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Possibility of a Decaying Vacuum Energy Density within Generalized Field Equations Frame

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Abstract
A Desitter universe accepts only a constant vacuum energy density otherwise in an expanding universe decaying vacuum energy density was allowed. Vacuum energy density was supposed to decay into matter. A decaying vacuum energy density was found to be related to the gravitational field also it depends on the kind of the universe whether open, closed or flat. A decaying vacuum energy depending on the inverse of the square of the scale factor is related to the weak gravitational field while a power series function in the inverse of the square of the scale factor was allowed in the strong gravitational field which means that the functional form of the vacuum energy density is highly affected by the strength of the gravitational field.

المستخلص
كون ديستر يتقبل فقط وجود كثافة طاقة فضاء لا تتغير مع الزمن أما فى الكون المتمدد فيسمح بكثافة طاقة فضاء مضمحلة. من المفترض أن كثافة طاقة الفضاء تضمحل منتجة مادة. وجد أن كثافة طاقة الفضاء المضمحلة ترتبط بالمجال التثاقلى كما تعتمد على نوع الكون مغلق ، مفتوح أو منبسط. كثافة طاقة الفضاء المضمحلة المعتمدة على مقلوب مربع معامل التمدد الكونى ترتبط بالمجال التثاقلى الضعيف بينما تلك المعتمدة على سلسلة اسية من مقلوب مربع معامل التمدد الكونى ترتبط بالمجال التثاقلى الشديد مما يعنى أن صورة كثافة طاقة الفضاء المضمحلة تتأثر بشدة المجال التثاقلى.

Keywords: strong gravitational field, cosmology, lagrangian, decaying vacuum energy.

Introduction
The accelerating expansion of the universe is the most surprising cosmological discovery in many decades. Empty space or vacuum contains energy which causes the observed accelerating expansion of the universe due to its negative pressure. The gravitational effect of this energy approximates that of Einstein's cosmological constant. Nowadays this energy can be possibly termed dark energy (Peebles et al., 2002), (Tajmar, 2006) and (Cohn, 1998). Dark energy may vary slowly with time and position as it decays by the emission of ordinary matter and radiation (Bronstein, 1933). Its density decreases less rapidly than that of matter and radiation as a result it dominates the universe at late times.

Einstein's General Relativity (GR) is a self-consistent theory which dynamically describes space, time and matter under the same standard. It is capable of explaining a huge number of gravitational phenomena, ranging from laboratory up to cosmological scales. Its predictions are well tested at Solar System scales but it is not the final theory of gravitational interaction as several shortcomings emerged in the last years. One of these shortcomings is that it is not applicable in a strong gravitational field vicinity. This leads to the conclusion that new theories of gravity are needed. Many competing models have surfaced (Bellini et al., 2010), (Capozziello et al., 2011), (Carroll, et al. 2006), (Brax, et al., 2011) and (Ali, 2013), the common goal is identifying the best possible description for gravitational phenomena, while remaining faithful to the observational and experimental constraints

Within the last decades attention was given to gravitational theories of nonlinear structures some of which propose to change the Einstein–Hilbert Lagrangian to a more general form able to reproduce the same general relativity tests on solar distance scales and further justify both inflationary and current acceleration of the universe.

One of these approaches is the Generalized Field Equations (GFE) (El_Tahir, 2002), (El_Tahir, 1991), (El_Tahir, 1992) and the model based on it (Mubarak et al., unpublished data).

It is a fourth order differential equation that governs the behavior of the gravitational field. GFE can be considered as a generalization of GR. It agrees with GR in the limit of weak gravitational field and it can be applied in the limit of strong gravitational field too. The presence of a strong gravitational field results in some interesting and unusual phenomena.

The objective of this paper is to find whether there is a relation between the decaying vacuum energy density and the strong gravitational field. Also, to find the appropriate functional form of the vacuum energy density within GFE. The contracted equation of GFE is presented in section2 considering different functional forms of vacuum energy density. First, assuming a constant vacuum energy density in a Desitter universe, then a decaying vacuum energy density in an expanding isotropic universe with $a \, \alpha \, ct^n$ (Lesgourgues, 2006). Unlike some literature (Basilakos, 2018) where only constant vacuum energy density was examined, here varying functional forms for the vacuum energy density were considered and the results were compared with previous work. Section3 is devoted for the discussion of results and conclusion.

Model and Suitable Vacuum Energy Density

The contracted equation of GFE is given by (Kamal et al., 2005)

$$-\ddot{R} = \frac{\beta R + 2\gamma}{6\alpha} \tag{1}$$

where α , β are constants associated with the gravitational field, γ stands for energy density, R is the scalar curvature given as usual by

$$R = -6\left(\frac{\ddot{a}a + \dot{a}^2 + k}{a^2}\right) \tag{2}$$

$k = 0, \pm 1$ and a is the scale factor of the universe expansion. The universe is described by the Robertson-Walker metric given by (Carroll, 1997):

$$ds^2 = c^2 dt^2 - a^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3}$$

where t is the proper time. Differentiating R from (2) twice, substituting (2) in (1) also identifying ρ_T for γ one gets

$$-6\left(\frac{-8\ddot{a}\dot{a}^2}{a^3} + \frac{6\dot{a}^4}{a^4} + \frac{\ddddot{a}}{a} + \frac{\ddot{a}^2}{a^2}\right) + 12k\left(\frac{\ddot{a}}{a^3} - \frac{3\dot{a}^2}{a^4}\right) = \frac{\beta}{\alpha}\left(\frac{\ddot{a}a + \dot{a}^2 + k}{a^2}\right) - \frac{\rho_T}{3\alpha} \tag{4}$$

where the over dot stands for ordinary differentiation w.r.t. time.

The above equation will be solved for different varying functions for the vacuum energy density, as the energy density can give an indication of the history and future of the universe under investigation. In what follows these cases are shown:

Assuming the presence of constant vacuum energy density in a de Sitter-type inflationary expansion of the universe with $a = be^{mt}$ where b and m are constants. Using this relation and its derivatives in (4), one gets

$$\rho_v = 6m^2\beta \ , \quad k=0 \tag{5}$$

The concept of a constant cosmic vacuum energy density is not accepted because this concept namely would claim for a force that acts upon space-time and matter dynamics without itself being acted upon by space-time or matter. That is why it has been suggested that the vacuum energy density ρ_v is actually decaying with time a possible explanation could be an exchange of energy between matter and vacuum (Moncy et al., 2003). Here we assume the possibility of a decaying vacuum in the form of

$$\rho_v = \frac{\sigma}{a^2} \tag{6}$$

σ is a constant, (Peracaula et al., 2019), (El-Nabulsi, 2010) unlike (Borges et al., 2005) where ρ_v is a power series of H and (Opher et al., 2005) where $\rho_v = \sigma H$ .Non relativistic matter with density ρ_m is assumed to be created from vacuum , hence $\rho_T = \rho_m + \rho_v$. The conservation equation

$$\dot{\rho}_m + 3H\rho_m = -\dot{\rho}_v \tag{7}$$

was used to describe the transfer of the vacuum energy density into matter where matter is produced from vacuum and it is not independently conserved. To find the matter energy density (7) and (4) were equated, also assuming $a = ct^n$, yields

$$\dot{\rho}_m = -54\alpha H[-8n^3(n-1) + 6n^4 + n(n-1)(n-2)(n-3) + n^2(n-1)^2]t^{-4} - 9H\beta\{[-n+2n^2]t^{-2} + \frac{k}{c^2}t^{-2n}\} + 108Hk\alpha[\frac{-n-2n^2}{c^2t^{2+2n}}] - \dot{\rho}_v + 3H\rho_v \quad (8)$$

Using $H = \frac{\dot{a}}{a}$ and (6) one gets

$$\dot{\rho}_m = -54\alpha n[-8n^3(n-1) + 6n^4 + n(n-1)(n-2)(n-3) + n^2(n-1)^2]t^{-5} - 9n\beta\left\{[-n+2n^2]t^{-3} + \frac{k}{c^2}t^{-2n-1}\right\} + 108nk\alpha\left[\frac{-n-2n^2}{c^2t^{3+2n}}\right] + n\frac{5\sigma t^{-2n-1}}{c^2} \quad (9)$$

Integrating yields

$$\rho_m = \frac{27}{2}\alpha n[-8n^3(n-1) + 6n^4 + n(n-1)(n-2)(n-3) + n^2(n-1)^2]t^{-4} + \frac{9}{2}n\beta\left\{[n(n-1)+n^2]t^{-2} + \frac{k}{-2nc^2}t^{-2n}\right\} + 108nk\alpha[\frac{-n-2n^2}{(-2-2n)c^2}]t^{-2n-2} - \frac{5\sigma t^{-2n}}{2c^2} \quad (10)$$

Inserting (10) in (4) one obtains

$$3\alpha\left[48n^3(n-1) - 36n^4 - 6n(n-1)(n-2)(n-3) - 6n^2(n-1)^2\right]t^{-4} + 36\alpha k\left[n(n-1) - 3n^2\right]t^{-2-2n} = 3\beta\left[(n(n-1)+n^2)t^{-2} + \frac{k}{c^2t^{2n}}\right] - \frac{27}{2}\alpha n\left[-8n^3(n-1) + 6n^4 + n(n-1)(n-2)(n-3) + n^2(n-1)^2\right]t^{-4} - \frac{9}{2}n\beta\{\left[n(n-1)+n^2\right]t^{-2} + \frac{\mathrm{k}}{\mathrm{-2nc^2}}\mathrm{t^{-2n}}\} - 108\mathrm{nk}\alpha[\frac{\mathrm{n+2n^2}}{\mathrm{(2+2n)c^2}}]\mathrm{t^{-2n-2}} + \frac{4\sigma t^{-2n}}{2c^2} \quad (11)$$

This is a fourth order differential equation in t, α is supposed to be associated with the strong gravitational field and β with the weak gravitational field. This equation can be satisfied for different values of *n* to get a specific relation between the scale factor and time. It is satisfied for *n=1*, resulting in

$$0 = 3\beta\left[1 + \frac{k}{c^2}\right] - \frac{9}{2}\beta + \frac{4\sigma}{2c^2} + \frac{9k\beta}{4c^2} \quad (12)$$

and

$$c^2 = \frac{3k}{1+4k} \quad (13)$$

Solving (12) gives $\sigma = \frac{3\beta}{4}\left[c^2 - \frac{7k}{2}\right]$. Using (13), for an open universe *k=-1*, gives $c = \pm 1$, and $\sigma = \frac{27\beta}{8}$ hence $\rho_v = \frac{27\beta}{8t^2}$ but for a closed universe *k=1*, hence $c = \pm\sqrt{\frac{3}{5}}$ and $\sigma = \frac{-87\beta}{40}$ hence $\rho_v = \frac{-87\beta}{40t^2}$. In this case the vacuum energy density is related only to the weak field and the strong gravitational field does not show up, it also depends on the curvature. For a flat universe *k=0*, *c=0*, $\sigma = 0$ hence no decaying vacuum is allowed. Using *n=1* in (10) results in

$$\rho_m = 81\alpha\left\{1 + \frac{k}{c^2}\right\}t^{-4} + \frac{9}{2}n\beta\left\{1 + \frac{k}{-2c^2}\right\}t^{-2} - \frac{5\sigma t^{-2}}{2c^2} \qquad (14)$$

In an open universe it simplifies to $\rho_m = \frac{-27}{16}\beta t^{-2}$ which looks like (Arbab, 2003) where ρ_m decays as t^{-2} in terms of a one obtains $\rho_m = \frac{-27}{16}\beta c^2 a^{-2}$. But in a closed universe $\rho_m = 216\alpha t^{-4} + \frac{321}{80}\beta t^{-2}$ which as time goes on will look like $\rho_m = \frac{321}{80}\beta t^{-2}$ as the first term is negligible. In terms of a it gives

$$\rho_m = 216\alpha c^4 a^{-4} + \frac{321}{80}\beta c^2 a^{-2} \qquad (15)$$

Equation (11) is also satisfied for $n = \frac{1}{2}$ giving $c^2 = \frac{1}{2}$ and $\sigma = \frac{-21}{8}\beta k$ (16). Substituting the value of σ in (6) yields $\rho_v = -21\frac{\beta k}{4t}$ (17). Again, the vacuum energy density depends on the weak gravitational field alone and the curvature.

Substituting in (10) gives $\rho_m = \left\{-\frac{141\beta k}{8}\right\}t^{-1} + 36k\alpha t^{-3}$ (18)

In terms of a equation (18) becomes $\rho_m = \left\{-\frac{141\beta k}{16}\right\}\frac{1}{a^2} + \frac{18k\alpha}{a^6}$ (19)

In the limit of a $\rightarrow$ 0 or at very early times of universe, it reduces to $\rho_m = \frac{18k\alpha}{a^6}$ as the first term will be negligible unlike (Borges et al. ,2005). The ratio of vacuum density to matter energy density is calculated from (17) and (18) resulting in $\frac{\rho_v}{\rho_m} = \frac{42}{141}$ which is greater than the value obtained by (Sol`a, 2013).

Otherwise, if vacuum energy decays as a power series in the form of (¨Ozer et al., 1998)

$$\rho_v = \frac{C_v}{a^2} + \frac{C_{v1}}{a^4} \qquad (20)$$

where C_v and C_{v1} are constants, unlike vacuum which evolves as a power series of H (Lima et al., 2013). Substituting in (8) one gets the following:

$$\dot{\rho}_m = -54\alpha n[-8n^3(n-1)+6n^4+n(n-1)(n-2)(n-3)+n^2(n-1)^2]t^{-5} - 9n\beta\left[(-n+2n^2)t^{-3}+\frac{k}{c^2}t^{-2n-1}\right]+108nk\alpha\left[\frac{-n-2n^2}{c^2t^{3+2n}}\right]+\frac{nC_v t^{-2n-1}}{c^2}-\frac{nC_{v1}t^{-4n-1}}{c^4} \qquad (21)$$

Integrating

$$\rho_m = \frac{27}{2}\alpha n[-8n^3(n-1)+6n^4+n(n-1)(n-2)(n-3)+n^2(n-1)^2]t^{-4} + \frac{9}{2}n\beta\left[-n+2n^2)t^{-2}+\frac{k}{(-2n)c^2}t^{-2n}\right]+108nk\alpha\left[\frac{-n-2n^2}{(2+2n)c^2t^{2+2n}}\right]-\frac{nC_v t^{-2n}}{2nc^2}+\frac{nC_{v1}t^{-4n}}{4nc^4} \qquad (22)$$

Inserting (22) in (4) one obtains

$$3\alpha\left[48n^3(n-1)-36n^4-6n(n-1)(n-2)(n-3)-6n^2(n-1)^2\right]t^{-4} + 36\alpha k\left[n(n-1)-3n^2\right]t^{-2-2n} = 3\beta\left[(n(n-1)+n^2)t^{-2}+\frac{k}{c^2t^{2n}}\right]- \frac{27}{2}\alpha n\left[-8n^3(n-1)+6n^4+n(n-1)(n-2)(n-3)+n^2(n-1)^2\right]t^{-4} -\frac{9}{2}n\beta\{\left[n(n-1)+n^2\right]t^{-2}+\frac{k}{(-2n)c^2}t^{-2n}\} -108nk\alpha\left[\frac{n+2n^2}{(2+2n)c^2}\right]t^{-2n-2}-\frac{C_v t^{-2n}}{2c^2}-\frac{5C_{v1}t^{-4n}}{4c^4} \qquad (23)$$

When $n=1$ $\quad \frac{3}{2}\beta = \frac{21k}{4c^2}-\frac{C_v}{2c^2}$ and $-27\alpha-\alpha k(108-\frac{81}{c^2}) = \frac{5C_{v1}}{4c^4}$

$$C_v = 3c^2\beta - \frac{21k}{2}\ , \qquad C_{v1} = -\frac{1}{5}[108\alpha c^4(1+k)+81\alpha kc^2] \qquad (24)$$

Discussion and Conclusion

From equation (5) we see that vacuum energy density is a constant in a Desitter universe. As this case is not allowed in reality because an interaction is needed between spacetime and matter dynamics and a specific force, a decaying vacuum energy density with time is suggested.

Equation (10) gives the matter energy density which is created from vacuum, one sees that it is related to both weak gravitational field symbolized by β and strong gravitational field symbolized by α. It also depends on the curvature of space. In a static universe it is equal to a constant $\rho_m = \frac{5\sigma}{2c^2}$

Equation (15) shows that the scale factor and the matter density do not have the same time dependence as in the standard model. In the limit a → 0, the matter energy density reduces to

$\rho_m = 216\alpha c^4 a^{-4}$. The first term in (15) gives the scaling of matter, the second term is related, to the production of matter, at the expenses of the vacuum decay.

Also, the vacuum energy density was found to be proportional to $\frac{1}{t^2}$ in agreement with (Arbab, 2003), but the ratio of vacuum energy density to matter energy density is approximately the double of that obtained by (Sol`a, 2013).

From equation (18) one sees that the matter energy density is related to both gravitational fields and depends on them and that the first term will dominate as time passes or if we set α to zero or as if we are far away from a black hole for example. Also, if we switch off momentarily the strong gravitational field the expression simplifies to that within GR. The ratio of vacuum energy density to matter energy density in an open universe is given $\frac{\rho_v}{\rho_m} = -2$ and in a closed universe after a long time passes is $\frac{\rho_v}{\rho_m} = -\frac{58}{107}$ which is not the same as (Singh et al., 2005) and (Borges et al., 2007) where it depends on the kind of energy density.
From equation (24) one deduces that the constant C_v depends on the weak gravitational field and the curvature while the constant C_{v1} depends on the strong gravitational field and curvature, hence in this case the vacuum energy density is related to both fields and is affected by both of them. If the strong gravitational approaches zero as in the case when one is far from a strong source then the decaying vacuum energy reduces to that in the previous case.

Hence, one sees that a decaying vacuum energy depending on the inverse of the square of the scale factor as in equation (6) is allowed in an expanding universe and is found to be related to the weak gravitational field only and the scalar curvature whereas a decaying vacuum energy as a series form in the inverse of the square of the scale factor as in equation (20) is related to both weak and strong gravitational fields and the scalar curvature. Thus, one may conclude that: first the decaying vacuum energy density depends on the kind of the universe whether open, closed or flat. Second the functional form of the vacuum energy density is highly affected by the strength of the gravitational field. Other functional forms of the decaying vacuum are open for investigation.

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