Sudan University of Science and Technology

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Symmetry for Extremal Functions in Subcritical and Fractional Caffarelli–Kohn–Nirenberg Inequalities

التماثل للدوال القصوى في الحرجة الجزئية ومتباينات كافاريلي – كوهن – نيرينبيرج الكسرية

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

We determine the best constants for Gagliardo–Nirenberg inequalities the applications to nonlinear diffusions, the first order interpolation inequalities with weights, the two subtle convex nonlocal approximations of the bounded variation norm, the limiting embedding theorems for Sobolev spaces and the BBM formula. We establish Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators with the nonlinear ground state representations and sharp Hardy inequalities with fractional Hardy-Sobolev-Maz'ya inequality for domains. The Caffarelli-Kohn-Nirenberg inequalities with remainder terms, sharp constants, existence, nonexistence and fractional order are investigated. The symmetry of optimizers of extremal functions in subcritical and fractional Caffarelli–Kohn–Nirenberg inequalities are obtained.

الخلاصة

تم تحديد الثوابت الأفضل لمتباينات جاجلياردو – نيرينبيرج وتطبيقات الأنتشار غير الخطي ومتباينات الأستكمال من الرتبة الأولى مع المرجحات والتقريبين غير الموضوعيين المحدبيين الدقيقين لنظم المتغير المحدود ومبرهنات الطمر المنتهي لفضاءات سوبوليف وصيغة BBM. قمنا بتأسيس متباينات هاردي – ليب – ثيرنج لمؤثرات شرودنجر الكسرية مع تمثيلات الحال الأرضية غير الخطية ومتباينات هاردي القاطعة مع متباينات هاردي – سوبوليف – ماريه الكسرية للمجالات. قمنا بتقصي متباينات كافاريلي – كوهن – نيرينبيرج مع الحدود المتبقية والثوابت القاطعة والوجود واللاوجود والرتبة الكسرية. تم الحصول على التماثل الأمثل والدوال القصوى في الحرج الجزئية ومتباينات كافاريلي – كوهن – نيرينبيرج الكسرية.

Introduction

We consider the following inequalities due to Caffarelli, Kohn, and Nirenberg [7]: $\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{\frac{2}{p}} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx$ where, for $N \geq 3$, $-\infty < a < (N-2)/2, a \leq b \leq a+1$, and $p = 2N/(N-2+2(b-a))$. We shall answer some fundamental questions concerning these inequalities such as the best embedding constants, the existence and nonexistence of extremal functions, and their qualitative properties. We find optimal constants of a special class of Gagliardo–Nirenberg type inequalities which turns out to interpolate between the classical Sobolev inequality and the Gross logarithmic Sobolev inequality.

We proved some interpolation inequalities between functions and their derivatives. We show that the Lieb-Thirring inequalities on moments of negative eigenvalues of Schrödinger-like operators remain true, with possibly different constants, when the critical Hardy-weight $C|x|^{-2}$ is subtracted from the Laplace operator. We discuss the Caffarelli‑Kohn‑Nirenberg inequalities with remainder terms

Inspired by the BBM formula and by work of G. Leoni and D. Spector, we analyze the asymptotic behavior of two sequences of convex nonlocal functionals $(\Psi_n(u))$ and $(\Phi_n(u))$ which converge formally to the BV-norm of u. We give some applications on the limiting embedding theorems for Hardy spaces.

In their simplest form, the Caffarelli-Kohn-Nirenberg inequalities are a two parameter family of inequalities. It has been known that there is a region in parameter space where the optimizers for the inequalities have broken symmetry. It has been shown recently that in the complement of this region the optimizers are radially symmetric. The ideas for the proof will be given. We use the formalism of the Rényi entropies to establish the symmetry range of extremal functions in a family of subcritical Caffarelli–Kohn–Nirenberg inequalities. By extremal functions we mean functions that realize the equality case in the inequalities, written with optimal constants. The method extends recent results on critical Caffarelli–Kohn–Nirenberg inequalities. Using heuristics given by a nonlinear diffusion equation, we give a variational proof of a symmetry result, by establishing a rigidity theorem: in the symmetry region, all positive critical points have radial symmetry and are therefore equal to the unique positive, radial critical point, up to scalings and multiplications. This result is sharp.

We determine the sharp constant in the Hardy inequality for fractional Sobolev spaces. To do so, we develop a non-linear and non-local version of the ground state representation, which even yields a remainder term. We prove a fractional version of the Hardy–Sobolev–Maz'ya inequality for arbitrary domains and L^p norms with $p \geq 2$. Let $0 < s < 1$ and $p > 1$ be such that $ps < N$. Assume that Ω is a bounded domain containing the origin. Starting from the ground state inequality by R. Frank and R. Seiringer in [166] to obtain: the Caffarelli–Kohn–Nirenberg type inequalities offractional order:

We establish improved versions of the Hardy and Caffarelli-Kohn-Nirenberg inequalities by replacing the standard Dirichlet energy with some nonlocal nonconvex functionals which have been involved in estimates for the topological degree of continuous maps from a sphere into itself and characterizations of Sobolev spaces.

The Contents

Chapter 1
The Caffarelli-Kohn and Gagliardo–Nirenberg Inequalities and Best Constants

We show that in the Caffarelli, Kohn, and Nirenberg inequalities while the case $a \geq 0$ has been studied extensively and a complete solution is known, little has been known for the case $a < 0$. Our results for the case $a < 0$ reveal some new phenomena which are in striking contrast with those for the case $a \geq 0$. Results for $N = 1$ and $N = 2$ are also given. We show that the inequalities provide an optimal decay rate (measured by entropy methods) of the intermediate asymptotics of solutions to nonlinear diffusion equations.

Section (1.1): Sharp Constants with Existence and Nonexistence and Symmetry of Extremal Functions

In [7], among a much more general family of inequalities, Caffarelli, Kohn, and Nirenberg established the following inequalities: For all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx \tag{1}$$

where, for $N \geq 3$,

$$-\infty < a < \frac{N-2}{2}, \qquad a \leq b \leq a+1\,,$$

and

$$p = \frac{2N}{N-2+2(b-a)} \tag{2}$$

The cases$N = 2$ and $N = 1$ will be treated. The conditions for these cases are, for $N = 2$,

$$-\infty < a < 0, \;\; a < b \leq a+1\,, \qquad and \quad p = \frac{2}{b-a}, \tag{3}$$

and ,for $N = 1$,

$$-\infty < a < -\frac{1}{2}, \;\; a + \frac{1}{2} < b \leq a+1\,, \;\; and \;\; p = \frac{2}{-1+2(b-a)}. \tag{4}$$

Let $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner product

$$(u,v) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v dx. \tag{5}$$

Then we see that (1) holds for $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$. We define

$$S(a,b) = \inf_{u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)/0} E_{a,b}(u), \tag{6}$$

to be the best embedding constants, where

$$E_{a,b}(u) = \frac{\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp}|\mathrm{u}|^b dx\right)^{2/p}}. \tag{7}$$

The extremal functions for $S(a,b)$are ground state solutions of the Euler equation

$$-div(|x|^{-2a}\nabla u) = |x|^{-pb}u^{p-1}, \qquad u \geq 0, \qquad in\ \mathbb{R}^N. \tag{8}$$

This equation is regarded as a prototype of more general nonlinear degenerate elliptic equations from physical phenomena (e.g., [3], [13]).

Note that the Caffarelli-Kohn-Nirenberg inequalities (1) (see also generalizations in [20] by Lin) contain the classical Sobolev inequality ($a = b = 0$) and the Hardy inequality($a = 0$, $b = 1$) as special cases, which have played important roles in many applications by virtue of the complete knowledge about the best constants, extremal functions, and their qualitative properties (see e.g., [7], [14], [16], [19]). Thus it is a fundamental task to study the best constants, existence (and nonexistence) of extremal functions, as well as their qualitative properties in inequality (1) for parameters a and b in the full parameter domain (2).

Much progress has been made for the parameter region

$$0 \le a < \frac{N-2}{2}, \quad a \le b \le a+1\,,$$

(to which we shall refer as the "a-nonnegative region"). In [2], [24], the best constant and the minimizers for the Sobolev inequality ($a = b = 0$) were given by Aubin and Talenti. In [19], Lieb considered the casea $a = 0, 0 < b < 1$, and gave the best constants and explicit minimizers. In [12], Chou and Chu considered the full a-nonnegative region and gave the best constants and explicit minimizers. Also for this a-nonnegative region, Lions in [23] (for $a = 0$) and Wang and Willem (for $a > 0$) in [26] established the compactness of all minimizing sequences up to dilations provided$a \le b < a + 1$. The symmetry of the minimizers has also been studied in [12] and [19]. In fact ,all nonnegative solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ for the corresponding Euler equation (8) areradially symmetric(in the case $a = b = 0$, they are radial with respect to some point) and explicitly given (see [2], [12], [19], [24]). This was established in [12], where a generalization of the moving plane method was used (e.g., [6], [11], [15]).

On the other hand, it seems that little is known for parameters in the a- negative region

$$-\infty < a < 0, \quad a \le b \le a+1\,.$$

This also applies to $N = 1$ and $N = 2$, withbin the corresponding intervals (4) and (3). The case $-1 < a < 0$ and $b = 0$ was treated recently by Caldiroli and Musina in [8], who gave the existence of ground states. We settle some of the fundamental questions concerning inequalities (1) with parameters in the a-negative region, such as the best constants, the existence and nonexistence of minimizers, and the symmetry properties of minimizers. For the a-negative region we shall reveal new phenomena that are strikingly different from those for the a-nonnegative region.

To state the results, let $S_P(\mathbb{R}^N)$ be the best embedding constant from $H^1(\mathbb{R}^N)$ in to $L^P(\mathbb{R}^N)$, i.e.,

$$S_P(\mathbb{R}^N) = \inf_{u \in H^1(\mathbb{R}^N)/\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dX}{\left(\int_{\mathbb{R}^N} |\nabla u|^P dX\right)^{\frac{2}{P}}}.$$

In the theorems stated below, we assume $N \ge 3$. Results for$N = 1$ and $N = 2$ will be given.

Our approach to the problem is quite different from that used in the quoted previous (see [2], [8], [12], [19], [23], [24], [26]) in which the problem was worked directly in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$, and we shall take a detour to convert the problem to an equivalent one defined on $H^1(\mathbb{R} \times \mathbb{S}^{N-1})$. While taking advantage of the two formulations, we shall work mainly with the equivalent one on $H^1(\mathbb{R} \times \mathbb{S}^{N-1})$. The reformulation enables us to make use of a combination of analytical tools such as a compactness argument, rescaling, the concentration compactness principle, bifurcation analysis, the moving plane method, etc. Moreover, our approach also gives a different proof of inequalities (1).

We shall introduce a transformation that transforms our problem in $\mathbb{R}^N$ to one on the space $\mathbb{R} \times \mathbb{S}^{N-1}$ on which we have a family of inequalities corresponding to (1) and an Euler equation corresponding to (8). The two problems will be shown to be equivalent, and we shall mainly work on the transformed one on $\mathbb{R} \times \mathbb{S}^{N-1}$ .The advantage in working on the latter is that the equation is an autonomous one and is defined in $H^1(\mathbb{R} \times \mathbb{S}^{N-1})$. Radial solutions (as we shall see, the only bound state radial solutions are the ground state solutions in the radially symmetric class) will be examined completely and their energy levels will be computed so that some comparison arguments can be done later., we show Theorem (1.1.7) , first establishing the continuity of $S(a,b)$in$(a,b)$and then giving the nonexistence result for the case $b = a$ with a combination of continuity and comparison arguments. The existence of a minimizer for the case $a < b < a = 1$ will be given by using a compactness argument; an asymptotic estimate for $S(a,b)$ $a \to -\infty$ will be given using a concentration compactness principle. We establish the symmetry-breaking result Theorem (1.1.11). First a bifurcation analysis will be done to claim the symmetry breaking for a away from 0. For a close to 0 it is much subtler, and some continuity and comparison arguments will be employed is devoted to establishing the modified inversion symmetry (up to a dilation) for all bound state solutions of (8) by using the moving plane method. We treat the cases $N = 1$ and $N = 2$. For $N = 1$ we have a complete solution for the problem including the identification of all bound state solutions. Finally, we state results for a related problem that can be solved using our results for (8), and we also point out some related open questions.

We start by introducing a family of transformations that will transform our original problem to one defined on a cylinder $\mathbb{R} \times \mathbb{S}^{N-1}$. The two problems will be shown to be equivalent in a sense that will be precisely specified. Then some preliminary results on the radial solutions will be given.

To problem (1) and equation (8) on $\mathbb{R}^N$mwe shall derive an equivalent minimization problem and corresponding Euler equation on $\mathbb{R} \times \mathbb{S}^{N-1}$ .We shall use the notation $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1}$.While working on both problems to take advantage of the two formulations, we shall get most of our results on the cylinder $\mathcal{C}$. For integrals over a domain included in $\mathcal{C}$, by $d\mu$ we denote the volume element on $\mathcal{C}$. Also, by $|\nabla u|^2$ we understand $g^{ij}u_i u_j$ and (g^{ij}) are the components of the inverse matrix to the metric induced from $\mathbb{R}^{N+1}$. For points on $\mathcal{C}$ we use either the notation y to identify a point in $\mathbb{R}^{N+1}$ or (t,θ) to identify a point in $\mathbb{R} \times \mathbb{S}^{N-1}$. To u, a smooth function with compact support in $\mathbb{R}^N/\{0\}$, we associate v, a smooth function with compact support on $\mathcal{C}$, by the transformation

$$u(x) = |x|^{-\frac{N-2-2a}{2}\mathrm{K}} v\left(-\ln|x|, \frac{x}{|x|}\right). \qquad (9)$$

Here for $x \in \mathbb{R}^N/\{0\}$, with $t = \ln|x|$ and $0 = x/|x|$, we have $(t,\theta) \in \mathcal{C}$.

Let us denote by $L_a^p(\mathbb{R}^N) = \left\{u: \int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, dx < \infty\right\}$ the weighted L^pspace. We need the following lemma.

Lemma (1.1.1)[1]: For $a < \frac{N-2}{2}$, $a \le b \le a + 1$, and $b = \frac{2N}{N-2+2(b-a)}$, it holds that

$$\mathcal{D}_a^{1,2}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N/\{0\})}^{\|\cdot\|},$$

where $\|\cdot\|$ is the norm in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ given by(5). Moreover, $L_a^p(\mathbb{R}^N)$is also given by the completion of $C_0^\infty(\mathbb{R}^N/\{0\})$ under its norm.

Proof: By the definition of $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$, it suffices to show

$$C_0^\infty(\mathbb{R}^N \subset \overline{C_0^\infty(\mathbb{R}^N/\{0\})}^{\|\cdot\|}.$$

Let $\rho(t)$ be a cutoff function that is 1 for $t \geq 2$ and 0 for $0 < t \leq 1$. For a fixed $u \in C_0^\infty(\mathbb{R}^N)$, we define $u_\varepsilon(x) = \rho(|x|/\varepsilon)u(x) \in C_0^\infty(\mathbb{R}^N/\{0\})$. Then it is easy to check that $|u_\varepsilon - u| \to 0$ as $\varepsilon \to 0$. The second part is similar.

Now for $u \in C_0^\infty(\mathbb{R}^N/\{0\})$, by a direct computation we have

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2(x)dx = \int_{\mathbb{R}^N} |x|^{-N}\left(|\nabla_\theta v|^2 + \left(v_t + \frac{N-2-2a}{2}v\right)^2\right)dx;$$

therefore

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2(x)dx = \int_{\mathcal{C}} |\nabla_\theta v|^2 \left(v_t + \frac{N-2-2a}{2}v\right)^2 d\mu$$

$$= \int_{\mathcal{C}} |\nabla_\theta v|^2 + v_t^2 + \left(\frac{N-2-2a}{2}\right)^2 v^2\, d\mu.$$

Also,

$$\int_{\mathbb{R}^N} |x|^{-bp}u^p(x)dx = \int_{\mathbb{R}^N} |x|^{-N}v^p\, dx = \int_{\mathcal{C}} v^p d\mu.$$

From these and Lemma (1.1.1) we immediately have the following:

Proposition (1.1.2)[1]: The mapping given in (9) is a Hilbert space isomorphism from $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ to $H^1(\mathcal{C})$ where the inner product on $H^1(\mathcal{C})$ is

$$(v,w) = \int_{\mathcal{C}} \nabla v^2 \cdot \nabla w + \left(\frac{N-2-2a}{2}\right)^2 v\, w\, d\mu.$$

Now we define an energy functional on $H^1(\mathcal{C})$:

$$F_{a,b}(v) = \frac{\int_{\mathcal{C}} |\nabla_\theta v|^2 + v_t^2 + \left(\frac{N-2-2a}{2}\right)^2 v^2\, d\mu}{\left(\int_{\mathcal{C}} |v|^p d\mu\right)^{2/p}}. \qquad (10)$$

If $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$ and $v \in H^1(\mathcal{C})$ are related through (9), then

$$E_{a,b}(v) = F_{a,b}(v).$$

Moreover, if u is a solution of (8), then v satisfies

$$-v_{tt} - \Delta_\theta v + \left(\frac{N-2-2a}{2}\right)^2 v = v^{p-1}, \quad v > 0, on\ \mathcal{C} \qquad (11)$$

where $t = -\ln|x|$ and Δ_θ is the Laplace operator on the $(N-1)$-sphere.

We collect these observations in the following:

Proposition (1.1.3)[1]: With $a, b,$ and p satisfying(1), we have

(i) If $u \in \mathcal{D}_a^{1,2}$ and $v \in H^1(\mathcal{C})$ are related through (9), then $E_{a,b}(u) = F_{a,b}(v)$.

(ii) For $S(a,b)$ defined in (6), it holds $(a,b) = \inf_{H^1(\mathcal{C})/\{0\}} F_{a,b}(v)$.

(iii) Solutions of (8) and (11) are in one-to-one correspondence, being related through (9).

In order to study the symmetry property of solutions, we examine the invariance of the problem under the transformation (9). As in the case of the Yamabe problem ($a = b = 0$), the group of transformations that leaves problem (8) invariant is noncompact. The group of translations in $\mathbb{R}^N$ is a symmetry group for (8) only in the case $a = b = 0$. On the other hand, the dilations

$$u_\tau(x) = \tau^{\frac{N-2-2a}{2}} u(\tau, x), \quad \tau > 0\,, \tag{12}$$

leave the problem invariant for all a and b; i.e., if u is a solution of (8), so is u_τ.This still holds for$N = 2$ and$N = 1$, but for $N = 1$ the situation is a bit different and there is a two-parameter family of dilations (see (44)). The group that leaves (11) invariant, corresponding to dilations in $\mathbb{R}^N$, is the group of translations in the t-direction. If v and v_τ in $H^1(\mathcal{C})$ are related to u and u_τ in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ through (9), then

$$u_\tau(t, \theta) = v(t - \ln\tau, \theta).$$

Finally, the following modified inversion invariance of (8),

$$\bar{u}(x) = |x|^{-(N-2-2a)} u\left(\frac{x}{|x|^2}\right), \tag{13}$$

translates on the cylinder to the following obvious symmetry of (11),

$$\bar{v}(t, \theta) = v(-t, \theta).$$

Let $\mathcal{D}_{a,R}^{1,2}(\mathbb{R}^N)$ be the subspace of $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ consisting of radial functions. Define

$$R(a, b) = \inf_{u \in \mathcal{D}_{a,R}^{1,2}(\mathbb{R}^N)/\{0\}} E_{a,b}(u). \tag{14}$$

By Proposition (1.1.3)(i) we also have

$$R(a, b) = \inf_{u \in H_R^1(\mathcal{C})/\{0\}} E_{a,b}(u).$$

where $H_R^1(\mathcal{C})$ consists of functions independent of θ. We shall find the exact value of $R(a, b)$ and the exact form of the radial solutions that achieve these constants when $a \leq b < a + 1$. We remark here that our method applies for the a-nonnegative region also and in fact gives a new approach for the a-nonnegative region; the results we get agree with [12] and [19] in this region.

In order to study the radial solutions of (8), we shall need the exact form of particular positive solutions for the following nonlinear second-order ODE:

$$-v_{tt} + \lambda^2 v = v^{p-1}, \qquad v < 0\,, in\ \mathbb{R} \tag{15}$$

with $p > 2$.The problem can be associated to the Hamiltonian system

$$\frac{d}{dt} v = w, \qquad \frac{d}{dt} w = \lambda^2 v - v^{p-1}.$$

We have the Hamiltonian

$$H(v, w) = \frac{1}{2} w^2 - \frac{\lambda^2}{2} v^2 + \frac{1}{P} v^p.$$

All solutions correspond to level curves of $H(v, w)$. Up to translations, only one homoclinic solution v that is on the level $H(v, w) = 0$. The levels below this one will give v positive, periodic, and bounded away from zero. For the levels above, v changes sign so we lose positivity. The only positive solutions that are in $H^1(\mathbb{R})$ are translates of

$$v(t) = \left(\frac{\lambda^2 p}{2}\right)^{\frac{1}{p-2}} \left(\cosh\left(\frac{p-2}{2}\lambda t\right)\right)^{-\frac{2}{p-2}}. \tag{16}$$

A direct calculation gives that for the v above,

$$\frac{\int_{\mathbb{R}} v_t^2+\lambda^2 v^2 dt}{\left(\int_{\mathbb{R}} v^p dt\right)^{2/p}} = 2p\frac{\lambda^{(p+2)/p}}{(p-2)^{(p-2)/p}}\left(\frac{\Gamma^2\left(\frac{p}{p-2}\right)}{\Gamma\left(\frac{2p}{p-2}\right)}\right)^{\frac{p-2}{p}}. \quad (17)$$

Now, when searching for radial solutions, equation (11) becomes

$$v_{tt}-\left(\frac{N-2-2a}{2}\right)^2 v+v^{p-1}=0, \quad v>0\,, \qquad on\ \mathbb{R}, \quad (18)$$

which corresponds to equation (15) with $\lambda=\frac{N-2-2a}{2}$. According to (16), the homoclinic solutions of (18) are translates of

$$v(t)=\left(\frac{N(N-2-a2)^2}{4\big(N-2(1+a-b)\big)}\right)^{\frac{N-2(1+a-b)}{4(1+a-b)}}$$

$$\left(\cosh\frac{(N-2-a2)(1+a-b)}{N-2(1+a-b)}t\right)^{-\frac{N-2(1+a-b)}{2(1+a-b)}}. \quad (19)$$

The radial solution in $\mathbb{R}^N$ for (8) corresponding to this v is

$$u(x)=\left(\frac{N(N-2-a2)^2}{N-2(1+a-b)}\right)^{\frac{N-2(1+a-b)}{4(1+a-b)}}\frac{1}{\left(1+|x|^{\frac{2(N-2-a2)\,(1+a-b)}{N-2(1+a-b)}}\right)^{\frac{N-2(1+a-b)}{2(1+a-b)}}}. \quad (20)$$

All radial solutions in $\mathbb{R}^N$ for (8) are dilations of this u. Note that by substituting in (7)

$$\lambda=\frac{N-2-2a}{2} \ \ and \ \ p=\frac{2N}{N-2(1+a-b)}\,,$$

we estimate the energy of any radial solution in $H^1(\mathcal{C})$,

$$R(a,b)=E_{a,b}(u)=F_{a,b}(v),$$

$$R(a,b)=\frac{N\omega_{N-1}^{\frac{2(1+a-b)}{N}}(N-2-2a)^{\frac{2(N-(1-a-b))}{N}}}{2^{\frac{2(1+a-b)}{N}}\big(N-2(1+a-b)\big)^{\frac{N-2(1+a-b)}{N}}(1+a-b)^{\frac{2(1+a-b)}{N}}}$$

$$\left(\frac{\Gamma^2\left(\frac{N}{2(1+a-p)}\right)}{\Gamma\left(\frac{N}{1+a-b}\right)}\right)^{\frac{2(1+a-p)}{N}}. \quad (21)$$

Proposition (1.1.4)[1]: Up to a dilation(12), all radial solutions of (8) are explicitly given in(20), and $R(a,b)$is given in (21).

To prove Theorem (1.1.7)(i), we need a couple of lemmas.

Lemma (1.1.5)[1]: Let $a_0<\frac{N-2}{2}, a_0\le b_0\le a_0+1$. Then

$$\limsup_{(a,b)\to(a_0,b_0)} S(a,b)\le S(a_0,b_0).$$

Proof: For any $\varepsilon>0$,there is a nonnegative function $v\in C_0^\infty(\mathcal{C})$ such that

$$F_{a_0,b_0}(v)\le S(a_0,b_0)+\frac{\varepsilon}{2}.$$

Note that as $(a,b)\to(a_0,b_0), v^2(x)\to v^{p_0}(x)$ for all x .For any $p\in[2,2^*]$, $v^p(x)\le w(x)$ where $w(x)=v^2(x)$ if $v(x)<1$ and $w(x)=v^{2^*}(x)$ if $v(x)\ge 1$. Clearly w is integrable; therefore by the dominated convergence theorem we have

$$\lim_{(a,b)\to(a_0,b_0)}\int_{\mathcal{C}} v^p d\mu=\int_{\mathcal{C}} v^{p_0} d\mu\,.$$

From this, and because λ is continuous in a,we get there is $\delta > 0$ such that $|(a,b) - (a_0, b_0)| < \delta$ implies

$$S(a,b) \leq F_{a,b}(v) \leq F_{a_0,b_0} + \frac{\varepsilon}{2} \leq S(a_0,b_0) + \varepsilon.$$

Let $\varepsilon \to 0$.

Lemma (1.1.6)[1]: Let.$(p_n) \subset [2, 2^*]$ be a sequence convergent to p. If a sequence.(u_n) is uniformly bounded by M in $H^1(\mathcal{C})$, then

(i) if $p \in (2, 2^*)$, we have

$$\lim_{n\to\infty} \int_{\mathcal{C}} ||u_n|^{p_n} - |u_n|^p| \, d\mu = 0;$$

(ii) if $p = 2$ or $p = 2^*$, we have

$$\limsup_{n\to\infty} \int_{\mathcal{C}} (|u_n|^{p_n} - |u_n|^p) \, d\mu \leq 0.$$

Proof: We first prove (i). By the mean value theorem, there are functions ξ_n defined on $\mathcal{C}$ with values between p_n and p such that

$$\int_{\mathcal{C}} ||u_n|^{p_n} - |u_n|^p| \, d\mu = \int_{\mathcal{C}} \left|\ln|u_n||u_n|^{\xi_n(x)} (p_n - p)\right| d\mu.$$

Since $p \in (2, 2^*)$ let $\varepsilon > 0$ such that $[p - \varepsilon, p + \varepsilon\,] \subset (2, 2^*)$. Let $n_\varepsilon$be such that for $n \geq n_\varepsilon$ we have $|p_n - p| < \varepsilon$; therefore

$$\int_{\mathcal{C}} ||u_n|^{p_n} - |u_n|^p| \, d\mu \leq$$
$$|p_n - p| \left(\int_{|u_n|>1} \ln|u_n||u_n|^{p+\varepsilon} d\mu + \int_{0<|u_n|<1} \ln\frac{1}{|u_n|} |u_n|^{p-\varepsilon} d\mu\right).$$

The key now is to show that the two integrals on the right-hand side are bounded as $n \to \infty$. There is a constant C depending only on p such that

$$\ln u \leq C u^{2^*-p-\varepsilon} \quad \text{for all } u > 1$$

and

$$\ln\frac{1}{u} \leq \frac{C}{u^{p-\varepsilon-2}} \quad \text{for all } 0 < u < 1.$$

With

$$S_p(\mathcal{C}) = \inf_{u\in H^1(\mathcal{C})/\{0\}} \frac{\int_{\mathcal{C}} |\nabla u|^2 + u^2 d\mu}{\left(\int_{\mathcal{C}} |u|^p d\mu\right)^{2/p}},$$

we obtain

$$\int_{|u_n|>1} \ln|u_n| \, |u_n|^{p+\varepsilon} d\mu \leq C \int_{|u_n|>1} |u_n|^{2^*} \, d\mu \leq C \left(\frac{M}{S_{2^*}(\mathcal{C})}\right)^{\frac{2^*}{2}}.$$

We also have that

$$\int_{0<|u_n|<1} \ln\frac{1}{|u_n|} |u_n|^{p-\varepsilon} d\mu \leq C \int_{0<|u_n|<1} |u_n|^2 \, d\mu \leq C \frac{M}{S_2(\mathcal{C})}.$$

This concludes the proof of (i).

For part (ii), we use the same method as above after we make the estimates as follows: For $p = 2$,

$$\int_{\mathcal{C}} |u_n|^{p_n} - |u_n|^2 \, d\mu \leq \int_{|u_n|>1} |u_n|^{p_n} - |u_n|^2 \, d\mu ,$$

and for $p = 2^*$,

$$\int_{\mathcal{C}} |u_n|^{p_n} - |u_n|^{2^*} \, d\mu \leq \int_{0<|u_n|<1} |u_n|^{p_n} - |u_n|^{2^*} \, d\mu .$$

Theorem (1.1.7)[1]: (Best Constants and Nonexistence of Extremal Functions)

(i) $S(a, b)$ is continuous in the full parameter domain (2).

(ii) For $b = a + 1$, we have $S(a, a + 1) = \left(\frac{N-2-2a}{2}\right)^2$, and $S(a, a - 1)$ is not achieved.

(iii) For $a < 0$ and $b = a$, we have $S(a, a) = S(0,0)$(the best S obolev constant) , and $S(a, a)$ is not achieved.

Proof: (i) According to Lemma (1.1.5), it suffices to show that

$$\liminf_{(a,b)\to(a_0,b_0)} S(a, b) \geq S(a_0, b_0).$$

Assume there is a sequence $(a_n, b_n) \to (a_0, b_0)$ such that

$$\lim_{n\to\infty} S(a_n, b_n) < S(a_0, b_0). \tag{22}$$

Then there are $\varepsilon > 0$ and functions $(v_n) \subset H^1(\mathcal{C})$ such that

$$\int_{\mathcal{C}} |v_n|^{p_n} d\mu = 1$$

and

$$S(a_0, b_0) - \varepsilon \geq F_{a_n,b_n}(v_n).$$

Clearly, (v_n) is bounded in $H^1(\mathcal{C})$. From Lemma (1.1.6), we get

$$F_{a_n,b_n}(v_n) + o(1) \geq F_{a_0,b_0}(v_n) \geq S(a_0, b_0).$$

This and (22) give the desired contradiction.

(ii) Clearly, $F_{a,a+1}(v)((N - 2 - 2a)/2)^2$ for all $v \in H^1(\mathcal{C})$. On the other hand, one can easily construct a sequence . $(v_n) \subset H^1(\mathcal{C})$ of radial functions such that $F_{a,a+1}(v_n) \to ((N - 2 - 2a)/2)^2$. Therefore,

$$S(a, a + 1) = \left(\frac{N - 2 - 2a}{2}\right)^2 .$$

For nonexistence of minimizers, one notes that for $\lambda \geq 1$, the equation

$$-\Delta v + \lambda^2 v = v$$

has no nonzero solution in $H^1(\mathcal{C})$. For $0 < \lambda < 1$, i.e., . $(N - 4)/2 < a < (N - 2)/2$, assume that. $S(a, a + 1)$ is achieved by some function $v \in H^1(\mathcal{C})$ the maximum principle, $v > 0$ everywhere. Denote by $f(t)$ the average of v on the spheres t =const. Then f is a positive function in $H^1(\mathbb{R})$ and satisfies the ODE

$$-f_u + \lambda^2 f = f.$$

The only nonnegative solution is $f \equiv 0$. Therefore for all $< \frac{N-2}{2}$, the infimum $S(a, a + 1)$ is not achieved.

(iii) The case $a = b = 0$ is well known (the Yamabe problem in $\mathbb{R}^N$). In this case, the minimize $S(0,0)$ is achieved only by functions

$$U_{\mu,y}(x) = C\frac{\mu^{(N-2)/2}}{(\mu^2+|x-y|^2)^{(N-2)/2}}, \quad \mu > 0, \quad \lambda \in \mathbb{R}^N.$$

Note that for $a \in (-N/2,(N-2)/2)$, $U_{\mu,y} \in \mathcal{D}_a^{1,2}$. For $y \neq 0$ by a direct computation we get for $a \in (-N/2,(N-2)/2)$

$$\mathrm{S}(0,0) = \lim_{\mu\to 0} E_{a,a}(U_{\mu,y}).$$

Due to this fact one concludes that for $a \in (-N/2,(N-2)/2)$,

$$(a,a) \le S(0,0). \tag{23}$$

On the other hand, by the expression (10), for any $v \in \frac{H^1(\mathcal{C})}{\{0\},F_{a,a}(v)>F_{0,0}(v)\ge S(0,0)}$. Hence, $S(0,0) = S(0,0)$ for all $a \in (-N/2,0)$. Next, we fix $a_1 \in (-N/2,0)$. For any $a \le -N/2$ fixed and any $\varepsilon > 0$, there is $v \in H^1(\mathcal{C})$ such that

$$F_{a_1,b_1}(v) \le S(0,0) + \frac{\varepsilon}{2(\lambda(a)^2-\lambda(a_1)^2)(\lambda(a_1)^2-\lambda(0)^2)},$$

where$\lambda(a) = N-2-2a)/2$ .Together with $S(0,0) \le F_{0,0}(v) \le F_{a_1,a_1}(v_n)$ we conclude

$$\frac{\int_{\mathcal{C}} v^2 d\mu}{\left(\int_{\mathcal{C}} |v|^{2^*} d\mu\right)^{2/2^*}} \le \frac{\varepsilon}{2(\lambda(a)^2-\lambda(a_1)^2}.$$

Then

$$F_{a,a}(v) = F_{a_1,a_1}(v_n) + (\lambda(a)^2-\lambda(a_1)^2)\frac{\int_{\mathcal{C}} v^2 d\mu}{\left(\int_{\mathcal{C}} |v|^{2^*} d\mu\right)^{2/2^*}} \le S(0,0)+\varepsilon.$$

That is, $S(a,a) = S(0,0)$ for all $a \le 0$.

Next we show $S(a,a)$ is not achieved for $a < 0$. If the conclusion is not true, for some $a < 0$ and $v \in H^1(\mathcal{C})$ we get $S(a,a) = F_{a,a}(v)$. But using $F_{a,a}(v) > F_{0,0}(v) \ge S(0,0)$, we get a contradiction to $S(a,a) = S(0,0)$.

We show the existence of a minimizer for $a < 0$ and $a < b < a+1$. We also give an asymptotic estimate of $S(a,a)$ as $-a \to \infty$, while $b-a \in (0,1)$ is a fixed constant.

We shall need the following lemma. It is analogous to a result on $\mathbb{R}^N$ due to P. L. Lions [22]. The proof is similar to the proof of lemma in [27].

Lemma (1.1.8)[1]: Let $r > 0$ and $2 \le q < 2^*$.If w_n is bounded in $H^1(\mathcal{C})$ and if

$$\sup_{y\in\mathcal{C}} \int_{B_r(y)\cap\mathcal{C}} |w_n|^q d\mu \to 0 \ \ as\ n \to \infty,$$

then $w_n \to 0$ in $L^p(\mathcal{C})$ for $2 < p < 2^*$. Here $B_r(y)$ denotes the ball in $\mathbb{R}^{N+1}$ with radius r centered at y.

Theorem (1.1.9)[1]: (Best Constants and Existence of Extremal Functions)

(i) For $a < b < a+1$, $S(a,b)$ is always achieved.

(ii) For $b-a \in (0,1)$ fixed, as $a \to -\infty, S(a,b)$ is strictly increasing, and

$$S(a,b) = \left(\frac{N-2-2a2}{2}\right)^{2(b-a)}\left[S_p(\mathbb{R}^N)+o(1)\right].$$

Proof: (i) Let $a < 0$ and $a < b < a+1$ be fixed. Consider a minimizing sequence. $(w_n) \subset H^1(\mathcal{C})$ such that

$$\int_{\mathcal{C}} |w_n|^p d\mu = 1 \quad for\ all\ n \geq 1$$

and

$$\int_{\mathcal{C}} |\nabla w_n|^2 + \left(\frac{N-2-a}{2}\right)^2 w_n^2 d\mu \to S(a,b) \ \ as\ \ n \to \infty.$$

According to Lemma (1.1.4),

$$\delta = \lim_{n\to\infty} \inf\left(\sup_{y\in\mathcal{C}} \int_{B_r(y)\cap\mathcal{C}} w_n^2 d\mu \right) > 0.$$

Eventually by passing to a subsequence, we may assume there are. $(y_n) \subset \mathcal{C}$ and $y_0 \in \mathcal{C}$ fixed such that the sequence $v_n(x) = w_n(x - y_n)$ has the property

$$\int_{B_r(y_0)\cap\mathcal{C}} |v_n|^2 d\mu > \frac{\delta}{2}.$$

Clearly,

$$\int_{\mathcal{C}} |v_n|^p d\mu = 1 \quad for\ all\ \ n \geq 1$$

and

$$\int_{\mathcal{C}} |\nabla v_n|^2 + \left(\frac{N-2-a}{2}\right)^2 v_n^2 d\mu \to S(a,b) \ \ as\ \ n \to \infty.$$

Without loss of generality we can assume

$$v_n \to v \quad \text{weakly in } H^1(\mathcal{C}),$$
$$v_n \to v \quad \text{in } L^2_{loc}(\mathcal{C}),$$
$$v_n \to v \quad \text{almost everywhere in } \mathcal{C}.$$

According to the Brezis-Lieb lemma [4], we have

$$1 = |v|^p_{Lp} + \lim_{n\to\infty} |v_n - v|^p_{Lp}.$$

Hence

$$\begin{aligned} S(a,b) &= \lim_{n\to\infty} \int_{\mathcal{C}} |\nabla v_n|^2 + \left(\frac{N-2-a}{2}\right)^2 v_n^2 d\mu \\ &= \int_{\mathcal{C}} |\nabla v|^2 + \left(\frac{N-2-a}{2}\right) v^2 d\mu \\ &\quad + \lim_{n\to\infty} \int_{\mathcal{C}} |\nabla v_n - v|^2 + \left(\frac{N-2-a}{2}\right)^2 (v_n - v)^2 d\mu\,. \\ &\geq S(a,b)\left(|v|^2_{Lp} + \left(1 - |v|^2_{Lp}\right)^{\frac{2}{p}} \right). \end{aligned}$$

Since $v \not\equiv 0$, we obtain $|v|_{Lp} = 1$, and so

$$\int_{\mathcal{C}} |\nabla v\ \ |^2 + \left(\frac{N-2-a}{2}\right) v^2\ \ d\mu = S(a,b).$$

Let $b-a \in (0,1)$ be fixed so that $p \in (2, 2^*)$ is also fixed. We shall consider the asymptotic behavior of $S(a,b)\ as - a \to \infty$.

(ii) We use a rescaling argument. Let $h_\lambda: \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$be the scaling map $h_\lambda(x) = \lambda x$. Denote $\mathcal{C}_{\lambda=}h_\lambda(\mathcal{C})$ and for $H^1(\mathcal{C})$,define $u \in H^1(\mathcal{C}_\lambda)$ by $u(\lambda x) = v(x)$. For definiteness, on $H^1(\mathcal{C}_\lambda)$ we use the norm $\|u\|^2 = \int_{\mathcal{C}_\lambda} |\nabla u|^2 + |u|^2 d\mu$. We have

$$\int_{\mathcal{C}} |\nabla v|^2 + \lambda^2 v^2 d\mu = \lambda^{2-N} \int_{\mathcal{C}_\lambda} |\nabla u|^2 + u^2 du$$

and

$$\int_{\mathcal{C}} |\upsilon|^p\ d\mu = \lambda^{-N} \int_{\mathcal{C}_\lambda} |\mathrm{u}|^p\ du.$$

Therefore,

$$F_{a,b}(v) = \lambda^{2(b-a)} \frac{\int_{\mathcal{C}_\lambda} |\nabla u|^2 + u^2 d\mu}{\left(\int_{\mathcal{C}_\lambda} |u|^p\ d\mu\right)^{\frac{2}{p}}}$$

Now it suffices to show that

$$I(\lambda) \coloneqq \inf_{u \in H^1(\mathcal{C}_\lambda)/\{0\}} \frac{\int_{\mathcal{C}_\lambda} |\nabla u|^2 + u^2 d\mu}{\left(\int_{\mathcal{C}_\lambda} |u|^p\ d\mu\right)^{2/p}} \to S_p(\mathbb{R}^N)$$

as $\lambda \to \infty$.

First we have that

$$\lim_{\lambda\to\infty} \sup I(\lambda) \le S_p(\mathbb{R}^N) \tag{24}$$

We get this through a cutoff procedure. Let $r\ > 0$; then for fixed λ large and $y \in \mathcal{C}_\lambda$, we have a projection $\psi = \psi_{y,r,\lambda}$ from $B_r(0) \subset \mathbb{R}^N$ to $\psi(B_r(0)) \subset \mathcal{C}_\lambda$ defined as follows: Identify $\mathbb{R}^N$ with the tangent space to $\mathcal{C}_\lambda$ at $y \in \mathcal{C}_\lambda$, and let ψ to be the projection from $B_r(0)$ into $\mathcal{C}_\lambda$ along directions parallel to ν_y, the exterior normal to $\mathcal{C}_\lambda$at y. Then is a diffeomorphism on its image and for fixed r, the Jacobian matrix of ψ tends uniformly to the identity matrix as$y \to \infty$.

Denote by $w \in H^1(\mathbb{R}^N)$ a function with support in $B_r(0) \subset \mathbb{R}^N$ .For $y \in \mathcal{C}_\lambda$, let $u_\lambda(\psi_{y,r,\lambda}(x)) = w(x)$ and 0 outside $\psi_{y,r,\lambda}(B_r(0))$; then

$$\int_{\mathcal{C}_\lambda} |\nabla u_\lambda|^2 + u_\lambda^2 d\mu = \int_{\mathbb{R}^N} |\nabla w|^2 + w^2 dx + o(1) \tag{25}$$

and

$$\int_{\mathcal{C}_\lambda} |u_\lambda|^p d\mu = \int_{\mathbb{R}^N} |w|^p\ dx + o(1) \tag{26}$$

where $o(1) \to 0$ as $\lambda \to \infty$ uniformly in y.

In $\mathbb{R}^N$, it is known that the infimum $S_p(\mathbb{R}^N)$ achieved by a positive function w, radially symmetric about some point, which satisfies

$$-\Delta w + w = w^{p-1} \text{ in } \mathbb{R}^N.$$

To prove (24), let $\varepsilon > 0$ and let $r > r_0 > 0$, sufficiently large, so that for a cutoff function $\rho(x)$, which is identically 1 in $B_{r_0}(0)$ and 0 outside $B_r(0)$, we have

$$\frac{\int_{\mathbb{R}^N} |\nabla(\rho w)|^2 + (\rho w)^2 dx}{\left(\int_{\mathbb{R}^N} (\rho w)^p dx\right)^{\frac{2}{p}}} < S_p(\mathbb{R}^N) + \frac{\varepsilon}{2}.$$

Then from (25) and (26), there is λ large enough such that when we consider

$$u(x) = (\rho w)\big(\psi^{-1}(x)\big) \in H^1(\mathcal{C}_\lambda),$$

we get

$$\frac{\int_{\mathcal{C}_\lambda} |\nabla u|^2 + u^2 d\mu}{\left(\int_{\mathcal{C}_\lambda} u^p d\mu\right)^{2/p}} < \frac{\int_{\mathbb{R}^N} |\nabla(\rho w)|^2 + (\rho w)^2 dx}{\left(\int_{\mathbb{R}^N} (\rho w)^p dx\right)^{2/p}} < S_p(\mathbb{R}^N) + \frac{\varepsilon}{2}.$$

From the two inequalities we conclude that

$$I(\lambda) \le \frac{\int_{\mathcal{C}_\lambda} |\nabla u|^2 + u^2 d\mu}{\left(\int_{\mathcal{C}_\lambda} u^p d\mu\right)^{2/p}} < S_p(\mathbb{R}^N) + \varepsilon.$$

Therefore,

$$\limsup_{\lambda\to\infty} I(\lambda) \le S_p(\mathbb{R}^N) + \varepsilon.$$

We now prove

$$\liminf_{\lambda\to\infty} I(\lambda) \ge S_p(\mathbb{R}^N). \tag{27}$$

If (27) does not hold, there are $\varepsilon_0 > 0$ and a sequence λ_n which tends to ∞ such that

$$I_0 := \lim_{n\to\infty} I(\lambda_n) \le S_p(\mathbb{R}^N) - \varepsilon_0.$$

Then there are functions $u_n \in H^1(\mathcal{C}_\lambda)$ (here $\mathcal{C}_n = \mathcal{C}_{\lambda_n}$) such that

$$\int_{\mathcal{C}_n} |u_n|^p d\mu = 1 \quad and \quad I(\lambda_n) \le \int_{\mathcal{C}_n} |\nabla u_n|^2 + u_n^2 \, d\mu \le S_p(\mathbb{R}^N) - \varepsilon_0.$$

Now we need a more detailed concentration-compactness lemma than the one in [22] and along the lines in [25]. The result in [25] is for the $H^1(\mathbb{R}^N)$ setting, but the proof carries over to our situation, too. We omit it here. For $r > 0$ and $y_{n,i} \in \mathcal{C}_n$, let $\Omega_{n,i}(r)$ be $\psi_{n,i,r,\lambda_n}\big(B_r(0)\big)$.

Lemma (1.1.10)[1]: Let $\lambda_n \to \infty$ and $u_n \in H^1(\mathcal{C}_n)$ be uniformly bounded (with norm given by $\|u\|^2 = \int_{\mathcal{C}_n} |\nabla u|^2 + |u_n|^2 d\mu$). Assume $\int_{\mathcal{C}_n} |u_n|^p \, d\mu = 1$. Then there is a sub sequence (still denoted by (u_n)), a nonnegative, non increasing sequence α_i satisfying $\sum_{i=1}^{\infty} \alpha_i = 1$, and sequences $(y_{n,i})_i \subset \mathcal{C}_n$ associated with each $\alpha_i > 0$ satisfying

$$\liminf_{n\to\infty} |y_{n,i} - y_{n,j}| = \infty \quad for\ any\ i \ne j \tag{28}$$

such that the following property holds: If $\alpha_s > 0$ for some $s \ge 1$, then for any $\varepsilon > 0$ there exist $R > 0$, for all $r \ge R$ and all $\acute{r} \ge R$, such that

$$\limsup_{n\to\infty}\sum_{i=1}^{s}\left|\alpha_i-\int_{\Omega_{n,i}(r)}|u_n|^p d\mu\right| + \left|\left(1-\sum_{i=1}^{s}\alpha_i\right)-\int_{\mathcal{C}_n/\cup_{i=1}^{s}\Omega_{n,i}(\acute{r})}|u_n|^{p+1}d\mu\right|$$
$$<\varepsilon \quad (29)$$

In Lemma (1.1.10), fix $s>0$ with $\alpha_s>0$ such that

$$\sum_{i=1}^{s}\alpha_i > \left(\frac{I_0}{S_p(\mathbb{R}^N)}\right)^{\frac{p}{2}}. \tag{30}$$

For$\alpha_s>\varepsilon>0$, let $R>0$ and $(y_{n,i})_i\subset\mathcal{C}_n$ such that for all $r,\acute{r}>R$,we have

$$\lim_{n\to\infty}\sum_{i=1}^{s}\left|\alpha_i-\int_{\Omega_{n,i}(r)}|u_n|^p d\mu\right| + \left|\left(1-\sum_{i=1}^{s}\alpha_i\right)-\int_{\mathcal{C}_n/\cup_{i=1}^{s}\Omega_{n,i}(\acute{r})}|u_n|^{p+1}d\mu\right|$$
$$<\varepsilon\,. \tag{31}$$

We now consider a cutoff function ρ on $\mathbb{R}^N$ that is identically 1 inside $B_R(0)$ and 0 outside $B_{2R}(0)$ and $|\nabla\rho|\le\frac{2}{R}$ at any point. For $1\le i\le s$, define $\psi=\psi_{y_{n,i,}2R,\lambda_n}$ as before, and let $w_{n,i}(x)=\rho(x)u_n(\psi(x)$ designate functions with compact support in $\mathbb{R}^N$ .By a direct computation, we get

$$\int_{\mathbb{R}^N}|\nabla w_{n,i}|^2+w_{n,i}^2dx\le\int_{\Omega_{n,i(2R)}}|\nabla u_n|^2+u_n^2d\mu+o(1)+\frac{C}{R}$$

with C independent of $n,\varepsilon,$ and R, and $o(1)\to\infty$ as $n\to\infty$. Also,

$$\int_{\mathbb{R}^N}|w_{n,i}|^p\,dx\ge\int_{\Omega_{n,i(R)}}|u_n|^p\,d\mu+o(1).$$

Since

$$\int_{\mathbb{R}^N}|\nabla w_{n,i}|^2+w_{n,i}^2dx\ge\left(\int_{\mathbb{R}^N}|\nabla w_{n,i}|^p\,dx\right)^{\frac{2}{p}}S_p(\mathbb{R}^N),$$

we obtain

$$\int_{\Omega_{n,i(2R)}}|\nabla u_n|^2+u_n^2d\mu+o(1)+\frac{C}{R}\ge\left(\int_{\Omega_{n,i(R)}}|u_n|^p\,d\mu+o(1)\right)^{\frac{2}{p}}S_p(\mathbb{R}^N).$$

Therefore,

$$\int_{\mathcal{C}}|\nabla u_n|^2+u_n^2d\mu\ge\left(\sum_{i=1}^{s}\int_{\Omega_{n,i(R)}}|u_n|^p\,d\mu\right)^{\frac{2}{p}}S_p(\mathbb{R}^N)+o(1)+\frac{sC}{R}.$$

Form (31) we get

$$\int_{\mathcal{C}}|\nabla u_n|^2+u_n^2d\mu\ge\left(\sum_{i=1}^{s}\alpha_i-\varepsilon\right)^{\frac{2}{p}}S_p(\mathbb{R}^N)+o(1)-\frac{sC}{R}.$$

Letting $n \to \infty$ and then $R \to \infty$, we obtain

$$I_0 \geq \left(\sum_{i=1}^{s} \alpha_i - \varepsilon\right)^{\frac{2}{p}} S_p(\mathbb{R}^N).$$

Now, let $\varepsilon \to 0$ to get

$$I_0 \geq \left(\sum_{i=1}^{s} \alpha_i\right)^{\frac{2}{p}} S_p(\mathbb{R}^N),$$

which contradicts (30).

For symmetry breaking, we have Theorem (1.1.11)(i) and (ii). The results of (i) and (ii) will be proved using different ideas. For Theorem (1.1.11)(i), the idea is to use bifurcation techniques and to show that for certain (a, b), by perturbing the radial solution $v_a va$ given in (10), there are directions in which the energy decreases. Since $S(a, b)$ is achieved, the minimizer cannot be radial. This approach has been used for other problems, for example, for bifurcation of positive solutions on annular domains in [21]. On the other hand, for Theorem (1.1.11)(ii), we shall employ an idea in [5] by Brezis and Nirenberg (in which they studied a problem with a nearly critical exponent on annular domains) to compare the radial least energy and $S(a, b)$. A continuity argument then gives the conclusion.

We first give the proof of Theorem (1.1.11)(i). We work in $H^1(\mathcal{C})$ here. The linearization of (11) at the radial solution v_a decomposes by separation of variables into infinitely many ODEs. Denote by $\alpha_k = k\,(N-1+k)$ the k^{th} eigenvalue of $-\Delta_\theta$ on $\mathbb{S}^{N-1}$ .For $k \geq 0$, we denote by μ_k and f_k the first eigenvalue and the corresponding (positive) eigenfunction in the eigenvalue problem of μ ,

$$-f_{tt} + \lambda^2 f + \alpha_k f - (p-1)v_a^{p-2} f = \mu f.$$

This eigenvalue problem is well defined since $v_a(t) \to 0$ as $|t| \to \infty$. First, we show that there area and a function $a < h(a) < a+1$ defined for $a < a_0$ such that $a < a_0$ and $a < b < h(a)$ imply $\mu_1 < 0$. Indeed,

$$\mu_k = \inf_{f \in H^1(\mathcal{C})/\{0\}} \frac{\int_{\mathbb{R}} f_1^2 + \lambda^2 f^2 + \alpha_k f^2 - (p-1)v_a^{p-1} f^2 dt}{\int_{\mathbb{R}} f^2 dt}.$$

We use v_a as a test function, and since

$$\int_{\mathcal{C}} v_{a,t}^2 + \lambda^2 v_a^2 d\mu = \int_{\mathcal{C}} v_a^p d\mu,$$

we obtain

$$\mu_k \leq -(p-2)\frac{\int_{\mathcal{C}} v_a^p d\mu}{\int_{\mathcal{C}} v_a^2 d\mu} + \alpha_k. \tag{32}$$

Since $\alpha_0 = 0$ clearly we have $\mu_0 < 0$. We also have $\alpha_1 = N-1$, and by a direct calculation using (19), (32) gives

$$\mu_1 \leq -\frac{N(1+a-b)(N-2-2a)^2}{\big(N-2(1+a-b)\big)(N-(1+a-b))} + N - 1. \tag{33}$$

Note that the right-hand side in (33) is negative for

$$a < a_0 := \frac{N-2}{2} - \frac{N-2}{2}\sqrt{\frac{N-2}{N}} \tag{34}$$

and

$$a \le b < h(a) := 1 + a - \frac{2N}{l(a) + \sqrt{l^2(a) - 8}}, \tag{35}$$

where

$$l(a) = \frac{(N-2-2a)^2}{N-1} + 3.$$

Hence μ_1 is negative for a and b in this range. Note also that $a + 1 - h(a) \to 0$ as $a \to -\infty$. The a_0 and $h(a)$ above will be shown to have the property stated in Theorem (1.1.11)(i).

Define $w_k = \phi_k(\theta) f_k$, where ϕ_k is an eigenfunction of $-\Delta_\theta$ on $\mathbb{S}^{N-1}$ with eigenvalue α_k . $(\phi_0(\theta))$ is just a positive constant and $\phi_1(\theta)$ is a first harmonic.) We get

$$-\Delta w_k + \lambda^2 w_k - (p-1) v_a^{p-2} w_k = \mu_k w_k. \tag{36}$$

We now have the following:

Theorem (1.1.11) [1]: (Symmetry Breaking) (i) There is $a_0 \le 0$ and a function $h(a)$ defined for $a \le a_0$, satisfying $h(a_0) = a_0$, $a < h(a) < a + 1$ for $a < a_0$, and $a + 1 - h(a) \to 0$ as $-a \to \infty$ such that for any (a, b) satisfying $a < a_0$ and $a < b < h(a)$, the minimizer for $S(a, b)$ is nonradial.

(ii) There is an open subset H inside the a-negative region containing the set $\{(a, a) \in \mathbb{R}^2 : a < 0\}$ such that for any $(a, b) \in H$ with $a < b$, the minimize for $S(a, b)$ is nonradial.

Though the minimizers may be nonradial, we still have the following:

Proof :

(i) By the above lemma, for s small $|v_a + \delta(s) w_0 + s w_1|_{L^p} = 1$. Then (37)shows $S(a, b) < R(a, b)$. Since $S(a, b)$ is achieved, the minimizer is nonradial.

(ii): First we note that by a direct computation using (21) we always have for all $a < 0$

$$R(a, a) > S(a, a) = S(0,0).$$

We argue that for any $a_0 < 0$, there is $\varepsilon_0 > 0$ such that for all $|(a, b) - (a_0, a_0)| < \varepsilon_0$ with $a < b$, $S(a, b)$ is achieved by a nonradial function. As $(a, b) \to (a_0, a_0)$, we have that $R(a, a) \to R(a_0, a_0) > S(a_0, a_0) = S(0,0)$. On the other hand, from Theorem (1.1.7)(i) we have that $S(a, b) \to S(a_0, a_0)$ as $(a, b) \to (a_0, a_0)$ Therefore for any $a_0 < 0$, there is $\varepsilon_0 > 0$ such that $S(a, b) < R(a, b)$ if $|(a, b) - (a_0, a_0) < \varepsilon_0|$ with $a \le b$. By Theorem (1.1.9)(i), $S(a, b)$ is achieved, and due to the strict inequality, the minimizer for $S(a, b)$ is nonradial.

Lemma (1.1.12)[1]: For s small, there is $\delta = \delta(s)$ such that $\delta(0) = \acute{\delta}(0) = 0$ and

$$\int_{\mathcal{C}} |v_a + \delta(s) w_0 + s w_1|^p \, d\mu = 1.$$

If, in addition, (a, b) is such that $\mu_1 < 0$ (which holds for $a < a_0$ and $a \le b < h(a)$, then for s sufficiently small,

$$F(v_a + \delta(s) w_0 + s w_1) < F(v_a). \tag{37}$$

Proof: Set

$$G(\delta, s) = \int_{\mathcal{C}} |v_a + \delta w_0 + s w_1|^p \, d\mu.$$

We have $G(0,0) = 1$ and $\frac{\partial G}{\partial \delta}(0,0) = p \int_{\mathcal{C}} v_a^{p-1} w_0 d\mu > 0$, since $w_0 > 0$. By the implicit function theorem, there is an open s-interval around 0 where $\delta = \delta(s)$ differentiable and

$$G(\delta(s), s) = 1. \tag{38}$$

Furthermore, by a direct computation and $\phi_1(-\theta) = -\phi_1(\theta)$, we have

$$\frac{\partial G}{\partial s}(0,0) = p \int_{\mathcal{C}} v_a^{p-1} w_1 d\mu = p \int_{\mathcal{C}} v_a^{p-1} \phi_1(\theta) f_1 d\mu = 0.$$

Differentiating (38) we get

$$\frac{\partial G}{\partial \delta}(\delta(s), s)\acute{\delta}(s) + \frac{\partial G}{\partial s}(\delta(s), s) = 0. \tag{39}$$

Hence

$$\frac{\partial G}{\partial \delta}(0,0)\acute{\acute{\delta}}(0) + \frac{\partial^2 G}{\partial s^2}(0,0) = 0.$$

We have

$$\frac{\partial^2 G}{\partial s^2}(0,0) = p(p-1) \int_{\mathcal{C}} u_a^{p-1} w_1^2 d\mu \quad and \quad \frac{\partial G}{\partial \delta}(0,0) = p \int_{\mathcal{C}} u_a^{p-1} w_0 d\mu.$$

Thus,

$$\acute{\acute{\delta}}(0) = \frac{p(p-1) \int_{\mathcal{C}} v_a^{p-1} w_1^2 d\mu}{p \int_{\mathcal{C}} v_a^{p-1} w_0 d\mu}.$$

Now,

$$F(v_a + \delta(s) w_0 + s w_1) =$$

$$F(v_a) + s^2 \int_{\mathcal{C}} |\nabla w_1|^2 + \lambda^2 w_1^2 \, d\mu + 2\delta(s) \int_{\mathcal{C}} \nabla v_a \cdot \nabla w_0 + \lambda^2 v_a w_0 d\mu$$

$$+2s \int_{\mathcal{C}} \nabla v_a \cdot \nabla w_1 + \lambda^2 v_a w_1 d\mu + \delta^2(s) \int_{\mathcal{C}} |\nabla w_0|^2 + \lambda^2 w_0^2 \, d\mu$$

$$+2s\delta(s) \int_{\mathcal{C}} \nabla w_0 \cdot \nabla w_1 + \lambda^2 w_0 w_1 d\mu.$$

Since v_a is radial,

$$\int_{\mathcal{C}} \nabla v_a \cdot \nabla w_1 + \lambda^2 v_a w_1 d\mu = \int_{\mathcal{C}} v_a^{p-1} w_1 d\mu = 0;$$

therefore the fourth term is 0. Also, the fifth and the sixth terms are higher order. Hence $F(v_a + \delta(s) w_0 + s w_1) =$

$$F(v_a) + s^2 \int_{\mathcal{C}} |\nabla w_1|^2 + \lambda^2 w_1^2 \, d\mu + 2\delta(s) \int_{\mathcal{C}} \nabla v_a \cdot \nabla w_0 + \lambda^2 v_a w_0 d\mu + o(s^2).$$

From (36) we get

$$\int_{\mathcal{C}} |\nabla w_1|^2 + \lambda^2 w_1^2 \, d\mu = (p-1) \int_{\mathcal{C}} v_a^{p-1} \, w_1^2 d\mu + \mu_1 \int_{\mathcal{C}} w_1^2 \, d\mu.$$

Since v_a is a solution of (11), we have

$$\int_{\mathcal{C}} \nabla v_a \cdot \nabla w_0 + \lambda^2 v_a \, w_0 d\mu = \int_{\mathcal{C}} v_a^{p-1} \, w_0 d\mu.$$

Using the equalities above and

$$\delta(s) = -s^2 \frac{(p-1) \int_{\mathcal{C}} v_a^{p-2} \, w_1^2 d\mu}{p \int_{\mathcal{C}} v_a^{p-1} \, w_0 d\mu} + o(s^2),$$

we obtain for s sufficiently small

$$F(v_a + \delta(s) w_0 + s w_1) = F(v_a) + s^2 \mu_1 \int_{\mathcal{C}} w_1^2 \, d\mu + o(s^2) < F(v_a).$$

We use the moving plane method [15] to show that for $a \le b < a+1$ any positive solution of (11) on the cylinder $\mathcal{C}$ is symmetric about some $t = \text{const}$, so up to a translation in the t-direction, the solution is even in t and satisfies the monotonicity property. Together with the discussion , we get that any solution of (8) satisfying $u(x) > 0$ for $x \in \mathbb{R}^N/\{0\}$, up to a dilation (12), satisfies the modified inversion symmetry in Theorem (1.1.14).Our argument follows closely the method in [11] though we have a differential equation defined on a manifold $\mathcal{C}$, while in [11] equations in $\mathbb{R}^N$ were treated.

Let v be a positive solution of (11). For $\mu < 0$ and $x = (t, \theta) \in \mathcal{C}$, denote $x^\mu = (2\mu - t, \theta) \in \mathcal{C}$, the reflection of x relative to the hyperplane $t = \mu$. We let

$$w_\mu(x) = v(x^\mu) - v(x),$$

a function defined on the region$\sum_\mu = \{(t,\theta) \in \mathcal{C} : t < \mu\}$. Clearly, $w(x) = 0$ for any $x \in T_\mu = \partial \sum_\mu = \{(t, \theta) \in \mathcal{C} : t = \mu\}$. We have the following :

Lemma (1.1.13)[1]: There is $R_0 > 0$ independent of μ such that if w_μ has a negative local minimum at (t_0, θ_0), then $|t_0| \le R_0$.

Proof: First, by elliptic regularity theory and the fact that

$$\int_{\tau \le t \le \tau+1} v^{2^*} \, d\mu \to 0 \qquad as \; |\tau| \to \infty,$$

we have $v(t,\theta) \to 0$ as $|\tau| \to \infty$. Then we take R_0 to be such that

$$v(t,\theta) < \left(\frac{\lambda^2}{p-1}\right)^{\frac{1}{p-2}}$$

for all $|t_0| \ge R_0$. Since v is a solution of (11),w_μ satisfies

$$-\Delta w_\mu + \lambda^2 w_\mu - a(x) w_\mu = 0 \tag{40}$$

in $\sum_\mu$, where

$$a(x) = (p-1) \int_0^1 \left[u(x) + s\big(u(x^\mu) - u(x)\big)\right]^{p-1} ds.$$

Assume $x_0 = (t_0, \theta_0) \in \sum_\mu$ is a minimum such that $w_\mu(x_0) < 0$ and $|t_0 > R_0|$. Then

$$v(x_0^\mu) < v(x_0) < \left(\frac{\lambda^2}{p-1}\right)^{\frac{1}{p-2}}.$$

Therefore

$$a(x_0) < \lambda^2. \tag{41}$$

Since $\Delta w_\mu(x_0) \geq 0$, we obtain

$$\lambda^2 w_\mu(x_0) - a(x_0)w_\mu(x_0) \geq 0,$$

which means $\lambda^2 \leq a(x_0)$, contradicting (41).

We shall need the following:

Theorem (1.1.14)[1]: (Symmetry Property) For $a \leq b < a+1$, any bound state solution u of (8) in $\mathcal{D}_a^{1,2}$ satisfying $u(x) > 0$ for $x \in \mathbb{R}^N/\{0\}$, possibly after a dilation $u(x) \to \tau^{(N-2-2a)/2}u(\tau x)$, satisfies the "modified inversion" symmetry:

$$u\left(\frac{x}{|x|^2}\right) = |x|^{N-2-2a}u(x).$$

Moreover, writing $|x| = e^{-t}$ and $\theta = x/|x|$, we have that for fixed θ,

$$e^{-\frac{N-2-2a}{2}}u(e^{-t}\theta)$$

is even in t and monotonically decreasing in t for $t > 0$.

Proof: Since for $t \to \infty$ we have $w_\mu(t,\theta) \to 0$ and $w(x) = 0$ for all $x \in T_\mu$, Lemma (1.1.13) implies $w_\mu(x) \geq 0$ for $x \in \sum_\mu$ with all $\mu \leq -R_0$. Let μ_0 be the largest μ with the property that w_μ is nonnegative on $\sum_\mu$. Clearly such μ_0 exists since $v(t,\theta) \to 0$ as $t \to \infty$. We argue that

(i) $\quad w_\mu(x) > 0$ for $x \in \sum_\mu$, $\mu < \mu_0$.

(ii) $\quad w_{\mu_0} \equiv 0$ on $\sum_{\mu_0}$.

Since $w_\mu \geq 0$ for all $\mu < \mu_0$, it follows that $v_t \geq 0$ for all $t < \mu_0$. To prove (i) , assume there is $\delta > 0$ such that for some (t_0,θ_0), we have $t_0 < \mu_0 - \delta$ and $w_{\mu_{0-\delta}}(t_0,\theta_0) = 0$. By the maximum principle it follows that $w_{\mu_{0-\delta}} \equiv 0$. This implies that $v(\mu_0 - 2\delta,\theta_0) = v(\mu_0,\theta_0)$. Since $\partial v/\partial t \geq 0$, it follows that

$$\frac{\partial v}{\partial t}(t,\theta_0) = 0 \quad for\ all\ \ t \in [\,\mu_0 - 2\delta, \mu_0\,].$$

There fore

$$\frac{\partial w_{\mu_{0-2\delta}}}{\partial t}(\,\mu_0 - 2\delta, \mu_0\,) = 0.$$

By the Hopf lemma we get $w_{\mu_{0-2\delta}} \equiv 0$. Continuing in this fashion, we obtain that v is independent of t, which is not possible. Therefore, $\partial w_\mu/\partial t < 0$ on T_μ for $\mu < \mu_0$ and then $v_t > 0$ on $\sum_\mu$.

For (ii), assume $w_{\mu_0} \not\equiv 0$. By the maximum principle and the Hopf lemma, $w_{\mu_0} > 0$ on $\sum_{\mu_0}$ and $\partial w_{\mu_0}/\partial t < 0$ on T_{μ_0}. From the definition of μ_0, there is a sequence $\mu_k \searrow \mu_0$ and there are point $x_k \in \sum_{\mu_k}$, absolute minima for w_{μ_k} , such that $w_{\mu_k}(x_k) < 0$. By Lemma (1.1.13) we have that x_k is a bounded sequence; hence (by passing to a subsequence) we can assume it converges to some point x_0. It follows that $x_0 \in T_{\mu_0}$ and $w_{\mu_{0,t}}(x_0) = 0$, which is a contradiction.

Eventually after a translation in the t-direction, we can assume μ_0. Therefore v is even in t and monotonically decreasing for $t > 0$.

Translations in t on $\mathcal{C}$ correspond to dilations in $\mathbb{R}^N$; hence up to a dilation $u(x) \to \tau^{\frac{N-2-2a}{2}}u(\tau x)$, positive solutions of (8) have the modified inversion symmetry as given in Theorem (1.1.14).

In one dimension, equation (8) becomes

$$-(|x|^{-2a}\acute{u}\,\acute{)} = |x|^{-bp}u^{p-1}, \quad u \geq 0, \quad in\, \mathbb{R}. \tag{42}$$

We have a rather complete answer for the problem. In fact, we can identify all solutions of (42). We look for solutions v that are critical points for the energy in $\mathcal{D}_a^{1,2}(\mathbb{R})$

$$E_{a,b}(u) = \frac{\int_{\mathbb{R}}|x|^{-2a}|\acute{u}|^2dx}{\left(\int_{\mathbb{R}}|x|^{-bp}|u|^p dx\right)^{2/p}}.$$

The parameter range is

$$a < -\tfrac{1}{2}, \quad a + \tfrac{1}{2} < b \leq a + 1, \quad and \quad p = \frac{2}{-1 + 2(b-a)}.$$

We first observe that $E_{a,b}(u)$is invariant under the following rather nonstandard dilations: for $(\tau_-, \tau_+) \in (0, \infty)^2$

$$u(x) \to u_{\tau_-,\tau_+}(x) = \begin{cases} \tau_-^{-\frac{1+2a}{2}}\, u(\tau_- x), & x < 0, \\ \tau_+^{-\frac{1+2a}{2}}\, u(\tau_+ x), & x > 0, \end{cases} \tag{43}$$

That is, dilations can be made independently for $x < 0$ and $x > 0$ so that $E_{a,b}(u)$ is still invariant.

Note that for $N = 1$ the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^0 = \mathbb{R} \cup \mathbb{R}$ the union of two real lines. We denote the two components by $\mathcal{C}_-$and $\mathcal{C}_+$corresponding to $\mathbb{R}_-$ and $\mathbb{R}_+$ in $\mathbb{R}$, respectively. The coordinates for $\mathcal{C}_-$and $\mathcal{C}_+$ are $y = (t, -1) \in \mathcal{C}_-$ and $y = (t, 1) \in \mathcal{C}_+$. For simplicity, we write them as t_1 (for $(t, -1)$) and t_2 (for $(t, 1)$). To be more precise, for a function $w(y)$defined on $\mathcal{C}$ we write $w(y) = w_1(t_1)$ when $y = t_1 \in \mathcal{C}_-$ and $w(y) = w_2(t_2)$ when $y = t_2 \in \mathcal{C}_+$. To a function $u \in \mathcal{D}_a^{1,2}(\mathbb{R})$, we associate a function w (corresponding to a pair of w_1, w_2) defined on $\mathcal{C}$ by

$$u(x) = (-x)^{(1+2a)/2}w_1(-\ln(-x)) \quad for\ x < 0, \tag{44}$$
$$u(x) = x^{(1+2a)/2}w_2(-\ln x) \quad for\ x > 0,$$

and $t_1 = -\ln|-x|$ for $x < 0$ and $t_2 = -\ln x$ for $x > 0$. Then equation (42) is equivalent to the system of autonomous equation: for $i = 1,2$,

$$-\frac{d^2w_i}{dt_i^2} + \left(\frac{1+2a}{2}\right)^2 w_i = |w_i|^{p-2}w_i. \tag{45}$$

Critical points of $E_{a,b}(u)$ on $\mathcal{D}_a^{1,2}(\mathbb{R})$ now correspond to critical points of a new energy functional on $H^1(\mathcal{C})$

$$F_{a,b}(w) = \frac{\int_{\mathcal{C}}|\nabla w|^2 + \left(\frac{1+2a}{2}\right)^2 |w|^2 d\mu}{\left(\int_{\mathcal{C}}|w|^p\, d\mu\right)^{2/p}}, \quad w \in H^1(\mathcal{C}).$$

It is easy to see that both integrals in the numerator and the denominator are decoupled as two integrals for w_1 and w_2 . Each of the two ODEs of (45) has the zero solution, and according to (15) with $\lambda = -(1+2a)/2$ the only (positive) homoclinic solutions are translates of

$$v(t)=\left(\frac{(1+2a)^2}{4\big(1-2(1+a-b)\big)}\right)^{\frac{1-2(1+a-b)}{4(1+a-b)}}\left(\cosh\ \frac{(1+2a)(1+a-b)}{1-2(1+a-b)}t\right)^{\frac{1-2(1+a-b)}{2(1+a-b)}} \quad (46)$$

The minimizers of $F_{a,b}(w)$are achieved by w, for which one of two components w_1 or w_2 is identically zero and the other is a translate of v_t given above. According to (17), the infimum is

$$S(a,b)=\frac{(-1-2a)^{2(b-a)}}{2^{2(1+a-b)}\big(-1+2(b-a)\big)^{-1+2(b-a)}(1+a-b)^{2(1+a-b)}}\left(\frac{\Gamma^2\left(\frac{1}{2(1+a-b)}\right)}{\Gamma\left(\frac{1}{1+a-b}\right)}\right)^{2(1+a-b)}. \quad (47)$$

We observe that as $b\searrow a+\frac{1}{2}$, we obtain $S(a,b)\to -1-2a$. Note that when both $w_1$and $w_1$are nonzero and are (possibly different) translates of $v(t)$ in (46) we get the energy $F_{a,b}(w)$ to be higher

$$R(a,b)=2^{2(1+a-b)}S(a,b),$$

which is the least energy in the radial class. On this energy level, there is a two-parameter family of positive solutions according to the two parameters that control by how much $w_1$and w_1 are translated from (46). Correspondingly ,$u(x)$ defined in (44) is a two-parameter family of solutions for (42), which after a dilation given by (43) for some $(\tau_-,\tau_+)\in(0,\infty)^2$ is radial in $\mathbb{R}$..

Summarizing all these, we can state the main results for $N=1$ now.

Theorem (1.1.15)[1]: (Best Constants and Nonexistence of Extremal Functions)

(i) $S(a,b)$ is continuous in the full parameter domain.

(ii) For $b=a+1$, we have $S(a,a+1)=\left(\frac{1+2a}{2}\right)^2$, and $S(a,a+1)$ is not achieved.

(iii) For $b\to\left(a+\frac{1}{2}\right)^+$, we get $S\to(a,b)-1-2a$.

Theorem (1.1.16)[1]: (Best Constants and Existence of Extremal Functions) For $a+\frac{1}{2}<b<a+1$, $S(a,b)$ is explicitly given in(47), and up to a dilation of the form (43) it is achieved at a function of the form(44)with either $w_1=0$ and w_2 given by (46), or vice versa. Consequently, the minimizer for $S(a,b)$ is never radial

Theorem (1.1.17)[1]: (Bound State Solutions and Symmetry) Up to a dilation (43), the only solution of (42) besides the ground state solutions is of the form of (44) with bothw1 andw2 given by(46). Consequently, all bound state solutions of (42), possibly after a dilation given in(43), satisfy the modified inversion symmetry.

The Case $N=2$

In this case the parameter range is

$$-\infty<a<0,\quad a<b\le a+1,\quad and\ p=\frac{2}{b-a}.$$

With no changes in the proofs for the case $N\ge 3$, we have the following results.

Theorem (1.1.18)[1]: (Best Constants and Nonexistence of Extremal Functions)

(i) $S(a,b)$ is continuous in the full parameter domain.

(ii) For $b=a+1$, we have $S(a,a+1)=a^2$,and $S(a,a+1)$ is not achieved.

Theorem (1.1.19)[1]: (Best Constants and Existence of Extremal Functions)

(i) For $a < b < a + 1$, $S(a, b)$ is always achieved.

(ii) For $b - a \in (0,1)$ fixed, as $a \to -\infty$, $S(a, b)$ is strictly increasing, and

$$S(a, b) = (-a)^{2(b-a)}\left[S_p(\mathbb{R}^2) + o(1)\right].$$

One notes in (34) that for $N = 2$ we have $a_0 = 0$. Therefore we also have the following:

Theorem (1.1.20)[1]: (Symmetry Breaking)There is a function $h(a)$ defined for $a < 0$, satisfying $a < h(a) < a + 1$ for $a < 0$ and $a = 1 - h(a) \to 0$ $as - a \to \infty$, such that for any (a, b) satisfying $a < 0$ and $a < b < h(a)$, the minimizer for $S(a, b)$ is nonradial.

Theorem (1.1.21)[1]: (Symmetry Property) For $a < b < a + 1$, the minimizer of $S(a, b)$, possibly after a dilation $u(x) \to \tau^{-a}u(\tau x)$, satisfies the modified inversion symmetry:

$$u\left(\frac{x}{|x|^2}\right) = |x|^{-2a}\, u(x).$$

we shall consider a related problem that can be solved by using our method and the results. For $0 \le a < (N - 2)/2$, special cases of the following problem were considered in [23] and [26]:

For $N \ge 3$, we consider the following problem:

$$-div(|x|^{-2a}\nabla w) + \gamma|x|^{-2(1+a)}w = |x|^{-bp}w^{p-1},\ \ u \ge 0,\ \ in\ \mathbb{R}^N, \tag{48}$$

where

$$a < \frac{N-2}{2},\quad a \le b < a + 1,\qquad \gamma > -\left(\frac{N-2-2a}{2}\right)^2,$$

$$p = \frac{2N}{N - 2 + 2(b - a)}.$$

The solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ of this problem are critical points of

$$E_{a,b,\gamma}(u) = \frac{\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2 + \gamma|x|^{-2(1+a)}u^2dx}{\left(\int_{\mathbb{R}^N}|x|^{-bp}|u|^p dx\right)^{2/p}}.$$

Proposition (1.1.22)[1]: The solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ of (48) are in one-to-one correspondence to solutions in $\mathcal{D}_{\bar{a}}^{1,2}(\mathbb{R}^N)$ of

$$-div(|x|^{-2\bar{a}}\nabla u) = |x|^{\bar{b}p}\, u^{p-1},\ \ u \ge 0,\qquad in\ \mathbb{R}^N,$$

where

$$\bar{a} = a + \lambda - \sqrt{\lambda^2 + \gamma},\quad \bar{b} = b + \lambda - \sqrt{\lambda^2 + \gamma},\quad \lambda = \frac{N-2-2a}{2}.$$

This correspondence is given by

$$u(x) = |x|^{\lambda - \sqrt{\lambda^2+\gamma}}w(x).$$

Direct computations verify the proof, which we omit here.

Due to this proposition, we can put equation (48) in the same frame of work as in (8), and we can translate all of our results for (8) to get corresponding results for (48). We note that even in the a-nonnegative region, for γ sufficiently large the minimizer of $E_{a,b,\gamma}(u)$ is nonradial. All of our main theorems are adapted in the obvious way. We leave the statements of these results to the reader.

Section (1.2): Applications to Nonlinear Diffusions

For $d \geq 3$, Sobolev's inequality [66] states the existence of a constant $A > 0$ suchthat for any function $u \in L^{2d/(d-2)}(\mathbb{R}^d)$ with $\nabla u \in L^2(\mathbb{R}^d)$,

$$\|w\|_{\frac{2d}{d-2}} \leq A\|\nabla w\|_2. \tag{49}$$

Here and in what follows, we define for $q > 0$

$$\|\upsilon\|_q = \left(\int_{\mathbb{R}^d} |\upsilon|^q dx\right)^{1/q}.$$

The value of the optimal constant is known to be

$$A = \frac{1}{\sqrt{\pi d(d-2)}}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{1/d}$$

as established by Aubin and Talenti in [32], [24]. This optimal constant is achieved precisely by constant multiples of the functions

$$w_{\sigma,\bar{x}}(x) = \left(\frac{1}{\sigma^2 + |x-\bar{x}|^2}\right)^{\frac{d-2}{2}}.$$

with $\sigma > 0, \bar{x} \in \mathbb{R}^d$. On the other hand, a celebrated logarithmic Sobolev inequality was found in 1975 by Gross [49]. In the case of Lebesgue measure it states that all functions $w \in H^1(\mathbb{R}^d), d \geq 2$ satisfy for any $\sigma > 0$

$$\int_{\mathbb{R}^d} w^2 \log(w^2/\|w\|_2^2)\, dx + d(1 + \log(\sqrt{\pi}\,\sigma))\|w\|_2^2 \leq \sigma^2\|\nabla w\|_2^2. \tag{50}$$

The extremals of this inequality (which is not stated here in a scaling invariant form) are constant multiples of the Gaussians:

$$w(x) = (\pi\sigma^2)^{-d/4} e^{-\frac{|x-\bar{x}|^2}{2\sigma^2}}, \tag{51}$$

with $\bar{x} \in \mathbb{R}^d$ [42], [67]. We will answer the naturally arising question of how these two classical inequalities are related. As we will see, these inequalities correspond to limiting cases of a one-parameter family of optimal Gagliardo–Nirenberg type inequalities [48], [61] which we shall describe next.

For $p > 0$, we define:

$$\mathcal{D}^p(\mathbb{R}^d) = \{w \in L^{1+p}(\mathbb{R}^d): \nabla w \in L^2(\mathbb{R}^d)\ and\ |w|^{2p} \in L^1(\mathbb{R}^d)\}.$$

Our first main result states the validity of the following optimal Gagliardo–Nirenberg inequality.

$$\|w\|_{2p} \leq A\|\nabla w\|_{p+1}^{1-\theta}, \tag{52}$$

where

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{b2}} \left(\frac{\Gamma(y)}{\Gamma(y-d/2)}\right)^{\frac{\theta}{d}},$$

with

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p}, \qquad y = \frac{p+1}{p-1}.$$

A is optimal, and (52)is reached with equality if and only ifwis a constant multiple of one of the functions

$$w_{\sigma,\bar{x}}(x) = \left(\frac{1}{\sigma^2 + |x-\bar{x}|^2}\right)^{\frac{1}{p-1}},$$

with $\sigma > 0$ and $\bar{x} \in \mathbb{R}^d$.

An analogous estimate takes place in the case $0 < p < 1$. In fact we have the following result.

$$\|w\|_{p+1} \leq A \|\nabla w\|_2^{\theta} \|w\|_{2p}^{1-\theta}, \tag{53}$$

where

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y}{2y+d}\right)^{\frac{1-\theta}{2p}} \left(\frac{\Gamma(d/2+1+y)}{\Gamma(1+y)}\right)^{\frac{\theta}{d}},$$

with

$$\theta = \frac{d(1-p)}{(1+p)(d-(d-2)p)}, \qquad y = \frac{p+1}{1-p}.$$

A is optimal, and (53)is reached with equality by the compactly supported functions

$$w_{\sigma,\bar{x}}(x) = (\sigma^2 + |x-\bar{x}|^2)_+^{\frac{1}{1-p}},$$

with $\sigma > 0$ and $\bar{x} \in \mathbb{R}^d$.

The above results are special cases of Gagliardo–Nirenberg inequalities, which are found here in optimal form. Theorem (1.2.3) contains the optimal Sobolev inequality when $p = \frac{d}{d-1}$. Moreover, it provides a direct proof of the Gross–Sobolev inequality with an optimal constant as $p \downarrow 1$. In fact, taking the logarithm of both sides of inequality (53) for any $w \in H^1(\mathbb{R}^d)$, we get

$$\frac{1}{\theta} \log\left(\frac{\|w\|_{2p}}{\|w\|_{p+1}}\right) \leq \frac{1}{\theta} \log A + \log\left(\frac{\|\nabla w\|_2}{\|w\|_{p+1}}\right).$$

Using that $\theta \sim \frac{d}{4}(p-1)$ as $p \downarrow 1$, we get then

$$\frac{2}{d} \int_{\mathbb{R}^d} \left(\frac{w}{\|w\|_2}\right)^2 \log\left(\frac{w}{\|w\|_2}\right) dx \leq \lim_{p\downarrow 1} \frac{1}{\theta} \log A + \log\left(\frac{\|\nabla w\|_2}{\|w\|_2}\right).$$

Since $\lim_{p\downarrow 1} A = 1$, it is enough to compute $\lim_{p\downarrow 1} \frac{A-1}{\theta}$. For that purpose, we choose for A the extremal function:

$$w_p(x) = \left(1 + \frac{p-1}{2}|x|^2\right)^{-\frac{1}{p-1}},$$

which converges to

$$e^{-\frac{|x|^2}{2}} = w_1(x) \quad as\ p \downarrow 1.$$

Thus

$$\lim_{p\downarrow 1} \frac{A-1}{\theta} = -\log\left(\frac{\|\nabla w_1\|_2}{\|w_1\|_2}\right) + \frac{4}{d} \lim_{p\downarrow 1} \frac{1}{p-1}\left(\frac{\|w_p\|_{2p}}{\|w_p\|_{p+1}}\right) = I + II.$$

Now,

$$II = \frac{2}{d} \int_{\mathbb{R}^d} \left(\frac{w_1}{\|w_1\|_2}\right)^2 \log\left(\frac{w_1}{\|w_1\|_2}\right) dx + III - IV,$$

where

$$III = \lim_{p\downarrow 1} \tfrac{1}{p-1} \log\left(\frac{\|w_p\|_{2p}}{\|w_1\|_{2p}}\right) \quad and \quad IV = \lim_{p\downarrow 1} \tfrac{1}{p-1} \log\left(\frac{\|w_p\|_{p+1}}{\|w_1\|_{p+1}}\right)$$

A straightforward computation yields

$$\lim_{p\downarrow 1} \frac{1}{p-1} \int_{\mathbb{R}^d} (w_p^{2p} - w_1^{2p}) dx = \lim_{p\downarrow 1} \frac{1}{p-1} \int_{\mathbb{R}^d} (w_p^{p+1} - w_1^{p+1}) dx = \tfrac{1}{4} \int_{\mathbb{R}^d} e^{-|x|^2} |x|^4 dx.$$

It follows that III−IV= 0, hence

$$\lim_{p\downarrow 1} \tfrac{A-1}{\theta} = -\log\left(\frac{\|\nabla w_1\|_2}{\|w_1\|_2}\right) + \tfrac{2}{d} \int_{\mathbb{R}^d} \frac{w_1^2}{\|w_1\|_2^2} + \log\left(\frac{w_1^2}{\|w_1\|_2^2}\right) dx$$
$$= \tfrac{1}{2} \log\left(\frac{2}{\pi d e}\right),$$

using the facts

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx = \pi^{\frac{d}{2}} \quad and \quad \int_{\mathbb{R}^d} e^{-|x|^2} |x|^2 dx = \tfrac{d}{2} \pi^{\frac{d}{2}}.$$

We have then reached the inequality

$$\int_{\mathbb{R}^d} \frac{w^2}{\|w\|_2^2} \log\left(\frac{w^2}{\|w\|_2^2}\right) dx \le \tfrac{d}{2} \log\left(\frac{2\|\nabla w\|_2^2}{\pi d e \|w\|_2^2}\right), \tag{54}$$

for any $w \in H^1(\mathbb{R}^N)$. But this inequality is precisely that obtained from (50), when optimizing in $\sigma > 0$. This inequality is the form of the logarithmic Sobolev inequality which is invariant under scaling [70], [56]. As a consequence, optimal functions for (54) are any of the Gaussians given by (51) with $\sigma > 0$, $\bar{x} \in \mathbb{R}^d$. We may also notice that, as a subproduct of the above derivation of (54),this inequality holds with optimal constants. See (54).

As an application of these optimal inequalities, we will derive some new results for the asymptotic behavior of solutions to the Cauchy problem:

$$u_t = \Delta u^m, \quad t > 0, \; x \in \mathbb{R}^d, \tag{55}$$
$$u(0,x) = u_0(x) \ge 0, \quad u_0 \in L^1(\mathbb{R}^d). \tag{56}$$

When $m > 0, m \ne 1$, this problem has been extensively studied. The case $m > 1$ is the so-called porous medium equation. When $0 < m < 1$ it is usually referred to as the fast diffusion equation . Both for $m > 1$ and for $0 < m < 1$, this problem is known to be well posed in weak sense. Moreover, it preserves mass whenever $m > \frac{d-2}{d}$, in the sense that $\int_{\mathbb{R}^d} u(x,t) dx$ is constant in $t > 0$. When $\frac{d-2}{d} < m < 1$, solutions are regular and positive for $t > 0$ [50], but this is no longer true when m is below this threshold: for instance, finite time vanishing may occur as simple examples show. For $m > 1$, solutions are at least Hölder continuous.

The qualitative behavior of solutions to these problems has been the subject of a large number. Since mass is preserved, it is natural to ask whether a scaling brings the solution into a certain universal profile as time goes to infinity. This is the case and the role of the limiting profiles is played by an explicit family of self-similar solutions known as the Barenblatt–Prattle solutions [34], characterized by the fact that their initial data is a Dirac mass. These solutions remain invariant under the scaling $u_\lambda(t,x) = \lambda^{d\alpha} u(\lambda^\alpha x, \lambda t)$ with

$\alpha = (2 - d(1-m))^{-1} > 0$, which leaves the equation invariant. They are explicitly given by:

$$U(t,x) = t^{-d\alpha} \cdot v_\infty\left(\frac{x}{t^\alpha}\right) \quad with\, v_\infty(x) = \left(\sigma^2 - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}} \quad , \tag{57}$$

provided $m > \frac{d-2}{d}, m \neq 1$. These solutions have a constant mass uniquely determined by the parameter σ.

If σ is chosen so that the mass of Ucoincides with that of u_0, it is known that the asymptotic behavior ofuitself is well described by U as $t \to +\infty$. This phenomenon was first rigorously described by Friedman and Kamin of $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, both in the cases $m > 1$ and $(d-2)/d < m < 1$ [47]. These results have been later improved and extended by. Vázquez and Kamin [52], [53]. Also see [69]. Thus far it is well known that if $u_0 \in L^1(\mathbb{R}^d)$ and either $m > 1$ or $(d-2)/d < m < 1$, then

$$\lim_{t\to+\infty} \|u(t,\cdot) - U(t,\cdot)\|_1 = 0, \quad \lim_{t\to+\infty} t^{d\alpha} \|u(t,\cdot) - U(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = 0. \tag{58}$$

On the other hand, for the heat equation $(m = 1)$, the following fact is classical:

$$\lim_{t\to+\infty} \sup \sqrt{t} \cdot \|u(t,\cdot) - U(t,\cdot)\|_{L^1(\mathbb{R}^d)} < +\infty,$$

with

$$U(t,x) = (2\pi t)^{-d/2} \|u_0\|_{L^1(\mathbb{R}^d)} e^{\frac{|x|^2}{2t}}.$$

Our next result extends the above asymptotic behavior to the range $\frac{d-2}{d} \le m < 2$ using an appropriate Lyapunov functional.

The main tool in deriving the above result turns out to be the optimal inequalities of Theorems (1.2.3) and (1.2.4), which are showing. We derive some further consequences of independent interest, including the key estimate for the show of Theorem (1.2.12), which we carry out.

The question of optimal constants has been the subject. In the case of critical Sobolev injections and scaling invariant inequalities with weights (Hardy– Littlewood–Sobolev and related inequalities), apart from [32], [24], one has to cite the remarkable explicit computation by Lieb [30] and various results based on concentration compactness methods [23], but the optimality of the constants in Gagliardo–Nirenberg inequalities (see [57] for an estimate) is a long standing question to which we partially answer here. The special case of Nash's inequality [59] has been solved by Carlen and Loss in [43]. This case, as well as Moser's inequality [58], does not enter in the subclass that we consider here, but it has the striking property that the minimizers are compactly supported, as in Theorem (1.2.2). For more details on the connection between Nash's inequality and the logarithmic Sobolev inequality, see [37], we will establish the validity of Theorems (1.2.3) and (1.2.4), and derive some consequences that will be useful for later purposes. First, in order to treat the case $p > 1$ of Theorem (1.2.3), we will establish Theorem (1.2.1) (which is actually equivalent). Let us consider the functional:

$$G(w) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2\, dx + \frac{1}{1+p}\int_{\mathbb{R}^d} |w|^{1+p}\, dx.$$

We define the minimization problem:

$$I_\infty \equiv \inf_{w\in X} G(w)$$

over the set X of all nonnegative functions $w \in \mathcal{D}^p(\mathbb{R}^d)$ that satisfy the constraint

$$\frac{1}{2p}\int_{\mathbb{R}^d}|w|^{2p}\,dx = J_{\infty,} \tag{59}$$

where for convenience we make the choice:

$$J_\infty := \frac{\pi^{d/2}}{2p}\left(\frac{2p}{d-p(-2)}\right)^{y+1}\frac{(d-y-1)^d}{p^{d/2}}\,\frac{\Gamma(y+1-d/2)}{\Gamma(\mathrm{y}+1)}$$

with $y=(p+1)/(p-1)$.The following result characterizes the minimizers of I_∞.

Theorem (1.2.1)[29]: Assume that $p>1$ and $p<\frac{d}{d-2}$ if $d\geq 3$.Then I_∞ is achieved . Moreover, for any minimize $\bar{w}\in X$, there exists $\bar{x}\in\mathbb{R}^d$ such that

$$\bar{w}(x)=\left(\frac{a}{b+|x-\bar{x}|^2}\right)^{\frac{1}{p-1}}\quad \forall x\in\mathbb{R}^d,$$

where

$$a = 2\,\frac{2p-d(p-1)}{(p-1)^2}\quad and\quad b=\frac{\big(2p-d(p-1)\big)^2}{p(p-1)^2}. \tag{60}$$

Proof: Using Sobolev's and Hölder's inequalities, it is immediately verified that $I_\infty>0$. For each $R>0$, we set B_R to be the ball centered at the origin with radius R and $X_R = X\cap H_0^1(B_R)$ (here we extend functions of H_0^1 outside of B_R by 0). Let us consider the family of infima

$$I_R=\inf_{w\in X_R} G(w);$$

I_R is decreasing with R. Besides, by density, $\lim_{R\to+\infty} I_R = I_\infty$. On the other hand, I_R is achieved since $p<\frac{d}{d-2}$ by some nonnegative, radially symmetric function w_R defined on B_R. The minimize w_R satisfies on B_R the equation:

$$-\Delta w_R + w_R^p = \mu_R w_R^{2p-1},$$

where μ_R is a Lagrange multiplier. Let us observe that

$$\int_{\mathbb{R}^d}|\nabla w_R|^2\,dx + \int_{\mathbb{R}^d}|w_R|^{1+p}\,dx = \mu_R\int_{\mathbb{R}^d}|w_R|^{2p}dx = 2p\mu_R\,J_\infty.$$

Thus

$$\frac{2p}{p+1}\mu_R\,J_\infty\leq I_R\leq p\mu_R\,J_\infty.$$

so that μ_R is uniformly controlled from above and from below as $R\to+\infty$, and converges up to the extraction of a subsequence to some limit $\mu_\infty>0$. Since $I_R$itself controls the H^1 norm of $w_R$over each fixed compact subset of B_R, from the equation satisfied by w_R and standard elliptic estimates, we deduce a uniform control over compacts in $C^{2,\alpha}$ norms. Passing to a convenient subsequence of $R\to+\infty$, we may then assume that w_R converges uniformly and in the C^2 sense over compact sets to a radial function w. We may also assume that $w_R\to w$ weakly in $L^{p+1}(\mathbb{R}^d)$ and $\nabla w_R\to\nabla w$ weakly in $L^2(\mathbb{R}^d)$. Besides, since w_R reaches its maximum at the origin, let us also observe from the equation that we get the estimate

$$I\leq\ \mu_R w_R^{p-1}(0).$$

This relation implies that $w_R$does not trivialize in the limit. The function w is thus a positive, radially decreasing solution of

$$-\Delta w + w^p = \mu_\infty w^{2p-1}.$$
in entire $\mathbb{R}^d$,and $w(|x|) \to 0$ as $|x| \to +\infty$. Now, since the convergence of w_R to w is uniform over compact sets, and w_R is radially decreasing, we may choose a sufficiently large, but fixed number ρsuch that on $\rho < |x| < R$, w_R satisfies an inequality of the form

$$-\Delta w + \frac{1}{2}w^2 \leq 0.$$

On the other hand, the fact that $p < \frac{d}{d-2}$ yields that the function

$$\zeta(x) = \frac{C}{|x|^{2/(p-1)}}$$

satisfies for any sufficiently large choice of C,

$$-\Delta w \zeta \frac{1}{2} \zeta^p \geq 0.$$

If we make this choice so that $w_R(\rho) < \zeta(\rho)$ for all large R, then by comparison we obtain that

$$w_R(x) < \frac{C}{|x|^{2/(p-1)}}, \quad |x| > \rho.$$

Now, if we notice that $\frac{2p}{p-1} > d$,then

$$\lim_{M\to+\infty} \sup_{R>M} \int_{M<|x|<R} |w_R|^{2p}\, dx = 0.$$

As a consequence, $w_R \to w$ strongly in $L^{2p}(\mathbb{R}^d)$. Hence $w \in X$ and since by weak convergence we have $G(w) \leq I_\infty$, the existence of a minimize is guaranteed.

The Lagrange multiplier μ_∞ is uniquely determined by the system:

$$\begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2\, dx \frac{1}{p+1}\int_{\mathbb{R}^d} |w|^{1+p}\, dx = I_{\infty,} \\ \int_{\mathbb{R}^d} |\nabla w|^2\, dx + \int_{\mathbb{R}^d} |w|^{1+p}\, dx = 2p\mu_\infty J_{\infty,} \\ \frac{d-2}{2d}\int_{\mathbb{R}^d} |\nabla w|^2\, dx \frac{1}{p+1}\int_{\mathbb{R}^d} |w|^{1+p}\, dx = \mu_\infty I_{\infty,} \end{cases}$$

which follows respectively from the definition of I_∞, and as a consequence of the equation multiplied by$w$and $(x \cdot \nabla w)$. The constantμ_∞therefore depends only on m , ρ and d.

Finally, let us consider any minimize w of G over X. It necessarily satisfies the equation

$$-\Delta w + w^p = \mu_\infty w^{2p-1}.$$

Ground state solutions of this equation are known to be radial around some point [15]. With no loss of generality, we take it to be the origin. On the other hand, there is a unique choice of a positive paramenter λ such that $\bar{w}(x) = \lambda^{2/(p-1)} w(\lambda x)$ satisfies

$$-\Delta \bar{w} + \bar{w}^p = \bar{w}^{2p-1}.$$

Invoking uniqueness results of positive solutions by Pucci and Serrin [63] and by Serrin and Tang for quasilinear elliptic equations [65], we deduce that the above equation has only one positive radial ground state. On the other hand, the function

$$\overline{w}(x) = \left(\frac{a}{b+|x|^2}\right)^{\frac{1}{p-1}},$$
where the values of a and b are precisely those given by (60), is an explicit solution, hence the unique one. Finally, the fact that

$$\int_{\mathbb{R}^d} w^{2p}\, dx = J_\infty$$
determines exactly what the value of λ is, in fact $\lambda = 1$. This ends to the show of Theorem (1.2.1).

Next we will state and show the analogue of Theorem (1.2.1) for the case $0 < p < 1$. We consider now the functional:

$$\tilde{G}(w) = \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2\, dx + \frac{1}{2p}\int_{\mathbb{R}^d} |w|^{2p}\, dx.$$
We shall denote by
$$\tilde{I}_\infty \equiv \inf_{w\in\tilde{X}} \tilde{G}\,(w)$$
the problem of minimizing $\tilde{G}$ over the class $\tilde{X}$ of all nonnegative functions w$w \in \mathcal{D}^p(\mathbb{R}^d)$ that satisfy the constraint

$$\tfrac{1}{p+1}\int_{\mathbb{R}^d} |w|^{p+1}\, dx = \tilde{J}_\infty,$$
where $\tilde{J}_\infty$ is now the number
$$\tilde{J}_\infty = \frac{\pi^{d/2}}{p+1}\left(\frac{2p}{d-p(d-2)}\right)^{1-y}\frac{(d+y-1)^d}{p^{d/2}}\frac{\Gamma(1+\mathrm{y})}{\Gamma(1+\mathrm{y}+\mathrm{d}/2)}$$
with $y = \frac{p+1}{1-p}$. Then we have the following result

Theorem(1.2.2)[29]: Assume that $0 < p < 1$. Then $\tilde{I}_\infty$ is achieved by the radially symmetric function
$$\overline{w}(x) = a^{-\frac{1}{1-p}}(b-|x|^2)_+^{\frac{1}{1-p}},$$
where a and b are given by(60)as in Theorem (1.2.1). Moreover, if $p > \frac{1}{2}$, for any minimize w, there exists $\tilde{x} \in \mathbb{R}^d$ such that $w(x) = \tilde{w}(x-\tilde{x}), \forall x \in \mathbb{R}^d$.

Proof: The proof goes similarly to that of Theorem (1.2.1). We consider the minimization problem on $\tilde{X}_R = \tilde{X} \cap H_1^0\,(B_R)$. By compactness, the minimizer is achieved. Moreover, using decreasing rearrangements, one finds that this minimizer $w_R$can be chosen radially symmetric and decreasing. It satisfies the equation
$$-\Delta w_R + w_R^{2p-1} = \mu_R w^p,$$
within the ball where w_R is strictly positive (we need to be careful with the fact that $2p - 1$ may be a negative quantity). Exactly the same analysis as above, yields that μ_R is uniformly controlled and approaches some positive mumber μ_∞. Moser's iteration provides us with a uniform L^∞ bound derived from the H^1 bound. We should observe at this point that the O.D.E. satisfied by w_R easily gives by itself an upper local estimate $C(R_0^2 - |x|^2)_+^{1/(1-p)}$ for some $C > 0$ in case the support corresponds to $|x| < R_0 \leq R$.If this is the case for some $R_0 > 0$, then the minimizer will be unchanged for any $R > R_0$ and in fact will be the solution of the minimization problem in $\mathbb{R}^d$. On the other hand, a

straightforward comparison with barriers of that type [44] actually yields that at some point the minimizer does get compactly supported in side B_R for all R sufficiently large. This minimizer is thus a ground state radial solution of

$$-\Delta w + w^{2p-1} = \mu_\infty w^p$$

and for the same reason as in the show of Theorem (1.2.1), μ_∞ is unique. According to the uniqueness results of Pucci and Serrin [63] and Serrin and Tang [65] again, such a radial minimizer is unique. A scaling argument (with $\overline{w}(x) = \lambda^{1/(p-1)} w(\lambda x)$) similar to the one employed in the show of Theorem (1.2.1) gives that $\mu_\infty = 1$ and w is then nothing but the explicit solution given in the statement of Theorem (1.2.2).

In case that $2p - 1 > 0$, it is known that all ground states are compactly supported and radially symmetric on each component of their supports [44]. We obtain then a complete classification of the minimizers as in Theorem (1.2.1). When $2p - 1 < 0$, the question arises of whether we do get out of the Euler–Lagrange equation a nice ground state solution, and whether such a solution is symmetric. This does not seem to be known.

We are now in a position to proceed with the proofs of Theorems (1.2.3) and (1.2.4).

Theorem (1.2.3)[29]: Let $d \geq 2$. If $p > 1$, and $p \leq \frac{d}{d-2}$ for $d \geq 3$, then for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$ the following inequality holds:

Proof : Let $w \in \mathcal{D}^p$ satisfy the constraint

$$J[w] := \frac{1}{2p} \int_{\mathbb{R}^d} |w(x)|^{2p}\, dx = J_\infty,$$

with J_∞ given in (59). For $\lambda > 0$, we consider the scaled function

$$w_\lambda(x) = \lambda^{\frac{d}{2p}} w(\lambda x),$$

which still satisfies $J[w_\lambda] = J_\infty$. Then for each $\lambda > 0$,

$$G(w_\lambda) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2\, dx \cdot \lambda^{d/p-(d-2)} + \frac{1}{1+p} \int_{\mathbb{R}^d} |w|^{1+p}\, dx \cdot \lambda^{-d(p-1)/2p} \geq I_\infty.$$

Minimizing the left hand side of the above expression in $\lambda > 0$ yields

$$C_*\left[\|\nabla w\|_2^\theta\, \|w\|_{p+1}^{1-\theta}\right]^\delta \geq I_\infty,$$

where

$$C_* = \frac{1}{2}\lambda_*^{d/p-(d-2)} + \frac{1}{p+1}\lambda_*^{-d(p-1)/2p}, \qquad \lambda_* = \frac{d}{d-p(d-2)}\,\frac{p-1}{p+1},$$

$$\delta = 2p\frac{d+2-(d-2)p}{4p-d(p-1)} \quad and \quad \theta = \frac{d(p-1)}{p(d+2-p(d-2))}.$$

Since $\|w\|_{2p} = 2pJ_\infty$, we may write:

$$\|\nabla w\|_2^\theta\, \|w\|_{p+1}^{1-\theta} \geq \left(\frac{I_\infty}{C_*}\right)^{1/\delta} \frac{\|w\|_{2p}}{(2pJ_\infty)^{1/(2p)}}.$$

Theorem (1.2.4)[29]: Let $d \geq 2$ and assume that $0 < p < 1$. Then for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$ the following inequality holds:

Proof: It is very similar to the proof of Theorem (1.2.3). For any $w \in \mathcal{D}^p$ satisfying the constraint

$$\tilde{J}[w] := \frac{1}{p+1} \int_{\mathbb{R}^d} |w(x)|^{p+1}\, dx = \tilde{J}_\infty$$

and for any $\lambda > 0$, we consider the scaling $w_\lambda(x) = \lambda^{d/(p+1)} w(\lambda x)$, which also satisfies $\tilde{J}[w_\lambda] = \tilde{J}_\infty$. Using now that $\tilde{G}[w_\lambda] \geq \tilde{J}_\infty$, we find, after optimizating on $\lambda > 0$,

$$\tilde{C}_*\left[\|\nabla w\|_2^{\theta}\ \|w\|_{2p}^{1-\theta}\right]^{\tilde{\delta}} \geq \tilde{I}_\infty,$$

where

$$\tilde{C} = \frac{1}{2}\lambda_*^{\frac{2d}{p+1}-(d-2)} + \frac{1}{2p}\lambda_*^{-d\frac{1-p}{p+1}}, \quad \lambda_* = \frac{p-1}{p}\frac{d}{d+2-p(d-2)},$$

$$\tilde{\delta} = \frac{(1+p)(d-(d-2)p)}{d+1-p(d-1)} \quad and \quad \theta = \frac{d(1-p)}{(1+p(d-(d-2)p)}.$$

Since $\|w\|_{1+p} = (p+1)\tilde{J}_\infty$, we may write:

$$\|\nabla w\|_2^{\theta}\ \|w\|_{2p}^{1-\theta} \geq \left(\frac{\tilde{I}_\infty}{\tilde{C}_*}\right)^{1/\tilde{\delta}} \frac{\|w\|_{p+1}}{((p+1)\tilde{J}_\infty)^{1/(p+1)}}.$$

By homogeneity and invariance under scaling, the above inequality is true for any $w \in \mathcal{D}^p$, with optimal constant

$$A = ((p+1)\tilde{J}_\infty)^{1/(p+1)} \left(\frac{\tilde{C}_*}{\tilde{I}_\infty}\right)^{1/\tilde{\delta}}.$$

See [33], [35], [36], [37], [38], [15], [65] [70].

Proposition (1.2.5)[29]: Let $d \geq 2$, $\tau > 0$ and $p > 0$ be such that $p \neq 1$, and $p \leq \frac{d}{d-2}$ if $d \geq 3$ .Then, for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$, the following inequality holds:

$$\frac{1}{2}\tau^{\frac{d}{p}-d+2}\|\nabla w\|_2^2 + \frac{\varepsilon}{p+1}\tau^{-d\frac{p-1}{2p}}\|w\|_{1+p}^{1+p} - \frac{\varepsilon}{2p}K\|\nabla w\|_{2p}^{\delta} \geq 0, \tag{61}$$

where ε is the sign of $(p-1)$,

$$\delta = 2p\frac{d+2-p(d-2)}{4p-d(p-1)}$$

and $K > 0$ is an optimal constant. For $p > \frac{1}{2}, p \neq 1$, optimal functions for inequality(61) are all given by the family of functions

$$x \mapsto \tau^{-\frac{d}{2p}}\,\overline{w}\left(\frac{x-\bar{x}}{\tau}\right).$$

For $0 < p \leq \frac{1}{2}$

, inequality(61)is also achieved by the same family of functions. Here

$$\overline{w}(x) = \left(\frac{a}{b+\varepsilon|x|^2}\right)_+^{\frac{1}{p-1}}$$

with a and b given by(60) (in both cases: $p > 1$ and $p < 1$) and K is explicitely given by(62) (see below).

Proof: Using the scaling

$$w \mapsto \tau^{-\frac{d}{2p}}\,w\left(\frac{\cdot}{\tau}\right),$$

it is clear that (61) holds for any $\tau > 0$ if and only if it holds at least for one. For $p > 1$, we take $\tau = 1$ and (61) is a direct consequence of the proof of Theorem (1.2.3), with $K = C_*\, A^{-\delta}$. The case $p < 1$ is slightly more delicate and we proceed as in the proof of Theorem (1.2.2). Let

$$w_\lambda(x) = \tau^{-\frac{d}{p+1}} w(\lambda x).$$

An optimization on $\lambda > 0$ of the quantity

$$\frac{1}{2}\tau^{\frac{d}{p}-d+2}\|\nabla w_\lambda\|_2^2+\frac{K}{2p}\|w_\lambda\|_{2p}^{\delta}$$
$$=\frac{1}{2}\|\nabla w\|_2^2\cdot\tau^{\frac{d}{p}-d+2}\lambda^{\frac{2d}{p+1}-d+2}+\frac{K}{2p}\|w_\lambda\|_{2p}^{\delta}\cdot\lambda^{d\frac{p-1}{p+1}\frac{\delta}{2p}}$$
shows that it is bounded from below by
$$K^{\frac{1}{2}\frac{4p-d(p-1)}{d-p(d-2)}}\left(C\|\nabla w\|_2^{\theta}\,\|w_\lambda\|_{2p}^{1-\theta}\right)^{p+1}\cdot\tau^{-d\frac{p-1}{2p}}$$
for some explicit constant $C>0$, which using Theorem (1.2.3) again allows to identify K.

Remark (1.2.6)[29]: The function $\overline{w}=\overline{w}_{a,b}$ is a (the unique up to a translation if $p>\frac{1}{2}$) nonnegative radial solution of $-\Delta w+\varepsilon\, w^p=\varepsilon\, w^{2p-1}$(on its support if $p\le\frac{1}{2}$), which allows us to compute K as

$$K=\frac{1}{2p}\|\overline{w}\|_{2p}^{2p-\delta}$$
$$=\begin{cases}\dfrac{1}{2p}a^{\frac{4p}{4p-d(p-1)}}b^{-1}\left(\pi^{d/2}\dfrac{\Gamma\left(\frac{2p}{p-1}-\frac{d}{2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}\right)^{\frac{2(p-1)}{4p-d(p-1)}} & if\ p>1,\\[2ex] \dfrac{1}{2p}a^{\frac{4p}{4p-d(p-1)}}b^{-1}\left(\pi^{d/2}\dfrac{\Gamma\left(\frac{2p}{p-1}-\frac{d}{2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}\right)^{\frac{2(p-1)}{4p-d(p-1)}} & if\ p>1,\end{cases}\qquad(62)$$

As noted in [35], the Gaussian weighted forms of the Poincaré inequality and logarithmic Sobolev inequalities may take very simple forms. If we denote by $d\mu$ the measure $(2\pi)^{-d/2}e^{|x|^2/2}dx$, these inequalities are respectively given by:

$$\int_{\mathbb{R}^d}|f|^2\,d\mu-\left(\int_{\mathbb{R}^d}|f|^2\,d\mu\right)^2\le\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\quad and$$
$$\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\int_{\mathbb{R}^d}|f|^2\,d\mu}\right)d\mu-\le 2\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu$$
and a whole family interpolates between both, for $1\le p<2$:
$$\int_{\mathbb{R}^d}|f|^2\,d\mu-\left(\int_{\mathbb{R}^d}|f|^p\,d\mu\right)^{2/p}\le(2-p)\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu$$
(the logarithmic Sobolev inequality appears as the derivative at $p=2$). However thisfamily is not optimal (except for $p=1$ or $p=2$). Here we will establish a family of optimal inequalities, to the price of weights that are slightly more complicated.

Corollary (1.2.7)[29]: Let $p>1$ and consider
$$w(x)=\left(\frac{a}{b+|x|^2}\right)^{\frac{1}{p-1}}$$
with a and b given by (60). Then for any measurable function f,
$$\frac{K}{p}\left(\int_{\mathbb{R}^d}|f|^{2p}w^{2p}dx\right)^{\frac{\delta}{2p}}-\int_{\mathbb{R}^d}|f|^2w^{2p}dx-\int_{\mathbb{R}^d}\left(\tfrac{2}{p+1}|f|^{p+1}-|f|^2\right)w^{p+1}dx\le\int_{\mathbb{R}^d}|\nabla f|^2w^2dx$$

provided all above integrals are well defined. Here K is an optimal constant, given by (62), and

$$\delta = 2p\frac{d+2-(d-2)p}{4p-d(p-1)}.$$

A similar result holds for $p < 1$.

Proof: It is a straightforward consequence of inequality (61) with $\tau = 1$ applied to (fw) and of:

$$\int_{\mathbb{R}^d} |\nabla(fw)|^2\, dx = \int_{\mathbb{R}^d} |\nabla f|^2 w^2 dx - \int_{\mathbb{R}^d} f^2\, \mathrm{w}\Delta\mathrm{w}\, dx$$

together with $\Delta w = w^p - w^{2p-1}$

As another straightforward consequence of Proposition (1.2.5), inequality (61) can be rewritten for

$$v = w^{2p}, \qquad m = \tfrac{p+1}{2p} \quad and$$

$$\tau^{-\frac{1}{2p}(4p-d(p-1))} = \frac{d-p(d-2)}{|p^2-1|}\ \left(for\ \ p < \frac{d}{d-2}\right)$$

Corollary (1.2.8)[29]: Let $\mathrm{d} \geq 2$, $m \geq \frac{d}{d-1}$($m > \frac{1}{2}$ if $d = 2$), $m \neq 1$ and v be a nonnegative function such that $\nabla v^{m-1/2} \in L^2(\mathbb{R}^d), x \mapsto |x|^2 v(x) \in L^1(\mathbb{R}^d)$ and

$$\begin{cases} v \in L^1(\mathbb{R}^d) & if\ m > 1, \\ v^m \in L^1(\mathbb{R}^d) & if\ m < 1. \end{cases}$$

Then

$$0 \leq L[v] - L[v_\infty] \leq \frac{1}{2}\int_{\mathbb{R}^d} v\left|x + \frac{m}{m-1}\nabla(v^{m-1})\right|^2 dx, \tag{63}$$

$$where\ \ L[v] = \int_{\mathbb{R}^d}\left(v\frac{|x|^2}{2} - \frac{1}{1-m}v^m\right)dx$$

and

$$v_m(x) = \left(\sigma^2 + \frac{1-m}{2m}|x|^2\right)_+^{\frac{1}{m-1}}$$

with σ defined in order that $M := \ \|v\|_1 = \|v_\infty\|_1$. This inequality is optimal and becomes an equality if and only if $v = v_\infty$.

Note that by convexity, v_∞. is the unique minimizer of $L[v]$ under the constraint $\|v\|_1 = M$. The constant σ arising in the expression of v_∞ is explicit:

$$\sigma^{\frac{2-d(1-m)}{1-m}} = \begin{cases} \dfrac{1}{M}\left(\dfrac{2m}{1-m}\pi\right)^{\frac{d}{2}} \dfrac{\Gamma\left(\frac{1}{1-\mathrm{m}} - \frac{\mathrm{d}}{2}\right)}{\Gamma\left(\frac{1}{1-m}\right)} & if\ m < 1, \\ \dfrac{1}{M}\left(\dfrac{2m}{1-m}\pi\right)^{\frac{d}{2}} \dfrac{\Gamma\left(\frac{\mathrm{m}}{\mathrm{m}-1}\right)}{\Gamma\left(\frac{m}{m-1} + \frac{d}{2}\right)} & if\ m > 1. \end{cases}$$

In what follows, we denote by $u(x,t)$ the solution of the Cauchy problem (55)–(56). We will also denote henceforth

$$M = \int_{\mathbb{R}^d} u_0\,(x)dx.$$

For $m \neq 1$, let us consider the solution of $\dot{R} = R^{(1-m)d-1}, R(0) = 1$:

$$R(t) = \left((1 + (2 - d(1-m)t\right)^{\frac{1}{2-d(1-m)}}, \tag{64}$$

and let $\tau(t) = \log R(t)$. The function $v(x,\tau)$ defined from u by the relation

$$u(t,x) = R(t)^{-d} \cdot v\left(\tau(t), \frac{x}{R(t)}\right) \tag{65}$$

satisfies the equation

$$v_\tau = \Delta(v^m) + \nabla \cdot (xv) \quad \tau > 0, \; x \in \mathbb{R}^d \tag{66}$$

which for $m = 1$ corresponds to the linear Fokker–Planck equation. Let us observe that $R(t) \to +\infty$ whenever $(d-2)/d < m$, which covers our entire range of interest. In (65), the L^1 norm is preserved:

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|v(\tau(t),\cdot)\|_{L^1(\mathbb{R}^d)}.$$

Since $R(0) = 1$and $\tau(0) = 0$, the initial data is preserved:

$$v(\tau = 0, x) = u_0(x) \qquad \forall x \in \mathbb{R}^d.$$

With the same notations as

$$t \to +\infty, \qquad R(t) \sim t^\alpha, \qquad u_\infty(t,\cdot) \sim U(t,\cdot)$$

and, according to (58), the known fact $\mathrm{u}(t,\cdot) \sim U(t,\cdot)$ when $\frac{d-2}{d} < m < 1$ or $m > 1$ readsin these new scales just as:

$$v(\tau, x) \to v_\infty(x) \qquad for\;\; \tau \to +\infty,$$

both uniformly and in the L^1 sense, with the notations of Corollary (1.2.8). It turns out that

$$v \mapsto L[v] = \int_{\mathbb{R}^d} \left(v\frac{|x|^2}{2} - \frac{1}{1-m}v^m\right)dx$$

defines a Lyapunov functional for Eq. (65) as we shall see below. The proof of Theorem (1.2.12) will be a consequence of Propositions (1.2.9) and (1.2.11) below, and of Corollary (1.2.8).

Proposition (1.2.9)[29]: Assume that $m > \frac{d+2}{2}$ and that u_0 is a nonnegative function such that $(1 + |x|^2)u_0$ and u_0^m belong to $L^1(\mathbb{R}^d)$. Let v be the solutions of Eq.(66)with initial data u_0. Then, with the above notations,

$$\frac{d}{dt}L[v(\tau,\cdot)] = \int_{\mathbb{R}^d} v(\tau,\cdot)\left|x + \frac{m}{m-1}\nabla v(\tau,\cdot)^{m-1}\right|^2 dx, \tag{67}$$

$$\lim_{\tau\to+\infty} L[v(\tau,\cdot)] = L[v_\infty], \tag{68}$$

and if $\frac{d-1}{d} \leq m < 1$ for d≥ 3, $\frac{1}{2} < m < 1$ if $d = 2$, or $m > 1$,then

$$0 \leq L[v(\tau,\cdot)] - L[v_\infty] \leq L[u_\infty] - L[v_\infty] \cdot e^{-2\tau} \qquad \forall \tau > 0. \tag{69}$$

Proof: Let us assume first that the initial data $u_0(x)$is smooth and compactly supported in say the ball $B(0,\rho)$ for some $\rho > 0$. Assume that

$$\frac{d}{d+2} < m < 1.$$

The solution is smooth thanks to the results in [50]. Let us consider the function:

$$w_\rho(x) = \left(\frac{1-m}{2m}\right)^{-\frac{1}{1-m}} (|x|^2 - \rho^2)^{-\frac{1}{1-m}}$$

It is easily checked that $w_\rho(x)$ is a steady state of (66), defined on the region $|x| > \rho$. Since this function takes infinite values on $\partial B(0,\rho)$, the comparison principle implies that $v(\tau, x) \leq w_\rho(x)$ for all $\tau > 0$. Hence

$$v(x,\tau) = O\left(|x|^{-2/(1-m)}\right)$$

uniformly in $\tau > 0$. Let us fix a number $R > 0$. Integrations by parts then give

$$\frac{d}{d\tau}\int_{B(0,R)} v\frac{|x|^2}{2}dx$$

$$= \int_{B(0,R)} \frac{|x|^2}{2}\nabla\cdot(\nabla v^m + xv)dx$$

$$= -\int_{B(0,R)} x\cdot(\nabla v^m + xv)dx + \frac{R}{2}\int_{\partial B(0,R)} (\nabla v^m + xv)\cdot x\tilde{d}x$$

$$= d\int_{B(0,R)} v^m\, dx - \int_{B(0,R)} |x|^2\, v dx + \frac{R}{2}\int_{\partial B(0,R)} (\nabla v^m + xv)\cdot x\tilde{d}x$$

where $\tilde{d}x$ is the measure induced by Lebesgue's measure on $\partial B(0,R)$. Integrating with respect to τ,we get:

$$\int_{B(0,R)} (v(x,\tau) - u_0(x))\frac{|x|^2}{2}d$$

$$= d\int_0^\tau\int_{B(0,R)} v^m\,(x,s)dxds + \frac{R}{2}\int_0^\tau\int_{\partial B(0,R)} (\nabla v^m(x,s)\cdot x + v(x,s)R^2)\,\tilde{d}x.$$

Now, for fixed τ, the rate of decay of $v(x,\tau)$ implies that, as $R \to +\infty$,

$$R^3\int_0^\tau\int_{\partial B(0,R)} v\,(x,s)\tilde{d}xds = O\left(R^{d+2-2/(1-m)}\right).$$

On the other hand,

$$R^{1-d}\int_{\partial B(0,R)}\int_0^\tau v^m\,(x,s)\tilde{d}xds = O\left(R^{-2m/(1-m)}\right)\ \ as\ R \to +\infty,$$

which means that

$$\int_{\partial B(0,1)}\int_0^\tau v^m\,(R_z,s)\tilde{d}xds = O\left(R^{-2m/(1-m)}\right).$$

Hence along a sequence $R_n \to +\infty$, weget:

$$\frac{\partial}{\partial R}\int_{\partial B(0,1)}\int_0^\tau v^m\,(R_z,s)\tilde{d}xds\Big|_{R=R_n} = O\left(R_n^{-2m/(1-m)-1}\right).$$

Equivalently

$$R_n^{-d} \int_{\partial B(0,R_n)} \int_0^{\tau} \nabla v^m (x,s) \cdot x \, \tilde{d}xds = O\left(R_n^{-2m/(1-m)-1}\right),$$

hence

$$R_n \int_0^{\tau} \int_{\partial B(0,R_n)} \nabla v^m(x,s) \cdot x \, \tilde{d}xds = O\left(R_n^{d-2m/(1-m)}\right).$$

The latter term goes to zero as $R_n \to +\infty$ since $m > \frac{d}{d+2}$. We conclude then that

$$\int_{\mathbb{R}^d} (v(x,\tau) - u_0(x)) \frac{|x|^2}{2} d = d \int_0^{\tau} \int_{\mathbb{R}^d} v^m(x,s) \, dxds.$$

Now, a similar argument leads us to

$$\tfrac{1}{1-m} \int_{\mathbb{R}^d} (v^m(x,\tau) - v_0^m(x)) \, dx \int_0^{\tau} \int_{\mathbb{R}^d} \left(\frac{4m^2}{(2m-1)^2} \left|\nabla\left(v^{m-2/2}\right)\right|^2 - dv^m \right) dxds$$

We conclude that $L[v(\tau,\cdot)]$ is well defined and decreasing according to (67).

In the case $m > 1$, the solution has compact support for any $\tau > 0$ and the computation leading to Eq. (67) can be carried out directly. Finally, the requirement that u_0 is smooth and compactly supported can be removed by a density argument. The proof of (67) is complete.

If $\frac{d-1}{d} \leq m < 1$ for $d \geq 3, \frac{1}{2} < m < 1$ if $d = 2$ or $m > 1$, combining relation (67) with estimate (63) of Corollary (1.2.8), we get the differential inequality:

$$\frac{d}{d\tau} L[v(\tau,\cdot)] \leq -2(L[v(\tau,\cdot)] - L[v_\infty]).$$

Since $L[v_\infty]$ minimizes $L[w]$ on

$$\{w \in L_+^1(\mathbb{R}^d) \colon \|w\|_1 = \|u_0\|_1\},$$

(69) immediately follows. In that case, (68) is trivial.

Let us establish (68) when $\frac{d}{d+2} < m < \frac{d-1}{d} 1$. We have proven that L defines a Lyapunov functional for Eq. (65). The mass of v is finite and preserved in time, $L[v(\cdot,\tau)]$ is decreasing and therefore uniformly bounded from above in τ. The quantities

$$\int_{\mathbb{R}^d} v(\tau,x) \, |x|^2 dx \quad \textit{and} \quad \int_{\mathbb{R}^d} v^m(\tau,x) \, dx$$

are uniformly bounded from above in τ, because of Hölder's inequality applied to $v^m v_\infty^{-m(1-m)} \cdot v_\infty^{m(1-m)}$:

$$\int_\omega v^m \, dx \leq \left[\int_\omega v \left(\sigma^2 + \frac{1-m}{2m} |x|^2 \right) dx \right]^m \cdot \left[\int_\omega v_\infty^m \, dx \right]^{m-1}. \tag{70}$$

for any domain $\omega \subset \mathbb{R}^d$, and because of the definition of $L[v]$:

$$\int_{\mathbb{R}^d} v \frac{|x|^2}{2} dx - \frac{1}{1-m} \left[\int_{\mathbb{R}^d} v \left(\sigma^2 + \frac{1-m}{2m} |x|^2 \right) dx \right]^m \leq L[v] \tag{71}$$

(with here $\omega = \mathbb{R}^d$), thus giving estimates on

$$\int_{\mathbb{R}^d} v\frac{|x|^2}{2}dx \quad and \quad \|v^m\|_{L^1(\mathbb{R}^d)}$$

which depend only on m, M and $L[v]$. Next we claim that

$$\int_{\mathbb{R}^d} v^m\,dx \to \int_{\mathbb{R}^d} v_\infty^m\,dx \quad as\ \tau \to +\infty.$$

However, we already know that $v^m(\tau,\cdot) \to v_\infty^m$ in $L^{1/m}(\mathbb{R}^d)$. To establish the result it suffices to show that

$$\int_{|x|>R} v^m\,(\tau,x)dx \to 0 \quad as\ R \to +\infty,$$

uniformly in τ, which is easily achieved by applying (70) with

$$\omega = \{x \in \mathbb{R}^d : |x| > R\}.$$

The latter integral is finite for $m > \frac{d}{d+1}$ and goes to 0 as $R \to +\infty$. Using the decay term

$$\int_{\mathbb{R}^d} v\left|x + \frac{m}{m-1}\nabla v^{m-1}\right|^2 dx$$

$$= \frac{4m}{(2m-1)^2}\int_{\mathbb{R}^d}\left|\nabla v^{m-1/2}\right|^2 dx + \int_{\mathbb{R}^d} v|x|^2\,dx \quad - 2d\int_{\mathbb{R}^d} v^m\,dx,$$

it is clear that at least for a subsequence $\tau_n \to +\infty$,

$$\int_{\mathbb{R}^d} |x|^2\,v(x,\tau_n)dx \to \int_{\mathbb{R}^d} |x|^2\,v_\infty(x)dx,$$

which proves (68).

An estimate of the difference between v and v_∞ in terms of L is given by the following result.

Lemma (1.2.10)[29]: Assume that Ω is a domain in $\mathbb{R}^d$ and that s is a convex nonnegative function on $\mathbb{R}^+$ such that $s(1) = 0$ and $\acute{s}(1) = 0$ (.If μ is a nonnegative measure on Ω and if f and g are nonnegative measurable functions on Ω with respect to μ ,then

$$\int_\Omega s\left(\frac{f}{g}\right)g d\mu \geq \frac{K}{\max\left\{\int_\Omega f d\mu\,,\int_\Omega g d\mu\right\}}\cdot \|f-g\|^2_{L^1(\Omega,\mathrm{d}\mu)}, \tag{72}$$

where $K = \frac{1}{2}\cdot\min\{K_1, K_2\}$,

$$K_1 = \min_{\eta\in]0,1[}\acute{\acute{s}}\,(\eta) \quad and \quad K_2 = \min_{\theta\in]0,1[,\ h>0}\acute{\acute{s}}\,(1+\theta h)(1+h), \tag{73}$$

provided that all the above integrals are finite.

Proof: We may assume without loss of generality that f and $g$are strictly positive functions. Let us set $h = (f-g)/g$,so that $f\,/g = 1 + h$. If ω is any subdomain of Ω and k a positive, integrable on ω, function, then Cauchy–Schwarz's inequality yields:

$$\int_\omega \frac{|f-g|^2}{k}d\mu \geq \frac{\left(\int_\omega |f-g|d\mu\right)^2}{\int_\omega k\,d\mu}. \tag{74}$$

The proof of inequality (72) is based on a Taylor's expansion of $s(t)$ around $t = 1$. Since $s(1) = \acute{s}(1) = 0$, we have

$$s\left(\frac{f}{g}\right) = s(1+h) = \frac{1}{2}\acute{\acute{s}}(1+\theta h)h^2$$

for some function $x \longmapsto \theta(x)$ with values in $]0,1[$. Thus we need to estimate from below the function

$$\int_\Omega \acute{\acute{s}}(1+\theta h)gh^2\,d\mu.$$

First, we estimate

$$\int_{f<g} \acute{\acute{s}}(1+\theta h)gh^2\,d\mu = \int_{f<g} \acute{\acute{s}}(1+\theta h)\frac{|f-g|^2}{g}\,d\mu \geq K_1 \int_{f<g} \frac{|f-g|^2}{g}\,d\mu$$

according to the definition (73) of K_1 . Using (74) with

$$\omega = \{x \in \Omega:\ f(x) < g(x)\} \quad and \quad k = g,$$

we obtain:

$$\int_{f<g} \acute{\acute{s}}(1+\theta h)gh^2\,d\mu \geq K_1 \frac{\left(\int_{f<g}|f-g|\,d\mu\right)^2}{\int_{f<g} g\,d\mu} \tag{75}$$

On the other hand, we have:

$$\int_{f>g} \acute{\acute{s}}(1+\theta h)gh^2\,d\mu = \int_{f>g} \acute{\acute{s}}(1+\theta h)(1+h)\frac{|f-g|^2}{f}\,d\mu \geq K_2 \int_{f>g} \frac{|f-g|^2}{f}\,d\mu$$

using the definition (73) of K_2. Now, using again (74) with

$$\omega = \{x \in \Omega:\ f(x) > g(x)\} \quad and \quad k = f,$$

we get:

$$\int_{f>g} \acute{\acute{s}}(1+\theta h)gh^2\,d\mu \geq K_2 \frac{\left(\int_{f>g}|f-g|\,d\mu\right)^2}{\int_{f>g} f\,d\mu} \tag{76}$$

Combining (75) and (76), we obtain (72).

Proposition (1.2.11)[29]: Assume that $d \geq 2$ .Let v is a nonnegative function such that $x \mapsto (1+|x|^2)v$ and v^m belong to $L^1(\mathbb{R}^d)$ and consider v_∞ defined as in Corollary(1.2.9).

(i) If $\frac{d-2}{d} \leq m < 1, m > \frac{1}{2}$, then there exists a constant $C > 0$ which depends only on m , $M = \int_{\mathbb{R}^d} v dx$ and $L[v]$ such that

$$C\|v^m - v_\infty^m\|^2_{L^1(\mathbb{R}^d)} \leq L[v] - L[v_\infty].$$

(ii) If $1 < m \leq 2$ and $R = \sqrt{2m/(m-1)\sigma^2}$, then

$$C\|(v - v_\infty)v_\infty^{m-1}\|^2_{L^1(\mathbb{R}^d)} \leq L[v] - L[v_\infty].$$

For the proof of this result, we need a lemma which is a variation of the Csiszár–Kullback inequality. We provide a proof for completeness and refer to [45], [54], [31] for related results.

Proof: The result is a direct consequence of Lemma (1.2.10). For $m < 1$, we take:

$$s(t) = \frac{mt^{1/m} - t}{1-m} + 1, \quad K_1 = K_2 = \frac{1}{m}, \quad d\mu(x) = dx \ and$$

$$L[v] = \int_{\mathbb{R}^d} s\left(\frac{v^m}{v_\infty^m}\right) v_\infty^m dx.$$

According to (70) and (71), the quantities

$$\int_{\mathbb{R}^d} v\frac{|x|^2}{2}dx \quad and \quad \|v^m\|_{L^1(\mathbb{R}^d)}^2$$

depend only on m, M and $L[v]$, which proves the statement on C. If $1 < m < 2$, we may write:

$$\frac{|x|^2}{2} = \frac{m}{m-1}(\sigma^2 - v_\infty^{m-1}) \le \frac{m}{m-1}\sigma^2 \quad for\ |x| < \sqrt{\frac{2m}{m-1}}\sigma,$$

$$\int_{\mathbb{R}^N} v v_\infty^{m-1} dx \le \frac{m}{m-1}\sigma^2 M$$

and apply Lemma (1.2.10) to

$$L[v] = \int_{\mathbb{R}^d} s\left(\frac{v}{v_\infty}\right) v_\infty d\mu(x) + \int_{B(0,R)^c} \left(v\frac{|x|^2}{2} + \frac{1}{m-1}v^m\right)dx,$$

with

$$s(t) = \frac{t^m - mt}{m-1} + 1, \quad K_1 = K_2 = m \ \ and \ \ d\mu(x) = v_\infty^{m-1}(x)dx.$$

Theorem (1.2.12)[29]: Assume that the initial datum u_0 is a nonnegative function with

$$\int_{\mathbb{R}^d} u_0(1+|x|^2)dx + \int_{\mathbb{R}^d} u_0^m dx < +\infty.$$

If u is the solution of (55)–(56), and U given by(57) satisfies

$$\int_{\mathbb{R}^d} U(t,x)dx = \int_{\mathbb{R}^d} u_0 dx,$$

then the following facts hold.

(i) Assume that $\frac{d-1}{d} < m < 1\ if d \ge 3$, and $\frac{1}{2} < m < 1$ if $d = 2$.Then

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m(t,\cdot) - U^m(t,\cdot)\|_{L^1(\mathbb{R}^d)} < +\infty.$$

(ii) Assume that $1 < m < 2$.Then

$$\limsup_{t\to+\infty} t^{\frac{1+d(1-m)}{2+d(1-m)}} \|[u(t,\cdot) - U(t,\cdot)]U^{m-1}(t,\cdot)\|_{L^1(\mathbb{R}^d)} < +\infty.$$

Proof : Estimate (69), Proposition (1.2.11) and relation (69) yield that for $m < 1$

$$C\|v^m(\cdot,\tau) - v_\infty^m\|_{L^1(\mathbb{R}^d)}^2 \le (L[u_0] - L[v_\infty]) \cdot e^{-2\tau} \quad \forall\tau > 0,$$

while for $m > 1$

$$C\|v(\cdot,\tau) - v_\infty)v_\infty^{m-1}\|_{L^1(\mathbb{R}^d)}^2 \le (L[u_0] - L[v_\infty]) \cdot e^{-2\tau} \quad \forall\tau > 0,$$

Recalling that in terms of the variable $t, \tau = \tau(t) \sim \log t$, and changing variables into the original definition of v in terms of $u(x,t)$, gives us exactly the relations seeked for in Theorem (1.2.12) with U replaced by

$$u_\infty(t,x) = R(t)^{-d} v_\infty\left(\log R(t), \frac{x}{R(t)}\right)$$

and R given by (64). A straightforward computation shows that U and u_∞ are asymptotically equivalent and this concludes the proof .

Finally, let us mention that the Lyapunov functional $L[v]$ had already been exhibited by Ralston and Newman in [60], [64]. An alternative approach for getting the decay of $L[v]$ is based on the entropy–entropy dissipation method, which has been used for the heat equation in [67], [68], [30] and generalized to nonlinear diffusions in [46], [41] (also see [62] by Otto on the gradient flow structure of the porous medium equation), providing another proof of inequality (63). More recents developments can be found in [40], [51], [55], [39]. See [56] and [33] for relations with Sobolev type inequalities.

Chapter 2
First Order Interpolation and Hardy-Lieb-Thirring Inequalities

We show first order interpolation inequalities with weights. We first establishing a Sobolev inequality for such operators. Similar results are true for fractional powers of the Laplacian and the Hardy-weight and, in particular, for relativistic Schrödinger operators. We also allow for the inclusion of magnetic vector potentials. As an application, we extend, for the first time, the proof of stability of relativistic matter with magnetic fields all the way up to the critical value of the nuclear charge $Z\alpha = 2/\pi$.

Section (2.1): Interpolation Inequalities with Weights

[72] proved certain interpolation inequalities. These are analogous to the standard interpolation between functions and their first derivatives in various L^pnorms on $\mathbb{R}^n$(see [73], [61]), but with each term weighted by a power of $|x|$. Instances of these inequalities have been studied previously [71], [75], [76], but the general case seems to have not yet been treated; we present it here, in the belief that such inequalities may show useful in other contexts. Lin [74] has generalized these result to include derivatives of any order.

For simplicity, we state our theorem for $u \in C_0^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support. Its extension to a more general class of functions is standard. In what follows $p, q, r; \alpha, \beta, \sigma$; and a are fixed real numbers (called parameters) satisfying

$$p, q \geq 1, \quad r > 0\,, \quad 0 \leq a \leq 1 \tag{1}$$

$$\frac{1}{p} + \frac{\alpha}{n}, \qquad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{2}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3}$$

Theorem (2.1.1)[7]: There exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(R^n)$

$$||x|^\gamma u|_{L^r} \leq C\left||x|^\alpha |Du|\right|_{L^p}^a \left||x|^\beta u\right|_{L^q}^{1-a} \tag{4}$$

if and only if the following relations hold :

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right) \tag{5}$$

(this is dimensional balance),

$$0 \leq \alpha - \sigma \quad \text{if} \quad a > 0,$$

and

$$\alpha - \sigma \leq 1 \qquad \text{if } a > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \tag{6}$$

Furthermore, on any compact set in parameter space in which (1),(2), (5) and $0 \leq \alpha - \sigma \leq 1$ hold, the constant C is bounded.

We emphasize the curious fact that one needs the conditions $\alpha - \sigma \leq 1$ only in case $a > 0$ and $1/p + (\alpha - 1)/n = 1/r + \gamma/n$.

The proof is rather long but elementary. We first verify necessity; then we verify the case $n = 1$, $\sigma = \alpha - 1$, using among other tools a weighted Hardy-type inequality showed by Bradley [71]. The case $n \geq 1$, $0 \leq \alpha - \sigma \leq 1$ is treated next; then finally the case $\alpha - \sigma > 1, 1/p + (\alpha - 1)/n \neq 1/r + \gamma/n$. Since when $a = 0$ there is nothing to show, we shall always assume $a > 0$.

Throughout, C denotes a constant, depending on the parameters, whose value may change from line to line. Although we will not estimate the constants explicitly, it will be clear from the arguments that the last assertion of the theorem holds.
Note first that the inequalities (1) are necessary in order for the norms in (4) to be finite. If (4) holds for $u(x)$ then it holds also for $u(\lambda x,),\ \lambda > 0$. Inserting this in (4) we obtain (5). This is merely dimension of $||x|^{\gamma} u|_{L^r}$ is $\gamma + n/r$, that of $\left||x|^{\alpha}|Du|\right|_{L^p}$ is $\alpha - 1 + n/p$, etc.

Next, for some fixed function $v \in C_0^{\infty}(|x| < 1),\ v \not\equiv 0$, let $u(x) = v(x - x_0)$ with $|x_0| = R$ large. Inserting this in (4) we see that

$$R^{\gamma} \leq C R^{a\alpha + (1-a)\beta}$$

so that

$$a\sigma + (1-a)\beta \leq a\alpha + (1-a)\beta.$$

Hence $\sigma \leq \alpha$. Next we show (6). Suppose

$$\frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}$$

$$= \frac{1}{q} + \frac{\beta}{n} \quad \text{by (5) if } a < 1. \tag{7}$$

We insert in (4) the function

$$u(x) = \begin{cases} 0 & for\ |x| \geq 1 \\ x^{-\gamma - n/r} \ \log\dfrac{1}{|x|} & for\ \epsilon \leq |x| \leq 1 \\ x^{-\gamma - n/r} \ \log\dfrac{1}{\epsilon} & for\ \ |x| \leq \epsilon \end{cases}$$

This function is not in C^{∞} but it is clear that (4) must also hold for it Straightforward calculation shows that, if ρ, θ polar coordinates, $\theta \in S^{n-1}$,

$$\int_{R^n} |x|^{\gamma r}\, |u|^r \geq C \int_{\epsilon < |x| < 1} \frac{1}{\rho} \log^r \frac{1}{\rho} d\rho \geq C \log^{1+r} \frac{1}{\epsilon}$$

Consequently (4) implies

$$\frac{1}{r} + 1 \leq a\left(\frac{1}{p} + 1\right) + (1-a)\left(\frac{1}{q} + 1\right)$$

$$\frac{1}{r} \leq \frac{a}{p} + \frac{1-a}{q}.$$

But according to (3) and (5)

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q} + \frac{a}{n}(\alpha - 1 - \sigma) \tag{8}$$

Hence $\alpha - 1 - \sigma \leq 0$, i.e. (6) holds. Necessity is showed.
We present some inequalities which will be useful in what follows. Several of these are special cases of (4).

(I) If $1 \leq p \leq r$, $\quad \delta \in \mathbb{R}$, and $\alpha = \delta + 1/r + (p-1)/p$ then for $u \in C_0^{\infty}(\mathbb{R})$

$$\left||x|^{\delta} u\right|_{L^r} \leq C ||x|^{\alpha} D u|_{L^p} \tag{9}$$

in case either

(i) $\delta + \frac{1}{r} > 0$

(ii) $\delta + \frac{1}{r} < 0$ and $u(0) = 0$.

The constant C in (9) stays bounded as $p, r,$ and δ range over any compact subset of $\{1 \le p \le r, \quad \delta r \ne -1\}$.

One easily deduces these facts from the weighted Hardy-type inequalities in [71].For $r = p$.

(II) Assume (1)-(3) and (5) hold; for any $\rho > 0$, let $R_\rho = \{\rho < |x| \le 2\rho\}$. If $u \in C_0^\infty(\mathbb{R}^n)$ and

$$\delta = \gamma + \frac{n}{r} - n \tag{10}$$

then

$$\int_{R_\rho} |x|^{\gamma r} |u|^r$$

$$\le C\left(\int_{R_\rho} |x|^{\alpha p} |Du|^p\right)^{ar/p} \left(\int_{R_\rho} |x|^{\beta q} |u|^q\right)^{ar/p} + C\left(\int_{R_\rho} |x|^{\delta} |u|\right)^r \qquad (11)$$

with C independent of ρ. If $\int_{R_\rho} u = 0$ then the latter term in (11) may be omitted.

It suffices to consider $\rho = 1$, since the general case follows by scaling. Writing $R_1 = R$, we consider first the case that

$$\frac{1}{m} = \frac{a}{p} + \frac{1-a}{q} - \frac{a}{n} > 0. \tag{12}$$

Using a standard interpolation inequality ([73], [61]), and writing $\bar{u} = (meas\ R)^{-1} \int_R u$,

$$\int_R |u - \bar{u}|^m \le C\left(\int_R |Du|^p\right)^{am/p} \left(\int_R |u - \bar{u}|^q\right)^{(1-a)m/q}. \tag{13}$$

Since $\alpha - \sigma \ge 0$, $r \le m$; applying Holder's inequality to (13) we find

$$\int_R |u - \bar{u}|^r \le C\left(\int_R |u - \bar{u}|^m\right)^{r/m}$$

$$\le C\left(\int_R |Du|^p\right)^{ar/p} \left(\int_R |u - \bar{u}|^q\right)^{(1-a)r/q} \tag{14}$$

If, on the other hand, (12) fails, then $a/p + (1-a)/q \le a/n$. It follows that

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{n}; \tag{15}$$

and if (15) holds then

$$\int_R |u - \bar{u}|^r \le C\left(\int_R |Du|^p\right)^{br/p} \left(\int_R |u - \bar{u}|^q\right)^{(1-b)r/q} \tag{16}$$

where

$$b = 0 \qquad if \quad r \le q$$

$$b\left(\frac{1}{q} + \frac{1}{n} - \frac{1}{p}\right) = \frac{1}{q} - \frac{1}{r} \qquad if \quad r \ge q,$$

and in particular $b \le a$. By Sobolev's inequality and (15),

$$\left(\int_R |u-\bar{u}|^q\right)^{1/q} \le C\left(\int_R |Du|^p\right)^{1/p}. \tag{17}$$

combining (16) and (17) yields (14) once again. Rescaling and multiplying by $\rho^{\gamma r}$, we conclude that if $\int_{R_\rho} u = 0$ then

$$\int_{R_\rho} |x|^{\gamma r}\,|u|^r \le C\left(\int_{R_\rho} |x|^{\alpha r}|Du|\ \right)^{ar/p}\left(\int_{R_\rho} |x|^{\beta q}|u|^q\ \right)^{(1-a)r/q}.$$

If $\bar{u} \neq 0$, we note that

$$\left(\int_R |u-\bar{u}|^q\right)^{1/q} \le C\left(\int_R |u|^q\right)^{1/q}.$$

Therefore, using (14),

$$\begin{aligned}\left(\int_R |u|^r\right)^{1/r} &\le \left(\int_R |u-\bar{u}|^r\right)^{1/r} + \left(\int_R |\bar{u}|^r\right)^{1/r}\\ &\le C\left(\int_R |Du|^p\right)^{a/p}\left(\int_R |u|^q\right)^{(1-a)/q} + C\int_R |u|.\end{aligned}$$

This shows (11) in case $\rho = 1$, and the general case follows once again by scaling.

(III) Suppose (1)-(3) and (5) hold, and $\sigma = \alpha - 1$; and suppose further that

$$a > (1+q-q/p)^{-1}. \tag{18}$$

Then (4) holds, with constant C uniform so long $as\ \gamma r + n$ stays bounded away from zero. Since $\sigma = \alpha - 1$ implies $1/r = a/p + (1-a)/q$, the condition (18) is equivalent to

$$a > 1/r. \tag{19}$$

we show (4) in this context using radial integration by parts:

$$\begin{aligned}\int\ |x|^{\gamma r}\,|u|^r &\le C\int\ |x|^{\gamma r+1}\,|u|^{r-1}|Du|\\ &\le C\int (|x|^{\alpha}|Du|)\left(|x|^{\beta}|u|\right)^{a^{-1}-1}\left(|x|^{\epsilon}|u|^{r-a^{-1}}\right)\end{aligned}$$

where $\epsilon = \gamma r + 1 - \alpha + \beta - \beta/a = \gamma(r - a^{-1})$. By (18) $a^{-1} - 1 \le q$, so

$$\int\ |x|^{\gamma r}\,|u|^r \le C\big||x|^{\alpha}|Du|\big|_{L^p}\big||x|^{\beta}u\big|_{L^q}^{a^{-1}-1}\big||x|^{\epsilon}|u|^{r-a^{-1}}\big|_{L^k} \tag{20}$$

with k chosen so that

$$\frac{1}{p} + \frac{1}{q}(a^{-1}-1) + \frac{1}{k} = 1.$$

One checks that

$$(r - a^{-1})k = r \text{ and } \epsilon k = \gamma r;$$

using this in (20) yields

$$||x|^{\gamma}u|_{L^r}^{a^{-1}} \le C\big||x|^{\alpha}|Du|\big|_{L^p}\big||x|^{\beta}u\big|_{L^q}^{a^{-1}-1}$$

from which (4) follows.

(IV) If $t, r \ge 1$; $\gamma + n/r\,, \epsilon + \epsilon\, n/t, \beta + n/q > 0$; and $0 \le b \le 1$ then

$$||x|^{\gamma}u|_{L^r} \ \le ||x|^{\epsilon}u|_{L^r}^{b}\big||x|^{\beta}u\big|_{L^q}^{1-b} \tag{21}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ provided that

$$\frac{1}{r}=\frac{b}{t}+\frac{1-b}{q} \tag{22}$$

and

$$\gamma=b\epsilon+(1-b)\rho. \tag{23}$$

This is an easy consequence of Hodder' s inequality .

For notation convenience we set

$$||x|^{\alpha}Du|_{L^p} = A, \quad \left||x|^{\beta}u\right|_{L^q} = B \tag{24}$$

we show

$$||x|^{\gamma}u|_{L^r} \le CA^aB^{1-a}.$$

Throughout, $\zeta(x)$will represent a fixed C_0^∞ function on $\mathbb{R}^n$ with the properties

$$0\le\zeta\le1, \quad \zeta\equiv1 \ if \ \left|x<\tfrac{1}{2}, \quad \zeta\right|\equiv0 \ if \ |x|>1. \tag{25}$$

When possible, we shall use (I) to verify the case $a=1$ and (IV) to interpolate between $a=0$ and $a=1$. Substantial complications arise, however, because (I) does not apply when $1/p+\alpha-1=0$ (this corresponds to the case $\delta+1/r=0$ in (9). Note that $\sigma=\alpha-1$ implies

$$\frac{1}{r}=\frac{a}{p}+\frac{1-a}{q}, \quad \gamma=a(\alpha-1)+(1-a)\beta. \tag{26}$$

(i) The case $\gamma+1/r=1/p+\alpha-1, \qquad 0\le a\le1$

By(IV),

$$||x|^{\gamma}u|_{L^r}\le||x|^{\alpha-1}u|_{L^p}^{a}\left||x|^{\beta}u\right|_{L^q}^{1-a}, \tag{27}$$

while by (I),

$$||x|^{\alpha-1}u|_{L^p}\le C||x|^{\alpha}Du|_{L^p}. \tag{28}$$

Combining (27) and (28) yields (4)

The remainder of addresses the case $\gamma+1/r\ne1/p+\alpha-1$.

In that event one may rescale u so that $A=B=1$; we henceforth assume such a normalization, so that our goal becomes show

$$||x|^{\gamma}u|_{L^r} \le C.$$

(ii) The case $1/p+\alpha-1>0$ and bounded away from zero

The argument used for part (i) applies here, too. Note however that as $1/p+\alpha-1\to0$, the constant in (28) tends to ∞.

The case $1/p+\alpha-1\approx0$ will be handled in (iii)-(v); part (vi) will treat the case $1/p+\alpha-1<0$ and handled away from zero.

We choose a real number v, depending on the parameters, such that $0<v<\frac{1}{2}$ and

$$2v\le\gamma+\frac{1}{r}, \quad 2v\le\beta+\frac{1}{q}\le(2v)^{-1}. \tag{29}$$

(iii) The case $-v^3\le1/p+\alpha-1\le v$ and $1/p<1-v$

Note that a and $(1+q-q/p)^{-1}$are bounded away from 1 in this case :

$$a\le1-2v^2 \tag{30}$$

$$(1+q-q/p)^{-1}<\frac{1}{1+v}. \tag{31}$$

Let $\mu=(1+2v^2)^{-1}$, and set $a_0=a/\mu$, so that $a<a_0\le1-4v^4$. By (III),

$$||x|^{\gamma}u|_{L^r} \le C||x|^{\epsilon}u|_{L^r}^{a_0}\left||x|^{\beta}u\right|_{L^q}^{1-a_0} \tag{32}$$

where ϵ and t are determined by

$$\frac{1}{t} = \frac{\mu}{p} + \frac{1-\mu}{q}$$
$$\epsilon = \mu(\alpha - 1) + (1-\mu)\beta.$$
Moreover we see that
$$\frac{1}{t} + \epsilon = \mu\left(\frac{1}{p} + \alpha - 1\right) + (1-\mu)\left(\beta + \frac{1}{q}\right) \geq 3\nu^3\mu$$
is bounded away from zero. Since $\nu < \frac{1}{2}$, (31) implies $\mu(1 + q - q/p) > 1$; hence by (III)
$$||x|^{\epsilon}u|_{L^t} \leq CA^{\mu}B^{1-\mu}; \tag{33}$$
substitution of (33) into (32) yields (4).

(iv) the case $-\nu^3 \leq 1/p + \alpha - 1 \leq \nu, \quad 1-\nu \leq 1/p \leq 1, a \geq \nu$

We set $\delta = \gamma + 1/r - 1$, and note that under the above hypotheses
$$\alpha \leq 2\nu \leq \delta + 1 \tag{34}$$
and
$$\delta - \beta \leq \frac{1}{\mathrm{q}} - 1 - \nu^2 \tag{35}$$
We assert that
$$\left(\int |x|^{\gamma r}\,|u|^r\right)^{1/r} \leq CA^aB^{1-a} + C\int |x|^{\delta}\,|u|\ . \tag{36}$$
Indeed, if $R_k = \{2^k < |x| \leq 2^{k+1}\}$ for any integer k, then (II) yields
$$\int_{R_k} |x|^{\gamma r}\,|u|^r$$
$$\leq C\left(\int_{R_k} |x|^{\alpha p}\,|Du|^p\right)^{ar/p}\left(\int_{R_k} |x|^{\beta q}\,|u|^q\right)^{(1-a)r/q} + C\left(\int_{R_k} |x|^{\delta}\,|u|\right)^r. \tag{37}$$
We add (37) for $K \in \mathbb{Z}$, using the inequalities
$$\sum x_k^c\, y_k^d \leq \left(\sum x_k\right)^c \left(\sum y_k\right)^d, \qquad c + d \geq 1 \tag{38}$$
$$\sum x_k^c \leq \left(\sum x_k\right)^c, \qquad c \geq 1, \tag{39}$$
valid for $x_k, y_k,\ c,\ d \geq 0$. Since $ar/p + (1-a)r/q = 1$ and $r \geq 1$ by (26), these inequalities apple and yields (36).

Thus we need only show that $\int |x|^{\delta}\,|u| \leq C$. With ζ as in (25) we write
$$\int |x|^{\delta}\,|u| = \int |x|^{\delta}\,\zeta\,|u| + \int |x|^{\delta}\,(1-\zeta)|u| \tag{40}$$
and estimate the two terms separately. Since δ is bounded away from -1, we may use radial integration by part in the first term:
$$\int |x|^{\delta}\,\zeta\,|u| \leq C\int |x|^{\delta+1}\,\zeta\,|Du| + C\int |x|^{\delta+1}\,|D\zeta|\,|u|$$
$$\leq C\int_{|x|<1} |x|^{\delta+1}\,|Du| + C\int_{\frac{1}{2}<|x|<1} |x|^{\delta+1}\,|u|.$$
$$\leq C\int_{|x|<1} |x|^{\delta+1}\,|Du| + CB$$
If $p = 1$ then

$$\int_{|x|<1} |x|^{\delta+1}\,|Du| \le CA \tag{41}$$

since $\delta + 1 \ge \alpha$ by (29). If $p > 1$ then

$$\int_{|x|<1} |x|^{(\delta+1)}\,|Du| \le A \cdot \left(\int_{|x|<1} |x|^{(\delta+1-\alpha)\acute{p}}\right)^{1/\acute{p}}$$

where $\acute{p} = p/(p-1)$. Since $\delta + 1 - \alpha \ge 0$, the integral converges; hence (41) holds also for $p > 1$. Thus

$$\int |x|^{\delta}\,\zeta\,|u| \le C. \tag{42}$$

We argue similarly for the second term in (40), but without integrating by parts:

$$\int |x|^{\delta}\,(1-\zeta)|u| \le \int_{|x|>\frac{1}{2}} |x|^{\delta}\,|u| \le B\left(\int_{|x|>\frac{1}{2}} |x|^{(\delta-\beta)\acute{q}}\right)^{1/\acute{q}}$$

assuming $q > 1$, and setting $\acute{q} = q/(q-1)$. The last integral converges, by (35), so

$$\int |x|^{\delta}\,(1-\zeta)|u| \le C. \tag{43}$$

If $q = 1$ we see from (35) that $\delta < \beta$, so

$$\int_{|x|>\frac{1}{2}} |x|^{\delta}\,|u| \le C\int_{|x|>\frac{1}{2}} |x|^{\beta}\,|u| \le CB,$$

which yields (43) also $q = 1$. We have shown

$$\int |x|^{\delta}\,|u| \le C,$$

with constant C uniform for fixed ν. By (36), the desired result (4) follows.

(v) The case $-\nu^3 \le 1/p + \alpha - 1 \le \nu, \quad 0 \le a < \nu$

We argue much as in part (iii). Let ϵ and t satisfy

$$\frac{1}{t} = \frac{\mu}{p} + \frac{1-\mu}{q}, \quad \epsilon = \mu(\alpha-1) + (1-\mu)\beta$$

with $\mu = \frac{1}{2}$, we recall from (32) (with $\mu = \frac{1}{2}$) that

$$\left||x|^{\gamma}u\right|_{L^r} \le C\left||x|^{\epsilon}u\right|_{L^t}^{2a}\left||x|^{\beta}u\right|_{L^q}^{1-2a}. \tag{44}$$

Since $\epsilon + 1/t \ge \frac{1}{2}(2\nu - \nu^3) \ge \frac{1}{2}\nu$, we have from cases (iii) and (v) that

$$\left||x|^{\epsilon}u\right|_{L^t} \le CA^{1/2}B^{1/2}. \tag{45}$$

Combining (44) and (45) yields (4).

(vi) The case $1/p + \alpha - 1 < -\nu^3$

Let $\tilde{u}(x0 - u(0)\zeta(x)$, with ζ as in (25). Arguing as in parts (i) and (ii), we obtain

$$\left||x|^{\gamma}\tilde{u}\right|_{L^r} \le C\left||x|^{\alpha}D\tilde{u}\right|_{L^p}^{a}\left||x|^{\beta}\tilde{u}\right|_{L^q}^{1-a} \le C(\left||x|^{\alpha}Du\right|_{L^p} + |u(0)|)^{a}\left(\left||x|^{\beta}u\right|_{L^p} + |u(0)|\right)^{1-a}$$
$$\le C(1 + |u(0)|).$$

Thus to show (4) we need only show that $|u(0)| \le C$.

Now

$$u(0) = -\int_0^{\infty} \frac{d}{dx}(u\zeta)$$

so that

$$|u(0)| \leq C\int_0^1 |Du| + C\int_{1/2}^1 |u|$$
$$\leq C\int_0^1 |Du| + C.$$
If $p > 1$, then
$$\int_0^1 |Du| \leq ||x|^\alpha Du|_{L^p} \cdot \left(\int_0^1 |x|^{-\alpha \acute{p}}\right)^{1/\acute{p}}$$
and the integral on the right converges, because $-\alpha\acute{p} \geq -1 + \nu^3\acute{p}$; hence
$$\int_{|x|<1} |Du| \leq CA \leq C. \tag{46}$$
If $p = 1$, we still conclude (46), since in that case $\alpha < 0$. Thus we have shown
$$|u(0)| \leq C. \tag{47}$$
The show of (4) for $n = 1,\ \sigma = \alpha - 1$is now complete.

Case when $n \geq 1, \alpha \geq \sigma \geq \alpha - 1$

Note that in this case
$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q} + \frac{a(\alpha - \sigma - 1)}{n} \leq \frac{a}{p} + \frac{1-a}{q} \tag{48}$$
We consider $u(x) = f(|x|)$, where f is smooth on $[0, \infty)$ and vanishes for $|x|$ large. For integers k, let $R_k = \{2^k < |x| \leq 2^{k+1}\}$; by (II) we have
$$\int_{R_k} |x|^{\gamma r}\,|u|^r \tag{49}$$
$$\leq C\left(\int_{R_\chi} |x|^{\alpha p}\,|Du|^p\right)^{ar/p}\left(\int_{R_\chi} |x|^{\beta q}\,|u|^q\right)^{(1-a)r/q} + C\left(\int_{R_\chi} |x|^{\delta}\,|u|\right)^r$$
with $\delta = \gamma + n/r - n$. Let s be defined by
$$\frac{1}{s} = \frac{a}{p} + \frac{1-a}{q} \tag{50}$$
so that $1/r \leq 1/s \leq 1$. By Holder's inequality,
$$\left(\int_{R_\chi} |x|^{\delta}\,|u|\right)^r \leq C\left(\int_{R_\chi} |x|^{\mu s}\,|u|^s\right)^{r/s}, \tag{51}$$
with
$$\mu = n\left(\frac{1}{r} - \frac{1}{s} + \frac{\gamma}{n}\right). \tag{52}$$
We add (49) for all k, using (51) and the inequalities (38), to obtain
$$\left(\int\ |x|^{\gamma r}\,|u|^r\right)^{1/r} \leq CA^a B^{1-a} + C\left(\int |x|^{\mu s}\,|x|^s\right)^{1/s}. \tag{53}$$

Now,
$$\left(\int_{\mathbb{R}^n} |x|^{\mu s}\,|u|^s\right)^{1/s} \leq C\left(\int_0^\infty \rho^{\overline{\mu} s}\,|f|^s d\rho\right)^{1/s}, \quad \bar{\mu} = \gamma + \frac{n}{r} - \frac{1}{s} \tag{54}$$
while

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha p} |Du|^p\right)^{1/p} \geq C\left(\int_0^\infty \rho^{\bar{\alpha} p} |Df|^p\right)^{1/p}, \qquad \bar{\alpha} = \alpha + \frac{n-1}{p} \tag{55}$$

and

$$\left(\int_{\mathbb{R}^n} |x|^{\beta q} |u|^q\right)^{1/q} \geq C\left(\int_0^\infty \rho^{\bar{\beta} q} |f|^q\right)^{1/q}, \qquad \bar{\beta} = \beta + \frac{n-1}{q}. \tag{56}$$

Since

$$\frac{1}{s} + \bar{\mu} = a\left(\frac{1}{p} + \bar{\alpha} - 1\right) + (1-a)\left(\frac{1}{q} + \bar{\beta}\right),$$

we conclude from that

$$||x|^{\bar{\mu}} f|_{L^s} \leq C||x|^{\bar{\alpha}} Df|_{L^p}^a \left||x|^{\bar{\beta}} f\right|_{L^q}^{1-a}. \tag{57}$$

(Strictly speaking, one must first extend f to a function on $(-\infty, \infty)$, and then apply the results . Alternatively, one may simply note that the remain valid for functions on $[0, \infty)$.)Combining (53)-(57) yields $||x|^\gamma u|_{L^r} \leq CA^a B^{1-a}$.

For any $u \in C_0(\mathbb{R}^n)$, let $U: (0, \infty) \to \mathbb{R}^n$ denote its spherical mean function

$$U(\rho) = ⨍_{|x|=\rho} u \tag{58}$$

and let u^* by the associated radial function on $\mathbb{R}^n$

$$u^*(x) = U(|x|). \tag{59}$$

We have

$$|DU(\rho)| \leq ⨍_{|x|=\rho} |Du|, \qquad |U(\rho)| \leq ⨍_{|x|=\rho} |u|$$

so that

$$||x|^\alpha Du^*|_{L^p} \leq A, \qquad \left||x|^\beta Du^*\right|_{L^q} \tag{60}$$

also, of course, $u - u^*$ has mean zero on each sphere $|x| = \rho$.

Let $R_k = \{2^k < |x| \leq 2^{k+1}\}$ for integers k; by (II) we have

$$\int_{R_k} |x|^{\gamma r} |u - u^*|^r \leq C\left(\int_{R_\kappa} |x|^{\alpha p} |Du - Du^*|^p\right)^{ar/p} \left(\int_{R_\kappa} |x|^{\beta q} |u - u^*|^q\right)^{(1-a)r/q} \tag{61}$$

for each k. We add the inequalities (61), using (38) and (48), to conclude

$$||x|^\gamma (u - u^*)|_{L^r} \leq C||x|^\alpha D(u - u^*)|_{L^p}^a \left||x|^\beta (u - u^*)\right|_{L^q}^{1-a}, \tag{62}$$

whence using (60) and (I),

$$||x|^\gamma u|_{L^r} \leq CA^a B^{1-a} + ||x|^\gamma u^*|_{L^r} \leq CA^a B^{1-a}.$$

Case $1/p + (\alpha - 1)/n \neq 1/r + \gamma/n$ and $\sigma < \alpha - 1$

Notice that in this case $a < 1$ necessarily. we may assume $A = B = 1$, since this normalization may be achieved by scaling. Since (4) has been showed for $\sigma = \alpha - 1$, we know that

$$\left||x|^{\delta}u\right|_{L^s} \le C, \quad ||x|^{\epsilon}u^*|_{L^t} \le C \tag{63}$$

provided that δ, s, ϵ, and t related by

$$\begin{aligned}
&\delta = b\alpha + (1-b)\beta \\
&\frac{1}{s} = \frac{b}{p} + \frac{1-b}{q} - \frac{b}{n} \\
&\epsilon = d(\alpha - 1) + (1-d)\beta \\
&\frac{1}{t} = \frac{d}{p} + \frac{1-d}{q}
\end{aligned} \tag{64}$$

for some choices of b and $d, 0 \le b, d \le 1$, and provided that

$$\frac{\delta}{n} + \frac{1}{s} > 0, \qquad \frac{\epsilon}{n} + \frac{1}{t} > 0. \tag{65}$$

Under certain conditions upon b and d we shall see that (63) implies a bound for $||x|^{\gamma}u|_{L^r}$. For ζ as in (23), we estimate

$$\left(\int |x|^{\gamma r} \zeta\, |u|^r\right)^{1/r} \le ||x|^{\epsilon}u|_{L^t} \left(\int_{|x|<1} |x|^{(\gamma-\epsilon)tr/(t-r)}\right)^{1/r-1/t} \tag{66}$$

and

$$\left(\int |x|^{\gamma r}(1-\zeta)|u|^r\right)^{1/r} \le \left||x|^{\delta}u\right|_{L^s} \left(\int_{|x|>\frac{1}{2}} |x|^{(\gamma-\delta)sr/(s-r)}\right)^{1/r-1/s} \tag{67}$$

be Holder's inequality, provided that

$$\frac{1}{t} \le \quad and \quad \frac{1}{s} \le \frac{1}{r}. \tag{68}$$

The integrals on the right in (66) and (67) converge if

$$\frac{1}{t} + \frac{\epsilon}{n} < \frac{1}{r} + \frac{\gamma}{n} < \frac{1}{s} + \frac{\delta}{n}. \tag{69}$$

One computes that

$$\begin{aligned}
\frac{1}{t} + \frac{\epsilon}{n} &= d\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-d)\left(\frac{1}{q} + \frac{\beta}{n}\right) \\
\frac{1}{r} + \frac{\gamma}{n} &= a\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right) \\
\frac{1}{s} + \frac{\delta}{n} &= b\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-b)\left(\frac{1}{q} + \frac{\beta}{n}\right)
\end{aligned}$$

so that (69) holds whenever

$$\begin{aligned}
&b < a < d \qquad if \quad \frac{1}{p} + \frac{\alpha-1}{n} < \frac{1}{q} + \frac{\beta}{n} \\
&d < a < b \qquad if \quad \frac{1}{p} + \frac{\alpha-1}{n} > \frac{1}{q} + \frac{\beta}{n},
\end{aligned}$$

and (65) holds too if $|d-a|$ and $|b-a|$ are sufficiently small. One computes furthermore that

$$\begin{aligned}
\frac{1}{r} - \frac{1}{s} &= (a-b)\left(\frac{1}{p} - \frac{1}{q} - \frac{1}{n}\right) + \frac{a}{n}(\alpha - \sigma) \\
\frac{1}{r} - \frac{1}{t} &= (a-d)\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{a}{n}(\alpha - \sigma - 1);
\end{aligned}$$

since $a > 0$ and $\sigma < \alpha - 1$,

$$0 < \frac{a}{n}(\alpha - \sigma - 1) < \frac{a}{n}(\alpha - \sigma);$$
therefore if $|b - a|$ and $|a - d|$ are small enough (68) will hold as well. For such choices of b and d, we use (63), (66), and (67) to conclude
$$||x|^{\gamma} u|_{L^r} \leq C.$$

Section (2.2): Fractional Schrodinger Operators

We shall generalize several well known inequalities about the negative spectrum of Schrodinger-like operators on $\mathbb{R}^d$. As an application of our results we shall give a proof of the 'stability of relativistic matter'-one which goes further than previous shows by permitting the inclusion of magnetic fields for values of the nuclear charge all the way up to $Z\alpha = 2/\pi$, which is the critical value in the absence of a field.

There are three main inequalities to which we refer. The first is Hardy's inequality, whose classical form for $d \geq 3$ is the following. (we shall not be precise about the space of functions in question, but will be precise later on.)
$$\int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 - \frac{(d-2)^2}{4|x|^2} |\mathrm{u}(x)|^2 \right) dx \geq 0. \quad (70)$$
The second is Sobolev's inequality for $d \geq 3$,
$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq S_{2,d} \left\{ \int_{\mathbb{R}^d} |u(x)|^{2d/(d-2)} dx \right\}^{1-2/d} = S_{2,d} \|u\|_{2d/(d-2)}^2. \quad (71)$$
The third is the Lieb-Thirring (LT) inequality [91] for the Schr¨odinger operator $H = -\Delta - V(x)$. If its negative eigenvalues are denoted by $-\lambda_1 \leq \lambda_2 \leq \cdots$, and if $\gamma \geq 0$, then
$$\sum_j \lambda_j^{\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_+^{\gamma+d/2} dx = L_{\gamma,d} \|V_+\|_{\gamma+d/2}^{\gamma+d/2}. \quad (72)$$
This holds if and only if $\gamma \geq \frac{1}{2}$ when $d = 1, \gamma > 0$ when $d = 2$ and $\gamma \geq 0$ when $d \geq 3$. (Here and in the sequel $t_- \coloneqq \max\{0, -t\}$ and $t_- \coloneqq \max\{0, t\}$ denote the negative and positive parts of t.)

By duality, (71) is equivalent to the fact that the Schrodinger operator $H = -\Delta - V$, with has no negative eigenvalues if $\|V_+\|_{d/2} \leq kV + \frac{kd}{2} \leq S_{2,d}$. On the other hand, (72) gives an upper bound to the number of negative eigenvalues in terms of $\|V_+\|_{d/2}$ when $\gamma = 0$ and it estimates the magnitude of these eigenvalues when $\gamma > 0$.

All three inequalities can be generalized by the inclusion of a magnetic vector potential A (related to the magnetic field B by $B = \text{curl } A$). That is, ∇ is replaced by $\nabla - iA(x)$, and Δ by $(\nabla - iA(x))^2$. The sharp constants in (70), (71) remain unchanged while the sharp constants in (72) that are independent of A might, in principle, be different from the ones for $A = 0$. However, the best constants known so far do not depend on A. The inclusion of A is easily done in (70), (71) by using the diamagnetic inequality, but the inclusion in (72) is more delicate; one uses the Feynman-Kac path integral formula to show that for each $x, y \in \mathbb{R}^d$ and $t, \tau > 0$, the A-field reduces the magnitude of the heat kernel et $e^{t(\nabla - iA)^2}(x, y)$ relative to $e^{t\Delta}(x, y)$, and hence reduces the resolvent kernel $|[-(\nabla - iA)^2 + \tau]^{-1}(\mathrm{x}, \mathrm{y})|$ relative to $[-\Delta + \tau]^{-1}(x, y)$.

From [80] one can deduce that (70) and (71) can be combined as follows: For $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 - \frac{(d-2)^2}{4|x|^2} |u(x)|^2 + |u(x)|^2 \right) dx \geq \acute{S}_{2,d,\varepsilon} \|u\|^2_{2d/(d-2+\varepsilon)} \tag{73}$$

with $\acute{S}_{2,d,\varepsilon} \to 0$ as $\varepsilon \to 0$. Note the extra term $|u(x)|^2$ on the left side to account for the fact that the left and right sides behave differently under scaling; examples show that it really is necessary to have $\varepsilon > 0$ here.

In [84], a parallel extension of (72) for the negative eigenvalues $-\lambda_j$ of the Schrodinger operator $H = -\Delta - (d-2)^2/(4|x|^2) - V$ is showed. For $\gamma > 0$ and $d \geq 3$,

$$\sum_j \lambda_j^\gamma \leq \acute{L}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_+^{\gamma+d/2} dx = \acute{L}_{\gamma,d} \|V_+\|^{\gamma+d/2}_{\gamma+d/2}, \tag{74}$$

with $\acute{L}_{\gamma,d} \geq L_{\gamma,d}$. Note that there is no need for an ε in (74); the fact that $\varepsilon \neq 0$ in (73) is reflected here in the fact that $\gamma > 0$ is needed. As before, a magnetic vector potential can easily be included in (73), but it does not seem easy to include a magnetic field in (74) by the methods in [84].

We extend these results in several ways. One extension is to include a magnetic field in (74). Another is to consider fractional powers of the (magnetic) Laplacian, i.e., to the case in which $|\nabla - iA|^2$ is replaced by $|\nabla - iA|^{\,2}$ s with $0 < s < \min\{1, d/2\}$ (which means that we can now include one- and two-dimensions). This is a significant generalization because the operator $(-\Delta)^s$ is not a differential operator and it is not 'local'. Really different techniques will be needed. We shall use the heat kernel to prove the analog of (74), in the manner of [87]. A bound on this kernel, in turn, will be derived from a Sobolev-like inequality (the analogue of (73)) by using an analogue of Nash's inequality, as explained in [90]. The appropriate inequalities are naturally formulated in a weighted space with measure $|x|^{-\beta} dx$ for $\beta > 0$. Therefore, a pointwise bound on the heat kernel for a weighted 'Hardy' operator $\exp\{-t|x|^\alpha((-\Delta)^s \, C_{s,d} \, |x|^{-2s} \, 1)|x|^\alpha\}$ for appropriate $\alpha > 0$ will be needed and will not be straightforward to obtain.

In the dimension most relevant for physics, $d = 3$, the earlier case $s = 1$ may be called the non-relativistic case, while the new result for $s = 1/2$ may be called the relativistic case. Indeed, the resulting LT inequality, together with some of the methodology in [92], yields a new proof of the stability of relativistic matter, which will be sketched. The main point, however, is that this new proof allows for an arbitrary magnetic vector potential A. Since the constant in the relativistic ($s = 1/2$) Hardy inequality that replaces $(d-2)^2/4$ is $2/\pi$ (which is the same as the critical value of $Z\alpha$ in the field-free relativistic case), we conclude that we can simultaneously have an arbitrary A-field and the critical value of $Z\alpha$, the nuclear charge times the fine-structure constant. Up to now it was not possible to have both an arbitrary field and $Z\alpha = 2/\pi$. Therefore, the proof of the analogue of (74) for $s = 1/2$ with an A-field opens a slightly improved perspective on the interaction of matter and radiation.

$$\int_{\mathbb{R}^d} |x|^{-2s} |u(x)|^2 dx \leq C_{s,d}^{-1} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi, \qquad u \in C_0^\infty(\mathbb{R}^d), \tag{75}$$

valid for $0 < 2s < d$. Here

$$\hat{u}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(x) e^{-i\xi \cdot x} dx, \qquad \xi \in \mathbb{R}^d,$$

denotes the Fourier transform of u. The sharp constant in (76),

$$C_{s,d} := 2^{2s}\frac{\Gamma^2((d+2s)/4)}{\Gamma^2((d-2s)/4)}, \tag{76}$$

has been found independently by Herbst [86] and Yafaev [98]. Moreover, it is shown that this constant is not achieved in the class of functions for which both sides are finite. In the case $0 < s < \min\{1, d/2\}$, this fact can also be deduced from our ground state representation in Proposition (2.2.9), which therefore represents an independent proof of (75).

We denote $D = -i\nabla$. Consider a magnetic vector potential $A \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ and the self-adjoint operator $(D-A)^2$ in $L^2(\mathbb{R}^d)$. For $0 < s \le 1$ we define the operator $|D-A|^{2s} := ((D-A)^2)^s$ by the spectral theorem. One form of the diamagnetic inequality states that if $0 < s \le 1$ and $u \in$dom $|D-A|^s$ then $|u| \in H^s(\mathbb{R}^d)$ and

$$\left\|(-\Delta)^{s/2}|u|\right\|^2 \le \||D-A|^s u\|^2. \tag{77}$$

Here and in the sequel, $\|\cdot\| = \|\cdot\|_2$ denotes the L^2-norm. We refer for more details concerning (77). Combining this inequality with the Hardy inequality (75) we find that the quadratic form

$$h_{s,A}[u] := \||D-A|^s u\|^2 - C_{s,d}\||x|^{-s}u\|^2 \tag{78}$$

is non-negative on dom $|D-A|^s$ if $0 < s < \min\{1, d/2\}$. We use the same notation for its closure and denote by

$$H_{s,A} = |D-A|^{2s} - C_{s,d}|x|^{-2s}$$

the corresponding self-adjoint operator in $L^2(\mathbb{R}^d)$. Our main result is

Theorem (2.2.11) will be showed. The main ingredient in its proof is a Sobolev-type inequality which might be of independent interest and which we shall present.

For the case $A = 0$ we shall drop the index A from the notation, i.e.,

$$h_{s,A}[u] = \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi - C_{s,d}\int_{\mathbb{R}^d} |x|^{-2s}\,|u(x)|^2 dx.$$

We denote the closure of this form by the same letter. In particular, its domain dom h_s is the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $(h_s[u] + \|u\|^2)^{1/2}$. Note that $H_s(\mathbb{R}^d) \subset$dom h_s with strict inclusion. In particular, there exist functions $u \in$ dom h_s for which both sides of (75) are infinite.

Hardy's inequality (75) implies that h_s is non-negative. The following theorem shows that for functions of compact support it even satisfies a Sobolev-type inequality; i.e., $h_s[u]$ can be bounded from below by an L^q-norm of u.

Note that (95) may be written in the scale-invariant form

$$\|u\|_q \le \acute{\acute{C}}_{q,d,s} h_s[u]^{\frac{d}{2s}\left(\frac{1}{2}-\frac{1}{q}\right)}\|u\|^{\frac{d}{s}\left(\frac{1}{q}-\frac{1}{2^*}\right)}, \tag{79}$$

where $\acute{\acute{C}}_{q,d,s}$ can be expressed explicitly in terms of $\acute{C}_{q,d,s}$. This inequality follows from applying (95) to functions of the form $u_\lambda(x) = u(\lambda x)$ and then optimizing over the choice of λ.

Although the sharp constants in Theorems (2.2.11) and (2.2.6), as well as in Corollary (2.2.8), are unknown, explicit upper bounds involving a certain variational expression can be deduced from our proof. We do evaluate explicit bounds on the constants in Theorem (2.2.6) in the special case $d = 3, s = 1/2$, which is the most interesting case from a physical point of view, as will be explained.

We shall now explain how the inequalities in Theorem (2.2.11) can be used to show stability of relativistic matter in the presence of an external magnetic field. The proof works up to and including the critical value $Z\alpha = 2/\pi$, which is a new result and solves a problem that has been open for a long time. See [88], [89].

We consider N electrons of mass $m \geq 0$ with q spin states ($q = 2$ for real electrons) and K fixed nuclei with (distinct) coordinates $R_1, \dots, R_k \in \mathbb{R}^3$ and charges $Z_1, \dots, Z_k > 0$. A pseudo-relativistic description of the corresponding quantum-mechanical system is given by the Hamiltonian

$$H_{N,K} = \sum_{j=1}^{N} \left(\sqrt{(D_j - \sqrt{\alpha} A(x_j))^2 + m^2} - m \right) + \alpha V_{N,K}(x_1, \cdots, x_N).$$

The Pauli exclusion principle dictates that $H_{N,K}$ acts on functions in the anti-symmetric N -fold tensor product of $L^2(\mathbb{R}^3;\ \mathbb{C}^q)$. Here we use units where $h = c = 1, \alpha > 0$ is the fine structure constant, and

$$V_{N,K}(x_1, \cdots, x_N) := \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1} - \sum_{j=1}^{N} \sum_{k=1}^{K} Z_k \, |x_j - R_k|^{-1}$$
$$+ \sum_{1 \leq k < i \leq k} Z_k Z_l |R_k - R_l|^{-1}.$$

Stability of matter means that $H_{N,K}$ is bounded from below by a constant times $(N + K)$, independently of the positions R_k of the nuclei.

By combining the methods in [92] and our Theorem (2.2.11), one can show the following

Theorem (2.2.1)[77]:(Stability of relativistic matter). There is an $\alpha_c > 0$ such that for all $N, K, q\alpha \leq \alpha_c$ and α max$\{Z_1, \dots, Z_K \leq 2/\pi$ one has

$$H_{N,K} \geq -mN.$$

The constant α_c can be chosen independently of m, A and $R_1, \dots, R_K$.

The constant α_c in Theorem (2.2.1) depends on the optimal constant in the Hardy-inequality (108) for $d = 3$ and $s = 1/2$. A bound on this constant, in turn, can be obtained from our proof in terms of the constant in the Sobolev-Hardy inequalities (91). We do derive a bound on the relevant constants in Appendix A, but these bounds are probably far from optimal. In particular, the available constants do not yield realistic values of α_c so far.

We discovered a different proof of the special case of the Hardy-LT inequality needed in the proof of Theorem (2.2.1), namely the case where the potential V is constant inside the unit ball, and infinite outside. In this special case a substantially improved constant can be obtained, and this permits the conclusion that Theorem (2.2.1) holds for the physical value of α, which equals $\alpha \approx 1/137$. See [85].

We briefly outline the proof of Theorem (2.2.1). An examination of the proof in [92] shows that there are two places that do not permit the inclusion of a magnetic vector potential A. (Localization of kinetic energy – general form) and (Lower bound to the short-range energy in a ball). Our Lemma (2.2.16.) is precisely the extension of to the magnetic case. This lemma implies that in [92] holds also in the magnetic case, without change except for replacing $|D|$ *by* $|D - A|$.

Our Theorem (2.2.11) can be used instead in [92]. In fact, the left side in [92] is bounded from below by $q\|\chi\|_\infty^2$ times the sum of the negative eigenvalues of $|D - A| - \frac{2}{\pi}|x|^{-1} - CR^{-1}\theta_R(x)$, where θ_R denotes the characteristic function of a ball of radius R.

By Theorem (2.2.11), this latter sum is bounded from below by a constant times C^4R^{-1}. The resulting bound is of the same form as the right side, except for the constant. It is this constant that determines the maximally allowed value of the fine structure constant, α_c .The rest of the show remains unchanged. Note that, in particular, the Daubechies inequality [81] remains true also in the presence of a magnetic field.

Before giving the proofs of our main results, we pause to outline the structure.

(i) we give the proof of the Sobolev-Hardy inequalities in Theorem (2.2.6) and Corollary (2.2.8).

(ii) In the following we show what is customarily called the "ground state representation" in Proposition (2.2.9) except that here the "ground state" fails to be an L^2 function. Such a representation for fractional differential operators does not seem to have appeared in the literature before.

(iii) we give the proof of our main Theorem (2.2.11) about Hardy-LT inequalities. We first consider the non-magnetic case $A = 0$. One of the key ingredients is the ground state representation obtained, which allows us to show a certain contraction property of the heat kernel in some weighted L^1-spaces. Nash's argument [90] then allows us to translate the Sobolev-Hardy inequalities in Corollary (2.2.8) into point wise bounds on the heat kernel in an appropriate weighted L^p-space. These bounds lead to the Hardy-LT inequalities via the trace formula in Proposition (2.2.12) in the spirit of [87].

(iv) Finally, we derive diamagnetic inequalities which will allow us to extend the show of Theorem (2.2.11) to the magnetic case.

We show Theorem (2.2.6) and Corollary (2.2.8) We start with a short outline of the structure of the proof.

Our proof is based on the fact that we can control the singularity of $H_s\psi$ near the origin if we know the singularity of ψ at that point (cf. Lemma 2.2.5). Theorem (2.2.6) follows by observing that the L^q-norm of a symmetric decreasing function can be bounded above by integrating the function against $|x|^{d(1/q-1)}$, see Lemma (2.2.5). Moreover, it is enough to restrict one's attention to symmetric decreasing functions. Corollary (2.2.8) follows from Theorem (2.2.6) by an IMS-type localization argument, see Lemma (2.2.7) We present some auxiliary results in the following. The next two contain the shows of Theorem (2.2.6) and Corollary (2.2.8), respectively.

We start with the following integral representation of the operator $(-\Delta)^s$.

Lemma (2.2.2)[77]: Let $d \geq 1$ and $0 < s < 1$. Then for all $u \in H^s(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi = a_{s,d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy, \tag{80}$$

where

$$a_{s,d} \coloneqq 2^{2s-1}\pi^{-d/2}\frac{\Gamma((d+2s)/2)}{|\Gamma(-s)|}. \tag{81}$$

Lemma (2.2.2) is well known; we sketch the proof.

Proof: For fixed y we change coordinates $z = x - y$ and apply Plancherel. Recalling that $(u(\cdot + z))\,^\wedge(\xi) = e^{i\xi\cdot z}\,\hat{u}(\xi)$ we obtain

$$\int\int \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy = \int\left(\int |z|^{-d-2s}\left|e^{i\xi\cdot z}-1\right|^2 dz\right)|\hat{u}(\xi)|^2 d\xi.$$

The integral in brackets is of the form $c_{s,d}|\xi|^{2s}$, with

$$c_{s,d} := \int_0^\infty \int_{\mathbb{S}^{d-1}} \left|e^{irw.\theta} - 1\right|^2 d\theta r^{-2s-1} dr$$
$$= 2\int_0^\infty (\left|\mathbb{S}^{d-1}\right| - (2\pi)^{d/2} r^{-(d-2)/2} J_{(d-2)/2}(r)) r^{-2s-1} dr.$$

Here, $J_{(d-2)/2}$ is the Bessel function of the first kind of order $(d-2)/2$ [78]. Recall that $\left|\mathbb{S}^{d-1}\right| = 2\pi^{d/2}/\Gamma(d/2)$. The formula (81) for $c_{s,d} = a_{s,d}^{-1}$ follows now from

$$\int_0^\infty r^{-z}\left(J_{(d-2)/2}(r) - 2^{-(d-2)/2}\Gamma(d/2)^{-1} r^{(d-2)/2}\right)dr = 2^{-z}\frac{\Gamma((d-2z)/4)}{\Gamma((d+2z)/4)}$$

for $d/2 < Re\, z < (d+4)/2$, see [98].

Let us recall that $|x|^{-\alpha}$ is a tempered distribution for $0 < \alpha < d$ with Fourier transform

$$b_\alpha(|\cdot|^{-\alpha})^\wedge(\xi) = b_{d-\alpha}|\xi|^{-d+\alpha}, \qquad b_\alpha := 2^{\alpha/2}\Gamma(\alpha/2) \tag{82}$$

(see, e.g., [90], where another convention for the Fourier transform is used, however). We assume now that $s < d/2$. Then $(-\Delta)^s|x|^{-\alpha}$ is an L^1_{Loc}-function for $0 < \alpha < d-2s$ and

$$\left((-\Delta)^s - C_{s,d}|x|^{-2s}\right)|x|^{-\alpha} = \Phi_{s,d}(\alpha)|x|^{-\alpha-2s}, \tag{83}$$

where $C_{s,d}$ is defined in (76) and

$$\Phi_{s,d}(\alpha) := \frac{b_{\alpha+2s}b_{d-\alpha}}{b_{d-\alpha-2s}\, b_\alpha} - C_{s,d}$$
$$= 2^{2s}\left(\frac{\Gamma((\alpha+2s)/2)\,\Gamma((d-\alpha)/2)}{\Gamma((d-\alpha-2s)/2)\,\Gamma(\alpha/2)} - \frac{\Gamma^2((d+2s)/4)}{\Gamma^2((d-2s)/4)}\right). \tag{84}$$

Later on we will need the following information about the α-dependence of $\Phi_{s,d}$.

Lemma (2.2.3)[77]: The function $\Phi_{s,d}$ is negative and strictly increasing in $(0,(d-2s)/2)$ with $\Phi_{s,d}\,((d-2s)/2) = 0$.

Proof. First one checks that

$$\lim_{\alpha\to 0}\Phi_{s,d}(\alpha) = -C_{s,d} < 0, \qquad \Phi_{s,d}((d-2s)/2) = 0. \tag{85}$$

Now we abbreviate $\beta := \alpha/2, r := d/2$ and write

$$f(\beta) := \Gamma(\beta)/\Gamma(r-\beta), \qquad g(\beta) := f(\beta+s)/f(\beta),$$

so that $\Phi_{s,d}(\alpha) = 2^{2s}g(\beta) - C_{s,d}$. In view of (85) it suffices to verify that $g(\beta)$ is strictly increasing with respect to $\beta \in (0,(r-s)/2)$. One finds that

$$\frac{f'(\beta)}{f(\beta)} = \frac{\Gamma'(\beta)}{\Gamma(\beta)} + \frac{\Gamma'(r-\beta)}{\Gamma(r-\beta)} = \psi(\beta) + \psi(r-\beta) \tag{86}$$

with $\psi = \Gamma'/\Gamma$ the Digamma function. Hence

$$g'(\beta) = g(\beta)\left(\frac{f'(\beta+s)}{f(\beta+s)} - \frac{f'(\beta)}{f(\beta)}\right) = g(\beta)\int_\beta^{\beta+s} h(t)\, dt$$

where, in view of (86), $h(t) := \psi'(t) - \psi'(r-t)$. Since ψ' is strictly decreasing (see [78]), one has $h(t) > 0$ for $t \in (0, r/2)$. This showes that $g'(\beta) > 0$ for all $\beta \in (0,(r-2s)/2)$. In the case $\beta \in ((r-2s)/2,(r-s)/2)$ one uses in addition the symmetry $h(t) = -h(r-t)$.

Our proof of Theorem (2.2.6) is close in spirit to [79] where a remainder term in the Sobolev inequality on bounded domains was found .We first exhibit functions $\psi \in$dom h_s

in the form domain which do not lie in the operator domain but for which the singularity of the distribution (indeed, function) $H_s\psi$ at $x = 0$ can be calculated explicitly

Lemma (2.2.4)[77]: Let $0 \le \chi \le 1$ be a smooth function on $\mathbb{R}_+$of compact support, with $\chi(r) = 1$ for $r \le 1$ Define

$$\psi_\lambda(x) := \chi(|x|/\lambda)|x|^{-\alpha} \tag{87}$$

for $0 < \alpha < (d-2s)/2$ and $\lambda > 0$. Then $\psi_\lambda \in$dom h_s for $0 < s < 1$ and, for every $\varepsilon > 0$, there exists a$\lambda_\varepsilon = \lambda_\varepsilon(\alpha, d, s, \chi)$ such that for any $\lambda \ge \lambda_\varepsilon$,

$$\left(\left((-\Delta)^2 - C_{s,d}|x|^{-2s}\right)\psi_\lambda\right)(x) \le -\left(\left|\Phi_{s,d}(\alpha)\right| - \varepsilon\right)|x|^{-\alpha-2s} \quad for\ all\ x \in B, \tag{88}$$

in the sense of distributions. Here, $\Phi_{s,d}$ (α) is given in (84), and B denotes the unit ball in $\mathbb{R}^d$.

Proof: It is not difficult to show that $\psi_\lambda \in H^s(\mathbb{R}^d)$, which implies that $\psi_\lambda \in$dom h_s . (Consult the proof of Proposition (2.2.9) for details.) Let $0 \le \varphi \in C_0^\infty(B)$. According to (83) one has

$$\left(\psi_\lambda, \left((-\Delta)^s - C_{s,d}|x|^{-2s}\right)\varphi\right) = \Phi_{s,d}(\alpha)(|x|^{-\alpha-2s}, \varphi) - \left(\tilde{\psi}_\lambda, (-\Delta)^s\varphi\right),$$

where $\tilde{\psi}_\lambda(x) := (1 - \chi(|x|/\lambda))|x|^{-\alpha}$. It follows from Lemma (2.2.2) (with the aid of polarization) that

$$\left(\tilde{\psi}_\lambda, (-\Delta)^s\varphi\right) = -2a_{s,d}\int\int \frac{\tilde{\psi}_\lambda(y)\varphi(x)}{|x-y|^{d+2s}}dxdy \ge -\rho(\lambda)\int_{\mathbb{R}^d}\frac{\varphi(x)}{|x|^{\alpha+2s}}dx,$$

with

$$\rho(\lambda) = \sup_{|x|\le 1} 2a_{s,d}\,|x|^{\alpha+2s}\int \frac{\tilde{\psi}_\lambda(y)}{|x-y|^{d+2s}}dy$$

$$= \sup_{|x|\le 1/\lambda} 2a_{s,d}\,|x|^{\alpha+2s}\int \frac{1-\chi(y)}{|y|^\alpha|x-y|^{d+2s}}dy. \tag{89}$$

Note that $\rho(\lambda)$ is finite for $\lambda \ge 1$, and monotone decreasing to 0 as $\lambda \to \infty$. Hence, for a given $\varepsilon > 0$ we can choose λ_ε such that $\rho(\lambda_\varepsilon) = \varepsilon$. Since $\Phi_{s,d}(\alpha)$ is negative by Lemma (2.2.3) we have established (88).

Lemma (2.2.5)[77]: Let $1 \le q < \infty$ and $u \in L^q(\mathbb{R}^d)$ a symmetric decreasing function. Then

$$\|u\|_q \le q^{-1}|B|^{-1/q'}\int_{\mathbb{R}^d} u(x)|x|^{-d/q'}\,dx \tag{90}$$

where $|B|$ is the volume of the unit ball B in $\mathbb{R}^d$, and $1/q + 1/q' = 1$.

Proof: First note that (90) is true (with equality) if u is the characteristic function of a centered ball. For general u we use the layer cake representation [90], $u(x) = \int_0^\infty \chi_t(x)\,dt$, where χ_t is the characteristic function of a centered ball of a certain t dependent radius. Then, by Minkowski's inequality [90],

$$\|u\|_q \le \int_0^\infty \|\chi_t\|_q dt = q^{-1}|B|^{-1/q'}\int_0^\infty\int \chi_t(x)|x|^{-d/q'}dxdt$$

$$= q^{-1}|B|^{-1/q'}\int u(x)|x|^{-d/q'}dx$$

showing (90)

Now we give

Theorem (2.2.6)[77]: (Local Sobolev-Hardy inequality). Let $0 < s < \min\{1, d/2\} and\ 1 \leq q < 2^* = 2d/(d-2s)$. Then there exists a constant $C_{q,d,s} > 0$ such that for any domain $\Omega \subset \mathbb{R}^d$ with finite measure $|\Omega|$ one has

$$\|u\|_q^2 \leq C_{q,d,s}\, |\Omega|^{2\left(\frac{1}{q}-\frac{1}{2^*}\right)} h_s\,[u], \qquad u \in C_0^\infty(\Omega). \tag{91}$$

Proof. We may assume Ω to be a ball and u to be a spherically symmetric decreasing function. Indeed, passing to the symmetric decreasing rearrangement of u leaves the left side of (91) invariant while it decreases the right side. The kinetic energy term on the right side is decreased by virtue of Riesz's rearrangement inequality (compare with and $\int |u|^2\, |x|^{-2s}\, dx$ increases Moreover, by scaling we may assume that $\Omega = B$, the unit ball.

Since B is bounded, Hölder's inequality implies that it suffices to show (91) for $d/(d-2s) < q < 2d/(d-2s)$. For such q let $\alpha := d/q' - 2s$ and note that $0 < \alpha < (d-2s)/2$. It follows from Lemmas (2.2.5) and (2.2.4) that for symmetric decreasing functions u on B

$$\|u\|_q \leq\ q^{-1}|B|^{-1/\acute{q}} \int_B u(x)|x|^{-d/\acute{q}}dx \leq 2q^{-1}|B|^{-1/\acute{q}}\left|\Phi_{s,d}(\alpha)\right|^{-1}|(u, H_s\psi)|.$$

Here $\psi = \psi\lambda_\varepsilon$ is chosen as in Lemma (2.2.4), with $\varepsilon = |\Phi_{s,d}(\alpha)|/2$. An application of Schwarz's inequality, $|(u, H_s\psi)|^2 \leq\ h_s\,[u]h_s[\psi]$, concludes the show of (91).

We shall give an upper bound on the constant appearing in the Sobolev inequality in the special case of $d = 3$ and $s = 1/2$, which is the case of interest in the application .

We will deduce Corollary (2.2.8) from Theorem (2.2.6) by a localization argument. For comparison, we recall first the IMS formula in the local case $s = 1$. If $\chi_0\,, \ldots, \chi_n$ are Lipschitz continuous functions on $\mathbb{R}^d$ satisfying $\sum_{j=0}^n \chi_j^2 \equiv 1$, then

$$\int_{\mathbb{R}^d}\left(|\nabla u|^2 - \frac{(d-2)^2}{4}\frac{|u|^2}{|x|^2}\right)dx$$
$$= \sum_{j=0}^n \int_{\mathbb{R}^d}\left(\left|\nabla(\chi_j u)\right|^2 - \frac{(d-2)^2}{4}\frac{\left|\chi_j u\right|^2}{|x|^2}\right)dx$$
$$- \int_{\mathbb{R}^d}\sum_{j=0}^n \left|\nabla\chi_j\right|^2 |u|^2 dx. \qquad (92)$$

The following analogous formula for the non-local case, suggested by Michael Loss, is given in [92]. The proof is an immediate consequence of Lemma (2.2.2). For a general ization to the magnetic case, see Lemma (2.2.16) below.

Lemma (2.2.7)[77]: Let $0 < s < \min\{1, d/2\}$ and let $\chi_0, \ldots, \chi_n$ be Lipschitz continuous functions on $\mathbb{R}^d$ satisfying $\sum_{j=0}^n \chi_j^2 \equiv 1$. Then

$$h_s[u] = \sum_{j=0}^n h_s\left[\chi_j u\right] - (u, Lu) \qquad u \in C_0^\infty(\mathbb{R}^d), \tag{93}$$

where L is the bounded operator with integral kernel

$$L(x,y) := a_{s,d}|x-y|^{-d-2s} \sum_{j=0}^n \left(\chi_j(x) - \chi_j(y)\right)^2.$$

Let us recall the following (non-critical) Sobolev embedding theorem, which is easy to show. (Cf., e.g., the proof of [90].) If $s < d/2$ and $2 \leq q < 2^* = 2d/(d-2s)$ then $H^s(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ and

$$\|u\|_q^2 \le S_{q,d,s}\left(\left\|(-\Delta)^{s/2}u\right\|^2 + \|u\|^2\right), \qquad u \in H^s(\mathbb{R}^d). \tag{94}$$

In combination with the localization Lemma (2.2.7) this allows us to give the

Corollary (2.2.8)[77]: (Global Sobolev-Hardy inequality). Let $0 < s < \min\{1, d/2\}$ and $2 \le q < 2^* = 2d/(d-2s)$. Then there exists a constant $C'_{q,d,s} > 0$ such that

$$\|u\|_q^2 \le C'_{q,d,s}\left(h_s[u] + \|u\|^2\right), \qquad u \in C_0^\infty(\mathbb{R}^d). \tag{95}$$

Proof. Let χ_0, χ_1 be smooth functions on $\mathbb{R}^d$ with $\chi_0^2 + \chi_1^2 \equiv 1$ such that $\chi_0(x) = 0$ if $|x| \ge 1$ and $\chi_1(x) = 0$ if $|x| \le \frac{1}{2}$. Let $2 \le q < 2d/(d-2s)$. Then, by Theorem (2.2.6),

$$\|\chi_0 u\|_q^2 \le C_{q,d,s}|B|^{2\left(\frac{1}{q}-\frac{1}{2^*}\right)} h_s[\chi_0 u],$$

and by (94)

$$\|\chi_1 u\|_q^2 \le S_{q,d,s}\left(\left\|(-\Delta)^{s/2}(\chi_1 u)\right\|^2 + \|\chi_1 u\|^2\right)$$
$$\le S_{q,d,s}\left(h_s[\chi_1 u] + \left(2^{2s}C_{s,d} + 1\right)\|\chi_1 u\|^2\right).$$

Hence Corollary (2.2.8) follows from Lemma (2.2.7) noting that L is a bounded operator.

Eq. (83) and Lemma (2.2.3) suggest that the function $|x|^{-(d-2s)/2}$ is a 'generalized ground state' of the operator Hs. Our next goal is to establish a ground state representation. Let us recall the analogous formula in the 'local' case $s = 1$. If $d \ge 3$ and $v(x) = |x|^{(d-2)/2}u(x)$ then

$$\int_{\mathbb{R}^d}\left(|\nabla u|^2 - \frac{(d-2)^2}{4}\frac{|u|^2}{|x|^2}\right)dx = \int_{\mathbb{R}^d}|\nabla v|^2 \frac{dx}{|x|^{d-2}}. \tag{96}$$

The corresponding formula in the non-local case $0 < s < 1$ is more complicated but close in spirit. It was derived some years ago by Michael Loss for the relativistic case $s = 1/2$ and $d = 3$.

Proposition (2.2.9)[77]: (Ground State Representation). Let $0 < s < \min\{1, d/2\}$. If $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ and $v(x) = |x|^{(d-2s)/2}\, u(x)$, then

$$h_s[u] = a_{s,d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}}\frac{dx}{|x|^{(d-2s)/2}}\frac{dy}{|y|^{(d-2s)/2}}, \tag{97}$$

with $a_{s,d}$ given in (81).

Proof: Let $0 < \alpha < (d-2s)/2$. We shall prove that if $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ and $v_\alpha(x) := |x|^\alpha u(x)$ then

$$\int_{\mathbb{R}^d}|\xi|^{2s}|\hat{u}(\xi)|^2\, d\xi - \left(C_{s,d} + \Phi_{s,d}(\alpha)\right)\int_{\mathbb{R}^d}|x|^{-2s}|u(x)|^2\, dx$$
$$= a_{s,d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|v_\alpha(x)-v_\alpha(y)|^2}{|x-y|^{d+2s}}\frac{dx}{|x|^\alpha}\frac{dy}{|y|^\alpha}. \tag{98}$$

The proposition follows by letting $\alpha \to (d-2s)/2$. Indeed, the constant in front of the second integral on the left side then converges to $C_{s,d}$, according to Lemma (2.2.3). By splitting the integral into four regions according to the support of u, it is easy to see that the right side is continuous in α and converges to the right side of (97).

For the proof of (98) we can suppose that the support of u is in the unit ball. We shall first prove the equality for mollified versions of $|x|^{-\alpha}$, namely functions $\omega_n(x) = |x|^{-\alpha}\,\chi(x/n)$, where $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi(x) = 1$ for $|x| \le 1$.

Let us first show that $\omega_n \in H^s(\mathbb{R}^d)$. It is clearly in $L^2(\mathbb{R}^d)$, hence it suffices to establish that $(-\Delta)^{s/2}\omega_n \in L^s(\mathbb{R}^d)$. According to [90] the Fourier transform of ω_n is given by the convolution of $\hat{\chi}$ and $|\xi|^{\alpha-d}$. Since χ is assumed to be smooth, $\hat{\chi}$ decays faster than any power of $|\xi|$. It is then easy to see that $\widehat{\omega}_n$ decays like $|\xi|^{\alpha-d}$, and hence $|\xi|^s\widehat{\psi} \in L^2(\mathbb{R}^d)$.

By polarization in Lemma (2.2.6), we get for any f and g in $H^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\xi|^{2s}\overline{\widehat{f}(\xi)}\widehat{g}(\xi)d\xi$$

$$= a_{s,d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\overline{f(x)}-\overline{f(y)}\right)\left(g(x)-g(y)\right)\frac{dxdy}{|x-y|^{d+2s}}. \quad (99)$$

We apply this formula to $g(x)=\omega_n(x)$ and $f(x)=|u(x)|^2/\omega_n(x)=|u(x)|^2|x|^\alpha$. In this case, the right side of (99) is given by

$$a_{s,d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(|u(x)-u(y)|^2-\left|\frac{u(x)}{\omega_n(x)}-\frac{u(y)}{\omega_n(y)}\right|^2\omega_n(x)\omega_n(y)\right)\frac{dxdy}{|x-y|^{d+2s}}. \quad (100)$$

Note that $u(x)/\omega_n(x)=u(x)|x|^\alpha=v_\alpha(x)$ is independent of n, and is a C_0^∞ function since the origin is not in the support of u by assumption. By dominated convergence,(100) converges to

$$a_{s,d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(|u(x)-u(y)|^2-|v_\alpha(x)-v_\alpha(y)|^2|x|^{-\alpha}|y|^{-\alpha}\right)\frac{dxdy}{|x-y|^{d+2s}} \quad (101)$$

as $n\to\infty$. The left side of (99) can be written as (compare with (82))

$$(2\pi)^{d/2}\frac{b_{d-\alpha}}{b_\alpha}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^{2s}\overline{\widehat{f}(\xi)}n^d\widehat{\chi}\left(n(\xi-\acute{\xi})\right)|\acute{\xi}|^{\alpha-d}d\xi d\acute{\xi} \quad (102)$$

Since $\widehat{f}$ decays faster than polynomially, $|\cdot|^{2s}\widehat{f}\in L^p(\mathbb{R}^d)$ for any $1\le p<\infty$. Hence its convolution with the approximate δ-function $(2\pi)^{d/2}n^d\,\widehat{\chi}(n\,\cdot)$ converges to $|\cdot|^{2s}\widehat{f}$ strongly in any L^p, for $1\le p<\infty$ [90]. Therefore, (102) converges to

$$\frac{b_{d-\alpha}}{b_\alpha}\int_{\mathbb{R}^d}|\xi|^{2s+\alpha-d}\overline{\widehat{f}(\xi)}d\xi=\frac{b_{\alpha+2s}\,b_{d-\alpha}}{b_{d-\alpha-2s}\,b_\alpha}\int_{\mathbb{R}^d}|u(x)|^2\,|x|^{-2s}dx. \quad (103)$$

Here we used (82) again. The equality of (101) and (103) proves (98).

The proof of our main result in Theorem (2.2.11) We consider here only the non-magnetic case $A=0$, the extension to non-zero A will be straightforward given the necessary diamagnetic inequalities which we derive in the next. We explain the necessary modifications in the proof of Theorem (2.2.11).

The ground state representation (97) suggests that it is more natural to regard $h_s[u]$ as a function of v given by $v(x)=|x|^{(d-2s)/2}u(x)$. In terms of this function v, the Sobolev-Hardy inequality in (95) can be formulated in the weighted space with measure $|x|^{-\beta}dx$, where $\beta=q(d-2s)/2$. Namely,

$$\|v\|^2_{L^q(\mathbb{R}^d,|x|^{-\beta}dx)}\le\acute{C}_{q,d,s}\left(v,B_\beta v\right)_{L^q(\mathbb{R}^d,|x|^{-\beta}dx)}, \quad (104)$$

where $B_\beta$is the operator on $L^q\left(\mathbb{R}^d,|x|^{-\beta}dx\right)$ defined by the quadratic form

$$\left(v,B_\beta v\right)_{L^2(\mathbb{R}^d,|x|^{-\beta}dx)}=h_s\left[|x|^{-(d-2s)/2}v\right]+\left\||x|^{-(d-2s)2}v\right\|^2_{L^2(\mathbb{R}^d,dx)}. \quad (105)$$

We suppress the dependence on s in B_β for simplicity. Note that the right side of (105) is independent of . The dependence of B_β on β comes from the measure $|x|^{-\beta}dx$ of the underlying L^2 space, which is determined by the value of q in the Sobolev inequality (104) as $\beta=q(d-2s)/2$. We emphasize again that the choice of the weight $|x|^{-(d-2s)/2}$ on the right side of (105) is determined by the ground state representation (97).

The proof of Theorem (2.2.11) proceeds in the following steps.

(i) From the ground state representation (2.2.11), we will deduce that B_β satisfies the Beurling-Deny criteria, which implies that e^{-tB_β} is a contraction on $L^1(\mathbb{R}^d, |x|^{-\beta}dx)$.

(ii) Together with the Sobolev-Hardy inequality (104) this yields a bound on the kernel of e^{-tB_β} via Nash's method.

(iii) This bound on the heat kernel can then be translated into a LT bound in the spirit of [87].

Let $\mathfrak{H}_\beta := L^2(\mathbb{R}^d, |x|^{-\beta}dx)$. We assume that $d-2s<\beta<d$, which corresponds to $2<q<2^*$ in (104). The quadratic form

$$b_\beta[v] \coloneqq h_s\left[|x|^{-(d-2s)/2}v\right] + \left\||x|^{-(d-2s)2}v\right\|^2,$$

considered in the Hilbert space $\mathfrak{H}_\beta$, is non-negative and closable on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$, and hence generates a self-adjoint operator B_β in $\mathfrak{H}_\beta$.

We shall deduce some positivity properties of the operator $\exp(-tB_\beta)$. By Proposition (2.2.9) the quadratic form b_β satisfies

(i) if $v, w \in \operatorname{dom} b_\beta$ are real-valued then $b_\beta[v+iw] = b_\beta[v] + b_\beta[w]$,

(ii) if v ∈dom bis real-valued then $|v|$ ∈dom b_β and $b_\beta[|v|] \le b_\beta[v]$,

(iii) if v ∈dom b_β is non-negative then $\min(v,1) \in \operatorname{dom} b_\beta$ and $b_\beta[\min(v,1)] \le b_\beta[v]$.

By a theorem of Beurling-Deny (see [82] or [94]) this implies that $\exp(-tB_\beta)$ is positivity-preserving and a contraction in $L^1(\mathbb{R}^d, |x|^{-\beta}dx)$. That is, it maps non-negative functions into non-negative functions, and it decreases L^1-norms.

From the contraction property derived above, we will deduce a point wise bound on the heat kernel, i.e., on the kernel of the integral operator $\exp(-tB_\beta)$. We emphasize that this kernel is defined by

$$\left(\exp\left(-tB_\beta\right)v\right)(x) = \int_{\mathbb{R}^d} \exp\left(-tB_\beta\right)(x,y)v(y)\frac{dy}{|y|^\beta}.$$

We shall use the Sobolev inequality (104) for this purpose.

Proposition (2.2.10)[77]: Let $d-2s<\beta<d$. Then $\exp(-tB_\beta)$is an integral operator on $\mathfrak{H}_\beta$ and its kernel satisfies

$$0 \le exp\left(-tB_\beta\right)(x,y) \le K_{\beta,d,s}t^{-p} \qquad t>0, x,y \in \mathbb{R}^d, \tag{106}$$

where $p := \beta/(\beta-d+2s)$. The constant can be chosen to be $K_{\beta,d,s} := (p\, C'_{q,d,s})^p$ where $C'_{q,d,s}$ is the constant from Corollary (2.2.8) with $q := 2\beta/(d-2s)$.

Proof: Let $\theta := (q-2)/(q-1) \in (0,1)$. Then Holder's inequality and Corollary (2.2.8) yield for any $v \in C_0^\infty(\mathbb{R}^d \setminus \{0\}$

$$\|v\|^2_{L^2(\mathbb{R}^d,|x|^{-\beta}dx)} \le \|v\|^{1-\theta}_{L^q(\mathbb{R}^d,|x|^{-\beta}dx)}\|v\|^\theta_{L^1(\mathbb{R}^d,|x|^{-\beta}dx)} \le \acute{C}_{q,d,s}^{(1-\theta)/2} b_\beta[v]^{(1-\theta)/2}\|v\|^\theta_{L^1(\mathbb{R}^d,|x|^{-\beta}dx)}.$$

Equivalently, if p is as in the proposition, then

$$\|v\|^\theta_{L^2(|x|^{-\beta}dx)} \le \acute{C}_{q,d,s}^{(1/2} b_\beta[v]^{1/2}\|v\|^{1/p}_{L^1(|x|^{-\beta}dx)}. \tag{107}$$

This is a Nash-type inequality in $\mathbb{R}^d$dwith measure $|x|^{-\beta}dx$. By Nash's argument (see [90] or [82]) this implies that $\exp(-tB_\beta)$ is an integral operator with kernel satisfying (106), with the constant $K_{\beta,d,s}$ given in the proposition. For the sake of completeness we sketch the proof of this claim.

Theorem (2.2.11)[77]: (Hardy-Lie b-Thirring inequalities). Let $\gamma>0$ and $0<s<min\{1,d/2\}$. Then there exists a constant $L_{\gamma,d,s}>0$ such that for all V and A

$$tr\left(|D-A|^{2s}-C_{s,d}|x|^{-2s}-V\right)_-^{\gamma} \leq L_{\gamma,d,s}\int_{\mathbb{R}^d} V(x)_+^{\gamma+d/2s}\,dx. \tag{108}$$

Proof: Step 1. As a first step, we seek an upper bound on the number of eigenvalues below $-\tau$ of the operator $H_s - V$, which we denote by $N\,(-\tau, H_s - V)$. By the variational principle we may assume that $V \geq 0$. Then the Birman-Schwinger principle (see, e.g., [94]) implies that for any increasing non-negative function F on $(0,\infty)$

$$N\,(-1, H_s - V\,) \leq F(1)^{-1} trF\left(V^{1/2}(H_s+I)^{-1}V^{1/2}\right).$$

Let $U\colon L_2(\mathbb{R}^d) \to \mathfrak{H}_\beta$ be the unitary operator which maps $u \mapsto |x|^{\beta/2}\,u$. Then

$$V^{1/2}(H_s+I)^{-1}V^{1/2}U^*W_\beta^{1/2}B_\beta^{-1}W_\beta^{1/2}U, \tag{109}$$

where W_β is the multiplication operator on $\mathfrak{H}_\beta$ which multiplies by the function $W_\beta(x) := |x|^{\,\beta+2s-d}V(x)$. Therefore

$$trF\left(V^{1/2}(H_s+I)^{-1}V^{1/2}\right) = tr_{\mathfrak{H}_\beta}F\left(W_\beta^{1/2}B_\beta^{-1}W_\beta^{1/2}\right). \tag{110}$$

We need the following trace estimate.

Proposition (2.2.12)[77]: Let f be a non-negative convex function on $[0,\infty)$, growing poly-nomially at infinity and vanishing near the origin, and let

$$F(\lambda) := \int_0^\infty f(\mu)\,e^{-\mu/\lambda}\mu^{-1}d\mu, \qquad \lambda > 0. \tag{111}$$

Then for any $d-2s < \beta < d$ and any multiplication operator $W \geq 0$

$$tr_{\mathfrak{H}_\beta}F\left(W^{1/2}B_\beta^{-1}W^{1/2}\right) \leq \int_0^\infty \int_{\mathbb{R}^d} \exp\left(-tB_\beta\right)(x,x) f(tW(x))\,\frac{dx}{|x|^\beta}\,\frac{dt}{t}. \tag{112}$$

Note that the heat kernel $\exp\left(-tB_\beta\right)(x,y)$ is well defined on the diagonal $x=y$ by the semigroup property. Namely, $\exp\left(-tB_\beta\right)(x,x) = \int\left|\exp\left(-tB_\beta/2\right)(x,y)\right|^2 |y|^{-\beta}\,dy$. For the proof of Proposition (2.2.12) one follows the proof of the CLR bound in [87] (see also [97] and [95]). As in the latter Trotter's product formula can be used in place of path integrals.

We shall now assume that F has the special form (111) in order to apply the trace estimate from Proposition (2.2.12). Given $d-2s < \beta < d$ and $p = \beta/(\beta-d+2s)$, Proposition (2.2.10) implies that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \exp\left(-tB_\beta\right)(x,x) f\left(tW(x)\right)\frac{dx}{|x|^\beta}\,\frac{dt}{t} \\
&\leq K_{\beta,d,s}\int_0^\infty \int_{\mathbb{R}^d} t^{-p} f\left(tW_\beta(x)\right)\frac{dx}{|x|^\beta}\,\frac{dt}{t} \\
&= K_{\beta,d,s}\left(\int_{\mathbb{R}^d} W_\beta(x)^p\,\frac{dx}{|x|^\beta}\right)\left(\int_0^\infty t^{-p-1} f(t)dt\right).
\end{aligned}$$

Note that $W_\beta(x)^p = V(x)^p|x|^\beta$. We conclude that for any $d/2s < p < \infty$,

$$N(-1, H_s - V) \leq \acute{K}_{p,d,s}\int_{\mathbb{R}^d} V\,(x)_+^p\,dx, \tag{113}$$

where the constant is given by

$$\acute{K}_{p,d,s} = K_{\beta,d,s}\inf_f F(1)^{-1}\left(\int_0^\infty t^{-p-1} f(t)dt\right).$$

Here $\beta = p(d-2s)/(p-1)$, and the infimum runs over all admissible functions f from Proposition (2.2.12) In order to obtain an explicit upper bound one may choose $f(x) := (x-a)_+$ and minimize over $a > 0$.

Step 2. Now we use the idea of [91] to deduce (108) from (113). Fix $\gamma > 0$ and choose some $d/(2s) < p < \gamma + d/(2s)$. First we note that by scaling we have, for any $\tau > 0$,

$$N(-\tau, H_s - V) = N(-1, H_s - V_\tau)$$

where $V_\tau(x) := \tau^{-1} V(\tau^{-1/2s}\, x)$. In view of (113) this yields

$$N(-\tau, H_s - V) \le \acute{K}_{p,d,s}\, \tau^{-p+\frac{d}{2s}} \int_{\mathbb{R}^d} V\,(x)_+^p dx. \tag{114}$$

Now, for any fixed $0 < \sigma < 1$, one has by the variational principle

$$N(-\tau, H_s - V) \le N(-(1-\sigma)\tau\; H_s - (V - \sigma\tau)_+).$$

Hence, by (114),

$$tr(H_s - V)_-^\gamma = \int_0^\infty N(-\tau, H_s - V)\, \tau^{\gamma-1} d\tau$$

$$\le \gamma \acute{K}_{p,d,s} (1-\sigma)^{-p+d/2s} \int_0^\infty \int_{\mathbb{R}^d} (V(x) - \sigma\tau)_+^p\, dx\, \tau^{\gamma-p+d/2s-1} d\tau.$$

We change the order of integration and calculate the τ -integral first. For fixed $x \in \mathbb{R}^d$,

$$\int_0^\infty (V(x) - \sigma\tau)_+^p\, \tau^{\gamma-p+d/2s-1} d\tau = \sigma^{\gamma-p-d/2s-1} V(x)_+^{\gamma+d/2s} B(\gamma + d/2s - p, p+1).$$

Here, B denotes the Beta-function $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Minimization over $\sigma \in (0,1)$ and $p \in (d/2s, \gamma + d/2s)$ yields

$$tr(H_s - V)_-^\gamma \le C_{d,s}(\gamma) \int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d}{2s}} dx \tag{115}$$

with

$$C_{d,s}(\gamma) := \min_{d/2s<p<\gamma+d/2s} \left\{ \begin{array}{l} \gamma^{\gamma+1} \acute{K}_{p,d,s} B(\gamma + d/2s - p, p+1) \\ \times (\gamma + d/2s - p)^{-\gamma-d/2s+p} (p - d/2s)^{-p+d/2s} \end{array} \right\}.$$

This concludes the proof of Theorem (2.2.11) in the case $A = 0$.

We prove certain diamagnetic inequalities which allow us to extend the proof of Theorem (2.2.11) to the case of non-zero magnetic fields. The main idea is contained in Proposition (2.2.13). The following contains some technical refinements we will need. We describe the necessary modifications in the proof of Theorem (2.2.11) to include magnetic fields. We devoted to the special case $s = 1$.

We shall assume that $A \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ and that $d \ge 2$. Note that in $d = 1$ any magnetic vector potential can be removed by a gauge transformation. In $\mathfrak{H}_\beta = L^2(\mathbb{R}^d, |x|^{-\beta} dx)$ consider the quadratic form

$$b_{\beta,A}[v] := h_{s,A}\big[|x|^{-(d-2s)/2}\big] + \big\||x|^{-(d-2s)/2} v\big\|^2, \qquad v \in C_0^\infty(\mathbb{R}^d/\{0\}). \tag{116}$$

We show that $b_{\beta,A}$, is closable and hence defines a self-adjoint operator $B_{\beta,A}$ in $\mathfrak{H}_\beta$. Our goal is to show that $\exp(-tB_{\beta,A})$ is an integral operator, whose kernel satisfies

$$\big|\exp(-tB_{\beta,A})(x,y)\big| \le \exp(-tB_\beta)(x,y). \tag{117}$$

We consider weighted magnetic operators $\omega|D-A|^{2s}\omega$ where $A \in L^2_{loc}(\mathbb{R}^d)$ and $\omega > 0$ with $\omega + \omega^{-1} \in L^\infty(\mathbb{R}^d)$. This is a self-adjoint operator in $L^2(\mathbb{R}^d)$ with form domain ω^{-1} dom $|D-A|^s$. It satisfies the following diamagnetic inequality.

Proposition (2.2.13)[77]: (Weighted diamagnetic inequality). Let $d \geq 2$ and $0 < s \leq 1$. Assume that $A \in L^2_{loc}(\mathbb{R}^d)$ and that $\omega > 0$ with $\omega + \omega^{-1} \in L^\infty(\mathbb{R}^d)$. Then for all $u \in L^2(\mathbb{R}^d)$ and all $t \geq 0$ one has

$$|\exp(-t\omega|D - A|^{2s}\omega)u| \leq \exp(-t\omega(-\Delta)^s\omega)|u| \tag{118}$$

Proof: First note that the assertion is true in the case $\omega \equiv 1$, i.e.,

$$|\exp(-t|D - A|^{2s})u| \leq \exp(-t(-\Delta)^s)|u|. \tag{119}$$

Indeed, for $s = 1$ this inequality is proved in [96] for all $A \in L^2_{loc}(\mathbb{R}^d)$. The general case $0 < s < 1$ follows from the fact that the function $\lambda \mapsto e^{-\lambda^s}$ sis completely monotone (i.e., its derivatives are alternating in sign) and hence is the Laplace transform of a positive measure by Bernstein's theorem [83]. This reduces the problem to the case $s = 1$.

Now assume that ω is as in the proposition and write $M_A := \omega|D - A|^{2s}\omega$. In view of the general relation

$$\exp(-tM_A) = s - \lim_{n\to\infty}(I + n^{-1} + tM_A)^{-n} \tag{120}$$

it suffices to prove the inequality $|(M_A + \tau)^{-1}u| \leq (M_0 + \tau)^{-1}|u|$. But since $(M_A + \tau)^{-1} = \omega^{-1}(|D - A|^{2s} + V)^{-1}\omega^{-1}$ with $V := \tau\omega^{-2}$ it suffices to prove $|(|D - A|^{2s} + V)^{-1}u| \leq (-\Delta^s + V)^{-1}\,|u|$. In view of the relation 'inverse' to (120),

$$(|D - A|^{2s} + V)^{-1} = \int_0^\infty \exp\big(-t(|D - A|^{2s} + V)\big)\,dt,$$

the assertion follows from (119) and Trotter's product formula.

In order to prove the desired inequality (117), we have to extend Proposition (2.2.13) in two directions. First, we want to use the singular weight $\omega(x) = |x|^\alpha, 0 < \alpha < s$, and second, we want to replace the operator $|D - A|^{2s}$ by $H_{s,A} + I$, i.e., we want to subtract the Hardy term.

Recall that $B_{\beta,A}$ was defined by (116). The main result is the following.

Proposition (2.2.14)[77]: (Weighted diamagnetic inequality, second version). Let $d \geq 2$ and $d - 2s < \beta < d$. Assume that $A \in L^2_{loc}(\mathbb{R}^d)$. Then for all $v \in \mathfrak{H}_\beta$ one has

$$\big|\exp\big(-tB_{\beta,A}\big)v\big| \leq \exp\big(-tB_\beta\big)|v|. \tag{121}$$

It follows from Proposition (2.2.10) that $\exp(-tB_\beta)$ is an integral operator that maps $L^1(\mathbb{R}^d, |x|^{-\beta}dx)$ to $L^\infty(\mathbb{R}^d, |x|^{-\beta}dx)$. Hence (121) implies that the same is true for $\exp(-tB_{\beta,A})$. Moreover, the kernels are related by the inequality (117).

We will need the following approximation result.

Lemma (2.2.15)[77]: Let T_n, T be closed, densely defined operators in a Hilbert space $\mathfrak{H}$ with $T_nT_n^* \to T\,T^*$in strong resolvent sense. Assume that there is a set $\mathcal{D} \subset \bigcap \text{dom }\ T_n \cap \text{dom T}$, dense in $\mathfrak{H}$, such that $T_n\varphi \to T\,\varphi$ for all $\varphi \in \mathcal{D}$. Then $T_n^*T_n \to T^*T$ in strong resolvent sense.

Proof: For $\gamma > 0$ and $\varphi, \psi \in \mathcal{D}$ one has

$$\gamma(\varphi, (T_n^*T_n + \gamma)^{-1}\psi) = (\varphi, \psi) - (T_n\varphi, (T_nT_n^* + \gamma)^{-1}T_n\psi).$$

By assumption the right side converges to $(\varphi, \psi) - (T\varphi, (TT^* + \gamma)^{-1}T\psi) = \gamma(\varphi, (T^*T + \gamma)^{-1}\psi)$, which proves that $T_n^*T_n \to T^*T$ in weak resolvent sense. However, the latter is the same as in strong resolvent sense, see [93].

Proof: Step 1. Consider again the unitary transformation $U: L^2(\mathbb{R}^d) \to \mathfrak{H}_\beta$, which maps $u \mapsto |x|^{\beta/2}u$. Then

$$Q_{\beta,A} := U^*B_{\beta,A}U \tag{122}$$

is a self-adjoint operator in the unweighted space $L^2(\mathbb{R}^d)$, whose quadratic form is given by

$$q_{\beta,A}[u] := h_{s,A}[|x|^\alpha u] + \||x|^\alpha u\|^2, \quad \alpha = (\beta + 2s - d)/2. \tag{123}$$

Eq. (121) is equivalent to

$$|\exp(-t\, Q_{\beta,A})u| \le \exp(-t\, Q_{\beta,0})|u|. \tag{124}$$

for $u \in L^2(\mathbb{R}^d)$.

Step 2. We begin by considering the case where the 'potential terms' in the definition of $Q_{\beta,A}$, are absent. We consider the operator $M_{\beta,A}$ in $L^2(\mathbb{R}^d)$, generated by the quadratic form $\||D-A|^s|x|^\alpha u\|^2$ on $C_0^\infty(\mathbb{R}^d\setminus\{0\})$We shall prove that for all $u \in L^2(\mathbb{R}^d)$ one has

$$|\exp(-t\, M_{\beta,A})u| \le \exp(-t\, M_{\beta,0})|u|. \tag{125}$$

Let ω_n be a family of smooth positive functions which decrease monotonically to $|x|^\alpha$ and agree with this function outside a ball a radius n^{-1} .Similarly, for fixed n let $\omega_{n,m}$, be a family of smooth positive and bounded functions which increase monotonically to $\omega_n$and agree with this function inside a ball a radius m.

The operators $|D-A|^s\omega_n$and $|D-A|^s\omega_{n,m}$,mare easily seen to be closable on $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ (as in Appendix C) and we denote their closures by T_n and $T_{n,m}$, respectively.One finds that $C_0^\infty(\mathbb{R}^d \subset \mathrm{dom}\, T_n^*$ with $T_n^*\upsilon = \omega_n|D-A|^s\,\upsilon$ for $\upsilon \in C_0^\infty(\mathbb{R}^d)$, and similarly for $T_{n,m}^*$. By construction of $\omega_{n,m}$ the operators $T_{n,m}T_{n,m}^*$ are monotonically increasing as $m \to \infty$ and hence converge in strong resolvent sense to $T_nT_n^*$ by [93]. Noting that $T_{n,m},\varphi \to T_n\varphi$ for any $\varphi \in C_0^\infty(\mathbb{R}^d\setminus\{0\})$ we conclude from Lemma (2.2.15) that $T_{n,m}^*, T_{\ n,m} \to T_n^*T_n$ in strong resolvent sense. One checks that $T_{n,m}^*$ coincides with the operator $\omega_{n,m}\,|D-A|^{2s}\omega_{n,m}$ and satisfies a diamagnetic inequality by Proposition (2.2.13). By the strong resolvent convergence the diamagnetic inequality is also valid for $T_n^*T_n$. Now we repeat the argument for $n \to \infty$ where we have monotone convergence from above. We apply [82] noting that $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ is a form core for all the operators involved, and conclude that $T_n^*T_n \to M_{\alpha,A}$ in strong resolvent sense. This proves the diamagnetic inequality (125).

Step 3. Now we use another approximation argument to include the Hardy term. We define $R_{\beta,A}$ via the quadratic form $h_{s,A}[|x|^\alpha u]$ on $C_0^\infty(\mathbb{R}^d\setminus\{0\})$. Moreover, for $n \in \mathbb{N}$ let $W_n(x) := C_{s,d}\min\{|x|^{-2(s-\alpha)}, n\}$. The boundedness of W_n, (125) and Trotter's product formula show that the diamagnetic inequality is valid for $M_{\ \beta,A} - W_n$. Since $M_{\beta,A} - W_n \to R_{\beta,A}$ in strong resolvent sense again by monotone convergence we find the diamagnetic inequality (125) with $M_{\beta,A}$ replaced by $R_{\beta,A}$.

Step 4. Finally, we note that $Q_{\beta,A} = R_{\beta,A} + |x|^{2\alpha}$ in the sense of quadratic forms. Moreover, $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ is a core for both quadratic forms involved. Equation (124) follows now from the diamagnetic inequality for $R_{\beta,A}$ by Kato's strong Trotter product formula [93].

In the case of non-vanishing magnetic field, the proof of Theorem (2.2.11) is essentially identical to the one presented. Although (112) does not necessarily hold with $B_{\beta,A}$ instead of B_β , it does hold if B_β is replaced by $B_{\beta,A}$ on the left side only! I.e.,

$$tr_{\mathfrak{H}_\beta} F\left(W^{1/2}B_{\beta,A}^{-1}W^{1/2}\right) \le \int_0^\infty \int_{\mathbb{R}^d} \exp(-tB_\beta)\ (x,x) f(tW(x)) \frac{dx}{|x|^\beta}\frac{dt}{t}.$$

For the proof, one uses the diamagnetic inequality (117) before applying Jensen's inequality. This leads to the conclusion that (113) also holds with magnetic fields, i.e.,

$$N\big(-1, H_{s,A} - V\big) \leq \acute{K}_{p,d,s} \int_{\mathbb{R}^d} V(x)_+^p \, dx, \tag{126}$$

with the same (A-independent) constant $\acute{K}_{p,d,s}$.

For the remainder of the proof, we note that for any $\tau > 0$,

$$N\big(-\tau, H_{s,A} - V\big) = N\big(-1, H_{s,A_\tau} - V_\tau\big)$$

where $V_\tau(x) := \tau^{-1} V(\tau^{-1/2s}\, x)$ and $A_\tau(x) := \tau^{-1/2s} A(\tau^{-1/2s}\, x)$. The scaling of A does not have any effect, however, since the constant in (126) is independent of A.Therefore (115) also holds with H_s replaced by $H_{s,A}$, with same constant $C_{d,s}(\gamma)$.

The special case $s = 1$. As noted, the proof of Theorem (2.2.11) just given works also in the case $s = 1$ in dimensions $d \geq 3$. We briefly comment on the necessary modifications.

The local Sobolev-Hardy inequalities for $s = 1$ have been proved in [80]. Alternatively, one can obtain them following our proof. Using the IMS formula (92) one can obtain the global Sobolev-Hardy inequalities (73). The rest of the proof goes through without change. To verify the Beurling-Deny criteria, one uses the ground-state representation (96) instead of Proposition (2.2.9) Note also that the weighted diamagnetic inequalities include the case $s = 1$.

We shall derive an explicit bound on the constants $C_{q,3,1/2}$for the Sobolev-Hardy inequalities (91) in the case $d = 3$ and $s = 1/2$, which is of interest for our theorem on stability of matter. Let $3/2 < q < 3$ and $\alpha := 2 - 3/q$. For $\lambda > 1$, let

$$\rho(\lambda) := \frac{1-\alpha}{\pi\lambda^{1+\alpha}} \int_1^\infty \frac{dr}{r^{(1+\alpha)/2}(r - \lambda^{-2})}. \tag{127}$$

We will show that

$$C_{q,3,1/2} \leq \frac{\pi^2}{3q^2}(1-\alpha)(3/4\pi)^{4/3} \inf_{\lambda>1} \frac{\lambda^{2(1-\alpha)}}{\big(\left|\Phi_{1/2,3}(\alpha)\right|\big)_+^2}. \tag{128}$$

We remark that in this special case

$$\left|\Phi_{1/2,3}(\alpha)\right| \frac{2}{\pi} - (1 - 3/q)\cos\left(\frac{\pi(1-3/q)}{2}\right).$$

The estimate (128) is a consequence of the following two facts. First, we claim that

$$\limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} h_{1/2}\left[\psi_\lambda^{\varepsilon,\delta}\right] \leq \lambda^{2(1-\alpha)} \frac{\pi^2}{3}(1-\alpha), \tag{129}$$

where $\psi_\lambda^{\varepsilon,\delta}(x) = \lambda^{-\alpha}\psi^{\varepsilon,\delta}(x/\lambda)$ and $\psi^{\varepsilon,\delta}$ is defined for $\varepsilon, \delta > 0$ by

$$\psi^{\varepsilon,\delta}(x) = \begin{cases} |x|^{-\alpha} & for\ |x| \leq 1, \\ |x|^{-1}\,(1 - \varepsilon^2(|x|)^2 - 1) & for\ \ 1 \leq |x|^2 \leq 1 + 1/\varepsilon, \\ 0 & for\ |x|^2 \geq\ 1 + 1/\varepsilon\,. \end{cases} \tag{130}$$

Note that $\psi^{\varepsilon,\delta}$ does not satisfy the smoothness assumption of Lemma (2.2.4), but it can be approximated by such functions in $h_{1/2}$-norm.

Secondly, we claim that $\rho^{\varepsilon,\delta}$ in (89) defined with the function $\psi_\lambda = \psi_\lambda^{\varepsilon,\delta}$ satisfies

$$\lim_{\varepsilon\to 0} \rho^{\varepsilon,\delta}(\lambda) = \rho(\lambda) \tag{131}$$

uniformly in $\delta > 0$ and $\lambda > 1$, with $\rho(\lambda)$ as in (127). Eq. (128) then follows from these two facts, proceeding as in the proof of Theorem (2.2.6). Instead of choosing λ such that $\rho(\lambda) = |\Phi_{1/2,3}(\alpha)|/2$ (we optimize now over the choice of λ.

For the proof of (129) we consider first an arbitrary radial function ψ and, with a slight abuse of notation, we write $\psi(x) = \psi(r)$ for $r = |x|$. Using the ground state representation in Proposition (2.2.9), introducing spherical coordinates, and integrating over the angles, we have

$$h_{1/2}[\psi] = 8\int_0^\infty dr \int_0^\infty ds \frac{rs}{(r^2 - s^2)^2} |r\psi(r) - s\psi(s)|^2.$$

By changing variables $r^2 \to r$ and $s^2 \to s$, this yields

$$h_{1/2}[\psi] = 2\int_0^\infty dr \int_0^\infty ds \frac{1}{(r-s)^2} \left|\sqrt{r}\psi(\sqrt{r}) - \sqrt{s}\psi(\sqrt{s})\right|^2 \tag{132}$$

Now assume that $\psi = \psi^{\varepsilon,\delta}$ as in (130). By scaling, if suffices to prove (129) for $\lambda = 1$. We split the integrals in (132) into several parts. First of all, we have

$$\int_0^1 dr \int_0^1 ds \frac{1}{(r-s)^2} \left(r^{(1-\alpha)/2} - s^{(1-\alpha)/2}\right)^2$$
$$= 2\int_0^1 ds \frac{1}{s^\alpha} \int_0^1 dt \frac{1}{(1-t)^2} \left(1 - t^{(1-\alpha)/2}\right)^2 \tag{133}$$

This identity can be obtained by noting that the integral on the left is the same as twice the integral over the region $r \le s$, and then writing $r = st$ for $0 \le t \le 1$. Simple computations then lead to

$$\int_0^1 dt \frac{1}{(1-t)^2} \left(1 - t^{(1-\alpha)/2}\right)^2$$
$$= \frac{(1-\alpha)^2}{4} \int_0^1 dt \frac{1}{(1-t)^2} \int_t^1 ds\, s^{-(1+\alpha)/2} \int_t^1 du\, u^{-(1+\alpha)/2}$$
$$= \frac{(1-\alpha)^2}{2} \int_0^1 ds \int_t^1 du\, (su)^{-(1+\alpha)/2} \int_0^s dt \frac{1}{(1-t)^2}$$
$$= (1-\alpha)\int_0^1 ds \frac{1}{1-s} s^{(1-\alpha)/2}\left(1 - s^{(1-\alpha)/2}\right). \tag{134}$$

We introduce the function

$$\eta(\lambda) = \int_0^\infty dt \left(\frac{e^{-t}}{t} - \frac{e^{-\lambda t}}{1 - e^{-t}}\right).$$

We note that $\eta(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda)$ is the Digamma-function. It is then easy to see that

$$\int_0^1 ds \frac{1}{1-s} s^{(1-\alpha)/2}\left(1 - s^{(1-\alpha)/2}\right) = \eta(2-\alpha) - \eta(3/2 - \alpha/2). \tag{135}$$

Altogether, we conclude that the contribution of $r \le 1$ and $s \le 1$ to the integral in (132) is given by

$$4(\eta(2-\alpha) - \eta(3/2 - \alpha/2)).$$

Similarly, we proceed with the other terms. We have

$$\lim_{\varepsilon\to 0} \int_0^1 dr \int_1^\infty ds \frac{1}{(r-s)^2} \left(r^{(1-\alpha)/2} - \left[1 - \varepsilon^\delta (s-1)^\delta\right]_+\right)^2$$
$$= \int_0^1 dr \int_1^\infty ds \frac{1}{(r-s)^2} \left(r^{(1-\alpha)/2} - 1\right)^2 = \int_0^1 dr \frac{1}{1-r} \left(r^{(1-\alpha)/2} - 1\right)^2 \tag{136}$$

Here we have used dominated convergence, noting that the integrand is bounded from above by the L^1 function $(r-s)^{-2}(2(1-r^{(1-\alpha)/2})^2+2\min\{1,(s-1)^{2\delta}\})$ for $\varepsilon\le 1$. The contribution of this term to (6) (noting that it appears twice) is thus given by

$$4(2\eta(3/2-\alpha/2)-\eta(1)-\eta(2-\alpha)).$$

We are left with calculating

$$\int_1^{1+1/\varepsilon}dr\int_1^{1+1/\varepsilon}ds\frac{1}{(r-s)^2}\left(\varepsilon^\delta(r-1)^\delta-\varepsilon^\delta(s-1)^\delta\right)^2$$
$$=\int_0^1 dr\int_0^1 ds\frac{1}{(r-s)^2}\left(r^\delta-s^\delta\right)^2=2(\eta(1+2\delta)-\eta(1+\delta)).$$

The last equality follows by proceeding as in (133)–(135). The last term to evaluate is

$$\int_1^{1+1/\varepsilon}dr\int_{1+1/\varepsilon}^{\infty}ds\frac{1}{(r-s)^2}\left(1-\varepsilon^\delta(r-1)^\delta\right)^2$$
$$=\int_0^1 dr\frac{1}{1-r}\left(1-r^\delta\right)^2=2\eta(1+\delta)-\eta(1)-\eta(1+2\delta).$$

We have thus shown that

$$\lim_{\varepsilon\to 0}h_{1/2}\left[\psi^{\varepsilon,\delta}\right]=4(\eta(3/2-\alpha/2)+\eta(1+\delta)-2\eta(1)).$$

Using concavity of η, together with $\eta'(1)=\pi^2/6$, yields the estimate

$$\lim_{\varepsilon\to 0}h_{1/2}\left[\psi^{\varepsilon,\delta}\right]\le\frac{2\pi^2}{3}\left(\frac{1-\alpha}{2}+\delta\right).$$

We proceed similarly for the calculation of $\rho^{\varepsilon,\delta}$. We have

$$\rho^{\varepsilon,\delta}(\lambda)=\frac{2}{\pi\lambda^{1+\alpha}}\int_1^{1+1/\varepsilon}\frac{r^{(1-\alpha)/2}}{(r-\lambda^{-2})^2}\left(1-\frac{1-\varepsilon^\delta(r-1)^\delta}{r^{(1-\alpha)/2}}\right)dr.$$

Eq. (131) then follows by dominated convergence and integration by parts.

We establish the analogue in the general case $A\neq 0$. As explained. First recall that for $s=1$ and $\sum_{j=1}^n\chi_j^2\equiv 1$ one has

$$\int_{\mathbb{R}^d}|D-Au|^2\,dx=\sum_{j=0}^n\int_{\mathbb{R}^d}\left|(D-Au)(\chi_j u)\right|^2dx-\int_{\mathbb{R}^d}\sum_{j=0}^n\left|\nabla\chi_j\right|^2|u|^2\,dx.$$

In this case the localization error $\sum_{j=0}^n\left|\nabla\chi_j\right|^2$ is local and independent of A. The analogue for $s<1$ is

Lemma (2.2.16)[77]: Let $d\ge 2, 0<s<1$ and $A\in L^2_{loc}(\mathbb{R}^d)$. Then there exists a function k_A on $\mathbb{R}^d\times\mathbb{R}^d$ such that the following holds. If $\chi_0,\dots,\chi_n$ are Lipschitz continuous functions on $\mathbb{R}^d$ satisfying $\sum_{j=0}^n\chi_j^2\equiv 1$, then one has

$$\||D-A|^s u\|^2=\sum_{j=0}^n\left\||D-A|^s\chi_j u\right\|^2-(u,L_A u),\qquad u\in dom|D-A|^s,\qquad(137)$$

where L_A is the bounded operator with integral kernel

$$L_A(x,y):=k_A(x,y)\sum_{j=0}^n\left(\chi_j(x)-\chi_j(y)\right)^2.$$

Moreover, for $a.e. x,y\in\mathbb{R}^d$

$$|k_A(x,y)|\le a_{s,d}|x-y|^{-d-2s},\qquad \textit{and hence}\qquad |L_A(x,y)|\le L(x,y)$$

with L defined in Lemma (2.2.7).

Proof: By the argument of [96] we can choose a form core for $|D-A|^{2s}$ which is invariant under multiplication by Lipschitz continuous functions. It suffices to prove (137) only for functions u from such a core.

We write $k_A(x,y,t) :=\exp(-t|D-A|)^{2s}(x,y)$ for the heat kernel and find

$$\sum_{j=0}^{n}\big(\chi_j\, u,(1-\exp(-t|D-A|^{2s})\chi_j\, u\big)=(u,(1-\exp(-t|D-A|^{2s}))u)$$

$$+\frac{1}{2}\sum_{j=0}^{n}\int\int k_A\,(x,y,t)\left(\chi_j(x)-\chi_j(y)\right)^2\overline{u(x)}\,u(y)dxdy.$$

Now we divide by t and note that by our assumption on u the left side converges to $\sum_{j=0}^{n}\left\||D-A|^s\chi_j\, u\right\|^2$ as $t\to 0$.Similarly the first term on the right side divided by t converges to $\||D-A|^s u\|^2$. Hence the last term divided by t converges to some limit $(u\,, L_A u)$. The diamagnetic inequality (119) yields the bound $|k_A(x,y,t)|\le exp(-t(-\Delta)^{2s})(x,y)$. This implies in particular that L_A is a bounded operator. Now it is easy to check that L_A is an integral operator and that the absolute value of its kernel is bounded pointwise by the one of L.

The following helps to clarify the role of the kernel k_A.

Corollary (2.2.17)[77]: Let $u\in \mathrm{dom}\,|D-A|^{2s}$ and assume that $\Omega:=\mathbb{R}^d\backslash supp\, u\ \neq\emptyset$. Then

$$(|D-A|^{2s}u)(x)=-\int_{\mathbb{R}^d}k_A\,(x,y)u(y)dy\quad for\ x\in\Omega.$$

Proof: Let $\varphi\in C_0^\infty(\Omega)$ and choose χ_0, χ_1 such that $\chi_0\ \equiv 1$ on supp $u,\chi_1\ \equiv 1$ on supp φ and $\chi_0^2+\ \chi_1^2\equiv 1$. By polarization, (137) implies $(\varphi,|D-A|^{2s}u)=-(\varphi,L_A u)=-\int\overline{\varphi(x)}\,k\,_A(x,y)u(y)\ dx\ dy$, whence the assertion.

Contain some technical details concerning the quadratic form $b_{\beta,A}$ defined in (116). In particular, we shall show its closability. Throughout we assume that $d\ge 2, 0<s\le 1$ and $A\in L_{loc}^2(\mathbb{R}^d)$.

Lemma (2.2.18)[77]: The sets $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ and $\mathcal{D}:=\{w\in\mathrm{dom}\ (D-A)^2\cap L^\infty(\mathbb{R}^d)$: supp w compact in $\mathbb{R}^d\backslash\{0\}\}$ are form cores for $|D-A|^{2s}$.

Proof: It suffices to prove the statement for $s=1$. In this case it is proved in [96] that $C_0^\infty(\mathbb{R}^d)$ and $\mathcal{D}^*:=\{w\in dom(D-A)^2\cap L^\infty(\mathbb{R}^d)$: supp w compact $\}$ are form cores for $(D-A)^2$. Hence the statement will follow if we can approximate every function in any of these two spaces by functions from the same space vanishing in a neighborhood of the origin. But for functions u from $C_0^\infty(\mathbb{R}^d)$ or $\mathcal{D}^*$ both functions Du and Au are square-integrable. This reduces the lemma to the case $A=0$ where it is well-known.

Now let $d-2s<\beta<d$ and recall that the quadratic form $q_{\beta,A}$ was defined in(123). Note that $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ is invariant under the unitary transformation in (122).Therefore, closability of $q_{\beta,A}$ and $b_{\beta,A}$ on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ are equivalent.

Lemma (2.2.19)[77]: The quadratic form $q_{\beta,A}$, defined in (123), is closable on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$.

Proof: It suffices to show closability of the form $r_{\beta,A}[u]:=h_{s,A}[|x|^\alpha u]$ on $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ for $0<\alpha=(\beta+2s-d)/2<s$.

Let $\mathcal{D}$ be as in Lemma (2.2.18) we shall show that the quadratic form $r_{\beta,A}$ on $\mathcal{D}$ is closable and that $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ is dense with respect to $(r_{\beta,A}[w] + \|w\|^2)^{1/2}$ in the closure of $\mathcal{D}$ with respect to this norm. This implies the assertion.

Let $w \in \mathcal{D}$. Since $|x|^\alpha$ is smooth on supp w and $\operatorname{dom}(D-A)^2$ is invariant under multiplication by smooth functions we have $|x|^\alpha w \in dom(D-A)^2$ and hence $|x|^\alpha w \in dom\,|D-A|^{2s}$. Since $|x|^\alpha$w has compact support it follows from Lemma (2.2.16) and Corollary (2.2.17) that $|(|D-A|^{2s}|\cdot|^\alpha w)(x)| \le C(w)|x|^{-d-2s}$ for all large $|x|$. In particular, $|x|^\alpha|D-A|^{2s}|x|^\alpha\, w \in L^2(\mathbb{R}^d)$. Moreover, $|x|^{-2(s-\alpha)}\, w \in L^2(\mathbb{R}^d)$. It follows that if $u_n \in \mathcal{D}$ such that $u_n \to 0$ in $L^2(\mathbb{R}^d)$., then the bilinear form associated with $r_{\beta,A}$ satisfies

$$r_{\beta,A}[u_n\,, v] = \left(u_n, |x|^\alpha|D-A|^{2s}|x|^\alpha\, w - C_{s,d}|x|^{-2(s-\alpha)}w\right) \to 0$$

as $n \to \infty$. By standard arguments, this proves that $r_{\beta,A}$ is closable on $\mathcal{D}$.

In order to show the density of $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ we let again $w \in \mathcal{D}$. Since $|x|^\alpha w \in$dom $|D-A|^s$, Lemma (2.2.18) yields a sequence $u_n \in C_0^\infty(\mathbb{R}^d\setminus\{0\})$ such that $\||D-A|^s(u_n - |x|^\alpha w)\| + \|u_n - |x|^\alpha w\| \to 0$ .Hence, if $w_n := |x|^{-\alpha}u_n$, then $w_n \to w\; in\; L^2(\mathbb{R}^d\setminus B)$ and $0 \le r_{\alpha,A}[w_n - w] \le \||D\, - A|^s|x|^\alpha(w_n - w)\|^2 \to 0$. Moreover, Hardy's inequality implies also that $\||x|^{-s+\alpha} + (w_n - w)\| \to 0$ and hence $w_n \to w$ in $L^2(B)$. This proves the density of $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ in the closure of $\mathcal{D}$ with respect to $(r_{\alpha,A}[w] + \|w\|^2)^{1/2}$.

For the sake of completeness we recall here how the Nash inequality (107) and the contraction property of $\exp(-tB_\beta)$ imply the heat kernel estimate (106). Let $k(t) := \|exp(-tB_\beta)v\|_{\mathfrak{H}_\beta}^2$. Then

$$\acute{k}(t) = -2b_\beta\left[\exp\left(-tB_\beta\right)v\right] \le -2\acute{C}_{q,d,s}^{-1}k(t)^{1+1/p}\left\|\exp\left(-tB_\beta\right)v\right\|_{L^1\left(|x|^{-\beta}dx\right)}^{-2/p}$$
$$\le -2\acute{C}_{q,d,s}^{-1}k(t)^{1+1/p}\|v\|_{L^1\left(|x|^{-\beta}dx\right)}^{-2/p},$$

where we used (107) and the contraction property. Hence

$$\left(k(t)^{-1/p}\right)' \ge 2p^{-1}\acute{C}_{q,d,s}^{-1}t\|v\|_{L^1\left(|x|^{-\beta}dx\right)}^{-2/p}$$

and, after integration,

$$k(t)^{-1/p} \ge 2p^{-1}\acute{C}_{q,d,s}^{-1}t\|v\|_{L^1\left(|x|^{-\beta}dx\right)}^{-2/p}.$$

This means that

$$\|exp(-tB_\beta)v\|_{\mathfrak{H}_\beta}^2 \le \left(p\,\acute{C}_{q,d,s}/2\right)^p t^{-p}\|v\|_{L^1\left(|x|^{-\beta}dx\right)}^2.$$

By duality, noting that $B_\beta$is self-adjoint in $L^2(|x|^{-\beta}\, dx)$, this also implies

$$\left\|\exp\left(-tB_\beta\right)v\right\|_{L^\infty\left(|x|^{-\beta}dx\right)}^2 \le \left(p\,\acute{C}_{q,d,s}/2\right)^p t^{-p}\|v\|_{\mathfrak{H}_\beta}^2.$$

Finally, by the semi-group property $\exp\left(-tB_\beta\right) = \exp\left(-tB_\beta/2\right)\exp\left(-tB_\beta/2\right)$

$$\left\|\exp\left(-tB_\beta\right)v\right\|_{L^\infty\left(|x|^{-\beta}dx\right)}^2 \le \left(p\,\acute{C}_{q,d,s}/2\right)^{2p}(t/2)^{-2p}\;\|v\|_{L^1\left(|x|^{-\beta}dx\right)}^2.$$

This is exactly the estimate (106) with the constant given in Proposition (2.2.10).

We sketch the argument leading to Proposition (2.2.12). We emphasize that we shall ignore several technical details. We assume that W is smooth with compact support in $\mathbb{R}^d\setminus\{0\}$, and we put $k(x,y,t) := \exp\left(-tB_\beta\right)(x,y)$. We claim that the trace formula

$tr_{\mathfrak{H}_\beta} F\left(W^{1/2} B_\beta^{-1} W^{1/2}\right)$

$$= \int_0^\infty \frac{dt}{d} \lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) f\left(\frac{t}{n}\sum_{j=1}^{n} W(x_j)\right) \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta} \quad (138)$$

holds true for any non-negative, lower semi-continuous function f vanishing near the origin. Here, F is related to f as in (111). In the integral in (138) we use the convention that $x_0 = x_n$. Indeed, by an approximation argument it suffices to prove this formula for

$$F(\lambda) = \lambda/(1+\alpha\lambda) \quad , \quad f(u) = \mu\, e^{-\alpha\mu},$$

where $\alpha > 0$ is a constant. Using the resolvent identity and Trotter's product formula, one easily verifies that

$$F\left(W^{1/2} B_\beta^{-1} W^{1/2}\right) = W^{1/2}\left(B_\beta + \alpha W\right)^{-1} W^{1/2}$$
$$= \int_0^\infty W^{1/2} \exp\left(-t(B_\beta + \alpha W)\right) W^{1/2} dt = \int_0^\infty \lim_{n\to\infty} T_n(t) dt$$

in this case. Here,

$$T_n(t) \coloneqq W^{1/2}\left(\exp\left(-tB_\beta/n\right)\exp(-t\alpha W/n)\right)^n W^{1/2}.$$

The latter is an integral operator. We evaluate its trace via integrating its kernel on the diagonal. Then we arrive at

$$tr_{\mathfrak{H}_\beta} T_n(t) = \int \cdots \int \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) W(x_n) e^{-\frac{\alpha t}{n}\sum_j W(x_j)} \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta}.$$

After symmetrization with respect to the variables this leads to

$$tr_{\mathfrak{H}_\beta} T_n(t) = \frac{1}{t}\int \cdots \int \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) f\left(\frac{t}{n}\sum_j W(x_j)\right) \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta}.$$

The claimed formula (138) follows if one interchanges the trace with the t-integration and the n-limit.

Now we assume in addition that f is convex. Then Jensen's inequality yields

$$\int \cdots \int \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) f\left(\frac{t}{n}\sum_j W(x_j)\right) \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta}$$
$$\leq \int \cdots \int \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) \frac{1}{n}\sum_j f\left(tW(x_j)\right) \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta}$$
$$= \int \cdots \int \prod_{j=1}^{n} k\left(x_j, x_{j-1}, \frac{t}{n}\right) f\left(tW(x_1)\right) \frac{dx_1}{|x_1|^\beta} \cdots \frac{dx_n}{|x_n|^\beta}$$

(Eq. (118) holds also in the magnetic case discussed. Before applying Jensen's inequality, one first has to use the diamagnetic inequality (117) to eliminate the magnetic field in the kernel k, however.) Finally, we use the semigroup property to integrate with respect to the variables $x_2, \ldots, x_n$. We find that the latter integral is equal to

$$\int_{\mathbb{R}^d} k(x,x,t)\, f\left(tW(x)\right) \frac{dx}{|x|^\beta}.$$

Plugging this into (138) leads to the estimate (112).

Details concerning the justification of the above manipulations can be found in [95].

Section (2.3): Caffarelli–Kohn–Nirenberg Inequalities with Remainder Terms

We concerned with Hardy and Hardy–Sobolev type inequalities with remainder terms. In particular, we shall focus on the following Hardy–Sobolev type inequalities due to [7]. For all $u \in C_0^\infty(\mathbb{R}^n)$ it holds

$$\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx \geq C_{a,b} \left(\int_{\mathbb{R}^N} |x|^{-bp} |u|^p dx \right)^{\frac{2}{p}}, \quad (139)$$

where

$$\left.\begin{array}{lll} \text{for } N \geq 3: a < \frac{N-1}{2}, & a \leq b \leq a+1, & p = \frac{2N}{N-2+2(b-a)}; \\ \text{for } N = 2: a < 0\,, & a < b \leq a+1, & p = \frac{2}{(b-a)}; \\ \text{for } N = 1: a < -\frac{1}{2}, & a + \frac{1}{2} < b \leq a+1, & p = \frac{2}{-1+2(b-a)}; \end{array}\right\} \quad (140)$$

Let $D_a^{1,2}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx, \quad (141)$$

which is given by the inner product $(u, v) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v dx$: Then (139) holds for $u \in C_a^{1,2}(\mathbb{R}^n)$. Define the best constant

$$S(a,b) = \inf_{D_a^{1,2}(\mathbb{R}^d \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp} |u|^p dx \right)^{\frac{2}{p}}}. \quad (142)$$

Then it is known that $S(a, a-1) = \left(\frac{N-2-2a}{2}\right)^2$ is never achieved and that for $N \geq 3, 0 \leq a < \frac{N-2}{2}, a \leq b < a+1, S(a,b)$ is achieved only by radial functions (in the case of $a = b = 0$, up to a translation in $\mathbb{R}^N$), which are given by

$$CU_\lambda(x) = C\lambda^{\frac{N-2}{2}} U(\lambda x), \quad (143)$$

where $C \in \mathbb{R}, \lambda > 0$ and

$$U(x) = k_0(1+|x|^\alpha)^{-\beta}, \alpha = \frac{2(N-2-2a)(1+a-b)}{N-2+2(b-a)}, \beta = \frac{N-2+2(b-a)}{2(1+a-b)} \quad (144)$$

with k_0 being chosen such that $\|U\|_a^2 = S(a,b)$ (see [12]).

To motivate our discussion, let us start with the Hardy inequality for the special case $a = 0, b = 1$. In this case (139) gives for $N \geq 3, u \in D^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx.$$

This inequality still holds for $u \in H_0^1(\Omega)$ for any bounded domain Ω. Using a very delicate argument, Brezis and Vazquez first discovered the following improved version of the inequality in bounded domains.

Motivated by and related to the above results, our first result here improves the above Theorems (2.3.4) and (2.3.6), and covers the weighted version as well. To avoid confusion of notations, we define

$$D_a^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|}, \quad (145)$$

where $\|\cdot\|$ is given in (141). Here Ω is a domain in $\mathbb{R}^N$ (not necessarily bounded). Note that when Ω is bounded, $D_a^{1,2}(\Omega) = H_0^1(\Omega)$. Whenever without confusion, we shall use $\|\cdot\|$ for the norm in (141) with a relevant $a < \frac{N-2}{2}$ in place and a domain $\Omega \subset \mathbb{R}^N$ in the context. The symbol $\|\cdot\|_p$ will be used to denote $L^p(\Omega)$ norm when Ω is clear.

We prove Theorems (2.3.3) and (2.3.5). The idea is to use a conformal transformation to convert the problem to an equivalent one defined in a domain on a cylinder $\wp = \mathbb{R} \times S^{N-1}$.This idea has been used in[1]to study the symmetry property of extremal functions for the Caffarelli–Kohn–Nirenberg inequalities (139). More precisely, to a function $u \in C_0^\infty(\Omega \backslash \{0\})$ we associate $v \in C_0^\infty(\widetilde{\Omega})$ by the transformation

$$u(x) = |x|^{\frac{N-2-2a}{2}} v\left(-\ln|x|, \frac{x}{|x|}\right), \tag{146}$$

where $\widetilde{\Omega}$ is a domain on $\wp$ defined by

$$(t, \theta) = \left(-\ln|x|, \frac{x}{|x|}\right) \in \widetilde{\Omega} \iff x \in \Omega \tag{147}$$

In [1], it was proved that when $\Omega = \mathbb{R}^N$, the above transformation defines a Hilbert space isomorphism between $D_a^{1,2}(\mathbb{R}^N)$ and $H^1(\wp)$ whose norm is given $\|v\|_{H^1(\wp)}^2 = \int_\wp \left(|\nabla v|^2 + \frac{N-2-2a}{2}\right) d\mu$.

Using the same computation, we have

Lemma (2.3.1)[99]: Let $N \geq 1$, $a < \frac{N-2}{2}$, $\Omega \subset \mathbb{R}^N$ a domain.Then under the transformation (146)

$$\int_\Omega |x|^{-2a} |\nabla u|^2 dx = \int_{\widetilde{\Omega}} \left[|\nabla v|^2 + \left(\frac{N-2-2a}{2}\right)^2 v^2\right] d\mu$$

and

$$\int_\Omega |x|^{-2(a+1)} u^2 dx = \int_{\widetilde{\Omega}} |v|^2 \, d\mu.$$

Let $\wp_+$($\wp_-$, resp.) denote the domain on $\wp$ with t component positive (negative, resp.).

Lemma (2.3.2)[99]: Let $N \geq 1$ and $\widetilde{\Omega} \subset \wp_+ or \ \widetilde{\Omega} \subset \wp_-$ be a domain.Then for all $v \in C_0^\infty(\widetilde{\Omega})$,

$$\int_{\widetilde{\Omega}} |\nabla v|^2 \, d\mu \geq \frac{1}{4} \int_{\widetilde{\Omega}} \frac{v^2}{t^2} d\mu. \tag{148}$$

Moreover $\frac{1}{4}$ is the best constant if $[L, \infty) \times S^{N-1} \subset \widetilde{\Omega}$ or $(-\infty, -L] \times S^{N-1} \subset \widetilde{\Omega}$ for $L > 0$.

Proof: This is a version of the classical Hardy inequality adapted for the cylinder case. For $v \in C_0^\infty(\widetilde{\Omega})$,

$$\int_0^\infty \frac{v^2(t,\theta)}{t^2} dt = -2 \int_0^\infty \frac{v v_t}{t} dt \leq \left(\int_0^\infty \frac{v^2}{t^2} dt\right)^{\frac{1}{2}} \left(\int_0^\infty v_t^2 \, dt\right)^{\frac{1}{2}}.$$

Thus

$$\int_0^\infty \frac{v^2(t,\theta)}{t^2} dt \leq 4 \int_0^\infty v_t^2 \, (t,\theta) dt.$$

Integrating on S^{N-1} gives the result. Since $\frac{1}{4}$ is the best constant for the classical one-dimensional Hardy inequality (see[103]), the optimality is proved by considering functions depending only on t.

Lemma (2.3.2) implies that if $\widetilde{\Omega} \subset \wp_+ or\ \widetilde{\Omega} \subset \wp_-$, the completion of $C_0^\infty(\widetilde{\Omega})$ under the norm $\sqrt{\int_{\widetilde{\Omega}} |\nabla v|^2 d\mu}$ is well defined, even for $N = 1$, and 2. We denote this space by $D_0^{1,2}(\widetilde{\Omega})$.

Theorem (2.3.3)[99]: Let $N \geq 1, a < \frac{N-2}{2}$. Assume $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then there exists $C = C(a, \Omega) > 0$ such that for all $u \in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|_2^2 - \left(\frac{N-2-2a}{2}\right)^2 \||x|^{-(a+1)}u\|_2^2 \geq C \left\|\left(\ln \frac{R}{|x|}\right)^{-1} |x|^{-a}\nabla u\right\|_2^2. \quad (149)$$

Moreover, when $0 \in \Omega$ the inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-1}$ cannot be replaced by $g(x)\ln\left(\frac{R}{|x|}\right)^{-1}$ with g satisfyin $|g(x)| \to \infty\ as|x| \to 0$.

In the case $a = 0$, by using Hölder inequality, we see (149) implies Theorem (2.3.6). Our approach is quite different from that in [80], [106], in some sense simpler and easier to be adapted for the weighted versions. Following the idea used in [1], we convert the problem from $\mathbb{R}^N$ to one defined on a cylinder $\wp = \mathbb{R} \times S^{N-1}$. From there an inequality similar to the classical one-dimensional Hardy inequality on $(0, \infty)$ is used to tackle the technical part of the proof. We also note that while the sharpness of Theorems (2.3.4) and (2.3.6) is open-ended (for $q < \frac{2N}{N-2}$ and $q < 2$, respectively), the sharpness in Theorem(2.3.3) is close-ended in the sense $\left(\ln \frac{R}{|x|}\right)^{-1}$ cannot be replaced by $\left(\ln \frac{R}{|x|}\right)^{-d}$ for $d < 1$.

We take $\Omega \subset\subset B_R(0)$ just to avoid the singularity of $\left(\ln \frac{R}{|x|}\right)^{-1}$ at $|x| = R$. Here we are interested in the singularity at zero. In fact, if we take $\delta > 0$ such that $B_\delta(0) \subset \Omega$, then it holds for all $u \in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|_{L^2(\Omega)}^2 - \left(\frac{N-2-2a}{2}\right)^2 \||x|^{-(a+1)}u\|_{L^2(\Omega)}^2 \geq C\||x|^{-a}\nabla u\|_{L^2(\Omega\backslash B_\delta(0))}^2,$$

for some $C > 0$.

Using similar ideas, we give another result of the same spirit, which works for bounded domains as well as exterior domains. It also takes into account the singularity of $\ln \frac{R}{|x|}$ at $|x| = R$.

Theorem (2.3.4)[99]: (Brezis and Vazquez [80]). Let $N \geq 3, \Omega \subset \mathbb{R}^N$ bounded. Then there exists $C = C(\Omega) > 0$ such that for all $u \in H_0^1(\Omega)$.

$$\|\nabla u\|_2^2 - \left(\frac{N-2}{2}\right)^2 \||x|^{-1}u\|_2^2 \geq C\|u\|_2^2. \quad (150)$$

From this result, they deduced that for any $2 \leq q < \frac{2N}{N-2}$,

$$\|\nabla u\|_2^2 - \left(\frac{N-2}{2}\right)^2 \||x|^{-1}u\|_2^2 \geq C\|u\|_q^2 \quad (151)$$

for some $C = C(q,\Omega) > 0$. and that q cannot be replaced by $2\frac{2N}{N-2}$. Very recently, Vazquez and Zuazua obtained an improved version of this result.

Proof: A simple scaling argument shows it suffices to take $R = 1$. Let $\gamma_0 = \max_{x\in\Omega}|x|$. Then $\gamma_0 < 1$. By Lemma (2.3.1), under transformation (146), it suffices to show that there exists $C > 0$ such that for all $v \in D_0^{1,2}(\widetilde{\Omega})$.

$$\int_{\widetilde{\Omega}} |\nabla v|^2 d\mu \geq C \int_{\widetilde{\Omega}} \frac{1}{t^2}\left[|\nabla_\theta v|^2 + \left(v_t + \frac{N-2-2a}{2}v\right)^2\right] d\mu.$$

But by Lemma (2.3.2)

$$\int_{\widetilde{\Omega}} \frac{1}{t^2}\left[|\nabla_\theta v|^2 + \left(v_t + \frac{N-2-2a}{2}v\right)^2\right] d\mu$$

$$\leq \quad 2\frac{1}{(\ln\gamma_0)^2}\int_{\widetilde{\Omega}} |\nabla v|^2 d\mu + 2\left(\frac{N-2-2a}{2}\right)^2 4\int_{\widetilde{\Omega}} v_t^2 d\mu$$

$$\leq \left(\frac{2}{(\ln\gamma_0)^2} + 2(N-2-2a)^2\right)\int_{\widetilde{\Omega}} |\nabla v|^2 d\mu.$$

To show the sharpness part of the theorem, assume $g(x)$ satisfies $|g(x)| \to +\infty$ as $|x| \to 0$. We may assume

$$\lim_{|x|\to 0}\frac{|g(x)|}{|\ln|x||} = 0.$$

Now it suffices to construct $v_n \in D_0^{1,2}(\widetilde{\Omega})$ such that

$$\frac{\int_{\widetilde{\Omega}}|\nabla v_n|^2 d\mu}{\int_{\widetilde{\Omega}}\frac{|\mathrm{g}(\mathrm{e}^{-\mathrm{t}})|^2}{t^2}\left(|\nabla_\theta v_n|^2 + \left(\frac{\partial v_n}{\partial t} + \frac{N-2-2a}{2}v_n\right)^2\right) d\mu} \to 0, \quad as\ n\to\infty.$$

Let $R_n \to \infty$, and η be a function defined on $[0,\infty)$ such that $\eta(t) = 1, 0 \leq t \leq 1$, $\eta(t) = 0$, $t \geq 2$, $|\acute{\eta}(t)| \leq 2$. Define

$$v_n(t,\theta) = \eta\left(\frac{|t-R_n|}{R_n}\right).$$

Then for n large, $v_n \in D_0^{1,2}(\widetilde{\Omega})$ since $\widetilde{\Omega}$ contains $[L,\infty)\times S^{N-1}$ for some L large. Then,

$$A_n := \int_{\widetilde{\Omega}} |\nabla v_n|^2 d\mu \leq C\int_{2R_n}^{6R_n}\frac{1}{R_n^2}(\acute{\eta})^2\, dt \leq \frac{C}{R_n}$$

$$B_n := \int_{\widetilde{\Omega}}\frac{|\mathrm{g}(\mathrm{e}^{-\mathrm{t}})|^2}{t^2}\left[|\nabla_\theta v_n|^2 + \left(\frac{\partial v_n}{\partial t} + \frac{N-2-2a}{2}v_n\right)^2\right] d\mu$$

$$\geq C\int_{3R_n}^{5R_n}\frac{|\mathrm{g}(\mathrm{e}^{-\mathrm{t}})|^2}{t^2}\left(\left(\frac{\partial v_n}{\partial t}\right)^2 + \left(\frac{N-2-2a}{2}\right)^2 v_n^2 + (N-2-2a)v_n\frac{\partial v_n}{\partial t}\right) dt.$$

Then

$$\int_{3R_n}^{6R_n}\frac{|\mathrm{g}(\mathrm{e}^{-\mathrm{t}})|^2}{t^2}\left(\frac{\partial v_n}{\partial t}\right)^2 dt = o\left(\frac{1}{R_n}\right), \qquad as\ n\to\infty,$$

and choosing $0 < \beta < 1$,

$$\left|\int_{3R_n}^{5R_n}\frac{|g(e^{-t})|^2}{t^2}v_n\frac{\partial v_n}{\partial t}d\right|\le\int_{3R_n}^{5R_n}\frac{|g(e^{-t})|^{2\beta}}{t^{2\beta}}\left(\frac{\partial v_n}{\partial t}\right)^2dt+\int_{3R_n}^{5R_n}\frac{|g(e^{-t})|^{2(2-\beta)}}{t^{2(2-\beta)}}v_n^2dt$$
$$=o\left(\frac{1}{R_n}\right)+o(1)\int_{3R_n}^{5R_n}\frac{|g(e^{-t})|^2}{t^2}v_n^2dt.$$

Then

$$B_n\ge C\int_{3R_n}^{5R_n}\frac{|g(e^{-t})|^2}{t^2}v_n^2dt-o\left(\frac{1}{R_n}\right)$$
$$\ge C\left(\inf_{t\ge 3R_n}|g(e^{-t})|^2\right).\frac{1}{R_n}-o\left(\frac{1}{R_n}\right).$$

Therefore,

$$\frac{A_n}{B_n}\le\frac{CR_n^{-1}}{C\left(\inf_{t\ge3R_n}|g(e^{-t})|^2\right)R_n^{-1}+o(1)R_n^{-1}}\to 0,\qquad n\to\infty.$$

The proof of Theorem (2.3.3) is complete.

Theorem (2.3.5)[99]: Let $N\ge 1, a\le\frac{N-1}{2}$. Assume $\Omega\subset B_R(0)\ or\ \Omega\subset B_R^C(0)=(\mathbb{R}^N\backslash B_R(0)$. Then for all $u\in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|_2^2-\left(\frac{N-2-2a}{2}\right)^2\||x|^{-(a+1)}u\|_2^2\ge\frac{1}{4}\left\|\left(\ln\frac{R}{|x|}\right)^{-1}|x|^{-(a+1)}u\right\|_2^2. \qquad (152)$$

This inequality is sharp in the sense that $\left(\ln\frac{R}{|x|}\right)^{-1}$ cannot be replaced by $g(x)\left(\ln\frac{R}{|x|}\right)^{-1}$ with $|g(x)|\to\infty$ as $|x|\to 0$ when $0\in\Omega$ (by $g(x)\left(\ln\frac{R}{|x|}\right)^{-1}$ with $|g(x)|\to\infty$ as $|x|\to\infty$ when $B_p^C(0)\subset\Omega$). The best constant $\frac{1}{4}$ is then also sharp.

For $a=0,$ this was proved recently in [100] (see also [102]) under condition $\Omega\subset B_{e^{-1}R}(0)$ and no estimate on the best constant is given there except for $a=0, N=2$.

Next, we turn to Hardy–Sobolev type inequalities which correspond to $a\le b<a+1$ in CKN inequality (139). Recall the norm on $L_w^q(\Omega)$ is defined by

$$\|u\|_{q,w}=\sup_S\frac{\int_S|u|dx}{|S|^{\frac{1}{\acute{q}}}},$$

where $\acute{q}$ is the conjugate exponent of $q, i.e.\frac{1}{q}+\frac{1}{\acute{q}}=1$ and $S\subset\Omega$ has a finite measure.

Theorem (2.3.6)[99]: (Vazquez and Zuazua [106]). Let $N\ge 3$, and $1\le q<2$. Assume Ω is bounded. Then there exists $C=C(q,\Omega)>0$ such that, for all $u\in H_0^1(\Omega)$,

$$\|\nabla u\|_2^2-\left(\frac{N-2}{2}\right)^2\||x|^{-1}u\|_2^2\ge C\|\nabla u\|_q^2. \qquad (153)$$

Here q cannot be replaced by 2.

Proof: Again we may assume $R=1$. Let us assume $a<\frac{N-2}{2}$ first. It suffices then to use Lemmas (2.3.1) and (2.3.2).

Since the constant on the right-hand side is $\frac{1}{4}$, which is independent of $a<\frac{N-2}{2}$,we may send $a\to\frac{N-2}{2}$ in the inequality. This can be done first for smooth functions, i.e. for all $u\in C_0^\infty(\Omega)$, with $a=\frac{N-2}{2}$, (152) is satisfied . This implies $D_a^{1,2}(\Omega)$ with $a=\frac{N-2}{2}$ is well defined and $\||x|^{-a}\nabla u\|_2$ can be taken as its norm. Now a density argument finishes the proof for the case $a = \frac{N-2}{2}$.

For the sharpness of the weight, we use the same test functions v_n as in the proof of Theorem (2.3.3). Then it is easy to see

$$\frac{\int_{\tilde{\Omega}} |\nabla v_n|^2 d\mu}{\int_{\tilde{\Omega}} \frac{|g(e^{-t})|^2}{t^2} d\mu} \le \frac{C}{\left(\inf_{t \ge R_n} |g(e^{-t})|^2\right)} \to 0, \quad as\ n \to \infty.$$

Finally, the constant $\frac{1}{4}$ on the right-hand side is the best constant by Lemma (2.3.2).

We consider the weighted Hardy–Sobolev inequality (139) on $D_a^{1,2}(\mathbb{R}^N)$,

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \ge 0$$

for the parameter range: $N \ge 3, 0 < a < \frac{N-2}{2}, a \le b < a+1,\ p = \frac{2N}{N-2+2(b-a)} \in (2, 2^*]$, where $2^* = \frac{2N}{N-2}$. Recall from introduction that the best constant $S(a,b)$ is achieved by the functions given in (143) and (144). Thus the minimizers for $S(a,b)$ consist of a two dimensional manifold $M \subset D_a^{1,2}(\mathbb{R}^N)$. Let us define

$$d(u, M) = \inf\{\||x|^{-a}\nabla(u - CU_\lambda)\|_2 :\ C \in \mathbb{R}, \lambda > 0\}.$$

We need the following result first which generalizes the results in [101], [79] for the case $a = 0$ to the case $a > 0$.

Lemma (2.3.7)[99]: Let $a > 0$, $a \le b < a+1$. The first two eigenvalues of (155) are given by $\lambda_1 = S(a,b)$ and $\lambda_2 = (p-1)S(a,b)$. The eigenspaces are spanned by U and $\left.\frac{d}{d\lambda}\right|_{\lambda=1} U_\lambda$, respectively.

Proof: It is easy to check that U and $\left.\frac{d}{d\lambda}\right|_{\lambda=1} U_\lambda$ are eigenfunctions corresponding to $S(a,b)$ and $(p-1)S(a,b)$, respectively. Then it suffices to show that any eigenfunction corresponding to an eigenvalue $\lambda \ge (p-1)S(a,b)$has to be radial. Let $C\Psi_i, i = 0,1, \dots$ the sequence of spherical harmonics, which are eigenfunctions of the Laplace–Beltrami operator on $S^{N-1}: -\Delta_{S^{N-1}} \Psi_i = \sigma_i \Psi_i,\ \sigma_0 = 0,\ \sigma_1 = \cdots \sigma_N = N-1,\ \ \sigma_{N+1} > \sigma_N$.Let u be an eigenfunction corresponding to an eigenvalue $\lambda \le (p-1)S(a,b)$. We shall show for all $i \ge 1$,

$$\int_{S^{N-1}} u(r,\theta)\,\Psi_i(\theta) d\theta \equiv 0.$$

Let $\varphi_i = \int_{S^{N-1}} u(r,\theta)\,\Psi_i(\theta)d\theta$. Then we can check

$$\begin{aligned}
\operatorname{div}(|x|^{-2a}\nabla\varphi_i) &= -2a|x|^{-2a-1}\frac{\partial}{\partial r}\varphi_i + |x|^{-2a}\Delta_r\varphi_i \\
&= \int_{S^{N-1}} \left[|x|^{-2a}\Delta_r u(r,\theta) - 2a|x|^{-2a-1}\frac{\partial u}{\partial r}(r,\theta)\right]\Psi_i(\theta)d\theta \\
&= \int_{S^{N-1}} \left[(\operatorname{div}|x|^{-2a}\nabla u) - \frac{|x|^{-2a}\Delta_\theta u}{r^2}\right]\Psi_i(\theta)d\theta \\
&= \int_{S^{N-1}} -\lambda|x|^{-bp}U^{p-2}u\,\Psi_i(\theta)d\theta + \frac{r^{-2a}\sigma_i}{r^2}\int_{S^{N-1}} u\,\Psi_i(\theta)d\theta \\
&= (r^{-2a-2}\sigma_i - \lambda r^{-bp}U^{p-2})\varphi_i.
\end{aligned}$$

Then for any $R > 0$,

$$0=\int_{B_R(0)}\left[\operatorname{div}(|x|^{-2a}\nabla\varphi_i)\frac{\partial U}{\partial r}+(\lambda r^{-bp}U^{p-2}-r^{-2a-2}\sigma_i)\varphi_i\,\frac{\partial U}{\partial r}\right]dx.$$

The first term can be calculated as follows:

$$\int_{B_R(0)}\operatorname{div}(|x|^{-2a}\nabla\varphi_i)U_r\,dx$$

$$=\int_{B_R(0)}\varphi_i\operatorname{div}\big(|x|^{-2a}\nabla(U_r)\big)\,dx-\int_{\partial B_R(0)}|x|^{-2a}\varphi_i\,\langle\nabla(U_r),\frac{x}{R}\rangle\,d\mu$$

$$+\int_{\partial B_R(0)}U_r\,\langle|x|^{-2a}\nabla\varphi_i\,(U_r),\frac{x}{R}\rangle\,d\mu$$

$$=\int_{B_R(0)}\varphi_i\operatorname{div}\big(|x|^{-2a}\nabla(U_r)\big)\,dx+\int_{\partial B_R(0)}R^{-2a}\left(U_r\frac{d\varphi_i}{dr}-U_{rr}\varphi_i\right)d\mu.$$

And using equation $-\operatorname{div}(|x|^{-2a}\nabla U)=S(a,b)|x|^{-bp}U^{p-2}$, we have

$$\int_{B_R(0)}\varphi_i\operatorname{div}\big(|x|^{-2a}\nabla(U_r)\big)\,dx=\int_{B_R(0)}\varphi_i\operatorname{div}\left(|x|^{-2a}U_{rr}\frac{x}{r}\right)dx$$

$$=\int_{B_R(0)}\varphi_i\,[Nr^{-2a}U_{rr}+|x|^{-2a}U_{rrr}-(2a+1)r^{-2a-1}U_{rr}]dx$$

$$=\int_{B_R(0)}\varphi_i\left[(N-2a-1)r^{-2a-1}U_{rr}\right.$$

$$\left.+r^{-2a}\frac{d}{dr}\left(\frac{2aU_r}{r^2}-\frac{N-1}{r}U_r-S(a,b)r^{-bp+2a}U^{p-2}\right)\right]dx$$

$$=\int_{B_R(0)}\varphi_i\left[(N-2a-1)r^{-2a-1}U_{rr}\right.$$

$$+r^{-2a}\left(2a\frac{rU_{rr}-U_r}{r^2}-\frac{(N-1)(rU_{rr}-U_r)}{r^2}+(bp-2a)S(a,b)\;r^{-bp+2a-1}U^{p-2}\right.$$

$$\left.\left.-r^{-bp+2a}(p-1)S(a,b)U^{p-2}U_r\right)\right]dx$$

$$=\int_{B_R(0)}\varphi_i\,r^{-2a}\frac{N-1-2a}{r^2}U_r+\int_{B_R(0)}(bp-2a)\,S(a,b)r^{-bp-1}U^{p-1}\varphi_i$$

$$-(p-1)S(a,b)\int_{B_R(0)}r^{-bp}U^{p-2}U_r\varphi_i.$$

Putting all these together, we get

$$0=\int_{\partial B_R(0)}R^{-2a}\left(U_r\frac{d\varphi_i}{dr}-U_{rr}\varphi_i\right)d\mu+\int_{B_R(0)}\varphi_i\,r^{-2a-2}\,(N-1-\sigma_i-2a)U_r dx$$

$$+\int_{B_R(0)}(bp-2a)S(a,b)\,r^{-bp-1}U^{p-1}\varphi_i dx$$

$$+\big(\lambda-(p-1)S(a,b)\big)\int_{B_R(0)}r^{-bp}U^{p-2}U_r\varphi_i dx.$$

Let R be the first zero of φ_i with $R=+\infty$ if φ_i is not zero anywhere. Without loss of generality assume $\varphi_i(r)>0$, $r\in(0,R)$. Then $\frac{d\varphi_i}{dr}(R)\le 0$. Thus the first and the forth terms are non-negative and the second and the third are positive unless $\varphi_i \equiv 0$ since $bp - 2a > 0$ for $a > 0$. The proof is finished.

Lemma (2.3.8)[99]: For any sequence $(u_n) \subset D_a^{1,2}(\mathbb{R}^N \backslash M$ such that $\inf_n \||x|^{-a}\nabla u_n\|_2^2 > 0$ and $d(u_n, M) \to 0$, it holds

$$\lim_{n\to\infty} \frac{\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}\nabla u_n\right\|_p^2}{d(u_n, M)} \geq 1 - \frac{\lambda_2}{\lambda_3}.$$

Proof: First we assume $d(u_n, M)\||x|^{-a}\nabla(u_n - U)\|_2$. Since $v_n = u_n - U$ is orthogonal to the tangent space of M,

$$T_U M = \operatorname{span}\left\{U, \frac{d}{d\lambda}\Big|_{\lambda=1} U_\lambda\right\},$$

we have by Lemma (2.3.7),

$$\lambda_3 \int |x|^{-bp}\, U^{p-1} v_n^2 dx \leq \||x|^{-a}\nabla v_n\|_2^2 = d^2(u_n, M).$$

Let $d_n = d(u_n, M)$. Using the equation $-\operatorname{div}(|x|^{-2a}\nabla U) = S(a,b)|x|^{-bp}U^{p-1}1$, we get

$$\int |x|^{-bp}\, |u_n|^p dx$$

$$= \int |x|^{-bp}\, U^p dx + p\int |x|^{-bp}\, U^{p-1}v_n dx + \frac{p(p-1)}{2}\int |x|^{-bp}\, U^{p-2}v_n^2 dx$$

$$+ o(d_n^2) = 1 + \frac{p}{2}\frac{\lambda_2}{S(a,b)\lambda_3} d_n^2 + o(d_n^2).$$

Then,

$$\left\||x|^{-b}u_n\right\|_p^2 \leq 1 + \frac{\lambda_2}{\lambda_3}\frac{d_n^2}{S(a,b)} + o(d_n^2).$$

By $\||x|^{-a}\nabla u_n\|_2^2 = S(a,b) + d_n^2$, we have

$$\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}u_n\right\|_p^2 \geq \left(1 - \frac{\lambda_2}{\lambda_3}\right) d_n^2 + o(d_n^2).$$

For the general case, $d(\nabla u_n, M) = \left\||x|^{-a}\nabla\left(u_n - C_n U_{\lambda_n}\right)\right\|_2$ for some $C_n \in \mathbb{R}, \lambda_n > 0$. We can use the invariance of the inequality by dilations to reduce it to the special case above.

Theorem (2.3.9)[99]: For $N \geq 3$, $0 < a < \frac{N-2}{2}$, $a \leq a + 1$, $p = \frac{2N}{N-2+2(b-a)}$, there exists $C = C(N, a, b)$ such that for all $u \in D_a^{1,2}(\mathbb{R}^N)$,

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \geq Cd(u, M)^2. \tag{154}$$

We first consider the eigenvalue problem

$$\begin{cases} -\operatorname{div}(|x|^{-a}\nabla u) = \lambda |x|^{-bp} U^{p-2} u \\ u \in D_a^{1,2}(\mathbb{R}^N) \end{cases} \tag{155}$$

Proof: If the theorem is false, we find $(u_n) \subset D_a^{1,2}(\mathbb{R}^N \backslash M$ such that

$$\frac{\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}u_n\right\|_p^2}{d(u_n, M)^2} \to 0.$$

We may assume $\||x|^{-a}\nabla u_n\|_2^2 = 1$ and thus $L = \lim_{n\to\infty} d(u_n, M) \in [0,1]$. Then

$$\left\||x|^{-b}u_n\right\|_p^2 \to S(a,b)^{-1}.$$

By a concentration-compactness argument [23], [26] we can find $\lambda_n > 0$,

$$\lambda_n^{\frac{N-2-2a}{2}} u_n(\lambda_n x) \to M \ \ in\ D_a^{1,2}(\mathbb{R}^N).$$

This implies $L=0$, a contradiction to Lemma (2.3.8).

For $a=b=0$, (156) was proved by Brezis and Lieb [79] (see also [5], and also by Bianchi and Egnell with a different proof [101]). For $a=0,\ 0<b<1$, (156) was proved by Radulescu et al.[104]. For $a=b=0$, (157) was proved in [79].

The approach to prove Theorem (2.3.10), though follows the idea in[104], [105], but improves theirs. Without using Schwarz symmetrization, our approach is easily adapted for the weighted versions. The method can be used to establish results like (156) in unbounded domains. This partially addresses a question raised by Brezis and Lieb [79].

In order to state our results for unbounded domains, let us define for a domain $\Omega \subset \mathbb{R}^N$,

$$\lambda_1(\Omega)=\inf_{D_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

We say Ω satisfies (Ω_0) condition if there exists an open cone with its vertex at $0, V_0$, such that for some $R>0$, $\Omega^C \supset (V_0 \backslash B_R(0))$. We say Ω satisfies (Ω_1) condition if there exists an open cone at $0, V_0$, such that for some $R>0$, for all $y\in\Omega$, $\Omega^C \supset (y+V_0)\backslash B_R(y)$.

Theorem (2.3.10)[99]: Let $N\geq 3,\ 0\leq a<\frac{N-2}{2},\ a\leq b<a+1,\ p=\frac{2N}{N-2+2(b-a)}$. Assume $\Omega\subset\mathbb{R}^N$ is bounded. Then there exists $C=C(a,b,\Omega)$ such that for all $u\in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \geq C\||x|^{-a}u\|^2_{\frac{N}{N-2-a},w} \tag{156}$$

and

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \geq C\||x|^{-a}\nabla u\|^2_{\frac{N}{N-1-a},w}. \tag{157}$$

Moreover, the weak norm on the right-hand side cannot be replaced by the strong norm.

Proof: Assume that (156) is not true. Then there exist $(u_n)\subset H_0^1(\Omega)$ such that

$$\frac{\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}u_n\right\|_p^2}{\||x|^{-a}u_n\|^2_{\frac{N}{N-2-a},w}} \to 0.$$

We assume $\||x|^{-a}\nabla u_n\|_2^2=1$ and $\||x|^{-a}u_n\|^2_{\frac{N}{N-2-a},w}$ is bounded by Sobolev's inequality. Then $\||x|^{-a}\nabla u_n\|_2^2 \to S(a,b)^{-1}$. By Theorem (2.3.9), there exist $(C_n,\lambda_n)\to(1,\infty)$ such that

$$d(u_n,M)=\left\||x|^{-a}\nabla(u_n-C_nU_{\lambda_n})\right\|_2 \to 0.$$

A direct computation shows

$$\begin{aligned} d(u_n,M)^2 &\geq C_n^2\int_{|x|\geq 1}|x|^{-a2}\left|\nabla U_{\lambda_n}\right|^2 dx \\ &= C\lambda_n^{N-2-2a}\int_1^\infty r^{-2a}\left(1+(\lambda_n r)^2\right)^{-2(\beta+1)}(\lambda_n r)^{2(\alpha-1)}\lambda_n^2\, r^{N-1}dr \\ &= C\int_{\lambda_n}^\infty S^{-2a}\left(1+S^\alpha\right)^{-2(\beta+1)}S^{2(\alpha-1)}S^{N-1}dS \geq C\lambda_n^{2a-(N-2)}, \end{aligned}$$

where $C>0$ is a constant independent of n.

Therefore,

$$
\begin{aligned}
&\| |x|^{-a} u_n \|_{L_w^{\frac{N}{N-2-a}}(\Omega)} \\
&\quad \le \left\| |x|^{-a} \left(u_n - C_n U_{\lambda_n} \right) \right\|_{L_w^{\frac{N}{N-2-a}}(\Omega)} + \left\| |x|^{-a} C_n U_{\lambda_n} \right\|_{L_w^{\frac{N}{N-2-a}}(\Omega)} \\
&\le C \left\| |x|^{-a} \left(u_n - C_n U_{\lambda_n} \right) \right\|_{L^{\frac{2N}{N-2}}(\Omega)} + \left\| C_n |x|^{-a} U_{\lambda_n} \right\|_{L_w^{\frac{N}{N-2-a}}(\mathbb{R}^N)} \\
&\le C \left\| |x|^{-a} \left(u_n - C_n U_{\lambda_n} \right) \right\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + C_n \lambda_n \left\| |x|^{\frac{2a-(N-2)}{2-a}} U \right\|_{\frac{N}{N-2-2a}, w} \\
&\le C d(u_n, M) + C_n \lambda_n^{\frac{2a-(N-2)}{2}} \| |x|^{-a} U \|_{\frac{N}{N-2-a}, w} \\
&\le C d(u_n, M).
\end{aligned}
$$

This is a contradiction with Theorem (2.3.9).

Since, by a direct computation

$$
\left\| |x|^{-a} C_n \nabla U_{\lambda_n} \right\|_{\frac{N}{N-1-a}, w} = C_n \lambda_n^{\frac{2a-(N-2)}{2}} \| |x|^{-a} \nabla U \|_{\frac{N}{N-1-a}, w},
$$

we obtain (157) by a similar argument.

We devoted to proving Theorems (2.3.15) and (2.3.16).

When $a = b = 0$, the manifold of minimizers for $S(0,0)$ is a $N + 2$ dimensional, given by

$$
M(0,0) = \{ C U_\lambda(\cdot + y) \mid C \in \mathbb{R}, \lambda > 0, y \in \mathbb{R}^2 \}
$$

U is given in (144) with $a = b = 0$.

Lemma (2.3.11)[99]: Let $N \ge 3, a = b = 0$. Assume Ω satisfies condition (Ω_1) .Then there exists $C = C(\Omega) > 0$, such that as $\lambda \to \infty$,

$$
\inf_{\lambda \in \Omega} \| \nabla U_\lambda(x + y) \|_{L^2(\Omega^C)}^2 \ge C \lambda^{2-N}.
$$

Proof: Just note that $|\nabla U_\lambda(x + y)|$ is radial in $|x + y|$ and there exists $C > 0$ such that as $\lambda \to \infty$,

$$
\| \nabla U_\lambda(x) \|_{L^2(B_R^C(0))}^2 \ge C \lambda^{2-N}.
$$

Similarly, we have

Lemma (2.3.12)[99]: Let $N \ge 3,\ 0 \le a < \frac{N-2}{2}, a \le b < a + 1, a + b \ne 0$. Assume Ω satisfies condition (Ω_0) .Then there exists $C = C(\Omega) > 0$ such that for $U_\lambda \in M(a, b)$ as $\lambda \to \infty$,

$$
\| |x|^{-a} \nabla U_\lambda \|_{L^2(\Omega^C)}^2 \ge C \lambda^{2a+2-N}.
$$

Lemma(2.3.13)[99]: Let $N \ge 3,\ 0 \le a < \frac{N-2}{2}, a \le b < a + 1$ .Let $\Omega \subset \mathbb{R}^N$ and $P: D_a^{1,2}(\mathbb{R}^N) \to D_a^{1,2}(\Omega)$ be the projection operator. Then for any $U \in M(a, b), 0 \le PU \le U$ in $\mathbb{R}^N$.

Proof: PU is given by $PU = U - v$ where v is the solution of

$$
\begin{cases} -\operatorname{div}(|x|^{-2a} \nabla v) = 0 & \text{in } \Omega, \\ v = U & \text{on } \partial\Omega. \end{cases}
$$

Then PU satisfies

$$
\begin{cases} -\operatorname{div}\left(|x|^{-2a} \nabla (PU) \right) = S(a, b) |x|^{-bp}\, U^{p-1} & \text{in } \Omega, \\ PU = 0 & \text{on } \partial\Omega. \end{cases}
$$

Then $P(U) \ge 0$ in Ω for otherwise, assume $P(U) < 0$ in $\Omega_- \subset \Omega$ Multiplying the equation by PU and integrating on Ω_-, we get

$$\int_{\Omega_-} |x|^{-2a} |\nabla(PU)|^2 = S(a,b) \int_{\Omega_-} |x|^{-bp}\, U^{p-1} P(U) \le 0,$$

which says $PU \equiv$ constant in Ω_-. Then $PU \equiv 0$ in Ω_-a contradiction. Also v satisfies $v \ge 0$ in Ω . Then $PU \le U$.

Lemma (2.3.14)[99]: Let $\lambda_1(\Omega) > 0$. Then $\exists C > 0$, for all $u \in D_a^{1,2}(\Omega)$,

$$\| |x|^{-a} u \|_{L^2(\Omega)} \le C \| |x|^{-a} \nabla u \|_{L^2(\Omega)}.$$

Proof: Since $D_a^{1,2}(\Omega) = \overline{C_0^\infty(\Omega \backslash \{0\})}^{\|\ \|_a}$, we need only consider $u \in C_0^\infty(\Omega\backslash\{0\})$. Then $|x|^{-a} u \in C_0^\infty(\Omega\backslash\{0\})$. But for all $v \in C_0^\infty(\Omega\backslash\{0\})$,

$$\int_\Omega v^2 \le \lambda_1 \int_\Omega |\nabla v|^2.$$

Therefore, using Hardy inequality,

$$\int_\Omega |x|^{-2a} u^2 \le \lambda_1 \int_\Omega |\nabla(|x|^{-a} u)|^2 = 2\lambda_1 \int_\Omega a^2 |x|^{-2(a+1)} u^2 + |x|^{-2a} |\nabla u|^2$$

$$\le C \int_\Omega |x|^{-2a} |\nabla u|^2.$$

Theorem (2.3.15)[99]: Let $N = 3,4$, $\Omega \subset \mathbb{R}^2$ satisfy (Ω_1) and $\lambda_1(\Omega) > 0$. Then there exists $C = C(\Omega) > 0$ such that for all $u \in D_0^{1,2}(\Omega)$

$$\|\nabla u\|_2^2 - S(0,0) \|u\|_{2^*}^2 \ge C \|u\|^2_{\frac{N}{N-2},w}$$

and

$$\|\nabla u\|_2^2 - S(0,0) \|u\|_{2^*}^2 \ge C \|\nabla u\|^2_{\frac{N}{N-1},w}.$$

Proof: Assume that Theorem (2.3.15) is not true. Then there exist $(u_n) \subset \mathrm{D}_0^{1,2}(\Omega)$ such that

$$\frac{\|\nabla u_n\|_2^2 - S(0,0) \|u_n\|_{2^*}^2}{\|u_n\|^2_{\frac{N}{N-2},w}} \to 0, \qquad n \to \infty.$$

We assume $\|\nabla u_n\|_2 = 1$. If $N = 4$, we have, by assumption,

$$\|u_n\|_{\frac{N}{N-2},w} \le \|u_n\|_{\frac{N}{N-2}} \le C \|\nabla u_n\|_2 = C.$$

If $N = 3$, by Hölder inequality and Sobolev inequality, we have

$$\|u_n\|_{\frac{N}{N-2},w} \le \|\nabla u_n\|_{\frac{N}{N-2}} \le \|u_n\|_2^{\lambda} \|u_n\|_{2^*}^{1-\lambda} \le C \|\nabla u_n\|_2 = C.$$

Then $\|u_n\|_{2^*}^2 \to S^{-1}(0,0)$. By the proof of Lemma (2.3.1) in [101], there exists $(C_n, \lambda_n) \to (1, \infty)$ and $(y_n) \subset \Omega$ such that

$$d(u_n, M) = \|\nabla(u_n - U_n)\|_{L^2(\mathbb{R}^N)} \to 0, \quad n \to \infty,$$

where $U_n = C_n U\big(\lambda_n(. - y_n)\big)$. By Lemma (2.3.11),

$$d(u_n, M)^2 \ge \int_{\Omega^C} |\nabla U_n|^2\, dx \ge C C_n^2 \lambda^{2-N}.$$

Using $P: D_0^{1,2}(\mathbb{R}^N) \to D_0^{1,2}(\Omega)$ as the projection operator, we have

$$\begin{aligned} \|u_n\|_{\frac{N}{N-2},w} &\le \|u_n - PU_n\|_{\frac{N}{N-2}} + \|PU_n\|_{\frac{N}{N-2},w} \\ &\le C \|\nabla(u_n - PU_n)\|_{L^2(\Omega)} + \|PU_n\|_{\frac{N}{N-2},w} \\ &\le C \|\nabla(u_n - U_n)\|_{L^2(\mathbb{R}^N)} + \|PU_n\|_{\frac{N}{N-2},w}. \end{aligned}$$

It follows from Lemma (2.3.13) that

$$\|PU_n\|_{\frac{N}{N-2},w} \le \|U_n\|_{L_w^{\frac{N}{N-2}}(\mathbb{R}^N)} \le C_n \lambda_n^{\frac{2-N}{2}} \|U\|_{L_w^{\frac{N}{N-2}}(\mathbb{R}^N)}.$$

Hence

$$\|u_n\|_{\frac{N}{N-2},w} \le Cd(u_n, M).$$

This is a contradiction with the Theorem in [101]. The proof of the second part of Theorem (2.3.15) is similar.

Theorem (2.3.16)[99]: Let $N \ge 3, \max\left\{0, \frac{N-4}{2}\right\} \le a < \frac{N-2}{2}, a \le b < a+1, a+b \ne 0, p = \frac{2N}{N-2+2(b-a)}$. Assume $\Omega \subset \mathbb{R}^N$ satisfies $\lambda_1(\Omega) > 0$ and condition (Ω_0). Then there exists $C = C(a, b, \Omega)$ such that for all $u \in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \ge C\||x|^{-a}u\|^2_{\frac{N}{N-2-a},w}$$

and

$$\||x|^{-a}\nabla u\|_2^2 - S(a,b)\left\||x|^{-b}u\right\|_p^2 \ge C\||x|^{-a}\nabla u\|^2_{\frac{N}{N-1-a},w}.$$

Typical domains that satisfy $\lambda_1(\Omega) > 0$ and (Ω_0) or (Ω_1) are strips or sub domains of strips. Here by strip we mean domains that are bounded in at least one direction.

Due to the translation invariance in Theorem(2.3.15), we need the stronger condition (Ω_1).

Proof: Assume that Theorem (2.3.16) is not true. Then there exist $(u_n) \subset D_a^{1,2}(\Omega)$ such that

$$\frac{\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}u_n\right\|_p^2}{\||x|^{-a}u_n\|^2_{\frac{N}{N-2-a},w}} \to 0, \qquad n \to \infty.$$

We assume $\||x|^{-a}\nabla u_n\|_2 = 1$.Using (139) and Lemma (2.3.14) 8, we obtain

$$\||x|^{-a}u_n\|_{\frac{N}{N-2-2a},w} \le \||x|^{-a}u_n\|^2_{\frac{N}{N-2-a}} \le \||x|^{-a}u_n\|_2^{\lambda}\, \||x|^{-a}u_n\|_{2^*}^{1-\lambda} \le C\||x|^{-a}\nabla u_n\|_2^2 = C.$$

Then $\left\||x|^{-b}u_n\right\|_p^2 \to S^{-1}(a,b)$. By Theorem (2.3.9), there exists $(C_n, \lambda_n) \to (0, \infty)$ such that

$$d(u_n, M) = \left\||x|^{-a}\nabla\left(u_n - C_n U_{\lambda_n}\right)\right\|_{L^2(\mathbb{R}^N)} \to 0, \quad n \to \infty.$$

By Lemma (2.3.12),

$$d(u_n, M)^2 \ge C_2^n \int_{\Omega^C} |x|^{-a}\left|\nabla U_{\lambda_n}\right|^2 dx \ge CC_n^2 \lambda_n^{2a+2-N}.$$

As in the proof of the preceding theorem, we obtain a contradiction with Theorem (2.3.9).

Chapter 3
Two Subtle Convex Nonlocal Approximations and Limiting Embedding Theorems

We show that pointwise convergence when u is not smooth can be delicate; by contrast, Γ-convergence to the BV-norm is a robust and very useful mode of convergence. We deal with the limiting embedding theorems.

Section (3.1): The BV -Norm

Ω denotes a smooth bounded open subset of $\mathbb{R}^d (d \geq 1)$. We first recall a formula (BBM formula) due to J. Bourgain, H. Brezis, and P. Mironescu [109] (with a refinement by J. Davila [118]).Let (ρ_n) be a sequence of radial mollifiers in the sense that

$$\rho_n \in L^1_{loc}(0, +\infty), \qquad \rho_n \geq 0, \tag{1}$$

$$\int_0^\infty \rho_n(r) r^{d-1} dr = 1 \qquad \forall n, \tag{2}$$

and

$$\lim_{n\to+\infty} \int_\delta^\infty \rho_n(r) r^{d-1} dr = 0 \qquad \forall \delta > 0. \tag{3}$$

Set

$$I_n(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_n(|x - y|) dx dy \leq +\infty, \quad \forall u \in L^1(\Omega) \tag{4}$$

and

$$I(u) = \begin{cases} \gamma_d \int_\Omega |\nabla u| & if\ u \in BV(\Omega), \\ +\infty & if\ u \in L^1(\Omega) \backslash BV(\Omega), \end{cases} \tag{5}$$

where, for any $e \in \mathbb{S}^{d-1}$,

$$\gamma_d = \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|\, d\sigma = \begin{cases} \frac{2}{d-1} |\mathbb{S}^{d-2}| & if d \geq 3, \\ 4 & if d = 2, \\ 2 & if d = 1. \end{cases} \tag{6}$$

Then

$$\lim_{n\to+\infty} I_n(u) = I(u) \qquad \forall u \in L^1(\Omega). \tag{7}$$

It has also been established by A. Ponce [130] that $I_n \to I$ as $n \to +\infty$ in the sense of Γ-convergence in $L^1(\Omega)$. For works related to the BBM formula, see [112], [113], [114], [122], [123]. Other functionals converging to the BV-norm are considered in [110], [115], [116], [124], [125], [126], [127], [128], [129].

We analyze the asymptotic behavior of sequences of functionals which "resemble" $I_n(u)$ and converge to $I(u)$ (at least when u is smooth). As we are going to see pointwise convergence of $I_n(u)$ when u is not smooth can be delicate and depends heavily on the specific choice of (ρ_n). By contrast, Γ-convergence to I is a robust concept which is not sensitive to the choice of (ρ_n). We first consider the sequence (Ψ_n) of functionals defined by

$$\Psi_n(u) = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{1+\varepsilon_n}}{|x - y|^{1+\varepsilon_n}} \rho_n(|x - y|) dx dy \right)^{\frac{1}{1+\varepsilon_n}} \leq +\infty, \quad \forall u \in L^1(\Omega), \tag{8}$$

where $(\varepsilon_n) \to 0_+$ and (ρ_n) is a sequence of mollifiers as above.

By choosing a special sequence of (ρ_n),one may greatly improve the conclusion of Proposition (3.1.2):

Proposition (3.1.1)[107]: We have

$$\Psi_n \to I \text{ in the sense of } \Gamma - \text{convergence in } L^1(\Omega), \quad as\ n \to +\infty. \tag{9}$$

Motivated by Image Processing (see, e.g., [108], [119], [120] [121], [132]), we set

$$E_n(u) = \int_\Omega |u-f|^q + \Psi_n(u) \quad for\ u \in L^q(\Omega), \tag{10}$$

and

$$E_0(u) = \int_\Omega |u-f|^q + I(u) \quad for\ u \in L^q(\Omega), \tag{11}$$

where $q > 1$ and $f \in L^q(\Omega)$.

Proposition (3.1.2)[107]: We have

$$\lim_{n\to+\infty} \Psi_n(u) = I(u) \quad \forall u \in \bigcup_{q>1} W^{1,q}(\Omega) \tag{12}$$

and

$$\liminf_{n\to+\infty} \Psi_n(u) \geq I(u) \quad \forall u \in L^1(\Omega). \tag{13}$$

Proof: We first establish (13). By Hölder's inequality, we have for every $u \in L^1(\Omega)$

$$I_n(u) \leq \Psi_n(u) \left(\int_\Omega \int_\Omega \rho_n (|x-y|) dxdy \right)^{\frac{\varepsilon_n}{1+\varepsilon_n}}. \tag{14}$$

From (2), we have

$$\int_\Omega \int_\Omega \rho_n (|x-y|) dxdy \leq |\mathbb{S}^{d-1}||\Omega|. \tag{15}$$

Note that

$$\lim_{n\to+\infty} \left(|\mathbb{S}^{d-1}||\Omega|\right)^{\frac{\varepsilon_n}{1+\varepsilon_n}} = 1.$$

Inserting (7) in (14) yields (13).

We next establish (12) for $u \in W^{1,q}(\Omega)$ with $q > 1$. Assuming n sufficiently large so that $1 + \varepsilon_n < q$, we may write using Hölder's inequality

$$\Psi_n(u) \leq I_n(u)^{2a} J_{n,q}^{b_n}, \tag{16}$$

where

$$\mathrm{J}_{n,q} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^q} \rho_n (|x-y|) dxdy \right)^{1/q}, \tag{17}$$

$$a_n + b_n = 1 \quad and \quad a_n = \frac{b_n}{q} = \frac{1}{1+\varepsilon_n}, \tag{18}$$

i.e.,

$$b_n \left(1 - \frac{1}{q}\right) = \frac{\varepsilon_n}{1+\varepsilon_n} \quad and \quad a_n = 1 - b_n. \tag{19}$$

From [109], we know that

$$\mathrm{J}_{n,q} \leq C\|\nabla u\|_{L^q}, \qquad \text{with } C \text{ independent of } n. \tag{20}$$

Combining (16), (19), (20), and using (7), we obtain

$$\limsup_{n\to+\infty} \Psi_n(u) \leq I(u).$$

This proves (12) since we already know (13).

Proposition (3.1.3)[107]: There exists *a* sequence (ρ_n) and a constant C such that

$$\Psi_n(u) \leq CI(u) \qquad \forall n, \forall u \in L^1(\Omega) \tag{21}$$

and

$$\lim_{n\to+\infty} \Psi_n(u) = I(u) \quad \forall u \in L^1(\Omega) \tag{22}$$

The proof of Propositions (3.1.2) and (3.1.3) is presented. By contrast, some sequences (ρ_n) may produce pathologies:

Proof: The sequence (ρ_n) is defined by

$$\rho_n(t) = \frac{1+d+\varepsilon_n}{\delta^{1+d+\varepsilon_n}} t^{1+\varepsilon_n} 1_{(0,\delta_n)}(t), \tag{23}$$

where 1_A denotes the characteristic function of the set A, and (δ_n) is a positive sequence converging to 0 and satisfying

$$\lim_{n\to+\infty} \delta_n^{\varepsilon_n} = 1; \tag{24}$$

one may take for example

$$\delta_n = e^{-1/\sqrt{\varepsilon_n}}. \tag{25}$$

We have

$$\Psi_n^{1+\varepsilon_n}(u) = \frac{1+d+\varepsilon_n}{\delta_n^{1+d+\varepsilon_n}} \int_\Omega \int_{\substack{\Omega \\ |x-y|<\delta_n}} |u(x)-u(y)|^{1+\varepsilon_n} \; dxdy \tag{26}$$

From the Sobolev embedding, we know that $BV(\Omega) \subset L^q(\Omega)$ with $q = d/(d-1)$ and moreover,

$$\left(\int_\Omega \int_\Omega |u(x)-u(y)|^q \, dxdy\right)^{1/q} \leq CI(u) \qquad \forall u \in L^1(\Omega). \tag{27}$$

Applying Holder's inequality as above, we find

$$\Psi_n(u) \leq \left(\frac{1+d+\varepsilon_n}{\delta_n^{1+d+\varepsilon_n}}\right)^{\frac{1}{1+\varepsilon_n}} X_n^{a_n} Y_n^{b_n}, \tag{28}$$

where

$$X_n = \int_\Omega \int_{\substack{\Omega \\ |x-y|<\delta_n}} |u(x)-u(y)| \; dxdy\,, \tag{29}$$

$$Y_n = \left(\int_\Omega \int_{\substack{\Omega \\ |x-y|<\delta_n}} |u(x)-u(y)| \; dxdy\right)^{1/q}, \tag{30}$$

and a_n and b_n are as in (18). From [109] (applied with $\rho_n(t) = \frac{1+d}{\delta_n^{1+d}} t 1_{(0,\delta_n)}(t)$), we know that

$$X_n \leq C\delta_n^{1+d} \; I(u). \tag{31}$$

Moreover, by (7), we have

$$\lim_{n\to+\infty} \frac{1+d}{\delta_n^{1+d}} \; X_n = I(u). \tag{32}$$

On the other hand, by (27), we obtain

$$Y_n \leq CI(u) := Y. \tag{33}$$

Inserting (31) and (33) in (28) gives

$$\Psi_n(u) \le C\frac{1}{\delta_n^{\alpha_n}} I(u), \tag{34}$$

where, by (19),

$$\alpha_n = \frac{1+d+\varepsilon_n}{1+\varepsilon_n} - (1+d)a_n = \frac{1+d+\varepsilon_n}{1+\varepsilon_n} - (1+d) + \frac{(1+d)q\varepsilon_n}{(q-1)(1+\varepsilon_n)}$$

$$= -\frac{\varepsilon_n d}{1+\varepsilon_n} + \frac{(1+d)q\varepsilon_n}{(q-1)(1+\varepsilon_n)} = \frac{\varepsilon_n d^2}{1+\varepsilon_n}.$$

From (34) and (24), we obtain (21).

We next prove (22). In view of (13), it suffices to verify that

$$\limsup_{n\to+\infty} \Psi_n(u) \le I(u) \qquad \forall u \in L^1(\Omega). \tag{35}$$

We return to (28) and write

$$\Psi_n(u) \le \left(\frac{1+d+\varepsilon_n}{\delta_n^{1+d+\varepsilon_n}}\right)^{\frac{1}{1+\varepsilon_n}} \left(\frac{\delta_n^{d+1}}{d+1}\right)^{a_n} \left(\frac{(1+d)X_n}{\delta_n^{1+d}}\right)^{a_n} Y^{b_n} = \gamma_n \delta_n^{-\alpha_n} \left(\frac{(1+d)X_n}{\delta_n^{1+d}}\right)^{a_n} Y^{b_n},$$

where $\gamma_n \to 1$, $a_n \to 1$, and $b_n \to 0$. Using (24) and (32), we conclude that (35) holds.

We establish Propositions (3.1.4) and (3.1.5)

Proposition (3.1.4)[107]: Assume $d = 1$. There exists a sequence (ρ_n) and some $v \in W^{1,1}(\Omega)$ such that

$$\Psi_n(v) = +\infty \quad \forall n \ge 1. \tag{36}$$

Proof : Take $\Omega = (-1/2, 1/2)$ and $\rho_n(t) = \varepsilon_n t^{\varepsilon_n - 1} - 1(0,1)(t)$. Then

$$\Psi_n^{1+\varepsilon_n}(u) \ge \varepsilon_n \int_0^{1/2} dx \int_{-1/2}^{0} \frac{|u(x)-u(y)|^{1+\varepsilon_n}}{|x-y|^2} dy.$$

If we assume in addition that $u(y) = 0$ on $(-1/2, 0)$, we obtain

$$\Psi_n^{1+\varepsilon_n}(u) \ge \varepsilon_n \int_0^{1/2} |u(x)|^{1+\varepsilon_n} \left(\frac{1}{x} - \frac{1}{x+1/2}\right) dx. \tag{37}$$

Choosing, for example,

$$u(x) = \begin{cases} |\ln x|^{-\alpha} & on\ 0 < x < 1/2, \\ 0 & on - 1/2, < x \le 0, \end{cases} \tag{38}$$

with $\alpha > 0$, we see that $u \in W^{1,1}(\Omega)$ while the RHS in (37) is $+\infty$ when $\alpha(1+\varepsilon_n) \le 1$; we might take, for example, $\alpha = \min_n\{1/(1+\varepsilon_n)\}$.

Proposition (3.1.5)[107]: Assume $d = 1$. Given any $M > 1$, there exists a sequence (ρ_n) and a constant C such that

$$\Psi_n(u) \le CI(u) \quad \forall n, \forall u \in L^1(\Omega), \tag{39}$$

$$\lim_{n\to+\infty} \Psi_n(u) = I(u) \quad \forall u \in W^{1,1}(\Omega), \tag{40}$$

and, for some nontrivial $v \in BV(\Omega)$,

$$\lim_{n\to+\infty} \Psi_n(v) = MI(v). \tag{41}$$

The proofs of Propositions (3.1.4) and (3.1.5) are presented., we return to a general sequence (ρ_n) and we establish the following results:

Proof : Take $\Omega = (-1, 1)$ and (ρ_n) as in (23) (but do not take δ_n as in (24)). Let

$$v(x) = \begin{cases} 0 & \text{for } x \in (-1,0), \\ 1 & \text{for } x \in (0,1). \end{cases}$$

Then

$$\Psi_n(v) = \frac{2+\varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_0^1 \int_{\substack{0 \\ x-y<\delta_n}}^1 dxdy = \frac{2+\varepsilon_n}{\delta_n^{\varepsilon_n}}.$$

Since $I(v) = 2$ (see (5) and (6)), we deduce that

$$\Psi_n(v) = \frac{2+\varepsilon_n}{2\delta_n^{\varepsilon_n}} I(v). \tag{42}$$

Given $M > 1$, let $A = \ln M > 0$ and $\delta_n = e^{-A/\varepsilon_n}$. Then

$$\lim_{n\to+\infty} \Psi_n(v) = MI(v).$$

On the other hand, we have, for every $u \in BV(\Omega)$,

$$\Psi_n(u) \le \frac{2+\varepsilon_n}{\delta_n^{2+\varepsilon_n}} \int_0 \int_{\substack{0 \\ |x-y|<\delta_n}} |u(x) - u(y)|^{1+\varepsilon_n} dxdy.$$

As in the proof of Proposition (3.1.3) (see (34)), we find

$$\Psi_n(u) \le C \frac{1}{\delta_n^{\alpha_n}} I(u),$$

Since $\delta_n = e^{-A/\varepsilon_n}$, we deduce that (39) holds.

In order to obtain (40), we recall (see (12)) that

$$\lim_{n\to+\infty} \Psi_n(\tilde{u}) = I(\tilde{u}) \quad \forall \tilde{u} \in C^1(\bar{\Omega}). \tag{43}$$

For $u \in W^{1,1}(\Omega)$, we write

$$\Psi_n(u) - I(u) = \Psi_n(u) - \Psi_n(\tilde{u}) + \Psi_n(\tilde{u}) - I(\tilde{u}) + I(\tilde{u}) - I(u),$$

and thus by (39),

$$|\Psi_n(u) - I(u)| \le CI(u - \tilde{u}) + |\Psi_n(\tilde{u}) - I(\tilde{u})|. \tag{44}$$

We conclude that $\lim_{n\to+\infty} |\Psi_n(u) - I(u)| = 0$ using (34) and the density of $C^1(\bar{\Omega})$ in $W^{1,1}(\Omega)$.

We devoted to the proof of Proposition (3.1.1) and a slightly stronger variant.

Recall that (see, e.g., [111], [117]), by definition, the sequence $(\Psi_n)\Gamma$-converges to Ψ in $L^1(\Omega)$ as $n \to \infty$ if the following two properties hold:

(G1) For every $u \in L^1(\Omega)$ and for every sequence $(u_n \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$

as $n \to \infty$, one has

Proof (G1): Going back to (14)–(16), we have

$$I_n(u) \le \beta_n \Psi_n(u) \quad \forall u \in L^1(\Omega),$$

where $\beta_n \to 1$. Thus

$$I_n(u_n) \le \beta_n \Psi_n(u_n) \quad \forall n,$$

and since $I_n \to I$ in the sense of Γ-convergence in $L^1(\Omega)$ (see [130] and also [114]), we conclude that

$$\liminf_{n\to+\infty} \Psi_n(u_n) \ge I(u).$$

(G2) For every $u \in L^1(\Omega)$, there exists a sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$

as $n \to \infty$, and

Proof (G2): Given $u \in BV(\Omega)$, we will construct a sequence (u_n) converging to u in $L^1(\Omega)$ such that

$$\limsup_{n\to+\infty} \Psi_n(u_n) \le I(u).$$

Let $v_k \in C^1(\bar{\Omega})$ be such that

$$v_k \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \mathrm{I}(v_k) \to \mathrm{I(u)}. \tag{45}$$

For each k, let n_k be such that

$$|\Psi_n(v_k) - I(v_k)| \leq 1/k \qquad \text{if } \mathrm{n} > n_k. \tag{46}$$

Without loss of generality, one may assume that (n_k) is an increasing sequence with respect to k. Define

$$u_n = v_k \quad \text{if } n_k < n \leq n_{k+1}.$$

Combining (45) and (46) yields

$$u_n \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \lim_{n\to+\infty} \Psi_n(u_n) = I(u).$$

In fact, a property stronger than (i) holds.

Proposition (3.1.6)[107]: For every $u \in L^1(\Omega)$ and for every sequence $(u_n) \subset L^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ as $n \to +\infty$, one has

$$\liminf_{n\to+\infty} \Psi_n(u_n) \geq I(u). \tag{47}$$

Proof: We adapt a suggestion of E. Stein (personal communication to H. Brezis) described in [112]. Let (μ_k) be a sequence of smooth mollifiers such that $\mu_k \geq 0$ and $\operatorname{supp} \ \mu_k \subset B_{1/k} = B_{1/k}(0) = B(0, 1/k)$. Fix D an arbitrary smooth open subset of Ω such that $\overline{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x, 1/k_0) \subset\subset \Omega$ for every $x \in D$. Given $v \in L^1(\Omega)$, define in D

$$v_k = \mu_k * v \qquad \text{for } k \geq k_0.$$

We have

$$\begin{aligned}
&\int_{\mathrm{D}}\int_{\mathrm{D}} \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy \\
&\quad = \int_{\mathrm{D}}\int_{\mathrm{D}} \frac{|\mu_k * v(x) - \mu_k * v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy \\
&\quad = \int_{\mathrm{D}}\int_{\mathrm{D}} \frac{\left|\int_{B(0,1/k)} \mu_k(z)\big(v(x-z) - v(y-z)\big)dz\right|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} dxdy \\
&\quad \leq \int_{\mathrm{D}}\int_{\mathrm{D}} \frac{\left|\int_{B(0,1/k)} \mu_k(z)|v(x-z) - v(y-z)|^{1+\varepsilon_n}\right|^{1+\varepsilon_n} dz}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy,
\end{aligned}$$

by Hölder's inequality. A change of variables implies, for $k \geq k_0$,

$$\begin{aligned}
&\int_{\mathrm{D}}\int_{\mathrm{D}} \frac{|v_k(x) - v_k(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy \\
&\qquad \leq \int_{\mathrm{D}}\int_{\mathrm{D}} \frac{|v(x) - v(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy.
\end{aligned} \tag{48}$$

Applying (48) to $v = u_n$ we find

$$\int_{\mathrm{D}}\int_{\mathrm{D}} \frac{\left|u_{k,n}(x) - u_{k,n}(y)\right|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}} \rho_n(|x-y|)dxdy \leq \Psi_n^{1+\varepsilon_n}(u_n), \tag{49}$$

where $u_{k,n} = \mu_k * u_n$ is defined in D for every n and every $k \geq k_0$. Since $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ we know that for each fixed k,

$$u_{k,n} \to \mu_k * u \qquad \text{strongly in } L^1 \text{ as } n \to +\infty.$$

Passing to the limit in (48) as $n \to +\infty$ (and fixed k) and applying Proposition (3.1.1) (Property (i)) we find that

$$\liminf_{n\to+\infty}\int_D\int_D \frac{|u_{k,n}(x)-u_{k,n}(y)|^{1+\varepsilon_n}}{|x-y|^{1+\varepsilon_n}}\rho_n(|x-y|)dxdy \ge \gamma_d\int_D |\nabla(\mu_k * u)|. \tag{50}$$

Combining (49) and (50) yields

$$\liminf_{n\to+\infty}\Psi_n(u_n)\ge\gamma_d\int_D|\nabla(\mu_k*u)| \qquad \forall k\ge k_0.$$

Letting $k\to+\infty$, we obtain

$$\liminf_{n\to+\infty}\Psi_n(u_n)\ge\gamma_d\int_D|\nabla_u|.$$

Since D is arbitrary, Proposition (3.1.6) follows.

Proposition (3.1.7)[107]: For each n, there exists a unique $u_n\in L^q(\Omega)$ such that

$$E_n(u_n)=\min_{u\in L^q(\Omega)}E_n(u).$$

Let υ be the unique minimizer of E_0 in $L^q(\Omega)\cap BV(\Omega)$. We have, as $n\to+\infty$,

$$u_n\to\upsilon \qquad in\ L^q(\Omega)$$

and

$$E_n(u_n)\to E_0(\upsilon).$$

We investigate similar questions for the sequence (Φ_n) of functionals defined by

$$\Phi_n(u)=\int_\Omega dx\left[\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(|x-y|)dy\right]^{1/p}\le+\infty, \quad \text{for } u\in L^1(\Omega),$$

where $p>1$. Such functionals were introduced and studied by G. Leoni and D. Spector [122], [123] (see also [133]); their motivation came by G. Gilboa and S. Osher [120] (where $p=2$) dealing with Image Processing.

Proof : For each fixed n, the functional E_n defined on $L^q(\Omega)$ by (10) is convex and lower semicontinuous (l.s.c.) for the strong L^q-topology (note that Ψ_n is l.s.c. by Fatou's lemma). Thus E_n is alsol.s.c. for the weak L^q -topology. Since $q>1, L^q$ is reflexive and $\inf_{u\in L^q(\Omega)}E_n(u)$ is achieved. Uniqueness of the minimizer follows from strict convexity.

We next establish the second statement. Since $q>1$, one may assume that $u_{n_k}\rightharpoonup u_0$ weakly in $L^q(\Omega)$ for some subsequence (u_{n_k}). We claim that

$$u_0=\upsilon. \tag{51}$$

By Proposition (3.1.1) (Property (ii)), there exists $(\upsilon_n)\subset L^1(\Omega)$ such that $\upsilon_n\to\upsilon$ in $L^1(\Omega)$ and

$$\limsup_{n\to\infty}\Psi_n(\upsilon_n)\le I(\upsilon). \tag{52}$$

Set, for $A>0$ and $s\in\mathbb{R}$,

$$T_A(s)=\begin{cases} s & \text{if } |s|\le A,\\ A & \text{if } s>A,\\ -A & \text{if } s<-A.\end{cases} \tag{53}$$

We have, since u_n is a minimizer of E_n,

$$E_n(u_n)\le E_n(T_A\upsilon_n)=\int_\Omega|T_A\upsilon_n-f|^q+\Psi_n(T_A\upsilon_n)\le\int_\Omega|T_A\upsilon_n-f|^q+\Psi_n(\upsilon_n). \tag{54}$$

Letting $n\to\infty$ and using (52), we derive

$$\limsup_{n\to+\infty}\mathrm{E}_n(u_n)\le\int_\Omega|T_A\upsilon-f|^q+I(\upsilon).$$

This implies, by letting $A\to+\infty$,

$$\limsup_{n\to+\infty} \mathrm{E}_n(u_n) \leq \mathrm{E}_0(v). \tag{55}$$

On the other hand, we have by Proposition(3.1.6),

$$\liminf_{n_k\to+\infty} \Psi_{n_k}(u_{n_k}) \geq I(v), \tag{56}$$

and therefore

$$\mathrm{E}_0(u_0) \leq \liminf_{n_k\to+\infty} E_{n_k}(u_{n_k}). \tag{57}$$

From (55) and (57), we obtain claim (51).

Next we write

$$\int_\Omega |u_n - f|^q = \mathrm{E}_n(u_n) - \Psi_n(u_n). \tag{58}$$

Combining (58) with (55) and (56) gives

$$\limsup_{n_k\to+\infty} \int_\Omega |u_{n_k} - f|^q \leq \mathrm{E}_0(v) - I(v) = \int_\Omega |v - f|^q. \tag{59}$$

Since we already know that $u_{n_k} \rightharpoonup v$ weakly in $L^q(\Omega)$, we deduce from (59) that $u_{n_k} \to v$ strongly in $L^q(\Omega)$. The uniqueness of the limit implies that $u_n \to v$ strongly in $L^q(\Omega)$, so that

$$\liminf_{n\to+\infty} \mathrm{E}_n(u_n) \geq \int_\Omega |v - f|^q + I(v) = \mathrm{E}_0(v).$$

Returning to (55) yields

$$\lim_{n\to+\infty} \mathrm{E}_n(u_n) = \mathrm{E}_0(v).$$

Proposition (3.1.8)[107]: Let (u_n) be a bounded sequence in $L^1(\Omega)$ such that

$$\sup_n \Psi_n(u_n) < +\infty. \tag{60}$$

When $d = 1$, we also assume that for each n the function $t \longmapsto \rho_n(t)$ is non-increasing. Then (u_n) is relatively compact in $L^1(\Omega)$.

Proof: From (14), (15) and (60), we have

$$I_n(u_n) \leq C \quad \forall n.$$

We may now invoke a result of J. Bourgain, H. Brezis, P. Mironescu in [109] when ρ_n is non-increasing. A. Poncein [131] established that the monotonicity of ρ_n is not necessary when $d \geq 2$.

Motivated by a suggestion of G. Gilboa and S. Osher in [120], G. Leoni and D. Spector [122], [123] studied the following functional

$$\Phi_n(u) = \int_\Omega dx \left[\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|)dy\right]^{1/p} \leq +\infty, \quad \text{for } u \in L^1(\Omega), \tag{61}$$

where $1 < p < +\infty$ and (ρ_n) satisfies (1)–(3). In [123], they established that (Φ_n) converges to J in the sense of Γ-convergence in $L^1(\Omega)$, where J is defined by

$$J(u) := \begin{cases} \gamma_{p,d} \int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases} \tag{62}$$

Here, for any $e \in \mathbb{S}^{d-1}$,

$$\gamma_{p,d} := \left(\int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma\right)^{1/p}. \tag{63}$$

In particular,

$$\gamma_{p,d} = 2^{1/p}. \tag{64}$$

When there is no confusion, we simply write γ instead of $\gamma_{p,d}$. [In fact, G. Leoni and D. Spector considered more general functionals involving a second parameter $1 \leq q < +\infty$ and they prove that it Γ-converges in $L^1(\Omega)$ to$\int_\Omega |\nabla u|^q$up to a positive constant. Here we are concerned only with the most delicate case $q = 1$ which produces the BV-norm in the asymptotic limit.]

Pointwise convergence of the sequence (Φ_n) turns out to be quite complex and not yet fully understood (which confirms again the importance of Γ-convergence). Several claims in [122] concerning the pointwise convergence of (Φ_n) were not correct as was pointed out in [123].

We describe various results (both positive and negative) concerning pointwise convergence. The case $d = 1$ is of special interest because the situation there is quite satisfactory. Our results for the case $d \geq 2$ are not as complete; see e.g. important open problems mentioned. We then present a new proof of Γ-convergence; as we already mentioned, this result is due to G. Leoni and D. Spector, but our proof is simpler. Finally, we discuss variational problems similar to (10) (where Ψ_n is replaced by Φ_n) with roots in Image Processing.

A general result concerning the pointwise convergence of $(\Phi_n)$is the following.

Proposition (3.1.9)[107]: We have

$$\lim_{n\to\infty} \Phi_n(u) = J(u) \qquad \forall u \in W^{1,p}(\Omega) \tag{65}$$

and

$$\liminf_{n\to\infty} \Phi_n(u) \geq J(u) \qquad \forall u \in L^1(\Omega). \tag{66}$$

Proof: The proof is divided into three steps.

Step 1: Proof of (65) for $u \in C^2(\overline{\Omega})$. We have

$$|u(x) - u(y) - \nabla u(x) \cdot (x-y)| \leq C|x-y|^2 \qquad \forall x, y \in \Omega,$$

for some positive constant C independent of x and y. It follows that

$$|u(x) - u(y)| \leq |\nabla u(x) \cdot (x-y)| + C|x-y|^2 \qquad \forall x, y \in \Omega \tag{67}$$

and

$$|\nabla u(x) \cdot (x-y)| \leq |u(x) - u(y)| + C|x-y|^2 \quad \forall x, y \in \Omega. \tag{68}$$

From (67), we derive that

$$\left(\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|)dy\right)^{1/p} \leq \left(\int_\Omega \frac{|\nabla u(x) \cdot (y-x)|^p}{|x-y|^p} \rho_n(|x-y|)dy\right)^{1/p} + C\left(\int_\Omega |x-y|^p \rho_n(|x-y|)dy\right)^{1/p}$$

which implies, by (1) and (3),

$$\left(\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|)dy\right)^{1/p} \leq \gamma|\nabla u(x)| + o(1). \tag{69}$$

Here and in what follows in this proof, $o(1)$ denotes a quantity which converges to 0 (independently of x) as $n \to +\infty$. We derive that

$$\Phi_n(u) \leq \gamma \int_\Omega |\nabla u(x)| + o(1). \tag{70}$$

For the reverse inequality, we consider an arbitrary open subset D of Ω such that $\overline{D} \subset \Omega$. For a fixed $x \in D$, using (1), (3) and (68) one can verify as in (69) that

$$\gamma|\nabla u(x)| \leq \left(\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n\,(|x-y|)dy\right)^{1/p} + o(1).$$

It follows that

$$\gamma \int_D |\nabla u(x)|dx \leq \Phi_n(u) + o(1). \tag{71}$$

Combining (70) and (71) yields

$$\gamma \int_D |\nabla u(x)|dx \leq \liminf_{n\to+\infty} \Phi_n(u) \leq \limsup_{n\to+\infty} \Phi_n(u) \leq \gamma \int_{\Omega} |\nabla u(x)|dx.$$

The conclusion of Step 1 follows since D is arbitrary,

Step 2: Proof of (66). We follow the same strategy as in the proof of Proposition(3.1.6). Let (μ_k) be a sequence of smooth mollifiers such that $\mu_k \geq 0$ and supp $\mu_k \subset B_{1/k}$. Fix D an arbitrary smooth open subset of Ω such that $\overline{D} \subset \Omega$ and let $k_0 > 0$ be large enough such that $B(x, 1/k_0) \subset\subset \Omega$ for every $x \in D$. Given $u \in L^1(\Omega)$, define in D

$$u_k = \mu_k * u \quad \text{ for } k \geq k_0.$$

We have, for $k \geq k_0$,

$$\int_D \left(\int_D \frac{|u_k(x)-u_k(y)|^p}{|x-y|^p}\rho_n\,(|x-y|)dy\right)^{1/p} dx \leq \Phi_n(u) \ \ \forall n. \tag{72}$$

Letting $n \to +\infty$ (for fixed k and fixed D), we find, using Step 1 on D, that, for $k \geq k_0$,

$$\lim_{n\to+\infty} \int_D \left(\int_D \frac{|u_k(x)-u_k(y)|^p}{|x-y|^p}\rho_n\,(|x-y|)dy\right)^{1/p} dx = \gamma \int_D |\nabla u_k(x)|dx.$$

We derive from (72) that

$$\liminf_{n\to+\infty} \Phi_n(u) \geq \gamma \int_D |\nabla u_k(x)|dx, \tag{73}$$

for $\geq k_0$. Letting $k \to +\infty$, we obtain

$$\liminf_{n\to+\infty} \Phi_n(u) \geq \gamma \int_D |\nabla u(x)|dx. \tag{74}$$

We deduce (66) since D is arbitrary.

Step 3: Proof of (65) for $u \in W^{1,p}(\Omega)$. By Hölder's inequality, we have

$$\Phi_n(u) \leq |\Omega|^{1-1/p}\left(\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n\,(|x-y|)dy\,dy\right)^{1/p}. \tag{75}$$

We may then invoke a result of [109] to conclude that

$$\Phi_n(u) \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \tag{76}$$

with $C > 0$ independent of n. We next write, using triangle inequality,

$$|\Phi_n(u) - \Phi_n(\tilde{u})| \leq \Phi_n(u-\tilde{u}) \leq C\|\nabla(u-\tilde{u})\|_{L^p(\Omega)} \quad \forall u, \tilde{u} \in W^{1,p}(\Omega).$$

This implies

$$\begin{aligned}|\Phi_n(u) - J(u)| &\leq |\Phi_n(u) - \Phi_n(\tilde{u})| + |\Phi_n(\tilde{u}) - J(\tilde{u})| + |J(\tilde{u}) - J(u)| \\ &\leq C\|\nabla(u-\tilde{u})\|_{L^p(\Omega)} + |\Phi_n(\tilde{u}) - J(\tilde{u})|.\end{aligned}$$

Using the density of $C^2(\overline{\Omega})$ in $W^{1,p}(\Omega)$, we obtain (65).

By choosing a special sequence (ρ_n), we may greatly improve the conclusion of Proposition (3.1.9). Moreprecisely, let (δ_n) be a positive sequence converging to 0 and define

$$\rho_n(t) = \frac{(p+d)}{\delta_n^{p+d}} t^p \mathbb{1}_{(0,\delta_n)}(t). \tag{77}$$

We have

The proof of Proposition(3.1.10) relies on the following inequality which is just a rescaled version of the standard Sobolev one. Let B_R be a ball of radius R, then for any $p \in [1, d/(d-1)]$,

$$\left(\int_{B_R} \left|u(y) - \int_{B_R} \bar{u}\right|^p dy\right)^{1/p} \leq CR^\alpha \int_{B_R} |\nabla u(z)|\, dz \quad \forall u \in L^1(B_R), \tag{78}$$

for some positive constant C depending only on d and p,where $\alpha := (d/p) + 1 - d \geq 0$.

Proposition (3.1.10) [107]: Let $d \geq 1$ and assume that either

$$1 < p \leq d/(d-1) \qquad \text{and } d \geq 2,$$

or

$$1 < p < +\infty \qquad \text{and } d = 2,$$

and let (ρ_n) be defined by (77). Then

$$\Phi_n(u) \leq C \int_\Omega |\nabla u| \qquad \forall n, \forall u \in L^1(\Omega), \tag{79}$$

for some positive constant C depending only on d, p, and Ω , and

$$\lim_{n\to+\infty} \Phi_n(u) = J(u) \;\; \forall u \in W^{1,1}(\Omega). \tag{80}$$

On the other hand, there exists some nontrivial $v \in BV(\Omega)$ such that

$$\lim_{n\to+\infty} \Phi_n(v) = \alpha_p J(v) \quad \text{ with } \alpha_p > 1. \tag{81}$$

Proof: Since $\Phi_n(u) = \Phi_n(u + c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u such that

$$\|u\|_{W^{1,1}(\mathbb{R}^d)} \leq C_\Omega \|u\|_{W^{1,1}(\Omega)} \leq C_\Omega \|\nabla u\|_{L^1(\Omega)}. \tag{82}$$

In view of (77), we have

$$\Phi_n(u) \leq \frac{(p+d)^{1/p}}{\delta_n^{1+d}} \int_\Omega \left(\int_{B(x,\delta_n)} |u(x) - u(y)|^p dy\right)^{1/p} dx. \tag{83}$$

We have, for $y \in B(x, \delta_n)$,

$$|u(x) - u(y)| \leq \left|u(x) - \int_{B(x,\delta_n)} u\right| + \left|u(y) - \int_{B(x,\delta_n)} u\right|. \tag{84}$$

It follows from the triangle inequality that

$$\left(\int_{B(x,\delta_n)} |u(x) - u(y)|^p dy\right)^{1/p}$$
$$\leq C\delta_n^{d/p} \left|u(x) - \int_{B(x,\delta_n)} u\right| + \left(\int_{B(x,\delta_n)} \left|u(y) - \int_{B(x,\delta_n)} u\right|^p dy\right)^{1/p}. \tag{85}$$

Here and in what follows in this proof, C denotes a positive constant depending only on d, p, and Ω. Inserting (78) in (85) yields

$$\left(\int_{B(x,\delta_n)} |u(x) - u(y)|^p dy\right)^{1/p} \leq C\delta_n^{d/p} \left|u(x) - \int_{B(x,\delta_n)} u\right| + C\delta_n^\alpha \int_{B(x,\delta_n)} |\nabla u(z)| dz. \tag{86}$$

We claim that

$$\int_{\Omega}\left|u(x)-\int_{B(x,\delta_n)}u\right|dx\leq C\delta_n\int_{\Omega}|\nabla u| \tag{87}$$

and

$$\int_{\Omega}dx\int_{B(x,\delta_n)}|\nabla u(z)|dz\leq C\delta_n^d\int_{\Omega}|\nabla u|. \tag{88}$$

Indeed, we have, for R large enough,

$$\int_{\Omega}\left|u(x)-\int_{B(x,\delta_n)}u\right|dx\leq C\delta_n^{d-1}\int_{B_R}\int_{\substack{B_R\\|x-y|<\delta_n}}|u(x)-u(y)|\ dxdy$$

$$\leq C\delta_n\int_{B_R}|\nabla u|\leq C\delta_n\int_{\Omega}|\nabla u|.$$

by the BBM formula applied to $\rho_n(t)=(d\ +\ 1)\delta_n^{-(d+1)}\,t\,\mathbb{1}_{(0,\delta_n)}$ and by (79). On the other hand,

$$\int_{\Omega}\int_{B(x,\delta_n)}|\nabla u(z)|dzdx\leq\int_{B_R}\int_{\substack{B_R\\|x-z|<\delta_n}}|\nabla u(z)|dzdx\leq C\delta_n^p\int_{\Omega}|\nabla u(x)|dx,$$

by (82). Combining (86)–(88) yields

$$\int_{\Omega}\left(\int_{B(x,\delta_n)}|u(x)-u(y)|^p dy\right)^{1/p}dx\leq C\delta_n^{1+d/p}\int_{\Omega}|\nabla u(z)|dz \tag{89}$$

(recall that $\alpha+d=1+d/p$). It follows from (83) that

$$\Phi_n(u)\leq C\|\nabla u\|_{L^1(\Omega);}$$

which is (79).

Assertion (80) is deduced from (79) via a density argument as in the proof of Proposition (3.1.9).

It remains to prove (81). For simplicity, take $\Omega=(-1/2,1/2)$ and consider $v(x)=\mathbb{1}_{(0,1/2)}(x)$. Then, for n sufficiently large,

$$\Phi_n(v)=2\frac{(p+1)^{1/p}}{\delta_n^{1/p}}\int_0^{\delta_n}\left(\int_0^{\delta_n-x}dy\right)^{1/p}dx=\frac{2(p+1)^{1/p}}{\delta_n^{1+1/p}}\int_0^{\delta_n}(\delta_n-x1)^{1/p}\,dx$$

$$=2\frac{(p+1)^{1/p}}{\delta_n^{1+1/p}}\frac{\delta_n^{1+1/p}}{1+1/p}=\frac{2p}{(p+1)^{1-1/p}}>2^{1/p}=J(v).$$

Indeed, since $p+1<2p$, it follows that $(p+1)^{1-1/p}<(2p)^{1-1/p}$ and thus

$$\frac{2p}{(p+1)^{1-1/p}}>(2p)^{1/p}>2^{1/p}.$$

We assume that $d=1$ and $\Omega=(-1/2,1/2)$.

Proposition (3.1.11)[107]: Assume that (ρ_n) satisfies (1)–(3). Then, for every $q>1$, we have

$$\Phi_n(u)\leq C_q\|\acute{u}\|_{L^q(\Omega)}\qquad\forall u\in W^{1,q}(\Omega),$$

for some positive constant C_q depending only on q. Moreover,

$$\lim_{n\to+\infty}\Phi_n(u)=J(u)\qquad\forall u\in\bigcup_{q>1}W^{1,q}(\Omega).$$

Proof: Since $\Phi_n(u) = \Phi_n(u + c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$.
Consider an extension of u to $\mathbb{R}$ which is still denoted by u, such that

$$\|u\|_{W^{1,q}(\mathbb{R})} \le C_q \|u\|_{W^{1,q}(\Omega)} \le C_q \|\acute{u}\|_{L^q(\Omega)}.$$

Let $M(f)$ denote the maximal function of f defined in $\mathbb{R}$, i.e.,

$$M(f)(x) := \sup_{r>0} \int_{x-r}^{x+r} |f(s)|\, ds.$$

From the definition of Φ_n, we have

$$\Phi_n(u) \le C \int_\Omega \left(\int_\Omega |M(\acute{u})(x)|^p \rho_n(|x-y|) dy \right)^{1/p} dx \le C \int_\Omega M(\acute{u})(x) dx.$$

The first statement now follows from the fact that $\| M(f) \|_{L^q(\mathbb{R})} \le C_q \| f \|_{L^q(\mathbb{R})}$ since $q > 1$. The second statement is derived from the first statement via a density argument as in the proof of Proposition (3.1.9).

The next result shows that Proposition (3.1.11) is sharp and cannot be extended to $q = 1$ (for a general sequence (ρ_n)).

Proposition (3.1.12)[107]: For every $p > 1$, there exist a sequence (ρ_n) satisfying (1)–(3) and some function $v \in W^{1,1}(\Omega)$ such that

$$\Phi_n(v) = +\infty \quad \forall n.$$

Proof: Fix $\alpha > 0$ and $\beta > 1$ such that

$$\alpha + \beta/p < 1. \tag{90}$$

Since $p > 1$ such α and β exist. Let (δ_n) be a sequence of positive numbers converging to 0 and consider

$$\rho_n(t) := A_n \frac{1}{t|\ln t|^\beta} \mathbb{1}_{(0,\delta_n)}.$$

Here A_n is chosen in such a way that (3) holds, i.e., $A_n \int_0^{\delta_n} \frac{1}{t|\ln t|^\beta} = 1$. Set

$$v(x) = \begin{cases} 0 & \text{if } -1/2 < x < 0, \\ |\ln x|^{-\alpha} & \text{if } 0 < x < 1/2. \end{cases}$$

Clearly, $v \in W^{1,1}(\Omega)$. We have

$$\Phi_n(v) = \int_{-1/2}^{1/2} \left(\int_{-1/2}^{1/2} \frac{|v(x) - v(y)|^p}{|x-y|^p} \rho_n(|x-y|) dy \right)^{1/p} dx$$
$$\ge \int_0^{\delta_n} A_n^{1/p} |v(x)| \left(\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y) dy \right)^{1/p} dx. \tag{91}$$

We have, for $0 < x < \delta_n/2$,

$$\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y) dy \ge \int_x^{\delta_n} \frac{dt}{t^{p+1}|\ln t|^\beta} \ge \int_x^{2x} \frac{dt}{t^{p+1}|\ln t|^\beta} \ge \frac{C_{p,\beta}}{x^p |\ln x|^\beta};$$

and thus

$$\left(\int_0^{\delta_n - x} \frac{1}{|x+y|^p} \rho_n(x+y) dy \right)^{1/p} \ge \frac{C_{p,\beta}}{x|\ln x|^{\beta/p}}. \tag{92}$$

Since, by (90),

$$\int_0^{\delta_n/2} \frac{1}{x|\ln x|^{\beta/p+\alpha}} dx = +\infty,$$

it follows from (91) and (92) that

$$\Phi_n(v) = +\infty \quad \forall n.$$

We present two "improvements" of (65) concerning the (pointwise) convergence of $\Phi_n(u)$ to $J(u)$. In the first one (Proposition (3.1.13) (ρ_n) is a general sequence (satisfying (1)–(3)), but the assumption on u is quite restrictive: $u \in W^{1,q}(\Omega)$ with $q > q_0$ where q_0 is defined in (93). In the second one (Proposition (3.1.15)) there is an additional assumption on (ρ_n), but pointwise convergence holds for a large (more natural) class of u's: $u \in W^{1,q}(\Omega)$ with $q > q_1$ where $q_1 < q_0$ is defined in (104).

Proposition (3.1.13)[107]: Let $p > 1$ and assume that (ρ_n) satisfies (1)–(3). Set

$$q_0 := pd/(d+p-1), \tag{93}$$

so that $1 < q_0 < p$. Then

$$\Phi_n(u) \le C\|\nabla u\|_{L^p} \qquad \forall u \in W^{1,q}(\Omega)\ with\ q > q_0\,, \tag{94}$$

for some positive constant $C = C_{p,q,\Omega}$ depending only on p, q, and Ω. Moreover,

$$\lim_{n\to+\infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,q}(\Omega)\ with\ q > q_0. \tag{95}$$

Proof: Since $\Phi_n(u) = \Phi_n(u+c)$ for any constant c, without loss of generality, one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u, such that

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \le C_{q,\Omega}\|u\|_{W^{1,q}(\Omega)} \le C_{q,\Omega}\|\nabla u\|_{L^p(\Omega)}.$$

For simplicity of notation, we assume that $\operatorname{diam}(\Omega) \le 1/2$. Then

$$\Phi_n(u) \le \int_\Omega \left[\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|u(x+r\sigma)-u(x)|^p}{r^p}\ \rho_n(r) r^{d-1} dr d\sigma\right]^{1/p} dx.$$

We have

$$|u(x+r\sigma)-u(x)| \le \left|u(x+r\sigma) - \int_{\mathbb{S}^{d-1}} u(x+r\acute{\sigma})d\acute{\sigma}\right| + \left|u(x) - \int_{\mathbb{S}^{d-1}} u(x+r\acute{\sigma})d\acute{\sigma}\right|$$

$$\le \int_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\acute{\sigma})|d\acute{\sigma} + \int_{\mathbb{S}^{d-1}} |u(x)-u(x+r\acute{\sigma})|d\acute{\sigma}.$$

It follows that

$$\Phi_n(u) \lesssim T_1 + T_2, \tag{96}$$

where

$$T_1 = \int_\Omega \left[\int_0^1 \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\acute{\sigma})|^p\, d\acute{\sigma}\, d\sigma\ \ \rho_n(r) r^{d-1-p} dr\right]^{1/p} dx$$

and

$$T_2 = \int_\Omega \left[\int_0^1 \left(\int_{\mathbb{S}^{d-1}} |u(x)-u(x+r\acute{\sigma})|\, d\acute{\sigma}\right)^p\ \rho_n(r) r^{d-1-p} dr\right]^{1/p} dx.$$

In this proof the notation $a \lesssim b$ means that $a \le Cb$ for some positive constant C depending only on p, q, and Ω.

We first estimate T_1. Let B_1 denotes the open unit ball of $\mathbb{R}^d$. By (93) we know that the trace mapping $u \longmapsto u|_{\partial B_1}$ is continuous from $W^{1,q_0}(B_1)$ into $L^q(\partial B_1)$. It follows that

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |u(x+r\sigma)-u(x+r\acute{\sigma})|^p\, d\acute{\sigma}\, d\sigma \lesssim \|\nabla u(x+r\,\cdot)\|^p_{L^{q_0}(B_1)} \lesssim r^p M^{p/q_0}(|\nabla u|^{q_0})(x)$$

(recall that $M(f)$ denotes the maximal function of a function f defined in $\mathbb{R}^d$). Using (1), we derive that

$$T_1 \lesssim \int_\Omega \left[\int_0^1 M^{p/q_0}(|\nabla u|^{q_0})(x)\rho_n(r)r^{d-1}dr\right]^{1/p} dx \lesssim \int_\Omega M^{1/q_0}(|\nabla u|^{q_0})(x)dx. \quad (97)$$

Since $q > q_0$, it follows from the theory of maximal functions that

$$\int_\Omega M^{1/q_0}(|\nabla u|^{q_0})(x)dx \lesssim \|\nabla u\|_{L^2(\Omega)}. \quad (98)$$

Combining (97) and (98) yields

$$T_1 \lesssim \|\nabla u\|_{L^q(\Omega).} \quad (99)$$

We next estimate T_2. We have

$$\int_{\mathbb{S}^{d-1}} |u(x) - u(x + r\acute{\sigma})|\, d\acute{\sigma} \le \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(x + s\acute{\sigma})| ds d\acute{\sigma}.$$

Applying Lemma (3.1.14), we obtain, for $0 < r < 1$ and $\in \Omega$,

$$\int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(x + s\acute{\sigma})| ds d\acute{\sigma} \le CrM(|\nabla u|)(x). \quad (100)$$

We derive that

$$T_2 \lesssim \int_\Omega M(|\nabla u|)(x)\, dx \lesssim \|\nabla u\|_{L^q} \quad (101)$$

by the theory of maximal functions since $q > 1$. Combining (96), (99) and (101) yields (94).

Assertion (95) follows from (94) via a density argument as in the proof of Proposition (3.1.9).

In the proof of Proposition (3.1.13), we used the following elementary.

Lemma (3.1.14)[107]: Let $d \ge 1, r > 0, x \in \mathbb{R}^d$, and $f \in L^1_{loc}(\mathbb{R}^d)$. We have

$$\int_{\mathbb{S}^{d-1}} \int_0^r |f(x + s\sigma)| ds d\sigma \le C_d r M(f)(x), \quad (102)$$

for some positive constant C_d depending only on d.

Proof: Set $\varphi(s) = \int_{\mathbb{S}^{d-1}} |f(x + s\sigma)| d\sigma$, so that, by the definition of $M(f)(x)$, we have

$$\int_{B_r(x)} |f(y)| dy \le M(f)(x) \quad \forall r > 0,$$

and thus

$$H(r) := \int_0^r \varphi(s) s^{d-1} ds \le |B_1| r^d M(f)(x) \quad \forall r > 0. \quad (103)$$

Then $H'(r) = \varphi(r) r^{d-1}$, so that

$$\int_0^r \varphi(s) ds = \int_0^r \frac{H'(s)}{s^{d-1}} ds = \frac{H(r)}{r^{d-1}} + (d-1)\int_0^r \frac{H(s)}{s^d} ds \le C_d r\, M(f)(x),$$

by (103); which is precisely (102). (The integration by parts can be easily justified by approximation.)

Under the assumption that ρ_n is non-increasing for every n, one can replace the condition $q > q_0$ in Proposition (3.1.13) by the weaker condition $q > q_1$, where

$$q_1 := \max\{pd/(p + d), 1\}, \quad (104)$$

so that $1 \le q_1 < q_0$. It is worth noting that the embedding $W^{1,q_1}(\Omega) \subset L^p(\Omega)$ is sharp and therefore q_1 is a natural lower bound for q. In fact, we prove a slightly more general result:

Proposition (3.1.15)[107]: Let $p > 1$ and assume that (ρ_n) satisfies (1)–(3). Suppose in addition that there exist $\Lambda > 0$ and a sequence of non-increasing functions $(\hat{\rho}_n) \subset L^1_{loc}(0, +\infty)$ such that

$$\rho_n \le \hat{\rho}_n \quad \text{and} \quad \int_0^\infty \hat{\rho}_n(t) t^{d-1} dt \le \Lambda \quad \forall n. \tag{105}$$

Then

$$\Phi_n(u) \le \|\nabla u\|_{L^q} \quad \forall u \in W^{1,q}(\Omega) \text{ with } q > q_1, \tag{106}$$

for some positive constant $C = C(p, q, \Lambda, \Omega)$ depending only on p, q, Λ, and Ω. Moreover,

$$\lim_{n\to+\infty} \Phi_n(u) = J(u) \quad \forall u \in W^{1,q}(\Omega) \text{ with } q > q_1. \tag{107}$$

Proof: For simplicity of notation, we assume that ρ_n is non-increasing for all n and work directly with ρ_n instead of $\hat{\rho}_n$.We first prove (106). As in the proof of Proposition (3.1.13), one may assume that $\int_\Omega u = 0$. Consider an extension of u to $\mathbb{R}^d$ which is still denoted by u such that

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \le C_{q,\Omega}\|u\|_{W^{1,q}(\Omega)} \le C_{q,\Omega}\|\nabla u\|_{L^p(\Omega)}.$$

For simplicity of notation, we assume that $\operatorname{diam}(\Omega) \le 1/2$. Then

$$\Phi_n(u) \le \int_\Omega \left[\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|u(x + r\sigma) - u(x)|^p}{r^p} \rho_n(r) r^{d-1} dr d\sigma\right]^{1/p} dx.$$

We claim that for a.e. $\in \Omega$,

$$Z(x) = \left[\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|u(x + r\sigma) - u(x)|^p}{r^p} \rho_n(r) r^{d-1} dr d\sigma\right]^{1/p}$$
$$\le C M^{1/q_1}(|\nabla u|^{q_1})(x). \tag{108}$$

Here and in what follows, C denotes a positive constant depending only on p, d, and Λ.

From (108), we deduce (106) via the theory of maximal functions since $> q_1$. Assertion (107) follows from (106) by density as in the proof of Proposition (3.1.9).

It remains to prove (108). Without loss of generality we establish (108) for $x = 0$. The proof relies heavily on two inequalities valid for all $R > 0$:

$$\left[\int_{B_R} \left|u(\xi) - \int_{B_R} u\right|^p d\xi\right]^{1/p} \le CRM^{1/q_1}(|\nabla u|^{q_1})(0) \tag{109}$$

and

$$\int_{B_R} |u(\xi) - u(0)| \, d\xi \le CRM^{1/q_1}(|\nabla u|^{q_1})(0), \tag{110}$$

where $B_R = B_R(0)$.

Inequality (109) is simply a rescaled version of the Sobolev inequality

$$\left\|u - \int_{B_1} u\right\|_{L^p(B_1)} \le C|\nabla u|_{L^{q_1}(B_1)},$$

which implies that

$$\left[\int_{B_R} \left|u(\xi) - \int_{B_R} u\right|^p d\xi\right]^{1/p} \le CR\left[\int_{B_R} |\nabla u|^{q_1}\right]^{1/q_1} \le CRM^{1/q_1}(|\nabla u|^{q_1})(0).$$

To prove (110), we write

$$\int_{B_R} |u(\xi) - u(0)|\, d\xi = \int_0^R \int_{\mathbb{S}^{d-1}} |u(r\sigma) - u(0)|\, r^{d-1} dr d\sigma$$
$$\leq C \int_0^R r^{d-1} dr \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(s\sigma)|\, ds d\sigma$$
$$\leq C \int_0^R r^d M(|\nabla u|)(0) \quad by\ Lamma\ (3.1.15).$$

Thus

$$\int_{B_R} |u(\xi) - u(0)|\, d\xi \leq CRM(|\nabla u|)(0) \leq CRM^{1/q_1}(|\nabla u|^{q_1})(0).$$

From (108), we obtain

$$Z(0)^p = \sum_{i=0}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} |u(r\sigma) - u(0)|^p \rho_n(r) r^{d-1-p} dr d\sigma\,,$$

so that

$$Z(0)^p \leq C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} |u(r\sigma) - u(0)|^p\, r^{d-1} dr d\sigma. \qquad (111)$$

We have

$$|u(r\sigma) - u(0)| \leq \left| u(r\sigma) - \int_{B_{2^{-i}}} u \right| + \left| \int_{B_{2^{-i}}} u - u(0) \right|. \qquad (112)$$

Inserting (112) into (111) yields

$$Z(0)^p \leq C \sum_{i=0}^{\infty} (U_i + V_i), \qquad (113)$$

where

$$U_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} \left| u(r\sigma) - \int_{B_{2^{-i}}} u \right|^p r^{d-1} dr d\sigma$$

and

$$V_i = \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_{2^{-(i+1)}}^{2^{-i}} \left| \int_{B_{2^{-i}}} u - u(0) \right|^p r^{d-1} dr d\sigma.$$

Clearly,

$$U_i \leq \rho_n(2^{-(i+1)}) 2^{ip} \int_{\mathbb{S}^{d-1}} \int_0^{2^{-i}} \left| u(r\sigma) - \int_{B_{2^{-i}}} u \right|^p r^{d-1} dr d\sigma \ \leq \rho_n(2^{-(i+1)}) 2^{-id} A, \qquad (114)$$

by (109), where $A = M^{p/q_1}(|\nabla u|^{q_1})(0)$. On the other hand,

$$V_i \leq \rho_n(2^{-(i+1)}) 2^{ip} \left[\int_{B_{2^{-i}}}^{R} |u(\xi) - u(0)| d\xi \right]^p 2^{-id}$$
$$\leq C \rho_n(2^{-(i+1)}) 2^{-id} A\ by\ (110) \qquad (115)$$

Combining (113)–(115), we obtain

$$Z(0)^p \leq C \sum_{i=0}^{\infty} \rho_n(2^{-(i+1)}) 2^{-id} A\,.$$

Finally, we observe that

$$\int_0^1 \rho_n(r) r^{d-1} dr \geq \sum_{i=0}^{\infty} \int_{2^{-(i+1)}}^{2^{-(i+1)}} \rho_n(r) r^{d-1} dr \geq C \sum_{i=0}^{\infty} \rho_n\left(2^{-(i+1)}\right) 2^{-id}$$

and thus

$$Z(0)^p \leq CM^{p/q_1}(|\nabla u|^{q_1})(0) \int_0^1 \rho_n(r) r^{d-1} dr \leq CM^{p/q_1}(|\nabla u|^{q_1})(0)\,.$$

Remark (3.1.16)[107]: Assume that $d \geq 2$ and $1 < p \leq d/(d-1)$, so that $q_1 = 1$. The conclusion of Proposition (3.1.15) fails in the borderline case $q = q_1 = 1$. More precisely, for every $\in (1, d/(d-1)]$, there exist a sequence (ρ_n) satisfying (1)–(3) and (105), and a function $v \in W^{1,1}(\Omega)$ such that $\Phi_n(v) = +\infty$ for all n. The construction is similar to the one presented in the proof of Proposition(3.1.10). Indeed, let $\Omega = B_{1/2}(0)$. Fix $\alpha > 0$ and $\beta > 1$ such that

$$\alpha + \beta/p < 1. \tag{116}$$

Since $p > 1$ such α and β exist. Let (δ_n) be a sequence of positive numbers converging to 0 and consider

$$\rho_n(t) := A_n \frac{1}{t^d |\ln t|^\beta} \mathbb{1}_{(0,\delta_n)}.$$

Note that the functions $t \longmapsto \rho_n(t)$ are non-increasing. Here Anis chosen in such a way that (3) holds,i.e., $A_n \int_0^{\delta_n} \frac{dt}{t|\ln t|^\beta} = 1$. Set

$$V(x) = v(x_1) := \begin{cases} 0 & \text{if } -1/2 < x_1 < 0, \\ |\ln x_1|^{-\alpha} & \text{if } 0 < x_1 < 1/2,. \end{cases}$$

Clearly, $V \in W^{1,1}(\Omega)$. We have

$$\Phi_n(V) = \int_\Omega \left(\int_\Omega \frac{|V(x) - V(y)|^p}{|x-y|^p} \rho_n(|x-y|) dy \right)^{1/p} dx$$

$$\gtrsim \int_{\substack{B_{1/4}(0) \\ 0 < x_1 < \delta_n/4}} \mathrm{A}_\mathrm{n}^{1/\mathrm{p}} \left(\int_{|y-x| \leq \delta_n} \frac{|v(x_1) - v(y_1)|^p}{|x-y|^{p+d} |\ln|x-y||^\beta} dy \right)^{1/p} dx.$$

Note that, for $0 < x_1 < \delta_n/4$,

$$\int_{|y-x| \leq \delta_n} \frac{|v(x_1) - v(y_1)|^p dy}{|x-y|^{p+d} |\ln|x-y||^\beta} \gtrsim \int_{\substack{|y_1 - x_1| \leq \delta_n/4 \\ |\acute{x} - \acute{y}| \leq \delta_n/4}} \mathrm{A}_\mathrm{n}^{1/\mathrm{p}} \frac{|v(x_1) - v(y_1)|^p d\acute{y} dy_1}{(|x_1 - y_1|^{p+d} + |\acute{x} - \acute{y}|^{p+d}) |\ln|x_1 - y_1||^\beta}.$$

$$\gtrsim \int_{|y_1 - x_1| \leq \delta_n/4} \frac{|v(x_1) - v(y_1)|^p dy_1}{|x_1 - y_1|^{p+d} |\ln|x_1 - y_1||^\beta}$$

We derive as in the proof of Proposition (3.1.12) that

$$\int_{|y-x| \leq \delta_n} \frac{|v(x_1) - v(y_1)|^p dy}{|x-y|^{p+d}\ |\ln|x-y||^\beta} \gtrsim \frac{v(x_1)^p}{x_1^p |\ln x_1|^\beta}.$$

It follows that

$$\Phi_n(V) \gtrsim \int_{\substack{\Omega \\ 0 < x_1 < \delta_n/4}} A_n^{1/p} \frac{v(x_1)}{x_1 |\ln x_1|^{\beta/p}} dx = \int_{\substack{\Omega \\ 0 < x_1 < \delta_n/4}} A_n^{1/p} \frac{1}{x_1 |\ln x_1|^{\alpha + \beta/p}} dx = +\infty,$$

(by (116)).

Concerning the Γ-convergence of Φ_n, G. Leoni and D. Spector proved in [123].

Proposition (3.1.17)[107]: For every $p > 1$ we have

$$\Phi_n \xrightarrow{\Gamma} \Phi_0(\cdot) := \gamma \int_\Omega |\nabla \cdot| \qquad in\ L^1(\Omega),$$
where γ is given in (63).

Their proof is quite involved. Here is a simpler proof.

Proof: For D an open subset of Ω such that $\overline{D} \subset \Omega$, set

$$\Phi_n(u, D) = \int_D dx \left[\int_D \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) dy \right]^{1/p} \quad \text{for } u \in L^1(D).$$

Let $u \in L^1(\Omega)$ and $(u_n) \subset L^1(\Omega)$ be such that $u_n \to u$ in $L^1(\Omega)$. We must prove that

$$\lim_{n\to\infty} \inf \Phi_n(u_n) \geq \gamma \int_\Omega |\nabla u|.$$

Let (μ_k) be a sequence of smooth mollifiers such that supp $\mu_k \subset B_{1/k}$. Let D be a smooth open subset of Ω such that $\overline{D} \subset \Omega$ and fix k_0 such that $D + B_{1/k_0} \subset \Omega$. We have as in (48), for $k \geq k_0$,

$$\Phi_n(\mu_k * u_n, D) \leq \Phi_n(u_n). \tag{117}$$

Using the fact that

$$|\Phi_n(u, D) - \Phi_n(v, D)| \leq C_D \|u - v\|_{W^{1,\infty}(D)} \quad \forall u, n \in W^{1,\infty}(D),$$

we obtain

$$|\Phi_n(\mu_k * u_n, D) - \Phi_n(\mu_k * u, D)| \leq C_{k,D} \|u_n - u\|_{L^1(\Omega)}.$$

Hence

$$\Phi_n(\mu_k * u, D) \leq \Phi_n(\mu_k * u_n, D) + C_{k,D} \|u_n - u\|_{L^1(\Omega)}. \tag{118}$$

Combining (117) and (118) yields

$$\gamma \int_D |\nabla(\mu_k * u)| \leq \liminf_{n\to+\infty} \Phi_n(u_n).$$

Letting $k \to \infty$, we reach

$$\gamma \int_D |\nabla u| \leq \liminf_{n\to+\infty} \Phi_n(u_n).$$

Since $D \subset\subset \Omega$ is arbitrary, we derive that

$$\gamma \int_D |\nabla u| \leq \liminf_{n\to+\infty} \Phi_n(u_n).$$

We next fix $u \in BV(\Omega)$ and construct a sequence (u_n) converging to u in $L^1(\Omega)$ such that

$$\limsup_{n\to+\infty} \Phi_n(u_n) \leq \gamma \int_\Omega |\nabla u|.$$

Let $v_k \in C^1(\overline{\Omega})$ be such that

$$v_k \to u \quad \text{in } L^1(\Omega) \text{ and } \quad \int_\Omega |\nabla v_k| \to \int_\Omega |\nabla u|. \tag{119}$$

For each k, let n_k be such that

$$\left| |\Phi_n(v_k) - \gamma \int_\Omega |\nabla v_k| \right| \leq 1/k \quad \text{if } n > n_k. \tag{120}$$

Without loss of generality, one may assume that (n_k) is an increasing sequence with respect to k. Define

$$u_n = v_k \quad \text{if } n_k < n \leq n_{k+1}.$$

We derive from (119) and (120) that

$$u_n \to u \quad \text{in } L^1(\Omega) \text{ and } \quad \lim_{n\to+\infty} \Phi_n(u_n) = \gamma \int_\Omega |\nabla u|.$$

The proof is complete.

Set

$$\hat{E}_n(u) := \int_\Omega |u-f|^q + \Phi_n(u),$$

and

$$\hat{E}_0(u) := \int_\Omega |u-f|^q + \gamma \int_\Omega |\nabla u|,$$

where $q > 1$ and $f \in L^q(\Omega)$ is a given function. Motivated by Image Processing, we study variational problems related to $\hat{E}_n$. More precisely, we establish

Proposition (3.1.18)[107]: For every n, there exists a unique $u_n \in L^q(\Omega)$ such that

$$\hat{E}_n(u_n) = \min_{u\in L^q(\Omega)} \hat{E}_n(u).$$

Let u_0 be the unique minimizer of $\hat{E}_0$. We have, as $n \to +\infty$,

$$u_n \to u_0 \quad \text{in } L^q(\Omega)$$

and

$$\hat{E}_n(u_n) \to \hat{E}_0(u_0).$$

Proof: The proof is similar to the one of Proposition (3.1.7). The details are left to the reader.

Section (3.2): $W^{s,p}$ When $s \uparrow 1$ and Applications

This is a follow-up of [109] where we establish that

$$\lim_{s\uparrow 1}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \sim \|\nabla f\|_{L^p(\Omega)}, \tag{121}$$

for any $p \in [1,\infty)$, where Ω is a smooth bounded domain in $\mathbb{R}^d, d \geq 1$.

On the other hand, if $0 < s < 1, p > 1$ and $sp < d$, the Sobolev inequality for fractional Sobolev spaces (see e.g. [135]) asserts that

$$\|f\|^p_{W^{s,p}(\Omega)} \geq C(s,p,d)\left\|f - ⨍ f\right\|_{L^q(\Omega)} \tag{122}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d}. \tag{123}$$

Here we use the standard semi-norm on $W^{s,p}$

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy. \tag{124}$$

When $s = 1$ the analog of (122) is the classical Sobolev inequality

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(p,d)\left\|f - ⨍ f\right\|^p_{L^{p^*}(\Omega)} \tag{125}$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \quad and \quad 1 \leq p < d.$$

The behaviour of the best constant $C\,(p,d)$ in (125) as $p \uparrow d$ is known (see e.g. [138], and also Remark (3.2.2) below); more precisely one has

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(d)(p-d)^{p-1}\left\|f - ⨍ f\right\|^p_{L^{p^*}(\Omega)}. \tag{126}$$

Putting together (121), (124) and (126) suggests that (122) holds with

$$C(s,p,d) = C(d)\,(d-sp)^{p-1}/(1-s), \tag{127}$$
for all $s < 1, s$ close to 1 and $sp < d$.

For simplicity we work with $\Omega =$ the unit cube Q in $\mathbb{R}^d$

Corollary (3.2.1)[134]: For all $0 < \varepsilon \le 1/2$,
$$|A|\,|{}^cA| \le \left(C(d)\varepsilon \int_A \int_{{}^cA} \frac{dxdy}{|x-y|^{d+1-\varepsilon}}\right)^{d/(d-1+\varepsilon)}. \tag{128}$$
Note that in the special case $d = 1$, (128) takes the simple form
$$|A|\,|{}^cA| \le \left(C^*\varepsilon \int_A \int_{{}^cA} \frac{dxdy}{|x-y|^{2-\varepsilon}}\right)^{1/\varepsilon} \tag{129}$$
for some absolute constant C^*. Estimate (129) is sharp as can be easily seen when A is an interval.

The conclusion of Corollary (3.2.1) is related to a result stated in [109] There is however an important difference. In [109] the set A was fixed (independent of ε) and the statement there provides a bound for $|A|\,|{}^cA|$ in terms of the limit, as $\varepsilon \to 0$, of the RHS in (128). The improved version - which requires a more delicate argument- is used we apply Corollary (3.2.1) (with $d = 1$) to give a proof
of a result announced in [136]. Namely , on $\Omega = (\ 1, +1)$ consider the function
$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for} \quad -1 < x < 0, \\ 2\pi x/\delta & \text{for} \quad 0 < x < \delta, \\ 2\pi & \text{for} \quad \delta < x < 1, \end{cases}$$
where $\delta = e^{-1/\varepsilon}, \varepsilon > 0$ small.

Set $u_\varepsilon = e^{i\varphi_\varepsilon}$. It is easy to check (by scaling) that
$$\|u_\varepsilon\|_{H^{1/2}} = \|u_\varepsilon - 1\|_{H^{1/2}} \le C$$
as $\varepsilon \to 0$ and consequently $\|u_\varepsilon\|_{H^{(1-\varepsilon)/2}} \le C$ as $\varepsilon \to 0$. On the other hand, a straight forward computation shows that $\|\varphi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \sim \varepsilon^{-1/2}$.

The result announced in [136] asserts that any lifting φ_ε of u_ε blows up in $H^{(1-\varepsilon)/2}$ (at least) in the same rate as φ_ε:

Remark (3.2.2)[134]: There are various versions of the Sobolev inequality (125). All these forms hold with equivalent constants:

Form (i): $\|\nabla f\|_{L^p(Q)} \ge A_1 \left\|f - ⨍_Q f\right\|_{L^q(\mathrm{Q})} \quad \forall f \in W^{1,p}(Q)$.

Form (ii): $\|\nabla f\|_{L^p(Q)} \ge A_2 \left\|f - ⨍_Q f\right\|_{L^q(\mathrm{Q})}$ for all Q-periodic functions $f \in W^{1,p}_{loc}(\mathbb{R}^d)$.

Form (iii): $\|\nabla f\|_{L^p(\mathbb{R}^d)} \ge A_3 \|f\|_{L^q(\mathbb{R}^d)} \quad \forall f \in C_0^\infty(\mathbb{R}^d)$.

Form (i)⟹ Form (ii): Obvious with $A_2 = A_1$.

Form (ii)⟹ Form (i): Given any function $f \in W^{1,p}(Q)$, it can be extended by reflections to a periodic function on a larger cube $\tilde{Q}$ so that form (ii) implies Form(i) with $A_1 \ge C\,A_2$, and C depends only on d.

Form (i)⟹ Form (iii): By scale invariance, form(i) holds with the same constant A_1 on the cube Q_R of side R. Fix a function $f \in C_0^\infty(\mathbb{R}^d)$ and let $R >$ diam (Supp).We have
$$\|\nabla f\|_{L^p(Q_R)} \ge A_1 \left\|f - ⨍_{Q_R} f\right\|_{L^q(Q_R)}.$$
As $R \to \infty$ we obtain form (iii) with $A_3 = A_1$.

Form (iii)⟹ Form (ii): Given a smooth periodic function f on $\mathbb{R}^d$, let ρ be a smooth cut-off function with $\rho = 1$ on Q and $\rho = 0$ outside $2Q$. Then

$$\|\nabla(\rho f)\|_{L^p(\mathbb{R}^d)} \geq A_3\|\rho f\|_{L^q(\mathbb{R}^d)}$$

and thus

$$A_3\|f\|_{L^q(Q)} \leq C\big(\|\nabla f\|_{L^p(Q)} + \|f\|_{L^p(Q)}\big)$$

where C depends only on d. Replacing f by $(f - ⨍_Q f)$ and applying Poincare's inequality (see e.g. [138]) yields

$$A_3\left\|f - ⨍ f\right\|_{L^q(Q)} \leq C\|\nabla f\|_{L^q(Q)}.$$

We check easily that the same considerations hold for the fractional Sobolev norms such as in (130). The proof of the last implication (Form (iii) $\Longrightarrow$ Form (ii)) involves a Poincare-type inequality . What we use here is the following

Fact: Let $1 \leq p < \infty, 1/2 \ \leq s < 1$, then

$$(1-s)\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \geq c(d)\left\|f - ⨍_Q f\right\|^p_{L^p(Q)}.$$

The proof of this fact is left. It is an adaptation of the argument in the beginning. In (123) one uses an obvious lower bound:

$$(123) \geq c\left(\sum_r \|f_r\|_{L^p}\right)^p \geq c\left\|f - ⨍ f\right\|^p_{L^p}.$$

For the convenience of the reader we have divided the proof of Theorem (3.2.3) into several cases.

Theorem (3.2.3)[134]: Assume $d \geq 1, p \geq 1, 1/2 \leq \ s < 1$ and $sp < 1$. Then

$$\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \geq C(d)\frac{(d-sp)^{p-1}}{1-s}\left\|f - ⨍ f\right\|^p_{L^q(Q)} \tag{130}$$

where q is given by (123) and $C(d)$ depends only on d.

As can be seen from (130) there are two phenomena that govern the behaviour of the constant in (130). As $s \uparrow 1$ the constant gets bigger, while as $s \uparrow d/p$ the constant deteriorates. This explains why the we consider several cases in the proof.

As an application of Theorem (3.2.3) with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$ we easily obtain

Proof (1): when $p = 1$ and $d = 1$.

For simplicity , we work with periodic functions of period 2π (for non-periodic functions see Remark (3.2.2)). All integrals, L^p norms, etc...., are understood on the interval $(0, 2\pi)$. We must prove that, (with $\varepsilon = 1 - s$), for all $\varepsilon \in (0, 1/2]$,

$$C\varepsilon\int\int \frac{|f(x)-f(y)|^p}{|x-y|^{2-\varepsilon}}dxdy \geq \left\|f - ⨍ f\right\|_{L^{1/\varepsilon}}. \tag{131}$$

Write the left side as

$$\varepsilon\int \frac{1}{|h|^{2-\varepsilon}}\|f - f_h\|_1 dh \sim \varepsilon\sum_{k\geq 0} 2^{k(2-\varepsilon)}\int_{|h|\sim 2^{-k}}\|f - f_h\|_1. \tag{132}$$

For $|h| \sim 2^{-k}$

$\|f - f_h\|_1 \geq$

$$\left\|(f - f_h) * F_{N_k}\right\|_1 = \left(N_k = 2^{k-100}, N_k(x) = \sum_{|n|\leq N}\frac{N-|n|}{N}e^{inx} = \text{Fejer kernel}\right)$$

$$\left\|\sum_{|n|<N_k}\frac{N_k-|n|}{N_k}\hat{f}(n)(e^{inh}-1)e^{inx}\right\|_1 \sim$$

$$2^{-k}\left\|\sum_{|n|<N_k}\frac{N_k-|n|}{N_k}n\hat{f}(n)e^{inx}\right\|_1 \qquad \text{(by the choice } of\ N_k).$$

This last equivalence is justified via a smooth truncation as in the following

Lemma(3.2.4)[134]: $\left\|\sum_{|n|<N}\hat{f}(n)(e^{inh}-1)e^{inx}\right\|_1 \gtrsim \frac{1}{N}\left\|\sum_{|n|<N}n\hat{f}(n)e^{inx}\right\|_1$ for $|h|<\frac{1}{100N}$.

Proof: Write

$$\left\|\sum_{|n|<N}n\hat{f}(n)e^{inx}\right\|_1 \le \left\|\sum_{|n|<N}\hat{f}(n)(e^{inh}-1)e^{inx}\right\|_1 \cdot \left\|\sum\varphi\left(\frac{n}{N}\right)\frac{n}{e^{inh}-1}e^{inx}\right\|_1$$

where $0\le\varphi\le 1$ is a smooth function with

$$\varphi(t)=\begin{cases}1 & \text{for} \quad |t|\le 1\\ 0 & \text{for} \quad |t|\ge 2\end{cases}$$

We have from assumption

$$\left\|\sum\varphi\left(\frac{n}{N}\right)\frac{n}{e^{inh}-1}e^{inx}\right\|_1 \sim N\left\|\sum\varphi\left(\frac{n}{N}\right)\frac{nh}{e^{inh}-1}e^{inx}\right\|_1$$

and the second factor remains uniformly bounded. This may be seen by expanding

$$\frac{y}{e^{iy}-1}\sim\frac{1}{i}+0(y)$$

for $|y|<\frac{1}{50}$ and using standard multiplier bounds.

We now return to the proof of Theorem (3.2.3) ($p=1, d=1$).

Substitution in (132) gives thus

$$\varepsilon\sum_{k\ge 0}2^{-\varepsilon k}\left\|\sum_{|n|<N_k}\frac{N_k-|n|}{N_k}n\hat{f}(n)e^{inx}\right\|_1. \tag{133}$$

Define

$$k_0=\frac{10}{\varepsilon}.$$

For $k_0<k<2k_0$, minorate (using Lemma (3.2.4))

$$\left\|\sum_{|n|<N_k}\frac{N_k-|n|}{N_k}n\hat{f}(n)e^{inx}\right\|_1 \gtrsim \left\|\sum_{|n|<N_{k_0}}\frac{N_{k_0}-|n|}{N_{k_0}}n\hat{f}(n)e^{inx}\right\|_1$$

and therefore

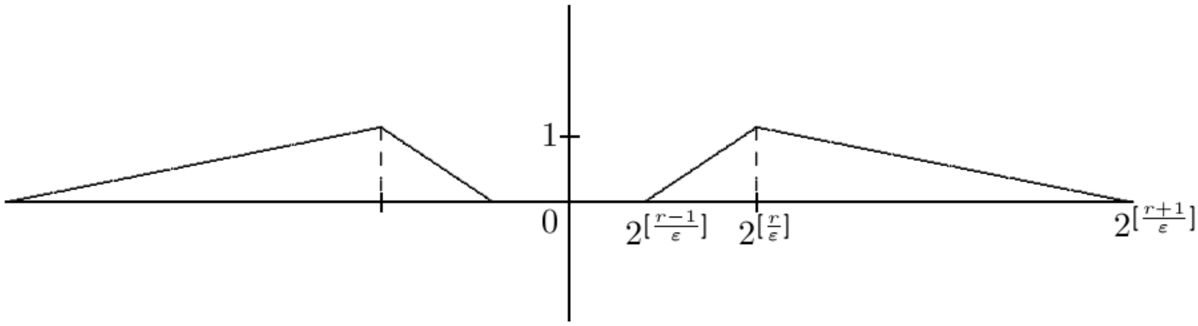
$$(133)\gtrsim\left\|\sum_{|n|<N_{k_0}}\frac{N_{k_0}-|n|}{N_{k_0}}n\hat{f}(n)e^{inx}\right\|_1 = \left\|\sum_{|n|<N_{k_0}}\frac{N_{k_0}-|n|}{N_{k_0}}n\hat{f}(n)e^{inx}\right\|_{W^{1,1}}$$

$$\ge\left\|\sum_{0<|n|<N_{k_0}}\frac{N_{k_0}-|n|}{N_{k_0}}n\hat{f}(n)e^{inx}\right\|_\infty. \tag{134}$$

Next write also

$$(133) \gtrsim \varepsilon \sum_{r\geq 1} 2^{-r} \sum_{\left[\frac{r+2}{\varepsilon}\right]\leq k<\left[\frac{r+3}{\varepsilon}\right]} \left\| \sum_{|n|<N_k} \frac{N_k-|n|}{N_k} n\hat{f}(n)e^{inx} \right\|_1$$

$$\gtrsim \sum_{r\geq 1} 2^{-r} \left\| \sum_{|n|<2^{\left[\frac{r+1}{\varepsilon}\right]}} \frac{2^{\left[\frac{r+1}{\varepsilon}\right]}-|n|}{2^{\left[\frac{r+1}{\varepsilon}\right]}} e^{inx} \right\|_1 . \qquad (135)$$



Denote for each r by $\lambda_r = \{\lambda_r(n) | n \in \mathbb{Z}\}$ the following multiplier
Thus

$$\lambda_r(n) = \lambda_r(-n)$$
$$\left\| \sum \lambda_r(n)e^{inx} \right\|_1 < C.$$

(This multiplier may be reconstructed from Fejer-kernels F_N with $N = 2^{\left[\frac{r+1}{\varepsilon}\right]}, 2^{\left[\frac{r}{\varepsilon}\right]}, 2^{\left[\frac{r-1}{\varepsilon}\right]}$).

Also

$$\left\| \sum_{|n|<2^{\left[\frac{r+1}{\varepsilon}\right]}} \frac{2^{\left[\frac{r+1}{\varepsilon}\right]}-|n|}{2^{\left[\frac{r+1}{\varepsilon}\right]}} n\hat{f}(n)e^{inx} \right\|_1 \gtrsim$$

$$\left\| \sum_{2^{\left[\frac{r-1}{\varepsilon}\right]}<|n|<2^{\left[\frac{r+1}{\varepsilon}\right]}} \lambda_r(n)n\hat{f}(n)e^{inx} \right\|_1 \qquad (136)$$

and

$$(135) \gtrsim \sum_{r\geq 1} 2^{-r} \left\| \sum_{2^{\left[\frac{r-1}{\varepsilon}\right]}<|n|<2^{2^{\left[\frac{r+1}{\varepsilon}\right]}}} \lambda_r(n)(\operatorname{sign} n)|n|\, \hat{f}(n)e^{inx} \right\|_1 . \qquad (137)$$

We claim that for $q > 2$

$$\left\| \sum_{N_1<|n|<N_2} \hat{g}(n)e^{inx} \right\|_q \leq C N_1^{-\frac{1}{q}} \left\| \sum_{N_1<|n|<N_2} |n|(\operatorname{sign} n)\, \hat{g}(n)e^{inx} \right\|_1 \qquad (138)$$

with the constant C independent of q.

Applying (138) with

$$q = \frac{1}{\varepsilon}, \qquad \hat{g}(n) = \lambda_r(n)\hat{f}(n), \quad N_1 = 2^{\left[\frac{r-1}{\varepsilon}\right]}, N_2 = 2^{\left[\frac{r+1}{\varepsilon}\right]}$$

we obtain the minoration

$$(137) \gtrsim \sum_{r\geq 1} \left\| \sum_{2^{\left[\frac{r-1}{\varepsilon}\right]}<|n|<2^{\left[\frac{r+1}{\varepsilon}\right]}} \lambda_r(n)\, \hat{f}(n) e^{inx} \right\|_q . \tag{139}$$

By construction

$$\sum_{r\geq 1} \lambda_r(n) = 1 \;\text{ for } |n| > 2^{\left[\frac{1}{\varepsilon}\right]} .$$

Using also minoration (134) together with the triangle-inequality yields

$$\text{LHS in } (131) \gtrsim (133) + (138) \gtrsim \left\| \sum_{n\neq 0} \hat{f}(n) e^{inx} \right\|_q$$

which proves the inequality .

Proof of (138).

Estimate

$$\left\| \sum_{N_1<|n|<N_2} \hat{g}(n) e^{inx} \right\|_q$$

$$\leq \left\| \sum_{N_1<|n|<N_2} |n|^{-1} (\operatorname{sign} n)\, e^{inx} \right\|_q \left\| \sum_{N_1<|n|<N_2} |n| (\operatorname{sign} n)\, \hat{g}(n) e^{inx} \right\|_1$$

where the first factor equals

$$\left\| \sum_{N_1<|n|<N_2} \frac{1}{n} \sin nx \right\|_q \lesssim$$

$$\left\| \sum_{\log_{N_1}<k<\log_{N_2}} \left| \sum_{n\sim 2^k} \frac{1}{n} \sin nx \right| \right\|_q \quad (\text{assume } N_1, N_2 \text{ powers of } 2)$$

$$\lesssim \left\| \sum_{\log_{N_1}<k<\log_{N_2}} \min(2^k|x|, 2^{-k}|n|^{-1}) \right\|_q \lesssim \left\| \frac{1}{1+N_1|x|} \right\|_q \lesssim N_1^{-1/q}. \tag{140}$$

This proves (138) and completes the proof of Theorem (3.2.3) when $p = 1$ and $d = 1$.

Theorem (3.2.5)[134]: Assume $d \geq 1, p \geq 1, 1/2 \leq s < 1$ and $sp < 1$. Then

$$\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} dxdy \geq C(d) \frac{(d-sp)^{p-1}}{1-s} \left\| f - \int f \right\|_{L^q(Q)}^p \tag{141}$$

where q is given by (123) and $C(d)$ depends only on d.

As can be seen from (130) there are two phenomena that govern the behaviour of the constant in (130). As $s \uparrow 1$ the constant gets bigger, while as $s \uparrow d/p$ the constant deteriorates. This explains why the we consider several cases in the proof.

As an application of Theorem (3.2.3) with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$ we easily obtain

Proof (2): when $p = 1$ and $d \geq 2$.

We have to prove that

$$\int\int \frac{|f(x)-f(y)|}{|x-y|^{d+s}}dxdy \geq \frac{C(d)}{1-s}\left\|f - ⨍f\right\|_q \tag{142}$$

where $q = d/(d-s)$. We assume $d = 2$. The case $d > 2$ is similar. Write

$$\int\int \frac{|f(x)-f(y)|}{|x-y|^{d+s}}dxdy \sim \sum_{0\leq k} 2^{k(d+s)} \int_{|h|\sim 2^{-k-10}} \|f(x+h)-f(x-h)\|_1 dh$$

$$\geq \sum 2^{k(d+s)} \int_{\substack{|h_1|\sim 2^{-k-10} \\ |h_1|\sim 2^{-k-10}}} \left\| \sum_{n\in\mathbb{Z}^d} \hat{f}(n)(\sin n.h)\, e^{in.x} \right\|_1 dh_1\, dh_2 \tag{143}$$

Let φ be a smooth function on $\mathbb{R}$ s.t. $0 \leq \varphi \leq 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{for} \quad |t| \leq 1 \\ 0 & \text{for} \quad |t| \geq 2 \end{cases}$$

As for $d = 1$, consider (radial) multipliers λ_0 and $\lambda_r, r \geq 1$

$$\lambda_0(n) = \varphi\left(2^{-\frac{1}{\varepsilon}}|n|\right)$$

$$\lambda_r(n) = \varphi\left(2^{-\frac{r+1}{\varepsilon}}|n|\right) - \varphi\left(2^{-\frac{r}{\varepsilon}}|n|\right) \tag{144}$$

where $\varepsilon = 1 - s$ and $\varepsilon \in (0, 1/2)$.

Hence

$$\sum \lambda_r(n) = 1$$

$$\|\lambda_r\|_{M(L^1,L^1)} \leq C \qquad \text{(multiplier norm)} \tag{145}$$

$$\text{supp}\, \lambda_0 \subset B\left(0, 2^{\frac{r}{\varepsilon}+1}\right) \tag{146}$$

$$\text{supp}\, \lambda_r \subset B\left(0, 2^{\frac{r+1}{\varepsilon}+1}\right) \backslash B\left(0, 2^{\frac{r}{\varepsilon}}\right). \tag{147}$$

Write

$$(143) \sum_{\frac{1}{\varepsilon}<k<\frac{2}{\varepsilon}} + \sum_{r\geq 1} \sum_{\frac{r+1}{\varepsilon}<k<\frac{r+2}{\varepsilon}}. \tag{148}$$

For $\frac{2}{\varepsilon} > k > \frac{1}{\varepsilon}$ and $|h| < 2^{-k-10}$, (145), (146) permit us to write

$$\left\| \sum_n \hat{f}(n) e^{in.x} \sin n.h \right\|_1 \gtrsim \left\| \sum_n \lambda_0(n)\, \hat{f}(n) e^{in.x} \sin n.h \right\|_1$$

$$\sim \left\| \sum_n \lambda_0(n)\, (n.h) \hat{f}(n) e^{in.x} \right\|_1$$

and thus

$$2^{k(d+1-\varepsilon)} \int_{|h_1|,|h_1|\sim 2^{-k-10}} \left\|\hat{f}(n)\, (\sin n.h) e^{in.x}\right\|_1 dh_1 dh_2 \gtrsim$$

$$2^{k(3+\varepsilon)} 8^{-k} \left(\left\| \sum \lambda_0(n) n_1\, \hat{f}(n) e^{in.x} \right\|_1 + \left\| \sum \lambda_0(n) n_2\, \hat{f}(n) e^{in.x} \right\|_1 \right)$$

$$= 2^{-k\varepsilon} \left(\left\| \partial_{x_1} \left(\sum \lambda_0(n)\, \hat{f}(n) e^{in.x} \right) \right\|_1 + \left\| \partial_{x_2}(\cdots) \right\| \right) \sim$$

$$\left\|\sum \lambda_0(n)\hat{f}(n)e^{in.x}\right\|_{W^{1,1}}. \tag{149}$$

Similarly , for

$$\frac{r+1}{\varepsilon} < k < \frac{r+2}{\varepsilon}$$

we have

$$2^{k(d+1-\varepsilon)}\int_{|h_1|,|h_1|\sim 2^{-k-10}}\left\|\sum \hat{f}(n)(\sin nh)e^{in.x}\right\|_1 \gtrsim$$

$$2^{-r}\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_{W^{1,1}}. \tag{150}$$

Since in the summation (148), each of the terms (149), (150) appear at least $\frac{1}{\varepsilon}$ times, we have

$$\varepsilon\cdot(143) \gtrsim \left\|\sum \lambda_0(n)\hat{f}(n)e^{in.x}\right\|_{W^{1,1}} + \sum_r 2^{-r}\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_{W^{1,1}}. \tag{151}$$

Write

$$\frac{2-s}{2} = 1-s+\frac{s}{2}$$

and by Holder's inequality

$$\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_{\frac{2}{2-s}}$$

$$\leq \left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_2^s \left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_1^{1-s}. \tag{152}$$

By the Sobolev embedding theorem ($d=2$)

$$\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_2 \leq C\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_{W^{1,1}}. \tag{153}$$

We estimate the last factor in (152).

Recalling (147).

$$2^{\frac{r+1}{\varepsilon}+1} > \max(|h_1|,|h_2|) > 2^{\frac{r}{\varepsilon}-1}$$

if $\lambda_r(n)\neq 0, r\geq 1$.

Hence, with φ as above

$$\lambda_r(n) = \lambda_r(n).(1-\varphi)(2^{-\frac{r-1}{\varepsilon}},n_1) + \lambda_r(n).\varphi(2^{-\frac{r-1}{\varepsilon}},n_1).(1-\varphi)(2^{-\frac{r-1}{\varepsilon}},n_2)$$

and thus

$$\left\|\sum \lambda_r(n)\hat{f}(n)e^{in.x}\right\|_1$$

$$\leq \left\|\sum \lambda_r(n)n_1\hat{f}(n)e^{in.x}\right\|_1 \left\|\sum \frac{1}{n_1}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}},n_1)e^{in.x}\right\|_1$$

$$+\left\|\sum \lambda_r(n) n_2 \hat{f}(n)\, e^{in.x}\right\|_1 \left\|\sum \frac{1}{n_2} \varphi(2^{-\frac{r-1}{\varepsilon}}, n_1)(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}, n_2)\, e^{in.x}\right\|_1 \leq$$

$$\leq \left(\left\|\sum_{n_1} \frac{1}{n_1}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}, n_1)\, e^{in.x}\right\|_{L^1_{x_1}} + \left\|\sum_{n_2} \frac{1}{n_2}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}, n_2)\, e^{in_2 x_2}\right\|_{L^1_{x_2}} \right).$$

$$\left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{W^{1,1}}. \qquad (154)$$

Since $(1-\varphi)\left(2^{-\frac{r-1}{\varepsilon}}, n_1\right) = 0$ for $|n_1| \leq 2^{-\frac{r-1}{\varepsilon}}$, one easily checks that

$$\left\|\sum_{n_1} \frac{1}{n_1}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}, n_1)\, \varepsilon^{e^{in_1 x_1}}\right\|_{L^1_{x_1}} \lesssim \sum_{\ell \geq \frac{r-1}{\varepsilon}} 2^{-\ell} < 2^{-\frac{r-2}{\varepsilon}}.$$

Similarly

$$\left\|\sum_{n_2} \frac{1}{n_1}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}, n_2)\, \varepsilon^{e^{in_2 x_2}}\right\|_{L^1_{x_2}} \leq 2^{-\frac{r-2}{\varepsilon}}.$$

Thus (154) implies that

$$\left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_1 \leq 2^{-\frac{r-2}{\varepsilon}} \left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{W^{1,1}}. \qquad (155)$$

Substitution of (153), (155) in (152) gives

$$\left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{\frac{2}{2-s}} \lesssim 2^{-\frac{r-2}{\varepsilon}}(1-s) \left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{W^{1,1}}$$

$$\sim 2^{-r} \left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{W^{1,1}}. \qquad (156)$$

By (153), (156)

$$\varepsilon.(143) \gtrsim \left\|\sum \lambda_0(n)\, \hat{f}(n)\, e^{in.x}\right\|_2 + \sum_{r \geq 1} \left\|\sum \lambda_r(n)\, \hat{f}(n)\, e^{in.x}\right\|_{\frac{2}{2-s}}$$

$$\geq \left\|f - ⨍ f\right\|_{\frac{2}{2-s}}$$

by (144).

This proves (142) and completes the proof of Theorem (3.2.5) when p = 1.

We present here some known inequalities used in the proof of Theorem (3.2.5) when $p > 1$. Let $\{\Delta_j f\}_{j=1,2,\cdots}$ be a Littlewood-Paley decomposition with $\Delta_j f$ obtained from a Fourier multiplier of the form $\varphi\left(2^{-j}|n|\right) - \varphi\left(2^{-j+1}|n|\right)$ with $0 \leq \varphi \leq 1$ a smooth function satisfying $\varphi(t) = 1$ for $|t| \leq 1$ and $\varphi(t) = 0$ for $|t| > 2$.

Recall the square-function inequality for $1 < q < \infty$

$$\frac{1}{C(q)}\left\|\left(\sum |\Delta_j f|^2\right)^{1/2}\right\|_q \leq \|f\|_q \leq C(q)\left\|\left(\sum |\Delta_j f|^2\right)^{1/2}\right\|_q. \qquad (157)$$

We will also consider square-functions wrt a martingale filtration. Denote thus $\{\mathbb{E}_j\}$ the expectation operators wrt a dyadic partition of $[0,1]^d$ and

$$\tilde{\Delta}_j f = (\mathbb{E}_j - \mathbb{E}_{j-1})f \qquad (158)$$

the martingale differences.

We will use the square-function inequality

$$\|f\|_q \leq C\sqrt{q}\left\|\left(\sum |\tilde{\Delta}_j f|^2\right)^{1/2}\right\|_q \qquad \text{for } \infty > q \geq 2 \qquad (159)$$

which is precise in terms of the behaviour of the constant for $q \to \infty$ (see [137]).

Remark (3.2.6)[134]: One should expect (144) also to hold if $\tilde{\Delta}_j$is replaced by Δ_j above but we will not need this fact.

We do use later on the following inequality .

Let

$$p < q \text{ and } s = d\left(\frac{1}{p} - \frac{1}{q}\right) \geq \frac{1}{2}.$$

Then, for $q \geq 2$

$$\|f\|_q \leq C\sqrt{q}\left[\sum_k \left(2^{ks}\|\Delta_k f\|_p\right)^2\right]^{1/2}. \qquad (160)$$

Proof of (144)

It follows from (144) that since $q \geq 2$

$$\|f\|_q \leq C\sqrt{q}\left(\sum_j \|\tilde{\Delta}_j f\|_q^2\right)^{1/2}. \qquad (161)$$

Write

$$\tilde{\Delta}_j f = \sum_{k \geq j} \tilde{\Delta}_j \Delta_k f + \sum_{k > j} \tilde{\Delta}_j \Delta_k f$$

$$\|\tilde{\Delta}_j f\|_q \lesssim \sum_{k \leq j} 2^{k-j}\|\Delta_k f\|_q + \sum_{k > j} 2^{js}\|\Delta_k f\|_p$$

$$\lesssim \sum_{k \leq j} 2^{k-j}\left(2^{ks}\|\Delta_k f\|_p\right) + \sum_{k > j} 2^{(j-k)s}\left(2^{ks}\|\Delta_k f\|_p\right). \qquad (162)$$

Substitution of (162) in (161) gives

$$\|f\|_q \leq C\sqrt{q}\left\{\left(\sum_{k \leq j} (j-k)^2 4^{k-j}\left(2^{ks}\|\Delta_k f\|_p\right)^2\right)^{1/2} + \left(\sum_{k > j} (k-j)^2 4^{(j-k)s}\left(2^{ks}\|\Delta_k f\|_p\right)^2\right)^{1/2}\right\}$$

$$\leq C\sqrt{q}\left(\sum_{k}\left(2^{ks}\|\Delta_k f\|_p\right)^2\right)^{1/2}. \tag{163}$$

Theorem (3.2.7)[134]: Assume $d \geq 1, p \geq 1, 1/2 \leq s < 1$ and $sp < 1$. Then

$$\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \geq C(d)\frac{(d-sp)^{p-1}}{1-s}\left\|f - ⨍f\right\|_{L^q(Q)}^p \tag{164}$$

where q is given by (123) and $C(d)$ depends only on d.

As can be seen from (130) there are two phenomena that govern the behaviour of the constant in (130). As $s \uparrow 1$ the constant gets bigger, while as $s \uparrow d/p$ the constant deteriorates. This explains why the we consider several cases in the proof.

As an application of Theorem (3.2.3) with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$ we easily obtain

Proof (3): when $1 < p < 2$.

Write

$$\int\int \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}dxdy \sim \sum_{k\geq 0} 2^{k(d+ps)}\int_{|h|\sim 2^{-k-10}}\|f(x+h)-f(x-h)\|_p^p dh$$

$$\geq \sum_{k\geq 0} 2^{k(d+ps)}\int_{|h|\sim 2^{-k-10}}\left\|\sum \hat{f}(n)(\sin n.h)\, e^{in.x}\right\|_p^p dh. \tag{165}$$

Following the argument (formula (151)), we get again for

$$s = d\left(\frac{1}{p}-\frac{1}{q}\right), 1-s = \varepsilon \tag{166}$$

$$\varepsilon.(165) \gtrsim \sum_r\left(2^{-r}\left\|\sum_n \lambda_r(n)\,\hat{f}(n)\, e^{in.x}\right\|_{W^{1,1}}\right)^p \tag{167}$$

where the multipliers λ_r are defined as before.

Case: $d = 1$

Define

$$f_r = \sum_n \lambda_r(n)\,\hat{f}(n)\, e^{in.x}.$$

We will make 2 estimates.

First write

$$f_r = \left(\sum n\lambda_r(n)\,\hat{f}(n)\, e^{in.x}\right) * \left(\sum_{2^{\frac{r}{\varepsilon}}<|n|<2^{\frac{r+1}{\varepsilon}}}\frac{1}{n}\, e^{in.x}\right)$$

implying

$$\|f_r\|_q \leq \|f_r\|_{W^{1,p}}\left\|\sum_{2^{\frac{r}{\varepsilon}}<n<2^{\frac{r+1}{\varepsilon}}}\frac{1}{n}\sin nx\right\|_{\left(\frac{1}{p}+\frac{1}{q}\right)^{-1}} \tag{168}$$

and by estimate (151)

$$\|f_r\|_q \lesssim 2^{-\frac{r}{\varepsilon}\left(\frac{1}{p}+\frac{1}{q}\right)}\|f_r\|_{W^{1,p}} = 2^{-\frac{r}{\varepsilon}(1-s)}\|f_r\|_{W^{1,p}} = 2^{-r}\|f_r\|_{W^{1,p}}. \tag{169}$$

Estimate then

$$\|f\|_q \le \sum_r \|f_r\|_q \le C \sum_r (2^{-r}\|f_r\|_{W^{1,p}}). \tag{170}$$

Next apply inequality (168). Observe that

$$|\Delta_k f| \le \sum_r |\Delta_k f_r|$$

where, by construction, there are, for fixed k, at most 2 nonvanishing terms.

Thus

$$\|\Delta_k f\|_p^2 \lesssim \sum_r \|\Delta_k f_r\|_p^2. \tag{171}$$

Also, for fixed r

$$\sum_k \left(2^{ks}\|\Delta_k f_r\|_p\right)^2 = \sum_r 4^{-ks}\|\Delta_k f_r\|_{W^{1,p}}^2 \lesssim \frac{1}{\varepsilon} 4^{-r}\|f_r\|_{W^{1,p}}^2. \tag{172}$$

Substituting (171), (172) in (168) gives

$$\|f\|_q \lesssim C\sqrt{q}\left[\sum_k \sum_r \left(2^{ks}\|\Delta_k f_r\|_p\right)^2\right]^{1/2} \le \left[C\sqrt{\frac{q}{\varepsilon}}\right)\left[\sum_r (2^{-r}\|f_r\|_{W^{1,p}})^2\right]^{1/2} \tag{173}$$

which is the second estimate.

Interpolation between (170) and (173) implies thus

$$\|f\|_q \le C\left(\sqrt{\frac{q}{\varepsilon}}\right)^{2\left(1-\frac{1}{p}\right)}\left[\sum_r (2^{-r}\|f_r\|_{W^{1,p}})^2\right]^{1/p}. \tag{174}$$

Recalling (167) and also (166) (which implies that $1-\varepsilon = \frac{1}{p}-\frac{1}{q} < \frac{1}{p}$, hence $\varepsilon > 1-\frac{1}{p}$) it follows that

$$\varepsilon.(165) \gtrsim \left(\frac{1}{q}\right)^{p-1}\|f\|_q^p \tag{175}$$

which gives the required inequality .

Case: $d > 1$

We will distinguish the further 2 cases

Case (i): $0 < \frac{1}{p} - \frac{1}{d}$ is not near 0

Case (ii): $\frac{1}{p} - \frac{1}{d}$ is near 0

Observe that case (ii) may only happen for $d = 2$ and p near 2 (we assumed $1 < p < 2$).

Case (i):

Define q_1 by

$$1 = d\left(\frac{1}{p} - \frac{1}{q_1}\right) \tag{176}$$

so that $q < q_1$ and q_1 is bounded from above by assumption.

Thus we have the Sobolev inequality

$$\|g\|_{q_1} \le C\|g\|_{W^{1,p}}. \tag{177}$$

Next, we make the obvious adjustment of the argument, (175)-(179).

Thus Holder's inequality gives

$$\|f_r\|_q \le \|f_r\|_{q_1}^{1-\theta}\|f_r\|_q^{\theta} \tag{178}$$

with

$$\frac{1}{q} - \frac{1-\theta}{q_1} + \frac{\theta}{p}, \ \ hence \ \ \theta = 1 - s = \varepsilon \ \ by \ (166), (176).$$

Hence, by (177)

$$\|f_r\|_q \leq C\|f_r\|_{W^{1,p}}^{1-\varepsilon}\|f_r\|_p^{\varepsilon} \ . \tag{179}$$

To estimate $\|f_r\|_p$, proceed as in (177). Thus

$$\|f_r\|_p \lesssim \left\|\sum \frac{1}{n} \ (1-\varphi)\,(2^{-\frac{r-1}{\varepsilon}} n) e^{in.x}\right\|_{L_x^1(\mathbb{T})} \|f_r\|_{W^{1,p}} \lesssim 2^{-\frac{r-1}{\varepsilon}}\|f_r\|_{W^{1,p}}. \tag{180}$$

Substitution of (180) in (179) gives

$$\|f_r\|_q \lesssim 2^{-r}\|f_r\|_{W^{1,p}}. \tag{181}$$

Substitution of (181) in (167) gives (since q is bounded by case (i) hypothesis)

$$\begin{aligned} \varepsilon.(165) \gtrsim \sum_r \|f_r\|_q^p \sim \sum_r \left\|\left(\sum_j \left|\Delta_j f_r\right|^2\right)^{1/2}\right\|_q^p \\ \gtrsim \left\|\left(\sum_{r,j} \left|\Delta_j f_r\right|^2\right)^{1/2}\right\|_q^p \\ \gtrsim \left\|\left(\sum_j \left|\Delta_j f_r\right|^2\right)^{1/2}\right\|_q^p \sim \|f\|_q^p \end{aligned} \tag{182}$$

(the second inequality requires distinction of the cases $q \geq 2$ and $p < q \leq 2$).

(182) gives the required inequality .

Case (ii):

Thus $d = 2$ and p is near 2.

Going back to (167) and applying (165), (168), we obtain

$$\begin{aligned} \varepsilon.(165) \gtrsim \sum_r (2^{-r}\|f_r\|_{W^{1,p}})^p \\ \gtrsim \left(\sum_r 4^{-r}\sum_j \|\Delta_j f_r\|_p^2\, 4^j\right)^{\frac{p}{2}} \gtrsim \left(\sum_j \left(2^{sj}\|\Delta_j f\|_p\right)^2\right)^{\frac{p}{2}} \\ \gtrsim q^{-\frac{p}{2}}\|f\|_q^p \end{aligned} \tag{183}$$

where

$$q^{-\frac{p}{2}} = \left(\frac{1}{p} - \frac{s}{2}\right)^{\frac{p}{2}} \sim (2-ps)^{p-1} \tag{184}$$

which again gives the required inequality .

Theorem (3.2.8)[134]: Assume $d \geq 1, p \geq 1, 1/2 \leq s < 1$ and $sp < 1$. Then

$$\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\, dxdy \geq C(d)\frac{(d-sp)^{p-1}}{1-s}\left\|f - ⨍ f\right\|_{L^q(Q)}^p \tag{185}$$

where q is given by (123) and $C(d)$ depends only on d.

As can be seen from (130) there are two phenomena that govern the behaviour of the constant in (130). As $s \uparrow 1$ the constant gets bigger, while as $s \uparrow d/p$ the constant deteriorates. This explains why the we consider several cases in the proof.

As an application of Theorem (3.2.3) with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$ we easily obtain

Proof (4): when $p \geq 2$.

From (167), we get now the minoration

$$\varepsilon.(186) \gtrsim \sum_j \left(2^{-sj}\|\Delta_j f\|_p\right)^p \tag{186}$$

which we use to majorize $\|f\|_q$.

We have already inequality (168), thus

$$\|f\|_q \leq C\sqrt{q}\left(\sum_j \left(2^{-sj}\|\Delta_j f\|_p\right)^2\right)^{1/2}. \tag{187}$$

Our aim is to prove that

$$\|f\|_q \leq Cq^{1-\frac{1}{p}}\left(\sum_j \left(2^{-sj}\|\Delta_j f\|_p\right)^p\right)^{1/2} \tag{188}$$

which will give the required inequality together with (186).

Using interpolation for $2 \leq p < \frac{d}{s}$, it clearly suffices to establish (188) for large values of q. To prove (188), we assume $2 \leq p \leq 4$ (other cases may be treated by adaption of the argument presented below). Assume further (taking previous comment into account)

$$q \geq 2p. \tag{189}$$

Again by interpolation, (188) will follow from (187) and the inequality

$$\|f\|_q \leq Cq^{\frac{3}{4}}\left(\sum_j \left(2^{sj}\|\Delta_j f\|_p\right)^4\right)^{1/4}. \tag{190}$$

We use the notation and start from the martingale square function inequality (188); thus

$$\|f\|_q \leq C\sqrt{q}\left\|\left(\sum_j |\tilde{\Delta}_j f|^2\right)^{1/2}\right\|_q. \tag{191}$$

Write

$$|\tilde{\Delta}_j f| \leq \sum_k |\tilde{\Delta}_j \Delta_k f| = \sum_{m\in\mathbb{Z}} |\tilde{\Delta}_j \Delta_{j+m} f|$$

(putting $\Delta_k = 0$ for $k < 0$).

Writing

$$\left\|\left(\sum_j |\tilde{\Delta}_j f|^2\right)^{1/2}\right\|_q \leq \sum_{m\in\mathbb{Z}}\left\|\left(\sum_j |\tilde{\Delta}_j \Delta_{j+m} f|^2\right)^{1/2}\right\|_q \tag{192}$$

we estimate each summand.

Fix m. Write

$$\left\| \left(\sum_j \left| \tilde{\Delta}_j \, \Delta_{j+m} \, f \right|^2 \right)^{1/2} \right\|_q^4 = \left\| \left(\sum_j \left| \tilde{\Delta}_j \, \Delta_{j+m} \, f \right|^2 \right)^{1/2} \right\|_{\frac{q}{4}}$$
$$\leq 2 \sum_{j_1 \leq j_2} \left\| \left| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right|^2 \left| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right|^2 \right\|_{\frac{q}{4}} \qquad (193)$$

and

$$\left\| \left| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right|^2 \left| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right|^2 \right\|_{\frac{q}{4}} = \left[\int \left| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right|^{\frac{q}{2}} . \, \mathbb{E}_{j_1} \left[\left| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right|^{\frac{q}{2}} \right] \right]^{\frac{4}{q}}$$
$$\leq \left\| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right\|_q^2 \left\| \left(\mathbb{E}_{j_1} \left[\left| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right|^{\frac{q}{2}} \right] \right)^{\frac{q}{2}} \right\|_q^2$$
$$\leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})} \left\| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right\|_q^2 \left\| \left(\mathbb{E}_{j_1} \left[\left| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right|^p \right] \right)^{1/p} \right\|_q^2$$
$$\leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})} 4^{d j_1 (\frac{1}{p}-\frac{2}{q})} \left\| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right\|_q^2 \left\| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right\|_q^2. \qquad (194)$$

Assume $m \leq 0$

Estimate

$$\left\| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right\|_q \lesssim 2^m \left\| \Delta_{j_1+m} \, f \right\|_q \leq 2^m 2^{d(j_1+m)\left(\frac{1}{p}-\frac{2}{q}\right)} \left\| \Delta_{j_1+m} \, f \right\|_p \qquad (195)$$
$$\left\| \tilde{\Delta}_{j_2} \, \Delta_{j_2+m} \, f \right\|_p \lesssim 2^m \left\| \Delta_{j_2+m} \, f \right\|_p. \qquad (196)$$

Substitution of (195), (196) in (194) gives

$$4^{\left(1-d\left(\frac{1}{p}-\frac{1}{q}\right)\right)m+m} 4^{-\frac{d}{q}(j_2-j_1)} \left[2^{d\left(\frac{1}{p}-\frac{1}{q}\right)(j_1+m)} \left\| \Delta_{j_1+m} \, f \right\|_q \right]^2 \left[2^{d\left(\frac{1}{p}-\frac{1}{q}\right)(j_2+m)} \left\| \Delta_{j_2+m} \, f \right\|_p \right]^2 \qquad (197)$$

where

$$d\left(\frac{1}{p}-\frac{1}{q}\right) = s.$$

Summing (193) for $j_1 < j_2$ and applying Cauchy-Schwartz implies for $m < 0$

$$(193) < 4^{(2-s)m} \left(\sum_{\ell \geq 0} 4^{-\frac{d}{q}\ell} \right) \left[\sum_j \left(2^{sj} \left\| \Delta_j \, f \right\|_p \right)^4 \right]$$
$$\lesssim 4^{(2-s)m} q \left[\sum_j \left(2^{sj} \left\| \Delta_j \, f \right\|_p \right)^4 \right]. \qquad (198)$$

Assume next $m > 0$.

Estimate

$$\left\| \tilde{\Delta}_{j_1} \, \Delta_{j_1+m} \, f \right\|_q \lesssim 2^{d j_1 \left(\frac{1}{p}-\frac{1}{q}\right)} \left\| \Delta_{j_1+m} \, f \right\|_p$$

and

$$(194) \leq 4^{d(j_2-j_1)\left(\frac{1}{p}-\frac{2}{q}\right)} 16^{d j_1 \left(\frac{1}{p}-\frac{1}{q}\right)} \left\| \Delta_{j_1+m} \, f \right\|_p^2 \left\| \Delta_{j_2+m} \, f \right\|_p^2$$
$$\leq 16^{ms} \, 4^{-(j_2-j_1)\frac{d}{q}} \left\| 2^{s(j_1+m)} \Delta_{j_1+m} \, f \right\|_p^2 \left\| 2^{s(j_2+m)} \, \Delta_{j_2+m} \, f \right\|_p^2. \qquad (199)$$

Summing over $j_1 < j_2$ implies that for $m > 0$

$$(193) \lesssim 16^{-m} q \left[\sum_j \left(2^{sj} \| \Delta_j f\|_p\right)^4\right]. \tag{200}$$

Summing (198), (200) in m implies that

$$(192) \leq \left(\sum_{m\leq 0} 2^{(1-\frac{s}{2})m} + \sum_{m>0} 2^{-sm}\right) q^{1/4} \left[\sum_j \left(2^{sj} \| \Delta_j f\|_p\right)^4\right]^{1/4}$$
$$\leq q^{1/4} \left[\sum_j \left(2^{sj} \| \Delta_j f\|_p\right)^4\right]^{1/4}. \tag{201}$$

To bound $\|f\|_q$, apply (191) which introduces an additional $q^{1/2}$-factor. This establishes (190) and completes the argument and the proof of Theorem (3.2.3).

Lemma (3.2.9)[134]: Let $I \subset \mathbb{R}$ be an interval and let $\psi: I \to \mathbb{Z}$ be any measurable function. Then, there is some $k \in \mathbb{Z}$ such that

$$|\{x \in I; \psi(x) \neq k\}| \leq 2\left(C^*\varepsilon \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{2-\varepsilon}} dxdy\right)^{1/\varepsilon}$$

for all $\varepsilon \in (0, 1/2]$. where C^* is the absolute constant in Corollary (3.2.1) (inequality(195)).

Proof: After scaling and shifting we may assume that $I = (1, +1)$. For each $k \in \mathbb{Z}$, set

$$A_k = \{x \in I\,;\ \psi(x) < k\}.$$

Note that $\mathrm{A_k}$ is nondecreasing, $\lim_{k\to-\infty} |A_k| = 0$ and $\lim_{k\to+\infty} |A_k| = 2$. Thus there exists exists some $k \in \mathbb{Z}$ such that

$$|A_k| \leq 1 \text{ and } |A_{k+1}| > 1. \tag{202}$$

Applying Corollary (3.2.1) with $A = A_k$ and with $A = A_{k+1}$ we find (using (202))

$$|A_k| \leq |A_k|\,|{}^cA_k| \leq \left(C^*\varepsilon \int_A \int_{{}^cA_k} \frac{dxdy}{|x-y|^{2-\varepsilon}}\right)^{1/\varepsilon} \tag{203}$$

and

$$|{}^cA_{k+1}| \leq |A_{k+1}|\,|{}^cA_{k+1}| \leq \left(C^*\varepsilon \int_{A_{k+1}} \int_{{}^cA_{k+1}} \frac{dxdy}{|x-y|^{2-\varepsilon}}\right)^{1/\varepsilon}. \tag{204}$$

On the other hand

$$|\psi(x) - \psi(y)| \geq 1 \text{ for a.e.} x \in A_k, y \in {}^cA_k$$

and

$$|\psi(x) - \psi(y)| \geq 1 \text{ for a.e.} x \in A_{k+1}, y \in {}^cA_{k+1}$$

therefore

$$|\{x \in I; \psi(x) \neq k\}| = |A_k| + |{}^cA_{k+1}|$$
$$\leq 2\left(C^*\varepsilon \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{2-\varepsilon}} dxdy\right)^{1/\varepsilon}.$$

Lemma (3.2.10)[134]: If $\alpha > 0$; $a < b < x, A \subset (a, b)$ is measurable, then

$$\int_{(a,b)\setminus A} \frac{dy}{(x-y)^\alpha} \geq \int_a^{b-|A|} \frac{dy}{(x-y)^\alpha}$$

and similarly, if $x < a < b$, then

$$\int_{(a,b)\setminus A} \frac{dy}{(y-x)^\alpha} \geq \int_{a+|A|}^{b} \frac{dy}{(y-x)^\alpha}.$$

The proof of Lemma (3.2.10) is elementary.

Theorem (3.2.11)[134]: Let $\psi_\varepsilon : \Omega \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$.Then

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \geq c\varepsilon^{-1/2}, \forall \varepsilon \in (0, 1/2),$$

for some absolute constant $c > 0$.

Proof: Let $\psi_\varepsilon :=(1,+1) \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$. We have to prove that for all $\varepsilon < 1/2$,

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} \geq c\varepsilon^{-1/2} \tag{205}$$

for some absolute constant c to be determined.

We argue by contradiction and assume that for some $\varepsilon < 1/2$

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} < \eta\varepsilon^{-1/2}. \tag{206}$$

We will reach a contradiction if η is less than some absolute constant. Set

$$\psi = \frac{1}{2\pi}(\psi_\varepsilon - \varphi_\varepsilon)$$

so that $\psi : \Omega \to \mathbb{R}$; recall that $u_\varepsilon = e^{i\varphi_\varepsilon}$ and the function φ_ε is defined by

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for} -1 < x < 0, \\ 2\pi x/\delta & \text{for} \quad 0 < x < \delta, \\ 2\pi & \text{for} \quad \delta < x < 1, \end{cases}$$

where $\delta^{-1/\varepsilon}$.

A straightforward computation (using the fact that ψ takes its values into $\mathbb{Z}$) shows that

$$|\psi(x) - \psi(y)| \leq |\psi_\varepsilon(x) - \psi_\varepsilon(y)| \text{ for } a.e.\ x, y \in \left(-1, \frac{2\delta}{3}\right) \tag{207}$$

and

$$|\psi(x) - \psi(y)| \leq |\psi_\varepsilon(x) - \psi_\varepsilon(y)| \text{ for } a.e.\ x, y \in \left(\frac{\delta}{3}, 1\right). \tag{208}$$

Applying Lemma (3.2.9) with $I = \left(-1, \frac{2\delta}{3}\right)$ and $I = \left(\frac{\delta}{3}, 1\right)$, together with (206), (207) and (208) yields the existence of $\ell, m \in \mathbb{Z}$ such that

$$\left|\left\{x \in \left(-1, \frac{2\delta}{3}\right); \psi(x) \neq \ell\right\}\right| \leq 2(C^*\eta^2)^{1/\varepsilon}$$

and

$$\left|\left\{x \in \left(x \in \frac{\delta}{3}, 1\right); \psi(x) \neq m\right\}\right| \leq 2(C^*\eta^2)^{1/\varepsilon}.$$

We choose η in such a way that

$$4(C^*\eta^2)^{1/\varepsilon} < \delta/3, \text{for } \varepsilon < 1/2,$$

for example

$$\eta^2 < 1/4e\, C^*. \tag{209}$$

It follows that $\ell = m$. Without loss of generality (after adding a constant to ψ_ε)we may assume that

$$\eta = m = 0. \tag{210}$$

Therefore

$$\psi_\varepsilon(x) = \varphi_\varepsilon(x) \text{for } x \in [(-1,0)\setminus A] \cup [(\delta, 1)\setminus B] \tag{211}$$

where

$$A = \{x \in (-1,0);\ \psi(x) \neq 0\}$$

and

$$B = \{x \in (\delta, 1);\ \psi(x) \neq 0\}$$

with

$$|A| < \delta/6\,, |B| < \delta/6. \tag{112}$$

From (211) and the definition of φ_ε we have

$$\varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dxdy \geq \varepsilon \int_{-1}^{0} dx \int_0^1 \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dy$$
$$\geq \varepsilon \int_{(-1,0)\setminus A} dx \int_{(\delta,1)\setminus B} \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dy$$
$$\geq \varepsilon \int_{(-1,0)\setminus A} dx \int_{(\delta,1)\setminus B} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}} dy.$$

Applying Lemma (3.2.10) and (206) we find

$$\eta^2 > \varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dxdy \geq \varepsilon \int_{-1}^{-|A|} dx \int_{\delta+|B|}^{1} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}}$$
$$\geq \varepsilon \int_{-1}^{-\delta/6} dx \int_{\delta+\delta/6}^{1} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}} = 4\pi^2(1-e^{-1}) + o(1)$$

as $\varepsilon \to 0$. We obtain a contradiction for an appropriate choice of η.

Let $\{\mathcal{F}_n\}_{n=0,1,2,\cdots}$ be refining finite partitions such that

$$\#\,\mathcal{F}_n = K^n$$
$$|Q| = K^{-n} \text{ if } Q \text{ is an } \mathcal{F}_n - \text{atom}$$

(If $\Omega = [0,1]^d\,, K = 2^d$).

Denote $\mathbb{E}_n$ the $\mathcal{F}_n$-expectation

$$\Delta_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f \qquad \text{(we used the notation } \tilde{\Delta}_n f\text{)}$$
$$Sf = \left(\sum |\Delta_n f|^2\right)^{1/2} \qquad \text{(the square function)}$$
$$|f| \leq f^* = \sup |\mathbb{E}_n f| \qquad \text{(the maximal function)}$$

Proposition (3.2.12)[134]:

$$\text{mes}\,[|f| > \lambda \|Sf\|_\infty] < e^{-c\lambda^2} \qquad (\lambda \geq 1) \tag{113}$$

where $c = c(K) > 0$ is a constant.

Proof: One verifies that there is a constant $A = A(K)$ such that if φ is $\mathcal{F}_n$-measurable and $\mathbb{E}_{n-1}\varphi = 0$,then

$$\mathbb{E}_{n-1}\left[e^{\varphi - A\varphi^2}\right] \leq 1. \tag{214}$$

Hence

$$\mathbb{E}_{n-1}\left[e^{\Delta_n f - A(\Delta_n f)^2 \varphi^2}\right] \leq 1 \tag{215}$$

and, denoting $S_n f = \left(\sum_{m \leq n} |\Delta_m f|^2\right)^{1/2}$,

$$\int e^{\mathbb{E}_n f - A(S_n f)^2} = \int e^{\mathbb{E}_{n-1} f - A(S_{n-1} f)^2}\, \mathbb{E}_{n-1}\left[e^{\Delta_n f - A(\Delta_n f)^2}\ \right]$$
$$\leq \int e^{\mathbb{E}_{n-1}}\, f - A(S_{n-1} f)^2 \qquad (by\ (215))$$
$$\leq 1.$$

thus

$$\int e^{f-A(Sf)^2} \leq 1. \tag{116}$$

Assume $\|Sf\|_\infty \leq 1$. Applying (116) to tf ($t > 0$ a parameter), we get

$$\int e^{tf} \leq e^{At^2}$$

$$\text{mes}\,[f > \lambda] \leq e^{At^2 - t\lambda}$$

and for appropriate choice of t

$$\text{mes}\,[f > \lambda] < e^{-\frac{\lambda^2}{4A}}.$$

This proves (213).

Proposition (3.2.13)[134]: (good- λ inequality)

$$\text{mes}\,[f^* > 2\lambda, Sf < \varepsilon\,\lambda, \sup \mathbb{E}_{n-1}\,[|\Delta_n f|] < \varepsilon\lambda]$$
$$< e^{-\frac{c}{\varepsilon^2}}\,\text{mes}\,[f^* > \lambda] \quad (0 < \varepsilon < 1)\ (217)$$

Proof: This is a standard stopping time argument.

Consider a collection of maximal atoms $\{Q_\alpha\} \subset \cup\mathcal{F}_n$ s.t. if Q_α is an $\mathcal{F}_n$-atom, then $|\mathbb{E}_n f| > \lambda$ on Q_α.Thus $Q_\alpha \cap Q_\beta = \phi$ for $\alpha \neq \beta$. Fix . From the maximality

$$|\mathbb{E}_{n-1} f| \leq \lambda \text{ on } Q_\alpha. \tag{218}$$

Therefore

$$\left[f^* > 2\lambda, Sf < \varepsilon\,\lambda, \sup \mathbb{E}_{m-1}\,[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda\right] \cap Q_\alpha \subset$$
$$\left[(f - \mathbb{E}_n f)^* > (1-\varepsilon)\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}\,[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda\right] \cap Q_\alpha = \tag{219}$$

For $m > n$, denote m the indicator function of the set

$$Q_\alpha \cap \left[\left(\sum_{\ell=n+1}^{m-1} |\Delta_\ell f|^2\right)^{1/2} < \varepsilon\lambda\right] \cap \left[\mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda\right] \cap$$
$$\bigcap_{n \leq \ell < m} [|\mathbb{E}_\ell f - \mathbb{E}_n f| \leq (1-\varepsilon)\lambda] = \qquad (220).$$

Thus

$$\chi_m = \mathbb{E}_{m-1}\ \chi_m$$

and

$$g = \sum_{m>n} \chi_m\,\Delta_m f$$

is an $\{\mathcal{F}_m | m \geq n\}$-martingale on Q_α.

From the definition of χ_m, we have clearly

$$S(g) = \left(\sum_{m>n} \chi_m |\Delta_m f|^2\right)^{1/2} < \varepsilon\lambda + \varepsilon\lambda \lesssim \varepsilon\lambda \tag{221}$$

and

$$|g| > (1-\varepsilon)\lambda \text{ on the set (219).}$$

From Proposition (3.2.12) and (221)

$$\text{mes}\,[x\ \varepsilon\ Q_\alpha|\ \ |g| > (1-\varepsilon)\lambda] < e^{-\frac{c}{\varepsilon^2}}\,|Q_\alpha| \tag{222}$$

hence

$$\text{mes (219)} \lesssim e^{-\frac{c}{\varepsilon^2}}\,|Q_\alpha|. \tag{223}$$

Summing (223) over α implies

$$\text{mes}\left[f^* > 2\lambda, Sf < \varepsilon\,\lambda, \sup \mathbb{E}_{m-1}\,[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda\right] < e^{-\frac{c}{\varepsilon^2}} \sum |Q_\alpha| \le e^{-\frac{c}{\varepsilon^2}}\,\text{mes}\,[f^* > \lambda]$$

which is (216).

Proposition (3.2.14)[134]:

$$\|f^*\|_q \le C\sqrt{q}\,\|Sf\|_q \quad \text{for } q \ge 2. \tag{224}$$

We follow essentially [137].

Proof:

$$\|f^*\|_q^q = q\int \lambda^{q-1}\ \text{mes}\,[f^* > \lambda]d\lambda = 2^q q\int \lambda^{q-1}\ \text{mes}\,[f^* > 2\lambda]d\lambda$$

$$\le 2^q q\int \lambda^{q-1}\ \left\{\text{mes}\,[Sf \ge \varepsilon\lambda] + \text{mes}\left[\sup\mathbb{E}_{n-1}[|\Delta_n f|] \ge \tfrac{\varepsilon}{K}\lambda\right] + e^{-\frac{c}{\varepsilon^2}}\,\text{mes}\,[f^* > \lambda]\right\}$$

$$< \left(\frac{2}{\varepsilon}\right)^q \left(\|Sf\|_q^q + K^q\|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q\right) + 2^q e^{-\frac{c}{\varepsilon^2}}\|f^*\|_q^q \tag{225}$$

Take $\frac{1}{\varepsilon}\sim\sqrt{q}$ so that the last term in (225) is at most $\frac{1}{2}\|f^*\|_q^q$. Thus

$$\|f^*\|_q < C\sqrt{q}\left(\|Sf\|_q + \|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q\right). \tag{226}$$

Also

$$\begin{aligned}\|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q &\le \left(\sum_n \|\mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q\right)^{1/q}\\ &\le \left(\sum_n \|\Delta_n f\|_q^q\right)^{1/q}\\ &\le \|Sf\|_q.\end{aligned} \tag{227}$$

and (224) follows from (226), (227).

Chapter 4
Subcritical Caffarelli–Kohn–Nirenberg Inequalities

We show some new symmetry results for the extremals of the Caffarelli-Kohn-Nirenberg inequalities, in any dimension larger or equal than 2. The condition on the parameters is indeed complementary of the condition that determines the region in which symmetry breaking holds as a consequence of the linear instability of radial optimal functions. Compared to the critical case, the subcritical range requires new tools. The Fisher information has to be replaced by Rényi entropy powers, and since some invariances are lost, the estimates based on the Emden–Fowler transformation have to be modified.

Section (4.1): On the Symmetry of Extremals

The Caffarelli-Kohn-Nirenberg inequality (see [7]) in space dimension $N \geq 2$, can be written as follows,

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{bp}} dx\right)^{2/p} \leq C_{a,b}^N \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}} dx \quad \forall u \in \mathcal{D}_{a,b} \tag{1}$$

with $a \leq b \leq a+1$ if $N \geq 3$, $a < b \leq a+1$ if $N = 2$, and $a \neq a_c$ defined by

$$a_c = a_c(N) := \frac{N-2}{2}.$$

The exponent,

$$p = \frac{2N}{N-2+2(b-a)}$$

is determined by scaling considerations. Furthermore,

$$\mathcal{D}_{a,b} := \{|x|^{-b} u \in L^p(\mathbb{R}^N, dx) : \ |x|^{-a} |\nabla u \in L^2(\mathbb{R}^N, dx)|\}$$

and $C_{a,b}^N$ denotes the optimal constant. Typically, inequality (1) is stated with $a < a_c$ (see [7]) so that the space $\mathcal{D}_{a,b}$ is obtained as the completion of $C_c^\infty(\mathbb{R}^N)$, the space of smooth functions in $\mathbb{R}^N$ with compact support, with respect to the norm $\|u\|^2 = \left\||x|^{-b} u\right\|_p^2 + \||x|^{-a} \nabla u\|_2^2$. Actually (1) holds also for $> a_c$, but in this case $\mathcal{D}_{a,b}$ is obtained as the completion with respect to $\|\cdot\|$ of the space $\{u \in C_c^\infty(\mathbb{R}^N) : \mathrm{supp}\,(u) \subset \mathbb{R}^N \setminus \{0\}\}$ that we shall denote by $C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Inequality (1) is sometimes called the Hardy-Sobolev inequality, as for $N > 2$ it interpolates between the usual Sobolev inequality ($a = 0$, $b = 0$) and the weighted Hardy inequalities (see [1]) corresponding to $b = a + 1$.

For $= a < 0$, $N \geq 3$, equality in (1) is never achieved in $\mathcal{D}_{a,b}$. For $b = a + 1$ and ≥ 2 , the best constant in (1) is given by $C_{a,a+1}^N = (N - 2 - 2\,a)^2/4$ and it is never achieved (see [1]). On the contrary, for $a < b < a + 1$ and ≥ 2 , the best constant in (1) is always achieved, say at some function $u_{\,a,b} \in \mathcal{D}_{a,b}$ that we will call an extremal function. However $u_{\,a,b}$ is not explicitly known unless we have the additional information that it is radially symmetric about the origin. In the class of radially symmetric functions, the extremals of (1) are all given (see [12], [17], [1]) up to a dilation, by

$$u_{a,b}^*(x) = \kappa^* \left(1 + |x|^{\frac{2(N-2-2a)(1+a-b)}{N-2(1+a-b)}}\right)^{\frac{N-2(1+a-b)}{2(1+a-b)}} \tag{2}$$

for an arbitrary normalization constant κ^*. See [1], [141] for more details and in particular for a “modified inversion symmetry” property of extremal functions, based on a generalized Kelvin transformation, which relates the parameter regions $a < a_c$ and $a > a_c$.

In the parameter region $0 \leq a < a_c$, $a \leq b \leq a + 1$, if $N \geq 3$, the extremals are radially symmetric (see [32], [24], [19] and more specifically [12], [17]); we give a simplified proof of the radial symmetry of all extremal functions in this range of parameters. Extremals are known to be non radially symmetric for a certain range of parameters (a, b) identified first in [1] and subsequently improved in [142], given by the condition $b < b^{FS}(a)$, $a < 0$ (see below). By contrast, few symmetry results are available for < 0. For instance, when ≥ 3 , for a fixed $b \in (a, a + 1)$, radial symmetry of the extremals has been proved for a close to 0 (see [146], [145]; also for an earlier but slightly less general result). In the particular case $= 2$, a symmetry result was proved in [141] for a in a neigbourhood of 0_{-}, which asymptotically complements the symmetry breaking region found in [1], [142], [141], as $a \to 0_{-}$.

In terms of a and b, we first prove that the symmetry region admits the half-line $b = a + 1$ as part of its boundary.

For completeness, let us state some already known symmetry results. We also provide a simplified proof in case $N \geq 3$, $a \geq 0$.

Lemma (4.1.1)[140]: If $N \geq 3$, $0 \leq a < a_c$ and $a \leq b < a + 1$, extremal functions for (1) are radially symmetric. If $N = 2$, for any > 0 , there exists $\eta > 0$ such that extremal functions for (1) are radially symmetric if $-\eta < a < 0$ and $-\varepsilon\, a \leq b < a + 1$.

Proof: The case $N = 2$ has been established in [141]. The result for $N \geq 3$ is also known; see [12], [17]. However, we give here a simpler proof (for $n \geq 3$),which goes as follows. Let $u \in C_c^{\infty}(\mathbb{R}^N \backslash \{0\})$ and consider $v(x) = |x|^{-a} u(x)$ for any $x \in \mathbb{R}^N$. Inequality (1) amounts to

$$\left(C_{a,b}^N\right)^{-1} \left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x|^{(b-a)p}} dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^N} \left| \nabla v + a \frac{x}{|x|^2} v \right|^2 dx$$
$$= \int_{\mathbb{R}^N} |\nabla v|^2\, dx + a^2 \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx + a \int_{\mathbb{R}^N} \frac{x}{|x|^2} \cdot \nabla(v^2)\, dx.$$

Integrating by parts, we find that $\int_{\mathbb{R}^N} \frac{x}{|x|^2} \cdot \nabla(v^2) dx = -(N-2) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx$. Hence, radial symmetry for the extremal functions of Inequality (1) is equivalent to prove that extremal functions for

$$\left(C_{a,b}^N\right)^{-1} \left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x|^{(b-a)p}} dx \right)^{\frac{2}{p}} + a[(N-2) - a] \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla v|^2\, dx$$

are radially symmetric. Since the coefficient $a\,[(N-2) - a] = a(2a_c - a)$ is positive in the considered range for a, the result follows from Schwarz's symmetrization. Both terms of the left hand side (resp. the term of the right hand side) are indeed increased (resp. is decreased) by symmetrization, and equality only occurs for radially symmetric decreasing functions; see [144] for details. The result can then be extended to $\mathcal{D}_{a,b}$ by density.

Notice that the proof is exactly the same for $N \geq 3$, $a_c < a \leq N - 2 = 2\,a_c$ and $a \leq b < a + 1$. For $N = 2$, a result similar to that of Lemma(4.1.1) has been achieved when $(2 + \varepsilon)\, a \leq b < a + 1$, $0 < a < \eta$. Radial symmetry has also been established for $N \geq 3$, $a < 0$, $|a|$ small, and $0 < b < a + 1$, see [105], [146].

It is convenient to formulate the Caffarelli-Kohn-Nirenberg inequality in cylindrical variables (see [1]). By means of the Emden-Fowler transformation

$$t = \log|x|, \quad \theta = \frac{x}{|x|} \in S^{N-1}, \;\; w(t,\theta) = |x|^{\frac{N-2-2a}{2}} u(x), \tag{3}$$

inequality (1) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequalit y on the cylinder $C := \mathbb{R} \times S^{N-1}$, that is

$$\|w\|^2_{L^p(C)} \leq C^N_{a,b}\left(\|\nabla w\|^2_{L^2(C)} + \Lambda\|w\|^2_{L^2(C)}\right), \tag{4}$$

for any $w \in H^1(C)$, with

$$\Lambda = \Lambda(N,a) := \frac{1}{4}(N-2-2a)^2, \qquad p = \frac{2N}{N-2+2(b-a)},$$

and the same optimal constant $C^N_{a,b}$ as in (1). In what follows, we will denote the cylinder variable by $y := (t,\theta) \in \mathbb{R} \times S^{N-1} = C$.

We may observe that if (4) holds for $a < a_c$, it also holds for $a > a_c$, with same extremal functions. Hence, the inequality

$$\|w\|^2_{L^p(C)} \leq C^N_{a,b}\left[\, \|\nabla w\|^2_{L^2(C)} + \Lambda(N,a)\|w\|^2_{L^2(C)} \,\right]$$

holds for any $a \neq a_c$, $b \in [a, a+1]$ and $p = 2\,N/(N-2+2\,(b-a))$ if $N \geq 3$,or any $a \neq 0 = a_c$, $b \in (a, a+1]$ and $p = 2/(b-a)$ if $N = 2$. Now there is no more need to make distinctions between the cases $a < a_c$ and $a > a_c$ as it was the case for inequality (1), in order to give the correct definition of the functional spaces $\mathcal{D}_{a,b}$. Moreover, as in [141], we may observe that $C^N_{a,b} = C^N_{\acute{a},\acute{b}}$ with $a' = N-2-a = 2\,a_c - a$ and $b' = b + N - 2 - 2\,a = b + 2(a_c - a)$. We shall therefore restrict a to $(-\infty, a_c)$ without loss of generality.

For simplicity, we shall reparametrize $\{(a, b \in \mathbb{R}^N : a < b < a+1\,, a < a_c\}$ in terms of $(\Lambda, p) \in (0,\infty) \times (2, 2^*)$ using the relations

$$\Lambda = \frac{1}{4}(N-2-2a)^2 \Leftrightarrow a\frac{N-2}{2} - \sqrt{\Lambda} \tag{5}$$

and

$$p = \frac{2N}{N-2+2(b-a)} \quad \text{with} \quad \begin{cases} b \in [a, a+1] & \text{if } N \geq 3 \\ b \in (a, a+1] & \text{if } N = 2 \end{cases} \tag{6}$$

$$\Leftrightarrow b = \frac{N}{p} - \sqrt{\Lambda} \quad \text{with} \quad \begin{cases} 2 \leq p \leq 2^* & \text{if } N \geq 3 \\ 2 \leq p < \infty & \text{if } N = 2 \end{cases}$$

so that, with the above rules, the constant $C^N_{\Lambda,b} := C^N_{a,b}$ is such that the minimum of the functional

$$\mathcal{F}_{\Lambda,p}[w] = \frac{\|\nabla w\|^2_{L^2(C)} + \Lambda\|w\|^2_{L^2(C)}}{\|w\|^2_{L^p(C)}} \tag{7}$$

on $H^1(C\backslash\{0\})$ takes the value $\left(C^N_{\Lambda,b}\right)^{-1}$.

For a given p , we are interested in the regime $a < a_c$, parametrized by $\Lambda > 0$. The function

$$\Lambda \mapsto \left(a = \frac{N-2}{2} - \sqrt{\Lambda}\,, b = \frac{N}{p} - \sqrt{\Lambda}\right)$$

parametrizes an open half-line contained in $a \leq b \leq a+1\,, a < a_c$ (and therefore parallel to the line $b = a$) in the (a,b)-plane. As a consequence of Lemma (4.1.1), we know that extremal functions are radially symmetric for $\Lambda > 0$, small enough. On the other hand, the region

$$a < 1, \quad a < b \leq b^{FS}(a) = \frac{N(N-2-2a)}{2\sqrt{(N-2-2a)^2 + 4(N-1)}} - \frac{N-2-2a}{2}$$

is given in terms of Λ and p by the condition $\Lambda > \Lambda^{FS}(p)$ where $\Lambda = \Lambda^{FS}(p)$ is uniquely defined by the condition

$$\frac{N}{p} - \sqrt{\Lambda} = b^{FS}(a) = \frac{N\sqrt{\Lambda}}{2\sqrt{\Lambda + N - 1}} - \sqrt{\Lambda},$$

that gives

$$\Lambda^{FS}(p) := \frac{4}{p^2 - 4}(N-1). \tag{8}$$

To interpret this condition in terms of the variational nature of the radial extremal, see Proposition (4.1.8) below.

We can summarize the above considerations as follows: For given $\Lambda > 0$ and $p \in (2, 2^*)$, the corresponding extremals of (4) are not radially symmetric if $\Lambda > \Lambda^{FS}(p)$. As a consequence, we can define

$$\Lambda^*(p) := \sup\left\{\Lambda > 0 : \mathcal{F}_{\Lambda,p} \text{ has a radially symmetric minimize r}\right\} \tag{9}$$

and observe that $0 < \Lambda^*(p) \leq \Lambda^{FS}(p)$ for any $p \in (2, 2^*)$.

For any $\Lambda > 0$, $p \in (2, 2^*]$ if $N \geq 3$, or $p \in (2, \infty)$ if $N = 2$, the inequality

$$\left(C^N_{\Lambda,p}\right)^{-1} \|w\|^2_{L^p(C)} \leq \|\nabla w\|^2_{L^2(C)} + \Lambda \|w\|^2_{L^2(C)} \tag{10}$$

is achieved in $H^1 \cap L^p(C)$ by at least one extremal positive function $w = w_{\Lambda,p}$ satisfying on C the Euler-Lagrange equation

$$-\Delta_y w + \Lambda \mathrm{w} = \mathrm{w}^{\mathrm{p}-1}. \tag{11}$$

For ≥ 2, we have

$$\left(C^N_{\Lambda,p}\right)^{-1} \left\|w_{\Lambda,p}\right\|^{p-2}_{L^p(C)} = \inf_{w \in H^1(C)\setminus\{0\}} \mathcal{F}_{\Lambda,p}[w].$$

According to [1], by virtue of the properties of the extremal function $w_{\Lambda,p}$ and the translation invariance of (10) in the t-variable, we can further assume that

$$\begin{cases} w_{\Lambda,p}(t,\theta) = w_{\Lambda,p}(-t,\theta) & \forall (t,\theta) \in \mathbb{R} \times S^{N-1} = C, \\ \left(w_{\Lambda,p}\right)_t(t,\theta) < 0 & \forall (t,\theta) \in (0,+\infty) \times S^{N-1}, \\ \max_C w_{\Lambda,p} = w_{\Lambda,p}(0,\theta_0). \end{cases} \tag{12}$$

for some $\theta_0 \in S^{N-1}$. A solution of (11) which does not depend on θ therefore satisfies on $\mathbb{R}$ the ODE

$$-w_{tt} + \Lambda \mathrm{w} = \mathrm{w}^{\mathrm{p}-1}.$$

Multiplying it by w_t and integrating with respect to , we find that

$$-\frac{1}{2}w_t^2 + \frac{\Lambda}{2}w^2 = \frac{1}{p}w^p + c$$

for some constant $c \in \mathbb{R}$. Due to the integrability conditions, namely the fact that w_t and w are respectively in $L^2(\mathbb{R})$ and $L^2 \cap L^p(\mathbb{R})$, it turns out that $c = 0$. Since we assume that w achieves its maximum at $t = 0$, this uniquely determines $w(0) > 0$ using the relation: $\Lambda\, w^2(0)/2 = w^p(0)/p$. In turn this yields a unique θ-independent solution $w^*_{\Lambda,p}$ defined by

$$w^*_{\Lambda,p}(t) := \left(\tfrac{1}{2}\Lambda p\right)^{\frac{1}{p-2}} \left(\cosh\left(\tfrac{1}{2}\sqrt{\Lambda}(p-2)t\right)\right)^{-\frac{2}{p-2}} \quad \forall t \in \mathbb{R}. \tag{13}$$

Such a solution is an extremal for (4) in the set of functions which are independent of the θ-variable, and satisfies:

$$\left(C_{\Lambda,p}^{N,*}\right)^{-1} := |S^{N-1}|^{1-2/p}\left\|w_{\Lambda,p}^{*}\right\|_{L^p(\mathbb{R})}^{p-2} = \inf_{f\in H^1(\mathbb{R})\setminus\{0\}} \mathcal{F}_{\Lambda,p}[f]. \quad (14)$$

where functions on $\mathbb{R}$ are considered as θ-independent functions on C.

By the coordinate change (3), w is independent of θ if and only if u is radially symmetric. This change of coordinates also tranforms the function $u_{a,b}^{*}$ defined in (2) into $w_{\Lambda,p}^{*}$, with a , b and p related by (5)-(6) and

$$\kappa^* = \left(\frac{N(N-2-2a)^2}{N-2(1+a-b)}\right)^{\frac{N-2(1+a-b)}{4(1+a-b)}}.$$

Lemma (4.1.2)[140]: Let $N \geq 2$, $p \in (2, 2^*)$. For any $\Lambda \neq 0$, we have

$$\left(C_{\Lambda,p}^{N}\right)^{-\frac{p}{p-2}} = \left\|w_{\Lambda,p}\right\|_{L^p(C)}^{p} \leq \left\|w_{\Lambda,p}^{*}\right\|_{L^p(C)}^{p} = 4|S^{N-1}|(2\Lambda p)^{\frac{p}{p-2}}\frac{c_p}{2p\sqrt{\Lambda}}$$

where c_p is an increasing function of p such that

$$\lim_{p\to 2_+} 2^{\frac{2p}{p-2}}\sqrt{p-2}\, c_p = \sqrt{2\pi}.$$

Proof: Observe that

$$\left\|w_{\Lambda,p}\right\|_{L^p(C)}^{p} = \left(C_{\Lambda,p}^{N}\right)^{-\frac{p}{p-2}} = \left(\mathcal{F}_{\Lambda,p}[w_{\Lambda,p}]\right)^{\frac{p}{p-2}} \leq \left(\mathcal{F}_{\Lambda,p}[w_{\Lambda,p}^{*}]\right)^{\frac{p}{p-2}} = \left\|w_{\Lambda,p}^{*}\right\|_{L^p(C)}^{p}$$

On the other hand,

$$\begin{aligned}\left\|w_{\Lambda,p}^{*}\right\|_{L^p(C)}^{p} &= |S^{N-1}|\left(\tfrac{1}{2}\Lambda p\right)^{\frac{p}{p-2}}\int_{-\infty}^{\infty}\left[\cosh\left(\tfrac{1}{2}\sqrt{\Lambda}(p-2)t\right)\right]^{-\frac{2p}{p-2}}dt\\ &= 2|S^{N-1}|\left(\tfrac{1}{2}\Lambda p\right)^{\frac{p}{p-2}}\int_{0}^{\infty}\frac{2^{\frac{2p}{p-2}}e^{-\sqrt{\Lambda}\,pt}}{\left(1+e^{-\sqrt{\Lambda}(p-1)t}\right)^{\frac{2p}{p-2}}}dt\\ &= 4|S^{N-1}|\left(\tfrac{1}{2}\Lambda p\right)^{\frac{p}{p-2}}\frac{2^{\frac{2p}{p-2}}}{2\sqrt{\Lambda}\,p}\int_{0}^{1}\frac{ds}{\left(1+s^{(p-2)/p}\right)^{\frac{2p}{p-2}}}.\end{aligned}$$

Hence by setting

$$c_p = \int_0^1 \frac{ds}{\left(1+s^{(p-2)/p}\right)^{\frac{2p}{p-2}}},$$

we easily check that c_p is monotonically increasing in p . The asymptotic behaviour of $c_p$as $p \to 2_+$ follows from the fact that c_p can be expressed as

$$c_p = 2^{-\frac{2p}{p-2}}\sqrt{\pi}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} \quad \text{with} \quad x = \frac{1}{2}+\frac{p}{p-2}.$$

Then we conclude using Sterling's formula that $\Gamma(x+\frac{1}{2})/\Gamma(x) \sim \sqrt{x}$ as $x \to +\infty$, which completes the proof.

Theorem (4.1.3)[140]: Let $N \geq 2$. For every $A < 0$, there exists $\varepsilon > 0$ such that the extremals of (1) are radially symmetric if $a+1-\varepsilon < b < a+1$ and $a \in (A, 0)$.So they are given by $u_{a,b}^{*}$ defined in (2), up to a scalar multiplication and a dilation.

We also prove that the regions of symmetry and symmetry breaking are separated by a continuous curve, that can be parametrized in terms of p. In fact, using that a , b and p satisfy the relation:

$$b = a + 1 + N\left(\frac{1}{p} - \frac{1}{2}\right) = \frac{N}{p} - \frac{N - 2 - 2a}{2}, \qquad (15)$$

the condition $a < b < a + 1$ can be expressed in terms of a and p, by requiring that $a \neq a_c$ and $p \in (2, 2^*)$, with $2^* := 2\,N/(N-2)$ if $N \geq 3$ or $2^* := +\infty$ if $N = 2$. Constant values of p define lines parallel to $b = a$ and in particular the line $b = a + 1$ coincides with $p = 2$.

Proof: We argue by contradiction. Because of (5), we may suppose that there exist sequences $(\Lambda_n)_{n \in \mathbb{N}}$ and $(p_n)_{n \in \mathbb{N}}$, with $\Lambda_n > 0$,

$$\lim_{n\to+\infty} \Lambda_n = \Lambda \geq (\mathrm{N}-2)^2/4, \qquad \lim_{n\to+\infty} p_n = 2_+ ,$$

such that the corresponding global minimizer, $w_n := w_{\Lambda_n, p_n}$ satisfies:

$$\mathcal{F}_{\Lambda,p}\left[w_{\Lambda_n,p_n}\right] < \mathcal{F}_{\Lambda,p}\left[w^*_{\Lambda_n,p_n}\right], \ -\Delta_y w_n + \Lambda_n w_n = w_n^{p-1} \quad \text{in } C,$$

together with (12), for each $n \in N$. In particular, $0 < \max_C w_n = w_n(0, \theta_0)$, for some fixed $\theta_0 \in S^{N-1}$.

Let us define $c_n > 0$ and W_n as follows:

$$c_n^2 = (\Lambda_n p_n)^{\frac{p_n}{2p_n-1}} 2^{\frac{p_n}{p_n-2}} \sqrt{p_n - 2} \ \ and \ \ W_n \coloneqq c_n w_n.$$

The function W_n satisfies

$$-\Delta W_n + \Lambda_n W_n = c_n^{2-p_n} W_n^{p-1} \ \text{ in } C,$$

and

$$\int_C |\nabla W_n|^2\, dy + \Lambda_n \int_C W_n^2 dy = c_n^2 \int_C w_n^{p_n} dy.$$

Note that $\lim_{n\to+\infty} \Lambda_n = 0$ is possible only if $N = 2$. In such a case, the conclusion follows from (i) in [141]. Hence assume from now on that $\lim_{n\to+\infty} \Lambda_n = \Lambda > 0$. By definition of c_n and Lemma (4.1.2), $\limsup_{n\to+\infty} c_n^2 \int_C w_n^{p_n} dy \leq |S^{N-1}| \sqrt{2\pi/\Lambda}$, so that $(W_n)_{n\in\mathbb{N}}$ is bounded in $H^1(C)$. Moreover, $\lim_{n\to+\infty} c_n^{2-p_n} = \Lambda$. Therefore, by elliptic regularity, up to subsequences, $(W_n)_{n\in\mathbb{N}}$ converges weakly in $H^1(C)$, and uniformly in every compact subset of C , towards a function . Again by definition of c_n , this function satisfies

$$-\Delta W + \Lambda W = \Lambda W \ \text{ in } C.$$

Hence W is constant but also in $H^1(C)$, and therefore $\equiv 0$. Let χ_n be any component of $\nabla_\theta\, W_n$. By differentiating both sides of the equation of W_n with respect to θ, we know that

$$-\Delta\chi_n + \Lambda_n \chi_n = (p_n - 1) c_n^{2-p_n} W_n^{p_n-2} \chi_n \quad \text{in } C.$$

So, multiplying this equation by χ_n and integrating by parts, we get

$$0 = \int_C |\nabla\chi_n|^2 dy + \Lambda_n \int_C |\chi_n|^2 dy - (p_n - 1) c_n^{2-p_n} \int_C W_n^{p_n-2} |\chi_n|^2\, dy$$

The function W_n is bounded by $W_n(0, \theta_0)$ and $\lim_{n\to+\infty} W_n(0, \theta_0) = 0$. Since $\int_{S^{N-1}} \nabla_\theta\ W_n(t, \theta) d\theta = 0$, an expansion of χ_n in spherical harmonics tells us that

$$\int_C |\nabla\chi_n|^2 dy \geq (N-1)\int_C |\chi_n|^2 dy.$$

By collecting these estimates, we get

$$0 \geq \left(N-1+\Lambda_n-(p_n-1)c_n^{2-p_n}W_n(0,\theta_0)^{p_n-2}\right)\int_C |\chi_n|^2 dy.$$

Since $\lim_{n\to+\infty}\Lambda_n=\Lambda$ and $\limsup_{n\to+\infty}(p_n-1)c_n^{2-p_n}W_n(0,\theta_0)^{p_n-2}\leq\Lambda$, for n large enough, $\chi_n\equiv 0$ and w_n is radially symmetric.

Theorem (4.1.4)[140]: For all $N\geq 2$, there exists a continuous function $a^*:(2,2^*)\to(-\infty,0)$ such that $\lim_{p\to 2^*}a^*(p)=0$, $\lim_{p\to 2_+}a^*(p)=-\infty$ and

(i) If $(a,p)\in(a^*(p),a_C)\times(2,2^*)$, (1) has only radially symmetric extremals.

(ii) If $(a,p)\in(-\infty,a^*(p))\times(2,2^*)$, none of the extremals of (1) is radially symmetric.

On the curve $\mapsto(p,a^*(p))$, radially symmetric and non radially sym-metric extremals for (1) may eventually coexist.

In a refinement of the results of [1], for ≥ 3, V. Felli and M. Schneider proved in [142] that in the region $a<b<b^{\mathrm{FS}}(a)$, $a<0$, extremals are non-radially symmetric, where

$$b^{FS}(a):=\frac{N(N-2-2a)}{2\sqrt{(N-2-2a)^2+4(N-1)}}-\frac{N-2-2a}{2}.$$

The proof is based on the linearization of a functional associated to (1) around the radial extremal $u_{a,b}^*$. Above the curve $b=b^{FS}(a)$, all corresponding eigenvalues are positive and $u_{a,b}^*$ is a local minimum, while there is at least one negative eigenvalue if $b<b^{FS}(a)$ and $u_{a,b}^*$ is then a saddle point. As $a\to-\infty$, $b=b^{FS}(a)$ is asymptotically tangent to $b=a+1$. But recalling (15), also the function $b^*(p):=a^*(p)+1+N\left(\frac{1}{p}-\frac{1}{2}\right)$ admits the same asymptotic behavior as $p\to 2_+$. Hence, it is natural to conjecture that the curve $p\longmapsto(a^*(p),b^*(p))$ coincides with the curve $a\longmapsto(a,b^{FS}(a))$.

Proof: We prove the existence of a function Λ^* which describes the boundary of the symmetry region (see Corollary (4.1.6)). Then we establish the upper semicontinuity of $p\longmapsto\Lambda^*(p)$, and, using spectral properties, its continuity (see Corollary (4.1.9)), which completes the proof of Theorem (4.1.2).

If $w\in H^1(C)\setminus\{0\}$, let $w_\sigma(t,\theta):=w(\sigma t,\theta)$ for any $\sigma>0$. A simple calculation shows that

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma)=\sigma^{1+2/p}\mathcal{F}_{\Lambda,p}(w)-\sigma^{-1+2/p}(\sigma^2-1)\frac{\int_C|\nabla_\theta w|^2dy}{\left(\int_C|w|^p dy\right)^{2/p}}. \tag{16}$$

As a consequence, we observe that

$$\left(C_{\sigma^2\Lambda,p}^{N,*}\right)^{-1}=\mathcal{F}_{\sigma^2\Lambda,p}\left(w_{\sigma^2\Lambda,p}^*\right)=\sigma^{1+2/p}\left(C_{\Lambda,p}^{N,*}\right)^{-1}=\sigma^{1+2/p}\mathcal{F}_{\Lambda,p}\left(w_{\Lambda,p}^*\right).$$

Lemma (4.1.5)[140]: If $N\geq 2$, $\Lambda>0$ and $p\in(2,2^*)$, the following properties hold.

(i) If $C_{\Lambda,p}^N=C_{\Lambda,p}^{N,*}$, then $C_{\lambda,p}^N=C_{\lambda,p}^{N,*}$, and $w_{\lambda,p}=w_{\lambda,p}^*$, for any $\lambda\in(0,\Lambda)$.

(ii) If there is a non radially symmetric extremal function $w_{\Lambda,p}$, then $C_{\lambda,p}^N>C_{\lambda,p}^{N,*}$ for all $\lambda>\Lambda$.

Proof: To prove (i), apply (16) with $w_\sigma = w_{\Lambda,p}, \lambda = \sigma^2\Lambda, \sigma < 1$ and $(t,\theta) = w_{\Lambda,p}(t/\sigma, \theta))$:

$$\left(C_{\lambda,p}^N\right)^{-1} = \mathcal{F}_{\sigma^2\Lambda,p}\left(w_{\Lambda,p}\right) = \sigma^{1+\frac{2}{p}}\mathcal{F}_{\Lambda,p}[w] + \sigma^{-1+\frac{2}{p}}(1-\sigma^2)\frac{\int_C |\nabla_\theta w|^2 dy}{\left(\int_C |w|^p dy\right)^{\frac{2}{p}}}$$

$$\geq \sigma^{1+\frac{2}{p}}\left(C_{\Lambda,p}^{N,*}\right)^{-1} + \sigma^{-1+\frac{2}{p}}(1-\sigma^2)\frac{\int_C |\nabla_\theta w|^2 dy}{\left(\int_C |w|^p dy\right)^{\frac{2}{p}}}$$

$$= \left(C_{\lambda,p}^{N,*}\right)^{-1} + \sigma^{-1+\frac{2}{p}}(1-\sigma^2)\frac{\int_C |\nabla_\theta w|^2 dy}{\left(\int_C |w|^p dy\right)^{\frac{2}{p}}}.$$

By definition, $C_{\lambda,p}^N \geq C_{\lambda,p}^{N,*}$ and from the above inequality the first claim follows.

Assume that $w_{\Lambda,p}$ is a non radially symmetric extremal function and apply (16) with $w = w_{\Lambda,p}, w_\sigma(t,\theta) := w(\sigma t, \theta), \lambda = \sigma^2\Lambda$ and $\sigma > 1$:

$$\left(C_{\lambda,p}^N\right)^{-1} \leq \mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+\frac{2}{p}}\left(C_{\Lambda,p}^N\right)^{-1} - \sigma^{-1+\frac{2}{p}}(\sigma^2-1)\frac{\int_C \left|\nabla_\theta w_{\Lambda,p}\right|^2 dy}{\left(\int_C \left|w_{\Lambda,p}\right|^p dy\right)^{\frac{2}{p}}}$$

$$\leq \sigma^{1+\frac{2}{p}}\left(C_{\Lambda,p}^{N,*}\right)^{-1} - \sigma^{-1+\frac{2}{p}}(\sigma^2-1)\frac{\int_C \left|\nabla_\theta w_{\Lambda,p}\right|^2 dy}{\left(\int_C \left|w_{\Lambda,p}\right|^p dy\right)^{\frac{2}{p}}} < \left(C_{\lambda,p}^{N,*}\right)^{-1},$$

since $\nabla_\theta w_{\Lambda,p} \not\equiv 0$. This proves the second claim with $\lambda = \sigma^2\Lambda$.

Lemma (4.1.5) implies the following properties for the function Λ^* defined in (9):

Corollary (4.1.6)[140]: Let $N \geq 2$. For all $p \in (2, 2^*), \Lambda^*(p) \in (0, \Lambda^{FS}(p)]$ and

(i) If $\lambda \in (0, \Lambda^*(p))$, then $w_{\lambda,p} = w_{\lambda,p}^*$ and clearly, $C_{\lambda,p}^N = C_{\lambda,p}^{N,*}$.

(ii) If $\lambda = \Lambda^*(p)$, then $C_{\lambda,p}^N = C_{\lambda,p}^{N,*}$.

(iii) If $\lambda > \Lambda^*(p)$, then $C_{\lambda,p}^N > C_{\lambda,p}^{N,*}$.

From the above results, note that Λ^* can be defined in three other equivalent ways:

$$\Lambda^*(p) = \max\left\{\Lambda > 0 : \ w_{\lambda,p} = w_{\lambda,p}^*\right\} = \max\left\{\Lambda > 0 : \ C_{\lambda,p}^N = C_{\lambda,p}^{N,*}\right\}$$
$$= \inf\left\{\Lambda > 0 : \ C_{\lambda,p}^N > C_{\lambda,p}^{N,*}\right\}.$$

Note also that for $p \in (2, 2^*)$, and $\Lambda = \Lambda^*(p)$, the equality $C_{\Lambda,p}^N = C_{\Lambda,p}^{N,*}$ holds, but there might be simultaneously a radially symmetric extremal function and a non radially symmetric one.

Lemma (4.1.7)[140]: Let $N \geq 2$. The function Λ^* is upper semicontinuous on $(2, 2^*)$.

Proof: Assume by contradiction that for some $\in (2, 2^*)$, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} p_n = p$ and

$$\Lambda^*(p) < \liminf_{n\to+\infty} \Lambda^*(p_n) := \overline{\Lambda}\,.$$

Let $\Lambda \in (\Lambda^*(p), \overline{\Lambda})$. The functions w_{Λ,p_n}^* are extremal and converge to $w_{\Lambda,p}^*$ which is also extremal by continuity of $C_{\Lambda,p}^N$ with respect to p. This contradicts Lemma (4.1.4), (ii).

On $H^1(C)$, let us define the quadratic form

$$Q[\psi] := \|\nabla\psi\|^2_{L^2(C)} + \Lambda\|\psi\|^2_{L^2(C)} - (p-1)\int_C \left|w^*_{\Lambda,p}\right|^{p-2} |\psi|^2 dy$$

and consider $\mu^1_{\Lambda,p} := \inf Q[\psi]$ where the infimum is taken over the set of all functions $\psi \in H^1(C)$ such that $\int_{S^{N-1}} \psi\,(t,\theta)\,d\theta = 0$ for $t \in \mathbb{R}$ a.e. and $\|\psi\|_{L^2(C)} = 1$

Proposition (4.1.8)[140]: Let $N \geq 2$, $\Lambda > (N-2)^2/4$ and $p \in (2, 2^*)$. Then $\mu^1_{\Lambda,p} = N - 1 - \frac{p^2-4}{4}\Lambda$ is positive for any $\Lambda \in (0, \Lambda^{FS}(p))$ and it is achieved by the function

$$\psi_1(t,\theta) = \left(\cosh\left(\tfrac{1}{2}(p-2)\sqrt{\Lambda}\,t\right)\right)^{-p/(p-2)} \varphi_1(\theta)$$

where $\varphi_1$is any eigenfunction of the Laplace-Beltrami operator on S^{N-1} corresponding to the eigenvalue $N-1$.

Proof: Let us analyze the quadratic form $Q[\psi]$ in the space of functions $\psi \in H^1(C)$ such that $\int_{S^{N-1}} \psi\,(t,\theta)\,d\theta = 0$ for a.e. $t \in \mathbb{R}$. To this purpose, we use the spherical harmonics expansion of ,

$$\psi(t,\theta) \sum_{k\in\mathbb{N}} f_k\,(t)\varphi_k(\theta),$$

and we take into account the zero mean average of ψ over S^{N-1} to write

$$Q[\psi] = \sum_{k=1}^{+\infty}\left(\|\acute{f}_k\|^2_{L^2(\mathbb{R})} + \gamma_k\|f_k\|^2_{L^2(\mathbb{R})} - (p-1)\int_{\mathbb{R}} \left|w^*_{\Lambda,p}\right|^{p-2} |f_k|^2 dt\right)$$

with $\gamma_k := \Lambda + k\,(k+N-2)$. The minimum is achieved for $k = 1$ and

$$\mu^1_{\Lambda,p} = \inf\left(\|\acute{f}\|^2_{L^2(\mathbb{R})} + \gamma_1\|f\|^2_{L^2(\mathbb{R})} - (p-1)\int_{\mathbb{R}} \left|w^*_{\Lambda,p}\right|^{p-2} |f|^2 dt\right),$$

where the infimum is taken over $\{\, f \in H^1(\mathbb{R}) : \|f\|^2_{L^2(\mathbb{R})} = 1\}$. In order to calculate $\mu^1_{\Lambda,p}$ and the corresponding extremal function f , we have to solve the ODE

$$-f'' - \beta\, V\, f \ = \lambda\, f,$$

in $H^1(\mathbb{R})$, with $\beta = \Lambda\, p(p-1)/2$ and $V(t) := \left(\cosh\left(\tfrac{1}{2}(p-2)\sqrt{\Lambda}\,t\right)\right)^{-2}$ .Finally, the eigenfunction $f(t) = V\,(t)^{\,p/(2(p-2))}$ corresponds to the first eigenvalue, $\lambda = -p^2\Lambda/4$. See [143], [142] for a more detailed discussion of the above eigenvalue problem.,

Corollary (4.1.9)[140]: Let $N \geq 2$ .The function Λ^* is continuous on $(2, 2^*)$ and $\lim_{q\to 2_+} \Lambda^*(q) \ = +\infty$.

Proof: We have to prove that for all $p \in (2, 2^*)$, for all $p_n \in (2, 2^*)$ converging to p , $\lim_{n\to+\infty} \Lambda^*(p_n) = \Lambda^*(p)$. Taking into account Lemma (4.1.7), assume by contradiction that there exists a sequence $(p_n)_{n\in N}$ such that $\lim_{n\to+\infty} p_n = p$. and $\lim_{n\to+\infty} \Lambda^*(p_n) = \bar{\Lambda} < \Lambda^*(p)$. Choose $\Lambda \in (\Lambda^*(p_n), \Lambda^*(p))$ for n large. By definition of Λ^*, the extremals $w_n := w_{\Lambda,p_n} > 0$ are not radially symmetric for n large enough. Now, by (14), the functions w_n are uniformly bounded in $H^1(C)$ and the functions $w_n^{p_n-1}$ are also uniformly bounded in $L^{p_n/(p_n-1)}(C)$, with $p_n \to p \in (2, 2^*)$. Hence, by elliptic regularity and the Sobolv embedding, we deduce that w_n is uniformly bounded in $C^{2,\alpha}_{loc}(C)$. So we can find a subsequence along which w_n converges pointwise, and uniformly in every compact subset of C. Since $\Lambda < \Lambda^*(p)$,by Corollary(4.1.6),this limit is $w^*_{\Lambda,p}$. Next, for any $\varepsilon > 0$ take

$R_\varepsilon > 0$ such that $w^*_{\Lambda,p}(R) < \varepsilon$ for all $R \geq R_\varepsilon$. By the decay in $|t|$ of w_n and $w^*_{\Lambda,p}$ we see that

$$\left\|w_n - w^*_{\Lambda,p}\right\|_{L^\infty(C)} \leq 2\left\|w_n - w^*_{\Lambda,p}\right\|_{L^\infty(|t|\leq R_\varepsilon)} + 2\left|w^*_{\Lambda,p}(R_\varepsilon)\right|,$$

and this, together with the uniform local convergence, proves that w_n converges towards $w^*_{\Lambda,p}$ uniformly in the whole cylinder C.

Let us now consider one of the components of $\nabla_\theta w_n$, that we denote by χ_n. Then $\chi_n \not\equiv 0$ satisfies

$$-\Delta\chi_n + \Lambda\chi_n = (p_n - 1)w_n^{p_n-2}\chi_n \quad \text{in} \quad C.$$

Multiplying the above equation by χ_n and integrating by parts we get

$$\int_C (|\nabla\chi_n|^2 + \Lambda|\chi_n|^2 - (p_n - 1)w_n^{p_n-2}|\chi_n|^2)dy = 0.$$

By Proposition (4.1.8), since $\Lambda < \Lambda^*(p) \leq \Lambda^{FS}(p)$, we have

$$\int_C (|\nabla\chi_n|^2 + \Lambda|\chi_n|^2 - (p_n - 1)w^*_{\Lambda,p_n} w_n^{p_n-2}|\chi_n|^2)dy \geq \mu^1_{\Lambda,p_n}\|\chi_n\|^2_{L^2(C)},$$

with $\liminf_{n\to+\infty} \mu^1_{\Lambda,p_n} > 0$. This contradicts the fact that

$$\int_C \left(\left|w^*_{\Lambda,p_n}\right|^{p_n-2} - |w_n|^{p_n-2}\right)|\chi_n|^2 dy = o\left(\|\chi_n\|^2_{L^2(C)}\right) \; as \; n \to +\infty,$$

which follows by the uniform convergence of w_n and w^*_{Λ,p_n} towards $w^*_{\Lambda,p}$,p,since, by assumption $\|\chi_n\|^2_{L^2(C)} \neq 0$ for n large enough.

The limit of $\Lambda^*(q) = +\infty$ as $q \to 2+$ follows from Theorem (4.1.3). Moreover in dimension $N = 2$ we know also the slope of the curve separating the symmetry and the symmetry breaking regions near the point $(a, b) = (0, 0)$, and as remarked before, it coincides with that of the Felli-Schneider curve $(a, b^{FS}(a)$. All this motivates our conjecture that the functions Λ^* and Λ^{FS} coincide over the whole range $(2, 2^*)$.

Section (4.2): Symmetry of Optimizers

Symmetries of optimizers in variational problems is a central theme in the calculus of variations. Sophisticated methods like rearrangement inequalities, reflection methods and moving plane methods belong now to the standard repertoire of any analyst. There are, however, examples where these methods cannot be applied. Variational problems that depend on parameters very often cannot be treated by such methods, simply because, depending on the parameters, the optimizers are symmetric and sometimes not. Famous examples are the minimizers of the Ginzburg-Landau functional in superconductivity, where, depending on the strength of the quartic interaction the minimizers form a single, symmetric vortex or a vortex lattice. Clearly such problems cannot be treated by general methods. For certain parameters they ought to work while in others they cannot. Thus, rather special techniques, tailored to the problems at hand, have to be developed to prove symmetry in the desired regions.

One class of such examples is given by the Caffarelli-Kohn-Nirenberg inequalities [7]. We shall specifically consider the case of the inequality

$$\int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2a}} dx \geq C^d_{a,b}\left(\int_{\mathbb{R}^d} \frac{|w|^2}{|x|^{bp}} dx\right)^{\frac{2}{p}} \qquad \text{(CKN)}$$

with $a \leq b \leq a + 1$ if $d \geq 3, a < b \leq a + 1$ if $d = 2$, and $a < a_c$ where

$$a_c := \frac{d-2}{2}, \qquad p = \frac{2d}{d-2+2(b-a)}.$$

The function w is in a suitable function space which contains, for instance, all smooth functions with compact support. The constant $C_{a,b}^{d}$ is, by definition, the best possible constant. Rotating the function w does not change the value of the various expressions in (CKN), i.e., the inequality is rotationally invariant. The special case where $a \geq 0$ has been treated (see [150]). Rearrangement inequalities can be used to reduce the problem to the set of radial functions, for which the optimality issue can then be solved explicitly.

For the case where $a < 0$ the problem is much more subtle. Catrina and Wang [1], proved that the optimizers, i.e., the functions that yield equality in (CKN), exist in the open strip $a < b < a + 1$. This result establishes the existence of non-negative solutions $w \in L^p(\mathbb{R}^d; |x|^{-bp}dx)$ of the equation

$$-\text{div}(|x|^{-2a}\nabla w) = |x|^{-bp} w^{p-1}. \tag{17}$$

Moreover, Catrina and Wang also showed that, in some region in the (a, b) plane, the rotational symmetry of the optimizers is broken. A more detailed analysis by Felli and Schneider [142] shows that the region where the optimizers have a broken symmetry contains the set $\mathcal{R}_{FS} := \{(a, b) : a < 0, b < b^{FS}(a)\}$ where

$$b^{FS}(a) := \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c.$$

We call this region $\mathcal{R}_{FS}$ the Felli-Schneider region.

In [142] more is shown. The optimizers in the radial class can be determined explicitly which allows to compute the second variation operator about these solutions. The lowest eigenvalue of this operator is strictly negative for $(a, b) \in \mathcal{R}_{FS}$, equals zero on the curve $b = b_{FS}(a)$ and is strictly positive in the open complement of the Felli-Schneider region: there, the radial optimizers are stable. Needless to say that positivity of the second variation does not imply the radial symmetry of the (global) optimizers for the (CKN) inequality. Thus, it is a natural question whether or not the optimizers possess rotational symmetry in the complement of $\mathcal{R}_{FS}$. Let $2^* := \frac{2d}{d-2}$ *if* $d \geq 3$ and $2^* := \infty$ if $d = 2$. The following theorem is proved in [150]:

Corollary (4.2.1)[147]: Let $d \geq 2, p \in (2, 2^*)$. Any non-negative solution $\phi \in L^p(\mathbb{R} \times \mathbb{S}^{d-1}; dz d\omega)$ of (20) is, up to translations, of the form

$$\phi_\Lambda(z) = \left(\frac{2}{p\Lambda}\cosh^2\left(\frac{p-2}{2}\sqrt{\Lambda}\, z\right)\right)^{-\frac{1}{p-2}},$$

if and only if

$$\Lambda \leq 4\frac{d-1}{p^2-4}.$$

In this range, equality in (21) is achieved if and only if $\phi(z) = \phi\Lambda(z + z_0)$ for some $z_0 \in \mathbb{R}$.

To put this result in perspective we compare it with a result in [148].

Theorem (4.2.2)[147]: Let $d \geq 2, p \in (2, 2^*)$. On the sphere $\mathbb{S}^d$ consider the equation

$$\Delta_u + \lambda_u = u^{p-1}$$

with $\lambda > 0$. Here Δ represents the Laplace-Beltrami operator on $\mathbb{S}^d$. Then the constant function $u \equiv \lambda_1/(p-2)$ is the only non-negative solution if and only if

$$\lambda \leq \frac{d}{p-2}.$$

Thus, Corollary (4.2.1) can be viewed as an extension of the above mentioned rigidity result to the non-compact case of a cylinder. As a special case, this also allows to identify the equality case in the interpolation inequality (21) on the cylinder.

We start by the simple case of the standard Sobolev inequality, explain how to recast (CKN) as a Sobolev type inequality in an artificial dimension n, where n is not necessarily an integer, and conclude by explaining how the main estimates can be produced using a fast diffusion flow.

Theorem (4.2.3)[147]: Let $d \geq 2, p \in (2, 2^*), a < 0$ and b in the complement of the Felli-Schneider region and such that $p = \frac{2\,d}{d-2+2\,(b-a)}$. Then any non-negative solution $w \in L^p(\mathbb{R}^d; |x|^{-bp} dx)$ of (17) must be of the form

$$(A + B|x|^{2\alpha})^{-\frac{n-2}{2}}$$

where A, B are positive constants,

$$\alpha = \frac{(1 + a - b)(a_c - a)}{a_c - a + b} \tag{18}$$

and

$$n = \frac{2p}{p-2}. \tag{19}$$

In particular this holds for the optimizers of (CKN).

There are some interesting consequences. Using the change of variables

$$w(r, \omega) = r^{a-a_c}\phi(\log r, \omega),$$

equation (17) can be cast in the form

$$-\partial_z^2\phi - \Delta_\omega\phi + \Lambda\phi = \phi^{p-1}. \tag{20}$$

Here, $\frac{x}{|x|} = \omega \in \mathbb{S}^{d-1}, r = |x|, z = \log r$, Δ_ω is the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$ and

$$\Lambda = (a - a_c)^2.$$

Thus, ϕ is a function on the cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$. Moreover, as noticed in [1], (CKN) is transformed into

$$\|\partial_z\phi\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} + \|\nabla_\omega\phi\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} + \Lambda\|\phi\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \geq C^d_{a,b}\|\phi\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}. \tag{21}$$

In order to avoid long computations it is best to explain the ideas in a 'simple' example. For any $d \geq 3$, the Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2\, dx \geq C_d \left(\int_{\mathbb{R}^d} |u|^2\, dx\right)^{2/p}, \quad \text{with} \quad p = 2^* = \frac{2d}{d-2} \tag{22}$$

is extremely well understood [32], [19], [24]. Once more C_d denotes the sharp constant. Note that this inequality appears as a special case of (CKN) if one sets $a = b = 0$, in which case $C_d = C^d_{0,0}$, . There is equality in (22) if and only if u is a translate of the Aubin-Talenti function

$$\left(c_\star\lambda + \frac{|x|^2}{\lambda}\right)^{-(d-2)/2},$$

where $c_\star$ and λ are positive constants. There have been some proofs using flow methods to understand this inequality [149], [41]. The flow used for the case at hand is a porous medium / fast diffusion flow. It is given by

$$\frac{\partial v}{\partial t} = \Delta v^{1-\frac{1}{d}} \tag{23}$$

and has the self-similar solutions

$$v_\star(x,t) = \left(c_\star t + \frac{|x|^2}{t}\right)^{-d}.$$

This function has slow decay in the x variable. The obvious similarity of the expressions of the Aubin-Talenti and self-similar functions suggests a reformulation of the Sobolev functional by setting

$$v = u^{\frac{2d}{d-2}}.$$

Let us define a pressure variable P by

$$v = P^{-d}.$$

A short computation shows

Lemma (4.2.4)[147]: The Sobolev inequality, written in terms of v and p, is given by

$$a_c^2 \int_{\mathbb{R}^d} v\,|\nabla P|^2 dx \geq C_d \left(\int_{\mathbb{R}^d} v\,dx\right)^{\frac{d-2}{d}}. \tag{24}$$

Assume now that v satisfies the fast diffusion equation (23). This implies that P evolves by the equation

$$\frac{\partial P}{\partial t} = \frac{d-1}{d} P\Delta P - d|\nabla P|^2.$$

The right side of (24) does not change if v evolves via (23). For the left side we have

Lemma (4.2.5)[147]: Assume that v evolves via (23). Then

$$\frac{d}{dt}\int_{\mathbb{R}^d}^2 v\,|\nabla P|^2 dx = -2\int_{\mathbb{R}^d}^2 \left[\tfrac{1}{2}\Delta|\nabla P|^2 - \nabla P\cdot\nabla\Delta P - \tfrac{1}{d}(\Delta P)^2\right] P^{1-d} dx$$

$$= -2\int_{\mathbb{R}^d} Tr\left[H_P - \tfrac{1}{d}(TrH_P)\ \mathrm{Id}\,2\right]^2 P^{1-d} dx$$

where $H_P = (\nabla\otimes\nabla)P$ denotes the Hessian matrix of P. Moreover,

$$H_P - \tfrac{1}{d}(TrH_P)\mathrm{Id} = 0$$

if and only if $P(x) = a + b\cdot x + c|x|^2$ for some $(a,b,c) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$.

The proof is a somewhat longish but straightforward computation. Note, that it is precisely the particular choice of v and P that renders the time derivative in such a simple form.

To summarize, while the right side of the Sobolev inequality stays fixed the left side diminishes under the flow. The idea is to use the fast diffusion flow to drive the functional towards its optimal value. Actually we use the fact that if v is optimal in (24), or if it is a critical point, the functional has to be stationary under the action of the flow, which allows to identify P, hence v. To exploit this idea for the (CKN) inequality we have to rewrite it in the form of a Sobolev type inequality.

The first step in the proof is to rewrite the problem in a form that resembles the Sobolev inequality. If we write

$$w(s,\omega) = u(s,\omega) \quad \text{with} \quad s = r^\alpha,$$

the inequality (CKN) takes the form

$$\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} \left[\alpha^2\left(\frac{\partial u}{\partial s}\right)^2 + \frac{|\nabla_\omega u|^2}{s^2}\right] s^{n-1} ds d\omega \geq C_{a,b}^d \alpha^{1-\frac{2}{d}} \left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} |u|^p\, s^{n-1} ds d\omega\right)^{\frac{2}{p}}$$

where $d\omega$ denotes the uniform measure on the sphere $\mathbb{S}^{d-1}$, ∇_ω denotes the gradient on $\mathbb{S}^{d-1}$ and where α and n are given by (18) and (19). We shall abbreviate

$$Du := \left(\alpha \frac{\partial u}{\partial s}, \frac{1}{s}\nabla_\omega u\right), \qquad |Du|^2 = \alpha^2 \left(\frac{\partial u}{\partial s}\right)^2 + \frac{|\nabla_\omega u|^2}{s^2}.$$

Our inequality is therefore equivalent to a Sobolev type inequality and takes the form

$$\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} |Du|^2 d\mu \geq C_{a,b}^d \alpha^{1-\frac{2}{d}} \left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} |u|^p \, d\mu\right)^{\frac{2}{p}}, \quad \text{with} \quad p = \frac{2n}{n-2}. \tag{25}$$

This inequality generalizes (22). Here the measure $d\mu$ is defined on $\mathbb{R}^+ \times \mathbb{S}^{d-1}$by

$$d\mu = s^{n-1} ds d\omega.$$

As we may consider $v = u^p$ and define a pressure variable P such that $v = P^{-n}$, so that $u = P^{-(n-2)/2}$. With these notations, (25) can be rewritten as

$$\tfrac{1}{4}(n-2)^2 \int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} v|DP|^2 d\mu \geq C_{a,b}^d \alpha^{1-\frac{2}{d}} \left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} v \, d\mu\right)^{\frac{2}{p}}. \tag{26}$$

We shall write $\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} f d\mu = \int_{\mathbb{R}^d} f d\mu$ and identify $L^p(\mathbb{R}_+ \times \mathbb{S}^{d-1};\, d\mu)$ with $L^p(\mathbb{R}^d;\, |x|^{n-d} dx)$ or simply $L^p(\mathbb{R}^d;\, d\mu)$.

One should note that n is, in general, not an integer and the above inequality reduces to Sobolev's inequality only if $n = d$. Of particular significance is that the curve

$$b = b_{FS}(a),$$

when represented in the new variables α and n, is given by the equation $\alpha = \alpha_{FS}$ with

$$\alpha_{FS} := \sqrt{\tfrac{d-1}{n-1}}.$$

Thus, for $\alpha > \alpha_{FS}$ the minimizers are not radial. The equation (17) transforms into the equation

$$-\mathcal{L}\, u = u^{p-1}, \tag{27}$$

where $\mathcal{L}$ is the Laplacian associated with the quadratic form given by the left side of (25), i.e., $\mathcal{L} = -D^* \cdot D$. Theorem (4.2.3) can be reformulated as

Theorem (4.2.6)[147]: Let $d \geq 2, p \in (2, 2^*), n = \frac{2p}{p-2} > d$ and $\alpha \leq \alpha_{FS}$. Then any non-negative solution $u \in L^p(\mathbb{R}^d;\, d\mu)$ of (27) must be of the form

$$(A + B|x|^2)^{-\frac{n-2}{2}} \tag{28}$$

where A, B are positive constants, and n is given by (19). *As* a special case, equality in (26) is achieved if and only if u is given by (28).

Any optimizer in the radial class that is not unstable under small perturbations is in fact a global minimizer for the (CKN) inequality.

We consider the fast diffusion flow

$$\frac{\partial v}{\partial t} \mathcal{L}\, v^{1-\frac{1}{n}}. \tag{29}$$

It is easily seen that the flow (29) has the self-similar solutions

$$v_\star(t; s, \omega) = t^{-n} \left(c_\star + \frac{s^2}{2(n-1)\alpha^2 t^2}\right)^{-n}.$$

The basic idea is now quite simple. We consider a non-negative solution $u \in L^p(\mathbb{R}^d;\, d\mu)$ of (27) and set $v = u^p$. We also consider the pressure variable P such that $v = P^{-n}$. The first thing to note is that the right side of (26) does not change if we evolve v and hence u under the flow (29). Further, if we differentiate the left side of (26) along the flow we obtain

$$\frac{d}{dt}\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} v\,|DP|^2 d\mu = -2\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} \left[\tfrac{1}{2}\mathcal{L}\,|DP|^2 - DP\cdot D\mathcal{L}P - \tfrac{1}{n}(\mathcal{L}P)^2\right] d\mu.$$

On the other hand simple computations show that

$$\tfrac{1}{4}(n-2)^2\frac{d}{dt}\left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} v\,|DP|^2 d\mu\right)\Big|_{t=0}$$

$$= -2\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} (\mathcal{L}u)u^{1-p}\left(\mathcal{L}u^{p(n-1)/n}\right)d\mu. \tag{30}$$

when expressed in terms of u. Now we take $v = u^p$, where u is the solution to (27), as initial datum for (17). With this choice, the right side in (30) is actually zero. Indeed, by multiplying both sides of (27) by $u^{1-p}\left(\mathcal{L}u^{p(n-1)/n}\right)$ one obtains

$$\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} (\mathcal{L}u)u^{1-p}\left(\mathcal{L}u^{p(n-1)/n}\right)d\mu = \int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} u^{p-1}u^{1-p}\left(\mathcal{L}u^{p(n-1)/n}\right)d\mu = 0.$$

The interesting point, and the heart of the argument, is that

$$0 = \int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} \left[\tfrac{1}{2}\mathcal{L}\,|DP|^2 - DP\cdot D\mathcal{L}P - \tfrac{1}{n}(\mathcal{L}P)^2\right] d\mu$$

can be written as a sum of non-negative terms precisely when α $\alpha \le \alpha_{FS}$, and the vanishing of these terms shows that u must be of the form $(A + B\,s^2)^{-(n-2)/2}$. In this way one obtains a classification of the non-negative solutions of (27) provided they are in $L^p(\mathbb{R}^d;\, d\mu)$. To simplify notations, we shall omit the index ω, so that from now on ∇ and Δ respectively refer to the gradient and to the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. With the notation$' = \partial_s$, our identity can be reworked as follows.

Lemma (4.2.7)[147]: Assume that $d \ge 3, n > d$ and let P be a positive function in $C^3(\mathbb{S}^{d-1})$. Then

$$\tfrac{1}{2}\mathcal{L}\,|DP|^2 - DP\cdot D\mathcal{L}P - \tfrac{1}{n}(\mathcal{L}P)^2$$

$$= \alpha^4\frac{n-1}{n}\left[P'' - \frac{P'}{r} - \frac{\Delta P}{\alpha^2(n-1)r^2}\right]^2 + \frac{2\alpha^2}{r^2}\left|\nabla P' - \frac{\nabla P}{r}\right|^2$$

$$+ \frac{1}{r^4}\left[\tfrac{1}{2}\Delta|\nabla P|^2 - \nabla P\cdot\nabla\Delta P - \tfrac{1}{n-1}(\Delta P)^2 - (n-2)\alpha^2|\nabla P|^2\right].$$

The only term in Lemma (4.2.7) that does not have a sign is the last one. When integrated against P^{1-n} over $\mathbb{S}^{d-1}$, however, this term can be written as a sum of squares. The following lemma holds for $d \ge 3$. For the case $d = 2$ see [150].

Lemma (4.2.8)[147]: Assume that $d \ge 3$ and that P is a positive function in $C^3(\mathbb{S}^{d-1})$. Then

$$\int_{\mathbb{S}^{d-1}}^{2} \left[\tfrac{1}{2}\Delta|\nabla P|^2 - \nabla P\cdot\nabla\Delta P - \tfrac{1}{n-1}(\Delta P)^2 - (n-2)\alpha^2|\nabla P|^2\right] P^{1-n} d\omega$$

$$= \frac{(n-2)(d-1)}{(n-1)(d-2)}\int_{\mathbb{S}^{d-1}}^{2} \left\| LP - \frac{3(n-2)(n-d)}{2(n-2)(d+1)}MP\right\|^2 P^{1-n} d\omega$$

$$+ \frac{n-d}{2(d+1)}\left[\frac{n+3}{2} + \frac{3(n-1)(n+1)(d-2)}{2(n-2)(d+1)}\right]\int_{\mathbb{S}^{d-1}}^{2} \frac{|\nabla P|^4}{P^2} P^{1-n} d\omega$$

$$+ (n-2)[\alpha_{FS}^2 - \alpha^2]\int_{\mathbb{S}^{d-1}}^{2} |\nabla P|^2\, P^{1-n} d\omega$$

where $LP:=(\nabla\otimes\nabla)P-\frac{1}{d-1}(\Delta P)g$ and $MP:=\frac{\nabla P\otimes\nabla P}{P}-\frac{1}{d-1}\frac{|\nabla P|^2}{P}g$. Here g is the standard metric on $\mathbb{S}^{d-1}$ and LP denotes the trace free Hessian of P.

The key device used for the proof of this lemma is the Bochner-Lichnerowicz-Weitzenbock formula. If $\mathcal{M}$ is a compact Riemannian manifold, then for any smooth function $f:\mathcal{M}\to\mathbb{R}$ we have

$$\frac{1}{2}\Delta|\nabla f|^2=\|H_f\|^2+\mathrm{Ric}(\nabla f,\nabla f)$$

where $\|H_f\|^2$ is the trace of the square of the Hessian of f and $\mathrm{Ric}(\nabla f,\nabla f)$ is the Ricci curvature tensor contracted against $\nabla f\otimes\nabla f$. If $\mathcal{M}=\mathbb{S}^{d-1}$, then $\mathrm{Ric}(\nabla f,\nabla f)=(d-2)|\nabla f|^2$. The main point in Lemma (4.2.8) is that, provided $\alpha\le\alpha_{FS}$, all terms are non-negative.

It is quite easy to see that the vanishing of these terms entails that P can only depend on the variable $s=|x|$ and must be of the form (28).

While the formal computations are straightforward there is the perennial issue of the boundary terms that occur in all the integration by parts. This is due to the fact that one is dealing with solutions of (27) and it is not at all clear that the boundary terms vanish. This requires a detailed regularity analysis of the solutions of (27). The task is non-trivial because the exponent P is critical for the scaling in the s variable. See [150].

The computations outlined above can be carried over to the case where $\mathbb{S}^{d-1}$ is replaced by a compact Riemannian manifold $\mathcal{M}$ of dimension $d-1$. The results are then expressed in terms of the Ricci curvature of the manifold. See [150] for details.

Section (4.3): Symmetry for Extremal Functions

With the norms

$$\|w\|_{L^{q,\gamma}(\mathbb{R}^d)}:=\left(\int_{\mathbb{R}^d}|x|^q\,|x|^{-\gamma}dx\right)^{1/q},\qquad \|w\|_{L^q(\mathbb{R}^d)}:=\|w\|_{L^{q,0}(\mathbb{R}^d)},$$

let us define $L^{q,\gamma}(\mathbb{R}^d)$ as the space of all measurable functions w such that $\|w\|_{L^{q,\gamma}(\mathbb{R}^d)}$ is finite. Our functional framework is a space $H^p_{\beta,\gamma}(\mathbb{R}^d)$ of functions $w\in L^{p+1,\gamma}(\mathbb{R}^d)$ such that $\nabla w\in L^{2,\beta}(\mathbb{R}^d)$, which is defined as the completion of the space $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ of the smooth functions on $\mathbb{R}^d$ with compact support in $\mathbb{R}^d\backslash\{0\}$, with respect to the norm given by $\|w\|^2:=(p_\star-p)\|w\|^2_{L^{p+1,\gamma}(\mathbb{R}^d)}+\|\nabla w\|^2_{L^{2,\beta}(\mathbb{R}^d)}$.

Now consider the family of Caffarelli–Kohn–Nirenberg interpolation inequalities given by

$$\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)}\le C_{\beta,\gamma,p}\|\nabla w\|^{\vartheta}_{L^{2,\beta}(\mathbb{R}^d)}\|w\|^{1-\vartheta}_{L^{p+1,\gamma}(\mathbb{R}^d)}\quad\forall w\in H^p_{\beta,\gamma}(\mathbb{R}^d).\tag{31}$$

Here the parameters β,γ and pare subject to the restrictions

$$d\ge 2,\ \ \gamma-2<\beta<\frac{d-2}{d}\gamma,\ \ \gamma\in(-\infty,d),\ \ p\in(1,p_\star]\ \ \text{with}\ \ p_\star:=\frac{d-\gamma}{d-\beta-2}\tag{32}$$

and the exponent ϑ is determined by the scaling invariance, i.e.,

$$\vartheta=\frac{(d-\gamma)(p-1)}{p\big(d+\beta+2-2\gamma-p(d-\beta-2)\big)}.$$

These inequalities have been introduced, among others, by L. Caffarelli, R. Kohn and L. Nirenberg in [7]. We observe that $\vartheta=1$ if $p=p_\star$, a case that has been dealt with in [150], and we shall focus on the sub-critical case $p<p_\star$. Throughout, $C_{\beta,\gamma,p}$ denotes the

optimal constant in (31). We shall say that a function $w \in H^p_{\beta,\gamma}(\mathbb{R}^d)$ is an extremal function for (31) if equality holds in the inequality.

Symmetry in (31) means that the equality case is achieved by Aubin–Talenti-type functions

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d$$

On the contrary, there is symmetry breaking if this is not the case, because the equality case is then achieved by a non-radial extremal function. It has been proved in [155] that symmetry breaking holds in (31) if

$$\gamma < 0 \quad \text{and} \quad \beta_{FS}(\gamma) < \beta < \frac{d-2}{d}\gamma, \tag{33}$$

where

$$\beta_{FS}(\gamma) := d - 2 - \sqrt{(\gamma - d)^2 - 4(d - 1}.$$

For completeness, we will give a short proof of this result of the set defined by(33), which means that (33) is the sharp condition for symmetry breaking. See Fig (1).

Theorem (4.3.1)[151]: Assume that (32) holds and that

$$\beta \le \beta_{FS}(\gamma) \quad \text{if} \;\; \gamma < 0. \tag{34}$$

Then the extremal functions for (31) are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star$.

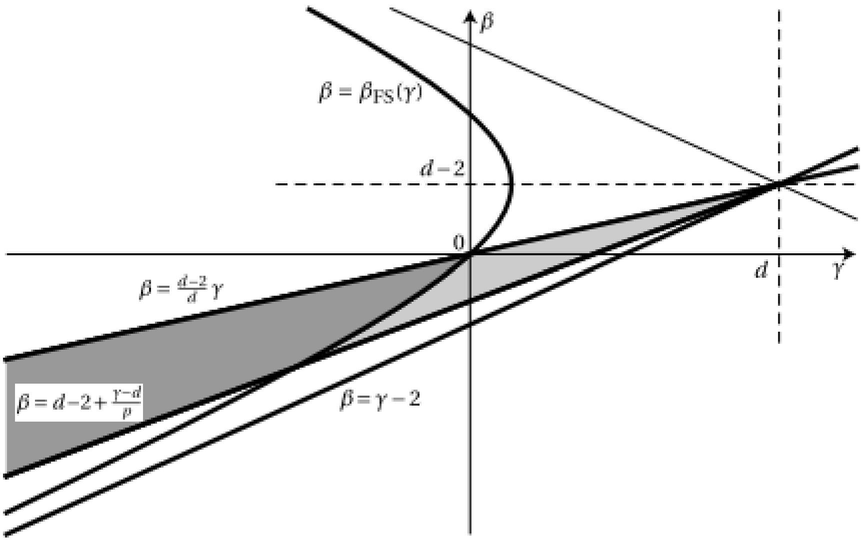


Fig (1)[151]:

In dimension $d = 4$, with $p = 1.2$, the grey area corresponds to the cone determined by $d - 2 + (\gamma - d)/p \le \beta < (d-2)\,\gamma/d$ and $\gamma \in (-\infty, d)$ in (32). The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola $(d-\gamma)^2 - (\beta - d + 2)^2 - 4\,(d-1) = 0$ or, equivalently $\beta = \beta_{FS}(\gamma)$. Notice that the condition $p \le p_\star$ induces the restriction $\beta \ge d - 2 + (\gamma - d)/p$, so that the region of symmetry is bounded. The largest possible cone is achieved as $p \to 1$ and is limited from below by the condition $\beta > \gamma - 2$.

The above result is slightly stronger than just characterizing the range of (β, γ) for which equality in (31) is achieved by radial functions. Actually our method of proof allows us to analyze the symmetry properties not only of extremal functions of (31), but also of all positive solutions in $H^p_{\beta,\gamma}(\mathbb{R}^d)$ of the corresponding Euler–Lagrange equations, that is, up to a multiplication by a constant and a dilation, of

$$-\mathrm{div}\left(|x|^{-\beta}\nabla w\right) = |x|^{-\gamma}(w^{2p-1} - w^{p}) \text{ in } \mathbb{R}^{d}\backslash\{0\}. \tag{35}$$

Corollary (4.3.2)[151]: Assume that α, n and p are such that

$$d \geq 2,\ \alpha > 0,\ n > d \text{ and } p \in (1, p_\star].$$

Then the inequality

$$\|v\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_\alpha v\|^{\vartheta}_{L^{2,d-n}(\mathbb{R}^d)} \|v\|^{1-\vartheta}_{L^{p+1,d-n}(\mathbb{R}^d)} \quad \forall v \in H^{p}_{d-n,d-n}(\mathbb{R}^d), \tag{36}$$

holds with optimal constant $K_{\alpha,n,p} = \alpha^{-\zeta}\, C_{\beta,\gamma,p}$ as above and optimality is achieved among radial functions if and only if

$$\alpha \leq \alpha_{FS} \text{ with } \alpha_{FS} := \sqrt{\frac{d-1}{n-1}}. \tag{37}$$

When symmetry holds, optimal functions are equal, up to a scaling and a multiplication by a constant, to

$$v_\star(x) := (1 + |x|^2)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d.$$

We may notice that neither α_{FS} nor β_{FS} depend on pand that the curve $\alpha = \alpha_{FS}$ determines the same threshold for the symmetry-breaking region as in the critical case $p = p_\star$. In the case $p = p_\star$, this curve was found by V. Felli and M. Schneider, who proved in [142] the linear instability of all radial critical points if $\alpha > \alpha_{FS}$. When $p = p_\star$, symmetry holds under Condition (37) as was proved in [150]. We extend this last result to the subcritical regime $p \in (1, p_\star)$.

The change of variables $s = r^\alpha$ is an important intermediate step, because it allows one to recast the problem as a more standard interpolation inequality in which the dimension n is, however, not necessarily an integer. Actually n plays the role of a dimension in view of the scaling properties of the inequalities and, with respect to this dimension, they are critical if $p = p_\star$ and sub-critical otherwise. The critical case $p = p_\star$ has been studied in [150] using tools of entropy methods, acritical fast diffusion flow and, in particular, a reformulation in terms of a generalized Fisher information. In the subcritical range, we shall replace the entropy by a Rényi entropy power as in [163], [161], and make use of the corresponding fast diffusion flow. As in [150], the flow is used only at the heuristic level in order to produce a well-adapted test function. The core of the method is based on the Bakry–Emery computation, also known as the carré du champ method, which is well adapted to optimal interpolation inequalities: see for instance [153] for a general exposition of the method and [158], [159] for its use in the presence of nonlinear flows. Also see [40] for earlier considerations on the Bakry–Emery method applied to nonlinear flows and related functional inequalities in unbounded domains. However, in non-compact manifolds and in the presence of weights, integrations by parts have to be justified. In the critical case, one can rely on an additional invariance to use an Emden–Fowler transformation and rewrite the problem as an autonomous equation on a cylinder, which simplifies the estimates a lot. Estimates have to be adapted, since after the Emden–Fowler transformation, the problem in the cylinder is no longer autonomous.

We recall the computations that characterize the linear instability of radially symmetric minimizers. We expose the strategy for proving symmetry in the subcritical regime when there are no weights. We devoted to the Bakry–Emery computation applied to Rényi entropy powers, in the presence of weights. This provides a proof of our main results, if we admit that no boundary term appears in the integrations by parts. To prove this last result, regularity and decay estimates of positive solutions to (35) are established, which indeed show that no boundary term has to be taken into account.

For completeness, we summarize known results on symmetry breaking for (31). Details can be found in [155]. With the notations of Corollary (4.3.2), let us define the functional

$$\mathcal{J}[v] := \vartheta \log\left(\|D_\alpha v\|_{L^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\log\left(\|v\|_{L^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathrm{K}_{\alpha,n,p} - \log\left(\|v\|_{L^{2p,d-n}(\mathbb{R}^d)}\right)$$

obtained by taking the difference of the logarithm of the two terms in (36). Let us define $d\mu_\delta := \mu_\delta(x)dx$, where

$$\mu_\delta(x) := \frac{1}{(1+|x|^2)^\delta}.$$

Since $v_\star$ as defined in Corollary (4.3.2) is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|D_\alpha v_\star\|^2_{L^{2,d-n}(\mathbb{R}^d)} \mathcal{J}[v_\star + \varepsilon\, \mu_{\delta/2} f] = \frac{1}{2}\varepsilon^2 \vartheta \mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |D_\alpha f|^2\, |x|^{n-d} d\mu_\delta - \frac{4p\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\, |x|^{n-d} d\mu_{\delta+1}.$$

The following Hardy–Poincaré inequality has been established in [155].

Proposition (4.3.3)[151]: Let $d \geq 2, \alpha \in (0,+\infty), n > d$ and $\delta \geq n$. Then

$$\int_{\mathbb{R}^d} |D_\alpha f|^2\, |x|^{n-d} d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2\, |x|^{n-d} d\mu_{\delta+1} \tag{38}$$

holds for any $f \in L^2(\mathbb{R}^d, |x|^{n-d} d\mu_{\delta+1})$, with $D_\alpha f \in L^2(\mathbb{R}^d, |x|^{n-d} d\mu_\delta)$, such that $\int_{\mathbb{R}^d} f\, |x|^{n-d} d\mu_{\delta+1} = 0$, with an optimal constant Λ given by

$$\Lambda = \begin{cases} 2\alpha^2(2\delta - n) & \text{if} \quad 0 < \alpha^2 \leq \frac{(d-1)\delta^2}{n(2\delta-n)(\delta-1)}, \\ 2\alpha^2\delta\eta & \text{if} \quad \alpha^2 > \frac{(d-1)\delta^2}{n(2\delta-n)(\delta-1)}, \end{cases}$$

where η is the unique positive solution to

$$\eta(\eta + n - 2) = \frac{d-1}{\alpha^2}.$$

Moreover, Λ is achieved by a non-trivial eigenfunction corresponding to the equality in (38). If $\alpha^2 > \frac{(d-1)\delta^2}{n(2\delta-n)(\delta-1)}$, the eigenspace is generated by $\varphi_i(s,\omega) = s^\eta \omega_i$, with $i = 1, 2, \ldots d$ and the eigenfunctions are not radially symmetric, while in the other case the eigenspace is generated by the radially symmetric eigenfunction $\varphi_i(s,\omega) = s^2 - \frac{n}{2\delta - n}$.

As a consequence, $\mathcal{Q}$ is a nonnegative quadratic form if and only if $\frac{4p\alpha^2}{p-1} \leq \Lambda$. Otherwise, $\mathcal{Q}$ takes negative values, and a careful analysis shows that symmetry breaking occurs in (31) if

$$2\alpha^2\delta\eta < \frac{4p\alpha^2}{p-1} \quad \Leftrightarrow \quad \eta < 1,$$

which means

$$\frac{d-1}{\alpha^2} = \eta(\eta + n - 2) < n - 1,$$

and this is equivalent to $\alpha > \alpha_{FS}$.

Before going into the details of the proof, we explain the strategy for the case of the Gagliardo–Nirenberg inequalities without weights. There are several ways to compute the optimizers, and see [29], [41], [156], [40], [153], [161]. The inequality is of the form

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le C_{0,0,p} \|\nabla w\|^{\vartheta}_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{L^{p+1}(\mathbb{R}^d)} \quad \text{with} \quad 1 < p < \frac{d}{d-2} \tag{39}$$

and

$$\vartheta = \frac{d(p-1)}{p\big(d+2-p(d-2)\big)}.$$

It is known through the work in [29] that the optimizers of this inequality are, up to multiplications by a constant, scalings and translations, given by

$$w_\star(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d.$$

The idea is to use a version of the carré du champ or Bakry–Emery method introduced in [152]: by differentiating a relevant quantity along the flow, we recover the inequality in a form that turns out to be sharp. The version of the carré du champ we shall use is based on the Rényi entropy powers whose concavity as a function of t has been studied by M. Costa in [157] in the case of linear diffusions (see [163]). In [165], C. Villani observed that the carré du champ method gives a proof of the logarithmic Sobolev inequality in the Blachman–Stam form, also known as the Weissler form: see[154], [70]. G. Savaré and G. Toscani observed in [163] that the concavity also holds in the nonlinear case, which has been used in [161] to give an alternative proof of the Gagliardo–Nirenberg inequalities, that we are now going to sketch.

The first step consists in reformulating the inequality in new variables. We set

$$u = w^{2p}.$$

which is equivalent to $w = u^{m-1/2}$, and consider the flow given by

$$\frac{\partial u}{\partial t} = \Delta u^m, \tag{40}$$

where m is related to p by

$$p = \frac{1}{2m-1}.$$

The inequalities $1 < p < \frac{d}{d-2}$ imply that

$$1 - \frac{1}{d} < m < 1. \tag{41}$$

For some positive constant $\kappa > 0$, one easily finds that the so-called Barenblatt–Pattle functions

$$u_\star(t,x) = \kappa^d t^{-\frac{d}{d\,m-d+2}} w_\star^{2p}\left(\kappa t^{-\frac{1}{d\,m-d+2}} x\right) = (a+b|x|^2)^{-\frac{1}{1-m}}$$

are self-similar solutions to (40), where $a = a(t)$ and $b = b(t)$ are explicit. Thus, we see that $w_\star = u_\star^{m-1/2}$ is an optimizer for (39) for all t and it makes sense to rewrite (39)in terms of the function u. Straightforward computations show that (39) can be brought into the form

$$\left(\int_{\mathbb{R}^d} u\,dx\right)^{(\sigma+1)\,m-1} \le C E^{\sigma-1} I \quad \text{where} \quad \sigma = \frac{2}{d(1-m)} - 1 \tag{42}$$

for some constant C which does not depend on u, where

$E := \int_{\mathbb{R}^d} u^m \, dx$ is a generalized Ralston–Newman entropy, also known in the literature as Tsallis entropy, and

$$I := \int_{\mathbb{R}^d} u \, |\nabla P|^2 dx$$

is the corresponding generalized Fisher information. Here we have introduced the pressure variable

$$P = \frac{m}{1-m} u^{m-1}.$$

The Rényi entropy power is defined by

$$F = E^{\sigma}$$

as in [163], [161]. With the above choice of σ, F is an affine function of t if $u = u_\star$. For an arbitrary solution to (40), we aim at proving that it is a concave function of t and that it is affine if and only if $u = u_\star$. For further references on related issues, see [29], [164]. Note that one of the motivations for choosing the variable P is that it has a particular simple form for the self-similar solutions, namely

$$P_\star = \frac{m}{1-m}(a + b|x|^2).$$

Differentiating E along the flow (40) yields

$$E' = (1-m) I,$$

so that

$$F' = \sigma(1-m) G \quad \text{with} \quad G = E^{\sigma-1} I.$$

More complicated is the derivative for the Fisher information:

$$I' = -2 \int_{\mathbb{R}^d} u^m \left[\mathrm{Tr}\left(\left(\mathrm{Hess}\, P - \tfrac{1}{d} \Delta P \, \mathrm{Id} \right)^2 \right) + \left(m - 1 + \tfrac{1}{d} \right) (\Delta P)^2 \right] dx.$$

Here Hess P and Id are respectively the Hessian of P and the (d × d) identity matrix. The computation can be found in [161]. Next we compute the second derivative of the Rényi entropy power F with respect to t:

$$\frac{(F)''}{\sigma E^{\sigma}} = (\sigma - 1) \frac{E'^2}{E^2} + \frac{E''}{E^2} = (\sigma - 1)(1-m)^2 \frac{I^2}{E^2} + (1-m) \frac{I'}{E} =: (1-m) \mathcal{H}.$$

With $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$, we obtain

$$\mathcal{H} = -2 \left\langle \mathrm{Tr}\left(\left(\mathrm{Hess}\, P - \tfrac{1}{d} \Delta P \, \mathrm{Id} \right)^2 \right) \right\rangle + (1-m)(1-\sigma) \langle (\Delta P - \langle \Delta P \rangle)^2 \rangle, \tag{43}$$

where we have used the notation

$$\langle A \rangle := \frac{\int_{\mathbb{R}^d} u^m \, A \, dx}{\int_{\mathbb{R}^d} u^m \, dx}.$$

Note that by (41), we have that $\sigma > 1$ and hence we find that $F'' = (E^{\sigma})'' \leq 0$, which also means that $G = E^{\sigma-1} I$ is a non-increasing function. In fact it is strictly decreasing unless P is a polynomial function of order two in x and it is easy to see that the expression (43) vanishes precisely when P is of the form $a + b\,|x - x_0|^2$, where $a, b \in \mathbb{R}, x_0 \in \mathbb{R}^d$ are constants (but a and b may still depend on t).

Thus, while the left side of (42) stays constant along the flow, the right side decreases. In [161] it was shown that the right side decreases towards the value given by the self-similar solutions $u_\star$ and hence proves (39) in the sharp form. The variational equation for the optimizers of (39) is given by

$$-\Delta w = a\, w^{2\,p-1} - b w^p.$$

A straightforward computation shows that this can be written in the form

$$2m\, u^{m-2}\ \operatorname{div}(u\nabla P) + |\nabla P|^2 + c_1 u^{m-1} = c_2$$

for some constants c_1, c_2 whose precise values are explicit. This equation can also be interpreted as the variational equation for the sharp constant in (42). Hence, multiplying the above equation by $\Delta u^m$and integrating yields

$$\int_{\mathbb{R}^d} [2mu^{m-2} \operatorname{div}(u\nabla P) + |\nabla P|^2\,]\,\Delta u^m dx + c_1 \int_{\mathbb{R}^d} u^{m-1}\,\Delta u^m\, dx = c_2 \int_{\mathbb{R}^d} \Delta u^m\, dx = 0.$$

We recover the fact that, in the flow picture, $\mathcal{H}$ is, up to a positive factor, the derivative of $\mathcal{G}$ and hence vanishes. From the observations made above, we conclude that P must be a polynomial function of order two in x. In this fashion, one obtains more than just the optimizers, namely a classification of all positive solutions to the variational equation. The main technical problem with this method is the justification of the integrations by parts, which in the case at hand, without any weight, does not offer great difficulties: see, for instance, [40]. This strategy can also be used to treat the problem with weights, which will be explained next. Dealing with weights, however, requires some special care, as we shall see.

Let us adapt the above strategy to the case where there are weights in all integrals entering into the inequality, that is, let us deal with inequality (36) instead of inequality (39). In order to define a new, well-adapted fast diffusion flow, we introduce the diffusion operator $L_\alpha := -\,D_\alpha^* D_\alpha$, which is given in spherical coordinates by

$$L_\alpha u = \alpha^2 \left(u'' + \frac{n-1}{s} u' \right) + \frac{1}{s^2} \Delta_\omega u,$$

where Δ_ω denotes the Laplace–Betrami operator acting on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ of the angular variables, and denotes here the derivative with respect to s. Consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = L_\alpha u^m \tag{44}$$

in the subcritical range $1 - \frac{1}{n} < m = 1 - \frac{1}{\nu} < 1$. The exponents m in (44) and p in (36) are related by

$$p = \frac{1}{2m-1} \quad \Leftrightarrow \quad m = \frac{p+1}{2p}$$

and ν is defined by

$$\nu := \frac{1}{1-m}.$$

We consider the Fisher information defined as

$$I[P] := \int_{\mathbb{R}^d} u|\mathrm{D}_\alpha P|^2 d\mu \quad \text{with} \quad P = \frac{m}{1-m} u^{m-1} \quad \text{and} \quad d\mu = s^{n-1} ds d\omega = s^{n-d} dx.$$

Here P is the pressure variable. Our goal is to prove that P takes the form $+bs^2$. It is useful to observe that (44) can be rewritten as

$$\frac{\partial u}{\partial t} = \mathrm{D}_\alpha^* (u \mathrm{D}_\alpha P)$$

and, in order to compute $\frac{dI}{dt}$, we will also use the fact that P solves

First step: computation of $\frac{dI}{dt}$

Let us define

$$K[P]:=A[P]-(1-m)(L_\alpha\, P)^2 \quad \text{where} \quad A[P]\coloneqq \tfrac{1}{2}L_\alpha|\mathrm{D}_\alpha P|^2-\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha L_\alpha\, P$$

and, on the boundary of the centered ball B_s of radius s, the boundary term

$$b(s):=\int_{\partial B_s}\left(\frac{\partial}{\partial s}\left(P^{\frac{m}{m-1}}|\mathrm{D}_\alpha P|^2\right)-2(1-m)P^{\frac{m}{m-1}}P' \;\; L_\alpha\, P\right)d\varsigma$$
$$=s^{n-1}\left(\int_{\mathbb{S}^{d-1}}\left(\frac{\partial}{\partial s}\left(P^{\frac{m}{m-1}}|\mathrm{D}_\alpha P|^2\right)-2(1-m)P^{\frac{m}{m-1}}P' \;\; L_\alpha\, P\right)d\omega\right)(s), \tag{46}$$

where by $d\varsigma=s^{n-1}\,d\omega$ we denote the standard Hausdorff measure on ∂B_s.

Lemma (4.3.4)[151]: If u solves (44) and if

$$\lim_{s\to 0_+} b(s)=\lim_{S\to+\infty} b(S)=0, \tag{47}$$

then ,

$$\frac{d}{dt}I[P]=-2\int_{\mathbb{R}^d}K[P]u^m d\mu. \tag{48}$$

Proof: For $0<s<S<+\infty$, let us consider the set $A_{(s,S)}:=\{x\in\mathbb{R}^d: s<|x|<S\}$, so that $\partial A_{(s,S)}=\partial B_s\cup\partial B_S$. Using (44) and (45), we can compute

$$\frac{d}{dt}\int_{A_{(s,S)}}u|\mathrm{D}_\alpha P|^2 d\mu$$
$$=\int_{A_{(s,S)}}\frac{\partial u}{\partial t}|\mathrm{D}_\alpha P|^2 d\mu+2\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha\frac{\partial P}{\partial t}d\mu$$
$$=\int_{A_{(s,S)}}L_\alpha(u^m)|\mathrm{D}_\alpha P|^2 d\mu+2\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha\big((1-\mathrm{m})PL_\alpha P-|\mathrm{D}_\alpha P|^2\big)d\mu$$
$$=\int_{A_{(s,S)}}u^m L_\alpha|\mathrm{D}_\alpha P|^2 d\mu+2\,(1-\mathrm{m})\int_{A_{(s,S)}}uP\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha L_\alpha P d\mu$$
$$+2(1-\mathrm{m})\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha P L_\alpha P d\mu-2\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha|\mathrm{D}_\alpha P|^2 d\mu$$
$$+\alpha^2\int_{\partial B_s}\left((u^m)'|\mathrm{D}_\alpha P|^2-u^m\tfrac{\partial}{\partial s}(|\mathrm{D}_\alpha P|^2)\right)d\varsigma-\alpha^2\int_{\partial B_s}\left((u^m)'|\mathrm{D}_\alpha P|^2-u^m\tfrac{\partial}{\partial s}(|\mathrm{D}_\alpha P|^2)\right)d\varsigma$$
$$=-\int_{A_{(s,S)}}u^m L_\alpha|\mathrm{D}_\alpha P|^2 d\mu+2\,(1-\mathrm{m})\int_{A_{(s,S)}}uP\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha L_\alpha P d\mu+2(1-\mathrm{m})\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha P L_\alpha P d\mu$$
$$+\alpha^2\int_{\partial B_s}\left((u^m)'|\mathrm{D}_\alpha P|^2+u^m\tfrac{\partial}{\partial s}(|\mathrm{D}_\alpha P|^2)\right)d\varsigma-\alpha^2\int_{\partial B_s}\left((u^m)'|\mathrm{D}_\alpha P|^2+u^m\tfrac{\partial}{\partial s}(|\mathrm{D}_\alpha P|^2)\right)d\varsigma,$$

where the last line is given by an integration by parts, upon exploiting the identity $u\,\mathrm{D}_\alpha P=-\mathrm{D}_\alpha((u^m)$:

$$\int_{A_{(s,S)}}u\mathrm{D}_\alpha P\cdot \mathrm{D}_\alpha|\mathrm{D}_\alpha P|^2 d\mu=-\int_{A_{(s,S)}}\mathrm{D}_\alpha(u^m)\cdot \mathrm{D}_\alpha|\mathrm{D}_\alpha P|^2 d\mu$$
$$=\int_{A_{(s,S)}}u^m L_\alpha|\mathrm{D}_\alpha P|^2 d\mu-\alpha^2\int_{\partial B_s}u^m\tfrac{\partial}{\partial s}L_\alpha(|\mathrm{D}_\alpha P|^2)d\varsigma$$

$$+\alpha^2 \int_{\partial B_s} u^m \frac{\partial}{\partial s}(|\mathrm{D}_\alpha P|^2) d\varsigma.$$

i) Using the definition of A $[P]$, we get that

$$= -\int_{A(s,S)} u^m L_\alpha |\mathrm{D}_\alpha P|^2 d\mu = -2\int_{A(s,S)} u^m \mathrm{A}[P] d\mu - 2\int_{A(s,S)} u^m \mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha L_\alpha P d\mu. \quad (49)$$

ii) Taking advantage again of $u\,\mathrm{D}_\alpha P = -\mathrm{D}_\alpha((u^m)$, an integration by parts gives

$$\int_{A(s,S)} u\,\mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha P L_\alpha P d\mu = -\int_{A(s,S)} \mathrm{D}_\alpha(u^m) \cdot \mathrm{D}_\alpha P L_\alpha P d\mu$$

$$= \int_{A(s,S)} u^m (L_\alpha P)^2 d\mu + \int_{A(s,S)} u^m \mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha L_\alpha P d\mu$$

$$-\alpha^2 \int_{\partial B_s} u^m \acute{P}\; L_\alpha P d\varsigma + \alpha^2 \int_{\partial B_s} u^m \acute{P}\; L_\alpha P d\varsigma$$

and, with $u\,P = \frac{m}{1-m} u^m$, we find that

$$2(1-m)\int_{A(s,S)} u\,P\mathrm{D}_\alpha \cdot \mathrm{D}_\alpha P L_\alpha P d\mu + 2(1-m)\int_{A(s,S)} u\,\mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha P L_\alpha P d\mu$$

$$= 2(1-m)\int_{A(s,S)} u^m (L_\alpha P)^2 d\mu + 2\int_{A(s,S)} u^m\,\mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha L_\alpha P d\mu$$

$$-2(1-m)\alpha^2 \int_{\partial B_s} u^m \acute{P}\; L_\alpha P d\varsigma + 2(1-m)\alpha^2 \int_{\partial B_s} u^m \acute{P}\; L_\alpha P d\varsigma. \quad (50)$$

Summing (49) and (50), using (46) and passing to the limits as $s \to 0_+, S \to +\infty$, establishes (48).

Let us define

$$k[P] := \frac{1}{2}\Delta_\omega |\nabla_\omega P|^2 - \nabla_\omega P \cdot \nabla_\omega \Delta_\omega P - \frac{1}{n-1}(\Delta_\omega P)^2 - (n-2)\alpha^2 |\nabla_\omega P|^2$$

and

$$R[P] := K[P] - \left(\frac{1}{n} - (1-m)\right)(L_\alpha P)^2.$$

We observe that

$$R[P] := \frac{1}{2} L_\alpha |\mathrm{D}_\alpha P|^2 - \mathrm{D}_\alpha P \cdot \mathrm{D}_\alpha L_\alpha P - \frac{1}{n}(L_\alpha P)^2$$

is independent of m. We recall the result of [150] and give its proof for completeness.

Lemma (4.3.5)[151]: Let $d \in \mathbb{N}, n \in \mathbb{R}$ such that $n > d \geq 2$, and consider a function $P \in C^3(\mathbb{R}^d \backslash \{0\})$. Then,

$$R[P] = \alpha^4 \left(1 - \frac{1}{n}\right)\left[\acute{\acute{P}} - \frac{\acute{P}}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2}\right]^2 + \frac{2\alpha^2}{s^2}\left|\nabla_\omega \acute{P} - \frac{\nabla_\omega P}{s}\right|^2 + \frac{k[P]}{s^4}.$$

Proof: By definition of $k[P]$, we have

$$R[P] = \frac{\alpha^2}{2}\left[\alpha^2 \acute{P}^2 + \frac{|\nabla_\omega P|^2}{s^2}\right]'' + \frac{\alpha^2}{2}\frac{n-1}{s}\left[\alpha^2 \acute{P}^2 + \frac{|\nabla_\omega P|^2}{s^2}\right]'$$

$$+ \frac{1}{2s^2}\Delta_\omega\left[\alpha^2 \acute{P}^2 + \frac{|\nabla_\omega P|^2}{s^2}\right] - \alpha^2 \acute{P}\left(\alpha^2 \acute{\acute{P}} + \alpha^2 \frac{n-1}{s}\acute{P} + \frac{\Delta_\omega P}{s^2}\right)'$$

$$-\frac{1}{s^2}\nabla_\omega P\cdot\nabla_\omega\left(\alpha^2 P''+\alpha^2\frac{n-1}{s}P'+\frac{\Delta_\omega P}{s^2}\right)-\frac{1}{n}\left(\alpha^2 P''+\alpha^2\frac{n-1}{s}P'+\frac{\Delta_\omega P}{s^2}\right)^2,$$

which can be expanded as

$$\begin{aligned}\mathcal{R}[P]= &\frac{\alpha^2}{2}\left[2\alpha^2 P''^2+2\alpha^2 P'P'''+2\frac{|\nabla_\omega P'|^2+\nabla_\omega P\cdot\nabla_\omega P''}{s^2}-8\frac{\nabla_\omega P\cdot\nabla_\omega P'}{s^3}+6\frac{|\nabla_\omega P|^2}{s^4}\right]\\ &+\alpha^2\frac{n-1}{s}\left[\alpha^2 P'P''+\frac{\nabla_\omega P\cdot\nabla_\omega P'}{s^2}-\frac{|\nabla_\omega P|^2}{s^3}\right]+\frac{1}{s^2}\left[\alpha^2 P'\Delta_\omega P'+\alpha^2|\nabla_\omega P'|^2+\frac{\Delta_\omega|\nabla_\omega P|^2}{2s^2}\right]\\ &-\alpha^2 P'\left(\alpha^2 P'''+\alpha^2\frac{n-1}{s}P''-\alpha^2\frac{n-1}{s}P'-2\frac{\Delta_\omega P}{s^3}+\frac{\Delta_\omega P'}{s^2}\right)\\ &-\frac{1}{s^2}\left(\alpha^2\nabla_\omega P\cdot\nabla_\omega P''+\alpha^2\frac{n-1}{s}\nabla_\omega P\cdot\nabla_\omega P'+\frac{\nabla_\omega P\cdot\nabla_\omega\Delta_\omega P}{s^2}\right)\\ &-\frac{1}{n}\left[\alpha^4 P''^2+\alpha^4\frac{(n-1)^2}{s^2}P'^2+\frac{(\Delta_\omega P)^2}{s^4}+2\alpha^4\frac{n-1}{s}P'P''+2\alpha^2\frac{P''\Delta_\omega P}{s^2}+2\alpha^2\frac{n-1}{s^3}P'\Delta_\omega P\right].\end{aligned}$$

Collecting terms proves the result.

Now let us study the quantity $k[P]$ which appears in the statement of Lemma (4.3.5). The following computations are adapted from [158] and [150]. For completeness, we give a simplified proof in the special case of the sphere $(\mathbb{S}^{d-1},g)$ considered as a Riemannian manifold with standard metric g. We denote by Hf the Hessianoff, which is seen as a $(d-1)\times(d-1)$ matrix, identify its trace with the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ and use the notation $\|A\|^2:=A:A$ for the sum of the squares of the coefficients of the matrix A. It is convenient to define the trace free Hessian, the tensor Zf and its trace free counterpart respectively by

$$Lf:=Hf-\frac{1}{d-1}(\Delta_\omega f)g,\quad Zf:=\frac{\nabla_\omega f\otimes\nabla_\omega}{f}\quad and\ Mf:=Zf-\frac{1}{d-1}\frac{|\nabla_\omega f|^2}{f}g$$

whenever $f\neq 0$. Elementary computations show that

$$\|Lf\|^2=\|Hf\|^2-\frac{1}{d-1}(\Delta_\omega f)^2\ and\ \|Mf\|^2=\|Zf\|^2-\frac{1}{d-1}\frac{|\nabla_\omega f|^4}{f^2}=\frac{d-2}{d-1}\frac{|\nabla_\omega f|^4}{f^2}. \qquad (51)$$

The Bochner–Lichnerowicz–Weitzenböck formula on $\mathbb{S}^{d-1}$ takes the simple form

$$\frac{1}{2}\Delta_\omega(|\nabla_\omega f|^2)=\|Hf\|^2+\nabla_\omega(\Delta_\omega f)\cdot\nabla_\omega f+(d-2)|\nabla_\omega f|^2 \qquad (52)$$

where the last term, *i.e.* $\mathrm{Ric}(\nabla_\omega f,\nabla_\omega f)=(d-2)|\nabla_\omega f|^2$, accounts for the Ricci curvature tensor contracted with $\nabla_\omega f\otimes\nabla_\omega f$.

We recall that $\alpha_{FS}:=\sqrt{\frac{d-1}{n-1}}$ and $\nu=1/(1-m)$. Let us introduce the notations

$$\delta:=\frac{1}{d-1}-\frac{1}{n-1}$$

and

$$B[P]:=\int_{\mathbb{S}^{d-1}}\left(\frac{1}{2}\Delta_\omega(|\nabla_\omega P|^2)-\nabla_\omega(\Delta_\omega P)\cdot\nabla_\omega P-\frac{1}{n}(\Delta_\omega P)^2\right)P^{1-\nu}d\omega,$$

so that

$$\int_{\mathbb{S}^{d-1}}K[P]P^{1-\nu}d\omega=B[P]-(n-2)\alpha^2\int_{\mathbb{S}^{d-1}}|\nabla_\omega P^2|\,P^{1-\nu}d\omega.$$

Lemma (4.3.6)[151]: Assume that $d\geq 2$ and $1/(1-m)=\nu>n>d$. There exists a positive constant $c(n,m,d)$ such that, for any positive function $P\in C^3(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} K[P]P^{1-\nu}d\omega \geq (n-2)(\alpha_{FS}^2-\alpha^2)\int_{\mathbb{S}^{d-1}} |\Delta_\omega P|^2\, P^{1-\nu}d\omega + c(n,m,d)\int_{\mathbb{S}^{d-1}} \frac{|\nabla_\omega P|^4}{P^2}P^{1-\nu}d\omega.$$

Proof: If $d=2$, we identify $\mathbb{S}^1$ with $[0,2\pi)\ni\theta$ and denote by P_θ and $P_{\theta\theta}$ the first and second derivatives of P with respect to θ. As in [150], a direct computation shows that

$$K[P] = \frac{n-2}{n-1}|P_{\theta\theta}|^2-(n-2)\alpha^2|P_\theta|^2=(n-2)\left(\alpha_{FS}^2|P_{\theta\theta}|^2-\alpha^2|P_\theta|^2\right).$$

By the Poincare inequality, we have

$$\int_{\mathbb{S}^1}\left|\frac{\partial}{\partial\theta}(P^{\frac{1-\nu}{2}}P_\theta)\right|^2 d\theta \geq \int_{\mathbb{S}^1}\left|(P^{\frac{1-\nu}{2}}P_\theta)\right|^2 d\theta.$$

On the other hand, an integration by parts shows that

$$\int_{\mathbb{S}^1} P_{\theta\theta}\frac{|P_\theta|^2}{P}P^{1-\nu}d\theta = \frac{1}{3}\int_{\mathbb{S}^1}\frac{\partial}{\partial\theta}(|P_\theta|^2P_\theta)\,P^{-\nu}d\theta = \frac{\nu}{3}\int_{\mathbb{S}^1}\frac{|P_\theta|^4}{P^2}P^{1-\nu}d\theta$$

and, as a consequence, by expanding the square, we obtain

$$\int_{\mathbb{S}^1}\left|\frac{\partial}{\partial\theta}\left(P^{\frac{1-\nu}{2}}P_\theta\right)\right|^2 d\theta = \int_{\mathbb{S}^1}\left|P_{\theta\theta}+\frac{1-\nu}{2}\frac{|P_\theta|^2}{P}\right|^2 P^{1-\nu}d\theta = \int_{\mathbb{S}^1}|P_{\theta\theta}|^2\,P^{1-\nu}d\theta$$
$$-\frac{(\nu-1)(\nu+3)}{12}\int_{\mathbb{S}^1}\frac{|P_\theta|^4}{P^2}P^{1-\nu}d\theta.$$

The result follows with $c(n,m,2)=\frac{n-2}{n-1}\,\frac{1}{12}(\nu-1)(\nu+3)=\frac{n-2}{n-1}\frac{m\,(4-3m)}{12\,(1-m)^2}$ from

$$\int_{\mathbb{S}^1}|P_{\theta\theta}|^2\,P^{1-\nu}d\theta \geq \int_{\mathbb{S}^1}|P_\theta|^2\,P^{1-\nu}d\theta+\frac{(\nu-1)(\nu+3)}{12}\int_{\mathbb{S}^1}\frac{|P_\theta|^4}{P^2}P^{1-\nu}d\theta.$$

Assume next that $d\geq 3$. We follow the method of [150]. Applying (52) with $f=P$ and multiplying by $P^{1-\nu}$ yields, after an integration on $\mathbb{S}^{d-1}$, that $B[P]$ can also be written as

$$B[P] = \int_{\mathbb{S}^{d-1}}\left(\|HP\|^2+(d-2)|\nabla_\omega P|^2-\tfrac{1}{n-1}(\Delta_\omega P)^2\right)P^{1-\nu}\,d\omega$$

.

We recall that $n>d\geq 3$ and set $P=f^\beta$ with $\beta=\frac{2}{3-\nu}$. A straightforward computation shows that $Hf^\beta=\beta f^{\beta-1}(Hf+(\beta-1)Zf)$ and hence

$$B[P]=\beta^2\int_{\mathbb{S}^{d-1}}\left(\|Hf+(\beta-1)Zf\|^2+(d-2)|\nabla_\omega f|^2-\tfrac{1}{n-1}\left(Tr(Hf+(\beta-1)Zf)\right)^2\right)d\omega$$
$$=\beta^2\int_{\mathbb{S}^{d-1}}\left(\|Lf+(\beta-1)Mf\|^2+(d-2)|\nabla_\omega f|^2+\delta\left(Tr(Hf+(\beta-1)Zf)\right)^2\right)d\omega.$$

Using (51), we deduce from

$$\int_{\mathbb{S}^{d-1}}\Delta_\omega f\,\frac{|\nabla_\omega f|^2}{f}d\omega = \int_{\mathbb{S}^{d-1}}\frac{|\nabla_\omega f|^4}{f^2}d\omega - 2\int_{\mathbb{S}^{d-1}}Hf{:}Zf\,d\omega$$

$$= \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} \|Mf\|^2 \, d\omega - 2 \int_{\mathbb{S}^{d-1}} Lf : Zf d\omega - \frac{2}{d-1} \int_{\mathbb{S}^{d-1}} \Delta_\omega \, f \frac{|\nabla_\omega f|^2}{f} d\omega$$

that

$$\int_{\mathbb{S}^{d-1}} \Delta_\omega \, f \frac{|\nabla_\omega f|^2}{f} d\omega = \frac{d-1}{d+1} \left[\frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} \|Mf\|^2 \, d\omega - 2 \int_{\mathbb{S}^{d-1}} Lf : Zf d\omega \right]$$

$$\frac{d-1}{d+1} \left[\int_{\mathbb{S}^{d-1}} \frac{d-1}{d-2} \|Mf\|^2 \, d\omega - 2 \int_{\mathbb{S}^{d-1}} Lf : Mf d\omega \right]$$

on the one hand, and from (52) integrated on $\mathbb{S}^{d-1}$ that

$$\int_{\mathbb{S}^{d-1}} (\Delta_\omega f)^2 d\omega = \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} \|Lf\|^2 \; d\omega + (d-1) \int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2 d\omega$$

on the other hand. Hence we find that

$$\int_{\mathbb{S}^{d-1}} (Tr(Hf + (\beta-1)Zf))^2 d\omega = \int_{\mathbb{S}^{d-1}} \left((\Delta_\omega f)^2 + 2(\beta-1)\Delta_\omega \, f \frac{|\nabla_\omega f|^2}{f} + (\beta-1)^2 \frac{|\nabla_\omega f|^4}{f^2} \right) d\omega$$

$$= \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} \|Lf\|^2 \, d\omega + (d-1) \int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2 d\omega$$

$$+2(\beta-1)\frac{d-1}{d+1} \left[\int_{\mathbb{S}^{d-1}} \frac{d-1}{d-2} \|Mf\|^2 \, d\omega - 2 \int_{\mathbb{S}^{d-1}} Lf : Mf d\omega \right]$$

$$+(\beta-1)^2 \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} \|Mf\|^2 \, d\omega.$$

Altogether, we obtain

$$\mathrm{B}[P] = \beta^2 \int_{\mathbb{S}^{d-1}} (a\|Lf\|^2 + 2bLf : Mf + c\|Mf\|^2) d\omega + \beta^2 \big(d - 2 + \delta(d-1)\big) \int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2 \, d\omega,$$

where

$$a = 1 + \delta\frac{d-1}{d-2}, \;\; b = (\beta-1)\left(1 - 2\delta\frac{d-1}{d+1}\right) \;\; and\ c = (\beta-1)^2 \left(1 + \delta\frac{d-1}{d-2}\right) + 2(\beta-1)\frac{\delta(d-1)^2}{(d+1)(d-2)}.$$

A tedious but elementary computation shows that

$$\mathrm{B}[P] = a\beta^2 \int_{\mathbb{S}^{d-1}} \left\| Lf + \frac{b}{a} Mf \right\|^2 d\omega (c - \frac{b^2}{a}) \, \beta^2 \int_{\mathbb{S}^{d-1}} \|Mf\|^2 d\omega + \; \beta^2 (n-1)\alpha_{FS}^2 + \int_{\mathbb{S}^{d-1}} |\nabla_\omega f|^2 \, d\omega$$

can be written in terms of P as

$$\mathrm{B}[P] = \int_{\mathbb{S}^{d-1}} \mathrm{Q}\,[P] P^{1-\nu} d\omega + (n-1)\alpha_{FS}^2 \int_{\mathbb{S}^{d-1}} |\nabla_\omega P|^2 \, P^{1-\nu} d\omega,$$

where

$$Q[P] := \alpha_{FS}^2 \frac{n-2}{d-2} \left\| LP + \frac{3(\nu-1)(n-d)}{(d+1)(n-2)(\nu-3)} MP \right\|^2 + \frac{(d-1)(\nu-1)(n-d)[((4d-5)\,n+d-8)\,\nu+9(n-d))]}{(d-2)(d+1)^2(\nu-3)^2\,(n-2)(n-1)} \|MP\|^2$$

is positive definite. This concludes the proof in the case $d \geq 3$ with $c(n,m,d) = \frac{m(n-d)[4(d+1)(n-2)-9m(n-d)]}{(d+1)^2(3m-2)^2(n-2)(n-1)}$.

Let us recall that

$$K[P] = R[P] + \left(\frac{1}{n} - (1-m)\right)(\mathcal{L}_\alpha P)^2.$$

We can collect the two results of Lemmas (4.3.5) and (4.3.6) as follows.

Corollary (4.3.7) [151]:. Let $d \in \mathbb{N}, n \in \mathbb{R}$ be such that $n > d \geq 2$, and consider a positive function $P \in C^3(\mathbb{R}^d \setminus \{0\})$. If u is related to P by $P = \frac{m}{1-m} u^{m-1}$ for some $m \in (1-\frac{1}{n}, 1)$, then there exists a positive constant $c(n,m,d)$ such that

$$\int_{\mathbb{R}^d} R[P] u^m \, d\mu \geq \alpha^4 \left(1 - \frac{1}{n}\right) \int_{\mathbb{R}^d} \left[\acute{\acute{P}} - \frac{\acute{P}}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2}\right]^2 u^m d\mu + 2\alpha^2 \int_{\mathbb{R}^d} \frac{1}{s^2} \left|\nabla_\omega \acute{P} - \frac{\nabla_\omega P}{s}\right|^2 u^m d\mu$$
$$+ (n-2)(\alpha_{FS}^2 - \alpha^2) \int_{\mathbb{R}^d} \frac{1}{s^4} |\nabla_\omega P|^2 u^m \, d\mu + c(n,m,d) \int_{\mathbb{R}^d} \frac{1}{s^4} \left|\frac{\nabla_\omega P}{P^2}\right|^2 u^m \, d\mu.$$

We keep investigating the properties of the flow defined by(44). Let us define the entropy as

$$E := \int_{\mathbb{R}^d} u^m \, d\mu$$

and observe that

$$\acute{E} = (1-m) I$$

if u solves(44), after integrating by parts. The fact that boundary terms do not contribute, i.e.

$$\lim_{s \to 0_+} \int_{\partial B_s} u^m \acute{P} d\varsigma = \lim_{S \to +\infty} \int_{\partial B_S} u^m \acute{P} d\varsigma = 0 \tag{53}$$

will be justified: see Proposition (4.3.12). Note that we use both for derivation $w.r.t.$ t and $w.r.t.$ s, at least when this does not create any ambiguity. We introduce the Renyi entropy power

$$F := E^\sigma$$

for some exponent σ to be chosen later, and find that $\acute{F} = \sigma(1-m)\, G$ where $G := E^{\sigma-1} I$. With $H := E^{-\sigma} \acute{G}$, by using Lemma (4.3.4), we also find that $E^{-\sigma} \acute{\acute{F}} = \sigma(1-m)\, H$ where.

$$E^2 H = E^{2-\sigma} \acute{G} = \frac{1}{\sigma(1-m)} E^{2-\sigma} \acute{\acute{G}} = (1-m)(\sigma-1)\left(\int_{\mathbb{R}^d} u |D_\alpha P|^2 d\mu\right)^2 - 2 \int_{\mathbb{R}^d} u^m d\mu \int_{\mathbb{R}^d} K[P] u^m d\mu$$
$$= (1-m)(\sigma-1)\left(\int_{\mathbb{R}^d} u |D_\alpha P|^2 d\mu\right)^2 - 2\left(\frac{1}{n} - (1-m)\right) \int_{\mathbb{R}^d} u^m d\mu \int_{\mathbb{R}^d} (\mathcal{L}_\alpha P)^2 u^m d\mu$$
$$-2 \int_{\mathbb{R}^d} u^m d\mu \int_{\mathbb{R}^d} R[P] u^m d\mu$$

if $\lim_{s\to 0_+} b(s) = \lim_{S\to+\infty} b(S) = 0$. Using $u\, D_\alpha P = -D_\alpha(u^m)$, we know that

$$\int_{\mathbb{R}^d} u|D_\alpha P|^2 d\mu \;= -\int_{\mathbb{R}^d} D_\alpha(u^m)\cdot D_\alpha P d\mu = \int_{\mathbb{R}^d} u^m \mathcal{L}_\alpha P d\mu$$

and so, with the choice

$$\sigma = \frac{2}{n}\;\frac{1}{1-m} - 1,$$

we may argue as and get that

$$E^2 H + (1-m)(\sigma-1)E\int_{\mathbb{R}^d} u^m \left|\mathcal{L}_\alpha P - \frac{\int_{\mathbb{R}^d} u|D_\alpha P|^2 d\mu}{\int_{\mathbb{R}^d} u^m d\mu}\right|^2 d\mu + 2E\int_{\mathbb{R}^d} R[P]u^m d\mu = 0$$

if $\lim_{s\to 0_+} b(s) = \lim_{S\to+\infty} b(S) = 0$. So, if $\alpha \le \alpha_{FS}$ and P is of class C^3, by Corollary (4.3.7), as a function of t, F is concave, that is, $G = E^{\sigma-1}\mathrm{I}$ is non-increasing in t. Formally, G converges towards a minimum, for which necessarily $\mathcal{L}_\alpha P$ is a constant and $R[P] = 0$, which proves that $P(x) = a + b|x|^2$ for some real constants a and b, according to Corollary (4.3.7). Since $\frac{2(1-\vartheta)}{\vartheta(p+1)} = \sigma - 1$, the minimization of G under the mass constraint $\int_{\mathbb{R}^d} u d\mu = \int_{\mathbb{R}^d} v^{2p} d\mu$ is equivalent to the Caffarelli–Kohn–Nirenberg interpolation inequalities (31), since for some constant κ ,

$$G = E^{\sigma-1} I = \kappa\left(\int_{\mathbb{R}^d} v^{p+1} d\mu\right)^{\sigma-1}\int_{\mathbb{R}^d} |D_\alpha v|^2 d\mu \;\; with \;\; v = u^{m-1/2}.$$

We emphasize that (44) preserves mass, that is, $\frac{d}{dt}\int_{\mathbb{R}^d} v^{2p} d\mu = \frac{d}{dt}\int_{\mathbb{R}^d} u d\mu = \int_{\mathbb{R}^d} \mathcal{L}_\alpha u^m d\mu = 0$ because, as we shall see in Proposition (4.3.12), no boundary terms appear when integrating by parts if v is an extremal function associated with(36). In particular, for mass conservation we need

$$\lim_{s\to 0_+}\int_{\partial B_s} u\acute{P} d\varsigma = \lim_{S\to+\infty}\int_{\partial B_S} u\acute{P} d\varsigma = 0. \tag{54}$$

The above remarks on the monotonicity of G and the symmetry properties of its minimizers can in fact be extended to the analysis of the symmetry properties of all critical points of G. This is actually the contents of Theorem (4.3.8).

Theorem (4.3.8)[151]: Assume that (32) and (34) hold. Then all positive solutions to (35) in $H^p_{\beta\cdot\gamma}(\mathbb{R}^d)$ are radially symmetric and, up to a scaling and a multiplication by a constant, equal to w_*.

Up to a multiplication by a constant, we know that all non-trivial extremal functions for (31) are non-negative solutions to (35). Non-negative solutions to (35) are actually positive by the standard Strong Maximum principle. Theorem (4.3.1) is therefore a consequence of Theorem (4.3.8). In the particular case when $\beta = 0$, the condition(23) amounts to $d \ge 2, \gamma \in (0,2), p \in (1,(d-\gamma)/(d-2)]$, and(31) can be written as

$$\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)} \le C_{0,\gamma,p}\|\nabla w\|^{\vartheta}_{L^2(\mathbb{R}^d)}\|w\|^{1-\vartheta}_{L^{p+1,\gamma}(\mathbb{R}^d)} \qquad \forall w \in H^p_{0,\gamma}(\mathbb{R}^d)\,.$$

In this case, we deduce from Theorem (4.3.1) that symmetry always holds. This is consistent with a previous result ($\beta = 0$ and $\gamma > 0$, close to 0) obtained in [160]. A few other cases were already known. The Caffarelli–Kohn–Nirenberg inequalities that were discussed in [150] correspond to the critical case $\theta = 1, p = p_*$ or, equivalently $\beta = d - 2 + (\gamma - d)/p$. Here by critical we simply mean that $\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)}$ scales like $\|\nabla w\|_{L^{2,\beta}(\mathbb{R}^d)}$. The limit case $\beta = \gamma - 2$ and $p = 1$, which is an endpoint for (32), corresponds to Hardy-type inequalities: there is no extremal function, but optimality is achieved among radial functions: see [140]. The other endpoint is $\beta = (d-2)\,\gamma/d$, in which case $p_* = d/(d-2)$. The results of Theorem (4.3.1) also hold in that case with $p = p_* = d/(d-2)$., up to existence issues: according to [1], either $\gamma \geq 0$, symmetry holds and there exists a symmetric extremal function, or $\gamma < 0$, and then symmetry is broken, but there is no optimal function.

Inequality (31) can be rewritten as an interpolation inequality with same weights on both sides using a change of variables. Here we follow the computations in [155] (also see [150], [147]). Written in spherical coordinates for a function

$$\widetilde{w}(r,\omega) = w(x), \qquad \text{with } r = |x| \quad \text{ and } \quad \omega = \frac{x}{|x|},$$

inequality (31) becomes

$$\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |\widetilde{w}|^{2p}\, r^{d-\gamma-1} dr d\omega\right)^{\frac{1}{2p}}$$

$$\leq C_{\beta,\gamma,p}\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |\nabla \widetilde{w}|^2\, r^{d-\beta-1} dr d\omega\right)^{\frac{\vartheta}{2}}\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |\widetilde{w}|^{p+1}\, r^{d-\gamma-1} dr d\omega\right)^{\frac{1-\vartheta}{p+1}},$$

where $|\nabla \widetilde{w}|^2 = \left|\frac{\partial \widetilde{w}}{\partial r}\right|^2 + \frac{1}{r^2}|\nabla_\omega \widetilde{w}|^2$ and $\nabla_\omega \widetilde{w}$ denotes the gradient of $\widetilde{w}$ with respect to the angular variable $\omega \in \mathbb{S}^{d-1}$. Next we consider the change of variables $r \mapsto s = r^{\alpha}$,

$$\widetilde{w}(r,\omega) = v(s,\omega) \qquad \forall (r,\omega) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}, \tag{55}$$

where α and n are two parameters such that

$$n = \frac{d-\beta-2}{\alpha} + 2 = \frac{d-\gamma}{\alpha}.$$

Our inequality can therefore be rewritten as

$$\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |v|^{2p}\, s^{n-1} ds d\omega\right)^{\frac{1}{2p}}$$

$$\leq K_{\alpha,n,p}\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\alpha^2 \left|\frac{\partial v}{\partial s}\right|^2 + \frac{1}{s^2}|\nabla_\omega v|^2\right)\, s^{n-1} ds d\omega\right)^{\frac{\vartheta}{2}}\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |v|^{p+1}\, s^{n-1} ds d\omega\right)^{\frac{1-\vartheta}{p+1}},$$

with

$$C_{\beta,\gamma,p} = \alpha^{\zeta} K_{\alpha,n,p} \quad and \;\; \zeta := \frac{\vartheta}{2} + \frac{1-\vartheta}{p+1} - \frac{1}{2p} = \frac{(\beta+2-\gamma)(p-1)}{2p(d+\beta+2-2\gamma-p(d-\beta-2))}$$

Using the notation

$$D_\alpha\, v = \left(\alpha \frac{\partial v}{\partial s}, \frac{1}{s}\nabla_\omega v\right),$$

with

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad and \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma},$$

Inequality (31) is equivalent to a Gagliardo–Nirenberg type inequality corresponding to an artificial dimension n or, to be precise, to a Caffarelli–Kohn–Nirenberg inequality with weight $|x|^{n-d}$ in all terms. Notice that

$$p_* = \frac{n}{n-2}.$$

Proof: Let w be a positive solution to equation (35). As pointed out above, by choosing

$$w(x) = u^{m-1/2}(r^{\alpha}, \omega),$$

we know that u is a critical point of G under a mass constraint on $\int_{\mathbb{R}^d} u dx$, so that we can write the corresponding Euler–Lagrange equation as $dG[u] = C$, for some constant C. That is, $\int_{\mathbb{R}^d} dG[u] \cdot \mathcal{L}_\alpha u^m d\mu = C \int_{\mathbb{R}^d} \mathcal{L}_\alpha u^m d\mu = 0$ thanks to (54). Using $\mathcal{L}_\alpha u^m$ as a test function amounts to apply the flow of (44) to G with initial datum u and compute the derivative with respect to t at $t = 0$. This means

$$0 = \int_{\mathbb{R}^d} dG[u] \cdot \mathcal{L}_\alpha d\mu = E^{\sigma} H$$

$$= -(1-m)(\sigma-1)E^{\sigma-1} \int_{\mathbb{R}^d} u^m \left| \mathcal{L}_\alpha P - \frac{\int_{\mathbb{R}^d} u|D_\alpha P|^2 d\mu}{\int_{\mathbb{R}^d} u^m d\mu} \right|^2 d\mu - 2E^{\sigma-1} \int_{\mathbb{R}^d} R[P] u^m d\mu$$

if $\lim_{s\to 0_+} b(s) = \lim_{S\to +\infty} b(S) = 0$ and (53) holds. Here we have used Lemma (4.3.4). We emphasize that this proof is purely variational and does not rely on the properties of the solutions to (44), although using the flow was very useful to explain our strategy. All we need is that no boundary term appears in the integrations by parts. Hence, in order to obtain a complete proof, we have to prove that (47), (53) and(54) hold with b defined by(46), whenever u is a critical point of G under mass constraint. This will be done in Proposition (4.3.12). Using Corollary (4.3.7), we know that $R[P] = 0, \nabla_\omega P = 0$ a.e.in $\mathbb{R}^d$ and $\mathcal{L}_\alpha P = \frac{\int_{\mathbb{R}^d} u|D_\alpha P|^2 d\mu}{\int_{\mathbb{R}^d} u^m d\mu} a.e.\, in\ \mathbb{R}^d, with\ P = \frac{m}{1-m} u^{m-1}$. We conclude as in [150] that P is an affine function of s^2.

We prove the regularity and decay estimates on w (or on P or u) that are necessary to establish the absence of boundary terms in the integrations

Lemma(4.3.9)[151]: Let β, γ and p satisfy the relations(32). Any positive solution w of (35) such that

$$\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)} + \|\nabla w\|_{L^{2,\beta}(\mathbb{R}^d)} + \|w\|^{1-\vartheta}_{L^{p+1,\gamma}(\mathbb{R}^d)} < +\infty. \quad (56)$$

is uniformly bounded and tends to 0 at infinity, uniformly in $|x|$.

Proof: The strategy of the first part of the proof is similar to the one in [160], which was restricted to the case $\beta = 0$.

Let us set $\delta_0 := 2(p_* - p)$. For any $A > 0$, we multiply(35) by $(w \wedge A)^{1+\delta_0}$ and integrate by parts (or, equivalently, plug it in the weak formulation of (35)): we point out that the latter is indeed an admissible test function since $w \in H^p_{\beta,\gamma}(\mathbb{R}^d)$. In that way, by letting $A \to +\infty$, we obtain the identity

$$\frac{4(1+\delta_0)}{(2+\delta_0)}\int_{\mathbb{R}^d}\left|\nabla w^{1+\delta_0/2}\right|^2|x|^{-\beta}dx+\int_{\mathbb{R}^d} w^{p+1+\delta_0}|x|^{-\gamma}dx=\int_{\mathbb{R}^d} w^{2p+\delta_0}|x|^{-\gamma}dx.$$

By applying (31) with $p = p_*$ (so that $\vartheta = 1$) to the function $w = w^{1+\delta 0/2}$, we deduce that

$$\|w\|^{2+\delta_0}_{L^{2p+\delta_1},\gamma(\mathbb{R}^d)} \le \frac{(2+\delta_0)^2}{4(1+\delta_0)} C^2_{\beta,\gamma,p_*}\|w\|^{2p+\delta_0}_{L^{2p+\delta_0},\gamma(\mathbb{R}^d)}$$

with $2p+\delta_1 = p_*(2+\delta_0)$. Let us define the sequence $\{\delta_n\}$by the induction relation $\delta_{n+1} := p_*(2+\delta_n) - 2\,p$ for any $n \in \mathbb{N}$, that is,

$$\delta_n = 2\frac{p_*-p}{p_*-1}(p_*^{n+1}-1)\ \ \forall n \in \mathbb{N},$$

and take $q_n = 2\,p + \delta_n$. If we repeat the above estimates with δ_0 replaced by δ_n and δ_1 replaced by δ_{n+1}, we get

$$\|w\|^{2+\delta_n}_{L^{q_{n+1}},\gamma(\mathbb{R}^d)} \le \frac{(2+\delta_n)^2}{4(1+\delta_n)} C^2_{\beta,\gamma,p_*}\|w\|^{q_n}_{L^{q_n},\gamma(\mathbb{R}^d)}.$$

By iterating this estimate, we obtain the estimate

$$\|w\|_{L^{q_{n+1}},\gamma(\mathbb{R}^d)} \le C_n\|w\|^{\zeta_n}_{L^{2p_*},\gamma(\mathbb{R}^d)} \quad \text{with} \quad \zeta_n := \frac{(p_*-1)p_*^n}{p-1+(p_*-p)p_*^n},$$

where the sequence $\{C_n\}$ is defined by $C_0 = 1$ and

$$C^{2+\delta_n}_{n+1} = \frac{(2+\delta_n)^2}{4(1+\delta_n)} C^2_{\beta,\gamma,p_*} C^{q_n}_n \quad \forall n \in \mathbb{N}.$$

The sequence $\{C_n\}$ converges to a finite limit C_∞. Letting $n \to \infty$ we obtain the uniform bound

$$\|w\|_{L^\infty(\mathbb{R}^d)} \le C_\infty\|w\|^{\zeta_\infty}_{L^{2p_*},\gamma(\mathbb{R}^d)} \le C_\infty\left(C_{\beta,\gamma,p_*}\|\nabla w\|_{L^{2,\beta}(\mathbb{R}^d)}\right)^{\zeta_\infty} \le C_\infty\left(C_{\beta,\gamma,p_*}\|w\|^p_{L^{2p,\gamma}(\mathbb{R}^d)}\right)^{\zeta_\infty}$$

where $\zeta_\infty := \frac{p_*-1}{p_*-p} = \lim_{n\to+\infty}\zeta_n$.

In order to prove that $\lim_{|x|\to\infty} w(x) = 0$, we can suitably adapt the above strategy. We shall do it as follows: we truncate the solution so that the truncated function is supported outside of a ball of radius R_0 and apply the iteration scheme. Up to an enlargement of the ball, that is, outside of a ball of radius $R_\infty = a\,R_0$ for some fixed numerical constant $a > 1$, we get that $\|w\|_{L^\infty(B^c_{R_\infty})}$ is bounded by the energy localized in $B^c_{R_0}$. The conclusion will hold by letting $R_0 \to +\infty$. Let us give some details.

Let $\xi \in C^\infty(\mathbb{R}^+)$ be a cut-off function such that $0 \le \xi \le 1, \xi \equiv 0$ in $[0,1)$ and $\xi \equiv 1$ in $(2,+\infty)$. Given $R_0 \ge 1$, consider the sequence of radii defined by

$$R_{n+1} = \left(1+\frac{1}{n^2}\right)R_n \quad \forall n \in \mathbb{N}.$$

By taking logarithms, it is immediate to deduce that $\{R_n\}$is monotone increasing and that there exists $a > 1$ such that

$$R_\infty := \lim_{n\to\infty} R_n = aR_0.$$

Let us then define the sequence of radial cut-off functions $\{\xi_n\}$ by

$$\xi_n(x) := \xi^2\left(\frac{|x|-R_n}{R_{n+1}-R_n}+1\right) \quad \forall x \in \mathbb{R}^d.$$

Direct computations show that there exists some constant $c > 0$, which is independent of n and R_0, such that

$$|\nabla\xi_n(x)| \le c\frac{n^2}{R_n}\chi_{B_{R_{n+1}}\setminus B_{R_n}}, \left|\nabla\xi_n^{1/2}(x)\right| \le c\frac{n^2}{R_n}\chi_{B_{R_{n+1}}\setminus B_{R_n}}, |\Delta\xi_n(x)| \le c\frac{n^4}{R_n^2}\chi_{B_{R_{n+1}}\setminus B_{R_n}} \quad \forall x \in \mathbb{R}^d . \qquad (57)$$

From here on we denote by $,\acute{c}$, etc. positive constants that are all independent of n and R_0. We now introduce the analogue of the sequence $\{\delta_n\}$ above, which we relabel $\{\sigma_n\}$ to avoid confusion. Namely, we set $\sigma_0 := 2p - 2$ and $\sigma_{n+1} = p_*(2+\sigma_n) - 2,$ so that $\sigma_n = 2(p\,p_*^n - 1)$. If we multiply (35) by $\xi_n w^{1+\sigma_n}$ and integrate by parts, we obtain:

$$\int_{\mathbb{R}^d} \nabla(\xi_n w^{1+\sigma_n}) \cdot \nabla w |x|^{-\beta} dx + \int_{\mathbb{R}^d} \xi_n w^{p+1+\sigma_n}|x|^{-\gamma} dx = \int_{\mathbb{R}^d} \xi_n w^{2p+\sigma_n}|x|^{-\gamma} dx ,$$

whence

$$\frac{4(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d} \xi_n \left|\nabla w^{1+\sigma_n/2}\right|^2 |x|^{-\beta} dx + \frac{1}{2+\sigma_n}\int_{\mathbb{R}^d} \nabla\xi_n \cdot \nabla w^{2+\sigma_n}|x|^{-\beta} dx \le \int_{B_{R_n}^c} w^{2p+\sigma_n}|x|^{-\gamma} dx .$$

By integrating by parts the second term in the l.h.s. and combining this estimate with

$$\int_{\mathbb{R}^d}\left|\nabla\left(\xi_n^{1/2}w^{1+\sigma_n/2}\right)\right|^2|x|^{-\beta}dx \le 2\int_{\mathbb{R}^d} \xi_n\left|\nabla w^{1+\sigma_n/2}\right|^2|x|^{-\beta}dx + 2\int_{\mathbb{R}^d}\left|\nabla\xi_n^{1/2}\right|^2 w^{2+\sigma_n}|x|^{-\beta}dx ,$$

we end up with

$$\frac{2(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d}\left|\nabla\left(\xi_n^{1/2}w^{1+\sigma_n/2}\right)\right|^2|x|^{-\beta}dx - \frac{4(1+\sigma_n)}{(2+\sigma_n)^2}\int_{\mathbb{R}^d}\left|\nabla\xi_n^{1/2}\right|^2 w^{2+\sigma_n}|x|^{-\beta}dx$$

$$-\frac{1}{2+\sigma_n}\int_{\mathbb{R}^d}\left(|x|^{-\beta}\Delta\xi_n - \beta|x|^{-\beta-2}x\cdot\nabla\xi_n\right)w^{2+\sigma_n}\,dx \le \int_{B_{R_n}^c} w^{2p+\sigma_n}|x|^{-\gamma}dx.$$

Thanks to (57), we can deduce that

$$\int_{\mathbb{R}^d}\left|\nabla\left(\xi_n^{1/2}w^{1+\sigma_n/2}\right)\right|^2|x|^{-\beta}dx \le \int_{B_{R_{n+1}}\setminus B_{R_n}}\left(\frac{2c^2+c}{R_n^2}n^4 + \frac{\beta\, c}{R_n}n^2|x|^{-1}\right)w^{2+\sigma_n}|x|^{-\beta}dx + \frac{(2+\sigma_n)^2}{2(1+\sigma_n)}\int_{B_{R_n}^c} w^{2p+\sigma_n}|x|^{-\gamma}dx.$$

In particular,

$$\int_{\mathbb{R}^d}\left|\nabla\left(\xi_n^{1/2}w^{1+\sigma_n/2}\right)\right|^2|x|^{-\beta}dx \le \acute{c}n^4\int_{B_{R_n}^c} w^{2+\sigma_n}|x|^{-\beta-2}dx + \frac{(2+\sigma_n)^2}{2(1+\sigma_n)}\|w\|_\infty^{2p-2}\int_{B_{R_n}^c} w^{2+\sigma_n}|x|^{-\gamma}dx.$$

Since (32) implies that $\beta + 2 > \gamma$, by exploiting the explicit expression of σ_n and applying(31) with $p = p_*$ (and $\vartheta = 1$) to the function $\xi_n^{1/2} w^{1+\sigma_n/2}$, we can rewrite our estimate as

$$\|w\|^{2+\sigma_n}_{L^{2+\sigma_{n+1},\gamma}\left(B^c_{R_{n+1}}\right)} \leq \acute{c} p_*^n \|w\|^{2+\sigma_n}_{L^{2+\sigma_n \gamma}\left(B^c_{R_n}\right)}.$$

After iterating the scheme and letting $n \to \infty$, we end up with

$$\|w\|_{L^\infty\left(B^c_{R_\infty}\right)} \leq \acute{\acute{c}}\, p_*^n \|w\|_{L^{2p,\gamma}\left(B^c_{R_0}\right)}.$$

Since w is bounded in $L^{2p,\gamma}(\mathbb{R}^d)$, in order to prove the claim, it is enough to let $R_0 \to +\infty$.

Lemma (4.3.10)[151]: Let β, γ and p satisfy the relations (32). Any positive solution w of (35) satisfying (56) is such that $w \in C^\infty(\mathbb{R}^d \backslash \{0\})$ and there exist two positive constants, C_1 and C_2 with $C_1 < C_2$, such that for $|x|$ large enough,

$$C_1 |x|^{(\gamma-2-\beta)/(p-1)} \leq w(x) \leq C_2 |x|^{(\gamma-2-\beta)/(p-1)}.$$

Proof: By Lemma (4.3.9) and elliptic bootstrapping methods we know that $w \in C^\infty(\mathbb{R}^d \backslash \{0\})$. Let us now consider the function $h(x) := C|x|^{(\gamma-2-\beta)/(p-1)}$, which satisfies the differential inequality

$$-\mathrm{div}\left(|x|^{-\beta} \nabla h\right) + (1-\varepsilon)|x|^{-\gamma} h^p \geq 0 \quad \forall x \in \mathbb{R}^d \backslash \{0\}$$

for any $\varepsilon \in (0,1)$ and C such that $C^{p-1} > \frac{2-\gamma+\beta}{1-\varepsilon} \frac{d-\gamma-p(d-2-\beta)}{(p-1)^2}$. On the other hand, by Lemma (4.3.9) , w^{2p-1} is negligible compared to w^p as $|x| \to \infty$ and, as a consequence, for any $\varepsilon > 0$ small enough, there is an $R_\varepsilon > 0$ such that

$$-\mathrm{div}\left(|x|^{-\beta} \nabla w\right) + (1-\varepsilon)|x|^{-\gamma} w^p \leq 0 \quad \text{if } |x| \geq R_\varepsilon.$$

With $q := (1-\varepsilon)\, |x|^{-\gamma}\ \frac{h^p - w^p}{h-w} \geq 0$, it follows that

$$-\mathrm{div}\left(|x|^{-\beta} \nabla(\mathrm{h} - w)\right) + q(h-w) \geq 0 \quad \text{if } |x| \geq R_\varepsilon.$$

Hence, for C large enough, we know that $h(x) \geq w(x)$ for any $x \in \mathbb{R}^d$ such that $|x| = R_\varepsilon$, and we also have that $\lim_{|x| \to +\infty} (h(x) - w(x)) = 0$. Using the Maximum Principle, we conclude that $0 \leq w(x) \leq h(x)$ for any $x \in \mathbb{R}^d$ such that $|x| \geq R_\varepsilon$. The lower bound uses a similar comparison argument. Indeed, since

$$-\mathrm{div}\ \left(|x|^{-\beta} \nabla \mathrm{w}\right) + |x|^{-\gamma} w^p \geq 0 \quad \forall x \in \mathbb{R}^d \backslash \{0\}$$

and

$$-\mathrm{div}\ \left(|x|^{-\beta} \nabla\, \mathrm{h}\right) + |x|^{-\gamma} h^p \leq 0 \quad \forall x \in \mathbb{R}^d \backslash \{0\},$$

if we choose C such that $C^{p-1} \leq (2-\gamma+\beta) \frac{d-\gamma-p(d-2-\beta)}{(p-1)^2}$, we easily see that

$$w(x) \geq \left(\min_{|x|=1} w(x) \wedge C\right) |x|^{(\gamma-2-\beta)/(p-1)} \quad \forall x \in \mathbb{R}^d \backslash B_1 .$$

This concludes the proof.

We obtain growth and decay estimates, respectively, on the functions P and u as they appear in the proof of Theorem (4.3.8), in order to prove Proposition (4.3.12). We also need to estimate their derivatives near the origin and at infinity. Let us start by reminding the change of variables (55), which in particular, by Lemma(4.3.10), implies that for some positive constants C_1 and C_2,

$$C_1 s^{2/(1-p)} \leq v(s,\omega) \leq C_2 s^{2/(1-p)} \quad as\ s \to +\infty.$$

Then we perform the Emden–Fowler transformation

$$\nu(s,\omega) \le s^a \varphi(z,\omega) \quad \text{with}\, z = -\log s, \quad a = \frac{2-n}{2}, \tag{58}$$

and see that φ satisfies the equation

$$-\alpha^2 \acute{\acute{\varphi}} - \Delta_\omega \varphi + a^2\alpha^2 \varphi = e^{((n-2)p-n)z}\varphi^{2p-1} - e^{((n-2)p-n-2)z/2}\varphi^p =: h \ \text{in}\ C := \mathbb{R} \times \mathbb{S}^{d-1} \ni (z,\omega). \tag{59}$$

From here on we shall denote by the derivative with respect either to z or to s, depending on the argument. By definition of φ and using Lemma (4.3.10), we obtain that

$$\varphi(z,\omega) \sim e^{\left(\frac{2-n}{2}+\frac{2}{p-1}\right)z} \quad as\ z \to -\infty,$$

where we say that $f(z,\omega) \sim g(z,\omega)$ as $z \to +\infty$ (resp. $z \to -\infty$) if the ratio f/g is bounded from above and from below by positive constants, independently of ω, and for z (resp. $-z$) large enough. Concerning $z \to +\infty$, we first note that Lemma (4.3.9) and (58) show that $\varphi(z,\omega) \le O(e^{az})$. The lower bound can be established by a comparison argument as in [150], after noticing that $|h(z,\omega)| \le O(e^{(a-2)z})$. Hence we obtain that

$$\varphi(z,\omega) \sim e^{az} = e^{\frac{2-n}{2}z} \quad \text{as}\ z \to +\infty.$$

Moreover, uniformly in ω, we have that

$$|h(z,\omega)| \le O\left(e^{-\frac{n+2}{2}z}\right) \quad \text{as}\ z \to +\infty, \quad |h(z,\omega)| \sim e^{\left(\frac{n+2}{2}+\frac{2p}{p-1}\right)z} \quad \text{as}\ z \to -\infty,$$

which in particular implies

$$|h(z,\omega)| = o(\varphi(z,\omega)) \quad \text{as}\ z \to +\infty \quad \text{and} \quad |h(z,\omega)| \sim \varphi(z,\omega) \quad \text{as}\ z \to -\infty.$$

Finally, using [162] on local $C^{1,\delta}$ estimates, as $|z| \to +\infty$ we see that all first derivatives of φ converge to 0 at least with the same rate as φ. Next, [162] provides local $W^{k+2,2}$ estimates which, together with [162], up to choosing k large enough, prove that

$$|\acute{\varphi}(z,\omega)|, |\acute{\acute{\varphi}}(z,\omega)|, |\nabla_\omega \varphi(z,\omega)|, |\nabla_\omega \acute{\varphi}(z,\omega)|, |\nabla_\omega \acute{\acute{\varphi}}(z,\omega)|, |\Delta_\omega \varphi(z,\omega)| \le O(\varphi(z,\omega)), \tag{60}$$

uniformly in ω. Here we denote by ∇_ω the differentiation with respect to ω. As a consequence, we have, uniformly in ω, and for $\ell \in \{0,1,2\}, t \in \{0,1\}$,

$$\left|\partial_z^\ell \nabla_\omega^t h(z,\omega)\right| \le O\left(e^{-\frac{n+2}{2}z}\right) \ \text{as}\ z \to +\infty, \left|\partial_z^\ell \nabla_\omega^t h(z,\omega)\right| \le O\left(e^{\left(-\frac{n+2}{2}+\frac{2p}{p-1}\right)z}\right) \ \text{as}\ z \to -\infty, \tag{61}$$

$$|\Delta_\omega h(z,\omega)| \le O\left(e^{-\frac{n+2}{2}z}\right) \ \text{as}\ z \to +\infty, |\Delta_\omega h(z,\omega)| \le O\left(e^{\left(-\frac{n+2}{2}+\frac{2p}{p-1}\right)z}\right) \text{as}\ z \to -\infty. \tag{62}$$

Lemma (4.3.11)[151]: Let β, γ and p satisfy the relations (32) and assume $\alpha \le \alpha_{FS}$. For any positive solution w of (35) satisfying (56), the pressure function $P = \frac{m}{1-m}u^{m-1}$ is such that $\acute{\acute{P}}, \acute{P}/s, P/s^2, \nabla_\omega \acute{P}/s, \nabla_\omega P/s^2$ and $\mathcal{L}_\alpha P$ are of class C^∞ and bounded as $s \to +\infty$. On the other hand, as $s \to 0_+$ we have

(i) $\int_{\mathbb{S}^{d-1}} |\acute{p}(s,\omega)|^2\, d\omega \le O(1)$,

(ii) $\int_{\mathbb{S}^{d-1}} |\nabla_\omega p(s,\omega)|^2\, d\omega \le O(s^2)$,

(iii) $\int_{\mathbb{S}^{d-1}} \left|\acute{\acute{p}}(s,\omega)\right|^2 d\omega \le O(1/s^2)$,

(iv) $\int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \acute{p}(s,\omega) - \frac{1}{s}\nabla_\omega p(s,\omega)\right|^2 d\omega \le O(1)$,

(v) $\int_{\mathbb{S}^{d-1}} \left|\frac{1}{s^2}\Delta_\omega p(s,\omega)\right|^2 d\omega \le O(1/s^2)$.

Proof: By using the change of variables (58), we see that

$$p(s,\omega) = \frac{p+1}{p-1} e^{-\frac{1}{2}(n-2)(p-1)z} \varphi^{1-p}(z,\omega), \quad z = -\log s.$$

From (60) we easily deduce that uniformly in $\omega, \acute{\acute{P}}, \acute{P}/s, P/s^2, \nabla_\omega \acute{P}/s, \nabla_\omega P/s^2$ and $\mathcal{L}_\alpha P$ are of class C^∞ and bounded as $s \to +\infty$. Moreover, as $s \to 0_+$, we obtain that

$$|\acute{p}(s,\omega)|^2 \le O\left(\frac{1}{s}\left(\frac{\acute{\varphi}(z,\omega)}{\varphi(z,\omega)} - a\right)\right) \quad \text{and} \quad \left|\frac{1}{s}\nabla_\omega p(s,\omega)\right| \le O\left(\frac{1}{s}\left(\frac{\nabla_\omega \varphi(z,\omega)}{\varphi(z,\omega)}\right)\right)$$

are of order at most $1/s$ uniformly in ω. Similarly we obtain that

$$\left|\acute{\acute{p}}(s,\omega)\right| \le O\left(\frac{1}{s^2}\left(\frac{\acute{\acute{\varphi}}(z,\omega)}{\varphi(z,\omega)} - p\frac{|\acute{\varphi}(z,\omega)|^2}{|\varphi(z,\omega)|^2} + (1-2a(1-p))\frac{\acute{\varphi}(z,\omega)}{\varphi(z,\omega)} + a^2(1-p) - a\right)\right),$$

$$\left|\frac{\nabla_\omega \acute{p}(s,\omega)}{s} - \frac{a(1-p)}{s^2}\nabla_\omega p(s,\omega)\right| \le O\left(\frac{1}{s^2}\left(\frac{\nabla_\omega \acute{\varphi}(z,\omega)}{\varphi(z,\omega)} - \frac{p\acute{\varphi}(z,\omega)\nabla_\omega \varphi(z,\omega)}{|\varphi(z,\omega)|^2}\right)\right),$$

$$\frac{1}{s^2}|\Delta_\omega p(s,\omega)| \le O\left(\frac{1}{s^2}\left(\frac{\Delta_\omega \varphi(z,\omega)}{\varphi(z,\omega)} - p\frac{|\nabla_\omega \varphi(z,\omega)|^2}{|\varphi(z,\omega)|^2}\right)\right),$$

are at most of order $1/s^2$ uniformly in ω. This shows that $|b(s)| \le O\,(s^{n-4})$ as $s \to 0+$ and concludes the proof if $4 \le d < n$. When $d = 2$ or 3 and $n \le 4$, more detailed estimates are needed. Properties (i)–(v) amount to prove that

(i) $\int_{\mathbb{S}^{d-1}} \left|\frac{\acute{\varphi}(z,\omega)}{\varphi(z,\omega)} - a\right|^2 d\omega \le O(e^{-2z})$,

(ii) $\int_{\mathbb{S}^{d-1}} \left|\frac{\nabla_\omega \varphi(z,\omega)}{\varphi(z,\omega)}\right|^2 d\omega \le O(e^{-2z})$,

(iii) $\int_{\mathbb{S}^{d-1}} \left|\frac{\acute{\acute{\varphi}}(z,\omega)}{\varphi(z,\omega)} - p\frac{|\acute{\varphi}(z,\omega)|^2}{|\varphi(z,\omega)|^2} + (1-2a(1-p))\frac{\acute{\varphi}(z,\omega)}{\varphi(z,\omega)} + a^2(1-p) - a\right|^2 d\omega \le O(e^{-2z})$,

(iv) $\int_{\mathbb{S}^{d-1}} \left|\frac{\nabla_\omega \acute{\varphi}(z,\omega)}{\varphi(z,\omega)} - \frac{p\acute{\varphi}(z,\omega)\,\nabla_\omega \varphi(z,\omega)}{|\varphi(z,\omega)|^2}\right|^2 d\omega \le O(e^{-2z})$,

(v) $\int_{\mathbb{S}^{d-1}} \left|\frac{\Delta_\omega p(z,\omega)}{\varphi(z,\omega)} - p\frac{|\nabla_\omega \varphi(z,\omega)|^2}{|\varphi(z,\omega)|^2}\right|^2 d\omega \le O(e^{-2z})$,

as $z \to +\infty$.

Step 1: **Proof** of **(ii)** and **(iv)**. If w is a positive solution to (35), then φ is a positive solution to (59). With $\ell \in \{0,1,2\}$, applying the operator $\nabla_\omega \partial_z^\ell \varphi$ to the equation (59) we obtain:

$$-\alpha^2(\nabla_\omega \partial_z^\ell \varphi)'' - \nabla_\omega \Delta_\omega \partial_z^\ell \varphi + a^2\alpha^2 \nabla_\omega \partial_z^\ell \varphi = \nabla_\omega \partial_z^\ell h(z,\omega) \quad \text{in } C.$$

Define

$$\chi_\ell(z) := \tfrac{1}{2}\acute{\chi}_\ell(z) = \int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell \varphi\, \nabla_\omega \partial_z^\ell \acute{\varphi}\, d\omega$$

which by(60) converges to 0 as $z \to \pm\infty$. Assume first that χ_ℓ is a positive function. After multiplying the above equation by $\nabla_\omega \partial_z^\ell \varphi$, integrating over $\mathbb{S}^{d-1}$, integrating by parts and using

$$\acute{\chi}_\ell = \int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell \varphi\, \nabla_\omega \partial_z^\ell \acute{\varphi}\, d\omega$$

and

$$\acute{\acute{\chi}}_\ell = \int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell \varphi\, \nabla_\omega \partial_z^\ell \acute{\acute{\varphi}}\, d\omega + \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \partial_z^\ell \acute{\varphi}\right|^2 d\omega,$$

we see that χ_ℓ satisfies

$$-\chi_\ell'' = \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \partial_z^\ell \varphi'\right|^2 d\omega \frac{1}{\alpha^2}\left(\int_{\mathbb{S}^{d-1}} \left|\Delta_\omega \partial_z^\ell \varphi\right|^2 d\omega - \lambda_1 \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \partial_z^\ell \varphi\right|^2 d\omega\right)$$
$$+ 2\left(a^2 + \frac{\lambda_1}{\alpha^2}\right)\chi_\ell = \frac{h_\ell}{\alpha^2},$$
with $h_\ell := \int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell h \nabla_\omega \partial_z^\ell \varphi d\omega$. Then, using $\int_{\mathbb{S}^{d-1}} \nabla_\omega \partial_z^\ell \varphi\, d\omega = 0$, by the Poincare inequality we deduce

$$\int_{\mathbb{S}^{d-1}} \left|\Delta_\omega \partial_z^\ell \varphi\right|^2 d\omega \geq \lambda_1 \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \partial_z^\ell \varphi\right|^2 d\omega$$
as e.g.in [158], where $\lambda_1 := d - 1$. A Cauchy–Schwarz inequality implies that
$$-\chi_\ell'' + \frac{\left|\chi_\ell'\right|^2}{2\chi_\ell} + 2\left(a^2 + \frac{\lambda_1}{\alpha^2}\right)\chi_\ell \leq \frac{|h_\ell|}{\alpha^2}.$$
The function $\zeta_\ell := \sqrt{\chi_\ell}$ satisfies
$$-\zeta_\ell'' + 2\left(a^2 + \frac{\lambda_1}{\alpha^2}\right)\zeta_\ell \leq \frac{|h_\ell|}{2\alpha^2\zeta_\ell}.$$
By the Cauchy–Schwarz inequality and(61) we infer that $|h_\ell/\zeta_\ell| = O\left(e^{(a-2)z}\right)$ for $z \to +\infty$, and $|h_\ell/\zeta_\ell| = O\left(e^{(a+2)/(p-1)z}\right)$ for $z \to -\infty$. By a simple comparison argument based on the Maximum Principle, and using the convergence of χ_ℓ to 0 at $\pm\infty$, we infer that
$$\zeta_\ell(z) \leq -\frac{e^{-\nu z}}{2\nu\alpha^2} \int_{-\infty}^{z} e^{\nu t} \frac{|h_\ell(t)|}{\zeta_\ell(z)} dt - \frac{e^{\nu z}}{2\nu\alpha^2} \int_{z}^{\infty} e^{-\nu t} \frac{|h_\ell(t)|}{\zeta_\ell(z)} dt$$
if $\nu := \sqrt{a^2 + \lambda_1/\alpha^2}$. This is enough to deduce that $\zeta_\ell(z) \leq O(e^{(a-1)z})$ as $z \to +\infty$ after observing that the condition
$$-\nu = -\sqrt{a^2 + \lambda_1/\alpha^2} \leq a - 1$$
is equivalent to the inequality $\alpha \leq \alpha_{FS}$. Hence we have shown that if χ_ℓ is a positive function, then for $\alpha \leq \alpha_{FS}$,
$$\chi_\ell(z) \leq O\left(e^{2(a-1)z}\right) \quad \text{as} \quad z \to +\infty\,. \tag{63}$$
In the case where χ_ℓ is equal to 0at some points of $\mathbb{R}$, it is enough to do the above comparison argument on maximal positivity intervals of χ_ℓ to deduce the same asymptotic estimate. Finally we observe that $\varphi(z,\omega) \sim e^{a z}$ as $z \to +\infty$, which ends the proof of **(ii)** considering the above estimate for χ_ℓ when $\ell = 0$. Moreover, the same estimate for $\ell = 0$ together with **(ii)** and (60) proves (**iv**).

Step 2: **Proof** of **(v)** . By applying the operator Δ_ω to (59), we obtain
$$-\alpha^2(\Delta_\omega\varphi)'' - \Delta_\omega^2\varphi + a^2\alpha^2\Delta_\omega\varphi = \Delta_\omega h \quad \text{as} \; C\,.$$
We proceed as in Step 1. With similar notations, by defining
$$\chi_3(z) := \frac{1}{2}\int_{\mathbb{S}^{d-1}} |\Delta_\omega \varphi|^2 d\omega\,,$$
after multiplying the equation by $\Delta_\omega \varphi$ and using the fact that
$$-\int_{\mathbb{S}^{d-1}} \Delta_\omega \varphi \Delta_\omega^2\varphi\, d\omega = \int_{\mathbb{S}^{d-1}} |\nabla_\omega\Delta_\omega^2\varphi|^2 d\omega \geq \lambda_1 \int_{\mathbb{S}^{d-1}} |\Delta_\omega\varphi|^2 d\omega,$$
we obtain
$$-\chi_3'' + \frac{\left|\chi_3'\right|^2}{2\chi_\ell} + 2\left(a^2 + \frac{\lambda_1}{\alpha^2}\right)\chi_3 \leq \frac{|h_3|}{\alpha^2}$$

with $h_3 := \int_{\mathbb{S}^{d-1}} \Delta_\omega h\, \Delta_\omega \varphi\, d\omega$. Again using the same arguments as above, together with (62), we deduce that

$$\chi_3(z) \le O\big(e^{2(a-1)z}\big) \quad \text{as} \;\; z \to +\infty\,.$$

This ends the proof of **(v)**, using **(ii)**,(60) and noticing again that $\varphi(z,\omega) \sim e^{az}$ as $z \to +\infty$.

Step 3: **Proof of (i) and (iii)** . Let us consider a positive solution φ to (59) and define on $\mathbb{R}$ the function

$$\varphi_0(z) := \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} \varphi(z,\omega)d\omega.$$

By integrating (59) on $\mathbb{S}^{d-1}$, we know that φ_0 solves

$$-\varphi_0'' \quad + a^2\varphi_0 = \frac{1}{\alpha^2|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} h(z,\omega)d\omega =: \frac{h_0(z)}{\alpha^2} \;\; \forall z \in \mathbb{R},$$

with

$$|h_0(z)| \le O\left(e^{-\frac{n+2}{2}z}\right) \quad \text{as} \quad z \to +\infty\,, \;\; |h_0(z)| \sim e^{-\left(\frac{n+2}{2}+\frac{2p}{p-1}\right)z} \;\; \text{as} \quad z \to -\infty\,.$$

From the integral representation

$$\varphi_0(z) = -\frac{e^{az}}{2a\,\alpha^2}\int_{-\infty}^{z} e^{-at}\, h_0(t)dt - \frac{e^{-az}}{2a\,\alpha^2}\int_{z}^{\infty} e^{at}\, h_0(t)dt,$$

we deduce that as $z \to +\infty, \varphi_0(z) \sim e^{az}$ and

$$\frac{\varphi_0'\;(z) - a\,\varphi_0(z)}{\varphi(z,\omega)} \sim e^{-2a\,z}\int_{z}^{\infty} e^{at}\, h_0(t)dt = O(e^{-2z}).$$

If we define the function $(z,\omega) := e^{-az}\,\varphi(z,\omega) - \varphi_0(z)$, we may observe that it is bounded for z positive and, moreover,

$$\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a = O(e^{-2z}) + \frac{\psi'(z,\omega)}{e^{-az}\varphi(z,\omega)} \quad \text{as} \quad z \to +\infty\,.$$

We recall that $e^{\,-az}\,\varphi(z,\omega)$ is bounded away from 0 by a positive constant as $z \to +\infty$. Hence we know that

$$\left|\frac{\varphi'(z,\omega)}{\varphi(z,\omega)} - a\right| \le O\big(|\psi'(z,\omega)|\big) + O(e^{-2z}). \tag{64}$$

By the Poincaré inequality and estimate (63) with $\ell = 0$, we have

$$\int_{\mathbb{S}^{d-1}} |\psi|^2 d\omega = \,e^{-2a\,z}\int_{\mathbb{S}^{d-1}} |\varphi - \varphi_0|^2 d\omega \le \frac{e^{-2a\,z}}{\lambda_1}\int_{\mathbb{S}^{d-1}} |\nabla_\omega \varphi|^2 d\omega \le O(e^{-2z}).$$

Moreover, by the estimate (63) with $\ell = 1$, we also obtain

$$e^{-2a\,z}\int_{\mathbb{S}^{d-1}} \left|\varphi' - \varphi_0'\;\right|^2 d\omega \le \frac{e^{-2az}}{\lambda_1}\int_{\mathbb{S}^{d-1}} |\nabla_\omega \varphi'|^2 d\omega \le O(e^{-2z}).$$

Hence, since $\psi' = -\,a\,\psi + e^{-az}(\varphi - \varphi_0)$, the above estimates imply that

$$\int_{\mathbb{S}^{d-1}} |\psi|^2 d\omega + \int_{\mathbb{S}^{d-1}} \left|\psi'\right|^2 d\omega \le O(e^{-2z}).$$

which together with (64) ends the proof of **(i)**.

To prove **(iii)**, we first check that

$$\frac{\varphi''}{\varphi} - p\frac{|\varphi'|^2}{|\varphi|^2} + \big(1 - 2a(1-p)\big)\frac{\varphi'}{\varphi} + a^2(1-p) - a = O\left(\left|\psi'\right| + \left|\psi'\right|^2 + \left|\psi''\right|\right) + O(e^{-2z}),$$

and so it remains to prove that $\int_{\mathbb{S}^{d-1}} \left|\acute{\acute{\psi}}\right|^2 d\omega +$ is of order $O(e^{-2z})$. Since

$$\acute{\acute{\psi}} = a^2\psi - 2ae^{-az}(\acute{\varphi} - \acute{\varphi_0}) + e^{-az}(\acute{\acute{\varphi}} - \acute{\acute{\varphi_0}}),$$

using the above estimates, we have only to estimate the term with the second derivatives. This can be done as above by the Poincare inequality,

$$e^{-2a\,z} \int_{\mathbb{S}^{d-1}} \left| \acute{\acute{\varphi}} - \acute{\acute{\varphi_0}} \right|^2 d\omega \leq \frac{e^{-2az}}{\lambda_1} \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega \acute{\acute{\varphi}}\right|^2 d\omega \leq O(e^{-2z}),$$

based on the estimate (63) with $\ell = 1$. This ends the proof of **(iii)**.

Proposition (4.3.12)[151] : Under Condition (32), if w is a positive solution in $H^p_{\beta,\gamma}(\mathbb{R}^d)$ of (35), then (47), (53) and (54) hold with bas defined by(46), $u = v^{2p}$ and v given by (55).

To prove this result, we split the proof in several steps: we will first establish a uniform bound and a decay rate for w inspired by [160] in Lemmas(4.3.9),(4.3.10), and then follow the methodology of [150] in the subsequent Lemma (4.3.11).

Proof: It is straightforward to verify that the boundedness of $\acute{\acute{P}}$, $\acute{P}/s$, P/s^2, $\nabla_\omega \acute{P}/s$, $\nabla_\omega P/s^2$, $\mathcal{L}_\alpha P$ as $s \to +\infty$ and the integral estimates (i)–(v) as $s \to 0^+$ from Lemma (4.3.11) are enough in order to establish (47), (53) and (54).

Chapter 5
Non-Linear Ground State Representations

We deduce from the sharp Hardy inequality the sharp constant in a Sobolev embedding which is optimal in the Lorentz scale. We characterize the cases of equality in the rearrangement inequality in fractional Sobolev spaces. The inequality combines the fractional Sobolev and the fractional Hardy inequality into a single inequality, while keeping the sharp constant in the Hardy inequality. We show improved Hardy inequalities for $p \geq 2$ of the optimal constant in the Hardy inequality, as a consequence of the improved Hardy inequality, we obtain that for all $q < p$, the inequalities can be understood as the fractional extension of the Callarelli–Kohn–Nirenberg inequalities in [7].

Section (5.1): Sharp Hardy Inequalities

Hardy's inequality plays an important role in many questions from mathematical physics, spectral theory, analysis of linear and non-linear PDE, harmonic analysis and stochastic analysis. It states that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx, \tag{1}$$

and holds for all $u \in C_0^\infty(\mathbb{R}^N)$ if $1 \leq p < N$, and for all $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$ if $p > N$.The constant on the right-hand side of (1) is sharp and, for $p > 1$, not attained in the corresponding homogeneous Sobolev spaces $\dot{W}_p^1(\mathbb{R}^N)$ and $\dot{W}_p^1(\mathbb{R}^N\backslash\{0\})$, respectively, i.e., the completion of $C_0^\infty(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N\backslash\{0\}$ with respect to the left-hand side of (1). If $p = 1$, equality holds for any symmetric decreasing function.

We are concerned with the fractional analog of Hardy's inequality (1), where the left-hand side is replaced by

$$\int_{\mathbb{R}^N} \int_{\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \tag{2}$$

for some $0 < s < 1$. By scaling the function $|x|^{-p}$ on the right-hand side has to be replaced by $|x|^{-ps}$. For $N \geq 1$ and $0 < s < 1$ we consider the homogeneous Sobolev spaces $\dot{W}_p^s(\mathbb{R}^N)$ and $\dot{W}_p^s(\mathbb{R}^N\backslash\{0\})$ defined as the completion with respect to (2) of $C_0^\infty(\mathbb{R}^N)$ for $1 \leq p < N/s$ and $C_0^\infty(\mathbb{R}^N\backslash\{0\}$ for $p > N/s$, respectively. Our main result is the optimal constant in the fractional Hardy inequality.

Theorem (5.1.1)[166]: (Sharp fractional Hardy inequality). Let $N \geq 1$ and $0 < s < 1$. Then for all $u \in \dot{W}_p^s(\mathbb{R}^N)$ in case $1 \leq p < N/s$, and for all $u \in \dot{W}_p^s(\mathbb{R}^N\backslash\{0\})$ in case $p > N/s$,

$$\int_{\mathbb{R}^N} \int_{\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \tag{3}$$

with

$$C_{N,s,p} := 2\int_0^1 r^{ps-1} \left|1 - r^{(N-ps)/p}\right|^p \Phi_{N,s,p}(r) dr, \tag{4}$$

and

$$\Phi_{N,s,p}(r) := |\mathbb{S}^{N-2}| \int_{-1}^{1} \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2rt+r^2)^{\frac{N+ps}{2}}}, \qquad N \geq 2,$$

$$\Phi_{1,s,p}(r) := \left(\frac{1}{(1-r)^{1+ps}} + \frac{1}{(1+r)^{1+ps}}\right), \qquad N = 1. \tag{5}$$

The constant $C_{N,s,p}$ is optimal. If $p = 1$, equality holds iff u is proportional to a symmetric decreasing function. If $p > 1$, the inequality is strict for any function $0 \not\equiv u \in \dot{W}_p^s(\mathbb{R}^N)$ or $\dot{W}_p^s(\mathbb{R}^N \backslash \{0\})$, respectively.

For $p = 1$ and, e.g., $N = 1$ or $N = 3$ one finds

$$C_{1,s,1} = \frac{2^{2-s}}{s}, \qquad C_{3,s,1} = 4\pi \frac{2^{1-s}}{s(s-1)}.$$

For general values of p and N the double integral is easily evaluated numerically or estimated analytically (see also (37) and (38) below for different expressions). For $p = 2$ one can evaluate $C_{N,s,p}$ via Fourier transform [77] and obtains the well-known expression

$$C_{N,s,2} = 2\pi^{N/2} \frac{\Gamma((N+2s)/4)^2}{\Gamma((N-2s)/4)^2} \frac{\Gamma((N+2s)/2)}{|\Gamma(-s)|}. \tag{6}$$

This was first derived by Herbst [86]; see also [171], [178], [98] for different proofs. Indeed, Herbst determined the sharp constants in the inequality

$$\left\|(-\Delta)^{s/2} u\right\|_p^p \geq \tilde{C}_{N,s,p} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \tag{7}$$

for arbitrary $1 < p < N/s$. For $p = 2$ the left-hand side is well-known to be proportional to the left-hand side in (3). For $p \neq 2$ and $0 < s < 1$, however, the expression on the left-hand side is not equivalent to (2). There is a one-sided inequality according to whether $1 < p < 2$ or $p > 2$; see, e.g., [188]. In particular, the sharp constant $\tilde{C}_{N,s,1}$ in (7) for $p = 1$ is zero, as opposed to (3).

We follow the recent work by Bourgain, Brezis, and Mironescu [109], [134] and by Maz'ya and Shaposhnikova [182]. Consider the case $N > ps$, and recall that the Sobolev embedding theorem asserts that $\dot{W}_p^s(\mathbb{R}^N) \subset L_{p^*}(\mathbb{R}^N)$ for $p^* = Np/(N-ps)$ with

$$\int_{\mathbb{R}^N} \int_{\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq S_{N,s,p} \|u\|_{p^*}^p, \tag{8}$$

see, e.g., [167]. The optimal values of the constants $S_{N,s,p}$ are unknown. In [134] Bourgain, Brezis, and Mironescu obtained quantitative estimates on the constants $S_{N,s,p}$ which reflect the correct behavior in the limits $s \to 1$ or $p \to N/s$. (They studied the corresponding problem for functions on a cube with zero average, but this problem is equivalent to the problem on the whole space, see [134] or [182]). The proof in [134] relies on advanced tools from harmonic analysis. It was simplified and extended by Maz'ya and Shaposhnikova [182] who showed that the sharp constant in (8) satisfies

$$S_{N,s,p} \geq c(N,p) \frac{(N-ps)^{p-1}}{s(1-s)}. \tag{9}$$

The key observation in [182] was that (9) follows from a sufficiently good bound on the constant in the fractional Hardy inequality. Maz'ya and Shaposhnikova did not, however, determine the optimal constants in this inequality. Their bound

$$C_{N,s,p} \geq \tilde{c}(N,p)\frac{(N-ps)^p}{s(1-s)}, \tag{10}$$

which leads to the Bourgain–Brezis–Mironescu result (9), is easily recovered from our explicit expression for $C_{N,s,p}$.

In fact, below we show that our sharp Hardy inequality implies an even stronger result. Namely, together with a symmetrization argument it yields a simple proof of the embedding

$$\dot{W}_p^s(\mathbb{R}^N) \subset L_{p^*,p}(\mathbb{R}^N), \quad 1 \leq p < N/s, \; p^* = Np/(N-ps), \tag{11}$$

due to Peetre [186]. Here $L_{p^*,p}(\mathbb{R}^N)$ denotes the Lorentz space, the definition of which is recalled. Embedding (11) is optimal in the Lorentz scale. Since $L_{p^*,p}(\mathbb{R}^N) \subset L_{p^*}(\mathbb{R}^N)$ with strict inclusion, (11) is stronger than (8). While we know only of non-sharp proofs of (11) via interpolation theory, our Theorem (5.1.13) below gives the optimal constant in this embedding and characterizes all optimizers. To do so, we need to characterize the optimizers in the rearrangement inequality by Almgren and Lieb for the functional (2), see Theorem (5.1.16). For another recent application of Lorentz norms in connection with Hardy–Sobolev inequalities see [183].

In contrast to the case $p = 2$, there seems to be no way to prove (3) via Fourier transform if $p \neq 2$. Instead, our proof is based on the observation that $|x|^{-(N-ps)/p}$ is a positive solution of the Euler–Lagrange equation associated with (3) (but fails to lie in $\dot{W}_p^s(\mathbb{R}^N)$ or $\dot{W}_p^s(\mathbb{R}^N\backslash\{0\})$, respectively). Writing $u = |x|^{-(N-ps)/p}v$, (3) becomes an inequality for the unknown function v. While it is well known and straightforward to prove (1) in this way, this approach seems to be new in the fractional case.

We automatically yields remainder terms. In particular, for $p \geq 2$ we obtain the following strengthening of (3).

Theorem (5.1.2)[166]: (Sharp Hardy inequality with remainder). Let $N \geq 1, 0 < s < 1$ and $p \geq 2$. Then for all $u \in \dot{W}_p^s(\mathbb{R}^N)$ in case $p < N/s$, and for all $u \in \dot{W}_p^s(\mathbb{R}^N\backslash\{0\})$ in case $p > N/s$, and $v = |x|^{(N-ps)/p}u$,

$$\int_{\mathbb{R}^N}\int_{\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - C_{N,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}dx$$

$$\geq c_p\int_{\mathbb{R}^N}\int_{\times\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\frac{dx}{|x|^{(N-ps)/2}}\frac{dy}{|y|^{(N-ps)/2}} \tag{12}$$

where $C_{N,s,p}$ is given by (4) and $0 < c_p \leq 1$ is given by

$$c_p := \lim_{0<\tau<1/2}((1-\tau)^p - \tau^p + p\tau^{p-1}). \tag{13}$$

If $p = 2$, then (12) is an equality with $c_2 = 1$.

We refer to the substitution of u by $v = \omega^{-1}u$, where ωis a positive solution of the Euler– Lagrange equation of the functional under consideration, as ‘ground state substitution.’ In the linear and local case, such representations go back at least to Jacobi and have numerous applications, among others, in the spectral theory of Laplace and Schrödinger operators (see [172], [177] and also [82]), constructive quantum field theory (especially in the work by Segal, Nelson, Gross, and Glimm–Jaffe; see, e.g., [175]) and Allegretto Piepenbrink theory (developed in particular by Allegretto, Piepenbrink and Agmon; see, e.g., [181], [187]).

We derive anon-local and non-linear analog of such a representation. Despite all these applications, even in the linear case a non-local version of the ground state representation has only recently been found [77]. While we were only interested in a special case in [77], we show that this formula holds in a much more general setting. Moreover, for $p > 2$ we will find a non-linear analog of this representation formula in the form of an inequality. This is the topic where we consider functionals of the form (2) with $|x-y|^{-N-ps}$ replaced by an arbitrary symmetric and non-negative, but not necessarily translation invariant kernel.

We derive Hardy inequalities and ground state representations in a general setting and we apply this method to prove Theorems (5.1.1) and (5.1.2). W show that Theorem (5.1.1) implies the optimal Sobolev embedding (11) by using some facts about rearrangement in fractional Sobolev spaces.

We fix $N \geq 1$, $p \geq 1$ and a non-negative measurable function k on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying $k(x,y) = k(y,x)$ for all $x, y \in \mathbb{R}^N$. We provide a condition under which a Hardy inequality for the functional

$$E[u] := \int_{\mathbb{R}^N} \int_{\times \mathbb{R}^N} |u(x) - u(y)|^p \, k(x,y) dx dy$$

holds. Our assumption is that there exists a positive function ω satisfying the equation

$$2 \int_{\mathbb{R}^N} \big(\omega(x) - \omega(y)\big) |\omega(x) - \omega(y)|^{p-2} \, k(x,y) dy = V(x) \omega(x)^{p-1} \qquad (14)$$

for some real-valued function V on $\mathbb{R}^N$. We emphasize that if k is too singular on the diagonal (for instance, in our case of primary interest $k(x,y) = |x-y|^{-N-ps}, s > 0$) the integral on the left-hand side will not be convergent and some regularization of principal value type will be needed. We think of ω as the 'virtual ground state' corresponding to the energy functional $E[u] - \int V \, |u|^p dx$.

We formulate the precise meaning of (14) as

Assumption (5.1.3)[166]: Let ω be a positive, measurable function on $\mathbb{R}^N$. There exists a family of measurable functions k_ε, $\varepsilon > 0$, on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying $k_\varepsilon(x,y) = k_\varepsilon(y,x)$, $0 \leq k_\varepsilon(x,y) \leq k(x,y)$ and

$$\lim_{\varepsilon \to 0} k_\varepsilon(x,y) = k(x,y) \qquad (15)$$

for a.e. $x, y \in \mathbb{R}^N$. Moreover, the integrals

$$V_\varepsilon(x) := 2\omega(x)^{-p+1} \int_{\mathbb{R}^N} \big(\omega(x) - \omega(y)\big) |\omega(x) - \omega(y)|^{p-2} \, k_\varepsilon(x,y) dy \qquad (16)$$

are absolutely convergent for a.e. x, belong to $L_{1,Loc}(\mathbb{R}^N)$ and $V := \lim_{\varepsilon \to 0} V_\varepsilon$ exists weakly in $L_{1,Loc}(\mathbb{R}^N)$ i.e., $\int V_\varepsilon \, g dx \to \int V g dx$ for any bounded g with compact support.

The following is a general version of Hardy's inequality.

Example (5.1.4)[166]: A typical application of the ground state representation (33) in mathematical physics concerns pseudo-relativistic Schrödinger operators $\sqrt{-\Delta + m^2} + V_0$ with a constant $m \geq 0$. Indeed, the kinetic energy can be put into the form considered,

$$\int \sqrt{|\xi|^2 + m^2} \, |\hat{u}(\xi)|^2 d\xi = \int \int |u(x) - u(y)|^2 \, k_m(|x-y|) dx dy$$

where $\hat{u}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi.x} \, u(x) dx$ is the Fourier transform of u and

$$k_m(r) = \begin{cases} \left(\frac{m}{2\pi}\right)^{(N+1)/2} r^{-(N+1)/2} K_{(N+1)/2}(mr) & \text{if } m > 0, \\ \pi^{-(N+1)/2} 2^{-1} \Gamma((N+1)/2) r^{-N-1} & \text{if } m = 0, \end{cases}$$

with K_ν a Bessel function; see [144].

More generally, one can consider non-negative functions t and k on $\mathbb{R}^N$ related by

$$t(\xi) = 4 \int_{\mathbb{R}^N} k(x) \sin^2(\xi \cdot x/2) dx \tag{17}$$

and introduce the self-adjoint operator $T = t(D), D = -i\nabla$, in $L_2(\mathbb{R}^N)$ with quadratic form

$$E[u] := \int_{\mathbb{R}^N} t(\xi)|\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^N} \int_{\times\mathbb{R}^N} |u(x) - u(y)|^2 \, k(x-y) dx dy. \tag{18}$$

The last identity is a consequence of Plancherel's identity and (17). We assume that t is locally bounded and satisfies $t(\xi) \leq \text{const}\, |\xi|^{2s}$ for some $0 < s < 1$ and all large ξ and, similarly, that $k(x)$ is bounded away from the origin and satisfies $k(x) \leq \text{const}\, |x|^{-N-2s}$ for all small x. Under these assumptions, $H^s(\mathbb{R}^N) = W_2^s(\mathbb{R}^N)$ is contained in the form domain of T and we can consider the Schrödinger-type operator $T + V_0$ with a real-valued function $V_0 \in L_{d/(2s)}(\mathbb{R}^N) + L_\infty(\mathbb{R}^N)$. Put $\lambda_0 = \inf \text{spec}\ (T + V_0)$ and assume that a positive function ω satisfies

$$(T + V_0)\omega = \lambda_0 \omega$$

in the sense of distributions. (Note that we do not require λ_0 to be an eigenvalue and ω an eigen-function.) If ω is Hölder continuous with exponent s, then one easily verifies Assumption (5.1.3) and one obtains the ground state representation

$$\int_{\mathbb{R}^N} t(\xi)|\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} V_0(x)|u(x)|^2 dx - \lambda_0 \int_{\mathbb{R}^N} |u(x)|^2 dx$$
$$= \int\int |v(x) - v(y)|^2 \, \omega(x) k(x-y) \omega(y) dx dy \tag{19}$$

for all u in the form domain of T and $v = \omega^{-1} u$.

Before proving Propositions (5.1.7) and (5.1.8) we would like to recall their 'local' analogs. Since these facts are essentially well known we shall ignore some technical details. Let g be a positive function on $\mathbb{R}^N$ and put

$$\hat{E}[u] := \int_{\mathbb{R}^N} g|\nabla u|^p dx$$

(with the convention that this is infinite if u does not have a distributional derivative or if this derivative is not in $L_p(\mathbb{R}^N, g)$). Moreover, assume that ω is a positive weak solution of the weighted p-Laplace equation

$$-\text{div}\,(g|\nabla\omega|^{p-2}\nabla\omega) = V\omega^{p-1}. \tag{20}$$

We claim that for any u with $\hat{E}[u]$ and $\int V_+|u|^p \, dx$ finite one has

$$\hat{E}[u] \geq \int_{\mathbb{R}^N} V(x)|u(x)|^p dx. \tag{21}$$

This is clearly the analog of (28). To prove (21) we write $u = \omega v$ and use the elementary convexity inequality

$$|a+b|^p \geq |a|^p + p|a|^{p-2} Re\, \bar{a} \cdot b \tag{22}$$
for vectors $a, b \in \mathbb{C}^N$ and $p \geq 1$. This yields

$$\tilde{E}[u] = \int_{\mathbb{R}^N} g|v\nabla\omega + \omega\nabla v|^p dx$$

$$\geq \int_{\mathbb{R}^N} g|v|^p |\nabla\omega|^p dx + p \int_{\mathbb{R}^N} g|\nabla\omega|^{p-2}\omega\, Re\, \bar{v}|v|^{p-2}\, \nabla v \cdot \nabla\omega dx.$$

Recognizing the integrand in the last integral as $p^{-1} g\omega|\nabla\omega|^{p-2}\nabla\omega \cdot \nabla(|v|^p)$ and integrating by parts using (20) we arrive at (21).

Next we show that for $p \geq 2$, (21) can be improved to

$$\tilde{E}[u] - \int_{\mathbb{R}^N} V(x)|u(x)|^p dx \geq c_p \int g\omega^p |\nabla v|^p dx =: c_p \tilde{E}_\omega[v] \tag{23}$$

for $u = \omega v$ with $\tilde{E}[u], \int V_+ |u| dx$, and $\tilde{E}\omega[v]$ finite. This follows by the same argument as before if one uses instead of (22) its improvement

$$|a+b|^p \geq |a|^p + p|a|^{p-2} Re\, \bar{a} \cdot b + c_p |b|^p \tag{24}$$

for $p \geq 2$. One can show that c_p given in (24) is the sharp constant in this inequality.

Since (24) is an equality for $p = 2$ and $c_2 = 1$, so is (23). This is the ground state representation which is familiar from the spectral theory of differential operators. In the case $p \geq 2$, (23) can be used to derive remainder terms in Hardy's inequality on domains; see, e.g., [170].

Remark (5.1.5)[166]: In the case $g \equiv 1, N \neq p$, and with $\omega(x) = |x|^{-(N-p)/p}$ and $v(x) = |x|^{(N-p)/p}\, u(x)$, the local Hardy inequality with remainder term yields the following improvement of (1),

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{|N-p|}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx + c_p \int_{\mathbb{R}^N} |\nabla v|^p \frac{dx}{|x|^{N-p}}. \tag{25}$$

The constant c_p in (25) is sharp for any $p \geq 2$. For $N > p$, this can be shown by using a trial function of the form $u(x) = |x|^{-(N-p)/p+\alpha}$ for $|x| \leq 1$ and $u(x) = |x|^{-(N-p)/p-\varepsilon}$ for $|x| \geq 1$, letting $\varepsilon \to 0$ and choosing $\alpha = (N-p)/(p\tau)$ where $0 < \tau < 1/2$ is the minimizer in (24). Similarly, for $N < p$, we choose $u(x) = |x|^{-(N-p)/p+\alpha}$ for $|x| \geq 1$ and $u(x) = |x|^{-(N-p)/p+\varepsilon}$ for $|x| \leq 1$.

Lemma (5.1.6)[166]: Let $p \geq 1$. Then for all $0 \leq t \leq 1$ and $a \in \mathbb{C}$ one has

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t). \tag{26}$$

For $p > 1$ this inequality is strict unless $a = 1$ or $t = 0$. Moreover, if $p \geq 2$ then for all $0 \leq t \leq 1$ and all $a \in \mathbb{C}$ one has

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t) + c_p t^{p/2}|a-1|^p \tag{27}$$

with $0 < c_p \leq 1$ given by (24). For $p = 2$, (27) is an equality with $c_2 = 1$. For $p > 2$, (27) is a strict inequality unless $a = 1$ or $t = 0$.

Proof: To prove the first assertion note that for fixed $|a|$ the minimum of the left-hand side is clearly achieved for a real and positive. Since for $|a|^p < t$ the inequality is trivial, one may thus assume that $a \geq t^{1/p}$. The assertion then follows from the fact that the derivative with respect to a of $(a-t)^p/(a^p - t)$ vanishes only at $a = 1$.

To prove the second assertion, we may assume that $p > 2$, since (27) is an equality if $p = 2$. We first prove the assertion for real a. The function

$$f(a,t) := \frac{|a-t|^p - (1-t)^{p-1}(|a|^p - t)}{t^{p/2}|a-1|^p}.$$

diverges at $a = 1$, and its partial derivative with respect to a is given by

$$\frac{\partial f}{\partial a}(a,t) = \frac{p(1-t)^{p-1}}{t^{p/2}(a-1)|a-1|^p}\left(\frac{|a-t|^{p-2}(t-a)}{(1-t)^{p-1}} + \frac{|a|^{p-2}a-t}{1-t}\right).$$

For $a > 1 > t$ this is negative, as follows from the first assertion with p replaced by $p-1$. Hence for all $a > 1$,

$$f(a,t) \geq f(+\infty,t) = t^{-p/2}(1-(1-t)^{p-1}).$$

An elementary calculation shows that the latter function is decreasing for $t \in (0,1)$. This proves that $f(a,t) \geq 1$ for $a > 1$.

Next, we claim that f does not attain its minimum in the interior of the region $\{(a,t): -\infty < a < 1, 0 < t < 1\}$. To see this, we write the partial derivative of f with respect tot as

$$\frac{\partial f}{\partial t}(a,t) = \frac{(1-t)^{p-1}}{2t^{(p+2)/2}|a-1|^p}\left(p(t+a)\left(\frac{|a-t|^{p-2}(t-a)}{(1-t)^{p-1}} + \frac{|a|^{p-2}a-t}{1-t}\right)\right.$$
$$\left. + \frac{t}{1-t}\left((|a|^p-1)(p-2) - ap(|a|^{p-2}-1)\right)\right).$$

The first line vanishes in case $\partial f/\partial a = 0$. Moreover, it is easy to see that the second line is nonzero for $a \in (-\infty,1)\backslash\{-1\}$. In fact, it is positive if $a \in (-\infty,-1)$ and negative if $a \in (-1,0\,]$. If $0 < a < 1$, it is negative in view of

$$\frac{a^p-1}{a^{p-1}-a} > \frac{p}{p-2}.$$

(The latter inequality holds since the left-hand side is strictly monotone decreasing.) To treat $a = -1$ one checks that $\partial f/\partial a\,(-1,t) \neq 0$ for $0 < t < 1$. This proves that f does not attain its minimum in the interior of the region $\{(a,t): -\infty < a < 1, 0 < t < 1\}$.

Now we examine f on the boundary of that region. Similarly as above, we have $\lim_{a\to-\infty} f(a,t) \geq 1$ uniformly in $t \in (0,1)$. Moreover, $\lim_{t\to 0} f(a,t) = +\infty$ uniformly in $a < 1$, and $\lim_{t\to 1} f(a,t=1)$ uniformly in $a \leq 1-\varepsilon$ for all $\varepsilon > 0$. Finally, $\lim_{a\to 1} f(a,t) = +\infty$ uniformly in $t \in (0,1-\varepsilon)$ for all $0 < \varepsilon < 1$. Thus it remains to study the limit $a \to 1$ and $t \to 1$. For given $\tau > 0$ we let $a \to 1$ and $t \to 1$ simultaneously with $1-t = \tau(1-a)$ and find

$$\lim_{a\to 1} f(a,1-\tau(1-a)) = |1-\tau|^p - \tau^p + p\tau^{p-1} \geq c_p.$$

The last inequality follows from the definition of c_p and the fact that the minimum over τ is attained for $\tau \in (0,1/2)$. This proves that $f(a,t) > c_p$ for all $a \in \mathbb{R}\backslash\{1\}$ and $0 < t < 1$.

Finally, we assume that a is an arbitrary complex number. We write $a - t = x + iy$ with x and y real and put $\beta := |a-t|$. What we want to prove is that for all $\beta \geq 0$ and $x \in (-\beta,\beta)$ one has

$$(1-t)^{p-1}(\beta^2+2tx+t^2)^{p/2} + c_p t^{p/2}(\beta^2 - 2(1-t)x + (1-t)^2)^{p/2}$$
$$\leq \beta^p + (1-t)^{p-1}t.$$

But for fixed β, the left-hand side is a convex function of x in the interval $(-\beta, \beta)$, so its maximum will be attained either at $x = \beta$ or $x = -\beta$, that is, for real values of $a - t$. This reduces the assertion in the complex case to the real case and completes the proof.

Proposition (5.1.7)[166]: Under Assumption (23), for any u with compact support and $\int V_+ |u|^p dx$ and finite one has

$$E[u] \geq \int_{\mathbb{R}^N} V(x)\, |u(x)|^p dx. \tag{28}$$

In applications where additional properties of k and V are available, the assumption that u has compact support can typically be removed by some limiting argument. It appears here because we want to work with the rather minimal Assumption (23).

Our next result improves this in the case $p \geq 2$ by giving an explicit remainder estimate. It involves the functional

$$E_\omega[v] := \int_{\mathbb{R}^N \times \mathbb{R}^N}\int |v(x) - v(y)|^p\, \omega(x)^{\frac{p}{2}} k(x,y) \omega(x)^{\frac{p}{2}} dx dy$$

and is a non-linear analog of what is known as 'ground state representation formula.'

Proof : We may assume that $\int V_- |u|^p dx < \infty$ for otherwise there is nothing to prove. Replacing u by u min $\{1, M|u|^{-1}\}$ and letting $M \to \infty$ using monotone convergence, we may assume that u is bounded. Recall also that u is assumed to have compact support.

We write $u = \omega v$, multiply (17) by $|v(x)|^p \omega(x)^p$ and integrate with respect to x. After symmetrizing with respect to x and y(recall that $k_\varepsilon(x,y) = k_\varepsilon(y,x)$) we obtain

$$\int_{\mathbb{R}^N \times \mathbb{R}^N}\int (|v(x)|^p\, \omega(x) - |v(y)|^p \omega(y))(\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k_\varepsilon(x,y) dx dy$$

$$= \int_{\mathbb{R}^N} V_\varepsilon\, (x) |u(x)|^p dx.$$

We write this as

$$\int_{\mathbb{R}^N \times \mathbb{R}^N}\int \Phi_u\, (x,y)\, k_\varepsilon(x,y) dx + \int_{\mathbb{R}^N} V_\varepsilon\, |u|^p dx$$

$$= \int_{\mathbb{R}^N \times \mathbb{R}^N}\int |\omega(x) - u(y)|^p k_\varepsilon(x,y) dx dy \tag{29}$$

where

$$\Phi_u(x,y) := |\omega(x)v(x) - \omega(y)v(y)|^p - (\omega(x)|v(x)|^p - \omega(y)|v(y)|^p)\big(\omega(x) - \omega(y)\big)|\omega(x) - \omega(y)|^{p-2}. \tag{30}$$

We claim that $\Phi_u \geq 0$ point wise. To see this, we may by symmetry assume that $\omega(x) \geq \omega(y)$. Putting $t = \omega(y)/\omega(x), a = v(x)/v(y)$ and applying (26) we deduce that $\Phi_u \geq 0$.

Now we pass to the limit $\varepsilon \to 0$ in (29). Since $|u|^p$ is bounded with compact support and$V\varepsilon \to V$weakly in $L_{1,Loc}$, the integral containingVεconverges. The other two terms converge by dominated convergence since $0 \leq k_\varepsilon \leq k$, and we obtain

$$\int_{\mathbb{R}^N\times \mathbb{R}^N}\int \Phi_u(x,y)\,k(x,y)dx + \int_{\mathbb{R}^N} V\,|u|^p dx = E[u]. \tag{31}$$

This implies the assertion since $\Phi_u \geq 0$.

We shall prove Propositions(5.1.7) and (5.1.8) after having discussed a typical application and having explained their analogs involving derivatives.

We are mostly interested in the case where $k(x,y) = |x-y|^{-N-ps}$which enters in (3). For this particular choice of the kernel and for $p = 2$, ground state representation (33) (with equality) was proved in [77]. The results for general kernels k seem to be new, even in the linear case $p = 2$.

Proposition (5.1.8)[166]: Let $p \geq 2$. Under Assumption (5.1.3), for any u with compact support write $u = \omega v$ and assume that $E[u], \int V_+\,|u|^p dx$, and $E_\omega[v]$ are finite. Then

$$E[u] - \int_{\mathbb{R}^N} V(x)\,|u(x)|^p dx \geq c_p E_\omega[v] \tag{32}$$

with c_p from(13).If $p = 2$, then (33) is an equality with $c_p = 1$.

Proof : The proof is similar to that of Proposition (5.1.7), using (27) instead of (26). We omit the details.

We fix $N \geq 1, 0 < s < 1$ and $p \neq N/s$ and abbreviate

$$\alpha := (N-ps)/p.$$

We will deduce the sharp Hardy inequality (3) using the general approach with the choice

$$\omega(x) = |x|^{-\alpha}\,, \quad k(x,y) = |x-y|^{-N-ps}\,, \quad V(x) = C_{N,s,p}|x|^{-ps}\,. \tag{33}$$

We begin the proof of Theorem (5.1.1) by verifying that ω solves the Euler–Lagrange equation associated with (3).

Lemma (51.9) [166]: One has uniformly for x from compacts in $\mathbb{R}^N\backslash\{0\}$

$$2\lim_{\varepsilon\to 0}\int_{||x|-|y||>\varepsilon}\big(\omega(x)-\omega(y)\big)\,|\omega(x)-\omega(y)|^{p-2}k(x,y)dy = \frac{C_{N,s,P}}{|x|^{ps}}\,\omega(x)^{P-1} \tag{34}$$

with $C_{N,s,P}$ from (4).

Proof: First note that it suffices to prove the convergence (34) for a fixed $x \in \mathbb{R}^N\backslash\{0\}$, since the uniformity will then follow by a simple scaling argument. Now the integral on the left-hand side of (34) is absolutely convergent for any $\varepsilon > 0$ and after integrating out the angles it can be written as

$$r^{-N+1}\int_{|\rho-r|>\varepsilon}\frac{\operatorname{sgn}(\rho^\alpha - r^\alpha)}{|\rho - r|^{2-p(1-s)}}\,\varphi(\rho,r)d\rho \tag{35}$$

where $r = |x|$,

$$\varphi(\rho,r) = \left|\frac{\rho^{-\alpha}-r^{-\alpha}}{r-\rho}\right|^{p-1}\cdot\begin{cases}\rho^{N-1}\left(1-\frac{r}{\rho}\right)^{1+ps}\Phi\left(\frac{\rho}{r}\right), & \text{if } \rho < r,\\ r^{N-1}\left(1-\frac{r}{\rho}\right)^{1+ps}\Phi\left(\frac{r}{\rho}\right), & \text{if } \rho > r,\end{cases} \tag{36}$$

and $\Phi = \Phi_{N,s,p}$ given in (5). Since $p(1-s) > 0$, the convergence of the integral in (35) for $\varepsilon \to 0$ will follow if we can show that $\varphi(\rho,r)$ is Lipschitz continuous as a function of

ρ at $\rho = r$. For this we only need to prove that $(1-t)^{1+ps}\Phi(t)$ and its derivative remain bounded as $t \to 1-$.

For $N = 1$ this is obvious and hence we restrict ourselves to the case $N \geq 2$ in the following. One can prove the desired property either directly using elementary estimates or, as we shall do here, deduce it from properties of special functions. According to [176]

$$\Phi(t) = |\mathbb{S}^{N-1}| B\left(\frac{N-1}{2}, \frac{1}{2}\right) F\left(\frac{N+ps}{2}, \frac{ps+2}{2}, \frac{N}{2}; t^2\right)$$

where $F(a,b,c;z)$ is a hypergeometric function. If $a+b-c>1$ the both $(1-z)^{a+b-c}F(a,b,c;z)$ and its derivative

$$\frac{d}{dz}\left((1-z)^{a+b-c} F(a,b,c;z)\right) = \frac{(c-a)(c-b)}{c}(1-z)^{a+b-c} F(a,b,c+1;z)$$

have a limit as $z \to 1-$; see [180]. Since $a+b-c = 1+ps > 1$ in our situation, one easily deduces that $(1-t)^{1+ps}\Phi(t)$ and its derivative have a limit as $t \to 1-$.

This argument gives (34) with $C_{N,s,P}$ replaced by the constant

$$\acute{C}_{N,s,P} \coloneqq 2 \lim_{\varepsilon \to 0} \int_{|\rho - r| > \varepsilon} \frac{\operatorname{sgn}(\rho^{\alpha} - 1)}{|\rho - r|^{2-p(1-s)}} \varphi(\rho, 1) d\rho. \tag{37}$$

To see that this constant coincides with (4), we change variables $\rho \mapsto \rho^{-1}$ in the integral on $(1+\varepsilon, \infty)$. Recalling the properties of φ we can pass to the limit $\varepsilon \to 0$ and obtain

$$\acute{C}_{N,s,P} = 2(\operatorname{sgn}\alpha) \int_0^1 \left(\rho^{-p(1-s)}\varphi(\rho^{-1}, 1) - (\rho, 1)\right) \frac{d\rho}{(1-\rho)^{2-p(1-s)}}$$

$$= 2 \int_0^1 \rho^{ps-1} |1-\rho^{\alpha}|^p \Phi(\rho) d\rho,$$

which is (4) for $N \geq 2$. The proof for $N = 1$ is similar.

Remark (5.1.10)[166]: It is possible to express the sharp Hardy constant as an N-dimensional double integral

$$C_{N,s,P} = \frac{|N-ps|}{p} \frac{2}{|\mathbb{S}^{N-1}|} \int_{\{|x|<1<|y|\}} \left| |x|^{-\frac{N-ps}{p}} - |y|^{-\frac{N-ps}{p}} \right|^{p-1} \frac{dxdy}{|x-y|^{N+ps}}. \tag{38}$$

To see this in the case $N > ps$, we multiply the integral in (34) by $\chi B(x)$, the characteristic function of the unit ball $B \subset \mathbb{R}^N$, and integrate with respect to x. After symmetrizing with respect to the variables x, $y$and passing to the limit $\varepsilon \to 0$, we find

$$\int_{\mathbb{R}^N} \int_{\times \mathbb{R}^N} \left(\chi B(x) - \chi B(y)\right)\left(\omega(x) - \omega(y)\right)|\omega(x) - \omega(y)|^{p-2} k(x,y) dxdy$$

$$= C_{N,s,P} \int_B \frac{\omega(x)^{p-1}}{|x|^{ps}} dx.$$

Performing the integration on the right-hand side yields (38) for $N > ps$. In the case $N < ps$,
we multiply (34) by $1 - \chi B(x)$ and proc ed similarly.

We apply the general approach with k, ω, Vas in (33) and

$$k_\varepsilon(x,y) := \begin{cases} k(x,y) & \text{if } \big||x|-|y|\big| > \varepsilon, \\ 0 & \text{if } \big||x|-|y|\big| \le \varepsilon, \end{cases} \tag{39}$$

For simplicity, let $\mathcal{Q} = \dot{W}_p^s(\mathbb{R}^N)$ if $N > ps$, and $\mathcal{Q} == \dot{W}_p^s(\mathbb{R}^N\backslash\{0\})$ if $N < ps$. In Lemma (5.1.19) we have verified that the modification of Assumption (5.1.3) mentioned is satisfied for $\Omega = \mathbb{R}^N\backslash\{0\}$. Inequalities (3) and (12) for $u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$ are an immediate consequence of Propositions (5.1.7) and (5.1.8). By density they extend to the homogeneous Sobolev space $\mathcal{Q}$.

Next we shall prove that for $p > 1$, inequality (3) is strict for all $0 \not\equiv u \in \mathcal{Q}$. (Note that for $p \ge 2$ this is an immediate consequence of (12).) We start from identity (31) which was proved for bounded functions u with compact support in $\mathbb{R}^N\backslash\{0\}$ and with $\int V\,|u|^p dx$ and $\int\int |u(x)-u(y)|^p\, k(x,y)dx$ finite. By a standard approximation argument, this identity extends to any $u \in \mathcal{Q}$ with Φ_u given by (30) and $u = \omega v$.

Assume that (28) holds with equality for some $u \in \mathcal{Q}$, and hence also for $|u|$. Since $\Phi|u|$ is non-negative and k is strictly positive, it follows from (31) that $\Phi|u| \equiv 0$. Since $p > 1$ this implies that $\omega(x)^{-p}|u(x)|^p$ is a constant (see Lemma (5.1.6)), whence $u \equiv 0$. This proves that inequality (28) is strict for any $0 \not\equiv u \in \mathcal{Q}$ if $p > 1$.

To prove that the constant $C_{N,s,P}$ in (28) is optimal we first assume that $N > ps$ and use a family of trial functions $u_n \in \dot{W}_p^s(\mathbb{R}^N)$ which approximate the 'virtual ground state' ω. For any integer $n \in \mathbb{N}$ we divide $\mathbb{R}^N$ into three regions,

$$I := \{x \in \mathbb{R}^N : 0 \le |x| < 1\},$$
$$M_n := \{x \in \mathbb{R}^N : 1 \le |x| < n\},$$
$$O_n := \{x \in \mathbb{R}^N : |x| \ge n\},$$

and define the functions

$$u_n(x) := \begin{cases} 1 - n^{-\alpha} & \text{if } x \in I, \\ |x|^{-\alpha} - n^{-\alpha} & \text{if } x \in M_n, \\ 0 & \text{if } x \in O_n, \end{cases}$$

These functions belong to $W_p^1(\mathbb{R}^N)$ and hence also to $\dot{W}_p^s(\mathbb{R}^N)$. Similarly as in the proof of Proposition (5.1.7) we integrate the right-hand side of (34) against $u_n(x)$ and symmetrize with respect to the variables. One easily shows that in the limit $\varepsilon \to 0$ one obtains

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \big(u_n(x)-u_n(y)\big)\big(\omega(x)-\omega(y)\big)|\omega(x)-\omega(y)|^{p-2}k(x,y)dxdy$$
$$= C_{N,s,P} \int_{\mathbb{R}^N} \frac{u_n(x)\omega(x)^{p-1}}{|x|^{ps}}dx. \tag{40}$$

Here we use the same abbreviations as in (33). The left-hand side of (40) can be rewritten as

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} |u_n(x)-u_n(y)|^p\, k(x,y)dxdy + 2\mathcal{R}_0$$

with

$$\mathcal{R}_0 := \int_{x\in I,\, y\in M_n}\int (1-\omega(y))\left(\omega(x)-\omega(y)\right)^{p-1} - (1-\omega(y)^{p-1})\;\; k(x,y)dxdy$$

$$+ \int_{x\in M_n,\, y\in O_n}\int (\omega(x)-n^{-\alpha})\left(\left(\omega(x)-\omega(y)\right)^{p-1} - (\omega(x)-n^{-\alpha})^{p-1}\right) k(x,y)dxdy$$

$$+ \int_{x\in I,\, y\in O_n}\int (1-n^{-\alpha})\left(\left(\omega(x)-\omega(y)\right)^{p-1} - (1-n^{-\alpha})^{p-1}\;\right) k(x,y)dxdy.$$

It follows from the explicit form of $\omega(x)$ that the integrands in all three integrals are pointwise non-negative, hence

$$\mathcal{R}_0 \geq 0. \tag{41}$$

On the other hand, the right-hand side of (40) divided by $C_{N,s,P}$ can be rewritten as

$$\int_{\mathbb{R}^N} \frac{u_n^p}{|x|^{ps}} dx + \mathcal{R}_1 + \mathcal{R}_2$$

with

$$\mathcal{R}_1 := \int_I (1-n^{-\alpha})\,(-\omega(x)^{p-1} - (1-n^{-\alpha})^{p-1})\;\; \frac{dx}{|x|^{ps}},$$

$$\mathcal{R}_2 := \int_{M_n} (\omega(x)-n^{-\alpha})\,(\omega(x)^{p-1} - (\omega(x)-n^{-\alpha})^{p-1})\;\; \frac{dx}{|x|^{ps}}.$$

Again both terms are non-negative and we shall show below that

$$\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{O}\,(1) \quad \text{ as } n \to \infty. \tag{42}$$

Since obviously $\int u_n^p\, |x|^{-ps}\, dx \to \infty$ as $n \to \infty$ we conclude from (41) and (42) that

$$\frac{\int\int_{\mathbb{R}^N\times\mathbb{R}^N} |u_n(x)-u_n(y)|^p k(x,y)dxdy}{\int_{\mathbb{R}^N} |u_n(x)|^p |x|^{-ps} dx}$$

$$= C_{N,s,P}\left(1 + \frac{\mathcal{R}_1+\mathcal{R}_2}{\int_{\mathbb{R}^N}|u_n(x)|^p|x|^{-ps}dx}\right) - \frac{2\mathcal{R}_0}{\int_{\mathbb{R}^N}|u_n(x)|^p|x|^{-ps}dx}$$

$$\leq C_{N,s,P}(1+o(1))$$

as $n \to \infty$. This shows that $C_{N,s,P}$ is sharp.

It remains to prove (42). Since the integrand in $\mathcal{R}_1$ is pointwise bounded by $\omega(x)^{p-1}|x|^{-ps} = |x|^{\alpha-N}$ we find that $\mathcal{R}_1 \leq \int_{|x|<1} |x|^{\alpha-N}\, dx < \infty$.To estimate $\mathcal{R}_2$ we use that $1-(1-t)^{\,p-1} \leq C_p t$ for $0 \leq t \leq 1$ with $C_p = 1$ for $1 \leq p \leq 2$ and $C_p = p^{-1}$ for $p > 2$. Hence the integrand in $\mathcal{R}_2$can be bounded according to

$$(\omega(x)-n^{-\alpha})(\omega(x)^{p-1} - (\omega(x)-n^{-\alpha})^{p-1}) \leq C_p n^{-\alpha}\omega(x)^{p-1}$$

and therefore after extending the integral to all $|x| < n$ and scaling $x \mapsto x/n$ we obtain $\mathcal{R}_2 \leq C_P \int_{|x|<1} |x|^{\alpha-N}\, dx < \infty$. This proves (42)

The case $N > ps$ is treated similarly, using a sequence of trial functions of the form

$$u_{n,m}(x) \coloneqq \begin{cases} 0 & \text{if } |x| \le 1/n, \\ |x|^{-\alpha} - n^{-\alpha} & \text{if } 1/n \le |x| \le 1, \\ (1-n^{-\alpha})\chi(|x|/m) & \text{if } |x| \ge 1, \end{cases}$$

where $0 \le \chi \le 1$ is a smooth, compactly supported function with $\chi(t) = 1$ for small t .After letting $m \to \infty$, the calculation proceeds along the same lines as above.

To conclude the proof of Theorem (5.1.1) we need to characterize the minimizers in the case $p = 1$. Actually, we present an alternative, simpler proof of inequality (3) in this case based on a symmetrization argument.

Note that the right-hand side of (3) remains unchanged if u is replaced by $|u|$, whereas the left-hand side does not increase. Indeed, it strictly decreases unless uis proportional to a non negative function. Moreover, under symmetric decreasing rearrangement the left-hand side of (3) does not increase (see [168] and also Theorem (5.1.16)), whereas the right-hand side does not decrease. Indeed, it strictly increases unless $|u|$ is symmetric decreasing (see [144]). This argument shows that any optimizer (provided it exists) will be proportional to a symmetric decreasing function. Below we show that (3) holds with equality for any symmetric decreasing u. By the previous argument this provides an alternative proof of Theorem(5.1.1) in the case $p = 1$.

A symmetric decreasing function u has a layer cake representation $u = \int_0^\infty \chi_t dt$ with χ_t the characteristic function of a ball centered at the origin with some radius $R(t)$. In this case the integral on the right-hand side of (3) equals

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{|x|^s} dx = \frac{|\mathbb{S}^{N-1}|}{N-s} \int_0^\infty R(t)^{N-1}\, dt,$$

and the integral on the left-hand side equals

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy = 2 \iint_{\{|x|<|y|\}} \frac{|\int(\chi_t(x)-\chi_t(x))|}{|x-y|^{N+s}} dxdy$$

$$= 2 \iiint_{\{|x|<R(t)<|y|\}} |x-y|^{-N-s}\, dxdydt$$

$$= 2 \iint_{\{|x|<I<|y|\}} |x-y|^{-N-s}\, dxdy \int_0^\infty R(t)^{N-s}\, dt.$$

This shows that (3) holds with equality for any symmetric decreasing function.

Let $1 \le q < \infty, 1 \le r \le \infty$ and recall that the Lorentz space $L_{q,r}(\mathbb{R}^N)$ consists of those measurable functions u on $\mathbb{R}^N$ for which the following quasinorm is finite,

$$\|u\|_{q,r} \coloneqq \left(q \int_0^\infty \mu_u(t)^{r/q} t^{r-1}\, dt \right)^{1/r} \quad \text{if } 1 < r < \infty, \qquad \|u\|_{q,\infty} \coloneqq \sup_{t>0} \mu_u(t)^{1/q}\, t$$

Here $\mu_u(t) := \{x \in \mathbb{R}^N : |u(x)| > t\}$denotes the distribution function of u. Note that $L_{q,q}(\mathbb{R}^N) = L_q(\mathbb{R}^N)$ and that one has strict inclusions $L_{q,r}(\mathbb{R}^N) \subset L_{q,s}(\mathbb{R}^N)$for $r < s$. A classical result by Peetre [186] states that the standard Sobolev embedding $\dot{W}_p^s(\mathbb{R}^N) \subset L_{p^*}(\mathbb{R}^N), p^* = Np/(N-ps)$ for $N > ps$, can be improved to $\dot{W}_p^s(\mathbb{R}^N) \subset L_{p^*,p}(\mathbb{R}^N)$.

Peetre's proof is based on interpolation and requires $p > 1$. We refer to [189] for more elementary interpolation arguments, including the case $p = 1$.

Here we give a direct proof of this embedding which avoids interpolation. It is based on symmetrization and leads to sharp constants.

Corollary (5.1.11)[166]: Let $N \geq 1, 0 < s < 1, 1 \leq p < N/s$ and $p \leq r \leq \infty$. *Put* $p^* = Np/(N - ps)$. Then $\dot{W}_p^s(\mathbb{R}^N) \subset L_{p^*,r}(\mathbb{R}^N)$ and

$$\|u\|_{p^*,r} \leq \left(\frac{p^*}{r}\right)^{1/r} \left(\frac{p}{p^*}\right)^{1/p} \left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{s/N} C_{N,s,P}^{-1/p} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\right)^{1/p} \quad (43)$$

Setting $r = p^*$ in (43) we recover the standard Sobolev inequality (8). Using the bound(10) on the constant, we recover the result (9) by Maz'ya and Shaposhnikova.

The link between Theorem (5.1.13) and the sharp Hardy inequality (3) is

Lemma (5.1.12)[166]: Let $0 < s \leq 1$ and $1 \leq p < N/s$. Then for any non-negative, symmetric decreasing u on $\mathbb{R}^N$

$$\|u\|_{p^*,p} = \left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{s/N} \left(\int_{\mathbb{R}^N} \frac{u^p}{|x|^{ps}} dx\right)^{1/p} \quad (44)$$

Proof: Introducing $w = u^p$ and $\mu = \mu_w$ we can rewrite the left-hand side of (44) as

$$\|u\|_{p^*,p}^p = \frac{p^*}{p} \int_0^\infty \mu(t)^{\frac{p}{p^*}} dt.$$

We write $w = \int_0^\infty \chi_t \, dt$ in its layer cake representation. Here χ_t is the characteristic function of $\{x: w(x) > t\}$, which is a ball of radius $(N\mu(t)/|\mathbb{S}^{N-1}|)^{1/N}$. Hence

$$\int_{\mathbb{R}^N} \frac{w}{|x|^{ps}} dx = \int_0^\infty \left(\int_{\mathbb{R}^N} \frac{\chi_t(x)}{|x|^{ps}} dx\right) dt = \frac{N^{\frac{N-ps}{N}}}{N - ps} |\mathbb{S}^{N-1}|^{\frac{ps}{N}} \int_0^\infty \mu(t)^{\frac{N-ps}{N}} dt,$$

proving (44) .

Theorem (5.1.13)[166]:(Sharp Sobolev inequality). Let $N \geq 1, 0 < s < 1, 1 \leq p < N/s$ and put $p^* = Np/(N - ps)$. Then $\dot{W}_P^s(\mathbb{R}^N) \subset L_{P^*,P}(\mathbb{R}^N)$ and

$$\|u\|_{P^*,P} \leq \left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{s/N} C_{N,s,P}^{-1/p} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\right)^{1/p} \quad (45)$$

for any $u \in \dot{W}_P^s(\mathbb{R}^N)$ with $C_{N,s,P}$ from (4). This constant is optimal. For $p = 1$ equality holds iff u is proportional to a non-negative function v such that the level sets $\{v > \tau\}$ are balls for a.e. τ. For $p > 1$ the inequality is strict for any $u \not\equiv 0$.

For $p = 1$ and u a characteristic function we obtain

$$|\Omega|^{(N-s)/N} \leq \frac{2(N - s)}{NC_{N,s,1}} \left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{s/N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{dxdy}{|x - y|^{N+s}}, \quad (46)$$

for any $\Omega \subset \mathbb{R}^N$ of finite measure, with equality iff Ω is a ball. Moreover, using that

$$\|u\|_{q,r} \leq \left(\frac{q}{r}\right)^{1/r} \left(\frac{p}{q}\right)^{1/p} \|u\|_{q,p}, \quad p < r$$

(which is easily proved using the layer cake representation for $\mu_u^{p/q}$ and Minkowski's inequality) one obtains

Proof : By symmetric decreasing rearrangement it suffices to prove (45) for symmetric decreasing u (see [168] and also Theorem (5.1.16)), for which it is an immediate consequence of Theorem (5.1.1) and Lemma (5.1.12). The sharpness of the constant and the non-existence of optimizers for $p > 1$ follows Theorem (5.1.1). For $p = 1$ one uses the characterization of equality in the rearrangement inequality in Theorem (5.1.16).

Remark (5.1.14)[166]: The 'local' analog of (45) for $s = 1$ is

$$\|u\|_{p^*,p} \leq \left(\frac{N}{|\mathbb{S}^{N-1}|}\right)^{1/N} \frac{p}{N-p}\left(\int_{\mathbb{R}^N} |\nabla u(x)|^p\,dx\right)^{1/p} \tag{47}$$

for $N \geq 2, 1 \leq p < N$ and $p^* = Np/(N-p)$. It is due to [185], [186]; the sharp constant in this case was found by Alvino [169]. Inequality (47) can be proved in the same way as Theorem (5.1.13), with the fractional Hardy inequality (3) replaced by the classical Hardy inequality (1).

Almgren and Lieb [168] have shown that the norm in $W_p^s(\mathbb{R}^N)$ does not increase under rearrangement. Since we have not found a characterization of the cases of equality in the literature, we include a proof. The special case $p = 1$ has been used in the proof of Theorem (5.1.13).

Lemma (5.1.15)[166]: Let J be a non-negative, convex function on $\mathbb{R}$ with $J(0) = 0$ and let $k \in L_1(\mathbb{R}^N)$ be a symmetric decreasing function. Then for all non-negative measurable u with $E[u]$ and $|\{u > \tau\}|$ finite for all $\tau > 0$ one has

$$E[u] \geq E[u^*]. \tag{48}$$

If, in addition, J is strictly convex and k is strictly decreasing, then equality holds iff u is a translate of a symmetric decreasing function. If $J(t) = |t|$, then equality holds iff the level sets $\{u > \tau\}$ are balls for $.e.\ \tau > 0$.

Inequality (48) under the additional assumptions $J(t) = J(-t)$ and $\int J\big(u(x)\big)dx < \infty$ is due to Almgren and Lieb [168]. The characterization of cases of equality seems to be new.

Proof: As in [144] we can write $J = J_+ + J_-$ with $J_+(t) = J(t)$ for $t \geq 0$ and $J_+(t) = 0$ for $t < 0$. We decompose $E = E_+ + E_-$ accordingly. Below we prove the assertion of the lemma with E replaced by E_+. The assertion for E_- (and hence for the original E) follows by exchanging the roles of x and y replacing $J(t)$ by $J(-t)$. Note that this argument yields a characterization of cases of equality under the weaker assumption that J is strictly convex on either $\mathbb{R}_+$ or $\mathbb{R}_-$.

Step 1. We first prove the assertion under the additional assumption that u is bounded. Since $J_+($ is convex it has a right derivative $\acute{J}_+$, which is non-negative and non-decreasing. Writing $J_+(t) = \int_0^t \acute{J}_+(\tau)d\tau$ one finds

$$J_+(u(x)-u(y)) = \int_0^\infty \acute{J}_+(u(x)-\tau)\,\chi_{\{u\leq\tau\}}(y)d\tau,$$

and hence by Fubini

$$E_+[u] = \int_0^\infty e_\tau^+[u]d\tau \tag{49}$$

where

$$e_\tau^+[u] := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \acute{J}_+(u(x)-\tau)k(x-y)\,\chi_{\{u\le\tau\}}(y)dxdy. \tag{50}$$

Since u is bounded and $|\{u>\tau\}|<\infty$ one has $\int_{\mathbb{R}^N} \acute{J}_+(u(x)-\tau)dx<\infty$. Writing $\chi_{\{u\le\tau\}} = 1-\chi_{\{u>\tau\}}$ we obtain

$$e_\tau^+[u] = \|k\|_1 \int_{\mathbb{R}^N} \acute{J}_+(u(x)-\tau)dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \acute{J}_+(u(x)-\tau)k(x-y)\,\chi_{\{u>\tau\}}(y)dxdy. \tag{51}$$

The first integral on the right-hand side of (51) does not change under rearrangement. Moreover, we note that $(\acute{J}_+(u-\tau))^* = \acute{J}_+(u^*-\tau)$. By Riesz's rearrangement inequality, the double integral on the right-hand side of (51) does not decrease under rearrangement, proving $e_\tau^+[u]\ge e_\tau^+[u^*]$ and hence $E_+[u]\ge E_+[u^*]$.

To characterize the cases of equality assume that k is strictly decreasing and $E_+[u] = E_+[u^*]$ for some bounded u. Then by (51) $e_\tau^+[u] = e_\tau^+[u^*]$ for $a.e.\,\tau$, and by Lieb's strict rearrangement inequality [179] for $a.e.\,\tau>0$ there is an $a_\tau\in\mathbb{R}^N$ such that $\chi_{\{u<\tau\}}(x) = \chi_{\{u^*<\tau\}}(x-a_\tau)$ and

$$\acute{J}_\pm(u(x)-\tau) = \acute{J}_\pm(u^*(x-a_\tau)-\tau) \tag{52}$$

for a.e. x. If $J_+(t) = t_+$ for all t, this means that $\{u>\tau\}$ is a ball for a.e. $\tau>0$. Now assume that J_+ is strictly convex on $\mathbb{R}_+$. Then J_+ is strictly increasing on $\mathbb{R}_+$ and we conclude that $(u(x)-\tau)_+ = (u^*(x-a_\tau)-\tau)_+$ for a.e. τ and x. This is easily seen to imply that a_τ is independent of τ, and hence u is a translate of a symmetric decreasing function.

Step 2. Now we remove the assumption that u is bounded, that is, we claim that (48) holds for any non-negative u with $E[u]$ and $|\{u>\tau\}|$ finite for all τ. To see this, replace u by $u_M = \min\{u, M\}$ and note that $(u_M)^* = (u^*)_M =: u_M^*$ and $E[u_M]\le E[u]$. By monotone convergence the claim follows easily from $E[u_M]\ge E[u_M^*]$.

Step 3. Finally, we characterize the cases of equality for general u. Assume that k is strictly decreasing and $E_+[u] = E_+[u^*]$ for some non-negative u with $E[u]$ and $|\{u>\tau\}|$ finite for all τ. For any $M>0$ we decompose

$$u = u_M + v_M, \quad u_M := \min\{u, M\},$$

and find

$$E_+[u] = E_+[u_M] + E_+[v_M] + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_M\big(v_M(x), u_M(y)\big)k(x-y)dxdy \tag{53}$$

with

$$F_M(v,u) := J_+(v+M-u) - J_+(v) - J_+(M-u).$$

Note that since J_+ is convex with $J_+(0)=0$, one has $F_M(v,u)\ge 0$ for $0\le u\le M$ and $v\ge 0$. Hence all three terms on the right-hand side of (53) are non-negative and finite. Note that replacing u by u^* amounts to replacing u_M and v_M by u_M^* and v_M^*, respectively. Below we shall prove that the double integral in (53) does not increase if both u_M and v_M are replaced by u_M^* and v_M^*. Moreover, by Step 2, $E^+[v_M]\ge E^+[v_M^*]$. Hence if $E^+[u] = E^+[u^*]$, then $E^+[u_M] = E^+[u_M^*]$ for all $M>0$. Using the characterization from Step 1 one easily concludes that u is of the form stated in the lemma.

It suffices to prove that the double integral in (53) does not increase under rearrangement. Since $\acute{J}_+$ is increasing, we have $\acute{J}_+(t) = \int_0^t d\mu(\tau)$ for a non-negative measure μ . Hence $J_+(t) = \int_0^\infty (t-\tau)_+ d\mu(\tau)$ and

$$F_M(v,u) = \int_0^\infty f_{M,\tau}(v,u)d\mu(\tau),$$

$$f_{M,\tau}(v,u) := (v+M-u-\tau)_+ - (v-\tau)_+ - (M-u-\tau)_+.$$

Since the integrand is non-negative for $0 \le u \le M$ and $v \ge 0$, it suffices to prove that for all τ the double integral

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int f_{M,\tau}\left(v_M(x), u_M(y)\right) k(x-y)dxdy$$

does not increase under rearrangement. We decompose further $f_{M,\tau} = f_{M,\tau}^{(1)} - f_{M,\tau}^{(2)}$ where $f_{M,\tau}^{(1)}(v) := v - (v-\tau)_+$ and

$$f_{M,\tau}^{(2)}(v,u) := v - (v+M-u-\tau)_+ + (M-u-\tau)_+ \min\{v, (u-M+\tau)_+\}.$$

Since $f_{M,\tau}^{(1)}$ is bounded and the support of v_M has finite measure, the integral

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int f_{M,\tau}^{(1)}\left(v_M(x)\right) k(x-y)dxdy = \|k\|_1 \int_{\mathbb{R}^N} f_{M,\tau}^{(1)}\left(v_M(x)\right)dx$$

is finite and invariant under rearrangement of v_M. Finally, by Fubini we can write

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int f_{M,\tau}^{(2)}\left(v_M(x), u_M(y)\right) k(x-y)dxdy$$

$$= \int_0^\infty \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \int \chi_{\{v_M > \tau\}}(x) k(x-y) \chi_{\{(u_M - M + \tau)_+ > t\}}(y)\, dxdy \right) dt.$$

By Riesz's rearrangement inequality, this does not decrease under rearrangement, completing the proof.

Theorem (5.1.16)[166]: Let $N \ge 1, 0 < s < 1, 1 \le p < N/s$ and $u \in \dot{W}_p^s(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \ge \int_{\mathbb{R}^N \times \mathbb{R}^N} \int \frac{|u^*(x)-u^*(y)|^p}{|x-y|^{N+ps}} dxdy. \quad (54)$$

If $p = 1$, then equality holds iff uis proportional to a non-negative function v such that the level set $\{v > \tau\}$ is a ball for $a.e.\tau > 0$.If $p > 1$, then equality holds iff u is proportional to a translate of a symmetric decreasing function.

Though we do not use the 'only if' statement for $p > 1$, we have included it since we think it is interesting in its own right. It might be compared with the result in the 'local case', namely, that if equality in $\int |\nabla u|^p dx \ge \int |\nabla u^*|^p dx$ is attained for a non-negative u, then the level sets of u are balls, but u is not necessarily a translate of a symmetric decreasing function; see [173].

We start by considering a slightly more general situation. For J a non-negative, convex function on $\mathbb{R}$ with $J(0) = 0$ and k a non-negative function on $\mathbb{R}^N$, we let

$$E[u] := \int_{\mathbb{R}^N \times \mathbb{R}^N} \int J\big(u(x)-u(y)\big)k(x-y)\,dxdy.$$

Proof: First note that $|u(x)-u(y)| \geq ||u(x)|-|u(y)||$, and that equality for all x, y holds iff u is proportional to a non-negative function. Hence we can restrict ourselves to non-negative functions. Writing as in [168]

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \int \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy = \frac{1}{\Gamma((N+ps)/2)} \int_0^\infty I_\alpha[u]\,\alpha^{(N+ps)/2-1}\,d\alpha \tag{55}$$

with

$$I_\alpha[u] := \int_{\mathbb{R}^N \times \mathbb{R}^N} \int |u(x)-u(y)|^p e^{-\alpha|x-y|^2}\,dxdy,$$

the assertion follows from Lemma (5.1.15).

Section (5.2): Fractional Hardy-Sobolev-Maz'ya Inequality for Domains

We are concerned here with the fractional Hardy inequality in an arbitrary domain $\Omega \subseteq \mathbb{R}^N$, which states that if $1 < p < \infty$ and $0 < s < 1$ with $ps > 1$, then

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy \geq \mathcal{D}_{N,p,s} \int_\Omega \frac{|u(x)|^p}{m_{ps}(x)^{ps}}\,dx \tag{56}$$

for all $u \in W^{os}_p(\Omega)$, the closure of $C_c^\infty(\Omega)$ with respect to the left side of (56).The pseudo distance $m_{ps}(x)$is defined in (59); its most important property for the present discussion is that for convex domain we have $m_{ps}(x) \leq \operatorname{dist}(x, \Omega^c)$. We denote by $\mathcal{D}_{N,p,s}$ the sharp constant in (56), which was recently found by Loss and Sloane [200] and is explicitly given in (67) below. This constant is independent of and coincides with that on the half space which was earlier found in [192], [197].

By the (well-known) Sobolev inequality the left side of (56) dominates an L_q-norm of u. Our main result, the fractional HSM inequality, states that the left side of (56), even after subtracting the right side, is still strong enough to dominate this L_q-norm.

We prove Theorem (5.2.8) in the particular case when $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$. Our starting point is the inequality

$$\iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy - \mathcal{D}_{N,p,s} \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{x_N^{ps}}\,dx \geq c_p J[v], \tag{57}$$

where c_p is an explicit, positive constant (for $p = 2$ this is an identity with $c_2 = 1$),

$$J[v] := \iint_{\mathbb{R}^N_+\times\mathbb{R}^N_+} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} (x_N y_N)^{ps-1}\,dxdy,$$

and $v(x) := x_N^{-(ps-1)/p} u(x)$. This inequality was derived in [197], using the `ground state representation' method from [166]. We note that $m_{ps}(x) = x_N$ in the case of a half space, as a quick computation shows (see also [200]).

In order to derive a lower bound on $J\,[v]$ we make use of the bound

$$(x_N y_N)^a \geq \min\{x_N^{2a}, y_N^{2a}\} = 2a \int_0^\infty \chi_{(t,\infty)}(x_N)\chi_{(t,\infty)}(y_N)t^{2a-1}dx$$

for $a > 0$. Combining this inequality with the fractional Sobolev inequality (see Lemma (5.2.1) below) and Minkowski's inequality, we can bound

$$J[v] \geq (ps-1)\int_0^\infty \iint_{x_N>t, y_N>t} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy\, t^{ps-2} dt$$

$$\geq (ps-1)C_{N,p,s}\int_0^\infty \left(\int_{x_N>t} |v(x)|^q\, dx\right)^{p/q} t^{ps-2}dt$$

$$\geq (ps-1)C_{N,p,s}\left(\int_{\mathbb{R}_+^N} |v(x)|^q \left(\int_0^{x_N} t^{ps-2}dt\right)^{q/p} dx\right)^{q/p}$$

$$= C_{N,p,s}\left(\int_{\mathbb{R}_+^N} |v(x)|^q\, x_N^{q(ps-1)/p} dx\right)^{p/q}.$$

Recalling the relation between u and v we arrive at (66). This completes the proof of Theorem (5.2.8) when $\Omega = \mathbb{R}_+^N$.

In the previous proof we used the Sobolev inequality on half-spaces for functions which do not necessarily vanish on the boundary. For the sake of completeness we include a short derivation of this inequality. The precise statement involves the closure $\dot{W}_p^s(\mathbb{R}_+^N)$ of $C_0^\infty(\overline{\mathbb{R}_+^N})$ with respect to the left side of (56).

Lemma (5.2.1)[190]: Let $N \geq 1, 1 \leq p < \infty$ and $0 < s < 1$ with $ps < N$. Then there is a constant $C_{N,p,s} > 0$ such that

$$\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq C_{N,p,s}\left(\int_{\mathbb{R}_+^N} |u(x)|^q\, dx\right)^{p/q}$$

for all $u \in \dot{W}_p^s(\mathbb{R}_+^N)$, where $q = Np/(N-ps)$.

Proof: If $\tilde{u}$ denotes the even extension of u to $\mathbb{R}^N$, then

$$\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} dxdy = 2\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy$$

$$+2\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N} \frac{|u(x)-u(y)|^p}{(|\acute{x}-\acute{y}|^2+(x_N+y_N)^2)^{(N+ps)/2}} dxdy$$

$$\leq 4\iint_{\mathbb{R}_+^N\times\mathbb{R}_+^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy.$$

On the other hand, by the `standard' fractional Sobolev inequality on $\mathbb{R}^N$ (see, e.g., [166] for explicit constants) the left side is an upper bound on

$$S_{N,p,s}\left(\int_{\mathbb{R}^N} |\tilde{u}(x)|^q dx\right)^{p/q} = 2^{p/q} S_{N,p,s}\left(\int_{\mathbb{R}_+^N} |u(x)|^q dx\right)^{p/q}.$$

Remark (5.2.2)[190]: The above proof of the fractional HSM inequality works analogously in the local case, that is, to show that

$$\int_{\mathbb{R}_+^N} |\nabla u|^q\, dx - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}_+^N} \frac{|\mathrm{u}|^p}{x_N^p} dx \geq \sigma_{N,p,1}\left(\int_{\mathbb{R}_+^N} |\mathrm{u}|^q\, dx\right)^{p/q}, q = \frac{Np}{N-p}, \quad (58)$$

for $u \in \dot{W}_p^1(\mathbb{R}_+^N)$ when $N \geq 3$ and $2 \leq p < N$. Again, the starting point [166] is to bound the left side from below by an explicit constant $c_p > 0$ times

$$\int_{\mathbb{R}_+^N} |\nabla v|^p x_N^{p-1} dx, \qquad v = x_N^{-(p-1)/p} u.$$

(For $p = 2$, this is an identity with $c_2 = 1$.) Next, we write $x_N^a = a\int_0^\infty \chi_{(t,\infty)}(x_N)t^{a-1}dt$ and use Sobolev's inequality on the half-space $\{x_N > t\}$ together with Minkowski's inequality. Note that the sharp constants in this half-space inequality are known explicitly (namely, given in terms of the whole-space constants via the reflection method of Lemma (5.2.1)).

The sharp constant in (58) for $p = 2$ and $N = 3$ was found in [191]. We think it would be interesting to investigate this question for the non-local inequality (66) and we believe that [202] is a promising step in this direction.

We show a fractional Hardy- Sobolev- Mażya inequality on the ball $B_r \subset \mathbb{R}^N$, $N \geq 2$, of radius r centered at the origin. The argument follows that from the previous, but is more involved.

This proves Theorem (5.2.8) in the special case $\Omega = B_r$ and $p = 2$ with $m_{2s}(x)$ replaced by $(r^2 - |x|^2)/2r$. We note that $(r^2 - |x|^2)/2r \leq \operatorname{dist}(x, B_r^c)$ for $x \in B_r$. (As an aside we note, however, that it is not always true that $(r^2 - |x|^2)/2r$ is greater than $m_{2s}(x)$. Indeed, take $x = 0$ and $N = 2$.)

We also note that Proposition (5.2.3) implies Theorem (5.2.8) for $\Omega = \mathbb{R}_+^N$ (and $p = 2$). Indeed, by translation invariance the proposition implies the inequality also on balls $B(a_r, r)$ centered at $ar = (0, \cdots, 0, r)$. We have $\operatorname{dist}(x, B(a_r, r)^c) \leq \operatorname{dist}(x, (\mathbb{R}_+^N)^c)$, and hence the result follows by taking $r \to \infty$.

Proposition (5.2.3)[190]: Let $N \geq 2, p = 2$ and $\frac{1}{2} < s < 1$. Then there is a constant $c = c(s, N) > 0$ such that for every $0 < r < 1$ and $u \in \dot{W}_2^s(B_r)$,

$$\int_{B_r}\int_{B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \mathcal{D}_{N,p,s}\int_{B_r} \frac{(2r)^{2s}}{(r^2 - |x|^2)^{2s}} |u(x)|^2 dx \geq c\left(\int_{B_r} |u(x)|^q dx\right)^{2/q}, \tag{59}$$

where $q = 2N/(N \quad 2s)$.

Proof: By scaling, we may and do assume that $r = 1$, that is, we consider only the unit ball $B_1 \subset \mathbb{R}^N$. We put $v = u/w_N$ with w_N defined in Lemma (5.2.5). According to that lemma, the left side of (59) is bound from below by

$$\tilde{J}[v] + c\int_{B_1} |v(x)|^2 w_N(x)^2 dx, \tag{60}$$

(Here we also used that $w_N \leq 1$.) For $x, y \in B_1$ we have

$$\begin{aligned} w_N(x)w_N(y) &\geq \min\{(1 - |x|^2)^{2s-1}, (1 - |y|^2)^{2s-1}\} \\ &= (2s - 1)\int_0^1 \chi_{(t,1]}(1 - |x|^2)\chi_{(t,1]}(1 - |y|^2)t^{2s-2}dt, \end{aligned}$$

and therefore,

$$\tilde{J}[v] + c\int_{B_1} |v(x)|^2 w_N(x)^2 dx$$
$$\geq (2s - 1)\int_0^1 \left(\int_{B\sqrt{1-t}}\int_{B\sqrt{1-t}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dxdy + c\int_{B\sqrt{1-t}} |v(x)|^2 dx\right) t^{2s-2}dt.$$

The fractional Sobolev inequality [193] and a scaling argument imply that there is a $\tilde{c} > 0$ such that for all $r > 0$,

$$r^{2s}\int_{B_r}\int_{B_r}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy+c\int_{B_r}|v(x)|^2dx\geq\tilde{c}r^{2s}\left(\int_{B_r}|v(x)|^q dx\right)^{2/q}.$$

Combining the last two relations and applying Minkowski's inequality, we may bound

$$\tilde{J}[v]+c\int_{B_1}|v(x)|^2 w_N(x)^2dx$$

$$\geq(2s-1)\tilde{c}\int_0^1\left(\int_{B\sqrt{1-t}}|v(x)|^q dx\right)^{\frac{2}{q}}\left(\sqrt{1-t}\right)^{2s}t^{2s-2}dt$$

$$\geq(1-t)^s t^{2s-2}dt\left(\int_{B_1}|v(x)|^q\left(\int_0^{1-|x|^2}(1-t)^s\,t^{2s-2}dt\right)^{\frac{q}{2}}dt\right)^{\frac{2}{q}}. \qquad (61)$$

We observe that

$$\int_0^{1-|x|^2}(1-t)^s t^{2s-2}dt\geq B\,(s+1,2s-1)(1-|y|^2)t^{2s-2},$$

which follows from the fact that $y\mapsto\int_0^y(1-t)^s t^{2s-2}dt/\int_0^y t^{2s-2}dt$ is decreasing on (0, 1). This allows us to bound the expression in (61) from below by

$$(2s-1)B(s+1,2s-1)\tilde{c}\left(\int_{B_1}|v(x)|^q(1-|x|^2)^{(s-1/2)q}\,dx\right)^{\frac{2}{q}}$$

$$=(2s-1)B(s+1,2s-1)\tilde{c}\left(\int_{B_1}|u(x)|^q\,dx\right)^{\frac{2}{q}},$$

and we are done.

This leaves us with proving Lemma (5.2.1).

The regional Laplacian (see, e.g., [199]) on an open set $\Omega\subset\mathbb{R}^N$ is, up to a multi plicative constant, given by

$$L_\Omega u(x)=\lim_{\epsilon\to0^+}\int_{\Omega\cap\{|y-x|>\epsilon\}}\frac{u(y)-u(x)}{|x-y|^{N+2s}}dy.$$

This operator appears naturally in our context since

$$\int_\Omega\overline{u(x)}\,(L_\Omega u)(x)dx=-\frac{1}{2}\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Our proof of Lemma (5.2.5) relies on a pointwise estimate for $L_{B_1}w_N$. In dimension $N=1$ this can be computed explicitly and we recall from [195] that

$$-L_{(-1,1)}w_1(x)=\frac{(1-x^2)^{\frac{-2s-1}{2}}}{2s}\left(B(s+\tfrac{1}{2},1-s)-(1-x)^{2s}+(1+x)^{2s}\right).$$

Hence, by [195],

$$-L_{(-1,1)}w_1(x)\geq c_1(1-x^2)^{\frac{-2s-1}{2}}+c_2(1-x^2)^{\frac{-2s+1}{2}}, \qquad (62)$$

where

$$c_1=\frac{B\left(s+\frac{1}{2},1-s\right)-2^{2s}}{2s},\qquad c_2=\frac{2^{2s}-2}{2s}.$$

Lemma (5.2.4)[190]: Let $N\geq2$ and let w_N be as in Lemma (5.2.5). Then

$$-L_{B_1}w_N(x) \geq \frac{c_1}{2}\int_{\mathbb{S}^{N-1}} |h_N|^{2s} dh \cdot (1-|x|^2)^{-\frac{2s+1}{2}} + \frac{c_2}{2}|\mathbb{S}^{N-1}| \cdot (1-|x|^2)^{-\frac{2s-1}{2}}$$

Proof: By rotation invariance we may assume that $\mathbf{X} = (0,0,\cdots:,0,x)$. With the notation $p = \frac{2s-1}{2}$ we have

$$-L_{B_1}w_N(\mathbf{X}) = p.v.\int_{B_1} \frac{(1-|\mathbf{X}|^2)^p - (1-|y|^2)^p}{|\mathbf{X}-y|^{N+2s}} dy$$

$$= \frac{1}{2}\int_{\mathbb{S}^{N-1}} dh p.v. \int_{-xh_N-\sqrt{x^2h_N^2-x^2+1}}^{-xh_N+\sqrt{x^2h_N^2-x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+2s}} dt.$$

We calculate the inner principle value integral by changing the variable $t = -xh_N + u\sqrt{x^2h_N^2 - x^2 + 1}$

$$g(x,h) := p.v.\int_{-xh_N-\sqrt{x^2h_N^2-x^2+1}}^{-xh_N+\sqrt{x^2h_N^2-x^2+1}} \frac{(1-|x|^2)^p - (1-|x+ht|^2)^p}{|t|^{1+2s}} dt$$

$$= p.v.\int_{-1}^{1} \frac{(1-x^2)^p - (1-u^2)^p(1-x^2+x^2h_N^2)^p}{\left|-xh_N + u\sqrt{x^2h_N^2-x^2+1}\right|^{1+2s}} \sqrt{x^2h_N^2-x^2+1}\, du$$

$$= (1-x^2+x^2h_N^2)^{p-s} p.v.\int_{-1}^{1} \frac{1-\left(1-\frac{x^2h_N^2}{1-x^2+x^2h_N^2}\right)^p - (1-u^2)^p}{\left|u-\frac{xh_N}{\sqrt{1-x^2+x^2h_N^2}}\right|^{1+2s}} du$$

$$= (1-x^2+x^2h_N^2)^{-1/2}\left(-L_{(-1,1)}w_1\right)\left(\frac{xh_N}{\sqrt{1-x^2+x^2h_N^2}}\right)$$

Hence by (62) we have

$$g(x,h) \geq (1-x^2+x^2h_N^2)^{-1/2}\Bigg(c_1\left(1-\frac{x^2h_N^2}{1-x^2+x^2h_N^2}\right)^{\frac{2s-1}{2}-2s}$$

$$+ c_2\left(1-\frac{x^2h_N^2}{1-x^2+x^2h_N^2}\right)^{\frac{2s-1}{2}-2s+1}\Bigg)$$

$$= c_1(1-x^2+x^2h_N^2)^s(1-x^2)^{-\frac{2s+1}{2}} + c_2(1-x^2+x^2h_N^2)^{s-1}(1-x^2)^{-\frac{2s-1}{2}}$$

$$\geq c_1|h_N|^{2s}(1-x^2)^{-\frac{2s+1}{2}} + +c_2(1-x^2)^{-\frac{2s-1}{2}}.$$

Thus

$$-L_{B_1}w_N(\mathbf{X}) = \frac{1}{2}\int_{\mathbb{S}^{N-1}} g(x,h)\, dh$$

$$\geq \frac{c_1}{2}\int_{\mathbb{S}^{N-1}} |h_N|^{2s}\, dh \cdot (1-x^2)^{-\frac{2s+1}{2}} + \frac{c_2}{2}|\mathbb{S}^{N-1}| \cdot (1-x^2)^{-\frac{2s-1}{2}},$$

and we are done.

Finally, we are in position to give the

Lemma (5.2.5)[190]: Let $N \geq 2, \frac{1}{2} < s < 1$ and define $w_N(x) = (1-|x|^2)^{\frac{2s-1}{2}}$ for $x \in B_1 \subset \mathbb{R}^N$. Then for all $u \in \dot{W}_2^s(B_1)$

$$\int_{B_1}\int_{B_1}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy-\mathcal{D}_{N,2,s}\int_{B_1}\frac{2^{2s}}{(1-|x|^2)^{2s}}|u(x)|^2dx$$
$$\geq \tilde{J}[v]+c\int_{B_1}|v(x)|^2dx, \tag{63}$$
where $v=u/w_N$,
$$\tilde{J}[v]=\int_{B_1}\int_{B_1}|v(x)-v(y)|^2\frac{w_N(x)-w_N(y)}{|x-y|^{N+2s}}dxdy$$
and $c=s^{-1}(2^{2s-1}-1)|\mathbb{S}^{N-1}|>0$.

This inequality is somewhat analogous to (57) in the previous proof. We emphasize, however, that there are two terms on the right side of (63) and we will need both of them. Accepting this lemma for the moment, we now complete the

Proof : We use the ground state representation formula [166], see also [195],
$$\int_{B_1}\int_{B_1}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy+2\int_{B_1}\frac{Lw_N(x)}{w_N(x)}|u(x)|^2dx=\tilde{J}[v]$$
with $u=w_N\,v$ and $\tilde{J}$ as defined in the lemma. The assertion now follows from Lemma (5.2.4), which implies that
$$-2\frac{Lw_N(x)}{w_N(x)}\geq \mathcal{D}_{N,2,s}\frac{2^{2s}}{(1-|x|^2)^{2s}}+c(1-|x|^2)^{-2s+1}$$
with $c=c_2|\mathbb{S}^{N-1}|>0$. Indeed, here we used $2^{2s-1}\mathcal{D}_{1,2,s}=c_1$ and
$$\mathcal{D}_{N,2,s}=\mathcal{D}_{1,2,s}\cdot\frac{1}{2}\int_{\mathbb{S}^{N-1}}|h_N|^{2s}\,dh.$$
as a quick computation shows.

We shall give a complete proof of Theorem (5.2.8). Our strategy is somewhat reminiscent of the proof of the Hardy-Sobolev-Maz'ya inequality in the local case in [196]. We use an averaging argument a la Gagliardo-Nirenberg to reduce the multi-dimensional case to the one-dimensional case. We describe this reduction and establish inequality.
The key ingredient in our proof of Theorem (5.2.8) is the following pointwise estimate of a function on an interval.

Lemma (5.2.6)[190]: Let $0<s<1, q\geq 1$ and $p\geq 2$ with $ps>1$. Then there is a $c=c(s,q,p)<\infty$ such that for all $f\in C_c^{\infty}(-1,1)$
$$\|f\|_{\infty}^{p+q(ps-1)}\leq$$
$$c\left(\int_{-1}^{1}\int_{-1}^{1}\frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}dxdy-\mathcal{D}_{1,p,s}\int_{-1}^{1}\frac{|f(x)|^p}{(1-|x|)^{ps}}dx\right)\|f\|_q^{q(ps-1)}. \tag{64}$$
Due to the particular form of the exponents this inequality has a scale-invariant form.

Corollary (5.2.7)[190]: Let $0<s<1, q\geq 1$ and $p\geq 2$ with $ps>1$. Then, with the same constant $c=c(p,s,q)<\infty$ *as* in Lemma (5.2.6), we have for all open sets $\Omega\subseteq\mathbb{R}$ and all
$f\in C_c^{\infty}(\Omega)$
$$\|f\|_{\infty}^{p+q(ps-1)}$$
$$\leq c\left(\int_{\Omega}\int_{\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}dxdy-\mathcal{D}_{1,p,s}\int_{\Omega}\frac{|f(x)|^p}{d(x)^{ps}}dx\right)\|f\|_q^{q(ps-1)} \tag{65}$$
where $d(x)=\mathrm{dist}(x,\Omega^c)$.

Proof: From Lemma (5.2.6), by translation and dilation, we obtain (65) for any interval and half-line. The extension to arbitrary open bounded sets is straight forward.

We prove Lemma (5.2.6). Now we show how this corollary allows us to deduce our main theorem. Taking advantage of an averaging formula of Loss and Sloane [200] the argument is almost the same as in [196], but we reproduce it here to make self-contained.

Theorem (5.2.8)[190]: Let $N \geq 2, 2 \leq p < 1$ and $0 < s < 1$ with $1 < ps < N$. Then there is a constant $\sigma_{N,p,s} > 0$ such that

$$\int\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{1+ps}} dxdy - \mathcal{D}_{N,p,s} \int_{\Omega} \frac{|u(x)|^p}{m_{ps}(x)^{ps}} dx$$

$$\geq \sigma_{N,p,s} \left(\int_{\Omega} |u(x)|^q dx \right)^{p/q} \qquad (66)$$

for all open $\Omega \subseteq \mathbb{R}^N$ and all $u \in \dot{W}_p^s(\Omega)$, where $q = Np/(N-ps)$.

Inequality (66) has been conjectured in [197] in analogy to the local HSM inequalities [201], [170]. Recently, Sloane [202] found a remarkable proof of (66) for $p = 2$ and Ω being a half-space. Our result generalizes this to any $p \geq 2$ and any .We emphasize that our constant $\sigma_{N,p,s}$ can be chosen independently of . Therefore Theorem (5.2.8) is the fractional analog of the main inequality of [196], which treats the local case.

We know explain the notation in (66). The sharp constant [200] in (56) is

$$\mathcal{D}_{N,p,s} = 2\pi^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{1+ps}{2}\right)}{\Gamma\left(\frac{N+ps}{2}\right)} \int_0^1 \left(1-r^{(ps-1)/p}\right)^p \frac{dr}{(1-r)^{1-ps}}. \qquad (67)$$

In the special case $p = 2$ we have

$$\mathcal{D}_{N,2,s} = 2\pi^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{1+ps}{2}\right)}{\Gamma\left(\frac{N+ps}{2}\right)} \frac{B\left(\frac{1+2s}{2},1-s\right)-2^{2s}}{2^{2s+1}s} = 2_{K,N,2s},$$

where $k_{N,2s}$is the notation used in [192], [200], [195]. We denote

$$d_w(x) = \inf\{|t| : x + tw \notin \Omega\}, \qquad x \in \mathbb{R}^N, \ w \in \mathbb{S}^{N-1}, \qquad (68)$$

where $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ is the $(N-1)$ -dimensional unit sphere. Following [200] we set for $\alpha > 0$

$$m_\alpha(x) = \left(\frac{2\pi^{\frac{N-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{S}^{N-1}} \frac{dw}{d_w(x)^\alpha} \right)^{\frac{1}{\alpha}}, \qquad (69)$$

which is analogous to the pseudo distance $m(x)$. We recall that for convex domains , we have $m\,(x) \leq d(x)$, see [200].

We present three independent proofs of (66), but only the last one in full generality. We use the ground state representation for half-spaces as the starting point. This allows us to obtain (66) for half-spaces and any $p \geq 2$. We derive a fractional Hardy inequality (34) for balls with two additional terms, and then deduce (66) in case when $p = 2$ and Ω is a ball or a half-space. We extend the method developed in [196] and use results from [198] and [200] to prove Theorem (5.2.8) for arbitrary domains.

Proof: Let $w_1, \dots, w_N$ be an orthonormal basis in $\mathbb{R}^N$. We write $x_j$for the j-th coordinate of $x \in \mathbb{R}^N$ in this basis, and $\tilde{x}_j = x - x_j w_j$. By skipping the j-th coordinate of $\tilde{x}_j$ (which is zero), we may regard $\tilde{x}_j$as an element of $\mathbb{R}^{N-1}$.For a given domain $\Omega \subseteq \mathbb{R}^N$ we write

$$d_j(x) = d_{wj}(x) = \inf\left\{|t| : x + tw_j \notin \Omega\right\}.$$

If $u \in C_c^\infty(\Omega)$, then Corollary (5.2.7) yields

$$|u(x)| \le C\big(g_j(\tilde{x}_j)h_j(\tilde{x}_j)\big)^{\frac{1}{p+q(ps-1)}}$$

for any $1 \le j \le N$, where

$$g_j(\tilde{x}_j) = \int_{\tilde{x}_j + aw_j \in \Omega} da \int_{\tilde{x}_j + bw_j \in \Omega} db \frac{\left|u(\tilde{x}_j + aw_j) - u(\tilde{x}_j + bw_j)\right|^p}{|a-b|^{1+ps}}$$

$$-\mathcal{D}_{1,p,s} \int_{\mathbb{R}} da \frac{\left|u(\tilde{x}_j + aw_j)\right|^p}{d_j(\tilde{x}_j + aw_j)^{ps}}$$

and

$$h_j(\tilde{x}_j) = \left(\int_{\mathbb{R}} da \left|u(\tilde{x}_j + aw_j)\right|^q\right)^{ps-1}.$$

Thus

$$|u(x)|^N \le C^N \prod_{j=1}^{N} \big(g_j(\tilde{x}_j)h_j(\tilde{x}_j)\big)^{\frac{1}{p+q(ps-1)}}.$$

We now pick $q = \frac{pN}{N-ps}$ and rewrite the previous inequality as

$$|u(x)|^q \le C^q \prod_{j=1}^{N} \big(g_j(\tilde{x}_j)h_j(\tilde{x}_j)\big)^{\frac{1}{ps(N-1)}}.$$

By a standard argument based on repeated use of Holder's inequality we deduce that

$$\int_{\mathbb{R}^N} |u(x)|^q \, dx \le C^q \prod_{j=1}^{N} \left(\int_{\mathbb{R}^{N-1}} g_j(y)^{\frac{1}{ps}} h_j(y)^{\frac{1}{ps}} dy\right)^{\frac{1}{N-1}}.$$

We note that

$$\left\| h_j^{\frac{1}{ps-1}} \right\|_{L^1(\mathbb{R}^{N-1})} = \|u\|_{L^q(\mathbb{R}^N)}^q \qquad \text{for } every \ j = 1, \cdots, N$$

and derive from the Hölder and the arithmetic- geometric mean inequality that

$$\prod_{j=1}^{N} \int_{\mathbb{R}^{N-1}} g_j(y)^{\frac{1}{ps}} h_j(y)^{\frac{1}{ps}} dy \le \prod_{j=1}^{N} \|g_j\|_1^{\frac{1}{ps}} \left\| h_j^{\frac{1}{ps-1}} \right\|_1^{\frac{ps-1}{ps}} = \|u\|_q^{\frac{q(ps-1)N}{ps}} \prod_{j=1}^{N} \|g_j\|_1^{\frac{1}{ps}}$$

$$\le \|u\|_q^{\frac{q(ps-1)N}{ps}} \left(N^{-1} \sum_{j=1}^{N} \|g_j\|_1\right)^{\frac{N}{ps}}.$$

To summarize, we have shown that

$$\|u\|_q^p \le C^{\frac{p^2 s(N-1)}{N-ps}} N^{-1} \sum_{j=1}^{N} \|g_j\|_1.$$

We now average this inequality over all choices of the coordinate system w_j. We recall the Loss-Sloane formula

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy$$

$$=\frac{1}{2}\int_{\mathbb{R}^{N-1}}dw\int_{\{x:x\,.w=o\}}d\mathcal{L}_w(x)\int_{x+aw\in\Omega}da\int_{x+aw\in\Omega}db\,\frac{|u(x+aw)-u(y+bw)|^p}{|a-b|^{1+ps}},$$

where $\mathcal{L}_w$ is $(N-1)$-dimensional Lebesgue measure on the hyperplane $\{x: x\cdot w=0\}$. Thus we arrive at

$$\|u\|_q^p\le\frac{2C^{\frac{p^2s(N-1)}{N-ps}}}{|\mathbb{S}^{N-1}|}\left(\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right.$$
$$\left.-\mathcal{D}_{1,p,s}\frac{\pi^{\frac{N-1}{2}}\Gamma\left(\frac{1+ps}{2}\right)}{\Gamma\left(\frac{N+ps}{2}\right)}\int_\Omega\frac{|u(x)|^p}{m_{ps}(x)^{ps}}dx\right).$$

Recalling the definition of $\mathcal{D}_{N,p,s}$ we see that this is the inequality claimed in Theorem(5.2.8).

Lemma (5.2.9)[190]: Let $0<s<1$ and $p\ge 2$ with $ps>1$. Then

$$\int_0^1\int_0^1\frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}dxdy-\mathcal{D}_{1,p,s}\int_0^1\frac{|f(x)|^p}{x^{ps}}dx$$
$$\ge c_p\int_0^1\int_0^1\frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}w(x)^{p/2}w(y)^{p/2}dxdy+\int_0^1 W_{p,s}(x)|v(x)|^p\,w(x)^p dx$$

for all f with $f(0)=0$ (and no boundary condition at $x=1$). Here $w(x)=x^{(ps-1)/p}$ and $f=wv$. The function $W_{p,s}$ is bounded away from zero and satisfies

$$W_{p,s}(x)\approx x^{-(p-1)(ps-1)/p}\qquad \text{for } x\in(0,1/2]$$

and

$$W_{p,s}(x)\approx\begin{cases}1 & \text{if } p-1-ps>0\,,\\ |\ln(1-x)| & \text{if } p-1-ps=0\,,\\ (1-x)^{-1-ps+p} & \text{if } p-1-ps<0\,,\end{cases}\qquad \text{for } x\in[1/2\,,1).$$

Proof: The general ground state representation [166] reads

$$\int_0^1\int_0^1\frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}dxdy\ge\int_0^1 V(x)|f(x)|^p$$
$$+c_p\int_0^1\int_0^1\frac{|v(x)-v(y)|^p}{|x-y|^{1+ps}}w(x)^{p/2}w(y)^{p/2}dxdy$$

with

$$V(x):=2w(x)^{-p+1}\int_0^1(w(x)-w(y))|w(x)-w(y)|^{p-1}\,|x-y|^{-1-ps}dy$$

(understood as principal value integral). We decompose

$$V(x)=2w(x)^{-p+1}\int_0^\infty\big(w(x)-w(y)\big)|w(x)-w(y)|^{p-2}\,|x-y|^{-1-ps}dy$$
$$-2w(x)^{-p+1}\int_1^\infty\big(w(x)-w(y)\big)|w(x)-w(y)|^{p-2}\,|x-y|^{-1-ps}dy$$
$$=\frac{-\mathcal{D}_{1,p,s}}{x^{ps}}+W_{p,s}(x).$$

(The computation of the first term is in [197].) For $x\in(0,1)$, the second term is positive, indeed,

$$W_{p,s}(x) = 2w(x)^{-p+1} \int_1^\infty \left(w(y) - w(x)\right)^{p-1} (y-x)^{-1-ps} dy.$$

Note that at $x = 0$

$$\int_1^\infty w(y)^{p-1} y^{-1-ps} dy = c_{p,s} < \infty$$

since $ps - (p-1)(ps-1)/p > 0$. Hence $W_{p,s}(x) \sim 2c_{p,s}\, x^{-(p-1)(ps-1)/p}$ as $x \to 0$. On the other hand, at $x = 1$, we have

$$\int_1^\infty (w(y) - 1)^{p-1} (y-1)^{-1-ps} dy = \tilde{c}_{p,s} < \infty \qquad \text{if} \quad p - 1 - ps > 0.$$

Hence $W_{p,s}(x) \to 2\tilde{c}_{p,s}$ as $x \to 1$ in that case. In the opposite case, one easily finds that for $x = 1 - \epsilon$, to leading order only y's with $y - 1$ of order ϵ contribute. Hence $W_{p,s}(x) \sim 2\tilde{c}_{p,s}\,(1-x)^{-1+ps+p}$ as $x \to 1$ if $p - 1 - ps < 0$ and $W_{p,s}(x) \sim 2\tilde{c}_{p,s}|\ln(x-1)|$ if $p - 1 - ps = 0$.

Corollary (5.2.10)[190]: Let $0 < s < 1$ and $p \geq 2$ with $ps > 1$. Then

$$\int_{-1}^1 \int_{-1}^1 \frac{|f(x) - f(y)|^p}{|x-y|^{1+ps}} dxdy - \mathcal{D}_{1,p,s} \int_{-1}^1 \frac{|f(x)|^p}{(1-|x|)^{ps}} dx$$
$$\geq c_p \left(\int_{-1}^0 \int_{-1}^0 + \int_0^1 \int_0^1 \right) \frac{|v(x) - v(y)|^p}{|x-y|^{1+ps}} w(x)^{p/2} w(y)^{p/2} dxdy$$
$$+ c_{p,s} \int_{-1}^1 |v(x)|^p w(x) dx$$

for all f with $f(-1) = f(1) = 0$. Here $w(x) = (1-|x|)^{(ps-1)/p}$ *and* $f = wv$.

Proof: The corollary follows by applying Lemma (5.2.9) to functions $f_1(x) = f(1+x)$ and $f_2(x) = f(1-x)$, where $x \in [0,1]$, and adding resulting inequalities.

The second ingredient besides Lemma (5.2.9) in our proof is the following bound due to Garsia, Rodemich and Rumsey [198].

Lemma (5.2.11)[190]: Let $p, s > 0$ with $ps > 1$. Then for any continuous function f on $[a, b]$

$$\int_a^b \int_a^b \frac{|f(x) - f(y)|^p}{|x-y|^{1+ps}} dxdy \geq c \frac{|f(b) - f(a)|^p}{(b-a)^{ps-1}} \tag{70}$$

with $c = (ps-1)^p\, (8(ps+1))^{-p}/4$.

Proof: This follows by taking $\Psi(x) = |x|^p$ and $p(x) = |x|^{s+1/p} x + 1 = p$ in [198].

Section (5.3): Applications of Caffarelli–Kohn–Nirenberg Type Inequalities of Fractional Order

In [7] proved the following result.

Theorem (5.3.1)[203]:(Caffarelli–Kohn–Nirenberg). Let $p, q, r, \alpha, \beta, \sigma$ and a be real constants such that $p, q \geq 1, r > 0, 0 \leq a \leq 1$, and

$$\frac{1}{p} + \frac{\alpha}{N}, \frac{1}{q} + \frac{\beta}{N}, \frac{1}{r} + \frac{m}{N} > 0,$$

where $m = a\sigma + (1-a)\beta$. Then there exists a positive constant C such that for all $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$\left\| |x|^m u \right\|_{L^r(\mathbb{R}^N)} \le C \left\| |x|^\alpha |\nabla u| \right\|^{a}_{L^p(\mathbb{R}^N)} \left\| |x|^\beta u \right\|^{1-a}_{L^q(\mathbb{R}^N)},$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{m}{N} = a\left(\frac{1}{p} + \frac{\alpha - 1}{N}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{N}\right),$$

with

$$0 \le \alpha - \sigma \quad \text{if} \quad a > 0\,,$$

and

$$\alpha - \sigma \le 1 \text{ if } a > 0 \text{ and } \frac{1}{r} + \frac{m}{N} = \frac{1}{p} + \frac{\alpha - 1}{N}.$$

This class of inequalities are related to the following local elliptic problem

$$-\mathrm{div}(|x|^{-p\gamma} |\nabla u|^{p-2} \nabla u) = 0. \tag{71}$$

As a consequence of Theorem (5.3.1), it follows that $|x|^{-\gamma}$, with $\gamma < \frac{N-p}{p}$, is an admissible weight in the sense that if u is a weak positive super solution to (71), then it satisfies a weak Harnack inequality.

There exists a positive constant $\kappa > 1$ such that for all $0 < q < \kappa(p-1)$,

$$\left(\int_{B_{2\rho}(x_0)} u^q(x) |x|^{-p\gamma} dx \right)^{\frac{1}{q}} \le C \inf_{B_\rho(x_0)} u,$$

where $B_{2\rho}(x_0) \subset\subset \Omega$, and $C > 0$ depends only on B.

See [212], [215] for a complete discussion and the proof of the weak Harnack inequality.

Notice that even the classical Harnack inequality holds for positive solution to (71). One of the main tools to get the weak Harnack inequality is a weighted Sobolev inequality that can obtained directly from Theorem (5.3.1).

An alternative argument to get the Sobolev inequality is to prove a weighted Hardy inequality as it was observed in [217].

We follow this approach in order to get a nonlocal version of the Caffarelli–Kohn–Nirenberg inequalities.

[166] proved the following Hardy inequality stating that for $p > 1$with $sp < N$ and for all $\emptyset \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\emptyset(x) - \emptyset(y)|^p}{|x-y|^{N+ps}} dxdy \ge \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|\emptyset(x)|^p}{|x|^{ps}} dx \tag{72}$$

where the constant $\Lambda_{N,p,s}$ is given by

$$\Lambda_{N,p,s} = 2 \int_0^\infty |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-1} K(\sigma) \tag{73}$$

and

$$K(\sigma) = \int_{|\acute{y}|=1} \frac{dH^{n-1}(\acute{y})}{|\acute{x} - \sigma \acute{y}|^{N+ps}}.$$

$$h_s(u) \equiv \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} dxdy \ge \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx,$$

they proved that for $p \ge 2$, there exists a positive constant $C = C(p, N, s)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$, if $v = |x|^{\frac{N-ps}{p}s} u$, it holds

$$h_s(u) \geq C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}. \tag{74}$$

The above inequality turns to be equality for $p = 2$ with $C = 1$.

As a consequence of (74), we easily get that $\Lambda_{N,p,s}$ is never achieved.

For $p = 2$, [205] proved the next result:

Theorem (5.3.2)[203]: Let $N \geq 1, 0 < s < 1$ and $N > 2s$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $1 < q < 2$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in C_0^\infty(\Omega)$,

$$a_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy - \Lambda_{N,2,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx$$
$$\geq C(\Omega, q, N, s) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+qs}} dxdy\,. \tag{75}$$

One of the main results of this work is to generalize Theorem (5.3.2) to the case $p > 2$.

It is clear that the condition imposed on β coincides in some sense with definition of admissible weight given in [215]. The proof of Theorem (5.3.14) is based on some weighted Hardy inequality given below.

As a direct application of the previous results, we will consider the problem

$$\begin{cases} L_{s,p}u - \lambda \dfrac{|u|^{p-2}u}{|x|^{ps}} = |u|^{q-1}u, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{76}$$

where

$$L_{s,p}u(x) := P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dx,$$

$0 < \lambda \leq \Lambda_{N,p,s}$ and $q > 0$.

In the local case, the problem is reduced to

$$\begin{cases} -\Delta_p u - \lambda \dfrac{|u|^{p-2}u}{|x|^p} = |u|^{q-1}u, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{77}$$

For $p = 2$, the authors in [208] proved that if $q > q_+(2)$, then problem (77) has no distributional super solution, however, if $q < q_+(2)$, there exists a positive super solution, with $q_+(2) = 1 + \frac{2}{\theta_1}, \theta_1 = \frac{N-2}{2} - \sqrt{\Lambda_{N,2} - \lambda}$ and $\Lambda_{N,2} = \frac{(N-2)^2}{4}$, the classical Hardy constant.

The case $p \neq 2$ was considered in [204] where the same alternative holds with $q_+(p) = p - 1 + \frac{p}{\theta_p}$, where θ_p is the smallest solution to the equation

$$\Xi(s) = (p-1)s^p - (N-1)s^{p-1} + \lambda.$$

The fractional case with $p = 2$ was studied in [213] and [206]. They proved the same alternative with $q_+(2\,,\ s) = 1 + \frac{2s}{\theta}$ where $\theta \equiv \theta(\lambda, s, N) > 0$

Our goal is to extend the results of [213] and [206] to the case $p \neq 2$.

We prove the main results, namely Theorems (5.312), (5.3.13) and (5.3.14). The starting point

The starting point will be the proof of a general version of the Picone inequality. As a consequence, we get a weighted version of the Hardy inequality for a class of "admissible weights".

Hence, following closely the arguments used in [205], taking in consideration the "weighted" Hardy inequality, we get the proof of Theorem (5.3.12).

Once Theorem (5.3.12) proved, we complete the proof of Theorem (5.3.13) using suitable Sobolev inequality.

At the end, and by using a weighted Hardy inequality, we are able to get a "fractional Caffarelli–Kohn–Nirenberg" inequality for admissible weights in $\mathbb{R}^N$ and then to proof Theorem (5.3.14).

We analyze problem (77). We prove the existence of a critical exponent $q_+(p\,,s)$ such that if $q > q_+(p\,,s)$, then problem (77) has no positive solution in a suitable sense. To show the optimality of the non-existence exponent, we will construct an appropriate super solution in the whole space.

We will use the next elementary inequality, see for instance [166].

Lemma (5.3.3)[203]: Assume that $p > 1$, then for all $0 \le t \le 1$ and $a \in \mathbb{C}$, we have

$$|a-t|^p \ge (1-t)^{p-1}(|a|^p - t). \tag{78}$$

Proof: Let us begin with some functional settings that will be used below, we refer to [211] and [217] for more details.

For $s \in (0,1)$ and $p \ge 1$, we define the fractional Sobolev spaces $W^{s,p}(\Omega), \Omega \subset \mathbb{R}^N$, by

$$W^{s,p}(\Omega) \equiv \{u \in L^p(\Omega): \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+qs}}dxdy < \infty\}.$$

It is clear that $W^{s,p}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}.$$

In the same way, we define the space $X_0^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{s,p}(\Omega)$.

Notice that, if $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (C\Omega \times C\Omega)$, then

$$\|\phi\|_{X_0^{s,p}(\Omega)} = \left(\int \int_Q \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}} + \|\phi\|_{L^p(\Omega)}.$$

Using the fractional Sobolev inequality we obtain $X_0^{s,p}(\Omega) \subset (\Omega)L^{p_s^*}$ with continuous inclusion, where $p_s^* = \frac{pN}{N-ps}$ for $ps < N$.

In the case where Ω is a bounded regular domain, the space $X_0^{s,p}(\Omega)$ can be endowed with the equivalent norm

$$\|\phi\|_{X_0^{s,p}(\Omega)} = \left(\int \int_Q \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}.$$

To prove the fractional Caffarelli–Kohn–Nirenberg inequality, we need to define fractional Sobolev spaces with weight. More precisely, let $0 < \beta < \frac{N-ps}{2}$ and $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$, the weighted Sobolev space $X^{s,p,\beta}(\Omega)$ is defined by

$$W^{s,p,\beta}(\Omega) := \left\{\phi \in L^p\left(\Omega, \frac{dx}{|x|^{2\beta}}\right) : \int_\Omega \int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta} < +\infty\right\}.$$

Thus $X^{s,p,\beta}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{X^{s,p,\beta}(\Omega)} = \left(\int_\Omega \frac{|\phi(x)|^p dx}{|x|^{2\beta}}\right)^{\frac{1}{p}} + \left(\int_\Omega \int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta}\right)^{\frac{1}{p}}.$$

Now, we define the weighted Sobolev space $X_0^{s,p,\beta}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the previous norm.

As in [135], see also [211], we can prove the following extension result.

Lemma (5.3.4)[203]: Assume that $\Omega \subset \mathbb{R}^N$ is a regular domain, then for all $w \in X^{s,p,\beta}(\Omega)$, there exists $\widetilde{w} \in X^{s,p,\beta}(\mathbb{R}^N)$ such that $\widetilde{w}_{|\Omega} = w$ and

$$\|\widetilde{w}\|_{X^{s,p,\beta}(\mathbb{R}^N)} \le C\|w\|_{X^{s,p,\beta}(\Omega)}$$

where $C \equiv C(N,s,p,\Omega) > 0$.

Lemma (5.3.5)[203]: (Picone inequality). Let $w \in X_0^{s,p,\beta}(\Omega)$ be such that $w > 0$ in Ω. Assume that $L_{s,p,\beta}(w) = \nu$ with $\nu \in L^1_{loc}(\mathbb{R}^N)$ and $\nu \gneqq 0$, then for all $u \in C_0^\infty(\Omega)$, we have

$$\frac{1}{2}\int \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta} \ge \langle L_{s,p,\beta} w, \frac{|u|^p}{w^{p-1}}\rangle.$$

Proof: The case $\beta = 0$ is obtained in [216] if $p = 2$ and in [207] if $p \ne 2$. For the reader convenience we include some details for the general case $\beta \ne 0$.

We set $\upsilon(x) = \frac{|u(x)|^p}{|w(x)|^{p-1}}$ and $k(x,y) = \frac{1}{|x-y|^{N+ps}|x|^\beta |y|^\beta}$, then

$$\langle L_{s,p,\beta}(w(x)), \upsilon(x)\rangle = \int_\Omega \upsilon(x) \int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))\,k(x,y)dxdy$$

$$= \int_\Omega \frac{|u(x)|^p}{|w(x)|^{p-1}} \int_{\mathbb{R}^N} |w(x)-w(y)|^{p-2}(w(x)-w(y))\,k(x,y)dxdy$$

Since k is symmetric, we obtain that

$$\langle L_{s,p,\beta}(w(x)), \upsilon(x)\rangle = \frac{1}{2}\int \int_Q \left(\frac{|u(x)|^p}{|w(x)|^{p-1}} - \frac{|u(y)|^p}{|w(y)|^{p-1}}\right)|w(x)-w(y)|^{p-2}(w(x) - w(y))k(x,y)dxdy.$$

Let $\upsilon_1 = \frac{u}{w}$, then

$$\langle L_{s,p,\beta}(w(x)), \upsilon(x)\rangle = \frac{1}{2}\int \int_Q (|\upsilon_1(x)|^p w(x) - |\upsilon_1(y)|^p w(y))\,|w(x)-w(y)|^{p-2}(w(x) - w(y))k(x,y)dydx.$$

Define

$$\Phi(x,y) = |u(x)-u(y)|^p - \left(|v_1(x)|^p\, w(x) - |v_1(y)|^p\, w(y)\right)|w(x)-w(y)|^{p-2}\left(w(x)-w(y)\right),$$

then

$$\langle L_{s,p,\beta}\left(w(x)\right), v(x)\rangle + \frac{1}{2}\int_Q \Phi(x,y)k(x,y)dydx = \frac{1}{2}\int\int_Q |u(x)-u(y)|^p k(x,y)dydx\,.$$

We claim that $\Phi \geq 0$. It is clear that, by a symmetry argument, we can assume that $w(x) \geq w(y)$. Let $t = w(y)/w(x), a = u(x)/u(y)$, then using inequality (78), the claim follows at once. Hence we conclude.

As a consequence, for $\beta = 0$, we have the next comparison principle that extends, to the fractional framework, the classical one obtained by Brezis-Kamin in [209].

Lemma (5.3.6)[203]: Let Ω be a bounded domain and let f be a nonnegative continuous function such that $f(\sigma) > 0$ if $\sigma > 0$ and $\frac{f(\sigma)}{\sigma^{p-1}}$ is decreasing. Let $u, v \in W_0^{s,p}(\Omega)$ be such that $u, v > 0$ in Ω and

$$\begin{cases} L_{s,p}u \geq f(u) \text{ in } \Omega\,, \\ L_{s,p}v \leq f(v) \text{ in } \Omega\,, \end{cases}$$

Then, $u \geq v$ in Ω.

Proof: Using an approximation argument, taking in consideration that $u, v > 0$, we can prove that

$$\frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}} \geq \left(\frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}\right). \tag{79}$$

We set $\xi = (v^p - u^p)_+$, then

$$\int_\Omega \left(\frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}\right)\xi dx \leq \int_\Omega \xi\left(\frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}}\right)dx. \tag{80}$$

Let us analyze each term in the previous inequality.

Using the definition of ξ we obtain that $\left(\frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}\right)\xi \geq 0$. On the other hand, we have

$$J \equiv \int_\Omega \xi\left(\frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}}\right)dx$$

$$= \frac{1}{2}\int\int_Q \frac{|u(x)-u(y)|^{p-2}\left(u(x)-u(y)\right)}{|x-y|^{N+ps}}\left(\frac{\xi(x)}{u^{p-1}(x)} - \frac{\xi(y)}{u^{p-1}(y)}\right)dxdy$$

$$-\frac{1}{2}\int\int_Q \frac{|v(x)-v(y)|^{p-2}\left(v(x)-v(y)\right)}{|x-y|^{N+ps}}\left(\frac{\xi(x)}{v^{p-1}(x)} - \frac{\xi(y)}{v^{p-1}(y)}\right)dxdy,$$

where $Q = \mathbb{R}^N \times \mathbb{R}^N \backslash (C\Omega \times C\Omega)$.

Notice that

$$|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\left(\frac{\xi(x)}{u^{p-1}(x)}-\frac{\xi(y)}{u^{p-1}(y)}\right)=$$
$$|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\left(\frac{v^{p}(x)}{u^{p-1}(x)}-\frac{v^{p}(y)}{u^{p-1}(y)}\right)-|u(x)-u(y)|^{p}.$$

In the same way, we obtain that

$$|v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)\left(\frac{\xi(x)}{v^{p-1}(x)}-\frac{\xi(y)}{v^{p-1}(y)}\right)$$
$$=-|v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)\left(\frac{u^{p}(x)}{v^{p-1}(x)}-\frac{u^{p}(y)}{v^{p-1}(y)}\right)$$
$$+|v(x)-v(y)|^{p}.$$

Thus

$$J=\frac{1}{2}\int\int_{Q}\frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+ps}}\left(\frac{v^{p}(x)}{u^{p-1}(x)}-\frac{v^{p}(y)}{u^{p-1}(y)}\right)dxdy$$
$$+\frac{1}{2}\int\int_{Q}\frac{|v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)}{|x-y|^{N+ps}}\left(\frac{u^{p}(x)}{v^{p-1}(x)}-\frac{u^{p}(y)}{v^{p-1}(y)}\right)dxdy$$
$$-\frac{1}{2}\int\int_{Q}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+ps}}-\frac{1}{2}\int\int_{Q}\frac{|v(x)-v(y)|^{p}}{|x-y|^{N+ps}}dxdy$$
$$=\int_{\Omega}\frac{L_{p,s}(u)}{u^{p-1}}v^{p}dx+\int_{\Omega}\frac{L_{p,s}(v)}{v^{p-1}}u^{p}dx-\frac{1}{2}\int\int_{Q}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+ps}}dxdy$$
$$-\frac{1}{2}\int\int_{Q}\frac{|v(x)-v(y)|^{p}}{|x-y|^{N+ps}}dxdy.$$

Now, using Picone's inequality, we conclude that $J\le 0$. Thus

$$\left(\frac{f(u)}{u^{p-1}}-\frac{f(v)}{v^{p-1}}\right)\xi\equiv 0$$

and then $\xi=0$ which implies that $u\le v$ in Ω.

Lemma (5.3.7)[203]: $Fix\ 0<\beta<\frac{N-ps}{2}$ and let $w(x)=|x|^{-\gamma}$ with $0<\gamma<\frac{N-ps-2\beta}{p-1}$, then there exists a positive constant $\Lambda(\gamma)>0$ such that

$$L_{s,p,\beta}(w)=\Lambda(\gamma)\frac{w^{p-1}}{|x|^{ps+2\beta}}\quad a.e.\ \text{in}\ \mathbb{R}^{N}\setminus\{0\}. \tag{81}$$

Proof: We set $r=|x|$ and $\rho=|y|$, then $x=r\acute{x}$, $y=\rho\acute{y}$ where $|\acute{x}|=|\acute{y}|=1$. Thus

$$L_{s,p,\beta}(w)=\frac{1}{|x|^{\beta}}\int_{0}^{+\infty}|r^{-\gamma}-\rho^{-\gamma}|^{p-2}\frac{(r^{-\gamma}-\rho^{-\gamma})\rho^{N-1}}{\rho^{\beta}r^{N+ps}}\left(\int_{|\acute{y}|=1}\frac{dH^{n-1}(\acute{y})}{\left|\acute{x}-\frac{\rho}{r}\right|^{N+ps}}\right)d\rho.$$

Let $\sigma=\frac{\rho}{r}$, then

$$L_{s,p,\beta}(w) = \frac{w^{p-1}}{|x|^{ps+2\beta}} \int_0^{+\infty} |1-\sigma^{-\gamma}|^{p-2}(1-\sigma^{-\gamma})\sigma^{N-\beta-1}\left(\int_{|\acute{y}|=1} \frac{dH^{n-1}(\acute{y})}{|\acute{x}-\sigma\acute{y}|^{N+ps}} \right) d\sigma.$$

Defining

$$K(\sigma) = \int_{|\acute{y}|=1} \frac{dH^{n-1}(\acute{y})}{|\acute{x}-\sigma\acute{y}|^{N+ps}},$$

as in [214], we obtain that

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma\frac{N-1}{2}} \int_0^{\pi} \frac{\sin^{N-2}(\theta)}{(1-2\sigma\cos(\theta)+\sigma^2)^{\frac{N+ps}{2}}} d\theta \tag{82}$$

Hence

$$L_{s,p,\beta}(w) = \frac{w^{p-1}}{|x|^{ps+2\beta}} \int_0^{+\infty} \psi(\sigma)d\sigma,$$

with

$$\psi(\sigma) = |1-\sigma^{-\gamma}|^{p-2}(1-\sigma^{-\gamma})\sigma^{N-\beta-1}K(\sigma). \tag{83}$$

Define $\Lambda(\gamma) \equiv \int_0^{+\infty} \psi(\sigma)\, d\sigma$, then to finish we just have to show that $0 < \Lambda(\gamma) < \infty$.

We have

$$\Lambda(\gamma) = \int_0^1 \psi(\sigma)d\sigma + \int_1^{\infty} \psi(\sigma)d\sigma = I_1 + I_2.$$

Notice that $K\left(\frac{1}{\xi}\right) = \xi^{N+ps} K(\xi)$ for any $\xi > 0$, then using the change of variable $\xi = \frac{1}{\sigma}$ in I_1, there results that

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^{\gamma}-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1}\right)d\sigma. \tag{84}$$

As $\sigma \to \infty$, we have

$$K(\sigma)(\sigma^{\gamma}-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1}\right) \simeq \sigma^{-1-\beta-ps} \in L^1(2,\infty).$$

Now, as, $\sigma \to 1$, we have

$$K(\sigma)(\sigma^{\gamma}-1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1}\right) \simeq (\sigma-1)^{p-1-ps} \in L^1(1,2).$$

Therefore, combining the above estimates, we get $|\Lambda(\gamma)| < \infty$. Now, using the fact that $0 < \gamma < \frac{N-ps-2\beta}{p-1}$, from (84), we reach that $\Lambda(\gamma) > 0$.

As a conclusion, we have proved that

$$L_{s,p,\beta}(w) = \Lambda(\gamma)\frac{w^{p-1}}{|x|^{ps+2\beta}} \quad a.e. \text{ in } \mathbb{R}^N \backslash \{0\}.$$

Hence the result follows.

As a consequence we have the following weighted Hardy inequality.

Theorem (5.3.8)[203]: Let $\beta < \frac{N-ps}{2}$, then for all $u \in C_0^{\infty}(\mathbb{R}^N)$, we have

$$2\Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\beta}} \frac{dy}{|y|^{\beta}}, \tag{85}$$

where $\Lambda(\gamma)$ is defined in (84).

Proof: Let $u \in C_0^\infty(\mathbb{R}^N)$ and $w(x) = |x|^{-\gamma}$ with $\gamma < \frac{N-ps-2\beta}{p-1}$. By Lemma (5.3.7), we have

$$L_{p,s,\beta}(w) = \Lambda(\gamma)\frac{w^{p-1}}{|x|^{ps+2\beta}}.$$

It is clear that $\frac{w^{p-1}}{|x|^{ps+2\beta}} \in L^1_{loc}(\mathbb{R}^N)$. Thus using Picone inequality in Lemma (5.3.5), it follows that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\frac{dx}{|x|^\beta}\frac{dy}{|y|^\beta} \geq \langle L_{p,s,\beta}w\,,\frac{|u|^p}{w^{p-1}}\rangle = \Lambda(\gamma)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps+2\beta}}dx.$$

Thus we conclude.

Remark (5.3.9)[203]: Let analyze the behavior of the constant $\Lambda(\gamma)$ in inequality (85). Recall that, for $\gamma < \frac{N-ps-2\beta}{p-1}$,

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^\gamma - 1)^{p-1}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1}\right)d\sigma,$$

then

$$\acute{\Lambda}(\gamma) = (p-1)\int_1^{+\infty} K(\sigma)\log(\sigma)\,(\sigma^\gamma - 1)^{p-2}\left(\sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps+\gamma-1}\right)d\sigma.$$

It is clear that if $\gamma_0 = \frac{N-\beta-ps}{p}$, then $\acute{\Lambda}(\gamma_0) = 0$, $\acute{\Lambda}(\gamma) > 0$ if $\gamma < \gamma_0$ and $\acute{\Lambda}(\gamma) < 0$ if $\gamma > \gamma_0$. Thus

$$\max_{\{0<\gamma<\frac{N-ps-2\beta}{p}\}}\Lambda(\gamma) = \Lambda(\gamma_0).$$

Hence

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\frac{dx}{|x|^\beta}\frac{dy}{|y|^\beta} \geq 2\Lambda(\gamma_0)\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps+2\beta}}dx. \tag{86}$$

Notice that for $\beta = 0$, then $2\Lambda(\gamma_0) = 2\Lambda\left(\frac{N-ps}{p}\right) \equiv \Lambda_{N,p,s}$ given in (73). Therefore, we have the next optimality result.

Theorem (5.3.10) [203]: Define

$$\Lambda_{N,p,s} = \inf_{\{\phi\in C_0^\infty(\mathbb{R}^N)\setminus 0\}}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}|x|^\beta|y|^\beta}dxdy}{\int_{\mathbb{R}^N}\frac{|\phi(x)|^p}{|x|^{ps+2\beta}}dx},$$

then $\Lambda_{N,p,s} = 2\Lambda(\gamma_0)$.

Proof: From (24), it follows that $\Lambda_{N,p,s,\gamma} \geq 2\Lambda(\gamma_0)$, hence to conclude we have just to prove the reverse inequality.

We closely follow the argument used in [166].

Let $w_0(x) = |x|^{-\gamma_0}$, by Lemma (5.3.7), we have

$$L_{p,s,\beta}(w_0) = \Lambda(\gamma_0)\frac{w_0^{p-1}}{|x|^{ps+2\beta}}.$$

We set

$$M_n = \{x \in \mathbb{R}^N : 1 \le |x| < n\} \quad \text{and} \quad O_n = \{x \in \mathbb{R}^N : |x| \ge n\},$$

and define

$$w_n = \begin{cases} 1 - n^{-\gamma_0} & \text{if } \ x \in B_1(0), \\ |x|^{-\gamma_0} - n^{-\gamma_0} & \text{if } \ x \in M_n, \\ 0 & \text{if } \ x \in O_n. \end{cases}$$

By a direct computation, we get easily that $w_n \in X_0^{s,p,\beta}(\mathbb{R}^N)$.

Hence

$$\langle L_{p,s,\beta}(w_0), w_n\rangle = \Lambda(\gamma_0)\int_{\mathbb{R}^N}\frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}}dx.$$

Thus

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(w_n(x)-w_n(y)\big)|w_0(x)-w_0(y)|^{p-2}\big(w_0(x)-w_0(y)\big)}{|x-y|^{N+ps}|x|^{\beta}|y|^{\beta}}dxdy$$

$$= 2\Lambda(\gamma_0)\int_{\mathbb{R}^N}\frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}}dx.$$

Let analyze each term in the previous identity. As in [166] we obtain that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(w_n(x)-w_n(y)\big)|w_0(x)-w_0(y)|^{p-2}\big(w_0(x)-w_0(y)\big)}{|x-y|^{N+ps}|x|^{\beta}|y|^{\beta}}dxdy$$

$$\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_n(x)-w_n(y)|^p}{|x-y|^{N+ps}|x|^{\beta}|y|^{\beta}}dxdy.$$

On the other hand we have

$$\int_{\mathbb{R}^N}\frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}}dx = \int_{\mathbb{R}^N}\frac{w_n^p}{|x|^{ps+2\beta}}dx + I_n + J_n,$$

where

$$I_n = \int_{B_{1(0)}}(1-n^{-\gamma_0})\left(w_0^{p-1} - (1-n^{-\gamma_0})^{p-1}\right)\frac{dx}{|x|^{ps+\beta}},$$

and

$$J_n = \int_{M_n}(w_0(x)-n^{-\gamma_0})\left(w_0^{p-1} - (w_0(x)-n^{-\gamma_0})^{p-1}\right)\frac{dx}{|x|^{ps+\beta}}.$$

It is clear that $I_n, J_n \ge 0$ using a direct computation we can prove that

$$I_n + J_n \le C \ \text{ for all } \ n \ge 1.$$

Thus, combining the above estimates, it holds

$$\Lambda_{N,p,s,\gamma} \leq \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^p}{|x-y|^{N+ps}|x|^\beta |y|^\beta} dxdy}{\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx} \tag{87}$$

$$\leq 2\Lambda(\gamma_0)\left(1 + \frac{I_n + J_n}{\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx}\right). \tag{88}$$

Since $\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx \uparrow \infty$ as $n \to \infty$, then passing to the limit in (87), it follows that

$$\Lambda_{N,p,s,\gamma} \leq 2\Lambda(\gamma_0)$$

and then the result follows.

We need to use a version of the Hardy inequality in bounded domains.

We have the next result.

Lemma (5.3.11)[203]: Let Ω be a bounded regular domain such that $0 \in \Omega$, then there exists a constant $C \equiv C(\Omega, s, p, N) > 0$ such that for all $u \in C_0^\infty(\Omega)$, we have

$$C\int_\Omega \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx \leq \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}. \tag{89}$$

Proof: Fix $u \in C_0^\infty(\Omega)$, and let $\tilde{u}$, be the extension of u to $\mathbb{R}^N$ defined in Lemma (5.3.4). Then from Theorem (5.3.8), we get

$$2\Lambda(\gamma)\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^p}{|x|^{ps+2\beta}} dx \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \leq \|\tilde{u}\|^p_{X^{s,p,\beta}(\mathbb{R}^N)} \leq C\|u\|^p_{X^{s,p,\beta}(\Omega)}.$$

Since $\tilde{u}_{|\Omega} = u$, we conclude that

$$2\Lambda(\gamma)\int_\Omega \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx \leq C\|u\|^p_{X^{s,p,\beta}(\Omega)}$$

$$\leq C_1 \||u|\|^p_{X_0^{s,p,\beta}(\Omega)} = C_1\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

Hence we reach the desired result.

Now, we are able to proof Theorem (5.3.12).

Theorem (5.3.12)[203]: Let $p > 2, 0 < s < 1$ and $N > ps$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $1 < q < p$, there exists a positive constant $C = C(\Omega, q, N, s)$ such that for all $u \in C_0^\infty(\Omega)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy - \Lambda_{N,p,s}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \geq C\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy. \tag{90}$$

As a consequence we get the next "fractional" Caffarelli–Kohn–Nirenberg inequality in bounded domain.

Proof: We follow closely the arguments used in [205]. Let $u \in C_0^\infty(\Omega)$, and define $\alpha = \frac{N-ps}{p}$, then $w(x) = |x|^{-\alpha}$ and $v(x) = \frac{u(x)}{w(x)}$.

Recall that from the result of [166], we have

$$h_n(u) \geq C \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dx}{|y|^{\frac{N-ps}{2}}}. \tag{91}$$

Let us analyze the right hand side of the previous inequality.

Notice that

$$\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} = \frac{|w(y)u(x)-w(x)u(y)|^p}{|x-y|^{N+ps}} \frac{1}{\left(w(x)w(y)\right)^{\frac{p}{2}}}$$

$$= \frac{\left|\left(u(x)-u(y)\right) - \frac{u(y)}{w(y)}\left(w(x)-w(y)\right)\right|^p}{|x-y|^{N+ps}} \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} = f_1(x,y).$$

In the same way, thanks to the symmetry of $f_1(x,y)$, it immediately follows that

$$\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \left(w(x)\right)^{\frac{p}{2}} \left(w(y)\right)^{\frac{p}{2}}$$

$$= \frac{\left|\left(u(y)-u(x)\right) - \frac{u(x)}{w(x)}\left(w(y)-w(x)\right)\right|^p}{|x-y|^{N+ps}} \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} = f_2(x,y).$$

Hence,

$$h_s(u) \geq \frac{1}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} f_1(x,y)dxdy + \frac{1}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} f_2(x,y)dxdy.$$

Since f_1 and f_2 are positive functions, it follows that

$$h_s(u) \geq \frac{1}{2} \int\limits_{\Omega} \int\limits_{\Omega} f_1(x,y)dxdy + \frac{1}{2} \int\limits_{\Omega} \int\limits_{\Omega} f_2(x,y)dxdy.$$

Using the fact that Ω is a bounded domain, we obtain that for all $(x,y) \in (\Omega \times \Omega)$ and $q < p$,

$$\frac{1}{|x-y|^{N+ps}} \geq \frac{C(\Omega)}{|x-y|^{N+qs}}$$

and

$$Q(x,y) \equiv \frac{\left(w(x)\, w(y)\right)^{\frac{p}{2}}}{w(x)^p + w(y)^p} \leq C.$$

Define

$$D(x,y) \equiv \left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}} + \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \equiv \frac{w(x)^p + w(y)^p}{\left(w(x)\, w(y)\right)^{\frac{p}{2}}},$$

then $Q(x,y)D(x,y) = 1$. Thus

$$f_1(x,y) \geq C(\Omega) Q(x,y) \left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}} \times$$

$$\left[\frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} - p\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+qs}} \langle u(x)-u(y)\,, \frac{u(y)}{w(y)}\,\left(w(x)-w(y)\right)\rangle\right.$$

$$+C(p)\frac{\left|\frac{u(y)}{w(y)}\left(w(x)-w(y)\right)\right|^{p}}{|x-y|^{N+qs}}\Bigg].$$

Hence

$$f_1(x,y)\geq\left[C(\Omega)Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+qs}}\right]$$

$$-\left[pC(\Omega)Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(x)}{w(x)}\right|\left|\left(w(x)-w(y)\right)\right|\right].$$

In the same way we reach that

$$f_2(x,y)\geq\left[C(\Omega)Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(y)-u(x)|^{p}}{|x-y|^{N+qs}}\right]$$

$$-\left[pC(\Omega)Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(x)}{w(x)}\right|\left|\left(w(x)-w(y)\right)\right|\right].$$

Therefore,

$$h_s(u)\geq C(\Omega)\int_{\Omega}\int_{\Omega}Q(x,y)\left(\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}+\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\right)\frac{|u(u)-u(y)|^{p}}{|x-y|^{N+qs}}\,dxdy$$

$$-pC(\Omega)\int_{\Omega}\int_{\Omega}[Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(y)}{w(y)}\right|\left|\left(w(x)-w(y)\right)\right|]\,dxdy$$

$$-pC(\Omega)\int_{\Omega}\int_{\Omega}[Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(x)}{w(x)}\right|\left|\left(w(x)-w(y)\right)\right|]\,dxdy$$

Thus

$$h_s(u)\geq C(\Omega)\int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+qs}}dxdy$$

$$-C_1(\Omega\,,\mathrm{p})\int_{\Omega}\int_{\Omega}\left(h_1(x\,,y)+h_2(x\,,y)\right)dxdy\,, \tag{92}$$

with

$$h_1(x\,,y)=Q(x,y)\left(\frac{w(y)}{w(x)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(y)}{w(y)}\right|\left|\left(w(x)-w(y)\right)\right|,$$

$$h_2(x\,,y)=Q(x,y)\left(\frac{w(x)}{w(y)}\right)^{\frac{p}{2}}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+qs}}\left|\frac{u(x)}{w(x)}\right|\left|\left(w(x)-w(y)\right)\right|.$$

Since $h_1(x,y)$ and $h_2(x,y)$ are symmetric functions, we just have to estimate $\int_\Omega\int_\Omega h_2(x,y)dxdy$.

Using Young inequality, we get

$$\int_\Omega\int_\Omega h_2(x,y)dxdy \le \varepsilon \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}}dxdy$$

$$+C(\varepsilon)\int_\Omega\int_\Omega G(x,y)dxdy, \qquad (93)$$

with

$$G(x,y) = \left(Q(x,y)\right)^p \left(\frac{w(x)}{w(y)}\right)^{\frac{p^2}{2}} \left|\frac{u(x)}{w(x)}\right|^p \frac{|w(x)-w(y)|^p}{|x-y|^{N+qs}}.$$

We claim that

$$I \equiv \int_\Omega\int_\Omega G(x,y)dxdy \le C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dx}{|y|^{\frac{N-ps}{2}}}.$$

Notice that

$$I \equiv \int_\Omega\int_\Omega \frac{\left(u(x)\right)^p}{|x-y|^{N+qs}} \frac{\left(w(x)\right)^{p^2-p}|w(x)-w(y)|^p}{(w(x)^p+w(y)^p)^p}dxdy,$$

then

$$I = \int_\Omega u^p(x)\left[\int_\Omega \frac{||x|^\alpha-|y|^\alpha|^p}{(|x|^{\alpha p}+|y|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x-y|^{N+qs}}dy\right]dx.$$

To compute the above integral, we closely follow the arguments used in [214]. We set $y = \rho\acute{y}$ and $x = \rho\acute{x}$ with $|\acute{x}| = |\acute{y}| = 1$, then taking in consideration that $\Omega \subset B_0(R)$, it follows that

$$I = \int_\Omega u^p(x)\left[\int_\Omega \frac{||x|^\alpha-|y|^\alpha|^p}{(|x|^{\alpha p}+|x|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x-y|^{N+qs}}dy\right]dx$$

$$\le \int_\Omega u^p(x)\int_0^R \frac{|r^\alpha-\rho^\alpha|^p\rho^{\alpha p(p-1)+N-1}}{(r^{p\alpha}-\rho^{p\alpha})^p}\left(\int_{\mathbb{S}^{N-1}} \frac{d\acute{y}}{|\rho\acute{y}-r\acute{x}|^{N+qs}}\right)d\rho dx.$$

We set $\rho = r\sigma$, then

$$I \le \int_\Omega \frac{u^p(x)}{|x|^{qs}}\int_0^{\frac{R}{r}} \frac{|1-\sigma^\alpha|^p\sigma^{\alpha p(p-1)+N-1}}{(1-\sigma^{\alpha p})^p}\left(\int_{\mathbb{S}^{N-1}} \frac{d\acute{y}}{|\sigma\acute{y}-\acute{x}|^{N+qs}}\right)d\rho dx$$

$$= \int_\Omega \frac{u^p(x)}{|x|^{qs}}\int_0^{\frac{R}{r}} \frac{|1-\sigma^\alpha|^p\sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p}K(\sigma)d\sigma dx \le \mu\int_\Omega \frac{u^p(x)}{|x|^{qs}}dx,$$

where

$$\mu = \int_0^\infty \frac{|1-\sigma^\alpha|^p \sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p} K(\sigma) d\sigma$$

and

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-1}(\theta)}{(1-2\sigma\cos\theta+\sigma^2)^{\frac{N+qs}{2}}} d\theta.$$

Let us show that $\mu < \infty$.

It is clear that, as $\sigma \to \infty$, we have

$$\frac{|1-\sigma^\alpha|^p \sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p} K(\sigma) \backsimeq \sigma^{-1-qs} \in L^1(1,\infty).$$

Now, taking in consideration that $K(\sigma) \leq C|1-\sigma|^{-1-ps}$ as $s \to 1$, and following the same computation as in Lemma (5.3.7), it follows that

$$\int_0^1 \frac{(1-\sigma^\alpha)^p \sigma^{\alpha p(p-1)+N-1}}{(1+\sigma^{\alpha p})^p} K(\sigma) d\sigma < \infty.$$

Thus $\mu < \infty$.

Hence combining the above estimates, there results that

$$I \leq C \int_\Omega \frac{u^p(x)}{|x|^{qs}} dx.$$

Since $u(x) = v(x)|x|^{-(\frac{N-ps}{p})}$, then

$$I \leq C \int_\Omega \frac{|v(x)|^p}{|x|^{N-s(p-q)}} dx.$$

Let $\beta_0 = \frac{N-ps}{2} + \frac{(q-p)s}{2}$, then $\beta_0 < \frac{N-ps}{2}$. Applying Lemma (5.3.11), we obtain that

$$\begin{aligned} I &\leq C(\Omega) \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}|x|^{\beta_0}|y|^{\beta_0}} dy dx \\ &\leq C_1(\Omega) \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}|x|^{\frac{N-ps}{2}}|y|^{\frac{N-ps}{2}}} dy dx \\ &\leq C_1(\Omega) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}|x|^{\frac{N-ps}{2}}|y|^{\frac{N-ps}{2}}} dy dx. \end{aligned}$$

Therefore, using again estimate (91), we reach that

$$I \leq C_2(\Omega) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}$$

and the claim follows.

As a direct consequence of the above estimates, we have proved that

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} dx dy \leq C_3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}} \tag{94}$$

Thus

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+qs}} dxdy \le C h_s(u),$$

and the result follows at once.

We are now in position to prove the Theorem (5.3.13).

Theorem (5.3.13)[203]: Let $p \ge 2, 0 < s < 1$ and $N > ps$. Assume that $\Omega \subset \mathbb{R}^N$ is abounded domain, then for all $1 < q < p$, there exists a positive constant $C = C(\Omega, q, N, s)$such that for all $u \in C_0^\infty(\Omega)$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} dxdy \ge C \left(\int_\Omega \frac{|u(x)|^{\frac{p}{p^*_{s,q}}}}{|x|^{2\beta \frac{p}{p^*_{s,q}}}} dx \right)^{\frac{p}{p^*_{s,q}}} \tag{95}$$

where $p^*_{s,q} = \frac{pN}{N-qs}$ and $\beta = \frac{N-ps}{2}$.

In the case where $\Omega = \mathbb{R}^N$,to get a natural generalization of the classical Caffarelli–Kohn–Nirenberg inequality obtained in [7], we have to consider a class of admissible weights in the sense of [215]. Precisely we obtain the following weighted Sobolev inequality.

Proof: Recall that $\alpha = \frac{N-ps}{p}$. Since $\alpha p^*_{s,q} = \frac{N(N-ps)}{N-qs} < N$, it follows that $\int_\Omega \frac{|u(x)|^{p^*_{s,q}}}{|x|^{\alpha p^*_{s,q}}} dx < \infty$, for all $u \in C_0^\infty(\mathbb{R}^N)$.

To prove (95), we will use estimate (94)and the fractional Sobolev inequality. Fix $u \in C_0^\infty(\Omega)$ and define $u_1(x) = \frac{u(x)}{|x|^\alpha}$. By (94), we obtain that

$$C(\Omega) \int_\Omega \int_\Omega \frac{|u_1(x)-u_1(y)|^p}{|x-y|^{N+qs}} dxdy \le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}.$$

Now, using Sobolev inequality, there results that

$$S\left(\int_\Omega |u_1(x)|^{p^*_{s,q}} dx \right)^{\frac{p}{p^*_{s,q}}} \le \int_\Omega \int_\Omega \frac{|u_1(x)-u_1(y)|^p}{|x-y|^{N+qs}} dxdy,$$

where $p^*_{s,q} = \frac{pN}{N-qs}$. Hence, substituting u_1 by its value, we get

$$\left(\int_\Omega \frac{|u(x)|^{p^*_{s,q}}}{|x|^{\alpha p^*_{s,q}}} dx \right)^{\frac{p}{p^*_{s,q}}} \le C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \tag{96}$$

If we set $\beta = \frac{N-ps}{2} = \alpha \frac{p}{2}$, then inequality (96) can be written in the form

$$\left(\int_\Omega \frac{|u(x)|^{p^*_{s,q}}}{|x|^{2\beta \frac{p^*_{s,q}}{p}}} dx \right)^{\frac{p}{p^*_{s,q}}} \le C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \tag{97}$$

As a consequence, we will prove the fractional Caffarelli–Kohn–Nirenberg inequality given in Theorem (5.3.14).

Theorem (5.3.14)[203]: Assume that $1 < p < \frac{N}{s}$ and let $0 < \beta < \frac{N-ps}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq S(\beta)\left(\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta\frac{p_s^*}{p}}} dx\right)^{\frac{p}{p_s^*}}, \tag{98}$$

where $S(\beta) > 0$.

Proof: Let $u \in C_0^\infty(\mathbb{R}^N)$, without loss of generality, we can assume that $u \geq 0$. Using the fact that $\beta < \frac{N-ps}{2}$, we easily get that $\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta\frac{p_s^*}{p}}} dx \leq \infty$.

From now and for simplicity of typing, we denote by $C, C_1, C_2, \cdots$ any universal constant that does not depend on uand can change from a line to another.

We set $\tilde{u}(x) = \frac{u(x)}{w_1(x)}$, where $w_1(x) = |x|^{\frac{2\beta}{p}}$, then

$$\left(\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta\frac{p_s^*}{p}}} dx\right)^{\frac{p}{p_s^*}} = \left(\int_{\mathbb{R}^N} |\tilde{u}|^{p_s^*}\, dx\right)^{\frac{p}{p_s^*}}. \tag{99}$$

Using Sobolev inequality, it follows that

$$S\left(\int_{\mathbb{R}^N} |\tilde{u}|^{p_s^*}\, dx\right)^{\frac{p}{p_s^*}} \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} dxdy. \tag{100}$$

To get the desired result we just have to show that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} dxdy \leq C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dx}{|x|^\beta} \tag{101}$$

for some positive constant C.

Using the definition of $\tilde{u}$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dx}{|x|^\beta} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_1(x)\tilde{u}(x)-w_1(y)\tilde{u}(y)|^p}{|x-y|^{N+ps}} \frac{dx}{w_1^{\frac{p}{2}}(x)} \frac{dy}{w_1^{\frac{p}{2}}(y)}.$$

Notice that

$$\frac{|w_1(x)\tilde{u}(x)-w_1(y)\tilde{u}(y)|^p}{|x-y|^{N+ps}} \frac{1}{w_1^{\frac{p}{2}}(x)} \frac{1}{w_1^{\frac{p}{2}}(y)} =$$

$$\frac{\left|\tilde{u}(x)-\tilde{u}(y)-w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+ps}} \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \equiv \tilde{f}_1(x,y).$$

In the same way we have

$$\frac{|w_1(x)\tilde{u}(x)-w_1(y)\tilde{u}(y)|^p}{|x-y|^{N+ps}} \frac{1}{w_1^{\frac{p}{2}}(x)} \frac{1}{w_1^{\frac{p}{2}}(y)} =$$

$$\frac{\left|\left(\tilde{u}(x)-\tilde{u}(y)\right)-w_1(x)\tilde{u}(x)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(x)}\right)\right|^p}{|x-y|^{N+ps}}\left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} \equiv \tilde{f}_2(x,y).$$

Since

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{f}_1(x,y)dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{f}_2(x,y)dxdy,$$

we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\frac{dx}{|x|^\beta}\frac{dx}{|x|^\beta} = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{f}_1(x,y)dxdy + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \tilde{f}_2(x,y)dxdy.$$

Notice that

$$\tilde{f}_1(x,y) \geq \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \times$$

$$\left[\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} - p\frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-2}}{|x-y|^{N+ps}}\langle \tilde{u}(x)-\tilde{u}(y)\,, w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(y)}\right)\rangle \right.$$

$$\left. +C(p)\frac{\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+ps}}\right].$$

Hence

$$\tilde{f}_1(x,y) \geq \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \times$$

$$\left[\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} - p\frac{|\tilde{u}(x)-\tilde{u}(y)|^{p-1}}{|x-y|^{N+ps}}\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|\right].$$

Using Young inequality, we get the existence of $C_1, C_2 > 0$ such that

$$\tilde{f}_1(x,y) \geq \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \times$$

$$\left[C_1\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} - C_2\frac{\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+ps}}\right].$$

In the same way and using that $\tilde{f}_1, \tilde{f}_2$ are symmetric functions, it holds

$$f_2(x,y) \geq \left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} \times$$

$$\left[C_1\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} - C_2\frac{\left|w_1(x)\tilde{u}(x)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(x)}\right)\right|^p}{|x-y|^{N+ps}}\right].$$

Thus we get the existence of positive constants C_1, C_2, C_3 such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\frac{dx}{|x|^\beta}\frac{dx}{|x|^\beta} \geq$$

$$C_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} \left[\left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} + \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \right] dxdy$$

$$-\ C_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \frac{\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+qs}} dxdy$$

$$-\ C_3 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} \frac{\left|w_1(x)\tilde{u}(x)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(x)}\right)\right|^p}{|x-y|^{N+qs}} dxdy.$$

Since

$$\left[\left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} + \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \right] \geq 1,$$

then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+ps}} dxdy \leq\ C_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\beta}}\ \frac{dx}{|x|^{\beta}}$$

$$+\ C_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \frac{\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+qs}} dxdy \tag{102}$$

$$+\ C_3 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} \frac{\left|w_1(x)\tilde{u}(x)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(x)}\right)\right|^p}{|x-y|^{N+qs}} dxdy\,.$$

We get

$$g_1(x,y) = \left(\frac{w_1(y)}{w_1(x)}\right)^{\frac{p}{2}} \frac{\left|w_1(x)\tilde{u}(x)\left(\frac{1}{w_1(y)}-\frac{1}{w_1(x)}\right)\right|^p}{|x-y|^{N+ps}}$$

and

$$g_2(x,y) = \left(\frac{w_1(x)}{w_1(y)}\right)^{\frac{p}{2}} \frac{\left|w_1(y)\tilde{u}(y)\left(\frac{1}{w_1(x)}-\frac{1}{w_1(y)}\right)\right|^p}{|x-y|^{N+ps}}$$

It is clear that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_1(x,y)dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_2(x,y)dxdy,$$

therefore, to get the desired result, we just have to show that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_1(x,y)dxdy \leq C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\beta}}\ \frac{dx}{|x|^{\beta}}.$$

Going back to the definition of $\tilde{u}$ and w_1, we reach that

$$g_1(x,y) = \frac{|u(x)|^p \left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|x|^{3\beta}|y|^{\beta}|x-y|^{N+ps}}.$$

We closely follow the same type of computation as in the proof of Lemma (5.3.7).

We have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_1(x,y)dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p \left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|x|^{3\beta}|y|^{\beta}|x-y|^{N+ps}} dxdy$$

$$= \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left(\int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^{\beta}|x-y|^{N+ps}} dy \right) dx.$$

We set $r = |x|$ and $\rho = |y|$, then $x = r\acute{x}$, $y = \rho\acute{y}$ with $|\acute{x}| = |\acute{y}| = 1$, then

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left(\int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^{\beta}|x-y|^{N+ps}} dy \right) dx$$

$$= \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left[\int_0^{+\infty} \frac{\left| r^{\frac{2\beta}{p}} - \rho^{\frac{2\beta}{p}} \right|^p \rho^{N-1}}{\rho^{\beta}} \left(\int_{|\acute{y}|=1} \frac{dH^{n-1}(\acute{y})}{|r\acute{x} - \rho\acute{y}|^{N+ps}} dy \right) d\rho \right] dx.$$

Let $\sigma = \frac{\rho}{r}$, then

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left(\int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^{\beta}|x-y|^{N+ps}} dy \right) dx$$

$$= \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{2\beta+ps}} \left[\int_0^{+\infty} \left| 1 - \sigma^{\frac{2\beta}{p}} \right|^p \sigma^{N-1-\beta} K(\sigma) d\sigma \right] dx,$$

where K is defined in (82). Since

$$\int_0^{+\infty} \left| 1 - \sigma^{\frac{2\beta}{p}} \right|^p \sigma^{N-1-\beta} K(\sigma) d\sigma \equiv C_3 < \infty,$$

it follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_1(x,y)dxdy = C_3 \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{2\beta+ps}} dx.$$

Now, using inequality (85), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_1(x,y)dxdy \le C_4 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^{\beta}} \frac{dx}{|x|^{\beta}}. \tag{103}$$

Combining (99), (100), (103) and (102), we reach the desired result.

In the case where Ω is a regular bounded domain containing the origin, we have the following version of Theorem (5.3.14).

Theorem (5.3.15)[203]: Assume that Ω is a regular bounded domain with $0 \in \Omega$, then there exists a positive constant $C \equiv C(\Omega, N, p, s, \beta)$ such that for all $\phi \in C_0^\infty(\Omega)$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dx}{|x|^\beta} \geq C\left(\int_\Omega \frac{|\phi(x)|^{p_s^*}}{|x|^{2\beta\frac{p_s^*}{p}}} dx\right)^{\frac{p}{p_s^*}}. \tag{104}$$

Proof: Let $\phi \in C_0^\infty(\Omega)$ and define $\tilde{\phi}$ to be the extension of ϕ to $\mathbb{R}^N$ given in Lemma (5.3.4), then using the fact that Ω is a regular bounded domain, we reach that

$$\|\tilde{\phi}\|_{X^{s,p,\beta}(\mathbb{R}^N)} \leq C_1 \|\phi\|_{X^{s,p,\beta}(\Omega)} \leq C_1 \left(\int_\Omega\int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta}\right)^{\frac{1}{p}}.$$

Now, applying Theorem (5.3.14) to $\tilde{\phi}$, it follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x)-\tilde{\phi}(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dx}{|x|^\beta} \geq S(\beta)\left(\int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x)|^{p_s^*}}{|x|^{2\beta\frac{p_s^*}{p}}} dx\right)^{\frac{p}{p_s^*}}.$$

Hence combining the above estimates we get the desired result.

We deal with the next problem

$$\begin{cases} L_{p,s}\ u = \lambda \dfrac{u^{p-1}}{|x|^{ps}} + u^q, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{105}$$

where

$$L_{s,p}\ u \coloneqq \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+ps}} dy,$$

and $0 < \lambda \leq \Lambda_{N,p,s}$.

In the case where $0 < q < p-1$, the existence result follows using variational arguments. More precisely we have:

(i) If $\lambda < \Lambda_{N,p,s}$, then the existence of a solution u to (105) follows using classical minimizing argument. In this case $u \in W_0^{s,p}(\Omega)$.

(ii) If $\lambda < \Lambda_{N,p,s}$, the existence result follows using the improved Hardy inequality in Theorem (5.3.12). In this case u satisfies $h_{s,\Omega}(u) < \infty$ where $h_{s,\Omega}$ is defined by

$$h_{s,\Omega}(u) \equiv \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - \Lambda_{N,p,s} \int_\Omega \frac{|u(x)|^p}{|x|^{ps}} dx. \tag{106}$$

This clearly implies that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \leq \infty \ \text{ for all } q < p.$$

We deal now with the case $q > p-1$.

Define $w(x) = |x|^{-\gamma}$ with $0 < \gamma < \frac{N-ps}{p-1}$, then we have previously obtained that

$$L_{s,\Omega}(w) = \Lambda(\gamma)\frac{w^{p-1}}{|x|^{ps}} \quad a.e. \text{ in } \mathbb{R}^N\backslash\{0\},$$

where

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^\gamma - 1)^{p-1}\left(\sigma^{N-1-\gamma(p-1)} - \sigma^{ps-1)}\right)\ d\sigma,$$

and K is given by (82). Let us begin by proving the next lemma.

Lemma (5.3.16)[203]: Assume that $0 < \lambda < \Lambda_{N,p,s}$, then there exist γ_1, γ_2 such that

$$0 < \gamma_1 < \frac{N-ps}{p} < \gamma_2,$$

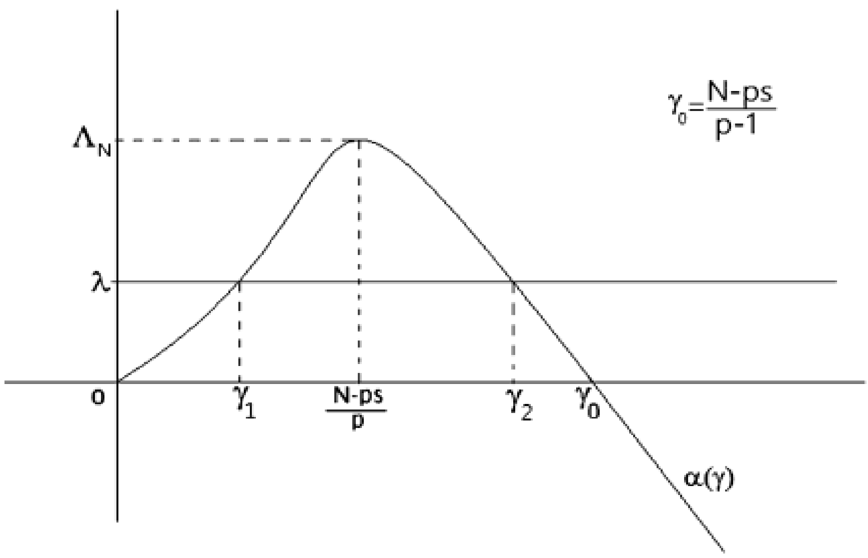
and $\Lambda(\gamma_1) = \Lambda(\gamma_2) = \lambda$.

Proof: We have $\Lambda(0) = 0$, $\Lambda\left(\frac{N-ps}{p}\right) = \Lambda_{N,p,s}$ $\Lambda(\gamma) < 0$ if $\gamma > \frac{N-ps}{p-1}$ and

$$\acute{\Lambda}(\gamma) = \int_1^{+\infty} K(\sigma)\log(\sigma)(\sigma^\gamma - 1)^{p-2}\left(\sigma^{N-1-\gamma(p-1)} - \sigma^{ps+\gamma-1)}\right)\ d\sigma.$$

It is clear that for $\gamma_0 = \frac{N-ps}{p}$, we have $\acute{\Lambda}(\gamma_0) = 0$, $\acute{\Lambda}(\gamma) > 0$ if $\gamma < \gamma_0$ and $\acute{\Lambda}(\gamma) < 0$ if $\gamma > \gamma_0$.

Hence, since $\lambda < \Lambda_{N,p,s}$, we get the existence of $0 < \gamma_1 < \frac{N-ps}{p} < \gamma_2 \frac{N-ps}{p-1}$ such that



$\Lambda(\gamma_1) = \Lambda(\gamma_2) = \lambda$.

Define $q + (p,s) = p - 1 + \frac{ps}{\gamma_1}$, it is clear that $p_s^* - 1 < q + (p,s)$. We have the next existence result

Theorem (5.3.17)[203]: Assume that $q < q_+(p,s)$, then

(i) If $p - 1 < q < p_s^* - 1$, problem (105) has a solution u. Moreover, $u \in W_0^{s,p}(\Omega)$ if $\lambda < \Lambda_{N,p,s}$ and $h_{s,\Omega}(u) < \infty$ if $\lambda = \Lambda_{N,p,s}$ where $h_{s,\Omega}$ is defined in (106).

(ii) If $p_s^* - 1 \le q < q_+(p,s)$, then problem (105) has a positive super solution u.

Proof: Let us begin with the case where $p - 1 < q < p_s^* - 1$. If $\lambda < \Lambda_{N,p,s}$, then using the Mountain Pass Theorem, see [218], we get a positive solution $u \in W_0^{s,p}(\Omega)$.

However, if $\lambda < \Lambda_{N,p,s}$, then using the improved Hardy inequality in Theorem (5.3.12) and the Mountain Pass Theorem, we reach a positive solution u to problem (105) with $h_{s,\Omega}(u) < \infty$.

Assume now that $p_s^* - 1 \le q < q_+(p,s)$ and fix $\lambda_1 \in (\lambda, \Lambda_{N,p,s})$ to be chosen later.

Let $\gamma_1 \in (0, \frac{N-ps}{p})$ be such that $\Gamma(\gamma_1) = \lambda_1$ and set $w(x) = |x|^{-\gamma_1}$, then

$$L_{s,p}(w) = \lambda_1 \frac{w^{p-1}}{|x|^{ps}} \quad a.e. \text{ in } \mathbb{R}^N \setminus \{0\}$$

with $\frac{w^{p-1}}{|x|^{ps}} \in L^1_{loc}(\mathbb{R}^N)$. Hence

$$L_{s,p}(w) = \lambda \frac{w^{p-1}}{|x|^{ps}} + (\lambda_1 - \lambda) \frac{w^{p-1}}{|x|^{ps}} \quad a.e. \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using the fact that $q < q_+(p,s)$, we can choose $\lambda_1 > \lambda$, very close to λ such that $\gamma_1(p-1) + ps > q\gamma_1$, thus, in any bounded domain Ω, we have

$$(\lambda_1 - \lambda) \frac{w^{p-1}}{|x|^{ps}} \ge C(\Omega) w^q.$$

Define $\widehat{w} = Cw$, by the previous estimates, we can choose $C(\Omega) > 0$ such that $\widehat{w}$ will be a supersolution to (105) in Ω. Hence the result follows..

Now, we show the optimality of the exponent $q_+(p,s)$. We have the following non existence result.

Lemma (5.3.18)[203]: Let $\Omega \subset \mathbb{R}^N$ be a regular domain such that $0 \in \Omega$. Define

$$\Lambda(\Omega) = \inf_{\{\phi \in C_0^\infty \Omega \setminus 0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} dxdy}{\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps}} dx},$$

then $\Lambda(\Omega) = \Lambda_{N,p,s}$ defined in (73).

Proof: Recall that

$$\Lambda_{N,p,s} = \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N) \setminus 0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} dxdy}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} dx},$$

thus $\Lambda(\Omega) \ge \Lambda_{N,p,s}$. It is clear that if $\Omega_1 \subset \Omega_2$, t hen $\Lambda(\Omega_1) \ge \Lambda(\Omega_2)$.

Now, using a dilatation argument we can prove that $\Lambda(B_{R_1}(0)) = \Lambda(B_{R_2}(0))$ for all $0 < R_1 < R_2$. Hence we conclude that $\Lambda(\Omega) \equiv \overline{\Lambda}$ does not depend of the domain Ω. For $\phi \in C_0^\infty(\mathbb{R}^N)$, we set

$$Q(\phi) \equiv \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} dxdy}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} dx}.$$

Let $\{\phi_n\}_n \subset C_0^\infty(\mathbb{R}^N)$ be such that $Q(\phi_n) \to \Lambda_{N,p,s}$. Without loss of generality and using a symmetrization argument we can assume that $\text{Supp}(\phi_n) \subset B_{R_2}(0)$. It is clear that $Q(\phi_n) \ge \Lambda(\text{Supp}(\phi_n)) = \overline{\Lambda}$, thus, as $n \to \infty$, it follows that $\overline{\Lambda} \le \Lambda_{N,p,s}$. As a conclusion we reach that $\Lambda = \Lambda_{N,p,s}$ and the result follows.

We need the next lemma.

Lemma (5.3.19) [203]: Let Ω be a bounded domain such that $0 \in \Omega$. Assume that $u \in W^{s,p}(\mathbb{R}^N)$ is such that $u \geq 0$ *in* $\mathbb{R}^N, u > 0$ in Ω and $L_{N,p,s}u \gneqq \lambda \frac{u^{p-1}}{|x|^{ps}}$ in Ω, then there exists $C > 0$ such that $u(x) \geq C|x|^{-\gamma_1}$ in $B_\eta(0)$ where γ_1 is defined in Lemma (5.3.16).

Proof: Without loss of generality we can assume that $B_1(0) \subset \Omega$.

Fixed $\lambda < \Lambda_{N,p,s}$ sand define

$$\tilde{w}(x) = \begin{cases} |x|^{-\gamma_1} - 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

It is clear that $\tilde{w} \in W_0^{s,p}(B_1(0))$ and

$$\begin{cases} L_{p,s}\tilde{w} = h(x)\dfrac{\tilde{w}^{p-1}}{|x|^{ps}} & \text{in } B_1(0), \\ \tilde{w} = 0 & \text{in } \mathbb{R}^N \backslash B_1\{0\} \end{cases} \tag{107}$$

where

$$h(x) = \int_0^{\frac{1}{|x|}} \left|1 - \sigma^{-\tilde{\gamma}}\right|^{p-2}\left(1 - \sigma^{-\tilde{\gamma}}\right)K(\sigma)d\sigma + \left(1 - |x|^{\tilde{\gamma}}\right) \int_{\frac{1}{|x|}}^{\infty} \sigma^{N-1} K(\sigma)d\sigma .$$

Using the definition of γ_1, see Lemma (5.3.16), we can prove that $h(x) \leq \lambda$ for all $x \in B_1(0)$.

Since $L_{p,s}\, u \gneqq 0$ and $u > 0$ in Ω, then using the nonlocal weak Harnack inequality in [210], we get the existence of $\varepsilon > 0$ such that $u \geq \varepsilon$ in $\bar{B}_1(0)$.

Therefore we obtain that

$$\begin{cases} L_{p,s}u \geq \lambda \dfrac{u^{p-1}}{|x|^{ps}} & \text{in } B_1(0), \\ L_{p,s}\tilde{w} \leq \lambda \dfrac{\tilde{w}^{p-1}}{|x|^{ps}}, & \text{in } B_1(0), \\ u \geq \tilde{w} & \text{in } \mathbb{R}^N \backslash B_1(0). \end{cases} \tag{108}$$

Thus by the comparison principle in Lemma (5.3.6), it follows that $\tilde{w} \leq u$ which is the desired result.

We are now in position to prove Theorem (5.3.20).

Theorem (5.3.20)[203]: Let $q_+(p,s) = p - 1 + \frac{ps}{\gamma_1}$. If $q > q_+(p,s)$, then the unique nonnegative supersolution $u \in W_{loc}^{s,p}(\Omega)$ to problem (105) is $u \equiv 0$.

We first prove the next lemma which shows that the Hardy constant is independent of the domain.

Proof: We argue by contradiction. Assume the existence of $u \gneqq 0$ such that $u \in W^{s,p}(\mathbb{R}^N)$ and u is a supersolution to problem (105) in Ω, then $u > 0$ in Ω. Let $\phi \in C_0^\infty(B_\eta(0))$ with $B_\eta(0) \subset\subset \Omega$ and $\eta > 0$ to be chosen later.

Using Picone's inequality in Lemma (5.3.5), it follows that

$$\|\phi\|^p_{X_0^{s,p}(\mathbb{R}^N)} \geq \int_{B_\eta(0)} \frac{L_{p,s}(u)}{u^{p-1}}|\phi|^p dx.$$

Thus

$$\|\phi\|^p_{X_0^{s,p}(\mathbb{R}^N)} \geq \int_{B_\eta(0)} u^{q-(p-1)}\,|\phi|^p dx.$$

Since $q > q_+(p,s)$, we get the existence of $\varepsilon > 0$ such that

$$(\gamma_1 - \varepsilon)(q-(p-1)) > ps + \rho$$

for some $\rho > 0$. Thus, using Lemma (5.3.19), we can choose $\eta > 0$ such that

$$u^{q-(p-1)} \geq C|x|^{-ps-\rho} \text{ in } B_\eta(0).$$

Therefore

$$\|\phi\|^p_{X_0^{s,p}(\mathbb{R}^N)} \geq C \int_{B_\eta(0)} \frac{|\phi|^p}{|u|^{ps+\rho}}dx,$$

which is a contradiction with the optimality of the Hardy inequality proved in Lemma (5.3.18).

Hence we conclude.

Corollary (5.3.21)[239]: (Picone inequality). Let $w \in X_0^{1-\epsilon,1+\epsilon,1+\epsilon}(\Omega)$ be such that $w > 0$ in Ω. Assume that $L_{1-\epsilon,1+\epsilon,1+\epsilon}(w) = \nu$ with $\nu \in L^1_{loc}(\mathbb{R}^N)$ and $\nu \gneqq 0$, then for all $u \in C_0^\infty(\Omega)$, we have

$$\frac{1}{2}\int_\Omega^2\int_Q^2 \sum_{n_0} \frac{\left|u(x_{n_0}) - u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+1-\epsilon^2}} \frac{dx_{n_0}dy_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}} \geq \langle L_{1-\epsilon,1+\epsilon,1+\epsilon}w, \frac{|u|^{1+\epsilon}}{w^\epsilon}\rangle.$$

Proof: The case $\epsilon = -1$ is obtained in [216] if $\epsilon = 2$ and in [207] if $\epsilon \neq 0$. For the reader convenience we include some details for the general case $\epsilon \neq -1$.

We set $v(x_{n_0}) = \frac{|u(x_{n_0})|^{1+\epsilon}}{|w(x_{n_0})|^\epsilon}$ and $k(x_{n_0}, y_{n_0}) = \frac{1}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}}$, then

$$\sum_{n_0} \langle L_{1-\epsilon,1+\epsilon,1+\epsilon}\left(w(x_{n_0})\right), v(x_{n_0})\rangle$$
$$= \int_\Omega^1 \sum_{n_0} v(x_{n_0}) \int_{\mathbb{R}^N}^1 \left|w(x_{n_0}) - w(y_{n_0})\right|^{\epsilon-1}\left(w(x_{n_0}) - w(y_{n_0})\right) k(x_{n_0}, y_{n_0}) dx_{n_0}dy_{n_0}$$
$$= \int_\Omega^1 \sum_{n_0} \frac{|u(x_{n_0})|^{1+\epsilon}}{|w(x_{n_0})|^\epsilon} \int_{\mathbb{R}^N}^1 \left|w(x_{n_0}) - w(y_{n_0})\right|^{\epsilon-1}\left(w(x_{n_0}) - w(y_{n_0})\right) k(x_{n_0}, y_{n_0}) dx_{n_0}dy_{n_0}$$

Since k is symmetric, we obtain that

$$\sum_{n_0} \langle L_{1-\epsilon,1+\epsilon,1+\epsilon}\left(w(x_{n_0})\right), v(x_{n_0})\rangle$$
$$= \frac{1}{2}\int_\Omega^2\int_Q^2 \sum_{n_0} \left(\frac{\left|u(x_{n_0})\right|^{1+\epsilon}}{|w(x_{n_0})|^\epsilon} - \frac{\left|u(y_{n_0})\right|^{1+\epsilon}}{|w(y_{n_0})|^\epsilon}\right)\left|w(x_{n_0}) - w(y_{n_0})\right|^{\epsilon-1}\left(w(x_{n_0}) - w(y_{n_0})\right) k(x_{n_0}, y_{n_0}) dx_{n_0}dy_{n_0}.$$

Let $v_1 = \frac{u}{w}$, then

$$\sum_{n_0} \langle L_{1-\epsilon,1+\epsilon,1+\epsilon} \left(w(x_{n_0})\right), v(x_{n_0})\rangle$$
$$= \frac{1}{2}\int_{\Omega}^{2} \int_{Q}^{2} \sum_{n_0} \left(|v_1(x_{n_0})|^{1+\epsilon} w(x_{n_0}) - |v_1(y_{n_0})|^{1+\epsilon} w(y_{n_0})\right) |w(x_{n_0}) - w(y_{n_0})|^{\epsilon-1} \left(w(x_{n_0}) - w(y_{n_0})\right) k(x_{n_0}, y_{n_0}) dy_{n_0} dx_{n_0}.$$

Define
$$\Phi(x_{n_0}, y_{n_0}) = \sum_{n_0} |u(x_{n_0}) - u(y_{n_0})|^{1+\epsilon} - \sum_{n_0} \left(|v_1(x_{n_0})|^{1+\epsilon} w(x_{n_0}) - |v_1(y_{n_0})|^{1+\epsilon} w(y_{n_0})\right) |w(x_{n_0}) - w(y_{n_0})|^{\epsilon-1} \left(w(x_{n_0}) - w(y_{n_0})\right),$$
then
$$\sum_{n_0} \langle L_{1-\epsilon,1+\epsilon,1+\epsilon} \left(w(x_{n_0})\right), v(x_{n_0})\rangle + \frac{1}{2}\int_{Q}^{2} \sum_{n_0} \Phi(x_{n_0}, y_{n_0}) k(x_{n_0}, y_{n_0}) dy_{n_0} dx_{n_0}$$
$$= \frac{1}{2}\int \int_{Q} \sum_{n_0} |u(x_{n_0}) - u(y_{n_0})|^{1+\epsilon} k(x_{n_0}, y_{n_0}) dy_{n_0} dx_{n_0}.$$

We claim that $\Phi \geq 0$. It is clear that, by a symmetry argument, we can assume that $w(x_{n_0}) \geq w(y_{n_0})$. Let $t = w(y_{n_0})/w(x_{n_0}), a = u(x_{n_0})/u(y_{n_0})$, then using inequality (78), the claim follows at once. Hence we conclude.

Corollary (5.3.22)[239]: Let Ω be a bounded domain and let f be a nonnegative continuous function such that $f(\sigma) > 0$ if $\sigma > 0$ and $\frac{f(\sigma)}{\sigma^{\epsilon}}$ is decreasing. Let $u, v \in W_0^{1-\epsilon,1+\epsilon}(\Omega)$ be such that $u, v > 0$ in Ω and
$$\begin{cases} L_{1-\epsilon,1+\epsilon} u \geq f(u) & \text{in } \Omega, \\ L_{1-\epsilon,1+\epsilon} v \leq f(v) & \text{in } \Omega, \end{cases}$$
Then, $u \geq v$ in Ω.

Proof: Using an approximation argument, taking in consideration that $u, v > 0$, we can prove that
$$\frac{L_{1-\epsilon,1+\epsilon} u}{u^{\epsilon}} - \frac{L_{1-\epsilon,1+\epsilon} v}{v^{\epsilon}} \geq \left(\frac{f(u)}{u^{\epsilon}} - \frac{f(v)}{v^{\epsilon}}\right). \tag{109}$$
We set $\xi = (v^{1+\epsilon} - u^{1+\epsilon})_+$, then
$$\int_{\Omega}^{2} \left(\frac{f(u)}{u^{\epsilon}} - \frac{f(v)}{v^{\epsilon}}\right) \xi dx_{n_0} \leq \int_{\Omega}^{2} \xi \left(\frac{L_{1-\epsilon,1+\epsilon} u}{u^{\epsilon}} - \frac{L_{1-\epsilon,1+\epsilon} v}{v^{\epsilon}}\right) dx_{n_0}. \tag{110}$$
Let us analyze each term in the previous inequality.

Using the definition of ξ we obtain that $\left(\frac{f(u)}{u^{\epsilon}} - \frac{f(v)}{v^{\epsilon}}\right)\xi \geq 0$. On the other hand, we have

$$J \equiv \int_{\Omega}^{2} \xi\left(\frac{L_{1-\epsilon,1+\epsilon}u}{u^{\epsilon}} - \frac{L_{1-\epsilon,1+\epsilon}v}{v^{\epsilon}}\right) dx_{n_0}$$
$$= \frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0} \frac{\left|u(x_{n_0}) - u(y_{n_0})\right|^{\epsilon-1}\left(u(x_{n_0}) - u(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+1-\epsilon^2}} \left(\frac{\xi(x_{n_0})}{u^{\epsilon}(x_{n_0})} - \frac{\xi(y_{n_0})}{u^{\epsilon}(y_{n_0})}\right) dx_{n_0} dy_{n_0}$$
$$-\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0} \frac{\left|v(x_{n_0}) - v(y_{n_0})\right|^{\epsilon-1}\left(v(x_{n_0}) - v(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+1-\epsilon^2}} \left(\frac{\xi(x_{n_0})}{v^{\epsilon}(x_{n_0})} - \frac{\xi(y_{n_0})}{v^{\epsilon}(y_{n_0})}\right) dx_{n_0} dy_{n_0},$$
where $Q = \mathbb{R}^N \times \mathbb{R}^N \backslash (C\Omega \times C\Omega)$.

Notice that

$$\sum_{n_0} \left|u(x_{n_0}) - u(y_{n_0})\right|^{\epsilon-1}\left(u(x_{n_0}) - u(y_{n_0})\right)\left(\frac{\xi(x_{n_0})}{u^{\epsilon}(x_{n_0})} - \frac{\xi(y_{n_0})}{u^{\epsilon}(y_{n_0})}\right) =$$
$$\sum_{n_0} \left|u(x_{n_0}) - u(y_{n_0})\right|^{\epsilon-1}\left(u(x_{n_0}) - u(y_{n_0})\right)\left(\frac{v^{1+\epsilon}(x_{n_0})}{u^{\epsilon}(x_{n_0})} - \frac{v^{1+\epsilon}(y_{n_0})}{u^{\epsilon}(y_{n_0})}\right) - \left|u(x_{n_0}) - u(y_{n_0})\right|^{1+\epsilon}.$$

In the same way, we obtain that

$$\sum_{n_0} \left|v(x_{n_0}) - v(y_{n_0})\right|^{\epsilon-1}\left(v(x_{n_0}) - v(y_{n_0})\right)\left(\frac{\xi(x_{n_0})}{v^{\epsilon}(x_{n_0})} - \frac{\xi(y_{n_0})}{v^{\epsilon}(y_{n_0})}\right)$$
$$= -\sum_{n_0} \left|v(x_{n_0}) - v(y_{n_0})\right|^{\epsilon-1}\left(v(x_{n_0}) - v(y_{n_0})\right)\left(\frac{u^{1+\epsilon}(x_{n_0})}{v^{\epsilon}(x_{n_0})} - \frac{u^{1+\epsilon}(y_{n_0})}{v^{\epsilon}(y_{n_0})}\right) + \sum_{n_0} \left|v(x_{n_0}) - v(y_{n_0})\right|^{1+\epsilon}.$$

Thus

$$J = \frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0} \frac{\left|u(x_{n_0}) - u(y_{n_0})\right|^{\epsilon-1}\left(u(x_{n_0}) - u(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+1-\epsilon^2}} \left(\frac{v^{1+\epsilon}(x_{n_0})}{u^{\epsilon}(x_{n_0})} - \frac{v^{1+\epsilon}(y_{n_0})}{u^{\epsilon}(y_{n_0})}\right) dx_{n_0} dy_{n_0}$$
$$+\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0} \frac{\left|v(x_{n_0}) - v(y_{n_0})\right|^{\epsilon-1}\left(v(x_{n_0}) - v(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+1-\epsilon^2}} \left(\frac{u^{1+\epsilon}(x_{n_0})}{v^{\epsilon}(x_{n_0})} - \frac{u^{1+\epsilon}(y_{n_0})}{v^{\epsilon}(y_{n_0})}\right) dx_{n_0} dy_{n_0}$$

$$-\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}}$$

$$-\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0}\frac{\left|v(x_{n_0})-v(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}}dx_{n_0}dy_{n_0}$$

$$=\int_{\Omega}^{2}\sum_{n_0}\frac{L_{1+\epsilon,1-\epsilon}(u)}{u^{\epsilon}}\,v^{1+\epsilon}dx_{n_0}$$

$$+\int_{\Omega}^{2}\sum_{n_0}\frac{L_{1+\epsilon,1-\epsilon}(v)}{v^{\epsilon}}\,u^{1+\epsilon}dx_{n_0}$$

$$-\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}}dx_{n_0}dy_{n_0}$$

$$-\frac{1}{2}\int_{\Omega}^{2}\int_{Q}^{2}\sum_{n_0}\frac{\left|v(x_{n_0})-v(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}}dx_{n_0}dy_{n_0}.$$

Now, using Picone's inequality, we conclude that $J\leq 0$. Thus

$$\left(\frac{f(u)}{u^{\epsilon}}-\frac{f(v)}{v^{\epsilon}}\right)\xi\equiv 0$$

and then $\xi=0$ which implies that $u\leq v$ in Ω.

Corollary (5.3.23)[239]: Fix $0<1+\epsilon<\frac{N-(1+\epsilon)^2}{2}$ and let $w(x_{n_0})=|x_{n_0}|^{-\gamma}$ with $0<\gamma<\frac{N-(1+\epsilon)^2-2(1+\epsilon)}{\epsilon}$, then there exists a positive constant $\Lambda(\gamma)>0$ such that

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w)=\Lambda(\gamma)\frac{w^{\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}\quad a.e.\ \text{in}\ \mathbb{R}^N\backslash\{0\}. \tag{111}$$

Proof: We set $1+\epsilon=|x_{n_0}|$ and $\rho=|y_{n_0}|$, then $x_{n_0}=(1+\epsilon)\acute{x}_{n_0}$, $y_{n_0}=\rho y'_{n_0}$ where $|\acute{x}_{n_0}|=|\acute{y}_{n_0}|=1$. Thus

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w)$$

$$=\sum_{n_0}\frac{1}{\left|x_{n_0}\right|^{1+\epsilon}}\int_0^{+\infty}|(1+\epsilon)^{-\gamma}$$

$$-\rho^{-\gamma}|^{\epsilon-1}\frac{((1+\epsilon)^{-\gamma}-\rho^{-\gamma})\rho^{N-1}}{\rho^{1+\epsilon}(1+\epsilon)^{N+(1+\epsilon)^2}}\left(\int_{|\acute{y}_{n_0}|=1}^{2}\frac{dH^{n-1}(\acute{y}_{n_0})}{\left|\acute{x}_{n_0}-\frac{\rho}{1+\epsilon}\right|^{N+(1+\epsilon)^2}}\right)d\rho.$$

Let $\sigma=\frac{\rho}{1+\epsilon}$, then

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w) = \sum_{n_0} \frac{w^{\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}} \int_0^{+\infty} |1-\sigma^{-\gamma}|^{\epsilon-1} (1 - \sigma^{-\gamma})\sigma^{N-\epsilon} \left(\int_{|\acute{y}_{n_0}|=1}^{2} \frac{dH^{n-1}(\acute{y}_{n_0})}{\left|\acute{x}_{n_0} - \sigma \acute{y}_{n_0}\right|^{N+(1+\epsilon)^2}} \right) d\sigma.$$

Defining

$$K(\sigma) = \int_{|\acute{y}_{n_0}|=1}^{2} \sum_{n_0} \frac{dH^{n-1}(\acute{y}_{n_0})}{\left|\acute{x}_{n_0} - \sigma \acute{y}_{n_0}\right|^{N+(1+\epsilon)^2}},$$

as in [214], we obtain that

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma\frac{N-1}{2}} \int_0^{\pi} \frac{1 \sin^{N-2}(\theta)}{(1-2\sigma\cos(\theta)+\sigma^2)^{\frac{N+(1+\epsilon)^2}{2}}} d\theta \quad (112)$$

Hence

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w) = \sum_{n_0} \frac{w^{\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}} \int_0^{+\infty} \psi(\sigma) d\sigma,$$

with

$$\psi(\sigma) = |1-\sigma^{-\gamma}|^{\epsilon-1}(1-\sigma^{-\gamma})\sigma^{N-\epsilon}K(\sigma). \quad (113)$$

Define $\Lambda(\gamma) \equiv \int_0^{+\infty} \psi(\sigma)\, d\sigma$, then to finish we just have to show that $0 < \Lambda(\gamma) < \infty$.

We have

$$\Lambda(\gamma) = \int_0^1 \psi(\sigma) d\sigma + \int_1^{\infty} \psi(\sigma) d\sigma = I_1 + I_2.$$

Notice that $K\left(\frac{1}{\xi}\right) = \xi^{N+(1+\epsilon)^2} K(\xi)$ for any $\xi > 0$, then using the change of variable $\xi = \frac{1}{\sigma}$ in I_1, there results that

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma)(\sigma^{\gamma}-1)^{\epsilon}\left(\sigma^{N-2-\epsilon-\gamma(\epsilon)} - \sigma^{\epsilon+(1+\epsilon)^2}\right) d\sigma. \quad (114)$$

As $\sigma \to \infty$, we have

$$K(\sigma)(\sigma^{\gamma}-1)^{\epsilon}\left(\sigma^{N-2-\epsilon-\gamma(\epsilon)} - \sigma^{\epsilon+(1+\epsilon)^2}\right) \simeq \sigma^{-2-\epsilon-(1+\epsilon)^2} \in L^1(2,\infty).$$

Now, as, $\sigma \to 1$, we have

$$K(\sigma)(\sigma^{\gamma}-1)^{\epsilon}\left(\sigma^{N-2-\epsilon-\gamma(\epsilon)} - \sigma^{1+3\epsilon+\epsilon^2}\right) \simeq (\sigma-1)^{-(1+\epsilon+\epsilon^2)} \in L^1(1,2).$$

Therefore, combining the above estimates, we get $|\Lambda(\gamma)| < \infty$. Now, using the fact that $0 < \gamma < \frac{N-(1+\epsilon)^2-2(1+\epsilon)}{\epsilon}$, from (114), we reach that $\Lambda(\gamma) > 0$.

As a conclusion, we have proved that

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w) = \Lambda(\gamma) \frac{w^{\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}} \quad a.e. \text{ in } \mathbb{R}^N\backslash\{0\}.$$

Hence the result follows.

Corollary (5.3.24)[239]: Let $1+\epsilon < \frac{N-(1+\epsilon)^2}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$2\Lambda(\gamma) \int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{|u(x_{n_0})|^{1+\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0}$$

$$\leq \int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} \frac{\left|u(x_{n_0}) - u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}} \frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}}, \qquad (115)$$

where $\Lambda(\gamma)$ is defined in (114).

Proof: Let $u \in C_0^\infty(\mathbb{R}^N)$ and $w(x_{n_0}) = |x_{n_0}|^{-\gamma}$ with $\gamma < \frac{N-(1+\epsilon)^2-2(1+\epsilon)}{\epsilon}$. By Corollary (5.3.23), we have

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w) = \Lambda(\gamma) \sum_{n_0} \frac{w^\epsilon}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}.$$

It is clear that $\frac{w^\epsilon}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}} \in L^1_{loc}(\mathbb{R}^N)$. Thus using Picone inequality in Corollary (5.3.21), it follows that

$$\frac{1}{2} \int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} \frac{\left|u(x_{n_0}) - u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}} \frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}} \geq \langle L_{1+\epsilon,1+\epsilon,1+\epsilon} w , \frac{|u|^{1+\epsilon}}{w^\epsilon} \rangle$$

$$= \Lambda(\gamma) \int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{|u(x_{n_0})|^{1+\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0}.$$

Thus we conclude.

Corollary (5.3.25)[239]: Define

$$\Lambda_{N,1+\epsilon,1+\epsilon} = \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N)\backslash 0\}} \sum_{n_0} \frac{\int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0}) - \phi(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2} \left|x_{n_0}\right|^{1+\epsilon} \left|y_{n_0}\right|^{1+\epsilon}} dx_{n_0} dy_{n_0}}{\int_{\mathbb{R}^N}^1 \frac{\left|\phi(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0}},$$

then $\Lambda_{N,1+\epsilon,1+\epsilon} = 2\Lambda(\gamma_0)$.

Proof: From (24), it follows that $\Lambda_{N,1+\epsilon,1+\epsilon,\gamma} \geq 2\Lambda(\gamma_0)$, hence to conclude we have just to prove the reverse inequality.

We closely follow the argument used in [166].

Let $w_0(x_{n_0}) = |x_{n_0}|^{-\gamma_0}$, by Corollary (5.3.23), we have

$$L_{1+\epsilon,1+\epsilon,1+\epsilon}(w_0) = \Lambda(\gamma_0) \sum_{n_0} \frac{w_0^{\epsilon^2}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}.$$

We set

$$M_n = \{x_{n_0} \in \mathbb{R}^N : 1 \leq \left|x_{n_0}\right| < n\} \quad \text{and} \quad O_n = \{x_{n_0} \in \mathbb{R}^N : \left|x_{n_0}\right| \geq n\},$$

and define

$$w_n = \begin{cases} 1 - n^{-\gamma_0} & \text{if } x_{n_0} \in B_1(0), \\ \left|x_{n_0}\right|^{-\gamma_0} - n^{-\gamma_0} & \text{if } x_{n_0} \in M_n, \\ 0 & \text{if } x_{n_0} \in O_n. \end{cases}$$

By a direct computation, we get easily that $w_n \in X_0^{1+\epsilon,1+\epsilon,1+\epsilon}(\mathbb{R}^N)$.
Hence

$$\langle L_{1+\epsilon,1+\epsilon,1+\epsilon}(w_0)\,, w_n\rangle = \Lambda(\gamma_0) \int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{w_n w_0^{\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0}.$$

Thus

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left(w_n(x_{n_0}) - w_n(y_{n_0})\right)\left|w_0(x_{n_0}) - w_0(y_{n_0})\right|^{\epsilon-1}\left(w_0(x_{n_0}) - w_0(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}} dx_{n_0} dy_{n_0}$$

$$= 2\Lambda(\gamma_0) \int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{w_n w_0^{\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0}.$$

Let analyze each term in the previous identity. As in [166] we obtain that

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left(w_n(x_{n_0}) - w_n(y_{n_0})\right)\left|w_0(x_{n_0}) - w_0(y_{n_0})\right|^{\epsilon-1}\left(w_0(x_{n_0}) - w_0(y_{n_0})\right)}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}} dx_{n_0} dy_{n_0}$$

$$\geq \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left|w_n(x_{n_0}) - w_n(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}} dx_{n_0} dy_{n_0}.$$

On the other hand we have

$$\int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{w_n w_0^{\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0} = \int_{\mathbb{R}^N}^{1} \sum_{n_0} \frac{w_n^{1+\epsilon}}{|x_{n_0}|^{(1+\epsilon)^2+2(1+\epsilon)}} dx_{n_0} + I_n + J_n,$$

where

$$I_n = \int_{B_{1(0)}}^{1} \sum_{n_0} (1 - n^{-\gamma_0})(w_0^{\epsilon} - (1 - n^{-\gamma_0})^{\epsilon}) \frac{dx_{n_0}}{|x_{n_0}|^{(1+\epsilon)^2+1+\epsilon}},$$

and

$$J_n = \int_{M_n}^{1} \sum_{n_0} \left(w_0(x_{n_0}) - n^{-\gamma_0}\right)\left(w_0^{\epsilon} - (w_0(x_{n_0}) - n^{-\gamma_0})^{\epsilon}\right) \frac{dx_{n_0}}{|x_{n_0}|^{(1+\epsilon)^2+1+\epsilon}}.$$

It is clear that $I_n, J_n \geq 0$ using a direct computation we can prove that

$$I_n + J_n \leq C \text{ for all } n \geq 1.$$

Thus, combining the above estimates, it holds

$$\Lambda_{N,1+\epsilon,1+\epsilon,\gamma} \leq \sum_{n_0} \frac{\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2} \frac{\left|w_n(x_{n_0}) - w_n(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0} - y_{n_0}\right|^{N+(1+\epsilon)^2}\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}} dx_{n_0} dy_{n_0}}{\int_{\mathbb{R}^N}^{1} \frac{\left|w_n(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+1+\epsilon}} dx_{n_0}} \tag{116}$$

$$\leq 2\Lambda(\gamma_0)\left(1+\frac{I_n+J_n}{\int_{\mathbb{R}^N}^1 \sum_{n_0} \frac{\left|w_n(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+1+\epsilon}}dx_{n_0}}\right). \tag{117}$$

Since $\int_{\mathbb{R}^N}^1 \sum_{n_0} \frac{\left|w_n(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+1+\epsilon}}dx_{n_0} \uparrow \infty$ as $n \to \infty$, then passing to the limit in (116), it follows that

$$\Lambda_{N,1+\epsilon,1+\epsilon,\gamma} \leq 2\Lambda(\gamma_0)$$

and then the result follows.

Corollary (5.3.26)[239]: Let Ω be a bounded regular domain such that $0 \in \Omega$, then there exists a constant $C \equiv C(\Omega, 1+\epsilon, 1+\epsilon, N) > 0$ such that for all $u \in C_0^\infty(\Omega)$, we have

$$C\int_\Omega^1 \sum_{n_0} \frac{\left|u(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}dx_{n_0}$$
$$\leq \int_\Omega^2\int_\Omega^2 \sum_{n_0} \frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}} \frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}}. \tag{118}$$

Proof: Fix $u \in C_0^\infty(\Omega)$, and let $\tilde{u}$, be the extension of u to $\mathbb{R}^N$ defined in Lemma (5.3.4). Then from Corollary (5.3.24), we get

$$2\Lambda(\gamma)\int_{\mathbb{R}^N}^1 \sum_{n_0} \frac{\left|\tilde{u}(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}dx_{n_0}$$
$$\leq \int_{\mathbb{R}^N}^2\int_{\mathbb{R}^N}^2 \sum_{n_0} \frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}} \frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}} \leq \|\tilde{u}\|^{1+\epsilon}_{X_0^{1+\epsilon,1+\epsilon,1+\epsilon}(\mathbb{R}^N)}$$
$$\leq C\|u\|^{1+\epsilon}_{X_0^{1+\epsilon,1+\epsilon,1+\epsilon}(\Omega)}.$$

Since $\tilde{u}_{|\Omega} = u$, we conclude that

$$2\Lambda(\gamma)\int_\Omega^1 \sum_{n_0} \frac{\left|u(x_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}\right|^{(1+\epsilon)^2+2(1+\epsilon)}}dx_{n_0} \leq C\|u\|^{1+\epsilon}_{X_0^{1+\epsilon,1+\epsilon,1+\epsilon}(\Omega)}$$
$$\leq C_1 |||u|||^{1+\epsilon}_{X_0^{1+\epsilon,1+\epsilon,1+\epsilon}(\Omega)}$$
$$= C_1\int_\Omega^2\int_\Omega^2 \sum_{n_0} \frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}} \frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}}.$$

Hence we reach the desired result.

Corollary (5.3.27)[239]: Let $\epsilon > 0$ and $N > (2+\epsilon)(1-\epsilon)$. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, then for all $\epsilon > 0$, there exists a positive constant $C = C(\Omega, 1+\epsilon, N, 1-\epsilon)$ such that for all $u \in C_0^\infty(\Omega)$,

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1-\epsilon)}}dx_{n_0}dy_{n_0}$$

$$-\Lambda_{N,1+2\epsilon,1-\epsilon}\int_{\mathbb{R}^N}^{1}\sum_{n_0}\frac{\left|u(x_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}\right|^{(1+2\epsilon)(1-\epsilon)}}dx_{n_0}$$

$$\geq C\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1-\epsilon)}}dx_{n_0}dy_{n_0}. \quad (119)$$

As a consequence we get the next “fractional” Caffarelli–Kohn–Nirenberg inequality in bounded domain.

Proof. We follow closely the arguments used in [205]. Let $u\in C_0^\infty(\Omega)$, and define $\alpha=\frac{N-(1+\epsilon)^2}{1+\epsilon}$, then $w(x_{n_0})=|x_{n_0}|^{-\alpha}$ and $v(x_{n_0})=\frac{u(x_{n_0})}{w(x_{n_0})}$.

Recall that from the result of [166], we have

$$h_n(u)\geq C\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|v(x_{n_0})-v(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{\frac{N-(1+\epsilon)^2}{2}}}\frac{dx_{n_0}}{\left|y_{n_0}\right|^{\frac{N-(1+\epsilon)^2}{2}}}. \quad (120)$$

Let us analyze the right hand side of the previous inequality.

Notice that

$$\sum_{n_0}\frac{\left|v(x_{n_0})-v(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}w(x_{n_0})^{\frac{1+\epsilon}{2}}w(y_{n_0})^{\frac{1+\epsilon}{2}}$$

$$=\sum_{n_0}\frac{\left|w(y_{n_0})u(x_{n_0})-w(x_{n_0})u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\frac{1}{\left(w(x_{n_0})w(y_{n_0})\right)^{\frac{1+\epsilon}{2}}}$$

$$=\sum_{n_0}\frac{\left|\left(u(x_{n_0})-u(y_{n_0})\right)-\frac{u(y_{n_0})}{w(y_{n_0})}\left(w(x_{n_0})-w(y_{n_0})\right)\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\left(\frac{w(y_{n_0})}{w(x_{n_0})}\right)^{\frac{1+\epsilon}{2}}$$

$$=\sum_{n_0}f_1(x_{n_0},y_{n_0}).$$

In the same way, thanks to the symmetry of $f_1(x_{n_0},y_{n_0})$, it immediately follows that

$$\sum_{n_0}\frac{\left|v(x_{n_0})-v(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\left(w(x_{n_0})\right)^{\frac{1+\epsilon}{2}}\left(w(y_{n_0})\right)^{\frac{1+\epsilon}{2}}$$

$$=\sum_{n_0}\frac{\left|\left(u(y_{n_0})-u(x_{n_0})\right)-\frac{u(x_{n_0})}{w(x_{n_0})}\left(w(y_{n_0})-w(x_{n_0})\right)\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\left(\frac{w(x_{n_0})}{w(y_{n_0})}\right)^{\frac{1+\epsilon}{2}}$$

$$=\sum_{n_0}f_2(x_{n_0},y_{n_0}).$$

Hence,

$$h_{1+\epsilon}(u) \geq \frac{1}{2}\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} f_1(x_{n_0}, y_{n_0})dx_{n_0}dy_{n_0} + \frac{1}{2}\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} f_2(x_{n_0}, y_{n_0})dx_{n_0}dy_{n_0}.$$

Since f_1 and f_2 are positive functions, it follows that

$$h_{1+\epsilon}(u) \geq \frac{1}{2}\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0} f_1(x_{n_0}, y_{n_0})dx_{n_0}dy_{n_0} + \frac{1}{2}\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0} f_2(x_{n_0}, y_{n_0})dx_{n_0}dy_{n_0}.$$

Using the fact that Ω is a bounded domain, we obtain that for all $(x_{n_0}, y_{n_0}) \in (\Omega \times \Omega)$ and $\epsilon > 0$,

$$\sum_{n_0} \frac{1}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \geq \sum_{n_0} \frac{C(\Omega)}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}}$$

and

$$Q(x_{n_0}, y_{n_0}) \equiv \sum_{n_0} \frac{\left(w(x_{n_0})\, w(y_{n_0})\right)^{\frac{1+2\epsilon}{2}}}{w(x_{n_0})^{1+2\epsilon} + w(y_{n_0})^{1+2\epsilon}} \leq C.$$

Define

$$D(x_{n_0}, y_{n_0}) \equiv \sum_{n_0} \left(\frac{w(x_{n_0})}{w(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} + \sum_{n_0} \left(\frac{w(y_{n_0})}{w(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}}$$

$$\equiv \sum_{n_0} \frac{w(x_{n_0})^{1+2\epsilon} + w(y_{n_0})^{1+2\epsilon}}{\left(w(x_{n_0})\, w(y_{n_0})\right)^{\frac{1+2\epsilon}{2}}},$$

then $Q(x_{n_0}, y_{n_0})D(x_{n_0}, y_{n_0}) = 1$. Thus

$$f_1(x_{n_0}, y_{n_0}) \geq \sum_{n_0} C(\Omega)Q(x_{n_0}, y_{n_0})\left(\frac{w(y_{n_0})}{w(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \times$$

$$\sum_{n_0}\left[\frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} - (1\right.$$

$$+ 2\epsilon)\sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{2\epsilon-1}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \langle u(x_{n_0})$$

$$- u(y_{n_0}), \frac{u(y_{n_0})}{w(y_{n_0})}\left(w(x_{n_0}) - w(y_{n_0})\right)\rangle$$

$$\left. +C(1+2\epsilon)\sum_{n_0} \frac{\left|\frac{u(y_{n_0})}{w(y_{n_0})}\left(w(x_{n_0}) - w(y_{n_0})\right)\right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \frac{2}{3}\right].$$

Hence

$$f_1(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left[C(\Omega) Q(x_{n_0}, y_{n_0}) \left(\frac{w(y_{n_0})}{w(x_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \right]$$
$$-\sum_{n_0} \left[(1+2\epsilon) C(\Omega) Q(x_{n_0}, y_{n_0}) \left(\frac{w(y_{n_0})}{w(x_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(x_{n_0}) - u(y_{n_0})|^{2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \left| \frac{u(x_{n_0})}{w(x_{n_0})} \right| \left| (w(x_{n_0}) - w(y_{n_0})) \right| \right].$$

In the same way we reach that

$$f_2(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left[C(\Omega) Q(x_{n_0}, y_{n_0}) \left(\frac{w(x_{n_0})}{w(y_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(y_{n_0}) - u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \right]$$
$$-\sum_{n_0} \left[(1+2\epsilon) C(\Omega) Q(x_{n_0}, y_{n_0}) \left(\frac{w(x_{n_0})}{w(y_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(x_{n_0}) - u(y_{n_0})|^{2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \left| \frac{u(x_{n_0})}{w(x_{n_0})} \right| \left| (w(x_{n_0}) - w(y_{n_0})) \right| \right].$$

Therefore,

$$h_{1+\epsilon}(u) \geq C(\Omega) \int_{\Omega}^{2} \int_{\Omega}^{2} \sum_{n_0} Q(x_{n_0}, y_{n_0}) \left(\left(\frac{w(y_{n_0})}{w(x_{n_0})} \right)^{\frac{1+2\epsilon}{2}} + \left(\frac{w(y_{n_0})}{w(x_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \right) \frac{|u(u) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0}$$
$$-(1+2\epsilon) C(\Omega) \int_{\Omega}^{2} \int_{\Omega}^{2} \sum_{n_0} \left[Q(x_{n_0}, y_{n_0}) \left(\frac{w(y_{n_0})}{w(x_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(x_{n_0}) - u(y_{n_0})|^{2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \left| \frac{u(y_{n_0})}{w(y_{n_0})} \right| \left| (w(x_{n_0}) - w(y_{n_0})) \right| \right] dx_{n_0} dy_{n_0}$$
$$-(1+2\epsilon) C(\Omega) \int_{\Omega}^{2} \int_{\Omega}^{2} \sum_{n_0} \left[Q(x_{n_0}, y_{n_0}) \left(\frac{w(x_{n_0})}{w(y_{n_0})} \right)^{\frac{1+2\epsilon}{2}} \frac{|u(x_{n_0}) - u(y_{n_0})|^{2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} \left| \frac{u(x_{n_0})}{w(x_{n_0})} \right| \left| (w(x_{n_0}) - w(y_{n_0})) \right| \right] dx_{n_0} dy_{n_0}$$

Thus

$$h_{1+\epsilon}(u) \geq C(\Omega) \int_{\Omega}^{2} \int_{\Omega}^{2} \sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0}$$

$$-C_1(\Omega\,,\mathrm{p})\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ \left(h_1\left(x_{n_0}\,,y_{n_0}\right)+h_2\left(x_{n_0}\,,y_{n_0}\right)\right)dx_{n_0}dy_{n_0}\,, \tag{121}$$

with

$$h_1\left(x_{n_0}\,,y_{n_0}\right)$$

$$=\sum_{n_0}\ Q\left(x_{n_0},y_{n_0}\right)\left(\frac{w\left(y_{n_0}\right)}{w\left(x_{n_0}\right)}\right)^{\frac{1+2\epsilon}{2}}\frac{\left|u\left(x_{n_0}\right)-u\left(y_{n_0}\right)\right|^{2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\left|\frac{u\left(y_{n_0}\right)}{w\left(y_{n_0}\right)}\right|\left|\left(w\left(x_{n_0}\right)-w\left(y_{n_0}\right)\right)\right|,$$

$$h_2\left(x_{n_0}\,,y_{n_0}\right)$$

$$=\sum_{n_0}\ Q\left(x_{n_0},y_{n_0}\right)\left(\frac{w\left(x_{n_0}\right)}{w\left(y_{n_0}\right)}\right)^{\frac{1+2\epsilon}{2}}\frac{\left|u\left(x_{n_0}\right)-u\left(y_{n_0}\right)\right|^{2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\left|\frac{u\left(x_{n_0}\right)}{w\left(x_{n_0}\right)}\right|\left|\left(w\left(x_{n_0}\right)-w\left(y_{n_0}\right)\right)\right|.$$

Since $h_1\left(x_{n_0}\,,y_{n_0}\right)$ and $h_2\left(x_{n_0}\,,y_{n_0}\right)$ are symmetric functions, we just have to estimate

$$\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ h_2\left(x_{n_0}\,,y_{n_0}\right)dx_{n_0}dy_{n_0}\,.$$

Using Young inequality, we get

$$\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ h_2\left(x_{n_0}\,,y_{n_0}\right)dx_{n_0}dy_{n_0}\ \le\varepsilon\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ \frac{\left|u\left(x_{n_0}\right)-u\left(y_{n_0}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}dx_{n_0}dy_{n_0}$$

$$+C(\varepsilon)\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ G\left(x_{n_0}\,,y_{n_0}\right)dx_{n_0}dy_{n_0}\,, \tag{122}$$

with

$$G\left(x_{n_0}\,,y_{n_0}\right)$$

$$=\sum_{n_0}\ \left(Q\left(x_{n_0},y_{n_0}\right)\right)^{1+2\epsilon}\left(\frac{w\left(x_{n_0}\right)}{w\left(y_{n_0}\right)}\right)^{\frac{(1+2\epsilon)^2}{2}}\left|\frac{u\left(x_{n_0}\right)}{w\left(x_{n_0}\right)}\right|^{1+2\epsilon}\frac{\left|w\left(x_{n_0}\right)-w\left(y_{n_0}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}.$$

We claim that

$$I\equiv\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\ G\left(x_{n_0}\,,y_{n_0}\right)dx_{n_0}dy_{n_0}$$

$$\le C\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|v\left(x_{n_0}\right)-v\left(y_{n_0}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}}\frac{dx_{n_0}}{\left|y_{n_0}\right|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}}.$$

Notice that

$$I\equiv\int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0}\frac{\left(u\left(x_{n_0}\right)\right)^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}\frac{\left(w\left(x_{n_0}\right)\right)^{2(\epsilon+2\epsilon^2)}\left|w\left(x_{n_0}\right)-w\left(y_{n_0}\right)\right|^{1+2\epsilon}}{\left(w(x_{n_0})^{1+2\epsilon}+w(y_{n_0})^{1+2\epsilon}\right)^{1+2\epsilon}}dx_{n_0}dy_{n_0},$$

then

$$I = \int_{\Omega}^{1} \sum_{n_0} u^{1+2\epsilon}(x_{n_0}) \left[\int_{\Omega}^{1} \frac{\left| |x_{n_0}|^{\alpha} - |y_{n_0}|^{\alpha} \right|^{1+2\epsilon}}{\left(|x_{n_0}|^{\alpha 1+2\epsilon} + |y_{n_0}|^{\alpha(1+2\epsilon)} \right)^{1+2\epsilon}} \frac{|y_{n_0}|^{\alpha(1+2\epsilon)(2\epsilon)}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dy_{n_0} \right] dx_{n_0}.$$

To compute the above integral, we closely follow the arguments used in [214]. We set $y_{n_0} = \rho \acute{y}_{n_0}$ and $x_{n_0} = \rho \acute{x}_{n_0}$ with $|\acute{x}_{n_0}| = |\acute{y}_{n_0}| = 1$, then taking in consideration that $\Omega \subset B_0(R)$, it follows that

$$I = \int_{\Omega}^{1} \sum_{n_0} u^{1+2\epsilon}(x_{n_0}) \left[\int_{\Omega}^{1} \frac{\left| |x_{n_0}|^{\alpha} - |y_{n_0}|^{\alpha} \right|^{1+2\epsilon}}{\left(|x_{n_0}|^{\alpha(1+2\epsilon)} + |x_{n_0}|^{\alpha(1+2\epsilon)} \right)^{1+2\epsilon}} \frac{|y_{n_0}|^{\alpha(1+2\epsilon)(2\epsilon)}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dy_{n_0} \right] dx_{n_0}$$

$$\leq \int_{\Omega}^{1} \sum_{n_0} u^{1+2\epsilon}(x_{n_0}) \int_{0}^{R} \frac{|(1+\epsilon)^{\alpha} - \rho^{\alpha}|^{1+2\epsilon} \rho^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{((1+\epsilon)^{(1+2\epsilon)\alpha} - \rho^{(1+2\epsilon)\alpha})^{1+2\epsilon}}$$

$$\times \left(\int_{\mathbb{S}^{N-1}}^{1} \frac{d\acute{y}_{n_0}}{|\rho \acute{y}_{n_0} - (1+\epsilon) \acute{x}_{n_0}|^{N+(1+\epsilon)^2}} \right) d\rho\, dx_{n_0}.$$

We set $\rho = (1+\epsilon)\sigma$, then

$$I \leq \int_{\Omega}^{1} \sum_{n_0} \frac{u^{1+2\epsilon}(x_{n_0})}{|x_{n_0}|^{(1+\epsilon)^2}} \int_{0}^{\frac{R}{1+\epsilon}} \frac{|1 - \sigma^{\alpha}|^{1+2\epsilon} \sigma^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{(1 - \sigma^{\alpha(1+2\epsilon)})^{1+2\epsilon}}$$

$$\times \left(\int_{\mathbb{S}^{N-1}}^{1} \frac{d\acute{y}_{n_0}}{|\sigma \acute{y}_{n_0} - \acute{x}_{n_0}|^{N+(1+\epsilon)^2}} \right) d\rho\, dx_{n_0}$$

$$= \int_{\Omega}^{1} \sum_{n_0} \frac{u^{1+2\epsilon}(x_{n_0})}{|x_{n_0}|^{(1+\epsilon)^2}} \int_{0}^{\frac{R}{1+\epsilon}} \frac{|1 - \sigma^{\alpha}|^{1+2\epsilon} \sigma^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{(1 + \sigma^{\alpha(1+2\epsilon)})^{1+2\epsilon}} K(\sigma)\, d\sigma\, dx_{n_0}$$

$$\leq \mu \int_{\Omega}^{1} \sum_{n_0} \frac{u^{1+2\epsilon}(x_{n_0})}{|x_{n_0}|^{(1+\epsilon)^2}} dx_{n_0},$$

where

$$\mu = \int_{0}^{\infty} \frac{|1 - \sigma^{\alpha}|^{1+2\epsilon} \sigma^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{(1 + \sigma^{\alpha(1+2\epsilon)})^{1+2\epsilon}} K(\sigma)\, d\sigma$$

and

$$K(\sigma) = 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \int_{0}^{\pi} \frac{\sin^{N-1}(\theta)}{(1 - 2\sigma \cos\theta + \sigma^2)^{\frac{N+(1+\epsilon)^2}{2}}} d\theta.$$

Let us show that $\mu < \infty$.

It is clear that, as $\sigma \to \infty$, we have

$$\frac{|1 - \sigma^{\alpha}|^{1+2\epsilon} \sigma^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{(1 + \sigma^{\alpha(1+2\epsilon)})^{1+2\epsilon}} K(\sigma) \simeq \sigma^{-1-(1+\epsilon)^2} \in L^1(1, \infty).$$

Now, taking in consideration that $K(\sigma) \le C|1-\sigma|^{-1-(1+2\epsilon)(1+\epsilon)}$ as $\epsilon \to 0$, and following the same computation as in Corollary (5.3.23), it follows that

$$\int_0^1 \frac{(1-\sigma^\alpha)^{1+2\epsilon}\sigma^{\alpha(1+2\epsilon)(2\epsilon)+N-1}}{(1+\sigma^{\alpha(1+2\epsilon)})^{1+2\epsilon}} K(\sigma)d\sigma < \infty.$$

Thus $\mu < \infty$.

Hence combining the above estimates, there results that

$$I \le C \int_\Omega^1 \sum_{n_0} \frac{u^{1+2\epsilon}(x_{n_0})}{|x_{n_0}|^{(1+\epsilon)^2}} dx_{n_0}.$$

Since $u(x_{n_0}) = \sum_{n_0} v(x_{n_0})|x_{n_0}|^{-(\frac{N-(1+2\epsilon)(1+\epsilon)}{1+2\epsilon})}$, then

$$I \le C \int_\Omega^1 \sum_{n_0} \frac{|v(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{N-(1+\epsilon)(\epsilon)}} dx_{n_0}.$$

Let $1+\epsilon = \frac{N-(1+2\epsilon)(1+\epsilon)}{2} + \frac{(-\epsilon)(1+\epsilon)}{2}$, then $1+\epsilon < \frac{N-(1+2\epsilon)(1+\epsilon)}{2}$. Applying Corollary (5.3.26), we obtain that

$$I \le C(\Omega) \int_\Omega^2 \int_\Omega^2 \sum_{n_0} \frac{|v(x_{n_0}) - v(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)} |x_{n_0}|^{1+\epsilon} |y_{n_0}|^{1+\epsilon}} dy_{n_0} dx_{n_0}$$

$$\le C_1(\Omega) \int_\Omega^2 \int_\Omega^2 \sum_{n_0} \frac{|v(x_{n_0}) - v(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)} |x_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}} |y_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} dy_{n_0} dx_{n_0}$$

$$\le C_1(\Omega) \int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \sum_{n_0} \frac{|v(x_{n_0}) - v(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)} |x_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}} |y_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} dy_{n_0} dx_{n_0}.$$

Therefore, using again estimate (120), we reach that

$$I \le C_2(\Omega) \int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \sum_{n_0} \frac{|v(x_{n_0}) - v(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} \frac{dy_{n_0}}{|y_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}}$$

and the claim follows.

As a direct consequence of the above estimates, we have proved that

$$\int_\Omega^2 \int_\Omega^2 \sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0}$$

$$\le C_3 \int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \sum_{n_0} \frac{|v(x_{n_0}) - v(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} \frac{dy_{n_0}}{|y_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} \quad (123)$$

Thus

$$\int_\Omega^2 \int_\Omega^2 \sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0} \le Ch_{1+\epsilon}(u),$$

and the result follows at once.

Corollary (5.3.28)[239]: Let $\epsilon \geq 0$ and $N > (2+\epsilon)(1-\epsilon)$. Assume that $\Omega \subset \mathbb{R}^N$ is abounded domain, then for all $\epsilon > 0$, there exists a positive constant $C = C(\Omega, 1+\epsilon, N, 1-\epsilon)$such that for all $u \in C_0^\infty(\Omega)$,

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1-\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}} \frac{dy_{n_0}}{|y_{n_0}|^{1+\epsilon}} dx_{n_0} dy_{n_0}$$

$$\geq C \sum_{n_0} \left(\int_{\Omega}^{1} \frac{|u(x_{n_0})|^{\frac{1+2\epsilon}{p^*_{1-\epsilon,1+\epsilon}}}}{|x_{n_0}|^{2(1+\epsilon)\frac{1+2\epsilon}{p^*_{1-\epsilon,1+\epsilon}}}} dx_{n_0} \right)^{\frac{1+2\epsilon}{p^*_{1-\epsilon,1+\epsilon}}} \tag{124}$$

where $p^*_{1-\epsilon,1+\epsilon} = \frac{(1+2\epsilon)N}{N+\epsilon^2-1}$ and $1+\epsilon = \frac{N-(1+2\epsilon)(1-\epsilon)}{2}$.

Proof. Recall that $\alpha = \frac{N-(1+2\epsilon)(1+\epsilon)}{1+2\epsilon}$. Since $\alpha p^*_{1+\epsilon,1+\epsilon} = \frac{N(N-(1+2\epsilon)(1+\epsilon))}{N-(1+\epsilon)^2} < N$, it follows that $\int_\Omega^1 \sum_{n_0} \frac{|u(x_{n_0})|^{p^*_{1+\epsilon,1+\epsilon}}}{|x_{n_0}|^{\alpha(1+2\epsilon)^*_{1+\epsilon,1+\epsilon}}} dx_{n_0} < \infty$, for all $u \in C_0^\infty(\mathbb{R}^N)$.

To prove (124), we will use estimate (123)and the fractional Sobolev inequality. Fix $u \in C_0^\infty(\Omega)$ and define $u_1(x_{n_0}) = \sum_{n_0} \frac{u(x_{n_0})}{|x_{n_0}|^\alpha}$. By (123), we obtain that

$$C(\Omega) \int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0} \frac{|u_1(x_{n_0}) - u_1(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0}$$

$$\leq \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}} \frac{dy_{n_0}}{|y_{n_0}|^{\frac{N-(1+2\epsilon)(1+\epsilon)}{2}}}.$$

Now, using Sobolev inequality, there results that

$$\sum_{n_0} S\left(\int_{\Omega}^{2} |u_1(x_{n_0})|^{p^*_{1+\epsilon,1+\epsilon}} dx_{n_0} \right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon,1+\epsilon}}} \leq \int_{\Omega}^{2}\int_{\Omega}^{2}\sum_{n_0} \frac{|u_1(x_{n_0}) - u_1(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+\epsilon)^2}} dx_{n_0} dy_{n_0},$$

where $p^*_{1+\epsilon,1+\epsilon} = \frac{(1+2\epsilon)N}{N-(1+\epsilon)^2}$. Hence, substituting u_1 by its value, we get

$$\sum_{n_0} \left(\int_{\Omega}^{1} \frac{|u(x_{n_0})|^{p^*_{1+\epsilon,1+\epsilon}}}{|x_{n_0}|^{\alpha(1+2\epsilon)^*_{1+\epsilon,1+\epsilon}}} dx_{n_0} \right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon,1+\epsilon}}}$$

$$\leq C \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}} \frac{dy_{n_0}}{|y_{n_0}|^{1+\epsilon}} \tag{125}$$

If we set $1+\epsilon = \frac{N-(1+2\epsilon)(1+\epsilon)}{2} = \alpha \frac{1+2\epsilon}{2}$, then inequality (125) can be written in the form

$$\sum_{n_0}\left(\int_{\Omega}^{1}\frac{\left|u(x_{n_0})\right|^{p^*_{1+\epsilon,1+\epsilon}}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p^*_{1+\epsilon,1+\epsilon}}{1+2\epsilon}}}dx_{n_0}\right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon,1+\epsilon}}}$$
$$\leq C\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}} \quad (126)$$

Corollary (5.3.29)[239]: Assume that $1<1+\epsilon<\frac{N}{1-\epsilon}$ and let $0<1+\epsilon<\frac{N+\epsilon^2-1}{2}$, then for all $u\in C_0^\infty(\mathbb{R}^N)$, we have
$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+1-\epsilon^2}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dy_{n_0}}{\left|y_{n_0}\right|^{1+\epsilon}}$$
$$\geq S(1+\epsilon)\sum_{n_0}\left(\int_{\mathbb{R}^N}^{1}\frac{\left|u(x_{n_0})\right|^{p^*_{1-\epsilon}}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p^*_{1-\epsilon}}{1+\epsilon}}}dx_{n_0}\right)^{\frac{1+\epsilon}{p^*_{1-\epsilon}}}, \quad (127)$$
where $S(1+\epsilon)>0$.

Proof. Let $u\in C_0^\infty(\mathbb{R}^N)$, without loss of generality, we can assume that $u\geq 0$. Using the fact that $1+\epsilon<\frac{N-(1+2\epsilon)(1+\epsilon)}{2}$, we easily get that $\int_{\mathbb{R}^N}^{1}\sum_{n_0}\frac{\left|u(x_{n_0})\right|^{p^*_{1+\epsilon}}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p^*_{1+\epsilon}}{1+2\epsilon}}}dx_{n_0}\leq\infty$.

From now and for simplicity of typing, we denote by $C, C_1, C_2,\cdots$ any universal constant that does not depend on uand can change from a line to another.

We set $\tilde{u}(x_{n_0})=\sum_{n_0}\frac{u(x_{n_0})}{w_1(x_{n_0})}$, where $w_1(x_{n_0})=|x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}$, then
$$\sum_{n_0}\left(\int_{\mathbb{R}^N}^{1}\frac{\left|u(x_{n_0})\right|^{p^*_{1+\epsilon}}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p^*_{1+\epsilon}}{1+2\epsilon}}}dx_{n_0}\right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon}}}=\sum_{n_0}\left(\int_{\mathbb{R}^N}^{1}|\tilde{u}|^{p^*_{1+\epsilon}}\,dx_{n_0}\right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon}}}. \quad (128)$$
Using Sobolev inequality, it follows that
$$\sum_{n_0}S\left(\int_{\mathbb{R}^N}^{1}|\tilde{u}|^{p^*_{1+\epsilon}}\,dx_{n_0}\right)^{\frac{1+2\epsilon}{p^*_{1+\epsilon}}}\leq\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}dx_{n_0}dy_{n_0}. \quad (129)$$
To get the desired result we just have to show that
$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}dx_{n_0}dy_{n_0}$$
$$\leq C\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}} \quad (130)$$
for some positive constant C.

Using the definition of $\tilde{u}$, we get

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$=\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})-w_1(y_{n_0})\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{w_1^{\frac{1+2\epsilon}{2}}(x_{n_0})}\frac{dy_{n_0}}{w_1^{\frac{1+2\epsilon}{2}}(y_{n_0})}.$$

Notice that

$$\sum_{n_0}\frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})-w_1(y_{n_0})\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{1}{w_1^{\frac{1+2\epsilon}{2}}(x_{n_0})}\frac{1}{w_1^{\frac{1+2\epsilon}{2}}(y_{n_0})}$$

$$=\sum_{n_0}\frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})-w_1(y_{n_0})\tilde{u}(y_{n_0})\left(\frac{1}{w_1(x_{n_0})}-\frac{1}{w_1(y_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}}$$

$$\equiv\sum_{n_0}\tilde{f}_1(x_{n_0},y_{n_0}).$$

In the same way we have

$$\sum_{n_0}\frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})-w_1(y_{n_0})\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{1}{w_1^{\frac{1+2\epsilon}{2}}(x_{n_0})}\frac{1}{w_1^{\frac{1+2\epsilon}{2}}(y_{n_0})}$$

$$=\sum_{n_0}\frac{\left|\left(\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right)-w_1(x_{n_0})\tilde{u}(x_{n_0})\left(\frac{1}{w_1(y_{n_0})}-\frac{1}{w_1(x_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}}$$

$$\equiv\sum_{n_0}\tilde{f}_2(x_{n_0},y_{n_0}).$$

Since

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\tilde{f}_1(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0}=\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\tilde{f}_2(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0},$$

we get

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$=\frac{1}{2}\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\tilde{f}_1(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0}+\frac{1}{2}\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\tilde{f}_2(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0}.$$

Notice that

$$\tilde{f}_1(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \left[\frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} - (1 + 2\epsilon) \frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{2\epsilon-1}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \langle \tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0}), w_1(y_{n_0})\tilde{u}(y_{n_0}) \left(\frac{1}{w_1(y_{n_0})} - \frac{1}{w_1(y_{n_0})}\right) \rangle + C(1+2\epsilon) \frac{\left| w_1(y_{n_0})\tilde{u}(y_{n_0}) \left(\frac{1}{w_1(x_{n_0})} - \frac{1}{w_1(y_{n_0})}\right) \right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} - \right].$$

Hence

$$\tilde{f}_1(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \left[\frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} - (1 + 2\epsilon) \frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \left| w_1(y_{n_0})\tilde{u}(y_{n_0}) \left(\frac{1}{w_1(x_{n_0})} - \frac{1}{w_1(y_{n_0})}\right) \right| \right].$$

Using Young inequality, we get the existence of $C_1, C_2 > 0$ such that

$$\tilde{f}_1(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \left[C_1 \frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} - C_2 \frac{\left| w_1(y_{n_0})\tilde{u}(y_{n_0}) \left(\frac{1}{w_1(x_{n_0})} - \frac{1}{w_1(y_{n_0})}\right) \right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \right].$$

In the same way and using that $\tilde{f}_1, \tilde{f}_2$ are symmetric functions, it holds

$$f_2(x_{n_0}, y_{n_0}) \geq \sum_{n_0} \left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \left[C_1 \frac{|\tilde{u}(x_{n_0}) - \tilde{u}(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} - C_2 \frac{\left| w_1(x_{n_0})\tilde{u}(x_{n_0}) \left(\frac{1}{w_1(y_{n_0})} - \frac{1}{w_1(x_{n_0})}\right) \right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \right].$$

Thus we get the existence of positive constants C_1, C_2, C_3 such that

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$\geq C_1 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\left[\left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} + \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}}\right]dx_{n_0}dy_{n_0}$$

$$- C_2 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \frac{\left|w_1(y_{n_0})\tilde{u}(y_{n_0})\left(\frac{1}{w_1(x_{n_0})}-\frac{1}{w_1(y_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}dx_{n_0}dy_{n_0}$$

$$- C_3 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})\left(\frac{1}{w_1(y_{n_0})}-\frac{1}{w_1(x_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}dx_{n_0}dy_{n_0}.$$

Since

$$\sum_{n_0}\left[\left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} + \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}}\right] \geq 1,$$

then

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left|\tilde{u}(x_{n_0})-\tilde{u}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}dx_{n_0}dy_{n_0}$$

$$\leq C_1 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \frac{\left|u(x_{n_0})-u(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$+ C_2 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \times \frac{\left|w_1(y_{n_0})\tilde{u}(y_{n_0})\left(\frac{1}{w_1(x_{n_0})}-\frac{1}{w_1(y_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}dx_{n_0}dy_{n_0} \quad (131)$$

$$+ C_3 \int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0} \left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})\left(\frac{1}{w_1(y_{n_0})}-\frac{1}{w_1(x_{n_0})}\right)\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+\epsilon)^2}}dx_{n_0}dy_{n_0}.$$

We get

$$g_1(x_{n_0}, y_{n_0}) = \sum_{n_0} \left(\frac{w_1(y_{n_0})}{w_1(x_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \frac{\left|w_1(x_{n_0})\tilde{u}(x_{n_0})\left(\frac{1}{w_1(y_{n_0})} - \frac{1}{w_1(x_{n_0})}\right)\right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}$$

and

$$g_2(x_{n_0}, y_{n_0}) = \sum_{n_0} \left(\frac{w_1(x_{n_0})}{w_1(y_{n_0})}\right)^{\frac{1+2\epsilon}{2}} \frac{\left|w_1(y_{n_0})\tilde{u}(y_{n_0})\left(\frac{1}{w_1(x_{n_0})} - \frac{1}{w_1(y_{n_0})}\right)\right|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}$$

It is clear that

$$\int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} g_1(x_{n_0}, y_{n_0}) dx_{n_0} dy_{n_0} = \int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} g_2(x_{n_0}, y_{n_0}) dx_{n_0} dy_{n_0},$$

therefore, to get the desired result, we just have to show that

$$\int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} g_1(x_{n_0}, y_{n_0}) dx_{n_0} dy_{n_0}$$

$$\leq C \int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} \frac{|u(x_{n_0}) - u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} \frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}} \frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}}.$$

Going back to the definition of $\tilde{u}$ and w_1, we reach that

$$g_1(x_{n_0}, y_{n_0}) = \sum_{n_0} \frac{|u(x_{n_0})|^{1+2\epsilon} \left||x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}} - |y_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)} |y_{n_0}|^{1+\epsilon} |x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}.$$

We closely follow the same type of computation as in the proof of Corollary (5.3.23).

We have

$$\int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} g_1(x_{n_0}, y_{n_0}) dx_{n_0} dy_{n_0}$$

$$= \int_{\mathbb{R}^N}^{2} \int_{\mathbb{R}^N}^{2} \sum_{n_0} \frac{|u(x_{n_0})|^{1+2\epsilon} \left||x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}} - |y_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)} |y_{n_0}|^{1+\epsilon} |x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} dx_{n_0} dy_{n_0}$$

$$= \int_{\mathbb{R}^N}^{2} \sum_{n_0} \frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)}} \left(\int_{\mathbb{R}^N}^{2} \frac{\left||x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}} - |y_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}}{|y_{n_0}|^{1+\epsilon} |x_{n_0} - y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}} dy_{n_0} \right) dx_{n_0}.$$

We set $1 + \epsilon = |x_{n_0}|$ and $\rho = |y_{n_0}|$, then $x_{n_0} = (1+\epsilon)\acute{x}_{n_0}$, $y_{n_0} = \rho \acute{y}_{n_0}$ with $|\acute{x}_{n_0}| = |\acute{y}_{n_0}| = 1$, then

$$\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)}}\left(\int_{\mathbb{R}^N}^{2}\frac{\left||x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}-|y_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}}{|y_{n_0}|^{1+\epsilon}|x_{n_0}-y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}dy_{n_0}\right)dx_{n_0}$$

$$=\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)}}\left[\int_0^{+\infty}\frac{\left|(1+\epsilon)^{\frac{2(1+\epsilon)}{1+2\epsilon}}-\rho^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}\rho^{N-1}}{\rho^{1+\epsilon}}\right.$$

$$\left.\times\left(\int_{|y'_{n_0}|=1}^{2}\frac{dH^{n-1}(\acute{y}_{n_0})}{|(1+\epsilon)x'_{n_0}-\rho\acute{y}_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}dy_{n_0}\right)d\rho\right]dx_{n_0}.$$

Let $\sigma=\frac{\rho}{1+\epsilon}$, then

$$\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{3(1+\epsilon)}}\left(\int_{\mathbb{R}^N}^{2}\frac{\left||x_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}-|y_{n_0}|^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}}{|y_{n_0}|^{1+\epsilon}|x_{n_0}-y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}dy_{n_0}\right)dx_{n_0}$$

$$=\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)+(1+2\epsilon)(1+\epsilon)}}\left[\int_0^{+\infty}\left|1\right.\right.$$

$$\left.\left.-\sigma^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}\sigma^{N-2-\epsilon}K(\sigma)d\sigma\right]dx_{n_0},$$

where K is defined in (112). Since

$$\int_0^{+\infty}\left|1-\sigma^{\frac{2(1+\epsilon)}{1+2\epsilon}}\right|^{1+2\epsilon}\sigma^{N-2-\epsilon}K(\sigma)d\sigma\equiv C_3<\infty,$$

it follows that

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}g_1(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0}=C_3\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})|^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)+(1+2\epsilon)(1+\epsilon)}}dx_{n_0}.$$

Now, using inequality (115), we get

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}g_1(x_{n_0},y_{n_0})dx_{n_0}dy_{n_0}$$

$$\le C_4\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{|u(x_{n_0})-u(y_{n_0})|^{1+2\epsilon}}{|x_{n_0}-y_{n_0}|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}}\frac{dx_{n_0}}{|x_{n_0}|^{1+\epsilon}}. \qquad (132)$$

Combining (128), (129), (132) and (131), we reach the desired result.

Corollary (5.3.30)[239]: Assume that Ω is a regular bounded domain with $0\in\Omega$, then there exists a positive constant $C\equiv C(\Omega,N,1+2\epsilon,1+\epsilon,1+\epsilon)$ such that for all $\phi\in C_0^\infty(\Omega)$, we have

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|\phi(x_{n_0})-\phi(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$\geq C\sum_{n_0}\left(\int_{\Omega}^{1}\frac{\left|\phi(x_{n_0})\right|^{p_{1+\epsilon}^*}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p_{1+\epsilon}^*}{1+2\epsilon}}}dx_{n_0}\right)^{\frac{1+2\epsilon}{p_{1+\epsilon}^*}}. \qquad (133)$$

Proof: Let $\phi\in C_0^\infty(\Omega)$ and define $\tilde{\phi}$ to be the extension of ϕ to $\mathbb{R}^N$ given in Lemma (5.3.4), then using the fact that Ω is a regular bounded domain, we reach that

$$\|\tilde{\phi}\|_{X^{1+\epsilon,1+2\epsilon,1+\epsilon}(\mathbb{R}^N)}\leq C_1\|\phi\|_{X^{1+\epsilon,1+2\epsilon,1+\epsilon}(\Omega)}$$

$$\leq C_1\sum_{n_0}\left(\int_{\Omega}^{2}\int_{\Omega}^{2}\frac{\left|\phi(x_{n_0})-\phi(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}dy_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}\left|y_{n_0}\right|^{1+\epsilon}}\right)^{\frac{1}{1+2\epsilon}}.$$

Now, applying Corollary (5.3.29) to $\tilde{\phi}$, it follows that

$$\int_{\mathbb{R}^N}^{2}\int_{\mathbb{R}^N}^{2}\sum_{n_0}\frac{\left|\tilde{\phi}(x_{n_0})-\tilde{\phi}(y_{n_0})\right|^{1+2\epsilon}}{\left|x_{n_0}-y_{n_0}\right|^{N+(1+2\epsilon)(1+\epsilon)}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}\frac{dx_{n_0}}{\left|x_{n_0}\right|^{1+\epsilon}}$$

$$\geq S(1+\epsilon)\sum_{n_0}\left(\int_{\mathbb{R}^N}^{1}\frac{\left|\tilde{\phi}(x_{n_0})\right|^{p_{1+\epsilon}^*}}{\left|x_{n_0}\right|^{2(1+\epsilon)\frac{p_{1+\epsilon}^*}{1+2\epsilon}}}dx_{n_0}\right)^{\frac{1+2\epsilon}{p_{1+\epsilon}^*}}.$$

Hence combining the above estimates we get the desired result.

Corollary (5.3.31)[239]: Assume that $0<\lambda<\Lambda_{N,2(1+\epsilon),1+\epsilon}$, then there exist γ_1,γ_2 such that

$$0<\gamma_1<\frac{N-2(1+\epsilon)^2}{2(1+\epsilon)}<\gamma_2,$$

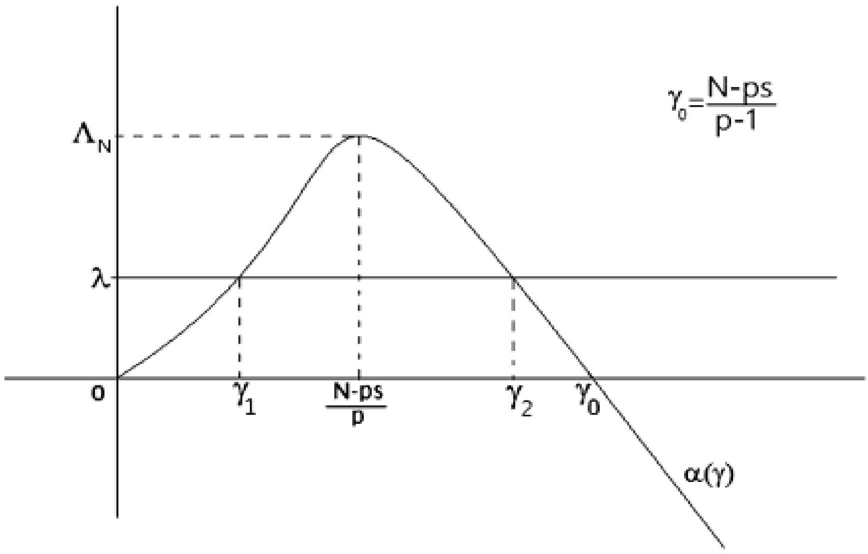
and $\Lambda(\gamma_1)=\Lambda(\gamma_2)=\lambda$.

Proof: We have $\Lambda(0)=0,\Lambda\left(\frac{N-2(1+\epsilon)^2}{2(1+\epsilon)}\right)=\Lambda_{N,2(1+\epsilon),1+\epsilon}\ \Lambda(\gamma)<0$ if $\gamma>\frac{N-2(1+\epsilon)^2}{1+2\epsilon}$ and

$$\acute{\Lambda}(\gamma)=\int_1^{+\infty}K(\sigma)\log(\sigma)(\sigma^\gamma-1)^{2\epsilon}\left(\sigma^{N-1-\gamma(1+2\epsilon)}-\sigma^{1+4\epsilon+2\epsilon^2+\gamma}\right)^2d\sigma.$$

It is clear that for $\gamma_0=\frac{N-2(1+\epsilon)^2}{2(1+\epsilon)}$, we have $\acute{\Lambda}(\gamma_0)=0,\acute{\Lambda}(\gamma)>0$ if $\gamma<\gamma_0$ and $\acute{\Lambda}(\gamma)<0$ if $\gamma>\gamma_0$.

Hence, since $\lambda<\Lambda_{N,2(1+\epsilon),1+\epsilon}$, we get the existence of $0<\gamma_1<\frac{N-2(1+\epsilon)^2}{2(1+\epsilon)}<\gamma_2\frac{N-2(1+\epsilon)^2}{1+2\epsilon}$ such that $\Lambda(\gamma_1)=\Lambda(\gamma_2)=\lambda$.



Define $3(1+\epsilon),0=\epsilon+\frac{2(1+\epsilon)^2}{\gamma_1}$, it is clear that $p_{1+\epsilon}^*-1<1+\epsilon+(2(1+\epsilon),1+\epsilon)$.

Corollary (5.3.32)[239]: Assume that $1+\epsilon<q_+(2(1+\epsilon),1+\epsilon)$, then

(i) If $1+2\epsilon<1+\epsilon<p_{1+\epsilon}^*-1$, problem (105) has a solution u. Moreover, $u\in W_0^{1+\epsilon,2(1+\epsilon)}(\Omega)$ if $\lambda<\Lambda_{N,2(1+\epsilon),1+\epsilon}$ and $h_{1+\epsilon,\Omega}(u)<\infty$ if $\lambda=\Lambda_{N,2(1+\epsilon),1+\epsilon}$ where $h_{1+\epsilon,\Omega}$ is defined in (106).

(ii) If $p_{1+\epsilon}^*-1\le 1+\epsilon<q_+(2(1+\epsilon),1+\epsilon)$, then problem (105) has a positive super solution u.

Proof: Let us begin with the case where $1+2\epsilon<1+\epsilon<p_{1+\epsilon}^*-1$. If $\lambda<\Lambda_{N,2(1+\epsilon),1+\epsilon}$, then using the Mountain Pass Theorem, see [218], we get a positive solution $u\in W_0^{1+\epsilon,2(1+\epsilon)}(\Omega)$. However, if $\lambda<\Lambda_{N,2(1+\epsilon),1+\epsilon}$, then using the improved Hardy inequality in Corollary (5.3.27) and the Mountain Pass Theorem, we reach a positive solution u to problem (105) with $h_{1+\epsilon,\Omega}(u)<\infty$.

Assume now that $p_{1+\epsilon}^*-1\le 1+\epsilon<q_+(2(1+\epsilon),1+\epsilon)$ and fix $\lambda_1\in(\lambda,\Lambda_{N,2(1+\epsilon),1+\epsilon})$ to be chosen later.

Let $\gamma_1\in(0,\frac{N-2(1+\epsilon)^2}{2(1+\epsilon)})$ be such that $\Gamma(\gamma_1)=\lambda_1$ and set $w(x_{n_0})=|x_{n_0}|^{-\gamma_1}$, then

$$L_{1+\epsilon,2(1+\epsilon)}(w)=\lambda_1\sum_{n_0}\frac{w^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}\quad a.e.\text{ in }\mathbb{R}^N\backslash\{0\}$$

with $\frac{w^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}\in L_{loc}^1(\mathbb{R}^N)$. Hence

$$L_{1+\epsilon,2(1+\epsilon)}(w)=\lambda\sum_{n_0}\frac{w^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}+(\lambda_1-\lambda)\sum_{n_0}\frac{w^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}\quad a.e.\text{ in }\mathbb{R}^N\backslash\{0\}.$$

Using the fact that $1+\epsilon<q_+(2(1+\epsilon),1+\epsilon)$, we can choose $\lambda_1>\lambda$, very close to λ such that $\gamma_1(1+2\epsilon)+2(1+\epsilon)^2>(1+\epsilon)\gamma_1$, thus, in any bounded domain Ω, we have

$$(\lambda_1-\lambda)\sum_{n_0}\frac{w^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}\ge C(\Omega)w^{1+\epsilon}.$$

Define $\hat{w}=Cw$, by the previous estimates, we can choose $C(\Omega)>0$ such that $\hat{w}$ will be a supersolution to (105) in Ω. Hence the result follows..

Corollary (5.3.34)[239]: Let $\Omega \subset \mathbb{R}^N$ be a regular domain such that $0 \in \Omega$. Define

$$\Lambda(\Omega) = \inf_{\{\phi \in C_0^\infty \Omega \setminus 0\}} \sum_{n_0} \frac{\int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0}) - \phi(y_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0} - y_{n_0}\right|^{N+2(1+\epsilon)^2}} dx_{n_0} dy_{n_0}}{\int_\Omega^2 \frac{\left|\phi(x_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0}\right|^{2(1+\epsilon)^2}} dx_{n_0}},$$

then $\Lambda(\Omega) = \Lambda_{N,2(1+\epsilon),1+\epsilon}$ defined in (73).

Proof: Recall that

$$\Lambda_{N,2(1+\epsilon),1+\epsilon} = \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N) \setminus 0\}} \sum_{n_0} \frac{\int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0}) - \phi(y_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0} - y_{n_0}\right|^{N+2(1+\epsilon)^2}} dx_{n_0} dy_{n_0}}{\int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0}\right|^{2(1+\epsilon)^2}} dx_{n_0}},$$

thus $\Lambda(\Omega) \geq \Lambda_{N,2(1+\epsilon),1+\epsilon}$. It is clear that if $\Omega_1 \subset \Omega_2$, t hen $\Lambda(\Omega_1) \geq \Lambda(\Omega_2)$.

Now, using a dilatation argument we can prove that $\Lambda(B_{R_1}(0)) = \Lambda(B_{R_2}(0))$ for all $0 < R_1 < R_2$. Hence we conclude that $\Lambda(\Omega) \equiv \overline{\Lambda}$ does not depend of the domain Ω. For $\phi \in C_0^\infty(\mathbb{R}^N)$, we set

$$Q(\phi) \equiv \sum_{n_0} \frac{\int_{\mathbb{R}^N}^2 \int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0}) - \phi(y_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0} - y_{n_0}\right|^{N+2(1+\epsilon)^2}} dx_{n_0} dy_{n_0}}{\int_{\mathbb{R}^N}^2 \frac{\left|\phi(x_{n_0})\right|^{2(1+\epsilon)}}{\left|x_{n_0}\right|^{2(1+\epsilon)^2}} dx_{n_0}}.$$

Let $\{\phi_n\}_n \subset C_0^\infty(\mathbb{R}^N)$ be such that $Q(\phi_n) \to \Lambda_{N,2(1+\epsilon),1+\epsilon}$. Without loss of generality and using a symmetrization argument we can assume that $\mathrm{Supp}(\phi_n) \subset B_{R_2}(0)$. It is clear that $Q(\phi_n) \geq \Lambda(\mathrm{Supp}(\phi_n)) = \overline{\Lambda}$, thus, as $n \to \infty$, it follows that $\overline{\Lambda} \leq \Lambda_{N,2(1+\epsilon),1+\epsilon}$. As a conclusion we reach that $\Lambda = \Lambda_{N,2(1+\epsilon),1+\epsilon}$ and the result follows.

Corollary (5.3.35)[239]: Let Ω be a bounded domain such that $0 \in \Omega$. Assume that $u \in W^{1+\epsilon,2(1+\epsilon)}(\mathbb{R}^N)$ is such that $u \geq 0$ *in* $\mathbb{R}^N, u > 0$ in Ω and $L_{N,2(1+\epsilon),1+\epsilon} u \gneqq \lambda \frac{u^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}$ in Ω, then there exists $C > 0$ such that $u(x_{n_0}) \geq C|x_{n_0}|^{-\gamma_1}$ in $B_{1+\epsilon}(0)$ where γ_1 is defined in Corollary (5.3.31).

Proof: Without loss of generality we can assume that $B_1(0) \subset \Omega$.

Fixed $\lambda < \Lambda_{N,2(1+\epsilon),1+\epsilon}$ sand define

$$\tilde{w}(x_{n_0}) = \begin{cases} |x_{n_0}|^{-\gamma_1} - 1 & \text{if } |x_{n_0}| < 1, \\ 0 & \text{if } |x_{n_0}| > 1. \end{cases}$$

It is clear that $\tilde{w} \in W_0^{1+\epsilon,2(1+\epsilon)}(B_1(0))$ and

$$\begin{cases} L_{2(1+\epsilon),1+\epsilon} \tilde{w} = \sum_{n_0} h(x_{n_0}) \frac{\tilde{w}^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}} & \text{in } B_1(0), \\ \tilde{w} = 0 & \text{in } \mathbb{R}^N \setminus B_1\{0\} \end{cases} \tag{134}$$

where

$$h(x_{n_0}) = \int_0^{\frac{1}{|x_{n_0}|}} \sum_{n_0} |1-\sigma^{-\tilde{\gamma}}|^{2\epsilon}(1-\sigma^{-\tilde{\gamma}})K(\sigma)d\sigma^2 + \sum_{n_0} \left(1-|x_{n_0}|^{\tilde{\gamma}}\right) \int_{\frac{1}{|x_{n_0}|}}^{\infty} \sigma^{N-1}\, K(\sigma)d\sigma.$$

Using the definition of γ_1, see Corollary (5.3.31), we can prove that $h(x_{n_0}) \le \lambda$ for all $x_{n_0} \in B_1(0)$.

Since $L_{2(1+\epsilon),1+\epsilon}\, u \gneqq 0$ and $u > 0$ in Ω, then using the nonlocal weak Harnack inequality in [210], we get the existence of $\varepsilon > 0$ such that $u \ge \varepsilon$ in $\bar{B}_1(0)$.

Therefore we obtain that

$$\begin{cases} L_{2(1+\epsilon),1+\epsilon} u \ge \lambda \sum_{n_0} \dfrac{u^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}} & \text{in } B_1(0), \\ L_{2(1+\epsilon),1+\epsilon} \tilde{w} \le \lambda \sum_{n_0} \dfrac{\tilde{w}^{1+2\epsilon}}{|x_{n_0}|^{2(1+\epsilon)^2}}, & \text{in } B_1(0), \\ u \ge \tilde{w} & \text{in } \mathbb{R}^N \backslash B_1(0). \end{cases} \tag{135}$$

Thus by the comparison principle in Corollary (5.3.22), it follows that $\tilde{w} \le u$ which is the desired result.

Corollary (5.3.36)[239]: Let $q_+(2(1+\epsilon),1+\epsilon) = 1+2\epsilon+\frac{2(1+\epsilon)^2}{\gamma_1}$. If $1+\epsilon > q_+(2(1+\epsilon),1+\epsilon)$, then the unique nonnegative supersolution $u \in W_{loc}^{1+\epsilon,2(1+\epsilon)}(\Omega)$ to problem (105) is $u \equiv 0$.

Proof. We argue by contradiction. Assume the existence of $u \gneqq 0$ such that $u \in W^{1+\epsilon,2(1+\epsilon)}(\mathbb{R}^N)$ and u is a supersolution to problem (105) in Ω, then $u > 0$ in Ω. Let $\phi \in C_0^\infty(B_{1+\epsilon}(0))$ with $B_{1+\epsilon}(0) \subset\subset \Omega$ and $\epsilon \ge 0$ to be chosen later.

Using Picone's inequality in Corollary (5.3.21), it follows that

$$\|\phi\|^{2(1+\epsilon)}_{X_0^{1+\epsilon,2(1+\epsilon)}(\mathbb{R}^N)} \ge \int_{B_{1+\epsilon}(0)}^{2} \sum_{n_0} \frac{L_{2(1+\epsilon),1+\epsilon}(u)}{u^{1+2\epsilon}} |\phi|^{2(1+\epsilon)} dx_{n_0}.$$

Thus

$$\|\phi\|^{2(1+\epsilon)}_{X_0^{1+\epsilon,2(1+\epsilon)}(\mathbb{R}^N)} \ge \int_{B_{1+\epsilon}(0)}^{2} \sum_{n_0} u^{-\epsilon}\, |\phi|^{2(1+\epsilon)} dx_{n_0}.$$

Since $1+\epsilon > q_+(2(1+\epsilon),1+\epsilon)$, we get the existence of $\varepsilon > 0$ such that

$$(\gamma_1-\varepsilon)(-\epsilon) > 3+5\epsilon+2\epsilon^2$$

for some $\epsilon \ge 0$. Thus, using Corollary (5.3.35), we can choose $\epsilon \ge 0$ such that

$$u^{-\epsilon} \ge C \sum_{n_0} |x_{n_0}|^{-(3+5\epsilon+2\epsilon^2)} \text{ in } B_{1+\epsilon}(0).$$

Therefore

$$\|\phi\|^{2(1+\epsilon)}_{X_0^{1+\epsilon,2(1+\epsilon)}(\mathbb{R}^N)} \geq C \int\limits_{B_{1+\epsilon}(0)}^{2} \sum_{n_0} \frac{|\phi|^{2(1+\epsilon)}}{|u|^{3+5\epsilon+2\epsilon^2}} dx_{n_0},$$

which is a contradiction with the optimality of the Hardy inequality proved in Corollary (5.3.34). Hence we conclude.

Chapter 6
Hardy and Fractional Caffarelli-Kohn-Nirenberg Inequalities

We revise the BBM formula due to J. Bourgain, H. Brezis, and P. Mironescu in [203]. We establish a full range of Caffarelli-Kohn-Nirenberg inequalities and their variants for fractional Sobolev spaces.

Section (6.1): The BBM Formula

We first recall the BBM formula due to J. Bourgain, H. Brezis, and P. Mironescu [109], see also [112], (with a refinement by J. Davila [118]). Let $d \geq 1$ be an integer. Throughout, (ρ_n) denotes a sequence of radial mollifiers in the sense that

$$\rho_n \in L^1_{loc}(0,+\infty), \quad \rho_n \geq 0, \tag{1}$$

$$\int_0^{\infty} \rho_n(r) r^{d-1} dr = 1 \ \ \forall n, \tag{2}$$

and

$$\lim_{n\to+\infty} \int_{\delta}^{\infty} \rho_n(r) r^{d-1} dr = 0 \qquad \forall \delta > 0. \tag{3}$$

Even though the next assumption is required only for a few results, it is convenient to assume that

$$\rho_n(r) = 0 \text{ for all } r > 1, n \in \mathbb{N}. \tag{4}$$

Set, for $p \geq 1$,

$$I_{n,p}(u) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dxdy \leq +\infty, \ \ \forall u \in L^1_{loc}(\mathbb{R}^d). \tag{5}$$

For $u \in L^1_{loc}(\mathbb{R}^d)$, define, for $p > 1$,

$$I_p(u) = \begin{cases} \gamma_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p & \text{if } \nabla u \in L^p(\mathbb{R}^d), \\ +\infty & \text{otherewise,} \end{cases} \tag{6}$$

and, for $p = 1$,

$$I_1(u) = \begin{cases} \gamma_{d,1} \int_{\mathbb{R}^d} |\nabla u| & \text{if } \nabla u \text{ is a finite measure,} \\ +\infty & \text{otherewise,} \end{cases} \tag{7}$$

where, for any $e \in \mathbb{S}^{d-1}$ and $p \geq 1$,

$$\gamma_{d,p} = \int_{\mathbb{S}^{d-1}} |\sigma. e|^p \, d\sigma. \tag{8}$$

In the case $p = 1$, we have

$$\gamma_{d,1} = \int_{\mathbb{S}^{d-1}} |\sigma. e|^p \, d\sigma = \begin{cases} \frac{2}{d-1}|\mathbb{S}^{d-1}| = 2|B^{d-1}| & \text{if } d \geq 3, \\ 4 & \text{if } d = 2, \\ 2 & \text{if } d = 1, \end{cases} \tag{9}$$

The BBM formula asserts that, for $p \geq 1$,

$$\lim_{n\to+\infty} I_{n,p}(u) = I_p(u) \qquad \forall u \in L^1_{loc}(\mathbb{R}^d). \tag{10}$$

Applying (10) with $p = 1, u = \mathbb{1}_E$ (the characteristic function of a measurable set E), and $\rho_n(r) = C_d n^{(d+1)/2}\, r e^{-nr^2}$, we obtain

$$\lim_{n\to+\infty} n^{(d+1)/2}\int_{E^c}\int_E e^{-n|x-y|^2}\,dxdy = A_d \operatorname{Per}(E).$$

By comparison the De Giorgi formula [221], [222] for the perimeter involves a derivative and asserts that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d} |\nabla W_n(x)|\,dx = B_d \operatorname{Per}(E),$$

where

$$W_n(x) = n^{d/2}\int_E e^{-n|x-y|^2}\,dy,$$

and $A_d, B_d,$ and C_d are positive constants depending only on d.

Define, for $p \geq 1, n \in \mathbb{N},$ and $u \in L^1_{loc}(\mathbb{R}^d)$,

$$D_{n,p}(u)(x) := \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(|x-y|)dy \quad \text{for a.}e.\ x \in \mathbb{R}^d. \tag{11}$$

Note that, see [109],

$$\int_{\mathbb{R}^d} D_{n,p}(u)(x)dx \leq C_{p,d}\int_{\mathbb{R}^d} |\nabla u|^p\,(x)dx \ \text{ for } n \in \mathbb{N},$$

and hence

$$D_{n,p}(x) < +\infty \text{ for a.}e.\ x \in \mathbb{R}^d \tag{12}$$

if $p > 1$ and $\nabla u \in L^p(\mathbb{R}^d)$ or $p = 1$ and ∇u is a finite measure. From the BBM formula, we have, for $p \geq 1$,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d} D_{n,p}(u)(x) = I_p(u) \quad \text{ for } u \in L^1_{loc}(\mathbb{R}^d).$$

On the other hand, an easy computation (see [109]) gives, for $p \geq 1,\ u \in C^1_c(\mathbb{R}^d),$ and $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x).$$

We investigate the mode convergence of $D_{n,p}(u)$ to $\gamma_{d,p}|\nabla u|^p$ as $n \to +\infty$ for non smooth u. We have following.

We will use the following elementary lemma (see [107]):

Lemma (6.1.1)[219]: Let $d \geq 1, r > 0, x \in \mathbb{R}^d$, and $f \in L^1_{loc}(\mathbb{R}^d)$. We have

$$\int_{\mathbb{S}^{d-1}}\int_0^r |f(x+s\sigma)|\,dsd\sigma \leq C_d r M(f)(x), \tag{13}$$

for some positive constant C_d depending only on d.

Here $M\,(f)$ denotes the maximal function of f. We now give the

Theorem (6.1.2)[219]: Let $d \geq 1, p \geq 1,$ and $u \in W^{1,p}_{loc}(\mathbb{R}^d)$. Then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d} \frac{|u(x+h)-u(x)-\nabla u(x).h|^p}{|h|^p}\rho_n(|h|)dh = \text{for a.}e.\ x \in \mathbb{R}^d. \tag{14}$$

Consequently,

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x) \ \text{ for a.}e.\ x \in \mathbb{R}^d. \tag{15}$$

Proof: We first present the proof for $u \in W^{1,p}(\mathbb{R}^d)$. We claim that, for all $u \in W^{1,p}(\mathbb{R}^d)$,

$$D_{n,p}(u)(x) \leq CM(|\nabla u|^p)(x) \text{ for a.}e.\ x \in \mathbb{R}^d \tag{16}$$

Here and in what follows, C denotes a positive constant depending only on d. We have, for $a.e. x \in \mathbb{R}^d, \sigma \in \mathbb{S}^{d-1}$, and $r > 0$,

$$u(x + r\sigma) - u(x) = \int_0^r \nabla u(x + s\sigma).\sigma ds.$$

Using polar coordinates, Holder's inequality, and Fubini's theorem, we obtain, for $a.e. x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_n(|h|)dh \leq \int_0^\infty \rho_n(r) r^{d-1} \frac{1}{r} \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla u(x + s\sigma).\sigma|^p \, ds d\sigma dr$$
$$= \int_0^\infty \rho_n(r) r^{d-1} \frac{1}{r} \int_{B(x,y)} |\nabla u(y)|^p |y|^{1-d} \, dy dr.$$

Applying Lemma (6.1.1), we obtain (16).

The proof of (14) now goes as follows. Set

$$\Omega(u) \coloneqq \left\{ x \in \mathbb{R}^d;\ \limsup_{n \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x).h|^p}{|h|^p} \rho_n(|h|)dh > \varepsilon \right\}.$$

Note that if $u \in C_c^1(\mathbb{R}^d)$ then (14) holds for all $x \in \mathbb{R}^d$. This implies

$$|\Omega(v)| = 0 \text{ for all } v \in C_c^1(\mathbb{R}^d).$$

It follows that

$$\Omega(v) = \Omega(u - v) \text{ for all } v \in C_c^1(\mathbb{R}^d). \tag{17}$$

Recall that, see e.g., [188], for $f \in L^1$ $(\mathbb{R}^d)$, we have

$$\left|\{x \in \mathbb{R}^d; M(f)(x) > \varepsilon\}\right| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |f|. \tag{18}$$

Using (16) and (18), we obtain

$$\left| \left\{ x \in \mathbb{R}^d \int_{\mathbb{R}^d} \frac{|(u-v)(x+h) - (u-v)(x) - \nabla(u-v)(x).h|^p}{|h|^p} \rho_n(|h|)dh > \varepsilon \right\} \right|$$
$$\leq \frac{C}{\varepsilon} \int_{\mathbb{R}^d} |\nabla(u-v)(x)|^p dx \quad \text{ for all } \varepsilon > 0. \tag{19}$$

Combining (17) and (19) yields (14). Assertion (15) follows from (14) by the triangle inequality after noting that, for every $V \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{|V \cdot h|^p}{|h|^p} \rho_n(|h|)dh = \int_0^\infty \int_{\mathbb{S}^{d-1}} |V \cdot \sigma|^p \rho_n(r) r^{d-1} d\sigma dr = \gamma_{d,p} |V|^p.$$

We now turn to the proof in the case $u \in W_{loc}^{1,p}(\mathbb{R}^d)$. Given $R > 1$, let $\varphi \in C_c^1(\mathbb{R}^d)$. be such that $\varphi = 1$ in $B(0, 2R)$. We have $\varphi u \in W^{1,p}(\mathbb{R}^d)$. Applying the above result to φu, we obtain

$$\lim_{n \to +\infty} D_{n,p}(\varphi u)(x) = \gamma_{d,p} |\nabla(\varphi u)|^p(x) \quad \text{for a.}e.\ x \in B(0, R).$$

Since $D_{n,p}(u)(x) = D_{n,p}(\varphi u)(x)$ for $x \in B_R$ by (4) and $\varphi(x)u(x) = u(x)$ in B_R, it follows that

$$\lim_{n \to +\infty} D_{n,p}(u)(x) = \gamma_{d,p} |\nabla(u)|^p(x) \quad \text{for a.}e.\ x \in B(0, R).$$

Since $R > 1$ is arbitrary, the conclusion follows. follows.

Here is a natural question related to Theorem (6.1.2). Suppose for example that $u \in W^{1,1}(\mathbb{R}^d)$ and u has compact support. Is it true that for every $1 < p < +\infty$,

$$\lim_{n \to +\infty} D_{n,p}(u)(x) = \gamma_{d,p} |\nabla u|^p(x) \quad \text{for a.}e.\ x \in \mathbb{R}^d?$$

Surprisingly, the answer is delicate and some pathologies may occur as seen in our next result.

Theorem (6.1.3)[219]: Let $d \geq 1$ and $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. We have

(i) If $d = 1$, then, for $p > 1$,

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{1,p}|\acute{u}|^p(x) \quad \text{for a.e. } x \in \mathbb{R}. \tag{20}$$

(ii) If $d \geq 2, p \leq d/(d-1)$, and ρ_n is non-increasing, then

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{21}$$

(iii) If $d \geq 2$ and $p > 1$, then

$$\liminf_{n\to+\infty} D_{n,p}(u)(x) \geq \gamma_{d,p}|\nabla u|^p(x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{22}$$

Moreover, strict inequality in (22) can occur:

(v) If $d \geq 2$, there exist $u \in W^{1,1}(\mathbb{R}^d)$ with compact support, a set $A \subset \mathbb{R}^d$ of positive measure, and a sequence of non-increasing functions (ρ_n) such that, for every $n \in \mathbb{N}$,

$$D_{n,p}(u)(x) = +\infty \quad \text{for a.e. } x \in A \qquad \text{for all } p > d/(d-1). \tag{23}$$

Proof: As in the proof of Theorem (6.1.2), one may assume that $u \in W^{1,1}(\mathbb{R}^d)$. We first prove (20). Since, for $a.e. x \in \mathbb{R}$ and $r > 0$,

$$|u(x+r) - u(x)| \leq \int_x^{x+r} |\acute{u}(s)|ds,$$

we have

$$D_{n,p}(u)^{1/p}(x) \leq CM\,(\acute{u})(x).$$

Assertion (20) now follows as in the proof of Theorem (6.1.2) by noting that, for $u \in C^1_c(\mathbb{R})$,

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{1,p}|\acute{u}|^p(x) \quad \text{for } x \in \mathbb{R}^d.$$

We next turn to the proof of (22). Using polar coordinates, we have, for $a.e. x \in \mathbb{R}^d$,

$$D_{n,p}(u)(x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} \left|\int_0^1 \nabla u(x+tr\sigma)\cdot\sigma dt\right|^p \rho_n(r) r^{d-1} d\sigma dr$$

$$\geq \int_{\mathbb{S}^{d-1}} \left|\int_0^\infty \int_0^1 \nabla u(x+tr\sigma)\cdot\sigma\rho_n(r) r^{d-1} dtdr\right|^p d\sigma. \tag{24}$$

We claim that, for $a.e. \sigma \in \mathbb{S}^{d-1}$ and for $a.e. x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty} \int_0^\infty \int_0^1 \nabla u(x+tr\sigma)\cdot\sigma\rho_n(r) r^{d-1} dtdr = \nabla u(x)\cdot\sigma. \tag{25}$$

Assuming this and applying Fatou's lemma, we derive from (24) and (25) that, for $a.e. x \in \mathbb{R}^d$,

$$\liminf_{n\to+\infty} D_{n,p}(u)(x) \geq \gamma_{p,d}|\nabla u|^p(x);$$

which is (22). To complete the proof of (22), it remains to prove (25). For $v \in W^{1,1}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}^{d-1}$, set

$$M(\nabla v, \sigma, x) = \sup_{r>0} \int_0^r |\nabla v(x+s\sigma)\cdot\sigma|\, ds. \tag{26}$$

Given $v \in W^{1,1}(\mathbb{R}^d)$, and $\sigma \in \mathbb{S}^{d-1}$, we claim that for all $\varepsilon > 0$, there exists a positive constant C independent of v, ε and σ such that

$$\left|\{x \in \mathbb{R}^d; M(\nabla v, \sigma, x) > \varepsilon\}\right| \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^d} |\nabla v(y)|dy. \tag{27}$$

Using Fubini's theorem, we derive from (27) that

$$\left|\{x \in \mathbb{R}^d \times \mathbb{S}^{d-1}\,; M(\nabla v, \sigma, x) > \varepsilon\}\right| \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^d} |\nabla v(y)| dy. \tag{28}$$

Using (28), one can now obtain assertion (25) as in the proof of Theorem (6.1.2) by noting that for all $u \in C_c^1(\mathbb{R}^d)$,

$$\lim_{n\to+\infty} \int_0^\infty \int_0^1 \nabla u(x + tr\sigma) \cdot \sigma \rho_n(r) r^{d-1} dt dr = \nabla u(x) \cdot \sigma \qquad \text{for all } x \in \mathbb{R}^d.$$

We next establish (27). For simplicity of notation, we assume that $\sigma = e_d := (0, \cdots, 0, 1)$. We have, by Fubini's theorem,

$$\left|\{x \in \mathbb{R}^d\,; M(\nabla v, e_d, x) > \varepsilon\}\right| = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \mathbb{1}_{\{x\in\mathbb{R}^d\,; M(\nabla v, e_d, x) > \varepsilon\}}\, dx_d d\acute{x}. \tag{29}$$

It follows from the theory of maximal functions (see (18)) that

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \mathbb{1}_{\{x\in\mathbb{R}^d\,; M(\nabla v, e_d, x) > \varepsilon\}}\, dx_d d\acute{x} \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left|\partial_{x_d} v(\acute{x}, x_d)\right| dx_d d\acute{x}. \tag{30}$$

Combining (29) and (30) yields

$$\left|\{x \in \mathbb{R}^d\,; M(\nabla v, e_d, x) > \varepsilon\}\right| \leq \frac{C}{\varepsilon}\int_{\mathbb{R}^d} |\nabla v(x)| dx\,;$$

which is (27). The proof of (22) is complete.

We finally establish (23). Let (δ_n) be a positive sequence converging to 0 such that $\delta_n < 1/2$ for all n, and define

$$\rho_n(t) = \delta_n t^{\delta_n - 1} \mathbb{1}_{(0,1)}(t). \tag{31}$$

Set $u(x) = \varphi(x) \mid x \mid^{(1-d)} \ln^{-2} |x|$ for some $\varphi \in C_c^1(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $\mid x \mid < 2$. It is clear that $u \in W^{1,1}(\mathbb{R}^d)$ and for $x \in \mathbb{R}^d$ with $1/4 < \mid x \mid < 1/2$,

$$\int_{|y|<1/8} |u(x) - u(y)|^p\, dy = +\infty$$

since $p > d/(d-1)$ and $\rho_n(\mid y - x \mid) \geq \delta_n (1/8)^{\delta_n - 1}$ for $\mid y \mid < 1/8$ and $1/4 < \mid x \mid < 1/2$. It follows that, for $1/4 < \mid x \mid < 1/2$,

$$D_{n,p}(u)(x) = +\infty \quad \forall n.$$

The proof is complete.

We present two proofs of Proposition (6.1.4).

Proposition (6.1.4)[219]: Let $d \geq 1, p \geq 1$, and $u \in L^1_{loc}(\mathbb{R}^d)$ with $\nabla u \in L^p(\mathbb{R}^d)$. Then

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|^p}{|h|^p} \rho_n(|h|) dh dx = 0. \tag{32}$$

Consequently,

$$\lim_{n\to+\infty} D_{n,p}(u) = \gamma_{d,p} |\nabla u|^p \text{ in } L^1(\mathbb{R}^d). \tag{33}$$

First proof : via Theorem (6.1.2). By Theorem (6.1.2), we have

$$\lim_{n\to+\infty} D_{n,p}(u)(x) = \gamma_{d,p} |\nabla u(x)|^p \text{ for } e.a.\ x \in \mathbb{R}^d. \tag{34}$$

On the other hand, by the BBM formula,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} D_{n,p}(u)(x) dx = \gamma_{d,p} \int_{\mathbb{R}^d} |\nabla u(x)|^p dx. \tag{35}$$

Recall that (see e.g., [220]) if $f_n(x) \to f(x)$ for $a.e. x \in \mathbb{R}^d$, and $\| f_n \|_{L^1(\mathbb{R}^d)} \to \| f \|_{L^1(\mathbb{R}^d)}$, then $f_n \to f$ in $L^1(\mathbb{R}^d)$. We deduce from (34) and (35) that

$$D_{n,p} \to \gamma_{d,p} |\nabla u|^p \quad \text{in } L^1(\mathbb{R}^d) \text{ as } n \to +\infty.$$

Direct proof: We have, see [109],

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla u(x) \cdot h|^p}{|h|^p} \rho_n(|h|) dh dx \le C_{p,d} \int_{\mathbb{R}^d} |\nabla u(x)|^p$$

and, for $v \in C_c^1(\mathbb{R}^d)$,

$$\lim_{n \to +\infty} D_{n,p}(v)(x) = \gamma_{d,p} |\nabla v(x)|^p \quad \text{in } L^1(\mathbb{R}^d) \text{ as } n \to +\infty.$$

The conclusion now follows by a standard approximation argument.

Let $d \ge 1, \mu$ be a Radon measure defined on $\mathbb{R}^d$, and $0 < R \le +\infty$. Denote

$$M_R(\mu)(x) = \sup_{0<s<R} \frac{|\mu|(B(x,s))}{|B(x,s)|} \quad \text{and} \quad M(\mu)(x) = M_\infty(\mu)(x).$$

Lemma (6.1.5)[219]: Let $d \ge 1$, μ be a positive Radon measure defined in $\mathbb{R}$, and let $(\chi_k)_{k \ge 1}$ be a sequence of mollifier such that supp $\chi_k \subset B(0, 1/k)$ and $0 \le \chi_k \le C k^d$ for some positive constant C depending only on d. Set $\mu_k = \mu * \chi_k$. We have, for $x \in \mathbb{R}^d$ and for $r > 0$,

$$\frac{1}{r} \int_{B(x,r)} |y - x|^{1-d} \, d\mu(y) \le C M_r(\mu)(x) \tag{36}$$

and, for every k,

$$\frac{1}{r} \int_{B(x,r)} |y - x|^{1-d} \, d\mu_k(y) \le C M(\mu)(x) \tag{37}$$

for some positive constant C depending only on d.

Proof: Without loss of generality, one may assume that $x = 0$. We have

$$\begin{aligned}
\frac{1}{r} \int_{B(0,r)} |y|^{1-d} \, d\mu(y) &= \frac{1}{r} \sum_{m=0}^{\infty} \int_{B(0,2^{-m}r)/B(0,2^{-(m+1)}r)} |y|^{1-d} \, d\mu(y) \\
&\le \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m(1-d)} r^{1-d} \int_{B(0,2^{-m}r)/B(0,2^{-(m+1)}r)} d\mu(y) \\
&\le \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m} r \, M_r(\mu)(0) = C M_r(\mu)(0);
\end{aligned}$$

which is (36).

We next prove (37). As above, we obtain

$$\frac{1}{r} \int_{B(0,r)} |y|^{1-d} \, d\mu_k(y) \le \frac{C}{r} \sum_{m=0}^{\infty} 2^{-m(1-d)} r^{1-d} \int_{B(0,2^{-m}r)/B(0,2^{-(m+1)}r)} d\mu_k(y). \tag{38}$$

We claim that

$$\int_{B(0,2^{-m}r)/B(0,2^{-(m+1)}r)} d\mu_k(y) \le C \, 2^{-md} \, r^d M(\mu)(0). \tag{39}$$

Combining (38) and (39) yields (37).

It remains to prove (38). We have

$$\int_{B(0,2^{-m}r)/B(0,2^{-(m+1)}r)} d\mu_k(y) \le \int_{B(0,2^{-m}r)/\overline{B(0,2^{-(m+2)}r)}} d\mu_k(y)$$

$$= \sup_{\varphi \in C_c\left(B(0,2^{-m}r)/\overline{B(0,2^{-(m+2)}r)}\right);|\varphi|\le 1} \int_{\mathbb{R}^d} \varphi \, d\mu_k. \quad (40)$$

We have

$$\int_{\mathbb{R}^d} \varphi \, d\mu_k = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(z)\chi_k(z-y) \, dz d\mu(y). \quad (41)$$

If $2^{-m}r < 1/k$, we have, for $\varphi \in C_c\left(B(0,2^{-m}r)/\overline{B(0,2^{-(m+2)}r)}\right)$ with $|\varphi| \le 1$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(z)\chi_k(z-y) \, dz d\mu(y)$$

$$\le \int_{|y|<2/k} \sup_y \int_{\mathbb{R}^d} |\varphi(z)|\chi_k(z-y) dz \, d\mu(y)$$

$$\le C(2^{-m}r)^d k^d \int_{|y|<\frac{2}{k}} d\mu(y) \le C 2^{-md} r^d \, M(\mu)(0). \quad (42)$$

Here we use the fact that supp $\chi_k \subset B(0,1/k)$ and $0 \le \chi_k \le C\, k^d$. Similarly, if $1/k < 2^{-m}r$, we have , for $\varphi \in C_c\left(B(0,2^{-m}r)/\overline{B(0,2^{-(m+2)}r)}\right)$ with $|\varphi| \le 1$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(z)\chi_k(z-y) \, dz d\mu(y) \le \int_{|y|<2^{-m+2}r} \sup_y \int_{\mathbb{R}^d} |\varphi(z)|\chi_k(z-y) dz \, d\mu(y)$$

$$\le \int_{|y|<2^{-m+2}r} d\mu(y) \le C 2^{-md} r^d \, M(\mu)(0). \quad (43)$$

Combining (40), (41), (42) and (4.8), we obtain (43). The proof is complete.

We recall that (see, e.g., [223])

$$\lim_{r\to 0} \frac{|\nabla^s u|(B(x,r))}{|B(x,r)|} = 0 \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (44)$$

As a consequence of (44), one obtains

$$M(|\nabla^s u|)(x) < +\infty \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (45)$$

Theorem (6.1.6)[219]: Let $d \ge 1$ and $u \in BV_{loc}(\mathbb{R}^d)$. Then

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - \nabla^{ac}u(x)\cdot h|}{|h|^p} \rho_n(|h|)dh = 0 \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (46)$$

Consequently,

$$\lim_{n\to+\infty} D_{n,1}(u)(x) = \gamma_{d,1}|\nabla^{ac}u|(x) \qquad \text{for a.e. } x \in \mathbb{R}^d. \quad (47)$$

Here and in what follows, for $u \in BV_{loc}(\mathbb{R}^d)$, we denote $\nabla^{ac}u$ and $\nabla^s u$ the absolutely continuous part and the singular part of ∇u.

Proof: As in the proof of Theorem(6.2.2), one may assume that $u \in BV(\mathbb{R}^d)$. Let $(\chi_k)_{k\ge 1}$ be a sequence of smooth mollifiers such that supp $\chi_k \subset B(0,1/k)$ and $0 \le \chi_k \le Ck^d$. Here and in what follows, C denotes a positive constant depending only on d. Set, for $k \in \mathbb{N}_+$,

$$u_k = u * \chi_k, \quad V_k^s = \nabla^s u * \chi_k, \quad \text{and} \quad V_k^{ac} = \nabla^{ac} u * \chi_k.$$

We have

$$\int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - V_k^{ac}(x)\cdot h|}{|h|^p} \rho_n(|h|)dh$$

$$= \int_0^\infty r^{d-1} \rho_n(r) \int_{\mathbb{S}^{d-1}} \frac{|u_k(x + r\sigma) - u_k(x) - rV_k^{ac}(x) \cdot \sigma|}{r} d\sigma dr. \tag{48}$$

Since

$$u_k(x + r\sigma) - u_k(x) - rV_k^{ac}(x) \cdot \sigma = \int_0^r \nabla u_k(x + s\sigma) \cdot \sigma \, ds - rV_k^{ac}(x) \cdot \sigma$$

and

$$\nabla u_k(x) = V_k^s(x) + V_k^{ac}(x),$$

it follows from (48) that

$$\int_{\mathbb{R}^d} \frac{|u_k(x + h) - u_k(x) - V_k^{ac}(x) \cdot h|}{|h|} \rho_n(|h|) dh$$
$$\leq \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^s(x + s\sigma)| ds d\sigma.$$
$$+ \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^{ac}(x + s\sigma) - V_k^{ac}(x)| ds d\sigma. \tag{49}$$

We claim that, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{k \to +\infty} \int_{\mathbb{R}^d} \frac{|u_k(x + h) - u_k(x) - V_k^{ac}(x) \cdot h|}{|h|} \rho_n(|h|) dh$$
$$= \int_{\mathbb{R}^d} \frac{|u(x + h) - u(x) - \nabla^{ac} u(x) \cdot h|}{|h|} \rho_n(|h|) dh, \tag{50}$$

$$\lim_{k \to +\infty} \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^s(x + s\sigma)| ds d\sigma$$
$$= \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{B(x,r)} |\nabla^s u(y)| \; |y - x|^{1-d} \, dy, \tag{51}$$

and

$$\lim_{k \to +\infty} \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |V_k^{ac}(x + s\sigma) - V_k^{ac}(x)| ds d\sigma$$
$$= \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla^{ac} u(x + s\sigma) - \nabla^{ac} u(x)| ds d\sigma \, . \tag{52}$$

Assuming these claims, we continue the proof. Combining (49), (50), (51), and (52) yields, for a.e. $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{|u(x + h) - u(x) - \nabla^{ac} u(x) \cdot h|}{|h|} \rho_n(|h|) dh,$$
$$\leq \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{B(x,r)} |\nabla^s u(y)| \; |y - x|^{1-d} \, dy$$
$$+ \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla^{ac} u(x + s\sigma) - \nabla^{ac} u(x)| ds d\sigma. \tag{53}$$

Hence it suffices to prove that, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{n \to +\infty} \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{B(x,r)} |\nabla^s u(y)| \; |y - x|^{1-d} \, dy = 0 \tag{54}$$

and

$$\lim_{n \to +\infty} \int_0^\infty r^{d-1} \rho_n(r) \frac{1}{r} dr \int_{\mathbb{S}^{d-1}} \int_0^r |\nabla^{ac} u(x + s\sigma) - \nabla^{ac} u(x)| ds d\sigma = 0. \tag{55}$$

Note that assertion (55) holds for every $x \in \mathbb{R}^d$ if $u \in C_c^1(\mathbb{R}^d)$ and, by Lemma (6.1.5),

$$\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|dsd\sigma \le CM(|\nabla^{ac}u|)(x).$$

As in the proof of Theorem (6.1.2), we have, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty}\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{\mathbb{S}^{d-1}}\int_0^r |\nabla^{ac}u(x+s\sigma)-\nabla^{ac}u(x)|dsd\sigma = 0;$$

which is (55).

We next establish (54). By Lemma (6.1.5), we have

$$\frac{1}{r}\int_{B(x,r)}|\nabla^s u(y)|\ |y-x|^{1-d}\,dy \le CM_r(|\nabla^s u|)(x).$$

It follows from (44) that

$$\lim_{n\to+\infty}\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)}|\nabla^s u(y)|\ |y-x|^{1-d}\,dy = 0 \quad \text{for a.e. } x \in \mathbb{R}^d;$$

which is (54).

It remains to prove claims (51), (52), and (53). We begin with claim (51). We have

$$\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-V_k^{ac}(x)\cdot h|}{|h|}\rho_n(|h|)dh$$
$$= \int_0^\infty \rho_n(r)r^{d-1}\frac{1}{r}dr\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma.$$

Using Lemma (6.1.5), we derive from (49) that

$$\frac{1}{r}\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma \le CM(|\nabla u|)(x).$$

Since for $a.e.\, x \in \mathbb{R}^d$,

$$\lim_{k\to+\infty}\frac{1}{r}\int_{\mathbb{S}^{d-1}}|u_k(x+r\sigma)-u_k(x)-rV_k^{ac}(x)\cdot\sigma|\,d\sigma$$
$$= \frac{1}{r}\int_{\mathbb{S}^{d-1}}|u(x+r\sigma)-u(x)-r\nabla^{ac}u(x)\cdot\sigma|\,d\sigma \quad \text{for a.e. } r>0,$$

it follows from the dominated convergence theorem that, for a.e. $x \in \mathbb{R}^d$,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-V_k^{ac}(x)\cdot h|}{|h|}\rho_n(|h|)dh$$
$$= \int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-\nabla^{ac}u(x)\cdot h|}{|h|}\rho_n(|h|)dh;$$

which is (50).

The proof of (52) follows similarly. We finally establish (51). Fix $\tau > 0$ (arbitrary). We have

$$\int_0^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{\mathbb{S}^{d-1}}\int_0^r |V_k^s u(x+s\sigma)|dsd\sigma$$
$$= \int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)/B(x,\tau)}|V_k^s(y)|\,|y-x|^{1-d}dy$$
$$+ \int_\tau^\infty r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,\tau)}|V_k^s(y)|\,|y-x|^{1-d}dy$$

$$+\int_0^{\tau} r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)}|V_k^s(y)|\,|y-x|^{1-d}dy. \qquad (56)$$

We have, for a.e. $r>0$,

$$\lim_{k\to+\infty}\frac{1}{r}\int_{B(x,r)/B(x,\tau)}|V_k^s(y)|\,|y-x|^{1-d}dy=\frac{1}{r}\int_{B(x,r)/B(x,\tau)}|\nabla^s u(y)|\,|y-x|^{1-d}dy$$

and ,by Lemma (6.1.5),

$$\frac{1}{r}\int_{B(x,r)/B(x,\tau)}|V_k^s(y)|\,|y-x|^{1-d}dy\le CM(|\nabla u|)(x).$$

It follows from the dominated convergence theorem that

$$\lim_{k\to+\infty}\int_{\tau}^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)/B(x,\tau)}|V_k^s(y)|\,|y-x|^{1-d}dy$$
$$=\int_{\tau}^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)/B(x,\tau)}|\nabla^s u(y)|\,|y-x|^{1-d}dy. \qquad (57)$$

On the other hand, by Lemma (6.1.5),

$$\int_{\tau}^{\infty} r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,\tau)}|V_k^s u(y)|\,|y-x|^{1-d}dy$$
$$\le CM(|\nabla u|)(x)\int_{\tau}^{\infty} r^{d-1}\rho_n(r)\,\tau/r\,dr \qquad (58)$$

and

$$\int_0^{\tau} r^{d-1}\rho_n(r)\frac{1}{r}dr\int_{B(x,r)}|V_k^s(y)|\,|y-x|^{1-d}dy\le CM(|\nabla u|)(x)\int_0^{\tau} r^{d-1}\rho_n(r)dr. \quad (59)$$

Since

$$\lim_{\tau\to 0}\left(\int_{\tau}^{\infty} r^{d-1}\rho_n(r)\,\tau/r\,dr+\int_0^{\tau} r^{d-1}\rho_n(r)dr\right)=0,$$

we obtain (51) from (56), (57), (58), and (59). The proof is complete.

The following result deals with a converse" of Proposition (6.1.4). It is due to D. Spector in [225] and [133] in the case $\rho_n(r)=d\varepsilon_n^{-d}\mathbb{1}_{(0,\varepsilon_n)})$ for a sequence of $(\varepsilon_n)\to 0_+$ and to A. Ponce and D. Spector [224] for a general sequence (ρ_n). The proof we present here is more direct.

Proposition (6.1.7)[219]: Let $d\ge 1$ and $u\in L^1(\mathbb{R}^d)$. Then $u\in W^{1,1}(\mathbb{R}^d)$ if and only if there exists $U\in[L^1(\mathbb{R}^d)]^d$ such that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x+h)-u(x)-U(x)\cdot h|}{|h|}\rho_n(|h|)dhdx=0. \qquad (60)$$

Proof: We already know that (60) holds for $u\in W^{1,1}(\mathbb{R}^d)$ with $\nabla u=U$ by Proposition (6.1.4). It remains to prove that if (60) holds, then $u\in W^{1,1}(\mathbb{R}^d)$. Let (χ_k) be a sequence of standard mollifiers. Define

$$u_k=u*\chi_k \quad and \quad U_k=U*\chi_k.$$

We have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u_k(x+h)-u_k(x)-U_k(x)\cdot h|}{|h|}\rho_n(|h|)dhdx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} u(x+h-y)\chi_k(y)dy - \int_{\mathbb{R}^d} u(x-y)\chi_k(y)dy - \int_{\mathbb{R}^d} U(x-y)\cdot h\chi_k(y)dy \right| |h|^{-1}\rho_n(|h|)dhdx.$$

This implies

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - U_k(x)\cdot h|}{|h|}\rho_n(|h|)dhdx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+h-y) - u(x-y) - U(x-y)\cdot h|}{|h|}\chi_k(y)dy\rho_n(|h|)dhdx.$$

A change of variables given

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - U_k(x)\cdot h|}{|h|}\rho_n(|h|)dhdx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x) - U(x)\cdot h|}{|h|}\rho_n(|h|)dhdx.$$

We derive from (60) that, for $k > 0$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_k(x+h) - u_k(x) - U_k(x)\cdot h|}{|h|}\rho_n(|h|)dhdx = 0.$$

Since u_k is smooth, we obtain

$$U_k = \nabla u_k.$$

As $k \to +\infty, u_k \to u$ and $U_k \to U$ in $L^1(\mathbb{R}^d)$, so that $u \in W^{1,1}(\mathbb{R}^d)$ and $\nabla u = U$.

Most of the above results hold when $\mathbb{R}^d$ is replaced by a smooth bounded domain Ω of $\mathbb{R}^d$. Define, for $p \ge 1, n \in \mathbb{N}$, and $u \in L^1_{loc}(\Omega)$,

$$D^{\Omega}_{n,p}(u)(x) := \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^p}\rho_n(|x-y|)dy \text{ for a.e. } x \in \Omega. \quad (61)$$

Here is a typical result:

Theorem (6.1.8)[219]: Let $d \ge 1, p \ge 1$ and $u \in W^{1,p}(\Omega)$. Then

$$\lim_{n\to+\infty} D^{\Omega}_{n,p}(u)(x) = \gamma_{d,p}|\nabla u|^p(x) \text{ for a.e. } x \in \Omega. \quad (62)$$

Proof: Let $\tilde{u}$ be an extension of u to $\mathbb{R}^d$ such that $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$. Let $w \subset\subset \Omega$. We have, for $x \in w$,

$$D^{\Omega}_{n,p}(u)(x) = D_{n,p}(\tilde{u})(x) - \int_{\mathbb{R}^d\setminus\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|}\rho_n(|x-y|)dy. \quad (63)$$

Applying Theorem (6.1.2) to $\tilde{u}$, we have for a.e. $x \in w$,

$$\lim_{n\to+\infty} D_{n,p}(\tilde{u})(x) = \gamma_{d,p}|\nabla\tilde{u}|^p(x) = \gamma_{d,p}|\nabla u|^p(x). \quad (64)$$

Since w is arbitrary, it suffices to prove that for a.e. $x \in w$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d\setminus\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|}\rho_n(|x-y|)dy = 0. \quad (65)$$

Let $\varphi \in C^1(\mathbb{R}^d)$ be such that $\varphi = 1$ in $\mathbb{R}^d \setminus \Omega$ and $\varphi = 0$ in w. Applying Theorem (6.1.2) to $\varphi\,\tilde{u}$, we obtain, for a.e. $x \in w$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d\setminus\Omega} \frac{|\tilde{u}(y)|}{|x-y|}\rho_n(|x-y|)dy = 0. \quad (66)$$

On the other hand, for a.e. $x \in w$,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d\setminus\Omega}\frac{|\tilde{u}(y)|}{|x-y|}\rho_n(|x-y|)dy = |u(x)|\lim_{n\to+\infty}\int_{\mathbb{R}^d\setminus\Omega}\frac{|1|}{|x-y|}\rho_n(|x-y|)dy = 0 \quad (67)$$

Section (6.2): Hardy and Caffarelli-Kohn-Nirenberrg in Equalities

In many branches of mathematical physics, harmonic and stochastic analysis, the classical Hardy inequality plays a central role. It states that, if $1 \le p < d$,

$$\left(\frac{d-p}{p}\right)^p \int_{\mathbb{R}^d}\frac{|u|^p}{|x|^p}dx \le \int_{\mathbb{R}^d}|\nabla u|^p\,dx,$$

for every $u \in C_c^1(\mathbb{R}^d)$ with optimal constant which, contrary to the Sobolev inequality, is never attained. Another class of relevant inequalities is given by the so called Caffarelli-Kohn-Nirenberg inequalities[7], [72]. Let $p \ge 1$, $q \ge 1$, $\tau > 0$, $0 < a \le 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{d},\quad \frac{1}{p}+\frac{\alpha}{d},\quad \frac{1}{q}+\frac{\beta}{d} > 0, \tag{68}$$

$$\frac{1}{\tau}+\frac{\gamma}{d} = a\left(\frac{1}{p}+\frac{\alpha-1}{d}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{d}\right),$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \le \alpha - \sigma$$

and

$$\alpha - \sigma \le 1 \ \text{if} \quad \frac{1}{\tau}+\frac{\gamma}{d} = \frac{1}{p}+\frac{\alpha-1}{d}.$$

Then, for every $u \in C_c^1(\mathbb{R}^d)$,

$$\||x|^\gamma u\|_{L^\gamma(\mathbb{R}^d)} \le C\||x|^\alpha \nabla u\|_{L^p(\mathbb{R}^d)}^a \||x|^\beta u\|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

for some positive constant C independent of u. This inequality has been an object of a large amount of improvement and extensions to more general frameworks.

In the non-local case, it was shown in [166], [182] that there exists $C > 0$, independent of $0 < \delta < 1$, such that

$$C\int_{\mathbb{R}^d}\frac{|u(x)|^p}{|x|^{p\delta}}dx \le J_\delta(u), \tag{69}$$

for all $u \in C_c^1(\mathbb{R}^d)$,where

$$J_\delta(u) := (1-\delta)\int\int_{\mathbb{R}^{2d}}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p\delta}}dxdy.$$

In light of the results of Bourgain, Brezis, and Mironescu [109], [134] and an tenement of Davila [118], it holds

$$\lim_{\delta\searrow 0} J_\delta(u) = K_{d,p}\int_{\mathbb{R}^d}|\nabla u|^p\,dx,\quad for\ u \in W^{1,p}(\mathbb{R}^d),\quad K_{d,p} := \frac{1}{p}\int_{\mathbb{S}^{d-1}}|e\cdot\sigma|^p\,d\sigma,$$

for some $e \in \mathbb{S}^{d-1}$, being $\mathbb{S}^{d-1}$the unit sphere in $\mathbb{R}^d$. This allows to recover the classical Hardy inequality from (69) by letting $\delta \searrow 0$. Various problems related to J_δ are considered in [112], [219], [107], [116], [130], [224].The full range of Caffarelli-Kohn-Nirenberg inequalities and their variants were established in [236] (see [203] for partial results in the case $a = 1$).

Set, for $p \ge 1$, Ω a measurable set of $\mathbb{R}^d$, and $u \in L^1_{loc}(\Omega)$,

$$I_\delta(u,\Omega) := \int_\Omega \int_\Omega \frac{\delta^p}{|x-y|^{d+p}} dxdy.$$
$$\{|u(x)-u(y)|>\delta\}$$

In the case, $\Omega = \mathbb{R}^d$, we simply denote $I_\delta(u,\mathbb{R}^d)$ by $I_\delta(u)$.The quantity I_δ with $p = d$ has its roots in estimates for the topological degree of continuous maps from a sphere into itself in [227], [232]. This also appears in characterizations of Sobolev spaces [110], [228], [116], [124], [126] and [115], [228], [116], [125], [128], [127], [234], [235]. It is known that for $p \geq 1$,

$$\lim_{\delta\searrow 0} J_\delta(u) = K_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \quad \text{for } u \in C_c^1 \,(\mathbb{R}^d)^1, \tag{70}$$

and, for $p > 1$,

$$I_\delta(u) \leq C_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \quad \text{for } u \in W^{1,p}(\mathbb{R}^d), \tag{71}$$

for some positive constant $C_{d,p}$ independent of u.

We improved versions of the local Hardy and Caffarelli-Kohn-Nirenberg type inequalities and their variants which involve nonlinear nonlocal nonconvex energies $I_\delta(u)$ and its related quantities. In what follows for $R > 0, B_R$ denotes the open ball of $\mathbb{R}^d$ centered at the origin of radius r. Our rst main result concerning Hardy's inequality is:

Theorem (6.2.1)[226]: (Improved Caffarelli-Kohn-Nirenberg's inequality for $a = 1$). Let $d \geq 2, 1 < p < d, > \tau > 0,\ 0 < r < R$, and $u \in L^p_{loc}\,(\mathbb{R}^d)$. Assume that

$$\frac{1}{r} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d} \quad and \quad 0 \leq \alpha - \gamma \leq 1.$$

We have

i) if $d - p + p\alpha > 0$ and supp $u \subset B_R$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} \, |u(x)|^\tau dx\right)^{p/\tau} \leq C(\, I_\delta(u,\alpha) + R^{d-p+p\alpha} \delta^p),$$

ii) if $d - p + p\alpha < 0$ and supp $u \subset \mathbb{R}^d \backslash B_1$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} \, |u(x)|^\tau dx\right)^{p/\tau} \leq C(\, I_\delta(u,\alpha) + r^{d-p+p\alpha} \delta^p),$$

iii) if $d - p + p\alpha = 0,\ \tau > 1$, and supp $u \subset B_r$, then

$$\left(\int_{\mathbb{R}^d \backslash B_1} \frac{|x|^{\gamma\tau} |u(x)|^\tau}{\ln^\tau(2R/|x|)} dx\right)^{p/\tau} \leq C(\, I_\delta(u,\alpha) + \ln(2R/r)\, \delta^p),$$

iv) if $d - p + p\alpha = 0,\ \tau > 1$, and supp $u \subset \mathbb{R}^d \backslash B_1$, then

$$\left(\int_{B_R} \frac{|x|^{\gamma\tau} |u(x)|^\tau}{\ln^\tau(2|x|/r)} dx\right)^{p/\tau} \leq C(\, I_\delta(u,\alpha) + \ln(2R/r)\, \delta^p).$$

Here C denotes a positive constant independent of u, r, and R.

Proposition (6.2.2)[226]: Let $p \geq 1\,, q \geq 1, \tau > 0,\ 0 < a \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1 - a)\left(\frac{1}{q} + \frac{\beta}{d}\right),$$

and, with $\gamma = a\sigma + (1 - a)\beta$,

$$0 \leq \alpha - \sigma$$

and

$$\alpha - \sigma \leq 1 \quad \text{ if } \quad \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}.$$

We have, for $u \in C_c^1(\mathbb{R}^d)$,

I) if $1/\tau + \gamma/d > 0$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau dx\right)^{1/\tau} \leq C \||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

II) if $1/\tau + \gamma/d < 0$, and supp $u \subset \mathbb{R}^d \setminus \{0\}$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau dx\right)^{1/\tau} \leq C \||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

Assume in addition that $\alpha - \sigma \leq 1$ and $\tau > 1$. We have

III) if $1/\tau + \gamma/d = 0$ and supp $u \subset B_R$ for some $R > 0$, then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau(2R/|x|)} |u|^\tau dx\right)^{1/\tau} \leq C \||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

IV) if $1/\tau + \gamma/d = 0$and supp $u \subset \mathbb{R}^d \setminus B_{R,}$ for some $r > 0$, then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^\tau(2|x|/r)} |u|^\tau dx\right)^{1/\tau} \leq C \||x|^\alpha \nabla u\|^a_{L^p(\mathbb{R}^d)} \||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)}.$$

Here C denotes a positive constant independent of u, r, and R.

Assertion **I**) is a slight improvement of the classical Caffarelli-Kohn-Nirenberg. Indeed, in the classical setting, Assertion **I**) is established under the additional assumptions

$$1/p + \alpha/d > 0 \quad \text{and} \quad 1/q + \beta/d > 0,$$

as mentioned in (68). Assertion **II**) with $a = 1$ and $\tau = p$ was known (see,e.g., [166]). Concerning Assertion **III**) with $a = 1$, this was obtained for $d = 2$ in [229] and [100] and, for $d \geq 3$, this was established in [100]. Assertion **IV**) with $a = 1$ might be known; however, we cannot find any references for it.

The ideas used in the proof of Theorems (6.2.5) and (6.2.1), and their general version (Theorem (6.2.10)) are as follows. On one hand, this is based on Poincare's and Sobolev inequalities related to $I_\delta(u\,,\Omega)$, (see Lemma (6.2.3) and Lemma (6.2.6)). These inequalities have their roots in [128]. Using these inequalities, we derive the key estimate (see Lemma (6.2.8) and also Lemma (6.2.3)), for an annulus D centered at the origin and for $\lambda > 0$,

$$\left(\int_{\lambda D} \left|u - \fint_{\lambda D} u\right|^\tau dx\right)^{1/\tau}$$
$$\leq C(\lambda^{p-d}\, I_\delta(u, \lambda D) + \delta^p)^{a/p} \left(\int_{\lambda D} \left|u - \fint_{\lambda D} u\right|^q dx\right)^{(1-a)/q} \quad (72)$$

for some positive constant C independent of u and λ. On the other hand, decomposing $\mathbb{R}^d$ into annuli $\mathcal{A}_k$which are defined by

$$\mathcal{A}_k := \{x \in \mathbb{R}^d : 2^k \leq |x| < 2^{k+1}\},$$

and applying (72) to each $\mathcal{A}_k$, we obtain

$$\left(\int_{A_k} \left|u - \fint_{A_k} u\right|^\tau dx\right)^{1/\tau} \leq C\left(2^{-(d-p)k}\, I_\delta(u, A_k) + \delta^p\right)^{a/p} \left(\int_{A_k} |u|^q\right)^{(1-a)/q}$$

Similar idea was used in [7]. Using (72) again in the cases i) and ii), we can derive an appropriate estimate for

$$2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^{\tau}.$$

This is the novelty in comparison with the approach in [7]. Combining these two facts, one obtains the desired inequalities. The other cases follow similarly. Similar approach is used to establish Caffarelli-Kohn-Nirenberg's inequalities for fractional Sobolev spaces in [236]
We now make some comments on the magnetic Sobolev setting. If $A: \mathbb{R}^d \to \mathbb{R}^d$ is locally bounded and $u: \mathbb{R}^d \to \mathbb{C}$, we set

$$\Psi_u(x,y) := e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y), \qquad x, y \in \mathbb{R}^d.$$

The following diamagnetic inequality holds

$$\|u(x) - u(y)\| \le |\Psi_u(x,x) - \Psi_u(x,y)|, \quad \text{for} \quad \text{e.a.}\, x, y \in \mathbb{R}^d.$$

In turn, by defining

$$I_\delta^A(u,\delta) = \underset{\{|\Psi_u(x,y)-\Psi_u(x,x)|>\delta\}}{\int_\Omega \int_\Omega} \frac{\delta^p |x|^{p\alpha}}{|x-y|^{d+p}}\, dxdy,$$

we have, for $\alpha \in \mathbb{R}$,

$$I_\delta(|u|,\alpha) \le I_\delta^A(u,\alpha) \quad \text{for all } \delta > 0.$$

Then, the assertions of Theorem (6.2.5) and (6.2.1) keep holding with $I_\delta^A(u,0)$ (resp. $I_\delta^A(u,\alpha)$)) on the right-hand side in place of $I_\delta(u)$ (resp. $I_\delta(u,\alpha)$). For the sake of completeness, see [233] for some recent results about new characterizations of classical magnetic Sobolev spaces in the terms of $I_\delta^A(u,0)$ (see [233], [237], [238] for the ones related to J_δ).
We prove Theorem (6.2.5)and (6.2.10) and Proposition (6.2.11) which imply Theorem (6.2.1) and Proposition (6.2.2). We present versions of Theorems (6.2.5) and (6.2.10) in a bounded domain Ω.

We first recall that a straight forward variant of [128] yields the following

Lemma (6.2.3)[226]: Let $d \ge 1, p \ge 1$ and set

$$D := \{x \in \mathbb{R}^d : r < |x| < R\}.$$

Then

$$\int_D \left|u(x) - \int_D u\right|^p dx \le C_{r,R}(\, I_\delta(u,D) + \delta^p), \quad \text{for all } u \in L^p(D).$$

As a consequence, we have, for $\lambda > 0$,

$$\int_{\lambda D} \left|u(x) - \int_{\lambda D} u\right|^p dx \le C_{r,R}(\, \lambda^{p-d} I_\delta(u,\lambda D) + \delta^p), \quad \text{for all } u \in L^p(\lambda D), \tag{73}$$

where $\lambda D := \{\lambda x : x \in D\}$.Here $C_{r,R}$ denotes a positive constant independent of u, δ and λ.
The following elementary inequality will be used several times.
Lemma (6.2.4)[226]: Let $\Lambda > 1$ and $\tau > 1$. There exists $C = C(\Lambda, \tau\,) > 0$, depending only on Λ and τ such that, for all $1 < c < \Lambda$,

$$(|x| + |b|)^\tau \le c|a|^\tau + \frac{C}{(c-1)^{\tau-1}}|b|^\tau, \quad \text{for all } \ a, b \in \mathbb{R}. \tag{74}$$

Proof: Since (74) is clear in the case $|b| \ge |a|$ and in the case $b = 0$, by rescaling and considering $x = |a|/|b|$, it suffices to prove, for $C = C(\Lambda,\ \tau)$ large enough, that

$$(x+1)^\tau \le cx^\tau + \frac{C}{(c-1)^{\tau-1}}, \quad \text{for all} \quad x \ge 1. \tag{75}$$

Set

$$f(x) = (x+1)^\tau - cx^\tau - \frac{C}{(c-1)^{\tau-1}} \quad \text{for} \ \ x > 0$$

We have

$$\acute{f}(x) = \tau(x+1)^{\tau-1} - c\tau\, x^{\tau-1} \quad \text{and} \quad \acute{f}(x) = 0 \text{ if and only if } x = x_0 := \left(c^{\frac{1}{\tau-1}} - 1\right)^{-1}.$$

One can check that

$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to 1} f(x) < 0 \text{ if } \ C = C(\Lambda, \quad \tau) \ \text{ is large enough.} \tag{76}$$

and

$$f(x_0) = cx_0^{\tau-1} - \frac{C}{(c-1)^{\tau-1}}. \tag{77}$$

If $c^{\frac{1}{\tau-1}} > 2$ then $x_0 < 1$ and (75) follows from (76). Otherwise $1 \le s := c^{\frac{1}{\tau-1}} \le 2$. By the mean value theorem, we have

$$s^{\tau-1} - 1 \le (s-1) \max_{1\le t\le 2} \ (\tau-1)\, t^{\tau-2} \text{ for } 1 \le s \le 2.$$

We derive from (77) that, with $C = \Lambda\left[\max_{1\le t\le 2}(\tau-1)t^{\tau-2}\right]^{\tau-1}$,

$$f(x_0) < 0.$$

The conclusion now follows from (76).

We now give

Theorem (6.2.5)[226]: (Improved Hardy inequality). Let $d \ge 1, p \ge 1, 0 < r < R$, and $u \in L^p(\mathbb{R}^d)$. We have

i) if $1 \le p < d$ and supp $u \subset B_R$, then

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \le C(I_\delta(u) + R^{d-p}\delta^p),$$

ii) if $p > d$ and supp $u \subset \mathbb{R}^d \backslash B_r$ then

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \le C(I_\delta(u) + r^{d-p}\delta^p),$$

iii) if $p = d \ge 2$ and supp $u \subset B_R$, then

$$\int_{\mathbb{R}^d\backslash B_r} \frac{|u(x)|^d}{|x|^d \ \ln^d(2R/|x|)} dx \le C(I_\delta(u) + \ln(2R/r)\,\delta^d),$$

iv) if $p = d \ge 2$ and supp $u \subset \mathbb{R}^d \backslash B_r$, then

$$\int_{B_r} \frac{|u(x)|^d}{|x|^d \ \ln^d(2|x|/r)} dx \le C(I_\delta(u) + \ln(2R/r)\,\delta^d),$$

where C denotes a positive constant depending only on p and d.

In light of (70), by letting $\delta \to 0$, one obtains variants of **i**), **ii**), **iii**), **iv**) of Theorem (6.2.5) where

the RHS is replaced by $C\int_{\mathbb{R}^d} |\nabla u|^p\, dx$; see Proposition (6.2.2) for a more general version. By (70) and (71), Theorem (6.2.5) provides improvement of Hardy's inequalities in the case $p > 1$.

We next discuss an improved version of Caffarelli-Kohn-Nirenberg in the case the exponent $a = 1$. The more general case is considered in Theorem (6.2.10) (see also Proposition (6.2.11)). Set, for $p \ge 1, \alpha \in \mathbb{R}$, and Ω a measurable subset of $\mathbb{R}^d$,

$$I_\delta(u,\Omega,\alpha) := \int_\Omega \int_{\Omega \atop \{|u(x)-u(y)|>\delta\}} \frac{\delta^p |x|^{p\alpha}}{|x-y|^{d+p}} dxdy, \quad \text{for } u \in L^1_{loc}(\Omega).$$

If $\Omega = \mathbb{R}^d$, we simply denote $I_\delta(u,\mathbb{R}^d,\alpha)$ by $I_\delta(u,\alpha)$. We have

Proof: Let $m,n \in \mathbb{Z}$ be such that

$$2^{n-1} \le R < 2^n \quad \text{and} \quad 2^m \le r < 2^{m+1}.$$

It is clear that $n-m \ge 1$. By (73) of Lemma (6.2.3), we have, for all $k \in Z$,

$$\int_{A_k} \left| u(x) - \fint_{A_k} u \right|^p dx \le C\left(2^{-(d-p)k} I_\delta(u,A_k) + \delta^p\right).$$

Here and in what follows in this proof, C denotes a positive constant independent of k, u, and δ. This implies

$$2^{-pk} \le \int_{A_k} \left| u(x) - \fint_{A_k} u \right|^p dx \le C\left(I_\delta(u,A_k) + 2^{(d-p)k}\delta^p\right).$$

It follows that

$$2^{-pk} \int_{A_k} |u(x)|^p \, dx \le C 2^{(d-p)k} \left| \fint_{A_k} u \right|^p + C\left(I_\delta(u,A_k) + 2^{(d-p)k}\delta^p\right). \tag{78}$$

Step 1: Proof of i): Summing (78) with respect to k from $-\infty$ to n, we obtain

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \le C \sum_{k=-\infty}^{n} 2^{(d-p)k} \left| \fint_{A_k} u \right|^p + C\, I_\delta(u) + C 2^{(d-p)n}\delta^n, \tag{79}$$

since $d > p$. We also have, by (73), for $k \in \mathbb{Z}$,

$$\left| \fint_{A_k} u - \fint_{A_{k+1}} u \right| \le C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right)^{1/p}$$

This implies

$$\left| \fint_{A_k} u \right| \le \left| \fint_{A_{k+1}} u \right| + C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right)^{1/p}.$$

Applying Lemma (6.2.4), we have

$$\left| \fint_{A_k} u \right|^p \le \frac{2^{d-p+1}}{1+2^{d-p}} \left| \fint_{A_{k+1}} u \right|^p + C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right).$$

It follows that, with $c = 2/1 + 2^{d-p} < 1$,

$$2^{(d-p)k} \left| \fint_{A_k} u \right|^p \le c 2^{(d-p)(k+1)} \left| \fint_{A_{k+1}} u \right|^p + C\left(I_\delta(u, A_k \cup A_{k+1}) + 2^{(d-p)k}\delta^p\right).$$

We derive that

$$\sum_{k=-\infty}^{n} 2^{(d-p)k} \left| \fint_{A_k} u \right|^p \le C \sum_{k=-\infty}^{n} I_\delta(u, A_k \cup A_{k+1}) + C 2^{(d-p)n}\delta^p. \tag{80}$$

A combination of (79) and (80) yields

$$\int_{\mathbb{R}^d} \frac{|u(x)|^d}{|x|^d} dx \le C I_\delta(u) + C 2^{(d-p)n}\delta^p,$$

The conclusion of **i**) follows.

Step 2: Proof of ii): Summing (78) with respect to k from m to $+\infty$, we obtain

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \le C \sum_{k=m}^{+\infty} 2^{(d-p)k} \left|\fint_{A_k} u\right|^p CI_\delta(u) + C2^{(d-p)m}\delta^p, \tag{81}$$

since $p > d$. We also have, by (73), for $k \in \mathbb{Z}$,

$$\left|\fint_{A_k} u - \fint_{A_{k+1}} u\right| \le C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right)^{1/p}.$$

This implies that

$$\left|\fint_{A_{k+1}} u\right| \le \left|\fint_{A_k} u\right| + C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right)^{1/p}.$$

Applying Lemma (6.2.4), we have

$$\left|\fint_{A_{k+1}} u\right|^p \le \frac{1 + 2^{d-p}}{2^{d-p+1}} \left|\fint_{A_k} u\right|^p + C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right).$$

It follows that, with $c = (1 + 2^{(d-p)})/2 < 1$,

$$2^{(d-p)(k+1)} \left|\fint_{A_{k+1}} u\right|^p \le c2^{(d-p)k} \left|\fint_{A_k} u\right|^p + C\left(I_\delta(u, A_k \cup A_{k+1}) + 2^{(d-p)k}\delta^p\right).$$

We derive that

$$\sum_{k=m}^{+\infty} 2^{(d-p)k} \left|\fint_{A_k} u\right|^p \le CI_\delta(u) + C2^{(d-p)m}\delta^p. \tag{82}$$

A combination of (81) and (82) yields

$$\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx \le CI_\delta(u) + C2^{(d-p)m}\delta^p$$

The conclusion of **ii**) follows.

Step 3: Proof of iii): Let $\alpha > 0$. Summing (78) with respect to k from m to n, we obtain

$$\int_{\{2^m<|x|<2^n\}} \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2R/|x|)} dx$$
$$\le C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{\alpha+1}} \left|\fint_{A_k} u\right|^p + CI_\delta(u) + C(n-m)\delta^p \tag{83}$$

We also have, by (73), for $k \in \mathbb{Z}$,

$$\left|\fint_{A_k} u\right| \le \left|\fint_{A_{k+1}} u\right| + C\left(I_\delta(u, A_k \cup A_{k+1})^{1/d} + \delta\right) \tag{84}$$

By applying Lemma (6.2.4) with

$$c = \frac{(n-k+1)^\alpha}{(n-k+1/2)^\alpha},$$

it follows from (84) that, for $m \le k \le n$,

$$\frac{1}{(n-k+1)^\alpha} \left|\fint_{A_k} u\right|^p \le \frac{1}{(n-k+1/2)^\alpha} \left|\fint_{A_{k+1}} u\right|^p$$
$$+C(n-k+1)^{d-1-\alpha}\left(I_\delta(u, A_k \cup A_{k+1}) + \delta^p\right) \tag{85}$$

We have, $m \le k \le n$,

$$\frac{1}{(n-k+1)^\alpha} - \frac{1}{(n-k+3/2)^\alpha} \sim \frac{1}{(n-k+1)^\alpha}. \tag{86}$$

Taking $\alpha = d - 1$ and combining (85) and (86) yield

$$\sum_{k=m}^{n} \frac{1}{(n-k+1)^d} \left| \int_{A_k} u \right|^d \leq C I_\delta(u) + C(n-m)\delta^d. \tag{87}$$

From (83) and (87), we obtain

$$\int_{\{|x|>2^m\}} \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2R/|x|)} dx \leq C I_\delta(u) + C(n-m)\delta^d.$$

This implies the conclusion of **iii**).

Step 4 Proof of iv): Let $\alpha > 0$. Summing (78) with respect to k from m to n, we obtain

$$\int_{\{2^m<|x|<2^n\}} \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2|x|/R)} dx$$
$$\leq \sum_{k=m}^{n} \frac{1}{(k-m+1)^{\alpha+1}} \left| \int_{A_k} u \right|^d C I_\delta(u) + C\delta^d \tag{88}$$

We have, by (73), for $k \in Z$,

$$\left| \int_{A_{k+1}} u \right| \leq \left| \int_{A_k} u \right| + C\left(I_\delta(u, A_k \cup A_{k+1})^{1/d} + \delta \right) \tag{89}$$

By applying Lemma (6.2.4) with

$$c = \frac{(n-k+1)^\alpha}{(n-k+1/2)^\alpha},$$

it follows from (89) that, for $m \leq k+1 \leq n$,

$$\frac{1}{(k-m+1)^\alpha} \left| \int_{A_{k+1}} u \right|^d \leq \frac{1}{(k-m+1/2)^\alpha} \left| \int_{A_k} u \right|^d$$
$$+ C(k-m+1)^{d-1-\alpha}\left(I_\delta(u, A_k \cup A_{k+1})^{1/d} + \delta^d \right). \tag{90}$$

We have, $m \leq k+1 \leq n$,

$$\frac{1}{(k-m+1)^\alpha} - \frac{1}{(k-m+3/2)^\alpha} \sim \frac{1}{(k-m+1)^{\alpha+1}}. \tag{91}$$

Taking $\alpha = d - 1$ and combining (90) and (91) yield

$$\sum_{k=m}^{n} \frac{1}{(k-m+1)^d} \left| \fint_{A_k} u \right|^d \leq C I_\delta(u) + C(n-m)\delta^d. \tag{92}$$

From (88) and (92), we obtain

$$\int_{\{2^m<|x|<2^n\}} \frac{|u(x)|^d}{|x|^d \ln^{\alpha+1}(2|x|/R)} dx \leq C I_\delta(u) + C(n-m)\delta^d$$

This implies the conclusion of **iv**).

The proof is complete.

In the proof of Theorem (6.2.1), we use the following result

Lemma (6.2.6)[226]: Let $1 < p < d$, be a smooth bounded open subset of $\mathbb{R}^d$, and $\upsilon \in L^p(\Omega)$. We have

$$\|u\|_{L^{p^*}(\Omega)} \leq C_\Omega\left(I_\delta(u)^{1/p} + \|u\|_{L^p} + \delta \right),$$

where $p^*:= dp/(d-p)$ denotes the Sobolev exponent of p.
Proof: For $\tau > 0$, let us set

$$\Omega_\tau := \{x \in \mathbb{R}^d:\ \mathrm{dist}(x,\Omega) < \tau\}.$$

Since is smooth, by [116], there exists $\tau > 0$ small enough and an extension U of u in Ω_τ such that

$$I_\delta(U,\Omega) \quad and \quad \|U\|_{L^p\Omega_\tau} \le C\|U\|_{L^p(\Omega)}, \tag{93}$$

for $0 < \delta < 1$. Fix such a τ . Let $\varphi \in C^1(\mathbb{R}^d)$ such that

$$\mathrm{supp}\,\varphi \subset \Omega_{2\tau/3}, \qquad \varphi = 1 \text{ in } \Omega_{\tau/3}, \qquad 0 \le \varphi \le 1 \text{ in } \mathbb{R}^d.$$

Define $v = \varphi U$ in $\mathbb{R}^d$. We claim that

$$I_{2\delta}(v) \le C\left(I_\delta(U,\Omega) + \|u\|_{L^p(\Omega)}^p \|u\|_{L^p}\right). \tag{94}$$

Indeed, set

$$f(x,y) = \frac{\delta^p}{|x-y|^{d+p}} \mathbb{1}_{\{|v(x)-v(y)|>2\delta\}}.$$

We estimate $I_{2\delta}(v)$. We have

$$\int\int_{\Omega\times\mathbb{R}^d} f(x,y)dxdy \le \int\int_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x,y)dxdy + \int\int_{\substack{\Omega_\tau\times\mathbb{R}^d \\ \{|x-y|>\tau/4\}}} f(x,y)dxdy,$$

and, since $\tau = 0$ in $\Omega_\tau \setminus \Omega_{2\tau/3}$,

$$\int\int_{(\mathbb{R}^d\setminus\Omega_\tau)\times\mathbb{R}^d} f(x,y)dxdy,$$

$$\le \int\int_{(\mathbb{R}^d\setminus\Omega_\tau)\times(\mathbb{R}^d\setminus\Omega_\tau)} f(x,y)dxdy + \int\int_{\substack{\Omega_\tau\times\mathbb{R}^d \\ \{|x-y|>\tau/4\}}} f(x,y)dxdy$$

$$\int\int_{(\Omega_\tau\setminus\Omega_\tau)\times\mathbb{R}^d} f(x,y)dxdy \le \int\int_{(\Omega_\tau\setminus\Omega)\times(\Omega_\tau\setminus\Omega)} f(x,y)dxdy$$

$$+\int\int_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x,y)dxdy + \int\int_{\substack{\Omega_\tau\times\mathbb{R}^d \\ \{|x-y|>\tau/4\}}} f(x,y)dxdy.$$

It is clear that, by (93),

$$\int\int_{\Omega_{\tau/3}\times\Omega_{\tau/3}} f(x,y)dxdy \le CI_\delta(u,\Omega), \tag{95}$$

by the fact that $\varphi = 0$ in $\mathbb{R}^d\setminus\Omega_\tau$,

$$\int\int_{(\mathbb{R}^d\setminus\Omega_\tau)\times(\mathbb{R}^d\setminus\Omega_\tau)} f(x,y)dxdy = 0, \tag{96}$$

and, by a straightforward computation,

$$\int\int_{\substack{\Omega_\tau\times\mathbb{R}^d \\ \{|x-y|>\tau/4\}}} f(x,y)dxdy \le C\delta^p. \tag{97}$$

We have, for $x,y \in \mathbb{R}^d$,

$$v(x) - v(y) = \varphi(x)\big(U(x) - U(y)\big) + U(y)\big(\varphi(x) - \varphi(y)\big).$$

It follows that if $|v(x) - v(y)| > 2\delta$ then either

$$|U(x) - U(y)| \ge \big|\varphi(x)\big(U(x) - U(y)\big)\big| > \delta$$

or

$$C|U(y)||x-y| \geq \left|U(y)\big(\varphi(x)-\varphi(y)\big)\right| > \delta.$$

We thus derive that

$$\int\int_{(\Omega_\tau\backslash\Omega)\times(\Omega_\tau\backslash\Omega)} f(x,y)dxdy \leq \int_{\substack{(\Omega_\tau\backslash\Omega)\\ \{|U(x)-U(y)|>\delta\}}} \int_{(\Omega_\tau\backslash\Omega)} \frac{\delta^p}{|x-y|^{d+p}}dxdy$$

$$+\int_{\substack{(\Omega_\tau\backslash\Omega)\\ \{|x-y|>C\delta/|U(y)|\}}} \int_{(\Omega_\tau\backslash\Omega)} \frac{\delta^p}{|x-y|^{d+p}}dxdy.$$

A straightforward computation yields

$$\int_{\substack{(\Omega_\tau\backslash\Omega)\\ \{|x-y|>C\delta/|U(y)|\}}} \int_{(\Omega_\tau\backslash\Omega)} \frac{\delta^p}{|x-y|^{d+p}}dxdy \leq \int_{\Omega_\tau} dy \int_{\{|x-y|>C\delta/|U(y)|\}} \frac{\delta^p}{|x-y|^{d+p}}dx$$

$$= C\int_{\Omega_\tau} |U(y)|^p dy.$$

Using (93), we deduce from (98) that

$$\iint_{(\Omega_\tau\backslash\Omega)\times(\Omega_\tau\backslash\Omega)} f(x,y) = dxdy \leq CI_\delta\,(u,\Omega) + C\|u\|_{L^p(\Omega)}^p \tag{99}$$

A combination of (95), (96), (3.5), and (99) yields Claim (94). By applying [128] and using the fact supp $v \subset \Omega_\tau$, we have

$$\|v\|_{L^{p^*}(\mathbb{R}^d)} \leq CI_{2\delta}(v)^{1/p} + C\delta. \tag{100}$$

The conclusion now follows from Claim (94).

Corollary (6.2.7)[226]: Let $d \geq 2, 1 < p < d, 0 < r < R$, and $\lambda > 0$, and set

$$\lambda D := \{\lambda x \in \mathbb{R}^d \ : \ r < |x| < R\}.$$

We have, for $1 \leq q \leq q^*$,

$$\left(\int_{\lambda D} u \left|\int_{\lambda D} u(x) - \right|^q dx\right)^{1/q} \leq C_{r,R}(\lambda^{p-1}I_\delta(u,\lambda D) + \delta^2)^{1/p}, \quad \text{for } u \in L^p(\lambda D),$$

where $C_{r,R}$ denotes a positive constant independent of u, δ, and λ.

Here is an application of Corollaries (6.2.7) which plays a crucial role in the proof of Theorem (6.2.10) below.

Lemma (6.2.8)[226]: Let $d \geq 1$, $1 < p < d, q \geq 1, \tau > 0$, and $0 \leq a \leq 1$ be such that

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{1}{d}\right) + \frac{1-a}{q}.$$

Let $0 < r < R$, and $\lambda > 0$ and set

$$\lambda D := \{\lambda x \in \mathbb{R}^d \ : \ r < |x| < R\}.$$

Then, for $u \in L^1(\lambda D)$,

$$\left(\int_{\lambda D} \left|u - \int_{\lambda D} u\right|^\tau dx\right)^{1/\tau} \leq C(\lambda^{p-d}I_\delta(u,\lambda D) + \delta^p)^{a/p}\left(\int_{\lambda D} \left|u - \int_{\lambda D} u\right|^q dx\right)^{(1-a)/q},$$

for some positive constant C independent of u, λ, and δ.

Proof: Let $\tau, \sigma, t > 0$, be such that

$$\frac{1}{\tau} \geq \frac{a}{\sigma} + \frac{1-a}{t}.$$

We have, by a standard interpolation inequality, that

$$\left(\int_{\lambda D}\left|u-\int_{\lambda D}u\right|^{\tau}dx\right)^{1/\tau}\leq\left(\int_{\lambda D}\left|u-\int_{\lambda D}u\right|^{\sigma}dx\right)^{a/\sigma}\left(\int_{\lambda D}\left|u-\int_{\lambda D}u\right|^{t}dx\right)^{(1-a)/t}.$$

Applying this inequality with $\sigma=p^{*}$ and $t=q$ and using Corollary (6.2.7), one obtains the conclusion.

Lemma(6.2.9)[226]:(Nirenberg's interpolation inequality).Let $d\geq 1,\ p\geq 1, q\geq 1, \tau\geq 0$, and $0\leq a\leq 1$ be such that

$$\frac{1}{\tau}\geq a\left(\frac{1}{p}-\frac{1}{d}\right)+\frac{1-a}{q}.$$

Let $0<r<R$, and $\lambda>0$ and set

$$\lambda D:=\{\lambda x\in\mathbb{R}^{d}\ :\ r<|x|<R\}.$$

Then, for $u\in L^{1}(\lambda D)$,

$$\left(\int_{\lambda D}\left|u-\int_{\lambda D}u\right|^{\tau}dx\right)^{1/\tau}\leq C\|\nabla u\|_{L^{p}(\lambda D)}^{a}C\|u\|_{L^{q}(\lambda D)}^{1-a},$$

for some positive constant C independent of u,λ, and δ .

We prove the following more general version of Theorem (6.2.1):

Theorem (6.2.10)[226]: Let $p\geq 1, q\geq 1,\ \tau>0, 0<a\leq 1, \alpha,\beta,\gamma\in\mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{d}=a\left(\frac{1}{p}+\frac{\alpha-1}{d}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{d}\right),\tag{101}$$

and, with $\gamma=a\sigma+(1-a)\beta$,

$$0\leq\alpha-\sigma\leq 1.$$

Set, for $k\in\mathbb{Z}$,

$$I_{\delta}(k,u):=\begin{cases}I_{\delta}(u,A_{k}\cup A_{k+1},\alpha)+2^{k(\alpha p+d-p)\delta p} & \text{if } 1<p<d,\\ \||x^{\alpha}|\nabla u\|_{L^{p}(A_{k}\cup A_{k+1})}^{p} & \text{otherwise.}\end{cases}\tag{102}$$

We have, for $u\in L_{loc}^{p}(\mathbb{R}^{d})$ and $m,n\in\mathbb{Z}$ with $m<n$,

i) if $1/\tau+=\gamma/d>0$ and supp $u\subset B_{2^{n}}$, then

$$\left(\int_{\mathbb{R}^{d}\backslash B_{2^{m}}}|x|^{\gamma\tau}|u|^{\tau}dx\right)^{1/\tau}\leq C\left(\sum_{k=m-1}^{n}I_{\delta}(k,u)\right)^{a/p}C\left\||x|^{\beta}u\right\|_{L^{q}(\mathbb{R}^{d})}^{(1-a)},$$

ii) if $1/\tau+=\gamma/d<0$ and supp $u\in\mathbb{R}^{d}\backslash B_{2^{m}}$, then

$$\left(\int_{B_{2^{n}}}|x|^{\gamma\tau}|u|^{\tau}dx\right)^{1/\tau}\leq C\left(\sum_{k=m-1}^{n}I_{\delta}(k,u)\right)^{a/p}C\left\||x|^{\beta}u\right\|_{L^{q}(\mathbb{R}^{d})}^{(1-a)},$$

iii) if $1/\tau+\gamma/d=0, \tau>1$, and supp $u\subset B_{2^{n}}$, then

$$\left(\int_{\mathbb{R}^{d}\backslash B_{2^{m}}}\frac{|x|^{\gamma\tau}}{\ln^{\tau}(2^{n+1}/|x|)}|x|^{\tau}dx\right)^{1/\tau}\leq C\left(\sum_{k=m-1}^{n}I_{\delta}(k,u)\right)^{a/p}\left\||x|^{\beta}u\right\|_{L^{q}(\mathbb{R}^{d})}^{(1-a)},$$

iv) if $1/\tau+\gamma/d=0, \tau>1$, and supp $u\in\mathbb{R}^{d}\backslash B_{2^{m}}$, then

$$\left(\int_{B_{2^{n}}}\frac{|x|^{\gamma\tau}}{\ln^{\tau}(2^{n+1}/|x|)}|x|^{\tau}dx\right)^{1/\tau}\leq C\left(\sum_{k=m-1}^{n}I_{\delta}(k,u)\right)^{a/p}\left\||x|^{\beta}u\right\|_{L^{q}(\mathbb{R}^{d})}^{(1-a)}.$$

Here C denotes a positive constant independent of u,δ,k,n, and m.

Proof: We only present the proof in the case $1 < p < d$. The proof for the other case follows similarly, however instead of using Lemma (6.2.8), one applies Lemma (6.2.9). We now assume that $1 < p < d$. Since $\sigma - \alpha \geq 0$, by Lemma (6.2.8), we have

$$\left(\int_{A_k}\left|u-\int_{A_k}u\right|^{\tau}dx\right)^{1/\tau} \leq C\left(2^{-(d-p)k}I_\delta(u,A_k)+\delta^p\right)^{a/p}\left(\int_{A_k}|u|^q\right)^{(1-a)/q}. \tag{103}$$

Using (101), we derive from (103) that

$$\int_{A_k}|x|^{\gamma\tau}|u|^{\tau}dx$$

$$\leq C2^{(\gamma\tau+d)k}\left|\int_{A_k}u\right|^{\tau}+C\left(I_\delta(u,A_k,\alpha)+2^{k(\alpha p+d-p)}\delta^p\right)^{a\tau/p}\left\||x|^{\beta}u\right\|_{L^q(A_k)}^{(1-a)\tau}. \tag{104}$$

Step 1: Proof of i). Summing (104) with respect to k from m to n, we obtain

$$\int_{\{|x|>2^m\}}|x|^{\gamma\tau}|u|^{\tau}dx \leq C\sum_{k=m}^{n}2^{(\gamma\tau+d)k}\left|\int_{A_k}u\right|^{\tau}$$

$$+C\sum_{k=m}^{n}\left(I_\delta(u,A_k,\alpha)+2^{k(\alpha p+d-p)}\delta^p\right)^{a\tau/p}\left\||x|^{\beta}u\right\|_{L^q(A_k)}^{(1-a)\tau}. \tag{105}$$

By Lemma (6.2.8), we have

$$\left|\int_{A_k}u\right| \leq \left|\int_{A_{k+1}}u\right|+C\left(2^{-(d-p)}I_\delta(u,A_k\cup A_{k+1})+\delta^p\right)^{a/p}\left(\int_{A_k\cup A_{k+1}}|u|^q\right)^{\frac{(1-a)}{a}}.$$

Applying Lemma (6.2.4), we derive that

$$\left|\int_{A_k}u\right|^{\tau} \leq \frac{2^{\gamma\tau+d+1}}{1+2^{\gamma\tau+d}}\left|\int_{A_{k+1}}u\right|^{\tau}$$

$$+C\left(2^{-(d-p)k}I_\delta(u,A_k\cup A_{k+1})+\delta^p\right)^{a/p}\left(\int_{A_k\cup A_{k+1}}|u|^q\right)^{\frac{(1-a)}{q}}.$$

It follows that, with $c = 2/(1+2^{\gamma\tau+d}) < 1$,

$$2^{(\gamma\tau+d)k}\left|\int_{A_k}u\right|^{\tau} \leq c2^{(\gamma\tau+d)(k+1)}\left|\int_{A_{k+1}}u\right|^{\tau}$$

$$+C\left(I_\delta(u,A_k\cup A_{k+1},\alpha)+2^{k(\alpha p+d-p)}\delta^p\right)^{\tau a/p}\left\||x|^{\beta}u\right\|_{L^q(A_k\cup A_{k+1})}^{(1-a)\tau}.$$

This yields

$$\sum_{k=m}^{n}2^{(\gamma\tau+d)k}\left|\int_{A_k}u\right|^{\tau}$$

$$\leq C\sum_{k=m}^{n}\left(I_\delta(u,A_k\cup A_{k+1},\alpha)+2^{k(\alpha p+d-p)}\delta^p\right)^{a\tau/p}\left\||x|^{\beta}u\right\|_{L^q(A_k\cup A_{k+1})}^{(1-a)\tau}. \tag{106}$$

Combining (105) and (106) yields

$$\int_{\{|x|>2^m\}}|x|^{\gamma\tau}|u|^{\tau}dx$$

$$\leq C \sum_{k=m-1}^{n} \left(I_\delta(u, A_k \cup A_{k+1}, \alpha) + 2^{k(\alpha p+d-p)}\delta^p \right)^{a\tau/p} \left\| |x|^\beta u \right\|_{L^q(A_k \cup A_{k+1})}^{(1-a)\tau}. \quad (107)$$

Applying the inequality, for $s \geq 0, t \geq 0$ with $s + t \geq 1$, and for $x_k \geq 0$ and $y_k \geq 0$,

$$\sum_{k=m}^{n} x_k^s y_k^t \leq C_{s,t} \left(\sum_{k=m}^{n} x_k \right)^s \left(\sum_{k=m}^{n} y_k \right)^t,$$

to $s = a\tau/p$ and $t = (1-a)\tau/q$, we obtain from (107) that

$$\int_{\{|x|>2^m\}} |x|^{\gamma\tau} |u|^\tau dx \leq C \left(\sum_{k=m}^{n} I_\delta(k, u) \right)^{\tau a/p} \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^d)}^{(1-a)\tau} \quad (108)$$

since $a/p + (1-a)/q \geq 1/\tau$ thanks to the fact $\alpha - \sigma - 1 \leq 0$.

Step 2: Proof of ii): The proof is in the spirit of the proof of ii) of Theorem (6.2.5).

Step 3: Proof of iii): Fix $\xi > 0$. Summing (104) with respect to k from m to n, we obtain

$$\int_{\{|x|>2^m\}} \frac{1}{\ln^{1+\xi}(\tau/|x|)} |x|^{\gamma\tau} |u|^\tau dx$$

$$\leq C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{1+\xi}} \left| \int_{A_k} u \right|^\tau C \sum_{k=m}^{n} \left(I_\delta(u, A_k, \alpha) + 2^{k(\alpha p+d-p)}\delta^p \right)^{a\tau/p} \left\| |x|^\beta u \right\|_{L^q(A_k)}^{(1-a)\tau}. \quad (109)$$

By Lemma (6.2.8), we have

$$\left| \int_{A_k} u \right| \leq \left| \int_{A_{k+1}} u \right| + C\left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p \right)^{a/p} \left(\int_{A_k \cup A_{k+1}} |u|^q \right)^{\frac{(1-a)}{q}}.$$

Applying Lemma (6.2.4) with

$$c = \frac{(n-k+1)^\xi}{(n-k+1/2)^\xi},$$

we deduce that

$$\frac{1}{(n-k+1)^\xi} \left| \int_{A_k} u \right|^\tau \leq \frac{1}{(n-k+1/2)^\xi} \left| \int_{A_{k+1}} u \right|^\tau + C(n-k+1)^{\tau-1-\xi} \left(2^{-(d-p)k} I_\delta(u, A_k \cup A_{k+1}) + \delta^p \right)^{a\tau/p} \left(\int_{A_k \cup A_{k+1}} |u|^q \right)^{\frac{(1-a)\tau}{q}}. \quad (110)$$

Recall that, for $k \leq n$ and $\xi > 0$,

$$\frac{1}{(n-k+1)^\xi} - \frac{1}{(n-k+3/2)^\xi} \sim \frac{1}{(n-k+1)^{\xi+1}}. \quad (111)$$

Taking $\xi = \tau - 1$, we derive from (110) and (111) that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k} \frac{1}{(n-k+1)^\tau} \left| \int_{A_k} u \right|^\tau \leq \sum_{k=m}^{n} C\left(I_\delta(k, u) \right)^{a\tau/p} \left\| |x|^\beta u \right\|_{L^q(A_k \cup A_{k+1})}^{(1-a)\tau}. \quad (112)$$

Combining (109) and (112), as in (108), we obtain

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^\tau(2^{n+1}/|x|)} |u|^\tau \, dx \le C \left(\sum_{k=m}^{n} I_\delta(k,u) \right)^{a\tau/p} \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^d)}^{(1-a)\tau}.$$

Step 4: Proof of iv): The proof is in the spirit of the proof of iv) of Theorem (6.2.5).

The proof is complete.

Proposition (6.2.11)[226]: Let $p \ge 1, q \ge 1, \tau > 0, 0 < a < 1, \alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right),$$

and, with $\gamma = a\sigma + (1+a)\beta$,

$$\alpha - \sigma > 1 \quad and \quad \frac{1}{\tau} + \frac{\gamma}{d} \ne \frac{1}{p} + \frac{\alpha - 1}{d}.$$

We have, for $u \in C_c^1(\mathbb{R}^d)$,

i) if $1/\tau + \gamma/d > 0$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau dx \right)^{1/\tau} \le C \left\| |x|^\alpha \nabla u \right\|_{L^p(\mathbb{R}^d)}^a \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

ii) if $1/\tau + \gamma/d < 0$ and supp $u \subset \mathbb{R}^d \setminus \{0\}$, then

$$\left(\int_{\mathbb{R}^d} |x|^{\gamma\tau} |u|^\tau dx \right)^{1/\tau} \le C \left\| |x|^\alpha \nabla u \right\|_{L^p(\mathbb{R}^d)}^a \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^d)}^{(1-a)},$$

for some positive constant C independent of u.

Proof: The proof is in the spirit of the approach in [7] (see also [236]). Since

$$\frac{1}{p} + \frac{\alpha - 1}{d} \ne \frac{1}{q} + \frac{\beta}{d}.$$

by scaling, one might assume that

$$\left\| |x|^\alpha \nabla u \right\|_{L^p(\mathbb{R}^d)} = 1 \quad \text{and} \quad \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^d)} = 1.$$

Let $0 < a_2 < 1$ be such that

$$|a_2 - a| \text{ is small enough,} \tag{113}$$

and set

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1 - a_2}{q} \quad \text{and} \quad \gamma_2 = a_2(\alpha - 1) + (1 - a_2)\beta.$$

We have

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} = a_2\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1 - a_2)\left(\frac{1}{q} + \frac{\beta}{d}\right). \tag{114}$$

Recall that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1 - a)\left(\frac{1}{q} + \frac{\beta}{d}\right). \tag{115}$$

Since $a > 0$ and $\alpha - \sigma > 1$, it follows from (113) that

$$\frac{1}{\tau} + \frac{1}{\tau_2} = a(1 - a_2)\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{a}{d}(\alpha - \sigma - 1) > 0. \tag{116}$$

We first choose a_2 such that

$$a_2 < a \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{117}$$

$$a < a_2 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{118}$$

Using (113), (117) and (118), we derive from (114), and (115) that

$$\frac{1}{\tau} + \frac{\gamma}{d} < \frac{1}{\tau_2} + \frac{\gamma_2}{d} \quad \text{and} \quad \left(\frac{1}{\tau} - \frac{\gamma}{d}\right)\left(\frac{1}{\tau_2} - \frac{\gamma_2}{d}\right) > 0. \tag{119}$$

It follows from (116), (119), and Holder's inequality that

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \leq C \||x|^{\gamma_2} u\|_{L^{\gamma_2}(\mathbb{R}^d)}.$$

Applying Theorem (6.2.10), we have

$$\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)} \leq C \||x|^{\alpha} \nabla u\|_{L^p(\mathbb{R}^d)}^{a_2} \||x|^{\beta} u\|_{L^q(\mathbb{R}^d)}^{(1-a_2)} \leq C,$$

which yields

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \leq C. \tag{120}$$

We next choose a_2 such that

$$a < a_2 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{121}$$

$$a_2 < a \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - 1}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{122}$$

Using (113), (121) and (122), we derive from (114), and (115) that

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} < \frac{1}{\tau} + \frac{\gamma}{d} \quad \text{and} \quad \left(\frac{1}{\tau} - \frac{\gamma}{d}\right)\left(\frac{1}{\tau_2} + \frac{\gamma_2}{d}\right) > 0. \tag{123}$$

It follows from (116), (123), and Holder's inequality that

$$\||x|^{\gamma} u\|_{L^{\tau}(B_1)} \leq C \||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)}.$$

Applying Theorem (6.2.10), we have

$$\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)} \leq C \||x|^{\alpha} \nabla u\|_{L^p(\mathbb{R}^d)}^{a_2} \||x|^{\beta} u\|_{L^q(\mathbb{R}^d)}^{(1-a_2)} \leq C,$$

which yields

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{R}^d \setminus B_1)} \leq C. \tag{124}$$

The conclusion now follows from (120) and (124).

Remark (6.2.12)[226]: Using the approach in the proof of [124], one can prove that, for $p > 1$,

$$I_\delta(u, \alpha) \leq C \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} |x|^{p\alpha} |\mathcal{M}(\sigma, \nabla u)(x)|^p d\sigma dx, \tag{125}$$

where

$$\mathcal{M}(\sigma, \nabla u)(x) := \sup_{r>0} \frac{1}{r} \int_0^r |\nabla u(x + s\sigma) \cdot \sigma| \, ds.$$

We claim that, for $-1/p < \alpha < 1 - 1/p$, it holds

$$\int_{\mathbb{R}^d} |x|^{p\alpha} |\mathcal{M}(\sigma, \nabla u)(x)|^p d\sigma dx \leq C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u(x) \cdot \sigma|^p dx, \quad for\ all\ \sigma \in \mathbb{S}^{d-1}. \tag{126}$$

for some positive constant C independent of σ and u. Then, combining (125) and (126) yields

$$I_\delta(u, \alpha) \leq C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u|^p dx. \tag{127}$$

For simplicity, we assume that $\sigma = e_d = (0, \cdots, 0, 1) \in \mathbb{R}^d$ and prove (126). We have, for any bounded interval (a, b) and for any $\acute{x} \in \mathbb{R}^{d-1}$

$$\int_a^b (|\acute{x}|+|s|)^{p\alpha}\, ds \left(\int_a^b (|\acute{x}|+|s|)^{-p\alpha/(p-1)}\, ds\right)^{p-1} \leq C, \qquad (128)$$

for some positive constant C independent of (a,b) and $\acute{x}$ since $-1/p < \alpha < 1-1/p$. Applying the theory of maximal functions with weights due to Muckenhoupt [231] (see also [230] , which holds whenever the weight satisfies (128), we obtain

$$\int_{\mathbb{R}^d} |x|^{p\alpha} |\mathcal{M}(e_d,\nabla u)(x)|^p dx \leq C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (|\acute{x}|+|x_d|)^{p\alpha} |\mathcal{M}(e_d,\nabla u)(\acute{x},x_d)|^p dx_d d\acute{x}$$
$$\leq C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (|\acute{x}|+|x_d|)^{p\alpha} \left|\partial_{x_d} u(\acute{x},x_d)\right|^p dx_d d\acute{x}$$
$$\leq C \int_{\mathbb{R}^d} |x|^{p\alpha} |\nabla u|^p dx.$$

The claim (126) is proved.

We present some results in the spirit of Theorems (6.2.5) and (6.2.10) for a smooth bounded domain Ω. As a consequence of Theorem (6.2.5) and the extension argument in the proof of Lemma (6.2.6), we obtain

Proposition (6.2.13)[226]: Let $d \geq 1, 1 \leq p \leq d$, $\Omega \Subset B_R$ a smooth open subset of $\mathbb{R}^d$, and $u \in L^p(\Omega)$.We have

i) if $1 \leq p < d$, then

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq C_\Omega \left(I_\delta(u,\Omega) + \|u\|^p_{L^p(\Omega)} + \delta^p\right),$$

ii) if $p > d$ and supp $u \subset \bar{\Omega} \setminus B_r$, then

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq C_\Omega \left(I_\delta(u,\Omega) + \|u\|^p_{L^p(\Omega)} + r^{d-p}\delta^p\right),$$

iii) if $p = d \geq 2$, then

$$\int_{\Omega \setminus B_r} \frac{|u(x)|^d}{|x|^d \ln^d(2R/|x|)} dx \leq C_\Omega \left(I_\delta(u,\Omega) + \|u\|^p_{L^p(\Omega)} + \ln(2R/r)\, \delta^d\right),$$

iv) if $p = d \geq 2$ and supp $u \subset \Omega \setminus B_r$, then

$$\int_{\Omega \cap B_R} \frac{|u(x)|^d}{|x|^d \ln^d(2|x|/r)} dx \leq C_\Omega \left(I_\delta(u,\Omega) + \|u\|^p_{L^p(\Omega)} + \ln(2R/r)\, \delta^d\right),$$

Here C_Ω denotes a positive constant depending only on p and Ω.

Using Theorem (6.2.1), we derive

Proposition (6.2.14)[226]: Let $d \geq 2, 1 < p < d, q \geq 1, \tau > 0$, $0 < a \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$, $0 \in \Omega \subset B_R$ a smooth bounded open subset of $\mathbb{R}^d$, and $u \in L^p(\Omega)$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - 1}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right),$$

and, with and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma \leq 1.$$

We have

i) if $1/\tau + \gamma/d > 0$, then

$$\left(\int_\Omega |x|^{\gamma\tau} |u|^\tau dx\right)^{1/\tau} \leq C\left(I_\delta(u,\Omega,\alpha) + \|u\|^p_{L^p(\Omega)} + \delta^p\right)^{a/p} \left\||x|^\beta u\right\|^{(1-a)}_{L^q(\Omega)},$$

ii) if $1/\tau + \gamma/d < 0$ and supp $u \subset \Omega \setminus \{0\}$, then

$$\left(\int_{\Omega} |x|^{\gamma\tau} |u|^{\tau} dx\right)^{1/\tau} \le C\left(I_{\delta}(u,\Omega,\alpha) + \|u\|^{p}_{L^{p}(\Omega)} + \delta^{p}\right)^{a/p} \left\||x|^{\beta}u\right\|^{(1-a)}_{L^{q}(\Omega)},$$

iii) if $1/\tau + \gamma/d = 0$ and $\tau > 1$, then

$$\left(\int_{\Omega \setminus B_r} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2R/|x|)} |u|^{\tau} dx\right)^{1/\tau}$$
$$\le C\left(I_{\delta}(u,\Omega,\alpha) + \|u\|^{p}_{L^{p}(\Omega)} + \delta^{d}\ln(2R/r)\right)^{a/p} \left\||x|^{\beta}u\right\|^{(1-a)}_{L^{q}(\Omega)},$$

iv) if $1/\tau + \gamma/d = 0$, $\tau > 1$, and supp $u \subset \Omega \setminus B_r$, then

$$\left(\int_{\Omega} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2|x|/r)} |u|^{\tau} dx\right)^{1/\tau}$$
$$\le C\left(I_{\delta}(u,\Omega,\alpha) + \|u\|^{p}_{L^{p}(\Omega)} + \delta^{d}\ln(2R/r)\right)^{a/p} \left\||x|^{\beta}u\right\|^{(1-a)}_{L^{q}(\Omega)}.$$

Here C denotes a positive constant independent of u and δ.

Proof: Let v be the extension of u in $\mathbb{R}^d$ as in the proof of Lemma (6.2.6). As in the proof of Lemma (6.2.6), we have, since $0 \in \Omega$,

$$I_{2\delta}(v,\alpha) \le C\left(I_{\delta}(u,\Omega,\alpha) + \|u\|_{L^{p}(\Omega)}\right).$$

We also have, since $0 \in \Omega$,

$$\left\||x|^{\beta}v\right\|_{L^{q}(\mathbb{R}^d)} \le C\left\||x|^{\beta}u\right\|_{L^{q}\ (\Omega)}.$$

The conclusion now follows from Theorem (6.2.10).

Corollary (6.2.15)[239]: Let $\epsilon > 0$. There exists $C = C(1+\epsilon, 1+\epsilon) > 0$, depending only on $1+2\epsilon$ and $1+\epsilon$ such that, for all $\epsilon > 0$,

$$(|x| + |a+\epsilon|)^{1+\epsilon} \le (1+\epsilon)|a|^{1+\epsilon} + \frac{C}{\epsilon^{\epsilon}}|a+\epsilon|^{1+\epsilon}, \quad \text{for all } a, a+\epsilon \in \mathbb{R}. \quad (129)$$

Proof: Since (129) is clear in the case $|a+\epsilon| \ge |a|$ and in the case $a+\epsilon = 0$, by rescaling and considering $x = |a|/|a+\epsilon|$, it suffices to prove, for $C = C(1+2\epsilon,\ 1+\epsilon)$ large enough, that

$$(x+1)^{1+\epsilon} \le (1+\epsilon)x^{1+\epsilon} + \frac{C}{\epsilon^{\epsilon}}, \quad \text{for all } x \ge 1. \quad (130)$$

Set

$$f(x) = (x+1)^{1+\epsilon} - (1+\epsilon)x^{1+\epsilon} - \frac{C}{\epsilon^{\epsilon}} \quad \text{for } x > 0$$

We have

$$\acute{f}(x) = (1+\epsilon)(x+1)^{\epsilon} - (1+\epsilon)^2 x^{\epsilon} \quad \text{and} \quad \acute{f}(x) = 0 \text{ if and only if } x = x_0$$
$$:= \left((1+\epsilon)^{\frac{1}{\epsilon}} - 1\right)^{-1}.$$

One can check that

$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to 1} f(x) < 0 \text{ if } C = C(1+2\epsilon, 1+\epsilon) \text{ is large enough.} \quad (131)$$

and

$$f(x_0) = (1+\epsilon)x_0^{\epsilon} - \frac{C}{\epsilon^{\epsilon}}. \quad (132)$$

If $(1+\epsilon)^{\frac{1}{\epsilon}} > 2$ then $x_0 < 1$ and (130) follows from (131). Otherwise $1 \le s := (1+\epsilon)^{\frac{1}{\epsilon}} \le 2$. By the mean value theorem, we have

$$s^{\epsilon} - 1 \leq (s-1) \max_{0\leq\epsilon\leq1} (\epsilon)\,(1+\epsilon)^{\epsilon-1} \text{ for } 1 \leq s \leq 2.$$

We derive from (132) that, with $C = (1+2\epsilon)\left[\max_{0\leq\epsilon\leq1}(\epsilon)(1+\epsilon)^{\epsilon-1}\right]^{\epsilon}$,

$$f(x_0) < 0.$$

The conclusion now follows from (131).

Corollary (6.2.16)[239]: (Improved Hardy inequality). Let $\epsilon \geq 0$, and $u_s \in L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})$. We have

(i) if $\epsilon \geq 0$ and supp $u_s \subset B_{1+2\epsilon}$, then

$$\int_{\mathbb{R}^{1+2\epsilon}}^{1} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon}} dx \leq C \sum_s \left(I_\delta(u_s) + (1+2\epsilon)^{\epsilon}\delta^{1+\epsilon}\right),$$

(ii) if $\epsilon > 0$ and supp $u_s \subset \mathbb{R}^{1+2\epsilon}\backslash B_{1+\epsilon}$ then

$$\int_{\mathbb{R}^{1+2\epsilon}}^{1} \sum_s \frac{|u_s(x)|^{1+2\epsilon}}{|x|^{1+2\epsilon}} dx \leq C \sum_s \left(I_\delta(u_s) + (1+\epsilon)^{-\epsilon}\delta^{1+2\epsilon}\right),$$

(iii) if $\epsilon \geq 0$ and supp $u_s \subset B_{1+2\epsilon}$, then

$$\int_{\mathbb{R}^{2+\epsilon}\backslash B_{1+\epsilon}}^{1} \sum_s \frac{|u_s(x)|^{2+\epsilon}}{|x|^{2+\epsilon}\ \ln^{2+\epsilon}(2(1+2\epsilon)/|x|)} dx$$
$$\leq C \sum_s \left(I_\delta(u_s) + \ln(2(1+2\epsilon)/(1+\epsilon))\,\delta^{2+\epsilon}\right),$$

(iv) if $\epsilon \geq 0$ and supp $u_s \subset \mathbb{R}^{2+\epsilon}\backslash B_{1+\epsilon}$, then

$$\int_{B_{1+\epsilon}}^{1} \sum_s \frac{|u_s(x)|^{2+\epsilon}}{|x|^{2+\epsilon}\ \ln^{2+\epsilon}(2|x|/(1+\epsilon))} dx$$
$$\leq C \sum_s \left(I_\delta(u_s) + \ln(2(1+2\epsilon)/(1+\epsilon))\,\delta^{2+\epsilon}\right),$$

where C denotes a positive constant depending only on $2+\epsilon$.

Proof: Let $m, n \in \mathbb{Z}$ be such that

$$2^{n-1} \leq 1+2\epsilon < 2^{n} \quad \text{and} \quad 2^{m} \leq 1+\epsilon < 2^{m+1}.$$

It is clear that $n - m \geq 1$. By (73) of Lemma (6.2.3), we have, for all $k \in Z$,

$$\int_{A_k} \sum_s \left| u_s(x) - \int_{A_k} u_s \right|^{1+\epsilon} dx \leq C \sum_s \left(2^{-(\epsilon)k}\, I_\delta(u_s, A_k) + \delta^{1+\epsilon}\right).$$

Here and in what follows in this proof, C denotes a positive constant independent of k, u_s, and δ. This implies

$$2^{-(1+\epsilon)k} \int_{A_k} \sum_s \left| u_s(x) - \int_{A_k} u_s \right|^{1+\epsilon} dx \leq C \sum_s \left(I_\delta(u_s, A_k) + 2^{(\epsilon)k}\delta^{1+\epsilon}\right).$$

It follows that

$$2^{-(1+\epsilon)k} \int_{A_k} \sum_s |u_s(x)|^{1+\epsilon}\, dx$$
$$\leq C 2^{\epsilon k} \sum_s \left|\left|\int_{A_k} u_s\right|\right|^{1+\epsilon} + C \sum_s \left(I_\delta(u_s, A_k) + 2^{(\epsilon)k}\delta^{1+\epsilon}\right). \qquad (133)$$

Step 1: Proof of i): Summing (133) with respect to k from $-\infty$ to n, we obtain

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon}} dx$$

$$\leq C \sum_{k=-\infty}^{n} 2^{(\epsilon)k} \sum_s \left| \int_{A_k} u_s \right|^{1+\epsilon} + C \sum_s I_\delta(u_s) + C 2^{(\epsilon)n} \delta^n, \tag{134}$$

since $\epsilon > 0$. We also have, by (73), for $k \in \mathbb{Z}$,

$$\sum_s \left| \int_{A_k} u_s - \int_{A_{k+1}} u_s \right| \leq C \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon}\right)^{1/1+\epsilon}$$

This implies

$$\left| \int_{A_k} \sum_s u_s \right| \leq \sum_s \left| \int_{A_{k+1}} u_s \right| + C \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon}\right)^{1/1+\epsilon}.$$

Applying Corollary (6.2.15), we have

$$\sum_s \left| \int_{A_k} u_s \right|^{1+\epsilon} \leq \frac{2^{1+\epsilon}}{1+2^\epsilon} \sum_s \left| \int_{A_{k+1}} u_s \right|^{1+\epsilon} + C \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon}\right).$$

It follows that, with $1+\epsilon = 2/1+2^\epsilon < 1$,

$$2^{\epsilon k} \sum_s \left| \int_{A_k} u_s \right|^{1+\epsilon}$$

$$\leq (1+\epsilon) 2^{\epsilon(k+1)} \sum_s \left| \int_{A_{k+1}} u_s \right|^{1+\epsilon} + C \sum_s \left(I_\delta(u_s, A_k \cup A_{k+1}) + 2^{\epsilon k} \delta^{1+\epsilon}\right).$$

We derive that

$$\sum_{k=-\infty}^{n} \sum_s 2^{\epsilon k} \left| \int_{A_k} u_s \right|^{1+\epsilon} \leq C \sum_{k=-\infty}^{n} \sum_s I_\delta(u_s, A_k \cup A_{k+1}) + C 2^{\epsilon n} \delta^{1+\epsilon}. \tag{135}$$

A combination of (134) and (135) yields

$$\int_{\mathbb{R}^{1+2\epsilon}} \sum_s \left| \frac{u_s(x)}{x} \right|^{1+2\epsilon} dx \leq C \sum_s I_\delta(u_s) + C 2^{\epsilon n} \delta^{1+\epsilon},$$

The conclusion of **i**) follows.

Step 2: Proof of ii): Summing (133) with respect to k from m to $+\infty$, we obtain

$$\int_{\mathbb{R}^{1+\epsilon}} \sum_s \left| \frac{u_s(x)}{x} \right|^{1+2\epsilon} dx$$

$$\leq C \sum_{k=m}^{+\infty} \sum_s 2^{-\epsilon k} \left| \int_{A_k} u_s \right|^{1+2\epsilon} C I_\delta(u_s) + C 2^{-em} \delta^{1+2\epsilon}, \tag{136}$$

since $\epsilon > 0$. We also have, by (73), for $k \in \mathbb{Z}$,

$$\sum_s \left| \int_{A_k} u_s - \int_{A_{k+1}} u_s \right| \leq C \sum_s \left(2^{\epsilon k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+2\epsilon}\right)^{1/1+2\epsilon}.$$

This implies that

$$\left| \int_{A_{k+1}} \sum_s u_s \right| \leq \sum_s \left| \int_{A_k} u_s \right| + C \sum_s \left(2^{\epsilon k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+2\epsilon}\right)^{1/1+2\epsilon}.$$

Applying Corollary (6.2.15), we have

$$\sum_s \left|\iint_{A_{k+1}} u_s\right|^{1+2\epsilon} \leq \frac{1+2^{-\epsilon}}{2^{1-\epsilon}} \sum_s \left|\iint_{A_k} u_s\right|^{1+2\epsilon} + C(2^{\epsilon k}\, I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+2\epsilon}).$$

It follows that, with $1+\epsilon = (1+2^{-\epsilon})/2 < 1$,

$$2^{-\epsilon(k+1)} \sum_s \left|\iint_{A_{k+1}} u_s\right|^{1+2\epsilon}$$
$$\leq (1+\epsilon)2^{-\epsilon k} \sum_s \left|\iint_{A_k} u_s\right|^{1+2\epsilon} + C \sum_s \left(I_\delta(u_s, A_k \cup A_{k+1}) + 2^{-\epsilon k} \delta^{1+2\epsilon}\right).$$

We derive that

$$\sum_{k=m}^{+\infty} \sum_s 2^{-\epsilon k} \left|\iint_{A_k} u_s\right|^{1+2\epsilon} \leq C \sum_s I_\delta(u_s) + C 2^{-\epsilon m} \delta^{1+2\epsilon}. \tag{137}$$

A combination of (136) and (137) yields

$$\int_{\mathbb{R}^{1+\epsilon}} \sum_s \left|\frac{u_s(x)}{x}\right|^{1+2\epsilon} dx \leq C \sum_s I_\delta(u_s) + C 2^{-\epsilon m} \delta^{1+2\epsilon}$$

The conclusion of **ii**) follows.

Step 3: Proof of iii): Let $\alpha > 0$. Summing (133) with respect to k from m to n, we obtain

$$\int_{\{2^m < |x| < 2^n\}} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon} \ln^{2+\epsilon}(2(1+2\epsilon)/|x|)} dx$$
$$\leq C \sum_{k=m}^{n} \sum_s \frac{1}{(n-k+1)^{2+\epsilon}} \left|\iint_{A_k} u_s\right|^{1+2\epsilon} + C \sum_s I_\delta(u_s) + C(n-m)\delta^{1+2\epsilon} \tag{138}$$

We also have, by (73), for $k \in \mathbb{Z}$,

$$\left|\iint_{A_k} \sum_s u_s\right| \leq \sum_s \left|\iint_{A_{k+1}} u_s\right| + C \sum_s \left(I_\delta(u_s, A_k \cup A_{k+1})^{1/1+\epsilon} + \delta\right) \tag{139}$$

By applying Corollary (6.2.15) with

$$1 = \frac{(1+\epsilon)^\epsilon}{((1+\epsilon)/2)^{1+\epsilon}},$$

it follows from (139) that, for $\epsilon \geq 0$,

$$\frac{1}{(\epsilon+1)^{1+\epsilon}} \sum_s \left|\iint_{A_{m+\epsilon}} u_s\right|^{1+2\epsilon} \leq \frac{1}{(\epsilon+1/2)^{1+\epsilon}} \sum_s \left|\iint_{A_{m+\epsilon+1}} u_s\right|^{1+2\epsilon}$$
$$+C(\epsilon+1)^{-1} \sum_s \left(I_\delta(u_s, A_{m+\epsilon} \cup A_{m+\epsilon+1}) + \delta^{1+2\epsilon}\right) \tag{140}$$

We have, $\epsilon \geq 0$,

$$\frac{1}{(\epsilon+1)^{1+\epsilon}} - \frac{1}{(\epsilon+3/2)^{1+\epsilon}} \sim \frac{1}{(\epsilon+1)^{1+\epsilon}}. \tag{141}$$

Taking $\epsilon = 0$ and combining (140) and (141) yield

$$\sum_{\epsilon=0}^{m+2\epsilon} \sum_s \frac{1}{(\epsilon+1)^{1+\epsilon}} \left|\iint_{A_{m+\epsilon}} u_s\right|^{1+\epsilon} \leq C \sum_s I_\delta(u_s) + C(2\epsilon)\delta^{1+\epsilon}. \tag{142}$$

From (138) and (142), we obtain

$$\int_{\{|x|>2^m\}} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon} \ln^{2+\epsilon}(2(1+2\epsilon)/|x|)} dx \leq C \sum_s I_\delta(u_s) + C(2\epsilon)\delta^{1+\epsilon}.$$

This implies the conclusion of **iii**).

Step 4 Proof of iv): Let $\alpha > 0$. Summing (133) with respect to $m+\epsilon$ from m to $m+2\epsilon$, we obtain

$$\int_{\{2^m<|x|<2^{m+2\epsilon}\}} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon} \ln^{2+\epsilon}(2|x|/(1+2\epsilon))} dx$$

$$\leq \sum_{\epsilon=0}^{m+2\epsilon} \sum_s \frac{1}{(\epsilon+1)^{2+\epsilon}} \left| \int_{A_{m+\epsilon}} u_s \right|^{1+\epsilon} C I_\delta(u_s) + C\delta^{1+\epsilon} \tag{143}$$

We have, by (73), for $m+\epsilon \in Z$,

$$\left| \int_{A_{m+\epsilon+1}} \sum_s u_s \right|$$

$$\leq \sum_s \left| \int_{A_{m+\epsilon}} u_s \right| + C \sum_s \left(I_\delta(u_s, A_{m+\epsilon} \cup A_{m+\epsilon+1})^{1/1+\epsilon} + \delta \right) \tag{144}$$

By applying Corollary (6.2.15) with

$$1 = \frac{1+\epsilon}{((1+\epsilon)/2)^{1+\epsilon}},$$

it follows from (144) that, for $\epsilon \geq 0$,

$$\frac{1}{(\epsilon)^{1+\epsilon}} \sum_s \left| \int_{A_{m+\epsilon}} u_s \right|^{1+\epsilon} \leq \frac{1}{(\epsilon/2)^{1+\epsilon}} \sum_s \left| \int_{A_{m+\epsilon-1}} u_s \right|^{1+\epsilon}$$

$$+C(\epsilon)^{-1} \sum_s \left(I_\delta(u_s, A_{m+\epsilon-1} \cup A_{m+\epsilon})^{1/1+\epsilon} + \delta^{1+\epsilon} \right). \tag{145}$$

We have, $\epsilon \geq 0$,

$$\frac{1}{(\epsilon)^{1+\epsilon}} - \frac{1}{(\epsilon+2/2)^{1+\epsilon}} \sim \frac{1}{(\epsilon)^{2+\epsilon}}. \tag{146}$$

Taking $\epsilon = 0$ and combining (145) and (146) yield

$$\sum_{\epsilon=1}^{m+2\epsilon} \frac{1}{(\epsilon)^{1+\epsilon}} \sum_s \left| \int_{A_{m+\epsilon-1}} u_s \right|^{1+\epsilon} \leq C \sum_s I_\delta(u_s) + C(2\epsilon)\delta^{1+\epsilon}. \tag{147}$$

From (143) and (147), we obtain

$$\int_{\{2^m<|x|<2^{m+2\epsilon}\}} \sum_s \frac{|u_s(x)|^{1+\epsilon}}{|x|^{1+\epsilon} \ln^{2+\epsilon}(2|x|/(1+2\epsilon))} dx \leq C \sum_s I_\delta(u_s) + C(2\epsilon)\delta^{1+\epsilon}$$

This implies the conclusion of **iv**).

The proof is complete.

Corollary (6.2.17)[239]: Let $\epsilon > 0$, be a smooth bounded open subset of $\mathbb{R}^{1+2\epsilon}$, and $v_s \in L^{1+\epsilon}(\Omega)$. We have

$$\left\| \sum_s u_s \right\|_{L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}(\Omega)} \leq C_\Omega \sum_s \left(I_\delta(u_s)^{1/1+\epsilon} + \|u_s\|_{L^{1+\epsilon}} + \delta \right),$$

where $\epsilon := 0$ denotes the Sobolev exponent of $1+\epsilon$.

Proof: For $\epsilon > 0$, let us set

$$\Omega_{1+\epsilon} := \{x \in \mathbb{R}^{1+2\epsilon}\colon\ \mathrm{dist}(x,\Omega) < 1+\epsilon\}.$$

Since is smooth, by [116], there exists $\epsilon > 0$ small enough and an extension U_s of u_s in $\Omega_{1+\epsilon}$ such that

$$\sum_s I_\delta(U_s,\Omega) \le C \sum_s I_\delta(u_s,\Omega) \quad \text{and} \quad \left\|\sum_s U_s\right\|_{L^{1+\epsilon}\Omega_{1+\epsilon}} \le C\|u_s\|_{L^{1+\epsilon}(\Omega)}, \tag{148}$$

for $0 < \delta < 1$. Fix such a $1+\epsilon$. Let $\varphi \in C^1(\mathbb{R}^{1+2\epsilon})$ such that

$$\operatorname{supp}\varphi \subset \Omega_{2(1+\epsilon)/3}, \qquad \varphi = 1 \text{ in } \Omega_{(1+\epsilon)/3}, \qquad 0 \le \varphi \le 1 \text{ in } \mathbb{R}^{1+2\epsilon}.$$

Define $v_s = \varphi U_s$ in $\mathbb{R}^{1+2\epsilon}$. We claim that

$$\sum_s I_{2\delta}(v_s) \le C \sum_s \left(I_\delta(U_s,\Omega) + \|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)} \|u_s\|_{L^{1+\epsilon}}\right). \tag{149}$$

Indeed, set

$$f(x,y) = \frac{\delta^{1+\epsilon}}{|x-y|^{2+3\epsilon}} \sum_s \mathbb{1}_{\{|v_s(x)-v_s(y)|>2\delta\}}.$$

We estimate $I_{2\delta}(v_s)$. We have

$$\int\int_{\Omega\times\mathbb{R}^{1+2\epsilon}} f(x,y)dxdy \le \int\int_{\Omega_{(1+\epsilon)/3}\times\Omega_{(1+\epsilon)/3}} f(x,y)dxdy + \int\int_{\substack{\Omega_{1+\epsilon}\times\mathbb{R}^{1+2\epsilon}\\ \{|x-y|>(1+\epsilon)/4\}}} f(x,y)dxdy,$$

and, since $\epsilon = -1$ in $\Omega_{1+\epsilon} \setminus \Omega_{2(1+\epsilon)/3}$,

$$\int\int_{(\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon})\times\mathbb{R}^{1+2\epsilon}} f(x,y)dxdy, \le \int\int_{(\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon})\times(\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon})} f(x,y)dxdy + \int\int_{\substack{\Omega_{1+\epsilon}\times\mathbb{R}^{1+2\epsilon}\\ \{|x-y|>(1+\epsilon)/4\}}} f(x,y)dxdy$$

$$\int\int_{(\Omega_{1+\epsilon}\setminus\Omega_{1+\epsilon})\times\mathbb{R}^{1+2\epsilon}} f(x,y)dxdy \le \int\int_{(\Omega_{1+\epsilon}\setminus\Omega)\times(\Omega_{1+\epsilon}\setminus\Omega)} f(x,y)dxdy + \int\int_{\Omega_{(1+\epsilon)/3}\times\Omega_{(1+\epsilon)/3}} f(x,y)dxdy + \int\int_{\substack{\Omega_{1+\epsilon}\times\mathbb{R}^{1+2\epsilon}\\ \{|x-y|>(1+\epsilon)/4\}}} f(x,y)dxdy.$$

It is clear that, by (148),

$$\int\int_{\Omega_{(1+\epsilon)/3}\times\Omega_{(1+\epsilon)/3}} f(x,y)dxdy \le C\sum_s I_\delta(u_s,\Omega), \tag{150}$$

by the fact that $\varphi = 0$ in $\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon}$,

$$\int\int_{(\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon})\times(\mathbb{R}^{1+2\epsilon}\setminus\Omega_{1+\epsilon})} f(x,y)dxdy = 0, \tag{151}$$

and, by a straightforward computation,

$$\int\int_{\substack{\Omega_{1+\epsilon}\times\mathbb{R}^{1+2\epsilon}\\ \{|x-y|>(1+\epsilon)/4\}}} f(x,y)dxdy \le C\delta^{1+\epsilon}. \tag{152}$$

We have, for $x, y \in \mathbb{R}^{1+2\epsilon}$,

$$v_s(x) - v_s(y) = \varphi(x)\big(U_s(x) - U_s(y)\big) + U_s(y)\big(\varphi(x) - \varphi(y)\big).$$
It follows that if $|v_s(x) - v_s(y)| > 2\delta$ then either
$$\sum_s |U_s(x) - U_s(y)| \geq \sum_s \left|\varphi(x)\big(U_s(x) - U_s(y)\big)\right| > \delta$$
or
$$C \sum_s |U_s(y)||x - y| \geq \sum_s \left|U_s(y)\big(\varphi(x) - \varphi(y)\big)\right| > \delta.$$
We thus derive that
$$\begin{aligned}
&\int\int_{(\Omega_{1+\epsilon}\backslash\Omega)\times(\Omega_{1+\epsilon}\backslash\Omega)} f(x,y)dxdy \\
&\leq \int_{\substack{(\Omega_{1+\epsilon}\backslash\Omega) \\ \{|U_s(x) - U_s(y)| > \delta\}}} \int_{(\Omega_{1+\epsilon}\backslash\Omega)} \sum_s \frac{\delta^{1+\epsilon}}{|x-y|^{2+3\epsilon}} dxdy \\
&+ \int_{\substack{(\Omega_{1+\epsilon}\backslash\Omega) \\ \{|x-y| > C\delta/|U_s(y)|\}}} \int_{(\Omega_{1+\epsilon}\backslash\Omega)} \sum_s \frac{\delta^{1+\epsilon}}{|x-y|^{2+3\epsilon}} dxdy. \qquad (153)
\end{aligned}$$
A straightforward computation yields
$$\begin{aligned}
&\int_{\substack{(\Omega_{1+\epsilon}\backslash\Omega) \\ \{|x-y| > C\delta/|U_s(y)|\}}} \int_{(\Omega_{1+\epsilon}\backslash\Omega)} \sum_s \frac{\delta^{1+\epsilon}}{|x-y|^{2+3\epsilon}} dxdy \\
&\leq \int_{\Omega_{1+\epsilon}} dy \int_{\{|x-y| > C\delta/|U_s(y)|\}} \sum_s \frac{\delta^{1+\epsilon}}{|x-y|^{2+3\epsilon}} dx \\
&= C \int_{\Omega_{1+\epsilon}} \sum_s |U_s(y)|^{1+\epsilon} dy.
\end{aligned}$$
Using (148), we deduce from (153) that
$$\iint_{(\Omega_{1+\epsilon}\backslash\Omega)\times(\Omega_{1+\epsilon}\backslash\Omega)} f(x,y) = dxdy \leq C \sum_s I_\delta(u_s, \Omega) + C\|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)} \qquad (154)$$
A combination of (150), (151), (152), and (154) yields Claim (149). By applying [128] and using the fact supp $v_s \subset \Omega_{1+\epsilon}$, we have
$$\sum_s \|v_s\|_{L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}(\mathbb{R}^{1+2\epsilon})} \leq C \sum_s I_{2\delta}(v_s)^{\frac{1}{1+\epsilon}} + C\delta. \qquad (155)$$
The conclusion now follows from Claim (149).

Corollary (6.2.18)[239]: Let $\epsilon \geq 0$, and $0 \leq a \leq 1$ be such that
$$1 \geq \frac{a\epsilon}{1+2\epsilon} + 1 - a.$$
Let $\epsilon > 0$, and $\lambda > 0$ and set
$$\lambda D := \{\lambda x \in \mathbb{R}^{1+2\epsilon} \ : \ 1+\epsilon < |x| < 1+2\epsilon\}.$$
Then, for $u_s \in L^1(\lambda D)$,
$$\begin{aligned}
&\sum_s \left(\int_{\lambda D} \left|u_s - \int_{\lambda D} u_s\right|^{1+\epsilon} dx\right)^{1/1+\epsilon} \\
&\leq C \sum_s \left(\lambda^{-\epsilon} I_\delta(u_s, \lambda D) + \delta^{1+\epsilon}\right)^{a/1+\epsilon} \left(\int_{\lambda D} \left|u_s - \int_{\lambda D} u_s\right|^{1+\epsilon} dx\right)^{(1-a)/1+\epsilon},
\end{aligned}$$

for some positive constant C independent of u_s, λ, and δ.

Proof: Let $1+\epsilon, \sigma, \epsilon > 0$, be such that

$$\frac{1}{1+\epsilon} \geq \frac{a}{\sigma} + \frac{1-a}{1+\epsilon}.$$

We have, by a standard interpolation inequality, that

$$\sum_s \left(\int_{\lambda D} \left| u_s - \int_{\lambda D} u_s \right|^{1+\epsilon} dx \right)^{1/1+\epsilon} \leq \sum_s \left(\int_{\lambda D} \left| u_s - \int_{\lambda D} u_s \right|^{\sigma} dx \right)^{a/\sigma} \left(\int_{\lambda D} \left| u_s - \int_{\lambda D} u_s \right|^{1+\epsilon} dx \right)^{(1-a)/(1+\epsilon)}.$$

Applying this inequality with $\sigma = \frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}$ and $\epsilon = 0$ and using Corollary (6.2.18), one obtains the conclusion.

Corollary (6.2.19)[239]: Let $\epsilon \geq 0$, $0 < a \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{a-\alpha}{1+\epsilon} + \frac{(a-\beta)(\alpha-1)-\gamma}{1+2\epsilon} = 0, \tag{156}$$

and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma \leq 1.$$

Set, for $k \in \mathbb{Z}$,

$$\sum_s I_\delta(k, u_s) := \begin{cases} \sum_s I_\delta(u_s, A_k \cup A_{k+1}, \alpha) + 2^{k(\alpha(1+\epsilon)+\epsilon)\delta^{1+\epsilon}} & \text{if } \epsilon > 0, \\ \sum_s \| |x^\alpha| \nabla u_s \|^{1+\epsilon}_{L^{1+\epsilon}(A_k \cup A_{k+1})} & \text{otherwise.} \end{cases} \tag{157}$$

We have, for $u_s \in L^{1+\epsilon}_{\text{loc}}(\mathbb{R}^{1+2\epsilon})$ and $m, n \in \mathbb{Z}$ with $m < n$,

i) if $1/1+\epsilon+\gamma/1+2\epsilon > 0$ and supp $u_s \subset B_{2^n}$, then

$$\sum_s \left(\int_{\mathbb{R}^{1+2\epsilon} \setminus B_{2^m}} |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx \right)^{1/1+\epsilon} \leq C \sum_s \left(\sum_{k=m-1}^{n} I_\delta(k, u_s) \right)^{a/1+\epsilon} C \left\| |x|^\beta u_s \right\|^{(1-a)}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})},$$

ii) if $1/1+\epsilon+\gamma/1+2\epsilon < 0$ and supp $u_s \in \mathbb{R}^{1+2\epsilon} \setminus B_{2^m}$, then

$$\sum_s \left(\int_{B_{2^n}} |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx \right)^{1/1+\epsilon} \leq C \sum_s \left(\sum_{k=m-1}^{n} I_\delta(k, u_s) \right)^{a/1+\epsilon} C \left\| |x|^\beta u_s \right\|^{(1-a)}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})},$$

iii) if $1/1+\epsilon+\gamma/1+2\epsilon = 0, \epsilon > 0$, and supp $u_s \subset B_{2^n}$, then

$$\sum_s \left(\int_{\mathbb{R}^{1+2\epsilon}\setminus B_{2^m}} \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\epsilon}(2^{n+1}/|x|)} |x|^{1+\epsilon} dx \right)^{1/1+\epsilon}$$

$$\leq C \sum_s \left(\sum_{k=m-1}^{n} I_\delta(k, u_s) \right)^{a/1+\epsilon} \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)},$$

iv) if $1/1+\epsilon+\gamma/1+2\epsilon = 0, \epsilon > 0$, and supp $u_s \in \mathbb{R}^{1+2\epsilon}\setminus B_{2^m}$, then

$$\sum_s \left(\int_{B_{2^n}} \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\epsilon}(2^{n+1}/|x|)} |x|^{1+\epsilon} dx \right)^{1/1+\epsilon}$$

$$\leq C \sum_s \left(\sum_{k=m-1}^{n} I_\delta(k, u_s) \right)^{a/1+\epsilon} \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)}.$$

Here C denotes a positive constant independent of u_s, δ, k, n, and m.

Proof: We only present the proof in the case $\epsilon > 0$. The proof for the other case follows similarly, however instead of using Corollary (6.2.18), one applies Lemma (6.2.9). We now assume that $\epsilon > 0$. Since $\sigma - \alpha \geq 0$, by Corollary (6.2.18), we have

$$\sum_s \left(\int_{A_k} \left| u_s - \int_{A_k} u_s \right|^{1+\epsilon} dx \right)^{1/1+\epsilon}$$

$$\leq C \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k) + \delta^{1+\epsilon} \right)^{a/1+\epsilon} \left(\int_{A_k} |u_s|^{1+\epsilon} \right)^{(1-a)/1+\epsilon}. \qquad (158)$$

Using (156), we derive from (158) that

$$\int_{A_k} \sum_s |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx$$

$$\leq C 2^{(\gamma(1+\epsilon)+1+2\epsilon)k} \sum_s \left| \int_{A_k} u_s \right|^{1+\epsilon}$$

$$+ C \sum_s \left(I_\delta(u_s, A_k, \alpha) + 2^{k(\alpha(1+\epsilon)+\epsilon)} \delta^{1+\epsilon} \right)^a \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(A_k)}^{(1-a)(1+\epsilon)}. \qquad (159)$$

Step 1: Proof of i). Summing (159) with respect to k from m to n, we obtain

$$\int_{\{|x|>2^m\}} \sum_s |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx \leq C \sum_{k=m}^{n} \sum_s 2^{(\gamma(1+\epsilon)+1+2\epsilon)k} \left| \int_{A_k} u_s \right|^{1+\epsilon}$$

$$+ C \sum_{k=m}^{n} \sum_s \left(I_\delta(u_s, A_k, \alpha) + 2^{k(\alpha(1+\epsilon)+\epsilon)} \delta^{1+\epsilon} \right)^a \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(A_k)}^{(1-a)(1+\epsilon)}. \qquad (160)$$

By Corollary (6.2.18), we have

$$\left|\int_{A_k}\sum_s u_s\right|$$

$$\leq \sum_s \left|\int_{A_{k+1}} u_s\right|$$

$$+ C\sum_s \left(2^{-(\epsilon)} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon}\right)^{a/1+\epsilon}\left(\int_{A_k\cup A_{k+1}} |u_s|^{1+\epsilon}\right)^{\frac{(1-a)}{a}}.$$

Applying Corollary (6.2.15), we derive that

$$\sum_s \left|\int_{A_k} u_s\right|^{1+\epsilon} \leq \frac{2^{\gamma(1+\epsilon)+2+2\epsilon}}{1+2^{\gamma(1+\epsilon)+1+2\epsilon}}\sum_s \left|\int_{A_{k+1}} u_s\right|^{1+\epsilon}$$

$$+C\sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon}\right)^{a/1+\epsilon}\left(\int_{A_k\cup A_{k+1}} |u_s|^{1+\epsilon}\right)^{\frac{(1-a)}{1+\epsilon}}.$$

It follows that, with $1+\epsilon = 2/(1+2^{\gamma(1+\epsilon)+1+2\epsilon}) < 1$,

$$2^{(\gamma(1+\epsilon)+1+2\epsilon)k}\sum_s \left|\int_{A_k} u_s\right|^{1+\epsilon} \leq (1+\epsilon)2^{(\gamma(1+\epsilon)+1+2\epsilon)(k+1)}\sum_s \left|\int_{A_{k+1}} u_s\right|^{1+\epsilon}$$

$$+C\sum_s \left(I_\delta(u_s, A_k \cup A_{k+1}, \alpha) + 2^{k(\alpha(\epsilon)+\epsilon)}\delta^{1+\epsilon}\right)^a \left\||x|^\beta u_s\right\|_{L^{1+\epsilon}(A_k\cup A_{k+1})}^{(1-a)(1+\epsilon)}.$$

This yields

$$\sum_{k=m}^{n}\sum_s 2^{(\gamma(1+\epsilon)+1+2\epsilon)k}\left|\int_{A_k} u_s\right|^{1+\epsilon}$$

$$\leq C\sum_{k=m}^{n}\sum_s \left(I_\delta(u_s, A_k \cup A_{k+1}, \alpha) + 2^{k(\alpha(1+\epsilon)+\epsilon)}\delta^{1+\epsilon}\right)^a \left\||x|^\beta u_s\right\|_{L^{1+\epsilon}(A_k\cup A_{k+1})}^{(1-a)(1+\epsilon)}. \quad (161)$$

Combining (160) and (161) yields

$$\int_{\{|x|>2^m\}}\sum_s |x|^{\gamma(1+\epsilon)}|u_s|^{1+\epsilon}dx$$

$$\leq C\sum_{k=m-1}^{n}\sum_s \left(I_\delta(u_s, A_k \cup A_{k+1}, \alpha)\right.$$

$$\left. + 2^{k(\alpha(1+\epsilon)+\epsilon)}\delta^{1+\epsilon}\right)^a \left\||x|^\beta u_s\right\|_{L^{1+\epsilon}(A_k\cup A_{k+1})}^{(1-a)(1+\epsilon)}. \quad (162)$$

Applying the inequality, for $s \geq 0, \epsilon \geq -1$ with $s+\epsilon \geq 0$, and for $x_k \geq 0$ and $y_k \geq 0$,

$$\sum_{k=m}^{n} x_k^s y_k^{1+\epsilon} \leq C_{s,1+\epsilon}\left(\sum_{k=m}^{n} x_k\right)^s\left(\sum_{k=m}^{n} y_k\right)^{1+\epsilon},$$

to $s = a$ and $\epsilon = -a$, we obtain from (162) that

$$\int_{\{|x|>2^m\}} \sum_s |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx$$

$$\leq C \sum_s \left(\sum_{k=m}^{n} I_\delta(k, u_s) \right)^a \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)(1+\epsilon)} \tag{163}$$

since $\epsilon \geq 0$ thanks to the fact $\alpha - \sigma - 1 \leq 0$.

Step 2: Proof of ii): The proof is in the spirit of the proof of ii) of Corollary (6.2.16). The details are left to the research.

Step 3: Proof of iii): Fix $\xi > 0$. Summing (159) with respect to k from m to n, we obtain

$$\int_{\{|x|>2^m\}} \sum_s \frac{1}{\ln^{1+\xi}(1+\epsilon/|x|)} |x|^{\gamma(1+\epsilon)} |u_s|^{1+\epsilon} dx$$

$$\leq C \sum_{k=m}^{n} \sum_s \frac{1}{(n-k+1)^{1+\xi}} \left| \int_{A_k} u_s \right|^{1+\epsilon}$$

$$+ C \sum_{k=m}^{n} \sum_s \left(I_\delta(u_s, A_k, \alpha) + 2^{k(\alpha(1+\epsilon)+\epsilon)} \delta^{1+\epsilon} \right)^a \left\| |x|^\beta u_s \right\|_{L^{1+\epsilon}(A_k)}^{(1-a)(1+\epsilon)}. \tag{164}$$

By Corollary (6.2.18), we have

$$\left| \int_{A_k} \sum_s u_s \right|$$

$$\leq \sum_s \left| \int_{A_{k+1}} u_s \right|$$

$$+ C \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) + \delta^{1+\epsilon} \right)^{a/1+\epsilon} \left(\int_{A_k \cup A_{k+1}} |u_s|^{1+\epsilon} \right)^{\frac{(1-a)}{1+\epsilon}}.$$

Applying Corollary (6.2.15) with

$$1 + \epsilon = \frac{(n-k+1)^\xi}{(n-k+1/2)^\xi},$$

we deduce that

$$\frac{1}{(n-k+1)^\xi} \sum_s \left| \int_{A_k} u_s \right|^{1+\epsilon} \leq \frac{1}{(n-k+1/2)^\xi} \sum_s \left| \int_{A_{k+1}} u_s \right|^{1+\epsilon}$$

$$+ C(n-k+1)^{\epsilon-\xi} \sum_s \left(2^{-(\epsilon)k} I_\delta(u_s, A_k \cup A_{k+1}) \right.$$

$$\left. + \delta^{1+\epsilon} \right)^a \left(\int_{A_k \cup A_{k+1}} |u_s|^{1+\epsilon} \right)^{(1-a)}. \tag{165}$$

Recall that, for $k \leq n$ and $\xi > 0$,

$$\frac{1}{(n-k+1)^\xi} - \frac{1}{(n-k+3/2)^\xi} \sim \frac{1}{(n-k+1)^{\xi+1}}. \tag{166}$$

Taking $\xi = \epsilon$, we derive from (165) and (166) that

$$\sum_{k=m}^{n}\sum_{s} 2^{(\gamma(1+\epsilon)+1+2\epsilon)k}\frac{1}{(n-k+1)^{1+\epsilon}}\left|\int_{A_k} u_s\right|^{1+\epsilon}$$
$$\leq \sum_{k=m}^{n}\sum_{s} C\big(I_\delta(k,u_s)\big)^{a}\left\||x|^{\beta}u_s\right\|_{L^{1+\epsilon}(A_k\cup A_{k+1})}^{(1-a)(1+\epsilon)}. \tag{167}$$

Combining (164) and (167), as in (163), we obtain

$$\int_{\{|x|>2^m\}}\sum_{s}\frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\epsilon}(2^{n+1}/|x|)}|u_s|^{1+\epsilon}\,dx$$
$$\leq C\sum_{s}\left(\sum_{k=m}^{n} I_\delta(k,u_s)\right)^{a}\left\||x|^{\beta}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)(1+\epsilon)}.$$

Step 4: Proof of iv): The proof is in the spirit of the proof of iv) of Corollary (6.2.16).

The proof is complete.

Corollary (6.2.20)[239]: Let $\epsilon \geq 0$, $0 < a < 1, \alpha , \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{a-\alpha}{1+\epsilon}+\frac{(a-\beta)(\alpha-1)-\gamma}{1+2\epsilon}=0,$$

and, with $\gamma = a\sigma + (1+a)\beta$,

$$\alpha-\sigma>1 \quad \text{and} \quad \gamma\neq\alpha-1.$$

We have, for $u_s \in C^1_{1+\epsilon}\,(\mathbb{R}^{1+2\epsilon})$,

i) if $1/1+\epsilon+\gamma/1+2\epsilon>0$, then

$$\sum_{s}\left(\int_{\mathbb{R}^{1+2\epsilon}}|x|^{\gamma(1+\epsilon)}\,|u_s|^{1+\epsilon}dx\right)^{1/1+\epsilon}\leq C\sum_{s}\left\||x|^{\alpha}\nabla u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{a}\left\||x|^{\beta}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)},$$

ii) if $1/1+\epsilon+\gamma/1+2\epsilon<0$ and supp $u_s\subset\mathbb{R}^{1+2\epsilon}\setminus\{0\}$, then

$$\sum_{s}\left(\int_{\mathbb{R}^{1+2\epsilon}}|x|^{\gamma(1+\epsilon)}\,|u_s|^{1+\epsilon}dx\right)^{1/1+\epsilon}\leq C\sum_{s}\left\||x|^{\alpha}\nabla u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{a}\left\||x|^{\beta}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a)},$$

for some positive constant C independent of u_s.

Proof: The proof is in the spirit of the approach in [7] (see also [236]). Since

$$\alpha-1\neq\beta.$$

by scaling, one might assume that

$$\sum_{s}\left\||x|^{\alpha}\nabla u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}=1 \quad\text{and}\quad \sum_{s}\left\||x|^{\beta}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}=1.$$

Let $0<a_2<1$ be such that

$$|a_2-a| \text{ is small enough,} \tag{168}$$

and set

$$\frac{1}{(1+\epsilon)_2}=\frac{1}{1+\epsilon}\quad\text{and}\quad\gamma_2=a_2(\alpha-1)+(1-a_2)\beta.$$

We have

$$\frac{1}{(1+\epsilon)_2}+\frac{\gamma_2}{1+2\epsilon}$$
$$=a_2\left(\frac{1}{1+\epsilon}+\frac{\alpha-1}{1+2\epsilon}\right)+(1-a_2)\left(\frac{1}{1+\epsilon}+\frac{\beta}{1+2\epsilon}\right). \tag{169}$$

Recall that

$$\frac{a-\alpha}{1+\epsilon}+\frac{(\alpha-\beta)(\alpha-1)-\gamma}{1+2\epsilon}=0. \quad (170)$$

Since $a>0$ and $\alpha-\sigma>1$, it follows from (168) that

$$\frac{1}{1+\epsilon}+\frac{1}{(1+\epsilon)_2}=a(1-a_2)\left(-\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right)+\frac{a}{1+2\epsilon}(\alpha-\sigma-1)$$

$$>0. \quad (171)$$

We first choose a_2 such that

$$a_2<a \quad \text{if} \quad \alpha-1<\beta, \quad (172)$$

$$a<a_2 \quad \text{if} \quad \alpha-1>\beta. \quad (173)$$

Using (168), (172) and (173), we derive from (169), and (170) that

$$\gamma,\alpha\ and\ \beta=-\frac{1+2\epsilon}{1+\epsilon}. \quad (174)$$

It follows from (171), (174), and Holder's inequality that

$$\left\|\sum_s |x|^{\gamma}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}\setminus B_1)} \le C\sum_s \||x|^{\gamma_2}u_s\|_{L^{\gamma_2}(\mathbb{R}^{1+2\epsilon})}.$$

Applying Corollary (6.2.19), we have

$$\left\|\sum_s |x|^{\gamma_2}u_s\right\|_{L^{(1+\epsilon)_2}(\mathbb{R}^{1+2\epsilon})} \le C\sum_s \||x|^{\alpha}\nabla u_s\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{a_2}\||x|^{\beta}u_s\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a_2)} \le C,$$

which yields

$$\left\|\sum_s |x|^{\gamma}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}\setminus B_1)} \le C. \quad (175)$$

We next choose a_2 such that

$$a<a_2 \quad \text{if} \quad \alpha-1<\beta, \quad (176)$$

$$a_2<a \quad \text{if} \quad \alpha-1>\beta. \quad (177)$$

Using (168), (176) and (177), we derive from (169), and (170) that

$$\frac{1}{(1+\epsilon)_2}+\frac{\gamma_2}{1+2\epsilon}<\frac{1}{1+\epsilon}+\frac{\gamma}{1+2\epsilon} \quad \text{and} \quad \left(\frac{1}{1+\epsilon}-\frac{\gamma}{1+2\epsilon}\right)\left(\frac{1}{(1+\epsilon)_2}+\frac{\gamma_2}{1+2\epsilon}\right)$$

$$>0. \quad (178)$$

It follows from (171), (178), and Holder's inequality that

$$\left\|\sum_s |x|^{\gamma}u_s\right\|_{L^{1+\epsilon}(B_1)} \le C\sum_s \||x|^{\gamma_2}u_s\|_{L^{(1+\epsilon)_2}(\mathbb{R}^{1+2\epsilon})}.$$

Applying Corollary (6.2.19), we have

$$\left\|\sum_s |x|^{\gamma_2}u_s\right\|_{L^{(1+\epsilon)_2}(\mathbb{R}^{1+2\epsilon})} \le C\sum_s \||x|^{\alpha}\nabla u_s\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{a_2}\||x|^{\beta}u_s\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}^{(1-a_2)} \le C,$$

which yields

$$\left\|\sum_s |x|^{\gamma}u_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}\setminus B_1)} \le C. \quad (179)$$

The conclusion now follows from (175) and (179).

Corollary (6.2.21)[239]: Let $\epsilon>0$, $0<a\le 1, \alpha,\beta,\gamma\in\mathbb{R}$, $0\in\Omega\subset B_{1+2\epsilon}$ a smooth bounded open subset of $\mathbb{R}^{2+\epsilon}$, and $u_s\in L^{1+\epsilon}(\Omega)$ be such that

$$\frac{a-\alpha}{1+\epsilon}+\frac{(a-\beta)(\alpha-1)-\gamma}{1+2\epsilon}=0,$$

and, with and, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \le \alpha - \sigma \le 1.$$

We have

i) if $1/1+\epsilon+\gamma/2+\epsilon > 0$, then

$$\sum_s \left(\int_\Omega |x|^{\gamma(1+\epsilon)}\, |u_s|^{1+\epsilon} dx\right)^{1/1+\epsilon} \le C \sum_s \left(I_\delta(u_s,\Omega,\alpha) + \|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)} + \delta^{1+\epsilon}\right)^{a/1+\epsilon} \left\||x|^\beta u_s\right\|^{(1-a)}_{L^{1+\epsilon}(\Omega)},$$

ii) if $1/1+\epsilon+\gamma/2+\epsilon < 0$ and supp $u_s \subset \Omega \setminus \{0\}$, then

$$\sum_s \left(\int_\Omega |x|^{\gamma(1+\epsilon)}\, |u_s|^{1+\epsilon} dx\right)^{1/1+\epsilon} \le C \sum_s \left(I_\delta(u_s,\Omega,\alpha) + \|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)} + \delta^{1+\epsilon}\right)^{a/1+\epsilon} \left\||x|^\beta u_s\right\|^{(1-a)}_{L^{1+\epsilon}(\Omega)},$$

iii) if $1/1+\epsilon+\gamma/2+\epsilon = 0$ and $\epsilon > 0$, then

$$\sum_s \left(\int_{\Omega \setminus B_{1+\epsilon}} \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\epsilon}(2(1+2\epsilon)/|x|)} |u_s|^{1+\epsilon} dx\right)^{1/1+\epsilon} \le C \sum_s \Big(I_\delta(u_s,\Omega,\alpha) + \|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)}$$
$$+ \delta^{2+\epsilon} \ln(2(1+2\epsilon)/(1+\epsilon))\Big)^{a/1+\epsilon} \left\||x|^\beta u_s\right\|^{(1-a)}_{L^{1+\epsilon}(\Omega)},$$

iv) if $1/1+\epsilon+\gamma/2+\epsilon = 0$, $\epsilon > 0$, and supp $u_s \subset \Omega \setminus B_{1+\epsilon}$, then

$$\sum_s \left(\int_{\Omega} \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\epsilon}(2|x|/(1+\epsilon))} |u_s|^{1+\epsilon} dx\right)^{1/1+\epsilon} \le C \sum_s \Big(I_\delta(u_s,\Omega,\alpha) + \|u_s\|^{1+\epsilon}_{L^{1+\epsilon}(\Omega)}$$
$$+ \delta^{2+\epsilon} \ln(2(1+2\epsilon)/(1+\epsilon))\Big)^{a/1+\epsilon} \left\||x|^\beta u_s\right\|^{(1-a)}_{L^{1+\epsilon}(\Omega)}.$$

Here C denotes a positive constant independent of u_s and δ.

Proof: Let v_s be the extension of u_s in $\mathbb{R}^{2+\epsilon}$ as in the proof of Corollary (6.2.17). As in the proof of Corollary (6.2.17), we have, since $0 \in \Omega$,

$$\sum_s I_{2\delta}(v_s,\alpha) \le C \sum_s \left(I_\delta(u_s,\Omega,\alpha) + \|u_s\|_{L^{1+\epsilon}(\Omega)}\right).$$

We also have, since $0 \in \Omega$,

$$\left\|\sum_s |x|^\beta v_s\right\|_{L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})} \le C \sum_s \left\||x|^\beta u_s\right\|_{L^{1+\epsilon}(\Omega)}.$$

The conclusion now follows from Corollary (6.2.19).

Section (6.3): Inequalities of Fractional Caffarelli-Kohn-Nirenberg

For $d \geq 1, p \geq 1, q \geq 1, \tau > 0, 0 \leq a \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{d}, \quad \frac{1}{p}+\frac{\alpha}{d}, \quad \frac{1}{q}+\frac{\beta}{d} > 0$$

and

$$\frac{1}{\tau}+\frac{\gamma}{d} = a\left(\frac{1}{p}+\frac{\alpha-1}{d}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{d}\right).$$

In the case $a > 0$, assume in addition that, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma$$

and

$$\alpha - \sigma \leq 1 \quad \text{if} \quad \frac{1}{\tau}+\frac{\gamma}{d} = \frac{1}{p}+\frac{\alpha-1}{d}.$$

Caffarelli, Kohn, and Nirenberg [182] (see also [72]) proved the following well-known inequality

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{R}^d)} \leq C\||x|^{\alpha}\nabla u\|^{a}_{L^{p}(\mathbb{R}^d)}\||x|^{\beta}\nabla u\|^{(1-a)}_{L^{q}(\mathbb{R}^d)} \quad \text{for } u \in C_c^1(\mathbb{R}^d). \tag{180}$$

We extend this family of inequalities to fractional Sobolev spaces $W^{s,p}$.In the case $a = 1, \tau = p$, the corresponding inequality was obtained for $\alpha = 0$ and $\gamma = -s$ in [166], [182] and for $\tau = pd/(d-sp), -(d-sp)/p < \alpha = \gamma < 0$, and $1 < p < d/s$ in [203]. For $p > 1, 0 < s < 1, \alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = \alpha$, and Ω a measurable subset of $\mathbb{R}^d$, set

$$|u|^{p}_{W^{s,p,\alpha(\Omega)}} = \int_{\Omega}\int_{\Omega}\frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x)-u(y)|^{p}}{|x-y|^{d+ps}}dxdy \leq +\infty \quad \text{for } u \in L^1(\Omega).$$

In the case $\alpha_1 = \alpha_2 = \alpha = 0$, we simply denote $|u|_{W^{s,p,0(\Omega)}}$ by $|u|_{W^{s,p(\Omega)}}$.

Let $d \geq 1, p > 1, q \geq 1, \tau > 0, 0 \leq a \leq 1, \alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{d} = a\left(\frac{1}{p}+\frac{\alpha-s}{d}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{d}\right). \tag{181}$$

In the case $a > 0$, assume in addition that, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma \tag{182}$$

and

$$\alpha - \sigma \leq s \quad \text{if} \quad \frac{1}{\tau}+\frac{\gamma}{d} = \frac{1}{p}+\frac{\alpha-s}{d}. \tag{183}$$

Then, we have

We first state a variant of Gagliardo-Nirenberg inequality for fractional Sobolev spaces.

Lemma (6.3.1)[236]: Let $d \geq 1, 0 < s < 1, p > 1, q \geq 1, \tau > 0$, and $0 < a \leq 1$ be such that

$$\frac{1}{\tau} = a\left(\frac{1}{p}-\frac{s}{d}\right)+\frac{1-a}{q}. \tag{184}$$

We have

$$\|u\|_{L^{\tau}(\mathbb{R}^d)} \leq C|u|^{a}_{W^{s,p}(\mathbb{R}^d)}\|u\|^{(1-a)}_{L^{q}(\mathbb{R}^d)} \quad \text{for } u \in C_c^1(\mathbb{R}^d),$$

for some positive constant C independent of u.

Proof: The result is essentially known. Here is a short proof of it. We first consider the case $1/p - s/d > 0$. Set $p^* := pd/(d-sp)$. We have, by Sobolev's inequality for fractional Sobolev spaces,

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C|u|_{W^{s,p}(\mathbb{R}^d)}.$$

In this proof, C denotes a positive constant independent of u. Inequality (185) is now a consequence of Holder's inequality. We next consider the case $1/p - s/d \leq 0$. Since

$$1/p - s/d \neq 1/q,$$

by a change of variables, one can assume that

$$|u|_{W^{s,p}(\mathbb{R}^d)} = \|u\|_{L^q(\mathbb{R}^d)} = 1.$$

Since $\tau > q \geq 1$ by (184), it follows from John-Nirenberg's inequality that

$$\|u\|_{L^\tau(\mathbb{R}^d)} \leq C.$$

The proof is complete.

The following result is a consequence of Lemma (6.3.1) and is used in the proof of Theorem(6.3.3).

Lemma(6.3.2)[236]: Let $d \geq 1, p > 1, 0 < s < 1, q \geq 1, \tau > 0$, and $0 < a \leq 1$ be such that

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{s}{d}\right) + \frac{1-a}{q}.$$

Let $\lambda > 0$ and $0 < r < R$ and set

$$D := \{x \in \mathbb{R}^d : \lambda r < |x| < \lambda R\}.$$

Then, for $u \in C^1(\bar{D})$,

$$\left(\int_D \left| u - \int_D u \right|^\tau dx\right)^{1/\tau} \leq C\left(\lambda^{sp-d}|u|^p_{W^{s,p}(D)}\right)^{a/p}\left(\int_D |u|^q dx\right)^{(1-\alpha)/q} \quad (185)$$

for some positive constant C independent of u and λ.

Proof: By scaling, one can assume that $\lambda = 1$. Let $0 < \acute{s} \leq s$ and $\acute{\tau} \geq \tau$ be such that

$$\frac{1}{\acute{\tau}} = a\left(\frac{1}{p} - \frac{\acute{s}}{d}\right) + \frac{1-a}{q}.$$

From Lemma (6.3.1), we derive that

$$\left\| u - \int_D u \right\|_{L^{\acute{\tau}}(D)} \leq C|u|^a_{W^{\acute{s},p}(D)}\|u\|^{1-a}_{L^q(D)}.$$

The conclusion now follows from Jensen's inequality and the fact $|u|_{W^{\acute{s},p}(D)} \leq C|u|_{W^{s,p}(D)}$.

We are ready to give

Theorem(6.3.3)[236]: Let $d \geq 1, p > 1, 0 < s < 1, q \geq 1, \tau > 0, 0 < a \leq 1, \alpha_1, \alpha_2, \alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha = \alpha_1 + \alpha_2$, and (181), (182), and (183) hold. We have

i) if $1/\tau + \gamma/d > 0$, then

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \leq C|u|^a_{W^{s,p,\alpha}(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d),$$

ii) if $1/\tau + \gamma/d < 0$, then

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \leq C|u|^a_{W^{s,p,\alpha}(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d \backslash \{0\}),$$

Assertion ii) was established in [166] for $a = 1, \tau = p, \alpha_1 = \alpha_2 = 0$, and $\gamma = -s$. Note that the conditions

$$\frac{1}{p} + \frac{\alpha}{d}, \quad \frac{1}{q} + \frac{\beta}{d} > 0$$

are not required in Theorem (6.3.3). Without these conditions, the RHSs in the estimates of Theorem (6.3.3) are finite for $u \in C_c^1(\mathbb{R}^d)$.The case $1/\tau + \gamma/d = 0$ will be considered. In contrast with the mentioned results on fractional Sobolev spaces where the condition $\alpha_1 = \alpha_2 = \alpha/2$ is used.

The idea of the proof is quite elementary and inspired by the work [182]. In the case $0 \le \alpha - \sigma \le s$, the proof uses a variant of Gagliardo-Nirenberg's interpolation inequality for fractional Sobolev spaces (Lemma(6.3.2)) and is as follows. We decompose $\mathbb{R}^d$ into annuli A_k defined by

$$A_k := \{x \in \mathbb{R}^d : 2^k \le |x| < 2^{k+1}\},$$

and apply the interpolation inequality to have

$$\left(\fint_{A_k} \left|u - \fint_{A_k} u\right|^\tau dx\right)^{1/\tau} \le C\left(2^{-(d-sp)k}|u|_{W^{s,p}(A_k)}\right)^{a/p}\left(\fint_{A_k}|u|^q dx\right)^{(1-a)/q}.$$

Here and in what follows, we denote

$$\fint_D v = \frac{1}{|D|}\int_D v dx$$

for a measurable subset D of $\mathbb{R}^d$ and for $v \in L^1(D)$. Using again the interpolation inequality in a slightly different way, we can obtain appropriate estimates for the averages and derive the desired conclusion. The proof in the case $\alpha - \sigma > s$ is via interpolation and has its roots in [182]. Similar ideas are used in [226] to obtain several improvements of (180) in the classical setting. In the case $1 < p < d, \alpha = 0$, and $\sigma > -1$, one can derive (180) using the results in [109], [134]and [182] (see Remark (6.3.4)).

We present the proof of Theorem (6.3.3).We discuss the case $1/\tau + \gamma/d = 0$.

We first state a variant of Gagliardo-Nirenberg inequality for fractional Sobolev spaces.

Proof (1): In the case $\alpha - \sigma \le s$. By Lemma (6.3.2),we have, for $k \in \mathbb{Z}$,

$$\left(\fint_{A_k} \left|u - \fint_{A_k} u\right|^\tau dx\right)^{1/\tau} \le C\left(2^{-(d-sp)k}|u|^p_{W^{s,p}(A_k)}\right)^{a/p}\left(\fint_{A_k}|u|^q dx\right)^{(1-a)/q}. \quad (186)$$

Using (181), we derive from (186) that

$$\int_{A_k}|x|^{\gamma\tau}|x|^\tau dx \le C2^{-(\gamma\tau+d)k}\left|\fint_{A_k} u\right|^\tau + C|u|^{a\tau}_{W^{s,p,\alpha}(A_k)}\left\||x|^\beta u\right\|^{(1-a)\tau}_{L^q(A_k)}. \quad (187)$$

Let $m, n \in \mathbb{Z}$ be such that $m \le n - 2$. Summing (187) with respect to k from m to n, we obtain

$$\int_{\{2^m<|x|<2^{n+1}\}}|x|^{\gamma\tau}|u|^\tau dx \le C\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^\tau + C\sum_{k=m}^{n}|u|^{a\tau}_{W^{s,p,\alpha}(A_k)}\left\||x|^\beta u\right\|^{(1-a)\tau}_{L^q(A_k)}. \quad (188)$$

Step 1: Proof of i). Choose n such that

$$\operatorname{supp} u \subset B_{2^n}.$$

We have

$$\left|\int_{A_k} u - \int_{A_{k+1}} u\right|^{\tau} \le C\left(2^{-(d-sp)k}|u|^p_{W^{s,p}(A_k\cup A_{k+1})}\right)^{a\tau/p}\left(\int_{(A_k\cup A_{k+1})}|u|^q dx\right)^{(1-a)\tau/q}.$$

It follows that, with $c = [(1+2^{\gamma\tau+d})/2]^{-1} < 1$,

$$2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^{\tau} \le c2^{(\gamma\tau+d)(k+1)}\left|\int_{A_{k+1}} u\right|^{\tau} + C|u|^{a\tau}_{W^{s,p,\alpha}(A_k\cup A_{k+1})}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(A_k\cup A_{k+1})}.$$

We derive that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^{\tau} \le C\sum_{k=m}^{n}|u|^{a\tau}_{W^{s,p,\alpha}(A_k\cup A_{k+1})}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(A_k\cup A_{k+1})}. \tag{189}$$

Combining (188) and (189) yields

$$\int_{\{|x|>2^m\}}|x|^{\gamma\tau}|u|^{\tau}dx \le C\sum_{k=m}^{n}|u|^{a\tau}_{W^{s,p,\alpha}(A_k\cup A_{k+1})}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(A_k\cup A_{k+1})}.$$

One has, for $s \ge 0, t \ge 0$ with $s + t \ge 1$, and for $x_k \ge 0$ and $y_k \ge 0$,

$$\sum_{k=m}^{n} x_k^s y_k^t \le \left(\sum_{k=m}^{n} x_k\right)^{s}\left(\sum_{k=m}^{n} y_k\right)^{t}. \tag{190}$$

Applying this inequality with $s = a\tau/p$ and $t = (1-a)\tau/q$, we obtain that

$$\int_{\{|x|>2^m\}}|x|^{\gamma\tau}|u|^{\tau}dx \le C|u|^{a\tau}_{W^{s,p,\alpha}(\cup_{k=m}^{\infty}A_k)}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(\cup_{k=m}^{\infty}A_k)}, \tag{191}$$

since $a/p + (1-a)/q \ge 1/\tau$ thanks to the fact $\alpha - \sigma - s \le 0$.

Step 2: Proof of ii). Choose m such that

$$\operatorname{supp} u \cap B_{2^m} = \emptyset.$$

We have

$$\left|\int_{A_k} u - \int_{A_{k+1}} u\right|^{\tau} \le C\left(2^{-(d-sp)k}|u|^p_{W^{s,p}(A_k\cup A_{k+1})}\right)^{a\tau/p}\left(\int_{(A_k\cup A_{k+1})}|u|^q\right)^{(1-a)\tau/q}.$$

It follows that, with $c = (1+2^{\gamma\tau+d})/2 < 1$,

$$2^{(\gamma\tau+d)(k+1)}\left|\int_{A_{k+1}} u\right|^{\tau} \le c2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^{\tau} + C|u|^{a\tau}_{W^{s,p,\alpha}(A_k\cup A_{k+1})}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(A_k\cup A_{k+1})}.$$

We derive that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\left|\int_{A_k} u\right|^{\tau} \le C\sum_{k=m-1}^{n-1}|u|^{a\tau}_{W^{s,p,\alpha}\left(A_k\cup A_{k+1}\right)}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q\left(A_k\cup A_{k+1}\right)}. \tag{192}$$

Combining (188) and (192) yields

$$\int_{\{|x|<2^{n+1}\}}|x|^{\gamma\tau}|u|^{\tau}dx \le C\sum_{k=m-1}^{n-1}|u|^{a\tau}_{W^{s,p,\alpha}\left(A_k\cup A_{k+1}\right)}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q\left(A_k\cup A_{k+1}\right)}.$$

As in Step 1, we derive from (190) that

$$\int_{\{|x|<2^{n+1}\}}|x|^{\gamma\tau}|u|^{\tau}dx \le C|u|^{a\tau}_{W^{s,p,\alpha}(\cup_{k=-\infty}^{n}A_k)}\left\||x|^{\beta}u\right\|^{(1-a)\tau}_{L^q(\cup_{k=-\infty}^{n}A_k)}.$$

The proof is complete in the case $\alpha - \sigma \le s$.

We next turn to

Proof (2): In the case $\alpha - \sigma > s$. We follows the strategy in [182]. Since

$$\frac{1}{p}+\frac{\alpha-s}{d} \neq \frac{1}{q}+\frac{\beta}{d}.$$

by scaling, one might assume that

$$|u|_{W^{s,p,\alpha}(\mathbb{R}^d)} = 1 \quad \text{and} \quad \|u\|_{L^q(\mathbb{R}^d)} = 1.$$

It is necessary from (183) that $0 < a < 1$. Let $0 < a_1, a_2 < 1$ (a_1, a_2 are close to a and are chosen later) and $\tau_1, \tau_2 > 0$ be such that

$$\begin{aligned} &\frac{1}{\tau_1} = \frac{a_1}{p} - \frac{a_1 s}{d} + \frac{1-a_1}{q} \quad \text{if} \quad \frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} > 0, \\ \frac{1}{\tau} > &\frac{1}{\tau_1} \geq \frac{a_1}{p} - \frac{a_1 s}{d} + \frac{1-a_1}{q} \quad \text{if} \quad \frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} \leq 0, \end{aligned} \tag{193}$$

and

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1-a_2}{q}.$$

Set

$$\gamma_1 = a_1\alpha + (1-a_1)\beta \quad \text{and} \quad \gamma_2 = a_2(\alpha - s) + (1-a_2)\beta.$$

We have

$$\frac{1}{\tau_1} + \frac{\gamma_1}{d} \geq a_1\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a_1)\left(\frac{1}{q} + \frac{\beta}{d}\right) \tag{194}$$

and

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} = a_2\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a_2)\left(\frac{1}{q} + \frac{\beta}{d}\right). \tag{195}$$

Recall that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\left(\frac{1}{p} + \frac{\alpha - s}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{d}\right). \tag{196}$$

We now assume that

$$|a_1 - a| \quad \text{and} \quad |a_2 - a| \text{ are small enough,} \tag{197}$$

$$a_1 < a < a_2 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - s}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{198}$$

$$a_2 < a < a_1 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - s}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{199}$$

Using (197), (198) and (199), we derive from (194), (195), and (196) that

$$0 < \frac{1}{\tau_2} + \frac{\gamma_2}{d} < \frac{1}{\tau} + \frac{\gamma}{d} < \frac{1}{\tau_1} + \frac{\gamma_1}{d}. \tag{200}$$

Since $a > 0$ and $\alpha - \sigma > s$, it follows from (199) that

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{a}{d}(\alpha - \sigma - s) > 0 \tag{201}$$

and, if $\frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} > 0$,

$$\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1)\left(\frac{1}{p} - \frac{s}{d} - \frac{1}{q}\right) + \frac{a}{d}(\alpha - \sigma) > 0. \tag{202}$$

Since, by (193), (201), and (202),

$$1/\tau > 1/\tau_1 \ \textit{and}\ 1/\tau > 1/\tau_2,$$

it follows from (200) and Holder's inequality that

$$\| |x|^\gamma u\|_{L^\tau(\mathbb{R}^d \setminus B_1)} \leq C \| |x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{R}^d)} \quad \text{and} \quad \| |x|^\gamma u\|_{L^\tau(B_1)} \leq C \| |x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)}.$$

Applying the previous case, we have

$$\||x|^{\gamma_1}u\|_{L^{\tau_1}(\mathbb{R}^d)} \leq C|u|^{a_1}_{W^{s,p,\alpha}(\mathbb{R}^d)} \left\||x|^{\beta}u\right\|^{(1-a_1)}_{L^q(\mathbb{R}^d)} \leq C$$

and

$$\||x|^{\gamma_2}u\|_{L^{\tau_2}(\mathbb{R}^d)} \leq C|u|^{a_2}_{W^{s,p,\alpha}(\mathbb{R}^d)} \left\||x|^{\beta}u\right\|^{(1-a_2)}_{L^q(\mathbb{R}^d)} \leq C.$$

The conclusion follows.

Remark (6.3.4)[236]: In the case $0 < p < d,$ one has, for $1/2 < s < 1$ (see [182]),

$$\left\|u - \int_D u\right\|_{L^{p^*}(D)} \leq C(1-s)^{1/p}|u|_{W^{s,p}(D)}.$$

The same proof yields, with $\alpha_1 = \alpha_2 = \alpha = 0, \sigma > -s,$ and $1/\tau + \gamma/d > 0,$

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{R}^d)} \leq C(1-s)^{a/p}|u|^{a}_{W^{s,p}(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

Using the results in [109], [134], one knows that

$$\lim_{s\to 1}(1-s)^{1/p}|u|_{W^{s,p}(\mathbb{R}^d)} = C_{d,p}\|\nabla u\|_{L^p(\mathbb{R}^d)} \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

We then derive that

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{R}^d)} \leq C\|\nabla u\|^{a}_{L^p(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

Remark (6.3.5)[236]: In the case $\alpha - \sigma \leq s,$ the proof also shows that if $1/\tau + \gamma/d > 0,$then

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{R}^d\backslash B_r)} \leq C|u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d\backslash B_r)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(\mathbb{R}^d\backslash B_r)} \quad \text{for } u \in C_c^1(\mathbb{R}^d).$$

and if $1/\tau + \gamma/d < 0,$ then

$$\||x|^{\gamma}u\|_{L^{\tau}(B_r)} \leq C|u|^{a}_{W^{s,p,\alpha}(B_r)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(B_r)} \quad \text{for } u \in C_c^1(\mathbb{R}^d\backslash\{0\}).$$

for any $r > 0.$ In fact, the proof gives the result with $r = 2^j$ with $j = m$ in the first case and $j = n + 1$ in the second case. However, a change of variables yields the result mentioned here.

Lemma (6.3.6)[236]: Let $\Lambda > 1$ and $\tau > 1.$There exists $C = C(\Lambda, \tau) > 0,$ depending only on Λ and τ such that, for all $1 < c < \Lambda,$

$$(|a| + |b|)^{\tau} \leq c|a|^{\tau} + \frac{C}{(c-1)^{\tau-1}}|b|^{\tau} \quad \text{for all } a, b \in \mathbb{R}^d.$$

Theorem (6.3.7)[236]: Let $d \geq 1, p > 1, 0 < s < 1, q \geq 1, \tau > 1, 0 < a \leq 1, \alpha_1, \alpha_2, \alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha = \alpha_1, \alpha_2,$ (181) holds, and

$$0 \leq a - \sigma \leq s.$$

Let $u \in C_c^1(\mathbb{R}^d),$ and $0 < r < R.$ We have

i) if $1/\tau + \gamma/d = 0$ and supp $u \subset B_R,$ then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2B/|x|)}|u|^{\tau}dx\right)^{1/\tau} \leq C|u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

ii) if $1/\tau + \gamma/d = 0$ and supp $u \cap B_R = \emptyset,$ then

$$\left(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2|x|/r)}|u|^{\tau}dx\right)^{1/\tau} \leq C|u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d)}\left\||x|^{\beta}u\right\|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

Proof: In this proof, we use the notations in the proof of Theorem (6.3.3). We only prove the first assertion. The second assertion follows similarly as in the spirit of the proof of Theorem (6.3.3.). Fix $\xi > 0.$ Summing (187) with respect to k from m to $n,$ we obtain

$$\int_{\{|x|>2^m\}} \frac{1}{\ln^{1+\xi}(\tau/|x|)} |x|^{\gamma\tau} |u|^\tau dx$$

$$\leq C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{1+\xi}} \left| \int_{A_k} u \right|^\tau + C \sum_{k=m}^{n} |u|_{W^{s,p,\alpha}(A_k)}^{a\tau} \left\| |x|^\beta u \right\|_{L^q(A_k)}^{(1-a)\tau} \quad . (203)$$

By Lemma (6.3.2),we have

$$\left| \int_{A_k} u - \int_{A_{k+1}} u \right| \leq C \left(2^{-(d-sp)k} |u|_{W^{s,p}(A_k \cup A_{k+1})}^p \right)^{a/p} \left(\int_{(A_k \cup A_{k+1})} |u|^q \right)^{(1-a)/q}$$

Applying Lemma (6.3.6) below with $c = (n-k+1)^\xi / (n-k+1/2)^\xi$, we deduce that

$$\frac{1}{(n-k+1)^\xi} \left| \int_{A_k} u \right|^\tau \leq \frac{1}{(n-k+1/2)^\xi} \left| \int_{A_{k+1}} u \right|^\tau$$

$$+ C(n-k+1)^{\tau-1-\xi} |u|_{W^{s,p,\alpha}(A_k \cup A_{k+1})}^{a\tau} \left\| |x|^\beta u \right\|_{L^q(A_k \cup A_{k+1})}^{(1-a)\tau}. \quad (204)$$

We have, for $\xi > 0$ and $k \leq n$,

$$\frac{1}{(n-k+1)^\xi} - \frac{1}{(n-k+3/2)^\xi} \sim \frac{1}{(n-k+1)^{\xi+1}}. \quad (205)$$

Taking $\xi = \tau - 1 > 0$, we derive from (204) and (205) that

$$\sum_{k=m}^{n} \frac{1}{(n-k+1)^{1+\xi}} \left| \int_{A_k} u \right|^\tau + C \sum_{k=m}^{n} |u|_{W^{s,p,\alpha}(A_k \cup A_{k+1})}^{a\tau} \left\| |x|^\beta u \right\|_{L^q(A_k \cup A_{k+1})}^{(1-a)\tau}. \quad (206)$$

Combining (203) and (206), as in (191), we obtain

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\xi}(2^{n-1}/|x|)} |u|^\tau dx \leq C \sum_{k=m}^{n} |u|_{W^{s,p,\alpha}(A_k \cup A_{k+1})}^{a\tau} \left\| |x|^\beta u \right\|_{L^q(A_k \cup A_{k+1})}^{(1-a)\tau}.$$

Applying inequality (190) with $s = a\tau/p$ and $t = (1-a)\tau/q$, we derive that

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\xi}(2^{n-1}/|x|)} |u|^\tau dx \leq C |u|_{W^{s,p,\alpha}(\cup_{k=m}^{\infty} A_k)}^{a\tau} \left\| |x|^\beta u \right\|_{L^q(\cup_{k=m}^{\infty} A_k)}^{(1-a)\tau}.$$

This yields the conclusion.

List of Symbols

Symbol		Page
L^p	: Lebesgue space	2
inf	: infimum	2
H^1	: Hardy space	2
sup	: Supremum	6
L^2	: Hilbert space	22
L^1	: Lebesgue on the real line	25
L^∞	: essential Lebesgue space	25
max	: maximum	50
min	: minimum	52
Loc	: locally	52
dom	: domain	52
L^q	: Dual of Lebesgue space	52
H^s	: Hardy space	54
tr	: trace	61
a. e	: Almost Everywhere	68
supp	: support	68
CKN	: Caffarelli-Kohn-Nirenberg	75
BV	: Bounded Variation	83
BBM	: J. Boargain – H. Brezis and P. Mironescu	83
$W^{1,p}$	: Sobolev space	84
l. s. c	: lower semi continuous	89
$W^{1,q}$	: Dual of Sobolev space	96
$W^{s,p}$	: Sobolev space	102
mes	: measure	139
FS	: Felli-Schneider	131
$\otimes$	: Tensor product	134
H_p	: Hessian matrix	134
Ric	: Ricci	137
$L^{q,r}$	: space of all measurable functions	137
Hess	: Hessian	142
$L_{p^+,p}$	: Lorentz spaces	163
sgn	: signature	170
$L_{q,r}$	: Lorentz spaces	173
dist	: distance	177
HSM	: Hardy-Soblev-Maźya	178
$X_o^{s,p}$	: weighted Sobolev space	191
per	: perimeter	239
$W_{\text{loc}}^{1,p}$	: Sobolev space	240

References

[1] Florin Catrina and Zhi-Qiang Wang, On the Caffarelli-Kohn-Nirenberg Inequalities: Sharp Constants, Existence (and Nonexistence), and Symmetry of Extremal Functions, 229-258.

[2] Aubin, T. Problèmes isopérimétriques de Sobolev. J. Differential Geometry 11 (1976), no. 4, 573–598.

[3] Berestycki, H.; Esteban, M. Existence and bifurcation of solutions for an elliptic degenerate problem. J. Differential Equations 134 (1997), no. 1, 1–25.

[4] Brezis, H.; Lieb, E. H. A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. 88 (1983), no. 3, 486–490.

[5] Brezis, H.; Nirenberg, L. Positive solutions of nonlinear elliptic equations involving critical exponents. Comm. Pure Appl. Math. 36 (1983), no. 4, 437–477.

[6] Caffarelli, L. A.; Gidas, B.; Spruck, J. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (1989), no. 3, 271–297.

[7] Caffarelli, L. A.; Kohn, R.; Nirenberg, L. First order interpolation inequalities with weights. Compositio Math. 53 (1984), no. 3, 259–275.

[8] Caldiroli, P.; Musina, R. On the existence of extremal functions for a weighted Sobolev embedding with critical exponent. Cal. Var. Partial Differential Equations 8 (1999), no. 4, 365–387.

[9] Catrina, F.; Wang, Z.-Q. On the Caffarelli-Kohn-Nirenberg inequalities. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 6, 437–442.

[10] Catrina, F.; Wang, Z.-Q. Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in $\mathbb{R}^N$. Ann. Inst. H. Poincaré Anal. Non Linéaire, in press.

[11] Chen, W.; Li, C. Classification of solutions of some nonlinear elliptic equations. Duke Math. J. 63 (1991), no. 3, 615–622.

[12] Chou, K. S.; Chu, C. W. On the best constant for a weighted Sobolev-Hardy inequality. J. London Math. Soc. (2) 48 (1993), no. 1, 137–151.

[13] Dautray, R.; Lions, J.-L. Mathematical analysis and numerical methods for science and technology. Vol. 1. Physical origins and classical methods. Spinger, Berlin, 1985.

[14] Davies, E. B. A review of Hardy inequalities. Preprint.

[15] Gidas, B.; Ni, W. M.; Nirenberg, L. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$. Mathematical analysis and applications, Part A, 369–402. Advances in Mathematics Supplemetary Studies, 7a. Academic Press, New York–London, 1981.

[16] Hardy, G. H.; Littlewood, J. E.; Pólya, G. Inequalities. Second edition. Cambridge, University Press, 1952.

[17] Horiuchi, T. Best constant in weighted Sobolev inequality with weights being powers of distance from the origin. J. Inequal. Appl. 1 (1997), no. 3, 275–292.

[18] Korevaar, N.; Mazzeo, R.; Pacard, F.; Schoen, R. Refined asymptotics for constant scalar curvature metrics with isolated singularities. Invent. Math. 135 (1999), no. 2, 233–272.

[19] Lieb, E. H. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. of Math. (2) 118 (1983), no. 2, 349–374.

[20] Lin, C. S. Interpolation inequalities with weights. Comm. Partial Differential Equations 11 (1986), no. 14, 1515–1538.
[21] Lin, S.-S. Existence of positive nonradial solutions for nonlinear elliptic equations in annular domains. Trans. Amer. Math. Soc. 332 (1992), no. 2, 775–791.
[22] Lions, P.-L. The concentration-compactness principle in the calculus of variations, The locally compact case, Part 1 and Part 2. Ann. Inst. H. Poincaré Anal. Non Linéaire. 1 (1984), 109–145, 223–283.
[23] Lio ns, P.-L. Concentration compactness principle in the calculus of variations. The limit case. I, II. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201; 1 (1985), no. 2, 45–121.
[24] Talenti, G. Best constant in Sobolev inequality. Ann. Mat. Pura Appl. (4) 110 (1976), 353–372.
[25] Wang, Z.-Q. Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations. J. Differential Equations 159 (1999), no. 1, 102–137.
[26] Wang, Z.-Q.; Willem, M. Singular minimization problems. J. Differential Equations 161 (2000), no. 2, 307–320.
[27] Willem, M. Minimax theorems. Progress in Nonlinear Differential Equations and Their Applications, 24. Birkhäuser, Boston, 1996.
[28] Willem, M. A decomposition lemma and critical minimization problems. Preprint.
[29] Manuel Del Pino, Jean Dolbeault, Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions, J. Math. Pures Appl. 81 (2002) 847–875
[30] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differential Equations 26 (2001) 43–100.
[31] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On generalized Csiszár–Kullback inequalities, Monatsh. Math. 131 (3) (2000) 235–253.
[32] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (4) (1976) 573–598.
[33] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in Disguise, Indiana Univ. Math. J. 44 (4) (1995) 1033–1074.
[34] G.I. Barenblatt, Ya.B. Zel'dovich, Asymptotic properties of self-preserving solutions of equations of unsteady motion of gas through porous media, Dokl. Akad. Nauk SSSR (N.S.) 118 (1958) 671–674.
[35] W. Beckner, A generalized Poincaré inequality for Gaussian measures, Proc. Amer. Math. Soc. 105 (2) (1989) 397–400.
[36] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, Ann. of Math. 138 (1993) 213–242.
[37] W. Beckner, Geometric proof of Nash's inequality, Internat. Math. Res. Notices 2 (1998) 67–71.
[38] W. Beckner, Geometric asymptotics and the logarithmic Sobolev inequality, Forum Math. 11 (1999) 105–137.
[39] P. Biler, J. Dolbeault, M.J. Esteban, Intermediate asymptotics in L^1 for general nonlinear diffusion equations, Appl. Math. Lett. 15 (1) (2001) 101–107.
[40] J.A. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani, A. Unterrreiter, Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities, Monatsh. Math. 133 (1) (2001) 1–82.

[41] J.A. Carrillo, G. Toscani, Asymptotic L^1-decay of solutions of the porous medium equation to selfsimilarity, Indiana Univ. Math. J. 49 (2000) 113–141.
[42] E.A. Carlen, Superadditivity of Fisher's information and logarithmic Sobolev inequalities, J. Funct. Anal. 101 (1991) 194–211.
[43] E. Carlen, M. Loss, Sharp constant in Nash's inequality, DukeMath. J., Internat. Math. Res. Notices 7 (1993) 213–215.
[44] C. Cortázar, M. Elgueta, P. Felmer, On a semilinear elliptic problem in $\mathbb{R}^N$ with a non-Lipschitzian nonlinearity, Adv. Differential Equations 1 (2) (1996) 199–218.
[45] I. Csiszár, Information-type measures of difference of probability distributions and indirect observations, Studia Sci. Math. Hungar. 2 (1967) 299–318.
[46] M. Del Pino, J. Dolbeault, Generalized Sobolev inequalities and asymptotic behaviour in fast diffusion and porous media problems, Preprint Ceremade 9905 (1999) 1–45.
[47] A. Friedmann, S. Kamin, The asymptotic behaviour of gas in a n-dimensional porous medium, Trans. Amer. Math. Soc. 262 (2) (1980) 551–563.
[48] E. Gagliardo, Proprietà di alcune classi di funzioni in più variabili, Ricerche Mat. 7 (1958) 102–137.
[49] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.
[50] M.A. Herrero, M. Pierre, The Cauchy problem for ut = _um when 0 < m < 1, Trans. Amer. Math. Soc. 291 (1) (1985) 145–158.
[51] A. Jüngel, P.A. Markowich, G. Toscani, Decay rates for solutions of degenerate parabolic equations, in: Proceedings of the USA–Chile Conference on Nonlinear Analysis (Conf. No. 06) Electron. J. Differential Equations (2001) 189–202.
[52] S. Kamin, J.-L. Vazquez, Fundamental solutions and asymptotic behaviour for the p-Laplacian equation, Rev. Mat. Iberoamericana 4 (2) (1988) 339–354.
[53] S. Kamin, J.-L. Vazquez, Asymptotic behaviour of solutions of the porous medium equation with changing sign, SIAM J. Math. Anal. 22 (1) (1991) 34–45.
[54] S. Kullback, A lower bound for discrimination information in terms of variation, IEEE Trans. Inform. Theory 13 (1967) 126–127.
[55] C. Lederman, P.A. Markowich, On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass, Comm. Partial Differential Equations (2002), in print.
[56] M. Ledoux, The geometry of Markov diffusion generators, Ann. Fac. Sci. Toulouse VI. Sér. Math. 9 (2) (2000) 305–366.
[57] H.A. Levine, An estimate for the best constant in a Sobolev inequality involving three integral norms, Ann. Mat. Pura Appl. (4) 124 (1980) 181–197.
[58] J. Moser, A Harnack' inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964) 101–134.
[59] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.
[60] W.I. Newman, A Lyapunov functional for the evolution of solutions to the porous medium equation to selfsimilarity, I, J. Math. Phys. 25 (1984) 3120–3123.
[61] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Pisa 13 (1959) 116–162.
[62] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, Comm. Partial Differential Equations 26 (1–2) (2001) 101–174.

[63] P. Pucci, J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, Indiana Univ. Math. J. 47 (2) (1998) 501–528.
[64] J. Ralston, A Lyapunov functional for the evolution of solutions to the porous medium equation to selfsimilarity, II, J. Math. Phys. 25 (1984) 3124–3127.
[65] J. Serrin, M. Tang, Uniqueness for ground states of quasilinear elliptic equations, Indiana Univ. Math. J. 49 (3) (2000) 897–923.
[66] S.L. Sobolev, On a theorem of functional analysis, Transl. Amer. Math. Soc. (2) 34 (1963) 39–68, Translated from Math. Sb. (N.S.) 4 (46) (1938) 471–497.
[67] G. Toscani, Sur l'inégalité logarithmique de Sobolev, C. R. Acad. Sci. Paris, Sér. I Math. 324 (1997) 689–694.
[68] G. Toscani, Entropy production and the rate of convergence to equilibrium for the Fokker–Planck equation, Quart. Appl. Math. 57 (3) (1999) 521–541.
[69] J.-L. Vazquez, Asymptotic behaviour for the Porous Medium Equation, I. In a Bounded Domain, the Dirichlet Problem, II. In the whole space, Notas del Curso de Doctorado "Métodos Asintóticos en Ecuaciones de Evolución", J.E.E., to appear.
[70] F.B. Weissler, Logarithmic Sobolev inequalities for the heat-diffusion semigroup, Trans. Amer. Math. Soc. 237 (1978) 255–269.
[71] J. Scott Badle, Hardy inequalities with mixed norms, canad. Math Bull 21 (1978), 405-408.
[72] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math, 35 (1982) 771-831.
[73] E. Gagliardo, Ulteriori proprieta di alcune classi di funzion in piu varibili. Ricerche di Mat. Napoli 8 (1959) 24-51.
[74] C.S. LIN: Interpolation inequalities with weighted norms, to appear.
[75] B. MUCKENHOUPT: Hardy's inequality with weights. Studia Math. 34 (1972) 31—38.
[76] B. MUCKENHOUPT and R. WHEEDEN: Weighted norm inequalities for fractional integrals. Trans. Amer. Math. Soc. ¡92 (1974) 261—274.
[77] Rupert L. Frank, Elliott H. Lieb, and Robert Seiringer Hardy-Lieb-Thirring Inequalities for Fractional Schr¨odinger Operators, arXiv:math/0610593v2[math.SP]27 Oct 2006.
[78] M. Abramowitz, I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables. Reprint of the 1972 edition. Dover Publications, New York, 1992.
[79] H. Brezis, E. H. Lieb, Sobolev inequalities with remainder terms. J. Funct. Anal. 62 (1985), no. 1, 73–86.
[80] H. Brezis, J.-L. V´azquez, Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Comp. Madrid 10 (1997), 443–469.
[81] I. Daubechies, An uncertainty principle fermions with a generalized kinetic energy. Comm. Math. Phys. 90 (1983), 511–520.
[82] E. B. Davies, Heat kernels and spectral theory. Cambridge Tracts in Mathematics 92, Cambridge University Press, Cambridge, 1990.
[83] W.F. Donoghue, Monotone matrix functions and analytic continuation. Springer, New York-Heidelberg, 1974.
[84] T. Ekholm, R. L. Frank, On Lieb-Thirring inequalities for Schr¨odinger operators with virtual level. Comm. Math. Phys. 264 (2006), no. 3, 725–740.

[85] R.L. Frank, E.H. Lieb, R. Seiringer, Stability of relativistic matter with magnetic fields for nuclear charges up to the critical value, in preparation.
[86] I. W. Herbst, Spectral theory of the operator (p2 + m2)1/2 − Ze2/r. Comm. Math. Phys. 53 (1977), no. 3, 285–294.
[87] E. H. Lieb, The number of bound states of one-body Schroedinger operators and the Weyl problem. Proc. Sympos. Pure Math., XXXVI, 241–252. Amer. Math. Soc., Providence, R.I., 1980.
[88] E. H. Lieb, The stability of matter: from atoms to stars. Bull. Amer. Math. Soc. (N.S.) 22 (1990), no. 1, 1–49.
[89] E. H. Lieb, The Stability of Matter and Quantum Electrodynamics, Proceedings of the Heisenberg symposium, Munich, Dec. 2001, Fundamental Physics–Heisenberg and Beyond, G. Buschhorn and J. Wess, eds., pp. 53–68, Springer (2004). A modified version appears in the Milan Journal of Mathematics 71, 199–217 (2003).
[90] E. H. Lieb, M. Loss, Analysis. Second edition. Graduate Studies in Mathematics 14, American Mathematical Society, Providence, RI, 2001.
[91] E. H. Lieb, W. Thirring, Inequalities for the moments of the eigenvalues of the Schr¨odinger Hamiltonian and their relation to Sobolev inequalities. Studies in Mathematical Physics, 269–303. Princeton University Press, Princeton, NJ, 1976.
[92] E. H. Lieb, H.-T. Yau, The stability and instability of relativistic matter. Comm. Math. Phys. 118 (1988), no. 2, 177–213.
[93] M. Reed, B. Simon, Methods of modern mathematical physics. I. Functional Analysis (Revised and enlarged edition). Academic Press, New York-London, 1980.
[94] M. Reed, B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York-London, 1978.
[95] G. Rozenblyum, M. Solomyak, The Cwikel-Lieb-Rozenblyum estimator for generators of positive semigroups and semigroups dominated by positive semigroups. St. Petersburg Math. J. 9 (1998), no. 6, 1195–1211.
[96] B. Simon, Maximal and minimal Schr¨odinger forms. J. Operator Theory 1 (1979), no. 1, 37–47.
[97] B. Simon, Functional integration and quantum physics. Second edition. AMS Chelsea Publishing, Providence, RI, 2005.
[98] D. Yafaev, Sharp constants in the Hardy-Rellich inequalities. J. Funct. Anal. 168 (1999), no. 1, 121–144.
[99] Zhi-Qiang Wanga and Michel Willem, Caffarelli‑Kohn‑Nirenberg inequalities with remainder terms, Journal of Functional Analysis 203 (2003) 550‑568.
[100] Adimurthi N. Chaudhuri, M. Ramaswamy, An improved Hardy‑Sobolev inequality and its applications, Proc. Amer. Math. Soc. 130 (2002) 489‑505.
[101] G. Bianchi, H. Egnell, A note on the Sobolev inequality, J. Funct. Anal. 100 (1991) 18‑24.
[102] X. Cabre, Y. Martel, Existence versus explosion instantane´e pour des e´quations de la chaleur line´ aires avec des potentiels singuliers, C. R. Acad. Sci. Paris 329 (1999) 973‑978.
[103] G.H. Hardy, J.E. Littlewood, G. Po´ lya, Inequalities, Cambridge University Press, Cambridge, 1934.
[104] V. Ra˘ dulescu, D. Smets, M. Willem, Hardy‑Sobolev inequalities with remainder terms, Topol. Meth. Nonlinear Anal. 20 (2002) 145‑149.

[105] D. Smets and M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems, Calc. Var. Partial Differential Equations, 18 (2003), pp. 57–75.
[106] J. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential, J. Funct. Anal. 173 (2000) 103-153.
[107] H. Brezis and H-M. Nguyen, Two subtle convex nonlocal approximations of the BV -norm, Nonlinear Anal. 137 (2016) 222-245.
[108] G. Aubert, P. Kornprobst, Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems? SIAM J. Numer. Anal. 47 (2009) 844-860.
[109] J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations, IOS Press, 2001, pp. 439-455. A volume in honour of A. Bensoussan's 60th birthday.
[110] J. Bourgain, H.-M. Nguyen, A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris 343 (2006) 75-80.
[111] A. Braides, Γ-convergence for beginners, in: Oxford Lecture Series in Mathematics and its Applications, vol. 22, Oxford University Press, Oxford, 2002.
[112] H. Brezis, How to recognize constant functions. Connections with Sobolev spaces, Uspekhi Mat. Nauk 57 (2002) 59-74 (English translation in Russian Math. Surveys 57 (2002), 693-708).
[113] H. Brezis, New approximations of the total variation and filters in Imaging, Rend. Lincei 26 (2015) 223-240.
[114] H. Brezis, P. Mironescu, Sobolev Maps with Values Into the Circle, Chapter 7, Birkh¨auser (in preparation).
[115] H. Brezis, H.-M. Nguyen, On a new class of functions related to VMO, C. R. Acad. Sci. Paris 349 (2011) 157-160.
[116] H. Brezis, H.-M. Nguyen, Non-local functionals related to the total variation and applications in image processing, preprint.
[117] G. Dal Maso, An introduction to Γ-convergence, in: Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkh¨auser Boston, Inc., Boston, MA, 1993.
[118] J. Davila, On an open question about functions of bounded variation, Calc. Var. Partial Differential Equations 15 (2002) 519-527.
[119] G. Gilboa, S. Osher, Nonlocal linear image regularization and supervised segmentation, Multiscale Model. Simul. 6 (2007) 595-630.
[120] G. Gilboa, S. Osher, Nonlocal operators with applications to image processing, Multiscale Model. Simul. 7 (2008) 1005-1028.
[121] S. Kindermann, S. Osher, P.W. Jones, Deblurring and denoising of images by nonlocal functionals, Multiscale Model. Simul. 4 (2005) 1091-1115.
[122] G. Leoni, D. Spector, Characterization of Sobolev and BV Spaces, J. Funct. Anal. 261 (2011) 2926-2958.
[123] G. Leoni, D. Spector, Corrigendum to "Characterization of Sobolev and BV spaces", J. Funct. Anal. 266 (2014) 1106-1114.

[124] H.-M. Nguyen, Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689‑720.
[125] H.-M. Nguyen, Γ-convergence and Sobolev norms, C. R. Acad. Sci. Paris 345 (2007) 679‑684.
[126] H.-M. Nguyen, Further characterizations of Sobolev spaces, J. Eur. Math. Soc. (JEMS) 10 (2008) 191‑229.
[127] H.-M. Nguyen, Γ-convergence, Sobolev norms, and BV functions, Duke Math. J. 157 (2011) 495‑533.
[128] H.-M. Nguyen, Some inequalities related to Sobolev norms, Calc. Var. Partial Differential Equations 41 (2011) 483‑509.
[129] H.-M. Nguyen, Estimates for the topological degree and related topics, J. Fixed Point Theory 15 (2014) 185‑215.
[130] A. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence, Calc. Var. Partial Differential Equations 19 (2004) 229‑255.
[131] A. Ponce, An estimate in the spirit of Poincar´e’ s inequality, J. Eur. Math. Soc. (JEMS) 6 (2004) 1‑15.
[132] L.I. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D 60 (1992) 259‑268.
[133] D. Spector, On a generalization of Lp-differentiability, preprint, Oct. 2015.
[134] Jean Bourgain, Haim Brezis and Petru Mironescu, Limiting Embedding Theorems for $W^{s,p}$ when s ↑ 1 and Applications, J. Anal. Math. 87 (2002) 77–101.
[135] R.A. Adams, Sobolev spaces, Acad. Press, 1975.
[136] J. Bourgain, H. Brezis and P. Mironescu, Lifting in Sobolev spaces, J. d'Analyse 80 (2000), 37-86.
[137] S. Chang, T. Wilson, T. Wol_, Some weighted norm inequalities concerning the Schrodinger operators , Comment. Math. Helv. 60 (1985), no 2, 217-246.
[138] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of Second order, Second edition, Springer-Verlag, Berlin, Heidelberg, New-York, 1983.
[139] H. Triebel, Theory of function spaces, Birkh • auser, Basel, Boston, 1983.
[140] J. Dolbeault, M.J. Esteban, M. Loss, G. Tarantello, On the symmetry of extremals for the Caffarelli–Kohn–Nirenberg inequalities, Adv. Nonlinear Stud. 9 (2009) 713–727.
[141] J. Dolbeault, M. J. Esteban, and G. Tarantello, The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 7 (2008), pp. 313–341.
[142] V. Felli and M. Schneider, Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type, J. Differential Equations, 191 (2003), pp. 121–142.
[143] L. Landau and E. Lifschitz, Physique th´eorique. Tome III: M´ecanique quantique. Th´eorie non relativiste. (French), Deuxi`eme ´edition. Translated from russian by E. Gloukhian. ´Editions Mir, Moscow, 1967.
[144] E. H. Lieb and M. Loss, Analysis, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.
[145] C.-S. Lin and Z.-Q. Wang, Erratum to, Proc. Amer. Math. Soc., 132 (2004), p. 2183.

[146] Symmetry of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities, Proc. Amer. Math. Soc., 132 (2004), pp. 1685–1691.
[147] J. Dolbeault and M.J. Esteban, Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities, arXiv:1603.03574v1[math.AP]11 Mar 2016.
[148] M.-F. Bidaut-V´eron and L. V´eron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, Invent. Math., 106 (1991), pp. 489–539.
[149] E. A. Carlen and M. Loss, Extremals of functionals with competing symmetries, J. Funct. Anal., 88 (1990), pp. 437–456.
[150] J. Dolbeault, M. J. Esteban, and M. Loss, Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces. Preprint, to appear in Inventiones Mathematicae, 2016.
[151] Jean Dolbeaulta, Maria J. Estebana, Michael Lossb, Matteo Muratori, Symmetry for extremal functions in subcritical Caffarelli–Kohn–Nirenberg inequalities, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 133–154
[152] D. Bakry, M. Émery, Diffusions hypercontractives, in: Séminaire de probabilités, XIX, 1983/84, in: Lecture Notes in Math., vol.1123, Springer, Berlin, 1985, pp.177–206.
[153] D. Bakry, I. Gentil, M. Ledoux, Analysis and Geometry of Markov Diffusion Operators, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol.348, Springer, Cham, Switzerland, 2014.
[154] N.M. Blachman, The convolution inequality for entropy powers, IEEE Trans. Inf. Theory IT-11 (1965) 267–271.
[155] M. Bonforte, J. Dolbeault, M. Muratori, B. Nazaret, Weighted fast diffusion equations (Part I): Sharp asymptotic rates without symmetry and symmetry breaking in Caffarelli–Kohn–Nirenberg inequalities, Kinet. Relat. Models 10(1) (2017) 33–59.
[156] J.A. Carrillo, J.L. Vázquez, Fine asymptotics for fast diffusion equations, Commun. Partial Differ. Equ. 28 (2003) 1023–1056.
[157] M.H.M. Costa, A new entropy power inequality, IEEE Trans. Inf. Theory 31 (1985) 751–760.
[158] J. Dolbeault, M.J. Esteban, M. Loss, Nonlinear flows and rigidity results on compact manifolds, J. Funct. Anal. 267 (2014) 1338–1363.
[159] J. Dolbeault, M.J. Esteban, M. Loss, Interpolation inequalities on the sphere: linear vs. nonlinear flows, Ann. Fac. Sci. Toulouse Math., to appear. Preprint hal-01206975, arXiv:1509.09099.
[160] J. Dolbeault, M. Muratori, B. Nazaret, Weighted interpolation inequalities: a perturbation approach, Math. Annal. (2016) 1–34, http://dx.doi.org/10.1007/s00208-016-1480-4(first online: 6 October 2016, http :/ /link.springer.com /article /10 .1007 /s00208-016-1480-4).
[161] J. Dolbeault, G. Toscani, Nonlinear diffusions: extremal properties of Barenblatt profiles, best matching and delays, Nonlinear Anal. 138 (2016) 31–43, Nonlinear Partial Differential Equations in honor of Juan Luis Vázquez for his 70th birthday.
[162] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.
[163] G. Savaré, G. Toscani, The concavity of Rényi entropy power, IEEE Trans. Inf. Theory 60 (2014) 2687–2693.

[164] J.L. Vázquez, Smoothing and Decay Estimates for Nonlinear Diffusion Equations, Equations of Porous Medium Type, vol.33, Oxford University Press, Oxford, 2006.
[165] C. Villani, A short proof of the “concavity of entropy power”, IEEE Trans. Inf. Theory 46 (2000) 1695–1696.
[166] Rupert L. Frank, Robert Seiringer, Non-linear ground state representations and sharp Hardy inequalities, Journal of Functional Analysis 255 (2008) 3407–3430.
[167] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second ed., Academic Press, New York, 2003.
[168] F.J. Almgren, E.H. Lieb, Symmetric decreasing rearrangement is sometimes continuous, J. Amer. Math. Soc. 2 (4) (1989) 683–773.
[169] A. Alvino, Sulla diseguaglianza di Sobolev in spazi di Lorentz, Boll. Unione Mat. Ital. Sez. A (5) 14 (1977) 148–156.
[170] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved Lp Hardy inequalities with best constants, Trans. Amer. Math. Soc. 356 (6) (2004) 2169–2196.
[171] W. Beckner, Pitt’s inequality and the uncertainty principle, Proc. Amer. Math. Soc. 123 (1995) 1897–1905.
[172] M.Sh. Birman, The spectrum of singular boundary problems, Amer. Math. Soc. Trans. (2) 53 (1966) 23–80.
[173] J.E. Brothers, W.P. Ziemer, Minimal rearrangements of Sobolev functions, J. Reine Angew. Math. 384 (1988) 153–179.
[174] R.L. Frank, B. Simon, T. Weidl, Eigenvalue bounds for perturbations of Schrödinger operators and Jacobi matrices with regular ground states, Comm. Math. Phys., in press.
[175] J. Glimm, A. Jaffe, Quantum Physics. A Functional Integral Point of View, Springer, New York, 1981.
[176] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, seventh ed., Elsevier/Academic Press, Amsterdam, 2007.
[177] J. Hersch, Correspondances entre des fonctions propres et valeurs propres de l’équation de Helmholtz dans des domaines de dimensions différentes, C. R. Acad. Sci. Paris 271 (1970) 445–448.
[178] V.F. Kovalenko, M.A. Perel’muter, Ya.A. Semenov, Schrödinger operators with Ll/2W (Rl)-potentials, J. Math. Phys. 22 (5) (1981) 1033–1044.
[179] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Stud. Appl. Math. 57 (2) (1977) 93–105.
[180] Y.L. Luke, Mathematical Functions and Their Approximations, Academic Press, New York, 1975.
[181] V. Maz’ya, Analytic criteria in the qualitative spectral analysis of the Schrödinger operator, in: F. Gesztesy, et al. (Eds.), Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon’s 60th Birthday, in: Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 257–288.
[182] V. Maz’ya, T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal. 195 (2) (2002) 230–238; Erratum, J. Funct. Anal. 201 (1) (2003) 298–300.
[183] V. Maz’ya, T. Shaposhnikova, A collection of sharp dilation invariant inequalities for differentiable functions, preprint, arXiv: 0802.3209.
[184] V. Maz’ya, I.E. Verbitsky, The form boundedness criterion for the relativistic Schrödinger operator, Ann. Inst. Fourier 54 (2) (2004) 317–339.

[185] R. O'Neil, Convolution operators and L(p, q) spaces, Duke Math. J. 30 (1963) 129–142.
[186] J. Peetre, Espaces d'interpolation et théorème de Soboleff, Ann. Inst. Fourier (Grenoble) 16 (1) (1966) 279–317 (in French).
[187] Y. Pinchover, Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations, in: F. Gesztesy, et al. (Eds.), Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday, in: Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 329–356.
[188] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.
[189] L. Tartar, Imbedding theorems of Sobolev spaces into Lorentz spaces, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (3) (1998) 479–500.
[190] Bart Lomiej Dyda and Rupert L. Frank, Fractional Hardy-Sobolev-Maz'ya Inequality for Domains,
[191] R. D. Benguria, R. L. Frank, and M. Loss. The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space. Math. Res. Lett., 15(4):613{622, 2008.
[192] K. Bogdan and B. Dyda. The best constant in a fractional Hardy inequality. Math. Nachr., 284(5-6):629{638, 2011.
[193] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d-sets. Stochastic Process. Appl., 108(1):27{62, 2003.
[194] E. B. Davies. Spectral theory and differential operators, volume 42 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.
[195] B. Dyda. Fractional Hardy inequality with a remainder term. Colloq. Math., 122(1):59{67, 2011.
[196] R. L. Frank and M. Loss. Hardy-Sobolev-Maz'ya inequalities for arbitrary domains. J. Math. Pures Appl., to appear, 2011.
[197] R. L. Frank and R. Seiringer. Sharp fractional Hardy inequalities in half-spaces. In Around the research of Vladimir Maz'ya. I, volume 11 of Int. Math. Ser. (N. Y.), pages 161{167. Springer, New York, 2010.
[198] A. M. Garsia, E. Rodemich, and H. Rumsey, Jr. A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. J., 20:565{578, 1970/1971.
[199] Q.-Y. Guan and Z.-M. Ma. Reected symmetric _-stable processes and regional fractional Laplacian. Probab. Theory Related Fields, 134(4):649{694, 2006.
[200] M. Loss and C. Sloane. Hardy inequalities for fractional integrals on general domains. J. Funct. Anal., 259(6):1369{1379, 2010.
[201] V. G. Maz'ja. Sobolev spaces. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.
[202] C. A. Sloane. A fractional Hardy-Sobolev-Maz'ya inequality on the upper halfspace. Proc. Amer. Math. Soc., 139:4003{4016, 2011.
[203] B. Abdellaoui, R. Bentifour, Caffarelli–Kohn–Nirenberg type inequalities of fractional order with applications, Journal of Functional Analysis 272 (2017) 3998–4029.

[204] B. Abdellaoui, V. Felli, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 9(2) (2006) 445–484.
[205] B. Abdellaoui, I. Peral, A. Primo, A remark on the fractional Hardy inequality with a remainder term, C. R. Acad. Sci. Paris, Ser. I 352 (2014) 299–303.
[206] B. Barrios, M. Medina, I. Peral, Some remarks on the solvability of non-local elliptic problems with the Hardy potential, Commun. Contemp. Math. 16(4) (2014) 1350046.
[207] B. Barrios, I. Peral, S. Vita, Some remarks about the summability of nonlocal nonlinear problems, Adv. Nonlinear Anal. 4(2) (2015) 91–107.
[208] H. Brezis, L. Dupaigne, A. Tesei, On a semilinear equation with inverse-square potential, Selecta Math. 11 (2005) 1–7.
[209] H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, Manuscripta Math. 74 (1992) 87–106.
[210] A. Di castro, T. Kuusi, G. Palatucci, Nonlocal Harnack inequalities, J. Funct. Anal. 267(6) (2014) 1807–1836.
[211] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136(5) (2012) 521–573.
[212] E.B. Fabes, C.E. Kenig, R.P. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7(1) (1982) 77–116.
[213] M.M. Fall, Semilinear elliptic equations for the fractional Laplacian with Hardy potential, preprint, arXiv:1109.5530v4[math.AP].
[214] F. Ferrari, I. Verbitsky, Radial fractional Laplace operators and Hessian inequalities, J. Differential Equations 253(1) (2012) 244–272.
[215] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equa-tions, Dover Publications, Inc., Mineola, NY, 2006.
[216] T. Leonori, I. Peral, A. Primo, F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, Discrete Contin. Dyn. Syst. 35(12) (2015) 6031–6068.
[217] V. Maz'ya, Sobolev Spaces with Applications to Elliptic Partial Differential Equations, second edition, Grundlehren der Mathematischen Wissenschaften, vol.342, Springer, Heidelberg, 2011.
[218] R. Servadei, E. Valdinocci, Mountain Pass solutions for non-local elliptic operators, J. Math. Anal. Appl. 389(2) (2012) 887–898.
[219] H. Brezis, H-M. Nguyen, The BBM formula revisited, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 27 (2016), 515-533.
[220] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, 2010.
[221] E. De Giorgi, Deffnizione ed espressione analitica del perimetro di un insieme, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali 14 (1953), 390{393.
[222] E. De Giorgi, Su una teoria generale della misura (r-1)-dimensionale in uno spazio ad r dimensioni, Annali di Matematica Pura ed Applicata 36 (1954), 191{213,
[223] L. C. Evans and R. F. Gariepy, Measure theory and _ne properties of functions, Revised edition, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 2015.

[224] A. Ponce, D. Spector, On formulae decoupling the total variation of BV functions, Nonlinear Anal. 154 (2017), 241{257.
[225] D. Spector, Lp-Taylor approximations characterize the Sobolev space W1;p, C. R. Math. Acad. Sci. Paris 353 (2015), 327{332.
[226] Hoai-Minh Nguyen and Marco Squassina, On Hardy and Caffarelli-Kohn-Nirenberg Inequalities, preprint.
[227] J. Bourgain, H. Brezis, H-M. Nguyen, A new estimate for the topological degree, C. R. Math. Acad. Sci. Paris 340 (2005), 787{791. 2
[228] H. Brezis, H-M. Nguyen, Non-convex, non-local functionals converging to the total variation, C. R. Acad. Sci. Paris 355 (2017), 24{27. 2
[229] X. Cabr_e, Y. Martin, Existence versus instantaneous blowup for linear heat equations with singular potentials, C. R. Acad. Sci. Paris S_er. I Math. 329 (1999), 973{978. 4
[230] R. R. Coifman, C. Fe_erman, Weighted norm inequalities for maximum functions and singular integrals, Studia Math. 51 (1974), 241{250. 4, 17
[231] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972), 207{226. 4, 17
[232] H-M. Nguyen, Optimal constant in a new estimate for the degree, J. Anal. Math. 101 (2007), 367{395. 2
[233] H-M. Nguyen, A. Pinamonti, M. Squassina, E. Vecchi, New characterization of magnetic Sobolev spaces, Adv. Nonlinear Anal., to appear, https://doi.org/10.1515/anona-2017-0239. 5
[234] H-M. Nguyen, M. Squassina, Logarithmic Sobolev inequality revisited, C. R. Math. Acad. Sci. Paris 355 (2017), 447{451. 2
[235] H-M. Nguyen, M. Squassina, Some remarks on rearrangement for nonlocal functionals, Nonlinear Anal. 162 (2017), 1{12. 2
[236] H-M. Nguyen, M. Squassina, Fractional Caffarelli-Kohn-Nirenberg inequalities, J. Funct. Anal. (2017), to appear, https://doi.org/10.1016/j.jfa.2017.07.007. 2, 5, 16
[237] A. Pinamonti, M. Squassina, E. Vecchi, Magnetic BV functions and the Bourgain-Brezis-Mironescu formula, Adv. Calc. Var. (2017), to appear, https://doi.org/10.1515/acv-2017-0019. 5
[238] M. Squassina, B. Volzone, Bourgain-Brezis-Mironescu formula for magnetic operators, C. R. Math. Acad. Sci. Paris 354 (2016), 825{831. 5
[239] Shawgy Hussein and Kamal Ibrahim, Symmetry for Extremal Functions in Subcritical and Fractional Caffarelli–Kohn–Nirenberg Inequalities, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2020).