\mathbb{G}\left(K_\pi, L^\infty(\mathbb{G})\right)$ which contains an orthonormal basis of $\operatorname{Hom}_\mathbb{G}(K_\pi, B)$denoted by $\{S_i\}_{i \in I_\pi^B}$.

We define the linear map $T_j : K_\pi \to L^\infty(\mathbb{G})$ by $T_j(\varepsilon_i) = v_{\pi_{ij}}$ for $j \in I_\pi$ . Then it is easy to see that $\{T_j\}_{j \in I_\pi}$ is an orthonormal basis of $\operatorname{Hom}_\mathbb{G}\left(K_\pi, L^\infty(\mathbb{G})\right)$ .Hence there exists a unitary matrix $\nu_{\pi_{ij}} := \{\nu_i\}_{i,j \in I_\pi}$ in $B(C^{|I_\pi|})$ such that for $i \in I_\pi$ ,

$$S_i = \sum_{j \in I_\pi} \nu_{\pi_{ji}} T_j.$$

We define the unitary representation $u_\pi := (1 \otimes \nu_\pi^*) v_\pi (1 \otimes \nu_\pi)$. Then we have

$$u_{\pi_{ij}} = \sum_{k,\ell \in I_\pi} (\nu_\pi^*)_{ik} v_{\pi_{\ell j}} = \sum_{k,\ell \in I_\pi} (\nu_\pi^*)_{ik} \nu_{\pi_{\ell j}} T_\ell(\varepsilon_k)$$

$$= \sum_{k \in I_\pi} (\nu_\pi^*)_{ik} S_j(\varepsilon_k) = S_j\left(\sum_{k \in I_\pi} (\nu_\pi^*)_{ik} \varepsilon_k\right),$$

and

$$S_j(\varepsilon_k) = \sum_{i\in I_\pi} v_{\pi_{ki}} u_{\pi_{ki}}.$$

Therefore $u_{\pi_{ij}} \in B$ for all $i \in I_\pi$ and $j \in I_\pi^B$ , and they span $B_\pi$.

Let $M$ be a von Neumann algebra and $\mathbb{G}$ a compact quantum group. Let $\alpha : M \to M\otimes L^\infty(\mathbb{G})$ be a unital faithful normal *-homomorphism. We say that a is an action of $\mathbb{G}$ on $M$ if $(\alpha\otimes\mathrm{id})\circ\alpha = (\mathrm{id}\otimes\delta)\circ\alpha$ . When the subspace $\{(\emptyset\otimes\mathrm{id})(\alpha(M))|\emptyset \in M_*\}$ is weakly dense in $L^\infty(\mathbb{G})$, we say that a has full spectrum. Set $M^\alpha := \{x \in M|\alpha(x) = x\otimes 1\}$. We recall the notion of minimality of an action which is introduced in [219], Definition (4.2.28) (i).

**Definition (4.2.13)[266]:** Let $\alpha : M \to M\otimes L^\infty(\mathbb{G})$ be an action. We say that a is minimal if $\alpha$ has full spectrum and satisfies $(M^\alpha)' \cap M = \mathbb{C}$ .

Let $\alpha$ be a minimal action of $\mathbb{G}$ on $M$. We set $N = M^\alpha$. Then the action a is dual when $N$ is infinite ( [277]) that is, there exists a $\mathbb{G}$ -equivariant embedding of $L^\infty(\mathbb{G})$ into $M$. We can prove this result in the same line as the proof of [279], where minimal actions of compact $K$ac algebras have been considered. In particular, [279], also holds for minimal actions of compact quantum groups. Hence for any $\pi \in \mathrm{Irr}(\mathbb{G})$, there exists a Hilbert space $\mathcal{H}_\pi \subset M$ with support 1 such that $\alpha(\mathcal{H}_\pi) \subset \mathcal{H}_\pi\otimes L^\infty(\mathbb{G})$. If $\{V_{\pi_i}\}_{i\in I_\pi}$ is an orthonormal basis of $\mathcal{H}_\pi$ , there exists $w_{\pi_{ij}} \in L^\infty(\mathbb{G})_\pi$ for $i,j \in I_\pi$ such that $(V_{\pi_i}^* \otimes 1)\,\alpha(V_{\pi_i}) = 1\otimes w_{\pi_{ji}}$ .Then we see that the matrix $(w_{\pi_{i,j}})_{i,j\in I_\pi}$ is a unitary representation equivalent to $\upsilon_\pi$. Hence we may assume that $w_{\pi_{ij}} = \upsilon_{\pi_{ij}}$ by taking the new $(V_{\pi_i})_{i\in I_\pi}$ if necessary, that is, we have

$$\alpha V_{\pi_i} = \sum_{j\in I_\pi} V_{\pi_j} x\otimes\upsilon_{\pi_{ji}}. \tag{17}$$

Let $\rho_{\mathcal{H}_\pi} \in \mathrm{End}\, M$ be the endomorphism implemented by the Hilbert space $\mathcal{H}_\pi$, that is,

$$\rho_{\mathcal{H}_\pi}(x) = \sum_{i\in I_\pi} V_{\pi_i} x V_{\pi_i}^* \quad for\; x \in M.$$

Then it is easy to see that $\rho_{\mathcal{H}_\pi}(N) \subset N,$ , and we denote $\rho_{\mathcal{H}_\pi}|N$ by $\rho_\pi$ ,which is irreducible. Note that $[\rho_\pi|N] \in \mathrm{Sect}(N)$ does not depend on the choice of $\mathcal{H}_\pi$. Let $\pi,\sigma \in \mathrm{Irr}\,(\mathbb{G})$. Then by minimality of $\alpha$, it is shown that $[\rho_\pi] = [\rho_\sigma] \in \mathrm{Sect}(N)$ if and only if $\pi = \sigma$.

Define the conditional expectation $E := (\mathrm{id}\otimes h)\circ\alpha$ a from $M$ onto $N$. Take a faithful state $\omega \in N_*$ and $\varphi := \omega\circ E \in M$. Let $[H_\varphi, \Lambda_\varphi]$ be the GNS representation of $M$ associated with $\varphi$. Let $N \subset M \subset M_1 := M\vee\{e_N\}''$ be the basic extension where the Jones projection $e_N \in M_1$ is defined by $e_N\Lambda_\varphi(x) = \Lambda_\varphi(E(x))$ for $x \in M$.We denote by $\hat{E} \in \mathcal{P}(M_1, M)$ the dual operator valued weight associated with $E$.

Now we assume that $N$ is infinite. Take $\mathcal{H}_\pi$ as before. Set $A_\pi := \mathcal{H}_\pi^*\, e_N\mathcal{H}_\pi$.Then $A_\pi$ is contained in $N' \cap M_1$ , and $\{A_\pi\}_{\pi\in\mathrm{Irr}\,(\mathbb{G})}$ are $d_\pi \times d_\pi$ -matrix algebras, respectively. Moreover on $A_\pi$, the weight $E\circ\hat{E}$ is finite. Let $z_\pi \in A_\pi$ be the unit projection.

**Lemma (4.2.14)[266]:** When Ma is infinite, the following statements hold:

**(i)** $1 = \sum_{\pi\in \mathrm{Irr}\,(\mathbb{G})} z_\pi$.

**(ii)** $N' \cap M_1 = \bigotimes_{\pi\in \mathrm{Irr}\,(\mathbb{G})} A_\pi$.

**(iii)** The inclusion $N \subset M$ is discrete.

**Proof:** (**i**) Take $\{W_{\pi_k}\}_{k=1}^{d_\pi}$ in $\mathcal{H}_\pi$ such that $E\left(W_{\pi_k} W_{\pi_k}^*\right) = \delta_{k,\ell}\, 1$. Then we have

$$z_\pi = \sum_{k=1}^{d_\pi} W_{\pi_k}^* e_N\, W_{\pi_k}.$$

Since $M$ is weakly spanned by $\mathcal{H}_\pi^* N, \pi \in \mathrm{Irr}\,(\mathbb{G})$, we have

$$H_\varphi = \overline{\mathrm{span}\{\Lambda_\varphi(\mathcal{H}_\pi^* N) | \pi \in \mathrm{Irr}\,(\mathbb{G})\}}.$$

For any $x \in N$ and $V_\sigma \in \mathcal{H}_\sigma$ with $\sigma \neq \pi$, we have

$$z_\pi \Lambda_\varphi(V_\pi^* x) = \sum_{k=1}^{d_\pi} W_{\pi_k}^* \Lambda_\varphi\,(E(W_{\pi_k} V_\sigma^*)x) = 0,$$

where we have used $E(\mathcal{H}_\pi \mathcal{H}_\sigma^*) = 0$ in the last equality. If $\sigma = \pi$, we have

$$z_\pi \Lambda_\varphi(V_\pi^* x) = \Lambda_\varphi(V_\pi^* x).$$

Hence the range space of $z_\pi$ coincides with $\Lambda_\varphi(\mathcal{H}_\pi^* N)$, and $\{z_\pi\}_{\pi\in \mathrm{Irr}\,(\mathbb{G})}$ is a partition of unity.

**(ii)** We first show that $z_\pi$ is a central projection in $N' \cap M_1$. For $\pi \in \mathrm{Irr}\,(\mathbb{G})$, take $\{W_{\pi_k}\}_{k=1}^{d_\pi}$ as above. It suffices to prove that $z_\pi(N' \cap M_1) z_\sigma = 0$ if $\pi \neq \sigma$ . Let $x \in N' \cap M_1$ and take $x_0 \in N$ such that $x_0 e_N = e_N W_{\pi_k} x W_{\sigma_\ell}^* e_N$. Then for any $y \in N$, we have

$$\begin{aligned} x_0 \rho_\sigma(y) e_N &= e_N W_{\pi_k} x W_{\sigma_\ell}^* \rho_\sigma(y) e_N = e_N W_{\pi_k} x y W_{\sigma_\ell}^* e_N \\ &= e_N W_{\pi_k} y x W_{\sigma_\ell}^* e_N = e_N \rho_\pi(y) W_{\pi_k} x W_{\sigma_\ell}^* e_N \\ &= \rho_\pi(y) x_0 e_N. \end{aligned}$$

This shows that $x_0$ intertwines $\rho_\sigma$ and $\rho_\pi$. So, we get $x_0 = 0$. Hence we have $e_N W_{\pi_k}(N' \cap M_1) W_{\sigma_\ell}^* e_N = 0$ and $z_\pi(N' \cap M_1) z_\sigma = 0$.

Second we show that each $p_k := W_{\pi_k}^*\, e_N W_{\pi_k}$ is a minimal projection in $N' \cap M_1$. This is because the reduced inclusion $N p_k \subset p_k M_1 p_k$ is isomorphic to the irreducible inclusion $\rho_\pi(N) \subset N$. Hence $N' \cap M_1$ is the direct sum of $A_\pi, \pi \in \mathrm{Irr}\,(\mathbb{G})$.

**(iii)** This is trivial by (ii) because $\hat{E}$ is finite on each $A_\pi$.

By the previous lemma, we can regard $\Xi = \mathrm{Irr}\,(\mathbb{G})$.

**Definition (4.2.15)[266]:** Let $\alpha: M \to M \otimes L^\infty(\mathbb{G})$ be a minimal action.

**(i)** For an intermediate subfactor $M^\alpha \subset L \subset M$, we define the weakly closed subspace $\mathcal{L}(L) \subset L^\infty(\mathbb{G})$ by

$$\mathcal{L}(L) = \overline{\mathrm{span}}^w\{(\omega \otimes \mathrm{id})(\alpha(L)) | \omega \in M_*\}.$$

**(ii)** For a left coideal $B \subset L^\infty(\mathbb{G})$, we define the intermediate subfactor

$$M^\alpha \subset \mathcal{M}(B) \subset M$$

by

$$\mathcal{M}(B) = \{x \in M | \alpha(x) \in M \otimes B\}.$$

We also denote by $\mathcal{L}_\alpha(L), \mathcal{M}_\alpha(B)$ for $\mathcal{L}(L), \mathcal{M}(B)$ when we want to specify the action a.

**Lemma (4.2.16)[266]:** For any intermediate subfactor $M^\alpha \subset L \subset M, \mathcal{L}(L)$ is a left coideal of $G$.

**Proof:** If we consider a minimal action $\beta := \mathrm{id} \otimes \alpha$ on $B(\ell_2) \otimes M$, then $\mathcal{L}_\beta(B(\ell_2) \otimes L) = \mathcal{L}_\alpha(L)$ .Therefore we may assume that Ma is infinite. We have to check that $\delta(\mathcal{L}(L)) \subset L^\infty(\mathbb{G}) \otimes \mathcal{L}(L)$ and $\mathcal{L}(L)$ is multiplicatively closed.

Set $\mathcal{K}_\pi = \mathcal{H}_\pi \cap L$. Then $L$ is weakly spanned by $\mathcal{K}_\pi^* N$ by Theorem (4.2.9). Recall two bases $\{V_{\pi_i}\}_{i \in I_\pi}$ and $\{W_{\pi_i}\}_{i \in I_\pi}$ in $\mathcal{H}_\pi$ as before, that is, we have the equalities $V_{\pi_i}^* V_{\pi_j} = \delta_{ij}$ ,(17) and $E\left(W_{\pi_i} W_{\pi_j}^*\right) = \delta_{ij}$ .We may assume that $\{W_{\pi_i}\}_{i \in I_\pi^L}$ is a basis of $\mathcal{K}_\pi$. There exist $c_{\pi_{ji}} \in \mathbb{C}, i, j \in I_\pi$ such that $V_{\pi_j}^* W_{\pi_i} = c_{\pi_{ji}}$ .Note that the matrix $(c_{\pi_{ji}})_{i,j \in I_\pi}$ is invertible. Since

$$\alpha(W_{\pi_i}) = \sum_{j \in I_\pi} \alpha\left(V_{\pi_j} V_{\pi_j}^* W_{\pi_i}\right) = \sum_{j \in I_\pi} c_{\pi_{ji}} \alpha\left(V_{\pi_j}\right) \tag{18}$$

$$= \sum_{j,k \in I_\pi} c_{\pi_{ji}} V_{\pi_k} \otimes v_{\pi k_j} = \sum_{j \in I_\pi} V_{\pi_k} \otimes \left(\sum_{j \in I_\pi} c_{\pi_{ji}} \otimes v_{\pi k_j}\right).$$

We have

$$\mathcal{L}(L) = \overline{\mathrm{span}}^w \left\{\sum_{j \in I_\pi} c_{\pi_{ji}} v_{\pi k_j} | \, i \in I_\pi^L, j, k \in I_\pi, \pi \in \Xi_L\right\}. \tag{19}$$

This implies that $\delta(\mathcal{L}(L)) \subset L^\infty(\mathbb{G}) \otimes \mathcal{L}(L)$. Let $\pi, \sigma \in (\mathbb{G})\mathrm{Irr}$ . Next we show that $\mathcal{L}(L)$ is multiplicatively closed. It suffices to show that the product of $\sum_{j \in I_\pi} c_{\pi_{rj}} v_{\pi i_{,j}}$ and $\sum_{j \in I_\sigma} c_{\sigma_{j\ell}} v_{\pi s_{,j}}$

Is contained in $\mathcal{L}(L)$ for all $(i, r) \in I_\pi^L \times I_\pi$ and $(\ell, s) \in I_\sigma^L \times I_\sigma$ .Then it is clear because the left-hand side of the following equality is contained in $\mathbb{C} \otimes \mathcal{L}(L)$:

$$\left(V_{\sigma s}^* V_{\pi_r}^* \otimes 1\right) \alpha(W_{\pi_i} W_{\sigma_\ell}) = 1 \otimes \left(\sum_{j \in I_\pi} c_{\pi_{ji}} \otimes v_{\pi r_j}\right)\left(\sum_{j \in I_\sigma} c_{\sigma_{j\ell}} v_{\sigma s_j}\right).$$

We present a Galois correspondence which is a generalization of [219] to minimal actions of compact quantum groups.

**Theorem (4.2.17)[266]:** (Galois correspondence). Let $\mathbb{G}$ be a compact quantum group and $M$ a factor. Let $\alpha: M \to M \otimes L^\infty(\mathbb{G})$ be a minimal action. Then there exists an isomorphism between the lattice of intermediate subfactors of $M^\alpha \subset M$ and the lattice of left coideals of $\mathbb{G}$. More precisely, the maps $\mathcal{M}$ and $\mathcal{L}$ are the mutually inverse maps, that is, for any intermediate subfactor $M^\alpha \subset L \subset M$ and any left coideal $B \subset L^\infty(\mathbb{G})$, one has

$$\mathcal{M}\left(\mathcal{L}(L)\right) = L, \qquad \mathcal{L}\left(\mathcal{M}(\mathrm{B})\right) = B.$$

**Proof:** If we consider a minimal action $\beta := \mathrm{id}\otimes\alpha$ on $B(\ell_2)\otimes M$, then we have $\mathcal{L}_\beta(B(\ell_2)\otimes L) = \mathcal{L}_\alpha(L)$ and $\mathcal{M}_\beta(B) = B(\ell_2)\otimes\mathcal{M}_\alpha(B)$. Hence we may and do assume That $M^\alpha$ is infinite.

By definition, we see that $L \subset \mathcal{M}(\mathcal{L}(L))$. We will show $\mathcal{M}(\mathcal{L}(L)) \subset L$. Set $\mathcal{K}_\pi := \mathcal{H}_\pi \cap L$ and $\widetilde{\mathcal{K}}_\pi := \mathcal{H}_\pi \cap \mathcal{M}(\mathcal{L}(L))$. Then $\mathcal{M}(\mathcal{L}(L))$ is $\sigma$-weakly spanned by $M^\alpha \widetilde{\mathcal{K}}_\pi$ for $\pi \in \mathrm{Irr}(\mathbb{G})$ by Theorem (4.2.9). We choose a basis $\{W_{\pi_i}\}_{i\in I_\pi}$ in $\mathcal{H}_\pi$ such that $E\left(W_{\pi_i}W_{\pi_j}^*\right) = \delta_{i,j}$ as before. We may assume that it contains bases of $\mathcal{K}_\pi$ and $\widetilde{\mathcal{K}}_\pi$, which are denoted by $\{W_{\pi_i}\}_{i\in I_\pi^L}$ and $\{W_{\pi_i}\}_{i\in J_\pi}$, respectively. We use the invertible matrix $\left(c_{\pi_{i,j}}\right)_{i,j\in I_\pi}$ as in the previous lemma.

Let $j \in J_\pi$. Since $W_j \in \mathcal{M}\left(\mathcal{L}(L)\right)$, $\alpha(W_j)$ is contained in $M\otimes\mathcal{L}(L)$, that is, $\sum_{k\in I_\pi} c_{\pi_{kj}} v_{\pi\ell_k} \in \mathcal{L}(L)$ for all $\ell \in I_\pi$ by (18). Recall that the $(v_{\pi_{k\ell}})_{k,\ell\in I_\pi}$ are linearly independent. By (19), there exists $d_{\pi_{ij}} \in \mathbb{C}$ for $i \in I_\pi^L$ such that for any $\ell \in I_\pi$,

$$\sum_{k\in I_\pi} c_{\pi_{kj}} v_{\pi_{\ell k}} = \sum_{i\in I_\pi^L} d_{\pi_{ij}}\left(\sum_{k\in I_\pi} c_{\pi_{ki}} v_{\pi_{\ell k}}\right),$$

That is,

$$c_{\pi_{kj}} = \sum_{i\in I_\pi^L} d_{\pi_{ij}}\, c_{\pi_{ki}} \qquad \text{for all } j \in J_\pi, k \in I_\pi. \tag{20}$$

Note that $d_{\pi_{ij}}$ does not depend on $\ell$. We know the matrix $C := (c_{\pi_{k\ell}})_{k\ell\in I_\pi}$ is invertible. Multiplying $(C^{-1})_{\ell k}(\ell \in I_\pi^L)$ to the both sides of the above equality, summing up with $k$, we have

$$\delta_{\ell j} = d_{\pi_{\ell j}} \qquad \text{for all } \ell \in I_\pi^L.$$

This yields $j \in I_\pi^L$. Indeed, if $j \neq I_\pi^L$, then $d_{\pi_{\ell j}} = 0$ for all $\ell \in I_\pi^L$ .Together with (20), we have $c_{\pi_{kj}} = 0$ for all $k \in I_\pi$. Then we have

$$W_{\pi_j} = \sum_{k\in I_\pi} V_{\pi_k}\left(V_{\pi_k}^* W_{\pi_j}\right) = \sum_{k\in I_\pi} V_{\pi_k} c_{\pi_{kj}} = 0,$$

but this is a contradiction. Therefore $W_{\pi_j} \in L$ for any $j \in I_\pi$, and $\mathcal{M}\left(\mathcal{L}(L)\right) \subset L$.

Next we will show that $\mathcal{L}(\mathcal{M}(B)) = B$. By definition, the inclusion $\mathcal{L}(\mathcal{M}(B)) \subset B$ holds. We prove $B \subset \mathcal{L}(\mathcal{M}(B))$. Since $B$ is $\sigma$-weakly spanned by subspaces

$$B_\pi = B \cap L^\infty(\mathbb{G})_\pi, \qquad \pi \in \mathrm{Irr}(\mathbb{G}),$$

it suffices to show that $B_\pi \subset \mathcal{L}(\mathcal{M}(B))$ for any $\pi \in \mathrm{Irr}(\mathbb{G})$. By Lemma (4.2.13), there exists a unitary matrix $\nu_\pi = (\nu_{\pi_{ij}})_{ij\in I_\pi} \in B\left(\mathbb{C}^{|I_\pi|}\right)$ such that $B_\pi$ is spanned by $u_{\pi_{ij}}, i \in I_\pi$ and $j \in I_\pi^B$ ,where $u_\pi = (1\otimes\nu_\pi^*)v_\pi(1\otimes\nu_\pi)$. For $i \in I_\pi$, we put $V'_{\pi_i} := \sum_{j\in I_\pi} \nu_{\pi_{ji}} V_{\pi_j}$ .Then we have

$$\alpha(V'_{\pi_i}) = \sum_{j\in I_\pi} \nu_{\pi_{ji}}\alpha(V_{\pi_j}) = \sum_{j,k\in I_\pi} \nu_{\pi_{ji}}(V_{\pi_k}\otimes v_{\pi_{kj}})$$

$$= \sum_{k\in I_\pi} (V_{\pi_k} \otimes (v_\pi(1\otimes v_\pi))_{ki}) = \sum_{k\in I_\pi} \left(V_{\pi_k} \otimes ((1\otimes v_\pi)_{u_\pi})_{ki}\right)$$
$$= \sum_{j,k\in I_\pi} \left(v_{\pi_{ki}} V_{\pi_k} \otimes u_{\pi_{ji}}\right) = \sum_{j\in I_\pi} V'_{\pi_j} \otimes u_{\pi_{ji}}.$$

Let $i \in I_\pi^B$ .Then $u_{\pi_{ji}} \in B_\pi$ for all $j \in I_\pi$ , and $V'_{\pi_i} \in \mathcal{M}(B)$ by the above equality. Again by the above equality, $u_{\pi_{ji}} \in \mathcal{L}(\mathcal{M}(B))$ for all $j \in I_\pi$ . This implies $B_\pi \subset \mathcal{L}(\mathcal{M}(B))$.

When $\mathbb{G}$ is of $K$ac type, it has been proved in [219] that there exists a normal conditional expectation from $M$ onto any intermediate subfactor of $M^\alpha \subset M$. However, if $\mathbb{G}$ is not of $K$ac type, then this is not the case in general as we will see below. We can characterize which intermediate subfactor has such a property. We recall the following notion introduced in [275], Definition (4.2.11) (ii).

**Definition (4.2.18)[266]:** Let $B \subset L^\infty(\mathbb{G})$ be a left coideal. We say that $B$ has the expectation property if there exists a faithful normal conditional expectation $E_B$ from $L^\infty(\mathbb{G})$ onto $B$ satisfying $h \circ E_B = h$.

The following lemma is probably well-known for specialists.

**Lemma (4.2.19)[266]:** Let $B \subset L^\infty(\mathbb{G})$ be $a$ left coideal. Then the following statements are equivalent:

**(i)** $B$ has the expectation property.

**(ii)** $\sigma_t^h(B) = B$ *for all* $t \in \mathbb{R}$.

**(iii)** $\tau_t(B) = B$ *for all* $t \in \mathbb{R}$.

**Proof:** (i) ⟹ (ii). This follows from Takesaki's theorem [272], p. 309.

(ii) ) ⟹ (iii). Since $B_\pi \in \mathrm{Irr}(\mathbb{G})$ spans a dense subspace of $B$, it suffices to show that $\tau_t(B_\pi) \subset B_\pi$ for all $t \in \mathbb{R}$ and $\pi \in \mathrm{Irr}(\mathbb{G})$. Recall the equality (16). Then for $x \in B_\pi$, we have

$\tau_t(x) = (f_{2it} \otimes \sigma_{-t}^h)(\delta(x))$. Since $B$ is a left coideal globally invariant under the modular group $\sigma^h$, we see that $\tau_t(x) \in B_\pi$. Hence $\tau_t(B_\pi) \subset B_\pi$ for all $t \in \mathbb{R}$ and $\pi \in \mathrm{Irr}(\mathbb{G})$.

(iii) ⟹ (i). Let $\pi \in \mathrm{Irr}(\mathbb{G})$ and $x \in B_\pi$ . By (16), we have $\sigma_t^h(x) = (f_{2it} \otimes \tau_{-t})(\delta(x))$. Since $B$ is a left coideal globally invariant under the scaling group $\tau$ , we see that $\sigma_t^h(x) \in B_\pi$. Hence $\sigma_t^h(B_\pi) \subset B_\pi$ for all $t \in \mathbb{R}$ and $\pi \in \mathrm{Irr}(\mathbb{G})$, and $B$ is globally invariant under the modular group $\sigma^h$. Again by Takesaki's theorem, there exists a faithful normal conditional expectation $E_B : L^\infty(\mathbb{G}) \to B$ preserving $h$. Hence $B$ has the expectation property.

**Theorem (4.2.20)[266]:** Let $\alpha$ be a minimal action of $\mathbb{G}$ on $a$ factor $M$. Let $M^\alpha \subset L \subset M$ be an intermediate subfactor. Then there exists $a$ faithful normal conditional expectation $E_L^M : M \to L$ if and only if the left coideal $\mathcal{L}(L)$ has the expectation property.

**Proof:** Let $\omega \in N_*$ be a faithful state. Put $\varphi\colon \omega \text{ o } E \in M$. We note that L is the image of a faithful normal conditional expectation of $M$ if and only if $\sigma_t^\varphi(L) \subset L$ for all $t \in \mathbb{R}$ . Indeed, if the former condition holds, there exists a faithful normal conditional expectation $E_L^M : M \to L$. Then the conditional expectation $E_N^L \text{o} E_L^M$ is equal to $E$ because $N \subset M$ is irreducible. Hence $\varphi \text{o} E_L^M = (\varphi \text{o} E_N^L) \text{o} E_L^M = \varphi \text{oE} = \varphi$. Then by Takesaki's theorem [272], p. 309, the latter condition holds. The converse implication also follows from his theorem.

Since $\varphi$ is invariant under the action $\alpha$, we have $\alpha \circ \sigma_t^{\varphi} = (\sigma_t^{\varphi} \otimes \tau_{-t}) \circ\alpha$ for all $t \in \mathbb{R}$ by [267]. Put $B := \mathcal{L}(L)$ . Then we have $L = \{x \in M | \alpha(x) \in M \otimes B\}$ by Theorem (4.2.18) .So, for $x \in L, \sigma_t^{\varphi}(x) \in L$ if and only if $\alpha(x) \in M \otimes \tau_t(B)$. The von Neumann subalgebra $\tau_t(B)$ is also a left coideal by the equality $\delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \delta$ . Hence $\sigma_t^{\varphi}(x) \in L$ if and only if $x \in \mathcal{M}(\tau_t(B))$. Therefore, $\sigma_t^{\varphi}(x) \alpha L$ if and only if $L \subset \mathcal{M}(\tau_t(B))$. Since $L = \mathcal{M}(B)$, this is equivalent with $B \subset \tau_t(B)$. Hence $L$ is the image of a faithful normal conditional expectation of $M$ iff $B = \tau_t(B)$ for all $t \in \mathbb{R}$ .By the previous lemma, this equivalently means that $B$ has the expectation property.

If $\mathbb{G}$ is of $K$ac type, the Haar state h is a faithful trace. Hence any left coideal has the expectation property. Then we have the following result which has been already shown in [219], Theorem (4.2.19).

**Corollary (4.2.21)[266]:** Let $\alpha$ be a minimal action of $\mathbb{G}$ on a factor $M$. Let $M^{\alpha} \subset L \subset M$ be an intermediate subfactor. If $\mathbb{G}$ is of $K$ac type, then there exists a faithful normal conditional expectation from $M$ onto $L$.

**Example (4.2.22)[266]:** We consider the twisted $SU_q$(ii) group, $SU_q$(ii) ( [278] ) and its minimal action a on a full factor $M$ as constructed by Ueda [276]. Then by minimality of $\alpha$, intermediate subfactors bijectively correspond to left coideals. By using Lemma (4.2.20), we can show the quantum spheres $L^{\infty}(S_{q,\theta}^2)$ with $0 < \theta \leq \pi/2$ ( [271], [167]) are left coideals without expectation property. By Theorem (4.2.21), there are no faithful normal conditional expectations from $M$ onto the corresponding subfactors.

**Corollary (4.2.23)[495]:** With the above settings, $(z_n)_{M-\epsilon}\mathcal{H}_{\varphi_\epsilon} = \overline{\Lambda_{\varphi_\epsilon}(M-\epsilon)}$ holds. In particular, one has $(z_n)_{M-\epsilon} \in M' - \epsilon \cap M_1$ and $e_{M-2\epsilon} \leqq (z_n)_{M-\epsilon}$.

**Proof:** First we note that the following holds:

$$(z_n)_{M-\epsilon} H_{\varphi_\epsilon} = \overline{\text{span}\{ \Lambda_{\varphi_\epsilon}(V^*_{(\xi_\epsilon)_\iota} M - 2\epsilon) | \, \xi_\epsilon \in \Xi_{M-\epsilon}, \iota \in I_{\xi_\epsilon}^{M-\epsilon} \}}. \tag{21}$$

Indeed, let $x_n \in M$. Then we have

$$(z_n)_{M-\epsilon}\Lambda_{\varphi_\epsilon}(x_n) = \sum_{\xi_\epsilon \in \Xi_{M-\epsilon}} \sum_{i \in I_{\xi_\epsilon}^{M-\epsilon}} V^*_{(\xi_\epsilon)_i} e_{M-2\epsilon} V_{(\xi_\epsilon)_i} \Lambda_{\varphi_\epsilon}(x_n)$$

$$= \sum_{\xi_\epsilon \in \Xi_{M-\epsilon}} \sum_{i \in I_{\xi_\epsilon}^{M-\epsilon}} (V^*_{(\xi_\epsilon)_i} E^M_{M-2\epsilon} \left( V_{(\xi_\epsilon)_i}(x_n) \right).$$

Hence the left-hand side of (21) is contained in the right-hand one. The converse inclusion follows from $E^M_{M-2\epsilon}\left( V_{(\xi_\epsilon)_i} V^*_{\xi_\epsilon \eta_i} \right) = \delta_{\xi_\epsilon \eta} \delta_{ij}$ for $\xi_\epsilon \eta \in \Xi$ and $i \in I_{\xi_\epsilon}, j \in I_\eta$.

In particular, this yields $(z_n)_{M-\epsilon} \mathrm{H}_{\varphi_\epsilon} \subset \overline{\Lambda_{\varphi_\epsilon}(M-\epsilon)}$. We will prove the equality by using the averaging technique presented in the proof of [219], as shown below. To prove it, we may and do assume that $M-2\epsilon, M-\epsilon$ and M are factors of type III by tensoring with a type III factor. Assume that there would exist $x_n \in M-\epsilon$ such that $\Lambda_{\varphi_\epsilon}(x_n) \notin (z_n)_{M-\epsilon} H_{\varphi_\epsilon}$. By the following equality:

$$(1-(z_n)_{M-\epsilon})\Lambda_{\varphi_\epsilon}(x_n) = \sum_{\xi_\epsilon \in \Xi \setminus \Xi_{M-\epsilon}} \sum_{i \in I_{\xi_\epsilon}} \Lambda_{\varphi_\epsilon}(V^*_{(\xi_\epsilon)_i} E^M_{M-2\epsilon}(V_{(\xi_\epsilon)_i} x_n)) +$$

$$= \sum_{\xi_\epsilon \in \Xi_{M-\epsilon}} \sum_{i \in I_{\xi_\epsilon} \setminus I_{\xi_\epsilon}^{M-\epsilon}} \Lambda_{\varphi_\epsilon}(V^*_{(\xi_\epsilon)_i} E^M_{M-2\epsilon} \left( V_{(\xi_\epsilon)_i} x_n \right),$$

the following two cases could occur: (I) there exists $\xi_\epsilon \in \Xi \backslash \Xi_{M-\epsilon}$ such that $E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} x_n) \neq 0$ for some $i \in I_{\xi_\epsilon}$ or (II) there exists $\xi_\epsilon \in \Xi_{M-\epsilon}$ such that $E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} x_n) \neq 0$for some $i \in I_{\xi_\epsilon} \backslash I_{\xi_\epsilon}^{M-\epsilon}$ . In case (I), we set $I_{\xi_\epsilon}^{M-\epsilon} = \emptyset$ and then proceed as with case (II). Assume that case (II) would occur. Take $\xi_\epsilon \in \Xi$ and $i \in I_{\xi_\epsilon} \backslash I_{\xi_\epsilon}^{M-\epsilon}$ such that $E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} x_n) \neq 0$ . Let $E_{\xi_\epsilon} : M-2\epsilon \rightarrow \rho_{\xi_\epsilon}(M-2\epsilon)$ be the faithful normal conditional expectation with respect to $\rho_{\xi_\epsilon}$. By using the equality $E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} a x_n (a + \epsilon)) = \rho_{\xi_\epsilon}(a) E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} x_n)(a+\epsilon)$ for $a, a+\epsilon \in M-2\epsilon$ , we may assume that $E_{\xi_\epsilon}(E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} x_n)) = 1$ since $M-2\epsilon$ is of type III.

We take a hyperfinite subfactor $R \subset M-2\epsilon$ which is simple in the sense of [248]. Then consider the weakly closed convex set

$$C := \overline{co}^{w_\epsilon}\{u_n x_n \rho_{\xi_\epsilon}(u_n^*) | u_n \in U(R)\} \subset M-\epsilon,$$

where $U(R)$ is the set of all unitaries in $R$. The hyperfiniteness of $R$ assures that there exists a point $w_\epsilon^* \in C$ such that w satisfies $w_\epsilon x_n = \rho_{\xi_\epsilon}(x_n) w_\epsilon x_n$ for all $x_n \in R$ and hence for all $x_n \in M-2\epsilon$ by [219]. This shows $w_\epsilon \in M-\epsilon \cap \mathcal{H}_{\xi_\epsilon} = \mathcal{K}_{\xi_\epsilon}$ . Since $i \in I_{\xi_\epsilon} \backslash I_{\xi_\epsilon}^{M-\epsilon}, V_{(\xi_\epsilon)_i}$ is orthogonal to $\mathcal{K}_{\xi_\epsilon}$, that is, $E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} w_\epsilon^*) = d(\xi_\epsilon)^{-1}(V_{(\xi_\epsilon)_i} w_\epsilon) = 0$. However $E_{\xi_\epsilon}(E_{M-2\epsilon}^{M}(V_{(\xi_\epsilon)_i} C)) = \{1\}$, and this is a contradiction. Therefore the cases (I) and (II) never occur, and for any $x_n \in M-\epsilon$, $(1-(z_n)_{M-\epsilon})\Lambda_{\varphi_\epsilon}(x_n) = 0$. This implies that $\overline{\Lambda_{\varphi_\epsilon}(M-\epsilon)} \subset (z_n)_{M-\epsilon} H_{\varphi_\epsilon}$.

**Corollary (4.2.24)[495]:** One has $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon} = \overline{(M-\epsilon) e_{M-2\epsilon} (M-\epsilon)}^{w_\epsilon} = (M-\epsilon)(z_n)_{M-\epsilon} \vee \{e_{M-2\epsilon}\}''$ .

**Proof:** Recall that $e_{M-2\epsilon} M_1 e_{M-2\epsilon} = M - 2\epsilon e_{M-2\epsilon}$. For $\xi_\epsilon, \eta \in \Xi$ , $i \in I_{\xi_\epsilon}^{M-\epsilon}$ and , $j \in I_\eta^{M-\epsilon}$ , we have

$$V_{(\xi_\epsilon)_i}^* e_{M-2\epsilon} V_{(\xi_\epsilon)_i} M_1 V_{\eta_j}^* e_{M-2\epsilon} V_{\eta_j} \subset V_{(\xi_\epsilon)_i}^* e_{M-2\epsilon} M_1 e_{M-2\epsilon} V_{\eta_j} \subset V_{(\xi_\epsilon)_i}^* M - 2\epsilon e_{M-2\epsilon} V_{\eta_j}$$
$$\subset (M-\epsilon) e_{M-2\epsilon} (M-\epsilon).$$

This implies that $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon} \subset \overline{(M-\epsilon) e_{M-2\epsilon} (M-\epsilon)}^{w_\epsilon}$ . By the previous lemma, $(z_n)_{M-\epsilon} \in M-\epsilon \cap M_1$ and $(z_n)_{M-\epsilon} e_{M-2\epsilon} = e_{M-2\epsilon}$ . Since $M_1$ contains $M-\epsilon$ and $e_{M-2\epsilon}$ , we have $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon} \supset (M-\epsilon)(z_n)_{M-\epsilon} \vee \{e_{M-2\epsilon}\}'' \supset (M-\epsilon) e_{M-2\epsilon} (M-\epsilon)$ . Hence we have $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon} = \overline{(M-\epsilon) e_{M-2\epsilon} (M-\epsilon)}^{w_\epsilon} = (M-\epsilon)(z_n)_{M-\epsilon} \vee \{e_{M-2\epsilon}\}''$.

Next we will show that the two-step inclusion $(M-2\epsilon)(z_n)_{M-\epsilon} \subset (M-\epsilon)(z_n)_{M-\epsilon} \subset (z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}$ is identified with the basic extension of $M-2\epsilon \subset M-\epsilon$. One might be able to prove this by using the abstract characterization of the basic extension [219], Corollary (4.2.26). To apply that result, we need to show that the restriction $\hat{E}_M^{M_1}$ on $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}$ is an operator valued weight from $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}$ to $(M-\epsilon)(z_n)_{M-\epsilon} \cong M-\epsilon$, but we do not have a proof for such a statement yet. We avoid using this method and directly compare the basic extension of $M-2\epsilon \subset M-\epsilon$ with $(M-2\epsilon)(z_n)_{M-\epsilon} \subset (M-\epsilon)(z_n)_{M-\epsilon} \subset (z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}$ instead.

We set $\psi_\epsilon := \omega_\epsilon \circ E_{M-2\epsilon}^{M-\epsilon} \in M_* - \epsilon$ .Then $\varphi_\epsilon|_{M-\epsilon} = \psi_\epsilon|_{M-\epsilon}$ holds trivially. Let $\{H_{\psi_\epsilon}, \Lambda_{\psi_\epsilon}\}$ be the GNS representation of $M-\epsilon$ associated with the state $\psi_\epsilon$. Let $f_{M-2\epsilon} \in B(H_{\psi_\epsilon})$ be the Jones projection defined by $f_{M-2\epsilon} \Lambda_{\psi_\epsilon}(x_n) = \Lambda_{\psi_\epsilon}(E_{M-2\epsilon}^{M-\epsilon}(x_n))$ for $x_n \in M-\epsilon$. We set $M_1 - \epsilon := (M-\epsilon) \vee \{f_{M-2\epsilon}\}'' \subset B(H_{\psi_\epsilon})$. Then we obtain the Jones' basic extension $M-2\epsilon \subset M-\epsilon \subset M_1-\epsilon$ associated with $E_{M-2\epsilon}^{M-\epsilon}$ .The dual operator valued weight of $E_{M-2\epsilon}^{M-\epsilon}$ is denoted by $\hat{E}_{M-\epsilon}^{M_1-\epsilon}$. Note that we do not know whether $\hat{E}_{M-\epsilon}^{M_1-\epsilon}$ is semifinite on $\acute{M}-2\epsilon \cap M_1-\epsilon$ or not. Set a weight $(\psi_\epsilon)_1 := \psi_\epsilon \circ \hat{E}_{M-\epsilon}^{M_1-\epsilon} \in \mathcal{P}(M_1-\epsilon, \mathbb{C})$. Let $\{H_{(\psi_\epsilon)_1}, \Lambda_{(\psi_\epsilon)_1}\}$ be the GNS representation of $M_1-\epsilon$ associated with the weight $(\psi_\epsilon)_1$. Recall the weight $(\varphi_\epsilon)_1 = \varphi_\epsilon \circ \hat{E}_M^{M_1}$ on $M_1$. Let $\{H_{(\varphi_\epsilon)_1}, \Lambda_{(\varphi_\epsilon)_1}\}$ be the GNS representation of $M_1$ associated with the weight $(\varphi_\epsilon)_1$. Then the following holds [219](see [31]):

$$H_{(\psi_\epsilon)_1} = \overline{\Lambda_{(\psi_\epsilon)_1}((M-\epsilon)f_{M-2\epsilon}(M-\epsilon))}, \qquad H_{(\varphi_\epsilon)_1} = \overline{\Lambda_{(\varphi_\epsilon)_1}(Me_{M-2\epsilon}M)}.$$

We introduce an isometry $U : H_{(\psi_\epsilon)_1} \to H_{(\varphi_\epsilon)_1}$ satisfying

$$U\Lambda_{(\psi_\epsilon)_1}(x_n f_{M-2\epsilon} y_n) = \Lambda_{(\varphi_\epsilon)_1}(x_n e_{M-2\epsilon} y_n), \qquad for\ x_n, y_n \in M-\epsilon.$$

The well-definedness is verified as follows. For $x_n, y_n, a, a+\epsilon \in M-\epsilon$, we have

$$\begin{aligned}\langle \Lambda_{(\varphi_\epsilon)_1}(x_n e_{M-2\epsilon} y_n), \Lambda_{(\varphi_\epsilon)_1}(a e_{M-2\epsilon}(a+\epsilon))\rangle &= (\varphi_\epsilon)_1(a^* e_{M-2\epsilon}(a+\epsilon)^* x_n e_{M-2\epsilon} y_n) \\ &= (\varphi_\epsilon)_1((a+\epsilon)^* E_{M-2\epsilon}^M(a^* x_n) e_{M-2\epsilon} y_n) \\ &= \varphi_\epsilon \circ \hat{E}_M^{M_1}((a+\epsilon)^* E_{M-2\epsilon}^M(a^* x_n) e_{M-2\epsilon} y_n) = \varphi_\epsilon((a+\epsilon)^* E_{M-2\epsilon}^M(a^* x_n) y_n) \\ = \psi_\epsilon((a+\epsilon)^* E_{M-2\epsilon}^{M-\epsilon}(a^* x_n) y_n) &= \langle \Lambda_{(\psi_\epsilon)_1}(x_n f_{M-2\epsilon} y_n), \Lambda_{(\psi_\epsilon)_1}(a f_{M-2\epsilon}(a+\epsilon))\rangle.\end{aligned}$$

**Corollary (4.2.25)[495]:** One has $x_n U = U x_n$ for $x_n \in M-\epsilon$ and $e_{M-2\epsilon} U = U f_{M-2\epsilon}$.

**Proof:** Since the subspace $\Lambda_{(\psi_\epsilon)_1}((M-\epsilon)f_{M-2\epsilon}(M-\epsilon)) \subset H_{(\psi_\epsilon)_1}$ is dense, it suffices to show the equalities on $\Lambda_{(\psi_\epsilon)_1}((M-\epsilon)f_{M-2\epsilon}(M-\epsilon))$. Let $x_n, a, a+\epsilon \in M-\epsilon$. Then we have

$$\begin{aligned}x_n U\Lambda_{(\psi_\epsilon)_1}(a f_{M-2\epsilon}(a+\epsilon)) &= x_n \Lambda_{(\varphi_\epsilon)_1}(a e_{M-2\epsilon}(a+\epsilon)) = \Lambda_{(\varphi_\epsilon)_1}(x_n a e_{M-2\epsilon}(a+\epsilon)) \\ &= U\Lambda_{(\psi_\epsilon)_1}(x_n a f_{M-2\epsilon}(a+\epsilon)) = U x_n \Lambda_{(\psi_\epsilon)_1}(a f_{M-2\epsilon}(a+\epsilon)).\end{aligned}$$

Hence $x_n U = U x_n$. Next $e_{M-2\epsilon} U = U f_{M-2\epsilon}$ is verified as follows:

$$\begin{aligned}e_{M-2\epsilon} U\Lambda_{(\psi_\epsilon)_1}(a f_{M-2\epsilon}(a+\epsilon)) &= e_{M-2\epsilon}\Lambda_{(\varphi_\epsilon)_1}(a e_{M-2\epsilon}(a+\epsilon)) \\ &= \Lambda_{(\varphi_\epsilon)_1}(e_{M-2\epsilon} a e_{M-2\epsilon}(a+\epsilon)) = \Lambda_{(\varphi_\epsilon)_1}(E_{M-2\epsilon}^M(a) e_{M-2\epsilon}(a+\epsilon)) \\ &= \Lambda_{(\varphi_\epsilon)_1}(E_{M-2\epsilon}^{M-\epsilon}(a) e_{M-2\epsilon}(a+\epsilon)) \\ = U\Lambda_{(\psi_\epsilon)_1}(E_{M-2\epsilon}^{M-\epsilon}(a) f_{M-2\epsilon}(a+\epsilon)) &= U f_{M-2\epsilon}\Lambda_{(\psi_\epsilon)_1}(a f_{M-2\epsilon}(a+\epsilon)).\end{aligned}$$

Set the range projection $p_{M-\epsilon} := UU^* \in B(H_{(\varphi_\epsilon)_1})$. It is clear that $p_{M-\epsilon} H_{(\varphi_\epsilon)_1} = \overline{\Lambda_{(\varphi_\epsilon)_1}((M-\epsilon)e_{M-2\epsilon}(M-\epsilon))}$. By the previous lemma or the definition of $p_{M-\epsilon}$, $p_{M-\epsilon}$ commutes with $M-\epsilon$ and $e_{M-2\epsilon}$. In particular, $p_{M-\epsilon} \in ((M-\epsilon)(z_n)_{M-\epsilon})' \cap \{e_{M-2\epsilon}\}' \subset B(H_{(\varphi_\epsilon)_1})$ .

The subspace $(z_n)_{M-\epsilon} H_{(\varphi_\epsilon)_1}$ plays a similar role to the GNS Hilbert space of $(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}$ associated with the restricted weight $(\varphi_\epsilon)_1|_{(z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon}}$ ,but $p_{M-\epsilon} H_{(\varphi_\epsilon)_1}$ may not coincide with the closure of $\Lambda_{(\varphi_\epsilon)_1}(n_{(\varphi_\epsilon)_1} \cap (z_n)_{M-\epsilon} M_1 (z_n)_{M-\epsilon})$ because the function $t \in \mathbb{R} \mapsto \sigma_t^{E_{M-2\epsilon}^M \circ \hat{E}_{M-2\epsilon}^{M_1}}((z_n)_{M-\epsilon}) \in (M-2\epsilon)' \cap M_1$ may not extend to the bounded analytic function on the strip $\{z_n \in \mathbb{C} | 0 \leqq \operatorname{Im}(z_n) \leqq 1/2\}$.

**Corollary (4.2.26)[495]:** *In* $B(H_{(\varphi_\epsilon)_1}), p_{M-\epsilon} \leqq (z_n)_{M-\epsilon}$ holds.

**Proof:** Using $(z_n)_{M-\epsilon} \in M' - \epsilon \cap M_1$ and $(z_n)_{M-\epsilon} e_{M-2\epsilon} = e_{M-2\epsilon}$, we have

$$(z_n)_{M-\epsilon}\, p_{M-\epsilon} H_{(\varphi_\epsilon)_1} = (z_n)_{M-\epsilon}\, \overline{\Lambda_{(\varphi_\epsilon)_1}((M-\epsilon)e_{M-2\epsilon}(M-\epsilon))}$$
$$= \overline{\Lambda_{(\varphi_\epsilon)_1}((M-\epsilon)(z_n)_{M-\epsilon}e_{M-2\epsilon}(M-\epsilon))} = \overline{\Lambda_{(\varphi_\epsilon)_1}((M-\epsilon)e_{M-2\epsilon}(M-\epsilon))}$$
$$= p_{M-\epsilon}H_{(\varphi_\epsilon)_1}.$$

Hence $p_{M-\epsilon} \leqq (z_n)_{M-\epsilon}$.

**Corollary (4.2.27)[495]:** There exists an isomorphism $\Psi_{M-\epsilon} : (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon} \to M_1 - \epsilon$ such that:

**(i)** $\Psi_{M-\epsilon}(x_n(z_n)_{M-\epsilon}) = x_n$ for $x_n \in M-\epsilon$.

**(ii)** $\Psi_{M-\epsilon}(e_{M-2\epsilon}) = f_{M-2\epsilon}$.

In particular, the inclusions $(M-2\epsilon)(z_n)_{M-\epsilon} \subset (M-\epsilon)(z_n)_{M-\epsilon} \subset (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon}$ and $M-2\epsilon \subset M-\epsilon \subset M_1-\epsilon$ are isomorphic.

**Proof:** We define the normal positive map $\Psi_{M-\epsilon}$: $(z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon} \to B(H_{(\psi_\epsilon)_1})$ by

$$\Psi_{M-\epsilon}(x_n) = U^*x_nU \quad \text{for } x_n \in (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon}.$$

Since $p_{M-\epsilon}$ commutes with $(M-\epsilon)(z_n)_{M-\epsilon}\vee\{e_{M-2\epsilon}\}'' = (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon}$ as is remarked after Corollary (4.2.26), we see that $\Psi_{M-\epsilon}$ is multiplicative. By the previous lemma, we have

$$\Psi_{M-\epsilon}((z_n)_{M-\epsilon}) = U^*(z_n)_{M-\epsilon}U = U^*(z_n)_{M-\epsilon}p_{M-\epsilon}U = U^*p_{M-\epsilon}U = 1,$$

that is, $\Psi_{M-\epsilon}$ is unital. Hence $\Psi_{M-\epsilon}$ is a unital *-homomorphism. By Corollary (4.2.26), the range of $\Psi_{M-\epsilon}$ is equal to $U^*((M-\epsilon)(z_n)_{M-\epsilon} \vee \{e_{M-2\epsilon}\}'')U = (M-\epsilon) \vee \{f_{M-2\epsilon}\}'' = M_1 - \epsilon$. Also we have $\Psi_{M-\epsilon}(x_n(z_n)_{M-\epsilon}) = x_n$ for $x_n \in M-\epsilon$ and $\Psi_{M-\epsilon}(e_{M-2\epsilon}) = f_{M-2\epsilon}$. Since $(z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon}$ is a factor, $\Psi_{M-\epsilon}$ is an isomorphism onto $M_1-\epsilon$.

**Corollary (4.2.28)[495]:** Let $M-2\epsilon \subset M$ be an irreducible inclusion of discrete type. Let $M-2\epsilon \subset M-\epsilon \subset M$ be an intermediate subfactor. Then one has the following:

**(i)** The inclusion $M-2\epsilon \subset M$ is discrete.

**(ii)** Suppose that $M-2\epsilon$ is infinite. Let $\gamma^M_{M-2\epsilon}$ and $\gamma^{M-\epsilon}_{M-2\epsilon}$ be the canonical endomorphisms for $M-2\epsilon \subset M$ and $M-2\epsilon \subset M-\epsilon$, respectively. Then $[\gamma^M_{M-2\epsilon}|M-2\epsilon]$contains $[\gamma^{M-\epsilon}_{M-2\epsilon}|M-2\epsilon]$ in Sect $(M-2\epsilon)$.

**(iii)** Suppose that $M-2\epsilon$ is infinite and $[\gamma^{M-\epsilon}_{M-2\epsilon}|M-2\epsilon]$ has in Sect $(M-2\epsilon)$ the following decomposition into irreducible sectors $[\rho_{\xi_\epsilon}], \xi_\epsilon \in \Xi_{M-\epsilon}$ :

$$[\gamma^{M-\epsilon}_{M-2\epsilon}|M-2\epsilon] = \bigoplus_{\xi_\epsilon\in\Xi_{M-\epsilon}} m_{\xi_\epsilon}[\rho_{\xi_\epsilon}]$$

Set $\mathcal{K}_{\xi_\epsilon} = \{V \in M-\epsilon | Vx_n = \rho_{\xi_\epsilon}(x_n)\, V \text{ for all } x_n \in M-2\epsilon\}$ for $\xi_\epsilon \in \Xi_{M-\epsilon}$ . Then one has $m_{\xi_\epsilon} = \dim(\mathcal{K}_{\xi_\epsilon})$ and $M-\epsilon$ is weakly spanned by $\mathcal{K}^*_{\xi_\epsilon}(M-2\epsilon), \xi_\epsilon \in \Xi$.

**Proof:** **(i)** We may and do assume that $M-2\epsilon$ is infinite by tensoring with an infinite factor if necessary. For $\xi_\epsilon \in \Xi_{M-\epsilon}$, we define the matrix algebra $B_{\xi_\epsilon} \subset M'-2\epsilon \cap M_1$ by $B_{\xi_\epsilon}\, \mathcal{K}^*_{\xi_\epsilon}e_{M-2\epsilon}\mathcal{K}_{\xi_\epsilon}$ .Then it is easy to see that $(z_n)_{M-\epsilon}A_{\xi_\epsilon}(z_n)_{M-\epsilon} = ((z_n)_{M-\epsilon})_{\xi_\epsilon}A_{\xi_\epsilon}((z_n)_{M-\epsilon})_{\xi_\epsilon} = B_{\xi_\epsilon}$. Hence we have

$$(z_n)_{M-\epsilon}((M-2\epsilon)'\cap M_1)_{(z_n)_{M-\epsilon}} = \bigoplus_{\xi_\epsilon\in\Xi_{M-\epsilon}} B_{\xi_\epsilon}.$$

Let $\Psi_{M-\epsilon} : (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon} \to M_1-\epsilon$ be the isomorphism constructed in the previous lemma. Using the equalities

$$\Psi_{M-\epsilon}((z_n)_{M-\epsilon}((M-2\epsilon)'\cap M_1)_{(z_n)_{M-\epsilon}})$$
$$=\Psi_{M-\epsilon}(((M-2\epsilon)(z_n)_{M-\epsilon})'\cap (z_n)_{M-\epsilon}M_1(z_n)_{M-\epsilon})$$
$$=(M-2\epsilon)'\cap M_1-\epsilon \text{ and } \Psi_{M-\epsilon}(B_{\xi_\epsilon})=\mathcal{K}^*_{\xi_\epsilon}f_{M-2\epsilon}\mathcal{K}_{\xi_\epsilon},$$

We have

$$M'-2\epsilon\cap M_1-\epsilon=\bigoplus_{\xi_\epsilon\in\Xi_{M-\epsilon}}\mathcal{K}^*_{\xi_\epsilon}f_{M-2\epsilon}\mathcal{K}_{\xi_\epsilon}.$$

Since $\hat{E}^{M_1-\epsilon}_{M-\epsilon}$ is finite on each matrix algebra $\mathcal{K}^*_{\xi_\epsilon}f_{M-2\epsilon}\mathcal{K}_{\xi_\epsilon}$, $\hat{E}^{M_1-\epsilon}_{M-\epsilon}$ is semi-finite on $M'-2\epsilon\cap M_1-\epsilon$. Therefore the inclusion $M-2\epsilon\subset M-\epsilon$ is discrete.

**(ii)** Take $V\subset\mathcal{K}_{\xi_\epsilon}$ such that $V^*f_{M-2\epsilon}V$ is a minimal projection in $\mathcal{K}^*_{\xi_\epsilon}f_{M-2\epsilon}\mathcal{K}_{\xi_\epsilon}$ .Note that $E^{M-\epsilon}_{M-2\epsilon}(V^*V)=1$. The projection $V^*f_{M-2\epsilon}V$ corresponds to an irreducible sector in Sect $(M-2\epsilon)$. The sector is actually equal to $[\rho_{\xi_\epsilon}]\in Sect(M-2\epsilon)$ as seen below. Set $W:=f_{M-2\epsilon}V\in M_1-\epsilon$. Using $WW^*=f_{M-2\epsilon}E^{M-\epsilon}_{M-2\epsilon}(V^*V)f_{M-2\epsilon}=f_{M-2\epsilon}$ and $W^*W=V^*f_{M-2\epsilon}V$ , we have

$$f_{M-2\epsilon}\rho_{\xi_\epsilon}(x_n)=WW^*\rho_{\xi_\epsilon}(x_n)=WV^*f_{M-2\epsilon}\rho_{\xi_\epsilon}(x_n)=WV^*\rho_{\xi_\epsilon}(x_n)f_{M-2\epsilon}$$
$$=Wx_nV^*f_{M-2\epsilon}=Wx_nW^*.$$

By [219] the minimal projection $V^*f_{M-2\epsilon}V$ corresponds to $[\rho_{\xi_\epsilon}]$, and the canonical endomorphism $\gamma^{M-\epsilon}_{M-2\epsilon}:M-\epsilon\to M-2\epsilon$ has the following decomposition in Sect $(M-2\epsilon)$:

$$[\gamma^{M-\epsilon}_{M-2\epsilon}|M-2\epsilon]=\bigoplus_{\xi_\epsilon\in\Xi_{M-\epsilon}}\dim(\mathcal{K}_{\xi_\epsilon})[\rho_{\xi_\epsilon}].$$

From this, we see that $[\gamma^{M-\epsilon}_{M-2\epsilon}|M-2\epsilon]$ is contained in $[\gamma^{M}_{M-2\epsilon}|M-2\epsilon]$ because each irreducible is contained in $[\gamma^{M}_{M-2\epsilon}|M-2\epsilon]$ and we trivially have dim $(\mathcal{K}_{\xi_\epsilon})\leqq\dim(\mathcal{H}_{\xi_\epsilon})$.

**(iii)** Apply [219] to the discrete inclusion $M-2\epsilon\subset M-\epsilon$.

**Corollary (4.2.29)[495]:** Let $B\subset L^\infty(\mathbb{G})$ be a left coideal and $\pi\in\mathrm{Irr}(\mathbb{G})$. Then there exist a unitary representation $(u_n)_\pi=((u_n)_{\pi_{i,j}})_{i,j\in I_\pi}$ and a subset $I^B_\pi\subset I_\pi$ such that:

**(i)** $(u_n)_\pi$ is equivalent to $(v_n)_\pi$.

**(ii)** $B_\pi=\mathrm{span}\left\{((u_n)_{\pi_{i,j}}|\ i\in I_\pi, j\in I^B_\pi\right\}$.

**Proof:** Let $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ be the set of $\mathbb{G}$ -linear maps from $K_\pi$ into $L^\infty(\mathbb{G})$, that is, it consists of linear maps $S: K_\pi\to L^\infty(\mathbb{G})$ such that $\delta\circ S=(id\otimes S)\circ(v_n)_\pi$ , where $(v_n)_\pi$ is regarded as a map from $K_\pi$ to $L^\infty(\mathbb{G})_\pi\otimes K_\pi$ . Similarly we define $\mathrm{Hom}_{\mathbb{G}}(K_\pi,B)$ , which is a subspace of $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ .Let $(\mathcal{E}_i)_{i\in I_\pi}$ be an orthonormal basis of $K_\pi$. We prepare the inner product of $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ defined by

$$\langle S|T\rangle 1:=\sum_{i\in I_\pi}T(\varepsilon_i)^*S(\varepsilon_i).$$

Then $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ is a Hilbert space of dimension $d_\pi$. We take an orthonormal basis $\{S_i\}_{i\in I_\pi}$ of $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ which contains an orthonormal basis of $\mathrm{Hom}_{\mathbb{G}}(K_\pi,B)$denoted by $\{S_i\}_{i\in I^B_\pi}$.

We define the linear map $T_j: K_\pi\to L^\infty(\mathbb{G})$ by $T_j(\varepsilon_i)=(v_n)_{\pi_{ij}}$ for $j\in I_\pi$ . Then it is easy to see that $\{T_j\}_{j\in I_\pi}$ is an orthonormal basis of $\mathrm{Hom}_{\mathbb{G}}(K_\pi,L^\infty(\mathbb{G}))$ .Hence there exists a unitary matrix $(\nu_n)_{\pi_{ij}}:=\{(\nu_n)_i\}_{i,j\in I_\pi}$ in $B(C^{|I_\pi|})$ such that for $i\in I_\pi$ ,

$$S_i = \sum_{j\in I_\pi} (\nu_n)_{\pi_{ji}} T_j.$$

We define the unitary representation $(u_n)_\pi := (1\otimes(\nu_n)_\pi^*)(\upsilon_n)_\pi(1\otimes(\nu_n)_\pi)$. Then we have

$$(u_n)_{\pi_{ij}} = \sum_{k,\ell\in I_\pi} ((\nu_n)_\pi^*)_{ik}(\upsilon_n)_{\pi_{\ell j}} = \sum_{k,\ell\in I_\pi} ((\nu_n)_\pi^*)_{ik}(\nu_n)_{\pi_{\ell j}} T_\ell(\varepsilon_k)$$
$$= \sum_{k\in I_\pi} ((\nu_n)_\pi^*)_{ik} S_j(\varepsilon_k) = S_j\left(\sum_{k\in I_\pi} ((\nu_n)_\pi^*)_{ik}\varepsilon_k\right),$$

and

$$S_j(\varepsilon_k) = \sum_{i\in I_\pi} (\nu_n)_{\pi_{ki}} (u_n)_{\pi_{ki}}.$$

Therefore $(u_n)_{\pi_{ij}} \in B$ for all $i \in I_\pi$ and $j \in I_\pi^B$ , and they span $B_\pi$.

**Corollary (4.2.30)[495]:** When $M^{\alpha_\epsilon}$ is infinite, the following statements hold:

**(i)** $1 = \sum_{\pi\in \mathrm{Irr}\,(\mathbb{G})} (z_n)_\pi$.

**(ii)** $(M-2\epsilon)' \cap M_1 = \bigotimes_{\pi\in \mathrm{Irr}\,(\mathbb{G})} A_\pi$.

**(iii)** The inclusion $M - 2\epsilon \subset M$ is discrete.

**Proof:** (**i**) Take $\{W_{\pi_k}\}_{k=1}^{d_\pi}$ in $\mathcal{H}_\pi$ such that $E(W_{\pi_k} W_{\pi_k}^*) = \delta_{k,\ell}\, 1$. Then we have

$$(z_n)_\pi = \sum_{k=1}^{d_\pi} W_{\pi_k}^* e_{M-2\epsilon} W_{\pi_k}.$$

Since $M$ is weakly spanned by $\mathcal{H}_\pi^*(M-2\epsilon), \pi \in \mathrm{Irr}\,(\mathbb{G})$, we have

$$H_{\varphi_\epsilon} = \overline{\mathrm{span}\{\Lambda_{\varphi_\epsilon}(\mathcal{H}_\pi^*(M-2\epsilon)) | \pi \in \mathrm{Irr}\,(\mathbb{G})\}}.$$

For any $x_n \in M - 2\epsilon$ and $V_\sigma \in \mathcal{H}_\sigma$ with $\sigma \neq \pi$, we have

$$(z_n)_\pi \Lambda_{\varphi_\epsilon}(V_\pi^* x_n) = \sum_{k=1}^{d_\pi} W_{\pi_k}^* \Lambda_{\varphi_\epsilon}(E(W_{\pi_k} V_\sigma^*) x_n) = 0,$$

where we have used $E(\mathcal{H}_\pi \mathcal{H}_\sigma^*) = 0$ in the last equality. If $\sigma = \pi$, we have

$$(z_n)_\pi \Lambda_{\varphi_\epsilon}(V_\pi^* x_n) = \Lambda_{\varphi_\epsilon}(V_\pi^* x_n).$$

Hence the range space of $(z_n)_\pi$ coincides with $\Lambda_{\varphi_\epsilon}(\mathcal{H}_\pi^*(M-2\epsilon))$, and $\{(z_n)_\pi\}_{\pi\in \mathrm{Irr}\,(\mathbb{G})}$ is a partition of unity.

**(ii)** We first show that $(z_n)_\pi$ is a central projection in $(M-2\epsilon)' \cap M_1$. For $\pi \in \mathrm{Irr}\,(\mathbb{G})$, take $\{W_{\pi_k}\}_{k=1}^{d_\pi}$ as above. It suffices to prove that $(z_n)_\pi((M-2\epsilon)' \cap M_1)(z_n)_\sigma = 0$ if $\pi \neq \sigma$ . Let $x_n \in M - 2\epsilon' \cap M_1$ and take $(x_n)_0 \in M - 2\epsilon$ such that $(x_n)_0 e_{M-2\epsilon} = e_{M-2\epsilon} W_{\pi_k} x_n W_{\sigma_\ell}^* e_{M-2\epsilon}$. Then for any $y_n \in M - 2\epsilon$, we have

$$(x_n)_0 \rho_\sigma(y_n) e_{M-2\epsilon} = e_{M-2\epsilon} W_{\pi_k} x_n W_{\sigma_\ell}^* \rho_\sigma(y_n) e_{M-2\epsilon} = e_{M-2\epsilon} W_{\pi_k} x_n y_n W_{\sigma_\ell}^* e_{M-2\epsilon}$$
$$= e_{M-2\epsilon} W_{\pi_k} y_n x_n W_{\sigma_\ell}^* e_{M-2\epsilon} = e_{M-2\epsilon} \rho_\pi(y_n) W_{\pi_k} x_n W_{\sigma_\ell}^* e_{M-2\epsilon}$$
$$= \rho_\pi(y_n)(x_n)_0 e_{M-2\epsilon}.$$

This shows that $(x_n)_0$ intertwines $\rho_\sigma$ and $\rho_\pi$. So, we get $(x_n)_0 = 0$. Hence we have $e_{M-2\epsilon} W_{\pi_k}((M-2\epsilon)' \cap M_1) W_{\sigma_\ell}^* e_{M-2\epsilon} = 0$ and $(z_n)_\pi((M-2\epsilon)' \cap M_1)(z_n)_\sigma = 0$.

Second we show that each $p_k := W_{\pi_k}^* e_{M-2\epsilon} W_{\pi_k}$ is a minimal projection in $(M-2\epsilon)' \cap M_1$. This is because the reduced inclusion $(M-2\epsilon)p_k \subset p_k M_1 p_k$ is isomorphic to the irreducible inclusion $\rho_\pi(M-2\epsilon) \subset M-2\epsilon$. Hence $(M-2\epsilon)' \cap M_1$ is the direct sum of $A_\pi, \pi \in \text{Irr}(\mathbb{G})$.

**(iii)** This is trivial by (ii) because $\hat{E}$ is finite on each $A_\pi$.

By the previous lemma, we can regard $\Xi = \text{Irr}(\mathbb{G})$.

**Corollary (4.2.31)[495]:** For any intermediate subfactor $M^{\alpha_\epsilon} \subset M-\epsilon \subset M, \mathcal{L}(M-\epsilon)$ is a left coideal of $G$.

**Proof:** If we consider a minimal action $\beta \coloneqq \text{id} \otimes \alpha_\epsilon$ on $B(\ell_2) \otimes M$, then $\mathcal{L}_\beta(B(\ell_2) \otimes (M-\epsilon)) = \mathcal{L}_{\alpha_\epsilon}(M-\epsilon)$ .Therefore we may assume that $M^{\alpha_\epsilon}$ is infinite. We have to check that $\delta(\mathcal{L}(M-\epsilon)) \subset L^\infty(\mathbb{G}) \otimes \mathcal{L}(M-\epsilon)$ and $\mathcal{L}(M-\epsilon)$ is multiplicatively closed.

Set $\mathcal{K}_\pi = \mathcal{H}_\pi \cap M-\epsilon$ . Then $M-\epsilon$ is weakly spanned by $\mathcal{K}_\pi^* (M-2\epsilon)$ by Corollary (4.2.29). Recall two bases $\{V_{\pi_i}\}_{i \in \text{I}_\pi}$ and $\{W_{\pi_i}\}_{i \in \text{I}_\pi}$ in $\mathcal{H}_\pi$ as before, that is, we have the equalities $V_{\pi_i}^* V_{\pi_j} = \delta_{ij}$ ,(17) and $E\left(W_{\pi_i} W_{\pi_j}^*\right) = \delta_{ij}$ .We may assume that $\{W_{\pi_i}\}_{i \in I_\pi^{M-\epsilon}}$ is a basis of $\mathcal{K}_\pi$. There exist $c_{\pi_{ji}} \in \mathbb{C}, i,j \in I_\pi$ such that $V_{\pi_j}^* W_{\pi_i} = c_{\pi_{ji}}$ .Note that the matrix $(c_{\pi_{ji}})_{i,j \in \text{I}_\pi}$ is invertible. Since

$$\alpha_\epsilon(W_{\pi_i}) = \sum_{j \in I_\pi} \alpha_\epsilon\left(V_{\pi_j} V_{\pi_j}^* W_{\pi_i}\right) = \sum_{j \in I_\pi} c_{\pi_{ji}} \alpha_\epsilon\left(V_{\pi_j}\right) \tag{22}$$

$$= \sum_{j,k \in I_\pi} c_{\pi_{ji}} V_{\pi_k} \otimes (v_n)_{\pi k_j} = \sum_{j \in I_\pi} V_{\pi_k} \otimes \left(\sum_{j \in I_\pi} c_{\pi_{ji}} \otimes (v_n)_{\pi k_j}\right).$$

We have

$$\mathcal{L}(M-\epsilon) = \overline{\text{span}}^{w_\epsilon} \left\{ \sum_{j \in I_\pi} c_{\pi_{ji}} (v_n)_{\pi k_j} \mid i \in I_\pi^{M-\epsilon}, j,k \in I_\pi, \pi \in \Xi_{M-\epsilon} \right\}. \tag{23}$$

This implies that $\delta(\mathcal{L}(M-\epsilon)) \subset L^\infty(\mathbb{G}) \otimes \mathcal{L}(M-\epsilon)$. Let $\pi, \sigma \in (\mathbb{G})\text{Irr}$ . Next we show that $\mathcal{L}(M-\epsilon)$ is multiplicatively closed. It suffices to show that the product of $\sum_{j \in I_\pi} c_{\pi_{rj}} (v_n)_{\pi i_{,j}}$ and $\sum_{j \in I_\sigma} c_{\sigma_{j\ell}} (v_n)_{\pi s_{,j}}$

Is contained in $\mathcal{L}(M-\epsilon)$ for all $(i,r) \in I_\pi^{M-\epsilon} \times I_\pi$ and $(\ell, s) \in I_\sigma^{M-\epsilon} \times I_\sigma$ .Then it is clear because the left-hand side of the following equality is contained in $\mathbb{C} \otimes \mathcal{L}(M-\epsilon)$:

$$\left(V_{\sigma s}^* V_{\pi_r}^* \otimes 1\right) \alpha_\epsilon(W_{\pi_i} W_{\sigma_\ell}) = 1 \otimes \left(\sum_{j \in I_\pi} c_{\pi_{ji}} \otimes (v_n)_{\pi r_j}\right)\left(\sum_{j \in I_\sigma} c_{\sigma_{j\ell}} (v_n)_{\sigma s_j}\right).$$

**Corollary (4.2.32)[495]:** (Galois correspondence). Let $\mathbb{G}$ be a compact quantum group and $M$ a factor. Let $\alpha_\epsilon: M \to M \otimes L^\infty(\mathbb{G})$ be a minimal action. Then there exists an isomorphism between the lattice of intermediate subfactors of $M^{\alpha_\epsilon} \subset M$ and the lattice of left coideals of $\mathbb{G}$. More precisely, the maps $\mathcal{M}$ and $\mathcal{L}$ are the mutually inverse maps, that is, for any intermediate subfactor $M^{\alpha_\epsilon} \subset M-\epsilon \subset M$ and any left coideal $B \subset L^\infty(\mathbb{G})$, one has

$$\mathcal{M}\left(\mathcal{L}(M-\epsilon)\right) = M-\epsilon, \qquad \mathcal{L}\left(\mathcal{M}(\mathrm{B})\right) = B.$$

**Proof:** If we consider a minimal action $\beta := \mathrm{id}\otimes\alpha_\epsilon$ on $B(\ell_2)\otimes M$, then we have $\mathcal{L}_\beta(B(\ell_2)\otimes M-\epsilon) = \mathcal{L}_{\alpha_\epsilon}(M-\epsilon)$ and $\mathcal{M}_\beta(B) = B(\ell_2)\otimes\mathcal{M}_{\alpha_\epsilon}(B)$. Hence we may and do assume That $M^{\alpha_\epsilon}$ is infinite.

By definition, we see that $M-\epsilon \subset \mathcal{M}(\mathcal{L}(M-\epsilon))$. We will show $\mathcal{M}(\mathcal{L}(M-\epsilon)) \subset M-\epsilon$. Set $\mathcal{K}_\pi := \mathcal{H}_\pi \cap M-\epsilon$ and $\widetilde{\mathcal{K}}_\pi := \mathcal{H}_\pi \cap \mathcal{M}(\mathcal{L}(M-\epsilon))$. Then $\mathcal{M}(\mathcal{L}(M-\epsilon))$ is $\sigma$-weakly spanned by $M^{\alpha_\epsilon}\widetilde{\mathcal{K}}_\pi$ for $\pi\in\mathrm{Irr}(\mathbb{G})$ by Corollary (4.2.29). We choose a basis $\{W_{\pi_i}\}_{i\in I_\pi}$ in $\mathcal{H}_\pi$ such that $E\left(W_{\pi_i}W_{\pi_j}^*\right) = \delta_{i,j}$ as before. We may assume that it contains bases of $\mathcal{K}_\pi$ and $\widetilde{\mathcal{K}}_\pi$, which are denoted by $\{W_{\pi_i}\}_{i\in I_\pi^{M-\epsilon}}$ and $\{W_{\pi_i}\}_{i\in J_\pi}$, respectively. We use the invertible matrix $\left(c_{\pi_{i,j}}\right)_{i,j\in I_\pi}$ as in the previous lemma.

Let $j\in J_\pi$. Since $W_j\in\mathcal{M}\left(\mathcal{L}(M-\epsilon)\right)$, $\alpha_\epsilon(W_j)$ is contained in $M\otimes\mathcal{L}(M-\epsilon)$, that is, $\sum_{k\in I_\pi} c_{\pi_{kj}}(v_n)_{\pi\ell_k}\in\mathcal{L}(M-\epsilon)$ for all $\ell\in I_\pi$ by (22). Recall that the $((v_n)_{\pi_{k\ell}})_{k,\ell\in I_\pi}$ are linearly independent. By (23), there exists $d_{\pi_{ij}}\in\mathbb{C}$ for $i\in I_\pi^{M-\epsilon}$ such that for any $\ell\in I_\pi$,

$$\sum_{k\in I_\pi} c_{\pi_{kj}}(v_n)_{\pi_{\ell k}} = \sum_{i\in I_\pi^{M-\epsilon}} d_{\pi_{ij}}\left(\sum_{k\in I_\pi} c_{\pi_{ki}}(v_n)_{\pi_{\ell k}}\right),$$

That is,

$$c_{\pi_{kj}} = \sum_{i\in I_\pi^{M-\epsilon}} d_{\pi_{ij}}\, c_{\pi_{ki}} \qquad \text{for all } j\in J_\pi, k\in I_\pi. \tag{24}$$

Note that $d_{\pi_{ij}}$ does not depend on $\ell$. We know the matrix $C := (c_{\pi_{k\ell}})_{k\ell\in I_\pi}$ is invertible. Multiplying $(C^{-1})_{\ell k}(\ell\in I_\pi^{M-\epsilon})$ to the both sides of the above equality, summing up with $k$, we have

$$\delta_{\ell j} = d_{\pi_{\ell j}} \qquad \text{for all } \ell\in I_\pi^{M-\epsilon}.$$

This yields $j\in I_\pi^{M-\epsilon}$. Indeed, if $j\neq I_\pi^{M-\epsilon}$, then $d_{\pi_{\ell j}} = 0$ for all $\ell\in I_\pi^{M-\epsilon}$ .Together with (24), we have $c_{\pi_{kj}} = 0$ for all $k\in I_\pi$. Then we have

$$W_{\pi_j} = \sum_{k\in I_\pi} V_{\pi_k}\left(V_{\pi_k}^* W_{\pi_j}\right) = \sum_{k\in I_\pi} V_{\pi_k} c_{\pi_{kj}} = 0,$$

but this is a contradiction. Therefore $W_{\pi_j}\in M-\epsilon$ for any $j\in I_\pi$, and $\mathcal{M}\left(\mathcal{L}(M-\epsilon)\right)\subset M-\epsilon$.

Next we will show that $\mathcal{L}(\mathcal{M}(B)) = B$. By definition, the inclusion $\mathcal{L}(\mathcal{M}(B))\subset B$ holds. We prove $B\subset\mathcal{L}(\mathcal{M}(B))$. Since $B$ is $\sigma$-weakly spanned by subspaces

$$B_\pi = B\cap L^\infty(\mathbb{G})_\pi, \qquad \pi\in\mathrm{Irr}(\mathbb{G}),$$

it suffices to show that $B_\pi\subset\mathcal{L}(\mathcal{M}(B))$ for any $\pi\in\mathrm{Irr}(\mathbb{G})$. By Corollary (4.2.30), there exists a unitary matrix $(\nu_n)_\pi = ((\nu_n)_{\pi_{ij}})_{ij\in I_\pi}\in B\left(\mathbb{C}^{|I_\pi|}\right)$ such that $B_\pi$ is spanned by $(u_n)_{\pi_{ij}}, i\in I_\pi$ and $j\in I_\pi^B$ ,where $(u_n)_\pi = (1\otimes(\nu_n)_\pi^*)(v_n)_\pi(1\otimes(\nu_n)_\pi)$. For $i\in I_\pi$, we put $V'_{\pi_i} := \sum_{j\in I_\pi}(\nu_n)_{\pi_{ji}} V_{\pi_j}$ .Then we have

$$\alpha_\epsilon(V'_{\pi_i}) = \sum_{j\in I_\pi}(v_n)_{\pi_{ji}}\alpha_\epsilon(V_{\pi_j}) = \sum_{j,k\in I_\pi}(v_n)_{\pi_{ji}}(V_{\pi_k}\otimes(v_n)_{\pi_{kj}})$$
$$= \sum_{k\in I_\pi}(V_{\pi_k}\otimes((v_n)_\pi(1\otimes(v_n)_\pi))_{ki}) = \sum_{k\in I_\pi}\left(V_{\pi_k}\otimes\left((1\otimes(v_n)_\pi)_{(u_n)_\pi}\right)_{ki}\right)$$
$$= \sum_{j,k\in I_\pi}\left((v_n)_{\pi_{ki}}V_{\pi_k}\otimes(u_n)_{\pi_{ji}}\right) = \sum_{j\in I_\pi}V'_{\pi_j}\otimes(u_n)_{\pi_{ji}}.$$

Let $i\in I_\pi^B$ .Then $(u_n)_{\pi_{ji}}\in B_\pi$ for all $j\in I_\pi$ , and $V'_{\pi_i}\in\mathcal{M}(B)$ by the above equality. Again by the above equality, $(u_n)_{\pi_{ji}}\in\mathcal{L}(\mathcal{M}(B))$ for all $j\in I_\pi$. This implies $B_\pi\subset\mathcal{L}\big(\mathcal{M}(B)\big)$.

When $\mathbb{G}$ is of $K$ac type, it has been proved in [219] that there exists a normal conditional expectation from $M$ onto any intermediate subfactor of $M^{\alpha_\epsilon}\subset M$. However, if $\mathbb{G}$ is not of $K$ac type, then this is not the case in general as we will see below. We can characterize which intermediate subfactor has such a property.

**Corollary (4.2.33)[495]:** Let $B\subset L^\infty(\mathbb{G})$ be $a$ left coideal. Then the following statements are equivalent:

**(i)** $B$ has the expectation property.

**(ii)** $\sigma_t^h(B)=B$ $for\ all$ $t\in\mathbb{R}$.

**(iii)** $\tau_t(B)=B$ $for\ all$ $t\in\mathbb{R}$.

**Proof:** (i) $\Longrightarrow$ (ii). This follows from Takesaki's theorem [272], p. 309.

(ii) ) $\Longrightarrow$ (iii). Since $B_\pi\in\mathrm{Irr}(\mathbb{G})$ spans a dense subspace of $B$, it suffices to show that $\tau_t(B_\pi)\subset B_\pi$ for all $t\in\mathbb{R}$ and $\pi\in\mathrm{Irr}(\mathbb{G})$. Recall the equality (16). Then for $x_n\in B_\pi$, we have $\tau_t(x_n)=(f_{2it}\otimes\sigma_{-t}^h)(\delta(x_n))$. Since $B$ is a left coideal globally invariant under the modular group $\sigma^h$, we see that $\tau_t(x_n)\in B_\pi$. Hence $\tau_t(B_\pi)\subset B_\pi$ for all $t\in\mathbb{R}$ and $\pi\in\mathrm{Irr}(\mathbb{G})$.

(iii) $\Longrightarrow$ (i). Let $\pi\in\mathrm{Irr}(\mathbb{G})$ and $x_n\in B_\pi$ . By (16), we have $\sigma_t^h(x_n)=(f_{2it}\otimes\tau_{-t})\big(\delta(x_n)\big)$. Since $B$ is a left coideal globally invariant under the scaling group $\tau$ , we see that $\sigma_t^h(x_n)\in B_\pi$. Hence $\sigma_t^h(B_\pi)\subset B_\pi$ for all $t\in\mathbb{R}$ and $\pi\in\mathrm{Irr}(\mathbb{G})$, and $B$ is globally invariant under the modular group $\sigma^h$. Again by Takesaki's theorem, there exists a faithful normal conditional expectation $E_B: L^\infty(\mathbb{G})\to B$ preserving $h$. Hence $B$ has the expectation property. Then we have the following (see [31]).

**Corollary (4.2.34)[495]:** Let $\alpha_\epsilon$ be a minimal action of $\mathbb{G}$ on $a$ factor $M$. Let $M^{\alpha_\epsilon}\subset M-\epsilon\subset M$ be an intermediate subfactor. Then there exists $a$ faithful normal conditional expectation $E_{M-\epsilon}^M: M\to M-\epsilon$ if and only if the left coideal $\mathcal{L}(M-\epsilon)$ has the expectation property.

**Proof:** Let $\omega_\epsilon\in M_*-2\epsilon$ be a faithful state. Put $\varphi_\epsilon$: $\omega_\epsilon\circ E\in M$. We note that L is the image of a faithful normal conditional expectation of $M$ if and only if $\sigma_t^{\varphi_\epsilon}(M-\epsilon)\subset M-\epsilon$ for all $t\in\mathbb{R}$ . Indeed, if the former condition holds, there exists a faithful normal conditional expectation $E_{M-\epsilon}^M: M\to M-\epsilon$. Then the conditional expectation $E_{M-2\epsilon}^{M-\epsilon}\circ E_{M-\epsilon}^M$ is equal to $E$ because $M-2\epsilon\subset M$ is irreducible. Hence $\varphi_\epsilon\circ E_{M-\epsilon}^M=(\varphi_\epsilon\circ E_{M-2\epsilon}^{M-\epsilon})\circ E_{M-\epsilon}^M=\varphi_\epsilon\circ E=\varphi_\epsilon$. Then by Takesaki's theorem [272], p. 309, the latter condition holds. The converse implication also follows from his theorem.

Since $\varphi_\epsilon$ is invariant under the action $\alpha_\epsilon$, we have $\alpha_\epsilon\,\mathrm{o}\,\sigma_t^{\varphi_\epsilon}=(\sigma_t^{\varphi_\epsilon}\otimes\tau_{-t})\,\mathrm{o}\alpha_\epsilon$ for all $t\in\mathbb{R}$ by [267]. Put $B:=\mathcal{L}(M-\epsilon)$ . Then we have $M-\epsilon=\{x_n\in$

$M|\alpha_\epsilon(x_n) \in M \otimes B\}$ by Corollary (4.2.33). So, for $x_n \in M - \epsilon$, $\sigma_t^{\varphi_\epsilon}(x_n) \in M - \epsilon$ if and only if $\alpha_\epsilon(x_n) \in M \otimes \tau_t(B)$. The von Neumann subalgebra $\tau_t(B)$ is also a left coideal by the equality $\delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \delta$. Hence $\sigma_t^{\varphi_\epsilon}(x_n) \in M - \epsilon$ if and only if $x_n \in \mathcal{M}(\tau_t(B))$. Therefore, $\sigma_t^{\varphi_\epsilon}(M - \epsilon) \subset M - \epsilon$ if and only if $M - \epsilon \subset \mathcal{M}(\tau_t(B))$. Since $M - \epsilon = \mathcal{M}(B)$, this is equivalent with $B \subset \tau_t(B)$. Hence $M - \epsilon$ is the image of a faithful normal conditional expectation of $M$ iff $B = \tau_t(B)$ for all $t \in \mathbb{R}$ .By the previous lemma, this equivalently means that $B$ has the expectation property.

## Section (4.3): Compact Group Actions on $C^*$-Algebras

For $(M, G, \delta\mathfrak{S})$ be a $W^*$-dynamical system with $M$ a von Neumann algebra, $G$ a compact group and $\delta : \mathrm{g} \in G \to \delta_\mathrm{g} \in Aut(M)$ a homomorphism of $G$ into the group $Aut(M)$ of all automorphisms of $M$ such that the mapping $\mathrm{g} \to \delta_\mathrm{g}(m)$ is simple-weakly continuous for every $m \in M$. Denote by $Aut\delta(M)$ the subgroup of $Aut(M)$ consisting of all automorphisms commuting with all $\delta_\mathrm{g}, \mathrm{g} \in G$. In [283] (see also [287]) it is proven that if $Aut\delta(M)$ contains an ergodic subgroup $S$, then there is a one-to-one correspondence between the set of normal, closed subgroups of $G$ and the set of all $G$ and $S$ globally invariant von Neumann subalgebras $N$ with $M^G \subset N \subset M$. This correspondence is given by: $N \leftrightarrow G^N$ where $G^N = \{\mathrm{g} \in G|\ \delta_\mathrm{g}(n) = n,\ n \in N\}$. The main technical tool in Kishimoto's approach is the method of Hilbert spaces inside a von Neumann algebra, as developed in [286]. Later, in [219], the case of irreducible actions was considered. They proved that if $M^G \subset M$ is an irreducible pair of factors, i.e. $(M^G)' \cap M = CI$ , then there is a one-to-one correspondence between the set of intermediate subfactors $M^G \subset N \subset M$ and the closed subgroups of $G$ given by $N \leftrightarrow G^N$, where, as above, $G^N = \{\mathrm{g} \in G|\delta_\mathrm{g}(n) = n, n \in N\}$. This correspondence is called Galois correspondence. In [219], an action of a compact group with the property $(M^G)' \cap M = CI$ is called minimal. Notice that in this case, $S = \{Ad(u)|\ u \in M^G, \text{ unitary}\}$ is ergodic on $M$ and that $N$ is obviously $S$-invariant but is not required to be $G$-invariant. This result was extended to the case of compact quantum group actions on von Neumann factors by Tomatsu [266]. See [219] and [266] make extensive use of the method of Hilbert spaces inside a von Neumann algebra and other methods specific for von Neumann algebras. We will prove a result that extends Kishimoto's result to the case of compact actions on $C*$-algebras commuting with minimal actions, as defined below. This is the first result of this kind for $C*$-dynamical systems. The notion of minimal action that will be used is different from the one used in [219]. Our methods are specific to $C*$-dynamical systems and give, in particular, a new proof of Kishimoto's result. We also give an example that shows that the result is not true if the commutant of the compact action satisfies a weaker ergodicity condition, that, in the case of von Neumann algebras is equivalent with the usual one.

If $M$ is a von Neumann algebra, a subgroup $S \subset Aut(M)$ is called ergodic if $M^S = CI$ , where $C$ is the set of complex numbers and $MS$ denotes the fixed point algebra, $M^S = \{m \in M|\ s(m) = m, s \in S\}$. In the case of $C*$-algebras there are several distinct notions of ergodicity that are all equivalent for von Neumann algebras. These notions are distinct even for abelian $C*$-algebras, the case of topological dynamics. Let $A$ be a $C*$-algebra and $S \subset Aut(A)$ a subgroup of the automorphism group of $A$. Denote by $\mathcal{H}^S(A)$ the set of all nonzero hereditary $C*$-subalgebras of $A$ that are globally $S$-invariant. We recall the following definitions from [60]:

**i)** $S$ is called weakly ergodic if $A^S$ is trivial.

**ii)** $S$ is called topologically transitive if for every $C_1, C_2 \in \mathcal{H}^S(A)$, their product $C_1C_2 = \{\sum_{\text{finite}} c_1^i c_2^i \mid c_1^i \in C_1, c_2^i \in C_2\}$ is not zero. In the particular case of topological dynamics this condition is equivalent to the usual topological transitivity of the flow.

In [44] it is noticed that our condition ii) is equivalent to the following:

**ii′**) If $x, y \in A$ are not zero, then there is an $s \in S$ such that $xs(y) \neq 0$.

**iii)** $S$ is called minimal if $\mathcal{H}^S(A) = \{A\}$.

We caution that in [219] and [266] the notion of minimality is used for compact actions $(M, G, \delta)$ such that $(M^G)' \cap M = CI$. A group of automorphisms is called minimal if it satisfies condition iii) above.

Obviously iii) ⇒ ii) ⇒ i). Also, since in the case of von Neumann algebras, $M$, the $S$-invariant, hereditary $W^*$-subalgebras, are of the form $pMp$ where $p$ is a projection in $M^S$ it follows that all the above conditions are equivalent. Another situation when all of the above conditions are equivalent for a $C^*$-dynamical system is when $S$ is compact [60].

In [60] there are also discussed several criteria for checking topological transitivity. In [44], a seemingly stronger notion than topological transitivity is introduced, namely the notion of strong topological transitivity:

$S$ is said to be strongly topologically transitive if for each finite sequence $\{(x_i, y_i) | i = 1, 2, \ldots, n\}$ of pairs of elements $x_i, y_i \in A$ for which $\sum x_i \otimes y_i \neq 0$ in the algebraic tensor product $A \otimes A$, there exists an $s \in S$ such that $\sum x_i s(y_i) \neq 0$ in $A$.

Further, in [41] it is shown that in the case of von Neumann algebras strong topological transitivity is equivalent with topological transitivity and hence with the rest of the above conditions.

In what follows we will need the following results from [281]:

**Proposition(4.3.1)[280]:** Let $(A, S)$ *be a* $C^*$*-dynamical system and* $B \subset A$ *an* $S$*-invariant* $C^*$-subalgebra. Then $\mathcal{H}^S(B) = \{C \cap B | C \in \mathcal{H}^S(A)\}$.

**Proof:** This is [281].

If $A$ is a $C^*$-algebra, we denote by $A_{sa}$ the set of selfadjoint elements of $A$ and by $(A_{sa})^m$ the set of elements in the bidual $A^{**}$ of $A$ that can be obtained as strong limits of bounded, monotone increasing nets from $A_{sa}$ (see also [35]). Then we can state [281]:

**Proposition (4.3.2)[280]:** Let $(A, S)$ *be a* $C^*$-dynamical system. Then the following conditions are equivalent:

**i)** $(A, S)$ is minimal.

**ii)** If $a \in (A_{sa})^m$ is such that $s^{**}(a) = a$ for every $s \in S$, then $a \in CI$. Here, $s^{**}$ denotes thedouble dual of the automorphism $s \in S$.

**Proof:** This is [281].

Let $(A, G, \delta)$ be a $C^*$-dynamical system with $G$ compact. Denote by $\hat{G}$ the set of all equivalence classes of irreducible, unitary representations of $G$. For each $\pi \in \hat{G}$, fix a unitary representation, $u^\pi$ in the class $\pi$ and a basis in the Hilbert space $H_\pi$ of $u^\pi$. If $\pi \in \hat{G}$, denote by $\chi_\pi(\mathrm{g}) = d_\pi \sum_{i=1}^{d_\pi} \overline{u_{ii}^\pi(\mathrm{g})}$ the character of the class $\pi$, where $d_\pi$ is the dimension of $H_\pi$. For $\pi \in \hat{G}$ we consider the following mappings from $A$ into itself:

$$P^{\pi,\delta}(a) = \int_G \chi_\pi(\mathrm{g})\delta_\mathrm{g}(a)d\mathrm{g}$$

$$P_{ij}^{\pi,\delta}(a) = \int_G \overline{u_{ji}^\pi}(\mathrm{g})\delta_\mathrm{g}(a)d\mathrm{g}$$

We define the spectral subspaces of the action $\delta$:

$$A_1^{\delta}(\pi) = \{a \in A | P^{\pi,\delta}(a) = a\}, \qquad \pi \in \hat{G}$$

In the particular case when $\pi = \pi_0$ is the trivial one-dimensional representation of $G$, $A_1^{\delta}(\pi_0) = A^G$ the $C^*$-subalgebra of fixed elements of the action $\delta$.

As in [284] and [19], the matricial spectral subspaces are defined as follows:

$$A_2^{\delta}(u^{\pi}) = \{X = [x_{ij}] \in A \otimes B(H_{\pi}) | [\delta_g(x_{ij})] = X(1 \otimes u^{\pi}(g))\}$$

Notice that $A_2^{\delta}(u^{\pi})$ depends on the representation $u^{\pi}$ but for two equivalent representations, $A_2^{\delta}(u^{\pi})$ are spatially isomorphic. Obviously, $A_2^{\delta}(u^{\pi})A_2^{\delta}(u^{\pi})^*$ is a two sided ideal of $A^G \otimes B(H_{\pi})$ and $A_2^{\delta}(u^{\pi})^* A_2^{\delta}(u^{\pi})$ is a two sided ideal of $(A \otimes B(H_{\pi}))^{\delta \otimes ad(u^{\pi})}$. The proof of the following remark is straightforward:

**Remark (4.3.3)[280]:** Let $(A, G, \delta)$ be a $C^*$-dynamical system with $G$ compact and $s \in Aut(A)$ be such that $s\delta_g = \delta_g s$ for every $g \in G$. Then $s(A_1^{\delta}(\pi)) \subset A_1^{\delta}(\pi)$ and $(s \otimes \iota)(A_2^{\delta}(u^{\pi})) \subset A_2^{\delta}(u^{\pi})$ for every $\pi \in \hat{G}$. Here $\iota$ stands for the identity automorphism of $B(H_{\pi})$.

We will use the following results from [19]:

**Lemma (4.3.4)[280]: i)** $\sum_{\pi \in \hat{G}} A_1^{\delta}(\pi)$ is dense in $A$.

**ii)** $A_2^{\delta}(u^{\pi}) = \{[P_{ij}^{\pi,\delta}(a)] | a \in A\}$.

**Proof:** i) This is [19].

ii) This is [19].

We also need the following known result:

**Lemma (4.2.5)[280]:** Let $(C, G, \delta)$ be a $C^*$-dynamical system with $G$ compact. Then every approximate unit of the fixed point algebra $C^G$ is an approximate unit of $C$.

**Proof:** See for instance [282] for the more general case of compact quantum group actions.

Finally, we recall that a $C^*$-dynamical system $(A, G, \delta)$ with $G$ compact is called saturated if the closed, two sided ideal of the crossed product, $A \rtimes_{\delta} G$, generated by $\chi_{\pi_0}$ equals the crossed product. In this definition we used the known fact that every character, $\chi_{\pi}$, of $G$ is an element of the multiplier algebra $M(A \rtimes_{\delta} G)$ of the crossed product [284], [19]. If the system is saturated then, the crossed product is strongly Morita equivalent, in the sense of Rieffel, with the fixed point algebra, $A^G$ [116]. Then, we have [19]:

**Lemma (4.3.6)[280]:** Let $(C, G, \delta)$ be a $C^*$-dynamical system with $G$ compact. Then the following conditions are equivalent:

**i)** The system is saturated.

**ii)** The two sided ideal $C_2^{\delta}(u^{\pi})^* C_2^{\delta}(u^{\pi})$ is dense in $(C \otimes B(H_{\pi}))^{\delta \otimes ad(u^{\pi})}$ for every $\pi \in \hat{G}$, where $(C \otimes B(H_{\pi}))^{\delta \otimes ad(u^{\pi})}$ is the fixed point algebra of $C \otimes B(H_{\pi})$ for the action $\delta\ ad(u^{\pi})$ of $G$.

We will prove the main results and give examples and counterexamples.

Let $(A, G, \delta)$ be a $C^*$-dynamical system with $G$ compact. Let $B \subset A$ be $G$-invariant $C^*$-subalgebra such that $A^G \subset B$.

**Lemma (4.3.7)[280]:** If $(B, G, \delta)$ is saturated and if $\delta|B$ is faithful, then $B = A$.

**Proof.** By Lemma (4.3.4)(ii), we have to prove that for every $\pi \in \hat{G}$, $A_2^{\delta}(u^{\pi}) \subset B_2^{\delta}(u^{\pi})$. Let $\pi \in \hat{G}$ be arbitrary. Since $(B, G, \delta)$ is saturated, by Lemma (4.3.6) we have

$\overline{B_2^\delta(u^\pi)^*B_2^\delta(u^\pi)} = (B \otimes B(H_\pi))^{\delta\otimes ad(u^\pi)}$ .Since $(A^G \otimes I_B(H_\pi)) \subset (B \otimes B(H_\pi))\delta \otimes ad(u^\pi)$, it follows from Lemma (4.3.5) that $(B \otimes B(H_\pi))^{\delta\otimes ad(u^\pi)}$ contains an approximate unit of $A \otimes B(H_\pi)$. Since $B_2^\delta(u^\pi)^*B_2^\delta(u^\pi)$ is a dense two sided ideal of $(B \otimes B(H_\pi))^{\delta\otimes ad(u^\pi)}$, by [4], this latter $C^*$-algebra contains an approximate unit $\{E_\lambda\} \subset B_2^\delta(u^\pi)^*B_2^\delta(u^\pi), E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$ where $X_{i,\lambda}, Y_{i,\lambda} \in B_2^\delta(u^\pi)$. Let now $X \in A_2^\delta(u^\pi)$ .Then $XE_\lambda = \sum XX_{i,\lambda}^* Y_{i,\lambda} = \sum (XX_{i,\lambda}^*) Y_{i,\lambda} \in (A^G \otimes B(H_\pi))B_2^\delta(u^\pi) = B_2^\delta(u^\pi)$. Since $\{E_\lambda\}$ is an approximate unit of $A \otimes B(H_\pi)$, it follows that $X = (\text{norm}) \lim(XE_\lambda) \in B_2^\delta(u^\pi)$. Therefore $B = A$.

Let $B$ be a $G$-invariant $C^*$-subalgebra of $A$ such that $A^G \subset B$. Denote $G^B = \{g \in G |\, \delta_g(b) = b, b \in B\}$. Then we have:

**Remark (4.3.8)[280]: i)** $G^B$ is a closed, normal subgroup of $G$.

**ii)** The quotient action $\delta^\bullet$ of $G/G^B$ on $B$ is faithful.

**Proof:** Straightforward.

**Corollary (4.3.9)[280]:** Let $(A, G, \delta)$ be a $C^*$-dynamical system. If $A^G \subset B \subset A$ and $B$ is a $G$-invariant $C^*$-subalgebra such that $(B, G/G^B, \delta^\bullet)$ is saturated, where $\delta^\bullet$ is the quotient action, then $B = A^{G^B}$.

**Proof:** By Remark (4.3.8)ii) the quotient action $\delta^\bullet$of $G/G^B$ on $B$ is faithful and therefore, if we apply Lemma (4.3.7) to $G/G^B$ instead of $G$, we get the desired result.

**Remark (4.3.10)[280]:** As we have noticed in the proof of the previous lemma, if $B$ is a $G$-invariant $C^*$-subalgebra of $A$ and $\pi \in \hat{G}$ is such that $B_1^\delta(\pi) \neq (0)$ and hence $B_2^\delta(u^\pi) \neq (0)$, it follows that the ideal $\overline{B_2^\delta(u^\pi)^*B_2^\delta(u^\pi)}$ contains an approximate unit $\{E_\lambda\}$ of the form $E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$ where $X_{i,\lambda}, Y_{i,\lambda} \in B_2^\delta(u^\pi)$.

**Lemma (4.3.11)[280]:** Let $A$ be a $C^*$-algebra and $B \subset A$ $a$ $C^*$-subalgebra If $S \subset Aut(A)$ acts minimally on $A$ and leaves $B$ globally invariant, then $S$ acts minimally on $B$.

**Proof:** Indeed, by Proposition 1 every $S$-invariant hereditary $C^*$-subalgebra of $B, C$, is the intersection of $B$ with an $S$-invariant hereditary subalgebra of $A$. Since $S$ acts minimally on $A$ it follows that $C = B$.

In what follows, we will need the following result from [285]:

**Proposition (4.3.12)[280]:** Let $(A, G, \delta)$ be a dynamical system with $G$ compact. Assume that the action $\delta$ is faithful and that there is a subgroup $S$ of $Aut_\delta(A)$ which acts minimally on $A$. Then $A_1^\delta(\pi) \neq (0)$ for every $\pi \in \hat{G}$.

**Proof:** This is [285].

The next lemma provides a class of $C^*$-dynamical systems $(A, G. \delta)$ that are saturated. We will denote by $Aut_\delta(A)$ the subgroup of the group of all automorphisms of $A$ consisting of all automorphisms that commute with $\delta_g$ for all $g \in G$.

**Lemma (4.3.13)[280]:** Let $(A, G, \delta)$ be a dynamical system with $G$ compact. Assume that the action $\delta$ is faithful and that there is a subgroup S of $Aut_\delta(A)$ which acts minimally on $A$. Then the system is saturated.

**Proof.** We will prove that $\overline{A_2^\delta(u^\pi)^*A_2^\delta(u^\pi)} = A \otimes B(H^\pi)^{\alpha\otimes ad u^\pi}$ for every $\pi \in \hat{G}$ and the result will follow from Lemma (4.3.6). Notice first that according to Proposition(4.3.12), $A_1^\delta(\pi) \neq (0)$ f or every $\pi \in \hat{G}$ and hence $A_2^\delta(u^\pi) \neq (0)$ for every $\pi \in \hat{G}$. Let $\pi \in \hat{G}$ be arbitrary. Let $E_\lambda = \sum_{i=1}^{n_\lambda} X_{i,\lambda}^* Y_{i,\lambda}$, where $X_{i,\lambda} Y_{i,\lambda} \in A_2^\delta(u^\pi)$, be an increasing approximateunit of $\overline{A_2^\delta(u^\pi)^*A_2^\delta(u^\pi)}$ .Then $E_\lambda \nearrow E$ in the strong operator topology, where $E$ is the unit of the von Neumann algebra generated by $\overline{A_2^{\delta}(u^{\pi})^* A_2^{\delta}(u^{\pi})}$ in $A^{**} \otimes B(H_\pi)$, where $A^{**}$ is the second dual of $A$. Let $H$ be the Hilbert space of the universal representation of $A$ so that $A^{**} \subset B(H)$. Since every $s \in S$ commutes with every $\delta_g, g \in G$, by Remark (4.3.3) it follows that $A_2^{\delta}(u^{\pi})^* A_2^{\delta}(u^{\pi})$ and its weak closure in $A^{**} \subset B(H_\pi)$ is globally invariant under the automorphisms $\{s^{**} \otimes \iota | s \in S\}$ where $s^{**}$ is the double dual of $s$ and $\iota$ is the identity automorphism of $B(H_\pi)$. This means, in particular, that $(s^{**} \otimes \iota)(E) = E$ for every $s \in S$. If we write $E_\lambda = [E_{ij}^{\lambda}]$, $i, j = 1, \ldots, d_\pi$, as a matrix with entries in $A$ and $E = [E_{ij}], E_{ij} \in A^{**}$, then $s^{**}(E_{ij}) = E_{ij}$ for every $s \in S$ and all $i, j$. Since $E$ is a projection, it is in particular a positive operator which is a strong limit of elements of $A \otimes B(H_\pi)$ Therefore every diagonal entry, $E_{ij}$, of $E$, is a positive operator which is the strong limit of an increasing net of positive elements of $A$, so $E_{ij} \in A^m$. By Proposition (4.3.2) it follows that there are scalars $\mu_{ii}$ such that $E_{ii} = \mu_{ii} I$ where $I$ is the unit of $B(H)$. Now, $H \otimes H_\pi \simeq \otimes_{i=1}^{i=d_\pi} H_i$ where $H_i = H$ for all $i = 1, \ldots, d_\pi$ with $d_\pi$ the dimension of $d_\pi$. Let $\zeta = \otimes_{\zeta_i} \in H \otimes H_\pi$ with $\zeta_{i_0} = \zeta_{j_0} = \xi \in H$ and $\zeta_i = 0$ if $j_0 \neq i \neq i_0$. Then we have

$$
\begin{aligned}
(E_\lambda \zeta, \zeta) &= \left(E_{i_0 i_0}^{\lambda} \xi, \xi\right) + \left(E_{i_0 j_0}^{\lambda} \xi, \xi\right) + \left(E_{i_0 j_0}^{\lambda_*} \xi, \xi\right) + \left(E_{j_0 j_0}^{\lambda} \xi, \xi\right) \\
&= ((E_{i_0 j_0}^{\lambda} + E_{i_0 j_0}^{\lambda_*} + E_{i_0 i_0}^{\lambda} + E_{j_0 j_0}^{\lambda}) \xi, \xi)
\end{aligned}
$$

Since $E_\lambda \nearrow E$, it follows that $((E_{i_0 j_0}^{\lambda} + E_{i_0 j_0}^{\lambda_*} + E_{i_0 i_0}^{\lambda} + E_{j_0 j_0}^{\lambda}) \xi, \xi) \nearrow (E_{i_0 j_0} + E_{i_0 j_0}^{*} + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I)$ in the weak operator topology and, since $\{E_\lambda\}$ is norm bounded, in the strong operator topology. Hence, $E_{i_0 j_0} + E_{i_0 j_0}^{*} + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I \in A^m$ .As we noticed before, $E_{i_0 j_0} + E_{i_0 j_0}^{*} + \mu_{i_0 i_0} I + \mu_{j_0 j_0} I$ is $s^{**}$ -invariant for every $s \in S$ and therefore, by Proposition (4.3.2) it is a scalar multiple of the identity. Hence $E_{i_0 j_0} + E_{i_0 j_0}^{*}$ is a scalar multiple of the identity. Similarly, considering $\zeta = \otimes_{\zeta_i} \in H \otimes H_\pi$ with $\zeta_{i_0} = \sqrt{-1}\xi$, $\zeta_{j_0} = -\xi$, $\xi \in H$ and $\zeta i = 0$ if $j_0 \neq i \neq i_0$ we infer that $E_{i_0 j_0} - E_{i_0 j_0}^{*}$ is a scalar multiple of the identity. Hence there are scalars $\mu_{ij}$ such that $E_{ij} = \mu_{ij} I$, so all entries of $E$ are scalar multiples of the identity. Since $E$ is an element of the weak closure of $A \otimes B(H_\pi)\delta \otimes ad(u^{\pi})$ it follows that $E$ intertwines $u^{\pi}$ with itself and therefore, since $u^{\pi}$ is irreducible, we have $E = I$ and we are done.

**Theorem (4.3.14)[280]:** Let $(A, G, \delta)$ be a dynamical system with $G$ compact. Assume that there is a subgroup $S$ of $Aut_\delta(A)$ which acts minimally on $A$. If $A^G \subset B \subset A$ and $B$ is a $G$ and $S$ globally Invariant $C^*$-subalgebra, then $B = A^{G^B}$. Conversely, if $G_0 \subset G$ is a closed, normal subgroup, Then $B = A^{G_0}$ is $a$ $G$ and S-invariant $C^*$-subalgebra such that $A^G \subset B \subset A$.

**Proof:** It is immediate to see that the quotient action $\delta^{\bullet}$acts faithfully on $B$. By Lemma (4.3.11), $S$ acts minimally on $B$. By Lemma (4.3.13), the system $(B, \delta^{\bullet}, G/G^B)$ is saturated. By Corollary (4.3.9), $B = A^{G^B}$ and we are done. The converse is easily checked.

Notice that for $W^*$-dynamical systems the proof of the above Theorem (4.3.14) is simpler since the discussion about lower semicontinuous elements in the bidual $A^{**}$ is not necessary.

A simple example of a $C^*$-dynamical system $(A, G, \delta)$ with $G$ compact satisfying the hypotheses of Theorem (4.3.14) is the following:

**Example (4.3.15)[280]:** Let $G$ be a compact group and $C(G)$ the $C^*$-algebra of continuous functions on $G$. Denote by $\lambda$ the action of $G$ on $C(G)$ by left translations and by $\rho$ the action by right translations. Let $H$ be a Hilbert space and $K(H)$ the algebra of compact operators on $H$. Let $A = C(G) \otimes K(H)$ and $\delta_g = \lambda_g \otimes \iota, g \in G$. Then the subgroup $S \subset Aut_\delta(A)$ generated by $\{\rho_g \otimes ad(u) | g \in G, u \in \widetilde{K(H)}$, unitary$\}$ where $\rho_g$ is the right translation by $g \in G$, acts minimally on $A$. Here $\widetilde{K(H)}$ denotes the $C^*$-algebra obtained from $K(H)$ by adjoining a unit if $H$ is infinite dimensional.

The next result provides a class of examples of $C^*$-dynamical systems $(A, G, \delta)$ that satisfy the hypotheses of Theorem (4.3.14).

**Theorem (4.3.16)[280]:** Let $(A, G, \delta)$ be a $C^*$-dynamical system with $G$ compact abelian. Assume that the fixed point algebra $A^G$ is simple. If $B$ is $a$ $G$-invariant $C^*$-subalgebra such that $A^G \subset B \subset A$, then $B = A^{G^B}$.

**Proof:** Denote by $\widetilde{A^G}$ the $C^*$-algebra obtained from $A^G$ by adjoining a unit. We will show that the subgroup $S \subset Aut(A)$ generated by $\delta_G = \{\delta_g | g \in G\}$ and $\{ad(u) | u \in \widetilde{A^G}$, unitary$\}$ is minimal. Since $G$ is abelian and $ad(u), u \in A^G$ commute with $\delta_g, g \in G$, we have that $S \subset Aut_\delta(A)$ .We prove next that $S$ acts minimally on $A$. Let $C \in HS(A)$. Then if $L = \overline{AC}$, we have that $L$ is an $S$-invariant, in particular $G$-invariant, closed, left ideal of $A$ and $C = L \cap L^*$. We show that $L = A$ and hence $C = A$. Since $L$ is $S$-invariant, it follows that $L^G = \{\int_G \delta_g \ \delta g(l) dg | \ l \in L\} \subset L \cap A^G$ is a left ideal of $A^G$. Since $L$ is $ad(u)$-invariant for every $u \in \widetilde{A^G}$, unitary, we have:

$$L^G u = uad(u^*)(L^G) \subset L^G, \qquad u \in \widetilde{A^G}, unitary.$$

Therefore $L^G$ is a two sided ideal of $A^G$. Since $A^G$ is simple, it follows that $L^G = A^G$ and thus by Lemma (4.3.5), $L^G$ and so $L$ contains an approximate unit of $A$. Hence $L = A$ and therefore

$C = L \cap L^* = A$ .Therefore $S \subset Aut_\delta(A)$ is minimal and the conclusion follows from Theorem (4.3.14).

An example of $C^*$-dynamical system satisfying the hypotheses of Theorem(4.3.16) can be constructed as follows:

**Example (4.3.17)[280]:** Let $(C, G, \lambda)$ be a $C^*$-dynamical system with $G$ compact abelian. Assume that $\lambda$ is weakly ergodic, and therefore minimal, by [60]. Let $H$ be a Hilbert space. Let $A = C \otimes K(H)$, where $K(H)$ is the algebra of compact operators on $H$ and $\delta_g = \lambda_g \otimes \iota, g \in G$ where $\iota$ is the trivial automorphism of $K(H)$. Then $(A, G, \delta)$ satisfies the hypotheses of Theorem (4.3.16).

**Proof:** Straightforward.

The next example shows that the conclusion of Theorem (4.3.14) may fail if the minimality condition on $S$ is replaced with a weaker ergodicity condition such as topological transitivity, or even with strong topological transitivity.

**Example (4.3.18)[280]:** Let $G$ be a compact abelian group and $H$ an infinite dimensional Hilbert space. Denote by $\tau$ the action of $G$, by translations, on $C(G)$, the $C^*$-algebra of continuous functions on $G$. Let $A = C(G) \otimes \widetilde{K(H)}$, where $\widetilde{K(H)}$ is the subalgebra of $B(H)$ generated by $K(H)$ and the unit $I \in B(H)$. Let $\delta_g = \tau_g \otimes \iota, g \in G$, where $\iota$ is the identity automorphism of $K(H)$.Consider the system $(A, G, \delta)$. Clearly, $A^G = I_{C(G)} \otimes \widetilde{K(H)}$. We will prove the following two facts:

**(i)** $Aut_\delta(A)$ contains a subgroup $S$ which acts strongly topologically transitively on $A$.

**(ii)** There is a $G$ and $S$-invariant $C^*$-subalgebra $B$ such that $A^G \subset B \subset A$ and $B \neq A^{G^B}$.

**Proof:** (i) Let $S = \{\tau_g \otimes ad(u) | g \in G, u \in \widetilde{K(H)}, \text{ unitary}\} \subset Aut(A)$.Obviously,every element $s \in S$ commutes with all $\delta_g = \tau_g \otimes \iota, g \in G$. We prove next that $S$ acts ergodically on the von Neumann algebra $L^\infty(G) \otimes B(H)$ and then applying [60] (respectively, [41]) it will follow that $S$ acts topologically transitively (strongly topologically transitively) on $A$. Notice first that $\tau_g$ is implemented by the unitary operator $\lambda_g \in B(L^2(G))$ of translation by g. Hence the fixed point algebra $(B(L^2(G)) \otimes B(H))^S$ is the commutant $(C^*(G)'' \otimes B(H))' = C^*(G)'' \otimes CI$ where $C^*(G)$ is the group $C^*$-algebra of $G$. Since $C^*(G)'' \cap L^\infty(G) = CI$ , i) is proven.

ii) Let $B \subset A$ be the $C^*$-subalgebra generated by $C(G) \otimes K(H)$ and $I_{C(G)} \otimes I_{B(H)}$. Then, $B$ is obviously $G$ and $S$-invariant and $A^G = I_{C(G)} \otimes \widetilde{K(H)} \subset B$. Clearly, $G^B = \{g \in G |\ \delta_g(b) = b, b \in B\} = \{e\}$ where $e$ is the identity element of $G$. If we show that $B \neq A$, ii) is proven. Let $f \in C(G)$ be a non constant function. Then there are $g_1, g_2 \in G$ such that $f(g_1) \neq f(g_2)$. We claim that $f \otimes I \notin B$. Assume to the contrary that $f \in B$. Then there is a function $\Phi: G \to K(H)$ and a scalar $\mu$ such that $f(g) \otimes I = \Phi(g) + \mu I$. In particular $(f(g_1) - f(g_2))I = \Phi(g_1) - \Phi(g_2) \in K(H)$, which is a contradiction since $f(g_1) - f(g_2)$ is a nonzero scalar.

# Chapter 5
# Free Actions of Compact Quantum Abelian Groups

We use the factor systems to show that all finite coverings of irrational rotation $C^*$-algebras are left. We provide a complete classification theory of these actions for compact Abelian groups and explain its relation to the classification of classical principal bundles. We are able to express the freeness of a compact Hausdorff topological group action on a compact Hausdorff topological space in algebraic terms. As an application, we also show that a field of free actions yields a global free action.

## Section (5.1): Compact Quantum Groups on $C^*$-Algebras

Free actions of classical groups on $C^*$-algebras were first introduced under the name saturated actions by Rieffel [123] (see also [116], [36]) and equivalent characterizations where given by Ellwood [161] and by Gottman, Lazar, and Peligrad [81], [19] (see also [148]). This class of actions does not admit degeneracies that may be present in general actions. For this reasons they are easier to understand and to classify. Indeed, for compact Abelian groups, free and ergodic actions, i. e., free actions with trivial fixed point algebra, were completely classified by Olesen, Pedersen and Takesaki in [64] and independently by Albeverio and Høegh–Krohn in [42]. This classification was generalized to compactnon-Abelian groups by the remarkable work of Wassermann [300], [301], [302]. According to [42], [64], [300], for a compact group $G$ there is a 1-to-1 correspondence between free and ergodic actions of $G$ and unitary 2-cocycles of the dual group. An analogous result in the context of compact quantum groups has been obtained by Bichon, De Rijdt and Vaes [290]. Extending this classification beyond the ergodic case is however not straightforward because, even for a commutative fixed point algebra, the action cannot necessarily be decomposed into a bundle of ergodic actions.

The study of non-ergodic free actions is also motivated by their role as noncommutative principal bundles in noncommutative geometry. In fact, by a classical result, having a free action of a compact group $G$ on a locally compact space $P$ is equivalent saying that $P$ carries the structure of a principal bundle over the quotient $X := P/G$ with structure group $G$. Moreover, Rieffel showed that there is a 1-to-1 correspondence between classical free actions of compact groups on locally compact spaces and free actions of compact groups on commutative $C^*$-algebras (cf. [116]). From this perspective, the notion of a free action on a $C^*$-algebra provides a natural framework for noncommutative principal bundles, which become increasingly prevalent in application to geometry and physics. Regarding classification, the case of locally trivial principal bundles, that is, if $P$ is glued together from spaces of the form $U \times G$ with an open subset $U \subseteq X$, is very well-understood. This gluing immediately leads to $G$-valued cocycles. The corresponding cohomology theory, called $\check{C}$ech cohomology, gives a complete classification of locally trivial principal bundles with base space $X$ and structure group $G$.

The present is a sequel of [296] and [297], where we studied free actions of compact Abelian groups and so-called cleft actions, respectively. We achieved in [296] a complete classification of free, but not necessary ergodic actions of compact Abelian groups on unital $C^*$-algebras. This classification extends the results of [42], [64] and relies on the fact that the corresponding isotypic components are Morita self-equivalence over the fixed point algebra. Moreover, we provided a classification of principal bundles with compact Abelian structure group which are not locally trivial. For free actions of non-Abelian

compact groups the bimodule structure of the corresponding isotypic components is more subtle. For this reason we concentrated in [297] on a simple class of free actions of non-Abelian compact groups, namely cleft actions. Regarded as noncommutative principal bundles, these actions are characterized by the fact that all associated noncommutative vector bundles are trivial. We turn to the general case of free actions of compact quantum groups. The main objective is to provide a complete classification of free actions of compact quantum groups on unital $C^*$-algebras. Besides an interesting characterization of freeness, our approach uses the fact that nonergodic actions of compact quantum groups can be described in terms of weak unitary tensor functors, i. e., functors from the representation category of the underlying compact quantum group into the category of $C^*$-correspondences over the corresponding fixed point algebra (cf. [130]).
We introduce in the notion of freeness for compact $C^*$-dynamical systems and prove its equivalence to the Ellwood condition (Theorem (5.1.2)). We also list a few examples and establish the basis for our later classification results. we show that every free compact $C^*$-dynamical system gives rise to a socalled factor system and that free compact $C^*$-dynamical systems can be classified up to equivalence by their associated factor system (Theorem (5.1.11)). This extends the results presented in part 2 of this series [297], which deals with the particular class of cleft actions. Moreover, we give a K-theoretic characterization of cleft actions based on factor systems.
We show that the information provided by a factor system is enough to explicitly reconstruct the $C^*$-dynamical system by adapting results of [130]. This completes our classification result showing that there is a 1-to-1 correspondence between free compact $C^*$-dynamical systems and factor systems up to equivalence and conjugacy, respectively (Theorem (5.1.23)). As an application, we show that finite coverings of generic irrational rotation $C^*$-algebras are always cleft (Theorem (5.1.27)).

We concerned with free actions of compact groups on unital $C^*$-algebras and their classification. As a consequence, we use and blend tools from operator algebras and representation theory. e provide some definitions and notations which are repeatedly used.

Let $\mathcal{A}$ be a unital $C^*$-algebra. For the unit of $\mathcal{A}$ we write $\mathbb{1}_{\mathcal{A}}$ or simply $\mathbb{1}$. We will frequently deal with partial isometries, i. e., elements $v \in \mathcal{A}$ such that $v^*v$ and $vv^*$ are projections. In this case $v^*v$ is called the cokernel projection and $vv^*$ the range projection. Moreover, we say that a projection $p$ is larger than the range of an element $x$ if $px = x$, and we say that $p$ is larger than the cokernel of $x$ if $xp = x$. All tensor products of $C^*$-algebras are taken with respect to the minimal tensor product. We will frequently deal with multiple tensor products of unital $C^*$-algebras $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$. If there is no ambiguity, we regard $\mathcal{A}, \mathcal{B}$,, and $\mathcal{C}$ as subalgebras of $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ and extend maps on $\mathcal{A}, \mathcal{B}$, or $C$ canonically by tensoring with the identity map. For sake of clarity we may occasionally use the leg numbering notation, e. g., for $x \in \mathcal{A} \otimes \mathcal{C}$ we write $x_{13}$ to denote the corresponding element in $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$.
Inner products $\langle \cdot,\cdot \rangle$ on a Hilbert space is always assumed to be linear in the second component. For a Hilbert space $\mathfrak{H}_1, \mathfrak{H}_2$ we denote by $\mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ the set of bounded linear operators $T: \mathfrak{H}_1 \to \mathfrak{H}_2$. If $\mathfrak{H}_1 = \mathfrak{H}_2$ we briefly write $\mathcal{L}(\mathfrak{H}_1)$. We use the Dirac notation to specify operators, i. e., for two vectors $v_1 \in \mathfrak{H}_1, v_2 \in \mathfrak{H}_2$ we write $|v_2\rangle\langle v_1|$ for the operator $v \longmapsto \langle v_1, v\rangle v_2$.
For a unital $C^*$-algebra $\mathcal{A}$ a right pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module $\mathfrak{H}$ equipped with a sesquilinear map $\langle\cdot,\cdot\rangle_{\mathcal{A}} : H \times H \to \mathcal{A}$ that satisfies the usual axioms of a definite

inner product with $\mathcal{A}$-linearity in the second component. We call $\mathfrak{H}$ a right Hilbert $\mathcal{A}$-module if $\mathfrak{H}$ is complete with respect to the norm $\|x\|_{\mathfrak{H}} := \|\langle x,x\rangle\mathcal{A}\|^{1/2}$. The right Hilbert $\mathcal{A}$-module is called full if the two-sided ideal $\langle\mathfrak{H},\mathfrak{H}\rangle_{\mathcal{A}} := \overline{\text{lin}\{\langle x,y\rangle_{\mathcal{A}} \mid x,y\in\mathfrak{H}\}}$ is dense in $\mathcal{A}$. Since every dense ideal of $\mathcal{A}$ meets the invertible elements, in this case we have $\langle\mathfrak{H},\mathfrak{H}\rangle_{\mathcal{A}} = \mathcal{A}$. Left (pre-) Hilbert $\mathcal{A}$-modules are defined in a similar way.

A correspondence over $\mathcal{A}$, or a right Hilbert $\mathcal{A}$-bimodule, is a $\mathcal{A}$-bimodule $\mathfrak{H}$ equipped with a $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ which turns it into a right Hilbert $\mathcal{A}$-module such that the left action of $\mathcal{A}$ on $\mathfrak{H}$ is via adjointable operators. For two correspondences $\mathfrak{H}$ and $\mathfrak{K}$ over $\mathcal{A}$ we denote by $\mathfrak{H}\otimes_{\mathcal{A}}\mathfrak{K}$ their tensor product, on which the inner product is given by $\langle x_1\otimes y_1, x_{21}\otimes y_2\rangle_{\mathcal{A}} = \langle y_1, \langle x_1,x_2\rangle_{\mathcal{A}}, y_2\rangle_{\mathcal{A}}$ for all $x_1,x_2\in\mathfrak{H}_1$ and $y_1,y_2\in\mathfrak{K}$.

We rely on the $C^*$-algebraic notion of compact quantum groups as introduced by Woronowicz [303]. For an introduction and further details we recommend [291], [142], [298]. A compact quantum group is given by a unital $C^*$-algebra $\mathcal{G}$ together with a (usually implicit) faithful, unital$^*$-homomorphism $\Delta:\mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ satisfying the identity $(\Delta\otimes\text{id})\circ\Delta = (\text{id}\otimes\Delta)\circ\Delta$ and such that $\Delta(\mathcal{G})(\mathbb{1}\otimes\mathcal{G})$ is dense in $\mathcal{G}\otimes\mathcal{G}$. It can be shown that there is a unique state $h:\mathcal{G}\to\mathbb{C}$ such that $(\text{id}\otimes h)\circ\Delta = h = (h\otimes\text{id})\circ\Delta$ (see [303]). This state is called the Haar state of $\mathcal{G}$. It is not faithful in general but via the GNS-construction we may replace $\mathcal{G}$ by its reduced version on which the Haar state is faithful. Since $\mathcal{G}$ and its reduce version behave identically with respect to their representation theory and their actions (see [291]), we will throughout the text assume that the Haar state on $\mathcal{G}$ is faithful.

A unitary representation of a compact quantum group $\mathcal{G}$ on a finite-dimensional Hilbert space $V$ is a unitary element $\pi\,\mathcal{L}(V)\otimes\mathcal{G}$ such that $\text{id}\otimes\Delta(\pi) = \pi_{12}\,\pi_{13}$ in $\mathcal{L}(V)\otimes\mathcal{G}\otimes\mathcal{G}$. Unless explicitly stated otherwise, all representations are assumed unitary and finite dimensional. We recall that the set of equivalence classes of irreducible representations $\hat{\mathcal{G}}$ is countable and that the matrix coefficients of all $\pi\in\hat{\mathcal{G}}$ generate a dense$^*$-subalgebra of $\hat{\mathcal{G}}$. Since all constructions behave naturally with respect to intertwiners we will not distinguish between a representation and its equivalence class. The tensor product of two representations $(\pi,V)$ and $(\rho,W)$ of $\mathcal{G}$ is the representation $(\pi\otimes\rho, V\otimes W)$ given by the unitary element $\pi\otimes\rho := \pi_{13}\rho_{23}$ in $\mathcal{L}(V)\otimes\mathcal{L}(W)\otimes\mathcal{G}$. We also recall that for a representation $(\pi,V)$ of $\mathcal{G}$ the contragradient representation is in general not unitary. Its normalization $(\bar{\pi},\bar{V})$ is called the conjugated representation.

Some care has to be taken in the case that the Haar state is not tracial. Then the matrix coefficients with respect to some chosen basis of $V$ are not orthogonal in general. However, if $\pi$ is irreducible, there is a unique positive, invertible operator $Q(\pi)\in\mathcal{L}(V)$ normalized to $Tr[Q(\pi)] = Tr[Q(\pi)^{-1}]$ with

$$\frac{Tr[Q(\pi)T]}{Tr[Q(\pi)]}\mathbb{1}_V = \text{id}\otimes h\big(\pi(T\otimes\mathbb{1}_{\mathcal{G}})\pi^*\big), \qquad \frac{Tr[Q(\pi)^{-1}T]}{Tr[Q(\pi)]}\mathbb{1}_V = \text{id}\otimes h\big(\pi^*(T\otimes\mathbb{1}_{\mathcal{G}})\pi\big)$$

for every $T\in\mathcal{L}(V)$. The number $d_\pi := Tr[Q(\pi)]$ is called the quantum dimension of $\pi$. The quantum dimension behaves nicely with respect to taking direct sums, tensor products, and conjugated representations. An important detail for us is the fact that we may fix intertwiners $R:\mathbb{C}\to V\otimes\bar{V}$ and $\bar{R}:\mathbb{C}\to\bar{V}\otimes V$ for all irreducible representation such that $(R^*\otimes\text{id}V)(\text{id}V\otimes\bar{R}) = \text{id}_V$. In terms of an orthonormal basis

$e_1, \ldots, e_n \in V$ and its respective conjugated basis $\bar{e}_1, \ldots, \bar{e}_n \in V$ we typically choose $R(1) = \sum_{i=1}^{n} Q(\pi)^{1/2} e_i \otimes \bar{e}_i$.
An action of a compact quantum group $\mathcal{G}$ on a unital $C^*$-algebra $\mathcal{A}$ is a faithful, unital $^*$-homomorphism $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ that satisfies $(\mathrm{id} \otimes \Delta) \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \alpha$ and such that $(\mathbb{1} \otimes \mathcal{G})\, \alpha(\mathcal{A})$ is dense in $\mathcal{A} \otimes \mathcal{G}$. Since we assume that the Haar state is faithful, the map $P_1 := (\mathrm{id} \otimes h) \circ \alpha$ is a faithful conditional expectation onto the fixed point algebra $\mathcal{A}^{\mathcal{G}} := \{x \in \mathcal{A} \mid \alpha(x) = x \otimes \mathbb{1}\}$. In particular, $\mathcal{A}$ turns into a right pre-Hilbert $\mathcal{A}^{\mathcal{G}}$-bimodule with the $\mathcal{A}^{\mathcal{G}}$-valued inner product $\langle x, y \rangle_{\mathcal{A}^{\mathcal{G}}} := P_1(x^* y)$ for $x, y \in \mathcal{A}$. For each irreducible representation $\pi \in \hat{\mathcal{G}}$ the projection $P_\pi : \mathcal{A} \to \mathcal{A}$ onto the $\pi$-isotypic component $A(\pi) := P_\pi(\mathcal{A})$ is given by

$$P_\pi(a) := d_\pi\, Tr \otimes \mathrm{id}_{\mathcal{A}} \otimes h(\bar{\pi}_{13}(a)_{23}\, Q(\bar{\pi})_1^{-1}), \qquad a \in \mathcal{A},$$

where the leg numbering refers to $\mathcal{L}(\bar{V}) \otimes \mathcal{A} \otimes \mathcal{G}$ (see [167]). The set $A(\pi)$ is in fact closed with respect to the inner product (see [133]) and hence a correspondence over $\mathcal{A}^{\mathcal{G}}$. Furthermore, isotypic components for different $\pi \in \hat{\mathcal{G}}$ are orthogonal with respect to the inner product and the sum $\sum_{\pi \in \hat{\mathcal{G}}} A(\pi)$ is dense in $\mathcal{A}$.
Throughout the presentation we discuss compact $C^*$-dynamical systems $(\mathcal{A}, \mathcal{G}, \pi)$, by which we mean a unital $C^*$-algebra $\mathcal{A}$, a compact quantum group $\mathcal{G}$, and an action $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$. Given such a system, we recall that $\mathcal{A}$ can be decomposed in terms of its isotypic components $A(\pi)$, $\pi \in \hat{\mathcal{G}}$, and that each $A(\pi)$ is a correspondence over the fixed point algebra $\mathcal{A}^{\mathcal{G}} := \{x \in \mathcal{A} \mid \alpha(x) = x \otimes \mathbb{1}_{\mathcal{G}}\}$. For each irreducible representation $(\pi, V) \in \hat{\mathcal{G}}$ we denote by $\Gamma(V)$ the multiplicity space of the conjugated representation $\bar{\pi}$, which can be written in the form

$$\Gamma(V) = \{x \in V \otimes \mathcal{A} \mid \pi\alpha(x) = x \otimes \mathbb{1}_{\mathcal{G}}\}.$$

This space is naturally a correspondence over $\mathcal{A}^{\mathcal{G}}$ with respect to the usual left and right multiplication and the restriction of the inner product $\langle v \otimes a, w \otimes b \rangle_{\mathcal{A}^{\mathcal{G}}} := \langle v, w \rangle\, a^* b$ for all $v, w \in V$ and $a, b \in \mathcal{A}$. The $\pi$-isotypic component $A(\pi)$ is then as a correspondence isomorphic to $V \otimes \Gamma(\bar{V})$ via the isomorphism

$$\varphi_\pi : V \otimes \Gamma(\bar{V}) \to A(\pi), \qquad \varphi_\pi := \frac{1}{\sqrt{d_\pi}} R^* \otimes \mathrm{id}_{\mathcal{A}}\,. \tag{1}$$

The mapping $(\pi, V) \longmapsto \Gamma(V)$ can be extended to an additive functor from the representation category of $\mathcal{G}$ into the category of $C^*$-correspondences over $\mathcal{A}^{\mathcal{G}}$. Since $\mathcal{A}$ is the closure of the direct sum of its isotypic components and every isotypic component $A(\pi)$ is isomorphic to $V \otimes \Gamma(\bar{V})$, this functor allows us to reconstruct the Hilbert $\mathcal{A}^{\mathcal{G}}$-bimodule structure of $\mathcal{A}$ and the action $\alpha$ up to a suitable closure. To recover the multiplication on $\mathcal{A}$ we may look at the family of maps

$$m_{\pi,\rho}\; \Gamma(V) \otimes_{\mathfrak{B}} \Gamma(W) \to \Gamma(V \otimes W), \qquad m_{\pi,\rho}(x \otimes y) := x_{13}\, y_{23},$$

for representations $(\pi, V), (\rho, W)$ of $\mathcal{G}$. Here the subindices on the right hand side refer to the leg numbering in $V \otimes W \otimes \mathcal{A}$, that is, for elementary tensors $x = (v \otimes a)$ and $y = (w \otimes b)$ we write $x_{13}\, y_{23} = v \otimes w \otimes ab$ $(v \in V, w \in W, a, b \in \mathcal{A})$. The functor $\Gamma: V \longmapsto \Gamma(V)$ and the transformations $(m_{\pi,\rho})_{\pi,\rho}$ constitute a so-called weak tensor functor and allow to recover the reduced form of the compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ up to isomorphisms (see [130]).

To obtain a more concrete representation we restrict ourselves to the class of free action in the following sense. In addition to the correspondence structure, we equip each multiplicity space $\Gamma(V)$ with the left $\mathcal{L}(V) \otimes \mathcal{A}$-valued inner product given by

$$_{\mathcal{L}(V)_{\otimes\mathcal{A}}}\langle v \otimes a, w \otimes b \rangle := |v\rangle\langle w \otimes ab^*$$

for $v, w \in V$ and $a, b \in \mathcal{A}$. A few moments thought show that this left inner product takes values in the $C^*$-algebra $\{x \in \mathcal{L}(V) \otimes \mathcal{A} |\ \alpha(x) = \pi(x \otimes \mathbb{1}_{\mathcal{G}})\pi^*\}$ and that the only missing feature for $\Gamma(V)$ to be a Morita equivalence bimodule is that in general the left inner product need not be be full. This requirement is what we demand for a free action:

**Definition (5.1.1)[288]:** A compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is called free if for every $(\pi, V) \in \hat{\mathcal{G}}$ we have $\mathbb{1} \in {}_{\mathcal{L}(V)\otimes\mathcal{A}}\langle \Gamma(V), \Gamma(V)\rangle$.

There are various non-equivalent notions of freeness around the literature (see e. g. [161], [293] or [36] and ref. therein). The one given here was introduced for actions of classical groups by Rieffel [123] under the term saturated actions (see also the discussion after [81]) and it was already used in the other parts of this series [296], [297], where some equivalent conditions are summarized. A seemingly different version of freeness for actions of compact quantum groups was recently exploited by Commer et al. [133], [148] and is due to D. A. Ellwood [161]. We recall that a compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is said to satisfy the Ellwood condition if $(\mathcal{A} \otimes \mathbb{1})\alpha(\mathcal{A})$ is dense in $\mathcal{A} \otimes \mathcal{G}$. For classical groups it is well-known that the Ellwood condition is equivalent to freeness in the sense of Definition (5.1.1) (see [296]). The following result shows that this holds for compact quantum groups, too.

**Theorem (5.1.2)[288]:** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a compact $C^*$-dynamical system. Then the following conditions are equivalent:

**(i)** The $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is free.

**(ii)** For all representations $(\pi, V), (\rho, W)$ of $\mathcal{G}$ the map

$$m_{\pi,\rho} : \Gamma(V) \otimes_{\mathfrak{B}} \Gamma(W) \to \Gamma(V \otimes W), \qquad m_{\pi,\rho}(x \otimes y) = x_{13}\, y_{23},$$

has dense range or, equivalently, is surjective.

**(iii)** The $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ satisfies the Ellwood condition.

**Proof:** The implication (iii) ⇒ (i) follows from the proof of [133]. The implication (i) ⇒ (ii) will follow immediately from the independent later results and from Lemma (5.1.21). Therefore, we here focus on the implication (ii)⇒ (iii). We first observe that for (iii) to hold it is sufficient to verify that $(\mathcal{A} \otimes \mathbb{1})\alpha(\mathcal{A})$ contains a dense subset of $\mathbb{1} \otimes \mathcal{G}$. For this purpose we choose an irreducible representation $(\pi, V)$ of $\mathcal{G}$ and consider the co-restricted Ellwood map

$$\Phi_\pi : A(\bar{\pi}) \otimes A(\pi) \to A(\bar{\pi} \otimes \pi) \otimes G(\pi), \qquad x \otimes y \longmapsto (x \otimes \mathbb{1})\alpha(y)$$

(cf. [133]), where $G(\pi)$ denotes the ($\pi$-isotypic component of $\mathcal{G}$ with respect to the comultiplication. Moreover, we consider the following map which is a version of the co-restricted Ellwood map $\Phi_\pi$ using the identifications made in Equation (1):

$$\Psi_\pi : \bar{V} \otimes \Gamma(V) \otimes V \otimes \Gamma(\bar{V}) \to \bar{V} \otimes V \otimes \Gamma(V \otimes \bar{V}) \otimes G(\pi),$$

$$\bar{e}_i \otimes x \otimes e_i \otimes y \longmapsto \sum_{k=1}^{n} \bar{e}_i \otimes e_k \otimes m_{\pi,\bar{\pi}}(x \otimes y) \otimes \pi_{kj},$$

where $e_1, \dots, e_n$ denotes an orthonormal basis of $V$ and $\bar{e}_1, \dots, \bar{e}_n$ the corresponding dual basis of $\bar{V}$ . A few moments thought then substantiate the equation

$$\Phi_\pi \circ (\varphi_{\bar{\pi}} \otimes \varphi_\pi) = \sum_{\ell=1}^{m} \varphi_{\sigma_\ell} \circ (S_\ell^* \otimes \Gamma(\bar{S}_\ell)^*) \circ \Psi_\pi, \qquad (2)$$

where $S_1, \dots, S_m$ is a complete set of isometric intertwiners $S_\ell : V_{\sigma_\ell} \to \bar{V} \otimes V, \sigma_\ell \in \hat{\mathcal{G}}$, with respective conjugates $\bar{S}_\ell : \bar{V}_{\sigma_\ell} \to V \otimes \bar{V}$ . Next, we apply condition (ii) to find elements $x_1, \dots, x_N \in \Gamma(\bar{V})$ and $y_1, \dots, y_N \in \Gamma(\bar{V})$ with $\sum_{k=1}^{N} m_{\pi,\bar{\pi}}(x_k \otimes y_k) = \Gamma(R)(\mathbb{1}_{\mathfrak{B}})$.It follows that the element

$$z_{ij} = \sum_{k=1}^{n} Q\,(\pi)^{1/2} e_i \otimes x_k \otimes e_i \otimes yx$$

satisfies

$$\left(d_\pi^{-1/2} \bar{R}^* \otimes \Gamma\left(d_\pi^{-1/2} R\right)^* \otimes \mathrm{id}_{\mathcal{G}}\right)\left(\Psi_\pi\left(z_{ij}\right)\right) = \mathbb{1}_{\mathfrak{B}} \otimes \pi_{ij}.$$

On the other hand, the particular choice of the element $z_{ij}$ implies that only the intertwiner $S_\ell = d_\pi^{-1/2} R$ contributes to the right hand side of Equation (2), that is, the image of $\Phi_\pi$ contains the element $\mathbb{1}_{\mathfrak{B}} \otimes \pi_{ij}$ . Since the set of matrix coefficients is dense in $\mathcal{G}$, we conclude that the Ellwood condition is satisfied.

We continue with the following reformulation of freeness which will be convenient for our classification approach.

**Lemma (5.1.3)[288]:** A compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is free if and only if for every representation $(\pi, V)$ of $\mathcal{G}$ there is a finite-dimensional Hilbert space H and a coisometry $s \in \mathcal{L}(\mathfrak{H}, V) \otimes \mathcal{A}$ with $\pi\alpha(s) = s \otimes \mathbb{1}_{\mathcal{G}}$.

**Proof:** For the “if”-implication let $(\pi, V) \in \hat{\mathcal{G}}$ and let $s \in \mathcal{L}(\mathfrak{H}, V) \otimes \mathcal{A}$ be a coisometry with $\pi\alpha(s) = s$. We fix an orthonormal basis of $\mathfrak{H}$ and denote by $s_k \in V \otimes \mathcal{A}$ the columns of $s$.Then

$\sum_{k=1}^{n} {}_{\mathcal{L}(V)\otimes\mathcal{A}}\langle s_k, s_k \rangle = ss^* = \mathbb{1}$ . For the converse implication, we first observe that freeness of the $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ implies that, for each representation $(\pi, V)$ of $\mathcal{G}$, the space $\Gamma(V)$ is a Morita equivalence bimodule between the $C^*$-algebras $C(\pi) := \{x \in \mathcal{L}(V) \otimes \mathcal{A} \mid \alpha(x) = \pi x \pi^*\}$ and $\mathcal{A}^{\mathcal{G}}$. Since $C(\pi)$ is unital, we find elements $s_1, \dots, s_n \in \Gamma(V)$ such that

$\sum_{k=1}^{n} {}_{\mathcal{L}(V)\otimes\mathcal{A}}\langle s_k, s_k \rangle = \mathbb{1}$ (see Lemma (5.1.28)).We put $\mathfrak{H} := \mathbb{C}^n$ and denote by $s \in \mathcal{L}(\mathfrak{H}, V) \otimes \mathcal{A}$ the element with columns $s_1, \dots, s_n$ in the canonical orthonormal basis. Then $\pi\alpha(s) = s \otimes \mathbb{1}_{\mathcal{G}}$ , since $s_k \in \Gamma(V)$, and further $ss^* = \sum_{k=1}^{n} {}_{\mathcal{L}(V)\otimes\mathcal{A}}\langle s_k, s_k \rangle = \mathbb{1}$.

**Remark (5.1.4)[288]:** For each representation $\pi$ of $\mathcal{G}$ there is a minimal dimension, say $n(\pi)$, that the Hilbert space $\mathfrak{H}$ in Lemma (5.1.3) can take. Clearly we have $n(1) = 1, n(\pi \oplus \rho) \le n(\pi) + n(\rho)$ , and $n(\pi \oplus \rho) \le n(\pi) \cdot n(\rho)$ , using a variant of the multiplication map $m_{\pi,\rho}$.

Suppose we fix a Hilbert space $\mathfrak{H}_\pi$ and a respective coisometry $s(\pi)$ for each irreducible representation $\pi \in \hat{\mathcal{G}}$. Then we may extend $\pi \longmapsto \mathfrak{H}_\pi$ to an additive functor and $\pi \longmapsto s(\pi)$ to a family of coisometries that satisfies the condition in Lemma (5.1.3) and behaves

naturally with respect to intertwiners. However, the functor $\pi \longmapsto \mathfrak{H}_\pi$ is in general not a tensor functor and $s(\pi \oplus \rho)$ has no immediate relation to $s(\pi)$ and $s(\rho)$.
In the remaining part of present a collection of examples. To begin with, we recall that Definition (5.1.1) extends the classical notion of free actions of compact groups. In fact, given a compact space $P$ and a compact group $G$, it is a consequence of [116] that a continuous group action $\sigma: P \times G \to P$ is free, i. e., its stabilizer groups vanish at each point, if and only if the induced $C^*$-dynamical system $(C(P), G, \alpha_\sigma)$ is free in the sense of Definition (5.1.1). Therefore, Definition (5.1.1) also a natural framework for noncommutative principal bundles. Furthermore, we would like to point out that Definition (5.1.1) characterizes classical free group actions in of associated vector bundles and the condition therein means that the associated bundles have non-degenerate fibres (see e. g. [299]).
**Example (5.1.5)[288]:** We would like to recall a $C^*$-algebraic version of the nontrivial Hopf Galois extension studied in [295] (see also [292]). Let $\theta \in R$ be fixed and let $\theta'$ be the skewsymmetric $4 \times 4$-matrix with $\theta'_{1,2} = \theta'_{3,4} = 0$ and $\theta'_{1,3} = \theta'_{1,4} = \theta'_{2,3} = \theta'_{2,4} = \theta/2$. We consider the universal unital $C^*$-algebra $\mathcal{A}(\mathbb{S}^7_{\theta'})$ generated by normal elements $z_1, \ldots, z_4$ satisfying the relations

$$z_i z_j = e^{2\pi_i \theta'_{i,j}} z_j z_i, \qquad z_j^* z_i = e^{2\pi_i \theta'_{i,j}} z_i z_j^*, \qquad \sum_{k=1}^{4} z_k^* z_k = \mathbb{1}$$

for all $1 \leq i, j \leq 4$. A few moments thought show that the group $G = SU(2)$ acts strongly continuously on $\mathcal{A}(\mathbb{S}^7_{\theta'})$ via the $^*$-automorphisms $(\alpha U)_{U \in SU(2)}$ given on generators by

$$\alpha_U: (z_1, \ldots, z_4) \longmapsto (z_1, \ldots, z_4) \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

Moreover, the fixed point algebra turns out to be the universal unital $C^*$-algebra $\mathcal{A}(\mathbb{S}^4_\theta)$ generated by normal elements $w_1, w_2$ and a self-adjoint element $x$ satisfying

$$w_1 w_2 = e^{2\pi_i \theta} w_2 w_1, \quad w_2^* w_1 = e^{2\pi_i \theta} w_1 w_2^*, \quad \text{and} \qquad w_1^* w_1 + w_2^* w_2 + x^* x = \mathbb{1}.$$

For $\theta = 0$ all algebras are commutative and we recover the classical 7-dimensional Hopf fibration of the 4-sphere, which is a well-known example of a non-trivial principal bundle.Many arguments from the classical case can be extended to arbitrary $\theta$. In particular, it is easily checked that for the fundamental 2-dimensional representation $(\pi_1, \mathbb{C}^2)$ of SU(2) a coisometry $s \in \mathcal{L}(\mathbb{C}^4, \mathbb{C}^2) \otimes \mathcal{A}(\mathbb{S}^7_{\theta'})$ with $U_{\alpha U}(s) = s$ for all $U \in$ SU(2) is given by

$$s := \begin{pmatrix} z_1^* & -z_2 & z_3^* & -z_4 \\ z_2^* & z_1 & z_4^* & z_3 \end{pmatrix}.$$

Since every irreducible representation of SU(2) can be obtained as a subrepresentation of a suitable tensor powers of $\pi_1$, we may take tensor products of $s$ with itself in order to find a suitable coisometry for every representation $\pi$ of SU(2). We conclude that the compact $C^*$-dynamical system $(\mathcal{A}(\mathbb{S}^7_{\theta'}), \mathrm{SU}(2), \alpha)$ is free.
**Example (5.1.6)[288]: (i)** Bichon, De Rijdt and Vaes introduce in [290] the notion of quantum multiplicity of an irreducible representation in an ergodic action of a compact quantum group and classify ergodic actions of so-called full quantum multiplicity in terms of unitary fiber functors. It follows from that these actions are free.

**(ii)** According to [290], for sufficiently small parameters $q$ the compact quantum group $\mathrm{SU}_q(2)$ admits an ergodic action of full quantum multiplicity such that the multiplicity of the fundamental representation is arbitrarily large. Hence, there are plenty of free and ergodic actions of $\mathrm{SU}_q(2)$.

**Example (5.1.7)[288]:** Let $\mathcal{G}$ be an $R^+$-deformation (see e. g. [289]) of a semisimple compact Lie group. Furthermore, let $(\pi, \mathbb{C}^d)$ be a faithful representation of $\mathcal{G}$. Then [294] implies that the induced action $\alpha$ of $\mathcal{G}$ on the Cuntz algebra $\mathcal{O}_d$ defined by $\alpha(S_i) := \sum_{j=1}^{d} S_i \otimes \pi_{j,i}$ is free, where $S_1, \ldots S_d$ denote the generators of $\mathcal{O}_d$. It is not hard to check that for the representation $(\pi, \mathbb{C}^d)$ a coisometry $s \in \mathcal{L}(\mathbb{C}, \mathbb{C}^d) \otimes \mathcal{O}_d$ with $\pi\alpha(s) = s \otimes \mathbb{1}_{\mathcal{G}}$ is given by

$$s := (S_1^*, S_2^*, \ldots, S_d^*)^{\mathsf{T}}.$$

We have seen in Lemma (5.1.3) that freeness of a compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ can be cast in form of a family of coisometries. These coisometries may be used to give a more explicit picture of the spectral subspaces of the $C^*$-dynamical system. In fact, let $(\pi, V)$ be a representation of $\mathcal{G}$ and let $s(\pi) \in \mathcal{L}(\mathfrak{H}, V) \otimes \mathcal{A}$ be a coisometry with $\pi\alpha(s(\pi)) = s(\pi) \otimes \mathbb{1}_{\mathcal{G}}$ in $\mathcal{L}(\mathfrak{H}, V) \otimes \mathcal{A} \otimes \mathcal{G}$. Then a few moments thought show that the multiplicity space $\Gamma(V) \subseteq V \otimes \mathcal{A}$ is the range of the element $s(\pi)$, i. e., we have

$$\Gamma(V) = s(\pi)(\mathfrak{H} \otimes \mathcal{A}^{\mathcal{G}}). \tag{3}$$

The explicit form allows us to phrase the correspondence structure and the multiplicative structure among the generalized isotypic components only in terms of the fixed point algebra $\mathcal{A}^{\mathcal{G}}$ and the quantum group $\mathcal{G}$. This fact was already exploited in the previous part of this series [297], where we carried out the analysis in the case of cleft dynamical systems with a classical compact group. With some adjustments we generalize the construction here to arbitrary free $C^*$-dynamical systems and quantum groups.

We start with a free compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ and we write briefly $\mathfrak{B} := \mathcal{A}^{\mathcal{G}}$ for the corresponding fixed point algebra. Furthermore, we choose a functorial version of the finite-densional Hilbert spaces $\mathfrak{H}_\pi$ and the coisometries $s(\pi)$ for each representation $\pi$ of $\mathcal{G}$ (see also the discussion after Lemma (5.1.3)). In particular, we assume without loss of generality that $\mathfrak{H}_1 = \mathbb{C}$ and $s(1) = \mathbb{1}_{\mathfrak{B}}$. Then we consider for each representation $\pi$ of $\mathcal{G}$ the $^*$-homomorphism

$$\gamma_\pi : \mathfrak{B} \to \mathcal{L}(\mathfrak{H}_\pi) \otimes \mathfrak{B}, \qquad \gamma_\pi(b) := s(\pi)^* (\mathbb{1}_{V_\pi} \otimes b) s(\pi)$$

and for each pair $\pi, \rho$ of representations of $\mathcal{G}$ the element

$$w(\pi, \rho) := s(\pi \otimes \rho)^* s(\pi) s(\rho) \in \mathcal{L}(\mathfrak{H}_\pi \otimes \mathfrak{H}_{\pi\otimes\rho}) \otimes \mathfrak{B},$$

where $s(\pi)$ and $s(\rho)$ are amplified to act trivially on $\mathfrak{H}_\rho$ and $\mathfrak{H}_\pi$, respectively.

**Definition (5.1.8)[288]:** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a free compact $C^*$-dynamical system. Then the system $(\mathfrak{H}, \gamma, w) = (\mathfrak{H}_\pi, \gamma_\pi, w(\pi, \rho))_{\pi, \rho \in \hat{\mathcal{G}}}$ constructed above is called a factor system of $(\mathcal{A}, \mathcal{G}, \alpha)$.

**Remark(5.1.9)[288]:** For some computations it is convenient to express the factor system in terms of fixed orthonormal bases of the Hilbert spaces $\mathfrak{H}_\pi, \pi \in \hat{\mathcal{G}}$. In this situation we denote by $s(\pi)_1, \ldots, s(\pi)_n \in \Gamma(V)$ the columns of $s(\pi)$. Then the$^*$-homomorphism $\gamma_\pi : \mathfrak{B} \to M_n \otimes \mathfrak{B}$ has the coefficients

$$\gamma_\pi(b)_{i,j} = \langle s(\pi)_i, b \cdot s(\pi)_j \rangle_{\mathfrak{B}}$$

for all $1 \leq i \leq \dim \mathfrak{H}_\pi$ and $1 \leq j \leq \dim \mathfrak{H}_\rho$. For the partial isometry $w(\pi,\rho)$ we first fix an irreducible subrepresentation $\sigma$ of $\pi \otimes \rho$. Then the coefficients on the corresponding subspace $\mathfrak{H}_\sigma \subseteq \mathfrak{H}_\pi \otimes \mathfrak{H}_\rho$ are given by

$$w(\pi,\rho)_{(i,j),k} = \langle m(s(\pi)_i \otimes s(\rho)_j, s(\sigma)_k \rangle_{\mathfrak{B}}$$

for all $1 \leq i \leq \dim \mathfrak{H}_\pi$ , $1 \leq j \leq \dim \mathfrak{H}_\rho$ and $1 \leq k \leq \dim \mathfrak{H}_\sigma$.

Of course, different choices of Hilbert spaces $\mathfrak{H}_\pi$ and coisometries $s(\pi)$ give rise to different factor systems. However, as the following lemma shows, those choices only effect the factor system by a conjugacy with partial isometries:

**Lemma (5.1.10)[288]:** (cf. in [297]). For a factor system $(\mathfrak{H},\gamma,w)$ of a free compact $C^*$-dynamical system $(\mathcal{A},\mathcal{G},\alpha)$ with fixed point algebra $\mathfrak{B}$ the following assertions hold:

**(i)** We have $w(1,1) = \mathbb{1}_{\mathfrak{B}}, \gamma_1 = \mathrm{id}_{\mathfrak{B}}$ and

$$w(\pi,\rho)w(\pi,\rho)^* = \gamma_{\pi\otimes\rho}(\mathbb{1}), \qquad w(\pi,\rho)^* w(\pi,\rho) = \mathrm{id} \otimes \gamma_\rho(\gamma_\pi(\mathbb{1})) \tag{4}$$

$$\gamma_{\pi\otimes\rho}(b)w(\pi,\rho) = w(\pi,\rho)\gamma_\rho(\gamma_\pi(b)), \tag{5}$$

$$w(\pi,\rho\otimes\sigma)(\mathbb{1}\otimes w(\rho,\sigma)) = w(\pi,\rho\otimes\sigma)\mathrm{id}\otimes\gamma_\sigma(w(\pi,\rho)) \tag{6}$$

for all representations $\pi,\rho$ of $\mathcal{G}$ and $b \in \mathfrak{B}$. We refer to the Equation (5) as the coaction condition and to Equation (6) as the cocycle condition.

**(ii)** Let $(\mathfrak{H}',\gamma',w')$ be another factor system of $(\mathcal{A},\mathcal{G},\alpha)$.. Then there is a family of partial isometries $v(\pi) \in \mathcal{L}(\mathfrak{H}'_\pi,\mathfrak{H}_\pi) \otimes \mathfrak{B}, \pi \in \hat{\mathcal{G}}$, such that

$$v(\pi)v(\pi)^* = \gamma_\pi(\mathbb{1}), \qquad v(\pi)^* v(\pi) = \gamma'_\pi(\mathbb{1}) \tag{7}$$

$$v(\pi)\,\gamma'_\pi(b) = \gamma_\pi(b)v(\pi) \tag{8}$$

$$v(\pi\otimes\rho)w'(\pi,\rho) = w(\pi,\rho)\mathrm{id}\otimes\gamma_\rho(v(\pi))\,(\mathbb{1}\otimes v(\rho)) \tag{9}$$

hold for all $\pi,\rho \in \hat{\mathcal{G}}$ and $b \in \mathfrak{B}$.

**(iii)** Conversely, let $v(\pi) \in \mathcal{L}(\mathfrak{H}'_\pi,\mathfrak{H}_\pi) \otimes \mathfrak{B}, \pi \in \hat{\mathcal{G}}$, be a family of partial isometries for finite-dimensional Hilbert spaces $\mathfrak{H}'_\pi$ such that $v(\pi)\,v(\pi)^* = \gamma_\pi(\mathbb{1})$ holds for each $\pi \in \hat{\mathcal{G}}$. Then the following system $(\mathfrak{H}',\gamma',w')$ is a factor system of $(\mathcal{A},\mathcal{G},\alpha)$:

$$\gamma'_\pi(b) := v(\pi)^*\,\gamma_\pi(b)v(\pi)$$

$$w'(\pi,\rho) := v(\pi\otimes\rho)^*\; w(\pi,\rho)\,\mathrm{id}\otimes\gamma_\rho(v(\pi))\,(\mathbb{1}\otimes v(\rho))$$

for all $\pi,\rho \in \hat{\mathcal{G}}$ and $b \in \mathfrak{B}$.

**Proof:** For sake of a concise notation we amplify all elements to a common domain specified by the context. Let $s(\pi) \in \mathcal{L}(\mathfrak{H}_\pi, V_\pi) \otimes \mathcal{A}, \pi \in \hat{\mathcal{G}}$, be the coisometries with $\pi\,\alpha(s(\pi)) = s(\pi) \otimes \mathbb{1}_{\mathcal{G}}$ that generate the factor system $(\mathfrak{H},\gamma,w)$.

**(i)** Let $\pi,\rho$ be representations of $\hat{\mathcal{G}}$. Using the coisometry property of $s(\pi), s(\rho)$, and $s(\pi\otimes\rho)$ we obtain for the range and cokernel projection of $w(\pi,\rho)$

$$\begin{aligned} w(\pi,\rho)w(\pi,\rho)^* &= s(\pi\otimes\rho)^*s(\pi)s(\rho)s(\rho)^*\,s(\pi)^*\;s(\pi\otimes\rho) \\ &= s(\pi\otimes\rho)^*s(\pi\otimes\rho) = \gamma_{\pi\otimes\rho}(\mathbb{1}), \\ w(\pi,\rho)^*\,w(\pi,\rho) &= s(\rho)^*s(\pi)^*s(\pi\otimes\rho)s(\pi\otimes\rho)^*\,s(\pi)s(\rho) \\ &= s(\rho)^*s(\pi)^*s(\pi)s(\rho) = \gamma_\rho(\gamma_\pi(\mathbb{1})). \end{aligned}$$

To show the other two asserted equations we compute the left and right hand side individually using the coisometry property and compare for all $b \in \mathcal{A}^{\mathcal{G}}$:

$$w(\pi,\rho)\gamma_\rho\big(\gamma_\pi(b)\big) = s(\pi\otimes\rho)^* s(\pi)s(\rho)s(\rho)^*\, s(\pi)^*\ \ bs(\pi)s(\rho)$$
$$= s(\pi\otimes\rho)^* bs(\pi)s(\rho),$$
$$\gamma_{\pi\otimes\rho}(b)w(\pi,\rho) = s(\pi\otimes\rho)^* bs(\pi\otimes\rho)s(\pi\otimes\rho)^* s(\pi)s(\rho)$$
$$= s(\pi\otimes\rho)^* bs(\pi)s(\rho),$$
$$w(\pi,\rho\otimes\sigma)\, w(\rho,\sigma) = s(\pi,\rho\otimes\sigma)^*\, s(\pi)s(\rho\otimes\sigma)s(\rho\otimes\sigma)^* s(\rho)s(\sigma)$$
$$= s(\pi\otimes\rho\otimes\sigma)^*\, s(\pi)s(\rho)s(\sigma),$$
$$w(\pi\otimes\ \rho,\sigma)\,\gamma_\sigma(w(\pi,\rho))$$
$$= s(\pi\otimes\ \rho\otimes\sigma)^*\, s(\pi\otimes\rho)s(\sigma)s(\sigma)^* s(\pi\otimes\rho)^* s(\pi)s(\rho)s(\sigma)$$
$$= s(\pi\otimes\ \rho\otimes\sigma)^*\, s(\pi)s(\rho)s(\sigma).$$

**(ii)** Let $s'(\pi)\in\mathcal{L}(\mathfrak{H}'_\pi,V'_\pi)\otimes\mathcal{A},\pi\in\hat{\mathcal{G}}$, be the coisometries with $\pi\ \alpha(s'(\pi)) = s'(\pi)\otimes\mathbb{1}_\mathcal{G}$ that generate the factor system $(\mathfrak{H}',\gamma\,',w')$. Then the coisometry property implies that for each $\pi\in\hat{\mathcal{G}}$ the element $\upsilon(\pi) := s(\pi)^*\cdot s'(\pi)$ is a partial isometry satisfying $\upsilon(\pi)\,\upsilon(\pi)^* = s(\pi)^*\, s(\pi) = \gamma_\pi(\mathbb{1})$ and $\upsilon(\pi)^*\upsilon(\pi) = s'(\pi)^*\, s'(\pi) = \gamma'_\pi\,(\mathbb{1})$. Similarly the asserted relation of the$^*$-homomorphisms $\gamma_\pi$ and $\gamma'_\pi$ and of the elements $w(\pi,\rho)$ and $w'(\pi,\rho)$ immediately follow from the coisometry property.

Next, we explain how the correspondence structure of the isotypic components of a free compact $C^*$-dynamical system can be expressed only in terms of quantities of an associated factor system. For this purpose, let $(\mathfrak{H},\gamma,w)$ be a factor system of a free compact $C^*$-dynamical system $(\mathcal{A},\mathcal{G},\alpha)$: with fixed point algebra $\mathfrak{B}$. Then, for a representation $(\pi,V)$ of $\mathcal{G}$, the left and right action of $\mathfrak{B}$ and the inner product on $\Gamma(V)$ are given by

$$b\,.(s(\pi)x) = s(\pi)\gamma_\pi(b)x, \qquad (10)$$
$$(s(\pi)x)\,.b = s(\pi)xb \qquad (11)$$
$$\langle s(\pi)x, s(\pi)\,y\rangle_\mathfrak{B} = \langle x,\gamma_\pi(\mathbb{1}_\mathfrak{B})y\rangle_\mathfrak{B} \qquad (12)$$

for all $b\in\mathfrak{B}$ and $x,y\in\mathfrak{H}_\pi\otimes\mathfrak{B}$. Moreover, for two representation $(\pi,V)$ and $(\rho,W)$ of $\mathcal{G}$ the multiplication map $m_{\pi,\rho}:\Gamma(V)\otimes_\mathfrak{B}\Gamma(W)\to\Gamma(V\otimes W)$ can be written as

$$m_{\pi,\rho}(s(\pi)x\otimes s(\rho)y) = s(\pi\otimes\rho)w(\pi,\rho)\gamma_\rho(x)y \qquad (13)$$

for all $x\in\mathfrak{H}_\pi\otimes\mathfrak{B}$ and $y\in\mathfrak{H}_\rho\otimes\mathfrak{B}$, where $\gamma_\rho(x)\,y$ is given by the linear extension of $\gamma_\rho(\upsilon\otimes b_1)(w\otimes b_2) = \upsilon\otimes(\gamma_\rho(b_1))(w\otimes b_2))$ for all $\upsilon\in\mathfrak{H}_\pi, w\in\mathfrak{H}_\rho,$ and $b_1,b_2\in\mathfrak{B}$. As a consequence, up to equivalence, the $C^*$-dynamical system is uniquely determined by its factor system and vice versa. More precisely, we say that two factor systems $(\mathfrak{H},\gamma,w)$ and $(\mathfrak{H}',\gamma\,',w')$ are conjugated if there is a family of partial isometries $\upsilon(\pi)\in\mathcal{L}(\mathfrak{H}'_\pi,\mathfrak{H}_\pi)\otimes\mathfrak{B},\pi\in\hat{\mathcal{G}}$, satisfying the Equations (7), (8), and (9) for all $\pi,\rho,\sigma\in\hat{\mathcal{G}}$ and $b\in\mathfrak{B}$. Then we have the following 1-to-1 correspondence:

**Theorem (5.1.11)[288]:** Let $(\mathcal{A},\mathcal{G},\alpha)$ and $(\mathcal{A}',\mathcal{G}',\alpha')$ be free compact $C^*$-dynamical systems with the same fixed point algebra $\mathfrak{B}$ and let $(\mathfrak{H},\gamma,w)$ and $(\mathfrak{H}',\gamma',w')$ be associated factor systems, respectively. Then the following statements are equivalent:

**(i)** The $C^*$-dynamical systems $(\mathcal{A},\mathcal{G},\alpha)$ and $(\mathcal{A}',\mathcal{G}',\alpha')$ are equivalent.

**(ii)** The factor systems $(\mathfrak{H},\gamma,w)$ and $(\mathfrak{H}',\gamma',w')$ are conjugated.

**Proof:** As a distinction we add a prime to all notions referring to $(\mathcal{A}',\mathcal{G}',\alpha')$.

**(I)** To prove that (i) implies (ii) it suffices to show that for the same $C^*$-dynamical system different choices of coisometries lead to conjugated factor systems. This is exactly the second statement of Lemma (5.1.10).

**(II)** For the converse implication let $s(\pi) \in \mathcal{L}(\mathfrak{H}_\pi, V_\pi) \otimes \mathcal{A}, \pi \in \hat{\mathcal{G}}$, be the coisometries with $\pi\, \alpha(s(\pi)) = s(\pi) \otimes \mathbb{1}_\mathcal{G}$ that generate the factor system $(\mathfrak{H}, \gamma, w)$, and likewise $s'(\pi) \in \mathcal{L}(\mathfrak{H}'_\pi, V_\pi) \otimes \mathcal{A}$ for $(\mathfrak{H}', \gamma', w')$. Furthermore, let $v(\pi), \pi \in \hat{\mathcal{G}}$, be the partial isometries which realize the conjugation of the factor systems. Then a few moments thought show that, due to Equations (7) and (8), for every representation $(\pi, V)$ of $\mathcal{G}$ the map

$$\phi_\pi : \Gamma'(V) \to \Gamma(V), \qquad s'(\pi)\, x \longmapsto s(\pi) v(\pi)\, x$$

for all $x \in \mathfrak{H}'_\pi \otimes \mathfrak{B}$ is a well-defined isomorphism of correspondences of $\mathfrak{B}$. Moreover, by Equation (9), these isomorphisms intertwine the multiplication maps, i. e., we have

$$m_{\pi,\rho}(\phi_\pi(x) \otimes \phi_\rho(y)) = \phi_{\pi\otimes\rho}(m'_{\pi,\rho}(x \otimes y))$$

for all representations $(\pi, V), (\rho, W) \in \hat{\mathcal{G}}$ and all elements $x \in \Gamma'(V)$ and $y \in \Gamma'(W)$. Since $(\mathcal{A}, \mathcal{G}, \alpha)$ can be reconstructed from the correspondences $\Gamma(V)$ and the multiplicative structure between them (cf. Lemma (5.1.21) or [130]), and likewise for $(\mathcal{A}', \mathcal{G}', \alpha')$ with $\Gamma'(V)$, it is now easily checked that the maps $\phi_\pi, \pi \in \hat{\mathcal{G}}$, give rise to an equivalence between $(\mathcal{A}, \mathcal{G}, \alpha)$ and $(\mathcal{A}', \mathcal{G}', \alpha')$ (cf. also [297]).

A particular simple class of free actions are so-called cleft actions (see [297]). Regarded as noncommutative principal bundles, these actions are characterized by the fact that all associated noncommutative vector bundles are trivial.

**Definition (5.1.12)[288]:** A compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is called cleft if for each irreducible representation $(\pi, V)$ of $\mathcal{G}$ the so-called generalized isotypic component

$$A_2(\pi) \coloneqq \{x \in \mathcal{L}(V)\otimes\mathcal{A} \mid \pi\, \alpha(x) = x\otimes\mathbb{1}_\mathcal{G}\} \subseteq \mathcal{L}(V)\otimes\mathcal{A}$$

contains a unitary element. It directly follows from Lemma (5.1.3) that cleft $C^*$-dynamical systems are free.

**Example (5.1.13)[288]:** Given a unital $C^*$-algebra $\mathfrak{B}$ and a compact quantum group $\mathcal{G}$, the most basic example of a cleft action is given by the $C^*$-dynamical system $(\mathfrak{B} \otimes \mathcal{G}, \mathcal{G}, \mathrm{id} \otimes \Delta)$. In fact, for an irreducible representation $(\pi, V)$ of $\mathcal{G}$ it is easily seen that the unitary element $U := \pi_{13}^* \in \mathcal{L}(V) \otimes \mathfrak{B} \otimes \mathcal{G}$ satisfies $\pi(\mathrm{id} \otimes \Delta)(U) = U \otimes \mathbb{1}_\mathcal{G}$.

**Example (5.1.14)[288]:** For $\mathcal{G} = \mathrm{SU}_q(2)$ the only cleft and ergodic action is the canonical action of $\mathrm{SU}_q(2)$ on itself (see [290]). For $q = 1$ this already follows from the seminal work of Wassermann [301].

**Example (5.1.15)[288]:** (cf. Example (5.1.6))**.** For an arbitrary compact quantum group, the authors of [290] provide a classification of unitary fiber functors which preserve the dimension in terms of unitary 2-cocycles on the dual quantum group. It is not hard to see that the corresponding actions are cleft.

**Example (5.1.16)[288]:** It can be shown that the free $C^*$-dynamical system $(\mathcal{A}(\mathbb{S}^7_{\theta'}), \mathrm{SU}(2), \alpha)$ discussed in Example (5.1.5) is not cleft (cf. [295]).

We continue with a characterization of cleft actions in terms of their factor systems and the $K_0$-groups of the underlying fixed point algebras.

**Lemma (5.1.17)[288]:** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a free compact $C^*$-dynamical system with fixed point algebra $\mathfrak{B}$. Then the following statements are equivalent:

**(i)** The system $(\mathcal{A}, \mathcal{G}, \alpha)$ is cleft.

**(ii)** For some and hence for every factor system $(\mathfrak{H},\gamma,w)$ and every $\pi\in\hat{\mathcal{G}}$ we have $[\gamma_\pi(\mathbb{1}_\mathfrak{B})]=\dim(\pi)\cdot[\mathbb{1}_\mathfrak{B}]$ in $K_0(\mathfrak{B})$.

**Proof:** If $(\mathcal{A},\mathcal{G},\alpha)$ is cleft, every generalized isotypic component $A_2(\pi),\pi\in\hat{\mathcal{G}}$, contains a unitary element $s(\pi)$. The corresponding factor system $(\mathfrak{H},\gamma,w)$ is then given by $\mathfrak{H}_\pi=\bar{V}_\pi$ and $\pi(x)=s(\pi)^*(x\otimes\mathbb{1})s(\pi)$ for all $\pi\in\hat{\mathcal{G}}$. In particular, in $K_0(\mathfrak{B})$ we have

$$[\gamma_\pi(\mathbb{1}_\mathfrak{B})]=[s(\pi)^*s(\pi)]=[\mathbb{1}_\mathfrak{B}\otimes\mathbb{1}_\pi]=\dim(\pi)\cdot[\mathbb{1}_\mathfrak{B}].$$

By Lemma (5.1.10) every other factor system $(\mathcal{A},\mathcal{G},\alpha)$ differs only by partial isometries in a respective amplification and therefore satisfies the same equation in $K_0(\mathfrak{B})$.

Conversely, suppose that $(\mathfrak{H},\gamma,w)$ is a factor system of $(\mathcal{A},\mathcal{G},\alpha)$ satisfying the equation $[\gamma_\pi(\mathbb{1}_\mathfrak{B})]=\dim(\pi)\cdot[\mathbb{1}_\mathfrak{B}]$ in $K_0(\mathfrak{B})$. for all $\pi\in\hat{\mathcal{G}}$. Since $[\gamma_\pi(\mathbb{1}_\mathfrak{B})]=\dim(\pi)\cdot[\mathbb{1}_\mathfrak{B}]=[\mathbb{1}_\mathfrak{B}\otimes\mathbb{1}_\pi]$ in $K_0(\mathfrak{B})$, we find, for every $\pi\in\hat{\mathcal{G}}$, a partial isometry $v(\pi)\in\mathfrak{B}\otimes\mathcal{L}(V_\pi,\mathfrak{H}_\pi)$ with $\gamma_\pi(\mathbb{1}_\mathfrak{B})=v(\pi)v(\pi)^*$. By conjugating with this family of partial isometries we may assume that $\mathfrak{H}_\pi=V_\pi$ and $\gamma_\pi(\mathbb{1}_\mathfrak{B})=\mathbb{1}_\mathfrak{B}\otimes\mathbb{1}_\pi$. Let $s(\pi),\pi\in\hat{\mathcal{G}}$, be the family of associated coisometries in $A_2(\pi)$ with $\pi(x)=s(\pi)^*(x\otimes\mathbb{1}_\pi)s(\pi)$ for all $x\in\mathfrak{B}$. Then for $x:=\mathbb{1}_\mathfrak{B}$ we obtain $\mathbb{1}_\mathfrak{B}\otimes\mathbb{1}_\pi=\gamma_\pi(\mathbb{1}_\mathfrak{B})=s(\pi)^*s(\pi)$, i. e., $s(\pi)$ is in fact unitary.

**Example (5.1.18)[288]:** Suppose we are in the context of Example (5.1.7) and denote by $\mathfrak{B}$ the fixed point algebra of the free compact $C^*$-dynamical system $(O_d,\mathcal{G},\alpha)$. Then a few moments thought show that the$^*$-homomorphism $\gamma_\pi:\mathfrak{B}\to\mathfrak{B}$ induced by the coisometry

$$s:=(S_1^*,S_2^*,\dots,S_d^*)^\intercal$$

is given by $\gamma_\pi(b)=S_1bS_1^*+\dots+S_dbS_d^*$ for all $b\in\mathfrak{B}$. In particular, we have $\gamma_\pi(\mathbb{1}_\mathfrak{B})=\mathbb{1}_\mathfrak{B}$. Therefore, Lemma (5.1.17) implies that $(\mathcal{O}_d,\mathcal{G},\alpha_\pi)$ is not cleft.

We have seen that a free compact $C^*$-dynamical system is uniquely determined by its factor system $(\mathfrak{H},\gamma,w)$ and under which equivalence relation this becomes 1-to-1 correspondence (Theorem (5.1.11)). We will show that in fact every factor system $(\mathfrak{H},\gamma,w)$ satisfying the algebraic relations of Lemma (5.1.10) gives rise to a free compact $C^*$-dynamical system. The construction is based on the fact, that the factor system $(\mathfrak{H},\gamma,w)$ allows us to completely reconstruct the correspondence structure of the multiplicity spaces $\Gamma(V)$ and their multiplicative structure, i. e., the factor system provides a unitary tensor functor $V\longmapsto\Gamma(V)$ and hence a compact $C^*$-dynamical system (see [134], [130]). We recall the major steps in order to show that this construction yields a free compact $C^*$-dynamical system with factor system $(\mathfrak{H},\gamma,w)$.

Throughout the following let $\mathfrak{H}$ be a fixed unital $C^*$-algebra and let $\mathcal{G}$ be a fixed reduced compact quantum group. Furthermore, let $(\mathfrak{H},\gamma,w)=(\mathfrak{H}_\pi,\gamma_\pi,w(\pi,\rho))_{\pi,\rho\in\hat{\mathcal{G}}}$ be a family of finite-dimensional Hilbert spaces $\mathfrak{H}_\pi$, $*$-homomorphisms $\gamma_\pi:\mathfrak{B}\to\mathcal{L}(\mathfrak{H}_\pi)\otimes\mathfrak{B}$, and partial isometries $w(\pi,\rho)\in\mathcal{L}(\mathfrak{H}_\pi\otimes\mathfrak{H}_\rho,\mathfrak{H}_{\pi\otimes\rho})\otimes\mathfrak{B}$. By taking direct sums of irreducible representations, we define $\mathfrak{H}_\pi,\gamma_\pi$ and $w(\pi,\rho)$ for arbitrary representations $\pi,\rho$ of $\mathcal{G}$. In particular, for each intertwiner $T:V_\pi\to V_\rho$ we have a linear map $H(T)\,\mathfrak{H}_\pi\to\mathfrak{H}_\rho$.

**Definition (5.1.19)[288]:** A system $(\mathfrak{H},\gamma,w)$ as described above is called a factor system for the pair $(\mathfrak{B},\mathcal{G})$ if it satisfies Equations (4), (5), (6) for all $\pi,\rho\in\hat{\mathcal{G}}$ and $b\in\mathfrak{B}$, and if the normalization conditions $\mathfrak{H}_1=\mathbb{C}_1,=\mathrm{id}_\mathfrak{B},w(1,1)=\mathbb{1}_\mathfrak{B}$ holds.

From now on we suppose that $(\mathfrak{H}, \gamma, w)$ is a factor system. Then, for each representation $(\pi, V)$ of $\mathcal{G}$, we consider the vector space

$$\Gamma(V) := \gamma_\pi(\mathbb{I})(\mathfrak{H}_\pi \otimes \mathfrak{B}). \tag{14}$$

A few moments thought show that this space caries a natural right Hilbert $\mathfrak{B}$ –module structure given by restricting the action $(v_1 1 \otimes b_1) \,.\, b_2 := v_1 \otimes b_1 b_2$ and the inner product $\langle v_1 \otimes b_1, v2 \otimes b_2 \rangle_{\mathfrak{B}} := \langle v_1, v_2 \rangle \, b_1^* b_2$ for $v_1, v_2 \in \mathfrak{H}_\pi$ and $b_1, b_2 \in \mathfrak{B}$. Moreover, we equip $\Gamma(V)$ with the left action $b\,.\,x := \gamma_\pi(b)x$ for $b \in \mathfrak{B}$ and $x \in \Gamma(V)$. Then it is easily checked that $\Gamma(V)$ is a correspondence over $\mathfrak{B}$ and that $V \longmapsto \Gamma(V)$ becomes an additive functor from the representation category of $\mathcal{G}$ into the category of $C^*$-correspondences over $\mathfrak{B}$.

For each pair $(\pi, V), (\rho, W)$ of representation of $\mathcal{G}$ we define a linear map

$$m_{\pi,\rho} : \Gamma(V) \otimes_{\mathfrak{B}} \Gamma(W) \to \Gamma(V \otimes W)$$
$$m_{\pi,\rho} : (x \otimes y) := w(\pi, \rho)\gamma_\rho(x)\mathrm{y}, \tag{15}$$

where for elementary tensors we write briefly $\gamma_\rho(v \otimes b_1)(w \otimes b_2) := v \otimes \gamma_\rho(b_1)(w \otimes b_2)$ for all $v \in \mathfrak{H}_\pi, w \in \mathfrak{H}_\rho$, and $b_1, b_2 \in \mathfrak{B}$. It is easily checked that the maps $m_{\pi,\rho}$ are well-defined and behave naturally with respect to intertwiners. In fact, we are going to show that $V \longmapsto \Gamma(V)$ together with the maps $m_{\pi,\rho}$ forms a unitary tensor functor in the sense of [130]. **Definition (5.1.20)[288]:** A linear functor $V \longmapsto \Gamma(V)$ from the representation category of $\mathcal{G}$ into the category of $C^*$-correspondences over $\mathfrak{B}$ together with $\mathfrak{B}$ -bilinear family of unitary maps $m_{\pi,\rho}: \Gamma(V) \otimes_{\mathfrak{B}} \Gamma(W) \to \Gamma(V \otimes W)$ for all representations $(\pi, V), (\rho, W)$ of $\mathcal{G}$ is called a unitary tensor functor if the following conditions hold:

**(i)** For the trivial representation $(1, \mathbb{C}) \in \hat{\mathcal{G}}$ we have $\Gamma(\mathbb{C}) = \mathfrak{B}$ and for all $(\pi, V) \in \hat{G}$ we have $m_{\pi,1}(x \otimes b) = x.b$ and $m_{1,\pi}(b \otimes x) = b\,.\,x$ for all $x \in \Gamma(V), b \in \mathfrak{B}$.

**(ii)** For every intertwiner $T: V \to W$ we have $\Gamma(T^*) = \Gamma(T)^*$.

**(iii)** The maps $m$ are associative in the sense that for all $\pi, \rho, \sigma \in \hat{G}$ we have

$$m_{\pi,\rho\otimes\sigma} \circ (\mathrm{id} \otimes m_{\rho,\sigma}) = m_{\pi\otimes\rho,\sigma} \circ (m_{\pi,\rho} \otimes \mathrm{id}).$$

**Lemma (5.1.21)[288]:** The functor $V \longmapsto \Gamma(V)$ and the maps $m_{\pi,\rho} : \Gamma(V) \otimes \mathfrak{B} : \Gamma(W) \to: \Gamma(V \otimes W)$ given by the Equations (14) and (15), respectively, for $(\pi, V), (\rho, W) \in \hat{\mathcal{G}}$ constitute a unitary tensor functor.

**Proof:** **(i)** The normalization $\Gamma(\mathbb{C}) = \mathfrak{B}$ as correspondence immediately follows from $\mathfrak{H}_1 = \mathbb{C}$ and $\gamma_1 = \mathrm{id}_{\mathfrak{B}}$. Moreover, the normalization $w(1,1) = 1$ together with Conditions (4) and (5) of the factor system imply $w(\pi, 1) = \mathbb{I}_{\mathfrak{H}_\pi} = w(1, \pi)$ thatfor every $(\pi, V) \in \hat{\mathcal{G}}$. Hence, we obtain $m_{\pi,1}(x \otimes b) = w(\pi, 1)\gamma_{\mathfrak{B}}(x)b = x\,.\,b$ and $m_{1,\pi}(b \otimes x) = w(1, \pi)\gamma_\pi(b)x = b\,.\,x$ for all $x \in \Gamma(V)$ and $b \in \mathfrak{B}$.

**(ii)** For any intertwiner $T: V \to W$ it is easily checked that $\mathfrak{H}(T^*) = \mathfrak{H}(T)^*$ which in turn implies that $\Gamma(T) = \mathfrak{H}(T) \otimes \mathrm{id}_{\mathfrak{B}}|_{\Gamma(V)}$ is adjointable with $\Gamma(T) = \Gamma(T)^*$.

**(iii)** Associativity is an immediate consequence of the coaction and cocycle condition of the factor system. More precisely for all representations $\pi, \rho, \sigma \in \hat{\mathcal{G}}$ and elements $x \in \Gamma(V_\pi), y \in \Gamma(V_\rho), z \in \Gamma(V_\sigma)$ we have

$$m_{\pi,\rho\otimes\sigma}(x \otimes m_{\rho,\sigma}(y \otimes z)) = w(\pi, \rho \otimes \sigma)\gamma_{\rho\otimes\sigma}(x)(w(\rho,\sigma)\gamma_\sigma(y)z)$$
(5) (6)
$$= w(\pi,\rho,\otimes \sigma)w(\rho,\sigma)\gamma_\sigma(\gamma_\rho\big((x)y\big)z = w(\pi \otimes$$

$$\rho,\sigma)\gamma_\sigma\big(w(\pi,\rho)\big)\gamma_\sigma(\gamma_\sigma(x)y)$$
$$= w(\pi \otimes \rho,\sigma)\gamma_\sigma\big(w(\pi,\rho)\gamma_\rho(x)y\big)z = m_{\pi\otimes\rho,\sigma}\big(m_{\pi,\rho}(x \otimes y) \otimes z\big).$$

**(iv)** It remains to show that the maps $m_{\pi,\rho} : \Gamma(V) \otimes_{\mathfrak{B}} \Gamma(W) \to \Gamma(V \otimes W)$ are unitary for all representations $(\pi, V), (\rho, W)$ of $\mathcal{G}$. To see that $m_{\pi,\rho}$ is isometric we observe that by Equation (4) the projection $w(\pi,\rho)^* w(\pi,\rho) = \gamma_\rho(\gamma_\pi(\mathbb{1}))$ is larger than than the subspace of $\mathfrak{H}_\pi \otimes \mathfrak{H}_\rho \otimes \mathfrak{B}$ generated by all $\rho(x)y$ with $x \in \Gamma(V), y \in \Gamma(W)$.Hence, we have

$$\langle m_{\pi,\rho}(x_1 \otimes y_1), m_{\pi,\rho}(x_2 \otimes y_2)\rangle_{\mathfrak{B}} = \langle \gamma_\rho(x_1)y_1, w(\pi,\rho)^*\, w(\pi,\rho)\gamma_\rho(x_2)y_1\rangle_{\mathfrak{B}}$$
$$= \langle \gamma_\rho(x_1)y_1, \gamma_\rho(x_2)y_1\rangle_{\mathfrak{B}} = \langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{\mathfrak{B}}$$

for all $x_1, x_2 \in \Gamma(V)$ and $y_1, y_2 \in \Gamma(W)$. To show that $m_{\pi,\rho}$ is surjective, we notice that $\Gamma(V)$ is linearly generated by all elements of the for $\gamma_\pi(\mathbb{1})(v \otimes b)$ with $v \in \mathfrak{H}_\pi$ and $b \in \mathfrak{B}$; and likewise for $\Gamma(W)$. By Equation (4) the projection $\mathbb{1}_{\mathfrak{H}_\pi} \otimes \gamma_\rho(\mathbb{1}_{\mathfrak{B}})$ is larger than the cokernel projection $w(\pi,\rho)^* w(\pi,\rho) = \gamma_\rho(\gamma_\pi(\mathbb{1}))$. Choosing the elements $x := v \otimes \mathbb{1}_{\mathfrak{B}}$ and $y := w \otimes b$, we therefore find that the range of $m_{\pi,\rho}$ contains all elements of the form

$$w(\pi,\rho)(v \otimes \gamma_\rho(\mathbb{1})(w \otimes b)) = w(\pi,\rho)(v \otimes w \otimes b)$$

with $v \in \mathfrak{H}_\pi$, $w \in \mathfrak{H}_\rho$, $b \in \mathfrak{B}$. Hence, the image of $m_{\pi,\rho}$ contains the range of $w(\pi,\rho)$, which by Equation (4) is given by $\gamma_{\pi\otimes\rho}(\mathbb{1})(\mathfrak{H}_{\pi\otimes\rho} \otimes \mathfrak{B}) = \Gamma(V \otimes W)$.

Having the unitary tensor functor in hands, we may construct a $C^*$-dynamical system as presented in [134], [130]. For convenience of the reader we briefly summarize the main steps. We consider the algebraic direct sum

$$A := \bigoplus_{(\pi,V)\in\hat{\mathcal{G}}} V \otimes \Gamma(\bar{V}).$$

We equip each summand of this space with its canonical $\mathfrak{B}$ -valued inner product given by $\langle v \otimes x, w \otimes y\rangle_{\mathfrak{B}} = \langle v, w\rangle\, \langle x, y\rangle_{\mathfrak{B}}$ for all $v, w \in V$ and $x, y \in \Gamma(\bar{V})$, and we extend the resulting inner product sesquilinearly to $A$. Moreover, we equip $A$ with the multiplication defined, for $\bar{v} \otimes x \in \bar{V} \otimes \Gamma(V)$ and $\bar{w} \otimes y \in \bar{W} \otimes \Gamma(W)$ with $(\pi, V), (\rho, W) \in \hat{\mathcal{G}}$, by the product

$$(v \otimes x) \bullet (w \otimes y) := \sum_{k=1}^{N} S_k^* \otimes \Gamma(\bar{S}_k)^*(v \otimes w \otimes m_{\bar{\pi},\bar{\rho}}(x \otimes y)) \in \sum_{k=1}^{N} V_{\sigma_k} \otimes \Gamma(\bar{V}_{\sigma_k}),$$

where $S_1, \dots, S_N$ is a complete set of isometric intertwiners $S_k: V_{\sigma_k} \to V \otimes W, \sigma_k \in \hat{\mathcal{G}}$, with respective conjugates $\bar{S}_k: \bar{V}_{\sigma_k} \to \bar{V} \otimes \bar{W}$. Extending this product bilinearly yields an associative multiplication on $A$. The algebra B can be regarded as the subalgebra of $A$ corresponding to the trivial representation, and the left and right module action of $\mathfrak{B}$ coincides with the multiplication on $A$.

The next step is to construct an involution on $A$. For this purpose we first recall that for an irreducible representation $(\pi, V)$ of $\hat{\mathcal{G}}$ there is a pair of intertwiners $R: \mathbb{C} \to V \otimes \bar{V}$ and

$\bar{R}: \mathbb{C} \to \bar{V} \otimes V$ such that $(R^* \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \bar{R}) = \mathrm{id}_V$ . With this we may define involutions $^+ : \Gamma(V) \to \Gamma(\bar{V})$ and $^+ : \bar{V} \to V$ by putting

$$x^+ := m[x]^*(\Gamma(R)(\mathbb{1}_{\mathfrak{B}})), \qquad \bar{v}^+ := i[\bar{v}]^* \bar{R}(1)$$

where we briefly write $m[x]: \Gamma(\bar{V}) \to \Gamma(V \otimes \bar{V})$ for the map $m[x](y) := m_{\pi,\bar{\pi}}(x \otimes y)$ and $i[\bar{v}] : V \to V \otimes \bar{V}$ for the map $i[\bar{v}](w) := \bar{v} \otimes w$. Then for $\bar{v} \otimes x \in \bar{V} \otimes \Gamma(V) \subseteq A$ we may put $(\bar{v} \otimes x)^+ := \bar{v}^+ \otimes x^+$ and extend this anilinearly to a map on $A$. It can be shown that this involution turns $A$ into a$^*$-algebra (see [130]).

**Remark(5.1.22)[288]:** Our conventions for the inner products and the involution slightly deviate from [130], but the reader may easily adapt the arguments of [130] to our conventions.

Every summand $\bar{V} \otimes \Gamma(V)$ admits a unitary representation of $\mathcal{G}$ by acting on the first tensor factor. Taking direct sums yields a map $\alpha : A \to A \otimes \mathcal{G}$. This map is in fact a $*$-homomorphism satisfying $(\alpha \otimes \mathrm{id}_{\mathcal{G}}) \circ \alpha = (\alpha \otimes \Delta) \circ \alpha$ . Altogether we have an algebraic action of the quantum group $\mathcal{G}$ on $the^*$-algebra $\mathcal{A}$. From this we may pass to a $C^*$-dynamical system by taking the completion $\mathfrak{H}_A$ of $A$ with respect to the norm $\|x\|_2 := \|\langle x, x\rangle_{\mathfrak{B}}\|^{1/2}$. Then the left multiplication of $A$ yields a faithful representation $\lambda : A \to \mathcal{L}(\mathfrak{H}_A)$ and a $C^*$-algebra $\mathcal{A} := \overline{\lambda(\mathcal{A})}$. The$^*$-homomorphism $\alpha$ can be extended to an action $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, which we denote by the same letter. See [134].

**Theorem (5.1.23)[288]:** The compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is free with fixed point algebra $\mathfrak{B}$ and factor system $(\mathfrak{H}, \gamma, w)$.

**Proof:** For an irreducible representation $(\pi, V) \in \hat{\mathcal{G}}$, the $\pi$-isotypic component of $(\mathcal{A}, \mathcal{G}, \alpha)$ is obviously given by $V \otimes \Gamma(\bar{V})$. Hence the $\pi$-multiplicity space of the $C^*$-dynamical system

$$\Gamma_{\mathcal{A}}(V) := \{x \in V \otimes \mathcal{A} \mid \pi\alpha(x) = x\} \subseteq V \otimes \bar{V} \otimes \Gamma(V)$$

is isomorphic to $\Gamma(V)$. as a correspondence over $\mathfrak{B}$. More precisely, a few moments thought show that an isomorphism is given by $\varphi : \Gamma(V) \to \Gamma_{\mathcal{A}}(V), x \longmapsto R(1) \otimes x$. For the computations it is convenient to fix an orthonormal basis $e_1, \dots, e_d$ of . We also fix an arbitrary orthonormal basis $f_1, \dots, f_n$ of $\mathfrak{H}_\pi$ and consider the elements

$$s_k := \gamma_\pi(\mathbb{1})(f_k \otimes \mathbb{1}_{\mathfrak{B}}).$$

It it easily checked that these elements satisfy $\sum_{k=1}^n s_k \langle s_k, x\rangle_{\mathfrak{B}} = x$ for every $x \in \Gamma(V)$ and hence $\sum_{k=1}^n m[s_k]m[s_k]^* = \mathrm{id}_{\Gamma(V \otimes \bar{V})}$. It follows that for the left inner product on $\Gamma_{\mathcal{A}}(V)$ we obtain

$$\sum_{k=1}^n \mathcal{L}(V) \otimes_{\mathcal{A}} \langle \varphi(s_k), \varphi(s_k)\rangle = \sum_{k=1}^n \mathcal{L}(V) \otimes_{\mathcal{A}} \langle R(1)_{\otimes_{s_k}}, R(1) \otimes_{s_k}\rangle$$

$$= \sum_{k=1}^n \sum_{i,j,r,s=1}^d [Q^{1/2}]_{i,j} [Q^{1/2}]_{s,r} |e_i\rangle \langle e_r| \otimes ((\bar{e}_j \otimes s_k) \bullet (\bar{e}_s \otimes s_k)^+)$$

$$= \sum_{k=1}^n \sum_{i,j,r,s=1}^d \sum_{\ell=1}^m [Q^{1/2}]_{i,j} [Q^{1/2}]_{s,r} |e_i\rangle \langle e_r| \otimes \left(\bar{S}_\ell^*(\bar{e}_j \otimes \bar{e}_s^+) \otimes \Gamma(S_\ell^*) m_{\pi,\bar{\pi}}(s_k \otimes s_k^+)\right), \quad (16)$$

where $Q := Q(\pi)$ is the modular operator and where $S_1, \dots, S_m$ is an orthonormal basis of intertwiners $\bar{S}_\ell : V_{\sigma_\ell} \to V \otimes \bar{V}$ with irreducible representations $\sigma_\ell \in \hat{\mathcal{G}}$, one of them being $S_\ell = d_\pi^{-1/2} R$. First summing over $k$ yields

$$\sum_{k=1}^{n} m_{\pi,\bar{\pi}} (s_k \otimes s_k^+) = \sum_{k=1}^{n} m\,[s_k] m[s_k]^* \Gamma(R)(\mathbb{1}_{\mathfrak{B}}) = \Gamma(R)(\mathbb{1}_{\mathfrak{B}}).$$

Therefore, only the intertwiner $S_\ell = d_\pi^{-1/2} R$ contributes in Equation (16). Moreover, a straightforward computation yields $R(v \otimes w^+) = \langle w, Q(\pi) v\rangle$ for all $v, w \in V$. We may hence simplify Equation (16) to

$$\sum_{k=1}^{n} \mathcal{L}(V)_{\otimes\mathcal{A}} \langle \varphi(s_k), \varphi(s_k)\rangle =$$

$$= \frac{1}{d_\pi} \sum_{i,j,r,s=1}^{d} [Q^{1/2}]_{i,j} [Q^{1/2}]_{s,r} |e_i\rangle\langle e_r| \otimes \left(\bar{R}^* (\bar{e}_j \otimes \bar{e}_s^+) \otimes \Gamma(R^*)\Gamma(R)(\mathbb{1}_{\mathfrak{B}})\right)$$

$$= \sum_{i,j,r,s=1}^{d} [Q^{1/2}]_{i,j} [Q^{1/2}]_{s,r} \big| [Q^{-1}]_{j,s}\, |e_i\rangle\langle e_r| \otimes \mathbb{1}_{\mathfrak{B}} = \mathbb{1}_{\mathfrak{B}}.$$

This proves that the compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is free. Furthermore, the collection of elements $s_k k (1 \le k \le n)$ for each $\pi \in \hat{\mathcal{G}}$ provide a factor system $(\tilde{\mathfrak{H}}, \tilde{\gamma}, \tilde{w})$ of $(\mathcal{A}, \mathcal{G}, \alpha)$ with Hilbert spaces $\tilde{\mathfrak{H}}_\pi = \mathfrak{H}_\pi$. In terms of the chosen basis $f_1, \dots, f_n$, the $^*$-homomorphism $\tilde{\gamma}_\pi : \mathfrak{B} \to M_n \otimes \mathfrak{B}$ for $\pi \in \hat{\mathcal{G}}$ is given by

$$\tilde{\gamma}_\pi(b)_{i,j} = \langle \varphi(s_i), b\,.\,\varphi(s_j)\rangle_{\mathfrak{B}} = \langle s_i, b\,.\,s_j\rangle_{\mathfrak{B}} = \langle f_i \otimes \mathbb{1}_{\mathfrak{B}}, \gamma_\pi(b)(f_i \otimes \mathbb{1}_{\mathfrak{B}})\rangle_{\mathfrak{B}} = \gamma_\pi(b)_{i,j}$$

for all $b \in \mathfrak{B}$ (see Remark (5.1.9)). That is, we have $\tilde{\gamma} = \gamma$ and similar computation shows that $\tilde{w} = w$, too. Summarizing, we find that $(\mathfrak{H}, \gamma\,, w)$ is indeed a factor system of the free compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$.

Given a unital $C^*$-algebra B, we call a free compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$. With a finite quantum group $\mathcal{G}$ and fixed point algebra $\mathfrak{B}$ a finite covering of $\mathfrak{B}$. The main purpose is to use factor systems to show that finite coverings of generic irrational rotation $C^*$-algebras are cleft (cf. Definition (5.1.12)).

**Lemma (5.1.24)[288]:** Let $\theta \in \mathbb{R}$. Then every positive group homomorphism of $\mathbb{Z} + \theta\mathbb{Z}$ is $a$ multiple of the identity.

**Proof:** Let $h : \mathbb{Z} + \theta\mathbb{Z} \to \mathbb{Z} + \theta\mathbb{Z}$ be a positive group homomorphism. Then for all $x, y \in \mathbb{Z}$ we have that $x + \theta y \ge 0$ implies $h(1)x + h(\theta)y \ge 0$ and $x + \theta y \le 0$ implies $h(1)x + h(\theta)y \le 0$. Considering $q := -x/y$, it follows that for all $q \in \mathbb{Q}$ we have that $q \ge \theta$ implies $h(1)q \ge h(\theta)$ and $q \le \theta$ implies $h(1)q \le h(\theta)$. Taking the limit $q \to \theta$ in rationals, we may conclude that $h(1)\theta = h(\theta)$. Finally, for every $z = x + \theta y \in \mathbb{Z} + \theta\mathbb{Z}$ we obtain $h(z) = xh(1) + yh(\theta) = h(1)z$ as asserted.

**Remark(5.1.25)[288]:** Extending the preceding proof, the equation $h(1)\theta = h(\theta)$ is a quadratic equation with integer coefficients. Thence for non-quadratic $\theta$ the factor $h(1)$ must be a positive integer.

Given a finite group $G$ and its representation ring $R(G)$, it is a well-known fact that there is only one ring homomorphism $r: R(G) \to \mathbb{R}$ with $r(\pi) > 0$ for every $\pi \in \hat{G}$, namely $r(\pi) = \dim \pi$ for every $\pi \in \hat{G}$.

The next result shows that this statement remains true in the context of finite quantum groups.

**Lemma (5.1.26)[288]:** Let $\mathcal{G}$ be a finite quantum group and denote by $R(\mathcal{G})$ its representation ring. Then there is only one ring homomorphism $r: R(\mathcal{G}) \to \mathbb{R}$ with $r(\pi) > 0$ for every $\pi \in \hat{\mathcal{G}}$, namely $r(\pi) = \dim \pi$ for every $\pi \in \hat{\mathcal{G}}$.

**Proof:** Let $r_1, r_2 2: R(G) \to \mathbb{R}$ be two such positive, non-zero ring homomorphisms and let us fix $\pi \in \hat{G}$. We consider the matrix $T(\pi)$ with rows and columns index by $\hat{G}$ given by

$$T(\pi)_{\rho,\sigma} := \frac{m(\sigma, \rho \otimes \pi) r_1(\sigma)}{r_1(\rho \otimes \pi)}$$

for all $\rho, \sigma \in \hat{\mathcal{G}}$, where $m(\sigma, \rho \otimes \pi)$ denotes the multiplicity of $\sigma$ in $\rho \otimes \pi$. A straightforward computation verifies that $T(\pi)$ is a stochastic matrix. Moreover, the vector $c = (c_\rho)_{\rho \in \hat{\mathcal{G}}}$ with $c_\rho := r_2(\rho)/r_1(\rho)$ is an eigenvector of $T(\pi)$ with eigenvalue $\lambda = r_2(\pi)/r_1(\pi)$, because the homomorphism property implies

$$(T(\pi)c)_\rho = \frac{1}{r_1(\rho \otimes \pi)} \sum_{\sigma \in \hat{\mathcal{G}}} m(\sigma, \rho \otimes \pi) r_2(\sigma) = \frac{r_2(\rho \otimes \pi)}{r_1(\rho \otimes \pi)} = \frac{r_2(\pi)}{r_1(\pi)} c_\rho$$

Since all eigenvalues of stochastic matrices lie in the unit disc, we now conclude that $r_2(\pi) \le r_1(\pi)$. Exchanging the role of $r_1$ and $r_2$ likewise yields $r_2(\pi) \le r_1(\pi)$ and consequently we obtain $r_1 = r_2$.

**Theorem (5.1.27)[288]:** Let $\theta \in \mathbb{R}$ be irrational and non-quadratic. Furthermore, let $\mathcal{G}$ be a finite quantum group. Then every free compact $C^*$-dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ with fixed point algebra $\mathcal{A}_\theta^2$ is cleft.

**Proof:** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a free compact $C^*$-dynamical system with $\mathcal{A}^{\mathcal{G}} = \mathcal{A}_\theta^2$ and let $(\mathfrak{H}, \gamma, w)$ be a factor system of $(\mathcal{A}, \mathcal{G}, \alpha)$. Then for every representation $\pi$ of $\mathcal{G}$ the $^*$homomorphism $\gamma_\pi : \mathcal{A}_\theta^2 \to \mathcal{A}_\theta^2 \otimes \mathcal{L}(\mathfrak{H}_\pi)$ induces a positive group homomorpism

$$K_0(\gamma_\pi) : \mathbb{Z} + \theta\mathbb{Z} \to: \mathbb{Z} + \theta\mathbb{Z},$$

where we have identified $K_0(\mathcal{A}_\theta^2 \otimes \mathcal{L}(\mathfrak{H}_\pi))$ with $K_0(\mathcal{A}_\theta^2) = \mathbb{Z} + \theta\mathbb{Z}$. By Remark (5.1.25), this group homomorphism must be a positive integer of the identity, say for some factor $r(\pi) > 0$. Given two representations $\pi, \rho$ of $\mathcal{G}$, we clearly have $r(\pi \oplus \rho) = r(\pi) + r(\rho)$. Moreover, the coaction condition of the factor system implies that $K_0(\gamma_\rho) \circ K_0(\gamma_\pi) = K_0(\gamma_{\pi \otimes \rho})$ and therefore that $r(\rho) \cdot r(\pi) = r(\pi \otimes \rho)$. As a consequence, we may extend the map $\pi \longmapsto r(\pi)$ to a ring-homomorphism $r: R(\mathcal{G}) \to \mathbb{R}$. Lemma (5.1.26) then shows that $r(\pi) = \dim(\pi)$ holds for every $\pi \in \hat{\mathcal{G}}$ and hence we obtain

$$[\gamma_\pi(\mathbb{1})] = K_0 \gamma_\pi)[\mathbb{1}] = r(\pi) \cdot [\mathbb{1}] = \dim(\pi) \cdot [\mathbb{1}]$$

in $K_0(\mathcal{A}_\theta^2)$. We finally conclude from Lemma (5.1.17) that $(\mathcal{A}, \mathcal{G}, \alpha)$ is cleft.

We show that Morita equivalence bimodules between unital $C^*$-algebras admit a so-called standard module frames. Although this might be well-known to experts, we have not found such a statement explicitly discussed in the literature.

**Lemma (5.1.28)[288]:** Let $M$ be a Morita equivalence between unital $C^*$-algebras $\mathcal{A}$ and $\mathfrak{B}$. Then there are elements $x_1, \dots, x_n \in M$ with $\sum_{i=1}^n {}_{\mathcal{A}}\langle x_i, x_i \rangle = \mathbb{1}$. In particular, for anycollection of such elements we have a Fourier decomposition given for all $x \in M$ by $x = \sum_{i=1}^n x_k \langle x_k, x \rangle_{\mathfrak{B}}$.

**Proof:** The linear span of left inner products $J := {}_{\mathcal{A}}\langle M, M\rangle$ is a dense ideal in $\mathcal{A}$. Since the invertible elements of $\mathcal{A}$ form an open subset, $J$ contains invertible elements and hence $J = \mathcal{A}$. That is, there are elements $x_1, \ldots, x_n \in M$ and $y_1, \ldots, y_n \in M$ with $\mathbb{1} = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle x_i, y_i\rangle$. Then the Morita equivalence property implies

$$y = \mathbb{1}\,.\,y = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle x_i, y_i\rangle\,.\,y = \sum_{i=1}^{n} x_i\,.\,\langle y_i, y\rangle_{\mathfrak{B}} \qquad (17)$$

for every $y \in M$. Now consider the matrix $Y \in \mathfrak{B} \otimes M_n$ given by $Y_{i,j} := \langle y_i, y_j\rangle_{\mathfrak{B}}$ for $1 \leq i, j \leq n$. Since $Y$ is positive, we find a matrix $R = (R_{i,j})_{i,j}$ in $\mathfrak{B} \otimes M_n$ with $Y = RR^*$. Putting $z_k := \sum_{i=1}^{n} x_i\,.\,R_{i,k}$ for all $1 \leq j \leq n$ we find

$$\sum_{k=1}^{n} {}_{\mathcal{A}}\langle z_k, z_k\rangle = \sum_{i,j,k=1}^{n} {}_{\mathcal{A}}\langle x_i, R_{i,k}, x_j.R_{j,k}\rangle = \sum_{i,j=1}^{n} {}_{\mathcal{A}}\langle x_i.\left(\sum_{k=1}^{n} R_{i,k}, R_{j,k}^{*}\right), x_j\rangle$$

$$= \sum_{i,j=1}^{n} {}_{\mathcal{A}}\langle x_i.\langle y_i, y_j\rangle_{\mathfrak{B}}, x_j\rangle \overset{(17)}{=} \sum_{j=1}^{n} {}_{\mathcal{A}}\langle y_j, x_j\rangle =.\,\mathbb{1}$$

## Section (5.2): Compact Abelian Groups on $C^*$-Algebras

We study free actions of compact groups on unital $C^*$-algebras. This class of actions was first introduced under the name saturated actionsby Rieffel [123] and equivalent characterizations where given by Ellwood [318] and by Gottman, Lazar, and Peligrad [81](see also [133],[341]). Other related notions of freeness were studied by Phillips [116] in connection with K-theory (see also [36]).

Free actions do not admit degeneracies that may be present in general actions. For this reasons they are easier to understand and to classify. For compact Abelian groups, free ergodic actions, i.e., free actions with trivial fixed point algebra, were completely classified by Olesen, Pedersen and Takesaki in [64] and independently by Albeverio and Høegh–Krohn in [42]. This classification was generalized to compact non-Abelian groups by the Wassermann [342]–[302]. According to [42], [64], [342], for a compact group Gthere is a 1-to-1 correspondence between free ergodic actions of Gand 2-cocycles of the dual group. An analogous result of compact quantum groups has been obtained by Bichon , De Rijdt and Vaes [290]. Extending these results beyond the ergodic case is however not straightforward because, even for a commutative fixed point algebra, the action cannot necessarily be decomposed into a bundle of ergodic actions. For compact Abelian groups our results about free, but not necessary ergodic actions may be regarded as a generalization of the classification given in [42], [64]. We would also like to point out that Neshveyev [130] obtained a classification of actions of compact quantum groups in terms of weak unitary tensor functors, which do unfortunatelynot have an obvious homological interpretation.

The study of non-ergodic free actions is also motivated by the established theory of principal bundles. In fact, by a classical result, having a free action of a compact group $G$ on a compact $P$ is equivalent saying that $P$ carries the structure of a principal bundle over the quotient $X := P/G$ with structure group $G$. Very well-understood is the case of locally trivial principal bundles, that is, if $P$ is glued together from spaces of the form $U \times G$ with an open subset $U \subseteq X$. This gluing immediatelyleads to $G$-valued cocycles. The corresponding cohomology theory, called Č ech cohomology, gives a complete classification of locally trivial principal bundles with base space $X$ and structure group $G$

(see [336]). For principal bundles that are not locally trivial, however, there is no obvious classification available. Our results provide such a classification in the case of a compact Abelian structure group.

Passing to noncommutative geometry poses the question how to extend the concept of principal bundles to noncommutative geometry. In the case of vector bundles the Theorem of $S$erre and Swan (cf. [174]) gives the essential clue: The category of vector bundles over a compact space $X$ is equivalent to the category of finitely generated and projective $C(X)$-modules. This observation leads to a notion of noncommutative vector bundles and is the connection between the topological $K$-theory based on vector bundles and the $K$-theory for $C^*$-algebras. For principal bundles, free and proper actions offer a good candidate for a notion of noncommutative principal bundles (see e.g.[308], [148], [318], [36]). Asimilar geometric approach based on transformation groups was developed by one of [339], [340]. In a purely algebraic setting, the well-established theory of Hopf–Galois extensions provides a wider framework comprising coactions of Hopf algebras (e.g.[319], [295], [170]). We also would like to mention the related notion of noncommutative principal torus bundles proposed by Echterhoff, Nest, and Oyono-Oyono [293](see also [320]), which relies on a noncommutative version of Green's Theorem. Considering free actions as noncommutative principal bundles, We characterize principal bundles in terms of associated vector bundles.

Extending the classical theory of principal bundles to noncommutative geometry is not of purely mathematical interest. In fact, noncommutative principal bundles become more and more prevalent in geometry and physics. For instance, Ammann and Bär [306], [307] study the properties of the Riemannian spin geometry of a smooth principal $U(1)$-bundle. Under suitable hypotheses, they relate the spin structure and the Dirac operator on the total space to the spin structure and the Dirac operator on the base space. Anoncom-mutative generalization of these results was developed by Dabrowski, Sitarz, and $Z$ucca in [314], [315] using spectral triples and the Hopf–Galois analogue of principal $U(1)$-bundles. Noncommutative principal bundles also appear in the study of 3-dimensional topological quantum field theories that are based on the modular tensor category of representations of the Drinfeld double (cf. [325]). In special types of Hopf–Galois extensions correspond to symmetries of the theory or, equivalently, to invertible defects. As such, they are connected to module categories and, in particular, to the Brauer–Picard group of pointed fusion categories. Furthermore, $T$-duality is considered to be an important symmetry of string theories ([305], [313]). It is known that a circle bundle with $H$-flux given by a Neveu–Schwarz 3-form admits a $T$-dual circle bundle with dual $H$-flux. However, it is also known that in general torus bundles with $H$-flux do not necessarily have a $T$-dual that is itself a classical torus bundle. Mathai and Rosenberg showed in [327],[328] that this problem is resolved by passing to noncommutative spaces. For example, it turns out that every principal $\mathbb{T}^2$-bundle with H-flux does indeed admit a $T$-dual but its $T$-dual is non-classical. It is a bundle of noncommutative 2-tori, which can (locally) be realized as a noncommutative principal $\mathbb{T}^2$-bundle in the sense of [293]. All these examples demand a better understanding of the geometry of noncommutative principal bundles. Although the classification relies purely on the topology of the bundle, we hope that parts of our classification extends in such a way that additional geometrical information of the space is comprised.

We investigate the structure of free actions on $C^*$-algebras, which is one framework for noncommutative principal bundles. We restrict ourselves to the compact setting, that is, to compact groups acting on unital $C^*$-algebras. The main objective is to provide a complete classification of free actions of compact Abelian groups on unital $C^*$-algebras. We achieve such a classification by inspecting the Morita equivalence bimodule structure of the isotypic components. It turns out that the resulting classification can be handled by methods of group cohomology.

We discuss the different equivalent characterizations of free-ness in the (Theorem (5.2.10)). We further provide some methods to construct new free actions from given ones. As an example we present a one-parameter family of free $SU(2)$-actions which is related to the Connes-Landi spheres (cf. [295]).

We construct a first invariant of a $C^*$-dynamical system, namely a group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ from the dual of the compact Abelian group $G$ to the Picard group of the unital fixed point algebra $\mathfrak{B}$. In general, this invariant neither distinguishes all free actions of $G$ nor is there a dynamical system for each group homomorphism $\varphi$. However, refining this invariant leads to our classifying data, to which we refer as factor system due to some similarity with the theory of group extensions. As one cornerstone of our classification, we show that every factor system can indeed be obtained from a $C^*$-dynamical system. The construction of this dynamical system forms the main.

The full classification of free actions of compact Abelian groups on unital $C^*$-algebras is discussed. We additionally fix a group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ and restrict our attention to $C^*$-dynamical system with the given $\varphi$ as invariant. The main result is that, if such dynamical systems exist, all free actions associated to the triple $(\mathfrak{B}, G, \varphi)$ are parametrized, up to 2-coboundaries, by 2-cocycles on the dual group $\hat{G}$ with values in the group $UZ(\mathfrak{B})$ of central unitary elements in $\mathfrak{B}$ (Theorem (5.2.52) and Corollary (5.2.53)). In other words, the set in question is a principal homogeneous space with respect to a classical cohomology group $H^2(\hat{G}, UZ(\mathfrak{B}))$. We provide a group theoretic criterion for the existence of free actions with invariant $\varphi$; that is, factor systems associated to the triple $(\mathfrak{B}, G, \varphi)$.

As already mentioned, for a compact group $G$ and a compact space $X$, locally trivial principal $G$-bundles over $X$ are classified by the Čech cohomology for the pair $(X, G)$. From the $C^*$-algebraic viewpoint, local triviality is not easy to capture. We provid instead a classification of not necessarily locally trivial principal bundles in case of a compact Abelian structure group. Finally, we discuss a few examples.

We point out that with little effort the arguments and the results which are presented for actions of compact Abelian groups extend to coactions of group $C^*$-algebras of finite groups. We also would like to mention that the first part of a larger program aiming at classifying more general free actions on $C^*$-algebras (cf. [297],[288]). This classification may be used to develop a fundamental group for $C^*$-algebras (cf. [334]). It could also serve as a starting point for an approach to quantum gerbes and a theory of $T$-duals for noncommutative principal torus bundles.

Our study is concerned with free actions of compact groups on unital $C^*$-algebras and their classification. As a consequence, we use and blend tools from geometry, representation theory and operator algebras.

For $P$ and $X$ be compact spaces. Let $G$ be a compact group. A locally trivial principal bundle is a quintuple $(P, X, G, q, \sigma)$, where $q: P \to X$ is a continuous map and $\sigma: P \times G \to P$ a continuous action, with the property of local triviality: Each point $x \in X$

has an open neighborhood $U$ for which there exists a homeomorphism $\varphi_U: U \times G \to q^{-1}(U)$ satisfying $q \circ \varphi_U = \mathrm{pr}_U$ and additionally the equivariance property

$$\varphi_U(x, gh) = \varphi_U(x, g).h$$

for $x \in U$ and $g, h \in G$. It follows that the map $q$ is surjective, that the action $\sigma$ is free and proper, and that the natural map $P/G \mapsto X, p.G \mapsto q(p)$ is a homeomorphism. In particular, we recall that the action $\sigma$ is called free if and only if all stabilizer groups $G_p := \{g \in G \mid \sigma(p, g) = p\}, p \in P$, are trivial. For a solid background on free group actions and principal bundles [25,26,36].

Tensor products of $C^*$-algebras are taken with respect to the minimal tensor product denoted by $\otimes$. Let $\mathcal{A}$ be a unital $C^*$-algebra and $G$ a compact group that acts on $\mathcal{A}$ by$^*$-automorphism $\alpha_g: \mathcal{A} \to \mathcal{A}(g \in G)$ such that the map $G \times \mathcal{A} \to \mathcal{A}, (g, a) \mapsto \alpha_g(a)$ is continuous. We call such a triple $(\mathcal{A}, G, \alpha)$ a $C^*$-dynamical system. We sometimes denote by $\mathfrak{B} := \mathcal{A}^G$ the corresponding fixed point $C^*$-algebra of the action αand we write $P_0: \mathcal{A} \to \mathcal{A}$ for the conditional expectation given by

$$P_0(x) := \int_G \alpha_g(x) dg.$$

At this point it is worth mentioning that all integrals over compact groups are understood to be with respect to probability Haar measure. More generally, for an irreducible representation $(\pi, V)$ of $G$ we write $P_\pi: \mathcal{A} \to \mathcal{A}$ for the continuous $G$-equivariant projection onto the isotypic component $A(\pi) := P_\pi(\mathcal{A})$ given by

$$P_\pi(x) := \dim\, V \cdot \int_G \mathrm{tr}_V(\pi_g^*) \cdot \alpha_g(x) dg.$$

where $\mathrm{tr}_V$ denotes the canonical trace on the algebra $\mathcal{L}(V)$ of linear endomorphisms of $V$. It is a consequence of the Peter–Wey l Theorem [321] that the algebraic direct sum $\bigoplus_{\pi \in \hat{G}} A(\pi)$ is a dense $^*$-subalgebra of $\mathcal{A}$. Here we write $\hat{G}$ for the set of equivalence classes of all irreducible representations. Finally, we point out that each continuous group action $\sigma: P \times G \to P$ of a compactgroup $G$ on a compact space $P$ gives rise to a $C^*$-dynamical system $(C(P), G, \alpha_\sigma)$ defined by

$$\alpha_\sigma: G \times C(P) \to C(P), \qquad (g, f) \mapsto f \circ \sigma_g.$$

A huge part is concerned with Hilbert module structures. We recall some of the central definitions. Let $\mathfrak{B}$ be a unital $C^*$-algebra. A right pre-Hilbert $\mathfrak{B}$-module is a vector space $M$ which is a right $\mathfrak{B}$-module equipped with a positive definite $\mathfrak{B}$-valued sesquilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{B}}$ satisfying

$$\langle x, y \cdot b \rangle_{\mathfrak{B}} = \langle x, y \rangle_{\mathfrak{B}} b \quad \text{and} \quad \langle x, y \rangle_{\mathfrak{B}}^* = \langle y, x \rangle_{\mathfrak{B}}$$

for all $x, y \in M$ and $b \in \mathfrak{B}$. A right Hilbert $\mathfrak{B}$-module is a right pre-Hilbert $\mathfrak{B}$-moduleMwhich is complete with respect to the norm given by $\|x\|^2 = \|\langle x, y \rangle_{\mathfrak{B}}\|$ for $x \in M$. It is called a full right Hilbert $\mathfrak{B}$-module if the right ideal $J := span\{\langle x, y \rangle_{\mathfrak{B}} \mid x, y \in M\}$ is dense in $\mathfrak{B}$. Since every dense ideal meets the invertible elements, in this case we have $J = \mathfrak{B}$. Left (pre-) Hilbert $\mathfrak{B}$-modules are defined in a similar way. Next, let $\mathcal{A}$ and $\mathfrak{B}$ be unital $C^*$-algebras. A right (pre-) Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule is a right (pre-) Hilbert $\mathfrak{B}$-module $M$ that is also a left $\mathcal{A}$-module satisfying

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad and \quad \langle a \cdot x, y \rangle_{\mathfrak{B}} = \langle x, a^* \cdot y \rangle_{\mathfrak{B}}$$

for all $x, y \in M$, $a \in \mathcal{A}$ and $b \in \mathfrak{B}$. We point out that right Hilbert $\mathcal{A} - \mathfrak{B}$-bimodules are sometimes called $\mathcal{A} - \mathfrak{B}$ correspondences in the literature. Given a right (pre-) Hilbert A −B-bimodule $M$ and a right (pre-) Hilbert $\mathfrak{B} - \mathfrak{B}$ –bimodule $M$, their algebraic $\mathfrak{B}$-tensor product $M \otimes_{\mathfrak{B}} N$ carries a natural right pre-Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule structure with right $\mathfrak{B}$-valued inner product given by

$$\langle x_1 \otimes_{\mathfrak{B}} y_1, x_2 \otimes_{\mathfrak{B}} y_2 \rangle_{\mathfrak{B}} := \langle y_1, \langle x_1, x_2 \rangle_{\mathfrak{B}} \cdot y_2 \rangle_{\mathfrak{B}} \tag{18}$$

for $x_1, x_2 \in M$ and $y_1, y_2 \in N$. In particular, its completion $M \widehat{\otimes}_{\mathfrak{B}} N$ with respect to the induced norm is a right Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule. Left (pre-) Hilbert $\mathcal{A} - \mathfrak{B}$-bimodules are defined in a similar way. A (pre-) Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule is an $\mathcal{A} - \mathfrak{B}$-bimodule $M$ which is a left (pre-) Hilbert $\mathcal{A}$-module and a right (pre-) Hilbert $\mathfrak{B}$-module satisfying

$$\langle a \cdot x, y \rangle_{\mathfrak{B}} = \langle x, a^* \cdot y \rangle_{\mathfrak{B}}, \quad {}_{\mathcal{A}}\langle x \cdot b, y \rangle = {}_{\mathcal{A}}\langle x, y \cdot b^* \rangle \quad and \quad {}_{\mathcal{A}}\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathfrak{B}}$$

for all $x, y, z \in M, a \in \mathcal{A}$ and $b \in \mathfrak{B}$. A Morita equivalence $\mathcal{A} - \mathfrak{B}$-bimodule is a Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule with full inner products. The algebraic $\mathfrak{B}$-tensor product $M \otimes_{\mathfrak{B}} N$ of a (pre-) Hilbert A −B-bimodule $M$ and a (pre-) Hilbert $\mathfrak{B} - \mathfrak{B}$ -bimodule $N$ carries a natural pre-Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule structure with inner products as in Equation (18). Its completion $M \widehat{\otimes}_{\mathfrak{B}} N$ is a Hilbert $\mathcal{A} - \mathfrak{B}$-bimodule. Finally, if $M$ is a Morita equivalence $\mathcal{A} - \mathfrak{B}$-bimodule and Na Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule, it is easily checked that the completion $M \widehat{\otimes}_{\mathfrak{B}} N$ is a Morita equivalence $\mathcal{A} - \mathfrak{B}$-bimodule. For a detailed background on Hilbert module structures we refer to [309], [310], [317], [139], [332].

We discuss some of the forms of free actions of compact groups on $C^*$-algebras that have been used. In particular, we give some indications of their strengths and relationships to each other. Furthermore, we provide some methods to construct new free actions from given ones. As an example we present a one-parameter family of free $SU(2)$-actions which is related to the Connes–Landi spheres.

We use the symbol $\otimes_{alg}$ to denote the algebraic tensor product of vector spaces and we write $\otimes$ for the minimal tensor product of $C^*$-algebras.

**Proposition (5.2.1)[304]:** ([116]). Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra Aand a compact group $G$. Then the following definitions make a suitable completion of $\mathcal{A}$ into a Hilbert $\mathcal{A} \rtimes_\alpha G - \mathcal{A}^G$-bimodule:

**(i)** $f \cdot x := \int_G f(g)\, \alpha_g(x)\, dg$ for $f \in L^1(G, \mathcal{A}, \alpha)$ and $x \in \mathcal{A}$.

**(ii)** $x \cdot b := xb$ for $x \in \mathcal{A}$ and $b \in \mathcal{A}^G$.

**(iii)** $\mathcal{A} \rtimes_\alpha G\langle x, y \rangle$ is the function $g \mapsto x\alpha_g(y^*)$ for $x, y \in \mathcal{A}$ .

**(iv)** $\langle x, y \rangle_{\mathcal{A}^G} := \int_G \alpha_g\, (x^* x)\, dg$ for $x, y \in \mathcal{A}$.

It is easily seen that the module under consideration in the previous statement is almost a Morita equivalence $\mathcal{A} \rtimes_\alpha G - \mathcal{A}^G$-bimodule. In fact, the only missing condition is that the range of $\mathcal{A}_{\rtimes_\alpha G}\langle \cdot, \cdot \rangle$ need not be dense. The imminent definition was originally introduced by $M$. Rieffel and has a number of good properties that resemble the classical theory of free actions of compact groups as we will soon see below.

**Definition(5.2.2)[304]:**( [116]). Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. We call $(\mathcal{A}, G, \alpha)$ free if the bimodule from Proposition (5.2.1) is a Morita equivalence bimodule, that is, the range of $\mathcal{A}_{\rtimes_\alpha G}\langle \cdot, \cdot \rangle$ is dense in the crossed product $\mathcal{A} \rtimes_\alpha G$.

**Remark (5.2.3)[304]:** We point out that $M$. Rieffel used the notion "saturated" instead of free, i.a., because of its relation to Fell bundles in the case of compact A belian group actions. Moreover, we recall that [123] provides a notion for free actions of locally compact groups which is consistent with Definition (5.2.2) for compact groups.

The next result shows that Definition (5.2.2) extends the classical notion of free actions of compact groups.

**Theorem (5.2.4)[304]:**( [116]). Let $P$ be a compact space and $G$ a compact group. A continuous group action $\sigma: P \times G \to P$ is free if and only if the corresponding $C^*$-dynamical system $(C(P), G, \alpha_\sigma)$ is free in the sense of Definition (5.2.2).

Another hint for the strength of Definition (5.2.2) comes from the following observation: Let $(P, X, G, q, \sigma)$ be a locally trivial principal bundle and $(\pi, V)$ a finite-dimensional unitary representation of $G$. Then it is a well-known fact that the isotypic component $C(P)(\pi)$ is finitely generated and projective as a right $C(X)$-module (cf. [337]). In the $C^*$-algebraic setting a similar statement is valid.

**Theorem (5.2.5)[304]:** ( [133]). Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. Furthermore, let $(\pi, V)$ be a finite-dimensional unitary representation of $G$. If $(\mathcal{A}, G, \alpha)$ is free, then the corresponding isotypic component $A(\pi)$ is finitely generated and projective as a right $\mathcal{A}^G$-module.

We proceed with introducing two more notions which will turn out to be equivalent characterizations of noncommutative freeness. The first notion is a $C^*$-algebraic version of the purely algebraic Hopf–Galois condition (cf. [319], [170]) and is due to $D.A.$ Ellwood.

**Definition (5.2.6)[304]:** ( [318]). Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. We say that $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition if the map

$$\Phi: \mathcal{A} \otimes_{alg} \mathcal{A} \to C(G, \mathcal{A}), \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range (with respect to the canonical $C^*$-norm on $C(G, \mathcal{A})$).

The second notion is of representation-theoretic nature and makes use of the so-called generalized isotypic components of a $C^*$-dynamical system.

**Definition (5.2.7)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. Furthermore, let $(\pi, V)$ be a finite-dimensional unitary representation of $G$. Then the space

$$A_2(\pi) := \{s \in \mathcal{A} \otimes \mathcal{L}(V) \mid \alpha_g(s) = s \cdot \pi_g \ \ for\ all\ \ g \in G\}$$

is called the generalized isotypic component of $(\pi, V)$. It is easily checked that the canonical right action of the unital $C^*$-algebra

$$C(\pi) := \{c \in \mathcal{A} \otimes \mathcal{L}(V) | \alpha_g(c) = \pi_g^* \cdot c \cdot \pi_g \quad \text{for all} \quad g \in G\}$$

turns $A_2(\pi)$ into a right $C(\pi)$-module.

The following statement describes the natural Hilbert bimodule structure of the generalized isotypic components. The arguments consist of straightforward computations.

**Proposition (5.2.8)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. Furthermore, let $(\pi, V)$ be a finite-dimensional unitary representation of $G$. Then the following definitions make $A_2(\pi)$ into a $\mathcal{A}^G \otimes \mathcal{L}(V) - \mathcal{C}(\pi)$-Hilbert bimodule:

**(i)** $b.s := bs$ for $b \in \mathcal{A}^G \otimes \mathcal{L}(V)$ and $s \in A_2(\pi)$.

**(ii)** $s.c := sc$ for $s \in A_2(\pi)$ and $c \in \mathcal{C}(\pi)$.

**(iii)** ${}_{\mathcal{A}^G \otimes \mathcal{L}(V)}\langle s, t\rangle := st^*$ for $s, t \in A_2(\pi)$.

**(iv)** $\langle s, t\rangle_{\mathcal{C}(\pi)} := s^*t$ for $s, t \in A_2(\pi)$.

We are now ready to present the second notion which is of major relevance in our attempt to classify free $C^*$-dynamical systems.

**Definition(5.2.9)[304]:** ([20)]). Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$ algebra $\mathcal{A}$ and $a$ compact group $G$. The $A$ $\nu$erson spectrum of $(\mathcal{A}, G, \alpha)$ is defined as

$$Sp(\alpha) := \{[(\pi, V)] \in \hat{G} \mid \text{span}\{ \langle s, t\rangle_{\mathcal{C}(\pi)} \mid s, t \in A_2(\pi)\} = \mathcal{C}(\pi) \}.$$

That is, $[(\pi, V)] \in Sp(\alpha)$ if and only if the corresponding generalized isotypic component $A_2(\pi)$ is a full right $\mathcal{A}^G \otimes \mathcal{L}(V) - \mathcal{C}(\pi)$-Hilbert bimodule.

As the following result finally shows, all the previous notions agree.

**Theorem (5.2.10)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and $a$ compact group $G$. Then the following statements are equivalent:

**(i)** The $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is free.

**(ii)** The $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition.

**(iii)** The $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ satisfies $Sp(\alpha) = \hat{G}$.

The equivalence between (i) and (ii) was proved quite recently in [133], although it has been known that these two conditions are closely related to each other (cf. [341] for the case of compact Lie group actions). A proof of the equivalence between (i) and (iii) can be found in [81].

We now focus our attention on compact Abelian groups. In fact, given a $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with a unital $C^*$-algebra $\mathcal{A}$ and a compact A belian group $G$, we first note that the definition of the isotypic component $A(\pi)$ corresponding to a character $\pi \in \hat{G} = \text{Hom}_{gr}(G, \mathbb{T})$ simplifies to

$$A(\pi) = \{a \in \mathcal{A} \mid \alpha_g(a) = \pi_g \cdot a \quad \text{for all} \quad g \in G\}.$$

Moreover, it is easily seen that $A_2(\pi) = A(\pi)$ and that $\mathcal{C}(\pi) = \mathcal{A}^G$. The next statement provides equivalent characterizations of a free action in the context of compact A belian groups. It directly follows from Theorem (5.2.10) and the previous observations by repeatedly using the fact that $A(-\pi) = A(\pi)^* := \{a^* \in \mathcal{A} \mid a \in A(\pi)\}$ holds for each $\pi \in \hat{G}$.

**Corollary (5.2.11)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and $a$ compact A belian group $G$. Then the following conditions are equivalent:

**(i)** The $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is free.

**(ii)** For each $\pi \in \hat{G}$ the space $A(\pi)$ is $a$ Morita equivalence $\mathcal{A}^G - \mathcal{A}^G$-bimodule.

**(ii)** For each $\pi \in \hat{G}$ the multiplication map on $\mathcal{A}$ induces an isomorphism between $A(-\pi) \widehat{\otimes}_{\mathcal{A}^G} A(\pi)$ and $\mathcal{A}^G$.

As we will see soon, Corollary (5.2.11) gives rise to a first invariant for free actions of compact A belian groups. For the time being, we continue with some examples to get more comfortable with free actions of compact A belian groups.

**Example (5.2.12)[304]:** Let $\theta$ be a real skew-symmetric $n \times n$ matrix. The noncommutative $n$-torus $T_\theta^n$ is the universal unital $C^*$-algebra generated by unitaries $U_1, \ldots, U_n$ with

$$U_r U_s = \exp(2\pi i \theta_{rs}) U_s U_r \quad \text{for all} \quad 1 \leq r, s \leq n.$$

It carries a continuous action $\alpha_\theta^n$ of the $n$-dimensional torus $\mathbb{T}^n$ by algebra automorphisms which is on generators defined by

$$\alpha_{\theta,z}^n (U^{\boldsymbol{k}}) := z^{\boldsymbol{k}} \cdot U^{\boldsymbol{k}},$$

where $z^{\boldsymbol{k}} := z_1^{k_1} \cdots z_n^{k_n}$ and $U^{\boldsymbol{k}} := U_1^{k_1} \cdots U_n^{k_n}$ for $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ and $\boldsymbol{k} := (k_1, \ldots, k_n) \in \mathbb{Z}^n$. The isotypic component $(\mathbb{T}_\theta^n)(\boldsymbol{k})$ corresponding to the character $\boldsymbol{k} \in$

$\mathbb{Z}^n$ is given by $\mathcal{C} \cdot U^{\boldsymbol{k}}$. In particular, each isotypic component contains invertible elements from which we conclude that the $C^*$-dynamical system $(\mathbb{T}^n_\theta, \mathbb{T}^n, \alpha^n_\theta)$ is free.

**Example (5.2.13)[304]:** Let $H$ be the discrete, 3-dimensional Heisenberg group and let $C^*(H)$ denote its group $C^*$-algebra. Then $C^*(H)$ is the universal $C^*$-algebra generated by unitaries $U, V$ and $W$ satisfying

$$UW = WU, VW = WV \quad \text{and} \quad UV = WVU.$$

It carries a continuous action αof the 2-dimensional torus $\mathbb{T}^2$ by algebra automorphisms which is on generators defined by

$$\alpha_{(z,w)}(U^{\boldsymbol{k}}V^l\, W^m) := z^{\boldsymbol{k}}w^l \cdot U^{\boldsymbol{k}}V^l\, W^m,$$

where $(z, w) \in \mathbb{T}$ and $k, l, m \in \mathbb{Z}$. The corresponding fixed point algebra $\mathfrak{B}$ is the center of $C^*(H)$ which is equal to the group $C^*$-algebra $C^*(Z)$ of the center $Z \cong \mathbb{Z}$ of H. Moreover, the isotypic component $C^*(H)_{(k,l)}$ corresponding to the character $(k, l) \in \mathbb{Z}^2$ is given by $\mathfrak{B} \cdot U^k V^l$. In particular, each isotypic component $C^*(H)_{(k,l)}$ contains invertible elements from which we conclude that the C∗-dynamical system $(C^*(H), \mathbb{T}^2, \alpha)$ is free. We point out that $C^*(H)$ serves as a "universal" noncommutative principal $\mathbb{T}^2$-bundle in [293] and that its $K$-groups are isomorphic to $\mathbb{Z}^3$.

**Example (5.2.14)[304]:** For $q \in [-1, 1]$ consider the $C^*$-algebra $SU_q(2)$ from [278]. We recall that it is the universal $C^*$-algebra generated by two elements $a$ and csubject to the five relations

$$a^*a + cc^* = 1, \qquad aa^* + q^2cc^* = 1, \qquad cc^* = c^*c, \qquad ac = qca \text{ and } ac^* = qc^*a.$$

It carries a continuous action $\alpha$ of the one-dimensional torus $\mathbb{T}$ by algebra automorphisms, which is on the generators defined by

$$\alpha_z(a) := z \cdot a \quad \text{and} \quad \alpha_z(c) := z \cdot c, \qquad z \in \mathbb{T}.$$

The fixed point algebra of this action is the quantum 2-sphere $S^2_q$ and we call the corresponding $C^*$-dynamical system $(SU_q(2), \mathbb{T}, \alpha)$ the quantum Hopf fibration. It is free according to [335]. In fact, the author shows that if $E$ is a locally finite graph with no sources and no sinks, then the natural gauge action on the graph $C^*$-algebra $C^*(E)$ is free .

**Remark (5.2.15)[304]:** We recall that Example (5.2.12) and Example (5.2.13) are special cases of so-called trivial noncommutative principal bundles as discussed in [337], [338], [340]. In fact, it is not hard to see that each trivial noncommutative principal bundle is free (cf. Remark (5.2.59) and [297]).

**Remark (5.2.16)[304]:** Let $\Bbbk$ be the algebra of compact operators on some separable Hilbert space. The $C^*$-algebra $SU_q(2)$ is described in [316] as an extension of $C(\mathbb{T})$ by $C(\mathbb{T}) \otimes \Bbbk$, i.e., by a short exact sequence

$$0 \to C(\mathbb{T}) \otimes \mathbb{K} \to SUq(2) \to C(\mathbb{T}) \to 0 \tag{19}$$

of $C^*$-algebras. If we consider $C(\mathbb{T})$ endowed with the canonical T-action induced by right-translation, then a few moments thought shows that the sequence (19) is in fact $\mathbb{T}$-equivariant. In particular, it induces the following short exact sequence of $C^*$-algebras:

$$0 \to (C(\mathbb{T}) \otimes \mathbb{K}) \rtimes \mathbb{T} \to SU_q(2) \rtimes_\alpha \mathbb{T} \to C(\mathbb{T}) \rtimes \mathbb{T} \to 0. \tag{20}$$

Since

$$(C(\mathbb{T}) \otimes \mathbb{K}) \rtimes \mathbb{T} \cong (C(\mathbb{T}) \rtimes \mathbb{T}) \otimes \mathbb{K} \cong \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$$

and $C(\mathbb{T}) \rtimes \mathbb{T} \cong \mathbb{K}$ by the well-known Stone–von Neumann Theorem, we conclude from [333] that the crossed product $SU_q(2) \rtimes_\alpha \mathbb{T}$ is stable. Moreover, the fact that the $C^*$-dynamical system $(SU_q(2), \mathbb{T}, \alpha)$ is free implies that the crossed product $SU_q(2) \rtimes \mathbb{T}$ is Morita equivalent to the corresponding fixed-point algebra $S_q^2$ and thus they are also stably isomorphic by a famous result of Brown, Green and Rieffel (cf. [311] or [332]), i.e., $SU_q(2) \rtimes \mathbb{T} \cong S_q^2 \otimes \mathbb{K}$. The previous result affirms a question of the second author in the context of a notion of freeness which is related to Green's Theorem (cf. [82] and [293]).

We provide some methods to construct new free actions from given ones. As an application, we present a one-parameter family of free $SU(2)$-actions which are related to the Connes–Landi spheres.

**Proposition (5.2.17)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. If $(\mathcal{A}, G, \alpha)$ is free and Ha closed subgroup of $G$, then also the restricted $C^*$-dynamical system $(\mathcal{A}, H, \alpha|_H)$ is free.

**Proof:** Since $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition, the surjectivity of the restriction map $C(G, \mathcal{A}) \to C(H, \mathcal{A}), f \mapsto f|_H$ implies that the map

$$\Phi: \mathcal{A} \otimes_{alg} \mathcal{A} \to C(H, \mathcal{A}), \qquad \Phi(x \otimes y)(h) := x\alpha_h(y)$$

has dense range. That is, $(\mathcal{A}, H, \alpha|_H)$ also satisfies the Ellwood condition.

**Proposition (5.2.18)[304]:** Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group G. If $(\mathcal{A}, G, \alpha)$ is free and Na closed normal subgroup of $G$, then also the induced $C^*$-dynamical system $(\mathcal{A}^N, G/N, \alpha|_{G/N}$ is free.

**Proof:** Since $(\mathcal{A}, G, \alpha)$ satisfies the Ellwood condition, the map

$$\Phi: \mathcal{A} \otimes_{alg} \mathcal{A} \to C(H, \mathcal{A}), \qquad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range. Moreover, the $C^*$-algebra $C(G/N, \mathcal{A}^N)$ is naturally identified with functions in $C(G, \mathcal{A})$ satisfying $f(g) = \alpha_{n_1}(f(gn_2))$ for all $g \in G$ and $n_1, n_2 \in N$. In other words, $C(G/N, \mathcal{A}^N)$ is the fixed point algebra of the action $\alpha \otimes \mathrm{rt}$ of $N \times N$ on $C(G) \otimes \mathcal{A} = C(G, \mathcal{A})$, where $\mathrm{rt}: N \times C(G) \to C(G), \mathrm{rt}(n, f)(g) := f(gn)$ denotes the right-translation action by $N$. Let $P_N: \mathcal{A} \to \mathcal{A}$ and $P_{N\times N}: C(G, \mathcal{A}) \to C(G, \mathcal{A})$ be the conditional expectations for the actions $\alpha|_N$ and $\alpha \otimes \mathrm{rt}$, respectively. Then we obtain for arbitrary $x, y \in \mathcal{A}$

$$\Phi\big(P_N(x) \otimes P_N(y)\big) = \int_{N\times N} \alpha_{n_1}(x)\, \alpha_{gn_2}(y)\, d_{n_1} d_{n_2}$$

$$= \int_{N\times N} \alpha_{n_1}(x)\, \alpha_{n_2 g}(y)\, d_{n_1} d_{n_2}$$

$$= \int_{N\times N} \alpha_{n_1}\left(x\alpha_{n_2 g}(y)\right)\, d_{n_1} d_{n_2} = \int_{N\times N} \alpha_{n_1}\left(x\alpha_{gn_2}(y)\right)\, d_{n_1} d_{n_2}$$

$$= P_{N\times N}\big(\Phi(x \otimes \mathrm{y})\big).$$

It follows that the restricted map

$$\Phi|_{\mathcal{A}^N} \otimes_{alg\, \mathcal{A}^N}: \mathcal{A}^N \otimes_{alg} \mathcal{A}^N \to C(G, A), \qquad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range in the $C^*$-subalgebra $C(G/N,\mathcal{A}^N)$. That is, $(\mathcal{A}^N,G/N,\alpha|_{G/N})$ also satisfies the Ellwood condition.

**Proposition (5.2.19)[304]:** Let $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$ be $C^*$-dynamical systems with unital $C^*$-algebras $\mathcal{A},C$ and compact groups $G,H$. If $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$ are free, then also their tensor product $(\mathcal{A}\otimes C,G\times H,\alpha\otimes\gamma)$ is free.

**Proof:** We first note that the map

$$\Phi:\mathcal{A}\otimes_{alg}C\otimes_{alg}\mathcal{A}\otimes_{alg}C\to C(G\times H,\mathcal{A}\otimes C),$$
$$\Phi:(x\otimes y\otimes u\otimes v)(h)\coloneqq x\alpha_h(y)\otimes u\,\gamma_h(v)$$

is, up to a permutation of the tensor factors, an amplification of the corresponding maps induced by $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$. Therefore, $(\mathcal{A}\otimes C,G\times H,\alpha\otimes\gamma)$ inherits the Ellwood condition from $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$.

**Remark (5.2.20)[304]:** Suppose that $(\mathcal{A},G,\alpha)$ is a free $C^*$-dynamical system with a unital $C^*$-algebra $\mathcal{A}$ and a compact group $G$. Furthermore, let $C$ be an arbitrary unital $C^*$-algebra. Then Proposition (5.2.19) applied to the trivial group $H$ implies that the $C^*$-dynamical system $(\mathcal{A}\otimes C,G,\alpha\otimes \mathrm{id}_C)$ is free. More generally, if $(C,G,\gamma)$ is an arbitrary $C^*$-dynamical system, it is not hard to check that the $C^*$-dynamical system $(\mathcal{A}\otimes C,G,\alpha\otimes\gamma)$ satisfies the Ellwood condition, i.e., $(\mathcal{A}\otimes C,G,\alpha\otimes\gamma)$ is free. This observation corresponds in the classical setting to the situation of endowing the $C$artesian product of a free and compact $G$-space $X$ and any compact $G$-space $Y$ with the free diagonal action of $G$.

**Theorem(5.2.21)[304]:** Let $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$ be free $C^*$-dynamical systems with unital $C^*$-algebras $\mathcal{A},C$ and compact groups $G,H$. Furthermore, let $(\mathcal{A},H,\beta)$ be any another $C^*$-dynamical system such that the actions $\alpha$ and $\beta$ commute. Then the following assertions hold:

**(i)** The $C^*$-dynamical system $(\mathcal{A}\otimes C),G\times H,(\alpha\circ\beta)\otimes\gamma$ is free.

**(ii)** The $C^*$-dynamical system $(\mathcal{A}\otimes C)^H,G,\alpha\otimes\mathrm{id}_C$ is free, where the fixed space $(\mathcal{A}\otimes C)^H$ is taken with respect to the tensor product action $\beta\otimes\gamma$ of $H$.

**Proof:** (i) We first note that $(\mathcal{A},G,\alpha)$ and $(C,H,\gamma)$ both satisfy the Ellwood condition from which we conclude that the maps

$$\Phi_1:\mathcal{A}\otimes_{alg}\mathcal{A}\otimes_{alg}C\to C(G,\mathcal{A}\otimes C),\quad \Phi_1(x_1\otimes x_2\otimes y)(g)\coloneqq x_1\alpha_g(x_2)\otimes y$$

and $\Phi_2:(\mathcal{A}\otimes_{alg}C)\otimes_{alg}(\mathcal{A}\otimes_{alg}C)\to C(H,\mathcal{A}\otimes\mathcal{A}\otimes C)$ given by

$$\Phi_2(x_1\otimes y_1)\otimes(x_2\otimes y_2)(h\coloneqq x_1\otimes\beta_h(x_2)\otimes y_1\,\gamma_h(y_2)$$

have dense range. It follows, identifying $C(H,C(G,\mathcal{A}\otimes C))$ with $C(G\times H,\mathcal{A}\otimes C)$, that also their amplified composition

$$\Phi:=(\mathrm{id}_{C(H)}\otimes\Phi_1)\circ\Phi_2:(\mathcal{A}\otimes_{alg}C)\otimes_{alg}(\mathcal{A}\otimes_{alg}C)\to C(G\times H,\mathcal{A}\otimes C)$$

given by

$$\Phi:=\big((x_1\otimes y_1)\otimes(x_2\otimes y_2)\big)(g,h)=x_1(\alpha_g\beta_h)(x_2)\otimes y_1\,\gamma_h(y_2)$$

has dense range. That is, $(\mathcal{A}\otimes C),G\times H,(\alpha\circ\beta)\otimes\gamma$ satisfies the Ellwood condition. (ii) To verify the second assertion we simply apply Proposition (5.2.18) to the $C^*$-dynamical system in part (i) and the normal subgroup $\{\mathbb{1}_G\}\times H$ of $G\times H$.

**Example(5.2.22)[304]:** The $C$ onnes–Landi spheres $S^n_\theta$ are extensions of the noncommutative tori $\mathbb{T}^n_\theta$ (cf. [295]). We are in particularly interested in the case $n = 7$. In this case there is a continuous action of the 2-torus $\mathbb{T}^2$ on the 7-sphere $\mathbb{S}^7 \subseteq \mathbb{C}^4$ given by

$$\sigma: \mathbb{S}^7 \times \mathbb{T}^2 \to \mathbb{S}^7, \qquad ((z_1, z_2, z_3, z_4), (t_1,\ t_2)) \mapsto (t_1 z_1, t_1 z_2, t_2 z_3, t_2 z_4).$$

Let $(C(\mathbb{S}^7), \mathbb{T}^2, \alpha_\sigma$ be the corresponding $C^*$-dynamical system. Furthermore, let $(\mathbb{T}^2_\theta, \mathbb{T}^2, \alpha^2_\theta)$ be the free $C^*$-dynamical system associated to the gauge action on the noncommutative 2-torus $\mathbb{T}^2_\theta$ (see Example (5.2.12) below). The $C$onnes–Landi sphere $S^7_\theta$ is defined as the fixed point algebra of the tensor product action $\alpha_\sigma \otimes \alpha^2_\theta$ of $\mathbb{T}^2$ on $C(\mathbb{S}^7, \mathbb{T}^2_\theta) = C(\mathbb{S}^7) \otimes \mathbb{T}^2_\theta$, i.e.,

$$\mathbb{S}^7_\theta = C\left(\mathbb{S}^7, \mathbb{S}^2_\theta\right)^{\mathbb{T}^2}.$$

Our intention is to use Theorem (5.2.21)(ii) to endow $\mathbb{S}^7_\theta$ with a free $SU(2)$-action. For this purpose, we consider the free and continuous $SU(2)$-action on the 7-sphere $\mathbb{S}^7$given by

$$\mu : \mathbb{S}^7 \times SU(2) \to \mathbb{S}^7, \qquad ((z_1, z_2, z_3, z_4), M)) \mapsto (z_1, z_2, z_3, z_4) \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

It follows from Theorem3.4that the induced $C^*$-dynamical system $(C(\mathbb{S}^7), SU(2), \alpha_\mu)$ is free. Moreover, it is easily verified that the actions $\alpha_\mu$ and $\alpha_\sigma$ commute. Therefore, Theorem (5.2.21)(ii) implies that the $C^*$-dynamical system $\mathbb{S}^7_\theta, SU(2), \alpha_\mu \otimes \mathrm{id}_{\mathbb{T}^2_\theta}$ is free.

We will specify the data on which our classification of free actions of compact a belian groups is based (see Definition (5.2.27)) and, moreover, we will show that it is complete in the sense that every classifying data indeed can be obtained from a free $C^*$-dynamical system.

Corollary (5.2.11) suggests the relevance of Morita equivalence $\mathfrak{B} - \mathfrak{B}$ bimodules for the classification of free actions with a given fixed point algebra $\mathfrak{B}$ and a given compact A belian group $G$. These objects have a natural interpretation as noncommutative line bundles "over" $\mathfrak{B}$. In particular, just like in the classical theory of line bundles, the set of their equivalence classes carry a natural group structure.

**Definition (5.2.23)[304]:** Let $\mathfrak{B}$ be a $C^*$-algebra. Then the set of equivalence classes of Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodules forms an A belian group with respect to the internal tensor product of Hilbert $\mathfrak{B} - \mathfrak{B}$ -bimodules. This group is called the Picard group of $\mathfrak{B}$ and it is denoted by Pic($\mathfrak{B}$).

**Remark (5.2.24)[304]:** For a unital $C^*$-algebra $\mathfrak{B}$, the group of outer automorphisms Out $(\mathfrak{B}) := \mathrm{Aut}(\mathfrak{B}) / \mathrm{Inn}(\mathfrak{B})$ is always a subgroup of $\mathrm{Pic}(\mathfrak{B})$. More precisely, for any $^*$-automorphism $\alpha$ of $\mathfrak{B}$ we may define an element of the Picard group $\mathrm{Pic}(\mathfrak{B})$ by the following Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodule. Let $M_\alpha$ be the vector space $\mathfrak{B}$ endowed with the canonical left Hilbert $\mathfrak{B}$-module structure. Moreover, let the right action be given by $m \cdot b := m\, \alpha(b)$ for $m \in M_\alpha$ and $b \in \mathfrak{B}$, and let the right $\mathfrak{B}$-valued inner product be given by $m_1, m_2\ \mathfrak{B} := \alpha^{-1}(m_1^* m_2)$ for $m_1, m_2 \in M_\alpha$. It is straightforwardly checked that $M_\alpha$ is a Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodule and that, for $\alpha, \beta \in \mathrm{Aut}(\mathfrak{B})$, we have $M_\alpha \widehat{\otimes}_{\mathfrak{B}} M_\beta \simeq M_{\alpha \circ \beta}$. A few moments thought also shows that $M_\alpha \simeq \mathfrak{B}$ iff $\alpha$ is inner. Summarizing we have the exact sequence of groups

$$1 \longrightarrow \mathrm{Inn}(\mathfrak{B}) \longrightarrow \mathrm{Aut}(\mathfrak{B}) \longrightarrow \mathrm{Pic}(\mathfrak{B}).$$

**Example (5.2.25)[304]:**

**(i)** For a finite-dimensional $C^*$-algebra $\mathfrak{B}$, the Picard group $\operatorname{Pic}(\mathfrak{B})$ is isomorphic to the group of permutations of the spectrum of $\mathfrak{B}$ (see [311]).

**(ii)** For some compact space $X$, the Picard group $\operatorname{Pic} C(X)$ is isomorphic to the semidirect product $\operatorname{Pic}(X) \rtimes \operatorname{Homeo}(X)$, where $\operatorname{Pic}(X)$ denotes the set of equivalence classes of complex line bundles over $X$ and $\operatorname{Homeo}(X)$ the group of homeomorphisms of $X$ (see [311], [312]).

**(iii)** Let $0 < \theta < 1$ be irrational and $\mathbb{T}_\theta^2$ the corresponding quantum 2-torus. Then $\operatorname{Pic}(\mathbb{T}_\theta^2)$ is isomorphic to $\operatorname{Out}(\mathbb{T}_\theta^2)$ in case $\theta$ is quadratic and isomorphic to $\operatorname{Out}(\mathbb{T}_\theta^2) \rtimes \mathbb{Z}$ otherwise (see [324]).

The next statement is a first step towards finding invariants, i.e., classification data, for free $C^*$-dynamical systems with a prescribed fixed point algebra.

**Proposition (5.2.26)[304]:** Each free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with unital $C^*$-algebra $\mathcal{A}$, compact Abelian group $G$ and fixed point algebra $\mathfrak{B} := \mathcal{A}^G$ gives rise to a group homomorphism $\varphi_{\mathcal{A}}: \hat{G} \to \operatorname{Pic}(\mathfrak{B})$ given by $\varphi_{\mathcal{A}}(\pi) := [A(\pi)]$.

**Proof:** To verify the assertion we choose $\pi, \rho \in \hat{G}$ and use Corollary (5.2.11) to compute

$$\varphi_{\mathcal{A}}(\pi+\rho) = [A(\pi+\rho)] = [A(\pi) \mathbin{\widehat{\otimes}}_{\mathcal{A}} A(\rho)] = [A(\pi)][A(\rho)] = \varphi_{\mathcal{A}}(\pi)\varphi_{\mathcal{A}}(\rho).$$

This shows that the map $\varphi_{\mathcal{A}}$ is a group homomorphism.

The group homomorphism $\varphi : G \to \operatorname{Pic}(\mathfrak{B})$ in Proposition (5.2.26) is not enough to uniquely determine the $C^*$-dynamical system up to equivalence. Loosely speaking it determines the linear structure but not the multiplication of $\mathcal{A}$. In order to see what is missing, we choose for each $\pi \in \hat{G}$ a Morita equivalence bimodule $M_\pi$ in the class $\varphi(\pi)$. The group homomorphism property of $\varphi$ guarantees that for each $\pi, \rho \in \hat{G}$ there exist isomorphisms of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodules

$$\Psi_{\pi,\rho}: M_\pi \mathbin{\widehat{\otimes}}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho}.$$

In general $\varphi$ does not impose any relation among the maps $\Psi_{\pi,\rho}$. But, comingfrom a free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with unital $C^*$-algebra $\mathcal{A}$, compact A belian group $G$ and fixed point algebra $\mathfrak{B} := \mathcal{A}^G$, we may choose canonically

$$M_\pi := A(\pi) \text{ and } \Psi_{\pi,\rho}(x \otimes_{\mathfrak{B}} y) := xy$$

for each $\pi, \rho \in \hat{G}$. In this case, the associativity of the multiplication in $\mathcal{A}$ implies

$$\Psi_{\pi+\rho,\sigma} \circ \left(\Psi_{\pi,\rho} \otimes_{\mathfrak{B}} \operatorname{id}_\sigma\right) = \Psi_{\pi,\rho+\sigma} \circ \left(\operatorname{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}\right). \tag{21}$$

This suggests to take the following notion of factor system as a classifying object.

**Definition (5.2.27)[304]:** Let $\mathfrak{B}$ be a unital $C^*$-algebra, $G$ be a compact A belian group, and let $\varphi: G \to \operatorname{Pic}(\mathfrak{B})$ be a group homomorphism. Furthermore, let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ be a family where

**(i)** for each $\pi \in G$ we have a Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule $M_\pi$ in the class $\varphi(\pi)$.

**(ii)** for each $\pi, \rho \in G$ we have an isomorphism $\Psi_{\pi,\rho}: M_\pi \mathbin{\widehat{\otimes}}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho}$ of Morita equivalences.

Then $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ is called a factor system (for the map $\varphi$) if it satisfies equation (21) for all $\pi, \rho \in \hat{G}$ and the normalization condition $(M_0, \Psi_{0,0}) = (\mathfrak{B}, \operatorname{id}_{\mathfrak{B}})$.

**Remark (5.2.28)[304]:**

**(i)** Up to the canonical isomorphism $M_\pi \widehat{\otimes}_{\mathfrak{B}} \mathfrak{B} \simeq M_\pi \simeq \mathfrak{B} \widehat{\otimes}_{\mathfrak{B}} M_\pi$, the normalization condition implies for all $\pi \in \hat{G}$

$$\Psi_{\pi,0} = \mathrm{id}_\pi = \Psi_{\pi,0} \quad \text{and} \quad \Psi_{\pi,-\pi} \otimes_{\mathfrak{B}} \mathrm{id}_\pi = \mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\pi,-\pi}.$$

**(ii)** The group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ can obviously be recovered from the factor system.

**(iii)** Each free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with unital $C^*$-algebra $\mathcal{A}$ and compact A belian group $G$ gives rise to a factor system $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for the group homomorphism $\varphi_{\mathcal{A}}$ (cf. Proposition (5.2.26)) with

$$m_{\pi,\rho}: A(\pi) \widehat{\otimes}_{\mathfrak{B}} A(\rho) \to A(\pi+\rho), \qquad x \widehat{\otimes}_{\mathfrak{B}} y \mapsto xy.$$

**(iv)** Given a group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$, the existence of a factor system imposes a non-trivial cohomological condition on $\varphi$. we will characterize this condition and provide cohomological ways to construct factor systems without a $C^*$-dynamical system at hand.

We will show how to construct a corresponding $C^*$-dynamical system from a given factor system. This is done by reverse engineering an adaption of the $GNS$-representation for $C^*$-dynamical systems with arbitrary fixed point algebra. For this reason, we briefly review the construction. Readers familiar with the material may skip.

Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system and $P_0: \mathcal{A} \to \mathcal{A}$ the conditional expectation onto the fixed point algebra $\mathfrak{B} := \mathcal{A}^G$. Then $\mathcal{A}$ can be equipped with the structure of a right pre-Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodule with respect to the usual multiplication and the inner product given by $\langle x, y\rangle_{\mathfrak{B}} := P_0(x^*y)$ for $x, y \in \mathcal{A}$. Since $P_0$ is faithful, this inner product on $\mathcal{A}$ is definite and we may take the completion of $\mathcal{A}$ with respect to the norm $\|x\|_2 := \|P_0(x^*x)\|^{1/2}$. This provides a right Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodule $L^2(\mathcal{A})$ with $\mathcal{A}$ as a dense subset (cf .Proposition (5.2.1)). For each $\pi \in \hat{G}$ the projection $P_\pi$ onto the isotypic component $A(\pi)$ can be continuously extended to a self-adjoint projection on $L^2(\mathcal{A})$. In particular, the sets $A(\pi)$ are closed, pairwise orthogonal right Hilbert $\mathfrak{B} - \mathfrak{B}$-subbimodules of $L^2(\mathcal{A})$. and $L^2(\mathcal{A})$. can be decomposed into

$$L^2(\mathcal{A}) = \overline{\bigoplus_{\pi\in\hat{G}} A(\pi)}^{\|\cdot\|_2}$$

as right Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodules. For each element $a \in \mathcal{A}$ the left multiplication operator $\lambda_a: \mathcal{A} \to \mathcal{A}, x \mapsto ax$, then extends to an adjointable operator on $L^2(\mathcal{A})$. The arising representation

$$\lambda : \mathcal{A} \to \mathcal{L}\,(L^2(\mathcal{A})), \qquad a \mapsto \lambda_a$$

is called the left regular representation of $\mathcal{A}$. For each $g \in G$ the automorphism $\alpha_g$ extends from $\mathcal{A}$ to an automorphism $U_g: L^2(\mathcal{A}) \to L^2(\mathcal{A})$ of right Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodules and the strongly continuous group $(U_g)_{g\in G}$ implements αgin the sense that

$$\alpha_g(\lambda_a) = U_g\, \lambda_a U_g^+$$

for all $a \in \mathcal{A}$. The vector $\mathbb{1}_{\mathfrak{B}} = \mathbb{1}_{\mathcal{A}} \in L^2(\mathcal{A})$ is obviously cyclic and separating for this representation. In particular, the left regular representation is faithful and we may identify $\mathcal{A}$ with the subalgebra $\lambda(\mathcal{A})$. Since the sum of the isotypic components is dense in $\mathcal{A}$, the $C^*$-algebra $\lambda(\mathcal{A})$. is in fact generated by the operators $\lambda_a$ with $a \in A(\pi), \pi \in \hat{G}$. For such elements $a$ in some fixed isotypic component $A(\pi), \pi \in \hat{G}$, the operator $\lambda_a$ maps each subset $A(\rho) \subseteq L^2(\mathcal{A}), \rho \in \hat{G}$, into $A(\pi+\rho)$ and therefore it is determined by the multiplication map

$$m_{\pi,\rho} : A(\pi) \otimes_{alg} A(\rho) \to A(\pi+\rho), m_{\pi,\rho}(x,y) := xy = \lambda_x(y).$$

It is easily verified that $m_{\pi,\rho}$ factors to an isometry of the right Hilbert $\mathfrak{B}-\mathfrak{B}$-bimodules $A(\pi) \widehat{\otimes}_{\mathfrak{B}} A(\rho)$ and $A(\pi+\rho)$.

We consider a fixed unital $C^*$-algebra $\mathfrak{B}$, a compact A belian group $G$, and a group homomorphism $\varphi : \hat{G} \to \mathrm{Pic}(\mathfrak{B})$. We choose for each $\pi \in \hat{G}$ a Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodule $M\pi$in the isomorphism class $\varphi(\pi) \in \mathrm{Pic}(\mathfrak{B})$, where for $\pi = 0$ we choose $M_0 := \mathfrak{B}$. Since $\varphi$ is a group homomorphism, the Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodules $M_\pi \widehat{\otimes}_{\mathfrak{B}} M_\rho$ and $M_{\pi+\rho}$ must be isomorphic for all $\pi, \rho \in \hat{G}$ with respect to some isomorphism

$$\Psi_{\pi,\rho} : M_\pi \widehat{\otimes}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho}$$

of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodules. We fix a choice of $\Psi_{\pi,\rho}$ for each $\pi, \rho \in \hat{G}$ where $\Psi_{0,0} := \mathrm{id}_{\mathfrak{B}}$. This provides a bilinearmap

$$m_{\pi,\rho}: M_\pi \times M_\rho \to M_{\pi+\rho}, \qquad m_{\pi,\rho}(x,y) := \Psi_{\pi,\rho}(x \otimes_{\mathfrak{B}} y).$$

The family of all such maps $(m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ now gives rise to a multiplication map $m$ on the algebraic vector space

$$A = \bigoplus_{\pi\in\hat{G}} M_\pi.$$

**Proposition (5.2.29)[304]:** The following statements are equivalent:

**(i)** $m$ is associative, i.e., $A$ is an algebra.

**(ii)** $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ is a factor system.

**Proof:** For given $\pi, \rho, \sigma \in \hat{G}$ we explicitly compute for all $x \in M_\pi$ $y \in M_\rho$ $z \in M_\sigma$:

$$m\,(x, m(y,z)) = m\,(x, \Psi_{\rho,\sigma}(y \otimes_{\mathfrak{B}} z)) = \Psi_{\pi,\rho+\sigma}\,(x, \Psi_{\rho,\sigma}(y \otimes_{\mathfrak{B}} z))$$

$$m(m(x,y),z) = m\left(\Psi_{\pi,\rho}(x \otimes_{\mathfrak{B}} y), z\right) = \Psi_{\pi,\rho+\sigma}\left(\Psi_{\pi,\rho}(x \otimes_{\mathfrak{B}} y), z\right).$$

Therefore, $m$ is associative if and only if equation (21) holds for all $\pi, \rho, \sigma \in \hat{G}$.

We continue with a fixed factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for the map $\varphi$ and we write $A$ for the associated algebra. Our goal is to turn $A$ into a $^*$-algebra and right pre-Hilbert $\mathfrak{B}-\mathfrak{B}$-bimodule. For this purpose we involve the right Hilbert $\mathfrak{B}-\mathfrak{B}$-bimodule structure of each direct summands $M_\pi$ of $A$, i.e., the right $\mathfrak{B}$-valued inner products $\langle\cdot,\cdot\rangle_\pi$ on $M_\pi$.

**Lemma (5.2.30)[304]:** The map $\langle\cdot,\cdot\rangle : A \times A \to \mathfrak{B}$ defined for $x = \bigoplus_\pi x_\pi, y = \bigoplus_\pi y_\pi \in A$ by

$$\langle x, y\rangle_{\mathfrak{B}} := \underbrace{\sum}_{\pi\in\hat{G}} \langle x_\pi, y_\pi\rangle_\pi$$

turns $A$ into a right pre-Hilbert $\mathfrak{B}-\mathfrak{B}$ -bimodule and satisfies

$$\langle m(b,x), m(b,x)\rangle_{\mathfrak{B}} \leq \|b\|^2\ \langle x, y\rangle_{\mathfrak{B}} \tag{22}$$

for all x ∈ A and $b \in \mathfrak{B}$.

**Proof:** The necessary computations are straightforward and thus left to the reader. We only point out that the inequality (22) is a consequence of the corresponding inequalities satisfied by the right $\mathfrak{B}$ -valued inner products $\langle\cdot,\cdot\rangle_\pi$.

**Lemma (5.2.31)[304]:** For each $y \in M_\pi$ and every $\rho \in \hat{G}$ the left multiplication operator

$$\ell_y : M_\rho \to M_\pi \otimes_{\mathfrak{B}} M_\rho, \qquad x \mapsto y \otimes_{\mathfrak{B}} x$$

is $a$djointable and hence bounded with $a$djoint given by

$$\ell_y^+ : M_\pi \widehat{\otimes}_{\mathfrak{B}} M_\rho \to M_\rho, \qquad z \otimes_{\mathfrak{B}} x \mapsto \langle y,z\rangle_{\mathfrak{B}} \cdot x.$$

**Proof:** To verify the assertion we first note that the linear span of elements $z \otimes_{\mathfrak{B}} z'$ with $z \in M_\pi$ and $z \in M_\rho$ is dense in $M_{\pi+\rho}$. For such an element and $x \in M_\rho$ we obtain

$$\langle \ell_y(x), z \otimes_{\mathfrak{B}} z'\rangle_{\mathfrak{B}} = \langle y \otimes_{\mathfrak{B}} x, z \otimes_{\mathfrak{B}} z'\rangle_{\mathfrak{B}} = \langle x, \langle y,z\rangle \cdot z'\rangle_{\mathfrak{B}} = \langle x, \ell_y^+(z \otimes_{\mathfrak{B}} z')\rangle_{\mathfrak{B}}$$

which implies that $\ell_y$ is adjointable with adjoint given by the map $\ell_y^+$.

**Proposition (5.2.32)[304]:** For each $y \in M_\pi$ and every $\rho \in \hat{G}$ the left multiplication operator

$$\lambda_y : M_\rho \to M_{\pi+\rho} \; \lambda_y(x) := m(y,x)$$

is $a$djointable and hence bounded and satisfies

$$\langle \lambda_y(x), \lambda_y(x)\rangle_{\mathfrak{B}} \leq \|\langle y,y\rangle_{\mathfrak{B}}\| \cdot \langle x,x\rangle_{\mathfrak{B}} \tag{23}$$

for all $x \in M_\rho$.

**Proof:** That the left multiplication operator $\lambda_y : M_\rho \to M_{\pi+\rho}$ is adjointable for each $y \in M_\pi$ and every $\rho \in \hat{G}$ is an immediate consequence of Lemma (5.2.31) and the unitarity of the map $\Psi_{\pi,\rho}$ because $\lambda_y = \Psi_{\pi,\rho} \circ \ell_y$. The asserted inequality (23) then easily follows from a short computation involving inequality (22). Indeed, we obtain

$$\begin{aligned}\langle \lambda_y(x), \lambda_y(x)\rangle_{\mathfrak{B}} &= \langle \Psi_{\pi,\rho}(y \otimes_{\mathfrak{B}} x), \Psi_{\pi,\rho}(y \otimes_{\mathfrak{B}} x)\rangle_{\mathfrak{B}} = \langle y \otimes_{\mathfrak{B}} x, y \otimes_{\mathfrak{B}} x\rangle_{\mathfrak{B}} \\ &= \langle x, \langle y,y\rangle_{\mathfrak{B}} \cdot x\rangle_{\mathfrak{B}} \leq \|\langle y,y\rangle_{\mathfrak{B}}\| \langle x,x\rangle_{\mathfrak{B}}\end{aligned}$$

for all $x \in M_\rho$.

**Corollary (5.2.33)[304]:** For each $a \in A$ the left multiplication operator

$$\lambda_a : A \to A, \qquad \lambda_a(x := m(a,x)$$

is adjointable and bounded.

We are now ready to introduce an involution on A which turns A into $a$ $^*$-algebra. Here we use the fact that the involution is determined by the inner product if we impose that the inner product on A takes it canonical form.

**Definition (5.2.34)[304]:** The adjoint map $i: A \to A, a \mapsto i(a)$ is given by

$$i(a) := \lambda_a^+(\mathbb{1}_{\mathfrak{B}}).$$

It is clearly antilinear and maps the subspace $M_\pi \; \pi \in \hat{G}$, into $M_{-\pi}$. Moreover, on the subspace $\mathfrak{B} \subseteq A$ the imap coincides with the usual adjoint, i.e., we have $i(b) = b^*$.

The following lemma shows that the adjoints of left multiplication operators commute with right multiplication operators.

**Lemma (5.2.35)[304]:** For all $x \in M_\pi, y \in M_\rho$ and $z \in M_{\pi+\sigma}$ with $\pi, \rho, \sigma \in \hat{G}$ we have

$$m(\lambda_x^+(z), y) = \lambda_x^+(m(z, y)).$$

**Proof:** It suffices to note that equation (21) implies that

$$\lambda_x \circ \Psi_{\sigma,\rho} = \Psi_{\pi+\sigma,\rho} \circ (\lambda_x \otimes_{\mathfrak{B}} \mathrm{id}_\rho).$$

Indeed, taking adjoints then leads to

$$\Psi_{\sigma,\rho}^+ \circ \lambda_x^+ = \left(\lambda_x^+ \otimes_{\mathfrak{B}} \mathrm{id}_\rho\right) \circ \Psi_{\pi+\sigma,\rho}^+$$

which verifies the asserted formula since the maps $\Psi_{\sigma,\rho}$ and $\Psi_{\pi+\sigma,\rho}$ are unitary.

**Theorem (5.2.36)[304]:** For all $x \in A$ we have $\lambda_x^+ = \lambda_{i(x)}$.

**Proof:** It suffices to show the assertion for elements in individual direct summands. For this, let $x \in M_\pi, y \in M_\rho$, and $z \in M_\sigma$ with $\pi, \rho, \sigma \in \hat{G}$. Then using Lemma (5.2.35) gives

$$\begin{aligned}\langle \lambda_{i(x)}(y), z\rangle_{\mathfrak{B}} &= \langle m(i(x), y), z\rangle_{\mathfrak{B}} = \langle m(\lambda_x^+(\mathbb{1}_{\mathfrak{B}}), y), z\rangle_{\mathfrak{B}} = \langle \lambda_x^+(m(\mathbb{1}_{\mathfrak{B}}, y), z\rangle_{\mathfrak{B}} \\ &= \langle m(\mathbb{1}_{\mathfrak{B}}, y), m(x, z)\rangle_{\mathfrak{B}} = \langle y, m(x, z)\rangle_{\mathfrak{B}} = \langle \lambda_x^+(y), z\rangle_{\mathfrak{B}}.\end{aligned}$$

We conclude with two useful corollaries, e.g., we finally verify that the map $i: A \to A$ from Definition (5.2.34) actually defines an involution.

**Corollary (5.2.37)[304]:** Let $P_0: A \to A$ be the canonical projection onto the subalgebra $\mathfrak{B}$. Then for all $x, y \in A$ we have

$$\langle x, y\rangle_{\mathfrak{B}} = P_0(m(i(y), x)).$$

**Proof:** Since the element $\mathbb{1}_{\mathfrak{B}}$ is fixed by $P_0$ we conclude from Theorem (5.2.36) that

$$\langle x, y\rangle_{\mathfrak{B}} = \langle m(i(y), x), \mathbb{1}_{\mathfrak{B}}\rangle_{\mathfrak{B}} = \langle P_0(m(i(y), x)), \mathbb{1}_{\mathfrak{B}}\rangle_{\mathfrak{B}} = P_0(m(i(y), x)).$$

**Corollary (5.2.38)[304]:** The algebra $A$ is involutive, i.e., for all $x, y \in A$ we have

$$i\big(i(x)\big) = x \quad \text{and} \quad i(m(x, y)) = m(i(y), i(x)).$$

**Proof:** Applying Theorem (5.2.36) twice gives

$$\langle i\, i((x))\, , z\, \rangle_{\mathfrak{B}} = \langle \mathbb{1}_{\mathfrak{B}}, m(i(x), z)\rangle_{\mathfrak{B}} = \langle x, z\rangle_{\mathfrak{B}},$$
$$\langle i(m(x, y)), z\rangle_{\mathfrak{B}} = \langle \mathbb{1}_{\mathfrak{B}}, m(m(x, y), z)\rangle_{\mathfrak{B}} = \langle i(x), m(y, z)\rangle_{\mathfrak{B}} = \langle m(i(y), i(x)), z\rangle_{\mathfrak{B}}$$

for all $z \in A$ which in turn implies that $i(i(x)) = x$ and $i(m(x, y)) = m(i(y), i(x))$.

We turned $A = \bigoplus_{\pi \in \hat{G}} M_\pi$ into a $^*$-algebra and right pre-Hilbert $\mathfrak{B}-\mathfrak{B}$-bimodule. We denote by $\bar{A}$ the corresponding completion of $A$ with respect to the norm

$$\|x\|_2 := \|\langle x, x\rangle_{\mathfrak{B}}\|^{1/2} = \|P_0(m(i(x), x))\|^{1/2}$$

(cf. Corollary (5.2.37)). By Corollary (5.2.33), left multiplication with an element $a \in A$ extends to an adjointable linear map on $\bar{A}$. Therefore, the map

$$\lambda: A \to \mathcal{L}(\bar{A}), \qquad a \mapsto \lambda_a$$

is well-defined. Moreover, the characterization of the norm implies that the vector $\mathbb{1}_{\mathfrak{B}} \in \bar{A}$ is separating the operators $\lambda(A) \subseteq \mathcal{L}(\bar{A})$, i.e., if $\lambda_a(\mathbb{1}_{\mathfrak{B}}) = 0$ for some $a \in A$ then $a = 0$. The intention is to finally construct a free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathfrak{B}$.

**Proposition (5.2.39)[304]:** The map $\lambda: A \to \mathcal{L}(\bar{A}), a \mapsto \lambda_a$ is a faithful representation of the $*$-algebra $A$ by adjointable operators on the right Hilbert $\mathfrak{B}-\mathfrak{B}$-bimodule $\bar{A}$. Moreover, its restriction to each $M_\pi, \pi \in \hat{G}$, is isometric.

**Proof:** The necessary algebraic conditions are easily checked using Corollary (5.2.38). Moreover, the injectivity of the map $\lambda$ is a consequence of the previous discussion about the separating vector $\mathbb{1}_{\mathfrak{B}} \in \bar{A}$. To verify that the restriction of $\lambda$ to each $M_\pi, \pi \in \hat{G}$, is isometric, we fix $\pi \in \hat{G}$ and use inequality (23) of Proposition (5.2.32) which implies that $\|\lambda_y\|_{\mathrm{op}}^2 \leq \|y\|_2^2$ holds for all $y \in M_\pi$. On the other hand, the inequality

$$\|y\|_2^2 = \left\|\lambda_y(\mathbb{1}_{\mathfrak{B}})\right\|_2^2 \leq \left\|\lambda_y\right\|_{\mathrm{op}}^2$$

Follows from the observation that $\mathbb{1}_{\mathfrak{B}} \in \bar{A}$ satisfies $\|\mathbb{1}_{\mathfrak{B}}\|_2 = 1$. We conclude that $\|\lambda_y\|_{\mathrm{op}} = \|y\|_2$ holds for each $y \in M_\pi$, which finally shows that the restriction of $\lambda$ to $M_\pi$ is isometric and thus completes the proof.

**Definition (5.2.40)[304]:** We denote by $\mathcal{A}$ the $C^*$-algebra which is generated by the image of $\lambda$, i.e., the closure of $\lambda(A)$ with respect to the operator norm on $\mathcal{L}(\bar{A})$. In particular, we point out that $\mathcal{A}$ contains $\mathcal{A}$ as a dense $^*$-subalgebra.

To proceed we need to endow the $C^*$-algebra $\mathcal{A}$ with a continuous action of the compact A belian group $G$ by $^*$-automorphisms. For this purpose we first construct a strongly continuous unitary representation of $G$ on the right Hilbert $\mathfrak{B}-\mathfrak{B}$ –bimodule $\bar{A}$.

**Lemma (5.2.41)[304]:** For each $\pi \in \hat{G}$ the map $U_\pi: G \to U(M_\pi),\ g \mapsto (U_\pi)_g$ given by

$$(U_\pi)_g(x) := \pi_g \cdot x$$

is a strongly continuous unitary representation of $G$ on the right Hilbert $\mathfrak{B}-\mathfrak{B}$ -bimodule $M_\pi$. Moreover, taking direct sums and continuous extensions then gives rise to a strongly continuous unitary representation $U: G \to U(\bar{A}), g \mapsto U_g$ of $G$ on the right Hilbert $\mathfrak{B}-\mathfrak{B}$ –bimodule $\bar{A}$.

**Proof:** The necessary computations are straightforward using the right $\mathfrak{B}$-valued inner products $\langle\cdot,\cdot\rangle_\pi$ and $\langle\cdot,\cdot\rangle$ of the spaces $M_\pi$ and $\bar{A}$, respectively.

**Lemma (5.2.42)[304]:** The map $\alpha: G \to \mathrm{Aut}(\mathcal{A}), g \mapsto \alpha_g$ given by

$$\alpha_g(\lambda_a) := U_g\, \lambda_a\, U_g^+$$

is $a$ continuous action of $G$ on $\mathcal{A}$ $by$ $^*$-automorphisms.

**Proof:** The action property is obviously satisfied due to the fact that $g \mapsto U_g$ is a representation (Lemma (5.2.41)). Continuity follows from the strong continuity of the map $U$ from Lemma (5.2.41).

**Theorem(5.2.43)[304]:** The $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ associated to the factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ is free and satisfies $A(\pi) = M_\pi$ for all $\pi \in \hat{G}$. In particular, its fixed point algebra is given by $\mathfrak{B}$.

**Proof:** **(i)** Let $\pi \in \hat{G}$. We first check that the corresponding isotypic component $A(\pi)$ is equal to $M_\pi$. Indeed, using the separating vector shows that $\alpha_g(a) = U_g(a)$ holds for elements $a \in A$. In particular, the elements of $M_\pi \subseteq \mathcal{A}$ are contained in $A(\pi)$. Moreover, the continuity of the projection $P_\pi: \mathcal{A} \to \mathcal{A}$ on to $A(\pi)$ implies that $M_\pi = P_\pi(A) \subseteq \mathcal{A}$ is dense in $A(\pi)$. Since the restriction of $\lambda$ to $M_\pi$ is isometric we conclude that $M_\pi$ is closed in $A(\pi)$ and hence that $A(\pi) = M_\pi$ as claimed. In particular, the fixed point algebra of $(\mathcal{A}, G, \alpha)$ is given by $\mathfrak{B}$.

**(ii)** Next we show that the $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is free. For this purpose, we again fix $\pi \in \hat{G}$. Since $A(\pi) = M_\pi$ holds by part (i) and

$$M_{-\pi} \cdot M_\pi := \operatorname{span}\{m(x,y) | x \in M_{-\pi}, y \in M_\pi\}$$

is dense in $\mathfrak{B}$ by construction, it follows that the multiplication map on $\mathcal{A}$ induces an isomorphism of $\mathfrak{B} - \mathfrak{B}$-Morita equivalence bimodules between $A(-\pi) \otimes_{\mathfrak{B}} A(\pi)$ and $\mathfrak{B}$. We therefore conclude from Corollary (5.2.11) that the $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is free.

We have seen how a factor system gives rise to a free $C^*$-dynamical system and vice versa. We finally establish a classification theory for free actions of compact a belian groups. If not mentioned otherwise, $\mathfrak{B}$ denotes a fixed unital $C^*$-algebra and $G$ a fixed compact A belian group.

**Definition (5.2.44)[304]:**

**(i)** Let $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G', \alpha')$ be two free $C^*$-dynamical systems with unital $C^*$-algebras $\mathcal{A}$ and $\mathcal{A}'$ such that $\mathcal{A}^G = (\mathcal{A}')^G = \mathfrak{B}$. We call $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G', \alpha')$ equivalent if there is a $G$- equivariant $^*$-isomorphism $T: \mathcal{A} \to \mathcal{A}'$ satisfying $T|_{\mathfrak{B}} = \mathrm{id}_{\mathfrak{B}}$.

**(ii)** We call two factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ equivalent if there is a family $(T_\pi: M_\pi \to M'_\pi)_{\pi \in \hat{G}}$ of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule isomorphisms satisfying for all $\pi, \rho \in \hat{G}$

$$\Psi'_{\pi,\rho} \circ \left(T_\pi \otimes_{\mathfrak{B}} T_\rho\right) = T_{\pi+\rho} \circ \Psi_{\pi,\rho}. \tag{24}$$

We are now in the position to state and prove on of the main classification theorems.

**Theorem(5.2.45)[304]:** Let $G$ be a compact A belian group and $\mathfrak{B}$ a unital $C^*$-algebra. Furthermore, let $\varphi : \hat{G} \to \operatorname{Pic}(\mathfrak{B})$ be a group homomorphism and $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ factor systems for the map $\varphi$. Then the following statements are equivalent:

**(i)** The factor systems are equivalent.

**(ii)** The associated free $C^*$-dynamical systems are equivalent.

**Proof:** Suppose first that the factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho \in \hat{G}}$ are equivalent and let $(T_\pi: M_\pi \to M'_\pi)_{\pi \in \hat{G}}$ be a family of $\mathfrak{B} - \mathfrak{B}$-Morita equivalence bimodule isomorphisms such that equation (24) holds for all $\pi, \rho \in \hat{G}$. Furthermore, let

$$A := \bigoplus_{\pi \in \hat{G}} M_\pi \qquad \textit{and} \qquad A' := \bigoplus_{\pi \in \hat{G}} M'_\pi$$

be the corresponding $^*$-algebras with involutions given by $i$ and $i'$, respectively. Then a few moments thought shows that the direct sum of the maps $T_\pi: M_\pi \to M'_\pi, \pi \in \hat{G}$, provides a $G$- equivariant $^*$-isomorphism $T: A \to A'$ of algebras. In fact, the map $T$ is clearly a $G$-equivariant isomorphism of right pre-Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodules by construction. Moreover, the assumption that equation (24) holds for all $\pi, \rho \in \hat{G}$ implies that Tis multiplicative. By Theorem (5.2.36), it is also $^*$-preserving, that is, $T(i(x)) = i'(T(x))$ holds for all $x \in A$. Passing over to the continuous extension of $T$ provides a $G$-equivariant isomorphism $\bar{T}: \bar{A} \to \bar{A}'$ of right Hilbert $\mathfrak{B} - \mathfrak{B}$-bimodules and it is easily checked with the help of the previous discussion that the relation

$$Ad[\bar{T}] \circ \lambda = \lambda' \circ T$$

holds, where $\lambda: A \to \mathcal{L}(\bar{A})$ and $\lambda': A' \to \mathcal{L}(\overline{A'}))$ denote the faithful $^*$-representations from Proposition (5.2.39). In particular, we conclude that the map $\mathrm{Ad}[\bar{T}]: \mathcal{L}(\bar{A}) \to \mathcal{L}(\overline{A'})$ restricts to a $G$-equivariant $^*$-isomorphism between the associated free $C^*$-dynamical systems $(\mathcal{A}, G, \alpha)$ and $(\mathcal{A}', G', \alpha')$ which completes the first part of the proof.

Suppose, conversely, that the associated free $C^*$-dynamical systems $((\mathcal{A}, m), G, \alpha)$ and $((\mathcal{A}', m'), G, \alpha')$ are equivalent and let $T: \mathcal{A} \to \mathcal{A}'$ be a $G$- equivariant $^*$-isomorphism. Then it is a consequence of the $G$-equivariance of the map Tthat the corresponding restriction maps $T_\pi := T|_{M_\pi}: M_\pi \to M'_\pi, \pi \in \hat{G}$, are well-defined and $\mathfrak{B} - \mathfrak{B}$ bimodule isomorphisms. Moreover, the multiplicativity of $T$ implies that equation (24) holds for all $\pi, \rho \in \hat{G}$. Hence it remains to show that the family $(T_\pi: M_\pi \to M'_\pi)_{\pi \in \hat{G}}$ preserves the $\mathfrak{B}$-valued inner products. To see that this is true, we first conclude from the $^*$-invariance of $T$ that $T_\pi(x)^* = T_{-\pi}(x^*)$ holds for all $\pi \in \hat{G}$ and all $x \in M_\pi$. It follows from a short computation involving equation (24) that

$$\langle T_\pi(x), T_\pi(y)\rangle_{\mathfrak{B}} = m'(T_\pi(x), T_\pi(y)^*) = m'(T_\pi(x), T_{-\pi}(y)^*) = m(x, y^*) = \langle x, y\rangle_{\mathfrak{B}}$$

holds for all $\pi \in \hat{G}$ and all $x, y \in M_\pi$. The corresponding computation for the left $\mathfrak{B}$-valued inner products can be verified in a similar way and completes the proof.

**Definition(5.2.46)[304]:** We write $\mathrm{Ext}(\mathfrak{B}, G)$ for the set of equivalence classes of free actions of $G$ with fixed point algebra $\mathfrak{B}$. The equivalence class of a free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathfrak{B}$ is denoted by $[(\mathcal{A}, G, \alpha)]$.

Recall that for a free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with fixed point algebra $\mathfrak{B}$ we have a group homomorphism $\varphi_{\mathcal{A}}: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ given by $\varphi_{\mathcal{A}}(\pi) := [A(\pi)]$ (cf. Proposition (26)). By Theorem (5.2.45), the map $\varphi_{\mathcal{A}}$ only depends on the equivalence class of $(\mathcal{A}, G, \alpha)$ and hence we have an invariant

$$I: Ext(\mathfrak{B}, G) \to \mathrm{Homg}_{gr}(\hat{G}, \mathrm{Pic}(\mathfrak{B})), \qquad I([(\mathcal{A}, G, \alpha)]) := \varphi_{\mathcal{A}}.$$

In particular, we may partition $\mathrm{Ext}(\mathfrak{B}, G)$ into the subsets

$$\mathrm{Ext}(\mathfrak{B}, G, \varphi) := I^{-1}(\varphi) = \{[(\mathcal{A}, G, \alpha)] \in \mathrm{Ext}(\mathfrak{B}, G) |_{\varphi_{\mathcal{A}}} = \varphi\}.$$

For a fixed group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$, set $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ may be empty. We postpone this problem until the end and concentrate first on characterizing the set $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ and its $C^*$-dynamical systems. We start with a useful statement about automorphisms of Morita equivalence bimodules. Although it might be well-known to experts, we have not found such a statement explicitly discussed in the literature.

**Proposition(5.2.47)[304]:** Let $T$ be an automorphism of the Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule $M$. Then there exists a unique unitary element $u$ of the center of $\mathfrak{B}$, i.e., an element $u \in UZ(\mathfrak{B})$, such that $T(m) = u \cdot m$ for all $m \in M$. In particular, the map

$$\psi: UZ(\mathfrak{B}) \to \mathrm{Aut}_{ME}(M), \qquad \psi(u)(m) := u \cdot m,$$

is an isomorphism of groups, where $\mathrm{Aut}_{ME}(M)$ denotes the group of automorphisms of the Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodule $M$.

**Proof:** We divide the proof of this statement into two steps:

**(i)** In the first step we show that the assertion holds for the canonical Morita equivalence $\mathfrak{B} - \mathfrak{B}$ –bimodule $\mathfrak{B}$. To see that this is true, we choose $u \in UZ(\mathfrak{B})$ and note that the map $T_u: \mathfrak{B} \to \mathfrak{B}, b \longmapsto u \cdot b$ defines an automorphism of the Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodule $\mathfrak{B}$. In particular, the assignment

$$\psi_1: UZ(\mathfrak{B}) \to \mathrm{Aut}_{ME}(\mathfrak{B}), \qquad u \longmapsto T_u$$

is an isomorphism of groups. In fact, given $T \in \mathrm{Aut}_{ME}(\mathfrak{B})$, a short calculation shows that Tis uniquely determined by $T(\mathbb{1}_{\mathfrak{B}})$ which is an element in $UZ(\mathfrak{B})$.

**(ii)** In the second step we show that Morita equivalence automorphisms of $\mathfrak{B}$ are in one-to-one correspondence with automorphisms of $M$. To begin with, we denote by $\bar{M}$ the conjugate module and recall that the map

$$\Psi: M \widehat{\otimes}_{\mathfrak{B}} \bar{M} \to \mathfrak{B}, \qquad \Psi(m \otimes_{\mathfrak{B}} \overline{m'}) := {}_{\mathfrak{B}}\langle m, m' \rangle$$

for $m, m' \in M$ defines an isomorphism of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodules. Therefore, given an element $T \in \mathrm{Aut}_{ME}(M)$, it is not hard to check that the composition map $T_\Psi := \Psi \circ (T \otimes_{\mathfrak{B}} \mathrm{id}_{\bar{M}}) \circ \Psi^{-1}$ defines an automorphism of the Morita equivalence $\mathfrak{B} - \mathfrak{B}$ -bimodule $\mathfrak{B}$. Next, we show that the map

$$\psi_2 : \mathrm{Aut}_{ME}(\mathfrak{B}) \to \mathrm{Aut}_{ME}(M), \qquad \psi_2(T)(m) := T(\mathbb{1}_{\mathfrak{B}}) \cdot m$$

is an isomorphism of groups. In fact, we first note that $\psi_2$ is a well-defined and injective group homomorphism. Since $M$ is a full right Hilbert $\mathfrak{B}$-module, there is a finite set of elements $m_i, m_i' \in M (1 \leq i \leq n)$ such that $\sum_{i=1}^{n} {}_{\mathfrak{B}}\langle m_i, m_i' \rangle = \mathbb{1}_{\mathfrak{B}}$. The surjectivity of $\psi_2$ is then a consequence of the equation

$$\psi_2(T_\Psi)(m) = T_\Psi(\mathbb{1}_{\mathfrak{B}}) \cdot m = T(m)$$

which holds for all $m \in M$. The assertion therefore follows from $\psi = \psi_2 \circ \psi_1$.

**Corollary (5.2.48)[304]:** Let $M$ be a Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule and $u \in UZ(\mathfrak{B})$. Then there exists a unique element $\Phi_M(u) \in UZ(\mathfrak{B})$ such that $\Phi_M(u) \cdot m = m \cdot u$ holds for all $m \in M$. Furthermore, the map

$$\Phi_M: UZ(\mathfrak{B}) \to UZ(\mathfrak{B}), u \mapsto \Phi_M(u)$$

is an automorphism of groups.

**Proof:** The first assertion is an immediate consequence of Proposition (5.2.47) applied to the automorphism of $M$ defined by $m \mapsto m \cdot u$. That the map $\Phi_M$ is an automorphism of groups follows from a short calculation.

**Proposition (5.2.49) [304]:** The map

$$\Phi: \mathrm{Pic}(\mathfrak{B}) \to \mathrm{Aut}(UZ(\mathfrak{B})), \qquad [M] \mapsto \Phi_M$$

is a group homomorphism.

**Proof:** **(i)** We first show that $\Phi$ is well-defined. Therefore let $\Psi: M \to N$ be an isomorphism of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodules and $u \in UZ(\mathfrak{B})$. Then

$$\Phi_M(u) \cdot \Psi(m) = \Psi(\Phi_M(u) \cdot m) = \Psi(m \cdot u) = \Psi(m) \cdot u = \Phi_N(u) \cdot \Psi(m)$$

holds for all $m \in M$ which implies that $\Phi_M = \Phi_N$.

**(ii)** To see that $\Phi$ is a group homomorphism, let $M$ and $N$ be Morita equivalence $\mathfrak{B} - \mathfrak{B}$ bimodules and $u \in UZ(\mathfrak{B})$. Then

$$\Phi_{M \widehat{\otimes}_{\mathfrak{B}} N}(u) \cdot (m \otimes_{\mathfrak{B}} n) = (m \otimes_{\mathfrak{B}} n) \cdot u = m \otimes_{\mathfrak{B}} (n \cdot u) = m \otimes_{\mathfrak{B}} (\Phi_N(u) \cdot n)$$
$$= (m \cdot \Phi_N(u)) m \otimes_{\mathfrak{B}} n = (\Phi_M(\Phi_N(u)) \cdot m) \otimes_{\mathfrak{B}} n = (\Phi_M \circ \Phi_N)(u) \cdot (m \otimes_{\mathfrak{B}} n)$$

holds for all $m \in M$ and $n \in N$ which shows that $\Phi_{M \widehat{\otimes}_{\mathfrak{B}} N} = \Phi_M \circ \Phi_N$.

**Remark(5.2.50)[304]:** We point out that the map $\Phi$ from Proposition (5.2.49) induces a map

$$\Phi_*: \mathrm{Hom}_{gr}(\hat{G}, \mathrm{Pic}(\mathfrak{B})) \to \mathrm{Hom}_{gr}(\hat{G}, \mathrm{Aut}(UZ(\mathfrak{B})), \qquad \Phi_*(\varphi) := \Phi \circ \varphi.$$

In particular, each $\varphi \in \mathrm{Hom}_{gr}(\hat{G}, \mathrm{Pic}(\mathfrak{B}))$ determines a $\hat{G}$-module structure on $UZ(\mathfrak{B})$ which enables us to make use of classical group cohomology. In fact, given an element $\varphi \in \mathrm{Hom}_{gr}(\hat{G}, \mathrm{Pic}(\mathfrak{B}))$, the cohomology groups

$$H_\varphi^n(\hat{G}, UZ(\mathfrak{B}) := H_{\Phi \circ \varphi}^n(\hat{G}, UZ(\mathfrak{B})$$

are at our disposal(cf. [326]).

**Theorem (5.2.51)[304]:** Let $G$ be a compact A belian group and $\mathfrak{B}$ a unital $C^*$-algebra. Furthermore, let $\varphi: \hat{G} \to \text{Pic}(\mathfrak{B})$ be a group homomorphism with $\text{Ext}(\mathfrak{B}, G, \varphi) \neq \emptyset$ and choose for all $\pi \in \hat{G}$ a Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule $M_\pi \in \varphi(\pi)$ such that $M_0 = \mathfrak{B}$. Then the
Following assertions hold:

**(i)** Each class in $\text{Ext}(\mathfrak{B}, G, \varphi)$ can be represented by a free $C^*$-dynamical system of the form $(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$.

**(ii)** Any other free $C^*$-dynamical system $((\mathcal{A}_{(M,\Psi')}, G, \alpha_{(M,\Psi')})$ representing an element of $\text{Ext}(\mathfrak{B}, G, \varphi)$ satisfies , $\Psi' = \omega\Psi$ with $(\omega\Psi)_{\pi,\rho} := \omega(\pi,\rho)\Psi_{\pi,\rho}$ for all $\pi, \rho \in \hat{G}$ for some 2-cocycle

$$\omega \in Z^2_\varphi(\hat{G}, UZ(\mathfrak{B})).$$

**(iii)** The free $C^*$-dynamical systems $(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$ and $\mathcal{A}_{(M,\omega\Psi)}, G,\ \alpha_{(M,\omega\Psi)}$ are equivalent if and only if

$$\omega \in B^2_\varphi(\hat{G}, UZ(\mathfrak{B})).$$

**Proof: (i)** Let $(\mathcal{A}, G, \alpha)$ be a free $C^*$-dynamical system representing an element in $\text{Ext}(\mathfrak{B}, G, \varphi)$ and recall that $(\mathcal{A}, G, \alpha)$ gives rise to a factor system for the map $\varphi$ of the form $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$. Then the assumption implies that there is a family $(T_\pi: M_\pi \to A(\pi))_{\pi\in\hat{G}}$ of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule isomorphisms which can be used to define another family $(\Psi''_{\pi,\rho}: M_\pi \widehat{\otimes}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\hat{G}}$ of Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule isomorphisms by

$$\Psi''_{\pi,\rho} \coloneqq T^+_{\pi+\rho} \circ m_{\pi,\rho} \circ (T_\pi \otimes_{\mathfrak{B}} T_\rho)$$

In particular, it is not hard to see that the later family gives rise to a factor system $(M_\pi, \Psi''_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for the map $\varphi$ which is equivalent to $(A(\pi),\ m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$. Therefore, the assertion is finally a consequence of Theorem (5.2.46).

**(ii)** Let $\mathcal{A}_{(M,\Psi')}, G,\ \alpha_{(M,\Psi)}$ be any other free $C^*$-dynamical system representing an element of $\text{Ext}(\mathfrak{B}, G, \varphi)$ and choose $\pi, \rho \in \hat{G}$. Then Proposition (5.2.48) implies that the automorphism $\Psi'_{\pi,\rho} \circ \Psi^{-1}_{\pi,\rho}$ of the Morita equivalence $\mathfrak{B} - \mathfrak{B}$-bimodule $M_{\pi+\rho}$ provides a unique element $\omega(\pi,\rho) \in UZ(\mathfrak{B})$ satisfying

$$\Psi'_{\pi,\rho} = \omega(\pi,\rho)\Psi_{\pi,\rho}.$$

Moreover, it is easily seen that the corresponding map $\omega: \hat{G} \times \hat{G} \to UZ(\mathfrak{B})$ is a normalized 2-c ochain. To see that $\omega$ actually defines a 2-cocycle, i.e., an element in $z^2_\varphi(\hat{G}, UZ(\mathfrak{B}))$, we repeatedly use the factor system condition equation (21) and Proposition (5.2.50). For example, we find that

$$\begin{aligned}\Psi'_{\pi,\rho+\sigma} \circ \left(\text{id}_\pi \otimes_{\mathfrak{B}} \Psi'_{\rho+\sigma}\right) &= \Psi'_{\pi,\rho+\sigma} \circ \left(\text{id}_\pi \otimes_{\mathfrak{B}} \omega(\rho,\sigma)\Psi_{\rho,\sigma}\right) \\ &= \Psi'_{\pi,\rho+\sigma} \circ \left(\text{id}_\pi \otimes_{\mathfrak{B}} \omega(\rho,\sigma)\Psi_{\rho,\sigma}\right) \\ &= \Psi'_{\pi,\rho+\sigma} \circ (\Phi_\pi\left(\omega(\rho,\sigma)\text{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}\right) \\ &= \Phi_\pi\left(\omega(\rho,\sigma)\right)\Psi'_{\pi,\rho+\sigma} \circ \left(\text{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}\right) \\ &= \Phi_\pi\left(\omega(\rho,\sigma)\right)\omega(\pi,\rho+\sigma)\Psi_{\pi,\rho+\sigma} \circ \left(\text{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}\right)\end{aligned}$$

holds for all $\pi, \rho, \sigma \in \hat{G}$, where $\text{id}_\pi\omega(\rho,\sigma) = \Phi_\pi(\omega(\rho,\sigma))\ \text{id}_\pi$ is understood in the sense of Corollary (5.2.49).

**(iii)** If $\omega = d_\varphi h$ holds for some element $h\ \in C^1(\hat{G}, UZ(\mathfrak{B}))$, then the factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ are equivalent. Hence, the assertion follows from Theorem (5.2.46). If, on the other hand, $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ are equivalent, then we conclude from Proposition (5.2.48) that there exists an element $h \in C^1(\hat{G}, UZ(B))$ which implements the equivalence given by a family $(T_\pi)_{\pi\in\hat{G}}$ of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodule isomorphisms $T_\pi: M_\pi \to M_\pi$, i.e., we have $T_\pi = T_{h(\pi)}$ for all $\pi \in \hat{G}$. Moreover, a few moments thought shows that $\omega = d_\varphi h \in B^2_\varphi(\hat{G}, UZ(\mathfrak{B}))$.

**Corollary (5.2.52)[304]:** Let $G$ be a compact A belian group and $\mathfrak{B}$ a unital $C^*$-algebra. Furthermore, let $\varphi : G \to \mathrm{Pic}(\mathfrak{B})$ be a group homomorphism with $\mathrm{Ext}(\mathfrak{B}, G, \varphi) = \varphi$. Then the map

$$H^2_\varphi(\hat{G}, UZ(\mathfrak{B})) \times Ext(\mathfrak{B}, G, \varphi) \to Ext(B, G, \varphi)$$
$$\left([\omega], [\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)}]\right) \mapsto [(\mathcal{A}_{(M,\omega\Psi)}, G, \alpha_{(M,\omega\Psi)})]$$

is a well-defined simply transitive action.

We conclude with a remark which shows that our constructions from are, up to isomorphisms, inverse to each other.

**Remark(5.2.53)[304]:** We first recall that each free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with unital $C^*$-algebra $\mathcal{A}$ and compact A belian group $G$ gives rise to a factor system for the map $\varphi_{\mathcal{A}}$ of the form $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ (cf. Remark (3.2.28)). Furthermore, it follows from Theo-rem (5.2.46) that the free $C^*$-dynamical system associated to this factor system is equivalent to $(\mathcal{A}, G, \alpha)$. On the other hand, given a group homomorphism $\varphi : \hat{G} \to \mathrm{Pic}\,(\mathfrak{B})$ and a factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for the map $\varphi$, it is easily seen that the associated free $C^*$-dynamical system $(\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$ in Theorem (5.2.43) recovers the original factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$. Indeed, Theorem (5.2.43) shows that $\mathcal{A}_{(M,\Psi)}(\pi) = M_\pi$ holds for all $\pi \in \hat{G}$. Moreover, the multiplication map of $\mathcal{A}_{(M,\Psi)}$ is by construction uniquely determined by the factor system, i.e., we have $m_{\pi,\rho} = \Psi_{\pi,\rho}$ for all $\pi, \rho \in \hat{G}$. We therefore conclude that our constructions, i.e., the procedure of associating a free $C^*$-dynamical system to a factor system and vice versa, are, up to isomorphisms, inverse to each other:

$$(\mathcal{A}, G, \alpha) \quad \underset{\mapsto}{F.S} \quad (A(\pi), m_{\pi,\rho})_{\pi,\rho\in\hat{G}}\ .$$
$$\left(M_\pi, \Psi_{\pi,\rho}\right)_{\pi,\rho\in\hat{G}} \quad \underset{\mapsto}{C^*} \quad (\mathcal{A}_{(M,\Psi)}, G, \alpha_{(M,\Psi)})$$

As we have already discussed before, each group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ gives rise to both a family $(M_\pi)_{\pi\in\hat{G}}$ of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodules and a family

$$(\Psi_{\pi,\rho} : M_\pi \mathbin{\widehat{\otimes}}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\hat{G}}$$

of Morita equivalence $\mathfrak{B}-\mathfrak{B}$ -bimodules isomorphisms. Given a group homomorphism $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ and such a family $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ satisfying $M_0 = \mathfrak{B}, \Psi_{0,0} = \mathrm{id}_{\mathfrak{B}}$ and $\Psi_{\pi,0} = \Psi_{0,\pi} = \mathrm{id}_\pi$ for all $\pi \in \hat{G}$ (which need not be a factor system), we can examine for all $\pi, \rho, \sigma \in \hat{G}$ the automorphism

$$d_M\Psi(\pi,\rho,\sigma) := \Psi_{\pi+\rho,\sigma} \circ (\Psi_{\pi,\rho} \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi^+_{\rho,\sigma}) \circ \Psi^+_{\pi,\rho+\sigma}$$

of the Morita equivalence $\mathfrak{B}-\mathfrak{B}$ -bimodule $M_{\pi+\rho+\sigma}$. The family of all such maps $(d_M\Psi(\pi,\rho,\sigma))_{\pi,\rho,\sigma\in\hat{G}}$ can be interpreted as an obstruction to the associativity of the multiplication (cf. Proposition (5.2.29)). On the other hand, it follows from the construction and from Proposition (5.2.48) that the map $d_M\Psi$ can also be considered as a normalized $UZ(\mathfrak{B})$-valued 3-cochain on $\hat{G}$, i.e., as an element in $C^3(\hat{G}, UZ(B))$. In fact, even more is true:

**Lemma (5.2.54)[304]:**The map $d_M\Psi$ defines an element in $Z^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$.

**Proof:** For the sake of brevity we omit the lengthy calculation at this point and refer instead .

**Lemma (5.2.55)[304]:**The class $[d_M\Psi]$ in $H^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$ is independent of all choices made.

**Proof:** (**i**) We first show that the class $[d_M\Psi]$ is independent of the choice of the family $(\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$.Therefore, let $(\Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ be another choice and note that Proposition (5.2.48) implies that there exists an element $h \in C^2(\hat{G}, UZ(\mathfrak{B}))$ satisfying $\Psi'_{\pi,\rho} = h(\pi,\rho)\Psi_{\pi,\rho}$ for all $\pi,\rho \in \hat{G}$. A short calculation then shows that

$$\Psi'_{\pi+\rho,\sigma} \circ (\Psi'_{\pi,\rho} \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) = h(\pi+\rho,\sigma)h(\pi,\rho)d_M\Psi(\pi,\rho,\sigma)(\Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}))$$

holds for all $\pi,\rho\sigma \in \hat{G}$. On the other hand, it follows from Proposition (5.2.50) that

$$\begin{aligned} &d_M\Psi'(\pi,\rho,\sigma)\left(\Psi'_{\pi,\rho+\sigma} \circ \left(\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi'_{\rho,\sigma}\right)\right) \\ &= d_M\Psi'(\pi,\rho,\sigma)h(\pi,\rho+\sigma)\pi.h(\rho,\sigma)(\Psi_{\pi,\rho+\sigma} \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi_{\rho,\sigma}))\end{aligned}$$

holds for all $\pi,\rho,\sigma \in \hat{G}$. From these observations we can now easily conclude that the 3-cocycles $d_M\Psi'$ and $d_M\Psi$ are cohomologous.

**Proof:** (**ii**) As a second step, we show that the class $[d_M\Psi]$ does not dependent on the choice of the family $(M_\pi)_{\pi\in\hat{G}}$. For this purpose, let $(\Psi'_\pi)_{\pi\in\hat{G}}$ be another choice and note that the construction implies that there is a family $(T_\pi: M_\pi \to M'_\pi)_{\pi\in\hat{G}}$ of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodule isomorphisms. This family can now be used to define another family $(\Psi''_{\pi,\rho}: M'_\pi \widehat{\otimes}_{\mathfrak{B}} M'_\rho \to M'_{\pi+\rho})_{\pi,\rho\in\hat{G}}$ of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodule isomorphisms by

$$M'_{\pi,\rho} := T_{\pi+\rho} \circ \Psi_{\pi,\rho} \circ (T^+_\pi \otimes_{\mathfrak{B}} T^+_\rho).$$

Then an explicit computation shows that

$$\begin{aligned} d_M\Psi'(\pi,\rho,\sigma) &= \Psi'_{\pi+\rho,\sigma} \circ \left(\Psi'_{\pi,\rho} \otimes_{\mathfrak{B}} \mathrm{id}_\sigma\right) \circ \left(\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi'^{+}_{\rho,\sigma}\right) \circ \Psi'^{+}_{\pi,\rho+\sigma} \\ &= T_{\pi+\sigma+\rho} \circ \Psi_{\pi+\rho,\sigma} \circ \left(T^+_{\pi+\rho} \otimes_{\mathfrak{B}} T^+_\sigma\right) \\ &\quad \circ (T_{\pi+\rho} \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) \circ (\Psi_{\pi,\rho} \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) \circ (T^+_\pi \otimes_{\mathfrak{B}} T^+_\rho \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) \\ &\quad \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} T_\rho \otimes_{\mathfrak{B}} \mathrm{id}_\sigma) \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} \Psi^+_{\rho,\sigma}) \circ (\mathrm{id}_\pi \otimes_{\mathfrak{B}} T^+_{\rho+\sigma}) \\ &\quad \circ \left(T_\pi \otimes_{\mathfrak{B}} T_{\rho+\sigma}\right) \circ \Psi^+_{\pi,\rho+\sigma} \circ T^+_{\pi+\rho+\sigma} \\ &= d_M\Psi(\pi,\rho,\sigma)\end{aligned}$$

holds for all $\pi,\rho,\sigma \in \hat{G}$. We conclude that $d_M\Psi' = d_M\Psi$, i.e., that the 3-cocycle $d_M\Psi$ is unchanged.

**Definition (5.2.56)[304]:** Let $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ be a group homomorphism. We call

$$\chi(\varphi) := [d_M\Psi] \in H^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$$

the characteristic class of $\varphi$.

The following result provides a group theoretic criterion for the non-emptiness of the set $\mathrm{Ext}(\mathfrak{B}, G, \phi)$.

**Theorem(5.2.57)[304]:**Let $G$ be $a$ compact A belian group and $\mathfrak{B}$ $a$ unital $C^*$-algebra. Furthermore, let $\varphi: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ be a group homomorphism. Then $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ is non-empty if and only if the class $\chi(\varphi) \in H^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$ vanishes.

**Proof:**(⇒) Suppose first that $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ is non-empty and let $(\mathcal{A}, G, \alpha)$ be a free $C^*$-dynamical system representing an element in $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$. Then $(\mathcal{A}$, G, α) gives rise to a factor system for the map $\varphi$ of the form $(A(\pi), m_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and the associativity of the multiplication implies that the corresponding characteristic class $\chi(\varphi) \in H^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$ vanishes.

($\Leftarrow$) Let $(M_\pi)_{\pi\in\hat{G}}$ be a family of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodules and

$$(\Psi_{\pi,\rho} : M_\pi \widehat{\bigotimes}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\hat{G}}$$

a family of Morita equivalence $\mathfrak{B}-\mathfrak{B}$ -bimodules isomorphisms as described in the introduction. Furthermore, suppose, conversely, that the class

$$\chi(\varphi) = [d_M\Psi] \in H^3_\varphi(\hat{G}, UZ(\mathfrak{B}))$$

vanishes. Then there exists an element $h \in C^2(\hat{G}, UZ(\mathfrak{B}))$ with $d_M\Psi = d_\varphi h^{-1}$ which can be use to define a family $(\Psi'_{\pi,\rho}: M_\pi \widehat{\bigotimes}_{\mathfrak{B}} M_\rho \to M_{\pi+\rho})_{\pi,\rho\in\hat{G}}$ of Morita equivalence bimodule $\mathfrak{B}-\mathfrak{B}$ -isomorphism by

$$\Psi'_{\pi,\rho} := h(\pi,\rho)\Psi_{\pi,\rho}.$$

The construction implies that $d_M\Psi' = \mathbb{1}_{\mathfrak{B}}$. In particular, it follows that $(M_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ is a factor system for the map $\varphi$ and we can finally conclude from Theorem (5.2.42) that the set $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ is non-empty.

**Remark (5.2.58)[304]:**The group of outer automorphisms Out $(\mathfrak{B})$ is always a subgroup of the Picard group $\mathrm{Pic}(\mathfrak{B})$. The intention of this remark is to describe the elements of the set $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ for a given group homomorphism $\varphi: \hat{G} \to Out(\mathfrak{B})$. For this purpose, let $(\mathcal{A}, G, \alpha)$ be a free $C^*$-dynamical system representing an element of $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ Then it is not hard to see that each isotypic component contains an invertible element in $\mathcal{A}$. In fact, it follows from Corollary (5.2.11) that the map

$$A(-\pi) \widehat{\bigotimes}_{\mathfrak{B}} A(\pi) \to \mathfrak{B}, \qquad x \otimes_{\mathfrak{B}} y \mapsto xy \tag{25}$$

is an isomorphism of Morita equivalence $\mathfrak{B}-\mathfrak{B}$-bimodules for all $\pi \in \hat{G}$. Moreover, the assumption on $\varphi$ implies that for each $\pi \in G$ we have $\varphi(\pi) = [A(\pi)]$; that is, there is an automorphism $S(\pi) \in \mathrm{Aut}(\mathfrak{B})$ and an isomorphism $T_\pi: M_{S(\pi)} \to A(\pi)$ of Morita equivalence $\mathfrak{B}-\mathfrak{B}$ -bimodules. If we now define $u_\pi := T_\pi(\mathbb{1}_{\mathfrak{B}})$, then a few moments thought shows that

$$A(\pi) = u_\pi\mathfrak{B} = \mathfrak{B}u_\pi,$$

from which we conclude together with equation (25) that

$$u_\pi\mathfrak{B}_{u-\pi} = u_{-\pi}\mathfrak{B}u_\pi = \mathfrak{B}.$$

Consequently, the element $u_\pi \in A(\pi)$ is invertible in $\mathcal{A}$. Conversely, let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system such that each isotypic component contains an invertible element. Then it is easily verified that $(\mathcal{A}, G, \alpha)$ is free and that the corresponding group homomorphism $\varphi_{\mathcal{A}}: \hat{G} \to \mathrm{Pic}(\mathfrak{B})$ from Proposition (5.2.26) takes values in $\mathrm{Out}(\mathfrak{B})$. Indeed, if for every $\pi \in \hat{G}$ we have an element $u_\pi \in A(\pi)$ that is invertible in $\mathcal{A}$, then $A(\pi) = u_\pi\mathfrak{B} = \mathfrak{B}u_\pi$, and the map (25) is an isomorphism of Morita equivalence $\mathfrak{B}-\mathfrak{B}$ -bimodules. In particular, $(\mathcal{A}, G, \alpha)$ represents an element in $\mathrm{Ext}(\mathfrak{B}, G, \varphi_{\mathcal{A}})$.

$C^*$-dynamical systems with the property that each isotypic component contains invertible elements have been studied, for example, in [170], [338], [340], [302] and may be considered as a noncommutative version of trivial principal bundles (cf. Remark (5.2.15)).

**Remark (5.2.59)[304]:** The aim of the following discussion is to explain how to classify the $C^*$-dynamical systems described. Indeed, let $(\mathcal{A}, G, \alpha)$ be a $C^*$-dynamical system such that each isotypic component contains an invertible element. Furthermore, let $(u_\pi)_{\pi\in\hat{G}}$ be a family of unitaries with $u_\pi \in A(\pi)$ and $u_0 = \mathbb{1}_{\mathfrak{B}}$. Then the maps $S: \hat{G} \to Aut(\mathfrak{B})$ and $\omega: \hat{G} \times \hat{G} \to U(\mathfrak{B})$ given by

$$S(\pi)(b) := u_\pi b_\pi^* \quad and \quad \omega(\pi, \sigma) := u_\pi \, u_\rho \, u_{\pi+\rho}^*,$$

give rise to an element $(S, \omega) \in C^1(\hat{G}, Aut(\mathfrak{B})) \times C^2(\hat{G}, U(\mathfrak{B}))$ satisfying for all $\pi, \rho, \sigma \in \hat{G}$ and $b \in \mathfrak{B}$ the relations

$$S(\pi)\big(\omega(\rho, \sigma)\big)\omega(\pi, \rho + \sigma) = \omega(\pi + \rho, \sigma)\omega(\pi, \rho) \tag{26}$$

$$S(\pi)\big(S(\rho)(b)\big) = \omega(\pi, \rho)S(\pi + \rho)(b)\omega(\pi, \rho)^* \,. \tag{27}$$

Conversely, each pair $(S, \omega) \in C^1(\hat{G}, Aut(\mathfrak{B})) \times C^2(\hat{G}, U(\mathfrak{B}))$ satisfying, for all $\pi, \rho, \sigma \in \hat{G}$ and $b \in \mathfrak{B}$, the relations (25) and (27) defines a factor system $(M_\pi)_{\pi\in\hat{G}}$ and $(\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ given by $M_\pi := M_{S(\pi)}$ and

$$\Psi_{\pi,\rho}(b \otimes_{\mathfrak{B}} b') := bS(\pi)(b')\omega(\pi, \rho).$$

The associated free $C^*$-dynamical system represents an element of $\mathrm{Ext}(\mathfrak{B}, G, \varphi)$ with $\varphi := \mathrm{pr}_{\mathfrak{B}} \circ S: \hat{G} \to \mathrm{Out}(\mathfrak{B})$ , where $\mathrm{pr}_{\mathfrak{B}}: \mathrm{Aut}(\mathfrak{B}) \to \mathrm{Out}(\mathfrak{B})$ denotes the canonical quotient homomorphism. It is worth pointing out that in this situation, the involution can be expressed explicitly in terms of the pair $(S, \omega)$.

Let $P$ and $X$ be compact spaces. Let $G$ be a compact group. Each locally trivial principal bundle $(P, X, G, q, \sigma)$ can be considered as a geometric object that is glued together from local pieces which are trivial, i.e., which are of the form $U \times G$ for some small open subset $U$ of $X$. This approach immediately leads to the concept of $G$-valued cocycles and therefore to a cohomology theory, called the Čech cohomology for the pair $(X, G)$. This cohomology theory gives a complete classification of locally trivial principal bundles with structure group $G$ and base space $X$ (see [336]). On the other hand, Theorem (5.2.4) implies that each locally trivial principal bundle $(P, X, G, q, \sigma)$ gives rise to a free $C^*$-dynamical system $(C(P), G, \alpha_\sigma)$ and it is therefore natural to ask how the Čech cohomology for the pair $(X, G)$ is related to our previous classification theory. But since our construction is global in nature, it is not obvious how to encode local triviality in our factor system approach (though we recall that in the smooth category there is a one-to-one correspondence between free actions and locally trivial principal bundles). For this reason we now focus our attention on topological principal bundles, a notion of principal bundles which need not be locally trivial.

**Definition (5.2.60)[304]:**

**i.** Let $P$ be a compact space and $G$ a compact group. We call a continuous action $\sigma: P \times G \to P$ which is free a topological principal bundle or, more precisely, a topological principal $G$-bundle over $X := P/G$.

**ii.** We call two topological principal bundles $\sigma: P \times G \to P$ and $\sigma': P' \times G \to P'$ over $X$ equivalent if there is a $G$-equivariant homeomorphism $h : P \to P'$such that the induced map on $X$ is the identity.

We now come back to our $C^*$-algebraic setting. Let $G$ be a compact A belian group and $\sigma: P \times G \to P$ a topological principal $G$-bundle over $X$. Then for $\pi \in \hat{G}$ the isotypic component $C(P)(\pi)$ is a finitely generated and projective $C(X)$-module ac-cording to Theorem (5.2.5). Therefore, the Theorem of Serre and Swan (cf.[174]) gives rise to a locally trivial complex line bundle $\mathbb{V}_\pi$ over $X$ such that $C(P)(\pi)$ as a right $C(X)$-module is isomorphic to the corresponding space $\Gamma\mathbb{V}_\pi$ of continuous. We point out that this isomorphism can be extended to an isomorphism between Morita equivalence $C(X) - C(X)$-bimodules. In what follows we identify the class of $C(P)(\pi)$ in $\mathrm{Pic}(C(X))$ with the class of the corresponding complex line bundle $\mathbb{V}_\pi$ in $\text{Pic}(X)$. Then the group homomorphism $\varphi: \hat{G} \to \text{Pic}(C(X))$ induced by the topological principal bundle $\sigma: P \times G \to P$ is given by

$$\varphi(\pi) = [\mathbb{V}_\pi] \in \text{Pic}(X) \subseteq \text{Pic}(C(X))$$

(cf. Example (5.2.25) and Proposition (5.2.26)). In particular, the $\hat{G}$-module structure on the unitary group $U(C(X)) = C(X, T)$ induced by this group homomorphism is trivial since the left and right action of $C(X)$ on $C(P)(\pi)$ commute. At this point it is useful to introduce the following definition.

**Definition (5.2.61)[304]:** Let $G$ be a compact A belian group and $X$ a compact space. Furthermore, let $\varphi : \hat{G} \to \text{Pic}(X)$ be a group homomorphism. We denote by $\text{Ext}_{\text{top}}(X, G, \varphi)$ the subset of $\text{Ext}(C(X), G, \varphi)$ describing the topological principal $G$-bundles over $X$ inducing the map $\varphi$.

We continue with a compact space $X$ and a group homomorphism $\varphi: \hat{G} \to \text{Pic}(X) \subseteq \text{Pic}(C(X))$. Given a factor system $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for the map $\varphi$, the canonical flip

$$fl_{\pi,\rho} : M_\pi \,\widehat{\otimes}_{C(X)}\, M_\rho \to M_\rho \,\widehat{\otimes}_{C(X)}\, M_\pi, \qquad x \otimes_{C(X)} y \mapsto y \otimes_{C(X)} x$$

defines an isomorphism of Morita equivalence $C(X) - C(X)$-bimodules for all $\pi, \rho \in \hat{G}$. In particular, we may examine, for all $\pi, \rho \in \hat{G}$, the automorphism

$$C_M\Psi(\pi, \rho) := \Psi_{\rho,\pi} \circ fl_{\pi,\rho} \circ \Psi^+_{\pi,\rho}$$

of the Morita equivalence $C(X) - C(X)$-bimodule $M_{\pi+\rho}$. According to Proposition (5.2.47), the map $C_M\Psi$ can be considered as a normalized 2-cochain on $\hat{G}$ with values in $C(X, \mathbb{T})$, i.e., as an element in $C^2(\hat{G}, C(X, T))$. In fact, even more is true:

**Lemma (5.2.62)[304]:** The map $C_M\Psi$ defines an $a$ntisymmetric element in $Z^2(\hat{G}, C(X, \mathbb{T}))$.

**Proof:** It is obvious that the map $C_M\Psi$ satisfies $C_M\Psi(\rho, \pi) = C_M\Psi(\pi, \rho)^*$ for all $\pi, \rho \in \hat{G}$. In order to verify that $C_M\Psi$ defines an element in $Z^2(\hat{G}, C(X, \mathbb{T}))$ we have to show that

$$C_M\Psi(\rho, \sigma)C_M\Psi(\pi, \rho + \sigma) = C_M\Psi(\pi + \rho, \sigma)C_M\Psi(\pi, \rho) \tag{28}$$

holds for all $\pi, \rho, \sigma \in \hat{G}$. Indeed, explicit computations using the factor system property, equation (21), show that

$$\Psi_{\sigma+\rho,\pi} \circ \left(\Psi_{\sigma,\rho} \otimes_{C(X)} \text{id}_\pi\right) = C_M\Psi(\rho, \sigma)C_M\Psi(\pi, \rho + \sigma)\Psi_{\pi,\sigma+\rho} \circ \left(\text{id}_\pi \otimes_{C(X)} \Psi_{\rho,\sigma}\right) \tag{29}$$

and

$$\Psi_{\sigma,\rho+\pi} \circ \left(\text{id}_\sigma \otimes_{C(X)} \Psi_{\rho,\pi}\right) = C_M\Psi(\pi, \rho)C_M\Psi(\pi + \rho, \sigma)\Psi_{\pi+\rho,\sigma} \circ \left(\Psi_{\pi,\rho} \otimes_{C(X)} \text{id}_\pi\right) \tag{30}$$

hold for all $\pi, \rho, \sigma \in \hat{G}$. Again using equation (21) to the right-hand side of the later expression finally leads to the desired 2-cocycle condition (28).

**Lemma(5.2.63)[304]:** For equivalent factor systems $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ we have

$$C_{M'}\Psi' = C_M\Psi.$$

**Proof:** By assumption there is a family $(T_\pi: M_\pi \to M'_\pi)_{\pi\in\hat{G}}$ of Morita equivalence $C(X) - C(X)$-bimodule isomorphisms satisfying

$$\Psi'_{\pi,\rho} := T_{\pi+\rho} \circ \Psi_{\pi,\rho} \circ \left(T^+_\pi \otimes_{C(X)} T^+_\rho\right).$$

Therefore, an explicit computation shows that for all $\pi, \rho \in \hat{G}$ we have

$$\begin{aligned} C_{M'}\Psi'(\pi, \rho) &= \Psi'_{\rho,\pi} \circ fl'_{\pi,\rho} \circ \Psi'^+_{\pi,\rho} \\ &= T_{\rho+\pi} \circ \Psi_{\rho,\pi} \circ (T^+_\rho \otimes_{C(X)} T^+_\pi) \circ fl'_{\pi,\rho} \circ (T_\pi \otimes_{C(X)} T_\rho) \circ \Psi^+_{\pi,\rho} \circ T^+_{\pi+\rho} \\ &= T_{\rho+\pi} \circ \Psi_{\rho,\pi} \circ fl_{\pi,\rho} \circ \Psi^+_{\pi,\rho} \circ T^+_{\pi+\rho} = C_M\Psi(\pi, \rho) \end{aligned}$$

**Lemma (5.2.64)[304]:** Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ be two factor systems for the map $\varphi$. Then there exists an element $\omega \in Z^2(\hat{G}, C(X,\mathbb{T}))$ satisfying for all $\pi,\rho \in \hat{G}$

$$C_{M'}\Psi'(\pi,\rho) = \omega(\pi,\rho)\omega(\rho,\pi)^* C_M\Psi(\pi,\rho).$$

**Proof:** To verify the assertion we use Theorem (5.2.51), which implies that the factor system $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ is equivalent to a factor system of the form $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ for some 2-cocycle $\omega \in Z^2(\hat{G}, C(X,\mathbb{T}))$. In particular, we conclude from Lemma (5.2.63) and a short calculation that

$$C_{M'}\Psi'(\pi,\rho) = C_M(\omega\Psi)(\pi,\rho) = (\omega\Psi)_{\rho,\pi} \circ fl_{\pi,\rho} \circ (\omega\Psi)^+_{\pi,\rho}$$
$$= \omega(\rho,\pi)\omega(\pi,\rho)^* C_M\Psi(\pi,\rho)$$

holds for all $\pi,\rho \in \hat{G}$.

Let us denote by $\mathrm{Alt}^2(\hat{G}, C(X,\mathbb{T}))$ the group of biadditive maps $\hat{G}\times\hat{G} \to C(X,\mathbb{T})$ that vanish on the diagonal. Then a few moments thought shows that each element $\omega \in Z^2(\hat{G}, C(X,\mathbb{T}))$ gives rise to an element $\lambda_\omega \in \mathrm{Alt}^2(\hat{G}, C(X,\mathbb{T}))$ defined by

$$\lambda_\omega(\pi,\rho := \omega(\pi,\rho)\,\omega(\rho,\pi)^*,$$

which only depends on the class $[\omega] \in H^2(\hat{G}, C(X,\mathbb{T}))$. In particular, we obtain a group homomorphism

$$\lambda: H^2(\hat{G}, C(X,\mathbb{T})) \to \mathrm{Alt}^2(\hat{G}, C(X,\mathbb{T})), \qquad [\omega] \longmapsto \lambda_\omega$$

whose kernel is given by the subgroup $H^2_{ab}(\hat{G}, C(X,\mathbb{T}))$ of $H^2(\hat{G}, C(X,\mathbb{T}))$ describing the A belian extensions of $\hat{G}$ by $C(X,\mathbb{T})$. We recall from [330] that the corresponding short exact sequence

$$0 \longrightarrow H^2_{ab}(\hat{G}, C(X,\mathbb{T})) \longrightarrow H^2(\hat{G}, C(X,\mathbb{T})) \xrightarrow{\lambda} \mathrm{Alt}^2(\hat{G}, C(X,\mathbb{T})) \longrightarrow 0$$

is split. Moreover, we write $\mathrm{pr}_{ab}: H^2(\hat{G}, C(X,\mathbb{T})) \to H^2_{ab}(\hat{G}, C(X,\mathbb{T}))$ for the induced projection map.

**Proposition (5.2.65)[304]:** The class $\mathrm{pr}_{ab}([C_M\Psi] \in H^2_{ab}(\hat{G}, C(X,\mathbb{T}))$ does not depend on the choice of the factor system and is therefore an invariant for the set $\mathrm{Ext}(C(X), G, \varphi)$.

**Proof:** Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ and $(M'_\pi, \Psi'_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ be two factor systems for the map $\varphi$. Then it follows from Lemma (5.2.64) and the construction of the map $\mathrm{pr}_{ab}$ that

$$\mathrm{pr}_{ab}([C_{M'}\Psi']) = \mathrm{pr}_{ab}([\lambda_\omega C_M\Psi]) = \mathrm{pr}_{ab}([\lambda_\omega][C_M\Psi]) = \mathrm{pr}_{ab}([C_M\Psi]).$$

**Definition (5.2.66)[304]:** Let $\varphi : \hat{G} \to \mathrm{Pic}(X)$ be a group homomorphism. Then we call

$$\chi_2(\varphi) := \mathrm{pr}_{ab}([C_M\Psi]) \in H^2_{ab}(\hat{G}, C(X,\mathbb{T}))$$

the secondary characteristic class of $\varphi$.

The next result provides a group theoretic criterion for the existence of topological principal $G$-bundle over $X$.

**Theorem (5.2.67)[304]:** Let $G$ be a compact A belian group and $X$ a compact space. Furthermore, let $\varphi: \hat{G} \to \mathrm{Pic}(X)$ be a group homomorphism. Then the following statements are equivalent:

**(a)** The set $\mathrm{Ext}_{\mathrm{top}}(X, G, \varphi)$ is non-empty, that is, there exists a topological principal bundle $\sigma: P \times G \to P$ over $X$ representing an element of $\mathrm{Ext}_{\mathrm{top}}(X, G, \varphi)$.

**(b)** The map $\varphi$ satisfies the following two conditions in the indicated order:

($b_1$) The class $\chi(\varphi) \in H^3(\hat{G}, C(X,\mathbb{T}))$ vanishes.

($b_2$) Furthermore, the class $H^3(\hat{G}, C(X,\mathbb{T}))$ vanishes.

**Proof:** Suppose first that the set $\mathrm{Ext}_{\mathrm{top}}(X, G, \varphi)$. is non-empty. Then Theorem (5.2.57) implies that the characteristic class $\chi(\varphi)$ in $H^3(\hat{G}, C(X,\mathbb{T}))$ vanishes. To verify that the class $\chi_2(\varphi) \in H^2_{ab}(\hat{G}, C(X,\mathbb{T}))$ vanishes, let $\sigma: P \times G \to P$ be a topological principal bundle over $X$ representing an element of $\text{Ext}_{\text{top}}(X, G, \varphi)$. Then the canonical factor system of the associated free $C^*$-dynamical system $(C(P), G, \alpha_\sigma)$(cf. Theorem (5.2.4)) is given by

$$\Psi_{\pi,\rho} : C(P)(\pi) \widehat{\otimes}_{C(X)} C(P)(\rho) \to C(P)(\pi + \rho), \quad f \otimes_{C(X)} g \to fg.$$

Therefore, the claim follows from the commutativity of $C(P)$ since we have

$$\Psi_{\pi,\rho}(f \otimes_{C(X)} g) = fg = gf = \Psi_{\rho,\pi}(g \otimes_{C(X)} f)$$

for all $f \in C(P)(\pi)$ and $g \in C(P)(\rho)$, i.e., $C_M\Psi = \mathbb{1}_{C(X)}$.

If, conversely, condition ($b_1$) is satisfied, then it follows from Theorem (5.2.57) that there is a free $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ representing an element of $\text{Ext}(C(X), G, \varphi)$. Let $(M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ be its associated factor system. We then use condition ($b_2$) to find an element $\omega \in Z^2(\hat{G}, C(X, \mathbb{T}))$ such that

$$\lambda_{\omega^*} = C_M\Psi \in \text{Alt}^2\left(\hat{G}, C(X, \mathbb{T})\right).$$

Consequently, the corresponding factor system $(M_\pi, \omega(\pi,\rho)\Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$ has the property $C_M(\omega\Psi) = \mathbb{1}_{C(X)}$, that is, its associated free $C^*$-dynamical system is equivalent to one of the form $(C(P), G, \alpha_\sigma)$ induced by some topological principal bundle $\sigma: P \times G \to P$ over $X$.

The following statement provides a classification of topological principal G-bundles over $X$. It is a consequence of Corollary (5.2.52) and Lemma (5.2.64).

**Corollary (5.2.68)[304]:** Let $G$ be a compact A belian group and $X$ a compact space. Furthermore, let $\varphi: \hat{G} \to \text{Pic}(X)$ be a group homomorphism with $\text{Ext}_{\text{top}}(X, G, \varphi) \neq \emptyset$. Then the map

$$H^2_{ab}(\hat{G}, C(X, \mathbb{T})) \times \text{Ext}_{\text{top}}(X, G, \varphi) \to \text{Ext}_{\text{top}}(X, G, \varphi), \qquad \left([\omega], \left[P_{(M,\Psi)}\right]\right) \mapsto \left[P_{(M,\omega\Psi)}\right]$$

is a well-defined simply transitive action, where $P_{(M,\Psi)}$ denotes the topological principal bundle associated to a given factor system $(M, \Psi) := (M_\pi, \Psi_{\pi,\rho})_{\pi,\rho\in\hat{G}}$.

We conclude with a few words on how our previous results relates to the Čech cohomology for the pair $(X, G)$ classifying locally trivial principal $G$-bundles over $X$.

**Example (5.2.69)[304]:** Let $G$ be a compact A belian group. Furthermore, let $\mathbb{M}_m(\mathbb{C})$ be the $C^*$-algebra of $m \times m$ matrices and recall that its natural representation on $\mathbb{C}^m$ is, up to equivalence, the only irreducible representation of $\mathbb{M}_m(\mathbb{C})$. Therefore, it follows from Example (5.2.25) that $\text{Pic}(\mathbb{M}_m(\mathbb{C}))$ is trivial. In particular, there is only the trivial group homomorphism from $\hat{G}$ to $\text{Pic}(\mathbb{M}_m(\mathbb{C}))$ and a realization is given by the free $C^*$-dynamical system

$$(C(G, (\mathbb{M}_m(\mathbb{C})), G, rt \otimes \text{id}_{\mathbb{M}_n(\mathbb{C})}),$$

where

$$rt: G \times C(G) \to C(G), rt(g, f)(h) := f(hg)$$

denotes the right-translation action by $G$. Moreover, we conclude from Corollary (5.2.52) that all free actions of $G$ with fixed point algebra $\mathbb{M}_m(\mathbb{C})$ are parametrized by the cohomology group $H^2(G, T)$. In the case $G = \mathbb{T}^n, n \in \mathbb{N}$, this cohomology group is isomorphic to $\mathbb{T}^{\frac{1}{2}n(n-1)}$ and parametrizes the free actions given by tensor products of the noncommutative n-tori endowed with their natural $\mathbb{T}^n$-action (cf. Example (5.2.12)) and the $C^*$-algebra $\mathbb{M}_m(\mathbb{C})$.

**Example (5.2.70)[304]:** Consider the 2-fold direct sum $\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ and notice that the group $(UZ\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}))$ is isomorphic to $\mathbb{T}^2$. Since the spectrum of $\mathbb{M}_2(\mathbb{C}) \oplus$

$\mathbb{M}_2(\mathbb{C})$ contains two elements, it follows from Example (5.2.25) that $\mathrm{Pic}\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ is isomorphic to $\mathbb{Z}_2$. If $\varphi : \mathbb{Z} \to \mathbb{Z}_2$ denote the canonical group homomorphism with kernel $2\mathbb{Z}$, then it is a consequence of [326] that the cohomology groups $H^2_\varphi(\mathbb{Z}, \mathbb{T}^2)$ and $H^3_\varphi(\mathbb{Z}, \mathbb{T}^2)$ are trivial. Therefore, Theorem (5.2.57) implies that the set $\mathrm{Ext}(\mathbb{M}_m(\mathbb{C}).\oplus \mathbb{M}_m(\mathbb{C}).,\mathbb{T},\varphi)$ is non-empty and contains according to Corollary (5.2.52) exactly one element, namely the class of the trivial system

$$(C(\mathbb{T}, \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}), \mathbb{T}, \mathrm{rt} \otimes \mathrm{id}).$$

**Example(5.2.71)[304]:** For the following discussion we recall the notation from Example (5.2.12). Let $\mathbb{T}^2_\theta$ be the noncommutative n-torus defined by the real skew-symmetric $n \times n$ matrix $\theta$ and let $\omega_1$be the corresponding $\mathbb{T}$-valued 2-cocycle on $\mathbb{Z}^n$ given for $k, k' \in \mathbb{Z}^n$ by

$$\omega_1(k, k') := exp(\mathbb{c}^n\langle \theta k, k' \rangle).$$

Furthermore, let $S: \mathbb{Z}^m \to \mathrm{Aut}(\mathbb{T}^n_\theta)$ be a group homomorphism leaving the isotypic components of $\mathbb{T}^n_\theta$ (with respect to the canonical gauge action by $\mathbb{T}^n$) invariant, i.e., such that for all $I \in \mathbb{Z}^m$ and $k \in \mathbb{Z}^n$

$$S(I)U_k = c_{I,k}U_k$$

for some $c_{I,k} \in \mathbb{T}$. Then, given another $\mathbb{T}$-valued 2-cocycle $\omega_2$ on $\mathbb{Z}^m$, that the pair $(S, \omega_2)$ gives rise to a factor system for the group homomorphism $\varphi := \mathrm{pr}_{\mathbb{T}^n_\theta} \circ S: \mathbb{Z}^m \to \mathrm{Pic}(\mathbb{T}^n_\theta)$. Moreover, it is easily seen that the associated free $C^*$-dynamical system is equivalent to the free $C^*$-dynamical system $(\mathbb{T}^{n+m}_{\theta'}, \mathbb{T}^m, \alpha)$, where $\mathbb{T}^{n+m}_{\theta'}$ denotes the noncommutative $(n + m)$-torus determined by the $\mathbb{T}$-valued 2-cocycle on $\mathbb{Z}^{n+m}$ given for $k, k' \in \mathbb{Z}^n$ and $\mathrm{I}, \mathrm{I}' \in \mathbb{Z}^m$ by

$$\omega\ (k, \mathrm{I}), (k', \mathrm{I}') :=\ c_{\mathrm{I},k'}\omega_1(k, k')\omega_2(\mathrm{I}, \mathrm{I}')$$

and $\alpha$ is the restriction of the gauge action $\alpha^{n+m}_{\theta'}$ to the closed subgroup $\mathbb{T}^m$ of $\mathbb{T}^{n+m}$. That $(\alpha^{n+m}_{\theta'}, \mathbb{T}^m, \alpha)$ is actually free is a consequence of Proposition (5.2.17). In particular, it represents an element in $\mathrm{Ext}(\mathbb{T}^n_\theta, \mathbb{T}^m, \varphi)$ which is, according to Corollary(5.2.52), parametrized by the cohomology group

$$H^2_\varphi(\mathbb{Z}^m, UZ(\mathbb{T}^n_\theta)).$$

**Example (5.2.72)[304]:** Let $\theta$ be an irrational number in $[0, 1]$ and $\mathbb{T}^2_\theta$ the corresponding noncommutative 2-torus from Example (5.2.12). We recall that in this case $UZ(\mathbb{T}^2_\theta)$ is isomorphic to $\mathbb{T}$. Furthermore, let $\varphi: \mathbb{Z}^2 \to \mathrm{Pic}(\mathbb{T}^2_\theta)$ be any group homomorphism (note that $\mathbb{T}^2 \subseteq \mathrm{Aut}(\mathbb{T}^2_\theta)$ to apply the construction in Example (5.2.71)). Then it is a consequence of [326] that the cohomology group $H^3_\varphi(\mathbb{Z}^2, \mathbb{T})$ is trivial. Therefore, Theorem (5.2.57) implies that the set $\mathrm{Ext}(\mathbb{T}^2_\theta, \mathbb{T}^2, \varphi)$ is non-empty and, according to Corollary (5.2.52), parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}^2, \mathbb{T})$. For its computation we refer, for example, to [338].

**Example (5.2.73)[304]:** Let $H$ be the discrete (three-dimensional) Heisenberg group and let $(C^*(H), \mathbb{Z}^2, \alpha)$ the corresponding free $C^*$-dynamical system from Example (5.2.13). If

$$\phi: \mathbb{Z}^2 \to\ \mathrm{Pic}(C(\mathbb{T})) \cong \mathrm{Pic}(\mathbb{T}) \rtimes\ \mathrm{Homeo}(\mathbb{T})$$

denotes the associated group homomorphism, then the class of $(C^*(H), \mathbb{T}^2, \alpha)$ is contained in the set $\mathrm{Ext}(C(\mathbb{T}), \mathbb{T}^2, \varphi)$ of equivalence classes of realizations of $\varphi$, which is by Corollary (5.2.52) parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}^2, C(\mathbb{T}, \mathbb{T}))$. For its computation we refer, again, to [338].

**Example(5.2.74)[304]:** For $q \in [-1, 1]$ let $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ be the quantum Hopf fibration from Example (5.2.14) and $L_q$ (1) the isotypic component corresponding to $1 \in \mathbb{Z}$. If

$$\varphi: \mathbb{Z} \to \mathrm{Pic}(S_q(2)), \qquad 1 \longmapsto [L_q(1)]$$
denotes the associated group homomorphism, then the class of $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ is contained in the set $\mathrm{Ext}(S_q(2), \mathbb{T}, \varphi)$ of equivalence classes of realizations of $\varphi$, which is by Corollary (5.2.52) parametrized by the cohomology group $H^2_\varphi(\mathbb{Z}, UZ(S_q(2))$. It follows, for example, from [326] that this cohomology group is trivial, i.e., up to isomorphism the quantum Hopf fibration $(\mathrm{SU}_q(2), \mathbb{T}, \alpha)$ is the unique realization of the group homomorphism $\varphi$.

## Section (5.3): Compact Quantum Groups on Unital $C^*$-Algebras

A compact quantum group [176], [177] is a unital $C^*$-algebra $H$ with a given unital injective $*$-homorphism $\Delta$ (referred to as comultiplication)
$$\Delta: H \to H \otimes_{min} H \tag{31}$$
which is coassociative i.e. there is commutativity in the diagram
$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \underset{\min}{\otimes} H \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id} \otimes \Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array} \tag{32}$$
such that the two-sided cancellation property holds:
$$\{(a \otimes 1) \triangle (b) | \, a, b \in H\}^{cls} = H \otimes_{min} H = \{\triangle (a)(1 \otimes_{min} b) | a, b \in H\}^{cls}. \tag{33}$$
Here $\otimes_{\min}$ denotes the spatial tensor product of $C^*$-algebras and cls denotes the closed linear span of a subset of a Banach space.

Let $A$ be a unital $C^*$-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism.We call $\delta$ a coaction of $H$ on $A$ (or an action of the compact quantum group $(H, \Delta)$ on $A$) if

(i) $(\delta \otimes id) \circ \delta = (id \otimes \Delta) \circ \delta$ (coassociativity),

(ii) $\{\delta(a)(1 \otimes h) \, | \, a \in A, h \in H\}^{clf} = A \otimes_{\min} H$ (counitality).

By definition [161], the coaction $\delta$ is free if and only if
$$\{(x \otimes 1)\delta(y) \, | \, x, y \in A\}^{cls} = A \otimes_{\min} H. \tag{34}$$
Given a compact quantum group $(H, \Delta)$, we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations [177], [166]. This is Woronowicz's Peter-Weyl theory in the case of compact quantum groups. Moreover, denoting by $\otimes$ the purely algebraic tensor product over the field C of complex numbers, wedefine the Peter-Weyl subalgebra of $A$ (cf. [167], [173]) as
$$\mathcal{P}_H(A) := \{a \in A \, | \, \delta(a) \in A \otimes \mathcal{O}(H) \}. \tag{35}$$
Using the coassociativity of $\delta$, one can check that $\mathcal{P}_H(A)$ is a right $\mathcal{O}(H)$-comodule algebra. In particular, $\mathcal{P}_H(H) = \mathcal{O}(H)$. The assignment $A \longmapsto \mathcal{P}_H(A)$ is functorial with respect to equivariant unital $*$-homomorphisms and comodule algebra maps. We call it the Peter-Wey functor.

**Theorem (5.3.1)[148]:** Let $A$ be a unital $C^*$-algebra equipped with an action of a compact quantum group $(H, \Delta)$ given by $\delta : A \to A \otimes_{\min} H$. Denote by $B = A^{coH} := \{a \in A \, | \, \delta(a) = a \otimes 1\}$ the unital $C^*$-subalgebra of coaction-invariants. Then the action is free if and only if the canonical map
$$can: \mathcal{P}_H(A) \otimes_B \mathcal{P}_H(A) \to \mathcal{P}_H(A) \otimes \mathcal{O}(H)$$

$$can: x \otimes y \mapsto (x \otimes 1)\delta(y) \tag{36}$$
is bijective. (Here the tensor product over an algebra denotes the purely algebraic tensor product over that algebra.)

**Definition (5.3.2)[148]:** The action of a compact quantum group $(H, \Delta)$ on a unital $C^*$-algebra A satisfies the Peter-Weyl-Galois (PWG) condition iff the canonical map (36) is bijective.

Our result generalizes Woronowicz's Peter-Weyl theory from compact quantum groups to compact quantum principal bundles. In the spirit of the Woronowicz theory, our result states that the original functional analysis formulation of free action is equivalent to the much more algebraic PWG-condition.

We now proceed to explain our main result in the classical setting. Let $G$ be a compact Hausdorff topological group acting (by a continuous right action) on a compact Hausdorff topological space $X$
$$X \times G \to X. \tag{37}$$
It is immediate that the action is free i.e. $xg = x \Rightarrow g = e$ (where $e$ the identity element of $G$) if and only
$$X \times G\, X \times_{X/G} X$$
$$(x, g) \mapsto (x, xg) \tag{38}$$
is a homeomorphism. $Here$ $X \times_{X/G} X$ is the subset of $X \times X$ consisting of pairs $(x_1, x_2)$ such that $x_1$ and $x_2$ are in the same $G$-orbit.

This is equivalent to the assertion that the $*$-homomorphism
$$C\left(X \times_{\frac{X}{G}} X\right) \to C(X \times G) \tag{39}$$
obtained from the above map $(x, g) \mapsto (x, xg)$ is an isomorphism. Here, as usual, $C(Y)$ denotes the commutative $C^*$-algebra of all continuous complex-valued functions on the compact Hausdorff space .

In turn, the assertion that $*$-homomorphism (39) is an isomorphism is readily proved equivalent to
$$\{(x \otimes 1)\delta(y) \mid x, y \in C(X)\}^{cls} = C(X) \otimes_{\min} C(G), \tag{40}$$
where
$$\delta: C(X) \to C(X) \otimes_{\min} C(G) \tag{41}$$
is the $*$-homomorphism obtained from the map $X \times G \to X$ via the formula $(\delta(f)(g))(x) = f(xg)$.

Hence in the case of a compact group acting on a compact space "free action" agrees with "free action" as defined in the setting of a compact quantum group acting on a unital $C^*$-algebra.

Thus we can formulate the classical case of Theorem (5.3.1) as follows.

**Theorem (5.3.3)[148]:** Let $G$ be a compact Hausdorff group acting continuously on a compact Hausdorff space $X$. Then the action is free if and only if the canonical map
$$can: \mathcal{P}_{C(G)}\big(C(X)\big) \otimes_{C(X/G)} \mathcal{P}_{C\,(G)}\big(C(X)\big) \to \mathcal{P}_{C(G)}(C(X)) \otimes \mathcal{O}(C(G)) \tag{42}$$
is and isomorphism.

Observe that even in the above special case of a compact group acting on a compact space, a proof is required for the equivalence of "free action" and the bijectivity of the canonical map (PWG-condition). Theorem (5.3.3) brings a new algebraic tool (strong connection) to the realm of compact principal bundles. In this setting, the Peter-Weyl algebra $\mathcal{P}_{C(G)}(C(X))$ is the algebra of continuous global of the associated bundle of algebras $X \times_G \mathcal{O}(C(G))$:

$$\mathcal{P}_{C(G)}(C(X)) = \Gamma(X \times_G \mathcal{O}(C(G))). \tag{43}$$

Here $\mathcal{O}(C(G))$ is the subalgebra of $C(G)$ generated by matrix coefficients of irreducible representations of $G$. The algebra $\mathcal{O}(C(G))$ is topologized as the direct limit of its finite dimensional subspaces. Multiplication and addition is pointwise.

Although Theorem (5.3.3) is a special case of Theorem (5.3.1), the proof we give of Theorem (5.3.3) is not a special case of the proof of Theorem (5.3.1). Therefore we treat Theorem (5.3.3) separately, and prove it. The proof uses the strong monoidality (i.e. preservation of tensor products) of the Serre-Swan Theorem. This is later reflected in the noncommutative setting of Theorem (5.3.14). We prove the main result (Theorem (5.3.1)) by taking advantage of an underlying Hilbert module structure.

We consider the general algebraic setting of principal coactions. We prove that principality of a comodule algebra $\mathcal{P}$ over a Hopf algebra $\mathcal{H}$ is equivalent to the exactness and strong monoidality of the cotensor product functor $\mathcal{P} \boxdot_{\mathcal{H}}$. This framework unifies free actions of compact Hausdorff groups on compact Hausdorff spaces and principal actions of affine algebraic groups on affine schemes. Thus the main result is somewhat analogous to the Atiyah-Hirzebruch transplantation of $K$-theory from algebraic geometry to topology [150], [151].

As an application, we prove that if a unital $C^*$-algebra $A$ equipped with an action of a compact quantum group can be fibred over a compact Hausdorff space $X$ with the PWG-condition valid on each fibre, then the PWG-condition is valid for the action on $A$. We end with an appendix observing that regularity of a finite covering is equivalent to bijectivity of the canonical map (42).

The main advantage of the proof over the approximation proof in [152] is that it does not rely on an approximation argument which can be used only in the classical case. To be consistent with general notation, we should only use $C^*$-algebras $C(G), C(X)$, etc., rather than spaces themselves. However, this would make formulas too cluttered, so that throughout we consistently omit writing $C(\,)$ in the subscript and the argument of the Peter-Weyl functor.

The implication "PWG-condition ⇒ freeness" is proved as follows. The PWG-condition immediately implies

$$(\mathcal{P}_G(X) \otimes \mathbb{C})\delta(\mathcal{P}_G(X)) = \mathcal{P}_G(X) \otimes \mathcal{O}(G). \tag{44}$$

As the right hand side is a dense subspace of $C(X) \otimes_{\min} C(G)$, we obtain the density condition (40). The latter is equivalent to freeness, as explained in the introduction.

For the converse implication "PWG-condition ⇐ freeness" we shall use the Serre-Swan theorem.

**Theorem (5.3.4)[148]:** ([174]). Let $Y$ be a compact Hausdorff topological space. Then a $C(Y)$-module is finitely generated and projective if and only if it is isomorphic to the module of continuous global sections of a vector bundle over $Y$.

For a compact Hausdorff topological space $Y$, we denote by Vect($Y$) the category of $\mathbb{C}$ vector bundle on $Y$. An object in Vect($Y$) is a $\mathbb{C}$ vector bundle $E$ with base space $Y$. The projection of $E$ onto $Y$ is denoted by $\pi : E \to Y$. A section of $E$ is a continuous map

$$s : Y \to E \text{ with } \pi \circ s = I_Y = \text{identity map of } Y. \tag{45}$$

A morphism in Vect($Y$) is a vector bundle map

$$\varphi : E \to F, \varphi : E_y \to F_y, \quad \forall y \in Y. \tag{46}$$

Note that $E_y$ and $F_y$ are both finite dimensional vector spaces over $C$ and $\varphi : E_y \to F_y$ is a linear transformation in the sense of standard linear algebra.

View the commutative $C^*$-algebra $C(Y)$ as a commutative ring with unit. Denote by FProj$(C(Y))$ the category of finitely generated projective (left) $C(Y)$-modules. An object in FProj$(C(Y))$ is a finitely generated projective (left) $C(Y)$-module. A morphism in FProj$(C(Y))$ is a map of $C(Y)$-modules $\psi : M \to N$.

If $E$ is a $\mathbb{C}$ vector bundle on $Y$, then $\Gamma(E)$ denotes the (left) $C(Y)$-module consisting of all (continuous) sections of $E$. The module structure is pointwise, i.e. for $s_1, s_2, s \in \Gamma(E), f \in C(Y), y \in Y$,

$$(s_1 + s_2)(y) = s_1(y) + s_2(y), \quad (fs)(y) = f(y)s(y). \tag{47}$$

According to the Serre-Swan theorem, the functor $\Gamma$

$$Vect(Y) \to FProj(C(Y)), E \longmapsto \Gamma(E), \tag{48}$$

is an equivalence of categories and preserves all the basic properties of the two categories. In particular, $E \longmapsto \Gamma(E)$ preserves $\oplus$ and $\otimes$:

$$\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F), \tag{49}$$

$$\Gamma(E \otimes F) = \Gamma(E) alg \otimes^{alg}_{C(Y)} \Gamma(F). \tag{50}$$

In order to prove action of $G$ on $X$ is free $\Rightarrow \mathcal{P}_G(X) \otimes^{alg}_{C(X/G)} \mathcal{P}G(X) \to \mathcal{P}_G(X) \otimes^{alg}_{\mathbb{C}} \mathcal{O}(G)$ is an isomorphism we shall use a (very slight) extension of Serre-Swan. The extension is that each category will be replaced by its minimal enlargement which admits countable direct sums.

On the $C(Y)$-module side, let FProj $\left(C(Y)\right)^{\oplus}$ be the minimal enlargement of FProj$(C(Y))$ which allows countable direct sums. An object in FProj$\left(C(Y)\right)^{\oplus}$ is a $C(Y)$-module $M$ such that

$\exists$ finitely generated projective $C(Y)$-modules

$$P_0, P_1, P_2, \ldots \text{ with } M \cong P0 \oplus P1 \oplus P2 \oplus \cdots. \tag{51}$$

A morphism in FProj $\left(C(Y)\right)^{\oplus}$ is a map of $C(Y)$-modules $\psi : M \to N$.

On the vector bundle side, we will have vector bundles $E$ on $Y$ such that each fiber $E_y$ is a $\mathbb{C}$ vector space which admits a countable (or finite) basis. The bundle $E$ is required to satisfy:

$$\exists\ E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E \tag{52}$$

with:

- Each $E_j$ is an ordinary (i.e. finite-dimensional fibers) vector bundle on $Y$ and is a subvector-bundle of $E$.
- $\cup_j E_j = E$.
- Each $E_j$ is a closed subset of $E$.
- $E$ has the direct limit topology, i.e. a subset $\Omega$ of $E$ is closed if and only if $\forall j = 0, 1, 2, \ldots \Omega \cap E_j$ is a closed subset of $E_j$.

Equivalently, the condition on E is: $\exists$ ordinary vector bundles (i.e. finite-dimensional fibers)

$F_0, F_1, F_2, \ldots$ with $E \cong F_0 \oplus F_1 \oplus F_2 \oplus \cdots$.

Denote the category of such vector bundles by Vect$(Y)^{\oplus}$. The category Vect$(Y)^{\oplus}$ is the minimal enlargement of Vect$(Y)$ which allows countable direct sums. A morphism in Vect$(Y)^{\oplus}$ is a vector bundle map $\varphi: E \to F$.

Note that $E_y$ and $F_y$ are vector spaces over $\mathbb{C}$ and $\varphi: E_y \to F_y$ is a linear transformation. As with ordinary vector bundles $\Gamma(E)$ is the set of all (continuous) sections of $E$, and $\Gamma(E)$ is a $C(Y)$-module. Since $E$ is topologized by the direct limit topology, given any $s \in \Gamma(E)$, there exists a sub-vector-bundle $F$ of $E$, $F \subseteq E$ such that:
- $F$ is an ordinary vector bundle on $E$, i.e. $F$ has finite-dimensonal fibers.
- $s(Y) \subseteq F$.

The Serre-Swan theorem implies that he functor $\Gamma$

$$\text{Vect}(Y)^{\oplus} \to FProj\big(C(Y)\big)^{\oplus}, E \mapsto \Gamma(E), \quad (53)$$

is an equivalence of categories and preserves all the basic properties of the two categories. In particular, $E \mapsto \Gamma(E)$ preserves $\oplus$ and $\otimes$:

$$\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F), \quad (54)$$

$$\Gamma(E \otimes F) = \Gamma(E) \otimes_{C(Y)}^{alg} \Gamma(F). \quad (55)$$

For a compact Hausdorff topological group $G, \hat{G}$ is the set of (equivalence classes of) irreducible representations of $G$. For simplicity, from now on will assume that $\hat{G}$ is countable. (If $\hat{G}$ is not countable, then the proof is the same except that the enlarged categories have to admit uncountable direct sums.) The examples of interest all satisfy this countability assumption. $\{F\ D \text{ Reps of } G\}$ denotes the category of finite dimensional representations of $G$.

The representations are on finite dimensional vector spaces over the complex numbers $\mathbb{C}$. If $X$ is a compact Hausdorff $G$-space, with the action of $G$ on $X$ free, then there is a functor

$$\{F\ D \text{ Reps } of\ G\} \to \{\text{Vector bundles } on\ X/G\} \quad (56)$$

$$V \mapsto X \times_G V \quad (57)$$

$$\oplus\ and\ \otimes \text{ are preserved.} \quad (58)$$

$\{L\ F\ D \text{ Reps of } G\}$ denotes the minimal enlargement of $\{F\ D \text{ Reps of } G\}$ which allows (purely algebraic) countable direct sums. If $X$ is a compact Hausdorff G-space, with the action of $G$ on $X$ free, then there is a functor

$$\{M\ F\ D \text{ Reps of } G\} \to \{M \text{ Vector bundles on } X/G\} \quad (59)$$

$$V \mapsto X \times_G V \quad (60)$$

$$\oplus\ and\ \otimes \text{ are preserved.} \quad (61)$$

$$X \times_G (V_1 \oplus V_2) = (X \times_G V_1) \oplus (X \times_G V_2)$$

$$X \times_G (V_1 \otimes V_2) = (X \times GV_1) \otimes (X \times GV_2)$$

For example

$$\mathcal{O}(G) \mapsto X \times_G \mathcal{O}(G) \mapsto \Gamma\big(X \times_G \mathcal{O}(G)\big). \quad (62)$$

**Lemma (5.3.5)[148]:** If $X$ is a compact Hausdorff $G$-space with free $G$-action, then

$$\Gamma(X \times_G \mathcal{O}(G)) = \mathcal{P}_G(X). \quad (63)$$

**Corollary (5.3.6)[148]:** $\Gamma(X \times_G \left(\mathcal{O}(G) \otimes_{C(X/G)}^{alg} \mathcal{O}(G)\right) = \mathcal{P}_G(X) \otimes_{C(X/G)}^{alg} \mathcal{P}_G(X)$

**Corollary (5.3.7)[148]:** Denote by $\mathcal{O}(G)$ trivial the vector space $\mathcal{O}(G)$ with the trivial action of $G$, i.e. every $g \in G$ is acting by the identity map of $\mathcal{O}(G)$. Then

$$\Gamma(X \times_G \left(\mathcal{O}(G) \otimes \mathcal{O}(G)^{trivial}\right) = \mathcal{P}_G(X) \otimes_{C\left(\frac{X}{G}\right)}^{alg} (C(X/G) \otimes \mathcal{O}(G)), \quad (64)$$

which implies

$$\Gamma(X \times_G (\mathcal{O}(G) \otimes \mathcal{O}(G)^{trivial}) = \mathcal{P}_G(X) \otimes \mathcal{O}(G). \quad (65)$$

**Lemma (5.3.8)[148]:** As representations of $G$,

$$\mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O} \otimes \mathcal{O}(G)^{trivial}. \quad (66)$$

**Proof.** The group $G$ acts on $G \times G$ and on $G \times (G)^{trivial}$ by

$$(a,b)g = (ag,bg)\ (a,b)g = (ag,b)\ a,b,g \in G \tag{67}$$

The map

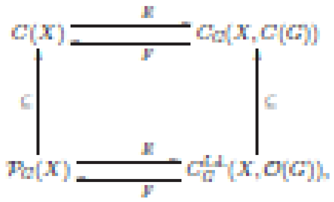
$$G \times G \leftarrow G \times (G)^{trivial} \tag{68}$$

$$(a,ba) \leftarrow (a,b) \tag{69}$$

is a $G$-equivariant homeomorphism. The resulting map $\mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)^{trivial}$ is then an isomorphism of representations of $G$.

Combining the Lemma (5.3.5) with Corollary (5.3.6) and Corollary (5.3.7) gives

$$\mathcal{P}_G(X) \otimes_{C\left(\frac{X}{G}\right)} \mathcal{P}_G(X) \cong \mathcal{P}_G(X) \otimes \mathcal{O}(G). \tag{70}$$



(71)

$$E(f)(x)(g) := f(xg), F(\alpha)(x) := \alpha(x)(e), E \circ F = id, F \circ E = id. \tag{72}$$

$$\mathcal{P}_G(X) \otimes^{alg}_{C\left(\frac{X}{G}\right)} \mathcal{P}_G(X) \xrightarrow{E \otimes E} C_G^{f.d.}\big(X, \mathcal{O}(G)\big) \times^{alg}_{C(X/G)} C_G^{f.d.}\left(X, \mathcal{O}(G)\right) \xrightarrow{diag}$$

$$C_G^{f.d.}\left(X, \mathcal{O}(G) \otimes_{alg} \mathcal{O}(G)\right) \xrightarrow{W^* \circ} C_{G,id}^{f.d.}\left(X, \mathcal{O}(G) \otimes_{alg} \mathcal{O}(G)\right) \xrightarrow{\sum_i (id \otimes e^i) \otimes e_i}$$

$$C_G^{f.d.}\left(X, \mathcal{O}(G)\right) \otimes_{alg} \mathcal{O}(G) \xrightarrow{F \otimes id} \mathcal{P}_G(X) \otimes_{alg} \mathcal{O}(G), \text{where } W(g,g') = (g,gg').$$

Why is the isomorphism obtained from this vector bundle point of view the same as the canonical map $\mathcal{P}_G(X) \otimes^{alg}_{C\left(\frac{X}{G}\right)} \mathcal{P}_G(X) \to \mathcal{P}_G(X) \otimes_{\mathbb{C}} \mathcal{O}(G)$

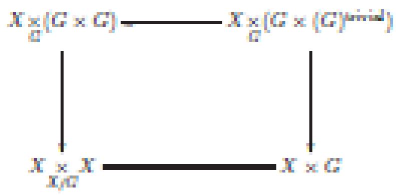
Will consistently take all actions of $G$ on spaces to be right actions. Modules and representations will be left. For example, the regular representation of $G$ on $L^2G$ is

$$(gf)(a) = f(ag)\ f \in L^2G\ g,a \in G \tag{73}$$

If $W$ and $Z$ are $G$ spaces, then $W \times GZ = \frac{W \times Z}{G}$ where $G$ acts on $W \times Z$ by

$$(w,z)g = (wg,zg).$$

(74)



The upper horizontal arrow is $(x,(a,ba)) \leftarrow (x,(a,b))$.

The lower horizontal arrow is $(x,xg) \leftarrow (x,g)$. The left vertical arrow is

$$(x,(a,b))$$
$$\downarrow$$
$$(xa^{-1}, xb^{-1}).$$

The right vertical arrow is

$$(x,(a,b))$$

$$\downarrow$$
$$(xa^{-1}, b^{-1}).$$
This diagram commutes and all four maps are homeomorphisms.
This is the desired isomorphism, which completes the proof.

The implication "PWG-condition ⇒ freeness" is proved as follows. The PWG-condition immediately implies

$$(\mathcal{P}_H(A) \otimes \mathbb{C})\delta(\mathcal{P}_H(A)) = \mathcal{P}_H(A) \otimes \mathcal{O}(H). \tag{75}$$

As the right hand side is a dense subspace of $A \otimes_{\min} H$ (see [167] and [173]), we obtain the density condition defining freeness.

For the converse implication "PWG-condition ⇐ freeness" we need some preparations. If $(V, \delta_V)$ is a finite-dimensional $H$-comodule, we write $H_V$ for the smallest vector subspace of $H$ such that $\delta_V(V) \subseteq V \otimes H_V$. We write

$$A_V := \{a \in A \mid \delta(a) \in A \otimes H_V\} \tag{76}$$

Note that in the case $(A, \delta) = (H, \Delta)$, we have $A_V = H_V$. Thus $H_V$ is a coalgebra.

One can define a continuous projection map $E_V$ from $A$ onto $A_V$ as follows [167]. Let us call two finite-dimensional comodules of $H$ disjoint if the set of morphisms between them only contains the zero map. Then $E_V$ is the unique endomorphism of $A$ which is the identity on $A_V$ and which vanishes on $A_W$ for W any finite-dimensional comodule disjoint from . In the special case of $(A, \delta) = (H, \Delta)$, we use the notation eV instead of $E_V$. The equivariance property

$$\delta \circ EV = (id \otimes eV) \circ \delta. \tag{77}$$

is proved by a straightforward verification. When $V$ is the trivial representation, we write $E_V = E_B$ and $e_V = \varphi_H$, where $B = A^{coH}$ is the algebra of coaction invariants and $\varphi_H$ is the invariant state on $H$. Then the formula (77) specializes to

$$EB = (id \otimes \varphi_H)\Delta\, \delta. \tag{78}$$

The key lemma in the proof of Theorem (5.3.1) is:

**Lemma (5.3.9)[148]:** (Theorem (5.3.5) in [133]). Let $\delta : A \to A \otimes_{\min} H$ be a free coaction, and let $V$ be a finite-dimensional $H$-comodule. Then $A_V$ is finitely generated projective as a right $B$-module.

In classical case $X \times G \to X$, we have $H = C(G)$ and $B = C(X/G)$.The $B$-module $A_V = \Gamma(X \times_G H_V)$ and thus is finitely generated projective.

Define a $B$-valued inner product on $A_V$ by

$$\langle a, b\rangle_B := E_B(a^* b). \tag{79}$$

**Lemma (5.3.10)[148]:** (Corollary 2.6 in [133]). The B-valued inner product (79) makes AV a (right) Hilbert B-module [139]. The Hilbert module norm $\|a\|_B := \|\langle a, a\rangle_B\|^{1/2}$ is equivalent to the $C^*$-norm of $A$ restricted to $A_V$.

We will need the following lemma concerning the interior tensor product of Hilbert modules.

**Lemma (5.3.11)[148]:** (cf. Proposition (5.3.19) in [139]). Let $C$ and $D$ be unital $C^*$-algebras, and let $(\mathcal{E}, \langle .,.\rangle_C)$ be a right Hilbert $C$-module which is finitely generated projective as a right $C$-module. Let $(\mathcal{F}, \langle .,.\rangle_D)$ be an arbitrary right Hilbert $D$-module, and $\pi : C \to \mathcal{L}(\mathcal{F})$ a unital $*$-homorphism of $C$ into the $C^*$-algebra of adjointable operators on $\mathcal{F}$. Then the algebraic tensor product $\mathcal{E} \otimes_C^{alg} \mathcal{F}$ is a right Hilbert $D$-module with respect to the inner product

$$\langle x \otimes y, z \otimes wi\rangle := \langle y, \pi(\langle x, z\rangle_C) w\rangle_D. \tag{80}$$

**Proof.** We are to show that the semi-norm $\|z\| = \|\langle z, z\rangle_D\|^{1/2}$ on $\mathcal{E} \otimes_C^{alg} \mathcal{F}$ is in fact a norm with respect to which the space is complete. The statement obviously holds for $\mathcal{E} = C^n$, the $n$-fold direct sum of the standard right $C$-module $C$. Since $\mathcal{E}$ is finitely generated projective, $\mathcal{E}$ can be realized as a direct summand of $C^n$, so that the conclusion also applies for this case.

We are now ready to prove the implication "PWG-condition ⇐ freeness". By the freeness assumption, the image of can is dense in $A \otimes H$. In particular, for a given finite-dimensional comodule $V$ and any $h \in H_V$, we can find a sequence $k_n \in \mathbb{N}$ and elements $p_{n,i}$ and $q_{n,i}$ in $\mathcal{P}_H(A)$ with $1 \leq i \leq k_n$ such that

$$\sum_{i=1}^{k_n} (p_{n,i} \otimes 1)\delta(q_{n,i}) \underset{n\to\infty}{\longrightarrow} 1 \otimes h \tag{81}$$

in the $C^*$-norm. Applying $id \otimes e_V$ to this expression, and using (77), we see that we can take $q_{n,i} \in A_V$.

Applying δ to the first leg of (81) and using coassociativity, we obtain

$$\sum_{i=1}^{k_n} \left(\delta(p_{n,i}) \otimes 1\right)(id \otimes \Delta)\left(\delta(q_{n,i})\right) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes h. \tag{82}$$

Observe now that, since $q_{n,i} \in A_V$, by (76) we obtain $(id \otimes \Delta)(\delta(q_{n,i})) \in A_V \otimes H_V \otimes H_V$. Hence the left hand side of (82) belongs to the tensor product $(A \otimes_{\min} H) \otimes H_V$. As $H_V$ is finite dimensional, the restriction of the antipode $S$ of $\mathcal{O}(H)$ to $H_V$ is continuous. Therefore, we can apply S to the third leg of (82) to conclude

$$\sum_{i=1}^{k_n} \left(\delta(p_{n,i}) \otimes 1\right)(id \otimes (id \otimes S)\circ\Delta)\left(\delta(q_{n,i})\right) \underset{n\to\infty}{\longrightarrow} 1 \otimes 1 \otimes S(h). \tag{83}$$

Again by the finite dimensionality of $H_V$, multiplying the second and third legs is a continuous operation, so that

$$\sum_{i=1}^{k_n} \delta(p_{n,i})(q_{n,i} \otimes 1) \underset{n\to\infty}{\longrightarrow} 1 \otimes S(h). \tag{84}$$

Since $(h) \in H_{\bar{V}}$, where $\bar{V}$ is the contragredient of $V$, applying $id \otimes e_{\bar{V}}$ to the above limit, and using the equivariance property (77), we infer that in the above limit we can choose $p_{n,i} \in A_{\bar{V}}$.

Consider now the right $B$-module map

$$G_V : A_{\bar{V}} \otimes_B^{alg} A_V \to A_{\bar{V}\otimes V} \otimes H_{\bar{V}}, a \mapsto \delta(a)(b \otimes 1). \tag{85}$$

By Lemma (5.3.9) and Lemma (5.3.11), the left hand side becomes an interior tensor product of right Hilbert $B$-modules for the inner product

$$\langle c \otimes d, a \otimes b\rangle_B = E_B(d^* E_B(c^* a) b). \tag{86}$$

On the other hand, equipping $H_{\bar{V}}$ with the Hilbert space structure $\langle h, k\rangle = \varphi_H(h^* k)$ the right hand side is a right Hilbert $B$-module by

$$\langle b \otimes h, a \otimes g\rangle_B = \varphi_H(h^* g) E_B(b^* a). \tag{87}$$

From these formulas and (78), it follows that $G_V$ is an isometry between these Hilbert modules. Hence the range of $G_V$ is closed.

From (84) and the equivalence of $C^*$- and Hilbert $C^*$-module norms in Lemma (5.3.10), it follows that the range of $G_V$ contains $1 \otimes S(h)$. Therefore, as the domain of $G_V$ is an algebraic tensor product, we can find a finite number of elements $p_i, q_i \in \mathcal{P}_H(A)$ such that

$$\sum_i \delta(p_i)(q_i \otimes 1) = 1 \otimes S(h). \tag{88}$$

Now applying the map $a \otimes g \mapsto (1 \otimes S^{-1}(g))\delta(a)$ to both sides yields

$$\sum_i (p_i \otimes 1)\delta(q_i) = 1 \otimes h. \tag{89}$$

As $h$ was arbitrary in $\mathcal{O}(H)$, it follows that can is surjective.
Finally, as the Hopf algebra $\mathcal{O}(H)$ is cosemisimple, according to [172], bijectivity of the canonical map can follows from surjectivity. This completes the proof of the implication "PWG-condition ⇐ freeness".

The framework of principal comodule algebras unifies in one category many algebraically constructed non-commutative examples and classical compact principal bundles.

**Definition (5.3.12)[148]:** ([154]). Let $\mathcal{H}$ be a Hopf algebra with bijective antipode, and let $\Delta_{\mathcal{P}} : \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$ be a coaction making $\mathcal{P}$ an $\mathcal{H}$-comodule algebra. We call $\mathcal{P}$ principal if and only if:

(i) $\mathcal{P} \otimes_{\mathcal{B}} \mathcal{P} \ni p \otimes q \longmapsto can(p \otimes q) := (p \otimes 1)\Delta_{\mathcal{P}}(q) \in \mathcal{P} \otimes \mathcal{H}$ is bijective, where $\mathcal{B} = \mathcal{P}^{coH} := \{p \in \mathcal{P} \,|\Delta_{\mathcal{P}}(p) = p \otimes 1\}$;

(ii) there exists a left $\mathcal{B}$-linear right $\mathcal{H}$-colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{P} \to \mathcal{P}$.

Here (i) is the Hopf-Galois condition and (ii) is right equivariant left projectivity of $\mathcal{P}$.
Alternatively, one can approach principality through strong connections:

**Definition (5.3.13)[148]:** Let $\mathcal{H}$ be a Hopf algebra with bijective antipode $S$, and $\Delta_{\mathcal{P}}: \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$ be a coaction making $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra. A strong connection $\ell$ on $\mathcal{P}$ is a unital linear map $\ell : \mathcal{H} \to \mathcal{P} \otimes \mathcal{P}$ satisfying:

(i) $(id \otimes \Delta_{\mathcal{P}}) \circ \ell = (\ell \otimes id) \circ \Delta$;

(ii) $({}_{\mathcal{P}}\Delta \otimes id) \circ \ell = (id \otimes \ell) \circ \Delta$, where ${}_{\mathcal{P}}\Delta := (S^{-1} \otimes id) \circ \text{flip} \circ \Delta_{\mathcal{P}}$;

(iii) $\widetilde{can} \circ \ell = 1 \otimes id$, where $\widetilde{can} : \mathcal{P} \otimes \mathcal{P} \ni p \otimes q \longmapsto (p \otimes 1)\Delta_{\mathcal{P}}(q) \in \mathcal{P} \otimes \mathcal{H}$.

One can prove (see [155]) that a comodule algebra is principal if and only if it admits a strong connection.
If $\Delta_M\colon M \to M \otimes C$ is a coaction making $M$ a right comodule over a coalgebra $C$ and $N$ is a left $C$- comodule via a coaction ${}_N\Delta\colon N \to C \otimes N$, then we define their cotensor product as

$$M \boxdot_C N := \{t \in M \otimes N \,|(\Delta_M \otimes id)(t) = (id \otimes {}_N\Delta)(t)\}. \tag{90}$$

In particular, for a right $H$-comodule algebra $\mathcal{P}$ and a left $\mathcal{H}$-comodule , we observe that $P \boxdot_{\mathcal{H}} V$ is a left $\mathcal{P}^{coH}$- module in a natural way. One of the key properties of principal comodule algebras is that, for any finite-dimensional left $H$-comodule $V$, the left $\mathcal{P}^{coH}$-module $\mathcal{P} \boxdot_{\mathcal{H}} V$ is finitely generated projective [154]. Here $\mathcal{P}$ plays the role of a principal bundle and $\mathcal{P} \boxdot_{\mathcal{H}} V$ plays the role of an associated vector bundle. Therefore, we call $\mathcal{P} \boxdot_{\mathcal{H}} V$ an associated module.
Principality can also be characterized by the exactness and strong monoidality of the cotensor functor. This characterisation uses the notion of coflatness of a comodule: a right

comodule is coflat if and only if cotensoring it with left comodules preserves exact sequences.

**Theorem (5.3.14)[148]:** Let $\mathcal{H}$ be a Hopf algebra with bijective antipode, and $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra. Then $\mathcal{P}$ is principal if and only if $\mathcal{P}$ is right $\mathcal{H}$-coflat and for all left $\mathcal{H}$-comodules $V$ and $W$ the map

$$\beta : (\mathcal{P} \boxdot V) \otimes_{\mathcal{B}} (\mathcal{P} \boxdot W) \to \mathcal{P} \boxdot (V \otimes W)$$
$$(a \otimes v) \otimes (b \otimes w) \longmapsto ab \otimes (v \otimes w)$$

is bijective. In other words, $\mathcal{P}$ is principal if and only if the cotensor product functor is exact and strongly monoidal with respect to the above map $\beta$.

Proof. The proof relies on putting together [172], [169], [154] and [171]. First assume that $\mathcal{P}$ is principal. Then $\mathcal{P}$ is right equivariantly projective, and it follows from [154] that $\mathcal{P}$ is faithfully flat. Now we can apply [169] to conclude that β is bijective. Furthermore, by [172], the faithful flatness of $\mathcal{P}$ implies the coflatness of $\mathcal{P}$ Conversely, assume that cotensoring with $\mathcal{P}$ is exact and strongly monoidal with respect to $\beta$. Then substituting $\mathcal{H}$ for $V$ and $W$ yields the Hopf-Galois condition. Now [171] implies the equivariant projectivity of $\mathcal{P}$.

Let $A$ be a unital $C^*$-algebra with center $Z(A)$, let $X$ be a compact Hausdorff space and let $\theta : C(X) \to Z(A)$ be a unital inclusion. The triple $(A, C(X), \theta)$ is called a unital $C(X)$-algebra ([165]). In the following, we simply consider $C(X)$ as a subalgebra of $A$. For $x \in X$, let $J_x$ be the closed 2-sided ideal in $A$ generated by the functions $f \in C(X)$ that vanish at $x$. Then we have quotient $C^*$-algebras $A_x = A/J_x$ with natural projection maps $\pi_x : A \to A_x$, and the triple $(X, A, \pi_x)$ is a field of $C^*$-algebras. For any $a \in A$, the map $n_x : X \to R, x \longmapsto \|\pi_x(a)\|$
is upper semi-continuous [160] (see also [168]). If the latter map is continuous, the field is called continuous, but this property will not be necessary to assume for our purposes.

**Lemma (5.3.15)[148]:** Let $X$ be a compact Hausdorff space, $A$ a unital $C(X)$-algebra, and $(H, \Delta)$ a compact quantum group acting on $A$ via $\delta : A \to A \otimes_{\min} H$. Assume that $C(X) \subseteq A^{coH}$. Then for each $x \in X$ there exists a unique coaction $\delta_x : A_x \to A_x \otimes_{\min} H$ such that for all $a \in A$

$$\delta_x(\pi_x(a)) = (\pi x \otimes id)(\delta(a)). \qquad (91)$$

**Proof.** Let $x \in X$ and $f \in C(X)$ with $f(x) = 0$. As $\delta(f) = f \otimes 1$ by assumption, it follows $that$ $(\pi_x \otimes id)(\delta(f)) = 0$. Hence $(\pi_x \otimes id)(\delta(a)) = 0$ for $a \in J_x$, so that $\delta_x$ can be defined by (91). It is straightforward to check that each $\delta_x$ satisfies the coassociativity and counitality conditions.

**Theorem (5.3.16)[148]:** Let $X$ be a compact Hausdorff space, A a unital $C(X)$-algebra, and $(H, \Delta)$ a compact quantum group acting on $A$ via $\delta : A \to A \otimes_{\min} H$. Assume that $C(X) \subseteq A^{coH}$. Then, if the coactions $\delta_x$ are free for each $x \in X$, so is the coaction $\delta$.

**Proof.** First note that $A \otimes_{\min}$ H is again a $C(X)$-algebra in a natural way. We will denote the quotient $(A \otimes_{\min} H)/(J_x \otimes_{\min} H)$ by $A_x \otimes_x H$. This will be a $C^*$ -completion of the algebraic will denote the quotient map at $x$ by $\pi_x \otimes_x id : A \otimes_{\min} H \to A_x \otimes_x H$.

Assume now that each $\delta_x$ is free. Fix $\varepsilon > 0$, and choose $h \in \mathcal{O}(H)$. By Theorem (5.3.1), for each $x \in X$ we can find an element $z_x \in (A \otimes 1)\delta(A)$ such that $(\pi_x \otimes_x id)(z_x) = 1 \otimes h$ in $A_x \otimes_x H$.

Consider the function

$$f_x : X \ni y \longmapsto \left\|(\pi_y \otimes_y id)(z_x - 1 \otimes h)\right\|$$

$$= \|(\pi y \otimes y\, id)(zx) - 1 \otimes h\| \in \mathbb{R}. \quad (92)$$

As the norm on the field $y \longmapsto A_y \otimes_y H$ is upper semi-continuous, the function $y \longmapsto f_x(y)$ is upper semi-continuous. Since $f_x(x) = 0$, we can find an open neighborhood $U_x$ of $x$ such that for all $y \in U_x$

$$f_x(y) = \left\|(\pi_y \otimes_y id)(z_x) - 1 \otimes h\right\|_{A_y \otimes_y H} < \varepsilon. \quad (93)$$

Let $\{f_i\}_i$ be a partition of unity subordinate to a finite subcover $\{U_{xi}\}_i$. An easy estimate shows that for $z := \sum_i (f_i \otimes 1)_{z_{x_i}}$ and all $y \in X$

$$\left\|(\pi_y \otimes_y id)(z - 1 \otimes h)\right\|_{A_y \otimes_y H} < \varepsilon. \quad (94)$$

Taking the supremum over all $y$, we conclude by [160] and the compactness of $X$ that $\|z - 1 \otimes h\| < \varepsilon$. Hence $(A \otimes 1)\delta(A)$ is dense in $A \otimes H$, i.e. the coaction $\delta$ is free.
As a particular case we consider:
**Definition (5.3.17)[148]:** ([353]). Let $(H, \Delta)$ be a compact quantum group acting on a unital $C^*$-algebra $A$ via $\delta : A \to A \otimes_{\min} H$. We call the unital $C^*$ -algebra

$$A * h := \{f \in C([0,1], A \otimes_{\min} H) f(0) \in \mathbb{C} \otimes H, f(1) \in \delta(A)\} \quad (95)$$

the noncommutative join of $A$ and $H$.
The $C^*$-algebra $A * H$ is obviously a $C([0,1])$-algebra with $(A * H)_x \cong A \otimes_{\min} H$ for $x \in (0,1), (A * H)_0 \cong H$ and $(A * H)_1 \cong A$. We identify $A * H$ as a subalgebra of $C([0,1]) \otimes_{\min} A \otimes_{\min} H$.
The following lemma shows that $A * H$ carries a natural coaction by $(H, \Delta)$.
**Lemma (5.3.18)[148]:** The compact quantum group $(H, \Delta)$ acts on the unital $C^*$-algebra $A * H$ via

$$\delta_{A*H} : A * H \ni f \longmapsto (id \otimes id \otimes \Delta) \circ f \in (A * H) \otimes_{\min} H. \quad (96)$$

**Proof.** Note that $\delta_{A*H}$ is the restriction of $(id \otimes id \otimes \Delta)$ to $A * H$. Let us first show that the range of $\delta_H$ is contained in $(A * H) \otimes_{\min} H$.
Consider an element $F \in A * H$ as an $A \otimes_{\min} H$-valued function on [0,1]. Since $F$ is uniformly continuous and $\mathcal{P}_H(A)$ is dense in $A$ by [167] and [173], an elementary partition of unity argument shows that $F$ can be approximated by a finite sum of functions of three kinds:
(i) $F_1 : [0,1] \ni t \longmapsto \xi_0(t)(1 \otimes h) \in \mathbb{C} \otimes \mathcal{O}(H)$, where $\xi_0 \in C([0,1],[0,1])$ , $\xi_0(1) = 0$, and $h$ $is$ $a$ $f$ixed element $of$ $\mathcal{O}(H)$.
(ii) $F_2 : [0,1] \ni t \longmapsto \xi(t)(a \otimes h) \in \mathcal{P}_H(A) \otimes_{alg} \mathcal{O}(H)$, where $\xi \in C([0,1],[0,1])$with $\xi(0) = \xi(1) = 0$, and $a$ and $h$ are fixed elements of $\mathcal{P}_H(A)$ and $\mathcal{O}(H)$ respectively.
(iii ) $F_3 : [0,1] \ni t \longmapsto \xi_1(t)\delta(a) \in \delta(\mathcal{P}_H(A))$, where $\xi_1 \in C([0,1],[0,1]), \xi_1(0) = 0$, and $a$ is afixed element of $\mathcal{P}_H(A)$.
It is clear that $\delta_{A*H}(Fi) \in C([0,1], A \otimes_{\min} H) \otimes_{alg} \mathcal{O}(H)$ for all $i$. Let $\omega$ be a functional on $\mathcal{O}(H)$. Then $(id \otimes \omega)(\delta A * H(F_i)) \in A * H$ for all $i$. This implies that $\delta_{A*H}(F_i) \in (A * H) \otimes_{alg} H$ for all $i$. It follows from the continuity of $\delta_{A*H}$ that $\delta_{A*H}(F) \in (A * H) \otimes_{\min} H$. Hence $\delta_{A*H}$ has range in $(A * H) \otimes_{\min} H$.
The coassociativity of $\delta_{A*H}$is immediate from the coassociativity of $\Delta$. The counitality condition follows from the same approximation argument as above.
**Corollary (5.3.19)[148]:** The coaction $\delta_{A*H} : A * H \to (A * H) \otimes_{\min} H$ is free.
**Proof.** The $C^*$-algebra $A * H$ is a unital $C([0,1])$-algebra with $C([0,1]) \in (A * H)^{coH}$. With the notation of Lemma (5.3.15), we have:

(i) $((A * H)_0, \delta_0) \cong (H, \Delta)$,
(ii) $((A * H)_x, \delta_x) \cong (A \otimes_{\min} H, id \otimes \Delta)$ $for\ x \in (0,1)$,
(iii) $((A * H)_1, \delta_1) \cong (A, \delta)$.
As each of the above actions are free, we infer from Theorem (5.3.16) that $\delta_{A*H}$ is free. Alternatively, one can use a direct approximation argument as in Lemma (5.3.18).

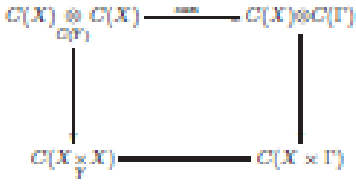
Let $X, Y$ be topological spaces and let $\pi : X \to Y$ be a covering map — i.e. Given any $y \in Y$ , $\exists$ an open set $U$ in $Y$ with $y \in U$ such that $\pi^{-1}(U)$ is a disjoint union of open sets each of which $\pi$ maps homeomorphically onto $U$. A deck transformation is a homeomorphism
$h: X \to X$ with $\pi \circ h = \pi$.

**Proposition (5.3.20)[148]:** Let $X$ and $Y$ be compact Hausdorff topological spaces.Let $\pi : X \to Y$ be a covering map, and let $\Gamma$ be the group of deck transformations of this covering. Assume that $\Gamma$ is finite. Then $X$ is a locally trivial principal $\Gamma$ bundle on $Y$ if and only if the canonical map

$$can: C(X) \otimes_{C(Y)} C(X) \to C(X) \otimes C(\Gamma)$$
$$can: f_1 \otimes f_2 \mapsto (f_1 \otimes 1)\delta(f_2)$$

is an isomorphism.

**Proof.** Consider the commutative diagram



(97)

in which each vertical arrow is the evident inclusion and the lower horizontal arrow is the $*$-homomorphism resulting from the map of topological spaces

$$X \times \Gamma \to X \times_Y X \qquad (x, \gamma) \mapsto (x, x\gamma) \tag{98}$$

$X$ is a (locally trivial) principal $\Gamma$ bundle on $Y$ if and only if this map of topological spaces is a homeomorphism — which is equivalent to bijectivity of the lower horizontal arrow.
Hence to prove the proposition, it will suffice to prove that the two vertical arrows are isomor-phisms.
The right vertical arrow is an isomorphism because $\Gamma$ is a finite group, so $C(\Gamma)$ is a finite dimensional vector space over the complex numbers $\mathbb{C}$.
For the left vertical arrow, let $E$ be the vector bundle on $Y$ whose fiber at $y \in Y$ is $\mathrm{Map}(\pi^{-1}(y), \mathbb{C})$ i.e. is the set of all set-theoretic maps from $\pi^{-1}(y)$ to $\mathbb{C}$. Note that $\pi^{-1}(y)$ is a discrete subset of the compact Hausdorff space $X$ and therefore is finite.
Let $S(E)$ be all the continuous sections of $E$. Then :

$$S(E) = C(X) \tag{99}$$

Similarly, let

$$\rho: X \times_Y X \to Y \ \text{ be } (x_1, x_2) \mapsto \pi(x_1) = \pi(x_2) \tag{100}$$

and let $F$ be the vector bundle on $Y$ whose fiber at $y \in Y$ is $\mathrm{Map}(\rho^{-1}(y), \mathbb{C})$ i.e. is the set of all set-theoretic maps from $\rho^{-1}(y)$ to $\mathbb{C}$. Then :

$$S(F) = C(X \times_Y X) \tag{101}$$

where $S(F)$ is all the continuous sections of $F$.

As vector bundles on $Y$

$$F = E \otimes E \tag{102}$$

This implies $S(F) = S(E) \otimes_{C(Y)} S(E)$ and thus proves bijectivity for the left vertical arrow.

<u>Special Case.</u> Let $X, Y$ be connected finite CW complexes. Let $\pi : X \rightarrow Y$ be a covering map. $\Gamma$ denotes the group of deck transformations. Then the action of $\Gamma$ on each fiber of $\pi$ is transitive if and only if the canonical map

$$can: C(X) \otimes_{C(Y)} C(X) \rightarrow C(X) \otimes C(\Gamma) \tag{103}$$

is an isomorphism.

**Example (5.3.21)[148]:** Without connectivity conditions the group of deck transformations can be infinite. Let $Y$ be the Cantor set $C$ and let $X$ be $C \times \{0,1\}$ where the two-element set $\{0,1\}$ has the discrete topology. Let $\pi : C \times \{0,1\} \rightarrow C$ be the projection

$$\pi(c,t) = c \qquad c \in C \; t \in \{0,1\} \tag{104}$$

Let $U$ be a subset of $C$ which is both open and closed. Define $h_U : C \times \{0,1\} \rightarrow C \times \{0,1\}$ by:

$$h_U(c,t) = \begin{cases} (c,t) & c \notin U \\ (c, 1-t) & c \in U \end{cases} \tag{105}$$

Then $h_U$ is a deck transformation and there are infinitely many $h_U$.

# Chapter 6
# Invariant Measures with Homoclinic Groups and Generic Points

We get that for every integer d > 1 and every Toeplitz flow (X, T), there exists a Toeplitz $\mathbb{Z}^d$ -subshift which is topologically orbit equivalent to (X, T). We show that for an expansive algebraic action of a polycyclic-by-finite group on X, the entropy of the action is equal to the entropy of the induced action on the Pontryagin dual of the homoclinic group, the homoclinic group is a dense subgroup of the IE group, the homoclinic group is nontrivial exactly when the action has positive entropy, and the homoclinic group is dense in X exactly when the action has completely positive entropy. We extend the Sigmund theorem and completes the result of Ren.

## Section (6.1): Orbit Equivalence for Generalized Toeplitz Subshifts

The Toeplitz subshifts are a rich class of symbolic systems introduced by Jacobs and Keane in [388], $Z$-actions. Since then, they have been extensively studied and used to provide series of examples with interesting dynamical properties (see for example [374], [375], [384], [394]). Generalizations of Toeplitz subshifts and some of their propertiesto more general group actions can be found in [370], [372], [376], [389], [390]. For instance, in [372] Toeplitz subshifts are characterized as the minimal symbolic almost 1-1 extensions of odometers (see [380]). We give an explicit construction that generalizes the result of Downarowicz in [374], to Toeplitz subshifts given by actions of groups which are amenable, countable and residually finite. The following is our main result.

By a topological dynamical system we mean a triple $(X, T, G)$, where T is a continuous left action of a countable group G on the compact metric space $(X, d)$. For every $g \in G$, we denote $T^g$ the homeomorphism that induces the action of g on $X$. The unit element of $G$ will be called e. The system $(X, T, G)$ or the action T is minimal if for every $x \in X$ the orbit $o_T(x) = \{T^g(x) : g \in G\}$ is dense in $X$. We say that $(X, T, G)$ is a minimal Cantor system or a minimal Cantor $G$-system if $(X, T, G)$ is a minimal topological dynamical system with X a Cantor set.

An invariant probability measure of the topological dynamical system $(X, T, G)$ is a probability Borel measure μ such that $\mu(T^g(A)) = \mu(A)$, for every Borel set $A$. We denote by $\mathcal{M}(X, T, G)$ the space of invariant probability measures of $(X, T, G)$.

For every $g \in G$, denote $L_g : G \to G$ the left multiplication by $g \in G$. That is, $L_g(h) = gh$ for every $h \in G$. Let $\sum$ be a finite alphabet. $\sum^G$ denotes the set of all the functions $x : G \to \sum$. The (left) shift action $\sigma$ of $G$ on $\sum^G$ is given by $\sigma^g(x) = x \circ L_{g^{-1}}$, for every $g \in G$. Thus $\sigma^g(x)(h) = x(g^{-1}h)$. We consider $\sum$ endowed with the discrete topology and $\sum^G$ with the product topology. Thus every $\sigma^g$ is a homeomorphism of the Cantor set $\sum^G$. The topological dynamical system $(\sum^G, \sigma, G)$ is called the full $G$-shift on $\sum$. For every finite subset $D$ of $G$ and $x \in \sum^G$, we denote $x|D \in \sum^G$ the restriction of $x$ to $D$. For $F \in \sum^G$ ($F$ is a function from $D$ to $\sum$) we denote by $[F]$ the set of all $x \in \sum^G$ such that $x|D = F$. The set $[F]$ is called the cylinder defined by $F$, and it is a clopen set (both open and closed). The collection of all the sets $[F]$ is a base of the topology of $\sum^G$ .

**Definition(6.1.1)[367]:** $A$ subshift or $G$-subshift of $\sum^G$ is a closed subset X of $\sum^G$ which is invariant by the shift action.

The topological dynamical system $(X, \sigma|X, G)$ is also called subshift or $G$-subshift. See [369] for details.

Toeplitz $G$-subshifts. An element $x \in \sum^{G}$ is a Toeplitz sequence, if for every $g \in G$ there exists a finite index subgroup $\Gamma$ of $G$ such that $\sigma^{\gamma}(x)(g) = x(\gamma^{-1}g) = x(g)$, for every $\gamma \in \Gamma$.
A subshift $X \subseteq \sum^{G}$ is a Toeplitz subshift or Toeplitz $G$-subshift if there exists a Toeplitz sequence $x \in \sum^{G}$ such that $X = \overline{o_{\sigma}(x)}$. It is shown in [372], [389] and [390] that a Toeplitz sequence x is regularly recurrent, i.e. for every neighborhood $V$ of $x$ there exists a finite index subgroup $\Gamma$ of $G$ such that $\sigma^{\gamma}(x) \in V$, for every $\gamma \in \Gamma$. This condition is stronger than almost periodicity, which implies minimality of the closure of the orbit of x (see [368] ).

Given a sequence of continuous maps $f_n : X_{n+1} \to X_n, n \geq 0$ on topological spaces $X_n$, we denote the associated inverse limit by

$$\lim_{\leftarrow n}(X_n, f_n) = X_0 \overset{f_0}{\leftarrow} X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} \cdots$$
$$:= \{(x_n)_{n;} x_n \in X_n, x_n = f_n(x_n + 1)\ \forall n \geq 0\}.$$

Let us recall that this space is compact when all the spaces $X_n$ are compact and the inverse limit spaces associated to any increasing subsequences$(n_i)_i$of indices are homeomorphic.
In a similar way, we denote for a sequence of maps $g_n : X_n \to X_{n+1}, n \geq 0$ the associated direct limit by

$$\lim_{\to n}(X_n, g_n) = X_0 \overset{g_0}{\to} X_1 \overset{g_1}{\to} X_2 \overset{g_2}{\to} \cdots$$
$$:= \{(x, n), x \in X_n, n \geq 0\}/\sim$$

where two elements are equivalent $(x, n) \sim (y, m)$ if and only if there exists $k \geq m, n$ such that $g_k \circ \ldots \circ g_n(x) = g_k \circ \ldots \circ g_m(x)$. We denote by $[x, n]$ the equivalence class of $(x, n)$. When the maps $g_n$ are homomorphisms on groups $X_n$, then the direct limit inherits a group structure.

A group $G$ is said to be residually finite if there exists a nested sequence $(\Gamma_n)_{n\geq 0}$ of finite index normal subgroups such that $\bigcap_{n\geq 0} \Gamma_n$ is trivial. For every $n \geq 0$, there exists then a canonical projection $\pi_n : G/\Gamma_{n+1} \to G/\Gamma_n$. The $G$-odometer or adding machine $O$ associated to the sequence $(\Gamma_n)_n$ is the inverse limit

$$O := \lim_{\leftarrow n}(G/\Gamma_n \pi_n) = G/\Gamma_0 \overset{\pi_0}{\leftarrow} G/\Gamma_1 \overset{\pi_1}{\leftarrow} G/\Gamma_2 \overset{\pi_2}{\leftarrow} \cdots$$

See [372] for the basic properties of such a space. Let us recall that it inherits a group structure through the quotient groups $G/\Gamma_n$n and it contains $G$ as a subgroup thanks the injection $G \ni g \mapsto ([g]_n) \in O$, where $[g]_n$ denotes the class of g in $G/\Gamma_n$.Thus the group $G$ acts by left multiplication on $O$. When there is no confusion, we call this action also odometer. It is equicontiuous, minimal and the left Haar measure is the unique invariant probability measure. Notice that this action is free: the stabilizer of any point is trivial. The Toeplitz $G$-subshifts are characterized as the subshifts that are minimal almost 1-1 extensions of $G$-odometers [372].

For more details about ordered groups and dimension groups we refer to [379] and [385]. An ordered group is a pair $(H, H^+)$, such that $H$ is a countable abelian group and $H^+$ is a subset of $H$ verifying $(H^+) + (H^+) \subseteq H^+, (H^+) + (-H^+) = H$ and $(H^+) \cap (-H^+) = \{0\}$ (we use 0 as the unit of $H$ when $H$ is abelian). An ordered group $(H, H^+)$, is a dimension group if for every $n \in \mathbb{Z}^+$ there exist $k_n \geq 1$ and a positive homomorphism $A_n : \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_{n+1}}$, such that $(H, H^+)$, is isomorphic to $(J, J^+)$, where $J$ is the direct limit

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z}^{k_0} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

And $J^+ = \{[v,n] : a \in (\mathbb{Z}^+)^{k_n}, n \in \mathbb{Z}^+\}$. The dimension group is simple if the matrices An can be chosen strictly positive.

An order unit in the ordered group $(H, H^+)$, is an element $u \in H^+$ such that for every $g \in H$ there exists $n \in \mathbb{Z}^+$ such that $nu - g \in H^+$. If $(H, H^+)$, is a simple dimension group then each element in $H^+ \setminus \{0\}$ is an order unit. $A$ unital ordered group is a triple $(H, H^+, u)$ such that $(H, H^+)$ is an ordered group and u is an order unit. An isomorphism between two unital ordered groups $(H, H^+, u)$ and $(J, J^+, v)$ is an isomorphism $\phi : H \to J$ such that $\phi(H^+) = J^+$ and $\phi(u) = v$. $A$ state of the unital ordered group $(H, H^+, u)$ is a homomorphism $\phi : H \to R$ so that $\phi(u) = 1$ and $\phi(H^+) \subseteq \mathbb{R}^+$. The infinitesimal subgroup of a simple dimension group with unit $(H, H^+, u)$ is

$$\inf(H) = \{a \in H : \phi(a) = 0 \text{ for all state } \phi\}.$$

It is not difficult to show that inf $(H)$ does not depend on the order unit. The quotient group $H/\inf(H)$ of a simple dimension group $(H, H^+)$, is also a simple dimension group with positive cone

$$(H/\inf(H))^+ = \{[a] : a \in H^+\}.$$

The next result is well-known.

**Lemma (6.1.2)[367]:** Let $(H, H^+)$, be a simple dimension group equals to the direct limit

$$\lim_{\to n}(\mathbb{Z}^{k_n}, M_n) = \mathbb{Z}^{k_0} \xrightarrow{M_0} \mathbb{Z}^{k_1} \xrightarrow{M_1} \mathbb{Z}^{k_2} \xrightarrow{M_2} \cdots.$$

Then for every $z = (z_n)n \geq 0$ in the inverse limit

$$\lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T) = (\mathbb{R}^+)^{k_0} \xleftarrow{,M_0^T} (\mathbb{R}^+)^{k_1} \xleftarrow{,M_1^T} (\mathbb{R}^+)^{k_2} \xleftarrow{,M_2^T} \cdots$$

the function $\phi_z : H \to \mathbb{R}$ given by $\phi([n,v]) = < v, z_n >$, for every $[n,v] \in H$, is well defined and is a homomorphism of groups such that $\phi_z(H^+) \subseteq \mathbb{R}^+$. Conversely, for every group homomorphism $\phi : H \to \mathbb{R}$ such that $\phi(H^+) \subseteq \mathbb{R}^+$, there exists a unique $z \in \lim_{\leftarrow n}((\mathbb{R}^+)^{k_n},, M_n^T)$ such that $\phi = \phi_z$.

The following lemma is a preparatory lemma to prove Theorem (6.1.21) and (6.1.23).

**Lemma (6.1.3)[367]:** Let $(H, H^+, u)$ be a simple dimension group with unit given by the following direct limit

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

with unit $u = [1, 0]$. Suppose that $A_n > 0$ for every $n \geq 0$. Then $(H, H^+, u)$ is isomorphic to

$$\mathbb{Z} \xrightarrow{\tilde{A}_0} \mathbb{Z}^{k_1+1} \xrightarrow{\tilde{A}_1} \mathbb{Z}^{k_2+1} \xrightarrow{\tilde{A}_2} \cdots,$$

where $\tilde{A}_0$ is the $(k_1 + 1) \times 1$-dimensional matrix given by

$$\tilde{A}_0 = \begin{bmatrix} A_0(1,\cdot) \\ A_0(1,\cdot) \\ A_0(2,\cdot) \\ \vdots \\ A_0(k_1,\cdot) \end{bmatrix},$$

and $\tilde{A}_n$ is the $(k_1 + 1 \times 1) \times (k_n + 1)$ dimensional matix given by

$$\tilde{A}_n = \begin{bmatrix} 1 & A_n(1,1)-1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(1,1)-1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(2,1)-1 & A_n(2,2) & \cdots & A_n(2,k_n) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & A_n(k_{n+1},1)-1 & A_n(k_{n+1},2) & \cdots & A_n(k_{n+1},k_n) \end{bmatrix}, \text{ for every } n \geq 0.$$

**Proof:** For $n \geq 1$, consider $M_n$ the $(k_n + 1) \times k_n$-dimensional matrix given by

$$M_n(\cdot, k) = \begin{cases} \vec{e}_{n,1} + \vec{e}_{n,2} & \text{if} \quad k = 1 \\ \vec{e}_{k+1} & \text{if} \quad 3 \leq k \leq k_n \end{cases},$$

where $\vec{e}_{n,1}, \cdots, \vec{e}_{n,k_{n+1}}$ are the canonical vectors in $\mathbb{R}^{k_{n+1}}$. Let $B_n$ be the $k_{n+1} \times (k_n + 1)$-dimensional matrix defined by

$$B_n(i,j) = \begin{cases} 1 & \text{if} \quad j = 1 \\ A_n(i,1) - 1 & \text{if} \quad j = 2 \\ A_n(i,j-1) & \text{if} \quad 3 \leq j \leq k_n + 1 \end{cases}$$

We have $A_n = B_n M_n$ and $\tilde{A}_n = M_{n+1} + B_n$ for every $n \geq 1$, and $\tilde{A}_0 = M_1 A_0$.Thus the Bratteli diagrams defined by the sequences of matrices $(A_n)_n \geq 0$ and $(\tilde{A}_n)_n \geq 0$ are contractions of the same diagram. This shows that the respective dimension groups with unit are isomorphic (see [383] or [377]).

Let $(X, T, G)$ be a topological dynamical system such that $X$ is a Cantor set and $T$ is minimal. The ordered group associated to $(X, T, G)$ is the unital ordered group

$$\mathcal{G}(X,T,G) = (D_m(X,T,G), D_m(X,T,G)^+, [368]),$$

where

$$D_m(X,T,G) = C(X,Z)/\{f \in C(X,\mathbb{Z}) : \int f d\mu = 0, \forall \mu \in M(X,T,G)\},$$
$$D_m(X,T,G)^+ = \{[f] : f \geq 0\},$$

and $[368] \in D_m(X,T,G)$ is the class of the constant function 1.

Two topological dynamical systems $(X_1, T_1, G_1)$ and $(X_2, T_2, G_2)$ are (topologically) orbit equivalent if there exists a homeomorphism $F: X_1 \to X_2$ such that $F(oT_1(x)) = oT_2(F(x))$ for every $x \in X_1$.

[382] show the following algebraic caracterization of orbit equivalence.

**Theorem (6.1.4)[367]:** [382]. Let $(X, T, \mathbb{Z}^d)$ and $(X', T', \mathbb{Z}^m)$ be two minimal actions on the Cantor set. Then they are orbit equivalednt if and only if

$$\mathcal{G}(X,T,\mathbb{Z}^d) \simeq \mathcal{G}(X',T',\mathbb{Z}^m)$$

as isomorphism of unital ordered group.

Let $G$ be a residually finite group, and let $(\Gamma_n)_{n\geq 0}$ be a nested sequence of finite index normal subgroup of $G$ such that $\bigcap_{n\geq 0} \Gamma_n = \{e\}$.

For technical reasons it is important to notice that since the groups $\Gamma_n$ are normal, we have $g\Gamma_n = \Gamma_n g$, for every $g \in G$.

To construct a Toeplitz $G$-subshift that is an almost 1-1 extension of the odometer defined by the sequence $(\Gamma_n)_n$, we need a "suitable" sequence $(F_n)_n$ of fundamental domains of

$G/\Gamma_n$. More precisely, each $F_{n+1}$ has to be tileable by translated copies of $F_n$. To control the simplex of invariant measures of the subshift, we need in addition the sequence $(F_n)_n$ to be Følner. We did not find in the specialized litterature a result ensuring these conditions.

Let $\Gamma$ be a normal subgroup of $G$. By a fundamental domain of $G/\Gamma$, we mean a subset $D \subseteq G$ containing exactly one representative element of each equivalence class in $G/\Gamma$.

**Lemma (6.1.5)[367]:** Let $(D_n)_{n\geq 0}$ be an increasing sequence of finite subsets of $G$ such that for every $n \geq 0, e \in D_n$ and $D_n$ is a fundamental domain of $G/\Gamma_n$. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be an increasing sequence. Consider $(F_i)_{i\geq 0}$ defined by $F_0 = D_{n_0}$ and

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1} \text{ for every } i \geq 1.$$

Then for every $i \geq 0$ we have the following:

**(i)** $F_i \subseteq F_{i+1}$ and $F_i$ is a fundamental domain of $G/\Gamma_{n_i}$.

**(ii)** $F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} v \in F_i$.

**Proof:** Since $\in \mathrm{D}_{n_i}$ , the sequence $(F_i)_{i\geq 0}$ is increasing.

$F_0 = \mathrm{D}_{n_0}$ is a fundamental domain of $G/\Gamma_{n_0}$ . We will prove by induction on i that $F_i$ is a fundamental domain of $G/\Gamma_{n_i}$ . Let $i > 0$ and suppose that $F_{i-1}$ is a fundamental domain of $G/\Gamma_{n_{i-1}}$.

Let $\in \mathrm{D}_{n_i}$ . There exist then $u \in F_{i-1}$ and $w \in \Gamma_{n_{i-1}}$ such that $v = wu$. Let $z \in \mathrm{D}_{n_i}$ and $\gamma \in \Gamma_{n_i}$ be such that $w = \gamma z$. Since $z \in F_{i-1} \cap \mathrm{D}_{n_i}$ and $v = \gamma zu$, we conclude that $F_i$contains one representing element of each class in $G/\Gamma_{n_i}$.

Let $w_1, w_2 \in F_i$ be such that there exists $\gamma \in \Gamma_{n_i}$ verifying $w_1 = \gamma w_2$. By definition, $w_1 = v_1 u_1$ and $w_2 = v_2 u_2$, for some $u_1, u_2 \in F_{i-1}$ and $v_1, v_2 \in \mathrm{D}_{n_i} \cap \Gamma_{n_{i-1}}$ . This implies that $u_1$ and $u_2$ are in the same class of $/\Gamma_{n_{i-1}}$ . Since $F_{i-1}$ is a fundamental domain, we have $u_1 = u_2$. From this we get $v_1 = \gamma v_2$, which implies that $v_1 = v_2$. Thus we deduce that $F_i$ contains at most one representing element of each class in $/\Gamma_{n_i}$ . This shows that $F_i$ is a fundamental domain of $G/\Gamma_{n_i}$.

To show that $\mathrm{D}_{n_i} \cap \Gamma_{n_{i-1}} \subseteq F_i \cap \Gamma_{n_{i-1}}$ , observe that the definition of $F_i$ implies that for every $v \in \mathrm{D}_{n_i} \cap \Gamma_{n_{i-1}}$ and $u \in \Gamma_{n_{i-1}}, vu \in F_i$. Then for $u = e \in F_{i-1}$ we get $v = ve \in F_i$. Now suppose that $v \in F_i \cap \Gamma_{n_{i-1}} \subseteq F_i$. The definition of $F_i$ implies there exist $u \in F_{i-1}$ and $\gamma \in \mathrm{D}_{n_i} \cap \Gamma_{n_{i-1}}$ such that $v = \gamma u$. Since v and $\gamma$ are in $\Gamma_{n_{i-1}}$ , we get that $u \in \Gamma_{n_{i-1}} \cap F_{i-1}$. This implies that $u = e$ because $\Gamma_{n_{i-1}} \cap F_{i-1} = \{e\}$.

By Følner sequences we mean right Følner sequences. That is, a sequence $(F_n)_{i\geq 0}$ of nonempty finite sets of $G$ is a Følner sequence if for every $g \in G$

$$\lim_{n\to\infty} \frac{|F_n g \Delta F_n|}{F_n} = 0.$$

Observe that $(F_n)_{n\geq 0}$ is a right Følner sequence if and only if $(F_n^{-1})_{n\geq 0}$ is a left Følner sequence.

**Lemma (6.1.6)[367]:** Suppose that $G$ is amenable. There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\in\mathbb{Z}^+}$, such that

**i)** $F_i \subseteq F_{i+1}$ and $F_i$ is a fundamental domain of $G/\Gamma_{n_i}$ , for every $i \geq 0$.
**ii)** $G = \bigcup_{i\geq 0} F_i$.
**iii)** $F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} v\, F_i$, for every $i \geq 0$.

**Proof:** From [393] (see [389] for a proof in our context), there exists an increasing sequence $(m_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(D_i)_{i\in\mathbb{Z}^+}$ such that for every $i \geq 0, D_i \subseteq D_{i+1}, D_i$ is a fundamental domain of $/\Gamma_{m_i}$ , and $G = \bigcup_{i\geq 0} D_i$. Up to take subsequences, we can assume that $D_i$ is a fundamental domain of $G/\Gamma_i$, for every $i \geq 0$, and that $e \in D_0$.
We will construct the sequences $(n_i)_{i\geq 0}$ and $(F_n)_{n\geq 0}$ as follows:
**Step** 0: We set $n_0 = 0$ and $F_0 = D_0$.
**Step** i: Let $i > 0$. We assume that we have chosen $n_i$ and $F_j$ for every $0 \leq j < i$. We take $n_i > n_{i-1}$ in order that the following two conditions are verified:

$$\frac{|D_{n_i} g \,\Delta\, D_{n_i}|}{D_{n_i}} < \frac{1}{i|F_{i-1}|}, \text{for every } g \in F_{i-1}. \tag{1}$$

$$D_{n_{i-1}} \subseteq \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1} \;. \tag{2}$$

Such integer ni exists because $(D_n)_{n\geq 0}$ is a Følner sequence and $F_{i-1}$ is a fundamental domain of $G/\Gamma_{n_{i-1}}$ (then $G = \bigcup_{v\in\Gamma_{n_{i-1}}} vF_{i-1}$).
We define

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}$$

Lemma (6.1.5) ensures that $(F_i)_{i\geq 0}$ verifies i) and iii) of the lemma. The equation (2) implies that $(F_i)_{i\geq 0}$ verifies ii) of the lemma.
It remains to show that $(F_i)_{i\geq 0}$ is a Følner sequence. By definition of $F_i$ we have

$$(F_i \backslash D_{n_i}) \subseteq \bigcup_{g\in F_{i-1}} (D_{n_i} g \backslash D_{n_i}) \,.$$

Then by equation (1) we get

$$\frac{|F_i \backslash D_{n_i}|}{D_{n_i}} \leq \sum_{g\in F_{i-1}} \left(\frac{|D_{n_i} g \backslash D_{n_i}|}{|D_{n_i}|}\right)$$
$$\leq |F_{i-1}| \frac{1}{i|F_{i-1}|} = \frac{1}{i}.$$

Since

$$|F_i \cap D_{n_i}| + |D_{n_i} \backslash F_i|) = |Dni| = |F_i| = |F_i \cap D_{n_i}| + |F_i \backslash D_{n_i}|,$$

we obtain

$$\frac{|D_{n_i} \backslash F_i|}{|D_{n_i}|} \leq \frac{1}{i}.$$

Let $g \in G$. Since

$$F_i g \backslash F_i = [(F_i \cap D_{n_i}) g \backslash F_i] \bigcup [(F_i \backslash D_{n_i}) g \backslash F_i]$$

$$\subseteq \left[(F_i \cap D_{n_i})g \backslash F_i\right] \bigcup (F_i \backslash D_{n_i})g$$
$$\subseteq \left[D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)\right] \bigcup (F_i \backslash D_{n_i})g,$$
we have
$$\frac{F_i g \backslash F_i}{|F_i|} \le \frac{\left|D_{n_i}g(F_i \cap D_{n_i})\right|}{\left|D_{n_i}\right|} + \frac{\left|(F_i \backslash D_{n_i})g\right|}{\left|D_{n_i}\right|} \le \frac{\left|D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)\right|}{\left|D_{n_i}\right|} + \tfrac{1}{i}. \tag{3}$$
On the other hand, the relation
$$D_{n_i}g \backslash D_{n_i} = D_{n_i}g \backslash \left[(D_{n_i} \cap F_i) \cup \left(D_{n_i} \backslash F_i\right)\right] = \left[D_{n_i}g \backslash \left(D_{n_i} \cap F_i\right)\right] \backslash \left(D_{n_i} \backslash F_i\right)$$
implies that
$$D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)$$
$$= \left[\left(D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)\right) \cap \left(D_{n_i} \backslash F_i\right)\right] \bigcup \left[\left(D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)\right) \backslash \left(D_{n_i} \backslash F_i\right)\right]$$
$$= \left[\left(D_{n_i}g \backslash \left(F_i \cap D_{n_i}\right)\right) \cap \left(D_{n_i} \backslash F_i\right)\right] \bigcup \left[D_{n_i}g \backslash D_{n_i}\right]$$
$$\subseteq \left(D_{n_i} \backslash F_i\right) \bigcup \left(D_{n_i}g \backslash D_{n_i}\right),$$
which ensures that
$$\frac{\left|D_{n_i}g \backslash (F_i \cap D_{n_i})\right|}{\left|D_{n_i}\right|} \le \frac{\left|D_{n_i} \backslash F_i\right|}{\left|D_{n_i}\right|} + \frac{\left|D_{n_i}g \backslash D_{n_i}\right|}{\left|D_{n_i}\right|}. \tag{4}$$
From equations (3) and (4), we obtain
$$\frac{|F_i g \backslash F_i|}{|F_i|} \le \tfrac{2}{i} + \frac{\left|D_{n_i}g \backslash D_{n_i}\right|}{\left|D_{n_i}\right|},$$

which implies
$$\lim_{i \to \infty} \frac{|F_i g \backslash F_i|}{|F_i|} = 0. \tag{5}$$
In a similar way we deduce that
$$F_i \backslash F_i g \subseteq \left[D_{n_i} \backslash (F_i \cap D_{n_i})g\right] \bigcup \left(F_i \backslash D_{n_i}\right),$$
$$D_{n_i} \backslash D_{n_i}g = \left[D_{n_i} \backslash (D_{n_i} \cap F_i)g\right](D_{n_i} \backslash F_i),$$
and
$$D_{n_i} \backslash (F_i \cap D_{n_i})g \subseteq (D_{n_i} \backslash F_i) \bigcup \left(D_{n_i} \backslash D_{n_i}g\right).$$
Combining the last three equations we get
$$\frac{|F_i \backslash F_i g|}{|F_i|} \le \tfrac{2}{i} + \frac{\left|D_{n_i} \backslash D_{n_i}g\ \right|}{\left|D_{n_i}\right|},$$
which implies
$$\lim_{i \to \infty} \frac{|F_i \backslash F_i g|}{|F_i|} = 0. \tag{6}$$

Equations (5) and (6) imply that $(F_i)_{i \ge 0}$ is Følner.
The following result is a direct consequence of Lemma (6.1.6).
**Lemma (6.1.7)[367]:** Let $G$ be an amenable residually finite group and let $(\Gamma_n)_{n \ge 0}$ be a decreasingmsequence of finite index normal subgroups of $G$ such that $\bigcap_{n \ge 0} \Gamma_n = \{e\}$.

There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and $a$ Følner sequence $(F_i)_{i\geq 0}$ of $G$ such that

**(i)** $\{e\} \subseteq F_i \subseteq F_{i+1}$ and $F_i$ is a fundamental domain of $G/F_{n_i}$ , for every $i \geq 0$.

**(ii)** $G = \bigcup_{i\geq 0} F_i$.

**(iii)** $F_i = \bigcup_{v\in Fj\cap F_{n_i},} vF_i$ for every $j > i \geq 0$.

**Proof:** The existence of the sequence of subgroups of $G$ and the Følner sequence verifying (i), (ii) and (iii) for $j = i + 1$ is direct from Lemma (6.1.6). Using induction, it is straightforward to show (iii) for every $j > i \geq 0$.

$G$ is an amenable, countable, and residually finite group.

Let $\sum$ be a finite alphabet and let $(\sum^G, \sigma, G)$ be the respective full G-shift.

For a finite index subgroup $\Gamma$ of $G, x \in \sum^G$ and $a \in \sum$, we define

$$Per(x,\Gamma,a) = \{g \in G\colon \sigma^{\gamma}(x)(g) = x(\gamma^{-1}g) = a, \forall\gamma \in \Gamma\},$$

and $Per(x,\Gamma) = \bigcup_{a\in\sum} Per(x,\Gamma,a)$.

It is straightforward to show that $x \in \sum^G$ is a Toeplitz sequence if and only if there exists an increasing sequence $(\Gamma_n)_{n\geq 0}$ of finite index subgroups of $G$ such that $G = \bigcup_{n\geq 0} Per(\mathrm{x},\Gamma_n)$ (see [372]).

A period structure of $x \in \sum^G$ is an increasing sequence of finite index subgroups $(\Gamma_n)_{n\geq 0}$ of $G$ such that $G = \bigcup_{n\geq 0} Per(\mathrm{x},\Gamma_n)$ and such that for every $n \geq 0$, $\Gamma_n$ is an essential group of periods: This means that if $g \in G$ is such that $Per(x,\Gamma_n,a) \subseteq Per(\sigma^g(x),\Gamma_n,a)$ for every $a \in \sum$, then $g \in \Gamma_n$.

It is known that every Toeplitz sequence has a period structure (see for example [372]). We construct, thanks the period structure, a KakutaniRokhlin partition and we deduce a characterization of its ordered group.

We suppose that $x_0 \in \sum^G$ is a non-periodic Toeplitz sequence ($\sigma^g(x_0) = x_0$ implies $g = e$) having a period structure $(\Gamma_n)_{n\geq 0}$ such that for every $n \geq 0$,

**(i)** $\Gamma_{n+1}$ is a proper subset of $\Gamma_n$,

**(ii)** $\Gamma_n$ is a normal subgroup of $G$.

Every non-periodic Toeplitz sequence has a period structure verifying (i) [372]. Condition (ii) is satisfied for every Toeplitz sequence whose Toeplitz subshift is an almost 1-1 extension of an odometer (in the general case these systems are almost 1-1 extensions of subodometers. See [372]).

By Lemma (6.1.7) we can assume there exists a Følner sequence $(\mathrm{F}_n)_{n\geq 0}$ of $G$ such that

(F1) $\{e\} \subseteq \mathrm{F}_n \subseteq \mathrm{F}_{n+1}$ and $\mathrm{F}_n$ is a fundamental domain of $G/\Gamma_n$, for every $n \geq 0$.

(F2) $G = \bigcup_{n\geq 0} F_n$.

(F3) $\mathrm{F}_n = \bigcup_{v\in F_n\cap\Gamma_i,} vF_i$, for every $n > i \geq 0$.

We denote by $X$ the closure of the orbit of $x_0$. Thus $(X,\sigma|X,G)$ is a Toeplitz subshift.

**Definition (6.1.8)[367]:** We say that a finite *clopen* partition $\mathcal{P}$ of $X$ is a regular Kakutani- Rokhlin partition ($r - K - R$ partition), if there exists a finite index subgroup $\Gamma$ of $G$ with a fundamental domain $F$ containing $e$ and $a$ clopen $C_k$, such that

$$\mathcal{P} = \left\{\sigma^{u^{-1}}(C_k)\colon u \in F, 1 \leq k \leq N\right\}$$

and

$$\sigma^{\gamma}\left(\bigcup_{k=1}^{N} C_k\right) = \bigcup_{k=1}^{N} C_k \text{ for every } \gamma \in \Gamma.$$

To construct a regular KakutaniRokhlin partition of X, we need the following technical lemma.

**Lemma (6.1.9)[367]:** Let $\mathcal{P}' = \{\sigma^{u^{-1}}(D_k): u \in F, 1 \le k \le N\}$ be $a$ $r$-$K$-$R$ partition of $X$ and $\mathcal{Q}$ any other finite clopen partition of $X$. Then there exists $a$ $r$-$K$-$R$ partition $\mathcal{P} = \{\sigma^{u^{-1}}(C_k) : u \in F, 1 \le k \le M\}$ of $X$ such that

**(i)** $\mathcal{P}$ is finer than $\mathcal{P}$ and $\mathcal{Q}$,

**(ii)** $\bigcup_{k=1}^{M} C_k = \bigcup_{k=1}^{N} D_k.$

**Proof:** Let $F = \{u_0, u_1, \cdots, u|F| - 1\}, with\, u_0 = e.$

We refine every set $D_k$ with respect to the partition $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \cdots, D_{1,l_1}\ ; \cdots ; D_{N,1}, \cdots, D_{N,l_N},$$

such that each of these sets is in an atom of $\mathcal{Q}$ and $D_k = \bigcup_{j=1}^{l_k} D_{k,j}$ for every $1 \le k \le N$ .Thus $\mathcal{P}_0 = \{\sigma^{u^{-1}}(D_k): u \in F, 1 \le j \le l_k, 1 \le k \le N\}$ is $a$ $r$-$K$-$R$ partition of $X$. For simplicity we write

$$\mathcal{P}_0 = \{\sigma^{u^{-1}}\left(D_k^{(0)}\right): u \in F, 1 \le k \le N_0\}.$$

We have that $\mathcal{P}_0$ verifies (2) and every $D_k^{(0)}$is contained in atoms of $\mathcal{P}'$ and $\mathcal{Q}$. Let $0 \le n < |F| - 1$. Suppose that we have defined $a$ $r$-$K$-$R$ partition of $X$

$$\mathcal{P}_n = \left\{\sigma^{u^{-1}}\left(D_k^{(n)}\right): u \in F, 1 \le k \le N_n\right\},$$

such that $\mathcal{P}_n$ verifies (2) and such that for every $0 \le j \le n$ and $1 \le k \le N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}\left(D_k^{(n)}\right) \subseteq A, B.$$

Now we refine every set $\sigma^{u_{n+1}^{-1}}\left(D_k^{(n)}\right)$ with respect to $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \cdots, D_{1,s_1}\ ; \cdots ; D_{N_n,1}, \cdots, D_{N_n,s_{N_n}},$$

such that each of these sets is in an atom of $\mathcal{Q}$ and $\sigma^{u_{n+1}^{-1}}\left(D_k^{(n)}\right) = \bigcup_{j=1}^{s_k} D_{k,j}$ , for every $1 \le k \le N_n$. For every $1 \le k \le N_n$. and $1 \le\ j \le s_k$, let $C_{k,j} = \sigma^{u_{n+1}}(D_{k,j}) \subseteq D_k^{(n)}$ . We have that

$$\mathcal{P}_{n+1} = \left\{\sigma^{u^{-1}}\left(C_{k,j}\right): u \in F, 1 \le j \le s_k, 1 \le k \le N_n\right\}$$

is a $r$-$K$-$R$ partition of $X$ verifying (2) and such that for every $0 \le i \le n+1, 1 \le j \le s_k$ and $1 \le k \le N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}\left(C_{k,j}\right) \subseteq A, B.$$

At the step $n = |F| - 1$ we get $\mathcal{P} = \mathcal{P}_{|F|-1}$ verifying (1) and (2).

**Proposition (6.1.10)[367]:** There exists a sequence $\left(\mathcal{P}_n = \left\{\sigma^{u^{-1}}\left(C_{n,k}\right): u \in F_n, 1 \le k \le k_n, 1 \le k \le k_n\right\}\right)_{n \ge 0}$ of $r$-$K$-$R$ partitions of $X$ such that for every $n\ \ge\ 0,$

**(i)** $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n,$

**(ii)** $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} = C_{n,k},$

**(iii)** $\bigcap_{n\geq 1} C_n = \{x_0\}$,
**(iv)** The sequence $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of $X$.

**Proof:** For every $n \geq 0$, let define

$$C_n = \{x \in X: Per(x, \Gamma_n, a) = Per(x_0, \Gamma_n, a) \forall a \in \sum \}.$$

From [372] we get

$$C_n = \overline{\{\sigma^{\gamma}(x_0): \gamma \in \Gamma_n\}},$$

and that $\mathcal{P}'_n = \{\sigma^{u^{-1}}(C_n): u \in F_n\}$ is a clopen partition of $X$ such that $\sigma^{\gamma}(C_n) = C_n$ for every $\gamma \in \Gamma_n$. Thus $\mathcal{P}'_n$ is a $r$-$K$-$R$ partition of $X$. Furthermore, the sequence $(\mathcal{P}'_n)_{n\geq 0}$ verifies (1), (2) and (3).

For every $n \geq 0$, let $\mathcal{Q}_n = \{[B] \cap X: B \in \sum^{F_n}, [B] \cap X \neq \emptyset\}$. This is a finite clopen partition of $X$ and $(\mathcal{Q}_n)_{n\geq 0}$ spans the topology of $X$.

We define $\mathcal{P}_0 = \{\sigma^{u^{-1}}(C_{0,k}) : u \in F_0, 1 \leq k \leq K_0\})$ the $r$-$K$-$R$ partition finer than $\mathcal{P}'_0$ and $\mathcal{Q}_0$ given by Lemma (6.1.9). Now we take $\mathcal{P}''_n$ the $r$-$K$-$R$ partition finer that $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$ given by Lemma (6.1.9), and we define

$$\mathcal{P}_n = \left\{\sigma^{u^{-1}}\left(C_{n,k}\right): u \in F_n, 1 \leq k \leq K_n\right\},$$

the $r$-$K$-$R$ partition finer than $\mathcal{P}' = \mathcal{P}'_n$ *and* $\mathcal{Q} = \mathcal{P}''_n$ given by Lemma (6.1.9). Thus $\mathcal{P}_n$ is finer

Sthan $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$. This implies that the sequence $(\mathcal{P}_n)_{n\geq 0}$ verifies (1) and (4). Since $\bigcup_{k=1}^{k_n} C_{n,k} = C_n$, we deduce that $(\mathcal{P}_n)_{n\geq 0}$verifies (2) and (3).

**Proposition (6.1.11)[367]:** Let $\left(\mathcal{P}_n = \left\{\sigma^{u^{-1}}\left(C_{n,k}\right): u \in F_n, 1 \leq k \leq K_n\right\}\right)_{n\geq 0}$ be a nested sequence of $r$-$K$-$R$ partitions of $X$ with an associated sequence of incidence matrices $(M_n)_{n\geq 0}$. Then

**(i)** $(X, \sigma|_X, G)$ is an almost 1-1 extension of the odometer $O = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$,

**(ii)** there is an affine homeomorphism between the set of invariant probability measures of $(X, \sigma|_X, G)$ and the inverse limit $\lim_{\leftarrow n}(\Delta(k_n, |F_n|), M_n)$,

**(iii)** the ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$, where $(H, H^+)$ is given by

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

where $M = |F_0|(1,\cdots,1)$ and $u = [M^T, 0]$.

**Proof:** **(i)** For every $x \in X$ and $n \geq 0$, let $v_n(x) \in F_n$ be such that $x \in \sigma^{v_n(x)^{-1}}(C_n)$. The map $\pi: X \to O$ given by $\pi(x) = (v_n(x)^{-1}\Gamma_n)_{n\geq 1}$ is well defined, is a factor map and verifies $\pi^{-1}(\pi(x_0)) = \{x_0\}$. This shows that $(X, \sigma|_X, G)$ is an almost 1-1 extension of $O$.

**(ii)** It is clear that for any invariant probability measure $\mu$ of $(X, \sigma|_X, G)$, the sequence $(\mu_n)_{n\geq 0}$, with $\mu_n = (\mu(C_{n,k}) : 1 \leq k \leq k_n)$, is an element of the inverse limit $\lim_{\leftarrow n}(\Delta(k_n, |F_n|), M_n)$. Conversely, any element $(\mu_{n,k} : 1 \leq k \leq k_n)_{m\geq 0}$ of such inverse limit, defines a probability measure $\mu$ on the $\sigma$-algebra generated by $(\mathcal{P}_n)_{n\geq 0}$, which is equal to the Borel $\sigma$-algebra of $X$ because $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of $X$ and is countable. Since the sequence $(F_n)$ is Følner, it is standard to check that the measure $\mu$ is invariant by the $G$-action.

The function $\mu \mapsto (\mu_n)_{n\geq 0}$ is thus an affine bijection between $\mathcal{M}(X, \sigma|_X, G)$ and the inverse limit $\lim_{\leftarrow n}(\Delta(k_n, |F_n|), M_n)$. Observe that this function is a homeomorphism with respect to the weak topology in $\mathcal{M}(X,\sigma|_X,G)$ and the product topology in the inverse limit.

**(iii)** We denote by $[k,-1]$ the class of the element $(k,-1)\in\mathbb{Z}\times\{-1\}$ in $H$.

Let $\phi: H\to D_m(X,\sigma|_X,G)$ be the function given by $\phi([v,n])=\sum_{k=1}^{k_n} v_i\,[1c_{n,k}]$, for every $v=(v_1,\cdots,v_{kn})\in\mathbb{Z}^{k_n}$ and $n\geq 0$, and $\phi([k,-1])=k1_X$ for every $k\in\mathbb{Z}$. It is easy to check that $\phi$ is a well defined homomorphism of groups that verifies $\phi(H^+)\subseteq D_m(X,\sigma|_X,G)^+$. Since $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of $X$, every function $f\in C(X,\mathbb{Z})$ isconstant on every atom of $\mathcal{P}_n$, for some $n\geq 0$. This implies that $\phi$ is surjective. Lemma (6.1.2) and (2) of Proposition (6.1.11), imply that $Ker(\phi)=\inf(H)$. Finally, $\phi$ induces a isomorphism $\hat{\phi}: H/\inf(H)\to D_m(X,\sigma|_X,G)$ such that $\hat{\phi}((H/\inf(H))^+)=D_m(X,\sigma|_X,G)^+$. Since $[1,-1]=[M^T,0]$, we get $\phi([M^T,0])=[1_X]$.

We say that a sequence of positive integer matrices $(M_n)_{n\geq 0}$ is managed by the increasing sequence of positive integers $(p_n)_{n\geq 0}$, if for every $n\geq 0$ the integer $p_n$ divides $p_{n+1}$, and if the matrix $M_n$ verifies the following properties:

**(i)** $M_n$ has $k_n\geq 2$ rows and $k_{n+1}\geq 2$ columns;

**(ii)** $\sum_{k=1}^{k_n} M_n(i,k)=\frac{p_{n+1}}{p_n}$, for every $1\leq k\leq k_{n+1}$.

If $(M_n)_{n\geq 0}$ is a sequence of matrices managed by $(p_n)_{n\geq 0}$, then for each $n\geq 0$, $M_n(\Delta(k_{n+1},p_{n+1}))\subseteq\Delta(k_n,p_n)$.

Observe that the sequences of incidence matrices associated to the nested sequences of $r$-$K$-$R$ partitions are managed by $(|F_n|)_{n\geq 0.}$

We construct Toeplitz subshifts with nested sequences of $r$-$K$-$R$ partitions whose sequences of incidence matrices are managed.

In the rest $G$ is an amenable and residually finite group. Let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroup of $G$ such that $\bigcap_{n\geq 0}\Gamma_n=\{e\}$, and let $(\mathrm{F}_n)_{n\geq 0}$ be a Følner sequence of $G$ such that

(F1) $\{e\}\subseteq\mathrm{F}_n\subseteq\mathrm{F}_{n+1}$ and $\mathrm{F}_n$ is a fundamental domain of $G/\Gamma_n$, for every $n\geq 0$.

(F2) $G=\bigcup_{n\geq 0}\mathrm{F}_n$ .

(F3) $\mathrm{F}_n=\bigcup_{v\in\mathrm{F}_n\cap\Gamma_i} v\mathrm{F}_i$, for every $n>i\geq 0$.

Lemma (6.1.7) ensures the existence of a Følner sequence verifying conditions (F1), (F2) and(F3).

For every $n\geq 0$, we call $\mathrm{R}_n$ the set $\mathrm{F}_n\cdot F_n^{-1}\cup F_n^{-1}\cdot\mathrm{F}_n$. This will enable us to define a "border" of each domain $\mathrm{F}_{n+1}$.

Let $\sum$ be a finite alphabet. For every $n\geq 0$, let $\mathrm{k}_n\geq 3$ be an integer. We say that the sequence of sets $(\{B_{n,1},\cdots,B_{n,k_n}\})_{n\geq 0}$ where for any $n\geq 0$, $\{B_{n,1},\cdots,B_{n,k_n}\}\subseteq\sum^{\mathrm{F}_n}$ is a collection of different functions, verifies conditions $(C1)-(C4)$ if it verifies the following four conditions for any $n\geq 0$:

(C1) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n}\in\{B_{n,i}: 1\leq i\leq k_n\}$, for every $\gamma\in\mathrm{F}_{n+1}\cap\Gamma_n$, $1\leq k\leq k_{n+1}$.

(C2) $B_{n+1,k})|_{F_n}=B_{n,1}$, for every $1\leq k\leq k_{n+1}$.

(C3) For any $g\in\mathrm{F}_n$ such that for some $1\leq k,k'\leq k_n B_{n,k}(gv)=B_{n,k'}(v)$ for all $v\in F_n\cap g^{-1}F_n$, then $g=e$.

(C4) $\sigma^{-1}(B_{n+1,k})|_{F_n}=B_{n,k_n}$ for every $\gamma\in(F_{n+1}\cap\Gamma_n)\cap\{F_{n+1}\setminus F_{n+1}g^{-1}\}$, forsome $g\in R_n$.

**Example (6.1.12)[367]:** To illustrate these conditions, let us consider the case $G=\mathbb{Z},\sum=\{1,2,3,4\}$ and $\Gamma_n=3^{2(n+1)}\mathbb{Z}$ for every $n\geq 0$. The set

$$F_n = \left\{-\left(\frac{3^{2(n+1)}-1}{2}\right), -\left(\frac{3^{2(n+1)}-1}{2}\right)+1, \dots, \left(\frac{3^{2(n+1)}-1}{2}\right)\right\}$$

is a fundamental domain of $\mathbb{Z}/\Gamma_n$. Furthermore we have

$$F_n = \bigcup_{v\in\{k3^{2n}:-4\le k\le 4\}} (F_{n-1}+v),$$

for every $n \geq 1$. This shows that sequence $(F_n)_{n\geq 0}$ satisfies (F1), (F2) and (F3). Now let us consider the case where $k_n = 4$ for every $n \geq 0$. We define $B_{0,k}(j) = k$ for Every $j \in F_0$ and $1 \leq k \leq 4$, and for $n \geq 1$,

$$B_{n,k}|_{F_{n-1}} = B_{n-1,1}, B_{n,k}|_{F_{n-1+v}} = B_{n-1,4}\ for\ v \in \{-l\cdot 3^{2n}, l\cdot 3^{2n}: l = 3, 4\}.$$

Thus they verify the conditions (C1) and (C4). We fill the rest of the $B_{n,k}|_{F_{n-1+v}}$ with $B_{n-1,3}$ and $B_{n-1,2}$ in order that $B_{n,1},\cdots,B_{n,4}$ are different. They satisfy conditions (C2) and (C4). The limit in $\sum^{\mathbb{Z}}$ of the functions $B_{n,1}$ is a $\mathbb{Z}$-Toeplitz sequence $x$. If $X$ denotes the closure of the orbit of $x$, then we prove in the next lemma (in a more general setting) that

$$(\mathcal{P}_n = \{\sigma^j([B_{n,k}]\cap X): j\in F_n, 1\leq k\leq 4\})_{n\geq 0}$$

is a sequence of nested Kakutani-Rokhlin partitions of the subshift $X$.

In the next lemma, we show that conditions (C1) and (C2) are sufficient to construct a Toeplitz sequence. The technical conditions (C3) (aperiodicity) and (C4) (also known as "forcing the border") will allow to construct a nested sequence of $r$-$K$-$R$ partitions of $X$.

**Lemma (6.1.13)[367]:** Let $(\{B_{n,1},\cdots,B_{n,k_n}\})_{n\geq 0}$ be a sequence that verifies conditions (C1) (C4).Then:

**(i)** The set $\bigcap_{n\geq 0}[B_{n,1}]$ contains only one element $x_0$ which is $a$ Toeplitz sequence.

**(ii)** Let $X$ be the orbit closure of x0 with respect to the shift action. For every $n \geq 0$, $let$

$$\mathcal{P}_n = \left\{\sigma^{u^{-1}}\left([B_{n,k}]\cap X\right):, 1\leq k\leq k_n, u\in F_n\right\}.$$

Then $(\mathcal{P}_n)_{n\geq 0}$ is a sequence of nested $r$-$K$-$R$ partitions of X.

Let $(M_n)_{n\geq 0}$be the sequence of incidence matrices of $(\mathcal{P}_n)_{n\geq 0}$. Thus we have

**(iii)** The Toeplitz subshift $(X, \sigma|_X, G)$ is an almost 1-1 extension of the odometer $O = \varprojlim_n(G/\Gamma_n, \pi_n)$.

**(v)** There is an affine homeomorphism between the set of invariant probability measures of $(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n(\Delta(k_n, |F_n|), M_n)$.

**(iv)** The ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$, where $(H, H^+)$ is given by

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

with $M = |F_0|(1,\cdots,1)$ and $u = [M^T, 0]$..

**Proof:** Condition (C2) implies that $\bigcap_{n\geq 0}[B_{n,1}]$ is non empty, and since $G = \bigcup_{n\geq 0} F_n$, there is only one element $x_0$ in this intersection. Let $X$ be the orbit closure of $x_0$. For every $n \geq 0$ and $1 \leq k \leq k_n$, we denote $C_{n,k} = [B_{n,k}]\cap X$.

Claim: For every $m > n \geq 0, 1 \leq k \leq k_m$ and $\gamma \in F_m \cap \Gamma_n$,

$$\sigma^{\gamma^{-1}}(B_{m,k})|_{F_n} \in \{B_{n,i}: 1\leq i\leq k_n\}. \tag{7}$$

Condition (C1) implies that (7) holds when $n = m - 1$. We will show the claim by induction on $n$.
Suppose that for every $1 \leq k \leq k_m$ and $\gamma \in \cap \Gamma_{n+1}$,

$$\sigma^{\gamma^{-1}}\left(B_{m,k}\right)|_{F_{n+1}} \in \left\{B_{n+1,i}: 1 \leq i \leq k_{n+1}\right\}.$$

Let $g \in \Gamma_n \cap F_m$. Condition (F3) implies there exist $\upsilon \in \Gamma_{n+1} \cap F_m$ and $u \in F_{n+1}$ such that $g = \upsilon u$. Thus we get

$$\sigma^{g^{-1}}\left(B_{m,k}\right)|_{F_{n+1}} = \sigma^{g^{-1}\upsilon^{-1}}\left(B_{m,k}\right) = \sigma^{\upsilon^{-1}}\left(B_{m,k}\right)|_{uF_n}.$$

Since $u \in \Gamma_n \cap \mathrm{F}_{n+1}$, condition (F3) implies that $uF_n \subseteq \mathrm{F}_{n+1}$. Then by hypothesis, there exists $1 \leq l \leq k_{n+1}$ such that

$$\sigma^{\upsilon^{-1}}\left(B_{m,k}\right)|_{uF_n} = B_{n+1,l}|_{uF_n},$$

which is equal to some $B_{n,s}$, by (C1). This shows the claim.
From (7) we deduce that $\sigma^{-1}(x_0)|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}$, for every $\gamma \in \Gamma_n$. Thus if $g$ is any element in $G$, and $u \in F_n$ and $\gamma \in \Gamma_n$ are such that $g = \gamma u$, then $\sigma^{g^{-1}}(x_0) = \sigma^{u^{-1}}(\sigma^{\gamma^{-1}}(x_0)) \in \sigma^{u^{-1}}(C_{n,k})$, for some $1 \leq k \leq k_n$. It follows that

$$\mathcal{P}_n = \left\{\sigma^{u^{-1}}(C_{n,k}):\ 1 \leq k \leq k_n, u \in F_n\right\}$$

is a clopen covering of $X$.
From condition (C2) and (7) we get that $\sigma^{\gamma^{-1}}(x_0)|_{F_{n-1}} = B_{n-1,1}$ for any $\gamma \in \Gamma_n$, which implies that $F_{n-1} \subseteq Per(x_0, \Gamma_n)$. This shows that $x_0$ is Toeplitz.
Now we will show that $\mathcal{P}_n$ is a partition. Suppose that $1 \leq k, l \leq k_n$ and $u \in F_n$ are such that $\sigma^{u^{-1}}(C_{n,k}) \cap C_{n,l} \neq \emptyset$. Then there exist $x \in C_{n,k}$ and $y \in C_{n,l}$ such that $\sigma^{u^{-1}}(x) = y$. From this we have $x(uv) = y(v)$ for every $v \in G$. In particular, $x(uv) = y(v)$ for every $v \in F_n \cap u^{-1}F_n$, which implies $B_{n,k}(uv) = B_{n,l}(v)$ for every $v \in F_n \cap u^{-1}F_n$. From condition (C3) we get $u = e$ and $k = l$. This ensures that the set of return times of $x_0$ to $\bigcup_{k=1}^{k_n} C_{n,k}$ , i.e. the set
$\{g \in G : \sigma^{g^{-1}}(x_0) \in \bigcup_{k=1}^{k_n} C_{n,k}\}$, is $\Gamma_n$. From this it follows that $\mathcal{P}_n$ is a r-K-R partition. From (C1) we have that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and that $C_{n+1} \subseteq \bigcup_{k=1}^{k_n} C_{n,k} = C_n$. By the definition of $x_0$ we have that $\{x_0\} = \bigcap_{n\geq 0} C_n$ .
Now we will show that $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of $X$. Since every $\mathcal{P}_n$ is a partition, for every $n \geq 0$ and every $x \in X$ there are unique $\upsilon n(x) \in F_n$ and $1 \leq k_n(x) \leq k_n$such that

$$x \in \sigma^{\upsilon_n(x^{-1})}(x_0) \in \left(C_{n,k_n(x)}\right).$$

The collection $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of $X$ if and only if $(\upsilon_n(x))_{n\geq 0} = (\upsilon_n(y))_{n\geq 0}$ and $(k_n(x))_{n\geq 0} = (k_n(y))_{n\geq 0}$ imply $x = y$.
Let x, $y \in X$ be two sequences such that $\upsilon n(x) = \upsilon n(y) = \upsilon n$ and $k_n(x) = k_n(y)$ for every $n \geq 0$. Let $g \in G$ be such that $x(g) \neq y(g)$.
We have then for any $n \geq 0$

$$\sigma^{\upsilon_n}(x)|_{F_n} = \sigma^{\upsilon_n}(y)|_{F_n} \in \left\{B_{n,i}: 1 \leq i \leq k_n\right\},$$

and then

$$x|_{\upsilon_n^{-1}F_n} = y|_{\upsilon_n^{-1}F_n}.$$

Thus by definition, we get $g \notin \upsilon_n^{-1} F_n$ for any n. We can take n sufficiently large in order that $g \in F_{n-1.}$
Let $\gamma \in \Gamma_n$ and $u \in F_n$ such that $\upsilon_n(x)g = \gamma u$. Observe that $ug^{-1} \notin F_n$. Indeed, if $ug^{-1} \in F_n$, then the relation $\upsilon_n(x) = \gamma u g^{-1}$ implies $\gamma = e$, but in that case we get

$v_n(x)g = u \in F_n$ which is not possible by hypothesis. By the condition (C1), there exists an index $1 \le i \le k_n$ such that $\sigma^{\gamma^{-1}}(\sigma^{v_n}(x))|_{F_n} = B_{n,i}$ and then

$$x(g) = \sigma^{\gamma^{-1}}\sigma^{v_n}(x)(\gamma^{-1}v_n g) = B_{n,i}(u).$$

Let $\gamma' \in \Gamma_{n-1} \cap F_n$ and $u' \in \mathrm{F}_{n-1}$ such that $u = \gamma' u'$. Since $\gamma' u' g^{-1} = ug^{-1} \notin F_n$, we get $\gamma' \in F_n \setminus F_n g u'^{-1}$. This implies that $\gamma' \in F_n \setminus F_n w$, for $w = gu'^{-1} \in R_{n-1}$ and $B_{n,i}(u) = B_{n-1,\mathrm{k}_{n-1}}(u')$. by the condition (C4). Thus $x(g) = B_{n-1,\mathrm{k}_{n-1}}(u')$. The same argument implies that $y(g) = B_{n-1,\mathrm{k}_{n-1}}(u') = x(g)$ and we obtain a contradiction. This shows that $(\mathcal{P}_n)_{n\ge0}$ is a sequence of nested r-K-R partitions of $X$.

The point (3), (4) and (5) follows from Propositions (6.1.11).

The next result shows that, up to telescope a managed sequence of matrices, it is possible to obtain a managed sequence of matrices with sufficiently large coefficient to satisfy the conditions of Lemma (6.1.13).

**Lemma (6.1.14)[367]:** Let $(M_n)_{n\ge0}$ be a sequence of matrices managed by $(|F_n|)_{n\ge0}$. Let $k_n$ be the number of rows of $M_n$, for every $n \ge 0$.

Then there exists an increasing sequence $(n_i)_{i\ge0} \subseteq \mathbb{Z}^+$ such that for every $i \ge 0$ and every $1 \le k \le k_{n_{i+1}}$,

**(i)** $R_{n_i} \subseteq F_{n_{i+1}}$,

**(ii)** For every $1 \le l \le k_{n_i}$,

$$M_{n_i}M_{n_{i+1}}\cdots M_{n_{i+1}-1}(l,k) > 1 + \left|\bigcup_{g\in R_{n_i}} F_{n_{i+1}} \setminus F_{n_{i+1}}g^{-1}\right|$$

If in addition there exists a constant $K > 0$ such that $k_{n+1} \le K\frac{|F_{n+1}|}{|F_n|}$ for every $n \ge 0$, then the sequence $(n_i)_{i\ge0}$ can be chosen in order that

**(iii)** $k_{n_{i+1}} < M_{n_i}\cdots M_{n_{i+1}-1}(l,k)$, for every $1 \le i \le k_{n_i}$.

**Proof:** We define $n_0 = 0$. Let $i \ge 0$ and suppose that we have defined $n_j$ for every $0 \le j \le i$. Let $m_0 > n_i$ be such that for every $m \ge m_0$,

$$R_{n_i} \subseteq F_m.$$

Let $0 < \varepsilon < 1$ be such that $\varepsilon|R_{n_i}| < 1$. Since $(F_n)_{n\ge0}$ is a Følner sequence, there exists $m_1 > m_0$ such that for every $m \ge m_1$,

$$\frac{|F_m \setminus F_m g^{-1}|}{|F_m|} < \frac{\varepsilon}{|F_{n_{i+1}}|}, \text{for every } g \in R_{n_i}. \tag{8}$$

Since $\varepsilon|R_{n_i}| < 1$, there exists $m_2 > m_1$ such that for every $m \ge m_2$,

$$1 - \frac{|F_{n_{i+1}}|}{|F_m|} > \varepsilon|R_{n_i}|.$$

Then

$$\frac{|F_m|}{|F_{n_{i+1}}|} - 1 > \varepsilon|R_{n_i}|\frac{|F_m|}{|F_{n_{i+1}}|}$$

Since the matrices $M_n$ are positive, using induction on m and condition (2) for managed sequences, we get

$$M_{n_i}\cdots M_{m-1}(l,j) \ge \frac{|F_m|}{|F_{n_{i+1}}|}, \text{for every } 1 \le l \le k_n, 1 \le j \le k_m$$

Combining the last two equations we get

$$M_{n_i}\cdots M_{m-1}(l,j)-1>\varepsilon|R_{n_i}|\frac{|F_m|}{|F_{n_{i+1}}|},$$

and from equation (8), we obtain

$$M_{n_i}\cdots M_{m-1}(l,j)-1>|F_m\backslash F_m g^{-1}||R_{n_i}|,\text{for every } g\in R_{n_i},$$

which finally implies that

$$M_{n_i}\cdots M_{m-1}(l,j)>\left|\cup_{g\in R_{n_i}}F_m\backslash F_m g^{-1}\right|+1,\text{for every } 1\le l\le k_n, 1\le j\le k_m.$$

Now, suppose there exists $K>0$ such that $k_{m+1}\le\frac{|F_{m+1}|}{|F_m|}$ for every $m\ge 0$. The property (2) for managed sequences of matrices implies

$$M_{n_i}\cdots M_m(l,j)\ge\frac{|F_{m+1}|}{|F_{n_{i+1}}|}\text{ for every } m>n_i.$$

Let $m_3>m_2$ be such that $K<\frac{|F_m|}{|F_{n_{i+1}}|}$ for every $m\ge m_3$. Then for every $m\ge m_3$ we have

$$k_{m+1}\le K\frac{|F_{m+1}|}{|F_{n_i}|}\le M_{n_i}\cdots M_m(l,j)\text{ for every } 1\le l\le k_n \text{ and } 1\le j\le k_{m+1}.$$

By taking $n_{i+1}\ge m_3$ we get the desired subsequence $(n_i)_{i\ge0}\subseteq\mathbb{Z}^+$ .The following proposition shows that given a managed sequence, there exists a sequence of decorations verifying conditions (C1)-(C4). The aperiodicity condition (C3) is obtained by decorating the center of $F_n$ in a unique way with respect to other places in $F_n$. $A$ restriction on the number of columns of the matrices gives enough choices of coloring to ensure conditions (C3) and (C4).

**Proposition (6.1.15)[367]:** Let $(M_n)_{n\ge0}$ be $a$ sequence of matrices which is managed by $(|F_n|)_{n\ge0}$.For every $n\ge0$, we denote by $k_n$ the number of rows of $M_n$. Suppose in addition there exists $K>0$ such that $k_{n+1}\le K\frac{|F_{n+1}|}{|F_n|}$ ,for every $n\ge0$. Then there exists a Toeplitz subshift $(X,\sigma|_X,G)$ verifying the following three conditions:

**(i)** The set of invariant probability measures of $(X,\sigma|_X,G)$ is affine homeomorphic to $\varprojlim_n(\triangle(k_n,|F_n|),M_n)$.

**(ii)** The ordered group $\mathcal{G}(X,\sigma|_X,G)$ is isomorphic to $(H/\inf(H),(H/\inf(H))^+,u+\inf(H))$, where $(H,H^+)$ is given by

$$\mathbb{Z}\xrightarrow{M^T}\mathbb{Z}^{k_0}\xrightarrow{M_0^T}\mathbb{Z}^{k_1}\xrightarrow{M_1^T}\mathbb{Z}^{k_2}\xrightarrow{M_2^T}\cdots,$$

with $M=|F_0|(1,\cdots,1)$ and $u=[M^T,0]$.

**(iii)** $(X,\sigma|_X,G)$ is an almost 1-1 extension of the odometer $O=\varprojlim_n(G/\Gamma_n,\pi_n)$.

**Proof:** Let $(n_i)_{i\ge0}\subseteq\mathbb{Z}^+$ be a sequence as in Lemma (6.1.14). Since $(M_n)_{n\ge0}$ and the sequence $(M_{n_i}\cdots M_{n_{i+1}-1})_{i\ge0}$ define the same inverse and direct limits, without loss of generality we can assume that for every $n\ge0$ we have:

$$R_n\subseteq F_{n+1},$$

$$M_n(i,k)>1+\left|\cup_{g\in R_n}F_{n+1}\backslash F_{n+1}g^{-1}\right|\text{ for every } 1\le i\le k_n, 1\le k\le k_{n+1},$$

and

$$k_{n+1}<\min\{M_n(i,j):1\le i\le k_n, 1\le j\le k_{n+1}\}.$$

Let $\widetilde{M}$ be the $1\times(k_0+1)$-dimensional matrix given by

$$\widetilde{M}(\cdot\,,1) = \widetilde{M}(\cdot\,,2) = M(\cdot\,,1),$$
and $\widetilde{M}(\cdot,k+1) = M(\cdot,k)$ for every $2 \le k \le k_0$. For every $n \ge 0$, consider the $(k_n + 1) \times (k_{n+1}+1)$-dimensional matrix given by

$$\widetilde{M}_n(\cdot\,,1) = \widetilde{M}_n(\cdot\,,2) = \begin{bmatrix} 1 \\ M_n(1,1)-1 \\ M_n(2,1) \\ \vdots \\ M_n(k_n,1) \end{bmatrix}$$

and

$$\widetilde{M}_n(\cdot,k+1) = \begin{bmatrix} 1 \\ M_n(1,k)-1 \\ M_n(2,k) \\ \vdots \\ M_n(k_n,k) \end{bmatrix} \text{ for every } 2 \le k \le k_{n+1}.$$

Lemma (6.1.3) implies that the dimension groups with unit given by

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

and

$$\mathbb{Z} \xrightarrow{\widetilde{M}^T} \mathbb{Z}^{k_0+1} \xrightarrow{\widetilde{M}_0^T} \mathbb{Z}^{k_1+1} \xrightarrow{\widetilde{M}_1^T} \mathbb{Z}^{k_2+1} \xrightarrow{\widetilde{M}_2^T} \cdots,$$

are isomorphic.

Thus from Lemma (6.1.2) we get that $\varprojlim_{n}(\Delta(k_n,|F_n|),M_n)$ and $\varprojlim_{n}(\Delta(k_n+1,|F_n|),\widetilde{M}_n)$ are affine homeomorphic. Observe that $(\widetilde{M}_n)_{\ge 0}$ is managed by $(|F_n|)_{n\ge 0}$ and verifies for every $n \ge 0$:

$\widetilde{M}_n(i,k) \ge 1 + \left|\bigcup_{g\in R_n} F_{n+1}\backslash F_{n+1}g^{-1}\right|$ for every $2 \le i \le k_n + 1$, $1 \le k \le k_{n+1} + 1$,

and

$$3 \le k_{n+1} + 1 \le \{\min\{M_n(i,j): 2 \le i \le k_n+1, 1 \le j \le k_{n+1}+1\}\}.$$

Thus, by Lemma (6.1.13), to prove the proposition it is enough to find a Toeplitz subshift having a sequence of r-K-R-partitions whose sequence of incidence matrices is $(\widetilde{M}_n)_{\ge 0}$. For every $n \ge 0$, we call $l_n$ and $l_{n+1}$ the number of rows and columns of $M_n$ respectively. For every $n \ge 0$, we will construct a collection of functions $B_{n,1},\cdots,B_n,l_n \in \sum^{F_n}$ as in Lemma (6.1.13), where $\sum = \{1,\cdots,l_0\}$.

For every $1 \le k \le l_0$ we define $B_{0,k} \in \sum^{F_n}$ by $B_{0,k}(g) = k$, for every $g \in F_0$. Observe that the collection $\{B_{0,1},\cdots,B_{0,l_0}\}$ verifies condition (C3).

Let $n \ge 0$. Suppose that we have defined $B_{n,1},\cdots,B_{n,l_n} \in \sum^{F_n}$ verifying condition (C3). For $1 \le k \le l_{n+1}$, we define

$$B_{n+1}|_{F_n} = B_{n,1},$$

and

$$\sigma^{s^{-1}} B_{n+1,k}\big|_{F_n} = B_{n,l_n} \text{ for every } s \in \bigcup_{g\in R_n} F_{n+1}\backslash F_{n+1g^{-1}} \cap \Gamma_n\ .$$

We fill the rest of the coordinates $\upsilon \in F_{n+1} \cap \Gamma_n$ in order that $\sigma^{\upsilon^{-1}}(B_{n+1,k})\big|_{F_n} \in \{B_{n,1},\cdots,B_{n,l_n}\}$ and such that

$$\left|\left\{\upsilon \in F_{n+1} \cap \Gamma_n : \sigma^{\upsilon^{-1}}(B_{n+1,k})\big|_{F_n} = B_{n,i}\right\}\right| = \widetilde{M}_n(i,k), \text{for every } 2 \le i \le l_n\ .$$

Since $\widetilde{M}_n(1,k)=1$, if $\sigma^{v^{-1}}(B_{n+1,k})\big|_{F_n}=B_{n,1}$ then $v=e$.
Notice that the number of $B\in\sum^{F_{n+1}}$ that we could choose to be equal to $B_{n+1,k}$ is at least $\widetilde{M}_n(2,k)+1$, because there are at least $\widetilde{M}_n(2,k)+1$ free coordinates to be filled with $\widetilde{M}_n(2,k)$ copies of $B_{n,2}$ and one copy of $B_{n,l_n}$. Since $\widetilde{M}_n(2,k)+1\geq l_{n+1}$, the number of columns of $\widetilde{M}_n$ which are equal to $\widetilde{M}_n(\cdot,k)$ does not exceed the number of possible choices of functions in $\sum^{F_{n+1}}$ in order that $B_{n+1,1},\cdots,B_{n+1,l_{n+1}}$ are pairwise different. By construction, every function $B_{n+1,k}$ verifies (C1), (C2) and (C4). Let us assume there are $g\in F_{n+1}$ and $1\leq k,k'\leq k_{n+1}$ such that $B_{n+1,k}(gv)=B_{n+1,k'}(v)$ for any v where it is defined, then by the induction hypothesis, $g\in\Gamma_n$. This implies $\sigma^{g^{-1}}(B_{n+1,k})\big|_{F_n}=B_{n+1,k'}\big|_{F_n}=B_{n+1}$ and then $g=e$. This shows that the collection $B_{n+1,1},\cdots,B_{n+1,l_{n+1}}$ verifies (C3). We conclude applying Lemma (6.1.13).
For positive integers $n_1,\cdots,n_k$, we denote by $(n_1,\cdots,n_k)!$ the corresponding multinomial coefficient. That is,

$$(n_1,\cdots,n_k)!=\frac{(n_1+\cdots+n_k)!}{n_1!\ \cdots\ n_k!}$$

A compact, convex, and metrizable subset $K$ of a locally convex real vector space is said to be a (metrizable) Choquet simplex, if for each $v\in K$ there is a unique probability measure $\mu$ supported on the set of extreme points of $K$ such that $\int x\,d\mu(x)=v$.

We show that any metrizable Choquet simplex is affine homeomorphic to the inverse limit defined by a managed sequence of matrices satisfying the additional restriction on the number of columns.

For technical reasons, we have to separate the finite and the infinite dimensional cases.

**Lemma (6.1.16)[367]:** Let $K$ be a finite dimensional metrizable Choquet simplex with exactly $d\geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n\geq 0$ the integer $p_n$ divides $p_{n+1}$, and let $k\geq\max\{2,d\}$. Then there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of square $k$-dimensional matrices which is managed by $(p_{n_i})_{i\geq 0}$ such that $K$ is affine homeomorphic to $\varprojlim_n(\triangle(k,p_{n_i}),M_i)$.

**Proof:** Let $k\geq\max\{3,d\}$, we will define the subsequence $(n_i)_{i\geq 0}$ by induction on $i$ through a condition explained later. For every $i\geq 0$, we define $M_i$ the $k$-dimensional matrix by

$$M(l,j)=\begin{cases}\frac{p_{n_{i+1}}}{p_{n_i}}-k(k-1) & \text{if}\quad 1\leq l=j\leq d\\ k & \text{if}\quad l\neq j, 1\leq l\leq k\ and\ 1\leq j\leq d\\ M_i(l,d) & \text{if}\quad d<j\leq k.\end{cases}$$

We always suppose that $n_{i+1}$ is sufficiently large in order to have $\frac{p_{n_{i+1}}}{p_{n_i}}-k(k-1)>0$.
By the very definition, $M_i$ is a positive matrix having $k\geq 3$ rows and columns; $\sum_{l=1}^{k}M_i(l,j)=\frac{p_{n_{i+1}}}{p_{n_i}}$ for every $1\leq j\leq k$ and the range of $M_i$ is at most d. Thus the convex set $\varprojlim_n(\triangle(k,p_{n_i}),M_i)$ has at most d extreme points.

If it has exactly d extreme points, it is affine homeomorphic to $K$. We will choose the sequence $(p_{n_i})_{i\geq 0}$ in order that $P = \bigcap_{i\geq 0} M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ has $d$ extreme points, which implies that $\lim\limits_{\leftarrow n}(\triangle(k, p_{n_i}), M_i)$ has exactly d extreme points.

For every $i \geq 0$, the set $P_i = M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ is the closed convex set generated by the vectors $v_{i,1}, \cdots, v_{i,d}$, where

$$v_{i,l} = \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(\cdot, l),, \text{for every } 1 \leq l \leq d.$$

Since every $v_{i,l}$ is in $\triangle(k, p_{n_0})$ there exists a sequence $(i_j)_{j\geq 0}$ such that for every $1 \leq l \leq d$, the sequence $(v_{i_j}, l)_{j\geq 0}$ converges to an element $v_l$ in $\triangle(k, p_{n_0})$. Observe that $P$ is the closed convex set generated by $v_1, \cdots, v_d$. Thus if $v_1, \cdots, v_d$ are linearly independent then $P$ has $d$ extreme points.

Since for every $1 \leq l \leq d$ we have $\sum_{j=1}^{k} \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \frac{1}{p_{n_0}}$, there exists a positive vector $\delta_l^{(i)} = (\delta_{1,l}^{(i)}, \cdots, \delta_{k,l}^{(i)})^T$ such that $\sum_{j=1}^{k} \delta_{j,l}^{(i)} = 1$ and such that for each $1 \leq j \leq k$

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \delta_{j,l}^{(i)} \frac{1}{p_{n_0}}.$$

Thus if $B_i$ is the matrix given by

$$B_i(\cdot, l) = \begin{cases} v_{i,l} & \text{if } 1 \leq l \leq d \\ \frac{1}{p_{n_0}} e_l^{(k)} & \text{if } d+1 \leq l \leq k. \end{cases}$$

then $B_i = DA_i$, where $D$ is the $k$-dimensional diagonal matrix given by

$$D_i(l, l) = \frac{1}{p_{n_0}}, \text{for every } 1 \leq l \leq k,$$

and $A_i$ is the $k$-dimensional matrix defined by

$$A_i(\cdot, l) = \begin{cases} \delta_l^{(i)} & \text{if } 1 \leq l \leq d \\ e_l^{(k)} & \text{if } d+1 \leq l \leq k. \end{cases}$$

If $\lim\limits_{j\to\infty} A_j = A$ is invertible ($A$ is the $k$-dimensional matrix whose columns are the vectors $\lim\limits_{j\to\infty} \delta_l^{(i_j)}$ and the canonical vectors $e_{d+1}^{(k)}, \cdots, e_k^{(k)}$), then $v_1, \cdots, v_l$ are linearly independent. For this it is enough to show that $A$ is strictly diagonally dominant (see the Levy-Desplanques Theorem in [387]).

Now we will define $(n_i)_{i\geq 0}$ in order that $A$ is strictly diagonally dominant. Let $\varepsilon \in (0, \frac{1}{4})$. Let $n_0 = 0$ and $n_1 > n_0$ such that for every $1 \leq l \leq d$,

$$\delta_{l,l}^{(0)} = 1 - \frac{p_{n_0}}{p_{n_1}} \sum_{j=1, j\neq l}^{k} M_0(j, l) = 1 - \frac{p_{n_0}}{p_{n_1}} k(k-1) \geq \frac{3}{4} + \varepsilon.$$

For $i \geq 1$ we choose $n_{i+1} > n_i$ in order that

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_{i-1}(l, l) < \varepsilon \frac{1}{p_{n_0} k(k-1)2^i}, \text{for every } 1 \leq l \leq d.$$

After a standart computation, for every $i \geq 1$ and $1 \leq l \leq d$ we get

$$\delta_{l,l}^{(i)} \geq \delta_{l,l}^{(i-1)} - \frac{p_{n_0}}{p_{n_{i+1}}} k(k-1) M_0 \cdots M_{i-1}(l, l),$$

which implies that

$$\delta_{l,l}^{(i)} \geq \delta_{l,l}^{(0)} - \varepsilon \sum_{j\geq 1} \tfrac{1}{2^j} \geq \frac{3}{4}.$$

It follows that $A(l,l) \geq \frac{3}{4}$ for every $1 \leq l \leq k$, and since the sum of the elements in a column of $A$ is equal to 1, we deduce that $A$ is strictly diagonally dominant.

We use the following characterization of infinite dimensional metrizable Choquet simplex.

**Lemma (6.1.17)[367]:** ([391], Corollary p.186). For every infinite dimensional metrizable Choquet simplex $K$, there exists a sequence of matrices $(A_n)_{n\geq 1}$ such that for every $n \geq 1$

**(i)** $A_n(\triangle\,(n+1,1)) = \triangle\,(n,1)$,

**(ii)** $K$ is affine homeomorphic to $\varprojlim_n(\triangle\,(n,1), A_n)$.

Our strategy is to approximate the sequence of matrices $(A_n)_n$ by a managed sequence. Then we show that the associated inverse limits are affine homeomorphic. For this, we need the following classical density result, whose proof follows from the fact that every non cyclic subgroup of $\mathbb{R}$ is dense.

**Lemma (6.1.18)[367]:** Let $r = (r_n)_{n\geq 0}$ be a sequence of integers such that $r_n \geq 2$ for every $n \geq 0$. Let $C_r$ be the subgroup of $(\mathbb{R},+)$ generated by $\{(r_0 \cdots r_n)^{-1} : n \geq 0\}$. Then

$$(C_r)^p \cap \Delta(p,1) \cap \{v \in \mathbb{R}^p : v > 0\}$$

is dense in $\triangle\,(p,1)$, for every $p \geq 2$, where $(C_r)^p$ is the Cartesian product $\prod_{i=1}^{p} C_r$.

**Lemma (6.1.19)[367]:** Let K be an infinite dimensional metrizable Choquet simplex, and let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer $p_n$ divides $p_{n+1}$. Then there exist an increasing subsequence $(n_i)_{i\geq 1}$ of indices and a sequence of matrices $(M_i)_{i\geq 1}$ managed by $(p_{n_i})_{i\geq 0}$ such that for every $i \geq 0$,

$$k_{i+1} \leq \min\{M_i(l,k) : 1 \leq l \leq k_i \; 1 \leq k \leq k_{i+1}\},$$

and such that $K$ is affine homeomorphic to the inverse limit $\varprojlim_n(\triangle\,(k_i, |p_{n_i}|), M_i)$, where $k_i$ is the number of rows of $M_i$, for every $i \geq 0$.

**Proof:** For every $n \geq 0$, let $r_n \geq 2$ be the integer such that $p_{n+1} = p_n r_n$. Let $(A_n)_{n\geq 1}$ be the sequence of matrices given in Lemma (6.1.18). We can assume that $A_n : \triangle\,(n+3,1) \to \triangle\,(n+2,1)$, for every $n \geq 1$. Now we define the subsequence $(n_i)_i$ by induction

We set $n_1 = 0$.

Let $i \geq 1$ and suppose that we have defined $n_i \geq 0$. We set $r^{(i)} = (r_n)_{n\geq n_i}$. For every $1 \leq j \leq i+3$, Lemma (6.1.19) ensures the existence of $v^{(i,j)} \in (C_{r^{(i)}})^{i+2} \cap \triangle\,(i+2,1) \cap \{v \in \mathbb{R}^{i+2} : v > 0\}$ such that

$$\left\|v^{(i,j)} - A_i(\cdot,j)\right\|_1 < \tfrac{1}{2^i}. \tag{9}$$

Let $B_i$ be the matrix given by

$$B_i(\cdot,j) = v^{(i,j)}, \text{for every } 1 \leq j \leq i+3.$$

Observe that (9) implies that

$$\sum_{n\geq 1} \sup\{\|A_n v - B_n v\|_1 : v \in \Delta_{n+3}\} < \infty.$$

It follows from [373] that $K$ is affine homeomorphic $\varprojlim_n(\triangle\,(i+2,1), B_i)$.

Let $n_{i+1} > n_i$ be such that $r_{n_i} \cdots r_{n_{i+1}} - 1v^{(i,j)}$ is an integer vector and such that $r_{n_i} \cdots r_{n_{i+1}} - 1v^{(i,j)} > i+3$, for every $1 \leq j \leq i+3$.

We define

$$M_i = \frac{p_{n_{i+1}}}{p_{n_i}} B_i.$$

Thus $M_i = P_i^{-1} B_i P_{i+1}$, where $P_i$ is the diagonal matrix given by $P_i(j,j) = p_{n_i}$ for every $1 \leq j \leq i+2$ and $i \geq 1$. This shows that $\lim\limits_{\leftarrow n}(\triangle\,(i+2,1), B_i)$ is affine homeomorphic to $\lim\limits_{\leftarrow n}(\triangle\,(i+2, p_{n_i}), B_i)$.

The proof conclude verifying that $(M_i)_{i\geq 0}$ is managed by $(p_{n_i})_{i\geq 0}$.

**Theorem (6.1.20)[367]:** Let $G$ be an infinite, countable, amenable and residually finite group. For every metrizable Choquet simplex $K$ and any $G$-odometer $O$, there exists a Toeplitz $G$- subshift whose set of invariant probability measures is affine homeomorphic to $K$ and such that it is an almost 1-1 extension of $O$.

Typical examples of the groups $G$ involved in this theorem are the finitely generated subgroups of upper triangular matrices in $GL(n, \mathbb{C})$.

The strategy of Downarowicz in [374], is to construct an affine homeomorphism between an arbitrary metrizable Choquet simplex $K$ and a subset of the space of invariant probability measures of the full shift $\{0,1\}^{\mathbb{Z}}$. Then he shows it coincides with the space of invariant probability measures of a Toeplitz subshift $Y \subseteq \{0,1\}^{\mathbb{Z}}$. To do this, he uses the structure of metric space of the space of measures. We consider the representation of $K$ as an inverse limit of finite dimensional simplices with linear transition maps $(M_n)_n$. Then we use this transition maps to construct Toeplitz $G$-subshifts having sequences of Kakutani-Rokhlin partitions with $(M_n)_n$ as the associated sequence of incidence matrices. Our approach is closer to the strategy used in [384] by Gjerde and Johansen, and deals with the combinatorics of Følner sequences.

We obtain, furthermore some consequences in the orbit equivalence problem. Two minimal Cantor systems are (topologically) orbit equivalent, if there exists an orbit-preserving homeomorphism between their phase spaces. Giordano, Matui, Putnam and Skau show in [382] that every minimal $\mathbb{Z}^d$-action on the Cantor set is orbit equivalent to a minimal $\mathbb{Z}$-action. It is still unknown if every minimal action of a countable amenable group on the Cantor set is orbit equivalent to a $\mathbb{Z}$-action. It is clear that the result in [382] can not be extended to any countable group. For instance, by using the notion of cost, Gaboriau [381] proves that if two free actions of free groups $F_n$ and $F_p$ are (even measurably) orbit equivalent then their rank are the same i.e. $n = p$. Another problem is to know which are the $\mathbb{Z}$-orbit equivalence classes that the $\mathbb{Z}^d$-actions (or more general group actions) realize. We give a partial answer for this question. As a consequence of the proof of Theorem (6.1.21) we obtain the following result.

**Proof :** Let $ext(K)$ be the set of extreme points of $K$. If $ext(K)$ is finite, then the proof is direct from Proposition (6.1.15) and Lemma (6.1.17). If $ext(K)$ is infinite, the proof follows from Proposition (6.1.15) and Lemma (6.1.20).

**Lemma (6.1.21)[367]:** Let $x_0 \in \sum^{\mathbb{Z}}$ be a Toeplitz sequence and let $(X, \sigma|_X, \mathbb{Z})$ be the associated Toeplitz $\mathbb{Z}$-subshift. There exist a period structure $(p_n)_{n\geq 0}$ of $x_0$ and a sequence of matrices $(A_n)_{n\geq 0}$ managed by $(p_n)_{n\geq 0}$ such that the dimension group associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to

$$\mathbb{Z} \xrightarrow{A_0^T} \mathbb{Z}^{k_1} \xrightarrow{A_1^T} \mathbb{Z}^{k_2} \xrightarrow{A_2^T} \cdots.$$

Furthermore, if $k_n$ is the number of rows of $A_n$ and $r_n = \frac{p_{n+1}}{p_n}$, then for every $m > n > 0$ and $1 \le k \le k_m$,

$$\left|\left\{1 \le k \le k_m : A_{n,m-1}(\cdot, l) = A_{n,m-1}(\cdot, k)\right\}\right|$$
$$\le \left(A_{n,m-1}(\cdot, k) - r_{n+2} \cdots r_{m-1}, \cdots, A_{n,m-1}(k_n, k) - r_{n+2} \cdots r_{m-1}\right)!,$$

where $A_{n,m-1} = A_n \cdots A_{m-1}$.

**Proof:** In the proof in [384] the authors show there exist a period structure $(p_n)_{n\ge1}$ of $x_0$ and a sequence $(\mathcal{P}_n)_{n\ge0}$ of nested Kakutani-Rokhlin partitions of $(X, \sigma|_X, \mathbb{Z})$ such that $\mathcal{P}_0 = \{X\}$ and $\mathcal{P}_n = \{T^j(C_n, k) : 0 \le j < p_n, 1 \le k \le k_n\}$, where

$$C_{n,k} = \left\{x \in X : x[0, p_n - 1] = w_{n,k}\right\} \text{ for every } 1 \le k \le k_n,$$

with $W_n = \{w_{n,1}, \cdots, w_{n,k_n}\}$ the set of the words $w$ of $x_0$ of length $p_n$ verifying $w[0, p_{n-1} - 1] = x_0[0, p_{n-1} - 1]$, for every $n \ge 1$ (with $p_0 = 1$).

Thus the dimension group with unit associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n^T) = \mathbb{Z} \overset{A_0^T}{\to} \mathbb{Z}^{k_1} \overset{A_1^T}{\to} \mathbb{Z}^{k_2} \overset{A_2^T}{\to} \cdots,$$

where $A_n(i, j)$ is the number of times that the word $w_{n,i}$ appears in the word $w_{n+1,j}$, for every $1 \le i \le k_n, 1 \le j \le k_{n+1}$ and $n \ge 1$, and the matrix $A_0^T$ is the vector in $\mathbb{Z}^{k_1}$ whose coordinates are equal to $p_1$.

Since $w_{n+1,i} \ne w_{n+1,j}$ for $i \ne j$, equal columns of the matrix $A_n$ produce different concatenations of words in $W_n$. This implies that for every $1 \le k \le k_{n+1}$, the number of columns of $A_n$ which are equal to $A_n(\cdot, k)$ can not exceed the number of different concatenations of $r_n$ words in $W_n$ using exactly $A_n(j, k)$ copies of $w_{n,j}$, for every $1 \le j \le k_n$. This means that the number of columns which are equal to $A_n(\cdot, k)$ is smaller or equal to $(A_n(1, k), \cdots, A_n(k_n, k))!$.

Now fix $n > 0$ and take $m > n$. The coordinate $(i, j)$ of the matrix $A_{n,m-1}$ contains the number of times that the word $w_{n,i} \in W_n$ appears in $w_{m,j} \in W_m$. Observe that every word $u$ in $W_m$ is a concatenation of $r_{n+2} \cdots r_{m-1}$ words in $W_{n+2}$. In addition, each word in $W_{n+2}$ starts with $x_0[0, p_{n+1} - 1] \in W_{n+1}$, which is a word containing every word in $W_n$ (we can always assume that the matrices $A_n$ are positive). Thus there exist $0 \le l_1 < \cdots < l_{r_{n+1}} \cdots r_{m-1} < p_m$ such that $u[l_s, l_s + p_n - 1] = w[l_s, l_s + p_n - 1] \in W_n$, for every $1 \le s \le r_{n+2} \cdots r_{m-1}$ and $u, w \in W_m$.

This implies that the number of all possible concatenations of words in $W_n$ producing a word in $W_m$, according to the column $k$ of the matrix $A_{n,m-1}$ is smaller or equal to

$$\left(A_{n,m-1}(1, k) - r_{n+2} \cdots r_{m-1}, \cdots, A_{n,m-1}(k_n, k) - r_{n+2} \cdots r_{m-1}\right)!$$

**Theorem (6.1.22)[367]:** Let $(X, \sigma|_X, \mathbb{Z})$ be a Toeplitz $\mathbb{Z}$-subshift. Then for every $d \ge 1$ thereexistsa Toeplitz $\mathbb{Z}^d$-subshift which is orbit equivalent to $(X, \sigma|_X, \mathbb{Z})$. We devoted to introduce the basic definitions. For an amenable discrete group $G$ and a decreasing sequence of finite index subgroups of $G$ with trivial intersection, we construct an associated sequence $(F_n)_{n\ge0}$ of fundamental domains, so that it is Følner and each $F_{n+1}$ is tilable by translated copies of $F_n$. We construct Kakutani-Rokhlin partitions for generalized Toeplitz subshifts, and we use the fundamental domains introduced to construct Toeplitz subshifts having sequences of Kakutani-Rokhlin partitions with a prescribed sequence of incidence matrices. This construction improves and generalizes that one given in [371] for $\mathbb{Z}^d$-actions, and moreover, allows to characterize the associated ordered group with unit. We give a characterization of any Choquet simplex as an inverse limit defined by sequences of matrices that we use (they are called "managed" sequences). Finally, we use the previous results to prove Theorems (6.1.20) and (6.1.22).

**Proof :** Let $x_0 \in X$ be a Toeplitz sequence. Let $(p_n)_{n\geq 1}$ and $(A_n)_{n\geq 0}$ be the period structure of $x_0$ and the sequence of matrices given by Lemma (6.1.22) respectively. It is straightforward to check that Lemma (6.1.21) is also true if we take a subsequence of $(p_n)_{n\geq 0}$. Thus we can assume that for every $n \geq 1$, the matrix $A_n$ has its coordinates strictly grater than 1 and that there exist positive integers $r_{n,1},\cdots, r_{n,d} > 1$ such that

$$\frac{p_{n+1}}{p_n} = r_n = r_{n,1}\cdots r_{n,d}.$$

Le define $q_{n+1,i} = r_{0,i}\cdots r_{n,i}$ for every $1 \leq i \leq d$, and $\Gamma_{n+1} = \prod_{i=1}^{d} q_{n+1,i}\mathbb{Z}$, for every $n \geq 0$. We have $\Gamma_{n+1} \subseteq \Gamma_n$, $\bigcap_{n\geq 1}\Gamma_n = \{0\}$ and $|\mathbb{Z}^d/\Gamma_n| = p_n$. Let $(F_n)_{n\geq 0}$ be a Følner sequence associated to $(\Gamma_n)_{n\geq 1}$as in Lemma (6.1.7). We denote $R_n$ (the set that defines "border").

Now, we define an increasing sequence $(n_i)_{i\geq 1}$ of integers as follows:

We set $n_1 = 1$. For $i \geq 1$, given $n_i$we chose $n_{i+1} > n_i + 1$ such that

$$\sum_{g\in R_{n_i}} \frac{|F_{n_{i+1}} \backslash F_{n_{i+1}} - g|}{|F_{n_{i+1}}|} < \frac{1}{|F_{n_i}| r_{n_i} r_{n_i+1}}.$$

Thus we have

$$\frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \backslash F_{n_{i+1}} - g| > \frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \frac{|F_{n_{i+1}}|}{|F_{n_i}| r_{n_i} r_{n_i+1}}$$
$$= r_{n_i}\cdots r_{n_{i+1}-1} - r_{n_i+2}\cdots r_{n_{i+1}-1}$$
$$> r_{n_i}\cdots r_{n_{i+1}-1} - k_{n_i} r_{n_i+2}\cdots r_{n_{i+1}-1}$$

Let $M_0 = A_0$ and $M_i = A_{n_i}\cdots A_{n_{i+1}-1}$ be for every $i \geq 1$. For every $1 \leq k \leq k_{n_{i+1}}$ we get

$$M_i(k_n, k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \backslash F_{n_{i+1}} - g| > M_i(k_n, k) - r_{n_i+2}\cdots r_{n_{i+1}-1},$$

which implies that

$$(M_i(1,k),\cdots, M_i(k_{n_i}-1,k), M_i(k_{n_i},k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \backslash F_{n_{i+1}} - g|)!$$

is grater than

$$\left(M_i(1,k) - r_{n_i+2}\cdots r_{n_{i+1}-1},\cdots, M_i(k_n,k) - r_{n_i+2}\cdots r_{n_{i+1}-1}\right)!$$

Then from the previous inequality and Lemma (6.1.21) we get that the number of columns of $M_i$ which are equal to $M_i(\cdot,k)$ is smaller than

$$(M_i(1,k),\cdots, M_i(k_{n_i}-1,k), M_i(k_{n_i},k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \backslash F_{n_{i+1}} - g|)!$$

As in the proof of Proposition (6.1.15), we define $\widetilde{M}_i$ and we call $l_i$ and $l_{i+1}$ the number of rows and columns of $\widetilde{M}_i$ respectively, for every $i \geq 0$. According to the notations of the proof of Proposition (6.1.15), in our case $M_0$ corresponds to the matrix $M$ and $\widetilde{M}_0$ corresponds to the matrix $\widetilde{M}$ . Observe that the bound on the number of columns which are equal to $M_i(\cdot,k)$ (and then to $\widetilde{M}_i(\cdot,k)$) ensures the existence of enough possibilities to fill

the coordinates of $F_{n_i}$ in order to obtain different functions $B_{i,1} \cdots, B_{i,l_i} \in \{1,\cdots, l_1\}^{F_{n_i}}$ as in the proof of Proposition (6.1.15), for every $i \geq 1$ (see Remark (6.1.16)).
Lemma (6.1.13) implies that the Toeplitz $\mathbb{Z}^d$-subshift $(Y, \sigma|_Y, \mathbb{Z}^d)$ defined from $(B_{i,1},\cdots, B_{i,l_i})_{i\geq 1}$ has an ordered group $\mathcal{G}(Y, \sigma|_Y, \mathbb{Z}^d)$ isomorphic to $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$, where $(H, H^+)$ is given by

$$\mathbb{Z} \xrightarrow{\widetilde{M}_0^T} \mathbb{Z}^{l_0} \xrightarrow{\widetilde{M}_1^T} \mathbb{Z}^{l_2} \xrightarrow{\widetilde{M}_2^T} \mathbb{Z}^{l_3} \xrightarrow{\widetilde{M}_3^T} \cdots,$$

with $M_0 = |F_1|(1,\cdots,1)$ and $u = [1,0]$.
Lemma (6.1.3) implies that $(H, H^+, u)$ is isomorphic to the dimension group with unit $(J, J^+, w)$ associated to $(X, \sigma|_X, \mathbb{Z})$. Thus $(J/\inf(J), (J/\inf(J))^+, w + \inf(J))$, the ordered group associated to $(X, \sigma|_X, \mathbb{Z})$., is isomorphic to $\mathcal{G}(Y, \sigma|_Y, \mathbb{Z}^d)$. We conclude the proof applying Theorem (6.1.14).

[392], shows that every minimal Cantor system $(Y, T, \mathbb{Z})$ having an associated Bratteli diagram which satisfies the equal path number property, is strong orbit equivalent to a Toeplitz subshift $(X, \sigma|_X, \mathbb{Z})$. Thus the next result is inmediat.

## Section (6.2): IE Groups and Expansive Algebraic Actions

Given a (continuous) action of a countable discrete amenable group $\Gamma$ on a compact metrizable space $X$, one has the topological entropy $\mathrm{h}_{\mathrm{top}}(X)$ of the action, lying in $[0, +\infty]$. Besides being an invariant of the action, the entropy also gives us a lot of information about the action itself. Indeed, the intuition about the entropy is that the larger the entropy is, themore complicated the action is. Thus it is very natural to ask for the relation between entropy properties of the action and the asymptotic behavior of orbits of the action, i.e. the asymptotics of $\rho(sx, sy)$ as elements $s$ of $\Gamma$ go to infinity, where $\rho$ is a compatible metric on $X$ and $x, y \in X$.

A well-known result in this direction is that of Blanchard et al. [404] in the case of $\Gamma = \mathbb{Z}$. They showed that positive entropy implies Li–Yorke chaos.That is, if the $\mathbb{Z}$-action is generated by a homeomorphism $T: X \to X$ and $\mathrm{h}_{\mathrm{top}}(X) > 0$, then there exists an uncountable subset $Z$ of $X$ such that for any distinct $x, y$ in $Z$ one has $\limsup_{n\to+\infty} \rho(T^n x, T^n y) > 0$ and $\liminf_{n\to+\infty} \rho(T^n x, T^n y) = 0$.

We concentrate on the phenomenon $\lim_{s\to\infty} \rho(sx, s\,y) = 0$ for points under the action of a general group $\Gamma$. A pair $(x, y)$ of points in $X$ satisfying $\lim_{s\to\infty} \rho(sx, s\,y) = 0$ is called asymptotic or homoclinic. In the case $\Gamma = \mathbb{Z}$ and the action is generated by a homeomorphism $T: X \to X$, a pair $(x, y)$ of points in $X$ satisfying $\lim_{n\to+\infty} \rho(T^n x, T^n y) = 0$ is called positively asymptotic or positively homoclinic. Thus, one of the questions we want to address is the relation between $\mathrm{h}_{\mathrm{top}}(X) > 0$ and the existence of non-diagonal asymptotic pairs.

A positive result on this question is that of Blanchard et al. [406]. They showed that in the case $\Gamma = \mathbb{Z}$, when $\mathrm{h}_{\mathrm{top}}(X) > 0$, there exist non-diagonal positively asymptotic pairs. On the other hand, Lind and Schmidt [445] constructed examples of $\mathbb{Z}$-actions (actually toral automorphisms) which have positive entropy but no non-diagonal asymptotic pairs. Thus one has to add further conditions.

The condition we are going to add is expansiveness. An action $\Gamma \curvearrowright X$ is called expansive if there exists $r > 0$ such that $\sup_{s\in\Gamma} \rho(sx, s\,y) \geq r$ for all distinct $x, y$ in $X$. For example, for any $k \in \mathbb{N}$ and $A \in M_k(\mathbb{Z})$ being invertible in $M_k(\mathbb{Z})$, the toral automorphism of $\mathbb{R}^k/\mathbb{Z}^k = (\mathbb{R}/\mathbb{Z})^k$ defined by $x + \mathbb{Z}^k \mapsto Ax + \mathbb{Z}^k$ for $x \in \mathbb{R}^k$ is expansive if and only if $A$ has no eigenvalues with absolute value 1 [474]. Bryant [408] showed that for expansive $\mathbb{Z}$ -actions, when $X$ is infinite, there are non-diagonal positively asymptotic pairs. Schmidt showed that when $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, every subshift of finite type (which is always expansive) with positive entropy has non-diagonal asymptotic pairs [469]. These results lead us to ask the following question:

**Question (6.2.1)[396]:** Let a countable discrete amenable group $\Gamma$ act on a compact metrizable space $X$ expansively. If $h_{top}(X) > 0$, then must there be a non-diagonal asymptotic pair in $X$?

Despite all the evidence above, we do not know the answer to Question (6.2.1) even in the case $\Gamma = \mathbb{Z}$. One of the main results of this article is that Question (6.2.1) has an affirmative answer for algebraic actions of polycyclic-by-finite groups.

Actions of countable discrete groups $\Gamma$ on compact (metrizable) groups $X$ by (continuous) automorphisms are a rich class of dynamical systems, and have drawn much attention since the beginning of ergodic theory. Among such actions, the so called algebraic actions, meaning that $X$ is abelian in which case the action is completely determined by the module structure of the Pontryagin dual $\hat{X}$ of $X$ over the integral group ring $\mathbb{Z}\Gamma$ of $\Gamma$, is especially important because of the beautiful interplay between dynamics, Fourier analysis, and commutative or noncommutative algebra.

The $\mathbb{Z}$-actions on compact groups by automorphisms are well understood now (cf. [444],[455],[476],[477]). After investigation during the last few decades, much is also known for such actions of $\mathbb{Z}^d$ (cf. [417],[432],[433],[438],[445]–[448],[465],[468],[470],[472]). The fact that the integral group ring of $\mathbb{Z}^d$ is a commutative factorialNoetherian ring plays a vital role for such study, as it makes the machinery of commutative algebra available. In the last several years, much progress has been made towards understanding the algebraic actions of general countable groups $\Gamma$ (cf. [400],[407],[411],[413],[414],[437],[443],[457]). It is somehow surprising that operator algebras, especially the group $C^*$-algebras or group von Neumann algebras of $\Gamma$, turn out to be important for such a study.

Let a countable group $\Gamma$ act on a compact group $X$ by automorphisms, and denote by $e_X$ the identity element of $X$. A point $x \in X$ is called homoclinic if the pair $(x, e_X)$ is asymptotic, i.e. $sx \to e_X$ when $\Gamma \ni s \to \infty$. When $\Gamma$ is amenable, a point $x \in X$ is called $IE$ if, for any neighborhoods $U_1$ and $U_2$ of $x$ and $e_X$ respectively, there exists $c > 0$ such that for any sufficiently left invariant nonempty finite set $F \subseteq \Gamma$ one can find some $F' \subseteq F$ with $|F'| \geq c|F|$ being an independence set for $(U_1, U_2)$ in the sense that for any map $\sigma: F \to \{1,2\}$ one has $\bigcap_{s \in F'} s \in F\, s^{-1} U_{\sigma(s)} \neq \emptyset$. The set of all homoclinic points (resp. IE points), denoted by $\Delta(X)$ (resp. $\mathrm{IE}(X)$), is a $\Gamma$ -invariant subgroup of $X$. It is easy to see that $\Delta(X)$ describes all the asymptotic pairs of $X$ in the sense that a pair $(x, y)$ of points in $X$ is asymptotic if and only if $xy^{-1}$ lies in $\Delta(X)$. A group $\Gamma$ is called polycyclic-by-finite [459] if there is a sequence of subgroups $\Gamma = \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_n = \{e_\Gamma\}$ such that $\Gamma_j/\Gamma_{j+1}$ is finite or cyclic for every $j = 1, \ldots, n-1$. The polycyclic-by-finite groups are exactly the virtually solvable groups each of whose subgroups is finitely generated (cf. [473]). One of our main results is

**Theorem (6.2.2)[396]:** Let $\Gamma$ be a polycyclic-by-finite group. Let $\Gamma$ act on a compact abelian group $X$ expansively by automorphisms. Then the following hold:

**(i)** Let $G$ be a $\Gamma$ -invariant subgroup of $\Delta(X)$ such that $G$ and $\Delta(X)$ have the same closure. Treat $G$ as a discrete abelian group and consider the induced $\Gamma$ -action on the Pontryagin dual $\hat{G}$. Then the actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright \hat{G}$ have the same entropy.

**(ii)** $\Delta(X)$ is a dense subgroup of $\text{IE}(X)$.
**(iii)** The action has positive entropy if and only if $\Delta(X)$ is nontrivial.
**(iv)** The action has completely positive entropy $(CPE)$ with respect to the normalized Haar measure of $X$ if and only if $\Delta(X)$ is dense in $X$.

Note that the assertion (iii) of Theorem (6.2.2) answers Question (6.2.1) affirmatively for algebraic actions of polycyclic-by-finite groups. In the case $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, the assertions (iii) and (iv) are the main results of Lind and Schmidt in [445], and the assertion (i) was proved by Einsiedler and Schmidt [415] for $G = \Delta(X)$. As we mentioned above, the work in [415],[445] relies heavily on the machinery of commutative algebra. When $\Gamma$ is nonabelian, we do not have such tools available anymore. Instead, the $\ell^1$-group algebra $\ell^1(\Gamma)$ and the group von Neumann algebra of $\Gamma$ play a crucial role.

Expansive algebraic actions of countable groups, the local entropy theory for actions of countable amenable groups on compact groups by automorphisms, and duality for algebraic actions of countable amenable groups. Theorem (6.2.2) is the outcome of these three parts.

We give algebraic characterizations for expansiveness of algebraic actions of a countable group $\Gamma$. Given a matrix $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$, the canonical $\Gamma$-action on the Pontryagin dual $X_A$ of the $\mathbb{Z}\Gamma$-module $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A$ is expansive. Dynamically, our characterization says that the expansive algebraic actions of $\Gamma$ are exactly the restriction to closed $\Gamma$-invariant subgroups of $X_A$ for all such $A$.

The notion of $p$-expansiveness is introduced in for algebraic actions of a countable group $\Gamma$, for $1 \leq p \leq +\infty$. This yields a hierarchy of expansiveness: $q$-expansiveness implies $p$-expansiveness for $p < q$ and $+\infty$-expansiveness is exactly the ordinary expansiveness. We show that, for an algebraic action of $\Gamma$ on $X$, if $\Gamma$ is amenable and $\hat{X}$ is a finitely presented $\mathbb{Z}\Gamma$-module, then the action has finite entropy if and only if it is 1-expansive.This relies on a result of Elek [418] about the analytic zero divisor conjecture, the proof of which uses the group von Neumann algebra of $\Gamma$.

We study the group $\Delta^p(X)$ of $p$-homoclinic points for an algebraic action of a countable group $\Gamma$ and $1 \leq p < +\infty$. They are subgroups of $\Delta(X)$. For expansive algebraic actions, using our characterization for such actions, we show that $\Delta^1(X) = \Delta(X)$. Gives another application of our characterization for expansive algebraic actions of a countable group $\Gamma$. We show that various specification properties are equivalent and imply that $\Delta(X)$ is dense in $X$ for such actions.

Initiated by Blanchard [402], the local entropy theory for continuous actions of a countable amenable group $\Gamma$ on compact spaces developed quickly during the last 2 decades (cf. [403]–[406],[420],[422],[427]–[431]), and has been found to be related to combinatorial independence [435],[436], which appeared first in Rosenthal's [463] work on Banach spaces containing $\ell^1$. We develop the local entropy theory for actions of a countable amenable group $\Gamma$ on compact groups $X$ by automorphisms in Sects. 7 and 8. It turns out that $\text{IE}(X)$ determines the local entropy theory and the Pinsker factor for such actions. Furthermore, when $X$ is abelian, one has $\Delta^1(X) \subseteq \text{IE}(X)$. In particular, the "if" parts of the assertions (iii) and (v) of Theorem(6.2.2) actually hold for all countable amenable groups $\Gamma$. This also enables us to give a partial answer to a question of Deninger about the Fuglede–Kadison determinant (see Corollary (6.2.52)), which is an application to the study of the group von Neumann algebra of $\Gamma$. We also show that, for finite entropy actions on compact groups by automorphisms, having CPE is equivalent to having a unique maximal measure.

For an algebraic action of a countable group $\Gamma$ on $X$, we treat $\hat{X}$ and $\Delta^p(X)$ as a dual pair of discrete abelian groups, with the expectation that the dynamical properties of the $\Gamma$-actions on $X$ and $\widehat{\Delta^p(X)}$ would be reflected in each other. We discuss the relation of the entropy properties for such pairs of actions, for countable amenable groups $\Gamma$. It turns out that $\Delta^1(X)$ and $\Delta^2(X)$ are more closely related to the entropy properties of $\Gamma \curvearrowright X$ than any other
$\Delta^p(X)$ or $\Delta(X)$. In particular, when $\hat{X}$ is a finitely presented $\mathbb{Z}\Gamma$-module, the entropy of $\Gamma \curvearrowright X$ is bounded below by that for $\Gamma \curvearrowright \widehat{\Delta^1(X)}$.This depends on the equivalence between 1-expansiveness and finite entropy mentioned above. Actually we establish a general result (see Theorem (6.2.70)) stating the relation between $\Delta^1(X)$ and the entropy properties of $\Gamma \curvearrowright X$, much as the assertions of Theorem (6.2.2). Then Theorem (6.2.2) is just a consequence of this result and our characterizations of expansive algebraic actions.

In fact Theorem (6.2.2) holds whenever $\Gamma$ is amenable and $\mathbb{Z}\Gamma$ is left Noetherian. It is known that a polycyclic-by-finite group is amenable and its integral group ring is left Noetherian [424]. On the other hand, it is a long standing open question whether $\mathbb{Z}\Gamma$ is left Noetherian implies that $\Gamma$ is polycyclic-by-finite.

We set up some notations and recall some basic facts about group rings, algebraic actions, entropy theory, and local entropy theory.

All compact spaces are assumed to bemetrizable and all automorphisms of compact groups are assumed to be continuous. For a group $G$, we write $e_G$ for the identity element of $G$. When $G$ is abelian, sometimes we also write $0_G$.
For a unital ring $R$, we denote by $R\Gamma$ the group ring of $\Gamma$ with coefficients in $R$. It consists of finitely supported $R$-valued functions $f$ on $\Gamma$, which we shall write as $\sum_{s\in\Gamma} f_s s$. The algebraic structure of $R\Gamma$ is defined by $(\sum_{s\in\Gamma} f_s s) + (\sum_{s\in\Gamma} g_s s) = \sum_{s\in\Gamma}(f_s + g_s)s$ and $(\sum_{s\in\Gamma} f_s s)\,(\sum_{s\in\Gamma} g_s s) = \sum_{s\in\Gamma}(\sum_{t\in\Gamma} f_t\, g_{t^{-1}s})s$.

We denote by $\ell^\infty(\Gamma)$ the Banach space of all bounded $\mathbb{R}$-valued functionson $\Gamma$, equipped with the $\ell^\infty$-norm $\|\cdot\|_\infty$.We also denote by $\ell^1(\Gamma)$ the Banach algebra of all absolutely summable $\mathbb{R}$-valued functions on $\Gamma$, equipped with the $\ell^1$-norm $\|\cdot\|_1$. Note that $\ell^1(\Gamma)$ has a canonical algebra structure extending that of $\mathbb{R}\Gamma$, and is a Banach algebra. We shall write $f \in \ell^1(\Gamma)$ as $\sum_{s\in\Gamma} f_s s$. Note that $\ell^1(\Gamma)$ has an involution $f \mapsto f^*$ defined by $(\sum_{s\in\Gamma} f_s s)^* = \sum_{s\in\Gamma} f_s s^{-1}$.

For each $k \in \mathbb{N}$, we endow $\mathbb{R}^k$ with the supremum norm $\|\cdot\|_\infty$. For each $1 \le p \le +\infty$, we endow $\ell^p(\Gamma, \mathbb{R}^k) = (\ell^p(\Gamma))^k$ with the $\ell^p$-norm

$$\|(f_1, \dots, f_k)\|_p = \left\|\Gamma \ni s \mapsto \left\|(f_1(s), \dots, f_k(s))\right\|_\infty\right\|_p. \tag{10}$$

We shall write elements of $(\ell^p(\Gamma))^k$ as row vectors.

The algebraic structures of $\mathbb{Z}\Gamma$ and $\ell^1(\Gamma)$ also extend to some other situations naturally. For example, $(\mathbb{R}/\mathbb{Z})^\Gamma$ becomes a right $\mathbb{Z}\Gamma$-module naturally.

For any $n, k \in \mathbb{N}$, we also endow $M_{n\times k}(\ell^1(\Gamma))$ with the norm

$$\left\|(f_{i,j})_{1\le i\le n, 1\le j\le k}\right\|_1 := \sum_{1\le i\le n, 1\le j\le k} \left\|f_{i,j}\right\|_1.$$

The involution of $\ell^1(\Gamma)$ also extends naturally to an isometric linear map $M_{n\times k}(\ell^1(\Gamma)) \to M_{k\times n}(\ell^1(\Gamma))$ by

$$(f_{i,j})^*_{1\le i\le n,1\le j\le k} := \left(f^*_{j,i}\right)_{1\le i\le k,1\le j\le n}.$$

To simplify the notation, we shall write $M_k(\cdot)$ for $M_{k\times k}(\cdot)$. Note that $M_k(\ell^1(\Gamma))$ is a Banach algebra.

By an algebraic action of $\Gamma$, we mean an action of $\Gamma$ on a compact abelian group by automorphisms.

For a locally compact abelian group $X$, we denote by $\hat{X}$ its Pontryagin dual. Then for any compact abelian group $X$, there is a natural one-to-one correspondence between algebraic actions of $\Gamma$ on $X$ and actions of $\Gamma$ on $\hat{X}$ by automorphisms. There is also a natural one-to-one correspondence between the latter and left $\mathbb{Z}\Gamma$ -module structure on $\hat{X}$. Thus, when we have an algebraic action of $\Gamma$ on $X$, we shall talk about the left $\mathbb{Z}\Gamma$ -module $\hat{X}$. And when we have a left $\mathbb{Z}\Gamma$ -module $W$, we shall treat $W$ as a discrete abelian group and talk about the algebraic action of $\Gamma$ on $\widehat{W}$.

Note that for each $k\in\mathbb{N}$, we may identify the Pontryagin dual $\widehat{(\mathbb{Z}\Gamma)^k}$ of $(\mathbb{Z}\Gamma)^k$ with $((\mathbb{R}/\mathbb{Z})^k)^\Gamma = ((\mathbb{R}/\mathbb{Z})^\Gamma)^k$ naturally. Under this identification, the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k}$ is just the left shift action on $((\mathbb{R}/\mathbb{Z})^{\mathrm{k}})^\Gamma$. If $J$ is a left $\mathbb{Z}\Gamma$ -submodule of $(\mathbb{Z}\Gamma)^k$, then $\widehat{(\mathbb{Z}\Gamma)^k/J}$ is identified with

$$\{(x_1,\ldots,x_k)\in((\mathbb{R}/\mathbb{Z})^\Gamma)^k : x_1\mathrm{g}_1^*+\ldots+xk\mathrm{g}_k^* = 0_{(\mathbb{R}/\mathbb{Z})^\Gamma},\quad \text{for all } (\mathrm{g}_1,\ldots,\mathrm{g}_k)\in J\}.$$

We denote by $\rho$ the canonical metric on $\mathbb{R}/\mathbb{Z}$ defined by

$$\rho(t+\mathbb{Z},s+\mathbb{Z}) := \lim_{m\in\mathbb{Z}}|t-s-m|.$$

For $k\in\mathbb{N}$, we denote by $\rho_\infty$ the metric on $(\mathbb{R}/\mathbb{Z})^\Gamma$ defined by

$$\rho_\infty\big((t_1,\ldots,t_k),(s_1,\ldots,s_k)\big) := \max_{1\le j\le k}\rho\big(t_j,s_j\big). \tag{11}$$

An action of $\Gamma$ on a compact space $X$ is called expansive if there is a constant $c>0$ such that $\sup_{s\in\Gamma}\rho(sx,sy)>c$ for all distinct $x,y$ in $X$, where $\rho$ is a compatible metric on $X$. It is easy to see that the definition does not depend on the choice of $\rho$. If $\hat{X}=(\mathbb{Z}\Gamma)^k/J$ for some $k\in\mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, then the $\mathbb{Z}\Gamma$ -action on $X$ is expansive exactly when there exists $c>0$ such that the only $x\in X$ satisfying

$$\sup_{s\in\Gamma}\rho_\infty(x_s,0_{(\mathbb{R}/\mathbb{Z})^{\mathrm{k}}})>c$$

is $0_X$.

We shall need the following result [468].

**Proposition (6.2.3)[396]:** Let $\Gamma$ act on a compact abelian group X expansively by automorphisms. Then $\hat{\mathrm{X}}$ is a finitely generated left $\mathbb{Z}\Gamma$ -module.

See [458],[108] for details on the entropy theory of countable amenable groups. Throughout $\Gamma$ will be a countable amenable group.

Let $\Gamma$ act on a compact space $X$ continuously. Fix a compatible metric $\rho$ on $X$ and a left Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ in $\Gamma$, i.e. each $F_n$ is a nonempty finite subset of $\Gamma$ and $|\frac{KF_n\backslash F_n|}{|F_n|}\to 0$ as $n\to\infty$ for every finite set $K\subseteq\Gamma$. For a finite subset $F$ of $\Gamma$ and $\varepsilon>0$, we say that a set $Z\subseteq X$ is $(\rho,F,\varepsilon)$-separated if for any distinct $y,z\in Z$ one has $\max_{s\in F}\rho(sy,sz)>\varepsilon$. Denote by $N_{\rho,F,\varepsilon}(X)$ the maximal cardinality of $(\rho,F,\varepsilon)$-separated subsets of $X$.Then the topological entropy of the action $\Gamma\curvearrowright X$ is defined as

$$\mathrm{h_{top}}(X) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{\log N_{\rho,F_n,\varepsilon}(X)}{|F_n|}, \tag{12}$$

and does not depend on the choice of the Følner sequence $\{\{F_n\}_{n\in\mathbb{N}}$ and the metric $\rho$.

For a measure-preserving action of $\Gamma$ on a probability measure space $(X, B_X, \mu)$, one also has the measure entropy or Kolmogorov–Sinai entropy $h_\mu(X)$ defined.

When $\Gamma$ acts on a compact group $X$ by automorphisms, the topological entropy and the measure entropy with respect to the normalized Haar measure $\mu_X$ on $X$ coincide [411]. Thus we shall simply denote by $h(X)$ this common value, and refer to it the entropy of the action.

One has the following Yuzvinskii addition formula [443]:

**Proposition (6.2.4)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Let $Y$ be a closed $\Gamma$-invariant normal subgroup of $X$. Consider the restriction of the $\Gamma$-action to $Y$ and the induced $\Gamma$-action on $X/Y$ . Then

$$h(X) = h(Y) + h(X/Y).$$

The local entropy theory was initiated by Blanchard [402]. See [420],[422] for a nice account for the case $\Gamma = \mathbb{Z}$. In [435],[436] Kerr and the second named author gave a systematic combinatorial approach to the local entropy theory for general countable amenable groups. Here we follow the terminologies in [435],[436]. Throughout, $\Gamma$ will be a countable amenable group.

**Definition (6.2.5)[396]:** Let $\Gamma$ act on a compact space $X$ continuously. For a tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of $X$, we say that a finite set $F \subseteq \Gamma$ is an independence set for $\boldsymbol{A}$ if for every function $\sigma: F \to \{1, 2, \dots, k\}$ one has $\bigcap_{s\in F} s^{-1} A_{\sigma(s)} \neq \emptyset$.We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ an *IE-tuple* if for every product neighborhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$, there exist a nonempty finite set $K \subseteq \Gamma$ and $c, \varepsilon > 0$ such that for any finite set $F \subseteq \Gamma$ with $|KF \backslash F| \leq \varepsilon |F|$ the tuple $U_1, \cdots, U_k$ has an independence set $F' \subseteq F$ with $|F'| \geq c|F|$. We denote the set of $IE$-tuples of length $k$ by $IE_k(X)$.

We need the following properties of IE-tuples [435]. For a continuous action of $\Gamma$ on a compact space $X$, we denote by $\mathcal{M}(X, \Gamma)$ the set of $\Gamma$ -invariant Borel probability measures on $X$.

**Theorem (6.2.6)[396]:** Let $\Gamma$ act on compact spaces $X$ and $Y$ continuously. Let $k \in \mathbb{N}$. Then the following hold:

**(i)** $IE_k(X)$. is $a$ closed $\Gamma$ -invariant subset of $X^k$ , for the product $\Gamma$ -action on $X^k$ .

**(ii)** $IE_2(X)$ has non-diagonal elements if and only if $h(X) > 0$.

**(iii)** $IE_1(X)$ is the closure of $\bigcup_{\mu\in\mathcal{M}(X,\Gamma)} \mathrm{supp}(\mu)$.

**(iv)** Let $\pi: X \to Y$ be a $\Gamma$-equivariant continuous surjective map. Then $(\pi \times \cdots \times \pi)(IE_k(X)) = IE_k(Y)$.

**(v)** $IE_k(X \times Y) = IE_k(X) \times IE_k(Y)$, where we take the product $\Gamma$-action on $X \times Y$, and identify $(X \times Y)^k$ with $X^k \times Y^k$ naturally.

**Definition (6.2.7)[396]:** Let $\Gamma$ act on a compact space $X$ continuously and let $\mu \in M(X, \Gamma)$. For a tuple $\boldsymbol{A} = (A_1, \dots, A_k)$ of subsets of $X$ and a subset $D$ of $X$, we say that a finite set $F \subseteq \Gamma$ is an independence set for $\boldsymbol{A}$ relative to $D$ if for every function $\sigma : F \to \{1, 2, \dots, k\}$ one has $D \cap \bigcap_{s\in F} s^{-1} A_{\sigma(s)} \neq \emptyset$ .We call a tuple $\boldsymbol{x} = (x_1, \dots, x_k) \in X^k$ a $\mu$-*IE-tuple* if for every product neighborhood $U_1 \times \cdots \times U_k$ of $\boldsymbol{x}$, there exist $c, \delta > 0$ such that for any nonempty finite set $K \subseteq \Gamma$ and $\varepsilon > 0$ one can find a nonempty finite set $K \subseteq \Gamma$ with $|KF \backslash F| \leq \varepsilon |F|$ such that for every Borel set $D \subseteq X$ with $\mu(D) \geq 1 - \delta$ the tuple

$(U_1, ..., U_k)$ has an independence set $F' \subseteq F$ relative to $D$ with $|F'| \geq c|F|$. We denote the set of $\mu$-IE-tuples of length $k$ by $IE_k^\mu(X)$.

We need the following properties of measure IE-tuples [436]:

**Theorem (6.2.8)[396]:** Let $\Gamma$ act on compact spaces $X$ and $Y$ continuously. Let $\mu \in \mathcal{M}(X,\Gamma)$ and $\nu \in \mathcal{M}(Y, \Gamma)$. Let $k \in \mathbb{N}$. Then the following hold:

**(i)** $IE_k^\mu(X)$ is a closed $\Gamma$-invariant subset of $X^k$, for the product $\Gamma$-action on $X^k$.

**(ii)** $IE_2^\mu(X)$ has non-diagonal elements if and only if $h_\mu(X) > 0$.

**(iii)** $IE_1^\mu(X) = \mathrm{supp}(\mu)$.

**(iv)** Let $\pi: X \to Y$ be a $\Gamma$-equivariant continuous surjective map. Then $(\pi \times \cdots \times \pi)(IE_k^\mu(X)) = IE_k^{\pi_*(\mu)}(Y)$.

**(v)** $IE_k^{\mu\times\nu}(X \times Y) = IE_k^\mu(X) \times IE_k^\nu(Y)$, where we take the product $\Gamma$-action on $X \times Y$, and identify $(X \times Y)^k$ with $X^k \times Y^k$ naturally.

**(vi)** $IE_k(X)$ is the closure of $\bigcup_{\mu\in\mathcal{M}(X,\Gamma)} IE_k^\mu(X)$.

We prove the following algebraic characterizations for expansiveness of algebraic actions. Throughout $\Gamma$ will be a countable discrete group. For a unital ring $R$, a right $R$-module $\mathfrak{M}$ and a left $R$-module $\mathfrak{N}$, we denote by $\mathfrak{M} \otimes_R \mathfrak{N}$ the tensor product of $\mathfrak{M}$ and $\mathfrak{N}$ [397], which is an abelian group.

**Theorem (6.2.9)[396]:** *Let* $\Gamma$ act on a compact abelian group $X$ by automorphisms. Then the following are equivalent:

**(i)** the action is expansive;

**(ii)** the left $\mathbb{Z}\Gamma$-module $\hat{X}$ is finitely generated, and if we identify $\hat{X}$ with $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, then there exists $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the rows of $A$ are contained in $J$;

**(iii)** there exist some $k \in \mathbb{N}$, some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, and some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the left $\mathbb{Z}\Gamma$-module $\hat{X}$ is isomorphic to $(\mathbb{Z}\Gamma)^k/J$ and the rows of $A$ are contained in $J$;

**(iv)** the left $\mathbb{Z}\Gamma$-module $\hat{X}$ is finitely generated, and $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} \hat{X} = \{0\}$.

Previously, characterizations of expansiveness for algebraic actions have been obtained in various special cases, such as the case $\Gamma = \mathbb{Z}^d$ for $d \in \mathbb{N}$ [467], the case $\Gamma$ is abelian [456], the case $\hat{X} = \mathbb{Z}\Gamma/J$ for a finitely generated left ideal $J$ of $\mathbb{Z}\Gamma$ [414], the case $X$ is connected and finite-dimensional [401], and the case $\hat{X} = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ for some $f \in \mathbb{Z}\Gamma$ [413].

When $\Gamma$ is abelian, we have the following characterization of expansive algebraic actions.

**Proof:** (i)⇒(ii): By Proposition (6.2.3) we know that $\hat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module.

Identify $\hat{X}$with $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$. Denote by $\Omega$ the set of finitely generated $\mathbb{Z}\Gamma$-submodules of $J$. We claim that there exists some $\omega \in \Omega$ such that the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k/\omega}$ is expansive. Suppose that this fails. Let $\omega \in \Omega$. By Lemma (6.2.15) we can find some nonzero $y^\omega = (y_1^\omega, ..., y_k^\omega) \in (\ell^\infty(\Gamma))^k$ such that $\langle y^\omega, f\rangle = 0$ for every $f$ in a finite generating set $W$ of $\omega$, where the pairing $\langle y^\omega, f\rangle$ is given by the Eq. (13). Since $\omega$ is a left $\mathbb{Z}\Gamma$-module, we get $y^\omega f^* = 0$ for all $f \in W$, and hence $y^\omega f^* = 0$ for all $f \in \omega$. Replacing $y^\omega$ by $sy^\omega$ for some $s \in \Gamma$, we may assume that $\|y_{e_\Gamma}^\omega\|_\infty \geq \|y^\omega\|_\infty/2$. Replacing $y^\omega$ by $\lambda y^\omega$ for some $\lambda \in \mathbb{R}$, we may assume that $\|y^{\omega}_{e_\Gamma}\|_\infty = 1/2$. Then $y^{\omega} \in ([-1,1]^k)^\Gamma$. Note that $\Omega$ is directed by inclusion. Since the space $([-1,1]^k)^\Gamma$ is compact under the product topology, we can take a limit point $z$ of the net $(y^{\omega})_{\omega\in\Omega}$. Then $z f^* = 0$ for every $f \in J$, and hence $P(\lambda z) \in \widehat{(\mathbb{Z}\Gamma)^k/J}$ for every $\lambda \in \mathbb{R}$, where $P$ denotes the canonical map from $(\ell^\infty(\Gamma))^k$ to $((\mathbb{R}/\mathbb{Z})^\Gamma)^{\Gamma k}$. We also have $\|z_{e_\Gamma}\|_\infty = 1/2$, and hence $P(\lambda z) \neq 0$ for all $0 < \lambda \leq 1$. This contradicts the expansiveness of the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k/J}$. Thus our claimholds.

So let $f_1, \dots, f_n \in J$ such that the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k/\omega}$ is expansive for $\omega = \mathbb{Z}\Gamma f_1 + \cdots + \mathbb{Z}\Gamma f_n$. By Lemma (6.2.15) we can find some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1 1(\Gamma))$ such that the rows of $A$ are contained in $\omega$, and hence in $J$. This proves (ii).

(i)⇒(iii) is trivial.

(iii)⇒(i) can be proved as in the proof of (iv)⇒(i) of Lemma (6.2.15).

(iii)⇔(iv) follows from Lemma (6.2.16).

**Corollary (6.2.10)[396]:** Suppose that $\Gamma$ is abelian. Let $\Gamma$ act on a compact abelian group by automorphisms. Then the action is expansive if and only if $\hat{X}$ is a finitely generated $\mathbb{Z}\Gamma$-module and there exists $f \in \mathbb{Z}\Gamma$ being invertible in $\ell^1(\Gamma)$ such that $f\hat{X} = \{0\}$.

**Proof:** Suppose that the action is expansive. By Theorem (6.2.9) we can write the $\mathbb{Z}\Gamma$-module $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, and find $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the rows of $A$ are contained in $J$.

Since $\Gamma$ is abelian, $\ell^1(\Gamma)$ is a commutative algebra. Thus we can talk about the determinant $\det(B) \in \ell^1(\Gamma)$ for any $B \in M_k(\ell^1(\Gamma))$, and $B$ is invertible in $M_k(\ell^1(\Gamma))$ exactly when $\det(B)$ is invertible in $\ell^1(\Gamma)$. Furthermore, the formula of $A^{-1}$ in terms of $\det(A)$ and the minors of $A$ shows that $A^{-1}$ is of the form $(\det(A))^{-1}B$ for some $B \in M_k(\mathbb{Z}\Gamma)$.

For any $b \in (\mathbb{Z}\Gamma)^k$, we have

$$\det(A)b = \det(A)bA^{-1}A = b(\det(A)\cdot A^{-1})A = bBA \in J.$$

Thus $\det(A)\hat{X} = \{0\}$. This proves the "only if" part.

Now suppose that $\hat{X}$ is a finitely generated $\mathbb{Z}\Gamma$-module and there exists $f \in \mathbb{Z}\Gamma$ being invertible in $\ell^1(\Gamma)$ such that $f\hat{X} = \{0\}$. For any $h \in \ell^1(\Gamma)$ and $a \in \hat{X}$ we have

$$h\otimes a = hf^{-1}\otimes fa = 0$$

in $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} \hat{X}$. Thus $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} \hat{X} = \{0\}$. By Theorem (6.2.9) the action is expansive. This proves the "if" part.

From Corollary (6.2.10) we have the following consequence, which can also be deduced from [456].

**Corollary (6.2.11)[396]:** Suppose that $\Gamma$ is abelian. If $\Gamma$ acts expansively on a compact abelian group X by automorphisms and Y is a closed $\Gamma$-invariant subgroup of X with $\widehat{X/Y}$ being a finitely generated $\mathbb{Z}\Gamma$-module, then the induced $\Gamma$-action on $X/Y$ is expansive.

**Corollary (6.2.12)[396]:** Let $d \in \mathbb{N}$. If $\mathbb{Z}^d$ acts expansively on a compact abelian group X by automorphisms and Y is a closed $\mathbb{Z}^d$-invariant subgroup of X, then the induced $\mathbb{Z}^d$-action on $X/Y$ is expansive.

We remark that, based on Corollary (6.2.12), Schmidt also showed that if $\mathbb{Z}^d$ acts expansively on a compact (not necessarily abelian) group $X$ by automorphisms and $Y$ is a closed $\mathbb{Z}^d$-invariant normal subgroup of $X$, then the induced $\mathbb{Z}^d$-action on $X/Y$ is expansive [468].

Recall that a unital ring $R$ is said to be left Noetherian if every left ideal of $R$ is finitely generated. In general, we have

**Conjecture (6.2.13)[396]:** Suppose that $\Gamma$ is amenable and $\mathbb{Z}\Gamma$ is left Noetherian. If $\Gamma$ acts expansively on a compact group $X$ by automorphisms and $Y$ is a closed $\Gamma$-invariant normal subgroup of $X$, then the induced $\Gamma$-action on $X/Y$ is expansive.

The proof of [468] shows that, in order to prove Conjecture (6.2.13), it suffices to consider the case $X$ is abelian.

We start preparation for the proof of Theorem (6.2.9). We describe first a class of expansive algebraic actions, which are analogues of the expansive principal algebraic actions studied in [413]. Conditions (ii) and (iii) of Theorem (6.2.9)state that these actions are the largest expansive algebraic actions in the sense that every expansive algebraic action is the restriction of one of these actions to a closed invariant subgroup.

**Lemma (6.2.14)[396]:** Let $k \in \mathbb{N}$, and $A \in M_k(\mathbb{Z}\Gamma)$ be invertible in $M_k(\ell^1(\Gamma))$,. Then the canonical action of $\Gamma$ on $X_A A := \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ is expansive.

**Proof:** Let $x \in X_A$ be nonzero. Take $y \in ([-1/2,1/2]^k)^\Gamma \subseteq (\ell^\infty(\Gamma))^k$ with $P(y) = x$, where $P$ denotes the canonical map $(\ell^\infty(\Gamma))^k \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$.Then $y \neq 0$, and $yA^* \in \ell^\infty(\Gamma, \mathbb{Z}^k)$. Since $A$ is invertible in $M_k(\ell^1(\Gamma))$, so is $A^*$. Thus we have $yA^* \neq 0$, and hence $\|yA^*\|_\infty \geq 1$. Note that $\|yA^*\|_\infty \leq \|y\|_\infty \cdot \|A^*\|_1 = \|y\|_\infty \cdot \|A\|_1$. Therefore $\|y\|_\infty \geq \|A\|_1^{-1}$. Then $\sup_{s\in\Gamma} \rho_\infty(x_s, 0_{(\mathbb{R}/\mathbb{Z})^k}) \geq \|A\|_1^{-1}$, where $\rho_\infty$ is the metric on $(\mathbb{R}/\mathbb{Z})^k$ defined in (11), and hence the canonical action of $\Gamma$ on $X_A$ is expansive.

The following lemma is inspired by a question raised by Doug Lind and Klaus Schmidt, and uses the technique of [413].

**Lemma (6.2.15)[396]:** Let $k, n \in \mathbb{N}$, and $B \in M_{n\times k}(\mathbb{Z}\Gamma)$. Then the following are equivalent:

**(i)** the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n B}$ is expansive;

**(ii)** the linear map $\varphi: (\ell^\infty(\Gamma))^k \to (\ell^\infty(\Gamma))^n$ sending y to $yB^*$ is injective;

**(iii)** the linear map $\psi : (\ell^1(\Gamma))^n \to (\ell^1(\Gamma))^k$ sending $z$ to $zB$ has dense image;

**(iv)** there exists $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the rows of $A$ are contained in $(\mathbb{Z}\Gamma)^n B$.

**Proof:** (i)⇒(ii): Let $y \in (\ell^\infty(\Gamma))^k$ with $yB^* = 0$. Then $(\lambda y)B^* = 0$ for every $\lambda \in \mathbb{R}$. If we denote by $P$ the canonical map $(\ell^\infty(\Gamma))^k \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$, then $P(\lambda y) \in \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n B}$ for every $\lambda \in \mathbb{R}$. Since the canonical action of $\Gamma$ on $\widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n B}$ is expansive, we conclude that $y = 0$.

(ii)⇒(iii): Note that we can identify $(\ell^\infty(\Gamma))^n$ *and* $(\ell^\infty(\Gamma))^k$ with the dual spaces of $(\ell^1(\Gamma))^n$ *and* $(\ell^1(\Gamma))^k$ respectively in a canonical way. For instance, for $y = (y_1, \dots, y_k) \in (\ell^\infty(\Gamma))^k$ and $z = (z_1, \dots, z_k) \in (\ell^1(\Gamma))^k$, the pairing is given by

$$\langle y, z\rangle := (yz^*)_{e_\Gamma} = \left(\sum_{1\leq j\leq k} y_j z_j^*\right)_{e_\Gamma}. \tag{13}$$

Under such identification $\varphi$ is the dual of $\psi$. If the image of $\psi$ is not dense in $(\ell^1(\Gamma))^k$, then by the Hahn–Banach theorem we can find some nonzero bounded linear functional $y$ on $(\ell^1(\Gamma))^k$, vanishing at the image of $\psi$. Since $y$ is an element of $(\ell^\infty(\Gamma))^k$, this means $yB^* = 0$, which contradicts (ii). Thus the image of $\psi$ is dense in $(\ell^1(\Gamma))^k$.

(iii)⇒(iv): Since $M_k(\ell^1(\Gamma)),.$ is a Banach algebra, the set of invertible matrices in $M_k(\ell^1(\Gamma)),.$ is open. Thus we can find $z = z_1,\dots,z_k \in (\ell^1(\Gamma))^n$ such that the matrix

$$\begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \cdot B$$

in $M_k(\ell^1(\Gamma))$ is close enough to the identity matrix for being invertible. Since $\mathbb{Q}\Gamma$ is dense in $\ell^1(\Gamma)$, we may require that $z_1,\dots,z_k \in (\mathbb{Q}\Gamma)^n$. Then we can find some $N \in \mathbb{N}$ such that $Nz_1,\dots,Nz_k \in (\mathbb{Z}\Gamma)^k$ . Now we can set

$$A = N\begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \cdot B.$$

iv)⇒(i): Define $X_A$ as in Lemma (6.2.14). Then $\widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n B} \subseteq X_A$ . Since the canonical action of $\Gamma$ on $X_A$ is expansive by Lemma (6.2.14), its restriction on $\widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n B}$ is also expansive.

**Lemma (6.2.16)[396]:** Let $k \in \mathbb{N}$ and $J$ be a left $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^k$ . Then the following are equivalent:

**(i)** there exists $A \in M_k(\mathbb{Z}\Gamma$ being invertible in $M_k(\ell^1(\Gamma))$ such that the rows of A are contained in $J$ ;

**(ii)** $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} ((\mathbb{Z}\Gamma)^k/J) = \{0\}$;

**(iii)** the left $\ell^1(\Gamma)$-module $(\ell^1(\Gamma))^k$ is generated by $J$ .

**Proof :**(i)⇒(ii): Write the $j$-th row of $A$ as $g_j$ . Denote by $e_1,\dots,e_k$ the standard basis of $(\mathbb{Z}\Gamma)^k$. Let $w \in \ell^1(\Gamma), 1 \le i \le k$, and $f \in \mathbb{Z}\Gamma$. Then in $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} ((\mathbb{Z}\Gamma)^k/J)$ we have

$$\begin{aligned} w \otimes (fe_i + J) &= \sum_{1\le m\le k} wf\delta_{i,m} \otimes (e_m + J) \\ &= \sum_{1\le m\le k} wf\left(\sum_{1\le j\le k} (A^{-1})_{i,j}A_{j,m}\right) \otimes (e_m + J) \\ &= \sum_{1\le j\le k} wf(A^{-1})_{i,j} \otimes \sum_{1\le m\le k} (A_{j,m}\, e_m + J) \\ &= \sum_{1\le m\le k} wf(A^{-1})_{i,j} \otimes \left(g_j + J\right) = 0. \end{aligned}$$

(ii)⇒(iii): Denote by $J'$ the left $\ell^1(\Gamma)$-submodule of $(\ell^1(\Gamma))^k$ generated by $J$ . Consider the map $\ell^1(\Gamma) \times ((\mathbb{Z}\Gamma)^k/J) \to (\ell^1(\Gamma))^k/J$ sending $(w, f + J)$ to $wf + J'$ . It induces a group homomorphism $\varphi : \ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} ((\mathbb{Z}\Gamma)^k/J) \to (\ell^1(\Gamma))^k/J'$ defined by $\varphi(w \otimes (f + J)) = wf + J'$ for all $w \in \ell^1(\Gamma)$ and $f \in (\mathbb{Z}\Gamma)^k$ . Clearly $\varphi$ is surjective. Since $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} ((\mathbb{Z}\Gamma)^k/J) = \{0\}$, we conclude that $(\ell^1(\Gamma))^k/J' = \{0\}$. That is, $(\ell^1(\Gamma))^k = J'$ .

(iii)⇒(i): The condition (iii) says that every $g \in (\ell^1(\Gamma))^k$ can be written as $a_1 f_1 + \cdots + a_n f_n$ for some $n \in N, a_1,\dots,a_n \in \ell^1(\Gamma)$, and $f_1 ,\dots, f_n \in J$ .Taking $g$ to the standard basis of the $\ell^1(\Gamma)$-module $(\ell^1(\Gamma))^k$, we find some $B \in M_{k\times n} (\ell^1(\Gamma))$ for some $n \in \mathbb{N}$, and some $f_1,\dots,f_n \in J$ such that $BC$ is the identity matrix in $M_k(\ell^1(\Gamma))$ , where

$$C = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

Since $M_k(\ell^1(\Gamma))$ is a Banach algebra, the set of invertible matrices in $M_k(\ell^1(\Gamma))$ is open. Approximating $B$ by some $B' \in M_{k\times n}(\mathbb{Q}\Gamma)$, we may assume that $B'C$ is invertible in $M_k(\ell^1(\Gamma))$ . Take some suitable $N \in \mathbb{N}$ such that $NB' \in M_{k\times n}(\mathbb{Z}\Gamma)$. Then we may set $A = (NB')C$.

We study $p$-expansiveness, a weak version of expansiveness. Throughout $\Gamma$ will be a countable discrete group.

**Notation (6.2.17)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. For every $x \in X$ and $\varphi \in \hat{X}$ , we denote by $\Psi_{x,\varphi}$ the function on $\Gamma$ defined by

$$\Psi_{x,\varphi}(s) = \langle sx, \varphi\rangle - 1, \quad s \in \Gamma,$$

where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and $\langle\cdot,\cdot\rangle : X \times \hat{X} \to \mathbb{T}$ denotes the canonical pairing between $X$ and $\hat{X}$ .

**Definition (6.2.18)[396]:** Let $1 \le p \le +\infty$. We say that an action of $\Gamma$ on a compact abelian group $X$ by automorphisms is $p$-expansive if there exist a finite subset $W$ of $\hat{X}$ and $\varepsilon > 0$ such that $e_X$ is the only point $x$ in $X$ satisfying

$$\sum_{\varphi\in W} \left\|\Psi_{x,\varphi}\right\|_p < \varepsilon.$$

In the following we collect some basic properties of $p$-expansiveness. Assertion (iv) below justifies our terminology of $p$-expansiveness. Recall that for a unital ring $R$, a left $R$-module $\mathfrak{M}$ is called finitely presented if $\mathfrak{M} = R^k/J$ for some $k \in \mathbb{N}$ and some finitely generated left $R$ submodule $J$ of $R^k$ [439]. If $R$ is left Noetherian, then every finitely generated left $R$-modul is finitely presented [439].

**Proposition (6.2.19)[396]:** Let α be an action of $\Gamma$ on a compact abelian group by automorphisms. Let $1 \le \mathrm{p} \le +\infty$. Then the following hold:

**(i)** If $\alpha$ is $p$-expansive, then it is $q$-expansive for all $1 \le q \le p$.

**(ii)** If $\alpha$ is $p$-expansive, then $\hat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module.

**(iii)** If $\alpha$ is $p$-expansive, then for any finite subset $W$ of $\hat{X}$ generating $\mathbb{Z}\Gamma$ as a left $\mathbb{Z}\Gamma$-module, there exists $\varepsilon > 0$ such that $e_X$ is the only point $x$ in $X$ satisfying $\sum_{\varphi\in W}\left\|\Psi_{x,\varphi}\right\|_p < \varepsilon$.

**(iv)** $\alpha$ is $\infty$-expansive if and only if it is expansive.

**(v)** Suppose that $\hat{X}$ is a finitely presented left $\mathbb{Z}\Gamma$-module. Write $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n A$ for some $k, n \in \mathbb{N}$ and $A \in M_{n\times k}(\mathbb{Z}\Gamma)$. Then $\alpha$ is $p$-expansive if and only if the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^n$ sending a to $a\, A^*$ is injective.

**Proof:** **(i)** This follows from the fact that for any $f \in \ell^q(\Gamma)$ with $\| f \|_q \le 1$, one has $\| f \|_p^p \le \| f \|_q^q$ when $p < +\infty$ and $\| f \|_\infty \le \| f \|_q$ when $p = +\infty$.

(**ii**) Suppose that $\alpha$ is $p$-expansive. Let $W$ and ε be as in Definition (6.2.18). Denote by $G$ the $\mathbb{Z}\Gamma$-submodule of $\hat{X}$ generated by $W$. If $x \in X$ satisfies $\langle x, \psi\rangle = 1$ for all $\psi \in G$, then $\Psi_{x,\varphi} = 0$ for all $\varphi \in W$ and hence $x = e_X$. By Pontryagin duality we have $G = \hat{X}$ .

(**iii**) This follows from the fact that for any $x \in X, \varphi, \psi \in \hat{X}$ , and $a, b \in \mathbb{Z}\Gamma$ one has

$$\left\|\Psi_{x,a\varphi+b\psi}\right\|_p \le \|a\|_1\left\|\Psi_{x,\varphi}\right\|_p + \|b\|_1\left\|\Psi_{x,\psi}\right\|_p. \tag{14}$$

To prove the assertions (iv) and (v), we observe a general fact first. Suppose that $\hat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module, and write $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left

$\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$. For each $x \in X$ denote by $\Phi_x$ the function $s \mapsto \rho_\infty(x_s, 0_{(\mathbb{R}/\mathbb{Z})^k})$ on $\Gamma$, where $\rho_\infty$ is the canonical metric on $(\mathbb{R}/\mathbb{Z})^k$ defined in the Eq. (11). Denote by $e_1, \dots, e_k$ the standard basis of $(\mathbb{Z}\Gamma)^k$. Set $W = \{e_1 + J, \dots, e_k + J\}$. Note that there exists $C > 0$ such that

$$C|t| \leq \left|e^{2\pi i t} - 1\right| \leq C^{-1}|t|$$

for all $t \in [-1/2, 1/2]$. It follows that there exists $C_1 > 0$ such that

$$C_1\|\Phi_x\|_p \leq \sum_{\varphi \in W} \left\|\Psi_{x,\varphi}\right\|_p \leq C_1^{-1}\|\Phi_x\|_p \tag{15}$$

for all $x \in X$.

(iv). By the assertion (ii) and Proposition (6.2.3) both $\infty$-expansiveness and expansiveness imply that $\widehat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module. Now the assertion (iv) follows from the $p = \infty$ case of the inequalities (15) and the assertion (iii).

(v). Denote by $P$ the canonical map $\ell^\infty(\Gamma, \mathbb{R}^k) \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. Suppose that the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^n$ sending $a$ to $aA^*$ is not injective. Take a nonzero $a \in (\ell^p(\Gamma))^k$ with $aA^* = 0$. Then for any $\lambda \in \mathbb{R}$, one has $\lambda\, aA^* = 0$ and hence $P(\lambda a) \in X$.When $\lambda \to 0$, one has $\left\|\Phi_{P(\lambda\, a)}\right\|_p \to 0$, andHence $\sum_{\varphi \in W}\left\|\Phi_{P(\lambda\, a),\varphi}\right\| \to 0$. Since $a \neq 0$, when $|\lambda|$ is sufficiently small and nonzero, $P(\lambda a) \neq e_X$. Thus $\alpha$ is not $p$-expansive. This proves the "only if" part.

Suppose that $\alpha$ is not $p$-expansive. Then we can find a nonzero $x \in X$ such that $\|\Phi_x\|_p < \|A\|_1^{-1}$. Take a lift $\tilde{x}$ of $x$ in $\ell^\infty(\Gamma, \mathbb{R}^k)$ with $\|\tilde{x}\|_p$.Then $\tilde{x}\, A^* \in \ell^\infty(\Gamma, \mathbb{Z}^n)$, and

$$\|\tilde{x}\, A^*\|_\infty \leq \|\tilde{x}\, A^*\|_p \leq \|\tilde{x}\,\|_p\|\, A^*\|_1 = \|\Phi_x\|_p\|\, A\|_1 < 1.$$

It follows that $\tilde{x}\, A^* = 0$. Since $P(\,\tilde{x}) = x$ is nonzero, $\tilde{x} \neq 0$. This proves the "if" part.

**Notation (6.2.20)[396]:** For $f \in \mathbb{Z}\Gamma$, we denote by $\alpha_f$ the canonical $\Gamma$-action on $X_f := \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$. We also denote by $C_0(\Gamma)$ the space of $C$-valued functions on $\Gamma$ vanishing at infinity.

By Proposition (6.2.19).(v), for any $f \in \mathbb{Z}\Gamma$ and $1 \leq p \leq +\infty$, the action $\alpha$ f is $p$-expansive if and only if for any nonzero g $\in \ell^p(\Gamma)$ one has g $f^* \neq 0$. The latter is related to the analytic zero divisor conjecture and has been studied extensively in [449]–[453],[462].

**Example (6.2.21)[396]:** Recall that the class of elementary amenable groups is the smallest class of groups containing all finite groups and all abelian groups and is closed under taking subgroups, quotient groups, extensions, and inductive limits. Suppose that $\Gamma$ is torsion free and elementary amenable. Then for any nonzero $f \in \mathbb{C}\Gamma$ and nonzero g $\in \ell^2_{\mathbb{C}}(\Gamma)$, one has g $f \neq 0$ [449]. Thus $\alpha$ f is 2-expansive for every nonzero $f \in \mathbb{Z}\Gamma$. On the other hand, by [413], for $f \in \mathbb{Z}\Gamma, \alpha\, f$ is expansive exactly when $f$ is invertible in $\ell^1(\Gamma)$.

**Example (6.2.22)[396]:** Suppose that $s \in \Gamma$ has infinite order. Denote by $\Gamma'$ the subgroup$\Gamma$ generated by $s$. For any nonzero $f \in \mathbb{C}\,\Gamma'$ and nonzero g $\in C_0(\Gamma)$, one has g $f \neq 0$ [452].Using the cosets decomposition of $\Gamma$, it follows that for any nonzero $f \in \mathbb{C}\,\Gamma'$ and nonzero g $\in C_0(\Gamma)$,, one has g $f \neq 0$. Thus, for any nonzero $f \in \mathbb{C}\,\Gamma'$ the action $\alpha_f$ is $p$-expansive for all $1 \leq p < +\infty$. Note that $s - 1$ is not invertible in $\ell^1(\Gamma)$., thus $\alpha_{s-1}$ is not expansive by [413].

**Example (6.2.23)[396]:** Let $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$. One may identify $\mathbb{Z}\Gamma$ with the ring $Z[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ of Laurent polynomials with integer coefficients in the variables $u_1, \ldots, u_d$. For any nonzero $f \in \mathbb{C}\,\Gamma$ and nonzero $g \in \ell^{\frac{2d}{d-1}}(\Gamma)$, one has $g f \neq 0$ [453]. Thus $\alpha_f$ is 2-expansive for every nonzero $f \in \mathbb{Z}\,\Gamma$. Set $h = 2d - 1 - \sum_{j=1}^{d}(u_j + u_j^{-1}) \in \mathbb{Z}\,\Gamma$. Then there exists a nonzero $g \in \ell^\infty(\Gamma)$ such that $gh = 0$ and $g \in \ell^p(\Gamma)$ for all $\frac{2d}{d-1} < p \leq +\infty$ [462]. Thus, for any $\frac{2d}{d-1} < p \leq +\infty$, the action $\alpha_h$ is not $p$-expansive.

**Example (6.2.24)[396]:** Let $\Gamma$ be a free group with canonical generators $s_1, \ldots, s_d$, for $d \geq 2$. For any nonzero $f \in \mathbb{C}\,\Gamma$, one has $g f \neq 0$ for every nonzero $g \in \ell^2_{\mathbb{C}}(\Gamma)$ [450]. Thus the action $\alpha_f$ is 2-expansive for every nonzero $f \in \mathbb{Z}\,\Gamma$. Endow $\Gamma$ with the word length with respect to $s_1, \ldots, s_d$. For each $n \in \mathbb{Z}_{\geq 0}$ denote by $\chi_n$ the sum of the elements in $\Gamma$ with length $n$. Set $g = \sum_{n=0}^{\infty}(-1)^n(2d-1)^{-n}\chi_{2n} \in \ell^\infty(\Gamma)$. Then $g\chi_1 = 0$ and $g \in \ell^p(\Gamma)$ for all $2 < p \leq +\infty$ [453]. Thus, for any $2 < p \leq +\infty$, the action $\alpha_{\chi_1}$ is not $p$-expansive.

**Example (6.2.25)[396]:** Let $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$. Denote by $P$ the natural projection $\mathbb{R}^d \to (\mathbb{R}/\mathbb{Z})^d = \hat{\Gamma}$. For each $f \in \mathbb{C}\,\Gamma$, via the pairing between $\Gamma$ and $\hat{\Gamma}$ we may think of $f$ as a function on $\hat{\Gamma}$. Denote by $Z(f)$ the zero set of $f$ as a function on $\hat{\Gamma}$. For $f = \sum_{s\in\Gamma} \lambda_s s \in C\Gamma$, set $\bar{f} = \sum_{s\in\Gamma} \overline{\lambda_s s}$. The set $Z(\bar{f})$ is contained in the image of a finite union of hyperplanes in $\mathbb{R}^d$ under $P$ if and only if $g f \neq 0$ for all nonzero $g \in C_0(\Gamma)$ [453]. Thus, For $f \in \mathbb{Z}\,\Gamma$, if $Z(f)$ is contained in the image of a finite union of hyperplanes in $\mathbb{R}^d$ under $P$ (for instance when $Z(f)$ is a finite set), then so is $Z(f^*)$, and hence $\alpha_f$ is $p$-expansive for all $1 \leq p < +\infty$.

A finitely generated elementary amenable group either contains a nilpotent subgroup with finite index or has a free subsemigroup with two generators [409].

**Example (6.2.26)[396]:** Suppose that $\Gamma$ has a free subsemigroup generated by two elements $s$ and $t$. Set $f = \pm 3 - (1 + s - s^2)t \in \mathbb{Z}\Gamma$. Then $f$ is not invertible in $\ell^1(\Gamma)$ [443], and thus by [413] $\alpha\, f$ is not expansive. An argument similar to that in [445] shows that for any $g \in C_0(\Gamma)$, if $g f^* = 0$, then $g = 0$. Thus $\alpha_f$ is $p$-expansive for all $1 \leq p < +\infty$.

It is well known that when $\Gamma$ is amenable, every continuous expansive $\Gamma$- action on a compact space has finite topological entropy (the case $\Gamma = \mathbb{Z}$ is proved in [474]; the proof there also works for general countable amenable groups $\Gamma$). Next we show that 1-expansiveness and 2-expansiveness characterize finite entropy for finitely presented algebraic actions of countable amenable groups. The definition of entropy is recalled.

**Theorem (6.2.27)[396]:** Suppose that _ is amenable. If $A \in M_k(\mathbb{C}\Gamma)$ for some $k \in \mathbb{N}$ and $aA = 0$ for some nonzero $a \in (\ell^2_{\mathbb{C}}(\Gamma))^k$, then $bA = 0$ for some nonzero $b \in (\mathbb{C}\Gamma)^k$ .

We remark that the proof of Theorem (6.2.27) uses the group von Neumann algebra of $\Gamma$.

**Lemma (6.2.28)[396]:** Let K and $F$ be nonempty finite subsets of $\Gamma$. Then there exists a finite subset $F_1$ of $F$ with $\frac{|F_1|}{|F|} \geq \frac{1}{2|K|+1}$ and $\left((F_1F_1^{-1})\backslash\{e_\Gamma\}\right) \subseteq \Gamma\backslash K$.

**Proof:** Let $F_1$ be a maximal subset of $F$ subject to the condition that $s' \notin Ks$ for all distinct $s, s \in F_1$. Then $F \subseteq (\{e_\Gamma\} \cup K \cup K^{-1})F_1$. Thus $|F| \leq |\{e_\Gamma\} \cup K \cup K^{-1}| \cdot |F_1| \leq (2|K|+1)|F_1|$.

**Lemma (6.2.29)[396]:** Suppose that $\Gamma$ is amenable. Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. Suppose that $\hat{X} = (\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, and that there exists $0 \neq a \in (\mathbb{R}\Gamma)^k$ satisfying $ab^* = 0$ for all $b \in J$. Then $h(X) = +\infty$.

**Proof:** Denote by $K$ the support of $a$ as a $\mathbb{R}^k$-valued function on $\Gamma$. Fix a compatible metric $\rho$ on $X$. Denote by $P$ the natural projection $\ell^\infty(\Gamma, \mathbb{R}^k) \to ((\mathbb{R}/\mathbb{Z})^k)$. Then $P(\lambda a) \in X$ for every $\lambda \in \mathbb{R}$. If $\lambda_1, \lambda_2 \in (0, 1/\|a\|_\infty)$ are distinct, then $P(\lambda_1 a) \neq P(\lambda_2 a)$.

Let $M \in \mathbb{N}$. Take distinct $\lambda_1, \ldots, \lambda_M \in (0, 1/\|a\|_\infty)$. For each $1 \leq j \leq M$, set

$$Y_j = \left\{x \in X \coloneqq x_s = \left(P(\lambda_i a)\right)_s \text{ for all } s \in K\right\}.$$

Then the sets $Y_1, \ldots, Y_M$ are pairwise disjoint closed subsets of $X$. Thus we can find $\varepsilon > 0$ such that if $x \in Y_i$ and $y \in Y_j$ for some $1 \leq i < j \leq M$, then $\rho(x, y) > \varepsilon$.

Let $F$ be a nonempty finite subset of $\Gamma$. By Lemma (6.2.28) we can find a finite subset $F_1$ of $F$ with $\frac{|F_1|}{|F|} \geq \frac{1}{2|KK^{-1}|+1}$ and $((F_1 F_1^{-1})\backslash\{e_\Gamma\}) \subseteq \Gamma\backslash KK^{-1}$. Then the sets $s^{-1}K$ for $s \in F_1$ are pairwise disjoint. For each $\sigma \in \{1, \ldots, M\}^{F_1}$ ,set

$$x_\sigma = \sum_{s \in F_1} s^{-1} P\left(\lambda_{\sigma(s)} a\right).$$

Then $sx_\sigma \in Y_\sigma(s)$ for every $s \in F_1$. Thus the set $\{x_\sigma : \sigma \in \{1, \ldots, M\}^{F_1}\}$ is $(\rho, F, \varepsilon)$-separated. Therefore

$$N_{\rho,F,\varepsilon}(X) \geq M^{|F_1|} \geq M^{|F|/(2|KK^{-1}|+1)}.$$

It follows that $h(X) \geq \frac{1}{2|KK^{-1}|+1} \log M$. Since $M$ is arbitrary, we get $h(X) = +\infty$.

The following lemma is well known, and can be proved by a simple volume comparison argument (see for example the proof of [461]).

**Lemma (6.2.30)[396]:** Let $V$ be a finite-dimensional normed space over $\mathbb{R}$. Let $\varepsilon > 0$.Then any $\varepsilon$-separated subset of the unit ball of $V$ has cardinality at most $\left(1 + \frac{2}{\varepsilon}\right)^{\dim V}$.

**Theorem (6.2.31)[396]:** Suppose that $\Gamma$is amenable. Let $\alpha$ be an action of $\Gamma$ on a compact abelian group $X$ by automorphisms. Suppose that $\hat{X}$ is a finitely presented left $\mathbb{Z}\Gamma$-module, and write $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n A$ for some $k, n \in \mathbb{N}$ and $A \in M_{n\times k}(\mathbb{Z}\Gamma)$. Then the following are equivalent:

(i) $h(X) < +\infty$.

(ii) $\alpha$ is 1-expansive.

(iii) $\alpha$ is 2-expansive.

(iv) The linear map $(\mathbb{R}\Gamma)^k \to (\mathbb{R}\Gamma)^n$ sending $a$ to $aA^*$ is injective.

(v) The additive map $(\mathbb{Z}\Gamma)^k \to (\mathbb{Z}\Gamma)^n$ sending $a$ to $aA^*$ is injective.

We need some preparation for the proof of Theorem (6.2.31). First we need the following result of Elek [418]. He assumed $\Gamma$ to be finitely generated and $k = 1$, which are unnecessary.

We are ready to prove Theorem (6.2.31).

**Proof:** (i)⇒(iv) follows from Lemma (6.2.29).

(iv)⇒(i): Note that the definitions of $(\rho, F, \varepsilon)$-separated sets and $N_{\rho,F,\varepsilon}(X)$ extend directly to any continuous pseudometric $\rho$ on $X$. The Eq. (12) holds for any continuous pseudometric $\rho$ on $X$ which is dynamically generating in the sense that for any distinct $x, y \in X$ one has $\rho(sx, sy) > 0$ for some $s \in \Gamma$ [411] [443]. Recall the canonical metric $\rho_\infty$ on $(\mathbb{R}/\mathbb{Z})^k$ defined by the Eq. (11). Define a continuous pseudometric $\rho'$ on $X$ by $\rho'(x, y) = \rho_\infty(x_{e_\Gamma}, y_{e_\Gamma})$ for all $x, y \in X$. Clearly $\rho'$ is dynamically generating.

Denote by $K$ the union of $\{e_\Gamma\}$ and the support of $A$ as a $M_{n\times k}(\mathbb{Z})$-valued function on $\Gamma$. Let $\varepsilon > 0$ and $F$ be a nonempty finite subset of $\Gamma$. Take a $(\rho', F, \varepsilon)$-separated subset $E \subseteq X$ with $|E| = N_{\rho\, F,\varepsilon}(X)$. For each $x \in E$ denote by $\tilde{x}$ the element in $([0,1)^k)^\Gamma$ such that $x$ is the image of $\tilde{x}$ under the natural map $([0,1)^k)^\Gamma \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. Then $\tilde{x}A^* \in \ell^\infty(\Gamma, \mathbb{Z}^n)$ and $\|\tilde{x}A^*\|_\infty \le \|\tilde{x}\|_\infty \|A^*\|_1 \le \|A\|_1$.

For a finite subset $W$ of $\Gamma$, we shall identify $(\mathbb{R}^k)^W$ and $(\mathbb{R}^n)^W$ as linear subspaces of $\ell^\infty(\Gamma, \mathbb{R}^k) = (\ell^\infty(\Gamma))^k$ and $\ell^\infty(\Gamma, \mathbb{R}^n)$ respectively naturally, and denote by $pw$ the restriction maps $\ell^\infty(\Gamma, \mathbb{R}^k) \to (\mathbb{R}^k)^W$ and $\ell^\infty(\Gamma, \mathbb{R}^n) \to (\mathbb{R}^n)^W$.

Set $F' = \{s \in F : s^{-1}K \subseteq F^{-1}\}$. We define a map $\psi: E \to ((\mathbb{Z} \cap [-\|A\|_1, \|A\|_1])^n)^{(F')^{-1}}$ Sending $x$ to $p_{(F')^{-1}}(\tilde{x}A^*)$. Let $a \in ((\mathbb{Z} \cap [-\|A\|_1, \|A\|_1])^n)^{(F')^{-1}}$. We shall give an upper bound for $|\psi^{-1}(a)|$.

Consider the linear map $\xi: (\mathbb{R}^k)^{F^{-1}} \to (\mathbb{R}^n)^{F^{-1}K^{-1}}$ sending $z$ to $zA^*$. By (iv) $\xi$ is injective. Thus

$$\dim\ker\left(p_{(F')^{-1}} \circ \xi\right) \le \dim\left((\mathbb{R}^n)^{F^{-1}K^{-1}\setminus(F')^{-1}}\right) = n|F^{-1}K^{-1}\setminus(F')^{-1}|.$$

For each $x \in E$, set $x' = p_{F^{-1}}(\tilde{x}) \in (\mathbb{R}^k)^{F^{-1}}$. Note that $\tilde{x}A^* = x'A^*$ on $(F')^{-1}$. Fix $y \in \psi^{-1}(a)$. Then $(x' - y')A^* = 0$ on $(F')^{-1}$ for all $x \in \psi^{-1}(a)$. Thus $x' - y' \in ker(p_{(F')^{-1}} \circ \xi)$ for all $x \in \psi^{-1}(a)$.. Since $E$ is $(\rho', F, \varepsilon)$-separated, we see that the set $\{\frac{x'-y'}{2} : x \in \psi^{-1}(a)\}$ is $\frac{\varepsilon}{2}$-separated under the $\ell^\infty$-norm, and is clearly contained in the unit ball of $(\mathbb{R}^k)^{F^{-1}}$ with respect to the $\ell^\infty$-norm. By Lemma (6.2.30) we have

$$|\psi^{-1}(a)| \le \left(1 + \frac{4}{\varepsilon}\right)^{\dim\ker(p_{(F')^{-1}} \circ \xi)} \le \left(1 + \frac{4}{\varepsilon}\right)^{n|F^{-1}K^{-1}\setminus(F')^{-1}|}.$$

Therefore

$$\begin{aligned} N_{\rho'\, F,\varepsilon}(X) &= |E| \\ &\le (2\|A\|_1 + 1)^{n|F'|}\left(1 + \frac{4}{\varepsilon}\right)^{n|F^{-1}K^{-1}\setminus(F')^{-1}|} \\ &\le (2\|A\|_1 + 1)^{n|F|}\left(1 + \frac{4}{\varepsilon}\right)^{n|F^{-1}K^{-1}\setminus(F')^{-1}|}. \end{aligned}$$

When $F$ becomes sufficiently left invariant, $|F^{-1}K^{-1}\setminus(F')^{-1}|/|F|$ becomes arbitrarily small. It follows that

$$h(X) \le n\log(2\|A\|_1 + 1) \le +\infty.$$

(iv)⇒(iii): Suppose that (iii) fails. By Proposition (6.2.19).(v) we have $aA^* = 0$ for some nonzero $a \in (\ell^2(\Gamma))^k$. Then $aA^*A = 0$. By Theorem (6.2.27)we have $bA^*A = 0$ for some nonzero $b \in (\mathbb{C}\Gamma)^k$. Replacing $b$ by its real part or imaginary part, we may assume that $b \in (\mathbb{R}\Gamma)^k$ .Note that for any $w \in (\ell^2(\Gamma))^n$ and $w' \in (\ell^2(\Gamma))^k$ we have

$$\langle wA, w'\rangle = \langle w, w'A^*\rangle,$$

where we take $(\ell^2(\Gamma))^k$ and $(\ell^2(\Gamma))^n$ as the direct sum of copies of the Hilbert space $\ell^2(\Gamma)$. Thus $\langle bA^*, bA^*\rangle = \langle bA^*A, b\rangle = 0$, and hence $bA^* = 0$. Therefore (v) fails.

(iii)⇒(ii)⇒(iv) follows from Proposition (6.2.19).

(iv)⇒(v) is trivial.

(v)⇒(iv): Suppose that (iv) fails. Then $aA^* = 0$ for some nonzero $a \in (\mathbb{R}\Gamma)^k$ .Denote by $F$ (resp. $K$) the support of $a$ (resp. $A^*$) as a $\mathbb{R}^k$-valued (resp. $M_{k\times n}(\mathbb{Z})$-valued) function on $\Gamma$. Consider the equation $bA^* = 0$ for $b \in (\mathbb{R}F)^k$.One can interpret it as a system of integer-coefficients homogeneous linear equations indexed by $FK \times \{1, \dots, n\}$ which has variables indexed by $F \times \{1, \dots, k\}$. These linear equations have a nonzero solution given by $a$, thus has a nonzero integral solution. Therefore (v) fails.

We study $p$-homoclinic points. Throughout $\Gamma$ will be a countable discrete group.

When $\Gamma$ acts on a compact group $X$ by automorphisms, a point $x \in X$ is said to be homoclinic [445] if $sx \to e_X$ as $\Gamma \ni s \to \infty$. The set of all homoclinic points, denoted by $\Delta(X)$, is a $\Gamma$-invariant normal subgroup of $X$. Note that when $X$ is abelian, a point $x \in X$ is homoclinic exactly when $\Psi_{x,\varphi} \in C_0(\Gamma)$ for every $\varphi \in \hat{X}$ , where $\Psi_{x,\varphi}$ and $C_0(\Gamma)$ are defined in Notations (6.2.17) and (6.2.20) respectively.

**Definition (6.2.32)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. Let $1 \le p < +\infty$. We say that $x \in X$ is $p$-homoclinic if $\Psi_{x,\varphi} \in \ell^p(\Gamma)$ for every $\varphi \in \hat{X}$ .We denote by $\Delta^p(X)$ the set of all $p$-homoclinic points of $X$ .We also say $x \in X$ is $\infty$-homoclinic if it is homoclinic, and set $\Delta^\infty(X) = \Delta(X)$.

The set $\Delta^1(X)$ was studied in [447],[471] for algebraic $\mathbb{Z}^d$-actions. We shall see that both $\Delta^1(X)$ and $\Delta^2(X)$ play important roles in the study of entropy theory for algebraic actions.

**Proposition (6.2.33)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. Let $1 \le p \le +\infty$. Then the following hold:

**(i)** For any $p < q \le +\infty$, one has $\Delta^p(X) \subseteq \Delta^q(X)$.

**(ii)** $\Delta^p(X)$ is $a$ $\Gamma$-invariant subgroup of $X$.

**(iii)** If $\Gamma$ acts on a compact abelian group $Y$ by automorphisms and $\Phi: X \to Y$ is a continuous $\Gamma$-equivariant group homomorphism, then $\Phi(\Delta^p(X)) \subseteq \Delta^p(Y)$.

**(iv)** If $\hat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module and we write $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, then $\Delta^p(X)$ consists exactly of the elements $x \in X$ for which the function $s \mapsto \rho_\infty(x_s, 0_{(\mathbb{R}/\mathbb{Z})^k})$ on $\Gamma$ is in $\ell^p(\Gamma)$ when $p < +\infty$ or in $C_0(\Gamma)$ when $p = +\infty$, where $\rho_\infty$ is the canonical metric on $(\mathbb{R}/\mathbb{Z})^k$ defined by (11).

**(v)** If $\alpha$ is $p$-expansive, then $\Delta^p(X)$ is countable.

**(vi)** If $\mathbb{Z}\Gamma$ is left Noetherian, and $\alpha$ is p-expansive, then $\Delta^p(X)$is a finitely generated left $\mathbb{Z}\Gamma$-module.

**Proof:** The assertion (i) follows from the facts that $\ell^p(\Gamma) \subseteq C_0(\Gamma)$ and $\ell^p(\Gamma) \subseteq \ell^q(\Gamma)$ when $q < +\infty$. The assertions (ii) and (iii) are obvious.

The assertion (iv) follows from the inequalities (14) and (15).

Now we prove the assertion (iv). The case $p = +\infty$ is [445]. So we may assume $p < +\infty$. Suppose that $\alpha$ is $p$-expansive. By Proposition (6.2.19).(ii) we may write $\hat{X}$ as $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$. Denote by $P$ the canonical map $\ell^\infty(\Gamma, \mathbb{R}^k) \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$ . For each $x \in \Delta^p(X)$, by the inequalities (15) we can take $\tilde{x} \in \ell^p(\Gamma, \mathbb{R}^k)$ such that $P(\tilde{x}) = x$. Since $\alpha$ is $p$-expansive, by Proposition

(6.2.19).(iii) and the inequalities (15) we can find some $\varepsilon > 0$ such that if $x, y \in \Delta^p(X)$ are distinct, then $\|\tilde{x} - \tilde{y}\|_p > \varepsilon$. As $\ell^p(\Gamma, \mathbb{R}^k)$ is separable under the norm $\|\cdot\|_p$, any $\varepsilon$-separated subset of $\ell^p(\Gamma, \mathbb{R}^k)$ is countable. Therefore $\Delta^p(X)$ is countable.

To prove the assertion (vi), we need the following two lemmas.

**Lemma (6.2.34)[396]:** Let $1 \le p < +\infty$. Let $\Gamma$ act $p$-expansively on a compact abelian group $X$ by automorphisms. Assume that $\widehat{X}$ is a finitely presented left $\mathbb{Z}\Gamma$-module, and write $\widehat{X}$ as $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n A$ for some $A \in M_{n\times k}(\mathbb{Z}\Gamma)$. Then $\Delta^p(X)$ is isomorphic to $a$ $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^n/(\mathbb{Z}\Gamma)^k A^*$.

**Proof:** Denote by $P$ the canonical map from $(\ell^\infty(\Gamma))^k = \ell^\infty(\Gamma, \mathbb{R}^k)$ to $((\mathbb{R}/\mathbb{Z})^\Gamma)^k$. For each $x \in \Delta^p(X)$, by the inequalities (15) we can take $\tilde{x} \in \ell^p(\Gamma, \mathbb{R}^k)$ such that $P(\tilde{x}) = .$ Then $\tilde{x} A^* \in \ell^\infty(\Gamma, \mathbb{Z}^n) \cap \ell^p(\Gamma, \mathbb{R}^n) = (\mathbb{Z}\Gamma)^n$.Thus we can define a map $\varphi : \Delta^p(X) \to (\mathbb{Z}\Gamma)^n/(\mathbb{Z}\Gamma)^k A^*$ sending $x$ to $\tilde{x} A^* + (\mathbb{Z}\Gamma)^k A^*$.

If $a \in \ell^p(\Gamma, \mathbb{R}^k)$ satisfies $P(a) = x$, then $a - \tilde{x} \in \ell^\infty(\Gamma, \mathbb{Z}^k) \cap \ell^p(\Gamma, \mathbb{R}^k) = (\mathbb{Z}\Gamma)^k$, and hence $aA^* + (\mathbb{Z}\Gamma)^k A^* = \tilde{x} A^* + (\mathbb{Z}\Gamma)^k A^*$. It follows easily that $\varphi$ is a $\mathbb{Z}\Gamma$-module homomorphism.

Since the action is $p$-expansive, by Proposition (6.2.19).(v) the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^n$ sending $a$ to $aA^*$ is injective. If $x \in \ker \varphi$, then $\tilde{x} A^* \in (\mathbb{Z}\Gamma)^k A^*$, and hence $\tilde{x} \in (\mathbb{Z}\Gamma)^k$, which implies that $x = P(\tilde{x}) = e_X$ .Thus $\varphi$ is injective.

The next lemma is more than needed for the proof of (vi), but will be useful later.

**Lemma (6.2.35)[396]:** Let $k \in \mathbb{N}$, and $A \in M_k(\mathbb{Z}\Gamma)$ such that the linear map $T: (\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^k$ sending $a$ to $aA^*$ is invertible for some $1 \le p < +\infty$. Set $X_A = \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$. Then $\Delta^p(X_A) = \Delta(X_A) = P(T^{-1}((\mathbb{Z}\Gamma)^k))$ is dense in $X_A$, where $P$ denotes the canonicalmap from $(\ell^\infty(\Gamma))^k = \ell^\infty(\Gamma, \mathbb{R}^k)$ to $((\mathbb{R}/\mathbb{Z})^\Gamma)^k$. Furthermore, $\Delta(X_A)$is isomorphic to $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A^*$ as left $\mathbb{Z}\Gamma$-modules.

**Proof:** From (i) and (iv) of Proposition (6.2.33) we have $\Delta(X_A) \supseteq \Delta^p(X_A) \supseteq P(T^{-1}((\mathbb{Z}\Gamma)^k))$.

Let $x \in \Delta(X_A)$. Take $\tilde{x} \in ([-1/2, 1/2]^k)^\Gamma \subseteq ((\ell^\infty((\Gamma))^k$ with $P(\tilde{x}) = x$. Then the function $s \mapsto \|\tilde{x}_s\|_\infty$ on $\Gamma$ vanishes at infinity. Since $A^* \in M_k(\mathbb{Z}\Gamma)$, it follows that the function $s \mapsto \|(\tilde{x}A^*)_s\|_\infty$ on $\Gamma$ also vanishes at infinity. As $\tilde{x}A^* \in (\mathbb{Z}^k)^\Gamma$, we conclude that $T(\tilde{x}) = \tilde{x} A^* \in (\mathbb{Z}\Gamma)^k$. Then $\tilde{x} \in T^{-1}((\mathbb{Z}\Gamma)^k)$, and hence $x \in P(T^{-1}((\mathbb{Z}\Gamma)^k))$.Therefore $\Delta^p(X_A) = \Delta(X_A) = P(T^{-1}((\mathbb{Z}\Gamma)^k))$.

Take $1 < q \le +\infty$ with $p^{-1} + q^{-1} = 1$. Note that $\ell^q(\Gamma, \mathbb{R}^k)$ can be identified with the dual space of $\ell^p(\Gamma, \mathbb{R}^k)$ naturally, as in the Eq. (13). Denote by $T^*$ the bounded linear map $\ell^q(\Gamma, \mathbb{R}^k) \to \ell^q(\Gamma, \mathbb{R}^k)$ dual to $T$. Let $\psi \in \widehat{X_A}$ such that $\langle \Delta(X_A), \psi\rangle = 1$. Write $\psi$ as $a + (\mathbb{Z}\Gamma)^k A$ for some $a \in (\mathbb{Z}\Gamma)^k$. Then

$$\begin{aligned}(\tilde{y}((T^*)^{-1}(a))^*)_{e_\Gamma} &= (\tilde{y}((T^{-1})^*(a))^*)_{e_\Gamma} = \langle \tilde{y}, (T^{-1})^*(a)\rangle \\ &= \langle T^{-1}(\tilde{y}), a\rangle = (T^{-1}(\tilde{y})a^*)_{e_\Gamma} \in \mathbb{Z}\end{aligned}$$

for all $\tilde{y} \in (\mathbb{Z}\Gamma)^k$. Replacing $\tilde{y}$ by $s\,\tilde{y}$ for all $s \in \Gamma$, we get $\tilde{y}((T^*)^{-1}(a))^* \in \ell^p(\Gamma, \mathbb{R}^k)$ for all $\tilde{y} \in (\mathbb{Z}\Gamma)^k$. Taking $\tilde{y}$ to be the canonical basis of $(\mathbb{Z}\Gamma)^k$, we get $(T^*)^{-1}a \in \ell^p(\Gamma, \mathbb{R}^k)$. When $q < +\infty$, we get $(T^*)^{-1}a \in \ell^p(\Gamma, \mathbb{R}^k) = (\mathbb{Z}\Gamma)^k$.When $q = +\infty$, from the surjectivity of $T$ we see that $A^*$ has a left inverse in $M_k(\ell^1(\Gamma))$ and hence $A$ has a right inverse in $M_k(\ell^1(\Gamma))$. Multiplying this right inverse to $a = T^*((T^*)^{-1}(a)) = ((T^*)^{-1}(a))A$, we conclude that $(T^*)^{-1}a \in \ell^1(\Gamma, \mathbb{R}^k)$.Therefore we still have $(T^*)^{-1}a \in$

$\ell^p(\mathbb{Z}\Gamma)^k$. Thus $a \in T^*((\mathbb{Z}\Gamma)^k)) = (\mathbb{Z}\Gamma)^k A$ and hence $\psi = 0_{\widehat{X_A}}$. By the Pontryagin duality $\Delta(X_A)$ is dense in $X_A$.

Denote $\varphi$ by the map $(\mathbb{Z}\Gamma)^k \to \Delta(X_A)$ sending $a$ to $P(T^{-1}(a))$. Clearly $\varphi$ is a left $\mathbb{Z}\Gamma$-module homomorphism, and $ker\,\varphi \supseteq (\mathbb{Z}\Gamma)^k A^*$. If $a \in ker\,\varphi$, then $T^{-1}(a) \in \ell^\infty(\Gamma, \mathbb{Z}^k) \cap \ell^\infty(\Gamma, \mathbb{R}^k) = (\mathbb{Z}\Gamma)^k$ and hence $a \in T((\mathbb{Z}\Gamma)^k) = (\mathbb{Z}\Gamma)^k A^*$. Therefore $ker\,\varphi = (\mathbb{Z}\Gamma)^k A^*$. By the first paragraph of the proof $\varphi$ is also surjective. Thus $\Delta(X_A)$ is isomorphic to $(\mathbb{Z}\Gamma)^k / ker\,\varphi = (\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A^*$ as left $\mathbb{Z}\Gamma$-modules.

When $p < +\infty$, the assertion (vi) follows from Lemma (6.2.34), Proposition (6.2.19).(ii) and the fact that if a unital ring $R$ is left Noetherian, then every finitely generated left $R$-module is finitely presented [439].When $p = +\infty$, the assertion (vi) follows from Proposition(6.2.19).(iv), Theorem (6.2.9) and Lemma (6.2.35). This finishes the proof of Proposition (6.2.33).

**Example(6.2.36)[396]:** Let $\Gamma, f$, and $Z(f)$ be as in Example (6.2.25). When $f$ is irreducible in the factorial ring $\mathbb{Z}\Gamma$ and $Z(f)$ is finite, $\Delta^1(X_f)$ is dense in $X_f$ [447] .Denote by $u_1, \ldots, u_d$ the canonical basis of $\mathbb{Z}^d$ .The group $\Delta^1(X_f)$ was calculated explicitly for $f = 2d - \sum_{j=1}^{d}(u_j + u_j^{-1})$ in [471] and for $f = 2 - u_1 - u_2$ when $d = 2$ in [447].

Lind and Schmidt [445] showed that when $\Gamma$ is finitely generated with sub-exponential growth, for any expansive action of $\Gamma$ on a compact abelian group $X$ by automorphisms, one has $\Delta(X) = \Delta^1(X)$. From Theorem (6.2.9), Lemma (6.2.35), and Proposition (6.2.33).(i) we conclude that this holds for all $\Gamma$:

**Proposition (6.2.37)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. Suppose that $\hat{X}$ is $a$ finitely generated left $\mathbb{Z}\Gamma$-module. Then there is $a$ compatible translation-invariant metric $\rho$ on $X$ such that $\sum_{s\in\Gamma} \rho(0_X, sx) < +\infty$ for every $x \in \Delta^1(X)$.

**Proof:** The case $\Gamma$ is finite is trivial. Thus we may assume that $\Gamma$ is infinite. Take a finite set $W \subseteq \hat{X}$ generating $\hat{X}$ as a left $\mathbb{Z}\Gamma$-module. List the elements of $\Gamma$ as $s_1, s_2, \ldots$ . Define a continuous translation-invariant metric $\rho$ on $X$ by

$$\rho(x, y) = \sum_{j=1}^{\infty} \sum_{\varphi\in W} 2^{-j} \left|\Psi_{x-y,\varphi}(s_j)\right|$$

for all $x, y \in X$, where $\Psi_{x-y,\varphi}$ is defined in Notation (6.2.17). If we denote by $\tau$ the original topology on $X$, and by $\tau'$ the topology on $X$ induced by $\rho$, then the identity map $\sigma: (X, \tau) \to (X, \tau')$ is continuous. Since $(X, \tau)$ is compact and $(X, \tau')$ is Hausdorff , $\sigma$ is a homeomorphism. Thus $\rho$ is compatible. For any $x \in \Delta^1(X)$, one has

$$\begin{aligned}\sum_{t\in\Gamma} \rho(0_X, tx) &= \sum_{t\in\Gamma} \sum_{j=1}^{\infty} \sum_{\varphi\in W} 2^{-j} \left|\Psi_{tx,\varphi}(s_j)\right| \\ &= \sum_{\varphi\in W} \sum_{j=1}^{\infty} 2^{-j} \sum_{t\in\Gamma} \left|\Psi_{x,\varphi}(s_j t)\right| \\ &= \sum_{\varphi\in W} \sum_{j=1}^{\infty} 2^{-j} \left\|\Psi_{x,\varphi}\right\|_1\end{aligned}$$

$$= \sum_{\varphi \in W} \|\Psi_{x,\varphi}\|_1 < +\infty.$$

We discuss pairs of algebraic actions. Let $G_1$ and $G_2$ be discrete left $\mathbb{Z}\Gamma$-modules. Consider a map $\Phi\ G_1 \times G_2 \to T$ which is *bi-additive* in the sense that

$$\Phi(\varphi_1 + \psi_1, \varphi_2) = \Phi(\varphi_1, \varphi_2)\Phi(\psi_1, \varphi_2)$$

and

$$\Phi(\varphi_1, \varphi_2 + \psi_2) = \Phi(\varphi_1, \varphi_2)\Phi(\varphi_1, \psi_2)$$

for all $\varphi_1, \psi_2 \in G_1$ and $\varphi_2, \psi_2 \in G_2$, and equivariant in the sense that

$$\Phi(s\varphi_1, s\varphi_2) = \Phi(\varphi_1, \varphi_2)$$

for all $\varphi_1 \in G_1$, $\varphi_2 \in G_2$, and $s \in \Gamma$. Then $\Phi$ induces $\Gamma$-equivariant group homomorphisms $\Phi_1: G_1 \to \widehat{G_2}$ and $\Phi_2: G_2 \to \widehat{G_1}$ such that

$$\langle \Phi_1(\varphi_1), \varphi_2 \rangle = \Phi(\varphi_1, \varphi_2) = \langle \varphi_1, \Phi_2(\varphi_2) \rangle$$

for all $\varphi_1 \in G_1$ and $\varphi_2 \in \varphi_2$.

**Lemma (6.2.38)[396]:** Let $G_1, G_2, \Phi, \Phi_1$, and $\Phi_2$ be as above. Then the following hold:

**(i)** $\Phi_1$ is injective if and only if $\Phi_2(G_2)$ is dense in $\widehat{G_1}$.

**(ii)** For any $1 \le p \le +\infty, \Phi_1(G_1) \subseteq \Delta^p(\widehat{G_2})$ if and only if $\Phi_2(G_2) \subseteq \Delta^p(\widehat{G_1})$.

**Proof:** (i) This follows from the Pontryagin duality and the fact that $\ker \Phi_1$consists of exactly those $\varphi_1 \in G_1$ satisfying $\langle \varphi_1, \Phi_2(G_2) \rangle = 1$.

(ii) Let $\varphi_1 \in G_1$ and $\varphi_2 \in G_2$. For each $s \in \Gamma$, we have

$$\begin{aligned}
\Psi_{\Phi_2(\varphi_2),\varphi_1}(s) &= \langle s\Phi_2(\varphi_2), \varphi_1 \rangle - 1 \\
&= \langle \Phi_2(s\varphi_2), \varphi_1 \rangle - 1 \\
&= \langle s\Phi_2, \Phi_1(\varphi_1) \rangle - 1 \\
&= \langle \varphi_2, s^{-1}, \Phi_1(\varphi_1) \rangle - 1 \\
&= \Psi_{\Phi_1(\varphi_1),\varphi_2}(s^{-1}).
\end{aligned}$$

Thus $\Psi_{\Phi_2(\varphi_2),\varphi_1} \in C_0(\Gamma)$ exactly when $\Psi_{\Phi_1(\varphi_1),\varphi_2} \in C_0(\Gamma)$, and when $1 \le p < +\infty, \Psi_{\Phi_2(\varphi_2),\varphi_1} \in \ell^p(\Gamma)$ exactly when $\Psi_{\Phi_1(\varphi_1),\varphi_2} \in \ell^p(\Gamma)$. It follows that $\Phi_1(G_1) \subseteq \Delta^p(\widehat{G_2})$ exactly when $\Phi_2(G_2) \subseteq \Delta^p(\widehat{G_1})$.

The above pairing has been studied by Einsiedler and Schmidt [415],[416] for algebraic actions of $\Gamma = \mathbb{Z}^d$ with $d \in \mathbb{N}$ on $X$, in the case $G_1 = \hat{X}$ and $G_2 = \Delta(X)$. In view of Lemma (6.2.38), $(G_1, G_2)$ should be thought of as a dual pair, and the dynamic properties of the $\Gamma$-actions on $\widehat{G_1}$ and $\widehat{G_2}$are reflected in each other. This point of view will play a central role.

Using Theorem (6.2.9), we discuss the relation between various specification properties and having dense homoclinic points for expansive algebraic actions. Throughout this section $\Gamma$ will be a countable discrete group.

Specification is a strong orbit tracing property. Ruelle [466] studied the extension of the notion to $\mathbb{Z}^d$-actions.We take the definition of various specification properties from [445], modified to the general group case.

**Definition (6.2.39)[396]:** Let $\alpha$ be a continuous $\Gamma$-action on a compact space $X$. Let $\rho$ be a compatible metric on $X$.

**(i)** The action $\alpha$ has weak specification if there exists, for every $\varepsilon > 0$, a nonempty finite subset $F$ of $\Gamma$ with the following property: for any finite collection $F_1, \ldots, F_m$ of finite subsets of $\Gamma$ with

$$FF_i \cap F_j = \emptyset \text{ for } 1 \leq i, j \leq m, i \neq j, \tag{16}$$

and for any collection of points $x^{(1)}, \ldots, x^{(m)}$ in $X$, there exists a point $y \in X$ with

$$\rho\left(sx^{(j)}, sy\right) \leq \varepsilon \text{ for all } s \in F_j, 1 \leq j \leq m. \tag{17}$$

**(ii)** The action $\alpha$ has strong specification if there exists, for every $\varepsilon > 0$, a nonempty finite subset $F$ of $\Gamma$ with the following property: for any finite collection $F_1, \ldots, F_m$ of finite subsets of $\Gamma$ satisfying (16) and any subgroup $\Gamma$ of $\Gamma'$with

$$FF_i \cap F_j(\Gamma' \backslash \{e_\Gamma\} = \emptyset \text{ for } 1 \leq i, j \leq m, \tag{18}$$

and for any collection of points $x^{(1)}, \ldots, x^{(m)}$ in $X$, there exists a point $y \in X$ satisfying (17) and $sy = y$ for all $s \in \Gamma'$.

**(iii)** When $X$ is a compact group and $\alpha$ is by automorphisms of $X$, the action $\alpha$ has homoclinic specification if there exists, for every $\varepsilon > 0$, a nonempty finite subset $F$ of $\Gamma$ with the following property: for any finite subset $F_1$ of $\Gamma$ and any $x \in X$, there exists $y \in \Delta(X)$ with

$$\rho(sx, sy) \leq \varepsilon \text{ for all } s \in F_1,$$

$$\rho(e_X, sy) \leq \varepsilon \text{ for all } s \in \Gamma \backslash FF_1.$$

.

The following lemma is a version of [408]. The same argument also appeared in the proof of [423].

**Lemma (6.2.40)[396]:** Let $\alpha$ be an expansive continuous action of $\Gamma$ on a compactmetric space $(X, \rho)$. Let $d > 0$ such that if $x, y \in X$ satisfy $\sup_{s\in\Gamma} \rho(sx, sy) \leq d$, then $x = y$. Let $x, y \in X$ satisfy $\rho(sx, sy) \leq d$ for all but finitely many $s \in \Gamma$. Then $(x, y)$ is an asymptotic pair in the sense that $\rho(sx, sy) \to 0$ as $\Gamma \ni s \to \infty$.

**Proof:** Take a finite subset $W$ of $\Gamma$ such that $\rho(sx, sy) \leq d$ for every $s \in \Gamma \backslash W$. Suppose that $(x, y)$ is not an asymptotic pair. Then there exist $\varepsilon > 0$ and a sequence of elements $\{s_n\}_{n\in\mathbb{N}}$ in $\Gamma$ such that $\rho(s_n x, s_n y) \geq \varepsilon$ for all $n \in \mathbb{N}$ and for any finite subset $F$ of $\Gamma$ one has $s_n \notin F$ for all sufficiently large $n \in \mathbb{N}$. Passing to a subsequence of $\{s_n\}_{n\in\mathbb{N}}$, we may assume that $s_n x$ and $s_n y$ converge to $x'$ and $y'$ in $X$ respectively, as $n \to \infty$. Then $\rho(x, y) \geq \varepsilon$ and hence $x' \neq y'$.

Let $s \in \Gamma$. When $n \in \mathbb{N}$ is sufficiently large, one has $ss_n \notin W$ and hence $\rho(ss_n x, ss_n y) \leq d$. Letting $n \to \infty$, we obtain $\rho(sx', sy') \leq d$. By the hypothesis on $d$ we conclude that $x' = y'$, which is a contradiction. Therefore $(x, y)$ is an asymptotic pair.

**Theorem (6.2.41)[396]:** Let α be an expansive action of Γ on a compact abelian group $X$ by automorphisms. Then the following are equivalent:

**(i)** $\alpha$ has weak specification;

**(ii)** $\alpha$ has strong specification;

**(iii)** $\alpha$ has homoclinic specification. Furthermore, these conditions imply

**(iv)** $\Delta(X)$ is dense in $X$.

**Proof:** By Theorem (6.2.9) we may assume that $X = \widehat{(\mathbb{Z}\Gamma)^k}/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule $J$ of $(\mathbb{Z}\Gamma)^k$, and find some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in

$M_k(\ell^1(\Gamma))$ such that the rows of $A$ are contained in $J$ . Denote by $W$ the union of $\{e_\Gamma\}$ and the support of $A^*$ as a $M_k(\mathbb{Z})$-valued function on $\Gamma$. Recall the canonical metric $\rho_\infty$ on $(\mathbb{R}/\mathbb{Z})^k$ defined by (11) and the norm $\|\cdot\|_\infty$ on $\ell^\infty(\Gamma,\mathbb{R}^k)$ defined by (10). Let $\rho$ be a compatible metric on $X$.

(iii)⇒(ii): Let $\varepsilon > 0$. Then we can find a nonempty finite subset $W_1$ of and $\|A\|_1^{-1} > \varepsilon' > 0$ such that if $x, y \in X$ satisfy $\max_{s\in W_1} \rho_\infty(xs\,, ys) 2\varepsilon'$ , then $\rho(x,y) < \varepsilon$. Take a finite subset $W_2$ of $\Gamma$ containing $e_\Gamma$ such that $\sum_{s\in\Gamma\backslash W_2^{-1}} \|((A *) - 1)s\_\|_1 < \varepsilon'/(2\|A\|_1)$, where $\|B\|_1$ denotes the sum of the absolute values of the entries of $B$ for $B \in M_k(\mathbb{R})$. If $\tilde{x}, \tilde{y} \in \ell^\infty$ $(\Gamma, \mathbb{R}^k)$ satisfy $\|\tilde{x}\|_\infty, \|\tilde{y}\|_\infty \le \|A\|_1$ and $\tilde{x}$ and $\tilde{y}$ are equal on $sW_2$ for some $s \in \Gamma$ , then $\|(\tilde{x}(A^*)^{-1})_s - (\tilde{y}(A^*)^{-1})_s\|_\infty \le \varepsilon'$ .Take $\delta > 0$ such that if $x, y \in X$ satisfy $\rho(x,y) \le \delta$, then $\rho_\infty(x_{e_\Gamma}, y_{e_\Gamma}) \le \varepsilon'$

By the condition (iii) we can find a finite subset $W_3$ of $\Gamma$ containing $e_\Gamma$ with the following property: for any finite subset $F_1$ of $\Gamma$ and any $x \in X$, there exists $y \in \Delta(X)$ with

$$\max_{s\in F_1} \rho(sx, sy) \le \delta \text{ and } \sup_{s\in\Gamma\backslash W_3F_1} \rho(e_X, sy) \le \delta.$$

By our choice of $\delta$, we then have.

$$\max_{s\in F_1^{-1}} \rho_\infty(x_s, y_s) \le \varepsilon' \text{ and } \sup_{s\in\Gamma(W_3F_1)^{-1}} \rho_\infty(0_{(\mathbb{R}/\mathbb{Z})^k}, y_s) \le \varepsilon'.$$

Now set $F = (W_1W_2W_3^{-1})(W_1W_2W_3^{-1})^{-1}$.

Let $F_1, \ldots, F_m$ be a finite collection of finite subsets of $\Gamma$ satisfying (16), $\Gamma'$ be a subgroup of $\Gamma$ satisfying (18), and $x^{(1)}, \ldots, x^{(m)}$ be points in $X$. Take $y^{(1)}, \ldots, y^{(m)} \in \Delta(X)$ with

$$\max_{s\in F_j^{-1}W_1W_2} \rho_\infty(x_s^{(j)}, y_s^{(j)}) \le \varepsilon' \text{ and } \sup_{s\in\Gamma\backslash(F_j^{-1}W_1W_2W_3^{-1})} \rho_\infty(0_{(\mathbb{R}/\mathbb{Z})^k}, y_s^{(j)}) \le \varepsilon'.$$

for each $1 \le j \le m$. Denote by $P$ the canonical map $\ell^\infty$ $(\Gamma, \mathbb{R}^k)\infty \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. For each $1 \le j \le m$, take a lift $\tilde{x}^{(j)}$ and $\tilde{y}^{(j)}$ for $x^{(j)}$ and $y^{(j)}$ in $(([-1,1])^k)^\Gamma$ respectively such that

$$\max_{s\in F_j^{-1}W_1W_2} \left\|\tilde{x}_s^{(j)} - \tilde{y}_s^{(j)}\right\|_\infty \le \varepsilon' \text{ and } \sup_{s\in\Gamma\backslash(F_j^{-1}W_1W_2W_3^{-1})} \left\|\tilde{y}_s^{(j)}\right\|_\infty \le \varepsilon'.$$

Then $\tilde{y}^{(j)}A^*$ belongs to $\ell^\infty(\Gamma, \mathbb{Z}^k)$. Note that, for any $s \in \Gamma\backslash(W_1W_2W_3^{-1}W)$, one has

$$\left\|(\tilde{y}_s^{(j)}A^*)_s\right\|_\infty \le \|A^*\|_1 \cdot \sup_{t\in\Gamma\backslash(F_j^{-1}W_1W_2W_3^{-1})} \left\|\tilde{y}_t^{(j)}\right\|_\infty \le \|A\|_1 \cdot \varepsilon' < 1.$$

It follows that, as a $\mathbb{Z}^k$-valued function on $\Gamma, \tilde{y}^{(j)}A^*$ has support contained in $F_j^{-1}W_1W_2W_3^{-1}W$. We can rewrite (16) and (18) as

$$(F_i^{-1}W_1W_2W_3^{-1}W) \cap (F_j^{-1}W_1W_2W_3^{-1}W) = \varnothing \text{ for } 1 \le i, j \le m, i \ne j,$$

and

$$(F_i^{-1}W_1W_2W_3^{-1}W) \cap (\Gamma'\backslash\{e_\Gamma\})(F_j^{-1}W_1W_2W_3^{-1}W) = \varnothing \text{ for } 1 \le i, j \le m,$$

respectively. Thus the elements $s\tilde{y}^{(j)}A^*$ of $\mathbb{Z}^k\Gamma$ for $s \in \Gamma'$ and $1 \le j \le m$ have pairwise disjoint support. Then we have the element $\tilde{z} := \sum_{s\in\Gamma'} \sum_{j=1}^m s\tilde{y}^{(j)}A^*$ of $\ell^\infty(\Gamma, \mathbb{Z}^k)$ with

$$\|\tilde{z}\|_\infty = \max_{1\le j\le m}\left\|\tilde{y}^{(j)}A^*\right\|_\infty \le \|A^*\|_1 \cdot \max_{1\le j\le m}\left\|\tilde{y}^{(j)}\right\|_\infty \le \|A\|_1 .$$

Set $\tilde{y} = \tilde{z}(A^*)^{-1} \in \ell^\infty(\Gamma, \mathbb{Z}^k)$ and $y = P(\tilde{y}) \in ((\mathbb{R}/\mathbb{Z})^k k)^\Gamma$.

We claim that $y \in X$. For each finite subset $K$ of $\Gamma'$, define $\tilde{z}K = \sum_{s\in K}\sum_{j=1}^m s\tilde{y}^{(j)}A^*$ , $\tilde{y}K = \tilde{z}K\,(A^*)^{-1}$, and $yK = P(\tilde{y}K)$. Then $\|\tilde{z}K\|_\infty \le \|A\|_1$ for every $K$. For each $s \in \Gamma$, when $K \to \Gamma'$, we have $(\tilde{z}K)_s \to \tilde{z}_s$ and hence $(\tilde{y}K)_s \to \tilde{y}_s$. It follows that, when $K \to \Gamma'$, $y_K$ converges to $y$. Clearly $y_K = \sum_{s\in K}\sum_{j=1}^m sy^{(j)} \in X$ for each $K$. Therefore $y \in X$.

For each $s \in \Gamma'$, we have $s\,\tilde{z} = \tilde{z}$ and hence $sy = y$.

Fix $1 \le j \le m$. Note that $\tilde{z}$ and $\tilde{y}^{(j)}A^*$ are equal on $F_j^{-1}W_1W_2W_3^{-1}W \supseteq F_j^{-1}W_1W_2$. Since $\|\tilde{z}\|_\infty, \left\|\tilde{y}^{(j)}A^*\right\|_\infty \le \|A\|_1$, by our choice of $W_2$ we have $\max_{s\in F_1^{-1}W_1}\left\|\tilde{y}_s - y_s^{(j)}\right\|_\infty \le \varepsilon'$. Thus $\max_{s\in F_1^{-1}W_1}\rho_\infty(y_s, y_s^{(j)}) \le \varepsilon'$, and hence

$$\begin{aligned}\max_{s\in F_j, t\in W_1}\rho_\infty((sy)_t, (sx^{(j)})_t) &= \max_{s\in F_j^{-1}W_1}\rho_\infty\left(y_s, x_s^{(j)}\right)\\ &\le \max_{s\in F_j^{-1}W_1}\rho_\infty(y_s, y_s^{(j)}) + \max_{s\in F_j^{-1}W_1}\rho_\infty\left(y_s^{(j)}, x_s^{(j)}\right)\\ &\le 2\varepsilon'.\end{aligned}$$

By our choice of $W_1$ and $\varepsilon$, we get $\max_{s\in F_j}\rho(sy, sx^{(j)}) < \varepsilon$ as desired.

(ii)⇒(i) and (iii)⇒(iv) are trivial.

(i)⇒(iii): By Lemma (6.2.40) there exists $\varepsilon > 0$ with the following property: if $y \in X$ satisfies $\rho(e_X, sy) \le \varepsilon'$ for all but finitely many $s \in \Gamma$, then $y \in \Delta(X)$.Let $\varepsilon' \ge \varepsilon > 0$. Take $F$ as in the definition of weak specification. Replacing $F$ by $F \cup F^{-1}$ if necessary, we may assume that $F = F^{-1}$. Let $F_1$ be a finite subset of $\Gamma$ and $x \in X$. For each finite subset $F_2$ of $\Gamma\backslash F\,F_1$, by the choice of $F$, taking $x^{(1)} = x$ and $x^{(2)} = e_X$ we can find $yF_2 \in X$ such that $\max_{s\in F_1}\rho(sx, syF_2) < \varepsilon$ and $\max_{s\in F_2}\rho(se_X, syF_2) < \varepsilon$. Note that the set of finite subsets of $\Gamma\backslash F\,F_1$ is partially ordered by inclusion. Take a limit point $y$ of the net $\{yF_2\}_{F_2}$. Then $\max_{s\in F_1}\rho(sx, sy) < \varepsilon$ and $\sup_{s\in\Gamma\backslash F\,F_1}\rho(se_X, sy) \le \varepsilon \le \varepsilon'$ .By our choice of $\varepsilon'$, we conclude that $y \in \Delta(X)$.

We give a new proof of the following result of Lind and Schmidt [445].

**Theorem (6.2.42)[396]:** Suppose that $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$. Let $\alpha$ be an expansive action of $\Gamma$ on a compact abelian group $X$ by automorphisms. Then the conditions (i), (ii), (iii) and (iv) in Theorem (6.2.41) are all equivalent.

**Proof:** We just need to show that (iv)⇒(iii).By Propositions (6.2.3) and (6.2.37), we can find a compatible translation-invariantmetric $\rho$ on $X$ such that $\sum_{s\in\Gamma}\rho(0_X, sx) < +\infty$ for all $x \in \Delta(X)$.

Since $\mathbb{Z}\Gamma$ is Noetherian [441], from Propositions (6.2.19).(iv) and (6.2.33).(vi) we see that $\Delta(X)$ is a finitely generated left $\mathbb{Z}\Gamma$-module. Take $z_1, \ldots, z_n \in \Delta(X)$ such that $\Delta(X) = \sum_{j=1}^n \mathbb{Z}\Gamma\, z_j$.

By Theorem (6.2. 9) $X$ is a closed $\Gamma$-invariant subgroup of $X_A := \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ for some $k \in \mathbb{N}$ and some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$. Treat $\Delta(X_A)$ as a discrete abelian group and consider the induced $\Gamma$-action on the Pontryagin dual $\widehat{\Delta(X_A)}$. By Lemmas (6.2.35) and (6.2.14) the $\Gamma$-action on $\widehat{\Delta(X_A)}$. is expansive. By Corollary

(6.2.10) there exists $f \in \mathbb{Z}\Gamma$ being invertible in $\ell^1(\Gamma)$ such that $f\Delta(X_A) = 0$. In particular, $f\Delta(X) = 0$.

Let $\varepsilon > 0$. Take a nonempty finite subset $W$ of $\Gamma$ such that $\sum_{j=1}^{n} \sum_{s\in\Gamma\backslash W} \rho(0_X, sz_j) < \varepsilon/(2\| f \|_1)$. $Set\ F = WW^{-1}$.

Let $F_1$ be a finite subset of $\Gamma$ and $x' \in X$. By the condition (iv) we can take some $x \in \Delta(X)$ with $\max_{s\in F_1} \rho(sx', sx) < \varepsilon/2$.

We have $x = \sum_j a_j z_j$ for some $a_1, \ldots, a_n \in \mathbb{Z}\Gamma$. Note that $a_j f^{-1}$ is in $\ell^1(\Gamma)$. Let $b_j$ be the integral part of $a_j f^{-1}$. That is, $b_j \in \mathbb{R}^\Gamma$ has integral coefficients, and $a_j f^{-1} - b_j$ has coefficients in $[-1/2, 1/2)$. Then $\| b_j \|_1 \le 2\| a_j f^{-1} \|_1 < +\infty$, and hence $b_j \in \mathbb{Z}\Gamma$. Note that $x = \sum_j (a_j - b_j f) z_j$, and

$$\| a_j - b_j f \|_\infty \le \| a_j f^{-1} - b_j \|_\infty \cdot \| f \|_1 \le \| f \|_1.$$

Thus, replacing $a_j$ by $a_j - b_j f$ if necessary, we may assume that $\| a_j \|_\infty \le \| f \|_1$ for all $1 \le j \le n$.

For each $1 \le j \le n$, define $c_j \in \mathbb{Z}\Gamma$ to be the same as $a_j$ on $F_1^{-1}W$ and 0 outside of $F_1^{-1}W$. Set $y = \sum_j c_j z_j \in \Delta(X)$. For each $s \in F_1$, since $\rho$ is translation-invariant, we have

$$\begin{aligned}\rho(sx, sy) &= \rho\left(\sum_j s(a_j - c_j) z_j, 0_X\right) \\ &\le \sum_j \sum_{t\in\Gamma} \left|\left(s(a_j - c_j)\right)_t\right| \rho(tz_j, 0_X) \\ &= \sum_j \sum_{t\in\Gamma\backslash W} \left|(a_j - c_j)_{s^{-1}t}\right| \rho(tz_j, 0_X) \le \sum_j \sum_{t\in\Gamma\backslash W} \|a_j\|_\infty \rho(tz_j, 0_X)\end{aligned}$$

$$\le \|f\|_1 \sum_j \sum_{t\in\Gamma\backslash W} \|a_j\|_\infty \rho(tz_j, 0_X) \le \varepsilon/2,$$

and hence

$$\rho(sx', sy) \le \rho(sx', sx) + \rho(sx, xy) \le \varepsilon/2$$

For each $s \in \Gamma\backslash FF_1$, noting that $s^{-1}W \cap F_1^{-1} W = \emptyset$, we have

$$\begin{aligned}\rho(sy, 0_X) &\le \sum_j \sum_{t\in\Gamma} \left|(sc_j)_t\right| \rho(tz_j, 0_X) \\ &= \sum_j \sum_{t\in\Gamma\backslash W} \left|(c_j)_{s^{-1}t}\right| \rho(tz_j, 0_X) \\ &\le \sum_j \sum_{t\in\Gamma\backslash W} \|c_j\|_\infty \rho(tz_j, 0_X) \\ &\le \|f\|_1 \sum_j \sum_{t\in\Gamma\backslash W} \rho(tz_j, 0_X) \le \varepsilon/2,\end{aligned}$$

In general, we have

**Conjecture (6.2.43)[396]:** Suppose that Γ is amenable and $\mathbb{Z}\Gamma$ is left Noetherian. Let $\alpha$ be an expansive action of Γ on a compact abelian group $X$ by automorphisms. Then the conditions (i), (ii, (iii) and (iv) in Theorem (6.2.41) are all equivalent.
We study the local entropy theory for Γ-actions on compact groups by automorphisms. The basics of local entropy theory are recalled .Throughout this section Γ will be a countable amenable group, unless specified.

For a continuous action of Γ on a compact space $X$, we denote by $\mathcal{M}(X,\Gamma)$ the set of all Γ-invariant Borel probability measures on $X$. For a compact group $X$, we denote by $\mu_X$ the normalized Haar measure on $X$, and by $e_3$ the identity element of $X$.

**Lemma (6.2.44)[396]:** Let Γ act on a compact group $X$ by automorphisms, and let $\nu \in \mathcal{M}(X,\Gamma)$. Then the product map $X \times X \to X$ sending $(x,y)$ to $x\,y$ is a Γ-equivariant surjective continuous map, for $X \times X$ equipped with the product action, and sends the Γ-invariant measures $\mu_X \times \nu$ and $\nu \times \mu_X$ to $\mu_X$.

**Proof:** Denote the product map by $\pi$. Then $\pi_*(\mu_X \times \nu)$ is a Borel probability measure on $X$. Since $\mu_X$ is left-shift invariant, so is $\pi_*(\mu_X \times \nu)$. Thus $\pi_*(\mu_X \times \nu) = \mu_X$ . Similarly $\pi_*(\nu \times \mu_X) = \mu_X$ . The other parts of the lemma are obvious.

**Definition (6.2.45)[396]:** Let Γ act on a compact group $X$ by automorphisms. We say that a point $x \in X$ is an *IE-point* if $(x, e_X) \in \mathrm{IE}_2(X)$.We denote the set of all IE-points by $\mathrm{IE}(X)$.

**Theorem (6.2.46)[396]:** Let Γ act on a compact group $X$ by automorphisms. Then the following hold:

**(i)** The set $IE(X)$ is $a$ Γ-invariant closed normal subgroup of $X$.

**(ii)** For any $k \in \mathbb{N}$, the set $IE_k(X)$ is $a$ Γ-invariant (under the product action of Γ on $X^k$) closed subgroup of the group $X^k$ , and

$$IE_k(X) = \{(x_1y, \dots, x_ky) : x_1, \dots, x_k \in IE(X), y \in X\}$$

$$= \{(yx_1, \dots, yx_k) : x_1, \dots, x_k \in IE(X), y \in X\}.$$

**(iii)** For any $\nu \in \mathcal{M}(X,\Gamma)$ and $k \in \mathbb{N}$, one has $IE_k^{\nu}(X) \subseteq IE_k^{\mu_X}(X) = IE_k(X)$.

**(iv)** $h(X) > 0$ if and only if $IE(X)$ is nontrivial.

**(v)** Let $Y$ be a closed Γ-invariant normal subgroup of $X$ and denote by q the quotient map $X \to X/Y$ . Consider the induced Γ-action on $X/Y$ . Then $q(IE(X)) = IE(X/Y)$.

**Proof:** Let $k \in \mathbb{N}$.

(i) and (ii). Since $\operatorname{supp}(\mu_X) = X$, from Theorem (6.2.6).(iii) we have $IE_1(X) = X$. It is clear from the definition of $IE$-tuples that if $1 \le m < k$ and $(x_1, \dots, x_m) \in IE_m(X)$, then $(x_1, \dots, x_m, x_{m+1}, \dots, x_k) \in IE_k(X)$ for $x_{m+1} = \cdots = x_k = x_1$. It follows that the length $k$ diagonal element $(x, \dots, x)$ is in $IE_k(X)$ for every $x \in X$. In particular, $IE_k(X)$ contains the identity element of the group $X^k$ .

Consider the product action of Γ on $X \times X$. Denote by $\pi$ the product map $X \times X \to X$, and by $\pi^k$ its $k$-fold $(X \times X)^k \to X^k$ . Note that $\pi^k$ can be identified with the product map of the group $X^k$ . By Theorem (6.2.6).(v) one has $IE_k(X \times X) = IE_k(X) \times IE_k(X)$. From Lemma (6.2.44) and Theorem (6.2.6).(iv), one gets $\pi^k(IE_k(X \times X)) = IE_k(X)$. Thus, $IE_k(X) \cdot IE_k(X) = IE_k(X)$. Also applying Theorem (6.2.6).(iv) to the inverse map $X \to X$, one get $(IE_k(X))^{-1} = IE_k(X)$. Therefore $IE_k(X)$ is a subgroup of $X^k$.

By Theorem (6.2.6).(i), the set $IE_k(X)$. is Γ-invariant and closed.

Since $IE_2(X)$. is a $\Gamma$-invariant closed subgroup of $X^2$, clearly $IE(X)$ is a $\Gamma$-invariant closed subgroup of $X$. If $x \in IE(X)$ and $y \in X$, then $(y,y),(y^{-1},y^{-1})$ and $(x,e_X)$ are all in $IE_2(X)$. ,and hence $(yxy^{-1},e_X) = (y,y)\cdot(x,e_X)\cdot(y^{-1},y^{-1}) \in IE_2(X)$ .Therefore $IE(X)$ is a normal subgroup of $X$.

Now we show that $IE_k(X) = \{(x_1y,\dots,x_k y) : x_1,\dots,x_k \in IE(X), y \in X\}$.The case $k = 1$ follows from $IE_1(X) = X$. Assume that $k \geq 2$ .Note that, by the definition of $IE$-tuples, $IE_k(X)$ is closed under taking permutation. If $x_1,\dots,x_k \in IE(X)$ and $y \in X$, then the $k$-tuples $(x_1, e_X,\dots, e_X),(e_X, x_2,\dots, e_X),\dots,(e_X, e_X,\dots,x_k)$ and $(y,\dots,y)$ are all in $IE_k(X)$, and hence

$$(x_1y,x_2y,\dots,x_ky) = (x_1, e_X,\dots, e_X)\cdot(e_X, x_2,\dots, e_X)\dots \times (e_X, e_X,\dots,x_k)\cdot(y,\dots,y)$$

is in $IE_k(X)$.

Note that, by the definition of $IE$-tuples, if $(x_1,\dots,x_k) \in IE_k(X)$ and $1 \leq m \leq k$, then $(x_1,\dots,x_m) \in IE_m(X)$. Suppose that $(y_1,\dots,y_k) \in IE_k(X)$. Let $2 \leq j \leq k$. Then $(y_1,y_j) \in IE_2(X)$ , and hence $(e_X, y_jy_1^{-1}) = (y_1,y_j)\cdot(y_1^{-1},y_1^{-1}) \in IE_2(X)$ .Thus $y_jy_1^{-1} \in IE(X)$. Set $x_1 = e_X, x_j = y_jy_1^{-1}$ for all $2 \leq j \leq k$, and $y = y_1$. Then $(y_1,\dots,y_k) = (x_1y,\dots,x_ky)$ . This proves $IE_k(X) = \{(x_1y,\dots,x_k y) : x_1,\dots,x_k \in IE(X), y \in X\}$. Similarly, one has $IE_k(X) = \{(yx_1,\dots,yx_k): x_1,\dots,x_k \in IE(X), y \in X\}$.

(iii). Let $\nu \in \mathcal{M}(X,\Gamma)$. From Theorem (6.2.8).(v) one gets $IE_k^{\mu_X\times\nu}(X\times X) = IE_k^{\mu_X}(X)\times IE_k^{\nu}(X)$ . By Lemma (6.2.44)and Theorem (6.2.8).(iv), one Has $\pi^k(IE_k^{\mu_X\times\nu}(X\times X)) = IE_k^{\pi_*(\mu_X\times\nu)}(X) = IE_k^{\mu_X}(X)$. Thus, for any $\boldsymbol{x} \in IE_k^{\mu_X}(X)$ and $\boldsymbol{y} \in IE_k^{\nu}(X)$, one has $\boldsymbol{x}\cdot\boldsymbol{y} \in IE_k^{\mu_X}(X)$.

By Theorem (6.2.8).(iii), we have $IE_1^{\mu_X}(X) = \mathrm{supp}(\mu_X) = X$. It is clear from the definition of $\mu_X$ -IE-tuples that if $1 \leq m < k$ and $(x_1,\dots,x_m) \in IE_m^{\mu_X}(X)$ , then $(x_1,\dots,x_m,x_{m+1},\dots,x_k) \in IE_k^{\mu_X}(X)$ for $x_{m+1} = \cdots = x_k = x_1$. It follows that the length $k$ diagonal element $(x,\dots,x)$ is in $IE_k^{\mu_X}(X)$ for every $x \in X$. In particular, $IE_k^{\mu_X}(X)$ contains the identity element $e_{X^k}$ of $X^k$.

For any $\boldsymbol{y} \in IE_k^{\nu}(X)$,wehave $\boldsymbol{y} = e_{X^k}\cdot\boldsymbol{y}\, IE_k^{\mu_X}(X)$. This proves $IE_k^{\nu}(X) \subseteq IE_k^{\mu_X}(X)$.

Now from parts (i) and (vi) of Theorem (6.2.8) we conclude $IE_k(X) = IE_k^{\mu_X}(X)$.

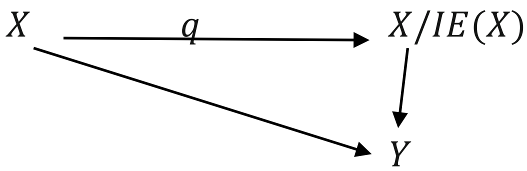
(iv). This follows from Theorem (6.2.6).(ii).

(v). By Theorem (6.2.6).(iv) we have $(q\times q)(IE_2(X)) = IE_2(X/Y)$. It follows that $q(IE(X)) \subseteq IE(X/Y)$. Furthermore, for any $z \in IE(X/Y)$, there exists $(x,y) \in IE_2(X)$ with $q(x) = z$ and $q(y) = e_X/Y$. Then $y \in Y$ . By Assertion (ii) we have $xy^{-1} \in IE(X)$. Thus $z = q(xy^{-1}) \in q(IE(X))$, and hence $IE(X/Y) \subseteq q(IE(X))$.

Under the assumptions of Theorem (6.2.46),$\Gamma$ has an induced action on $X/IE(X)$ by automorphisms. Under the quotient map $X \to X/IE(X), X/IE(X)$ is a topological factor of $X$.

**Theorem (6.2.47)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Denote by $q$ the quotient map $X \to X/IE(X)$. Then the following hold:

**(i)** X/IE(X) is the largest topological factor $Y$ of $X$ satisfying $\mathrm{h_{top}}(Y) = 0$, in the sense that $\mathrm{h_{top}}(X/IE(X)) = 0$ and if $Y$ is a topological factor of $X$ with $\mathrm{h_{top}}(Y) = 0$, then there is a topological factor map $X/IE(X) \to Y$ such that the diagram




commutes.

**(ii)** X/IE(X) is also the largest topological factor $Y$ of $X$ satisfying $h_{\pi_*}(\mu_X)(Y) = 0$ for $\pi : X \to Y$ denoting the factor map, in the sense that $h_{q_*}(\mu_X)(X/IE(X)) = 0$ and if $Y$ is a topological factor of $X$ with $h_{\pi_*}(\mu_X)(Y) = 0$, then there is a topological factor map $X/IE(X) \to Y$ such that the above diagram in (i) commutes.

**Proof:** (i). Let $(x, y) \in IE_2(X/IE(X))$. By Theorem (6.2.6).(iv) we can find $(\tilde{x}, \tilde{y}) \in IE_2(X)$ with $q(\tilde{x}) = x$ and $q(\tilde{y}) = y$. By Theorem (6.2.46) we have $\tilde{x}\, y^{-1} \in IE(X)$. Thus $x = q(\tilde{x}) = q(\tilde{x}\, y^{-1})q(\tilde{y}) = y$. Therefore $IE_2(X/IE(X))$ consists of only diagonal elements, and hence by Theorem (6.2.6).(ii) one has $\mathrm{h_{top}}(X/IE(X)) = 0$.

Now let $Y$ be a topological factor of $X$ such that $\mathrm{h_{top}}(Y) = 0$. Denote by $\pi$ the factor map $X \to Y$. To show that there is a topological factor map $X/IE(X) \to Y$ making the diagram in Assertion (i) commute, it suffices to show for any $x', y' \in X$ with $q(x') = q(y')$ one has $\pi(x') = \pi(y')$.

Let $x_1 \in IE(X)$ and $y_1 \in X$. From Theorem (6.2.46) we get $(y_1, x_1y_1) \in IE_2(X)$. By Theorem (6.2.6).(iv) we have $(\pi \times \pi)(IE_2(X)) = IE_2(Y)$. Thus $(\pi(y_1), \pi(x_1y_1)) \in IE_2(Y)$. Since $\mathrm{h_{top}}(Y) = 0$, by Theorem (6.2.6).(ii) the set $IE_2(Y)$ consists of only diagonal elements. Thus $\pi(y_1) = \pi(x_1y_1)$. If $x', y' \in X$ and $q(x') = q(y')$, then $x'(y')^{-1} \in IE(X)$ and hence $\pi(x') = \pi((x'(y')^{-1})y') = \pi(y')$.

(ii). This can be proved using arguments similar to that for Assertion (i), using Theorem (6.2.8)instead of Theorem (6.2.6).

From Theorem (6.2.47)we get

**Corollary (6.2.48)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Then the following are equivalent:

**(i)** $IE(X) = X$.

**(ii)** The only topological factor $Y$ of $X$ with $\mathrm{h_{top}}(Y) = 0$ is the trivial fact consisting of one point.

**(iii)** The only topological factor $Y$ of $X$ with $h_{\pi_*(\mu_X)}(Y) = 0$ for $\pi : X \to Y$ denoting the factor map is the trivial factor consisting of one point. Next we show that taking the IE group is an idempotent operation.

**Lemma (6.2.49)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Let $Y$ be a closed $\Gamma$-invariant normal subgroup of $X$. Then $IE(Y)$ is a normal subgroup of $X$.

**Proof:** Consider the conjugation map $\pi : X \times Y \to Y$ sending $(x, y)$ to $xyx^{-1}$. Clearly $\pi$ is surjective and $\Gamma$-equivariant for $X \times Y$ equipped with the product action. By (iv) and (v) of Theorem (6.2.6) we have $(\pi \times \pi)(IE_2(X) \times IE_2(Y)) = (\pi \times \pi)(IE_2(X \times Y)) = IE_2(Y)$. Let $x \in X$ and $y \in IE(Y)$. Then $(x, x) \in IE_2(X)$ and $(y, e_Y) \in IE_2(Y)$. Thus

$$(xyx^{-1}, e_Y) = (xyx^{-1}, e_xY\, x^{-1}) = (\pi \times \pi)((x, x) \times\ y, e_Y)) \in IE_2(Y),$$

and hence $xyx^{-1} \in IE(Y)$.

**Proposition (6.2.50)[396]:** Let $\Gamma$ act on a compact group X by automorphisms. Then $\mathrm{IE(IE(X)) = IE(X)}$.

**Proof:** By Theorem (6.2.46).(i) and Lemma (6.2.49)the group IE(IE(X)) is closed and normal in $X$ .By Theorem (6.2.47)we have $h(X/IE(X)) = h(IE(X)/IE(IE(X))) = 0$ .In virtue of Proposition (6.2.4) one has

$$h(X/IE(IE(X))) = h(X/IE(X)) + h(IE(X)/IE(IE(X))) = 0.$$

By Theorem (6.2.47) we get $IE(IE(X)) \supseteq IE(X)$. Thus $IE(IE(X)) = IE(X)$.

Now we describe the relation between $\Delta^1(X)$ and $IE(X)$ for algebraic actions.

**Theorem (6.2.51)[396]:** Let Γ act on a compact abelian group X by automorphisms. Suppose that $\widehat{X}$ is a finitely generated left $\mathbb{Z}\Gamma$-module. Then $\Delta^1(X) \subseteq IE(X)$.

Before giving the proof of Theorem (6.2.51), we use it to give a partial answer to a question of Deninger. For any countable discrete (not necessarily amenable) group Γ, and any invertible element $f$ in the group von Neumann algebra $\mathcal{L}\Gamma$ of Γ, the Fuglede-Kadison determinant $\det_{\mathcal{L}\Gamma} f$ is defined [419], which is a positive real number. We refer the reader to [412],[454] and [443] for background on $\mathcal{L}\Gamma$ and $\det_{\mathcal{L}\Gamma} f$. Deninger asked [412] if $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1(\Gamma)$ and has no left inverse in $\mathbb{Z}\Gamma$, then whether $\det_{\mathcal{L}\Gamma} f > 1$. This was answered affirmatively by Deninger and Schmidt [413] in the case Γ is residually finite and amenable. Now we answer Deninger's question for all countable amenable groups:

**Corollary (6.2.52)[396];** Suppose that $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1(\Gamma)$ and has no left inverse in $\mathbb{Z}\Gamma$. Then $\det_{\mathcal{L}\Gamma} f > 1$.

**Proof:** Let $X_f$ be as in Notation (6.2.20). Since $f$ has no left inverse in $\mathbb{Z}\Gamma$, the left $\mathbb{Z}\Gamma$-module $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ is nontrivial, and hence $X_f$ consists of more than one point. As $f$ is invertible in $\ell^1(\Gamma)$, by Lemma (6.2.35) $\Delta^1(X_f)$ is dense in $X_f$ and hence is nontrivial. In virtue of Theorem (6.2.51), $IE(X_f)$ is nontrivial. By Theorem (6.2.47)one has $h(X_f) > 0$. states that for any $g \in \mathbb{Z}\Gamma$ being invertible in $\mathcal{L}\Gamma$, one has $h(X_g) = \log \det_{\mathcal{L}\Gamma} g$. As invertibility in $\ell^1(\Gamma)$ implies invertibility in $\mathcal{L}\Gamma$ ,we get $\log \det_{\mathcal{L}\Gamma} f = h(X_f) > 0$. Therefore $\det_{\mathcal{L}\Gamma} f > 1$.

Theorem (6.2.51)follows from Proposition (6.2.37) and the next result, which is inspired by the proof of [413].

**Proposition (6.2.53)[396]:** Let Γ act on a compact group $X$ by automorphisms. Let $x \in X$ such that $\sum_{s\in\Gamma} \rho(sx, e_X) < +\infty$ for some compatible translation-invariant metric $\rho$ on $X$. Then $x \in IE(X)$.

**Proof:** Let $U_1$and $U_0$ be neighborhoods of $x$ and $e_X$ in $X$ respectively. Then there exists $\varepsilon > 0$ such that $U_1 \supseteq \{y \in X : \rho(y, x) < \varepsilon\}$ and $U_0 \supseteq \{y \in X : \rho(y, e_X) < \varepsilon\}$ . Since $\sum_{s\in\Gamma} \rho(sx, e_X) < +\infty$ , we can find a nonempty finite $K$ of Γ such that $\sum_{s\in\Gamma\backslash K} \rho(sx, e_X) < \varepsilon$.

Let $F$ be a nonempty finite subset of Γ. By Lemma (6.2.28) there exists $F_1 \subseteq F$ With $\frac{|F1|}{|F|} \geq \frac{1}{2|K|+1}$ and $((F_1F_1^{-1})\backslash\{e_\Gamma\}) \subseteq \Gamma\backslash K$. Say, $F_1 = \{s_1, \ldots, s_{|F_1|}\}$. For each $\sigma \in \{0,1\}^{F_1}$ , set

$$y_\sigma = (s_1^{-1}x)^{\sigma(s_1)}(s_2^{-1}x)^{\sigma(s_1)} \cdots (s_{|F_1|}^{-1}x)^{\sigma(s_{F_1})}.$$

We claim that $s(y_\sigma) \in U_{\sigma(s)}$ for every $s \in F_1$. Let $s \in F_1$. Since $\rho$ is translation invariant, we have

$$\rho(w_1w_2\dots w_k, z_1z_2\cdots z_k) \le \sum_{i=1}^{k}\rho(w_i, z_i)$$

for all $k \in \mathbb{N}$ and $w_1,\dots,w_k, z_1,\cdots,z_k \in X$. Thus

$$\rho\left(s(y_\sigma), x^{(s_1^{-1}x)^{\sigma(s)}}\right) \le \rho\big((ss^{-1}x)^{\sigma(s)}, x^{\sigma(s)}\big) + \sum_{s'\in F_1\setminus\{s\}}\rho\big((s(s')^{-1}x)^{\sigma(s')}, e_X\big)$$

$$= \sum_{s'\in F_1\setminus\{s\}}\rho\big((s(s')^{-1}x)^{\sigma(s')}, e_X\big).$$

Since $s(s')^{-1} \in \Gamma\setminus K$ for every $s \in F_1\setminus\{s\}$, we get

$$\rho\big(s(y_\sigma), x^{\sigma(s)}\big) \le \sum_{s''\in\Gamma\setminus K}\rho(s''x, e_X) < \varepsilon.$$

Therefore $(y_\sigma) \in U_{\sigma(s)}$. This proves the claim. Thus $F_1$ is an independence set for $(U_1, U_0)$. Then $(x, e_X) \in IE_2(X)$ and hence $x \in IE(X)$.

Throughout $\Gamma$ will be a countable amenable group.

Let $(X, \mathfrak{B}_X, \mu)$ be a standard probability space. That is, $(X, \mathfrak{B}_X)$ is a standard Borel space [434] and $\mu$ is a probability measure on $\mathfrak{B}_X$. Let $\Gamma$ act on $(X, \mathfrak{B}_X, \mu)$ via measure-preserving automorphisms. The Pinsker algebra of this action, denoted by $\prod(X)$ or $\prod(X, \mathfrak{B}_X, \mu)$, is the $\sigma$-algebra on $X$ consisting of $A \in \mathfrak{B}_X$ such that $h_\mu(\{A, X\setminus A\}) = 0$. For two sub-$\sigma$-algebras $B_1$ and $\mathfrak{B}_1$ of $\mathfrak{B}_X$, we write $\mathfrak{B}_1 = \mathfrak{B}_2$ mod $\mu$ if for every $A_1 \in \mathfrak{B}_1$ there exists $A_2 \in \mathfrak{B}_2$ with $\mu(A_1\Delta A_2) = 0$, and vice versa.

For a compact space $X$ (recall that all compact spaces are assumed to be metrizable), we denote by $\mathfrak{B}_X$ the $\sigma$-algebra of Borel subsets of $X$. Note that if $X$ is a compact space and $\mu$ is a probability measure on $\mathfrak{B}_X$, then $(X, \mathfrak{B}_X, \mu)$ is a standard probability space.

Recall that we denote by $\mu_X$ the normalized Haar measure of a compact group $X$. Also recall that when $\Gamma$ acts on a compact space $X$ continuously, we denote by $\mathcal{M}(X,\Gamma)$ the set of all $\Gamma$-invariant Borel probability measures on $X$.

The following theorem is the main result, saying that $IE(X)$ determines the Pinsker algebra with respect to $\mu_X$.

**Theorem (6.2.54)[396]:** Let $\Gamma$ act on two standard probability spaces $(X, \mathfrak{B}_X, \mu_X)$ and $(Y, \mathfrak{B}_Y, \mu_Y)$ via measure-preserving automorphisms. For the product action of $\Gamma$ on $(X\times Y, \mathfrak{B}_X\times\mathfrak{B}_Y, \mu_X\times\mu_Y)$, one has $\prod(X\times Y) = \prod(X)\times\prod(Y)$ mod $\mu_X\times\mu_Y$.

**Lemma (6.2.55)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Let $\nu \in \mathcal{M}(X,\Gamma)$. Then $x\cdot\prod(X, \mathfrak{B}_X, \mu_X), \prod(X, \mathfrak{B}_X, \mu_X)\cdot x \subseteq \prod(X, \mathfrak{B}_X, \nu)$ for all $x \in X$.

**Proof:** Denote by $\pi$ the product map $X\times X\to X$ sending $(x,y)$ to $xy$. By Lemma (6.2.44) this is a measure-theoretic factor map $(X\times X, \mathfrak{B}_X\times\mathfrak{B}_X, \mu_X\times\nu)\to(X, \mathfrak{B}_X, \mu_X)$.Then $\pi^{-1}(\prod(X, \mathfrak{B}_X, \mu_X)) \subseteq \prod(X\times X, \mathfrak{B}_X\times\mathfrak{B}_X, \mu_X\times\nu)$. Let $A\in\prod(X, \mathfrak{B}_X, \mu_X)$. By Theorem (6.2.54) we can find $B\in\prod(X, \mathfrak{B}_X, \mu_X)\times\prod(X, \mathfrak{B}_X, \nu)$ with $(\mu_X\times\nu)(\pi^{-1}(A)\Delta B) = 0$. Denote by $\chi_{\pi^{-1}}(A)$ the characteristic function of $\pi^{-1}(A)$. Then $\chi_{\pi^{-1}}(A) \in L^1(X\times X, \prod(X, \mathfrak{B}_X, \mu_X)\times\prod(X, \mathfrak{B}_X, \nu), \mu_X\times\nu)$. By the Fubini Theorem [464], the function $y\mapsto\chi_{\pi^{-1}}(A)(x,y) = \chi_A(xy) = \chi_{x^{-1}}A(y)$ is in $L^1(X, \prod(X, \mathfrak{B}_X, \nu), \nu)$ for $\mu_X$ almost all $x\in X$. That is, there exists $E\in\mathfrak{B}_X$ with $\mu_X(E) = 0$ such that the function $\chi_{x^{-1}}A$ is in $L^1(X, \prod(X, \mathfrak{B}_X, \nu), \nu)$ for every $x\in X\setminus E$.

Since $\mathrm{supp}(\mu_X) = X$, the set $X\setminus E$ is dense in $X$. Let $x_0\in X$. Since $X$ is metrizable, we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X\setminus E$ with $x_n\to x_0$ as $n\to\infty$. For each $n\in\mathbb{N}$, since the function $\chi^{n}_{x^{-1}}$ is in $L^1(X,\prod(X, \mathfrak{B}_X,\nu),\nu)$, we can find some $A_n \in \prod(X, \mathfrak{B}_X,\nu)$ such that $\nu(An\_(x_n^{-1}A)) = 0$. When $n \to \infty$, since $x_n \to x_0$ and $\nu$ is regular [474], we have $\nu((x_n^{-1}A)\Delta(x_0^{-1}A)) \to 0$ and hence $\nu(A_n\Delta(x_0^{-1}A)) \to 0$. Passing to a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ if necessary, we may assume that $\sum_{n\in\mathbb{N}}\nu(A_n\Delta(x_0^{-1}A)) < +\infty$ . It follows that $\lim_{k\to\infty}\nu((x_0^{-1}A))\Delta(\bigcup_{n\geq k}A_n)) = 0$ and hence $\nu((x_0^{-1}A)\Delta(\bigcap_{k\in\mathbb{N}}\bigcup_{n\geq k}A_n)) = 0$. Note that if $A' \in \mathfrak{B}_X$ satisfies $\nu(A') = 0$, then $A' \in \prod(X, \mathfrak{B}_X,\nu)$. Since $\bigcap_{k\in\mathbb{N}}\bigcup_{n\geq k}A_n$ is in $(X, \mathfrak{B}_X,\nu)$,we conclude that $x_0^{-1}A$ is in $\prod(X, \mathfrak{B}_X,\nu)$. This proves $x\cdot\prod(X, \mathfrak{B}_X,\mu_X) \subseteq \prod(X, \mathfrak{B}_X,\nu)$ for all $x \in X$. Similarly, one has $\prod(X, \mathfrak{B}_X,\mu_X)\cdot x \subseteq \prod(X, \mathfrak{B}_X,\nu)$ for all $x \in X$.

**Theorem (6.2.56)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Denote by $q$ the quotient map $X \to X/IE(X)$. Then the following hold:

**(i)** $q^{-1}(\mathfrak{B}_{X/IE(X)}) = \prod(X, \mathfrak{B}_X,\mu_X)$ mod $\mu_X$ .

**(ii)** For any $\nu \in \mathcal{M}(X,\Gamma)$, one has $q^{-1}(\mathfrak{B}_{X/IE(X)}) \subseteq \prod(X, \mathfrak{B}_X,\mu_X) \subseteq \prod(X, \mathfrak{B}_X,\nu)$.

We shall need the following result of Danilenko [410], which was proved first by Glasner et al. [421] in the case that the actions of $\Gamma$ on both $(X, \mathfrak{B}_X,\mu_X)$ and $(Y, \mathfrak{B}_Y,\mu_Y)$ are free and ergodic. Though Danilenko assumed $\Gamma$ to be infinite in [410], the following result holds trivially when $\Gamma$ is finite, since in such case the Pinsker algebra consists of measurable sets with measure 0 or 1 .

**Proof:** (i) By Lemma (6.2.55) we have $x\cdot\prod(X, \mathfrak{B}_X,\mu_X), \prod(X, \mathfrak{B}_X,\mu_X)\cdot x \subseteq \prod(X, \mathfrak{B}_X,\mu_X)$ for all $x \in X$. Thus, by [468], is a closed $\Gamma$ -invariant normal subgroup $Y$ of $X$ such that $q_1^{-1}(\mathfrak{B}_{X/Y}) = \prod(X, \mathfrak{B}_X,\mu_X)$ mod $\mu_X$, where $q_1$ denotes the quotient map $X \to X/Y$ .In particular, $h_{(q_1)_*(\mu_X)}(X/Y) = 0$. By Theorem (6.2.47), there is a surjective continuous map $q' : X/IE(X) \to X/Y$ such that $q'\circ q = q_1$ . Clearly $q$ is a group homomorphism and hence is open.

Every continuous open surjective map between compact metrizable spaces has a Borel [398]. Thus we can find a Borel map $\psi : X/Y \to X/IE(X)$ such that $q'\circ\psi$ is the identity map on $X/Y$ . It is easily verified that the map $\phi : X/Y \times \ker q' \to X/IE(X)$ sending $(z,y)$ to $\psi(z)y$ is an isomorphism from the measurable space $(X/Y \times \ker q', \mathfrak{B}_{X/Y} \times \mathfrak{B}_{\ker q'})$ onto the measurable space $(X/IE(X), \mathfrak{B}_{X/IE(X)})$.We claim that $\phi_*(\mu_{X/Y} \times \mu_{\ker q'})$ is left-translation invariant. Let $A \in \mathfrak{B}_{X/Y}\ \mathfrak{B}_{\ker q'}$ and $(z_1,y_1) \in (X/Y) \times \ker q'$. For each $z \in X/Y$ , denote by $A_z$ the set $\{y \in \ker q' : (z,y) \in A\}$. Note that $A_z \in \mathfrak{B}_{\ker q'}$ for every $z \in X/Y$. For any $(z_2,y_2) \in (X/Y) \times \ker q'$, we have

$$\phi(z_1,y_1)\phi(z_2,y_2) = \phi(z_1z_2,\psi(z_1z_2)^{-1}\psi(z_1)y_1\psi(z_2)y_2).$$

Thus, for any $z_2 \in X/Y$ , one has $(\phi^{-1}(\phi(z_2,y_1)\phi(A)))z_1z_2 = \psi(z_1z_2)^{-1}\psi(z_1)y_1\psi(z_2)A_{z_2}$ and hence $\mu_{\ker q'}((\phi^{-1}(\phi(z_1,y_1)\phi(A)))z_1z_2) = \mu_{\ker q'}(A_{z_2})$. Therefore,

$$(\mu_{X/Y} \times \mu_{\ker q'})(\phi^{-1}(\phi(z_1,y_1)\phi(A))) = \int_{X/Y}\mu_{\ker q'}((\phi^{-1}(\varphi(z_1,y_1) \times \phi(A)))_{z_1z_2})d\mu_{X/Y}(z_1z_2)$$

$$= \int_{X/Y}\mu_{\ker q'}(A_{z_2})d\mu_{X/Y}(z_2)$$

$$= (\mu_{X/Y} \times \mu_{\ker q'})(A).$$

This proves our claim. Therefore $\phi_*(\mu_{X/Y} \times \mu_{\ker q'}) = \mu_{X/IE(X)}$. Also note that the measures $(q_1)_*(\mu_X), (q')_*\mu_{X/IE(X)})$, and $q_*(\mu_X)$ are all translation invariant, and hence $(q_1)_*(\mu_X) = (q')_*\mu_{X/IE(X)}) = \mu_{X/Y}$ and $q_*(\mu_X) = \mu_{X/IE(X)}$.

We claim that $q'$ is an isomorphism. Suppose that $q'$ is not injective. Then we can find disjoint nonempty open subsets $U$ and $V$ of $\ker q'$. Since $\operatorname{supp}(\mu_{\ker q'}) = \ker q'$, we have $0 < \mu_{\ker q'}(U) < 1$. By Theorem (6.2.47) we have $h_{q_*(\mu_X)}(X/IE(X)) = 0$, and hence $q^{-1}(\mathfrak{B}_{X/IE(X)}) \subseteq \prod(X, \mathfrak{B}_X, \mu_X)$. Note that $(X/Y) \times U \in \mathfrak{B}_{X/Y} \times \mathfrak{B}_{\ker q'}$, and hence $\phi((X/Y) \times U) = \psi(X/Y)U$ is in $\mathfrak{B}_{X/IE(X)}$. As $q_1^{-1}(\mathfrak{B}_{(X/Y)} = \prod(X, \mathfrak{B}_X, \mu_X) \bmod \mu_X$, we can find some $A \in \mathfrak{B}_{(X/Y)}$ with $\mu_X(q_1^{-1}(A)\Delta q^{-1}(\psi(X/Y)U)) = 0$. Then

$$\begin{aligned}\mu_{X/IE(X)}((q')^{-1}(A)\Delta(\psi(X/Y)U)) &= q_*(\mu_X)((q')^{-1}(A)\Delta(\psi(X/Y)U)) \\ &= \mu_X(q^{-1}((q')^{-1}(A))\Delta q^{-1}(\psi(X/Y)U)) \\ &= \mu_X(q_1^{-1}(A)\Delta q^{-1}(\psi(X/Y)U)) = 0. \qquad (19)\end{aligned}$$

Note that

$$\begin{aligned}\mu_{X/IE(X)}(\psi(X/Y)U) &= \phi_*(\mu_{X/Y} \times \mu_{\ker q'})(\phi((X/Y) \times U)) \\ &= \mu_{X/Y}(X/Y) \cdot \mu_{\ker q'}(U) \\ &= \mu_{\ker q'}(U) > 0,\end{aligned}$$

and hence

$$\begin{aligned}\mu_{X/Y}(A) = q'_*(\mu_{X/IE(X)})(A) &= \mu_{X/IE(X)}((q')^{-1}(A)) \\ &= \mu_{X/IE(X)}(\psi(X/Y)U) > 0.\end{aligned}$$

Then

$$\begin{aligned}&\mu_{X/IE(X)}((q')^{-1}(A) \cap (\psi(X/Y)U)) \\ &= \mu_{X/IE(X)}((\psi(A)\ker q') \cap (\psi(X/Y)U)) \\ &= \phi_*(\mu_{X/Y} \times \mu_{ker\, q'})(\phi(A \times \ker q') \cap \phi((X/Y) \times U)) \\ &= \phi_*(\mu_{X/Y} \times \mu_{ker\, q'})(\phi((A \times \ker q') \cap ((X/Y) \times U)) \\ &= \phi_*(\mu_{X/Y} \times \mu_{ker\, q'})(\phi(A \times U)) \\ &= \mu_{X/Y}(A) \cdot \mu_{ker\, q'}(U) \\ &< \mu_{X/Y}(A) = \mu_{X/IE(X)}(\psi(X/Y)U),\end{aligned}$$

contradict to the equality (19). Therefore $q'$ is an isomorphism. Then $q^{-1}(\mathfrak{B}_{X/IE(X)}) = (q_1)^{-1}(\mathfrak{B}_{X/Y}) = \prod(X, \mathfrak{B}_X, \mu_X) \bmod \mu_X$.

(ii). Since $\prod(X, \mathfrak{B}_X, \mu_X)$ contains all $A \in \mathfrak{B}_X$ with $\mu_X(A) = 0$, from the assertion (i) we conclude that $q^{-1}(\mathfrak{B}_{X/IE(X)}) \subseteq \prod(X, \mathfrak{B}_X, \mu_X)$. Taking $x = e_X$ in Lemma (6.2.55), we get $\prod(X, \mathfrak{B}_X, \mu_X) \subseteq \prod(X, \mathfrak{B}_X, \nu)$.

We say that an action of $\Gamma$ on a compact group $X$ by automorphisms has *CPE* if $\prod(X, \mathfrak{B}_X, \mu_X)$ consists of Borel sets with $\mu_X$-measure 0 or 1. From Theorem (6.2.56) we get

**Corollary (6.2.57)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Then $IE(X) = X$ if and only if the action has *CPE*.

In the case $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ and $X$ is abelian, the following corollary was proved by Lind et al. [448], [468].

**Corollary (6.2.58)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms and let $Y$ be a closed $\Gamma$-invariant normal subgroup of $X$. Suppose that both the restriction of the action on $Y$ and the induced action on $X/Y$ have *CPE*. Then the action itself has *CPE*.

**Proof:** By Corollary (6.2.57) we have $IE(Y) = Y$ and $IE(X/Y) = X/Y$, and it suffices to show that $IE(X) = X$. From the definition of $IE$ tuples we have $IE_2(Y) \subseteq IE_2(X)$, and hence $Y = IE(Y) \subseteq IE(X)$. By Theorem (6.2.46).(v) one has $IE(X)/Y = IE(X/Y) = X/Y$ .Therefore $IE(X) = X$ as desired.

We discuss when a $\Gamma$-action on a compact group by automorphisms has a unique maximal measure.

**Theorem (6.2.59)[396]:** Let $\Gamma$ act on a compact group $X$ by automorphisms. Consider the following conditions:

**(i)** the action has $CPE$;

**(ii)** $h_\nu(X) < h_{\mu_X}(X)$ for every $\nu \in \mathcal{M}(X,\Gamma)$ not equal to $\mu_X$ .Then (ii)⇒(i). If furthermore $h(X) < \infty$, then (i)⇔(ii).

For the case $\Gamma = \mathbb{Z}$, Theorem (6.2.59)was proved by Berg [399]. Yuzvinskiˇı [476] showed that when $\Gamma = \mathbb{Z}$ and the action has finite entropy, the condition (i) is also equivalent to that $\mu_X$ is ergodic.

When $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, Lind et al. [448] proved Theorem (6.2.59) for the case $X$ is abelian, and later Schmidt [468] established the general case. Ledrappier [442] showed that, for $\Gamma = \mathbb{Z}^2$, the canonical $\Gamma$-action on $X = \widehat{\mathbb{Z}\Gamma/J}$ is mixing with respect to $\mu_X$ and has zero entropy, where $J = 2\mathbb{Z}\Gamma + (1 - u_1 - u_2)\mathbb{Z}\Gamma$ and $u_1, u_2$ denote the canonical basis of $\mathbb{Z}^2$.

For a standard probability space $(X, \mathfrak{B}_X)$, we say that two $\sigma$-algebras $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}_X$ are independent if $\mu(A \cap B) = \mu(A)\mu(B)$ for all $A \in \mathfrak{B}_1$ and $\mathfrak{B} \in \mathfrak{B}_2$. We need the following result of Danilenko [410] (in the statement of the condition $h_\mu(X) < +\infty$ is missing).

**Theorem (6.2.60)[396]:** Let $\Gamma$ act on a standard probability space $(X, \mathfrak{B}_X, \mu)$ via measure-preserving automorphisms. Suppose that $h_\mu(X) < +\infty$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be $\Gamma$-invariant sub-$\sigma$-algebras of $\mathfrak{B}_X$. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are independent if and only if $\prod(X, \mathfrak{B}_1, \mu)$ and $\prod(X, \mathfrak{B}_2, \mu)$ are independent and

$$h_\mu(\mathfrak{B}_1 \vee \mathfrak{B}_2) = h_\mu(\mathfrak{B}_1) + h_\mu(\mathfrak{B}_2).$$

We are ready to prove Theorem (6.2.59).

**Proof:** Assume that the condition (ii) holds. By Proposition (6.2.4) and Theorem (6.2.47) we have

$$h(X) = h(IE(X)) + h(X/IE(X)) = h(IE(X)) = h_{\mu IE(X)}(IE(X)) = h_{\mu IE(X)}(X).$$

Thus $\mu_{IE(X)} = \mu_X$ , and hence $IE(X) = X$. By Corollary (6.2.57)the condition (i) holds.

Now assume that $h(X) < \infty$ and that the condition (i) holds. Let $\nu \in \mathcal{M}(X,\Gamma)$ with $h_\nu(X) = h_{\mu_X}(X)$. We shall show that $\nu = \mu_X$ . Denote by $\pi_1$ and $\pi$ the first coordinate map $X \times X \to X$ sending $(x, y)$ to $x$ and the product map $X \times X \to X$ sending $(x, y)$ to $xy$ respectively. Set $\mathfrak{B}_1 = \pi_1^{-1}(\mathfrak{B}_X)$ and $\mathfrak{B}_2 = \pi^{-1}(\mathfrak{B}_X)$. As $X$ is compact metrizable, $\mathfrak{B}_{X\times X} = \mathfrak{B}_X \times \mathfrak{B}_X$ . By [399], both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\Gamma$-invariant sub-$\sigma$-algebras of $\mathfrak{B}_{X\times X}$, and $\mathfrak{B}_1 \vee \mathfrak{B}_2 = \mathfrak{B}_X \times \mathfrak{B}_X$ . The condition (i) says that $\prod(X, \mathfrak{B}_X, \mu_X)$ consists of elements in $\mathfrak{B}_X$ with $\mu_X$ -measure 0 or 1 . Then $\prod(X \times X, \mathfrak{B}_1, \mu_X \times \nu) = \pi_1^{-1}(\_(X, \mathfrak{B}_X, \mu_X))$ consists of elements in $\mathfrak{B}_{X\times X}$ with $\mu_X \times \nu$-measure 0 or 1. Thus $\prod(X \times X, \mathfrak{B}_1, \mu_X \times \nu)$ and $\prod(X \times X, \mathfrak{B}_2, \mu_X \times \nu)$ are independent under $\mu_X \times \nu$. Note that

$$h_{\mu_X\times\nu}(\mathfrak{B}_1\vee\mathfrak{B}_2)=h_{\mu_X\times\nu}(\mathfrak{B}_X\times\mathfrak{B}_X)=h_{\mu_X}(\mathfrak{B}_X)+h_\nu(\mathfrak{B}_X)=2h_{\mu_X}(\mathfrak{B}_X),$$
and by Lemma (6.2.44) ,

$$h_{\mu_X\times\nu}(\mathfrak{B}_1)+h_{\mu_X\times\nu}(\mathfrak{B}_2)=h_{\mu_X}(\mathfrak{B}_X)+h_{\mu_X}(\mathfrak{B}_X)=2h_{\mu_X}(\mathfrak{B}_X).$$
Thus
$$h_{\mu_X\times\nu}(\mathfrak{B}_1\vee\mathfrak{B}_2)=h_{\mu_G\times\nu}(\mathfrak{B}_1)+h_{\mu_X\times\nu}(\mathfrak{B}_2).$$
By Theorem (6.2.60) we see that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are independent with respect to $\mu_X\times\nu$. From [399] or [468] we conclude that $\mu_X=\nu$.
Throughout $\Gamma$ will be a countable amenable group.

Let $\Gamma$ act on a compact abelian group $X$ by automorphisms, and $1\le p\le+\infty$.We shall treat $\Delta^p(X)$ and its $\Gamma$-invariant subgroups $G$ as discrete abelian groups, thus consider the induced $\Gamma$-action on the Pontryagin dual $\widehat{\Delta^p(X)}$ and $\hat{G}$ by automorphisms. The pair $(\hat{X},G)$ will be treated as a dual pair.

We first give some conditions for $h(X)$ to be bounded below by $h(\widehat{\Delta^p(X)})$. The definition of entropy is recalled.

**Lemma (6.2.61)[396]:** Let $1\le p<+\infty$. There exists some universal constant $C_p>0$ such that for any $\lambda>1$, there is some $\delta>0$ so that for any nonempty finite set $Y$, any positive integer $n$ with $|Y|\le\delta n$, and any $M\ge1$ one has

$$|\{x\in\mathbb{Z}^Y:\|x\|_p\le M\cdot n^{1/p}\}|\le C_p\lambda^nM^{|Y|}.$$
**Proof:** Let $\delta>0$ be a small number less than $e^{-1}$ which we shall determine in a moment. Let $Y$ be a nonempty finite set, $n$ be a positive integer with $|Y|\le\delta n$, and $M\ge1$. For each $x\in\mathbb{Z}^Y$, denote $\{z\in\mathbb{R}^Y:0\le z_y-x_y\le1$ for all $y\in Y\}$ by $D_x$. Denote $\{x\in\mathbb{Z}^Y:\|x\|_p\le M\cdot n^{1/p}\}$ by $S$ and denote the union of $D_x$ for all $x\in S$ by $D_S$. Then the (Euclidean) volume of $D_S$ is equal to $|S|$. Note that for any $z\in D_S$, say $z\in D_x$, one has

$$\|z\|_p\le\|x\|_p+\|z-x\|_p\le M\cdot n^{1/p}+n^{1/p}\le 2Mn^{1/p}.$$
A simple calculation shows that the function $\varsigma(t):=(n/t)^{t/p}$ is increasing for $0<t\le ne^{-1}$. The volume of the unit ball of R$Y$ under $\|\cdot\|_p$ is $\frac{(2/p)^{|Y|}(\Gamma(1/p))^{|Y|}}{(|Y|/p)\Gamma(|Y|/p)}$, where $\Gamma$ denotes the gamma function. By Stirling's formula [440] there exists some constant $C'>0$ such that $\Gamma(t)\ge C\sqrt{2\pi}t^{t-1/2}e^{-t}$ for all $t\ge1/p$. Thus the volume of $D_S$ is no bigger than
$$\frac{(2/p)^{|Y|}(\Gamma(1/p))^{|Y|}(2Mn^{1/p})^{|Y|}}{(|Y|/p)\Gamma(|Y|/p)}\le\frac{(2/p)^{|Y|}(\Gamma(1/p))^{|Y|}(2Mn^{1/p})^{|Y|}}{(|Y|/p)C'\sqrt{2\pi}\,(|Y|/p)^{|Y|/p-1/2}e^{-|Y|/p}}$$

$$\le|Y|^{-1/2}C_p\tilde{C}^{|Y|}(n/|Y|)^{|Y|/p}M^{|Y|}$$
$$\le C_p\tilde{C}^{|Y|}\varsigma(|Y|)M^{|Y|}$$
$$\le C_p\tilde{C}^{\delta n}\varsigma(\delta n)M^{|Y|}=C_p\tilde{C}^{\delta n}\delta^{-\delta n/p}M^{|Y|}$$
Where $C_p=\sqrt{p/(2\pi)}/C'$ and $\tilde{C}=\max(4e^{1/p}p^{(1-p)/p}\Gamma(1/p),1)$. Take $\delta>0$ so small that $\tilde{C}^\delta\delta^{-\delta/p}\le\lambda$. Then the volume of $D_S$ is no bigger than $C_p\lambda^nM^{|Y|}$. Consequently, $|S|\le C_p\lambda^nM^{|Y|}$.

Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. For any nonempty finite subset $E$ of $\hat{X}$, the function $F \mapsto \log|\sum_{s\in F} s^{-1}E|$ defined on the set of nonempty finite subsets of $\Gamma$satisfies the conditions of the Ornstein-Weiss lemma [205], thus $\frac{\log|\sum_{s\in F} s^{-1}E|}{|F|}$ converges to some real number $c$, denoted by $\lim_F \frac{\log|\sum_{s\in F} s^{-1}E|}{|F|}$, when $F$ becomes more and more left invariant. That is, for any $\varepsilon > 0$, there exist a nonempty finite subset $K$ of $\Gamma$ and $\delta > 0$ such that for any nonempty finite subset $F$ of $\Gamma$ satisfying $|KF\backslash F| \le \delta|F|$ one has $\left|\frac{\log|\sum_{s\in F} s^{-1}E|}{|F|} - c\right| < \varepsilon$. We need the following beautiful result of Peters [460]:

**Theorem (6.2.62)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms. Then

$$h(X) = \sup_E \lim_F \frac{\log|\sum_{s\in F} s^{-1}E|}{|F|},$$

where $E$ ranges over all nonempty finite subsets of $\hat{X}$.

In [460], Theorem (6.2.62) was stated and proved only for the case $\Gamma = \mathbb{Z}$, but the proof there works for general countable amenable groups.

**Theorem (6.2.63)[396]:** Let $k, n \in \mathbb{N}$ and $A \in M_{n\times k}(\mathbb{Z}\Gamma)$. Let $X$ be a closed $\Gamma$-invariant subgroup of $\widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^n A}$. Let $1 \le p < +\infty$. Suppose that one of the following conditions holds:

**(i)** $p = 1$ and the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^n$ sending a to $aA^*$ is injective.

**(ii)** There exists $C > 0$ such that $\|a\|_p \le C\|aA^*\|_p$ for all $a \in (\ell^p(\Gamma))^k$, where the norm $\|\cdot\|_p$ is defined by the Eq. (10).

Then

$$h(X) \ge h\left(\widehat{\Delta^p(X)}\right).$$

To prove Theorem (6.2.63), we need the following lemma, of which the case $p = 2$ appeared in [443].

**Proof:** Fix a compatible translation-invariant metric $\rho$ on $X$. Denote by $K$ the support of $A$ as a $M_{n\times k}(\mathbb{Z})$-valued function on $\Gamma$. When $\Gamma$ is finite and acts on a compact space $Y$ continuously, one has $h_{\text{top}}(Y) = |Y|/|\Gamma|$ when $Y$ is a finite set and $h_{\text{top}}(Y)| = +\infty$ otherwise. Thus we may assume that $\Gamma$ is infinite.

By Theorem (6.2.62) it suffices to show

$$\lim_F \frac{\log|\sum_{s\in F} s^{-1}E|}{|F|} \le h(X) + \delta$$

for every nonempty finite subset $E$ of $\Delta^p(X)$ and every $\delta > 0$. Fix such $E$ and $\delta$. Recall the canonical metric $\rho_\infty$ on $(\mathbb{R}/\mathbb{Z})^k$ defined in (11). Take $\varepsilon > 0$ such that for any $x \in X$ with $\rho(x, 0_X) \le \varepsilon$ one has $\rho_\infty(x_{e_\Gamma} e, 0_{(\mathbb{R}/\mathbb{Z})^k}) \le (2\|A\|_1)^{-1}$.It suffices to show

$$\left|\sum_{s\in F} s^{-1}E\right| \le N_{\rho,F,\varepsilon}(X)\exp(\delta|F|)$$

for all sufficiently left invariant nonempty finite subsets $F$ of $\Gamma$.

Set $E' = E - E \subseteq \Delta^p(X)$. Denote by $B_{F,\varepsilon}$ the set of all $x \in X$ satisfying $\max_{s\in F} \rho(sx, 0_X) \le \varepsilon$. Take a maximal $(\rho, F, \varepsilon)$-separated subset $V_F$ of $\sum_{s\in F} s^{-1}E$. Then for any $x \in \sum_{s\in F} s^{-1}E$, since $\rho$ is translation-invariant, one can find some $y \in V_F$ with $x - y \in B_{F,\varepsilon}$. Note that $x - y \in \sum_{s\in F} s^{-1}E'$. It follows that

$$|\sum_{s\in F} s^{-1}E\,| \le |V_F||B_{F,\varepsilon}\cap \sum_{s\in F} s^{-1}E'\,| \le N_{\rho,F,\varepsilon}(X)|B_{F,\varepsilon}\cap \sum_{s\in F} s^{-1}E'\,|.$$

Thus it suffices to show

$$|B_{F,\varepsilon}\cap \sum_{s\in F} s^{-1}E'\,| \le \exp(\delta|F|) \tag{20}$$

for all sufficiently left invariant nonempty finite subsets $F$ of $\Gamma$.

Denote by $P$ the canonical projection map $\ell^\infty(\Gamma,\mathbb{R}^k)\to((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. For each $w\in E$, take $\tilde{w}\in\ell^\infty(\Gamma,\mathbb{R}^k)$ with $P(\tilde{w})=w$ and $\|\tilde{w}_s\|_\infty=\rho_\infty(w_s,0_{(\mathbb{R}/\mathbb{Z})^k})$ for all $s\in\Gamma$. Since $w\in\Delta^p(X)$, by Proposition (6.2.33).(iv) one has $\tilde{w}\in\ell^\infty(\Gamma,\mathbb{R}^k)$. Set $\tilde{E}=\{\tilde{w}: w\in E'\}$. For each $\tilde{w}\in\tilde{E}$, one has $\tilde{w}A^*\in\ell^\infty(\Gamma,\mathbb{Z}^n)\cap\ell^p(\Gamma,\mathbb{R}^n)=\mathbb{Z}^n\Gamma$. Denote by $K_1$ the finite subset $\bigcup_{\tilde{w}\in\tilde{E}}\operatorname{supp}(\tilde{w}A^*)$ of $\Gamma$. Note that for any nonempty finite subset $F$ of $\Gamma$ and any $\tilde{x}\in\sum_{s\in F}s^{-1}E'$, one has $\operatorname{supp}(\tilde{x}A^*)\subseteq F^{-1}K_1$. Let $F$ be a nonempty finite subset of $\Gamma$. Let $x\in B_{F,\varepsilon}\cap\sum_{s\in F}s^{-1}E'$. Take $x'\in\ell^\infty(\Gamma,\mathbb{R}^k)$ with $P(x)=x$ and $\|x'_s\|_\infty=\rho_\infty(x_s,0_{(\mathbb{R}/\mathbb{Z})^k})$ for all $s\in\Gamma$. Since $x\in\Delta^p(X)$, by Proposition (6.2.33).(iv) one has $x'\in\ell^p(\Gamma,\mathbb{R}^k)$. Set $F':=\{s\in F: s^{-1}K\subseteq F^{-1}\}$. As $x\in B_{F,\varepsilon}$, by our choice of $\varepsilon$ one has $\|x'_t\|_\infty=\rho_\infty(x_t,0_{(\mathbb{R}/\mathbb{Z})^k})\le(2\|A\|_1)^{-1}$ for every $t\in F^{-1}$, and hence

$$\|x'A^*\|_\infty\le(\max_{t\in F^{-1}}\|x'_t\|_\infty)\|A^*\|_1\le 1/2$$

for all $s\in(F')^{-1}$. Since $x\in X, x'A^*$ has integral coefficients. Thus $x'A^*=0$ on $(F')^{-1}$.

Since $P(\tilde{E})=E'$, we can find $\tilde{x}\in\sum_{s\in F}s^{-1}\tilde{E}$ with $P(\tilde{x})=x$. Define $x\in\ell^p(\Gamma,\mathbb{R}^k)$ to be the same as $x'$ on $F^{-1}$ and the same as $\tilde{x}$ on $\Gamma\backslash F^{-1}$. Then $\tilde{x}A^*=x'A^*=0$ on $(F')^{-1}$. Also, $\tilde{x}A^*=\tilde{x}A^*$ on $\Gamma\backslash(F^{-1}K^{-1})$, and hence $\tilde{x}A^*=\tilde{x}A^*=0$ on $\Gamma\backslash(F^{-1}(K_1\cup K^{-1}))$. Therefore $\operatorname{supp}(\tilde{x}A^*)\subseteq(F^{-1}(K_1\cup K^{-1}))\backslash(F')^{-1}$.

Since $P(\tilde{x})=P(x')=x$, we have $P(\tilde{x})=x\in X$, and hence $\tilde{x}A^*$ has integral coefficients.

Now we separate two cases.

Assume first that the condition (i) holds. Set $D=\sum_{\tilde{w}\in\tilde{E}}\|\tilde{w}\|_1$. Note that $\|\tilde{x}\|_\infty\le D$ and hence

$$\|\tilde{x}\|_\infty\le\max(\|\tilde{x}\|_\infty,\|x'\|_\infty)\le D+1.$$

Thus

$$\|\tilde{x}A^*\|_\infty\le\|\tilde{x}\|_\infty\cdot\|A\|_1\le(D+1)\|A\|_1.$$

Then the number of possible $\tilde{x}A^*$ is at most $(2(D+1)\|A\|_1+1)^{n|(F^{-1}(K_1\cup K^{-1}))\backslash(F')^{-1}|}$. Since
the map $(\ell^1(\Gamma))^k\to(\ell^1(\Gamma))^n$ sending $a$ to $aA^*$ is injective, the number of possible $\tilde{x}$ is also bounded above by $(2(D+1)\|A\|_1+1)^{n|(F^{-1}(K_1\cup K^{-1}))\backslash(F')^{-1}|}$. As $P(\tilde{x})=x$, we obtain

$$|B_{F,\varepsilon}\cap\sum_{s\in F}s^{-1}E'\,|\le(2(D+1)\|A\|_1+1)^{n|(F^{-1}(K_1\cup K^{-1}))\backslash(F')^{-1}|}.$$

When $F$ is sufficiently left invariant, the right hand side of the above inequality is bounded above by $\exp(\delta|F|)$, and hence (20) holds.

Next we assume that the condition (ii) holds. Set $D' = \sum_{\tilde{w}\in\tilde{E}} \|\tilde{w}A^*\|_1$. Note that $\|\tilde{x}\, A^*\|_\infty \le D'$, and hence $\|\tilde{x}\,\|_p \le C\|\tilde{x}\, A^*\|_p \le CD'|F^{-1}K_1|^{1/p}$.Define $\hat{x} \in \ell^p(\Gamma, \mathbb{R}^k)$ to be the same as $\tilde{x}$ on $((F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1})K$ and 0 on all other points of $\Gamma$. Then $\hat{x}\, A^* = \tilde{x}\, A^*$ on $(F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1})$. Note that

$$\|\hat{x}\,\|_p \le \|\tilde{x}\,\|_p + |((F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1}K|^{1/p}$$
$$\le CD'|F^{-1}K_1|^{1/p} + |((F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1}K|^{1/p}.$$

Since $\tilde{x}\, A^*$ has support in $(F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1}$, we get

$$\begin{aligned}\|\tilde{x}\, A^*\,\|_p &\le \|\hat{x}\, A^*\,\|_p \le \|\hat{x}\,\ \|_p\|A^*\|_1\\ &\le \left(CD'|F^{-1}K_1|^{1/p} + |((F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1}K|^{1/p}\right)\|A^*\|_1\\ &\le (2CD'+1)\|A\|_1|F|^{1/p},\end{aligned}$$

when $F$ is sufficiently left invariant. By Lemma (6.2.61), for any $\lambda > 1$, when $F$ is sufficiently left invariant, the number of $y \in \mathbb{Z}^n\Gamma$ with support in $(F^{-1}(K_1 \cup K^{-1}))\backslash(F')^{-1}$ and $\|y\,\|_p \le (2CD'+1)\|A\|_1|F|^{1/p}$ is at most

$$C_p^n\lambda^{n|F|}\left((2CD+1)\|A\|_1\right)^{n|(F^{-1}(K_1\cup K^{-1}))\backslash(F')^{-1}|}.$$

Since $\Gamma$ is infinite, it follows that when $F$ is sufficiently left invariant, the number of $x\, A^*$ is at most $\exp(\delta|F|)$. As in the first case, one concludes that the inequality (20) holds.

**Question (6.2.64)[396]:** Could one weaken the conditions (i) and (ii) of Theorem (6.2.63) to that the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^n$ sending $a$ to $a\, A^*$ is injective?

From Theorems (6.2.31)and (6.2.63), and Proposition (6.2.19).(iv) we get

**Corollary (6.2.65)[396]:** Let $\Gamma$ act on a compact abelian group $X$ by automorphisms such that $\widehat{X}$ is a finitely presented left $\mathbb{Z}\Gamma$-module. Then

$$h(X) \ge h(\widehat{\Delta^1(X)}).$$

From Theorems (6.2.9) and (6.2.63), we get

**Corollary (6.2.66)[396]:** Let $\Gamma$ act on a compact abelian group $X$ expansively by automorphisms.
Then

$$h(X) \ge h(\widehat{\Delta(X)}).$$

Let $A \in M_k(\mathbb{Z}\Gamma)$ for some $k \in \mathbb{N}$. Let $1 < p, q < +\infty$ with $p^{-1} + q^{-1} = 1$. One may identify $\ell^q(\Gamma, \mathbb{R}^k)$ with the dual space of $\ell^p(\Gamma, \mathbb{R}^k)$ naturally, as using the pairing in (13). For the bounded linear map $T: \ell^p(\Gamma, \mathbb{R}^k) \to \ell^p(\Gamma, \mathbb{R}^k)$ sending $a$ to $a\, A^*$, its dual $T^*: \ell^q(\Gamma, \mathbb{R}^k) \to \ell^q(\Gamma, \mathbb{R}^k)$ sends $b$ to $bA$. Thus $T$ is invertible exactly when $T^*$ is invertible. From Theorem (6.2.63) and Lemma (6.2.35) we get

**Corollary (6.2.67)[396]:** Let $k \in \mathbb{N}$, and $A \in M_k(\mathbb{Z}\Gamma)$ such that the linear map $(\ell^p(\Gamma))^k \to (\ell^p(\Gamma))^k$ sending a to $a\, A^*$ is invertible for some $1 < p < +\infty$. Set $X_A = (\mathbb{Z}\Gamma)^{\widehat{k/(\mathbb{Z}\Gamma)}^k}\, A$ and $X_{A^*} = (\mathbb{Z}\Gamma)^{\widehat{k/(\mathbb{Z}\Gamma)}^k}\, A^*$. Then

$$h(X_A) = h\left(\widehat{\Delta^p(X_A)}\right) = h(X_{A^*}).$$

Recall our convention of CPE before Corollary (6.2.57).

**Theorem (6.2.68)[396]:** Suppose that $\mathbb{Z}\Gamma$ is left Noetherian. Let $\Gamma$ act on a compact abelian group $Y$ 1-expansively by automorphisms such that $\Delta^1(Y)$ is dense in $Y$. Let $X$ be a closed $\Gamma$-invariant subgroup of $Y$. Then the following hold:

**(i)** For any $\Gamma$-invariant subgroup $G$ of $\Delta^1(X)$ with $\bar{G} = \overline{\Delta^1(X)}$, one has $h(X) = h(\hat{G})$.
**(ii)** $\Delta^1(X)$ is a dense subgroup of $IE(X)$.
**(iii)** The action $\Gamma \curvearrowright X$ has positive entropy if and only if $\Delta^1(X)$ is nontrivial.
**(iv)** The action $\Gamma \curvearrowright X$ has $CPE$ if and only if $\Delta^1(X)$ is dense in $X$.

**Proof:** (i). We show first $h(Y) = h(\widehat{G_1})$ for any $\Gamma$-invariant subgroup $G_1$ of $\Delta^1(Y)$ satisfying $G_1 = Y$. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$. The canonical pairing $Y \times \hat{Y} \to \mathbb{T}$ restricts to a pairing $G_1 \times \hat{Y} \to \mathbb{T}$ which is bi-additive and equivariant in the sense defined before Lemma (6.2.38). Since $G_1$ is dense in $Y$, by Lemma (6.2.38), the induced $\Gamma$-equivariant group homomorphism $\Phi : \hat{Y} \to \widehat{G_1}$ is injective and maps $\hat{Y}$ into $\Delta^1\widehat{(G_1)}$.

Since the $\Gamma$-action on $Y$ is 1-expansive, by Proposition (6.2.19).(ii) and Proposition (6.2.33).(vi) both $\hat{Y}$ and $\Delta^1(Y)$ are finitely generated left $\mathbb{Z}\Gamma$-modules. As $\mathbb{Z}\Gamma$ is left Noetherian, every left finitely generated $\mathbb{Z}\Gamma$-module is Noetherian and finitely presented [439]. Thus both $\hat{Y}$ and $G_1$ are finitely presented left $\mathbb{Z}\Gamma$-modules. In virtue of Corollary (6.2.65), we have

$$h(Y) \geq h\left(\widehat{\Delta^1(Y)}\right) \geq h\widehat{(G_1)}.$$

and

$$h\widehat{(G_1)} \geq h\left(\Delta^1\widehat{(\widehat{G_1})}\right) \geq h\left(\hat{\hat{Y}}\right) = h(Y).$$

Therefore $h(Y) = h(\widehat{\Delta^1(Y)}) = h\widehat{(G_1)}$ as desired.

Next we show $h(X) = h(\widehat{\Delta^1(X)})$. As above, both $\hat{X}$ and $\widehat{Y/X}$ are finitely presented left $\mathbb{Z}\Gamma$-modules.ByProposition (6.2.33).(iii) the quotient map $Y \to Y/X$ induces an embedding $\Delta^1(Y)/\Delta^1(X) \hookrightarrow \Delta^1(Y/X)$. In virtue of Corollary (6.2.65) we have

$$h(Y) \geq h\left(\widehat{\Delta^1(Y)}\right),$$

and

$$h(Y/X) \geq h(\Delta^1\widehat{(Y/X)}) \geq h(\Delta^1\widehat{(Y)/\Delta^1}(X)).$$

From Proposition (6.2.4) we then obtain

$$h(Y) = h(X) + h(Y/X) \geq h(\widehat{\Delta^1(X)}) + h(\Delta^1\widehat{(Y)/\Delta^1}(X)) = h(\Delta^1(Y)).$$

From the last paragraph we have $h(Y) = h(\widehat{\Delta^1(Y)})$. Since the $\Gamma$-action on $Y$ is 1-expansive and $\hat{Y}$ is a finitely presented left $\mathbb{Z}\Gamma$-module, by Theorem (6.2.31) one has $h(Y) < +\infty$. Thus we conclude that $h(X) = h(\widehat{\Delta^1(X)})$.

Finally we show $h(\widehat{\Delta^1(X)}) = h(\hat{G})$. For this purpose we may assume that $\overline{\Delta^1(X)} = \bar{G} = X$. Since the $\Gamma$-action on $Y$ is 1-expansive, its restriction on $X$ is also 1-expansive. From the first part of the proof we conclude that $h(\hat{G}) = h(X) = h(\widehat{\Delta^1(X)})$.

(ii). By Theorem (6.2.51) we have $\Delta^1(X) \subseteq IE(X)$. From Assertion (i) we have

$$h(Y) = h\left(\widehat{\Delta^1(Y)}\right) = h\left(\overline{\Delta^1(X)}\right).$$

In the above we have seen that $h(X) \leq h(Y) < +\infty$. Thus, by Proposition (6.2.4) we have

$$h(X/\overline{\Delta^1(X)}) = h(X) - h(\overline{\Delta^1(X)}) = 0.$$

In virtue of Theorem (6.2.47).(i), we conclude that $IE(X) \subseteq \overline{\Delta^1(X)}$. Therefore $IE(X) = \overline{\Delta^1(X)}$.

The assertion (iii) follows from the assertion (ii) and Theorem (6.2.46).(iv). The assertion (iv) follows from the assertion (ii) and Corollary (6.2.57).

**Example (6.2.69)[396]:** Let $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$. Let $f \in \mathbb{Z}\Gamma$ be irreducible such that $\mathbb{Z}(f)$ (as defined in Example (6.2.25)) is nonempty but finite. For instance, one may take f as $2d - \sum_{j=1}^{d}(u_j + u_j^{-1})$ or $d - \sum_{j=1}^{d} u_j$ for $u_1, \dots, u_d$ being the canonical basis of $\mathbb{Z}^d$. As pointed out in Examples (6.2.25) and (6.2.36), $\alpha$ f (as defined in Notation (6.2.20)) is 1-expansive and $\Delta^1(X_f)$ is dense in $X_f$. On the other hand, $\alpha_f$ is not expansive [467].

From Theorems (6.2.9), (6.2.68), parts (i) and (iv) of Proposition (6.2.19), and Lemma (6.2.35) we have

**Corollary (6.2.70)[396]:** Suppose that $\mathbb{Z}\Gamma$ is left Noetherian. Let $\Gamma$ act on a compact abelian group $X$ expansively by automorphisms. Then the following hold:

**(i)** For any $\Gamma$-invariant subgroup $G$ of $\Delta(X)$ with $\bar{G} = \overline{\Delta(X)}$, one has $h(X) = h(\hat{G})$.

**(ii)** $\Delta(X)$ is a dense subgroup of $IE(X)$.

**(iii)** The action has positive entropy if and only if $\Delta(X)$ is nontrivial.

**(iv)** The action has $CPE$ if and only if $\Delta(X)$ is dense in $X$.

Theorem (6.2.2) follows from Corollary (6.2.70) and the fact that when $\Gamma$ is polycyclic-by-finite, $\mathbb{Z}\Gamma$ is left Noetherian [424] [459].

## Section (6.3): Generic Points of Invariant Measures for an Amenable Residually Finite Group Actions with the Weak Specification Property

Generic points are a powerful tool of ergodic theory, allowing for example to quantify the difference between two measures. Recall that for a $T$-invariant measure $\mu$, a point $x$ is generic if its orbit is uniformly distributed, that is, for every continuous real valued function $f$ defined on the phase space $X$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^m(x)) = \int_X f d\mu.$$

If a measure is ergodic, then its generic points reflect the most typical behaviour of points from the phase space: it follows from the Birkhoff ergodic theorem that they form a set of the full measure. On the other hand, non-ergodic measures do not have to have any generic points. In fact, there is even a topologically mixing dynamical system with exactly two ergodic measures such that no non-ergodic measure has a single generic point (see [485]). Moreover, it follows from the ergodic decomposition theorem that even if a non-ergodic measure possesses some generic points, they form a measure zero set. Therefore, an important question is, under what assumptions every invariant measure has a generic point.

One of conditions which implies such a phenomenon is specification. This property was introduced by Rufus Bowen in [481] to study Axiom A diffeomorphisms. If a dynamical system satisfies specification, then we can find a point that traces an arbitrary collection of orbit segments, if the time between consecutive segments is large enough. Specification implies a very rich dynamics. It is known for instance that it is stronger than chain mixing.

Interestingly, the class of dynamical systems with this property is very wide and contains for example mixing graph maps, mixing sofic shifts (including shifts of finite type) and mixing interval maps. There are also many interesting systems which shows specification-like behaviour although they do not have the specification property in the sense of Bowen. Therefore many generalizations of this notion have been developed [482],[178], [187], [483], [485], [487], [486],[489],[490]. As far as smooth maps are considered, specification is closely related to hyperbolicity.

A dynamical system given by an iteration of a homeomorphism can be regarded as the action of the group of integers on the phase space. The group $\mathbb{Z}$ is an important example of a wide class of amenable residually finite groups.

In the literature one can find some specification-like properties for $\mathbb{Z}^d$-actions or, more generally, amenable groups actions. Among them there are for example topological Markov shifts, strongly irreducible shifts, semi-strongly irreducible shifts [484] or shifts satisfying the uniform filling property [479]. We use the approach introduced in [396] (see also [445],[466]). Recently Ren showed in [492] that if we assume that $G$ is an amenable residually finite group acting on a compact space $X$ and the dynamical systems $(X, G)$ has the specification property in this sense, then the simplex of $G$-invariant Borel probability measures supported at $X$ is either trivial (that is consists of only one element) or equal to the Poulsen simplex. The latter is a unique (up to an affine homeomorphism) Choquet simplex possessing a dense set of extreme points and has many remarkable properties [488]. Every Choquet simplex can be embedded into the Poulsen simplex as its face. What is more, the set of extreme points of the Poulsen simplex is arcwise connected. The first example of such a simplex was given in [494]. The result of Ren is a generalization of the first part of Sigmund's theorem who showed an analogous claim for actions of the group of integers which satisfy the specification property. The second part of the Sigmund theorem says that in this setting every invariant measure has a generic point.

It is known that even ergodic measures not necessarily have generic points with respect to any Følner sequence. The Lindenstrauss pointwise ergodic theorem guarantees that this is the case if we assume that the Følner sequence is tempered, that is it is growing in a certain way that we describe in more details. What is important, every amenable group admits a tempered Følner sequence.

We prove that every measure invariant for an amenable residually finite group action satisfying the weak specification property has a generic point. This extends the theorem of Sigmund and completes the result of Ren.

Through $(X, \rho)$ is a compact metric space. To simplify notation we assume that the diameter of $X$ with respect to $\rho$ is equal to 1. An infinite countable group $G$ acts on $X$ via homeomorphisms. We denote by $|A|$ the cardinality of the set $A$. Given $A, B \subset G$ we define

$$AB = \{ab : a \in A, b \in B\} \text{ and } A^{-1} = \{a^{-1} : a \in A\}.$$

Moreover, we denote by $A \Delta B$ the symmetric difference of $A$ and $B$. Let $\mathrm{Fin}(G)$ denote the family of finite, non-empty subsets of $G$. A fundamental domain of a finite index subgroup $H$ of a group $G$ is a set $F$ such that for every $g \in G$ we have $|Hg \cap F| = 1$. By $|G : H|$ we denote the index of a subgroup $H$.

We say that the group $G$ is amenable, if it admits a Følner sequence, that is a sequence $(F_n)_{n\in\mathbb{N}} \subset F\mathrm{in}(G)$ such that

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0 \text{ for every } g \in G.$$

Note that every subsequence of a Følner sequence is Følner itself. It is known that every countable abelian group (in particular $\mathbb{Z}^d$ with $d \in \mathbb{N}$) is amenable.

A countable group is residually finite if there exists a nested sequence of finite index normal subgroups which intersect trivially. Such a sequence is called askeleton of the group. A classical example of a residually finite group is $\mathbb{Z}^d$.

For $x \in X$ and $F \in \mathrm{Fin}(G)$ let

$$m(x, F_n) := \frac{1}{|F_n|} \sum_{f \in F_n} \hat{\delta}(fx),$$

where $\hat{\delta}(z)$ is a probability measure supported at $z$. Let $\mathcal{M}(X)$ be the family of all Borel probability measures on $X$. We equip $\mathcal{M}(X)$ with the Prokhorov metric $D: \mathcal{M}(X) \times \mathcal{M}(X) \to \mathbb{R}_+$ defined as

$$\mathrm{D}(\mu, \nu) = \inf\{\varepsilon > 0 : \text{fore very Borel set } B \subset X \text{ one has } \mu(B) \leq \nu(B^\varepsilon) + \varepsilon\},$$

where $B^\varepsilon = \{y \in X : \text{there exists } x \in B \text{ such that } \rho(x, y) < \varepsilon\}$. It is known that this metric gives the weak* topology on $\mathcal{M}(X)$ (see [491]), that is a topology such that a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if and only if for every continuous function $f: X \to \mathbb{R}$ one has

$$\int_X f d\mu_n \to \int_X f d\mu .$$

In particular, $(\mathcal{M}(X), D)$ is a compact metric space. Moreover, the following property of the Prokhorov metric is easy to be verified (see also [485]).

**Lemma (6.3.1)[478]:** Assume that $x, z \in X$ and $F \in \mathrm{Fin}(G)$ satisfy $\rho(gx, gz) < \varepsilon$ for some $\varepsilon > 0$ and all $g \in F$. Then

$$D(m(x, F), m(z, F)) \leq \varepsilon.$$

A dynamical system $(X, G)$ has the weak specification property if for every $\varepsilon > 0$ there exists $Z = Z(\varepsilon) \in \mathrm{Fin}\,(G)$ such that for any $m \in \mathbb{N}$ and $F_1, \ldots, F_m \in \mathrm{Fin}\,(G)$ satisfying $ZF_i \cap F_j = \emptyset$ for all $1 \leq i = j \leq m$ and for all $x_1, \ldots, x_m \in X$ there exists $z \in X$ such that for every $1 \leq i \leq m$ and $g \in F_i$ one has $\rho(gx_i, gz) \leq \varepsilon$.

A Borel probability measure $\mu$ is $G$-invariant if for every $g \in G$ and every measurable set $A$ one has $\mu(g^{-1}A) = \mu(A)$. Let $\mathcal{M}_G(X)$ be the simplex of Borel probability $G$-invariant measures. Amenability of $G$ implies that $\mathcal{M}_G(X)$ is nonempty. $A$ measure $\mu \in \mathcal{M}_G(X)$ is ergodic if every $G$-invariant measurable set is either of zero or full $\mu$-measure. We denote by $\mathcal{M}_G^e(X)$ the family of all $G$-invariant Borel probability ergodic measures. It is known that $\mathcal{M}_G^e(X)$ is equal to the set of extreme points of $\mathcal{M}_G(X)$. In particular it follows from the Krein-Milman theorem that

$$\overline{\mathrm{conv}\mathcal{M}_G^e(X)} = \mathcal{M}_G(X)$$

and so every invariant measure can be approximated by finite convex combinations of ergodic measures.

A point $x \in X$ is generic for $\mu \in \mathcal{M}_G(X)$ with respect to a Følner sequence $(F_n)_{n\in\mathbb{N}}$ if $m(x, F_n) \to \mu$ as $n \to \infty$ with respect to the weak*topology. It is worth to mention that even for $\mathbb{Z}$ action ergodic measures do not need to have generic points with respect to any Følner sequence. However, if we choose it in a suitable way, they do. In particular, Lindenstrauss in [194] proved the following theorem:

**Theorem (6.3.2)[478]:** Let $(F_n)_{n\in\mathbb{N}} \subset \mathrm{Fin}(G)$ be a Følner sequence such that there exists $C > 0$ satisfying for every $n \in \mathbb{N}$ the inequality
$$\left|\bigcup_{k\le n} F_k^{-1}\ F_{n+1}\right| \le C \cdot |F_{n+1}|\,.$$
Then every $G$-invariant ergodic measure has a generic point with respect to $(F_n)_{n\in\mathbb{N}}$. Moreover, every Følner sequence has a subsequence satisfying this property.

Following Shulman we call a Følner sequence with the above property tempered. Note that it follows from Theorem (6.3.2) that every amenable group has a tempered Følner sequence.

Cortez and Petite in [367] proved the following (note that the formulation of Lemma (6.3.4) in [367] is slightly different from ours as the authors consider right Følner sequences while we use left ones).

**Lemma (6.3.3)[478]:** Let $G$ be an amenable residually finite group and let $(H_n)_{n\in\mathbb{N}}$ be its skeleton. There exists a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that

**(i)** $F_n \subset F_{n+1}$,

**(ii)** $G = \bigcup_{n\in\mathbb{N}} F_n$,

**(iii)** $F_{n+1} = \coprod_{v\in F_{n+1}\cap H_n} F_n v$ for every $n \in \mathbb{N}$,

**(iv)** $F_n$ is $a$ fundamental domain of $G/H_n$ for every $n \in \mathbb{N}$.

Note that a subsequence of a skeleton $G$ is a skeleton of $G$ and a subsequence of a Følner sequence satisfying the above properties will fulfill them with respect to some skeleton of $G$. Therefore it follows from Theorem(6.3.2) that every amenable residually finite group has a tempered Følner sequence satisfying the above. We fix a countable amenable residually finite group $G$, its skeleton $(H_n)_{n\in\mathbb{N}}$ and a Følner sequence $(F_n)_{n\in\mathbb{N}}$which satisfies the conditions (i), (ii) and (iii) of Lemma(6.3.3). To avoid some uninteresting complications we assume also that for every $n \in \mathbb{N}$ one has $F_n \subset F_{n+1}$ and that $(|F_{n+1}\setminus F_n|)_{n\in\mathbb{N}}$ and $(|F_{n+1}/ F_n|)_{n\in\mathbb{N}}$ are increasing sequences.

**Lemma (6.3.4)[478]:** Let $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k \in [0,1]$ be positive numbers such that $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i = 1$ . If $\mu_1,\dots,\mu_k,\nu_1,\dots,\nu_k \in \mathcal{M}(X)$, then

$$D\left(\sum_{i=1}^k \alpha_i\mu_i\,,\sum_{i=1}^k \beta_i\,\nu_i\right) \le \tfrac{1}{2}\sum_{i=1}^k |\alpha_i - \beta_i| + \max_{1\le i\le k}\{D(\mu_i,\nu_i)\}\ .$$

**Proof:** Let $\mathcal{K} = \{j \in \{1,\dots,k\} : \alpha_j > \beta_j\}$. We can assume without loss of generality that

$$\sum_{j\in\mathcal{K}} |\alpha_j - \beta_j| \le \tfrac{1}{2}\sum_{j=1}^k |\alpha_i - \beta_i|,$$
as the proof in the other case is analogous. Fix a Borel set $B$. One has $\mu_j(B) \le \nu_j(B^\delta) + \delta$ for $j = 1,\dots,k$ and hence
$$\sum_{j=1}^k \alpha_j\mu_j\,(B) \le \sum_{j=1}^k \alpha_j\,\nu_j\,(B^\delta) + \delta$$

$$\leq \sum_{j=1}^{k} \beta_j \nu_j \left(B^{\delta+\sum_{j\in\mathcal{K}}|\alpha_j-\beta_j|}\right) + \delta \sum_{j\in\mathcal{K}} |\alpha_j - \beta_j|$$
$$\leq \sum_{j=1}^{k} \beta_j \nu_j \left(B^{\delta+\frac{1}{2\sum_{j=1}^{k}|\alpha_j-\beta_j|}}\right) + \delta + \frac{1}{2}\sum_{j\in 1}^{k} |\alpha_j - \beta_j|.$$

This completes the proof.

**Lemma (6.3.5)[478]:** Let $\varepsilon \in (0, 1/2)$ and $M, N \in \mathbb{N}$ satisfy $N(1-\varepsilon) \leq M \leq N(1+\varepsilon)$. Denote $m = \max\{M, N\}$. Choose $w_1, \ldots, w_m, z_1, \ldots, z_m \in X$ such that $\rho(w_i, z_i) < \varepsilon$ for every $1 \leq i \leq m$ and define

$$\mu = \frac{1}{M}\sum_{i=1}^{M} \hat{\delta}_{w_i} \text{ and } \nu = \frac{1}{N}\sum_{i=1}^{N} \hat{\delta}_{z_i}.$$

Then $D(\mu, \nu) \leq 2\varepsilon$.

**Proof:** Fix a Borel set $B$. We have

$$\mu(B) = \frac{1}{M}\ |\{1 \leq i \leq M : w_i \in B\}| \leq \frac{N}{M} \cdot \frac{1}{N}\ |\{1 \leq i \leq M : z_i \in B^{\varepsilon}\}| \leq$$
$$\leq \nu(B^{2\varepsilon}) + \frac{\varepsilon}{1-\varepsilon} \leq \nu(B^{2\varepsilon}) + 2\varepsilon.$$

Since $B$ is arbitrary we get the claim.

**Lemma (6.3.6)[478]:** $Fix\ \mu \in \mathcal{M}_G(X)$. Let $(\varepsilon_m)_{m\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}} \subset X$ be such that $\varepsilon_m \searrow 0$ as $m \nearrow \infty$ and for every $m \in \mathbb{N}$ there exits $N \in \mathbb{N}$ such that for all $n \geq N$ one has $D(m(z_n, F_m), \mu) \leq \varepsilon_m$. Then any accumulation point of the sequence $(z_n)_{n\in\mathbb{N}}$ is generic for $\mu$.

**Proof:** Let $z \in X$ be such that $z_{n_k} \to z$ for some $n_k \nearrow \infty$. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ in such a way that $\varepsilon_m < \varepsilon/2$. Pick $\delta > 0$ such that for all $x, y \in X$ satisfying $\rho(x, y) < \delta$ one has $\rho(fx, fy) < \varepsilon/2$ for all $f \in F_m$. It follows from Lemma (6.3.1) that then also $D(m(x, F_m), m(y, F_m)) \leq \varepsilon/2$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$ one has $D(m(z_n, F_m), \mu) < \varepsilon_m$. Let $k \in \mathbb{N}$ be such that $n_k \geq N$ and $\rho(z_n, z_{n_k}) < \delta$. Using the triangle inequality we get that

$$D(m(z, F_m), \mu) \leq D(m(z, F_m), m(z_{n_k}, F_m)) + D(m(z_{n_k}, F_m), \mu) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon$ is arbitrary the proof is completed.

**Theorem (6.3.7)[478]:** For every measure $\mu \in \mathcal{M}_G(X)$ there exists a point $x \in X$ which is generic for $\mu$ with respect to $(F_i)_{i\in\mathbb{N}}$.

**Proof:** Fix $\mu \in \mathcal{M}_G(X)$. It follows from the Krein–Milman theorem that

$$\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{p_i^{(n)}}{q_i^{(n)}}\ \nu_i^{(n)}$$

for some $p_i^{(n)} \in \mathbb{N} \cup \{0\}, q_i^{(n)} \in \mathbb{N}$ and $\nu_i^{(n)} \in \mathcal{M}_G^e(X)$. To simplify notation denote for every $n \in \mathbb{N}$

$$\tilde{\nu}_n := \sum_{i=1}^{n} \frac{p_i^{(n)}}{q_i^{(n)}} \nu_i^{(n)}.$$

Passing to a subsequence if necessary we can assume additionally that for every $n \in N$ one has

$$D(\tilde{\nu}_n, \mu) < \frac{1}{2^n}. \tag{21}$$

We will construct an increasing sequence $(K(n))_{n=0}^{\infty} \subset \mathbb{N} \cup \{0\}$ and a sequence of points $(z_n)_{n=0}^{\infty} \subset X$ such that:

**(I)** for every $n \in \mathbb{N} \cup \{0\}, s < n$ and $K(s-1) \leq m < K(s)$ one has

$$(D(m(z_n, F_m), \mu) \leq \frac{1}{2^{s-4}},$$

**(II)** for every $n \in \mathbb{N} \cup \{0\}$ and $K(n-1) \leq l < K(n)$ the following inequality is satisfied:

$$D(m(z_n, F_l), \mu) \leq \frac{1}{2^{s-3}}. \tag{22}$$

Define

$$A(n) = \prod_{i=1}^{n} q_i^{(n)} \text{ and } B(n,i) = \frac{p_i^{(n)}}{q_i^{(n)}} \cdot A(n) \text{ for } n \in \mathbb{N}, 1 \leq i \leq n.$$

Note that $A(n), B(n,i) \in \mathbb{N}$ and

$$\tilde{\nu}_n = \frac{1}{A(n)} \sum_{i=1}^{n} B(n,i)\, \nu_i^{(n)}.$$

Therefore it follows from (22) and Lemma (6.3.4) that for every $m \geq M(n)$ one has

$$D\left(\sum_{i=1}^{n} \frac{B(n,i)}{A(n)} m\left(x_i^{(n)}, F_m\right), \tilde{\nu}_n\right) < \frac{1}{2^{n+3}}. \tag{23}$$

For every $n \in \mathbb{N}$ let $Z_n \in \mathrm{Fin}\,(G)$ be provided from the definition of the weak specification property for $\varepsilon_n = 1/2^{n+2}$. Let $P(n) \geq M(n)$ be such that for every $p \geq P(n)$ one has

$$\frac{\left|Z_n \cup Z_n^{-1} F_p \Delta F_p\right|}{F_p} \leq \frac{1}{2^{n+3}}. \tag{24}$$

Pick $z_0 \in X$ in an arbitrary way and put $K(0) = 0$. Choose a sequence $(K(n))_{n \in \mathbb{N}}$ growing fast enough so that the following conditions are satisfied:

**(i)** $K(n) \geq max\,\{P(n+1)+1, K(n-1)\}$,

**(ii)** $\left|F_{K(n-1)+1} \setminus F_{K(n-1)}\right| \geq 2^{n+4} A(n) \cdot |F_{P(n)}|$,

**(iii)** $\frac{|F_{K(n)+1} \setminus F_{K(n)}|}{|F_{K(n)}|} \geq 2^{n+3}$.

Fix $n \geq 1$. Let $K(n-1) \leq l < K(n)$. Note that $F_l \setminus F_{l-1}$ consists of $c_l := |H_{P(n)} : H_l| - |H_{P(n)} : H_{l-1}|$ disjoint shifted copies of $F_{P(n)}$. Denote them by

$$T_1^{(l)} = F_{P(n)} g_1^{(l)}, \ldots, T_{c_l}^{(l)} = F_{P(n)} g_{c_l}^{(l)}.$$

Call this family $\mathcal{P}_l$ and divide it into $A(n)$ subfamilies $\mathcal{P}_1^{(l)}, \ldots, \mathcal{P}_{A(n)}^{(l)}$ with almost the same cardinality , that is in such a way that for every $1 \leq i \leq A(n)$ one has

$$\frac{c_l}{A(n)} + 1 \geq \left|\mathcal{P}_i^{(l)}\right| \geq \frac{c_l}{A(n)} - 1. \tag{25}$$

Obviously, $|\mathcal{P}_l| = c_l$. For every $1 \le i \le A(n)$ let $\pi(i) = x_k^{(n)}$, where $1 \le k \le n$ is such that

$$\sum_{j=1}^{k-1} B(n,j) < i \le \sum_{j=1}^{k} B(n,j).$$

Note that for every $1 \le k \le n$ one has $|\pi^{-1}(x_k^{(n)})| = B(n,k)$. Put also $\prod(T) = \pi(i)$ for every $\in \mathcal{P}_i^{(l)}$. For all $K(n-1) \le l < K(n)$ and $1 \le j \le c_l$ define

$$S_j^{(l)} = \left( T_j^{(l)} \setminus \bigcup_{i \ne j, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F_{l-1} \cup Z_n^{-1} F_{l-1}).$$

We need the following lemma.

**Lemma (6.3.8)[478]:** The family

$$\Xi = F_{K(n-1)-1} \cup \left\{ S_j^{(l)} : K(n-1) \le l < K(n), 1 \le j \le c_l \right\}$$

satisfies the condition from the definition of the weak specification property for $\varepsilon = 1/2^{n+2}$.

**Proof:** Pick $\xi_1, \xi_2 \in \Xi$ such that $\xi_1 \ne \xi_2$. We should show that $Z_n \xi_1 \cap \xi_2 = \emptyset$. We divide reasoning into cases:

**(i)** Assume that $\xi_1 = F_{K(n-1)-1}$ or $\xi_2 = F_{K(n-1)-1}$. If $\xi_1 = F_{K(n-1)-1}$, then

$$\xi_2 = \left( T_j^{(l)} \setminus \bigcup_{i \ne j, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F_{l-1} \cup Z_n^{-1} F_{l-1})$$

for some $K(n-1) \le l < K(n)$ and $1 \le j \le c_l$. This means that $Z_n \xi_1 = Z_n F_{K(n-1)-1} \subset Z_n F_{l-1}$ is disjoint from $\xi_2$. Analogous reasoning shows that if $\xi_2 = F_{K(n-1)-1}$ then for arbitrary $\xi_1$ one has $Z_n \xi_1 \cap \xi_2 = \emptyset$.

**(ii)** If

$$\xi_1 = \left( T_{j_1}^{(l)} \setminus \bigcup_{i \ne j_1, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F_{l-1} \cup Z_n^{-1} F_{l-1})$$

and

$$\xi_2 = \left( T_{j_2}^{(l)} \setminus \bigcup_{i \ne j_2, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F_{l-1} \cup Z_n^{-1} F_{l-1})$$

for some $K(n-1) \le l < K(n)$ and $1 \le j_1, j_2 \le c_l, j_1 \ne j_2$, then the claim is obvious.

**(iii)** Assume that

$$\xi_1 = \left( T_{j_1}^{(l_1)} \setminus \bigcup_{i \ne j_1, i \le c_{l_1}} Z_n T_i^{(l_1)} \right) \setminus \left( Z_n F_{l_1-1} \cup Z_n^{-1} F_{l_1-1} \right)$$

and

$$\xi_2 = \left( T_{j_2}^{(l_2)} \setminus \bigcup_{i \ne j_2, i \le c_{l_2}} Z_n T_i^{(l_2)} \right) \setminus \left( Z_n F_{l_2-1} \cup Z_n^{-1} F_{l_2-1} \right)$$

for some $K(n-1) \le l_1, l_2 < K(n), l_1 \neq l_2, 1 \le j_1 \le c_{l_1}$, and $1 \le j_2 \le c_{l_2}$. If $l_1 > l_2$, then $Z_n \xi_1 \subset Z_n F_{l_1} \subset Z_n F_{l_2 - 1}$ and hence $Z_n \xi_1 \cap \xi_2 = \emptyset$. Similarly we can show that the claim holds if $l_1 < l_2$.
This shows that the lemma holds.

Let $z_n \in X$ be a point such that:

**(i)** $\rho(gz_n, gz_{n-1}) < \frac{1}{2^{n+2}}$ for every $g \in F_{K(n-1)-1}$,

**(ii)** $\rho\left(gz_n, g(g_i^{(l)})^{-1} \prod(T_i^{(l)})\right) < \frac{1}{2^{n+2}}$ for all $K(n-1) \le l < K(n), 1 \le i \le c_l$, and $g \in S_i^{(l)}$.

We will show that the sequences $(K(n))_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ satisfy requested conditions. The claim for $n = 0$ is easy to verify (in fact, there is nothing to show here). Fix $n \ge 1$. We need the following lemma, which follows from Lemma (6.3.4) and (6.3.5).

**Lemma (6.3.9)[478]:** For every $K(n-1) \le l < K(n)$ one has

$$D\left(\frac{1}{c_l}\sum_{T\in\mathcal{P}_l} m\big(\Pi(T), F_{P(n)}\big), \sum_{i=1}^{n} \frac{B(n,i)}{A(n)} m\left(x_i^{(n)}, F_{P(n)}\right)\right) \le \frac{1}{2^{n+4}}. \tag{26}$$

**Proof:** Fix $K(n-1) \le l < K(n)$. Note that both of the above measures are linear combinations of the Dirac deltas supported at points from the set

$$fx_i^{(n)} : 1 \le i \le n, f \in F_{P(n)}.$$

For $1 \le i \le n$ and $f \in F_{P(n)}$ let $\alpha_{fi},, \beta_{f,i}$ denote the coefficients with which $\hat{\delta}_{fx_i^{(n)}}$ appear in

$$\frac{1}{c_l}\sum_{T\in\mathcal{P}_l} m\big(\Pi(T), F_{P(n)}\big) \text{ and} \sum_{i=1}^{n} \frac{B(n,i)}{A(n)} m\left(x_i^{(n)}, F_{P(n)}\right),$$

respectively. Fix $1 \le i \le n$ and $f \in F_{P(n)}$. Clearly

$$\alpha_{f,i} = \frac{\left|\Pi^{-1} x_i^{(n)}\right|}{c_l \left|F_{P(n)}\right|} \text{ and } \beta_{f,i} = \frac{B(n,i)}{A(n)\cdot\left|F_{P(n)}\right|}.$$

Note also that it follows from (25) that

$$\frac{B(n,i)}{\left|F_{P(n)}\right|}\cdot\left(\frac{1}{A(n)} - \frac{1}{c_l}\right) \le \frac{\left|\Pi^{-1} x_i^{(n)}\right|}{c_l\left|F_{P(n)}\right|} \le \frac{B(n,i)}{\left|F_{P(n)}\right|}\cdot\left(\frac{1}{A(n)} - \frac{1}{c_l}\right).$$

Hence

$$|\alpha_{f,i} - \beta_{f,i}| \le \frac{B(n,i)}{c_l\left|F_{P(n)}\right|}.$$

Therefore using (ii) we get that

$$\sum_{1\le i\le n, f\in F_{P(n)}} |\alpha_{f,i} - \beta_{f,i}| \le \frac{A(n)}{c_l} = \frac{A(n)\left|F_{P(n)}\right|}{|F_l \backslash F_{l-1}|} \le \frac{A(n)\left|F_{P(n)}\right|}{\left|F_{K(n-1)-1} \backslash F_{K(n-1)}\right|} \le \frac{1}{2^{n+3}}.$$

Hence it follows from Lemma (6.3.4) that the claim holds.

Lemma (6.3.9), the triangle inequality and (23) yield to:

$$D\left(\frac{1}{c_l}\sum_{T\in\mathcal{P}_l} m\big(\Pi(T),F_{P(n)}\big),\tilde{\nu}_n\right)\leq \tag{27}$$

$$\leq D\left(\frac{1}{c_l}\sum_{T\in\mathcal{P}_l} m\big(\Pi(T),F_{P(n)}\big),\sum_{i=1}^{n}\frac{B(n,i)}{A(n)}m\left(x_i^{(n)},F_{P(n)}\right)\right)+$$

$$+D\left(\sum_{i=1}^{n}\frac{B(n,i)}{A(n)}m\left(x_i^{(n)},F_{P(n)}\right),\tilde{\nu}_n\right)\leq\frac{1}{2^{n+4}}+\frac{1}{2^{n+3}}<\frac{1}{2^{n+2}}.$$

What is more, condition (ii), Lemma (6.3.1) and Lemma (6.3.4) imply that

$$D\left(\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left(z_n,S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|},\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left((g_i^{(l)})^{-1}\Pi\left((T_i^{(l)}\right),S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|}\right)\leq\frac{1}{2^{n+2}}. \tag{28}$$

Moreover, condition (24) gives that

$$\left|\coprod_{i=1}^{c_l}T_i^{(l)}\right|\geq\left|\coprod_{i=1}^{c_l}S_i^{(l)}\right|\geq\left(1-\frac{1}{2^{n+3}}\right)\left|\coprod_{i=1}^{c_l}T_i^{(l)}\right|-\frac{|F_{l-1}|}{2^{n+3}}\geq\left(1-\frac{1}{2^{n+2}}\right)\cdot\left|\coprod_{i=1}^{c_l}T_i^{(l)}\right|.$$

Hence, (28) and Lemma (6.3.5) give that

$$D\left(\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left(z_n,T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|},\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left(g_i^{(l)})^{-1}\Pi\left((T_i^{(l)}\right),T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|}\right) \tag{29}$$

$$\leq D\left(\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left(z_n,T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|},\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left(z_n,S_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|S_i^{(l)}\right|}\right)+$$

$$+D\left(\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left(z_n,S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|},\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left((g_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right),S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|}\right)+$$

.

$$+D\left(\frac{\sum_{i=1}^{c_l}|S_i^{(l)}|m\left((g_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right),S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|},\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left((g_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right),T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}\right)$$

$$\leq$$

$$\leq\frac{2}{2^{n+2}}+\frac{1}{2^{n+2}}+\frac{2}{2^{n+2}}<\frac{1}{2^{n-1}}.$$

Note also that

$$\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left(z_n,T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}=m(z_n,F_l\backslash F_{l-1}).$$

Moreover, one has:

$$\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left((g_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right),T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}=\frac{1}{c_l}\sum_{T\in\mathcal{P}_l}m(\Pi(T),F_{P(n)}).$$

Therefore conditions (21), (27) and (29) give that

$$D(m(z_n,\ F_l\setminus F_{l-1}),\mu) \leq \tag{30}$$

$$\leq D\left(\frac{\sum_{i=1}^{c_l}|T_i^{(l)}|m\left((z_n,T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|},\frac{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|m\left((g_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right),T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|}\right)+$$

$$+D\left(\frac{1}{c_l}\sum_{T\in\mathcal{P}_l}m\big(\Pi(T),F_{P(n)}\big),\tilde{v}_n\right)+D(\tilde{v}_n,\mu)<\frac{2}{2^{n-1}}+\frac{1}{2^{n+2}}+\frac{2}{2^n}<\frac{1}{2^{n-2}}.$$

What is more, $c$ gives that

$$\frac{|\,F_l\setminus F_{l-1}|}{|F_{l-1}|}\geq\frac{\left|\,F_{K(n-1)-1}\setminus F_{K(n-1)}\right|}{\left|\,F_{K(n-1)}\right|}\geq 2^{n+2}$$

and hence as a consequence of Lemma (6.3.5) we get that

$$D(m(z_n,\ F_l),m(z_n,\ F_l\setminus F_{l-1}))\leq\frac{1}{2^{n+1}}.$$

Using the above inequality and (30) we obtain that

$$D(m(z_n,\ F_l),\mu)\leq D(m(z_n,\ F_l),m(z_n,\ F_l\setminus F_{l-1}))+D(m(z_n,\ F_l\setminus F_{l-1}),\mu))\leq$$
$$\leq\frac{1}{2^{n+1}}+\frac{1}{2^{n-2}}\leq\frac{1}{2^{n-3}}. \tag{31}$$

This shows (II). Note also that condition (i) and Lemma (6.3.1) imply that for every $s<n$ and $K(s-1)\leq m<K(s)$ one has

$$D(m(z_n,\ F_m),\mu)\leq D(m(z_n,\ F_m),m(z_s,\ F_m))+D(m(z_s,F_m),\mu)\leq$$
$$\leq\sum_{i=s}^{n-1}\frac{1}{2^{i+3}}+\frac{1}{2^{s-3}}\leq\frac{2}{2^{s+2}}+\frac{1}{2^{s-3}}<\frac{1}{2^{s-4}}, \tag{32}$$

which proves (I).

**Corollary (6.3.10)[495]:** Let $\alpha_1^r,\ldots,\alpha_k^r,\beta_1^r,\ldots,\beta_k^r\in[0,1]$ be positive numbers such that $\sum_{i=1}^k\sum_r\ \alpha_i^r=\sum_{i=1}^k\sum_r\ \beta_i^r=1$ . If $\mu_1^r,\ldots,\mu_k^r,\nu_1^r,\ldots,\nu_k^r\in\mathcal{M}(X)$, then

$$\sum_r\ D\left(\sum_{i=1}^k\alpha_i^r\mu_i^r,\sum_{i=1}^k\beta_i^r\,\nu_i^r\right)\leq\tfrac{1}{2}\sum_{i=1}^k\sum_r\ |\alpha_i^r-\beta_i^r|+\max_{1\leq i\leq k}\sum_r\ \{D(\mu_i^r,\nu_i^r)\}\ .$$

**Proof:** Let $\mathcal{K}=\left\{j\ \in\{1,\ldots,k\}:\alpha_j^r>\ \beta_j^r\right\}$. We can assume without loss of generality that

$$\sum_{j\in\mathcal{K}}\sum_r\ |\alpha_j^r-\beta_j^r|\ \leq\tfrac{1}{2}\sum_{j=1}^k\sum_r\ |\alpha_i^r-\beta_i^r|,$$

as the proof in the other case is analogous. Fix a Borel set $A+\epsilon$. One has $\sum_r\ \mu_j^r(A+\epsilon)\leq\sum_r\ \nu_j^r((A+\epsilon)^\delta)+\ \delta$ for $j\ =\ 1,\ldots,k$ and hence

$$\sum_{j=1}^{k}\sum_{r} \alpha_j^r \mu_j^r (A+\epsilon) \leq \sum_{j=1}^{k}\sum_{r} \alpha_j^r \nu_j^r ((A+\epsilon)^{\delta}) + \delta$$
$$\leq \sum_{j=1}^{k}\sum_{r} \beta_j^r \nu_j^r \left((A+\epsilon)^{\delta+\sum_{j\in\mathcal{K}}|\alpha_j^r - \beta_j^r|}\right) + \delta \sum_{j\in\mathcal{K}}\sum_{r} |\alpha_j^r - \beta_j^r|$$
$$\leq \sum_{j=1}^{k}\sum_{r} \beta_j^r \nu_j^r \left((A+\epsilon)^{\delta+\frac{1}{2\sum_{j=1}^{k}\left|\alpha_j^r - \beta_j^r\right|}}\right) + \delta + \frac{1}{2}\sum_{j\in 1}^{k}\sum_{r} \left|\alpha_j^r - \beta_j^r\right|.$$

This completes the proof.

**Corollary (6.3.11)[495]:** Let $\varepsilon \in (0, 1/2)$ and $M, N \in \mathbb{N}$ satisfy $N(1-\varepsilon) \leq M \leq N(1+\varepsilon)$ . Denote $m = \max\{M, N\}$ . Choose $w_1^r, \ldots, w_m^r, z_1^r, \ldots, z_m^r \in X$ such that $\sum_r \rho(w_i^r, z_i^r) < \varepsilon$ for every $1 \leq i \leq m$ and define

$$\mu = \frac{1}{M}\sum_{i=1}^{M}\sum_{r} \hat{\delta}_{w_i^r} \text{ and } \nu = \frac{1}{N}\sum_{i=1}^{N}\sum_{r} \hat{\delta}_{z_i^r}.$$

Then $D(\mu, \nu) \leq 2\varepsilon$.

**Proof:** Fix a Borel set $A+\epsilon$. We have

$$\mu(A+\epsilon) = \frac{1}{M} |\{1 \leq i \leq M : w_i^r \in A+\epsilon\}| \leq \frac{N}{M}\cdot\frac{1}{N} |\{1 \leq i \leq M : z_i^r \in (A+\epsilon)^{\varepsilon}\}| \leq$$
$$\leq \nu((A+\epsilon)^{2\varepsilon}) + \frac{\varepsilon}{1-\varepsilon} \leq \nu((A+\epsilon)^{2\varepsilon}) + 2\varepsilon.$$

Since $A+\epsilon$ is arbitrary we get the claim.

**Corollary (6.3.12)[495]:** Fix $\mu \in \mathcal{M}_G(X)$. Let $(\varepsilon_m)_{m\in\mathbb{N}}$ and $(z_n^r)_{r,n\in\mathbb{N}} \subset X$ be such that $\varepsilon_m \searrow 0$ as $m \nearrow \infty$ and for every $m \in \mathbb{N}$ there exits $N \in \mathbb{N}$ such that for all $n \geq N$ one has $\sum_r D(m(z_n^r, F_m^r), \mu) \leq \varepsilon_m$. Then any accumulation point of the sequence $(z_n^r)_{r,n\in\mathbb{N}}$ is generic for $\mu$.

**Proof:** Let $z^r \in X$ be such that $z_{n_k}^r \to z^r$ for some $n_k \nearrow \infty$ . Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ in such a way that $\varepsilon_m < \varepsilon/2$. Pick $\delta > 0$ such that for all $x, y \in X$ satisfying $\rho(x, y) < \delta$ one has $\sum_r \rho(f_r x, f_r y) < \varepsilon/2$ for all $f_r \in F_m^r$. It follows from Lemma 1 that then also $\sum_r D(m(x, F_m^r), m(y, F_m^r)) \leq \varepsilon/2$ . Choose $N \in \mathbb{N}$ such that for all $n \geq N$ one has $\sum_r D(m(z_n^r, F_m^r), \mu) < \varepsilon_m$. Let $k \in \mathbb{N}$ be such that $n_k \geq N$ and $\sum_r \rho(z_n^r, z_{n_k}^r) < \delta$. Using the triangle inequality we get that

$$\sum_r D(m(z^r, F_m^r), \mu) \leq \sum_r D(m(z^r, F_m^r), m(z_{n_k}^r, F_m^r)) + \sum_r D(m(z_{n_k}^r, F_m^r), \mu)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon$ is arbitrary the proof is completed.

**Corollary (6.3.13)[495]:** For every measure $\mu \in \mathcal{M}_G(X)$ there exists a point $x \in X$ which is generic for $\mu$ with respect to $(F_i^r)_{r,i\in\mathbb{N}}$.

**Proof:** Fix $\mu \in \mathcal{M}_G(X)$. It follows from the Krein–Milman theorem that

$$\mu = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{r} \frac{p_i^{(n)}}{q_i^{(n)}} \nu_i^{r(n)}$$

for some $p_i^{(n)} \in \mathbb{N} \cup \{0\}, q_i^{(n)} \in \mathbb{N}$ and $\nu_i^{r(n)} \in \mathcal{M}_G^e(X)$. To simplify notation denote for every $n \in \mathbb{N}$

$$\tilde{\nu}_n^r := \sum_{i=1}^{n} \sum_{r} \frac{p_i^{(n)}}{q_i^{(n)}} \nu_i^{r(n)}.$$

Passing to a subsequence if necessary we can assume additionally that for every $n \in N$ one has

$$\sum_{r} D(\tilde{\nu}_n^r, \mu) < \frac{1}{2^n}. \quad (33)$$

We will construct an increasing sequence $(K(n))_{n=0}^{\infty} \subset \mathbb{N} \cup \{0\}$ and a sequence of points $(z_n^r)_{r,n=0}^{\infty} \subset X$ such that:

**(I)** for every $n \in \mathbb{N} \cup \{0\}, s < n$ and $K(s-1) \leq m < K(s)$ one has

$$\sum_{r} (D(m(z_n^r, F_m^r), \mu) \leq \frac{1}{2^{s-4}},$$

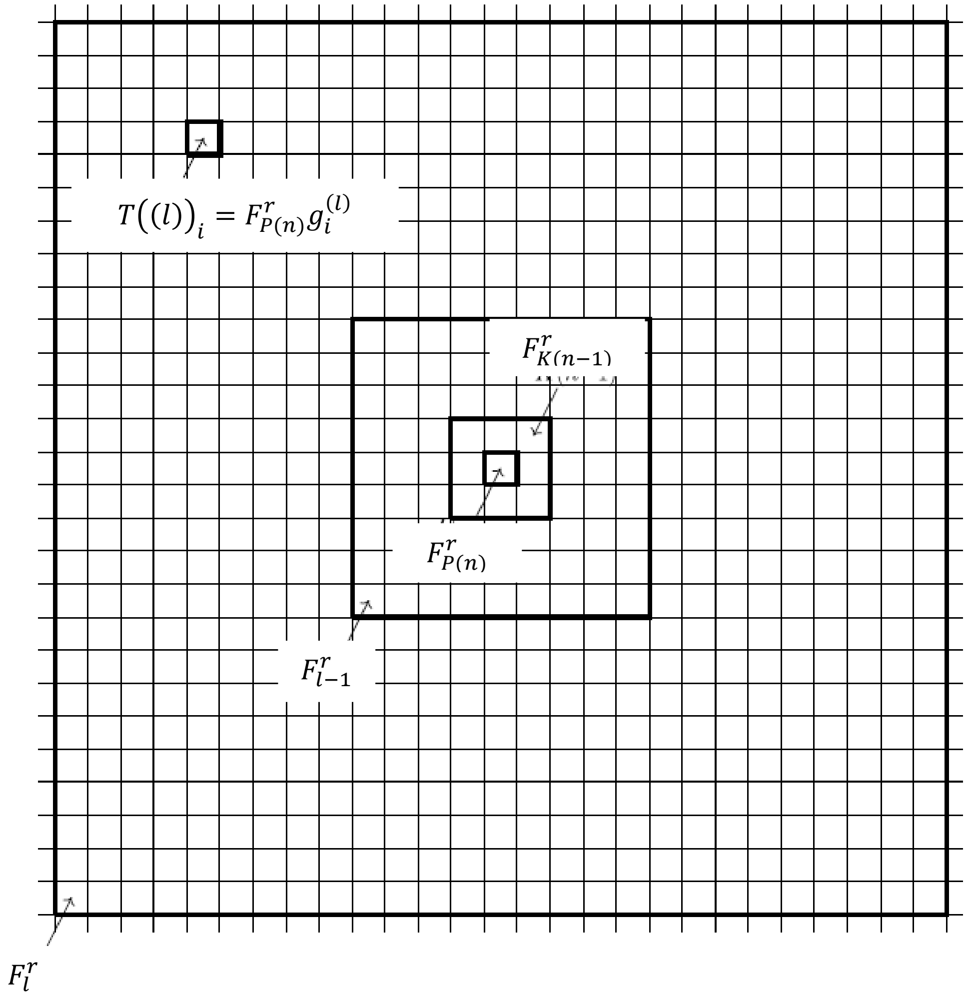



**Fig. 2[478].** Our choices of Følner sets

**(II)** for every $n \in \mathbb{N} \cup \{0\}$ and $K(n-1) \leq l < K(n)$ the following inequality is satisfied:

$$\sum_{r} D(m(z_n^r, F_l^r), \mu) \leq \frac{1}{2^{s-3}}. \quad (34)$$

D $F_l^r$ e

$A(n) = \prod_{i=1}^{n} q_i^{(n)}$ and $(A+\epsilon)(n,i) = \frac{p_i^{(n)}}{q_i^{(n)}} \cdot A(n)$ for $n \in \mathbb{N}, 1 \leq i \leq n$.

Note that $A(n), (A+\epsilon)(n,i) \in \mathbb{N}$ and

$$\tilde{\nu}_n^r = \frac{1}{A(n)} \sum_{i=1}^{n} \sum_r \; (A+\epsilon)(n,i)\, \nu_i^{r(n)}.$$

Therefore it follows from (34) and Corollary (6.3.10) that for every $m \geq M(n)$ one has

$$\sum_r D\left(\sum_{i=1}^{n} \frac{(A+\epsilon)(n,i)}{A(n)} m\left(x_i^{(n)}, F_m^r\right), \tilde{\nu}_n^r\right) < \frac{1}{2^{n+3}}. \tag{35}$$

For every $n \in \mathbb{N}$ let $Z_n \in \operatorname{Fin}(G)$ be provided from the definition of the weak specification property for $\varepsilon_n = 1/2^{n+2}$. Let $P(n) \geq M(n)$ be such that for every $p \geq P(n)$ one has

$$\sum_r \frac{\left|Z_n \cup Z_n^{-1} F_p^r \Delta F_p^r\right|}{F_p^r} \leq \frac{1}{2^{n+3}}. \tag{36}$$

Pick $z_0^r \in X$ in an arbitrary way and put $K(0) = 0$. Choose a sequence $(K(n))_{n\in\mathbb{N}}$ growing fast enough so that the following conditions are satisfied:

**(i)** $K(n) \geq max\,\{P(n+1)+1, K(n-1)\}$,

**(ii)** $\sum_r \left|F_{K(n-1)+1}^r \setminus F_{K(n-1)}^r\right| \geq \sum_r \; 2^{n+4} A(n) \cdot |F_{P(n)}^r|$,

**(iii)** $\sum_r \frac{|F_{K(n)+1}^r \setminus F_{K(n)}^r|}{|F_{K(n)}^r|} \geq 2^{n+3}$.

Fix $n \geq 1$. Let $K(n-1) \leq l < K(n)$. Note that $F_l^r \setminus F_{l-1}^r$ consists of $c_l := |H_{P(n)} : H_l| - |H_{P(n)} : H_{l-1}|$ disjoint shifted copies of $F_{P(n)}^r$. Denote them by

$$T_1^{(l)} = \sum_r F_{P(n)}^r (g_r)_1^{(l)} \;, \dots, T_{c_l}^{(l)} = \sum_r F_{P(n)}^r (g_r)_{c_l}^{(l)}.$$

Call this family $\mathcal{P}_l$ and divide it into $A(n)$ subfamilies $\mathcal{P}_1^{(l)} \;, \dots, \mathcal{P}_{A(n)}^{(l)}$ with almost the same cardinality , that is in such a way that for every $1 \leq i \leq A(n)$ one has

$$\frac{c_l}{A(n)} + 1 \geq \left|\mathcal{P}_i^{(l)}\right| \geq \frac{c_l}{A(n)} - 1. \tag{37}$$

Obviously, $|\mathcal{P}_l| = c_l$. For every $1 \leq i \leq A(n)$ let $\pi(i) = x_k^{(n)}$ , where $1 \leq k \leq n$ is such that

$$\sum_{j=1}^{k-1} (A+\epsilon)(n,j) \; < i \; \leq \sum_{j=1}^{k} (A+\epsilon)(n,j).$$

Note that for every $1 \leq k \leq n$ one has $|\pi^{-1}(x_k^{(n)})| = (A+\epsilon)(n,k)$. Put also $\prod(T) = \pi(i)$ for every $\in \mathcal{P}_i^{(l)}$ . For all $K(n-1) \leq l < K(n)$ and $1 \leq j \leq c_l$ define

$$S_j^{(l)} = \sum_r \left(T_j^{(l)} \setminus \bigcup_{i\neq j, i\leq c_l} Z_n T_i^{(l)}\right) \setminus (Z_n F_{l-1}^r \cup Z_n^{-1} F_{l-1}^r).$$

**Corollary (6.3.14)[495]:** The family

$$\Xi = \sum_r F^r_{K(n-1)-1} \cup \left\{ S_j^{(l)} : K(n-1) \le l < K(n), 1 \le j \le c_l \right\}$$

satisfies the condition from the definition of the weak specification property for $\varepsilon = 1/2^{n+2}$.

**Proof:** Pick $\xi_1, \xi_2 \in \Xi$ such that $\xi_1 \neq \xi_2$. We should show that $Z_n\xi_1 \cap \xi_2 = \emptyset$. We divide reasoning into cases:

**(i)** Assume that $\xi_1 = \sum_r F^r_{K(n-1)-1}$ or $\xi_2 = \sum_r F^r_{K(n-1)-1}$. If $\xi_1 = \sum_r F^r_{K(n-1)-1}$, then

$$\xi_2 = \sum_r \left( T_j^{(l)} \setminus \bigcup_{i \neq j, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F^r_{l-1} \cup Z_n^{-1} F^r_{l-1})$$

for some $K(n-1) \le l < K(n)$ and $1 \le j \le c_l$. This means that $Z_n\xi_1 = \sum_r Z_n F^r_{K(n-1)-1} \subset Z_n F^r_{l-1}$ is disjoint from $\xi_2$. Analogous reasoning shows that if $\xi_2 = F^r_{K(n-1)-1}$ then for arbitrary $\xi_1$ one has $Z_n\xi_1 \cap \xi_2 = \emptyset$.

**(ii)** If

$$\xi_1 = \sum_r \left( T_{j_1}^{(l)} \setminus \bigcup_{i \neq j_1, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F^r_{l-1} \cup Z_n^{-1} F^r_{l-1})$$

and

$$\xi_2 = \sum_r \left( T_{j_2}^{(l)} \setminus \bigcup_{i \neq j_2, i \le c_l} Z_n T_i^{(l)} \right) \setminus (Z_n F^r_{l-1} \cup Z_n^{-1} F^r_{l-1})$$

for some $K(n-1) \le l < K(n)$ and $1 \le j_1, j_2 \le c_l, j_1 \neq j_2$, then the claim is obvious.

**(iii)** Assume that

$$\xi_1 = \sum_r \left( T_{j_1}^{(l_1)} \setminus \bigcup_{i \neq j_1, i \le c_{l_1}} Z_n T_i^{(l_1)} \right) \setminus \left(Z_n F^r_{l_1-1} \cup Z_n^{-1} F^r_{l_1-1}\right)$$

and

$$\xi_2 = \sum_r \left( T_{j_2}^{(l_2)} \setminus \bigcup_{i \neq j_2, i \le c_{l_2}} Z_n T_i^{(l_2)} \right) \setminus \left(Z_n F^r_{l_2-1} \cup Z_n^{-1} F^r_{l_2-1}\right)$$

for some $K(n-1) \le l_1, l_2 < K(n), l_1 \neq l_2, 1 \le j_1 \le c_{l_1}$, and $1 \le j_2 \le c_{l_2}$. If $l_1 > l_2$, then $Z_n \xi_1 \subset Z_n F^r_{l_1} \subset Z_n F^r_{l_2 - 1}$ and hence $Z_n \xi_1 \cap \xi_2 = \emptyset$. Similarly we can show that the claim holds if $l_1 < l_2$.

This shows that the Corollary holds

**Corollary (6.3.15)[495]:** For every $K(n-1) \le l < K(n)$ one has

$$\sum_r D\left( \frac{1}{c_l} \sum_{T \in \mathcal{P}_l} m\big(\Pi(T), F^r_{P(n)}\big), \sum_{i=1}^{n} \frac{(A+\epsilon)(n,i)}{A(n)} m\left(x_i^{(n)}, F^r_{P(n)}\right) \right) \le \frac{1}{2^{n+4}} . \quad (38)$$

**Proof:** Fix $K(n-1) \le l < K(n)$. Note that both of the above measures are linear combinations of the Dirac deltas supported at points from the set

$$\sum_{r} \left\{ f_r x_i^{(n)} \colon 1 \ \le i \ \le \ n, f_r \in F_{P(n)}^r \right\}.$$

For $1 \le i \ \le n$ and $f_r \in F_{P(n)}^r$ let $\alpha_{f_r i}^r, , \beta_{f_r,i}^r$ denote the coefficients with which $\hat{\delta}_{f_r x_i^{(n)}}$ appear in

$$\frac{1}{c_l} \sum_{T \in \mathcal{P}_l} \sum_{r} \ m\big(\Pi(T), F_{P(n)}^r \ \big) \text{ and} \sum_{i=1}^{n} \sum_{r} \ \frac{(A+\epsilon)(n,i)}{A(n)} m \ \left( x_i^{(n)} \ , F_{P(n)}^r \right),$$

respectively. Fix $1 \le \ i \le n$ and $f_r \in F_{P(n)}^r$. Clearly

$$\alpha_{f_r,i}^r = \sum_{r} \ \frac{\left| \Pi^{-1} x_i^{(n)} \right|}{c_l \left| F_{P(n)}^r \right|} \text{ and } \ \beta_{f_r,i}^r \ = \sum_{r} \ \frac{(A+\epsilon)(n,i)}{A(n) \cdot \left| F_{P(n)}^r \right|}.$$

Note also that it follows from (37) that

$$\sum_{r} \ \frac{(A+\epsilon)(n,i)}{\left| F_{P(n)}^r \right|} \cdot \left( \frac{1}{A(n)} - \frac{1}{c_l} \right) \le \sum_{r} \ \frac{\left| \Pi^{-1} x_i^{(n)} \right|}{c_l \left| F_{P(n)}^r \right|} \le \sum_{r} \ \frac{(A+\epsilon)(n,i)}{\left| F_{P(n)}^r \right|} \cdot \left( \frac{1}{A(n)} - \frac{1}{c_l} \right).$$

Hence

$$\sum_{r} \ |\alpha_{f_r,i}^r - \ \beta_{f_r,i}^r| \ \le \sum_{r} \ \frac{(A+\epsilon)(n,i)}{c_l \left| F_{P(n)}^r \right|}.$$

Therefore using (ii) we get that

$$\sum_{1 \le i \ \le n \ , f_r \in F_{P(n)}^r} \ \sum_{r} \ |\alpha_{f_r,i}^r - \ \beta_{f_r,i}^r| \le \frac{A(n)}{c_l} = \sum_{r} \ \frac{A(n) \left| F_{P(n)}^r \right|}{\left| F_l^r \backslash F_{l-1}^r \right|} \le \sum_{r} \ \frac{A(n) \left| F_{P(n)}^r \right|}{\left| F_{K(n-1)-1}^r \backslash F_{K(n-1)}^r \right|}$$
$$\le \frac{1}{2^{n+3}}.$$

Hence it follows from Corollary (6.3.10) that the claim holds.

Corollary (6.3.15), the triangle inequality and (35) yield to:

$$\sum_{r} \ D \left( \frac{1}{c_l} \sum_{T \in \mathcal{P}_l} m\big(\Pi(T), F_{P(n)}^r \ \big), \tilde{\nu}_n^r \right) \le \tag{39}$$

$$\le \sum_{r} \ D \left( \frac{1}{c_l} \sum_{T \in \mathcal{P}_l} m\big(\Pi(T), F_{P(n)}^r \ \big) \ , \sum_{i=1}^{n} \frac{(A+\epsilon)(n,i)}{A(n)} m \ \left( x_i^{(n)} \ , F_{P(n)}^r \right) \right) +$$

$$+ \sum_{r} \ D \left( \sum_{i=1}^{n} \frac{(A+\epsilon)(n,i)}{A(n)} m \ \left( x_i^{(n)} \ , F_{P(n)}^r \right) \ , \tilde{\nu}_n^r \right) \le \frac{1}{2^{n+4}} + \frac{1}{2^{n+3}} < \frac{1}{2^{n+2}}.$$

What is more, condition (ii), Lemma (6.3.1) and Corollary (6.3.10) imply that

$$\sum_{r} \ D \left( \frac{\sum_{i=1}^{c_l} |S_i^{(l)} \ | m \ \left( z_n^r, S_i^{(l)} \right)}{\sum_{i=1}^{c_l} |S_i^{(l)}|} \ , \frac{\sum_{i=1}^{c_l} |S_i^{(l)} \ | m \ \left( \left( (g_r)_i^{(l)} \right)^{-1} \Pi \left( \left( T_i^{(l)} \right), S_i^{(l)} \right) \right)}{\sum_{i=1}^{c_l} |S_i^{(l)}|} \right)$$
$$\le \frac{1}{2^{n+2}}. \tag{40}$$

Moreover, condition (36) gives that

$$\left|\coprod_{i=1}^{c_l} T_i^{(l)}\right| \geq \left|\coprod_{i=1}^{c_l} S_i^{(l)}\right| \geq \left(1-\frac{1}{2^{n+3}}\right)\left|\coprod_{i=1}^{c_l} T_i^{(l)}\right| - \sum_r \frac{|F_{l-1}^r|}{2^{n+3}} \geq \left(1-\frac{1}{2^{n+2}}\right)\cdot\left|\coprod_{i=1}^{c_l} T_i^{(l)}\right|.$$

Hence, (40) and Corollary (6.3.11) give that

$$\sum_r D\left(\frac{\sum_{i=1}^{c_l} |T_i^{(l)}| m\left(z_n^r, T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}, \frac{\sum_{i=1}^{c_l}|T_i^{(l)}| m\left( (g_r)_i^{(l)})^{-1}\Pi\left((T_i^{(l)}\right), T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}\right) \tag{41}$$

$$\leq \sum_r D\left(\frac{\sum_{i=1}^{c_l}|T_i^{(l)}| m\left(z_n^r, T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|}, \frac{\sum_{i=1}^{c_l}|S_i^{(l)}| m\left(z_n^r, S_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|S_i^{(l)}\right|}\right)$$

$$+$$

$$+\sum_r D\left(\frac{\sum_{i=1}^{c_l}|S_i^{(l)}| m\left(z_n^r, S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|}, \frac{\sum_{i=1}^{c_l}|S_i^{(l)}| m\left( ((g_r)_i^{(l)})^{-1}\Pi\left(T_i^{(l)}\right), S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|}\right)$$

$$+$$

.

$$+\sum_r D\left(\frac{\sum_{i=1}^{c_l}|S_i^{(l)}| m\left( ((g_r)_i^{(l)})^{-1}\Pi(T_i^{(l)}), S_i^{(l)}\right)}{\sum_{i=1}^{c_l}|S_i^{(l)}|}, \frac{\sum_{i=1}^{c_l}|T_i^{(l)}| m\left( (g_i^{(l)})^{-1}\Pi(T_i^{(l)}), T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|}\right)$$

$$\leq \frac{2}{2^{n+2}} + \frac{1}{2^{n+2}} + \frac{2}{2^{n+2}} < \frac{1}{2^{n-1}}.$$

Note also that

$$\sum_r \frac{\sum_{i=1}^{c_l}|T_i^{(l)}| m\left(z_n^r, T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|} = \sum_r m(z_n^r, F_l^r \setminus F_{l-1}^r).$$

Moreover, one has:

$$\frac{\sum_{i=1}^{c_l}\sum_r |T_i^{(l)}| m\left(\left((g_r)_i^{(l)}\right)^{-1}\Pi\left(T_i^{(l)}\right), T_i^{(l)}\right)}{\sum_{i=1}^{c_l}|T_i^{(l)}|} = \frac{1}{c_l}\sum_{T\in\mathcal{P}_l}\sum_r m(\Pi(T), F_{P(n)}^r).$$

Therefore conditions (33), (39) and (41) give that

$$\sum_r D(m(z_n^r, F_l^r \setminus F_{l-1}^r), \mu) \leq \tag{42}$$

$$\leq \sum_r D\left(\frac{\sum_{i=1}^{c_l}|T_i^{(l)}| m\left( (z_n^r, T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|}, \frac{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right| m\left(\left((g_r)_i^{(l)}\right)^{-1}\Pi\left(T_i^{(l)}\right), T_i^{(l)}\right)}{\sum_{i=1}^{c_l}\left|T_i^{(l)}\right|}\right) +$$

$$+\sum_r D\left(\frac{1}{c_l}\sum_{T\in\mathcal{P}_l} m\left(\Pi(T), F_{P(n)}^r\right), \tilde{v}_n^r\right) + \sum_r D(\tilde{v}_n^r, \mu) < \frac{2}{2^{n-1}} + \frac{1}{2^{n+2}} + \frac{2}{2^n} < \frac{1}{2^{n-2}}.$$

What is more, $c$ gives that

$$\sum_r \frac{|F_l^r \setminus F_{l-1}^r|}{\left|F_{l-1}^r\right|} \geq \sum_r \frac{\left|F_{K(n-1)-1}^r \setminus F_{K(n-1)}^r\right|}{\left|F_{K(n-1)}^r\right|} \geq 2^{n+2}$$

and hence as a consequence of Corollary (6.3.11) we get that

$$\sum_{r} D(m(z_n^r, F_l^r), m(z_n^r, F_l^r \setminus F_{l-1}^r)) \leq \frac{1}{2^{n+1}}.$$

Using the above inequality and (42) we obtain that

$$\begin{aligned}
&\sum_{r} D(m(z_n^r, F_l^r), \mu) \\
&\qquad \leq \sum_{r} D(m(z_n^r, F_l^r), m(z_n^r, F_l^r \setminus F_{l-1}^r)) + \sum_{r} D(m(z_n^r, F_l^r \setminus F_{l-1}^r), \mu)) \\
&\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n-2}} \leq \frac{1}{2^{n-3}}.
\end{aligned} \tag{43}$$

This shows (II). Note also that condition (i) and Lemma (6.3.1) imply that for every $s < n$ and $K(s-1) \leq m < K(s)$ one has

$$\begin{aligned}
&\sum_{r} D(m(z_n^r, F_m^r), \mu) \leq D(m(z_n^r, F_m^r), m(z_s^r, F_m^r)) + D(m(z_s^r, F_m^r), \mu) \\
&\qquad \leq \sum_{i=s}^{n-1} \frac{1}{2^{i+3}} + \frac{1}{2^{s-3}} \leq \frac{2}{2^{s+2}} + \frac{1}{2^{s-3}} < \frac{1}{2^{s-4}},
\end{aligned} \tag{44}$$

which proves (I). This finishes the proof of the Corollary.

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